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and Their Influence On

N=2 Supersymmetric Systems

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Preface

This thesis summarises work done by the author between October 1995 and May 1999 at the Department of Mathematical Sciences, University of Durham, England under the supervision of Professor David Fairlie and Dr. Valya Khoze. No part of this work has been previously submitted for any degree at this or any other university.

Chapter one serves as an introduction, and no claim is made for originality. Chapter two is also a review of previously published material. Chapter three contains original work by the author in collaboration with Valya Khoze and Michael Mattis. The material of chapter four is also claimed to be original, and was carried out in collaboration with Ivo Sachs. Chapter five was original work, but during the analysis a paper [19] making use of the same techniques appeared superceding it. As such, it should be regarded as a review chapter. Work that is not that of the author wil be properly acknowledged. The material of chapter three is published in Physics Letters B408 (1997) 213, and the work presented in chapter four is currently being prepared for publication.

Abstract

This thesis is concerned with so-called higher derivative terms which arise in low energy approximations to certain physical models. In particular, the aim is to investigate the role that such terms play in low energy N=2 supersymmetric gauge theories in 4 dimensions, with gauge group $SU(2)$.

Chapter one serves as an introduction to the notions of supersymmetry and superfields. The problem of constructing an effective action which describes the low energy dynamics is introduced, and the construction of the Wilsonian action in terms of light and heavy modes is developed. The concept on a derivative expansion is also described.

Chapter two introduces N=2 supersymmetric gauge theories with spontaneous symmetry breaking. It is observed that such systems always have a Bogomolnyi bound, and the consequences are discussed. We then develop a derivative expansion of this system in terms of N=2 superfields, drawing particular attention to the next-to-leading order derivative term (that is, those with 4 derivatives/8 fermions). The duality properties of such a term are reviewed, and their impact on the mass formula discussed. Conclusions are drawn as to their influence on the results of Seiberg and Witten.

Chapter three deals with a non-renormalisation theorem for the next-to-leading order higher derivative term proposed by Dine and Seiberg. This states that instanton contributions to such a term in massless N=2 $SU(N_c)$ gauge theories vanish when the number of flavours $N_f = 2N_c$. We prove this result using the ADHM formalism for multi-instantons in the case $N_c = 2$.

Chapter four studies the relationship between the microscopic and effective coupling in the N=2 $SU(2)$ gauge theory with 4 massless flavours.

Chapter five then examines the correspondence between N=2 gauge theories and the dynamics of fields on an M-theory 5-brane wrapped on a Riemann surface. It is shown that at quadratic order the 5-brane action is identical to that of the gauge theory, and the Riemann surface has a natural interpretation as the curve arising in Seiberg and Witten's analysis of N=2 gauge theories. We then proceed to investigate whether the higher order terms also correspond.

The final chapter contains a brief summary and a hint of further directions for research that were outside the scope of this thesis.

Acknowledgements

Contents

Chapter 1

Introduction

In the middle of the 1990's, the mathematical physics community underwent a period of great activity. It was believed that the use of certain discrete transformations – dualities – offered an opportunity to solve some of the most difficult problems in field theory utilising a conceptually satisfying and mathematically elegant procedure. The paradigm for these efforts was the study of the Coulomb branch of gauge theories with N=2 extended supersymmetry. By use of dualities in the mass spectrum and a moduli space of inequivalent vacua, Seiberg and Witten were able to deduce information concerning the strong coupling regime and, as a consequence, justify a solution to the confinement problem of (super)QCD by means of the mechanism of monopole condensation.

A crucial ingredient in this programme was the use of an effective action to describe the low energy behaviour of the system. In particular, Seiberg and Witten considered the leading order term of a low energy expansion of an SU(2) gauge theory with N=2 supersymmetry in a 4d spacetime. As will be shown below, this term is governed by a prepotential which is holomorphic in (N=2 super)fields and as such has sufficient mathematical structure to allow a thorough analysis of their nature. Beyond this leading order the scenario is more complicated. In particular the next-to-leading-order Lagrangian is given by a prepotential which is real and analytic rather than holomorphic. This lack of holomorphicity means, for example, that certain non-

renormalisation theorems valid for the holomorphic leading order term are no longer satisfied for the higher order terms.

It is the purpose of this thesis to investigate various properties of these so-called higher derivative terms, but this chapter will first expand upon some of the concepts introduced in the foregoing paragraphs which will recur in this work.

1.1 Supersymmetry: An Outline

Since its introduction in the 1970's, supersymmetry – which at its most basic level can be thought of as a symmetry between fermions and bosons – has become one of the central tenets of modern mathematical physics. Models with supersymmetric invariance appear as extensions to familiar field theories, since they typically exhibit an improved behaviour under quantisation due to cancellations between fermionic and bosonic loops. In this work, we work almost exclusively in the arena of supersymmetric extensions to Yang-Mills systems with local SU(2) gauge invariance which, by means of a Higgs-mechanism, has been broken to the Coulomb branch. It is found that this model can be concisely presented by introducing so-called superfields which contain, as co-efficients in an expansion of Grassmannian co-ordinates, component fields which can be interpreted as the higgs scalar and, in N=2 theories, the gauge field strength. As such, we shall present a general introduction to supersymmetry sufficient for our purposes in subsequent chapters. More encompassing reviews of the extremely sizeable literature concerning supersymmetry can be found in [43, 51, 94, 85]. Our conventions will be those of Wess and Bagger [92].

1.1.1 Supersymmetry Algebra

Relativistic quantum field theories are postulated to be invariant under translations, boosts and rotations. Collectively, these are termed Poincaré transformations. It is well known that boosts and rotations by themselves arise from the Lorentz group which leave the spacetime metric invariant. If we denote the generator of translations as P_μ and the generator of the Lorentz group as $M_{\mu\nu}$ then the Poincaré algebra of

infinitesimal transformations is given by

$$[P_\mu, P_\nu] = 0 \tag{1.1.1a}$$

$$[P_\mu, M_{\nu\rho}] = i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) \tag{1.1.1b}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}) \tag{1.1.1c}$$

where η is the flat (4d) spacetime metric tensor diag$(-1, 1, 1, 1)$ and $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$. The operators P_μ and $M_{\mu\nu}$ have a physical significance, since P is the 4-momentum and M is related to spin. In addition to this Poincaré symmetry, the action for a quantum field theory is typically constructed to exhibit invariance under an internal symmetry group such as isospin. These groups G are of the Lie type, and have associated Lie algebra

$$[T^a, T^b] = if^{abc}T^c \tag{1.1.1d}$$

where T^a are the generators of the internal symmetry, and f^{abc} are the structure constants for the algebra with $a = 1, \cdots, \dim G$.

In a celebrated paper, Coleman and Mandula were able to show, subject to reasonable physical assumptions on the S-matrix, that the Poincaré symmetry and the internal symmetry group can only, by themselves, appear in the same Lagrangian as a direct product structure. Mathematically, this "no-go theorem" is exhibited in the relations

$$[P_\mu, T^a] = [M_{\mu\nu}, T^a] = 0 \tag{1.1.1e}$$

A method for avoiding this non-mixing was found by Haag, Lopuszański and Sohnius. This was to introduce additional generators into the algebra which obeyed anti-commutation relations. This is achieved on the introduction of a Graded Lie-algebra. Schematically, these algebra have relations of the form

$$[B, B] = B \qquad [B, F] = F \qquad \{F, F\} = B$$

where the symbols B and F stand for bosonic and fermionic generators respectively. In our case, the F generators are elements in the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representations of the Lorentz group – more correctly of $Sl(2, \mathbb{C})$ – and B in the $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ or

(1, 1) representations. In addition, one must introduce a graded Jacobi identity of the form $[B, \{F_1, F_2\}] = \{F_1, [B, F_2]\} + \{F_2, [B, F_1]\}$. The F generators are denoted as Q^i_α and its complex conjugate $\bar{Q}^i_{\dot\alpha}$ where $i = 1, \cdots, N$ and $\alpha, \dot\alpha = 1, 2$. In this work we shall be particularly interested in the case $N = 2$. Both Q and $\bar{Q}$ are 2-component Weyl spinors. We now have all the elements required to present the supersymmetry algebra. In addition to the numbered relations above, the most general N-super-Poincaré algebra consists of

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\alpha}\} = 2\delta^{ij}\sigma^\mu_{\alpha\dot\alpha}P_\mu \tag{1.1.1f}$$

$$[Q^i_\alpha, P_\mu] = 0 \tag{1.1.1g}$$

$$[Q^i_\alpha, M_{\mu\nu}] = i\left(\sigma_{\nu\mu}Q^i\right)_\alpha \tag{1.1.1h}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta}Z^{ij} \qquad Z^{ij} = -Z^{ji} \tag{1.1.1i}$$

$$[Z^{ij}, \text{anything}] = 0 \tag{1.1.1j}$$

$$[Q^i_\alpha, T^a] = (R^a)^i_j\, Q^j_\alpha \tag{1.1.1k}$$

along with the complex conjugate relations involving $\bar{Q}$. In the above, σ^μ is a Pauli matrix, $\sigma^{\mu\nu} = \frac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)$, Z^{ij} are termed central charges and R is termed the R-symmetry matrix which rotates the fermionic generators into each other. Those relations without the Lorentz generator M form the supersymmetry algebra. In chapter 2, we shall examine relation (1.1.1i) in more detail.

A simplification of the supersymmetry algebra occurs in the case where $N = 1$. Firstly, since there are no antisymmetric matrices of dimension 1, the $N = 1$ algebra contains no central charges. Next, on studying the commutator of Q_α with relation (1.1.1d) and using the graded Jacobi identity, one sees that the $N = 1$ algebra can only support an Abelian R-symmetry. Further, use of the Jacobi identity in relation (1.1.1f) with the Abelian generator T shows that $[Q_\alpha, T] = -Q_\alpha$ and $[\bar{Q}_{\dot\alpha}, T] = \bar{Q}_{\dot\alpha}$.

To conclude this subsection we note some facts. Since in relativistic theories it is known that the mass m arises in $P^\mu P_\mu = -m^2$, relation (1.1.1g) shows that irreducible representations of the above algebra will be degenerate in mass. On the other hand, due to relation (1.1.1h) the same representations will include particles with different spin. Lastly we show that in an irreducible representation of supersymmetry, the number of fermions equals the number of bosons. We introduce a fermion

number operator $(-)^F$ defined via the eigenvalue equations $(-)^F|B\rangle = +|B\rangle$ and $(-)^F|F\rangle = -|F\rangle$ where $|B\rangle$ and $|F\rangle$ are bosonic and fermionic states respectively. By the definitions of the supersymmetry algebra above we have $(-)^F Q^i_\alpha = -Q^i_\alpha(-)^F$ with a similar relation for $\bar{Q}^i_{\dot\alpha}$. Considering relation (1.1.1f) we see $(-)^F\{Q,\bar{Q}\} = -Q(-)^F\bar{Q} + (-)^F\bar{Q}Q$ where we have neglected the indices for convenience. Now, for a non-trivial finite representation of the supersymmetry algebra we can take the trace and use the cyclicity property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ to see $\mathrm{tr}\left[(-)^F\{Q,\bar{Q}\}\right] = 0$. By (1.1.1f) this means that $\mathrm{tr}\left[(-)^F P\right] = 0$ which implies, for fixed momenta – for example the rest frame – that $\mathrm{tr}(-)^F = 0$. As such, $(-)^F$ must have an equal number of positive and negative eigenvalues meaning that the number of bosons equals the number of fermions in an irreducible representation of supersymmetry.

1.1.2 Superspace and Superfields

For physical applications it is necessary to determine representations of the supersymmetry algebra with component fields corresponding to physical degrees of freedom. These component fields can then be used to construct actions. One method to achieve this is to start with familiar Lagrangians with bosons and fermions, introduce transformations on the fields which leave the Lagrangian invariant up to total derivatives, and which also close to the supersymmetry algebra. This was the original approach of Wess and Zumino and is reviewed in great detail in [85]. A much more elegant and compact method of developing supersymmetric Lagrangians is provided by the superspace method. This construct is a generalisation of Minkowski spacetime to include anti-commuting co-ordinates, and provides a natural arena to discuss supersymmetry. Since this concept will arise throughout this thesis, we shall briefly review it.

We shall consider the case of $N = 1$. It was pointed out in the previous subsection that the supersymmetry algebra could be considered a generalisation of a Lie algebra using anti-commuting generators. In analogy with the familiar case, this motivates the exponentiation of the algebra to form a group. That is, we consider a group

element

$$\Gamma(x, \theta, \bar{\theta}) = e^{i(-x \cdot P + \theta \cdot Q + \bar{\theta} \cdot \bar{Q})}$$

where x^{μ} can be considered a spacetime co-ordinate and the parameters θ_{α} and $\bar{\theta}_{\dot{\alpha}}$ are Grassmannian co-ordinates obeying the anti-commutation relations $\{\theta_{\alpha}, \theta_{\beta}\} = \{\theta_{\alpha}, \bar{\theta}_{\dot{\alpha}}\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0$. We can now investigate the product of two of these elements. Consider multipication on the left given by

$$\Gamma(y, \zeta, \bar{\zeta})\Gamma(x, \theta, \bar{\theta}) = \Gamma(x', \theta', \bar{\theta}') \tag{1.1.2}$$

Operators do not, in general, obey $e^a e^b = e^{a+b}$. Instead one must use the Hausdorff relation $e^a e^b = e^c$ where $c = a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] + [b, [b, a]]) + \cdots$. However, relations (1.1.1a), (1.1.1f), (1.1.1g), and (1.1.1i) mean that we only have to consider up to single commutators in (1.1.2). It is then simple to see that the co-ordinates

$$x^{\mu} \mapsto x^{\mu'} = x^{\mu} + y^{\mu} - i\zeta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} - i\bar{\zeta}_{\dot{\alpha}}\sigma^{\mu}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\theta^{\alpha}$$

$$\theta_{\alpha} \mapsto \theta'_{\alpha} = \theta_{\alpha} + \zeta_{\alpha}$$

$$\bar{\theta}_{\dot{\alpha}} \mapsto \bar{\theta}'_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}} + \bar{\zeta}_{\dot{\alpha}}$$

where we have used the commutator identity $[AB, CD] = A\{B, C\}D - AC\{B, D\} + \{A, C\}DB - C\{A, D\}B$.

On substitution of the co-ordinates x^m, θ, and $\bar{\theta}$ we see that the operator

$$\zeta Q + \bar{\zeta}\bar{Q} = \zeta^{\alpha}\left(\frac{\partial}{\partial\theta^{\alpha}} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu}\right) + \bar{\zeta}_{\dot{\alpha}}\left(\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_{\mu}\right) \tag{1.1.3}$$

generates the co-ordinate transformations without space-time translations. It is a matter of simple algebra, remembering that derivatives with respect to Grassmannian co-ordinates anti-commute, to show that the operators in (1.1.3) obey

$$\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}$$

$$\{Q_{\alpha}, Q_{\beta}\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$$

so that the identification $P_{\mu} = -i\partial_{\mu}$ shows that (1.1.3) is a representation of the N=1 superalgebra in terms of differential operators. A more rigorous approach using

the concept of cosets gives the same results. In fact, superspace is the coset of the super-Poincaré group by the Lorentz group [51].

We are now in a position to introduce the notion of a superfield. A superfield $f(x, \theta, \bar{\theta})$ is a function of the co-ordinates x, θ and $\bar{\theta}$ which transforms in the fundamental representation of the supersymmetry algebra. Since θ and $\bar{\theta}$ are defined to be Grassmannian, the series expansion of the superfield will terminate at finite order. In the case of N=1 supersymmetry, the most general superfield is given by

$$
\begin{aligned}
f(x, \theta, \bar{\theta}) = {} & a(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} + \theta^2 m(x) + \bar{\theta}^2 n(x) \\
& + \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} v_\mu(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \kappa_\alpha(x) + \theta^2 \bar{\theta}^2 d(x) \quad (1.1.4)
\end{aligned}
$$

where we have not given any physical interpretation to the component space-time fields which arise as co-efficients at each order of the expansion with respect to the Grassmannian co-ordinates. By imposing constraints which are invariant under supersymmetry, one can reduce the number of degrees of freedom in (1.1.4) to any required by physical considerations. A simple example of a constraint is to impose reality on the superfield (1.1.4). That is $f = f^\dagger$. In this case, one sees that the above superfield reduces to

$$
\begin{aligned}
& C(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}(x) \bar{\psi}^{\dot{\alpha}} + \theta^2 \left[M(x) + iN(x) \right] + \bar{\theta}^2 \left[M(x) - iN(x) \right] \\
& + \theta \sigma^\mu \bar{\theta} v_\mu(x) + \bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \theta^2 \bar{\theta}^2 D(x)
\end{aligned}
$$

where now C, M, N, v_μ, and D are real. Other possible constriants involve derivatives which are covariant under supersymmetry, and lead on to chiral superfields. These shall be introduced in later chapters when required. The supersymmetry transformation of the component fields which arise as co-efficients of the expansion of the superfield are given by reading off the coefficients of the transformation

$$
\left[\zeta Q + \bar{\zeta} \bar{Q} \right] f(x, \theta, \bar{\theta}) \tag{1.1.5}
$$

using the differential operators in (1.1.3). In (1.1.5) we see how superfields transform in the vector representation of supersymmetry.

For physical applications, one is interested in constructing supersymmetry invariant actions. To this end, one can consider functionals of (constrained) superfields as the

integrands of an integral over the entire superspace. This involves integration over the Grassmannian co-ordinates θ and $\bar{\theta}$, a process known to mathematical physicists as Berezin integration. The properties of these integrals, as well as the description of supersymmetric actions developed from superfields is undertaken in chapter 3.

Finally, we remark that this discussion can be generalised to N=2 supersymmetry in 4d, but the essential idea is as above. One introduces the requisite number of Grassmannian co-ordinates, develops a differential operator representation of the supersymmetry algebra, and defines N=2 superfields as expansions in the Grassmannian co-ordinates. This is a long exercise, which is adequately dealt with in standard textbooks [94, 51]. In this thesis, we shall content ourselves with stating the relevant results. Often we shall simply work with the N=1 superspace formulation of N=2 supersymmetry, as in chapter 3. Those interested in further details of N=2 superfields should also consult that chapter.

1.2   Low Energy Derivative Expansion

An effective theory is an idea implicit in all of physics [20]. A student measuring the acceleration of an object due to the Earth's gravitational field does not need to know that the spacetime is – however slightly – curved. Instead, the postulates of Newtonian dynamics suffice. Likewise, at energies currently attainable by particle accelerators, one is not required to understand the details of string theory – or any other proposed GUT – in order to calculate the cross-section of a particular process: the standard model suffices. More generally, a basic feature of physical models is that processes occurring at low energies (or equivalently large distances) do not depend on the dynamics at high energies (small distances). Put another way, the influence of any high energy degrees of freedom on low energy processes is negligible and can be taken into account by absorbing their effect into effective vertices. All that is important to the low energy observer is that one has adequate degrees of freedom to model the system, obeying any symmetries relevant at the chosen energy scale. As in all particle physics problems, this notion can be cast into the language of Lagrangians and actions.

In applications to supersymmetry with spontaneous breaking of a gauge symmetry, a particular construction of the effective theory is useful. This is the "Wilsonian Effective Action," and it is used, for example, in Seiberg and Witten's analysis of N=2 systems. In this section, we shall develop the notion of this effective action.

1.2.1 One Particle Irreducible (1PI) Effective Action

As a prelude to describing the low energy Wilsonian effective action, we shall quickly remind the reader of another effective action: the one particle irreducible (1PI) effective action familiar from canonical presentations [24, 65, 67] of quantum field theories. This will also allow us to explain the meaning of a derivative expansion.

Let ϕ denote a field. Whilst the following analysis holds for fermions and gauge potentials, we shall consider ϕ to be a scalar. This will circumvent the need to introduce anti-commuting quantities and ghosts into the path integral: elements which only obscure the main issues. The generating functional for the field theory is defined in the path integral formalism to be

$$Z[j] \equiv \int \mathcal{D}\phi \exp\left(i \int d^4x L[\phi, \partial\phi] + j(x)\phi(x)\right) \tag{1.2.6}$$

where $j(x)$ is a source coupled to the ϕ-field and L is the Lagrangian functional. The generating functional is at the heart of analysing quantum field theories since it can be used to calculate transition amplitudes for particular processes. More technically, differentiating Z m-times with respect to the source j leads on to the m-leg Green function

$$G_m(x_1, x_2, \ldots x_m) = \langle 0|T\{\phi(x_1)\ldots\phi(x_m)\}|0\rangle \tag{1.2.7}$$

where $T\{\}$ means the ϕ are time ordered and $|0\rangle$ is the vacuum state.

The generating functional can equivalently be defined as a series in sources. That is

$$Z[j] \equiv \sum_{m=0}^{\infty} \frac{i^m}{m!} \int \left(\prod_{i=1}^{m} d^4x_i\right) j(x_1)\cdots j(x_m)G_m(x_1, \ldots, x_m)$$

Normalisation of the ground state $|0\rangle$ to unity in (1.2.7) shows that $\langle 0|0\rangle = Z[0] = 1$. Also $G_m(x_1, \ldots x_m) = G_m(x_1 + a, \ldots, x_m + a)$ due to translation invariance of the ground state. As such G_m only depends on the differences $x_i - x_j$ of the co-ordinates.

It is common to introduce a functional $W[j]$ defined as $Z[j] = \exp(iW[j])$. This $W[j]$ can be shown to generate the connected Green functions, where in this case connected means that the Feynman diagram has no subdiagrams which are not joined to the others by a propagator. Using this definition, the vacuum expectation value (vev) in the presence of an external source $j(x)$ is given by

$$u(x) \equiv \frac{\langle 0|\phi(x)|0\rangle}{\langle 0|0\rangle} = \frac{\delta W[j]}{\delta j(x)} \tag{1.2.8}$$

and the vev $\langle\phi\rangle$ is given by $\lim_{j\to 0} u(x)$. In (1.2.8), $u(x)$ is determined by the external source $j(x)$. To consider which source $j(x)$ will produce a given $u(x)$ it is convenient to introduce the Legendre transformation

$$\Gamma[u] = W[j] - \int d^4x j(x)u(x) \tag{1.2.9}$$

where $\Gamma[u]$ is termed the **effective action**. Since

$$\begin{aligned}
\frac{\delta\Gamma[u]}{\delta u(x)} &= \frac{\delta W[j]}{\delta u(x)} - j(x) - \int d^4y \frac{\delta j(y)}{\delta u(x)} u(y) \\
&= -j(x) + \int d^4y \left[\frac{\delta W}{\delta j(y)} \frac{\delta j(y)}{\delta u(x)} - \frac{\delta j(y)}{\delta u(x)} u(y) \right]
\end{aligned}$$

we can use (1.2.8) to see

$$\frac{\delta\Gamma[u]}{\delta u(x)} = -j(x)$$

Translation invariance of the vacuum state means that whenever $j = 0$, $u(x)$ must take the constant value $\langle\phi\rangle$. This means that the vev $\langle\phi\rangle$ is a root of the equation $\left. \frac{\delta\Gamma[u]}{\delta u} \right|_{u=\langle\phi\rangle}$.

One may now develop $\Gamma[u]$ as a generating functional

$$\Gamma[u] = \sum_{n=1}^{\infty} \int \left(\prod_{i=1}^{n} d^4x_i \right) u(x_1) \cdots u(x_n) \Gamma^{(n)}(x_1, \ldots, x_n) \tag{1.2.10}$$

It can be shown that $\Gamma[u]$ is the generating functional for the one particle irreducible [1PI] or proper Feynman diagrams. These are diagrams which have the property that the diagram remains connected whenever an internal line is cut. They are also defined to have no external propagators. Such diagrams are the building blocks of the quantum field theory. Without too much detail, integrations over momenta can be

carried out independently in each 1PI subdiagram of a given diagram. Connected diagrams are formed by linking 1PI diagrams by propagators, and m-leg greens functions by summing connected diagrams. 1PI diagrams are also central to the renormalisation program since if each 1PI subdiagram can be made finite, then one removes all ultraviolet divergences.

An alternative expansion of $\Gamma[u]$ is that of the **derivative expansion**

$$\Gamma[u] = \int d^4x \left[-V_{eff}(u(x)) + \frac{1}{2}\partial_\mu u(x)\partial^\mu u(x)Z_{eff}(u(x)) + \text{terms with } \geq 4 \text{ derivatives} \right]$$
(1.2.11)

which can be related to the alternative definition (1.2.10) on using the fourier transform

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \int \left(\prod_{j=1}^n d^4k_j (2\pi)^{-4} \right) \tilde{\Gamma}^{(n)}(k_1, \ldots, k_n)e^{i\sum_m k^m x_m}$$
$$\times \underbrace{\int d^4x e^{-ix\sum_m k_m}}_{\text{conservation of momenta/translation invariance}}$$

and expanding $\tilde{\Gamma}$ about vanishing momenta. Odd powers of momenta in the expansion (1.2.11) vanish due to symmetry under $k_i \leftrightarrow -k_i$, and the off-diagonal second derivative terms vanish on suitably choosing the basis of momenta vectors. The expression (1.2.11) is the position space representation of this momentum expansion. It should be pointed out that the expression (1.2.10) for the effective action is highly non-local. This means that the derivative expansion (1.2.11), which has the appearance of being localised, would be preferred when one wishes to consider scattering processes.

The standard references at the start of this subsection then show how this object, and in particular the effective potential V_{eff}, is used to define physically useful quantities in the renormalised theory such as the mass and coupling constants. It is particularly useful when considering systems in which spontaneous symmetry breaking occurs. This is the situation in which the vacuum of the theory does not share a symmetry of the lagrangian. If this symmetry is global, it leads on to massless spin-0 states: if it is local and there are gauge fields, it leads on to the Higgs mechanism and yields massive gauge particles. We shall not require this formalism in this thesis.

1.2.2 Wilsonian Effective Action

As indicated in the general introduction to this section, physical models are often implicitly low energy approximations to a more general theory. In order that this takes place, one must remove explicit dependence on the high energy degrees of freedom. This procedure is termed "integrating out" the high energy modes, and can be achieved by constructing the Wilsonian low-energy effective action as shall be schematically described below.

As previously, we shall limit our attentions to a single scalar field $\phi(x)$. We begin by observing that expressions (1.2.9) and (1.2.6) imply that, in the path integral formalism, the 1PI effective action $\Gamma[u]$ obeys

$$\exp i\Gamma[u] = \int \mathcal{D}\phi \exp\left(i\int d^4x L(\phi, \partial_\mu \phi) + j(x)\left(\phi(x) - u(x)\right)\right)$$

where again L is the Lagrangian density, and $u(x)$ the vev in the presence of the external source j. One can now split the spectrum of particle states into light and heavy modes. A mode is called "heavy" if it satisfies

$$k^2 + m^2 \geq \mu^2 \tag{1.2.12}$$

where k is the 4-momentum, m the particle mass and μ some cutoff consistent with regularisation – for example, in a regularisation scheme with a UV cutoff Λ, one must take $\mu < \Lambda$ – such that μ is much less than scale associated with the high energy physics. It should be noted that the cutoff in (1.2.12) is imposed in Euclidean space. This avoids difficulties in Minkowski space where the components of k in lightlike directions is massive, whilst k^2 remains small. Obviously, light fields are defined to be those that are not heavy.

We shall denote heavy fields as ϕ_h, and light fields as ϕ_l. The low energy Wilsonian action S_w is then defined from

$$\exp i\Gamma_w[u] = \int \mathcal{D}\phi_l e^{iS_w(\phi_l, \mu) - i\int^{1/\mu} d^4x j(x)(\phi_l(x) - u(x))}$$

where

$$\exp iS_w(\phi_l, \mu) = \int \mathcal{D}\phi_h \exp i \int_{1/\mu} d^4x L(\phi_l, \phi_h)$$

In addition to the cutoff, an approximation has been made by demanding that the heavy modes decouple from the sources. This has the consequence that heavy modes only appear in loops, and are therefore absorbed into vertices. These ideas are perhaps best illustrated by a concrete example.

Consider the massive ϕ^4 theory with a cutoff regularisation. This theory has a generating functional given by

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} \equiv \int \mathcal{D}\phi e^{-\int d^4 x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4}$$

where we have Wick rotated into Euclidean space, and ignore source terms for the sake of simplicity. We can now define the heavy and light modes as follows

$$\phi_h(k) = \begin{cases} \phi(k) & ; \epsilon\Lambda \le |k| < \Lambda \\ 0 & ; |k| < \epsilon\Lambda \end{cases} \tag{1.2.13a}$$

$$\phi_l(k) = \begin{cases} 0 & ; |k| \ge \epsilon\Lambda \\ \phi(k) & ; |k| < \epsilon\Lambda \end{cases} \tag{1.2.13b}$$

so that $\phi = \phi_l + \phi_h$. In these expressions we have introduced the momentum space representation of the fields, and have $0 < \epsilon < 1$. Then

$$Z = \int \mathcal{D}\phi_h \mathcal{D}\phi_l e^{-\int d^4 x \frac{1}{2}(\partial_\mu \phi_h + \partial_\mu \phi_l)^2 + \frac{1}{2}m^2(\phi_h + \phi_l) + \frac{\lambda}{4!}(\phi_h + \phi_l)^4}$$

Now,

$$\int d^4 x \phi_l(x)\phi_h(x) = \int d^4 x \int d^4 k_1 (2\pi)^{-4} \int d^4 k_2 (2\pi)^{-4} e^{i(k_1 + k_2)\cdot x} \phi_h(k_1)\phi_l(k_2)$$

$$= \int d^4 k_1 (2\pi)^{-4} \int d^4 k_2 \phi_h(k_1)\phi_l(k_2)\delta^{(4)}(k_1 + k_2)$$

$$= 0$$

due to the definitions in (1.2.13a, 1.2.13b). Likewise, $\int d^4 x \partial_\mu \phi_h \partial_\mu \phi_l = 0$. Using these identities, we see that the generating functional separates into

$$Z = \int \mathcal{D}\phi_l e^{-S[\phi_l]} \int \mathcal{D}\phi_h e^{-\int d^4 x \frac{1}{2}(\partial\phi_h)^2 + \frac{1}{2}m^2\phi_h^2 + \lambda\left(\frac{1}{6}\phi_h^3\phi_l + \frac{1}{4}\phi_h^2\phi_l^2 + \frac{1}{6}\phi_h\phi_l^3 + \frac{1}{24}\phi_h^4\right)}$$

$$\equiv \int \mathcal{D}\phi_l e^{-\int d^4 x L_{eff}(\phi_l, \mu)} \tag{1.2.14}$$

where L_{eff} contains only those modes with $|k| < \epsilon\Lambda$. The expression (1.2.14) makes it apparent that $L_{eff}[\phi] = L[\phi] + $ corrections. These corrections compensate for the removal of the high-energy modes by the introduction of interactions amongst the

light modes ϕ_l that were previously mediated by fluctuations of the heavy modes ϕ_h.

The integration over the heavy modes ϕ_h can be done as in other field theories. In particular, one can develop the Feynman rules for the ϕ_h fields. As such, we can define a free Lagrangian to be $\frac{1}{2}(\partial \phi_h)^2$ and the interactions to be given by the other terms involving ϕ_h in (1.2.14), including the quadratic (mass) term. Then, the free action for the heavy fields is given by

$$\int d^4k (2\pi)^{-4} \phi(k) k^2 \phi(-k)$$

where we have used the fourier transform. This can be shown in the standard way to give rise to the heavy particle propagator

$$\phi_h(k_1)\phi_h(k_2) = \frac{1}{k^2}(2\pi)^4 \delta^{(4)}(k_1 + k_2)\Theta(k_1)$$

where the step-function $\Theta(k) = \begin{cases} 1 \text{ if } \epsilon\Lambda \leq |k| < \Lambda \\ 0 \text{ otherwise} \end{cases}$ arises due to the definition of ϕ_h in (1.2.13a). The interactions can then be dealt with in perturbation theory. Since this shall not be important for the rest of this thesis, we merely point out the the light fields ϕ_l merely behave like co-efficients in the path integration over ϕ_h. As such, one will obtain contributions to the coupling constants of the ϕ_l fields due to the ϕ_h loops. These are the effective vertices, and encode the influence of the heavy modes. Full details of this analysis can be found in [67]. Finally we remark that we can treat S_w in much the same manner as in the 1PI effective action in (1.2.11). In particular, one can construct a derivative expansion which will contain only the low energy fields and powers of their derivatives. This is what we mean by the low energy derivative expansion.

Chapter 2

Influence of Next to Leading

Order Terms

The existence of topological (Bogomolnyi) bounds is a general feature of physical systems with extended supersymmetry (SUSY). It arises due to the special nature of the supersymmetric algebra as will be shown soon. Mathematically, these bounds are due to the fact that there exists a subspace – the space of BPS states – of the (graded) complex Hilbert space associated with the physical system for which every element is annihilated by a particular combination of the supercharges of the algebra. Physically they arise due to the presence of a Higgs mechanism which breaks the semi-simple gauge group of the theory to some (typically Abelian) subgroup. This mechanism leads on to a minimal energy for particles in the way highlighted by Bogomolnyi. As such, an important feature of any model we consider will be so-called flat directions; manifolds of minimal potential energy.

This chapter hopes to clarify the various issues surrounding the nature of topological bounds in theories with N=2 extended SUSY, and a semi-simple gauge group which is broken to an Abelian subalgebra by means of a Higgs mechanism. In particular the role of the Abelian sector in determining the BPS states will be exemplified, with a consequent discussion of the veracity of using a low energy effective theory to determine the quantum Bogomolnyi bound. Eventually an evaluation of the use of Bogomolnyi bounds in establishing Montonen–Olive duality in Abelianised N=2

super Yang–Mills (SYM) shall be presented.

2.1 Topological Bounds in SUSY are algebraic

A typical feature of classical field theories is that they possess greatest lower bounds on the energy of the system. These relations are termed Bogomolnyi bounds, and are usually derived by a clever manipulation of the Lagrangian associated with the model. In theories with extended SUSY, such bounds are generic since they appear as a natural consequence of the SUSY algebra. To see this consider the usual two component superalgebra introduced in chapter 1

$$\{Q_\alpha^i, (Q_\beta^j)^\dagger\} = 2\sigma_{\alpha\beta}^\mu \epsilon^{\dot\alpha\dot\beta} \delta_j^i P_\mu = 2m\delta_\alpha^\beta \delta_j^i \tag{2.1.1a}$$

$$\{Q_\alpha^i, Q_\beta^j\} = \epsilon_{\alpha\beta} Z^{ij} \quad \text{where } Z^{ij} = -Z^{ji} \tag{2.1.1b}$$

$$\{(Q_\alpha^i)^\dagger, (Q_\beta^j)^\dagger\} = \epsilon^{\alpha\beta} Z_{ij}^* \tag{2.1.1c}$$

where we are assuming that the four momentum P_μ is related to the mass m through $p^2 = -m^2$. Since the particles have non-vanishing rest mass, relation (2.1.1a) is presented in the rest frame defined by $P_\mu = (-m, 0, 0, 0)$. In the above relations $\mu = 1, \ldots, 4$ is a Lorentz index, $\alpha, \beta = 1, 2$ are $Sl(2, \mathbb{C})$ indices and $i, j = 1, \ldots, N$ are the internal symmetry indices describing SUSY. As before, the supercharge Q_α^i is a two component Weyl spinor whilst Z^{ij} is a central charge of the algebra. We have also used a representation of the Pauli matrices in which $\sigma^0 = -\mathbb{I}_2$.

We follow Wess and Bagger [92] to investigate the representation of this algebra. Since the central charges Z^{ij} commute with all of the generators of the algebra, and in particular amongst themselves, there exists a similarity transformation which simultaneously diagonalises these operators such that they have eigenvalues Z^{ij}. Due to the anti-symmetry relation in (2.1.1b) we can construct an $N \times N$ anti-symmetric matrix whose entries are these eigenvalues. Standard texts on linear algebra then tell us that we can choose a basis such that this anti-symmetric matrix takes a standard form. This is achieved, in the usual manner, by an orthogonal transformation

$$\tilde{Z}^{ij} = U_m^i U_n^j Z^{mn}$$

In the case where N is even, this standard form is given by the direct product

$$\tilde{Z} = \epsilon \otimes D$$

where ϵ is a 2×2 antisymmetric matrix with $\epsilon^{12} = 1$ and D is a diagonal matrix with real entries Z_A, $A = 1, \ldots \frac{1}{2}N$. This result can be justified once we realise that every anti-symmetric matrix can be written in the form $\epsilon \otimes M$ and then we perform the diagonalisation procedure to M. In component form

$$\tilde{Z}^{ij} = (\epsilon \otimes D)^{ij} = (\epsilon \otimes D)^{aA,bB} \equiv \epsilon^{ab} D^{AB}$$

where we have decomposed the indices $i = (a, A)$ and $j = (b, B)$ with $a, b = 1, 2$ and $A, B = 1, \ldots, \frac{1}{2}N$. Obviously $\tilde{Z}^{ij} = -\tilde{Z}^{ji}$.

In order to be consistent, we must also transform the supercharges as

$$\tilde{Q}^i_\alpha = U^i_j Q^j_\alpha$$

Rewriting the superalgebra in this equivalent representation, we have

$$\{\tilde{Q}^{aA}_\alpha, (\tilde{Q}^{bB}_\beta)^\dagger\} = 2m\delta^\beta_\alpha \delta^a_b \delta^A_B \tag{2.1.2a}$$

$$\{\tilde{Q}^{aA}_\alpha, \tilde{Q}^{bB}_\beta\} = \epsilon_{\alpha\beta} \epsilon^{ab} \delta^{AB} Z_B \tag{2.1.2b}$$

$$\{(\tilde{Q}^{aA}_\alpha)^\dagger, (\tilde{Q}^{bB}_\beta)^\dagger\} = \epsilon^{\alpha\beta} \epsilon_{ab} \delta_{AB} Z_B \tag{2.1.2c}$$

We can now introduce an isomorphic algebra by defining the supercharges to be a linear combination of

$$r^A_\alpha \equiv \frac{1}{\sqrt{2}} \left[\tilde{Q}^{1A}_\alpha + \epsilon_{\alpha\beta} (\tilde{Q}^{2A}_\beta)^\dagger \right]$$

$$s^A_\alpha \equiv \frac{1}{\sqrt{2}} \left[\tilde{Q}^{1A}_\alpha - \epsilon_{\alpha\beta} (\tilde{Q}^{2A}_\beta)^\dagger \right]$$

and their hermitean conjugates $(r^A_\alpha)^\dagger$ and $(s^A_\alpha)^\dagger$. Using the fact that

$$(r^A_\alpha)^\dagger = \frac{1}{\sqrt{2}} \left[(\tilde{Q}^{1A}_\alpha)^\dagger - \epsilon^{\alpha\rho} \tilde{Q}^{2A}_\rho \right]$$

and $\epsilon^{12} = 1$ we see that the commutation relations (2.1.2a–2.1.2c) give

$$\begin{aligned}
\{r^A_\alpha, (r^B_\beta)^\dagger\} &= m\delta^A_B \delta^\alpha_\beta - m\epsilon_{\alpha\rho}\epsilon^{\beta\sigma}\delta^\rho_\sigma \delta^A_B - \frac{1}{2}\epsilon^{12}\delta^{AB} Z_B \epsilon_{\alpha\sigma}\epsilon^{\beta\sigma} + \frac{1}{2}\epsilon_{\alpha\rho}\epsilon^{\rho\beta}\epsilon_{21}\delta^{AB} Z_B \\
&= (2m + Z)\delta^A_B \delta^\beta_\alpha \tag{2.1.3a}
\end{aligned}$$

Similarly

$$\{r_\alpha^A, r_\beta^B\} = \{s_\alpha^A, s_\beta^B\} = \{r_\alpha^A, s_\alpha^B\} = 0 \qquad (2.1.3b)$$

$$\{s_\alpha^A, (s_\beta^B)^\dagger\} = \delta_\alpha^\beta \delta_B^A (2m - Z_n) \qquad (2.1.3c)$$

$$(2.1.3d)$$

Taking $\alpha = \beta$ and $A = B$ in (2.1.3a) and (2.1.3c) we see that the inequalities $2m + Z_n \geq 0$ and $2m - Z_n \geq 0$ arise. Multiplying these together, we wee that

$$2m \geq |Z_n|^2 \quad \text{for all } n \qquad (2.1.4)$$

Since m is the mass, we seen that the supersymmetry algebra (2.1.1a–2.1.1c) generically gives rise to a Bogomolnyi bound. Thus, as promised, we have shown that a Bogomolnyi bound arises in supersymmetry without the need to recourse to a particular Lagrangian in a manner to be exemplified later. In theories with extended supersymmetry and their attendant central charges, we see the Bogomolnyi bound is an algebraic consequence. In the next section we shall look at the more interesting case of a physical example, and shall find that it is rigorous to talk of a quantum bound in systems with extended supersymmetry.

Finally we remark that the relations presented in (2.1.3a–2.1.3c) have the form of an algebra of 2N fermionic creation and annihilation operators. The representations of such a construct are well known, and are formed by acting on the vacuum $|0\rangle$ defined by $r_\alpha^A |0\rangle = s_\alpha^A |0\rangle = 0$. Non-vacuum states are formed by the application of the creation operators $(r_\alpha^A)^\dagger$ and $(s_\alpha^A)^\dagger$ leading on to a set of 2^{2N} states due to the anticommutivity shown in (2.1.3b). However, if the Bogomolnyi bound is saturated, some of these states will be represented trivially and the dimension of the representation is correspondingly less. In particular, if there are r relations of the type $2m = \pm Z_n$ the dimension of the representation is reduced to $2^{2(N-r)}$.

2.2 Quantum Bogomolnyi Bounds I

Having shown that a Bogomolnyi bound arises in systems with extended SUSY on purely algebraic grounds, it is of interest to investigate a particular model. This will

justify our occasional use of the term "topological bound" and also allow a discussion of the persistence such bounds show under perturbative and non–perturbative quantum effects.

Due to its application to Montonen–Olive duality in Abelianised N=2 SYM [80], we shall consider an N=2 Super-Yang-Mills (SYM) theory with semi–simple gauge group $\mathcal{G}$ broken by a Higgs mechanism to an Abelian subgroup $\mathcal{H}$. In particular we take $\mathcal{G} = SU(2)$ and $\mathcal{H} = U(1)$. In this context N=2 SYM means that the field content forms a representation for both the gauge group (colour) and N=2 supersymmetry. In practice this means the model contains an N=1 chiral superfield Φ, containing the Higgs boson, in the same (adjoint) representation as a field strength superfield W_α, containing the field strength $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + ig\,[v_\mu, v_\nu]$ with v_μ the gauge connection of the theory. This latter superfield is related to a (real) vector superfield V by $W_\alpha = -\frac{1}{4}\bar{D}^2 \exp(-V)D_\alpha \exp(V)$. The classical Lagrangian expressed in N=1 superspace for such a system is known [80] to be:

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im} \int d^2\theta d^2\bar{\theta}\, \mathrm{tr}_\mathcal{G}\Big(A \exp(2gV)\,\bar{A} + \frac{\tau}{2}W^\alpha W_\alpha\, \delta^2(\bar{\theta})\Big) \qquad (2.2.5)$$

where the complexified coupling constant $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$, Θ being the theta vacuum angle and g the usual gauge coupling. Using the techniques of chapter 3 the above expression can be reduced into component form, but the only important aspect of this for our purposes is the appearance of a superpotential Σ:

$$\Sigma \sim \mathrm{tr}_\mathcal{G}\left[\bar{\phi}, \phi\right]^2 \quad \text{where} \quad \phi(x) = A(x, \theta)|_{\theta=0} \qquad (2.2.6)$$

This object describes a vacuum manifold defined by the vanishing of Σ. In the case of $SU(2)$ this occurs only when ϕ and $\bar{\phi}$ both lie in the Cartan subalgebra. Such directions in the group manifold along which the superpotential vanish are generically termed "flat directions." That these classical manifolds of minimal energy persist even on quantisation is well known. The proof relies on the fact that the only way to generate a superpotential is to break the N=2 SUSY, a situation which cannot occur dynamically [53]. This powerful result will allow us to speak of a quantum Bogomolnyi bound.

Following Olive and Witten [75] it is a matter of tedious manipulation to proceed from the component form of equation (2.2.5) to derive explicit expressions for the

central charge. We shall outline the method. Beginning with the component form of (2.2.5) one constructs the Noether current J_α^μ associated with N=1 SUSY transformations in the usual way, remembering to differentiate out the spinor parameter $\zeta_{(1)}$ associated with the supersymmetry. One then proceeds to get (2.1.1b) by means of:

$$\left\{ Q_\alpha^2, Q_\beta^1 \right\} = \frac{\partial}{\partial \zeta_{(2)}^\alpha} \left[\zeta_{(2)} \cdot Q^2 + \bar{\zeta}_{(2)} \cdot \bar{Q}^2, Q_\beta^1 \right] = \frac{\partial}{\partial \zeta_{(2)}^\alpha} \int d^3x \Delta_{\zeta_{(2)}} J_\beta^0 \equiv \epsilon_{\alpha\beta} Z$$

where Δ_ζ represents the infinitesimal transformation. The eventual result, on reading off terms, is that the central charge

$$Z = \text{tr}_{SU(2)} \int d^3x\, \partial_i \left(E_i \phi + B_i \bar{\phi} \right) \tag{2.2.7}$$

where the "electric field" $E_i^a = v_{0i}^a$ and the "magnetic field" $B_i^a = \frac{1}{2}\epsilon_{0ijk} v_{jk}^a$, with v the $su(2)$–valued curvature. We shall provide a more detailed calculation in the following section.

Equation (2.2.7) has several important consequences. Firstly, it is a divergence term and so we may use Stokes theorem to consider a surface integral at spatial infinity. Since E and B both decrease asymptotically as $\frac{1}{|x|^2}$ for large $|x|$, non-zero Z requires $\phi(\infty) \neq 0$. This in turn implies that the vacuum given by the necessary condition $\Sigma = 0$ is determined by the flat directions with non–zero vacuum expectation value. So central charges imply non–zero vacuum expectation value.

Secondly, the only components of the electric and magnetic fields important to the central charge are precisely those that permeate the vacuum. These are the gauge fields associated with the Cartan subalgebra of the original gauge group, since only these remain massless under the Higgs mechanism. In the case of $SU(2)$ this subgroup is $U(1)$ and so expression (2.2.7) has an interpretation as the electric and magnetic charges of the theory along the direction picked out[1] by the Higgs mechanism. This is the meaning of "topological bound": at least part of the central charge has a topological origin (the magnetic charge). In fact it has been written that the central charge is always associated with a topological current [86]. This is due

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[1] In the sense that, for example, the electric charge is associated with U(1) rotations about the axis picked out by the Higgs at spatial infinity

to an observation that the two supercurrents, the energy–momentum tensor and a topological current (all non-anomalous) form a representation of the SUSY algebra, and the time–component of these currents can be integrated over space to obtain conserved charges; $Q^1_\alpha, Q^2_\alpha, P_\mu$ and Z. Although it may be objected that the electric charge is not topological in origin, it can at least be written as a boundary term. Note that the electric charge must be calculated using a Noether procedure (the global symmetry being $U(1)$ rotations about the Higgs at large spatial distances), and is not the same as the gauge coupling.

We now come on to the most fundamental part of the arguments to be presented in this chapter; the consideration of what happens to the expression for the central charge and the topological bound when one quantises the system. The classic answer was given by Witten and Olive [80] and is based upon the representation theory of SUSY algebras. In the case of N=2, representations can have either $2^{2N} = 16$ states when massive or $2^N = 4$ when massless. The difference arises since in the latter case half of the SUSY generators are represented trivially. That states saturating the Bogomolnyi bound (2.1.4) form a short (that is massless) representation is seen by forming particular complex linear combinations of the supercharges, constructing an algebra isomorphic to (2.1.1b) et cetera, and noting that certain anticommutators vanish [92] when the mass is related to the central charge as in (2.1.4) with the inequality replaced by equality. There are then fewer non-trivial SUSY generators, and so the dimension of the representation on states is correspondingly less. This was shown in section 2.1.

It follows that this must be true even when quantisation has occurred, a process which is not expected to generate twelve new states. The Abelian fields automatically form a massless representation of SUSY, and so present themselves as a natural choice to describe the quantum BPS bound. The same cannot be said for those containing the massive $W^\pm$ particles which tend to zero at large distances. This idea has great implications for this chapter. In determining properties related to the topological bound we need only consider processes whose only external particles are the Abelian fields and their super-partners. This means we can realistically consider an Abelianised version of the full gauge theory – that is a low energy effective action

with the cutoff below the mass of the other gauge fields – to give us quantitative information of the Bogomolnyi bound of the model. We consider this later, but first give a more direct proof of the ubiquity of the Abelian fields in the Bogomolnyi bound.

2.3 Quantum Bogomolnyi Bounds II

It is possible to conclude that the Bogomolnyi bound for the N=2 SYM theory with gauge group SU(2) arises solely from the Abelian fields by more conventional means. This shall be done by constructing the bound directly. Our starting point is the most general N=2 supersymmetric action with local gauge symmetry [79]. This obviously includes the quantised system, and is:

$$S = \frac{1}{4\pi} \text{Im} \int d^4y \, d^2\theta \, d^2\bar{\theta} \left[\frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^a} (\exp 2gV)^{ab} \, \bar{\Phi}_b + \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial \Phi^a \partial \Phi^b} W^{a|\alpha} W_\alpha^b \delta^2(\bar{\theta}) \right]$$

(2.3.8)

where the $N = 1$ superfields have component content:

$$\Phi(y) = \phi(y) + \sqrt{2}\theta \cdot \psi(y) + \theta^2 F$$

$$\bar{\Phi}(y) = \phi^\star - 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^\star + \ldots$$

$$V(y) = V_{wz}(y) = -\theta\sigma^\mu\bar{\theta}v_\mu + i\theta^2\bar{\theta} \cdot \bar{\lambda} - i\bar{\theta}^2\theta \cdot \lambda + \frac{1}{2}\theta^2\bar{\theta}^2(\mathcal{D} - i\partial_\mu v^\mu)$$

$$W_\alpha = -\frac{1}{4}\bar{D}^2 \exp(-V)D_\alpha \exp(V) = -\frac{1}{4}\bar{D}^2(D_\alpha + \frac{1}{2}[D_\alpha V, V])$$

$$= -i\lambda_\alpha + \left(\delta_\alpha^\beta \mathcal{D} - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha^\beta v_{\mu\nu}\right)\theta_\beta + \theta^2\sigma_{\alpha\dot\alpha}^\mu\partial_\mu\bar{\lambda}^{\dot\alpha}$$

We have worked with chiral co-ordinates $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ for ease, and the Wess–Zumino gauge [92] since it provides the simplification $(V_{wz})^n = 0$ for all integers $n \geq 3$. Each of the fields is $su(2)$ valued. For instance, $A = A^a T_a$ where T_a is a generator of the Lie algebra associated with the gauge group. Lower case latin letters denote isospin indices. Also, the field strength $v_{\mu\nu}^a = \partial_\mu v_\nu^a - \partial_\nu v_\mu^a + g\epsilon^{abc}v_\mu^b v_\nu^c$ for SU(2) with vector potential v_μ and gauge coupling g. Of the other component fields ϕ is a scalar "Higgs", ψ is the higgsino, λ the gaugino, and $\mathcal{D}$, F auxiliary (non-propagating) fields introduced to ensure that the number of bosons equals the number of fermions off-shell.

Using these expressions it is possible to reduce (2.3.8) to the component action simply by looking for terms which transform as total derivatives under SUSY. It is well known that these are the $\theta^2\bar{\theta}^2$ terms in (2.3.8):

$$S = -\frac{1}{8\pi}\text{Im}\int d^4x \, \frac{\partial^2\mathcal{F}(\phi)}{\partial\phi^a\partial\phi^b}\left[\frac{1}{4}\left(v + i\star v\right)^a_{\mu\nu}\cdot\left(v + i\star v\right)^{\mu\nu|b} + \nabla_\mu\phi^a\bar{\nabla}^\mu\phi^{\star\,b}\right] \quad (2.3.9)$$

with covariant derivative $\nabla^{ab}_\mu = \partial_\mu\delta^{ab} + ig\epsilon^{abc}v^c_\mu$ and $\star v_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}v^{\lambda\kappa}$ the Hodge dual of $v_{\mu\nu}$ such that $\star^2 = -1$. In (2.3.9) we have written down only the contributions of the bosonic degrees of freedom[2]. This is because once one knows these terms, those parts with the fermionic and auxiliary degrees of freedom can be deduced using supersymmetric transformations. We have also neglected the expression for the superpotential since we will be interested in the physics on the vacuum manifold.

The derivation of a Hamiltonian from (2.3.9) can be determined by the usual method of coupling the lagrangian density to a non-flat background metric tensor h and varying with respect to this field to get the energy–momentum tensor $T_{\mu\nu}$ as in general relativity:

$$
\begin{aligned}
S[h] &= \int d^4x \, \sqrt{\det h} \, \mathcal{L}[h; v_\mu, \phi, \phi^\star] \quad \text{so} \\
\Delta_h S &= \int d^4x \, \sqrt{\det h} \, \Delta h^{\mu\nu} \, T_{\mu\nu}
\end{aligned}
$$

Performing this well–known calculation gives a result extremely similar to the classical case since $\mathcal{F}(\phi)$ does not depend on the spacetime metric. This fact will lead to many similarities between the classical and quantum versions of the Bogomolnyi bound. Introducing the electric vector $E_i = v_{0i} = -E^i$ and the magnetic vector $B_i = (\star v)_{0i} = -B^i$ with $i = 1, 2, 3$ we see, since the Hamiltonian $H = \int d^3x \, T_{00}$:

$$H = \frac{1}{16\pi}\text{Im}\int d^3x \, \frac{\partial^2\mathcal{F}}{\partial\phi^a\partial\phi^b}\left[E^a_i E^b_i + B^a_i B^b_i + 2\nabla_i\phi^a\bar{\nabla}_i\phi^{\star b} + 2\nabla_0\phi^a\bar{\nabla}_0\phi^{\star b}\right] \quad (2.3.10)$$

From this energy functional it is possible to construct a Bogomolnyi bound in the usual manner. First we must attempt to construct a first order differential equation (the Bogomolnyi equation). It is usual to consider the gauge $\nabla_0\phi = 0$ and we see

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[2]It should be noted that in deriving this action we have used the result $\nabla_\mu\mathcal{G}(\phi) = \frac{\partial\mathcal{G}}{\partial\phi^a}\nabla_\mu\phi^a$. This is clearly a covariant version of the usual chain rule for derivatives, and is derived in appendix A.

no reason to break this habit. Unlike the classical case this **does not** mean we are considering configurations with $\nabla_i E_i = 0$, since the presence of the $\partial^2 \mathcal{F} / \partial \phi \partial \phi$ prefactor modifies the equations of motion [21]. From (2.3.10) we get:

$$
\begin{aligned}
H = \; & \frac{1}{16\pi} \mathrm{Im} \int d^3 x \, \frac{\partial^2 \mathcal{F}}{\partial \phi^a \phi^b} \Omega_i^a \bar{\Omega}_i^b \\
& - \frac{\sqrt{2}}{16\pi} \mathrm{Im} \int d^3 x \frac{\partial^2 \mathcal{F}}{\partial \phi^a \partial \phi^b} \left((B + iE)_i^a \, \bar{\nabla}_i \phi^{\star b} + (B - iE)_i^a \, \nabla_i \phi^b \right)
\end{aligned}
$$

with $\Omega_i^a = B_i^a + iE_i^a + \sqrt{2} \nabla_i \phi^a$ the Bogomolnyi operator for this system. The Bogomolnyi saturated states are then given by $\Omega_i^a = 0$, $\nabla_0 \phi = 0$ and vanishing superpotential. It remains to determine what their energy is. After some tedious manipulation of the last term in the above equation using the trick of "adding zero" one eventually obtains:

$$
H_\infty = \frac{1}{\sqrt{2}} \mathrm{Im} \int d^3 x \partial_i \left[\pi_{i|a}(\phi^a) + \frac{1}{4\pi} B_{i|a} \left(\frac{\partial \mathcal{F}}{\partial \phi^a} \right) \right] \tag{2.3.11}
$$

which relies on the fact that $\pi_{i|a} = \frac{\partial \mathcal{L}}{\partial(\partial_0 v_i^a)}$, the momentum density conjugate to v_i^a, obeys $\nabla_i \pi_i^a = 0$ when $\nabla_0 \phi^a = 0$. The symbol H_∞ is intended to express that this is the contribution to the Hamiltonian from the boundary terms at large distances. Using Stokes theorem on (2.3.11) we see that it is a surface integral on the boundary at spatial infinity. This means that the only fields which contribute to the Bogomolnyi bound are those with infinite range: precisely the states which remain massless under the Higgs mechanism. For $SU(2)$ this is the degree of freedom in the Cartan subalgebra, justifying our assertion that the Bogomolnyi bound is given by the abelian fields.

Lastly, we introduce the (quantum) vacuum expectation values of ϕ and the "dual" field $\phi_D = \frac{\partial \mathcal{F}}{\partial \phi^a} T_a$ as:

$$
\lim_{|x| \to \infty} \phi(x) = a \qquad\qquad \lim_{|x| \to \infty} \phi_D(x) = a_D
$$

so that we obtain the useful expression for the central charge

$$
Z = H_\infty = \sqrt{2} \left(n_e a + n_m a_D \right) \tag{2.3.12}
$$

where $n_e a = -\frac{1}{2} \mathrm{Im} \int dS^i \pi_{i|a} \phi^a$ and $n_m a_D = -\frac{1}{8\pi} \mathrm{Im} \int dS^i B_{i|a} \phi_D^a$ so that the Bogomolnyi bound on the mass M is as in (2.1.4).

2.4 Abelianised N=2 SYM with Gauge Group SU(2)

As pointed out previously, the Bogomolnyi bound for N=2 super-Yang-Mills arises from the Abelian fields in the Cartan subalgebra of the subgroup. It therefore makes sense to consider a model derived from the full N=2 super-Yang-Mills lagrangian, formally obtained by integrating out all of the massive modes and their related super-partners in a manner similar to that presented in chapter 1. The calculation of the effective action by this method would be extremely lengthy, involving a large number of terms. However, a simple counting method will allow us to construct the derivative expansion of the low energy Wilsonian action.

The on-shell component fields in an N=2 super-Yang-Mills theory are the Higgs scalar ϕ, its higgsino superpartner ψ, the gauge potential v_μ, and its gaugino super-partner λ. The N=2 transformations assemble them into a "diamond" structure as in figure 3.1. Off-shell one has to introduce auxiliary fields D and F to ensure that the number of bosonic equals the number of fermionic degrees of freedom, thereby preserving supersymmetry. The off-shell representation of N=2 supersymmetry can then be encoded in an N=2 superfield, much as illustrated in chapter 1. In particular, the component fields enter as the coefficients of an expansion in two Grassmannian co-ordinates θ_1 and θ_2. This N=2 superfield shall be denoted as $\mathbb{A}$ and its properties – in particular any constraints – detailed in chapter 3. Then

$$\mathbb{A} \supset \{[\phi, \psi, F], [D, \lambda, v_\mu]\} \tag{2.4.13}$$

where the groupings are in terms of N=1 superfields, as will be shown later.

We now determine an expansion in terms of derivatives of the component fields in (2.4.13). Since we require that the expansion be N=2 supersymmetric, this is most easily done working with Abelian N=2 superfield in N=2 superspace. The general action in such a superspace is given by

$$S(\mathbb{A}, \bar{\mathbb{A}}) = \int d^4x \int d^2\theta_1 d^2\theta_2 d^2\bar{\theta}_1 d^2\bar{\theta}_2 L(\mathbb{A}, \bar{\mathbb{A}})$$

where L is a lagrangian density depending on fields and their derivatives. Let us introduce a counting scheme which will allow us to express this action in its

derivatives. If we denote $[\alpha]$ as the order of the operator α, then it is clear we must take $[\partial_\mu] = 1$. It is now possible to construct the order of any other operator. The differential representation (1.1.3) of the superalgebra (2.1.1a) indicates

$$\left[\frac{\partial}{\partial\theta^i_\alpha}\right] = \left[\frac{\partial}{\partial\theta^{\dot{\alpha}}_i}\right] = \frac{1}{2} = -\left[\theta^\alpha_i\right] = -\left[\bar{\theta}^i_{\dot{\alpha}}\right]$$

Likewise, since $\mathbb{A}$ is an expansion in θ_i with lowest component ϕ, we see $[\mathbb{A}] = [\phi]$. Since ϕ has no derivatives, we take $[\phi] = 0$. For consistency with the dimension of the supersymmetry generators, we require $[\psi] = [\lambda] = \frac{1}{2}$ and $[v_\mu] = 1$. For ease of reference, we tabulate the order of derivatives of various quantities

Operator	Dimension	Operator	Dimension
∂_μ	1	d^4x	-4
θ	$-\frac{1}{2}$	$\bar{\theta}$	$-\frac{1}{2}$
$d^2\theta$	1	$\frac{\partial}{\partial\theta}$	$\frac{1}{2}$
$\mathbb{A}, \bar{\mathbb{A}}$	0	ϕ	0
ψ, λ	$\frac{1}{2}$	v_μ	1

Using this counting scheme, we see that the most general term of lowest order in the derivative expansion is given by

$$\int d^4x \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} F(\mathbb{A}) + \text{c.c.} \qquad (2.4.14)$$

as found by Seiberg [79]. If we let n be the order, then this term has $n = -2$. This leading order term is that investigated by Seiberg and Witten. The functional $F(\mathbb{A})$ is said to be holomorphic due to the lack of any dependence on $\bar{\mathbb{A}}$, and is often termed the "holomorphic prepotential".

Now, a consequence of N=2 supersymmetry is that there are no odd n terms in the expansion. As such, the next to leading order term expressed in N=2 superspace is at $n = 0$, and is given by

$$\int d^4x \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} d^2\bar{\theta}_{\underline{1}} \bar{\theta}_{\underline{2}} H(\mathbb{A}, \bar{\mathbb{A}}) \qquad (2.4.15)$$

where now $H(\mathbb{A}, \bar{\mathbb{A}})$ is a real functional. In particular it is not holomorphic. This object is referred to as the "Henningson prepotential," and it encodes all of the next-to-leading order corrections in the derivative expansion of low energy effective N=2 super-Yang-Mills.

One can now proceed to construct the derivative expansion order by order, but as this thesis will be concerned primarily with the next-to-leading order term we shall simply note that the expansion becomes one in terms of (super)derivatives of the N=2 superfield $\mathbb{A}$, and that a Bianchi identity (2.4.20) means that (2.4.14) is the only holomorphic term that is possible.

2.4.1   Leading Order Term and Seiberg Witten

The leading order term (2.4.14) was used by Seiberg and Witten in their examination of N=2 supersymmetric gauge theories [80, 81]. As such, we review some features of this term and their use in Seiberg and Witten's analysis. The interested reader eager for more details should consult one of the many pedagogical reviews of this topic [17], and in particular the rigorous appraisal given in [45].

The generality of the leading order term in the low energy effective action (2.4.14) means that it can be used to determine any quantum corrections to the classical action which arises from the prepotential

$$F(\mathbb{A}) = \frac{1}{2}\tau \mathbb{A}^2$$

where τ is the complexified coupling constant. As such, one sees that

$$\tau(\langle \phi \rangle) = \left. \frac{\partial^2 F}{\partial \mathbb{A}^2} \right|_{\mathbb{A}=\langle \phi \rangle} \tag{2.4.16}$$

where $\langle \phi \rangle$ is the vacuum expectation value (vev) of the Higgs scalar. This can be shown, using the method of reduction introduced in chapter 3, to generalise directly to the quantum case. As such, the holomorphic prepotential F encodes information relating to the coupling constant.

One can now proceed to evaluate the quantum corrections to this classical result [14]. As noted in chapter 1, all supersymmetric theories have a (classical) $U(1)_R$ symmetry. In terms of the component fields, this acts as

$$(\psi_\alpha, \lambda_\alpha) \mapsto e^{i\alpha}(\psi_\alpha, \lambda_\alpha) \qquad \phi \mapsto e^{2i\alpha}\phi$$

where α is the $U(1)_R$ charge. In terms of the N=2 superfield $\mathbb{A}$ this symmetry looks

like

$$\mathbb{A} \mapsto \mathbb{A}' = e^{2i\alpha} \mathbb{A} \left(x, e^{-i\alpha} \theta_{\underline{1},\underline{2}} \right) \tag{2.4.17}$$

so that the classical action will be invariant if the grassmannian co-ordinates transform as $\theta \mapsto e^{i\alpha}\theta$ as in chapter 1. Consider then the Lagrangian which has the correct form to reproduce the component form of the N=2 super-Yang-Mills action

$$L(\mathbb{A}) = \frac{1}{16\pi} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} F(\mathbb{A})$$

Under the symmetry (2.4.17) we see this transforms as

$$\begin{aligned} L(\mathbb{A}') &= \frac{1}{16\pi} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} F \left(e^{2i\alpha} \mathbb{A}(x, e^{-i\alpha}\theta) \right) \\ &= \frac{1}{16\pi} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} e^{-4i\alpha} F \left(e^{2i\alpha} \mathbb{A}(x, \theta) \right) \end{aligned}$$

which for an infinitesimal transformation means

$$\begin{aligned} L + \delta L &= \frac{1}{16\pi} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} (1 - 4i\alpha) \left(F(\mathbb{A}) + 2i\alpha \mathbb{A} \frac{\partial F}{\partial \mathbb{A}} \right) + O(\alpha^2) \\ &= \frac{1}{16\pi} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} \\ &\qquad \left[1 + 4i\alpha \left(-1 + \mathbb{A}^2 \frac{\partial}{\partial(\mathbb{A}^2)} \right) + O(\alpha^2) \right] F(\mathbb{A}) \tag{2.4.18} \end{aligned}$$

where we have used the result $\frac{\partial}{\partial \mathbb{A}} = \frac{\partial \mathbb{A}^2}{\partial \mathbb{A}} \frac{\partial}{\partial \mathbb{A}^2} = 2\mathbb{A}\frac{\partial}{\partial \mathbb{A}^2}$. It has been shown in the literature that the $U(1)_R$ symmetry has an axial anomaly, and in particular that the one-loop result for $SU(2)$ is $\delta L = -\frac{\alpha}{4\pi^2} v_{\mu\nu} v^{\mu\nu}$. Using the method of reduction, this arises from an N=2 superspace expression $-\frac{\alpha}{8\pi^2} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} \mathbb{A}^2$. Comparing this with (2.4.18) we see

$$\frac{\alpha}{4\pi} \text{Im} \int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} \left[i \left(-1 + \mathbb{A}^2 \frac{\partial}{\partial(\mathbb{A}^2)} \right) F + \frac{1}{2\pi} \mathbb{A}^2 \right] = 0$$

so that the holomorphic prepotential $F(\mathbb{A})$ obeys the differential equation

$$\frac{\partial}{\partial(\mathbb{A}^2)} \left[\frac{1}{\mathbb{A}^2} F(\mathbb{A}) \right] = \frac{1}{2\pi} \frac{1}{\mathbb{A}^2}$$

almost everywhere. This equation has solution

$$F(\mathbb{A}) = \frac{i}{2\pi} \mathbb{A}^2 \ln \frac{\mathbb{A}^2}{\Lambda^2} \tag{2.4.19}$$

where the integration constant Λ can be fixed by the value of the coupling constant at a subtraction point. As such, it is renormalisation scheme dependent [30]. The expression (2.4.19) is the perturbative correction to the holomorphic prepotential F. A theorem of Novikov et. al. [17] shows that there are no higher loop corrections to the holomorphic prepotential of the *Wilsonian* effective action. As such, all that remains are the non-trivial non-perturbative corrections to the prepotential. These arise from instantons and can be put in the form of an (instanton) series

$$\sum_{n=1}^{\infty} F_n \left(\frac{\Lambda}{\mathbb{A}} \right)^{4n} \mathbb{A}^2$$

where the co-efficients F_n are determined by direct instanton calculation. This series can be justified by noting that the regularisation scale Λ obeys $\Lambda^b = \mu^b e^{-\frac{8\pi^2}{g^2(\mu)}}$ where b is the negative of the beta-function which, in the theories under consideration, is given by $\beta = N_f - 2N_c$ where $N_c = 2$ is the number of colours and $N_f = 0$ is the number of flavours. As such, $b = 4$. The exponential is the typical weighting factor for a single instanton, and μ is a characteristic scale of the theory which is taken to be the vev $\langle \phi \rangle$. Supersymmetrisation introduces $\mathbb{A}$ in place of $\langle \phi \rangle$ and the $\mathbb{A}^2$ factor appears from the semi-classical saddle-point approximation used in instanton calculations.

This then gives a complete characterisation of the quantum holomorphic prepotential in the low energy effective action of $N = 2$ super-Yang-Mills.

Seiberg and Witten Theory: A Brief Summary

Seiberg and Witten's analysis of N=2 supersymmetric gauge theories with spontaneous breaking of gauge symmetry is, at its core, really a characterisation of the quantum moduli space of inequivalent vacua. In such systems, the classical theory has a potential (2.2.6) for the complex scalar Higgs field ϕ which is in the adjoint representation of the gauge group. As remarked earlier, the vacuum is defined by the vanishing of this potential, a condition which is satisfied when the field ϕ is in the Cartan subalgebra of the gauge group. In the case of $SU(2)$, this subalgebra

has rank 1 so that the vacuum – or flat direction – can be chosen to be

$$\phi \equiv \frac{1}{2}a\sigma^3 \qquad \text{where } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } a \in \mathbb{C}$$

An examination of the kinetic term for the Higgs scalars in the microscopic theory $\frac{1}{2}|D_\mu\phi|^2$, where D_μ is the covariant derivative, on expansion about the vacuum state $a \neq 0$ shows that the gauge bosons lying outside the Cartan subalgebra develop a mass. Further, the gauge symmetry is broken to $U(1)$. In the case of $SU(2)$ this is the familiar Higgs mechanism giving rise to the massless photon and two massive gauge bosons $W^\pm$.

The Weyl group links physically equivalent vacua, and in the case of $SU(2)$ this means that the vacua states with a and $-a$ are indistinguishable. This requires the the co-ordinate of the classical moduli space of inequivalent vacua in $SU(2)$ super-Yang-Mills to be

$$u = \langle \text{tr } \phi^2 \rangle = \frac{1}{2}a^2$$

where $\langle \cdot \rangle$ means the vacuum value. In the classical theory, the point $u = 0$ corresponds to reinstating the full $SU(2)$ gauge symmetry and the massive particles becoming massless. As such, there exists a singularity in the moduli space at this point, so that the classical moduli space has the topology of a complex plane with a puncture at the origin.

As was remarked earlier, the flat directions persist under quantisation due to the general properties of N=2 supersymmetric gauge theories. As such, it is valid to enquire as to what way perturbative and non-perturbative corrections affect the classical moduli space. Of particular interest are the existence, or otherwise, of a global parameterisation of the moduli space and whether the punctures in the moduli space alter location and/or number. This is at the heart of Seiberg and Witten theory. Since the analysis of that work [80] will not form a crucial role in this thesis, we shall content ourselves with stating their main conclusions.

They first observe that the σ-model metric on the moduli space appearing as the co-efficient of the kinetic terms for the scalars in the effective theory is given by Imτ, where τ has been defined previously as the complexified coupling. The holo-

morphicity of τ in (2.4.16) means that this metric is a harmonic function and cannot then be positive definite. As such, the moduli space will have to be reparameterised in the strong coupling regime. Their observation is that a natural candidate for this reparameterisation is in terms of the dual variable $a_D = F'(a)$. Next they note the appearance of a singularity in the weak coupling regime of the quantum moduli space. Due to asymptotic freedom in N=2 $SU(N_c)$ gauge theories with N_f flavours and $2N_c > N_f$ they note that this regime corresponds to large a. In this range, the logarithm (2.4.19) in $F(a)$ dominates, and so circuits about the point $u = \infty$ lead on to non-trivial monodromy in the variables a and in particular a_D. This then shows a modification to the classical moduli space. Topological arguments, especially homotopy, then imply that there are at least two other singularities in the moduli space. Due to the axial anomaly in the classical $U(1)_R$ invariance, there is a symmetry $u \leftrightarrow -u$ in the moduli space. As such, Seiberg and Witten make the minimal assumption that there are precisely two more punctures in the quantum moduli space. The physical interpretations of these singularities is that they arise due to massive particles which were integrated out of the weak coupling regime of the effective theory becoming massless at values of u which correspond to the punctures. Seiberg and Wittens proposal is that the magnetic monopole which arises in Yang-Mills-Higgs systems becomes massless at one of the strong coupling singularities. Using arguments involving the monodromy group, they then show that the other puncture arises from a dyon.

The moduli space can then be given a geometrical interpretation. It has been shown [45] that the moduli space is uniquely given by the quotient manifold $\mathcal{H}/\Gamma_2$ where $\mathcal{H}$ is the upper half plane and

$$\Gamma_2 = \{M \in \mathrm{Sl}(2, \mathbb{Z}) : M \equiv \mathbb{I}_2(\bmod 2)\}$$

a modular subgroup. This leads on naturally to a Riemann surface which gives a positive definite metric τ as the ratio of the periods of this curve.

2.4.2 The Role of Higher Derivative Terms

There are at least two ways in which the higher order terms can affect the analysis of Seiberg and Witten, or they can fail to exhibit an $Sl(2, \mathbb{Z})$ duality. They can affect the Bogomolnyi bound in such a way that the conjecture that singularities in the moduli space of vacua comes from monopoles/dyons becoming massless is violated. We shall indicate that neither of these scenarios arise.

An N=2 chiral superfield $\mathbb{A}$ is constrained by the Bianchi identity

$$D^{\alpha|i}D^j_\alpha \mathbb{A} = \bar{D}^i_{\dot\alpha}\bar{D}^{\dot\alpha|j}\bar{\mathbb{A}} \tag{2.4.20}$$

which has been shown to be solved by the Mezincescu relation

$$\mathbb{A} = \bar{D}^4 D^{\alpha|i}D^j_\alpha \mathcal{V}_{ij} \tag{2.4.21}$$

in the abelian case. Here $\mathcal{V}_{ij}$ is a superfield constrained to be real and symmetric. The $i, j = 1, 2$ are internal SUSY indices, and the D_α are the usual N=1 supercovariant derivatives (since we are in a massless theory). We also have the chirality relation $\bar{D}\mathbb{A} = 0$. We examine these constraints in chapter /refch3.

The above constraint (2.4.20) can be imposed in the partition function on the introduction of a Lagrange multiplier in the path integral[3]

$$
\begin{aligned}
Z &= \int \mathcal{D}\mathcal{V}_{ij} \exp iS(\mathbb{A}, \bar{\mathbb{A}}) \\
&= \int \mathcal{D}\mathbb{A}\,\mathcal{D}\bar{\mathbb{A}}\,\mathcal{D}\mathcal{V}'_{ij} \\
&\quad \exp i\left(S(\mathbb{A}, \bar{\mathbb{A}}) + \int d^4x d^4\theta d^4\bar\theta\, \mathcal{V}'_{ij}\left(D^{\alpha|i}D^j_\alpha \mathbb{A} - \bar{D}^i_{\dot\alpha}\bar{D}^{\dot\alpha|j}\bar{\mathbb{A}}\right)\right)
\end{aligned} \tag{2.4.22}
$$

---

[3]In general, the action $S(\mathbb{A}, \bar{\mathbb{A}})$ can contain as many orders of derivatives as we please. That is we can have derivative interactions. In the context of classical mechanics, a Lagrangian L with functional dependence $L(\phi, \partial_{\mu(1)}\phi, \ldots, \partial_{\mu(1)\ldots\mu(n)}\phi; x)$ has phase space co-ordinates $\{\phi, \partial_{\mu(1)}\phi, \ldots, \partial_{\mu(1)\ldots\mu(n-1)}\phi\}$ and momenta co-ordinates. Quantum mechanically, the path integral measure could contain terms associated with these degrees of freedom, and the question arises what the actual measure will be. Letting $q_i = \partial_{\mu(1)\ldots\mu(i)}\phi$, these co-ordinates are not independent. We have derivative constraints of the form $q_{i+1} = \partial_{\mu(i+1)}q_i$ so we introduce these into the path integral via Lagrange multipliers and get

$\int \left(\prod_{i=0}^n \mathcal{D}q_i\right) \exp\left(L\left(\{q_i\}; x\right)\right) \prod_{i=1}^n \delta(q_i - \partial_{\mu(i)}q_{i-1})$

so that the measure reduces down to $\mathcal{D}\phi$ and the Lagrangian is as above.

where $\mathcal{V}'_{ij}$ is the N=2 superfield lagrange multiplier which is symmetric (since $D^{\alpha|i}D^j_\alpha = D^{\alpha|j}D^i_\alpha$ by the usual commutation rule for supercovariant derivatives) and real. This insertion plays much the same role as when one wishes to integrate over field strengths in the path integral rather than gauge potentials in gauge theories. Performing integration by parts:

$$\int d^4x d^4\theta d^4\bar{\theta} \; \mathcal{V}'_{ij} \left(D^{\alpha|i}D^j_\alpha \mathbb{A} - \bar{D}^i_{\dot{\alpha}}\bar{D}^{\dot{\alpha}|j}\bar{\mathbb{A}} \right) = \int d^4x d^2\theta \; \bar{D}^4 D^{\alpha|i} D^j_\alpha \mathcal{V}'_{ij} \, \mathbb{A} + \text{c.c.}$$

so that the partition function (2.4.22) reduces to

$$Z = \int \mathcal{D}\mathcal{V}'_{ij} \exp iS'(\mathbb{A}', \bar{\mathbb{A}}')$$

where we have introduced the new field $\mathbb{A}' = \bar{D}^4 D^{\alpha|i} D^j_\alpha \mathcal{V}'_{ij}$ which is an N=2 chiral superfield in comparison with (2.4.21). The action S' is given by a variant on the usual Fourier transform

$$\exp iS'(\bar{\mathbb{A}}', \bar{\mathbb{A}}') = \int \mathcal{D}\mathbb{A}\mathcal{D}\bar{\mathbb{A}} \exp i \left(S(\mathbb{A}, \bar{\mathbb{A}}) + \int d^4x d^2\theta \mathbb{A}'\mathbb{A} + \text{c.c.} \right) \qquad (2.4.23)$$

Such an operation is often termed an S-transformation of the action. It should be noted that this is not a symmetry of the theory. The fields are different, as is the functional form of the action. In fact, there is no non-trivial field redefinition which reinstates the original action. The S-transformation is often termed a weak-strong duality since it often relates a theory at strong coupling with another at weak coupling. Since the S–duality is a Fourier transform we might expect that applying two such transformations will give us back the original theory. This indeed occurs. Instigating the S operation twice on (2.4.22) yields

$$\exp iS''(\mathbb{A}'', \bar{\mathbb{A}}'') = \int \mathcal{D}\mathbb{A}\mathcal{D}\bar{\mathbb{A}}\mathcal{D}\mathbb{A}'\mathcal{D}\bar{\mathbb{A}}' \exp i \left(S(\mathbb{A}, \bar{\mathbb{A}}) + \int d^4x d^2\theta \mathbb{A}'\mathbb{A} + \mathbb{A}'\mathbb{A}'' \right)$$

Doing the integration over $\mathbb{A}'$ and $\bar{\mathbb{A}}'$ gives the result that

$$\exp iS''(\mathbb{A}'', \bar{\mathbb{A}}'') = \exp iS(-\mathbb{A}'', -\bar{\mathbb{A}}'')$$

so that apart from the trivial field definition $\mathbb{A} \to -\mathbb{A}$ (which are related by a Weyl reflection in the full gauge theory and so are physically equivalent), we see that $S^2 \simeq I$ where I is the identity transformation.

We now introduce the notion of a T-transformation. In ordinary Yang-Mills this corresponds to shifting the Θ vacuum angle by 2π. This is unobservable. Its analogue in the N=2 superfield formalism is to add into the action in (2.4.22) the term

$$\int d^4x d^4\theta \frac{1}{2} \mathbb{A}\mathbb{A} + \int d^4x d^4\bar{\theta} \frac{1}{2} \bar{\mathbb{A}}\bar{\mathbb{A}}$$

since the reduction of this expression to the N=0 component fields has in it a part $\frac{1}{32\pi} v^{\mu\nu} \star v_{\mu\nu}$ with $v_{\mu\nu}$ the abelian field strength. From ordinary Yang-Mills this is precisely the term which shifts the vacuum angle by 2π. It is now possible to consider the effect of the composition of S and T transformations on equation (2.4.22)

$$\begin{aligned}
\text{T} &:= \exp i\left(S(\mathbb{A}, \bar{\mathbb{A}}) + \int d^4x d^2\theta \frac{1}{2} \mathbb{A}\mathbb{A} + \text{c.c.} \right) \\
\text{ST} &:= \exp iS_1(\mathbb{A}_1, \bar{\mathbb{A}}_1) \\
&= \int \mathcal{D}\mathbb{A}\mathcal{D}\bar{\mathbb{A}} \exp i\left(S(\mathbb{A}, \bar{\mathbb{A}}) + \int d^4x d^4\theta \left(\frac{1}{2}\mathbb{A}\mathbb{A} + \mathbb{A}\mathbb{A}_1 \right) + \text{c.c.} \right) \\
(\text{ST})^3 &:= \exp iS_3(\mathbb{A}_3, \bar{\mathbb{A}}_3) \\
&= \int \mathcal{D}\mathbb{A}\mathcal{D}\bar{\mathbb{A}}\mathcal{D}\mathbb{A}_1\mathcal{D}\bar{\mathbb{A}}_1\mathcal{D}\mathbb{A}_2\mathcal{D}\bar{\mathbb{A}}_2 \\
&\qquad \exp i\left[S(\mathbb{A}, \bar{\mathbb{A}}) + \int d^4x d^4\theta \ldots \right. \\
&\qquad \left. \ldots \left(\frac{1}{2} (\mathbb{A}\mathbb{A} + \mathbb{A}_1\mathbb{A}_1 + \mathbb{A}_2\mathbb{A}_2) + \mathbb{A}\mathbb{A}_1 + \mathbb{A}_1\mathbb{A}_2 + \mathbb{A}_2\mathbb{A}_3 \right) + \text{c.c.} \right]
\end{aligned}$$

Doing the $\mathbb{A}_1$ followed by the $\mathbb{A}_2$ integration implies that

$$\exp iS_3(\mathbb{A}_3, \bar{\mathbb{A}}_3) = \exp iS(\mathbb{A}_3, \bar{\mathbb{A}}_3)$$

so $(ST)^3 \simeq I$.

Now, the two relations $S^2 = 1$ and $(ST)^3 = 1$ are the defining relations of the group $Sl(2, \mathbb{Z}) = \{m \in M_2(\mathbb{Z}) | \det(m) = 1\}$ which describes the modular transformations on the upper half plane of complex numbers. If we were speaking in terms of the matrix group, the S and T transformations would be the generators of the group. As a result of this, the action in (2.4.22) exhibits an $Sl(2, \mathbb{Z})$ duality. Since this action was general enough to contain higher derivatives, we have shown that one of our mechanisms for violating the results of Seiberg and Witten does not arise.

We now turn to the influence of higher order terms on the Bogomolnyi bound of the system. Since the low energy effective action is also N=2 supersymmetric it has such

a bound by the results of section 2.1. Furthermore, since these higher derivatives give the results for processes up to a scale $1/\mu^n$ for n-th order term, this Bogomolnyi bound must agree with that in (2.3.12) at least to the same accuracy. We remind the reader that μ is the energy above which modes are integrated out of the theory. A direct calculation of this bound[4] as in section 2.3 or of the central charge as outlined in section 2.2 using component fields is not feasible. This is because the higher derivative action has an extremely large number of terms. For example the next to leading order term has O(500) pieces in terms of N=0 component fields. It is for this reason that we look for a short cut. This will be the fact that the higher derivative terms exhibit an $Sl(2, \mathbb{Z})$ duality.

First of all we indicate that the term associated with the electric degrees of freedom are precisely the same as those in (2.3.12) so that higher derivatives do not contribute. We follow a clever trick of Seiberg and Witten. Consider coupling matter minimally to the super Yang-Mills fields appearing in (2.3.8). This is done by introducing an N=2 hypermultiplet which transforms in the fundamental representation of the gauge group. Such superfields contain (s)quarks in the N=0 language. In the N=1 superfield formalism, they contain two chiral superfields M_f and $\tilde{M}^f$. Here we shall specialise to the number of flavours $N_f = 1$ and so drop the f label. Of course $\tilde{M}$ transforms in the conjugate representation of M. In addition to the canonical kinetic terms, there is an N=2 superpotential

$$\Sigma_{matter} = \sqrt{2} A M \tilde{M} + \eta M \tilde{M}$$

where A is the N=1 chiral superfield in the N=2 vector multiplet. In this equation, the first term looks like a Yukawa coupling between the Higgs and the quarks and the latter is a mass term for the hypermultiplet introduced by hand. We can arrange

---

[4]One has to take care in calculating the Hamiltonian for higher derivative Lagrangians. As a simple example consider the classical Lagrangian in 1+1 Cartesian space with explicit dependence $L(x, x', x''; t)$ with prime denoting time derivative. This has degrees of freedom $q_1 = x$ and $q_2 = x'$. The corresponding equation of motion derived from a variational principle is $\frac{\partial L}{\partial x} = \frac{\partial}{\partial t} \frac{\partial L}{\partial x'} - \frac{\partial^2}{\partial t^2} \frac{\partial L}{\partial x''}$. This means that the conjugate momenta are clearly $p_1 = \frac{\partial L}{\partial q_2'}$ and $p_2 = \frac{\partial L}{\partial q_2} - \frac{\partial}{\partial t} \frac{\partial L}{\partial q_2'}$. A suitable Hamiltonian is these $H = p_1 q_1' + p_2 q_2' - L$ with the constraint $q_1' = q_2$. This may affect the naive expectation of the Hamiltonian being given by $H = \frac{\partial L}{\partial x'} x' - L$. Care must be taken should you be so inclined to want the Hamiltonian.

the mass η in such a way that the matter is not integrated out when we construct the corresponding Wilsonian theory below the energy of the massive gauge bosons. This being the case, the mass of the quarks can be read off the Lagrangian as $|\sqrt{2}a + \eta|$ where the vacuum expectation value a is as in (2.3.12). Putting η to zero we see that the fundamental electric particles have mass $|\sqrt{2}a|$ when we perform the Higgs mechanism so that the Bogomolnyi bound for electrically charged particles in the Wilsonian theory is

$$M \geq \sqrt{2}|n_e a|$$

where the number of electric units is n_e. Note that higher derivative terms cannot affect this result since we have read the mass term off of the Lagrangian directly.

It is not possible to perform such a manipulation for the magnetically charged particles since these are not elementary excitations that explicitly appear in the Lagrangian. We must try to derive them by alternative means.

Decomposing the general Wilsonian action $S(\mathbb{A}, \bar{\mathbb{A}})$ with all orders of derivatives into

$$S(\mathbb{A}, \bar{\mathbb{A}}) = S_h(\mathbb{A}, \bar{\mathbb{A}}; \frac{1}{\mu}) + \int d^4x d^4\theta \mathcal{F}(\mathbb{A}) + \text{c.c.}$$

where $\mathcal{F}$ is the leading order prepotential in the low energy expansion in powers of $\frac{1}{\mu}$ and S_h contains all other terms, we perform an S-transformation as in (2.4.23) to get

$$\exp iS'(\mathbb{A}', \bar{\mathbb{A}}') = \exp iS_h(-i\frac{\delta}{\delta \mathbb{A}'}, -i\frac{\delta}{\delta \bar{\mathbb{A}}'}) \exp i \int d^4x d^4\theta \mathcal{F}'(\mathbb{A}') + \text{c.c.} \qquad (2.4.24)$$

The dual prepotential $\mathcal{F}'(\mathbb{A}')$ follows from

$$\exp i \int d^4x d^4\theta \mathcal{F}'(\mathbb{A}') = \int \mathcal{D}\mathbb{A} \exp i \int d^4x d^4\theta \left(\mathcal{F}(\mathbb{A}) + \mathbb{A}\mathbb{A}' \right) \qquad (2.4.25)$$

In (2.4.24) we have performed a manipulation much as in the derivation of perturbative Feynman rules in field theories. Introducing the dual N=2 chiral superfield

$$\mathbb{A}_D = \frac{\delta \mathcal{F}}{\delta \mathbb{A}}(\mathbb{A})$$

we assume that $\mathbb{A}_D$ assumes every value once and only once. It then possible to evaluate (2.4.25) by expansion about the unique stationary point given by

$$\frac{\delta}{\delta \mathbb{A}} (\mathcal{F}(\mathbb{A}) + \mathbb{A}\mathbb{A}') = 0 \quad \Rightarrow \quad \mathbb{A}_D = \frac{\delta \mathcal{F}}{\delta \mathbb{A}} = -\mathbb{A}' \qquad (2.4.26)$$

Since $S^2 \simeq I$ we let the value of $\mathcal{A}$ corresponding to this point be $\mathbb{A} = \mathbb{A}'_D$ so that $\frac{\delta \mathcal{F}}{\delta \mathbb{A}}(\mathbb{A}'_D) + \mathbb{A}' = 0$. Usually an expansion about the stationary point would factorise the value of the integrand[5] and there would be a non-trivial integral to perform.

In the case of (2.4.25) the presence of supersymmetry gives a simplification. We can make a holomorphic change of variables such that the associated *super*jacobian is unity. The super-jacobian is used to preserve the supersymmetric nature of the integrand. To illustrate this point we consider two simpler cases. First consider a model with one Grassmann superspace co-ordinate θ. A chiral superfield Φ will have the θ expansion

$$\Phi = \phi + \sqrt{2}\theta\psi$$

with ϕ a boson and ψ a fermion. We instigate a holomorphic change of variable by

$$\Phi' = g(\Phi) = g(\phi) + \sqrt{2}\theta\psi\frac{\delta g}{\delta\phi} = \phi' + \sqrt{2}\theta\psi'$$

where g is a holomorphic functional. The super-jacobian for this is given by

$$\text{Sdet}\left(\frac{\delta\Phi'}{\delta\Phi}\right) = \text{Sdet}\begin{vmatrix} \frac{\delta g}{\delta\phi} & \psi\frac{\delta^2 g}{\delta\phi^2} \\ 0 & \frac{\delta g}{\delta\phi} \end{vmatrix} = 1$$

by the formula $\text{Sdet}^{-1}\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{\det(D - CA^{-1}B)}{\det A}$ where A, B, C, and D are respectively bose-bose, fermi-bose, bose-fermi and fermi-fermi submatrices.

Now let Φ be an N=1 chiral superfield with the usual θ expansion $\Phi = \phi + \sqrt{2}\theta^\alpha\psi_\alpha + \theta^2 F$ using chiral co-ordinates. The Grassmann parameter θ is a two component Weyl spinor, and F is an auxiliary field. Performing the change of variables

$$\Phi' = g(\Phi) = g(\phi) + \sqrt{2}\theta^\alpha\psi_\alpha\frac{\delta g}{\delta\phi} + \theta^2\left(F\frac{\partial g}{\partial\phi} + \frac{\delta^2 h}{\partial\phi^2}\psi^\alpha\psi_\alpha\right)$$

we see

$$\text{Sdet}\left(\frac{\delta\Phi'}{\delta\Phi}\right) = \text{Sdet}\begin{vmatrix} \begin{pmatrix} \frac{\delta g}{\delta\phi} & 0 \\ \left(F\frac{\delta^2 g}{\delta\phi^2} + \frac{\delta^3 g}{\delta\phi^3}\psi^\alpha\psi_\alpha\right) & \frac{\delta h}{\delta\phi} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ \frac{\delta^2 g}{\delta\phi^2}\psi_1 & \frac{\delta^2 g}{\delta\phi^2}\psi_2 \end{pmatrix} \\ \begin{pmatrix} \frac{\delta^2 g}{\delta\phi^2}\psi_1 & 0 \\ \frac{\delta^2 g}{\delta\phi^2}\psi_2 & 0 \end{pmatrix} & \begin{pmatrix} \frac{\delta g}{\delta\phi} & 0 \\ 0 & \frac{\delta g}{\delta\phi} \end{pmatrix} \end{vmatrix} = 1$$

---

[5]For instance in 1D Cartesian space $\int dx \exp f(x) = \exp(f(x_0))\int dx \exp\left(\sum_{n=2}^\infty \frac{d^n f}{dx^n}\big|_{x_0}\frac{(x-x_0)^n}{n!}\right)$ where $\frac{df}{dx}\big|_{x_0} = 0$ and x_0 is the stationary point.

as above.

By the extension of these results to the N=2 superfield formalism we can transform the seemingly intractable (2.4.25) to a Gaussian integral. This has the same number of extrema and can be given the same numerical value at this point. Consequently the result at the stationary point $\mathbb{A} = \mathbb{A}'_D$ is all that matters. The dual potential then becomes

$$\mathcal{F}'(\mathbb{A}') = \mathcal{F}(\mathbb{A}'_D) + \mathbb{A}'_D \mathbb{A}'$$

with $\frac{\delta \mathcal{F}'}{\delta \mathbb{A}'} = \mathbb{A}'_D = \mathbb{A}$. It follows that $\frac{\delta \mathcal{F}}{\delta \mathbb{A}}(\frac{\delta \mathcal{F}'}{\delta \mathbb{A}'}(\mathbb{A}')) = \frac{\delta \mathcal{F}}{\delta \mathbb{A}}(\mathbb{A}) = -\mathbb{A}'$. This can be represented more succinctly as

$$\begin{pmatrix} \mathbb{A} \\ \mathbb{A}_D \end{pmatrix} \xrightarrow{\text{S}} \begin{pmatrix} \mathbb{A}' \\ \mathbb{A}'_D \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{A} \\ \mathbb{A}_D \end{pmatrix} = \begin{pmatrix} -\mathbb{A}_D \\ \mathbb{A} \end{pmatrix}$$

Inspecting (2.4.24) a simple dimensional analysis [60] shows the higher derivative terms in the Wilsonian action can only be transformed to similar terms in the dual theory. This is extremely important for us since it implies that the mass terms in the dual Lagrangian will come purely from the leading order term $\mathcal{F}'(\mathbb{A}')$. The higher derivatives do not contribute. By the previous analysis on the mass formula of the elementary electric excitations we can say that the corresponding particles in the dual theory will have a mass given by $M = \sqrt{2}|a_D|$ with a_D as in (2.3.12). These modes are the "magnetic monopoles" so that the mass formula

$$M = \sqrt{2}|n_e a + n_m a_D|$$

for dyons in the low energy Lagrangian is precisely the same as in the full theory. The higher order derivative terms do not affect the mass formula at all. As a result our second route to violating Seiberg and Witten's analysis does not bear fruit.

2.5 Conclusions

This chapter has introduced the derivative expansion in $N = 2$ supersymmetric gauge theories, and used the duality properties of such a construct to consider possible modifications to the conclusions of Seiberg and Witten's work on the quantum

moduli space of the low energy effective model. A few papers [70, 97, 95] have attempted to study the higher derivative terms in a more systematic manner. Due to calculational difficulties, their attention focused on the next-to-leading order term. Perhaps the most ambitious attempt to synthesise the knowledge about this term was undertaken by Matone [70]. That author took the known behaviour of the Henningson prepotential $H(\mathbb{A}, \bar{\mathbb{A}})$ at weak coupling – that its one-loop contribution [95] was proportional to $\ln \mathbb{A} \ln \bar{\mathbb{A}}$ and that it should have m-instanton/n-anti-instanton contributions [97] of the form $(\Lambda/\mathbb{A})^{-4m}(\Lambda/\bar{\mathbb{A}})^{-4n}$ where Λ is the Wilsonian cut-off. This information arises from direct calculations, and there is no reason to doubt its validity. In addition, Matone also assumed that the weak coupling singularity due to the logarithm was the only one for $H(\mathbb{A}, \bar{\mathbb{A}})$ and that the zeroes of that functional are precisely at the locations in the moduli space where the singularities in the leading order term appeared (that is, where the once massive monopole becomes massless). His justification implicitly assumed that this was required to ensure that the results of Seiberg and Witten were not altered by the higher derivative term. However, the results of this chapter seem to indicate that this is not the case: the higher derivative terms do not, a priori, affect the conclusions of Seiberg and Witten, even if the Henningson prepotential H does not vanish at the punctures in the quantum moduli space. Likewise, it is debatable whether the behaviour at weak coupling is enough to determine the global form of H. This is because the prepotential at next-to-leading order is non-holomorphic, meaning that the concept of analytic continuation is not well-defined. As such, whilst Matone's conjecture as to the form of $H(\mathbb{A}, \bar{\mathbb{A}})$ may be correct, this would be more fortuitous than one might hope.

As it stands, one might hope to check Matone's results by comparing the expansion of his formula with higher loop contributions to H. We note, however, that there have been indications that these quantum effects are null. In particular, it has been shown that the two-loop contribution vanishes [52].

Finally, we remark that due to the calculational difficulties and their lack of influence on the physics, there is no interest in determining the explicit form of the higher order terms other than for completeness.

Chapter 3

Higher Derivatives and Instantons

One of the most satisfying features of supersymmetric theories is that they often yield non-renormalisation theorems. By considering the symmetries of the problem, Dine and Seiberg [36] were able to develop such a theorem for the next to leading order term in the low energy expansion of certain supersymmetric theories on the Coulomb branch. In particular, they were able to demonstrate that the non-holomorphic prepotential $\mathcal{H}(\mathbb{A}, \bar{\mathbb{A}})$ in $N = 2$ supersymmetric gauge theories with vanishing β-function is one-loop exact: it recieves no quantum corrections from higher orders in perturbation theory, nor from non-perturbative phenomena such as instantons.

This chapter aims to validate this non-renormalisation theorem for the instanton contributions to $\mathcal{H}$. As such, we shall be required to obtain the interaction vertices which appear in the component form of the next to leading order term of the low energy expansion. Once these are obtained, we shall be able to investigate instanton contributions to this prepotential utilising the ADHM method for multi-instantons.

3.1   Reduction of the Superfield Action

In this section we shall derive component terms in the low energy expansion of the N=2 super Yang–Mills action. This information will enable us to investigate the role of instantons in such theories. We shall utilise the method of projection (REF:Rocek

et al) to remove all explicit dependence on fermionic super-co-ordinates that appear in the Lagrangian originally expressed in terms of superfields.

3.1.1 Normalisation of the Superspace Measure

We shall follow the conventions of Wess and Bagger [92] when dealing with indices. Berezin integration of a single Grassmannian variable is defined as

$$\int d\theta \, \theta = 1$$

Now consider the equivalent expression in an N=1 superspace

$$\int d^2\theta \, \theta\theta = N \int d\theta^\alpha \epsilon_{\alpha\beta} d\theta^\beta \, \theta^\gamma \epsilon_{\gamma\delta} \theta^\delta$$

where N is a normalisation constant to be fixed, and $\epsilon_{\gamma\delta}$ is the anti-symmetric tensor which relates equivalent spinor representations. The indices $\alpha, \beta, \dots$ take the values 1 or 2. In essence, it can be thought of as an artifice for raising and lowering spinor indices. It is defined such that $\epsilon_{12} = -1$. Then

$$\int d^2\theta \, \theta\theta = 4N \int d\theta^1 d\theta^2 \, \theta^1\theta^2 = -4N$$

where we have used the anti-commuting property of Grassmannian variables. If we now demand the convention that $\int d^2\theta \, \theta\theta = 1$ [92, pp 62] then we must normalise the measure via $N = -1/4$. In an abuse of notation we may write

$$d^2\theta = -\frac{1}{4} d\theta^\alpha \epsilon_{\alpha\beta} d\theta^\beta$$

Similary

$$d^2\bar\theta = -\frac{1}{4} d\bar\theta_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} d\bar\theta_{\dot\beta}$$

$$d^2\theta d^2\bar\theta = \frac{1}{16} d\theta^\alpha \epsilon_{\alpha\beta} d\theta^\beta d\bar\theta_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} d\bar\theta_{\dot\beta}$$

These formulas generalise in an obvious way when we have more than one super-symmetry. In particular, the measure in N=2 superspace will be normalised as

$$d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} = \frac{1}{16} d\theta_{\underline{1}}^\alpha \epsilon_{\alpha\beta} d\theta_{\underline{1}}^\beta d\theta_{\underline{2}}^{\alpha'} \epsilon_{\alpha'\beta'} d\theta_{\underline{2}}^\beta$$

$$d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} d^2\bar\theta_{\underline{1}} d^2\bar\theta_{\underline{2}} = \frac{1}{256} d\theta_{\underline{1}}^\alpha \epsilon_{\alpha\beta} d\theta_{\underline{1}}^\beta d\theta_{\underline{2}}^{\alpha'} \epsilon_{\alpha'\beta'} d\theta_{\underline{2}}^{\beta'} d\bar\theta_{\underline{\dot\alpha}}^1 \epsilon^{\dot\alpha\dot\beta} d\bar\theta_{\dot\beta}^{\underline{1}} d\bar\theta_{\underline{\dot\alpha'}}^2 \epsilon^{\dot\alpha'\dot\beta'} d\bar\theta_{\dot\beta'}^{\underline{2}}$$

where the underlined numbers indicate the relevant supersymmetry transformation.

3.1.2 The Method of Projection

The method of projection is a procedure which facilitates the reduction of actions expressed in terms of superfields to that where the dependence on component fields is explicit (or in the case of extended supersymmetry, dependence of superfields from a superspace of lower dimension).

We shall investigate a simple, non-trivial example which will illustrate the method of projection. Consider an Abelian $N = 1$ vector field strength superfield $W_\alpha(x; \theta, \bar{\theta})$. This has the constraints [92]

$$\bar{D}_{\dot{\alpha}} W_\alpha = 0 \qquad D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \qquad (3.1.1)$$

where we have introduced the conjugate field $\bar{W}$ as well as the superderivatives

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m$$
$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \partial_m$$

The first of the constraints in equation (3.1.1) means, in the parlance of supersymmetry, that W is "chiral". A simple change of co-ordinates can now be introduced to simplify subsequent calculations. We redefine

$$(x^m, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \longrightarrow (y^m = x^m + i\theta\sigma^m\bar{\theta}, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$$

with the effect that

$$D_\alpha \longrightarrow \frac{\partial}{\partial \theta^\alpha} + 2i\sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m$$
$$\bar{D}_{\dot{\alpha}} \longrightarrow -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}$$

It can then be shown [92] that

$$W_\alpha(y; \theta) = -i\lambda_\alpha(y) + \left[\delta^\beta_\alpha D(y) - \frac{i}{2} (\sigma^{mn})^\beta_\alpha v_{mn}(y) \right] \theta_\beta + \theta\theta\sigma^m_{\alpha\dot{\alpha}} \partial_m \bar{\lambda}^{\dot{\alpha}}(y)$$

where $v_{mn} = \partial_m v_n - \partial_n v_m$ is the usual abelian field strength, with v_m the vector potential. The equation of motion for $D(y)$ is polynomial, and as such $D(y)$ auxiliary and can be eliminated by use of the solution to the equation of motion. The field λ

is the supersymmetric partner of v_m and is often termed the gaugino. The remaining symbols are δ, the Kroneker delta, and $\sigma^{mn} = \frac{1}{4}(\sigma^m\sigma^n - \sigma^n\sigma^m)$ with σ^m a Pauli[1] matrix.

Consider then the expression

$$\frac{1}{4}\int d^4x \int d^2\theta \int d^2\bar{\theta}\, W(x;\theta,\bar{\theta})W(x;\theta,\bar{\theta}) = \frac{1}{4}\int d^4y \int d^2\theta\, W(y;\theta)W(y;\theta)$$

where we have used the fact that the Jacobian of the co-ordinate transformation is unity.

We now proceed to reduce this action by the method of projection. This is essentially a supersymmetric generalisation of a property of Berezin integration of Grassmanian variables. It is well know that such an operation is equivalent to differentiation as follows

$$\int d\theta\, \theta = 1 \text{ is equivalent to } \frac{\partial}{\partial\theta}\theta = 1 \text{ and } \int d\theta\, 1 = 0 \text{ is equivalent to } \frac{\partial}{\partial\theta}1 = 0$$

To simply replace the integration measure with ordinary differentiation with respect to Grassmanian variables would not guarantee that the result of the reduction would be supersymmetric. Instead, one must use the superderivatives D_α and $\bar{D}_{\dot\alpha}$ since these operators are **defined** to commute with the generators of the supersymmetry transformation. After performing manipulations with these derivatives, one sets to zero all Grassmanian parameters. This step extracts the correct term from a more general expansion. Following this recipe, the specific details for (3.1.4) are

$$
\begin{aligned}
\frac{1}{4}\int d^2\theta\, WW &= -\frac{1}{16}d\theta^\alpha\epsilon_{\alpha\beta}d\theta^\beta WW \\
&= \frac{1}{16}D^\alpha D_\alpha\,(WW)\,| \\
&= -\frac{1}{8}\left(D^\alpha W^\beta|\, D_\alpha W_\beta| - W^\beta|\, D^\alpha D_\alpha W_\beta|\right) \\
&= -\frac{1}{8}\frac{\partial}{\partial\theta_\alpha}W_\kappa|\,\frac{\partial}{\partial\theta_\beta}W_\mu|\,\epsilon_{\beta\alpha}\epsilon^{\mu\kappa} + \frac{1}{8}\epsilon^{\mu\kappa}W_\kappa|\,\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha}W_\mu| \\
&= -\frac{1}{8}\epsilon_{\beta\alpha}\epsilon^{\mu\kappa}\left(\delta^\alpha_\kappa D - i\,[\sigma^{mn}]^\alpha_\kappa\, v_{mn}\right)\left(\delta^\beta_\mu D - i\,[\sigma^{pq}]^\beta_\mu\, v_{pq}\right) - \frac{i}{2}\lambda^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m\bar\lambda^{\dot\alpha} \\
&= \frac{1}{4}D^2 - \frac{i}{2}\lambda\sigma^m\partial_m\bar\lambda - \frac{1}{8}v^{mn}v_{mn} + \frac{i}{16}v^{mn}\star v_{mn} \quad\quad (3.1.2)
\end{aligned}
$$

---

[1]Although not required, for those who like to see a specific representation of the Pauli matrices, we can take $\sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The vertical bars indicate that one should set all Grassmanian co-ordinates to zero; that is $\theta_\alpha = \bar{\theta}_{\dot{\alpha}} = 0$. We have used the identities

$$
\begin{aligned}
\mathrm{Tr}\,\sigma^{mn} &= 0 \\
\mathrm{Tr}\left(\sigma^{mn}\sigma^{pq}\right) &= -\frac{1}{2}\left(\eta^{mp}\eta^{nq} - \eta^{mq}\eta^{np}\right) - \frac{i}{2}\epsilon^{mnpq} \\
[\sigma^{mn}]^\beta_\mu \, \epsilon_{\beta\alpha} &= [\sigma^{mn}]^\beta_\alpha \, \epsilon_{\beta\mu}
\end{aligned}
$$

and that D_α anti-commutes with W_β. One finds that this agrees with the more direct procedure of extracting the $O(\theta^2)$ terms from the series expansion of $W^\alpha W_\alpha$. This therefore indicates that the method of projection illustrated above is valid.

A similar calculation can be performed on $\frac{1}{4}\int d^2\bar{\theta}\,\bar{W}\bar{W}$ with the result that, when combined with (3.1.2)

$$
\frac{1}{4}\int d^4x \int d^2\theta \int d^2\bar{\theta}\,WW + \bar{W}\bar{W} = \frac{1}{2}D^2(x) - i\lambda\sigma^m\partial_m\bar{\lambda} - \frac{1}{4}v^{mn}v_{mn} \qquad (3.1.3)
$$

which is precisely the component form of the N=1 supersymmetric generalisation of abelian gauge theories which appears in the literature [10].

In later sections we shall utilise information derived from the Lagrangian to perform instanton calculations. As such, a great deal of importance is attached to self–dual and anti–self–dual gauge fields, where (anti–)self–duality is defined by $\epsilon_{mnpq}B^{pq} = (-)B_{mn}$ where B is an arbitary 2–tensor having the relevant property. In anticipation of this, we can rewrite the result (3.1.3) using a more useful notation.

The field strength v_{mn} in Minkowski space can be decomposed into self–dual (SD) and anti–self–dual (ASD) components using the identity

$$
\begin{aligned}
v^{SD}_{mn} &= \frac{1}{4}\left(\eta_{mp}\eta_{nq} - \eta_{mq}\eta_{np} + \frac{i}{2}\epsilon_{mnpq}\right)v^{pq} \\
&= \frac{1}{2}\left(\eta_{mp}\eta_{nq} + i\epsilon_{mnpq}\right)v^{pq} \\
&\equiv \frac{1}{2}\left(v_{mn} + \star v_{mn}\right)
\end{aligned}
$$

and $v^{ASD}_{mn} = \left(v^{SD}_{mn}\right)^\dagger$.

It then follows that $v_{mn} = \left(v^{SD}_{mn} + v^{ASD}_{mn}\right)$ and $\star v_{mn} = \left(v^{SD}_{mn} - v^{ASD}_{mn}\right)$.

Using this decomposition the action (3.1.3) can be rewritten as

$$\frac{1}{4}d^4x \int d^2\theta \int d^2\bar{\theta}\, WW + \bar{W}\bar{W} =$$
$$\frac{1}{2}D^2(x) - i\lambda\sigma^m\partial_m\bar{\lambda} - \frac{1}{4}\left(v^{SD}\right)^2 - \frac{1}{4}\left(v^{ASD}\right)^2 \qquad (3.1.4)$$

which is now more transparent for the purpose of investigating contributions from (anti)self-dual solutions, for example if we wish to consider the situation of an instanton background.

As a consequence of the above calculations we can write down a heuristic recipe for the method of reduction

$$d^2\theta \quad \longrightarrow \quad \frac{1}{4}D^\alpha D_\alpha|_{\theta=\bar{\theta}=0}$$
$$d^2\bar{\theta} \quad \longrightarrow \quad \frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}|_{\theta=\bar{\theta}=0}$$

3.1.3 The Reduction of the N=2 Effective Action

Using the techniques presented in the previous section we shall now reduce both the leading and next-to-leading order terms of the N=2 effective action of supersymmetric Yang-Mills.

N=2 Chiral Superfields

When dealing with physical models invariant under two independent supersymmetric transformations it is extremely useful to enlarge the $N = 1$ superspace to contain two fermionic co-oridinates. Such a construct is termed an $N = 2$ superspace [56] and fields defined therein $N = 2$ superfields. The general expansion of such a (constrained) superfield contains the relevant physical fields, and can be used to introduce $N = 2$ invariant Lagrangians and so, by integration over the superspace, actions.

There are essentially two N=2 superfields: the chiral superfield containing component fields which transmit the gauge field, and hypermultiplets which are typically
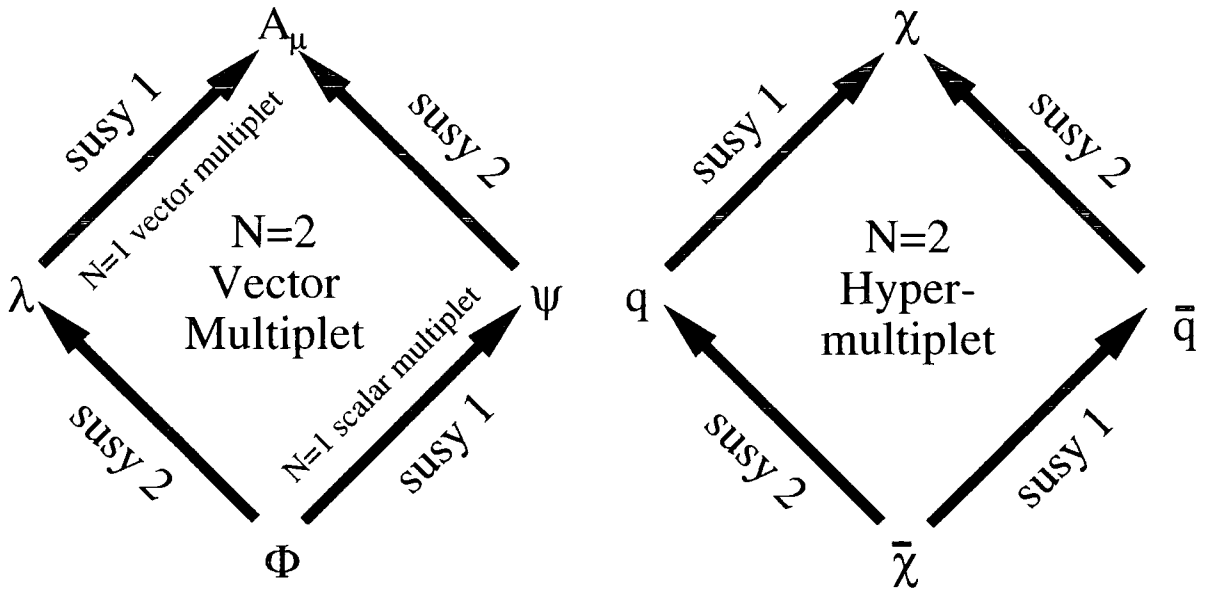
Figure 3.1: On-Shell Field Representations of N=2 Supersymmetry

used to represent matter. They can be presented pictorially as in figure 3.1. This diagram shows the relation of the various component fields. In this chapter, we shall be particularly concerned with the N=2 vector multiplet.

The manipulation of $N = 2$ quantities is aided by the use of superspace techniques. In particular, the information regarding the $N = 2$ vector multiplet is encoded in an $N = 2$ chiral superfield $\mathbb{A}$ which has been determined [51] to satisfy the constraints

$$\bar{D}_i \mathbb{A} \;=\; D_i \bar{\mathbb{A}} = 0 \tag{3.1.5a}$$

$$D^i D^j \mathbb{A} \;=\; \bar{D}^i \bar{D}^j \bar{\mathbb{A}} \tag{3.1.5b}$$

where contractions over fermionic indices is understood. In these equations $\underline{i} \in \{1, 2\}$ labels the supersymmetry.

In general, the supercovariant derivatives appearing in the above constraints can have an operational dependence on the central charge Z of the $N = 2$ superalgebra. In the case of the chiral superfield $\mathbb{A}$ we can effectively neglect such a dependence since

$$0 = \left\{ \bar{D}^i_{\dot{\alpha}}, \bar{D}^j_{\dot{\beta}} \right\} \mathbb{A} = \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{ij} D_Z \mathbb{A}$$

where we have used the condition (3.1.5a). Here D_Z is the differential operator related to the central charge. It should be apparent that $D_Z = \partial_Z$, thereby illustrating that $\mathbb{A}$ is independent of the central charge, and that we can ignore the central charge differential that would usually appear in the superderivatives.

The former constraint (3.1.5a) is the counterpart of the $N = 1$ chirality condition, whereas the second (3.1.5b) enforces the correct number of degrees of freedom. Whilst aesthetically pleasing, the $N = 2$ superspace formalism is unwieldy for practical calculations. As such, we shall reduce down to an $N = 1$ language and thereby extract component expressions from the action.

The $N = 2$ superfield can be thought of as an expansion in the Grassmannian coordinate $\theta_{\underline{2}}$, where the co-efficients are familiar $N = 1$ superfields. It is usual to take the following definitions [94]

$$\Phi = \mathbb{A}|_{\theta_{\underline{2}}=0} \qquad W^\alpha = D_{\underline{2}}^\alpha \mathbb{A}|_{\theta_{\underline{2}}=0} \qquad J = -\frac{1}{2} D_{\underline{2}} D_{\underline{2}} \mathbb{A}|_{\theta_{\underline{2}}=0}$$

where J is a field which we shall show can be replaced by an expression in terms of $\mathbb{A}$ using the constraints.

We can derive the $N = 1$ chirality condition from (3.1.5b)

$$\begin{aligned} \bar{D}_{\underline{1}\dot\alpha}\mathbb{A} = 0 &\implies \bar{D}_{\dot\alpha}\Phi = 0 \\ \bar{D}_{\underline{1}\dot\alpha}\mathbb{A} = 0 &\implies D_{\underline{2}\alpha}\bar{D}_{\underline{1}\dot\alpha}\mathbb{A} = 0 \\ &\implies \bar{D}_{\underline{1}\dot\alpha}D_{\underline{2}\alpha}\mathbb{A} \\ &\implies \bar{D}_{\dot\alpha}W_\alpha = 0 \end{aligned} \qquad (3.1.6)$$

We next consider the second constraint (3.1.5b). Firstly we investigate the case $\underline{i} = 1, \underline{j} = 2$.

$$D^{\underline{1}}D^{\underline{2}}\mathbb{A} = \epsilon^{\underline{12}}\epsilon^{\underline{21}}D_{\underline{2}}D_{\underline{1}}\mathbb{A} = D_{\underline{1}}D_{\underline{2}}\mathbb{A} = \bar{D}^{\underline{1}}\bar{D}^{\underline{2}}\bar{\mathbb{A}}$$

and taking $\theta_{\underline{2}} = 0$ we derive an additional constraint on the $N = 1$ vector superfield

$$D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

which taken together with (3.1.6) therefore shows that W_α is the usual $N = 1$ vector superfield containing the gauge field strength, and that Φ is a chiral N=1 superfield.

The fact that the internal supersymmetry indices are raised and lowered by the antisymmetric epsilon-tensor is related to the fact that for $N = 2$ theories, there is an internal symmetry group $SU_R(2)$. In fact, it can be shown [29] that this so-called "R–symmetry" together with $N = 1$ supersymmetry automatically gives a theory with $N = 2$ supersymmetry.

Now consider $\underline{i} = \underline{j} = 2$ in (3.1.5b). Then

$$D^2 \bar{D}^2 \mathbb{A} = \cdots = D^\alpha D_\alpha \mathbb{A} = \bar{D}^2 \bar{D}^2 \bar{\mathbb{A}}$$

with the result on setting $\theta_{\underline{2}} = 0$

$$J = -\frac{1}{2} D^\alpha D_\alpha \Phi$$

The conjugated result comes from considering the $\underline{i} = \underline{j} = 1$ constraint.

It is now possible to use these results to reduce actions in a similar manner to that done in the previous section. The simplest case, and one which is very instructive, is to attempt the reduction of an action constructed from an Abelian holomorphic $N = 2$ prepotential $\mathcal{F}(\mathbb{A})$

$$
\begin{aligned}
S_0 &= \mathrm{Im} \int d^4x d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} \mathcal{F}(\mathbb{A}) \\
&= \mathrm{Im} \int d^4x d^2\theta_{\underline{1}} \left[-\frac{1}{4} D_{\underline{2}} D_{\underline{2}} \mathcal{F}(\mathbb{A}) \right]_{\theta_{\underline{2}}} \\
&= -\frac{1}{4}\mathrm{Im} \int d^4x d^2\theta_{\underline{1}} [D_{\underline{2}} (D_{\underline{2}} \mathbb{A}\mathcal{F}(\mathbb{A}))]_{\theta_{\underline{2}}} \\
&= -\frac{1}{4}\mathrm{Im} \int d^4x d^2\theta_{\underline{1}} \left([D_{\underline{2}} D_{\underline{2}} \mathbb{A}]_{\theta_{\underline{2}}} \left[\frac{\partial \mathcal{F}}{\mathbb{A}} \right]_{\theta_{\underline{2}}} \right) \\
&= -\frac{1}{4}\mathrm{Im} \int d^4x d^2\theta_{\underline{1}} \left([D_{\underline{2}} D_{\underline{2}} \mathbb{A}]_{\theta_{\underline{2}}} \left[\frac{\partial \mathcal{F}}{\partial \mathbb{A}} \right]_{\theta_{\underline{2}}} + [D_{\underline{2}} \mathbb{A}]_{\theta_{\underline{2}}} [D_{\underline{2}} \mathbb{A}]_{\theta_{\underline{2}}} \left[\frac{\partial^2 \mathcal{F}}{\partial \mathbb{A} \partial \mathbb{A}} \right]_{\theta_{\underline{2}}} \right) \\
&= -\frac{1}{4}\mathrm{Im} \int d^4x d^2\theta_{\underline{1}} \left([\bar{D}_{\underline{1}} \bar{D}_{\underline{1}} \mathbb{A}]_{\theta_{\underline{2}}} \left[\frac{\partial \mathcal{F}}{\partial \mathbb{A}} \right]_{\theta_{\underline{2}}} + \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi \partial \Phi} W^\alpha W_\alpha \right) \\
&= \mathrm{Im} \left(\int d^4x d^2\theta d^2\bar{\theta} \bar{\Phi} \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} - \frac{1}{4} \int d^4x d^2\theta \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi \partial \Phi} W^\alpha W_\alpha \right) \quad (3.1.7)
\end{aligned}
$$

where the $]_{\theta_{\underline{2}}}$ indicates that one sets the co-ordinates $\theta_{\underline{2}} = 0$. This result is an F-term since we are integrating over only half of the superspace Grassmannians.

The second term in this final expression should be familiar as the action for the $N = 1$ massless super-Yang-Mills theory. This should be even more apparent if we

introduce the specific holomorphic prepotential

$$\mathcal{F}(\mathbb{A}) = \frac{1}{2}\tau \mathbb{A}^2$$

which gives the correct classical action. In this expression τ is the complexified coupling constant. More generally we see that the $N = 2$ superspace expression (3.1.7) leads on to kinetic terms for the gauge fields, the Higgs and their supersymmetric partners. Further, we can identify $\partial_{\phi\phi}\mathcal{F}(\phi)$ as a generalised complexified coupling constant which for suitable $\mathcal{F}$ will encode loop and instanton quantum corrections [79].

As such, the above action is recognised as that of the $N = 2$ supersymmetric $U(1)$ action expressed in terms of $N = 1$ superfields. It is also ubiquitous in Seiberg–Witten theory since this is precisely the lowest order (2 derivative/4 fermion) term in the Wilsonian effective action.

It is possible to reduce this action into component fields using methods similar to those used previously, but we choose not to duplicate that analysis. We note, however, that it can be reduced to the component form of the $N = 2$ super–Yang–Mills theory.

3.1.4 Non–Holomorphic Prepotential

In chapter 2 we introduced the next–to–leading order term in the low energy derivative expansion of N=2 super–Yang–Mills theories in the absence of matter. It was pointed out that this term could be encoded in an N=2 non–holomorphic prepotential $\mathcal{H}(\mathbb{A}, \bar{\mathbb{A}})$. In chapter 2 we investigated the duality properties of this construct and indicated that it has no effect on the results of Seiberg and Witten. In particular it has no influence on the BPS mass formula.

Using the technique of reduction introduced previously, we can derive the component action of the 4–derivative and up to 8–fermion real D–term

$$S_1 = \int d^4x d^4\theta_{\underline{1}} d^4\theta_{\underline{2}} \mathcal{H}(\mathbb{A}, \bar{\mathbb{A}})$$

As a subset of the (many) terms this expression has, we are particularly interested

in the nine effective vertices

$$\circ \quad \frac{1}{16} \left(v^{SD}\right)^2 \left(v^{ASD}\right)^2 \partial_{aa\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8a}$$

$$\circ \quad \frac{1}{2\sqrt{2}} \left(v^{SD}\right)^2 \bar{\lambda}\bar{\sigma}^{mn} v^{ASD}_{mn} \bar{\psi} \partial_{aa\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8b}$$

$$\circ \quad \frac{1}{2\sqrt{2}} \left(v^{ASD}\right)^2 \psi\sigma^{mn} v^{SD}_{mn} \lambda \partial_{aaa\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8c}$$

$$\circ \quad \frac{1}{8} \left(v^{SD}\right)^2 \bar{\psi}^2 \bar{\lambda}^2 \partial_{aa\bar{a}\bar{a}\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8d}$$

$$\circ \quad \frac{1}{8} \left(v^{ASD}\right)^2 \psi^2 \lambda^2 \partial_{aaaa\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8e}$$

$$\circ \quad \frac{1}{2} \psi\sigma^{mn} v^{SD}_{mn} \lambda \bar{\lambda}\bar{\sigma}^{pq} v^{ASD}_{pq} \bar{\psi} \partial_{aaa\bar{a}\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8f}$$

$$\circ \quad -\frac{1}{4\sqrt{2}} \psi\sigma_{mn} v^{SD}_{mn} \lambda \bar{\psi}^2 \bar{\lambda}^2 \partial_{aaa\bar{a}\bar{a}\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8g}$$

$$\circ \quad -\frac{1}{4\sqrt{2}} \bar{\lambda}\bar{\sigma}_{mn} v^{ASD}_{mn} \bar{\psi}\psi^2 \lambda^2 \partial_{aaaa\bar{a}\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8h}$$

$$\circ \quad \frac{1}{16} \psi^2 \lambda^2 \bar{\psi}^2 \bar{\lambda}^2 \partial_{aaaa\bar{a}\bar{a}\bar{a}\bar{a}} \mathcal{H}(a, \bar{a}) \tag{3.1.8i}$$

where a denotes the vacuum expectation value of the Higgs field which is the lowest component of the N=1 chiral superfield Φ which in turn is the lowest component of the N=2 chiral superfield $\mathbb{A}$. Notice that (3.1.8a) leads on to terms of the form v^4.

3.2 Instanton Calculation

Following a general approach originally developed by Seiberg and co-workers [1], Dorey, Khoze and Mattis [29, 30, 32] were able to investigate the instanton series arising in the Seiberg and Witten's holomorphic prepotential $\mathcal{F}(\mathbb{A})$ without recourse to duality arguments. This provided the first independent checks of Seiberg and Witten's results [80, 81].

The essential methodology involved the consideration of certain chirality violating Green's functions to study the instanton contributions to particular vertices which arise in the Lagrangian for $N = 2$ SUSY gauge theories with and without matter. More specifically, they studied instanton correlators with field insertions replaced by their values in the classical instanton background, they projected onto the unbroken $U(1)$ direction in colour space, and then they integrated over all bosonic

and fermionic collective co-ordinates which arise from zero-modes in the instanton background. To give a precise justification of this method would take us too far afield from the purpose of this chapter. Instead, we refer the interested reader to the excellent papers [29, 30, 32, 33] which develop the construction in great detail.

As an application of their formalism, we can consider the instanton contribution to the 8 fermion vertex

$$\frac{1}{16}\psi^2\lambda^2\bar\psi^2\bar\lambda^2\frac{\partial^4}{\partial a^4}\frac{\partial^4}{\partial\bar a^4}\mathcal{H}(a,\bar a) \tag{3.2.9}$$

which arose from the reduction of the next to leading order term in the low energy expansion of $N = 2$ supersymmetric gauge theories with gauge group $SU(2)$. The other vertices above (3.1.8i) can be analysed in a similar manner to that detailed below.

For reasons explained later, one must study the correlator

$$\langle\bar\psi_{\dot\alpha}(x_1)\bar\psi_{\dot\beta}(x_2)\bar\lambda_{\dot\gamma}(x_3)\bar\lambda_{\dot\delta}(x_4)\psi_\alpha(x_5)\psi_\beta(x_6)\lambda_\gamma(x_7)\lambda_\delta(x_8)\rangle \tag{3.2.10}$$

in order to consider instanton contributions to (3.2.9).

Now, since we are dealing with the effective theory in (3.2.9) we must use the relevant long-distance effective $U(1)$ fields which match into the short-distance singular superinstanton. The explicit form of these solutions was evaluated by Dorey et al. [29, 30] and we shall tabulate the relevant expressions here

$$\bar\psi_{\dot\alpha}(x_i) = i\sqrt{2}\zeta^{1\alpha}S_{\alpha\dot\alpha}(x_i,x_0)\frac{\partial}{\partial a} + \text{ other terms} \tag{3.2.11a}$$

$$\bar\lambda_{\dot\alpha}(x_i) = i\sqrt{2}\zeta^{2\alpha}S_{\alpha\dot\alpha}(x_i,x_0)\frac{\partial}{\partial a} + \text{ other terms} \tag{3.2.11b}$$

$$\psi_\alpha(x_i) = 4i\pi^2 S_{\alpha\dot\alpha}(x_i,x_0)\sum_{i=1}^n \bar w_k^{\dot\alpha\beta}\left(\tau^3\right)_\beta^\gamma \nu_{k\gamma} + \text{ other terms} \tag{3.2.11c}$$

$$\lambda_\alpha(x_i) = 4i\pi^2 S_{\alpha\dot\alpha}(x_i,x_0)\sum_{i=1}^n \bar w_k^{\dot\alpha\beta}\left(\tau^3\right)_\beta^\gamma \mu_{k\gamma} + \text{ other terms} \tag{3.2.11d}$$

where the "other terms" are those independent of the supersymmtry zero modes $\zeta^{1,2}$ or are those which fall of faster than $(x_i - x_0)^{-3}$.

In the above long distance approximations of the fields we have taken the unbroken $U(1)$ direction to lie in the τ^3 direction of colour space. The parameters $\bar w_k$, $\mu_{k\gamma}$,

and $\nu_{k\gamma}$ are quaternionic valued quantities which arise in the ADHM formalism for multi-instantons as detailed in the technical appendix B. Indeed, the upper bound n on the summation is defined as the instanton winding number. Finally, the fermionic propagators

$$S_{\alpha\dot{\alpha}}(x_i, x_0) = \sigma^m_{\alpha\dot{\alpha}} \partial_m G(x_i, x_0)$$

where $G(x_i, x_0) = \frac{1}{4\pi^2(x_i - x_0)^2}$ is the massless Euclidean propagator. One should note that the fermionic propagators $S_{\alpha\dot{\alpha}}$ link fermions to anti-fermions. It is for this reason, when one also takes into consideration the saturation of the Grassmannian integrals developed below, that the correlator (3.2.10) yields information concerning the vertex (3.2.9)

The n-instanton contribution to (3.2.9) is thus given, in the usual manner, by

$$\int d^4 x_0 d^2 \zeta^1 d^2 \zeta^2$$

$$\times \int d\tilde{\mu} \bar{\psi}_{\dot{\alpha}}(x_1) \bar{\psi}_{\dot{\beta}}(x_2) \bar{\lambda}_{\dot{\gamma}}(x_3) \bar{\lambda}_{\dot{\delta}}(x_4) \psi_\alpha(x_5) \psi_\beta(x_6) \lambda_\gamma(x_7) \lambda_\delta(x_8) e^{-S_{inst}} \quad (3.2.12)$$

where $d\tilde{\mu}$ represents the properly normalised integration measure for all collective co-ordinates arising from the zero modes in the solution space [34]. It excludes the $N = 2$ superspace position variables (x_0, ζ^1, ζ^2) which have been seperated and written explicitly. These latter fermionic collective co-ordinates are so-called "exact modes" and can be shown not to appear in the instanton action S_{inst}. As such, saturation of the ζ^i integrals takes place by explicit insertions of suitable fields from (3.2.11a), (3.2.11b). The fermionic modes in $d\tilde{\mu}$ do appear in S_{inst} and are "lifted" by the occurence of Yukawa terms.

Now we have seen that

$$\int d^2 \zeta^i \zeta^{i\alpha} \zeta^{i\beta} = -\frac{1}{2} \epsilon^{\alpha\beta} \int d^2 \zeta^i \left(\zeta^i \right)^2 = -\frac{1}{2} \epsilon^{\alpha\beta}$$

so that the collective co-ordinate integration 3.2.12 becomes

$$\int d^4 x_0 \epsilon^{\kappa\lambda} S_{\kappa\dot{\alpha}}(x_1, x_0) S_{\lambda\dot{\beta}}(x_2, x_0) \epsilon^{\rho\sigma} S_{\rho\dot{\gamma}}(x_3, x_0) S_{\sigma\dot{\delta}}(x_4, x_0)$$

$$\times \epsilon^{\dot{\kappa}\dot{\lambda}} S_{\alpha\dot{\kappa}}(x_5, x_0) S_{\beta\dot{\lambda}}(x_6, x_0) \epsilon^{\dot{\rho}\dot{\sigma}} S_{\gamma\dot{\rho}}(x_7, x_0) S_{\delta\dot{\sigma}}(x_8, x_0)$$

$$\times \frac{\partial^4}{\partial a^4} \int d\tilde{\mu} e^{-S_{inst}} \sum_{k,k',l,l'=1}^{n} (4i\pi^2)^4 (\nu_k \tau^3 w_k \bar{w}_{k'} \tau^3 \nu_{k'}) (\mu_l \tau^3 w_l \bar{w}_{l'} \tau^3 \mu_{l'})$$

This expression should be recognisable as the position space Feynman integral for a local $\psi^2 \lambda^2 \bar{\psi}^2 \bar{\lambda}^2$ vertex with an effective coupling given by

$$\frac{\partial^4}{\partial a^4} \int d\tilde{\mu} e^{-S_{inst}} \sum_{k,k',l,l'=1}^{n} (4i\pi^2)^4 (\nu_k \tau^3 w_k \bar{w}_{k'} \tau^3 \nu_{k'})(\mu_l \tau^3 w_l \bar{w}_{l'} \tau^3 \mu_{l'})$$

Comparing this with the vertex (3.2.9), which is a generic expression of an n-instanton contribution to such a localised vertex, ones sees that to leading semi-classical order

$$\frac{\partial^4}{\partial \bar{a}^4} \mathcal{H}(a,\bar{a}) = 64\pi^8 \int d\tilde{\mu} e^{-S_{inst}}$$
$$\times \sum_{k,k',l,l'=1}^{n} \frac{1}{4}(4i\pi^2)^4 (\nu_k \tau^3 w_k \bar{w}_{k'} \tau^3 \nu_{k'})(\mu_l \tau^3 w_l \bar{w}_{l'} \tau^3 \mu_{l'}) \quad (3.2.13)$$

As pointed out previously, it is possible to obtain formal expressions for $\frac{\partial^2}{\partial \bar{a}^2}\mathcal{H}$ and $\frac{\partial^3}{\partial \bar{a}^3}\mathcal{H}$ using the other vertices above (3.1.8i). Exchanging a and $\bar{a}$ in (3.2.13) gives the anti-instanton contribution. There may also be mixed n-instanton/m-anti-instanton contributions to $\mathcal{H}$ due to the non-holomorphicity of this prepotential. However, such configurations shall not be considered in this work.

We now present the simplest situation in which the above formal expression can be calculated explicitly. Let us consider the 1-instanton contribution to the non-holomorphic prepotential $\mathcal{H}(\mathbb{A}, \bar{\mathbb{A}})$ in the case of pure $N = 2$ super-Yang-Mills.

In this case, the instanton action S_{inst} is given to leading order [29] by

$$S_{inst} = 4\pi^2 a\bar{a}w^m w_m - 2\sqrt{2}i\pi^2 \bar{a}\mu^\alpha (\tau^3)_\alpha^\beta \nu_\beta \quad (3.2.14)$$

where the 1-instanton values of the various ADHM parameters (collective co-ordinates) are presented in a technical appendix. Also, [34] the correctly normalised instanton measure over the collective co-ordinates is given by

$$\int d\tilde{\mu} = \frac{\Lambda^4}{2\pi^4} \int d^4w d^2\mu d^2\nu$$

with Λ the dynamically generated (Pauli-Villars) scale. Then

$$\frac{\partial^4}{\partial \bar{a}^4}\mathcal{H} =$$
$$64\pi^8 \frac{\Lambda^4}{2\pi^4} \int d^4w \int d^2\mu d^2\nu e^{(-4\pi^2 a\bar{a}w^m w_m)} \left[(\nu\tau^3 w\bar{w}\tau^3 \nu)(\mu\tau^3 w\bar{w}\tau^3 \mu) \right] \quad (3.2.15)$$

where the second term in (3.2.14) does not appear since the Grassmannian integrations over μ and ν are already saturated. We now recognise

$$\int d^2\nu(\nu\tau^3 w\bar{w}\tau^3\nu) = -\frac{1}{2}w^m w^n \text{tr}_2(\tau^3\sigma_m\bar{\sigma}_n\tau^3)$$

But using the cyclicity of the trace and the results in appendix B we have $\text{tr}_2(\ldots) = \delta_{mn}\mathbb{I}$ so that (3.2.15) becomes

$$32\pi^4\Lambda^4\int d^4w(w^m w_m)^2 e^{-4\pi a\bar{a}w^m w_m}$$

We now change to polar co-ordinates where the measure $\int d^4w \mapsto \int d\Omega \int_0^\infty dr r^3$. It is easy to show that the angular integration in 4-d has the value $\int d\Omega = 2\pi^2$, and so all that remains to be evaluated in (3.2.15) is the radial contribution. By means of the substitution $r^2 = R$ and the standard integral [39] $\int_0^\infty dR R^n \exp(-\alpha R) = \frac{n!}{\alpha^{n+1}}$ we therefore see (3.2.15)

$$\frac{\partial^4}{\partial\bar{a}^4}\mathcal{H}(a,\bar{a}) = \frac{3}{4\pi^2}\frac{1}{a^4\bar{a}^4}$$

Integrating this expression four times, and on comparison with the other vertices above (3.1.8i) one finds that at the one instanton level

$$\mathcal{H}(a,\bar{a}) = -\frac{1}{8\pi^2}\frac{\Lambda^4}{a^4}\ln\bar{a} \qquad (3.2.16)$$

which agrees with an earlier prediction made by Yung [98] which was obtained using a completely different reasoning.

It is possible to evaluate analagous quantities in models where the number of flavours $N_f > 0$. It should be pointed out [81] that a discrete $\mathbb{Z}_2$ symmetry in the moduli space forbids all odd-instanton contributions to vertices. This can be seen in the instanton method. The presence of hypermultiplets introduces the collective co-ordinates $\mathcal{K}_{li}$ and $\tilde{\mathcal{K}}_{li}$ as detailed in appendix B. As such, we must introduce the measure for the n-instanton

$$\int d\mu_{hyp} \approx \int \prod_{i=1}^{N_f} d\mathcal{K}_{1i}\cdots d\mathcal{K}_{ni}d\tilde{\mathcal{K}}_{1i}\cdots d\tilde{\mathcal{K}}_{ni}e^{-S_{hyp}} \qquad (3.2.17)$$

where S_{hyp} is the action arising from the presence of hypermultiplets. For massless hypermultiplets, it can be shown [30] that S_{hyp} is even under $\mathcal{K}_{li} \leftrightarrow \tilde{\mathcal{K}}_{li}$. However,

the measure collects a factor of $(-1)^n$ under this transposition when we remember that the entries of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are Grassmannian. Thus we see that (3.2.17) vanishes when n is odd, and so only even instanton contributions survive in the massless theory. As such, one should begin by looking at the 2-instanton contribution using the ADHM method detailed in Dorey et als papers [29, 30]. We shall not explicitly pursue this line of investigation in this chapter, but the methodology is the same as that just presented.

3.2.1 Nonrenormalisation Theorem for the $N = 2$, $N_f = 4$ Model

Four dimensional $N = 2$ models with vanishing β-functions are finite and conformally invariant [64]. This proves a strong constraint on the behaviour of the next to leading order terms in the low energy expansion of such models, as we shall review below. This theoretical prediction based on the symmetries of the model [36] shall be confirmed by an analysis of the scaling properties of this term.

We have seen in section 3.1.3 that gauge fields in $N = 2$ models arise in the expansion of the $N = 2$ superfield $\mathbb{A}$. Concisely, such a superfield encodes an $N = 1$ chiral superfield Φ containing a scalar field ϕ often interpreted as the Higgs particle in a certain representation of the gauge group (in this case the adjoint). It also has as a sub-field an $N = 1$ field-strength superfield W_α which contains the field-strength $\mathbf{v}_{\mu\nu}$. In addition, we have seen by reducing the superfield action that the quadratic term

$$\int d^2\theta_1 d^2\theta_2 \tau \mathbb{A}\mathbb{A} \tag{3.2.18}$$

gives rise to kinetic terms for the gauge fields and the adjoint Higgs. The co-efficient τ can be considered a superfield in its own right. In fact, we have shown (3.1.7) that generically $\tau = \frac{\partial^2 \mathcal{F}(\mathbb{A})}{\partial \mathbb{A} \partial \mathbb{A}}$ where $\mathcal{F}$ is a holomorphic prepotential which, by virtue of being a function of the chiral superfield $\mathbb{A}$ is invariant under $\frac{1}{2}$ of the supercharges in the theory. In fact, for the model we are considering, τ is a constant function of $\mathbb{A}$.

The fact that the theory we are considering is scale invariant coupled with the

knowledge that $\mathbb{A}$ has dimension 1 under scaling – since $\mathbb{A}$ contains the scalar ϕ as its lowest ($N = 0$) component – tells us that (3.2.18) remains quadratic in $\mathbb{A}$ after quantum corrections are taken into consideration. Further, assuming that τ is compatible with Seiberg and Witten duality, it has been shown [81] that the co-efficient in this quadratic term is not affected either. In particular, τ is not replaced[2] by a function of τ.

The next co-efficient in the low energy expansion is given by

$$\int d^2\theta_{\underline{1}} d^2\theta_{\underline{2}} d^2\bar{\theta}_{\underline{1}} d^2\bar{\theta}_{\underline{2}} \mathcal{H}(\mathbb{A}, \bar{\mathbb{A}}, \tau, \bar{\tau}) \tag{3.2.19}$$

where the τ and $\bar{\tau}$ appear because in a scale invariant theory, there can obviously be no dynamically generated scale (other than that associated with $\mathbb{A}$ itself). Further, the non-holomorphic prepotential $\mathcal{H}$ must respect all of the symmetries of the model. In particular, it must respect the $U(1)_R$ symmetry which is non-anomolous in the model under consideration [79]. This R-symmetry acts in the following way

$$R \circ \mathbb{A}(\theta_{\underline{i}}, x) = \exp(2i\alpha)\mathbb{A}(\exp(-i\alpha)\theta_{\underline{i}}, x)$$

For fixed τ it has been shown [95] that there is a unique non-trivial form permitted by the symmetries. It is

$$\begin{aligned}
\mathcal{H} &\approx \ln\left(\frac{\mathbb{A}}{\Lambda}\right) \ln\left(\frac{\bar{\mathbb{A}}}{\Lambda}\right) \\
&= \frac{1}{2}\left(\ln \mathbb{A} + \ln \bar{\mathbb{A}}\right)^2 - \left(\ln^2 \mathbb{A} + -\frac{1}{2}\ln^2 \Lambda + \ln \Lambda \ln \mathbb{A} + \text{c.c.}\right) \\
&= \frac{1}{2}\ln^2(\mathbb{A}\bar{\mathbb{A}}) + (G(\mathbb{A}) + \text{c.c.})
\end{aligned} \tag{3.2.20}$$

Since we integrate over the entire superspace, the second term in 3.2.20 does not contribute to the action. Obviously, the scale Λ is fake as expected.

Using the form of the action (3.2.19) Dine and Seiberg showed – by means of promoting the constant τ to a superfield – that in order for $\mathcal{H}$ to be consistent with all of the symmetries in the model, including scale invariance, that the prepotential $\mathcal{H}$ has no explicit τ dependence. Hence there can be no perturbative or non-perturbative corrections to $\mathcal{H}$.

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[2]It will transpire that for the low energy theory, τ is the effective coupling. See chapter 4 for more details

We should be able to check the validity of this non-renormalisation theorem by considering a rescaling of the quantities appearing in the instanton calculation above.

Using the equation for the instanton action S_{inst} detailed in [30] it becomes apparent that when the masses of the hypermultiplets vanish, all dependence on a and $\bar{a}$ can be eliminated from S_{inst} by means of the ADHM collective co-ordinate rescaling

$$a \mapsto \frac{a}{\sqrt{a\bar{a}}} \quad \mathcal{M} \mapsto \frac{\mathcal{M}}{\sqrt{a}} \quad \mathcal{N} \mapsto \frac{\mathcal{N}}{\sqrt{\bar{a}}} \quad \mathcal{K} \mapsto \frac{\mathcal{K}}{\sqrt{a}} \quad \tilde{\mathcal{K}} \mapsto \frac{\tilde{\mathcal{K}}}{\sqrt{a}} \tag{3.2.21}$$

Consider the case of $N = 2$ models with $0 \leq N_f \leq 4$ flavours of massless fundamental hypermultiplets and no adjoint hypermultiplets. In these cases, the integration measure over the zero modes (excluding those arising from translations in superspace) taking into account the scaling behaviour in (3.2.21) shows

$$d\tilde{\mu} \mapsto (\sqrt{a\bar{a}})^{4-8n}(\sqrt{\bar{a}})^{8n-4}(\sqrt{a})^{2nN_f} \cdot d\tilde{\mu} = a^{2-(4-N_f)n}d\tilde{\mu}$$

where the first $8n - 4$ arises from the bosonic zero modes in a. This is seen since a **single** instanton has 4 translational zero modes, 1 scaling zero mode and 3 zero modes arising from iso-orientations in the colour space. In the far seperated limit, we expect the n-instanton solution to be approximated by n 1-instantons. Thus there are a total of $8n$ bosonic zero modes for the n-instanton. We can now identify four of these with a centre of mass co-ordinate and seperate it from the measure to agree with (3.2.12) giving a total of $8n - 4$ degrees of freedom in the ADHM collective co-ordinate matrix a. Likewise, by supersymmetry, we expect the matrices $\mathcal{M}$ and $\mathcal{N}$ developed in the appendix B to give a total of $8n - 4$, where on this occasion the 4 arises due to the seperation of the exact modes from the measure $d\tilde{\mu}$. Note, however, that the properties of Grassmanian integration – $\int d\theta\theta = 1$ ensure that the measure must scale inversely to the fermions. Lastly, the fundamental hypermultiplets give rise to a term $\int \prod_{i=1}^{N_f} \prod_{j=1}^{n} d\mathcal{K}_{ij}d\tilde{\mathcal{K}}_{ij}$ which leads on to the factor with $2nN_f$.

Now, in (3.2.13) we see

$$\sum_{k,k',l,l'=1}^{n} (\nu_k \tau^3 \ldots \mu_{l'}) \mapsto \frac{1}{a^2\bar{a}^4} \sum_{k,k',l,l'=1}^{n} (\ldots)$$

since w_k is an entry in the ADHM matrix a, μ in $\mathcal{M}$, and ν in $\mathcal{N}$. Applying all of the above arguments to the generic vertex (3.2.13) and integrating the requisite

number of times, we see that the n-instanton contribution to the non-holomorphic prepotential

$$\mathcal{H}(a, \bar{a}) \sim a^{(N_f - 4)n} \cdot \ln \bar{a}$$

which agrees with (3.2.16) in the special case $n = 1$, $N_f = 0$.

In the case of $N_f = 4$ massless hypermultiplets we see that

$$\mathcal{H}_{N_f = 4} \sim \ln \bar{a}$$

and is therefore an (anti-)holomorphic function in $\bar{a}$. Examining the possible vertices, one sees that the effective component vertices contained in the next-to-leading order Lagrangian vanish since the prepotential always comes with a mixture of a and $\bar{a}$ derivatives. Similar comments also hold for anti-instanton contributions which arise from swapping $a \leftrightarrow \bar{a}$. Thus we confirm the nonrenormalisation theorem of Dine and Seiberg outlined previously. That is, the non-holomorphic prepotential – which arises at next to leading order in the low energy expansion – in scale invariant theories does not get contributions from n (anti)instantons.

In S_{inst} the presence of massive hypermultiplets ruin this argument since the mass m_i rescales to $\frac{m_i}{a}$ which can then be pulled down from the exponent in S_{inst}.

3.3 Conclusions

Using the method of reduction, we were able to extract interaction vertices from the prepotentials which encode the first two terms in the low energy expansion of $N = 2$ supersymmetric gauge theories. In particular we extracted nine vertices from the non-holomorphic prepotential $\mathcal{H}$ which appears as the leading irrelevant operator on the Coulomb branch of the model. Utilising the ADHM method, these vertices were used to investigate the (multi)-instanton contributions to this functional. In particular, the ADHM method was shown to reproduce the results obtained by Yung [98] for the case of a single instanton in pure $N = 2$ super-Yang-Mills. Further, using scaling arguments, we were able to demonstrate the existence of a non-renormalisation

theorem in the case of a model with vanishing β-function: the $N = 2$ supersymmetric $SU(2)$ gauge theory with four (massless) hypermultiplets. This confirmed the results of Dine and Seiberg who originally proposed that in such models the prepotential $\mathcal{H}$ was one loop exact. In particular it obtained no contribution from instantons.

Using similar techniques to those detailed in this chapter, it is possible to show that the non-renormalisation theorem is also valid for the finite $N = 4$ supersymmetric gauge theory. We have not detailed it here, but the analysis can be found in the reference [33].

Chapter 4

Couplings in Scale Invariant Theories

In a series of papers, Dorey, Khoze, and Mattis [29, 30, 32] tested the results of N=2 supersymmetric gauge theories with gauge group $SU(2)$ as proposed by Seiberg and Witten [80, 81] against first-principles instanton calculations. In doing so they found perfect agreement when the number of flavours $N_f \leq 2$. When $N_f = 3, 4$ a discrepancy was found which they resolved by reinterpreting the parameters which appear in the Seiberg-Witten elliptic curve which encodes information about the vacuum in the theory. In particular, for the case $N_f = 4$ they showed that the effective coupling τ_{eff}, which arises on integrating out massive modes, is the parameter which appears in the curve; not the classical coupling τ classical. This is despite the absence of a running coupling in this scenario, which one might naively assume guarantees $\tau_{\text{eff}} = \tau$.

In the course of this analysis, Dorey et al [30] found a relationship between τ and τ_{eff}

$$\tau_{\text{eff}} = \tau + \sum_{n \geq 0} c_n q^n \qquad \text{where } q = \exp(2i\pi\tau)$$

with c_n non-zero for all even n. In this expression, it is clear that the T-symmetry of the modular group (or a subgroup) is preserved by this series. That is, $\tau \mapsto \tau + 1$ implies $\tau_{\text{eff}} \mapsto \tau_{\text{eff}} + 1$. One would hope that other modular symmetries would be

realised as some linear transformation $\tau_{\mathrm{eff}} \mapsto \frac{A + B\tau_{\mathrm{eff}}}{C + D\tau_{\mathrm{eff}}}$, with the instanton series above allowing us, in some way, to investigate this.

It is the aim of this section to propose an explicit formula relating τ and τ_{eff}. In the first instance this shall be attempted in the form of an ansatz. When this is shown to be unsatisfactory, we formulate the problem in a more systematic way, and propose a relation which should be verifiable using instanton techniques.

4.1 Scale Invariant $N_f = 4$ Theories

The massless $N_f = 4$ model has zero β-function and an identically vanishing $U(1)_R$ anomaly. This means that the microscopic coupling g and the vacuum angle θ combine into an holomorphic scale-independent coupling $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ with an effective Lagrangian which reduces to a classical $N = 2$ free field theory with suitably defined coupling constant. The relevant coupling is the one that appears in the BPS formula for the masses of dyons.

Seiberg and Witten [81] assumed that the absence of a running coupling constant guaranteed that the effective Abelian coupling τ_{eff}, arising from the integrating out of massive states, was identical to the classical $SU(2)$ coupling τ. That is $\tau_{\mathrm{eff}} = \tau$.

In fact, performing an explicit instanton calculation, Dorey et al [31] found that

$$\tau_{\mathrm{eff}} = \tau + \sum_{n \geq 0} c_n q^n \qquad \text{where } q = \exp(2i\pi\tau)$$

with $c_0 = \frac{i}{\pi} 4 \ln 2$ and $c_1 = -\frac{i}{\pi} \frac{7}{2^5 \cdot 3^6}$.

Since the β-function vanishes, with the consequence that no dynamical scale appears in the theory, the curve which controls the low energy behaviour has coefficients which are functions of the effective coupling τ_{eff}. This is in contrast to models with non-vanishing β-function which then have dynamical scales which appear as polynomials in the relevant curve.

Using the above observations, we give a brief outline of how the correct effective coupling τ_{eff} is used.

It is possible to show [81] that the cubic

$$y^2 = \prod_{i=1}^{3} (x - e_i(\tau_{\text{eff}})\tilde{u}) = x^3 - \frac{1}{4}g_2(\tau_{\text{eff}})x\tilde{u}^2 - \frac{1}{4}g_3(\tau_{\text{eff}})\tilde{u}^3 \qquad (4.1.1)$$

is the elliptic curve which encodes the low energy physics. In (4.1.1) we have $e_1 + e_2 + e_3 = 0$ and the Eisenstein series

$$g_2 = \frac{60}{\pi^4} \sum_{m,n \in \mathbb{Z} - \{0\}} (m\tau_{\text{eff}} + n)^{-4} = -4 \sum_{i<j} e_i e_j$$

$$g_3 = \frac{140}{\pi^6} \sum_{m,n \in \mathbb{Z} - \{0\}} (m\tau_{\text{eff}} + n)^{-6} = 4 \prod e_i$$

so the coefficients in (4.1.1) are modular forms of weight 4 and 6 respectively [82]. These depend on the complexified coupling constant τ_{eff}.

The holomorphic one-form $\omega = \frac{\sqrt{2}}{8\pi} \frac{dx}{y}$ has periods [48]

$$\frac{\partial a_D}{\partial \tilde{u}} = \int_{C_1} \omega \qquad \text{where } C_1 \text{ is a contour from 0 to } \pi$$

$$\frac{\partial a}{\partial \tilde{u}} = \int_{C_2} \omega \qquad \text{where } C_2 \text{ is a contour from 0 to } \pi\tau$$

Then

$$a = \sqrt{2}\tilde{u} \qquad\qquad a_D = \tau_{\text{eff}}a \qquad\qquad (4.1.2)$$

which is exactly the defining equations of a classical free field theory with prepotential $\mathcal{F}(a) = \frac{1}{2}\tau_{\text{eff}}a^2$. It is obvious that $\tilde{u} = \frac{1}{2}a^2$.

Now, it is a basic fact that for large fields the theory is weakly coupled and the quantum moduli space is well approximated by the classical moduli space. As such, the limit $\text{Im}\tau_{\text{eff}} \to \infty$ – that is $g \to 0$ – should reduce the cubic 4.1.1 to the classical curve.

Using the series expansion of the e_i [15]

$$e_1 = \frac{2}{3} + 16q^2 + 16q^4 + \dots$$

$$e_2 = -\frac{1}{3} - 8q - 8q^2 - \dots$$

$$e_3 = -\frac{1}{3} + 8q - 8q^2 + \dots$$

we see the curve $y^2 \to \left(x - \frac{2\tilde{u}}{3}\right)\left(x + \frac{\tilde{u}}{3}\right)\left(x + \frac{\tilde{u}}{3}\right)$ which under the reparameterisation $x \mapsto x - \frac{\tilde{u}}{3}$ gives

$$y^2 = x^2(x - \tilde{u})$$

which is the classical curve, an indication that the above prescription is correct.

4.2 An ansatz

In this section we consider a possible relation between the effective and microscopic theories in N=2 supersymmetric gauge theories with vanishing β-function, and in particular the theory with gauge group $SU(N_c)_{N_c=2}$ and $N_f = 4$, where N_f is the number of flavours. That this has vanishing β-function follows from the well known relation $\beta = N_f - 2N_c$.

We introduce the following relation as an ansatz between the microscopic coupling τ and the effective coupling τ_{eff}:

$$\tau_{\text{eff}} = eM_1^{-1}\left(\frac{a + bM_2(f\tau)}{c + dM_2(f\tau)}\right) \qquad (4.2.3)$$

where M_1 and M_2 are two modular functions[1] of two (not necessarily identical) non-identity subgroups of the modular group $\text{Sl}(2, \mathbb{Z})$, and $a, b, \ldots, f$ are constants to be determined.

In order to allow a facilitate a sensible classical limit, we presume that the relation between the two couplings has the property that it maps one weak coupling regime into another. Symbolically, we demand that $\text{Im}(\tau) \to \infty$ iff $\text{Im}(\tau_{\text{eff}}) \to \infty$.

We also take e and f to be positive and real. This is to restrict the couplings, which appear as the argument of the modular functions, to the upper half plane.

These assumptions can be used to attempt to fix the constants in the ansatz for specific choices of M_1 and M_2.

---

[1] In an abuse of notation we will sometimes use M_1 and M_2 to represent the subgroups. Hopefully the context should make this obvious.

4.2.1 $M_1 = M_2 = \mathbf{Sl}(2, \mathbb{Z})$

As a first guess we attempt to resolve the constants in the ansatz (4.2.3) using the modular group $\mathrm{Sl}(2, \mathbb{Z})$. The modular function of this group is commonly denoted as $J(z)$ in mathematical literature, with z in the upper half plane.

It shall suit our purposes to write J in terms of the modular function λ of the subgroup $\Gamma(2) < \mathrm{Sl}(2, \mathbb{Z})$. From [25] it is known that

$$\lambda(z) = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8 \text{ with } q = \exp(i\pi z) \tag{4.2.4}$$

Physically, z is the complexified coupling constant. Using (4.2.4) we can write

$$J(z) = \frac{4}{27} \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2 (1 - \lambda) 2}$$

Thus $J(z)$ has singularities when $\lambda = 0, 1$ or ∞; that is when $z = 0, 1,$ or ∞ [3].

Using [41, pp 23] we know that the inverse of this automorphic function in the regime of weak coupling is given by

$$2\pi i J^{-1}(w) = -\ln w - 3\ln 12 + \frac{G(\frac{1}{12}, \frac{5}{12}; 1; \frac{1}{w})}{{}_2F_1(\frac{1}{12}, \frac{5}{12}; 1; \frac{1}{w})} \quad ; |w| > 1 \text{ and } |\arg(1 - w)| < \pi$$

where

$$G(a, b; 1; \theta) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \left[\psi(a + n) + \psi(b + n) - 2\psi(n + 1) + \psi(a) + \psi(b) - 2\psi(1) \right] \theta^n$$

${}_2F_1$ is a hypergeometric function, $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$ is the digamma function, and the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Now consider the ansatz (4.2.3) for the case $M_1 = M_2 = \mathrm{Sl}(2, \mathbb{Z})$

$$\tau_{\text{eff}} = e J^{-1} \left[\frac{a + b J(f\tau)}{c + d J(f\tau)} \right]$$

If we demand that the weak coupling regimes of τ and τ_{eff} correspond to each other, then since $J(\infty) = \infty$ we see that

$$\lim_{w \to \infty} J(w) = \lim_{\mathrm{Im}(w) \to \infty} \frac{a + b J(w)}{c + d J(w)}$$

which implies $\frac{d}{b} = 0$, that is $d = 0$. Then

$$\tau_{\text{eff}} = e J^{-1} \left(a' + b' J(f\tau) \right) \tag{4.2.5}$$

where $a' = \frac{a}{c}$ and $b' = \frac{b}{c}$.

We are now in a position to expand the RHS of (4.2.5) and fix the free parameters a', b', e and f on comparison with the known result [31]

$$\tau_{\text{eff}} = \tau + \sum_{n \geq 0} c_n q^n \qquad ; q = \exp(2i\pi\tau) \qquad (4.2.6)$$

with $c_0 = \frac{i}{\pi} 4 \ln 2$ and $c_1 = -\frac{i}{\pi} \frac{7}{2^6 3^5}$.

In doing so, one must use the formulae [40] for the digamma function

$$\psi(\theta + n) = \psi(\theta) \sum_{j=0}^{n-1} \frac{1}{\theta + j}$$

and also [40, pp 19]

$$\psi\left(\frac{p}{q}\right) = \psi(1) - \ln q - \frac{\pi}{2} \cot\left(\frac{p}{q}\pi\right) + \sum_{r=1}^{\leq \frac{1}{2}q} \cos\left(2\pi r \frac{p}{q}\right) \ln\left[2 - 2\cos\left(2\pi \frac{r}{q}\right)\right]$$

which allows us to calculate

$$\psi\left(\frac{5}{12}\right) + \psi\left(\frac{1}{12}\right) = 2\psi(1) - 2\pi - 3\ln 3 - 8\ln 2$$

Performing the relevant expansion of (4.2.5) and equating terms with (4.2.6) one determines that $e = f = 1$ due to the absence of odd-instanton terms. Examining constant $O(q^0)$ terms fixes $a' = 256$, and study of the $O(q^2)$ terms yields a value for b' which involves logarithms and powers of π. This then leads on to instanton coefficients which are irrational, a situation which is unlikely to occur in reality. We therefore believe that either we have used the wrong modular groups, or the ansatz is wrong. For now, we try some other situations.

4.2.2 $M_1 = \text{Sl}(2, \mathbb{Z})$, $M_2 = \Gamma(2)$

With the inverse of the modular function of $\Gamma(2)$ defined as in (4.3.16), one finds that the above procedure also results in instanton coefficients dependent on logarithms and powers of π.

4.2.3 $M_1 = M_2 = \Gamma(2)$

Pursuing a similar method to that above, one fixes the constants in the ansatz as

$$\tau_{\text{eff}} = \frac{1}{2}\lambda^{-1}\left(\frac{1}{2^8}\frac{\lambda(2\tau)}{\left(1 - \frac{15493}{2^7 \cdot 3^5}\lambda(2\tau)\right)}\right) \tag{4.2.7}$$

which yields a prediction $c_2 = \frac{i}{\pi}\frac{11 \cdot 201881 \cdot 8713}{2^{15} \cdot 3^{10}} \approx 10\frac{i}{\pi}$. Whilst this appears to be satisfactory in that it provides a rational and finite value for c_2, it is believed that the above result is not the functional relationship between τ_{eff} and τ. We believe this because the corners of the fundamental polygons, defined later, of the two couplings are not sent to sensible values. For example, $\lambda(2\tau) = 1$ maps to $\lambda(2\tau_{\text{eff}}) \approx 7.78 \times 10^{-3}$.

4.2.4 $M_1 = \Gamma(2)$, $M_2 = \text{Sl}(2, \mathbb{Z})$

In this case, the ansatz becomes

$$\tau_{\text{eff}} = \lambda^{-1}\left(\frac{1}{2^{10} \cdot 3^3}\frac{1}{J(\tau) - \frac{7 \cdot 307 \cdot 673}{2^9 \cdot 3^8}}\right)$$

with instanton coefficient $c_2 = \frac{i}{\pi}\frac{190477161233707}{2^{15} \cdot 3^{10}} \approx 10^5\frac{i}{\pi}$ which seems unfeasibly large.

In summary, the ad hoc procedure of attempting to fix the parameters of the ansatz[2] (4.2.3) does not appear to be fruitful. Problems arise in the size or content of the instanton coefficients predicted by this method, or in the mapping properties of the singularities. In the next section we shall attempt to resolve this by using a more analytical method.

4.3 Relating τ_{eff} and τ using Schwarzians

4.3.1 Introduction

Having tried, and failed, to find a relation between the effective coupling τ_{eff} and the microscopic coupling τ using the ansatz (4.2.3), we now attempt another method.

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[2]We have also looked at the possibility that the τ in 4.2.7 is shifted by a constant value, and in particular by $\frac{i}{\pi}4\ln 2$ – so that the τ is replaced by the one loop corrected coupling – but similar problems to those outlined in the main text arise when the calculation is done.

We shall utilise the Schwarz-Christoffel transformation [62] which is concerned with mappings from simply connected domains $\mathcal{D}$ in the complex plane into the upper half plane $\mathcal{U}$. In particular the $\mathcal{D}$ shall be polygons bounded by circular arcs, and the Schwarz-Christoffel transformations correspond to the couplings mapping into the space of modular functions. This method will automatically detect any relation of the form (4.2.3). Central to this work is the theorem [62]

Theorem 4.1. *The function $y(x)$ mapping $Imx > 0$ onto a curvilinear polygon with n vertices $A_1, A_2, \ldots, A_n$ bounded by circular arcs which form angles $\pi\alpha_1, \ldots \pi\alpha_n$ and such that the vertices correspond to $a_1, \ldots, a_n$ satifies*

$$\{y(x), x\} = \sum_{i=1}^{n} \left(\frac{1}{2} \frac{1 - \alpha_i^2}{(x - a_i)^2} + \frac{\beta_i}{x - a_i} \right)$$

with conditions

$$\sum_{i=1}^{n} \beta_i = 0$$

$$\sum_{i=1}^{n} \left[2a_i\beta_i + 1 - \alpha_i^2 \right] = 0$$

$$\sum_{i=1}^{n} \left[\beta_i a_i^2 + a_i \left(1 - \alpha_i^2 \right) \right] = 0$$

where the "Schwarzian"

$$\{y(x), x\} = \frac{\partial_{xxx} y}{\partial_x y} - \frac{3}{2} \left(\frac{\partial_{xx} y}{\partial_x y} \right)^2 = \partial_x \left(\frac{\partial_{xx} y}{\partial_x y} \right) - \frac{1}{2} \left(\frac{\partial_{xx} y}{\partial_x y} \right)^2$$

Using this definition, it is straightforward to see that the Schwarzian is invariant under conformal transformations $y(x) \to \frac{Ay(x)+B}{Cy(x)+D}$ where $A, B, C, D \in \mathbb{C}$. The conditions on the parameters α_i, β_i arise by demanding that $\tau(\lambda)$ be regular at large λ and an assumption on the form of $\tau(\lambda)$ near singularities. Details can be found in [62].

The above conditions simplify if one of the singularities is at ∞. Starting with

$$\sum_{i=1}^{n} \left(2a_i\beta_i + 1 - \alpha_i^2 \right) = 0$$

so

$$\sum_{i=1}^{n-1} \left(2a_i\beta_i - \alpha_i^2 \right) = \alpha_\infty^2 - n - 2I\beta_\infty$$

where I is a symbol representing infinity in this case, and

$$\sum_{i=1}^{n-1} \left[\beta_i a_i^2 + a_i \left(1 - \alpha_i^2 \right) \right] = -I^2 \beta_\infty - I \left(1 - \alpha_\infty^2 \right)$$

We can now eliminate $O(I^2)$ terms and take $I \longrightarrow \infty$ to obtain the general case

$$\sum_{i=1}^{n-1} \left(2a_i \beta_i - \alpha_i^2 \right) = (2 - n) - \alpha_\infty^2$$

However, we are concerned only with curvilinear regions with "parallel sides" at infinity, and so we take $\alpha_\infty = 0$ so that

$$\sum_{i=1}^{n-1} \left(2a_i \beta_i - \alpha_i^2 \right) = (2 - n)$$

We can now solve the constraints for the case

$$a_1 = -a_2 = -u \qquad a_3 = \infty \qquad \alpha_1 = \alpha_2 = \alpha \qquad \alpha_3 = 0$$

which correspond to a symmetric region with 3 singularities, one of which is at infinity.

It is a matter of straightforward algebra to find that

$$\beta_\infty = 0 \tag{4.3.8a}$$

$$\beta_{-1} = -\beta_1 = \frac{1}{4u} \left(1 - 2\alpha^2 \right) \tag{4.3.8b}$$

This information shall be used later.

4.3.2 A Relation Between τ and τ_{eff}

It has been shown in [31] that

$$\tau_{\text{eff}}(\tau) = \tau + \sum_{n \geq 0} c_n q^n \qquad \text{where } q = e^{2\pi i \tau} \tag{4.3.9}$$

where

$$c_0 = \frac{i}{\pi} 4 \ln 2 \qquad c_1 = -\frac{i}{\pi} \frac{7}{2^6} 3^5 \tag{4.3.10}$$

We shall now attempt to invert this to determine $\tau(\tau_{\text{eff}})$. We have on exponentiating (4.3.9) that

$$q_{\text{eff}} e^{-2\pi i c_0} = q \prod_{n \geq 1} e^{2\pi i c_n} q^n \qquad \text{where } q_{\text{eff}} = e^{2\pi i \tau_{\text{eff}}} \qquad (4.3.11)$$

We now expand (4.3.11) around the weak coupling point $q = 0$ to get

$$q_{\text{eff}} e^{-2\pi i c_0} = q + q^2 \left(2\pi i c_1\right) + q^3 \left(2\pi i c_2 + \frac{1}{2} \left(2\pi i c_1\right)^2\right) + O(q^4) \qquad (4.3.12)$$

Now consider the general expansion $q = \sum_{n \geq 0} A_{n-1} q_{\text{eff}}^n$. Since we assume that the weak coupling regimes in the microscopic and effective theories correspond to each other we see that $A_{-1} = 0$. Then

$$q = q_{\text{eff}} \left(A_0 + A_1 q_{\text{eff}} + A_2 q_{\text{eff}}^2 + \dots\right) \qquad (4.3.13)$$

Substituting this into (4.3.12) and equating powers of q_{eff} yields

$$A_0 = e^{-2\pi i c_0}$$
$$A_1 = -e^{-4\pi i c_0} \left(2\pi i c_1\right)$$
$$A_2 = -e^{-6\pi i c_0} \left[2\pi i c_2 - \frac{3}{2} \left(2\pi i c_1\right)^2\right]$$

Taking logarithms of (4.3.13) with the values for A_i substituted in, and dividing the final result by $2\pi i$ gives the result

$$\tau = \tau_{\text{eff}} - c_0 - c_1 e^{-2\pi i c_0} q_{\text{eff}} + \left(2\pi i c_1^2 - c_2\right) e^{-4\pi i c_0} q_{\text{eff}}^2 + O(q_{\text{eff}}^4) \qquad (4.3.14)$$

as the inverted formula defining $\tau(\tau_{\text{eff}})$.

4.3.3   Modular Function with 3 Singularities at $\{-1, 1, \infty\}$ and its Inverse

We shall find it useful to investigate the relationship between modular functions with singularities at $\{-1, 1, \infty\}$ and $\{0, 1, \infty\}$, that is, between the modular groups $\Gamma(2)$ and $\Gamma_0(2)$ which are both index six subgroups of $Sl(2, \mathbb{Z})$.

4.3.4   The modular function of $\Gamma(2)$

This is well known in the literature where it is conventionally written as λ. From this point on, we shall denote it by λ_0 where the subscript 0 indicates that we have

a singularity at the origin in the complex λ_0 plane under the map $\tau_{\text{eff}}(\lambda_0)$. We shall take

$$\lambda_0(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8 \qquad \text{where } q = e^{i\pi z}, z \in \mathbb{C} \qquad (4.3.15)$$

with inverse [42]

$$\lambda_0^{-1}(z) = i \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - z)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; z)} \qquad (4.3.16)$$

which is a Schwarz triangle function, and Schwarzian

$$\{z, \lambda_0\} = \frac{1}{2} \left[\frac{1}{(1 - \lambda_0)^2} + \frac{1}{\lambda_0^2} + \frac{1}{(1 - \lambda_0)\lambda_0} \right] \qquad (4.3.17)$$

Lastly we note that λ_0 satisfies the so-called "Picard–Fuchs" equation [23]

$$\lambda_0 (1 - \lambda_0) \frac{d^2\Omega}{d\lambda_0^2} + (1 - 2\lambda_0) \frac{d\Omega}{d\lambda} - \frac{1}{4}\Omega = 0 \qquad (4.3.18)$$

where following the usual theory [62] a solution to the Schwarzian 4.3.17 is a ratio of two linearly independent solutions of this Pichard–Fuchs equation.

The Modular Function of $\Gamma_0(2)$

For this modular subgroup, we know that there are three singularities in the complex plane located at $\{-1, 1, \infty\}$ with zero open angles at the vertices of the fundamental polygon. This appears extremely similar to the case of $\Gamma(2)$. As such, we can utilise the modular function λ_0 of $\Gamma(2)$ to determine that of $\Gamma_0(2)$, which we shall denote by λ_{-1}.

We note that there are six transformations taking the points $\{0, 1, \infty\}$ onto the set $\{-1, 1, \infty\}$. They are

λ_{-1}	$\frac{2}{\lambda_0} - 1$	$2\lambda_0 - 1$	$\frac{1+\lambda_0}{1-\lambda_0}$
	$\{\infty, 1, -1\}$	$\{-1, 1, \infty\}$	$\{1, \infty, -1\}$

and their negatives. We are interested in the particular endomorphism

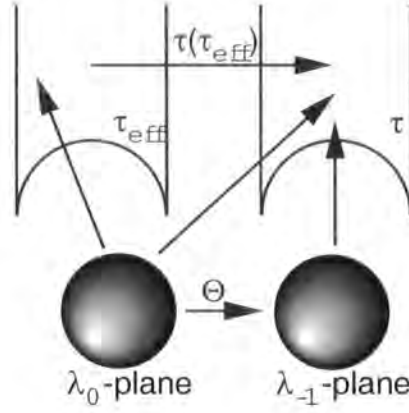$$\lambda_{-1} = \frac{2}{\lambda_0} - 1 \qquad (4.3.19)$$

Figure 4.1: Showing the relation between $\Gamma(2)$ and $\Gamma_0(2)$ with particular emphasis on how the functions compose. Θ is the map $\Theta(\lambda_0) = -1 + 2/\lambda_{-1}$

which has the effect

$$
\begin{array}{ccccc}
x & & \lambda_0 & & \lambda_{-1} \\
\hline
\infty & \longmapsto & 0 & \longmapsto & \infty \\
0 & \longmapsto & 1 & \longmapsto & 1 \\
1 & \longmapsto & \infty & \longmapsto & -1
\end{array}
\tag{4.3.20}
$$

The situation we are considering is illustrated in diagram 4.1

Under this mapping, the fundamental regions of $\Gamma(2)$ and $\Gamma_0(2)$ are transformed injectively into each other. For example the line Re $\lambda_0 = 0$ is sent onto the line $\frac{2}{1+\lambda_{-1}} + \frac{2}{1+\lambda_{-1}} = 0 \implies$ Re $\lambda_{-1} = -1$, and likewise Re $\lambda_0 = 1$ is mapped to the circle $|\lambda_{-1}|^2 = 1$ and the circle $|\frac{1}{2} - \lambda_0|^2 = \frac{1}{4}$ is taken to the line Re $\lambda_{-1} = 1$. Further, it is also apparent that since Re $\lambda_0 > 0 \mapsto$ Re $\lambda_{-1} > -1$, Re $\lambda_0 < 1 \mapsto |\lambda_{-1}|^2 > 1$ and $|\frac{1}{2} - \lambda_0|^2 > \frac{1}{4} \mapsto$ Re $\lambda_{-1} < 1$ we see that the interior of the fundamental region of $\Gamma(2)$ is sent to the interior of that of $\Gamma_0(2)$. These relations can be seen in figure 4.2.
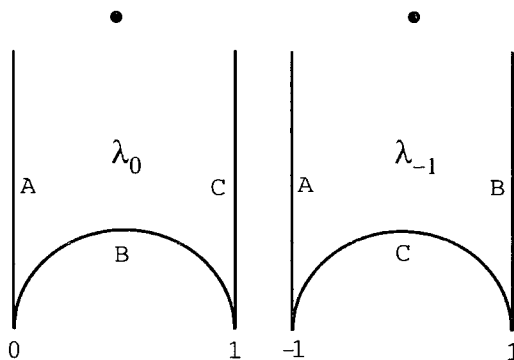
Figure 4.2: The relation between the fundamental polygons of the two modular groups $\Gamma(2)$ and $\Gamma_0(2)$ under the mapping $\lambda_{-1} = -1 + 2/\lambda_0$

Now consider the Schwarzian

$$
\begin{aligned}
\{x, \lambda_{-1}\} &= \{x, \lambda_0\} \left(\frac{\partial \lambda_0}{\partial \lambda_{-1}}\right)^2 + \{\lambda_0, \lambda_{-1}\} \\
&= \{x, \lambda_0\} \left(\frac{\partial \left(\frac{2}{1+\lambda_{-1}}\right)}{\partial \lambda_{-1}}\right)^2 \\
&= \{x, \lambda_0\} \frac{4}{(1 + \lambda_{-1})^4} \\
&= \frac{2}{(1 + \lambda_{-1})^4} \left[\frac{1}{\lambda_0^2} + \frac{1}{(1 - \lambda_0)^2} + \frac{1}{\lambda_0(1 - \lambda_0)}\right] \\
&= \frac{2}{(1 + \lambda_{-1})^4} \left[\frac{(1 + \lambda_{-1})^2}{4} + \frac{(1 + \lambda_{-1})^2}{(-1 + \lambda_{-1})^2} + \frac{(1 + \lambda_{-1})^3}{(-1 + \lambda_{-1})}\right] \\
&= \frac{1}{2} \frac{1}{(1 + \lambda_{-1})^2} + \frac{2}{(1 + \lambda_{-1})^2(-1 + \lambda_{-1})^2} + \frac{1}{(1 + \lambda_{-1})(-1 + \lambda_{-1})} \\
&= \frac{\lambda_{-1}^2 + 3}{2(-1 + \lambda_{-1})^2(1 + \lambda_{-1})^2}
\end{aligned}
\tag{4.3.21}
$$

Thus

$$
\{\tau, \lambda_{-1}\} = \frac{1}{2} \left[\frac{1}{(-1 + \lambda_{-1})^2} + \frac{1}{(1 + \lambda_{-1})^2} + \frac{1}{(1 + \lambda_{-1})(1 - \lambda_{-1})}\right]
\tag{4.3.22}
$$

as expected from general considerations of the fundamental region of $\Gamma_0(2)$ and using the formulae in [40]. In this analysis we have used the inverse relation $\lambda_0 = \frac{2}{1+\lambda_{-1}}$, the consequent identity $1 - \lambda_0 = \frac{-1+\lambda_{-1}}{1+\lambda_{-1}}$, and the fact that that $\{\lambda_0, \lambda_{-1}\} = 0$ since quantities related by a fractional linear mapping have vanishing Schwarzian.

Comparing with (4.3.16) we see that

$$x = i\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{-1+\lambda_{-1}}{1+\lambda_{-1}}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2}{1+\lambda_{-1}}\right)} = i\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\lambda_0\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda_0\right)} \tag{4.3.23}$$

It is now possible to derive the counterpart to the Picard–Fuchs equation (4.3.18). We shall calculate this in some detail since some of the calculation shall be useful in determining an ordinary differential equation for the microscopic coupling τ.

Consider the canonical hypergeometric equation with regular singularities at $0, 1,$ and ∞

$$z(1-z)\frac{d^2}{dz^2}u(z) + (c - (a+b+1)z)\frac{d}{dz}u(z) - abu(z) = 0$$

where a, b and c are constants related to the open angles in a fundamental polygon bounded by arcs to be described later.

We map this to an equation with singularities at $-1, 1$ and ∞ using the active transformation $z \mapsto \frac{2}{1+z}$ met previously. Performing this mapping leads to

$$(1+z)(1-z)\frac{d^2}{dz^2}u(z) + ((c-2)z + c - 2a - 2b)\frac{d}{dz}u(z) + \frac{2ab}{1+z} = 0 \tag{4.3.24}$$

By means of the substitution

$$u(z) = v(z)\exp\left(-\frac{1}{2}\int^z dw \frac{(c-2)w + (c-2a-2b)}{(1+z)(1-z)}\right)$$

we can reduce this to the form

$$\frac{d^2}{dz^2}v(z) + I(z)v(z) = 0 \tag{4.3.25}$$

where

$$I(z) = \frac{\alpha z^2 + \beta z + \gamma}{(z+1)^2(z-1)^2} \tag{4.3.26}$$

with

$$\alpha = \frac{1}{2}c - \frac{1}{4}c^2$$
$$\beta = -2ab + ca - \frac{1}{2}c^2 + cb$$
$$\gamma = 1 - \frac{1}{2}c + ca - cb - a^2 - \frac{1}{4}c^2 - b^2$$

However, it is known from the general theory of hypergeometric equations that the function $I(z)$ in (4.3.25) makes an appearance in the Schwarzian

$$\{G(z), z\} = 2I(z)$$

where G(z) is the ratio of two linearly independent solutions of equation (4.3.25), or equivalently (4.3.24). Now, from (4.3.21) we have seen

$$\{\tau_{\text{eff}}, z\} = \frac{z^2 + 3}{(z+1)^2(z-1)^2}$$

and on comparison with (4.3.26) we obtain the solution set

$$
\begin{aligned}
c &= 1 \\
a &= b = \frac{1}{2}
\end{aligned}
$$

which on substitution into (4.3.26) gives the relevant equation of hypergeometric type

$$(1-z)(1+z)\frac{d^2}{dz^2}u(z) - (1+z)\frac{d}{dz}u(z) + \frac{1}{2(1+z)} = 0 \tag{4.3.27}$$

which is important, since the ratio of two linearly independent solutions of this equation will yield a τ_{eff} relevant for $\Gamma(2)$.

4.4 Relating τ to τ_{eff}

We now seek to combine the findings of [31] with the analysis of the preceeding sections to propose an ansatz for the microscopic coupling τ. This will then allow us to predict the unknown constants which appears in the expansion

$$\tau_{\text{eff}} = \tau + \sum_{n \geq 0} c_n q^n \qquad \text{where } q = \exp(2\pi i \tau) \tag{4.4.28}$$

with c_n the instanton coefficients to be fixed when $n \geq 2$. In particular, we shall evaluate c_2. In what follows we shall denote λ_{-1} as z.

Consider the schwarzian

$$
\begin{aligned}
\{\tau_{\text{eff}}, \tau\} &= \{\tau_{\text{eff}}, z\}\left(\frac{\partial z}{\partial \tau}\right)^2 + \{\tau, z\} \\
&= \left(\frac{\partial z}{\partial \tau}\right)^2 [\{\tau_{\text{eff}}, z\} - \{\tau, z\}]
\end{aligned}
\tag{4.4.29}
$$

where we have used the identity

$$\{g\left(f(x)\right),x\} = \{g,f\}\left(\frac{\partial f}{\partial x}\right)^2 + \{f,x\}$$

which can be used, on letting $g = x$ and $f(x) = y(x)$, to derive the inverse relation

$$\{y,x\} = -\{x,y\}\left(\frac{\partial x}{\partial y}\right)^2$$

We now assume that the fundamental domains of the inequivalent couplings τ and τ_{eff} are polygons in the upper half plane which are bounded by circular[3] arcs. As such, the two Schwarzians on the right hand side of (4.4.29) have the standard form

$$\{\tau_{\text{eff}}, z\} = \sum_{i=1}^{n-1} \frac{1}{2} \frac{1-\alpha_i^2}{(z-a_i)^2} + \frac{\beta_i}{(z-a_i)} \tag{4.4.30}$$

and similarly

$$\{\tau, z\} = \sum_{i=1}^{n-1} \frac{1}{2} \frac{1-\tilde{\alpha}_i^2}{(z-\tilde{a}_i)^2} + \frac{\tilde{\beta}_i}{(z-\tilde{a}_i)} \tag{4.4.31}$$

where n is the number of corners (singularities) of the polygon. Substituting these into (4.4.29) then gives

$$\{\tau_{\text{eff}}, \tau\}\left(\frac{\partial \tau}{\partial z}\right)^2 = \sum_{i=1}^{n-1} \frac{1}{2} \frac{1-\alpha_i^2}{(z-a_i)^2} + \frac{\beta_i}{(z-a_i)} - \sum_{i=1}^{\tilde{n}-1} \frac{1}{2} \frac{1-\tilde{\alpha}_i^2}{(z-\tilde{a}_i)^2} + \frac{\tilde{\beta}_i}{(z-\tilde{a}_i)} \tag{4.4.32}$$

We now insist that both fundamental domains have the same number of singularities $(n = \tilde{n})$ and moreover that they are mapped into each other. We can relabel to ensure $a_i = \tilde{a}_i$. This is defined up to a linear fractional mapping on the z-plane, which itself is equivalent to an appropriate redefinition of the (α_i, a_i) parameters.

We have seen previously that the parameters in (4.4.32) are not free: rather they are constrained as in theorem 4.1. If we now accept that the domain representative of τ_{eff} is that of $\Gamma(2)$ as suggested by Seiberg and Witten [81] then we have

$$a_1 = -a_{-1} = 1 \qquad \alpha_i = 0 \ \forall i \qquad \beta_1 = -\beta_{-1} = -\frac{1}{4}$$

as calculated previously (4.3.8a, 4.3.8b).

---

[3]In the usual way, straight lines are considered as arcs of zero curvature.

The situation for the microscopic coupling τ is less certain. To be compatible with the singularity structure we must have that the fundamental polygon of τ has three singularities located at $-1, 1$ and ∞. We also assume that the domain is symmetric. This is a natural consequence of demanding that the singularities of τ_{eff} and τ map to each other and that they be joined by arcs of constant curvature. Due to the constraints in theorem 4.1 we have seen (4.3.8a, 4.3.8b) that this situation removes all but one degree of freedom which we shall write as a parameter $\tilde{\alpha}$. More explicitly

$$\tilde{\alpha}_1 = \tilde{\alpha}_{-1} = \tilde{\alpha} \qquad \tilde{\beta}_1 = -\tilde{\beta}_{-1} = -\frac{1}{4}(1 - 2\tilde{\alpha}^2)$$

Substituting these values into (4.3.20) and simplifying gives

$$\{\tau_{\text{eff}}, \tau\} \left(\frac{\partial \tau}{\partial z}\right)^2 = \frac{2\tilde{\alpha}^2}{(1 + z)^2(1 - z)^2}$$

We can now expand this about the point $z = \infty$, which from (4.3.20) corresponds to the weak coupling regime. Then, using a standard McLaurin Series

$$\{\tau_{\text{eff}}, \tau\} \left(\frac{\partial \tau}{\partial z}\right)^2 = \frac{2\tilde{\alpha}^2}{z^4} \sum_{n=0}^{\infty} (n + 1)z^{-n} \tag{4.4.33}$$

We know from (4.3.23) that

$$\tau_{\text{eff}} = i \frac{{}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{-1+z}{1+z}\right)}{{}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2}{1+z}\right)} \tag{4.4.34}$$

and can expand

$$\frac{-1 + z}{1 + z} = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{-1}{z}\right)^n$$

Now, from [40, pp 95] we have

$$\frac{\pi}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \theta\right) + \frac{1}{2} \ln(1 - \theta) \, {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \theta\right) = \sum_{n=0}^{\infty} B_n (1 - \theta)^n$$

with

$$B_n = \left[\frac{\left(\frac{1}{2}\right)_n}{n!}\right]^2 \left[\psi(n + 1) - \psi(n + \frac{1}{2})\right] \tag{4.4.35}$$

Using the transformation $\theta \mapsto 1 - \theta$ allows us to write

$$_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \theta\right) = -\frac{1}{\pi} \ln \theta_2 F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \theta\right) + \frac{2}{\pi} \sum_{n=0}^{\infty} B_n \theta^n \tag{4.4.36}$$

which allows $_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\theta\right)$ to be expanded about $\theta = 0$.

Since we shall be interested in terms beyond the zeroth order in θ it is useful to write

$$B_0 = 1 \quad ; \quad B_n = \prod_{j=0}^{n-1} \frac{(j+\frac{1}{2})^2}{(j+1)^2}\left(2\ln 2 - \sum_{k=1}^{n}\frac{1}{k(2k-1)}\right) \quad \forall n \qquad (4.4.37)$$

where we have used $\psi(1) - \psi(\frac{1}{2}) = 2\ln 2$ and $\psi(x+n) = \psi(x) + \sum_{j=0}^{n-1}\frac{1}{x+j}$. We note in particular that

$$B_0 = 1 \qquad B_1 = \frac{1}{2}\ln 2 - \frac{1}{4} \qquad B_2 = \frac{3}{128}(12\ln 2 - 7) \qquad (4.4.38)$$

Using these relations with $\theta = -2\sum_{n=1}^{\infty}(-z)^{-n}$ to see that the numerator in (4.4.34) is

$$\frac{1}{\pi}\left(3\ln 2 + \ln z + \frac{1}{2z}\left(3\ln 2 + \ln z\right) + \frac{1}{16z^2}\left(3\ln 2 - \ln x - 5\right)\right) + O(z^{-3}) \quad (4.4.39)$$

To determine the denominator we use the power series [13] for the hypergeometric function

$$_2F_1(a,b,;c;\theta) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{c_n n!}\theta^n$$

with

$$\theta = \frac{2}{1+z} = \frac{2}{z}\sum_{n=0}^{\infty}(-z)^{-n}$$

Evaluating this power series and combining it with (4.4.39) in (4.4.34) gives

$$\tau_{\text{eff}}(z) = \frac{i}{\pi}\left[3\ln 2 + \ln z - \frac{5}{16}\frac{1}{z^2}\right] + O(z^{-3}) \qquad (4.4.40)$$

That this is correct – at least to this order – can be seen by substituting τ_{eff} into

$$z = \frac{2-\tilde{z}}{\tilde{z}} \text{ where } \tilde{z} = 16\tilde{q}\prod_{n=1}^{\infty}\left(\frac{1+\tilde{q}^{2n}}{1+\tilde{q}^{2n-1}}\right)^8 \text{ with } \tilde{q} = \exp(i\pi\tau_{\text{eff}})$$

with $\tilde{z}$ is the modular function (4.3.15) for $\Gamma(2)$, and we have used the relation (4.3.19).

Now, from section 4.3.1 we know

$$\tau = \tau_{\text{eff}} - c_0 - c_1 e^{-2\pi i c_0}q_{\text{eff}} + \left(2\pi i c_1^2 - c_2\right)e^{-4\pi i c_0}q_{\text{eff}}^2 + O(q_{\text{eff}}^4) \qquad (4.4.41)$$

with $q_{\text{eff}} = \exp(2i\pi\tau_{\text{eff}})$. Using (4.4.39), (4.4.40) and (4.4.41) it is easy to see, after some manipulation, that

$$-\{\tau, \tau_{\text{eff}}\}\left(\frac{\partial\tau_{\text{eff}}}{\partial z}\right)^2 = \frac{7}{2\cdot 3^5}\frac{1}{z^4} + \left(\frac{7\cdot 7213}{2^5\cdot 3^{10}} + 2^{10}i\pi c_2\right)\frac{1}{z^6} \tag{4.4.42}$$

Comparing this expansion with (4.4.33) we can extract

$$\tilde{\alpha}^2 = \frac{7}{2^2\cdot 3^5} \tag{4.4.43}$$

which then fixes the erstwhile free parameter in $\{\tau, z\}$.

Taking (4.4.43) and equating $O(z^{-6})$ between (4.4.33) and (4.4.43) we can therefore make a prediction that the four instanton coefficient c_2 in the series (4.4.28) is

$$c_2 = \frac{i}{\pi}\frac{7\cdot 17\cdot 421}{2^6\cdot 3^{10}\cdot 521}$$

In principal the above method provides an algorithm for predicting all of the instanton coefficients, but we shall not pursue that route in this chapter.

Lastly, we can attempt to evaluate an ordinary differential equation which has the property that the ratio of two linearly independent solutions yields a functional form for τ in terms of the modular function z of $\Gamma_0(2)$.

We know that

$$\begin{aligned}
\{\tau, z\} &= \frac{1}{2}\frac{\frac{1}{2}z^2 + \frac{3}{2} - 2^2}{\tilde{\alpha}} \\
&= \frac{\frac{361}{243} + \frac{1}{2}z^2}{(1-z)^2(1+z)^2}
\end{aligned}$$

on substituting (4.4.43) into (4.4.31). Using a similar technique to that applied in the previous section, we can determine what the parameters $a, b,$ and c are in the equation of hypergeometric type (4.3.24)

$$(1+z)(1-z)\frac{d^2}{dz^2}u(z) + ((c-2)z + c - 2a - 2b)\frac{d}{dz}u(z) + \frac{2ab}{1+z} = 0$$

One finds that

$$\begin{aligned}
c &= 1 \\
b(1-a) + a(1-b) &= 1/2 \\
(a-b)^2 &= \frac{7}{972}
\end{aligned}$$

which can obviously be solved for a and b. We shall not do this, since it merely gives parameters with square roots. We note, however, that all of a, b and c are real and positive. We can also see that $a = b$ due to the interchange symmetry in the above system as expected due to the nature of the fundamental polygon proposed for τ.

4.5 Conclusions

In this chapter we have investigated the relationship between the effective coupling constant τ_{eff} and the microscopic coupling τ. Our derivations relied crucially on the instanton series calculated by Dorey et al (4.4.28) which gave a power series relation between the two complexified couplings.

Our first attempt to construct an analytic expression utilised an ansatz which we hoped would allow us to investigate any subsequent duality properties. Ultimately this failed when we demanded that it should predict sensible instanton coefficients and map singularities into sensible values.

We then changed the direction of our analysis, and drew on the methods of Schwarz-Christoffel transformations and their relationship with second order differential equations of hypergeometric type. Making several apparently sensible assumptions about the nature of the fundamental polygons, we were able to predict the instanton coefficient c_2 in the series (4.3.9). This is verifiable by utilising the ADHM method and would be the aim of future analysis. However, since it involves a four instanton process, this would more than likely be an involved calculation. A more serious obstacle is that, to our knowledge, no group has constructed a solution to the ADHM constraints in the case where $n = 4$. Finally, we have an implicit relation between τ_{eff} and τ. In the above text we have $\tau(z)$ and $z(\tau_{\text{eff}})$. It is obvious that one can the determine $\tau(\tau_{\text{eff}})$ using the analysis in the text.

As such, we conclude that if τ has three singularities located at -1, 1 and ∞ then it must be related to τ_{eff} as found in this chapter.

Chapter 5

Higher Derivatives from Branes

In the mid-1990's, a revolution occurred in the point of view of the string theory community. Up until that time, it was believed that there were 5 consistent and distinct string theories which were candidates for providing a unifying theory for modern particle physics. These were the type I, type IIA, type IIB and the two heterotic theories. There was a great deal of dissatisfaction that no one model could assert itself as <u>the</u> master theory which provided a description of the world as we see it. Eventually, it was discerned that each of these models, when one adds in 11-dimensional supergravity, were different facets of the same theory. The presence of so-called dualities between the parameters of the models gave rise to the notion of a moduli space of string theories, with each of the distinct models listed above lying in particular regions of this complicated web. This is M-theory.

An essential ingredient in realising this encompassing theory was the observation that there existed extended objects which generalised the notion of particle world-lines and string world-sheets. These were termed p-branes, and had (p+1)-dimensional world-volumes. In this scheme, world-lines are termed 0-branes and world-sheets 1-branes.

In M-theory, the low energy effective theory indicated the presence of 5-branes. Using a particular embedding of this 5-brane in spacetime, Witten was able to show that the results of 4d N=2 supersymmetric gauge theories were derivable from the

M-theory model. It is the aim of this chapter to review this method, demonstrating that the correct kinetic terms for the scalar fields arise. We will then examine[1] whether the higher derivative terms derived from M-theory coincide with those of the field theory.

5.1 Seiberg Witten Action from M–Theory

In this section we shall derive an action which corresponds to the kinetic terms for the scalar fields arising in the N=2 supersymmetric $SU(k)$ gauge theories studied by Seiberg-Witten [80, 81]. This shall be done in the framework of M–theory. A thorough discussion of the features of M–theory would take us too far afield from the purpose of this chapter. Instead we refer the reader to several of the excellent reviews which have appeared [38, 78, 89] and shall content ourselves with using the relevant results.

5.1.1 An M-Theory System with N=2 Worldvolume Supersymmetry

We wish to compare models with N=2 supersymmetry with respect to an observer in 4d. Since we will eventually consider physics on an M5 brane, and it is known that these objects satisfy a topological bound with the result that they preserve $\frac{1}{2}$ of the spacetime supersymmetries, we shall give an example of how to construct a background spacetime which has $N = 4$ supersymmetries to the 4d observer.

Let the background spacetime $\mathcal{M}_{11}$ be 11-dimensional, and to allow comparison with field theories in a Minkowski spacetime $\mathbb{R}^{1,3}$ let it be a product manifold of the form $\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathcal{M}_7$ with M_7 a 7d manifold. We wish there to be global supersymmetries in the $\mathbb{R}^{1,3}$, a condition which requires us to find spinors which are invariant under parallel transport in $\mathcal{M}_{11}$. Another way to state this is that the spinors must be covariantly constant in $\mathcal{M}_7$. If we want the background spacetime to

---

[1]Whilst undertaking this analysis, several papers were published which encompassed my work to this end. As such, this chapter should be seen as a short review of already published material, indicating the relationship between higher derivatives in different physical systems.

have N=4 global supersymmetry, the problem finally reduces to discerning 4 spinors which are in a trivial representation of the holonomy group, a mathematical structure which encodes how an object will transform under parallel transport around a closed loop in a manifold.

The largest [91] holonomy group for a 7-manifold is $SO(7)$. This has an associated semi-simple Lie algebra written as B_3 in the Dynkin classification. The spinor representation for this algebra is 8-dimensional [26] and is written as $\underline{8}$. This can be seen[2] by considering the general formula [69] for the dimension of the clifford algebra $Cl(2n+1)$ in an (2n+1)-dimensional manifold: $\dim[Cl(2n+1)] = 2^n$. We are required to find an embedding of a proper subalgebra in B_3 such that this representation has four trivial representations of the subalgebra. The exponentiation of this subalgebra will then give the holonomy group which will guarantee $N = 4$ supersymmetry. Using data concerning branching rules [71] to investigate which algebra leads on to the desired situation, one finds that only $SU(2)$, whose Lie algebra is A_1 in the Dynkin classification, yields the desired result

$$\underline{8} \xrightarrow{A_3} \bar{\underline{4}} \oplus \underline{4} \xrightarrow{A_2} \underline{3} \oplus \bar{\underline{3}} \oplus \underline{1} \oplus \underline{1} \xrightarrow{A_1} \underline{2} \oplus \underline{2} \oplus \underline{1} \oplus \underline{1} \oplus \underline{1} \oplus \underline{1}$$

Thus, in order to have a background spacetime with four supersymmetries, the manifold $\mathcal{M}_7$ is required to have $SU(2)$ holonomy. We can take[3] $\mathcal{M}_7 = \mathbb{R}^3 \times Q^4$ where Q^4 is a 4-manifold with $SU(2)$ holonomy. Thus, our spacetime is the product

$$\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathbb{R}^3 \times Q^4$$

Complexifying Q^4 means [55] that it is a so-called hyper-Kähler manifold, a special case of a Calibi-Yau manifold which are complex m-manifolds with $SU(m)$ holonomy.

Into this background spacetime, one can now immerse the 6-dimensional surface – the M5 brane – which arises in M-theory. In general, such an immersion generically

---

[2]It is a standard result [69] that the representations of the a Clifford algebra are the (spinor) representations of the universal cover of the special orthogonal groups.

[3]This can be made more rigorous when one considers the decomposition of the vector representation of B_3 under A_1. Using branching rules [71] we have

$$\underline{7} \xrightarrow{A_3} \underline{6} \oplus \underline{1} \xrightarrow{A_2} \underline{3} \oplus \bar{\underline{3}} \oplus \underline{1} \xrightarrow{A_1} \underline{2} \oplus \underline{2} \oplus \underline{1} \oplus \underline{1} \oplus \underline{1}$$

and then identify an $\mathbb{R}^3$ with the orbits of the singlets under the group action of $SU(2)$.

breaks all of the supersymmetry in the spacetime, but there are situations in which some is preserved [12]. This can be seen in the following discussion.

It is known [2] that the action for the M5 brane can be written in a form which is analogous to the construction of the supersymmetric string action in the Green–Schwarz formalism [54]. Let a flat[4] 11d spacetime have co-ordinates y^M where $M = 0, 1, \cdots 10$ and introduce a 32 component Majorana spinor Θ which will be used to construct an $N = 1$ superspace. If we let Γ be the Gamma matrices of the 11d Clifford algebra, then we have the defining relation $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$ where g^{MN} is the metric of our spacetime which has signature $(-, +, +, \ldots +)$.

The action for the 5-brane in 11d is written in terms of the differentials

$$d\Theta \qquad \text{and} \qquad \Pi^M = dy^M - \bar{\Theta}\Gamma^M d\Theta \qquad (5.1.1)$$

where the exterior derivative $d = dx^m \partial_m$ is the pull back to the brane worldvolume, and we consider x^m to represent the co-ordinates on the worldvolume. To an observer in the brane, the (super)co-ordinates of the background spacetime manifest themselves as fields. This should be familiar to those who have a aquaintance with string theory. A supersymmetry transformation of these fields is defined as

$$\delta_\epsilon \Theta = \epsilon \qquad \text{and} \qquad \delta_\epsilon y^M = \bar{\epsilon}\Gamma^M \Theta \qquad (5.1.2)$$

which obviously leave (5.1.1), and therefore the action, invariant. In (5.1.2) ϵ is clearly a constant 32-component Majorana spinor. Although the precise form of the 5-brane action shall not concern us, we note that it was discerned that one must include an additional (local) symmetry of the action if one is to obtain the correct number of components in Θ to guarantee supersymmetry. This is the so-called kappa symmetry familiar from string theory. It acts as

$$\delta_\kappa \Theta = 2P_+\kappa(x) \qquad \text{and} \qquad \delta_\kappa y^M = 2i\bar{\Theta}\Gamma^M P_+\kappa(x)$$

---

[4]The analysis in a curved spacetime is more complicated, but yields the same conclusions. A quick way to justify this is to consider the curved space as being immersed in $\mathbb{R}^{1,3}$. The Clifford algebra above (5.1.1) is then replaced by the pull back into the curved space. The analysis then proceeds as in the main text.

where

$$P_\pm = \frac{1}{2}\left(1 \pm \frac{1}{6!}\epsilon^{m_0\cdots m_5}\partial_{m_0}y^{M_0}\cdots\partial_{m_5}y^{M_5}\Gamma_{M_0\cdots M_5}\right) = \frac{1}{2}\left(1 \pm \frac{1}{6!}\epsilon^{m_0\cdots m_5}\Gamma_{m_0\cdots m_5}\right)$$

In this expression $\Gamma_{m_0\cdots m_5} = \frac{1}{6!}\Gamma_{[m_0}\Gamma_{m_1}\cdots\Gamma_{m_5]}$. Using the results in the appendix to [85] we have $(\prod_{i=0}^{5}\Gamma_i)^2 = \mathbb{I}$ so that $(P_\pm)^2 = P_\pm$ and $P_+P_- = P_-P_+ = 0$. This means that $P_\pm$ give a decomposition of unity.

Now, the insertion of a bosonic brane into a spacetime will generically break supersymmetry. This can be thought of as occurring since the spacetime is no-longer invariant under (super)translations. However, if it is possible to compensate this effect by a kappa transformation, then the brane preserves supersymmetry. For this to happen, we must have

$$\delta\Theta = \delta_\epsilon\Theta + \delta_\kappa\Theta = 0$$

which by the above formulae means $\epsilon + 2P_+\kappa = 0$. Applying the projection operator P_- we therefore have the final condition

$$P_-\epsilon = 0 \tag{5.1.3}$$

This projection operator halves the number of degrees of freedom in Θ and so the supersymmetry on the worldvolume is halved[5]. This then shows that we have 4d $N = 2$ supersymmetry on the worldvolume of the brane, provided the solution to (5.1.3) is compatible with our demands of trivial holonomy.

Now to have Poincare invariance in to a 4d observer in the brane worldvolume, it is required that the 5-brane fills all of the $\mathbb{R}^{1,3}$ in $\mathcal{M}_{11}$. This means that the M5-brane has geometry $\mathbb{R}^{1,3} \times \Sigma$ where Σ is a 2d surface. It is possible to consider Σ to be a complex curve in the canonical way. Furthermore, the fact that the brane preserves half of the supersymmetry is related to the presence of a topological bound arising from minimising the area of the brane. It has been shown [61] that this bound leads on to the condition that Σ is holomorphically embedded in Q^4.

---

[5]One may now use these conditions to investigate whether it is possible to preserve supersymmetries when more than one brane is present. This problem essentially reduces to finding configurations whose chirality operators $P_\pm$ commute with each other. We shall not be required to pursue this avenue of investigation in this chapter.

As such, we have illustrated how one can obtain a configuration in M-theory which allows comparison with 4d field theories with N=2 supersymmetry. One starts with an 11d spacetime of the form $\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathbb{R}^3 \times Q^4$ where Q^4 is a complex surface with $SU(2)$ holonomy. Into this spacetime one immerses a 5-brane with geometry $M5 = \mathbb{R}^{1,3} \times \Sigma$ where Σ is a complex curve holomorphically embedded in Q^4, and the factors of $\mathbb{R}^{1,3}$ are identified.

5.1.2   Leading Order Terms in Seiberg-Witten from M-Theory

We use the above configuration to develop a general method for constructing the kinetic terms for the scalars apparent to a 4d observer in the $\mathbb{R}^{1,3}$ of the M5 world-volume. We then examine a specific choice for Q^4 which was considered by Witten [96], and show how this leads on to the familiar Seiberg-Witten action for $N = 2$ supersymmetric gauge theories.

We begin by introducing co-ordinates to represent the configuration developed above. Let $y^0, \cdots y^{10}$ be the co-ordinates on $\mathcal{M}_{11}$ and $x^0, \cdots, x^5$ be those of the five-brane. We identify $y^i = x^i$ for $i = 0, 1, 2, 3$ and introduce $X^1 = y^4 + iy^5$ and $X^2 = y^6 + iy^{10}$ as the complex co-ordinates for Q^4. We understand $X^{\bar{i}} = \overline{X^i}$. Further, we demand that $y^7 = y^8 = y^9 = 0$ for the fivebrane. This is associated with an identification of the universal cover of the Lorentz group of the $\mathbb{R}^3$ in $\mathcal{M}^{11}$ with an R-symmetry of the supersymmetry algebra. On the M5 worldvolume we write $z = x^4 + ix^5$. Then, the condition of holomorphic embedding of the five-brane in $\mathcal{M}_{11}$ is expressed as $X^i = X^i(z, u_\alpha)$ and $X^{\bar{i}} = X^{\bar{i}}(\bar{z}, \bar{u}_\alpha)$ for $i = 1, 2$; where the u_α are moduli of the curve Σ which in general will have genus g. This will be seen in more detail later.

The bosonic action on the worldvolume of a 5-brane has been derived in [11, 2, 16], and has a term[6] of the form

$$S_5 = \int d^6 x \sqrt{-g} \qquad \text{where} \ \ g = \det(g_{ab}) \tag{5.1.4}$$

---

[6]There are, of course, other terms in this action, but they are either related to the given term via supersymmetry, or do not contribute to the subsequent analysis in this chapter. We have also suppressed various constants in this subsection. They shall be reinstated later.

where the metric tensor on the worldvolume g_{ab} is an embedding of the spacetime metric since, by equations of motion, it was discerned that $g_{ab} = G_{MN}\partial_a y^M \partial_b y^N$ with y^M the co-ordinates of the background spacetime. With the configuration outlined above, we have the spacetime metric, $G_{MN}dy^M dy^N$,

$$ds^2 = \eta_{mn}dy^m dy^n + 2G_{i\bar{j}}dX^i dX^{\bar{j}} + \delta_{pq}dy^p dy^q \tag{5.1.5}$$

with $m, n = 0, 1, 2, 3$ and $p, q = 7, 8, 9$. As noted above, $i, j = 1, 2$.

Let us now analyse (5.1.4) with this choice of metric. Due to the above identifications we see the measure

$$\int d^6 x \mapsto 2 \int d^4 x \, dz \, d\bar{z}$$

Now, since the M5 is at a point in the $\mathbb{R}^3$ and $G_{i\bar{j}} = G_{\bar{j}i}$ we can calculate

$$
\begin{aligned}
g_{mn} &= \eta_{mn} + G_{i\bar{j}}\frac{\partial X^i}{\partial x^m}\frac{\partial X^{\bar{j}}}{\partial x^n} + G_{i\bar{j}}\frac{\partial X^i}{\partial x^n}\frac{\partial X^{\bar{j}}}{\partial x^m} \\
&= \eta_{mn} + \left[G_{i\bar{j}}\frac{\partial X^i}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^m}\frac{\partial X^{\bar{j}}}{\partial \bar{u}^\beta}\frac{\partial \bar{u}^\beta}{\partial x^m} + (m \leftrightarrow n) \right] \\
&\equiv \eta_{mn} + \left[g_{\alpha\bar{\beta}}\partial_m u^\alpha \partial_n u^{\bar{\beta}} + (m \leftrightarrow n) \right]
\end{aligned}
\tag{5.1.6a}
$$

Likewise, we can write

$$g_{mz} = G_{i\bar{j}}\frac{\partial X^i}{\partial z}\frac{\partial X^{\bar{j}}}{\partial \bar{u}^\beta}\frac{\partial \bar{u}^\beta}{\partial x^m} \equiv g_{z\bar{\beta}}\partial_m \bar{u}^\beta = g_{zm} \tag{5.1.6b}$$

$$g_{m\bar{z}} = G_{i\bar{j}}\frac{\partial X^{\bar{j}}}{\partial \bar{z}}\frac{\partial X^i}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^m} \equiv g_{\alpha\bar{z}}\partial_m \bar{u}^\alpha = g_{\bar{z}m} \tag{5.1.6c}$$

$$g_{zz} = g_{\bar{z}\bar{z}} = 0 \text{ by holomorphic embedding} \tag{5.1.6d}$$

$$g_{z\bar{z}} = G_{i\bar{j}}\frac{\partial X^i}{\partial z}\frac{\partial X^{\bar{j}}}{\partial \bar{z}} \tag{5.1.6e}$$

Putting these expressions into a determinant and using the well known formula

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{pmatrix}$$

with $P = (g_{mn})$, $Q = \begin{pmatrix} g_{mz} & g_{m\bar{z}} \end{pmatrix}$, $R = \begin{pmatrix} g_{zm} \\ g_{\bar{z}m} \end{pmatrix}$ and $S = \begin{pmatrix} 0 & g_{z\bar{z}} \\ g_{\bar{z}z} & 0 \end{pmatrix}$ yields

$$\det g = g_{z\bar{z}}^2 \det \left(\eta_{mn} + \left\{ g_{\alpha\bar{\beta}} - \frac{g_{z\bar{\beta}}g_{\alpha\bar{z}}}{g_{z\bar{z}}} \right\} \left\{ \partial_m u^\alpha \partial_n \bar{u}^\beta + (m \leftrightarrow n) \right\} \right) \tag{5.1.7}$$

Keeping only the lowest order terms in the expansion $\det(\mathbb{I} + A) = 1 + \mathrm{tr}A + \dots$ and using $\det(A + B) = \det(A)\det(\mathbb{I} + A^{-1}B)$ and the Maclaurin series $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon + \dots$ we expand (5.1.7) about a flat background, a situation which corresponds to a low energy approximation

$$(-\det g)^{\frac{1}{2}} \approx g_{z\bar{z}} + \left(g_{z\bar{z}}g_{\alpha\bar{\beta}} - g_{z\bar{\beta}}g_{\alpha\bar{z}}\right)\partial_m u^\alpha \partial^m \bar{u}^\beta$$

Ignoring the $g_{z\bar{z}}$ term we then have

$$\int d^6 x(-\det g)^{\frac{1}{2}} \approx 2\int d^4 x \partial_m u^\alpha \partial^m \bar{u}^\beta \int dz d\bar{z}\left(g_{z\bar{z}}g_{\alpha\bar{\beta}} - g_{\alpha\bar{z}}g_{z\bar{\beta}}\right) \qquad (5.1.8)$$

as the kinetic term for the scalars u^α. Using the definitions in (5.1.6a)–(5.1.6e) yields

$$
\begin{aligned}
\mathcal{G}_{\alpha\bar{\beta}} &= 2\int dz d\bar{z}\left(g_{z\bar{z}}g_{\alpha\bar{\beta}} - g_{\alpha\bar{z}}g_{z\bar{\beta}}\right) \\
&= 2\int dz d\bar{z}\partial_z X^m \partial_{\bar{z}}X^{\bar{n}}\frac{\partial X^i}{\partial u^\alpha}\frac{\partial X^{\bar{j}}}{\partial \bar{u}^\beta}\left(G_{m\bar{n}}G_{i\bar{j}} - G_{i\bar{n}}G_{m\bar{j}}\right) \qquad (5.1.9)
\end{aligned}
$$

where we have introduced $\mathcal{G}_{\alpha\bar{\beta}}$ as the field theory Kähler metric for scalars as is standard since systems with $N = 2$ supersymmetry generically give rise to a Kähler manifold whose co-ordinates are the scalar fields [99, 4].

Now, for a general complex manifold, the Kähler form Ω is [73]

$$
\begin{aligned}
\Omega &= -J_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu \text{ where J is the complex structure} \\
&= -g_{\mu\bar{\lambda}}dz^\mu \wedge d\bar{z}^\nu J_\nu^{\bar{\lambda}} = ig_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\nu
\end{aligned}
$$

where in this expression, the $\{z^\mu; \mu = 1, \cdots, d\}$ are the co-ordinates of an arbitary d-dimensional complex manifold with hermitean metric $g_{\mu\bar{\nu}}$. Thus, in general,

$$\Omega \wedge \bar{\Omega} = \frac{1}{2}(g_{\mu\bar{\nu}}g_{\alpha\bar{\beta}} - g_{\mu\bar{\beta}}g_{\alpha\bar{\nu}})dz^\mu \wedge d\bar{z}^\nu \wedge dz^\alpha \wedge d\bar{z}^\beta$$

This demonstrates that the factor $G_{m\bar{n}}G_{i\bar{j}} - G_{i\bar{n}}G_{m\bar{j}}$ in (5.1.9) is proportional to $\Omega \wedge \bar{\Omega}$ in Q^4. Now it is a general result [73, pp 283] that for Hermitean manifolds $\Omega = \bar{\Omega}$ so that we have in principle calculated $\Omega \wedge \Omega$ which can now be used as a volume form for the Kähler manifold. This is due to the fact that we are considering a complex manifold of real dimension 4, and the Betti number $b_{(2,2)}(Q^4) = 1$, meaning that the space of $(2, 2)$ forms in our 2d complex Kähler manifold has dimension 1 [66].

Now consider the curve Σ in Q^4 to be given by

$$A(v)t^2 + B(v)t + C(v) = 0 \qquad (5.1.10)$$

where t, v are co-ordinates of Q^4. We shall relate t and v to the X^i later. Also, $A(v)$ and $C(v)$ are non-zero constant functions, so that they have no zeroes by the fundamental theorem of algebra. Mapping $t \mapsto \left(\frac{C}{A}\right)^{\frac{1}{2}} t$ and $B(v) \mapsto -2(AC)^{\frac{1}{2}} B(v)$ reduces (5.1.10) to

$$t^2 - 2B(v)t + 1 = 0 \qquad (5.1.11)$$

Obviously these transformations leave $\frac{dt}{t}$ invariant.

Let the polynomial $B(v)$ be of degree k in v. The most general such polynomial is

$$B(v) = a_0 v^k + a_1 v^{k-1} + \ldots + a_k \quad ; a_0 \neq 0$$

We can now scale and shift v. Let $v \mapsto a_0^{-1/k}(v + \Delta)$, with Δ a constant. It is clear that the differential $dv \mapsto a^{-1/k} dv$. Then

$$
\begin{aligned}
B(v) \; = \; & v^k + k\Delta v^{k-1} + \binom{k}{2}\Delta^2 v^{k-2} + \ldots + \Delta^k + \\
& \frac{a_1}{a_0^{1-1/k}}\left(v^{k-1} + (k-1)\Delta v^{k-2} + \ldots + \Delta^{k-1}\right) + \\
& \frac{a_2}{a_0^{1-2/k}}\left(v^{k-2} + \ldots + \Delta^{k-2}\right) + \ldots + a_k
\end{aligned}
$$

We now collect terms

$$
\begin{aligned}
B(v) \; = \; & v^k + v^{k-1}\left(k\Delta + \frac{a_1}{a_0^{1-1/k}}\right) + v^{k-2}\left(\binom{k}{2}\Delta^2 + \frac{(k-1)a_1}{a_0^{1-1/k}} + \frac{a_2}{a_0^{1-2/k}}\right) \\
& + \ldots + \left(\frac{a_1}{a_0^{1-1/k}}\Delta + \frac{a_2}{a_0^{1-2/k}}\Delta^2 + \ldots + \Delta^k\right)
\end{aligned}
$$

We now set $\Delta = -\frac{a_1}{ka_0^{1-1/k}}$ so that the v^{k-1} term vanishes.

We now transform $t \mapsto t - B$, and the curve (5.1.11) becomes

$$t^2 = B(v)^2 - 1 \qquad (5.1.12)$$

with

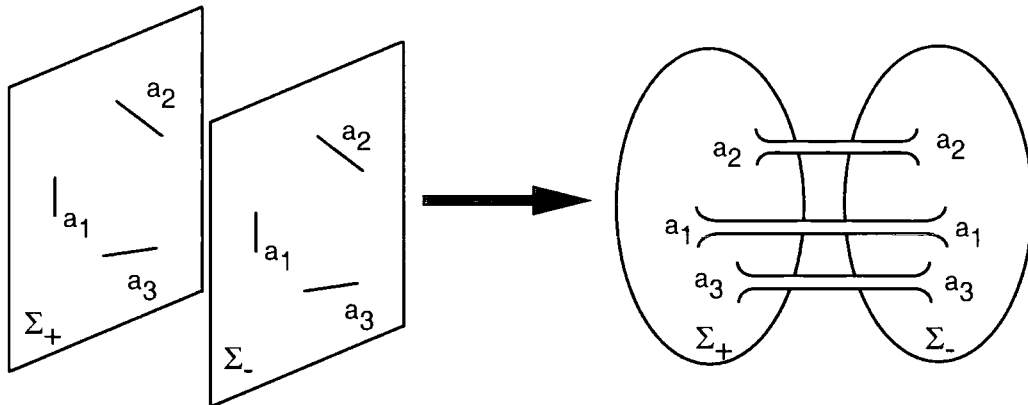$$B(v) = v^k + u_2 v^{k-2} + \ldots + u_{k-1}v + u_k \qquad (5.1.13)$$

Figure 5.1: Construction of branched cover (Riemann surface) for the polynomial $t^2 = B(v)^2 - 1$. The left diagram shows the two Riemann sheets with branch-cuts, the right diagram how these are patched together to form a smooth surface. We have illustrated the genus 2 case.

Comparison with [7, 58] shows that this is the (hyper)elliptic curve governing the coloumb phase of the $N = 2$ supersymmetric gauge theory without matter, and with gauge group $SU(k)$. From equation (5.1.13) we also see the meaning of the moduli u_α introduced previously, with $\alpha = 1, \cdots, k$ and $u_1 = \sum_{i=1}^{k} e_i = 0$ where e_i are the roots of k-degree polynomial $B(v)$.

Geometrically, we can picture the curve Σ embedded in Q^4 as follows. Each of the two roots of (5.1.12) in the t-plane lead on to two Riemann sheets Σ_+ and Σ_-, each sheet having branch points where the discriminant, $\sqrt{B(v)^2 - 1}$, of the curve vanishes. Since $B(v)$ is a complex polynomial of degree k, there are $2k$ such points. One now forms a surface by joining the branch points with cuts and then patching the Riemann sheets with k-circles. This situation is the canonical method of constructing a genus $(k - 1)$ Riemann surface [68], and is illustrated pictorially in figure 5.1.

We now leave the generalities of the problem behind and consider a specific model. Let the complex manifold Q^4 be parameterised by the co-ordinates

$$v = y^4 + iy^5 \text{ and } t = \exp(-s) = \exp(-y^6 - iy^{10})$$

This therefore gives Q^4 the topology[7] $R^3 \times S^1$. In this case, the Kähler form Ω is

---

[7]The astute reader will realise that $\mathbb{R}^{1,3} \times S^1$ has trivial holonomy. Naively one would expect

given by

$$\Omega = \frac{i}{2} \left(dv \wedge d\bar{v} + \frac{1}{|t|^2} dt \wedge d\bar{t} \right)$$

Using the foregoing analysis, we consider the curve in Q^4

$$t^2 - 2B(z; u_\alpha)t + \Lambda^{2k} = 0 \qquad (5.1.14)$$

where we have reinstated a renormalisation scale Λ, and have identified $v = z$ to conform with the configuration[8] originally considered by Witten [96]. Then, differentiating (5.1.14) with respect to u_α and using (5.1.13) we have

$$\frac{\partial t}{\partial u_\alpha}(t - B(z)) = t \frac{\partial B(z; u_\alpha)}{\partial u_\alpha} = t z^{k-\alpha}$$

so

$$\frac{\partial t}{\partial u_\alpha} = \frac{z^{k-\alpha}t}{t - B(z)} = \frac{z^{k-\alpha}t}{(B^2 - \Lambda^{2k})^{\frac{1}{2}}} \qquad (5.1.15)$$

Now, from (5.1.9) we see that the Kähler metric

$$\begin{aligned}
\mathcal{G}_{\alpha\bar{\beta}} &= 2 \int dz d\bar{z} \partial_z X^m \partial_{\bar{z}} X^{\bar{n}} \frac{\partial X^i}{\partial u^\alpha} \frac{\partial X^{\bar{j}}}{\partial u^{\bar{\beta}}} \left(G_{m\bar{n}} G_{i\bar{j}} - G_{i\bar{n}} G_{m\bar{j}} \right) \\
&= 2 \int_\Sigma d^2 z |t|^2 \frac{\partial t}{\partial u_\alpha} \overline{\left(\frac{\partial t}{\partial u_\beta} \right)} \left(G_{z\bar{z}} G_{t\bar{t}} - G_{t\bar{z}} G_{z\bar{t}} \right) \\
&= \int_\Sigma d^2 z \frac{z^{k-\alpha}}{t - B} \overline{\left(\frac{z^{k-\beta}}{t - B} \right)} \qquad (5.1.16)
\end{aligned}$$

the 4d observer in the worldvolume of the M5 to see N=4 supersymmetry using the argument that the spinor representation $\underline{8}$ of B_3 transforms as 8 $\underline{1}$'s under trivial holonomy group, and that half of these are preserved by the presence of the 5-brane in the spacetime. However, a detailed analysis, which is outside the scope of this chapter, shows that this choice for Q^4 does indeed give $N = 2$ supersymmetry. As pointed out previously, it has to be shown that solutions to (5.1.3) are compatible with the demands of trivial holonomy. In the case of $\mathbb{R}^{1,3}$ it is found that half of the 8 invariant spinors do not globally satisfy the demand of positive chirality under parallel transport. For more details see [61, 50].

[8]In Witten's paper, the initial analysis was in IIA string theory which is known to be the limit of M-theory with vanishing radius of the eleventh dimension. In this dimensional reduction, the M5-brane can be reduced into a configuration of so-called Neveu-Schwarz (NS) 5-branes or Dirichilet (D) 4-branes depending on whether the M5-brane is wrapped on the circular dimension or not. To describe this procedure in detail would take a great amount of time, and so we refer the reader to Witten's seminal paper [96]
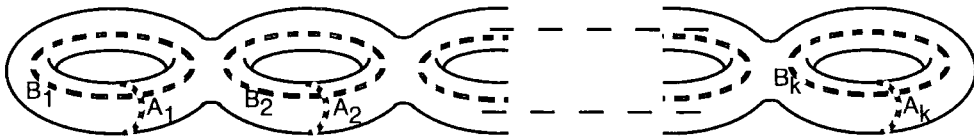
Figure 5.2: The symplectic basis of homology cycles [77] on a $(k-1)$-genus hyperelliptic curve.

where we have used the fact that $G_{z\bar{z}} = \frac{1}{2}$ and $G_{t\bar{t}} = \frac{1}{2}|t|^{-2}$ with all other entries zero.

Now, it is known [58] that the one-forms $\omega_\alpha = \frac{z^{k-\alpha}}{t-B}dz$ form a basis of holomorphic one forms for a curve Σ of genus $k-1$. Thus

$$\mathcal{G}_{\alpha\bar{\beta}} = \int_\Sigma \omega_\alpha \wedge \bar{\omega}_\beta \qquad (5.1.17)$$

This is precisely [37] the same result as is obtained in field theory. We have therefore demonstrated that the dynamics on a 5-brane with suitable identifications yields the correct Kähler metric for $SU(k)$ super-Yang-Mills without matter.

Further, the one-forms ω_α obey the Riemann bilinear relation [77, pp51]

$$\int_\Sigma \omega_\alpha \wedge \bar{\omega}_\beta = \sum_{i=1}^{\text{Genus}=k-1} \left(\int_{A_i} \omega_\alpha \int_{B_i} \bar{\omega}_\beta - \int_{A_i} \bar{\omega}_\beta \int_{B_i} \omega_\alpha \right) \qquad (5.1.18)$$

where A_i and B_i are the cycles on the curve as shown in figure 5.2.

Introducing the derivatives of the periods (a^i, a_D^i) of Σ

$$\frac{\partial a_i}{\partial u_\alpha} = \int_{A_i} \omega_\alpha \qquad \frac{\partial (a_D)_i}{\partial u_\alpha} = \int_{B_i} \omega_\beta$$

and placing these into (5.1.18) we see (5.1.17) becomes

$$\mathcal{G}_{\alpha\bar{\beta}} = \sum_{i=1}^{k-1} \left[\frac{\partial a_i}{\partial u_\alpha} \frac{\partial \bar{a}_D^i}{\partial \bar{u}_\beta} - \frac{\partial \bar{a}_i}{\partial \bar{u}_\beta} \frac{\partial a_D^i}{\partial u_\alpha} \right] \qquad (5.1.19)$$

which agrees with the formula of Seiberg and Witten [80]. We can now derive the kinetic action for $SU(k)$ $N=2$ super-Yang-Mills.

We have from (5.1.8) that the low energy dynamics of the scalar fields on the brane

worldvolume are given by

$$
\begin{aligned}
S &= \int d^4 x \eta^{mn} \frac{\partial u^\alpha}{\partial x^m} \frac{\partial \bar{u}^\beta}{\partial x^n} \mathcal{G}_{\alpha\bar\beta} = \int d^4 x \eta^{mn} \partial_m u^\alpha \partial_n \bar{u}^\beta \sum_{i=1}^{k-1} \left(\frac{\partial a_i}{\partial u_\alpha} \frac{\partial \bar{a}_D^i}{\partial \bar{u}_\beta} - \frac{\partial \bar{a}_i}{\partial \bar{u}_\beta} \frac{\partial a_D^i}{\partial u_\alpha} \right) \\
&= \int d^4 x \eta^{mn} \sum_{i=1}^{k-1} \partial_m a^i \partial_n \bar{a}_D^i - \overline{(\partial_m a^i \partial_n \bar{a}_D^i)} \\
&= 2i \mathrm{Im} \int d^4 x \sum_{i=1}^{k-1} \partial_m a^i \partial^m \bar{a}_D^i
\end{aligned}
$$

which is the canonical action for the scalar fields of $SU(k)$ $N = 2$ super-Yang-Mills. We have therefore illustrated how to obtain the Seiberg-Witten curve from M-theory, and in particular how the action for N=2 Super-Yang-Mills arises.

5.1.3 Higher Derivatives

We will now attempt to derive the higher derivative action using this brane dynamics formalism. We still consider the spacetime to be

$$
\mathcal{M}_{11} = \mathbb{R}^{1,3} \times \mathbb{R}^3 \times \underbrace{\mathbb{R}^3 \times S^1}_{Q^4}
$$

This was shown to lead on to the kinetic terms for the scalars of N=2 supersymmetric Yang-Mills and gave a natural interpretation for the Seiberg-Witten curve that appears so mysteriously in its original derivation [80]. It shall be illustrated that despite the agreement for holomorphic terms, the next to leading order correction derived from branes cannot be that of the field theory.

The metric for our spacetime was written in (5.1.5) as

$$
ds^2 = \eta_{mn} dx^m dx^n + 2G_{z\bar z} dz d\bar z + 2G_{t\bar t} dt d\bar t + \sum_{i=7,8,9} dy^i dy^i
$$

Whilst in the foregoing analysis it was possible to use this general form of the metric, it is now important that we reinstate any constants relevant to the physical problem at hand. It can be shown [19, Appendix A] that the correct metric is

$$
ds^2 = \eta_{mn} dx^m dx^n + |l_{st}^2 dv|^2 + |R\frac{dt}{t}|^2 + \cdots \tag{5.1.20}
$$

where we now have $t = e^{-(y^6 + iy^{10})/R}$ with R the radius of the circle in the 11th dimension. The $\cdots$ signify terms related to the first $\mathbb{R}^3$ factor in $\mathcal{M}_{11}$, but since

we consider our 5-brane to be at a point in this submanifold, we do not need their specific form. The parameter l_{st} has its interpretation in the $10d$ type IIA string theory which was originally used by Witten to study this model. It arises in (5.1.20) to ensure that the mass of the particles in the field theory are given by the tensions of strings linking the NS 4-branes; objects which arise on wrapping the M5 brane on the 11-th dimension. Indeed, the strings themselves arise on wrapping a 2-brane on the same circle. Comparing the actions of IIA string theory and the dimensionally reduced action of 11d supergravity (which is appropriate model for low energy M-theory) it was discerned [90] that there is a relation between the parameters l_{st}, R and the 11-dimensional Planck length l_{11}. This is

$$(l_{st})^2 R = (l_{11})^3 \qquad (5.1.21)$$

Now, due to the identifications made in the previous subsection

$$g_{z\bar{z}} = G_{i\bar{j}}\frac{\partial X^i}{\partial z}\frac{\partial X^{\bar{j}}}{\partial \bar{z}} = G_{z\bar{z}} + G_{t\bar{t}}\frac{\partial t}{\partial z}\frac{\partial \bar{t}}{\partial \bar{z}}$$

so that using the result $\partial_z t = \frac{t\partial_z B}{(t-B)}$ derived from (5.1.14) we see that on the M5 worldvolume

$$g_{z\bar{z}} = l_{st}^4 + \frac{R^2|\partial_z B|^2}{|t-B|^2} = \frac{l_{st}^4|t-B|^2 + R^2|\partial_z B|^2}{|t-B|^2}$$

Also, $g_{\alpha\bar{\beta}} = G_{i\bar{j}}\frac{\partial X^i}{\partial u_\alpha}\frac{\partial X^{\bar{j}}}{\partial \bar{u}^\beta}$ but as previously, our holomorphic embedding conditions mean that only $\frac{\partial t}{\partial u_\alpha}$ – and the complex conjugate – are non-zero. Thus

$$g_{\alpha\bar{\beta}} = G_{t\bar{t}}\frac{\partial t}{\partial u_\alpha}\frac{\partial \bar{t}}{\partial \bar{u}_\beta} = R^2\frac{z^{k-\alpha}\bar{z}^{k-\beta}}{|t-B|^2}$$

Also

$$g_{z\bar{\beta}} = G_{t\bar{t}}\frac{\partial t}{\partial z}\frac{\partial \bar{t}}{\partial \bar{u}^\beta} = R^2\frac{\partial_z B\bar{z}^{k-\beta}}{|t-B|^2}$$

and lastly

$$g_{\alpha\bar{z}} = R^2\frac{\overline{\partial_z B}z^{k-\alpha}}{|t-B|^2}$$

Using the above formulae it is a matter of simple algebra to show

$$g_{\alpha\bar{\beta}} - \frac{g_{z\bar{\beta}}g_{\alpha\bar{z}}}{g_{z\bar{z}}} = \frac{R^2 l_{st}^4 z^{k-\alpha}\bar{z}^{k-\beta}}{l_{st}^4|t-B|^2 + R^2|\partial_z B|^2}$$

which can then be placed in the general formula for the determinant derived in (5.1.7) to give

$$-\det g = g_{z\bar{z}}^2 \det \left[\delta_q^p + \Xi z^{k-\alpha} \bar{z}^{k-\beta} \left(\partial^p u^\alpha \partial_q u^{\bar{\beta}} + \partial^p u^{\bar{\beta}} \partial_q u^\alpha \right) \right] \qquad (5.1.22)$$

where we are denoting $\Xi = \frac{R^2 l_{st}^4}{l_{st}^4 |t-b|^2 + R^2 |\partial_z B|^2}$. Now, letting A be the difference from the unit matrix within the $[\cdots]$ in equation (5.1.22), one can derive

$$\text{tr } A = 2\Xi z^{k-\alpha} \bar{z}^{k-\beta} \partial^m u^\alpha \partial_m u^{\bar{\beta}}$$

$$\text{tr } A^2 = 2\Xi^2 z^{2k-\alpha-\alpha'} \bar{z}^{k-\beta\beta'} \left[\partial^m u^\alpha \partial_m u^{\alpha'} \partial^n u^{\bar{\beta}} \partial_n u^{\bar{\beta}'} + \partial^m u^{\alpha'} \partial_m u^{\bar{\beta}'} \partial^n u^{\bar{\alpha}} \partial_n u^{\bar{\beta}} \right]$$

We can now make use of the identity $\det (\mathbb{I} + A) = 1 + \text{tr } A - \frac{1}{2} \text{tr } A^2 + \frac{1}{2}(\text{tr } A)^2 + \cdots$ to expand (5.1.22) to next to leading order about a flat spacetime. We obtain

$$\sqrt{-\det g} = g_{z\bar{z}} \left[1 + \underbrace{2\Xi z^{k-\alpha} \bar{z}^{k-\beta} \partial^m u^\alpha \partial_m u^{\bar{\beta}}}_{\text{Leading order, derived previously}} \right.$$

$$+ \Xi^2 z^{2k-\alpha-\alpha'} \bar{z}^{2k-\beta-\beta'} \partial^m u^\alpha \partial_m u^{\bar{\beta}} \partial^n u^{\alpha'} \partial_n u^{\bar{\beta}'}$$

$$\left. - \Xi^2 z^{2k-\alpha-\alpha'} \bar{z}^{2k-\beta-\beta'} \partial^m u^\alpha \partial_m u^{\alpha'} \partial^n u^{\bar{\beta}} \partial_n u^{\bar{\beta}'} + \cdots \right]^{\frac{1}{2}}$$

and subsequently we expand the square root using $(1 + \epsilon)^{\frac{1}{2}} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots$ to see

$$\sqrt{-\det g} = g_{z\bar{z}} \left(1 - \Xi z^{k-\alpha} \bar{z}^{k-\beta} \partial^m u^\alpha \partial_m u^{\bar{\beta}} \right.$$

$$\left. - \frac{1}{2} \Xi^2 z^{2k-\alpha-\alpha'} \bar{z}^{2k-\beta-\beta'} \partial^m u^\alpha \partial_m u^{\alpha'} \partial^n u^{\bar{\beta}} \partial_n u^{\bar{\beta}'} + \cdots \right)$$

where the $\cdots$ are terms with more than four derivatives. In particular, the total contribution at four derivative order is given by

$$-\frac{1}{2} \Xi^2 g_{z\bar{z}} z^{2k-\alpha-\alpha'} \bar{z}^{2k-\beta-\beta'} \partial^m u^\alpha \partial_m u^{\alpha'} \partial^n u^{\bar{\beta}} \partial_n u^{\bar{\beta}'} \qquad (5.1.23)$$

Now, using the method of reduction used in chapter 3 it is possible to show that the general form of the higher derivative action (here expressed in N=2 superspace)

$$S_H = \int d^4x d^4\theta_1 d^4\theta_2 \mathcal{H}(\mathbb{A}, \bar{\mathbb{A}})$$

has as its purely scalar component term

$$S_H = \int d^4x \mathcal{H}_{ij\bar{k}\bar{l}} \partial^m \phi^i \partial_m \phi^j \partial^n \bar{\phi}^k \partial_n \bar{\phi}^l \qquad (5.1.24)$$

where $\mathcal{H}_{ij\bar{k}\bar{l}} = \frac{\partial^4 \mathcal{H}}{\partial \phi^i \partial \phi^j \partial \bar{\phi}^k \partial \bar{\phi}^l}$. On the worldvolume of the M5, the scalar fields are the u^α and so if the M-theory is to give rise to the higher derivative terms in a 4d field theory, we must have that the worldvolume action (5.1.4)

$$S_5 = \frac{2}{(l_{11})^6} \int d^4x d^2z \sqrt{-\det g}$$

– where we have reinstated a constant related to the brane tension – has an identical expression as in (5.1.24). On comparison between (5.1.23) and (5.1.24), we can see that this is indeed the case, with

$$\mathcal{H}_{\alpha\alpha'\bar{\beta}\bar{\beta}'} = - \int_\Sigma d^2z \frac{R^2 (l_{11})^6 z^{2k-\alpha-\alpha'} \bar{z}^{2k-\beta-\beta'}}{|t-B|^2 \left((l_{11})^6 |t-B|^2 + R^4 |\partial_z B|^2 \right)} \tag{5.1.25}$$

for the non-holomorphic prepotential $\mathcal{H}$ which encodes information on the next to leading order terms in the low energy expansion of supersymmetric field theories. In deriving (5.1.25) we have made use of the relation (5.1.21).

As a specific example, consider the case of $SU(2)$. From the analysis of the foregoing subsection, this corresponds to the polynomial (5.1.14) being $B(z;u) = z^2 + u$. Substituting this into (5.1.25) we see

$$\mathcal{H}_{uu\bar{u}\bar{u}} = - \int_\Sigma d^2z \frac{(l_11)^6 R^2}{\left| (z^2+u)^2 - \Lambda^4 \right| \left((l_{11})^6 \left| (z^2+u)^2 - \Lambda^4 \right| + 4R^4 |z|^2 \right)}$$

in which we have made use of $(t-B)^2 = B^2 - \Lambda^{2k}$. We can discern an upper bound on the magnitude of this integral by making use of the Cauchy inequality, that is

$$|\mathcal{H}_{uu\bar{u}\bar{u}}| \leq (l_{11})^6 R^2 \left| \int_\Sigma d^2z \frac{1}{\left| (z^2+u)^2 - \Lambda^4 \right|} \right| \left| \int_\Sigma d^2z \frac{1}{(l_{11})^6 \left| (z^2+u)^2 - \Lambda^4 \right| + 4R^4 |z|^2} \right|$$

$$\leq R^2 \left| \int_\Sigma d^2z \frac{1}{\left| (z^2+u)^2 - \Lambda^4 \right|} \right|^2$$

where use has been made of the triangle inequality $|a| + |b| \geq |a+b|$. However, an examination of the result (5.1.16) shows that the Kähler metric

$$\mathcal{G}_{u\bar{u}} = R \int_\Sigma d^2z \frac{1}{\left| (z^2+u)^2 - \Lambda^4 \right|}$$

so that

$$|\mathcal{H}_{uu\bar{u}\bar{u}}| \leq |\mathcal{G}_{u\bar{u}}|^2$$

Making use of (5.1.19) it is apparent that for gauge group $SU(2)$

$$
\begin{aligned}
\mathcal{G}_{u\bar{u}} &= \frac{\partial a}{\partial u}\frac{\partial \bar{a}_D}{\partial \bar{u}} - \frac{\partial a_D}{\partial u}\frac{\partial \bar{a}}{\partial \bar{u}} \\
&= \frac{\partial a}{\partial u}\frac{\partial \bar{a}}{\partial \bar{u}}\frac{\partial \bar{a}_D}{\partial \bar{a}} - \frac{\partial a}{\partial u}\frac{\partial \bar{a}}{\partial \bar{u}}\frac{\partial a_D}{\partial a} \\
&= -2i\left|\frac{\partial a}{\partial u}\right|\mathrm{Im}\,\frac{\partial a_D}{\partial a}
\end{aligned}
$$

Thus we see that

$$
|\mathcal{H}_{uu\bar{u}\bar{u}}| \leq 4\left|\frac{\partial a}{\partial u}\right|^2\left(\mathrm{Im}\,\frac{\partial a_D}{\partial a}\right)^2
$$

One can now use the results of Seiberg and Witten [80] which calculate explicit relations between a and u, and a_D and u for particular values of the moduli u which correspond to the singularities in the curve (5.1.13). This is a simple exercise which we shall not pursue here.

We now enquire whether the higher derivative terms arising from the theory on the worldvolume of a 5-brane are the same as that arising from field theory. We first note that whilst it may appear satisfactory that the brane formalism led on to a higher derivative term of the same type as that arising in field theory; this was to be expected. This statement can be justified by the fact that both the field theory and the physics on the worldvolume have $N = 2$ supersymmetry, and the action (5.1.24) arises from general considerations of systems with this amount of supersymmetry. Given this, any identification between the higher derivative terms in M-theory and field theory would be nothing other than a chimera. As is well known, the field theory is governed by the parameters Λ and u. In giving an interpretation to the Seiberg-Witten curve in M-theory, we identified the Λ of the surface Σ with the physical renormalisation scale of the super-Yang-Mills theory. Furthermore, the parameter u which arose as a scalar field on the brane worldvolume has its identification in the field theory as being the moduli related to vacuum expectation value of the Higgs field a. As far as the non-derivative terms are concerned, this would allow us to identify the physics on the brane with that in field theory. However, the higher derivative terms derived from M-theory contain extra parameters. These are R and l_{11}, relics of the fact that we are looking at a string theory with all the complications that entails. Due to this parameter mis-match, it is not possible for

the higher derivatives in super-Yang-Mills to be the same as those for the theory on the M5 worldvolume, and as such the two theories are distinct.

However, an examination of (5.1.25) shows that the integrand is singular whenever $t - B = 0$ or $|t - B|^2 + R^4 |\partial_z B|^2 = 0$. The second condition can be rephrased on noting that it is a sum of positive quantities, meaning that both $(t - B)$ and $\partial_z B$ must vanish at the same point. In either case $(t - B) = 0$, which means that the discriminant of the polynomial (5.1.14) vanishes. This is precisely the same condition as for the leading order holomorphic term in (5.1.16). At this leading order, the match to field theory is appropriate and, as is well-known, this situation corresponds to points in the moduli space where the dyons become massless [80]. As such, it is reasonable to claim that the two distinct theories have the same general properties, and can therefore be said to lie in the same universality class of models with $N = 2$ supersymmetry.

5.2 Conclusions

In this chapter we have investigated the relationship between the physics on the worldvolume of a 5-brane in 11d M-theory and super-Yang-Mills. It was demonstrated that to be permitted to do this, one had to choose the spacetime carefully and also immerse the 5-brane into the spacetime in a particular way. We then considered the configuration originally proposed by Witten [96] to show how the action for the scalar field on the 5-brane worldvolume led in a natural way to the action for the Higgs particles in Seiberg-Witten models of SU(k) super-Yang-Mills. In doing so, we illustrated the use of a (hyper)elliptic curve of genus (k-1) in the M-theory as a sub-manifold of the 5-brane worldvolume. This curve is related to the Seiberg-Witten curve of field theory. We then considered higher derivative terms arising in the brane theory, and showed that these had the expected generic form for N=2 supersymmetric theories. Lastly, it was pointed out that these could not be the same as those of super-Yang-Mills due to the appearance of extra parameters beyond those which arose in the identifications at leading order.

Chapter 6

Conclusions and Future Work

This thesis has attempted to investigate higher derivative terms and their influence on N=2 supersymmetric systems. In particular, we were concerned with those arising from the supersymmetric generalisation of the Yang-Mills-Higgs model with and without matter. It was demonstrated that these terms do not influence the conclusions of Seiberg and Witten in their analysis of such terms. We then proceeded to investigate multi-instanton contributions to the non-holomorphic prepotential which arises at next-to-leading order in the derivative expansion. It was then possible to prove a non-renormalisation theorem: in scale invariant models, instantons do not contribute to the prepotential. As a separate development, we then examined the relationship between the microscopic and effective coupling constants in these scale invariant theories making use of the Schwarzian derivative. Finally, we gave consideration to the exciting developments in string theory which seemed to relate the dynamics on an M-theory 5-brane to gauge theories in 4d. Comparing the low energy expansions we were able to demonstrate that the higher derivative terms had the same functional form, but that a careful examination of the parameters arising in the brane model showed that the two models were inherently different.

One of the limitations of this thesis was our reliance on the gauge group $SU(2)$. The Seiberg and Witten programme has been directly extended to other gauge groups [9, 58], for example the case of $SU(N_c)$ broken to it's Cartan subalgebra $U(1)^{N_c-1}$ by

a Higgs mechanism. As such, it would be of interest to generalise the investigations in this thesis to that case. For example, one could envisage looking at the instanton contributions to the non-holomorphic prepotential in this case, making use of the results in [35].

Another avenue of investigation might be to attempt to show higher derivative terms do not contribute to the mass formula directly by making use of the relation $\{Q_\alpha^{\text{eff}}, Q_\beta^{\text{eff}}\} = \epsilon_{\alpha\beta} Z^{\text{eff}}$ where the operators are determined from the low energy effective action.

One might also attempt to study the perturbative contributions to the non-holomorphic prepotential, due to an interesting result [52] which showed that for this object the contribution from two loops is zero. It would be interesting to see whether all higher loop contributions also vanish, since this is not guaranteed by the general results derived by Shifman et al – in particular there is no holomorphy.

Finally, we note that Matone [70] attempted to construct an explicit expression for the non-holomorphic prepotential in N=2 super-Yang-Mills using some minimal assumptions. In fact, the authors interest in higher derivatives in M-theory was instigated in the hope of independently verifying Matone's formula. Crucial to Matone's conjecture was the behaviour at large distances and that the non-holomorphic prepotential vanished at points in the moduli space where the holomorphic prepotential develops singularities. The former condition is contentious since the non-holomorphicity of the prepotential means that there is no concept of analytic continuation. The second condition is not required by physical arguments. As such, it would be of interest to develop methods to investigate the veracity of Matone's result. One could attempt to begin with the microscopic theory and integrate out the massive modes, but this would probably be inconclusive and/or too difficult. Rather, we would hope that there exists a more compact method to study this problem.

Appendix A

A Covariant Chain Rule

In chapter 2 we used the result

$$\nabla_\mu \mathcal{G}(\phi) = \nabla_\mu \phi^a \frac{\partial \mathcal{G}}{\partial \phi^a} \tag{1.0.1}$$

in deriving the action (2.3.9). In (1.0.1) ϕ is in the adjoint representation of the gauge group. Although we usually take this group to be $SU(2)$ we keep the discussion in this appendix more general. As such, we write $\phi = \sum_a \phi^a T^a$ where the sum has the dimension of the Lie algebra as the upper bound. In the case of $SU(2)$, we would take $T^a = \frac{1}{2}\tau^a$ with τ the Pauli matrices. The representation matrices obey the defining equation of a Lie algebra $[T^a, T^b] = \sum_c f^{abc} T^c$ where f^{abc} are the structure constants of the algebra (we have absorbed any factors into these co-efficients).

The group valued function $\mathcal{G}(\phi)$ is defined in terms of a power series in the matrix ϕ. As such, it will suffice to consider the term $\phi^n = \prod_{i=1}^n \phi$.

Now,

$$\partial_\mu \phi^n = \sum_{i=1}^n \phi^{a_1} \cdots \phi^{a_{i-1}} \partial_\mu \phi^{a_i} \phi^{a_{i+1}} \cdots \phi^{a_n} \prod_{j=1}^n T^{a_j}$$

and

$$\sum_i \partial_\mu \phi^{a_i} \frac{\partial \phi^n}{\partial \phi^{a_i}} = \sum_i \partial_\mu \phi^{a_i} \sum_j \phi^{a_1} \cdots \phi^{a_{j-1}} \delta_{a_i}^{a_j} \phi^{a_{j+1}} \cdots \phi^{a_n} \prod_{k=1}^n T^{a_k}$$

which on use of the Kronecker delta $\delta_b^a = \begin{cases} 1 \text{ if } a = b \\ 0 \text{ if } a \neq b \end{cases}$ shows

$$\partial_\mu \phi^n = \sum_i \partial_\mu \phi^{a_i} \frac{\partial \phi^n}{\partial \phi^{a_i}} \tag{1.0.2}$$

Next, consider the commutator

$$[v_\mu, \phi^n] = \sum_a v_\mu^a \prod_{k=1}^n \phi^{a_k} \sum_i T^{a_1} \cdots T^{a_{i-1}} \left(\sum_b f^{a a_i b} T^b \right) T^{a_{i+1}} \cdots T^{a_n}$$

where we have used the commutator identity $[A, BC] = B[A, C] + [A, B]C$. Comparing this with

$$\sum_i [v_\mu, \phi]^{a_i} \frac{\partial \phi^n}{\partial \phi^{a_i}} = \sum_i \sum_{a,b} v_\mu^a \phi^b f^{aba_i} \sum_j \phi^{a_1} \cdots \phi^{a_{j-1}} \delta_{a_i}^{a_j} \phi^{a_{j+1}} \cdots \phi^{a_n} \prod_{k=1}^n T^{a_k}$$

and recalling that the summed indices can be relabelled (that is, they are dummy indices) we find that

$$[v_\mu, \phi^n] = \sum_i [v_\mu, \phi]^{a_i} \frac{\partial \phi^n}{\partial \phi^{a_i}} \tag{1.0.3}$$

Now since the covariant derivative acting on fields A in the adjoint representation of the gauge group is defined as

$$\nabla_\mu A = \partial_\mu A + [v_\mu, A]$$

we see that on combining, (1.0.2) with (1.0.3), the stated result (1.0.1) is proved.

Appendix B

Instantons and the ADHM Construction

B.1 Instantons: a very brief introduction

An instanton is properly defined as a finite action solution to the Euler–Lagrange equations in a spacetime with Euclidean signature. Such a spacetime is often deduced from a Minkowski manifold which has been transformed by a Wick rotation (in a rough sense, time is complexified.) These instantons often have associated with them a topological invariant which pure mathematicians call the Pjontriagin index, but which physicists loosely term the "winding number."

To illustrate some of these ideas consider a pure Yang–Mills system with no supersymmetries. We let $\mathbf{A}_\mu$ be the vector potential with values in the Lie algebra of the gauge group. The associated field strength we write as $\mathbf{F}_{\mu\nu} = \partial_{(\mu}\mathbf{A}_{\nu)} + \mathbf{A}_{(\mu}\mathbf{A}_{\nu)}$ with $(\dots)$ being an antisymmetrisation operation. Notice that we have suitable rescaled the fields such that the coupling constant in this classical theory is unity, and also we work in a spacetime with signature $(1,1,1,1)$ with co-ordinates $x^\mu = (x_1, x_2, x_3, x_4)$. As we are working in such spacetime, there is no distinction between co- and contravariant tensors. This means we can raise and lower indices with no consequence.

The action, S, for this model is given by the canonical

$$S = \int d^4x \frac{1}{2} \text{tr} \left(\mathbf{v}_{\mu\nu} \mathbf{v}_{\mu\nu} \right)$$

$$= \text{tr} \int d^4x \frac{1}{4} \left(\mathbf{v}_{\mu\nu} \mp \star\mathbf{v}_{\mu\nu} \right) \left(\mathbf{v}_{\mu\nu} \mp \star\mathbf{v}_{\mu\nu} \right) \pm \frac{1}{2} \left(\mathbf{v}_{\mu\nu} \star \mathbf{v}_{\mu\nu} \right)$$

with the trace being taken over the group and $\star\mathbf{v}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{F}_{\rho\sigma}$ being the dual field strength, since ϵ is the anti–symmetric Levi-Cevita tensor.

It is straightforward to show that

$$\frac{1}{2}\text{tr}\left(\mathbf{v}_{\mu\nu} \star \mathbf{v}_{\mu\nu} \right) = \partial_\mu \text{tr}\epsilon_{\mu\nu\rho\sigma}\left(\mathbf{A}_\nu \partial_\rho \mathbf{A}_\sigma + \frac{2}{3}\mathbf{A}_\rho \mathbf{A}_\nu \mathbf{A}_\sigma \right)$$

and so

$$\pm\frac{1}{2}\text{tr} \int d^4x \mathbf{v}_{\mu\nu} \star \mathbf{v}_{\mu\nu} = \pm\frac{1}{2}\text{tr} \int_{\Sigma(\infty)} dS^\mu \epsilon_{\mu\nu\rho\sigma}\left(\mathbf{A}_\mu \partial_\rho \mathbf{A}_\sigma + \frac{2}{3}\mathbf{A}_\rho \mathbf{A}_\nu \mathbf{A}_\sigma \right)$$

is a boundary integral by the divergence theorem, with the surface $\Sigma(\infty)$ being that at infinity in the 4-dimensional spacetime.

At this point we note that a vector potential at ∞ need not decay to zero: it can be pure gauge. Therefore we will get a finite action solution if

$$\mathbf{v}_{\mu\nu} = \star\mathbf{v}_{\mu\nu} \qquad (2.1.1)$$

$$\lim_{|x|\to\infty} = g^{-1}\partial_\mu g \qquad (2.1.2)$$

where g is a (gauge) group transformation. For this solution one can observe

$$\pm\frac{1}{2}\text{tr} \int_{\Sigma(\infty)} dS^\mu \epsilon_{\mu\nu\rho\sigma}\left(\mathbf{A}_\nu \partial_\rho \mathbf{A}_\sigma + \frac{2}{3}\mathbf{A}_\rho \mathbf{A}_\nu \mathbf{A}_\sigma \right) = \pm\frac{2}{3} \int_{\Sigma(\infty)} dS^\mu \epsilon_{\mu\nu\rho\sigma} g^{-1}\partial_\nu g g^{-1}\partial_\rho g g^{-1}\partial_\sigma g$$

Using spherical polars it is then a tedious algerbraic exercise [65] to see that the action is bounded by:

$$S \geq 8\pi^2|n|$$

where $n \in \mathbb{Z}$ is identified with the instanton (winding) number. The sign of n determines whether one is dealing with instanton or anti-instantons. The above bound is saturated in the case of instantons wherein $\mathbf{v}_{\mu\nu} = \star\mathbf{v}_{\mu\nu}$.

B.2 Some Notation

We introduce some quaternion notation which will prove useful.

We shall denote the field of quaternions by $\mathbb{H}$. Our quaternions will be defined as

$$e_\mu \equiv (i\tau, \mathbb{I}_2) \tag{2.2.3}$$

where the τ are 3 anti-commuting quantities. We take these to be in a representation where (τ_1, τ_2, τ_3) are hermitian Pauli matrices which obey the usual anti-commutation relations. For our purposes, a specific representation is not important. It then follows that $e_i e_i = -e_4$ and $e_4 e_4 = e_4$ as is usual for a quaternionic algebra. We also have

$$e_\mu \bar{e}_\nu + e_\nu \bar{e}_\mu = 2\delta_{\mu\nu}\mathbb{I}_2 \tag{2.2.4}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix. We can then introduce quaternion valued vectors as in

$$x = x^\mu e_\mu = i\mathbf{x} \cdot \tau + x^4 \mathbb{I} \tag{2.2.5}$$

It will also be useful to introduce the t'Hooft symbols [63]

$$\eta_{\mu\nu} = \frac{1}{2}(e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu) \tag{2.2.6a}$$

$$\bar{\eta}_{\mu\nu} = \frac{1}{2}(\bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu) \tag{2.2.6b}$$

It is then apparent that $\epsilon_{\mu\nu\rho\sigma}\eta_{\rho\sigma} = \eta_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}\bar{\eta}_{\rho\sigma} = -\bar{\eta}_{\mu\nu}$. That is, η and $\bar{\eta}$ are respectively self-dual and anti-self dual.

B.3 The ADHM Construction of Multi-Instantons

B.3.1 Introduction

The general construction of instantons with arbitary winding number is a very difficult problem, but a significant advance was made by Atiyah, Drinfeld, Hitchen and

Martin (ADHM) when they proposed [6] a generic method for finding such solutions. This difficult work is grounded in algebraic geometry, but was brought to the attention of physicists by the "translations" into a more accessible form by several groups [76, 22, 27, 28]. Here we shall simply state the algorithm and outline several useful results.

First of all it should be noted that we are restricting our interests to the gauge group $Sp(N)$, and in particular we shall eventually focus on the case $N = 1$ which can be shown to be isomorphic to the special unitary group $SU(2)$. The elements of $Sp(N)$ are most simply represented by utilising quaternion notation. The elements of the $N \times N$ matrix representation g of $Sp(N)$ are then considered as quaternions. That is $i\alpha \cdot \tau + \alpha^4 \mathbb{I}$ and $g\bar{g} = I_{2N}$.

B.3.2 The ADHM Construction

Rule one

The vector potential is demanded to be

$$\mathbf{A}_\mu = \bar{H}\partial_\mu H \qquad\qquad \bar{H}H = I_{2n} \qquad\qquad (2.3.7)$$

This of course looks like a pure gauge, but in this construction the matrix H is <u>not</u> square. In fact $H \in M_{N+n,N}(\mathbb{H})$ where the M means matrix group with quaternion entries. It will eventually transpire that n is the winding number of the solution.

Using the usual definition of the field strength we then have

$$\begin{aligned}
\mathbf{v}_{\mu\nu} &= \partial_\mu\left(\bar{H}\partial_\nu H\right) + \bar{H}\partial_\mu H\bar{H}\partial_\nu H - (\mu \leftrightarrow \nu) \\
&= \bar{H}\left[\partial_\mu\left(H\bar{H}\right)\partial_\nu\left(H\bar{H}\right) - \partial_\nu\left(H\bar{H}\right)\partial_\mu\left(H\bar{H}\right)\right]
\end{aligned}$$

which obviously has the property of anti-symmetry in $\mu \leftrightarrow \nu$. It should be noted that we have <u>not</u> imposed the condition $H\bar{H} = I_{2(N+n)}$. We implement gauge transformations via $H \to Hg$, $g \in Sp(N)$. It then follows that the quantity $H\bar{H}$ is gauge invariant due to the condition $g^{-1} = \bar{g}$. We can then see that $\mathbf{v}_{\mu\nu} \to \bar{g}\mathbf{v}_{\mu\nu}g$ which is the usual gauge transformation of the field strength.

We should also observe that since $(H\bar{H})^2 = H(\bar{H}H)\bar{H} = H\bar{H}$ and $\overline{(H\bar{H})} = H\bar{H}$ that $H\bar{H}$ is a projection operator. Observing that $(H\bar{H})H = H$ we see that it acts as a projection operator on the columns of H.

We now seek the complementary operator to $H\bar{H}$. This we denote by Δ, and so

Rule 2

We demand that

$$\bar{\Delta}H = 0 \tag{2.3.8}$$

Requiring that the columns of Δ together with the columns of V span the relevant $(N + n)$–dimensional space we introduce the decomposition of unity

$$I_{2(N+n)} = H\bar{H} + \Delta \left(\bar{\Delta}\Delta\right)^{-1}\bar{\Delta} \tag{2.3.9}$$

where we notice that the operator $f = \left(\bar{\Delta}\Delta\right)^{-1}$ is hermitean. It should also be clear that Δ is an $(N + n) \times N$ quaternion valued matrix, and that the above condition is consistent with the various projections. Utilising the above ansatz gives

$$\begin{aligned}
\mathbf{v}_{\mu\nu} &= \bar{H}\left[\partial_\mu\left(\Delta f\bar{\Delta}\right)\partial_\nu\left(\Delta f\bar{\Delta}\right) - (\mu \leftrightarrow \nu)\right]H \\
&= \bar{H}\left[\partial_\mu\left(\Delta\right)f\partial_\nu\left(\bar{\Delta}\right) - (\mu \leftrightarrow \nu)\right]H
\end{aligned}$$

since any term with subexpressions $\bar{\Delta}H$ and $\bar{H}\Delta$ vanish due to the nature of projections and the definition of $f^{-1} = \bar{\Delta}\Delta$

Rule three

It is demanded that

$$\Delta = a + bx \tag{2.3.10}$$

In words, Δ be linear in the "co-ordinate" $x = x^\mu e_\mu$ expressed as a quaternion. Both a and b are constant $(N + n) \times N$ dimensional quaternion valued matrices. Then

$$\mathbf{v}_{\mu\nu} = \bar{H}b\left(e_\mu f\bar{e}_\nu - e_\nu f\bar{e}_\mu\right)\bar{b}H$$

If we make one final demand that $fe_\mu = e_\mu f$, that is f commutes[1] with the quaternion e_μ, then we have

$$\mathbf{v}_{\mu\nu} = \bar{H}bf\left(e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu\right)\bar{b}H$$
$$= 2\bar{H}bf\eta_{\mu\nu}\bar{b}H \tag{2.3.11}$$

and also that f is real in the quaternion sense: it has no entries proportional to $i\sigma_{1,2,3}$. From the above expression we can see that the solution has the required property $\mathbf{v}_{\mu\nu} = \star\mathbf{v}_{\mu\nu}$ thanks to the duality properties of the η.

We can now reduce the problem of finding a multi-instanton solution to three conditions. Since we require $f^{-1} = \overline{(a+bx)}(a+bx)$ to commute with each of the quaternions e_μ and for all x^μ we see that $f_{jk}|^{\dot\alpha}_{\dot\beta} = f_{jk}\delta^{\dot\alpha}_{\dot\beta}$, where $j, k \in (1, \dots, n)$ and $\dot\alpha, \dot\beta$ are the indices relating to the quaternionic entries. Further, $f_{jk}(x) = f_{jk}(0) + x^\mu \partial_\mu f_{jk}(0) + \frac{1}{2}x^\mu x^\nu \partial_\mu \partial_\nu f_{jk} + \dots$ by Taylor's theorem, and $f^{-1} = \bar{a}a + \bar{x}\bar{b}a + \bar{a}bx + \bar{x}\bar{b}bx$. Equating co-efficients of x^μ we see the following

$$\left(\bar{a}a\right)^t = \bar{a}a \propto e_4 \tag{2.3.12a}$$

$$\left(\bar{b}b\right)^t = \bar{b}b \propto e_4 \tag{2.3.12b}$$

$$\left(\bar{b}a\right)^t = \bar{b}a \tag{2.3.12c}$$

where the t superscript indicates transposition in the non-quaternionic indices. The first expression is trivial to derive, and so we show

$$\bar{x}\bar{b}bx = \frac{1}{2}x^\mu x^\nu \partial_{\mu\nu} f^{-1}(0)$$

$$\implies \quad \bar{e}_\mu \bar{b}b e_\nu = \frac{1}{2}\partial_{\mu\nu} f^{-1}(0)$$

$$\implies \quad \bar{b}b \propto e_\mu \bar{e}_\nu \partial_{\mu\nu} f^{-1}(0) \propto \{e_\mu, \bar{e}_\nu\}\partial_{\mu\nu} f^{-1}(0) \propto e_4$$

---

[1] Clearly this also means that f^{-1} also commutes with each of the quaternions. In fact, it is a general rule that $[A, B] = 0 \Leftrightarrow [A^{-1}, B] = 0$ provided that A is invertible.

and in a similar vein

$$\bar{e}_\mu \bar{b} a + \bar{a} b e_\mu = \partial_\mu f^{-1}(0)$$

$$\implies \quad e_\mu \bar{e}_\mu \bar{b} a + e_\mu \bar{a} b e_\mu = e_\mu \partial_\mu f^{-1}(0)$$

$$\text{and} \quad \bar{e}_\mu \bar{b} a \bar{e}_\mu + \bar{a} b e_\mu \bar{e}_\mu = \bar{e}_\mu \partial_\mu f^{-1}(0)$$

$$\implies \quad 4\bar{b}a - 2\left(\bar{a}b\right)^q = e_\mu \partial_\mu f^{-1}(0) \qquad (2.3.13a)$$

$$\text{and} \quad -2\left(\bar{b}a\right)^q + 4\bar{a}b = \bar{e}_\mu \partial_\mu f^{-1}(0) \qquad (2.3.13b)$$

The last equation (2.3.13b) can be made more useful using

$$-2e_\nu \left(\bar{b}a\right)^q e_\nu + 4e_\nu \bar{a} b e_\nu \; = \; e_\nu \bar{e}_\mu e_\nu \partial_\mu f^{-1}(0)$$

$$\implies \quad -2\bar{b}a + 4\left(\bar{a}b\right)^q \; = \; e_\mu \partial_\mu f^{-1}(0) \qquad (2.3.14)$$

meaning that, on taking (2.3.13a) from (2.3.14)

$$\bar{b}a - \left(\bar{a}b\right)^q = \bar{b}a - \left(\bar{b}a\right)^t = 0$$

where we have used the fact that co-efficients of quaternions are real, and also the result $e_\mu e_\nu e_\mu = \bar{e}_\mu e_\nu \bar{e}_\mu = -2\bar{e}_\nu$. The superscript q indicates conjugation in the quaternion indices only. This then derives the constraints (2.3.12a, 2.3.12b, 2.3.12c).

The above formulation has some invariances which further reduce the degrees of freedom of the problem. First of all, if S is a constant element of the group $Sp(N+n)$ then we can note that $v_\mu = \bar{H} \partial_\mu H$ is invariant under the transformation $H \to SH$. We also see that $\overline{(S\Delta)}(SH) = \bar{\Delta} H$ so that there is an ambiguity with Sa and Sb giving the same field. There is a similar ambiguity if we replace a and b with aT and bT respectively, where $T \in Gl(n, \mathbb{R})$.

Following the timbre of this work, we shall now concentrate on the case where the gauge group is $Sp(1) \simeq SU(2)$. In this case we can use the above invariances of the solution space in (2.3.8) to reduce the parameter matrices a and b to a more useful form. Let $S_1 \in Sp(n + 1)$ such that the row $[S_1]_{0i}$ is a vector in the orthogonal complement to the columns of b. Then $b \mapsto S_1 b$ has

$$b \mapsto \begin{pmatrix} 0 \\ b' \end{pmatrix}$$

where $\underline{0}$ is a null-vector and $b' \in M_{n,n}(\mathbb{H})$. From the constraint (2.3.12b) we see that $\bar{b}'b'$ must be real and symmetric. As such, the general theory of such matrices [57, 18] tells us that it can be expressed as a matrix product $\Omega^t \mathcal{E} \Omega$ where $\Omega \in O(n)$ and $\mathcal{E} = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n)$ where β_i is an eigenvalue of $\bar{b}'b'$. In fact, it is simple to see that Ω must be formed of the eigenvectors – made orthogonal by a Gram-Schmidt procedure – of the symmetric matrix. We can now transform b via $b \mapsto bT_1$ with $T_1 \in Gl(n)$ defined as $T_1 = \Omega \cdot \text{diag}(1/\sqrt{\beta_1}, \ldots 1/\sqrt{\beta_n})$, and follow this with $b \mapsto S_2 b$ where $[S_2]_{00} = 1$, $[S_2]_{0i} = [S_2]_{i0} = 0$, and $[S_2]_{ij} = [\bar{b}']_{ij}$. This then yields a canonical form for b as

$$b = \begin{pmatrix} \underline{0} \\ e_4 \mathbb{I}_n \end{pmatrix} \tag{2.3.15}$$

Applying the same transformations to a, we write

$$a = \begin{pmatrix} \underline{w} \\ a' \end{pmatrix} \tag{2.3.16}$$

where $\underline{w}$ is a n-tuple of quaternions. Hence, this canonical arrangement has $\Delta_{0l} = w_l$ and also all of the degrees of freedom are contained in the matrix a. Finally, we note that the constraint $(\bar{b}a)^t = \bar{b}a$ tells us that $a'^t = a'$.

Before we leave the theoretical underpinnings of the ADHM method to consider how it is applied to physical systems, we briefly calculate some asymptotic formulae.

Since equation (2.3.9) has

$$H\bar{H} = \mathbb{I} - \Delta f \bar{\Delta}$$

it is straightforward to show – remembering we are considering $SU(2)$ gauge group

$$|H_0|^2 = 1 - \frac{1}{2} \text{tr}_2 w f \bar{w} \tag{2.3.17a}$$

$$H_\kappa = -\frac{1}{|H_0|^2} \Delta f \bar{w} H_0 \quad \text{with } \kappa \neq 0 \tag{2.3.17b}$$

Using these expressions, it should be clear that at large distances,

$$\Delta = a + bx \asymp bx$$

$$f \asymp \frac{1}{|x|^2} \mathbb{I}_n$$

$$H_k \asymp -\frac{1}{|x|^2} x \bar{w}_k H_0$$

where the first and last expressions are obvious. The second relation can be derived by recalling $f^{-1} = \bar{\Delta}\Delta \mapsto \bar{x}\bar{b}bx$ and our canonical b (2.3.15). We note that specific gauge choices lead on to the values for H_0.

B.4 Some Fields

In the previous section, we derived the ADHM representation of the gauge field v_m. This was defined to be $v_m = \bar{H}\partial_m H$ which we remarked looked pure gauge, but was not due to the demand that $H \in M_{N+n,N}(\mathbb{H})$. It is possible to attempt to construct other fields within the framework of ADHM. We do this for the fundamental and adjoint fermions which arise in $N = 2$ supersymmetric models with matter.

It was established in [27] that the fundamental zero modes χ_i and $\tilde{\chi}_i$ with $i \in \{1, \ldots, N_f\}$ are given by

$$(\chi_i^\alpha)^{\dot\beta} = \bar{H}_\lambda^{\dot\beta\alpha} b_{\lambda k} f_{kl} \mathcal{K}_{li} \qquad (\chi_i^\alpha)^{\dot\beta} = \bar{H}_\lambda^{\dot\beta\alpha} b_{\lambda k} f_{kl} \tilde{\mathcal{K}}_{li} \qquad (2.4.18)$$

where we have explicitly written the indices appearing in the ADHM quantities. Of the spinor indices, α is a Weyl and $\dot\beta$ a gauge group – which we take to be $Sp(1) \simeq SU(2)$ – index. The matrices $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are constructed from grassmannian numbers **not** spinors. It is these matrices which encode the collective co-ordinates of the fermions obtained by seperating out the zero modes of the fluctuation operator from the semi-classical action, and so they appear in the measure introduced in chapter 3. Considering the definition of the covariant derivative ∇_m in the fundamental representation contracted with a spin matrix[2]

$$\bar{\sigma}^{m\dot\gamma\beta}\epsilon_{\beta\alpha}\left[\partial_m\left(\chi_i^\alpha\right)^{\dot\beta} + \bar{H}_\lambda^{\dot\beta\delta}\partial_m H_{\lambda\delta\dot\delta}\left(\chi_i^\alpha\right)^{\dot\delta}\right]$$

and manipulating this with the identities

$$\partial_m \bar{H} \cdot \Delta = -\bar{H} \cdot \partial_m \Delta \qquad (2.4.19a)$$

$$\partial_m f = -f\partial_m f^{-1} f = -f\partial_m(\bar{\Delta}\Delta)f = -f(\bar{e}_m\bar{b}\Delta + \bar{\Delta}be_m)f \qquad (2.4.19b)$$

$$\partial_m \bar{H} = -\bar{H}\partial_m H \bar{H} - \bar{U}\partial_m \Delta f \bar{\Delta} \qquad (2.4.19c)$$

---

[2]Essentially the spin matrix arises from the representation of the Dirac algebra which underpins fermions.

one finds that the spinors defined in (2.4.18) satisfy the Dirac equation $\bar{\sigma}_\alpha^{m\dot{\gamma}}(\nabla_m\chi_i^\alpha)^{\dot{\beta}} = 0$ without constraint.

Similarly, the adjoint fermion zero modes can be found in [22, 27, 28]. For example, the higgsino

$$(\psi_\alpha)_{\dot{\gamma}}^{\dot{\beta}} = \bar{H}^{\dot{\beta}\gamma}\mathcal{N}_\gamma f\bar{b}H_{\alpha\dot{\gamma}} - \bar{H}_\alpha^{\dot{\beta}}bf\mathcal{N}^{\gamma t}H_{\gamma\dot{\gamma}} \qquad (2.4.20)$$

where once more α is a Weyl index, and the others arise from the gauge group $SU(2)$. The $(n+1) \times n$ matrix $\mathcal{N}$ consists of elements which are grassmannian **spinors**. This time the covariant derivative ∇_m appears in the Dirac equation

$$\bar{\sigma}^{\dot{\alpha}\alpha}(\nabla_m\psi_\alpha)_{\dot{\gamma}}^{\dot{\beta}} = \bar{\sigma}^{\dot{\alpha}\alpha}\left(\partial_m(\psi_\alpha)_{\dot{\gamma}}^{\dot{\beta}} + \bar{H}^{\dot{\beta}\beta}\partial_m H_{\beta\dot{\delta}}(\psi_\alpha)_{\dot{\gamma}}^{\dot{\delta}} - (\psi_\alpha)_{\dot{\delta}}^{\dot{\beta}}\bar{H}^{\dot{\delta}\beta}\partial_m H_{\beta\dot{\gamma}}\right) = 0$$

Using the relations (2.4.19a, 2.4.19b, 2.4.19c) it is a matter of tedious algebra to show that the solution (2.4.20) only obeys the Dirac equation when the constraints

$$\bar{a}^{\dot{\alpha}\gamma}\mathcal{N}_\gamma = -\mathcal{N}^{\gamma t}a_\gamma^{\dot{\alpha}} \qquad (2.4.21a)$$

$$\bar{b}_\alpha^\gamma\mathcal{N}_\gamma = \mathcal{N}^{\gamma t}b_{\gamma\alpha} \qquad (2.4.21b)$$

are satisfied. The superscript t denotes transpose in the $(n+1) \times n$ "ADHM" indices – ie the non-grassmannian indices.

Similarly, one can introduce an $(n+1) \times n$ matrix $\mathcal{M}^\gamma$ of grassmannian spinors to encode the collective co-ordinates of the gaugino ψ. The formula for ψ follows from (2.4.20) by replacing $\mathcal{N}$ with $\mathcal{M}$.

B.5   An Illustration: The 1-instanton from ADHM

As a final illustration of the ADHM method, we shall derive the n-instanton solution when $n=1$ and when the gauge group is $Sp(1) \simeq SU(2)$. First we use the canonical ADHM matrices (2.3.15, 2.3.16) so that all of the degrees of freedom are contained in $a \in M_{2,1}(\mathbb{H})$ and we shall define $a = \begin{pmatrix} \rho \\ \tilde{a} \end{pmatrix}$. In this situation, we see that the ADHM constraints (2.3.12a, 2.3.12b, 2.3.12c) are trivially satisfied.

Using $\Delta = a + bx$ with $x = x^m e_m$ we see $\Delta = \begin{pmatrix} \rho \\ \tilde{a} + x \end{pmatrix}$. Since a is a constant quaternion, we are permitted to express it as $a = -x_0$ where the interpretation of

x_0 follows. Thus, we see

$$\Delta = \begin{pmatrix} \rho \\ x - x_0 \end{pmatrix}$$

In terms of components $[\Delta_0]_{\alpha\dot\alpha} = w_{\alpha\dot\alpha} = \rho_{\alpha\dot\alpha}$ and $[\Delta_1]_{\alpha\dot\alpha} = [x - x_0]_{\alpha\dot\alpha}$. Now, the ADHM construction demands that $f^{-1} = \bar\Delta\Delta$ commutes with all quaternions. It must therefore be proportional to $e_4 = \mathbb{I}$. That is $[f^{-1}]_{\dot\alpha}^{\dot\beta} = f^{-1}\delta_{\dot\alpha}^{\dot\beta}$. Thus

$$
\begin{aligned}
f^{-1}\delta_{\dot\alpha}^{\dot\beta} &= [\bar\Delta_0]^{\dot\beta\beta}[\Delta_0]_{\beta\dot\alpha} + [\bar\Delta_1]^{\dot\beta\beta}[\Delta_1]_{\beta\dot\alpha} \\
&= \bar\rho^{\dot\beta\beta}\rho_{\beta\dot\alpha} + [\bar x - \bar x_0]^{\dot\beta\beta}[x - x_0]_{\beta\dot\alpha} \\
&= \left(\rho^m\rho^n + (x - x_0)^m(x - x_0)^n\right)(\bar e_m e_n)_{\dot\alpha}^{\dot\beta} \\
&= \left[\rho^2 + (x - x_0)^2\right]\delta_{\dot\alpha}^{\dot\beta}
\end{aligned}
$$

Thus, since in the 1-instanton sector f is just a number when we seperate off the quaternion

$$f = \frac{1}{\rho^2 + (x - x_0)^2} \tag{2.5.22}$$

Consider the quantity

$$
\begin{aligned}
H_0 &= \sigma_{\alpha\dot\alpha}^0 \left[1 - \frac{1}{2}f\,\mathrm{tr}(w\bar w)\right]^{\frac{1}{2}} \\
&= \sigma_{\alpha\dot\alpha}^0 \left[1 - \frac{1}{2}\frac{\rho^2}{\rho^2 + (x - x_0)^2}\right]^{\frac{1}{2}} \\
&= \sigma_{\alpha\dot\alpha}^0 \left[\frac{(x - x_0)^2}{\rho^2 + (x - x_0)^2}\right]^{\frac{1}{2}} \tag{2.5.23}
\end{aligned}
$$

since $\mathrm{tr}(e^m\bar e^n) = 2\delta^{mn}\mathbb{I}$. It is easy to see that this matrix H_0 satisfies (2.3.17a) and so is a suitable candidate for the upper row of the matrix appearing in the definition of the vector potential. Clearly

$$|H_0|^2 = 1 - \frac{1}{2}f\,\mathrm{tr}(w\bar w) = \frac{(x - x_0)^2}{\rho^2 + (x - x_0)^2} \tag{2.5.24}$$

and so using (2.3.17b) the remaining entry in H is

$$
\begin{aligned}
[H_1]_{\alpha\dot\alpha} &= -\frac{1}{|H_0|^2}\Delta_1 f\bar{w} H_0 \\
&= -\frac{\rho^2 + (x-x_0)^2}{(x-x_0)^2}(x-x_0)^m e^m_{\alpha\dot\beta}\frac{1}{\rho^2 + (x-x_0)^2}\rho^n \bar{e}^{\dot\beta\beta}_n \sigma^0_{\beta\dot\alpha}\left[\frac{(x-x_0)^2}{\rho^2 + (x-x_0)^2}\right]^{\frac{1}{2}} \\
&= -\frac{(x-x_0)^m \rho^n}{\sqrt{(x-x_0)^2(\rho^2 + (x-x_0)^2)}}e^m_{\alpha\dot\beta}\bar{e}^{\dot\beta\gamma}_n \sigma^0_{\gamma\dot\alpha} \\
&= -\frac{|\rho|}{(x-x_0)^2}\sqrt{\frac{(x-x_0)^2}{\rho^2 + (x-x_0)^2}}(x-x_0)_{\alpha\dot\beta}\bar{\mathcal{O}}^{\dot\beta\gamma}\sigma^0_{\gamma\dot\alpha}
\end{aligned}
$$

where $\bar{\mathcal{O}}^{\dot\beta\gamma} = |\rho|^{-1}\bar{\rho}^{\dot\beta\gamma}$ is an iso-orientation matrix in the spin-$\frac{1}{2}$ representation of $SU(2)$. It obeys $\bar{\mathcal{O}}\mathcal{O} = \mathbb{I}_2$.

One can easily verify that $\sum_{\lambda=0}^{1}\bar{H}_\lambda H_\lambda = \mathbb{I}_2$. Further, we must investigate the expression for the vector potential

$$
v_m = \sum_{\lambda=0}^{1}\bar{H}_\lambda \partial_m H_\lambda
$$

as prescribed by the ADHM method. Performing the relevant differentiation, and collecting terms, one finds

$$
v_m = \frac{\rho^2}{(x-x_0)x^2\left((x-x_0)^2 + \rho^2\right)}\left\{-(x-x_0)^m + \bar{\sigma}_0\mathcal{O}(\bar{x}-\bar{x}_0)e_m\bar{\mathcal{O}}\sigma_0\right\}
$$

However, we recall that $\bar{x} = x^n\bar{e}_n$ and $\bar{e}_n e_m = \delta_{mn} + \bar{\eta}_{mn}$. Hence

$$
v_m = \frac{\rho^2(x-x_0)^n}{(x-x_0)^2 + \rho^2}\bar{\sigma}_0\mathcal{O}\bar{\eta}_{mn}\bar{\mathcal{O}}\sigma_0 \tag{2.5.25}
$$

Notice that the matrix term at the end of (2.5.25) is of the form $\bar{\eta}' = \bar{B}\bar{\eta}B$ where B is a constant matrix in $SU(2)$. This is simply a change of basis through rotation. Hence we have the final result

$$
v_m = \frac{\rho^2}{\rho^2 + (x-x_0)^2}\bar{\eta}'_{mn}(x-x_0)^n \tag{2.5.26}
$$

Comparison with standard references [5] show that this is exactly the form of the self-dual vector potential with winding number n=1 in the singular gauge. Therefore the ADHM method has reproduced the expected result. We also note that the parameters arising from our canonical form of the ADHM matrices have the usual interpretation in the 1-instanton sector. The collective co-ordinate x_0 is the centre of the instanton, $|\rho|$ is the size of the instanton and the quaternion "phase" of ρ yield the iso-orientation in colour space.

Appendix C

# $SU(2)$ **SUSY with** $N_f = 4$

In $N = 2$ $SU(N_c)$ super-Yang-Mills (SYM) theories with N_c colours and N_f flavours, it is a known result that the β-function is proportional to $2N_c - N_f$. This means that theories with $N_f = 2N_c$ are scale invariant. In this chapter we shall consider such a theory, in particular the case $N_c = 2$ and $N_f = 4$.

The classical theory has an $N = 2$ invariant – when one includes kinetic terms – superpotential

$$W = \sum_{i=1}^{N_f=4} \sqrt{2}\tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i \tilde{Q}_i \tag{3.0.1}$$

where Q_i and $\tilde{Q}_i$ are matter hypermultiplets (see figure 3.1) and m_i are the masses of these particles. The $N = 1$ superfield Φ was also introduced previously, and is the lowest $N = 1$ component of an $N = 2$ vector superfield. Physically, it contains the scalar Higgs field and its superpartners. The massless theory, which we consider from this point onwards, has global symmetry which is a quotient of the group $O(8) \times SU(2)_R \times U(1)_R$. As remarked earlier, the last two symmetries ensures $N = 2$ invariance. The $O(8)$ is a flavour symmetry. It arises in place of the usual $SU(N_f) \times U(1)$ since for gauge group $SU(2)$ the quarks Q and anti-quarks $\tilde{Q}$ are in isomorphic representations [26] of the gauge group. Under $O(8) \times SU(2)_R \times U(1)_R$ the quarks $(\tilde{Q}, Q)$ transform as $(\underline{8}, \underline{2}, 0)$ and the scalar ϕ in Φ as $(\underline{1}, \underline{1}, 2)$.

In this model, there is a flat direction with non-zero ϕ wherein the $SU(2)$ gauge group is broken in a Higgs-type mechanism to the Abelian subgroup $U(1)$. In addition, due
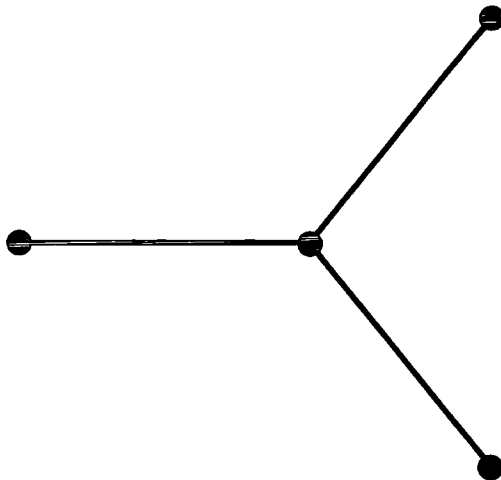
Figure C.1: Dynkin Diagram for $SO(8)$.

to the Yukawa couplings in (3.0.1) the quarks develop a mass. When this occurs, we say that the physics is on the Coulomb branch[1] of the theory. When one considers the quantum moduli space, one finds that the $O(8)$ is replaced with $SO(8)$ due to the spontaneous breaking of a parity $\mathbb{Z}_2 \subset O(8)$.

When $SU(2)$ is broken to $U(1)$ on the Coulomb branch, magnetic monopoles arise in the usual way (see chapter 2). In the background of this magnetic monopole, the fermions (and by implication their superpartners) develop zero modes. An index theorem [59] tells us that each fundamental fermion has one zero mode in this background. For 4 hypermultiplets each with an $SU(2)_R$ doublet of fermions we therefore have 8 zero modes transforming in a vector representation of a subgroup of $O(8)$. In fact, the presence of spinors indicates that we must consider the spin cover of $SO(8)$. This is the group $Spin(8)$ with centre $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The Dynkin diagram C.1 of the $spin(8) = D_4$ algebra indicates the presence of an outer automorphism. This is "triality" and is isomorphic to the permutation group S_3. If the particle states are distinguished by the quantum numbers (n_m, n_e) where n_m is the magnetic charge of the state, and n_e is the electric charge[2] then the quantum numbers under the centre of $Spin(8)$ are determined as

---

[1]There is another branch - when $N_f = 3, 4$ and all $m_i = 0$ – wherein the gauge symmetry is completely broken. This is the Higgs branch, and we shall not consider it in this chapter.

[2]It is these quantum numbers which appear, for example, in the BPS mass formula

(n_m, n_e)	Representation	Symbol	Interpretation
$(0,0)$	trivial	$\underline{8}_o$	elementary gauge fields
$(0,1)$	vector	$\underline{8}_v$	elementary quark
$(1,0)$	spinor	$\underline{8}_s$	fundamental monopole
$(1,1)$	spinor	$\underline{8}_c$	first excited monopole (dyon)

In general, there exist so-called "curves of marginal stability" in the moduli space of N=2 supersymmetric gauge theories. These loci occur in the strong coupling regime and are described by the equation $\mathrm{Im}\frac{a_D}{a} = 0$. Mathematically this corresponds to a collapse of the lattice Λ which appears in the quotient $\mathbb{C}/\Lambda$ which yields the Seiberg-Witten curve. Physically [8] this condition makes the decay of erstwhile stable BPS particles more likely. In essence [44] this means that the fundamental particles in the strong coupling core are different to those in the weak coupling regime. It is this phenomena that prevents the spectra of particles from being $\mathrm{Sl}(2,\mathbb{Z})$ invariant over the whole moduli space.

The existence of triality in $N_f = 4$ theories indicates that this decay is less likely than otherwise one would expect. This therefore lends probability to the $N_f = 4$ theory having an $\mathrm{Sl}(2,\mathbb{Z})$ invariant BPS spectra over the entire moduli space. Essentially one should be able to transform to another representation of $SO(8)$ using this outer automorphism. It is this realisation that makes study of the $N_f = 4$ theories interesting to mathematical physicists.

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