The inclusion of ghosts in landau gauge schwinger-dyson studies of infrared QCD

Watson, peter

The Inclusion of Ghosts in Landau Gauge Schwinger-Dyson Studies of Infrared QCD

A thesis presented for the degree of
Doctor of Philosophy
by

Peter Watson

Physics Department
University of Durham
July 2000

Abstract

It is widely believed that Quantum Chromodynamics (QCD) is the theory that describes the strong interaction. In the infrared region of the theory, the perturbative expansion breaks down and so, other techniques must be used. One such technique is the study of the Schwinger-Dyson equations.

In this thesis is presented such a study. It is shown that the ghost sector of QCD may be crucial to the understanding of the infrared behaviour. Conventionally, the Slavnov-Taylor identity is used to truncate the Schwinger-Dyson equations but it is found that for the ghost-gluon vertex, such an identity cannot be used in an appropriate manner. In order to extract information, a new technique is presented, based on the powerlaw behaviour of the two-point functions in the infrared. By demanding consistency in the full equations in Landau gauge and multiplicative renormalisability, it is found that in general, the gluon propagator dressing function cannot diverge and the ghost propagator function cannot vanish in the infrared. Further, it is shown that the powerlaw behaviour depends on a certain kinematical limit of only one function connected with the ghost-gluon vertex.

Acknowledgements

First and foremost, I would like to thank Mike Pennington, my supervisor. Although a very (important and) busy man, he has found time to inspire, challenge and aid me in my quest. From the moment he told me that "There are many integrals equal to zero." after I solved QCD for the first time, I knew that this was a clever man indeed. Thank you and good luck in ruling the empire.

I would also like to thank the staff and students of the CPT group in Durham. Special praise goes to Matt (who still can't beat me at cricket), John and Ania for coping with me in the office. For extraneous activities, I must thank Big Johnny C. (on me 'ead son), Ricardo and Matt Heyssler (my partner in crime outside the department). To the rest of the students goes a hearty farewell. For much sense and sensibility, I must also acknowledge Nigel Glover, whose helpful comments (both on physics and football) kept the wolves at bay.

Slightly further afield, I wish to thank my new found friends in Tuebingen, especially Reinhard Alkofer for giving me a wonderful opportunity to see not only another view on physics but another side of life. Thanks also to Steven, Christian and the rest of the group for looking after me.

Out with the physics, I have a vast multitude of people to thank. At Trevs, my tutor, mentor and friend Phil (and Paula too) gets a big mention for all his support. Also, for keeping an eye on my sanity, thanks to Maggie. Of course, I can't mention all the students who have been my friends over the last eight years but thank you one and all. Oh, and thanks to the BMC. On the German side, Joe, Nina, Bob, the Janes and the WG get a huge thank you for taking care of me.

Finally, I have to thank my parents. Firstly, they provided the genes. Then they provided the upbringing into the academic world. And then they gave me the cash to make it happen. At some point, my dad will want an explanation of my Schwinger-Dyson studies, so I have to make something up soon. Thanks to my sister too for never agreeing. Last but not least, I have to thank Tumble, the love of my life, to whom this thesis is dedicated.

The work in Durham has been supported by a PPARC studentship. In Germany, where the last chapter was born, the work was supported by the DFG (German Research Council) grant no. Al 279/3-2.

Declaration

I declare that I have previously submitted no material in this thesis for a degree at this or any other university.

Contents

Chapter 1

Introduction

This thesis presents work done on the study of the Schwinger-Dyson equations for covariant gauge Quantum Chromodynamics in the infrared region with specific reference to the ghost sector. The Schwinger-Dyson equations provide a natural framework from which it is hoped that the low energy (infrared) behaviour of Quantum Chromodynamics may be derived.

In this first chapter, a brief introduction to the theory and Schwinger-Dyson equations will be given. Many topics pertinent to later chapters will be motivated and introduced. The second chapter then goes on in more detail to introduce recent relevant work in the field and to motivate the research in the rest of the thesis. The third chapter builds up some technical results that will be of use to subsequent calculations.

One of the central elements of any Schwinger-Dyson study is what is known as the truncation scheme (see later for a more complete explanation). Loosely translated, this boils down to making an ansatz for unknown functions in the Schwinger-Dyson equations in order to close the system such that it may be solved either analytically (at best) or numerically (more usual). Such an ansatz is constrained by physical requirements such as gauge invariance and chapter four is concerned with an attempt to find an identity which constrains what is known as the ghost-gluon vertex. The technique is to use the perturbation expansion (again see later) in order to derive this identity. It is found that there does indeed exist a simple identity at the lowest order in the perturbation expansion, but this does not hold at higher orders.

In the light of such a failure to find an appropriate identity, chapter five goes on to look at another proposed identity. This identity was originally studied under a very simple truncation scheme [1, 2, 3] and was almost certainly the first work in standard Schwinger-

Dyson studies to realise the importance of the ghost sector in the infrared. Without the truncation, it is only possible to derive from this identity a small amount of information and this will be presented.

The final chapter pulls together the experience gained from all the previous chapters. It is realised that although no specific truncation scheme can reliably be made, there is however a way of looking at the equations in a consistent manner. This is the so-called powerlaw approach. By demanding consistency in the Schwinger-Dyson equations and observing the multiplicative renormalisability of the theory, it is possible to utilise what limited knowledge one has of the unknown functions to make definite conclusions.

1.1 QCD as a Gauge Theory

It is widely accepted that Quantum Chromodynamics (QCD) is the theory that describes the strong interaction (see for example one of the many standard textbook such as [4, 5, 6]). In its basic form, QCD is a locally gauge invariant quantum field theory whose matter (spin-$\frac{1}{2}$) constituents, the quarks, transform in the fundamental representation of the non-Abelian group $SU(3)$[1]. What this means in practice is that the quarks are represented by vectors ψ in a 3-dimensional (colour) space and that the observable physics is oblivious to local (gauge) transforms of the type

$$\psi(x) \to \psi'(x) = U(x)\psi(x) \tag{1.1.1}$$

where $U(x)$ is a unitary 3×3 matrix with determinant one ($U^\dagger U = 1$, $\det|U| = 1$). This matrix has 8 free parameters and can be written in the form

$$U(x) = \exp\{-\imath\theta_a(x)T^a\} \tag{1.1.2}$$

where there is a summation over the index a $(= 1,\ldots,8)$, the 3×3 hermitian matrices T^a $(= T^{a\dagger})$ are the generators of $SU(3)$ and $\theta_a(x)$ specifies the angle of rotation in the colour space. The T^a obey

$$\left[T^a, T^b\right] = \imath f^{abc}T^c \tag{1.1.3}$$

where the f^{abc} are the (completely antisymmetric) structure constants of the group. The T^a are traceless, since

$$\det|U| = \exp\{Tr\ln U\} = \exp\{-\imath\theta_a(x)Tr\,T^a\} = 1. \tag{1.1.4}$$

[1]Actually, for this thesis we use $SU(N_c)$ where it is understood that $N_c = 3$.

In addition, the generators are normalised such that

$$Tr\left[T^a T^b\right] = T^a_{ij} T^b_{ji} = \frac{1}{2}\delta^{ab}. \tag{1.1.5}$$

The quarks are spin-$\frac{1}{2}$ particles and must obey the Dirac equation. Thus, the Lagrange density must contain

$$\mathcal{L}_q = \overline{\psi}(x)\left(\imath\gamma^\mu \partial_\mu - m\right)\psi(x), \qquad \overline{\psi} = \psi^\dagger \gamma^0, \tag{1.1.6}$$

where the γ^μ are Dirac matrices which obey the (anticommuting) Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. However, this form alone is not invariant under the gauge transform above (1.1.1) so it is necessary to introduce a covariant derivative D_μ where

$$D_\mu = \partial_\mu - \imath g T^a A^a_\mu = \partial_\mu - \imath g A_\mu. \tag{1.1.7}$$

A^a_μ are the so-called gauge fields (the gluons) and g represents the coupling strength between the ψ and the A^a_μ. The Lagrange density is now invariant if D_μ transforms as

$$D_\mu \to D'_\mu = U D_\mu U^\dagger \tag{1.1.8}$$

so A_μ must transform as

$$A_\mu \to A'_\mu = U A_\mu U^\dagger - \frac{\imath}{g}(\partial_\mu U)U^\dagger. \tag{1.1.9}$$

In order to give the gauge field meaning, it is necessary to add a kinetic term to the Lagrange density. This is constructed from the field strength tensor $F_{\mu\nu}$

$$F_{\mu\nu} = \frac{\imath}{g}\left[D_\mu, D_\nu\right] = T^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\right) \tag{1.1.10}$$

such that $F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$. The (gauge invariant) kinetic term is (with prefactors chosen in analogy with QED and Maxwell's Equations)

$$\mathcal{L}_{gl} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{1.1.11}$$

giving the basic QCD Lagrange density

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \overline{\psi}(x)\left(\imath\gamma^\mu D_\mu - m\right)\psi(x). \tag{1.1.12}$$

1.2 The Functional Approach and Ghosts

In order to solve QCD, one must find all the Green's functions of the theory. There is more than one way to express a quantum field theory in terms of it's basic Green's functions. Two commonly used methods are the canonical approach (see for example [7]) and the functional approach (see [8] for a good review). The advantage of the functional approach is that it is based on the Lagrangian and preserves symmetries explicitly.

The functional method is based on the path integral formalism introduced by Feynman [9, 10]. The idea behind the path integral is that the probability amplitude for an event to occur is given by summing over all ways that the event may take place weighted with with the likelihood for that configuration given by a term $\exp\{\imath\mathcal{S}\}$ where $\mathcal{S}$ is the action. For a field theory, this translates to the following (see for example [8, 11, 12]). The central quantity is the generating functional – the vacuum to vacuum transition amplitude in the presence of external sources. This is given by

$$Z[J] = N^{-1}\int\mathcal{D}\phi\exp\left\{\imath\mathcal{S} + \imath\int d^4x\, J(x)\phi(x)\right\}, \qquad \mathcal{S} = \int d^4\,\mathcal{L}(x) \tag{1.2.1}$$

where N is a normalisation constant such that $Z[0] = 1$, $J(x)$ is the external source, $\phi(x)$ is the field and $\mathcal{D}\phi$ is the measure representing integration over all field configurations (which at $x_0 = \pm\infty$ approach the vacuum[2]). The n-point Green's function is derived by functional differentiation of Z,

$$G^n(x_1,\ldots,x_n) = (-\imath)^n\left.\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}. \tag{1.2.2}$$

In order to eliminate disconnected vacuum to vacuum diagrams, one must use the quantity $W[J]$ where

$$Z[J] = \exp\{\imath W[J]\}. \tag{1.2.3}$$

Now consider the purely gauge field part of the QCD Lagrange density (the pure Yang-Mills sector of the theory). It is clear that if one puts this into the generating functional, then one will be integrating not only over all different field configurations but over infinitely many identical configurations related by a gauge transform leading to a badly defined divergence. The way to deal with the overcounting of gauge equivalent

[2]This is necessary to ensure that for instance, any surface terms when one uses integration by parts vanish. However, this may not be the case if one considers topologically non-trivial solutions of the theory such as solitons and instantons [13].

field configurations is to use a trick introduced by Faddeev and Popov [14]. The idea is to reparameterise the integral into a form where the infinity is expressed as an overall coefficient (which can be absorbed into the normalisation factor N) whilst the rest is constrained by a gauge fixing condition. The presentation here follows [11] and will be discussed in some detail.

First consider integration over a group space. The integration measure dg is invariant under group transformations since in the integration, every group element is considered only once (even for continuous groups). Now, the gauge field A_μ transforms as

$$A_\mu \to A'_\mu = UA_\mu U^\dagger - \frac{\imath}{g}(\partial_\mu U)U^\dagger \tag{1.2.4}$$

and if we re-express U in terms of an infinitesimal transform parameterised by θ_a then we can expand as follows

$$U(\theta) = 1 - \imath\theta_a T^a + O(\theta^2), \quad U^\dagger(\theta) = 1 + \imath\theta_a T^a + O(\theta^2) \tag{1.2.5}$$

so

$$A_\mu \to A^\theta_\mu = T^a\left(A^a_\mu - f^{abc}A^b_\mu\theta_c - \frac{1}{g}\partial_\mu\theta_a\right) + O(\theta^2). \tag{1.2.6}$$

The integration measure for the integration over the group space can be taken as $\mathcal{D}\vec{\theta} = \prod_a \prod_x d\theta^a(x)$. Now one writes the identity

$$1 = \Delta[A]\int \mathcal{D}\vec{\theta}\delta(F^a(A^\theta)) \quad \text{or} \quad \Delta^{-1}[A] = \int \mathcal{D}\vec{\theta}\delta(F^a(A^\theta)). \tag{1.2.7}$$

From the second form of the above, it is clear that $\Delta[A]$ is invariant under gauge transforms. It is a Jacobian and can be written as $\Delta[A] = \det M$ where

$$M^{ab}(x,y) = \left.\frac{\partial F^a(A(x))}{\partial\theta^b(y)}\right|_{F=0}. \tag{1.2.8}$$

The argument of the delta function defines the gauge condition. Now consider the insertion of this into the following functional integral involving only the pure Yang-Mills sector (no quarks)

$$V[A] = \int \mathcal{D}A_\mu \exp\{\imath\mathcal{S}_{YM}\} = \int \mathcal{D}A_\mu\Delta[A]\int\mathcal{D}\vec{\theta}\delta(F^a(A^\theta))\exp\{\imath\mathcal{S}_{YM}\}. \tag{1.2.9}$$

Since $\mathcal{S}_{YM}$ is gauge invariant, one can do a transform to get

$$V[A] = \int\mathcal{D}\vec{\theta}\int\mathcal{D}A_\mu\Delta[A]\delta(F^a(A))\exp\{\imath\mathcal{S}_{YM}\}. \tag{1.2.10}$$

It is now apparent that the divergent $\int \mathcal{D}\vec{\theta}$ is just an overall factor that can be absorbed into the normalisation. With this, it is possible to re-express $\Delta[A]$ and $\delta(F^a(A))$ in a useful way. They are converted into additional terms in the Lagrange density. The gauge condition F^a is arbitrary and it is possible to write

$$F^a(A) \to F^a(A) - r^a(x) \tag{1.2.11}$$

for some function r^a. Changing r^a simply alters the centre of the delta function and so the rest of the functional integral is unaltered. It is further possible to supplement the delta function with a gaussian weighting factor (since this only changes the normalisation) so one can now write

$$\begin{aligned} \delta(F^a(A)) &\to \int \mathcal{D}r^a \exp\left\{-\frac{\imath}{2\xi}\int d^4x\,(r^a(x))^2\right\}\delta(F^a(A) - r^a(x)) \\ &= \exp\left\{-\frac{\imath}{2\xi}\int d^4x\,F^a(A)^2\right\}. \end{aligned} \tag{1.2.12}$$

The delta function can thus be expressed completely generally as a new term in the Lagrange density – the gauge-fixing term. In this thesis, we shall be considering only one class of gauge, the linear covariant gauges. These are obtained by setting

$$F^a(A) = \partial^\mu A^a_\mu \tag{1.2.13}$$

and the parameter ξ specifies which particular gauge (eg. $\xi = 1$ is the Feynman gauge) one is working in.

$\Delta[A]$ is a determinant and this can be re-expressed as a functional integral over Grassmann (anti-commuting) fields (see [11] for a detailed explanation). The appropriate form is

$$\Delta[A] = \det M \sim \int \mathcal{D}\bar{c}\mathcal{D}c \exp\{-\imath\int d^4x\, d^4y\, \bar{c}^a(x)M^{ab}(x,y)c^b(y)\} \tag{1.2.14}$$

Now consider an infinitessimal transform on $F^a(A)$,

$$F^a(A) \to F^a(A^\theta) = F^a(A) + \frac{\partial F^a}{\partial A^b_\mu}\delta A^b_\mu. \tag{1.2.15}$$

Using (1.2.6) it is possible to rewrite this as

$$F^a(A^\theta) = F^a(A) - \frac{1}{g}\frac{\partial F^a}{\partial A^b_\mu}D^{bc}_\mu\theta^c. \tag{1.2.16}$$

One can choose the gauge condition to be $F^a(A) = 0$ and using the definition of the matrix M

$$M^{ab}(x,y) = -\frac{1}{g}\frac{\partial F^a}{\partial A^d_\mu} D^{db}_\mu \delta(x-y). \tag{1.2.17}$$

The factor $-\frac{1}{g}$ is only a multiplicative constant in $\Delta[A]$ which itself is an overall factor in the functional integral, so one can absorb this into the normalisation with impunity. Replacing $F^a(A)$ in the derivative with it's form for the linear covariant gauges gives

$$\Delta[A] \sim \int \mathcal{D}\overline{c}\mathcal{D}c \exp\{-\imath \int d^4x\, \overline{c}^a(x)\partial^\mu D^{ab}_\mu c^b(x)\}. \tag{1.2.18}$$

The determinant $\Delta[A]$ has thus been expressed in a way that can be dealt with at the level of the Lagrange density. Two new sets of Grassmann fields fields have been introduced. They have a kinetic term $\overline{c}^a\partial^\mu\partial_\mu\delta^{ab}c^b$ and an interaction term $-gf^{abc}\overline{c}^a\partial^\mu A^c_\mu c^b$. The functional integral V originally considered can now be written as

$$V[A] = \int \mathcal{D}\overline{c}\mathcal{D}c\mathcal{D}A_\mu \exp\left\{\imath S_{YM} - \frac{\imath}{2\xi}\int d^4x\,(\partial^\mu A_\mu)^2 - \imath \int d^4x\, \overline{c}^a(x)\partial^\mu D^{ab}_\mu c^b(x)\right\}. \tag{1.2.19}$$

Thus, the QCD Lagrange density can be efficitively written as

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 - \overline{c}^a\partial^\mu D^{ab}_\mu c^b + \overline{\psi}\left(\imath\gamma^\mu D_\mu - m\right)\psi. \tag{1.2.20}$$

These new terms in the Lagrange density deserve a little further discussion. The Grassmann fields are known as ghosts and their introduction arises from the desire to consistently factor out an infinity from the functional integral due to the gauge symmetry, whilst preserving the theory. There are however more intuitive ways of introducing these terms. Firstly, in the basic QCD Lagrange density, the kinetic term for the gauge fields implies that there are four polarisation states for the gluon. However, just like the photon, the gluon should have only two physical polarisation states. The effect of the new terms is to kill off these unphysical degrees of freedom. Secondly, without the gauge fixing term, the gluon propagator (the 2-point Green's function with two external gluon legs) would have no inverse [11].

The ghost fields are also very unusual. They are Grassmann (anti-commuting) fields and so obey Fermi-Dirac statistics, but they are scalar and transform in the adjoint representation of the group just as the gluon fields do. Finally, the ghosts are not physical fields and so cannot appear as initial or final states in any physical process.

Although the Faddeev-Popov technique is used in standard QCD, there are however problems associated with it as pointed out by Gribov [15]. The Faddeev-Popov trick eliminates overcounting due to infinitesimal gauge transforms and then integrates over the group space, but it does not account for identical field configurations related by finite transforms. The Gribov problem will not be discussed in this thesis.

1.3 Schwinger-Dyson Equations and Slavnov-Taylor Identities

Now that we have a Lagrange density for QCD, it is possible to construct the full generating functional. For each field there is an external source term and repeated functional differentiation with respect to these sources gives the various Green's functions. As with any other theory built from a Lagrangian, there exists the Euler-Lagrange equations of motion. In a quantum field theory, these equations are known as the Schwinger-Dyson equations and they relate the different Green's functions. The Schwinger-Dyson equations can be derived in two ways: the functional method [16] or by a Dyson resummation [17]. The Dyson resummation essentially reorganises perturbative corrections (see later) into subdiagrams whereas the more rigorous functional method derives the equations directly from the invariance of the generating functional under variations of the field. The derivation of the Schwinger-Dyson equations will not be presented here – it would simply be too long. Indeed it can be characterised by the following quote:

> "This sort of thing should only be done in a locked room with the lights turned out." *M.R. Pennington, Durham graduate lecture series, 1997.*

The Schwinger-Dyson equations used in this thesis will be presented at the end of this chapter.

As mentioned before, the Lagrangian based construction allows the symmetries of the theory to be explicit. QCD has been built from the assertion that the theory be invariant under gauge transforms. By demanding that the generating functional is stationary with respect to gauge transforms one can derive the generalised Ward-Takahashi identities for QCD (see for example [11]). These equations however are not particularly enlightening due to the presence of ghost terms in the full (effective) Lagrange density. It was found that there does exist a symmetry in the full Lagrange density which places the ghost fields on the same footing as the gauge fields [18]. The BRS symmetry leads to the Slavnov-Taylor identities [19] and any solution to the Schwinger-Dyson equations must obey these too.

1.4 Schwinger-Dyson Equations, Green's Functions, the Perturbative Expansion, Feynman Rules and all that...

With the Lagrange density, it is possible to turn the theory into a physically meaningful formalism. The formalism relates scattering amplitudes (physically observable quantities) to the Green's functions. The probability of a transition from some initial state (a configuration of quarks) into a final state is given via the S-matrix. This matrix embodies all information about the scattering process and can be related to the Green's functions (this will not be reproduced here but can be found in many standard textbooks, eg. [6]). In the limit when the coupling (g) is small, the whole formalism can be expanded in powers of g, giving rise to the perturbation expansion and the Feynman rules. The Feynman rules are essentially the lowest perturbative order Green's functions corresponding to the propagation and interaction terms in the Lagrange density supplemented by symmetry factors, appropriate minus signs (for fermionic fields) and integral measures. From these rules it is (in principle) possible to derive both the matrix elements for scattering processes and the Green's functions themselves at an arbitrary order in the perturbative expansion.

It is also possible to write the full one-particle irreducible (1PI) Green's functions[3] in momentum space in terms of dressing functions (functions of the momenta) multiplied by prefactors containing the colour (group) content and basic kinematical structure of the quantity. The tree-level (lowest perturbative order) form is obtained by setting the dressing function to either unity (in the case of a simple propagator) or zero (in the case of some more complex vertex functions)[4]. One natural consequence of this is in the diagrammatic notation of the Schwinger-Dyson equations. The perturbative expansion can be expressed as a series of Feynman diagrams. The Green's functions are similarly expressed, but now all possible insertions of sub-diagrams (the dressings) are replaced by 'blobs'.

[3]These are related to the connected Green's function (generated from $W[J]$) by a Legendre transform and essentially the 'building blocks' of the theory.

[4]These forms are presented in full in the last section of the chapter.

Above, it was stated that the Green's functions could be calculated at arbitrary order in the perturbative expansion. In fact, this stems from the Schwinger-Dyson equations. The Schwinger-Dyson equations contain *all* information about the theory, not just the low-coupling limit. There can be other solutions to the theory, which cannot be described by perturbative means and so the blobs have more significance than merely being a set of sub-diagrams.

1.5 Renormalisation

In perturbation theory, if one calculates say the gluon self-energy (the 2-point 1PI Green's function with two external gluon legs) at the first non-trivial ('one-loop') level, one encounters what seems to be a major problem – 'ultra-violet' divergences. The one-loop calculation involves integration over an unconstrained internal loop momentum and as this momentum becomes large, a divergence occurs[5]. These UV divergences are ubiquitous in the Schwinger-Dyson equations and in order that meaningful physics be extracted, the theory must be renormalised.

The process of renormalisation stems from the realisation that since arbitrarily short distances (high momenta – the UV region) cannot be probed, it is only meaningful to compare physics at some finite scale with physics at another finite scale. If the theory can be re-expressed this way, such that all physical processes are free from UV divergences, it is said to be renormalisable. All physical quantum field theories must have this property and this was proved for QCD by 't Hooft [20].

The first step is to characterize the divergence in a systematic way, a procedure called regularisation. There are two methods used in this thesis: the UV-cutoff and dimensional regularisation. The UV-cutoff method takes the 4-dimensional Euclidean phase space to have a finite, but arbitrarily large size given by the radius $\sqrt{\Lambda}$. At the end of the calculation, one takes the limit $\sqrt{\Lambda} \to \infty$. This method has many disadvantages, but is a natural way to regularise full Schwinger-Dyson equations. Dimensional regularisation is applicable to perturbation theory. Here, all integrals are done in $d = 4 - 2\varepsilon$ dimensions for which they are finite. The $d = 4$ divergences show up as simple poles $1/\varepsilon$ as $\varepsilon \to 0$.

Having regularised the theory, it is now necessary to renormalise. This is done by

[5]This is not always the case. For instance, in the one-loop ghost-gluon vertex in Landau gauge ($\xi = 0$) the divergence is not present.

rewriting all the fields and parameters in the Lagrange density[6] in terms of a divergent part (denoted generically by $\sqrt{Z}$) and a finite part (the renormalised quantity, denoted by an overbar). Consider the following simple example of a scalar theory with the unrenormalised Lagrange density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - g\phi^4. \tag{1.5.1}$$

Now, multiplicative renormalisability (as it is known) requires that for every unrenormalised operator there exists a momentum independent (dimensionless) scale factor (the renormalisation coefficient Z) that ensures that the operator becomes independent of the regularisation (and thus the divergence) when expressed in terms of renormalised quantities. Thus, one must be able to write

$$\mathcal{L} = \frac{1}{2}Z_\phi(\partial_\mu \overline{\phi})(\partial^\mu \overline{\phi}) - \sqrt{Z_g}\,\overline{g}\overline{\phi}^4 \tag{1.5.2}$$

and identify

$$\phi = \sqrt{Z_\phi}\,\overline{\phi}, \qquad g = \sqrt{Z_g}\,Z_\phi^{-2}\overline{g}. \tag{1.5.3}$$

The Lagrange density can now be rewritten as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \overline{\phi})(\partial^\mu \overline{\phi}) - \overline{g}\overline{\phi}^4 + \frac{1}{2}(Z_\phi - 1)(\partial_\mu \overline{\phi})(\partial^\mu \overline{\phi}) - (\sqrt{Z_g} - 1)\overline{g}\overline{\phi}^4. \tag{1.5.4}$$

The first part of this is expressed solely in terms of renormalised quantities. The second part is explicitly divergent and the factors Z are to be calculated such that they cancel the divergences of the first part (this part of $\mathcal{L}$ defines the so-called counterterms). For an n-point 1PI Green's function $\Gamma^{(n)}$ under the scaling, only the external fields change so one can define

$$\overline{\Gamma}^{(n)} = Z_\phi^{\frac{n}{2}}\Gamma^{(n)} \tag{1.5.5}$$

where the renormalised quantity now does not depend on the regularisation and is explicitly free of UV-divergences. In order to completely specify the renormalised Green's function it is necessary to define its physical value for a certain momentum configuration, eg. all external momentum scales being $-\mu$. This is called the renormalisation prescription and once all the renormalised quantities have been identified with their physical counterparts (including the coupling, masses and the gauge parameter) then it is possible to derive the coefficients Z and the theory has been renormalised. The scale μ is called the renormalisation scale and is arbitrary.

[6]The quantities in $\mathcal{L}$ are not themselves observable.

One of the first consequences of the above when applied to QCD is the uniqueness of the coupling. There is a single physical (observable) coupling in the theory, but four basic interactions that could be used to derive it. This leads to the conclusion that there must be some relationship between the renormalisation coefficients of the corresponding vertices (the 3 and 4-point Green's functions that describe the interactions). The relationship does indeed exist and is one particular form of the Slavnov-Taylor identity. This will be an important piece of information later on.

The second important consequence of renormalisation is the existence of the renormalisation group (see [21] for a good review). The renormalisation coefficients (Z) depend only on μ, the regularisation, g and ξ (the latter two being unrenormalised[7]). The unrenormalised Green's functions depend on the regularisation (generically denoted here by Λ), g and ξ but are explicitly independent of μ so for fixed external momenta (g, ξ and Λ fixed too)

$$\mu\frac{d}{d\mu}\Gamma^{(n)}(\Lambda,g,\xi)=0. \tag{1.5.6}$$

Now, if one re-expresses the unrenormalised Green's function in terms of renormalised quantities, then one gets an equation describing how all these renormalised quantities (which also have an implicit dependence on μ through $\overline{g}$ and $\overline{\xi}$) vary with respect to μ whilst leaving the unrenormalised function invariant. The total derivative is thus rewritten as

$$\begin{aligned}\mu\frac{d}{d\mu} &= \mu\frac{\partial}{\partial\mu}+\mu\left.\frac{\partial\overline{g}}{\partial\mu}\right|_{g,\xi,\Lambda\text{fixed}}\frac{\partial}{\partial\overline{g}}+\mu\left.\frac{\partial\overline{\xi}}{\partial\mu}\right|_{g,\xi,\Lambda\text{fixed}}\frac{\partial}{\partial\overline{\xi}}\\ &= \mu\frac{\partial}{\partial\mu}+\beta(\overline{g},\overline{\xi})\frac{\partial}{\partial\overline{g}}+\delta(\overline{g},\overline{\xi})\frac{\partial}{\partial\overline{\xi}}.\end{aligned} \tag{1.5.7}$$

Applying this to

$$\overline{\Gamma}^{n}(\mu,\overline{g},\overline{\xi})=Z_{\phi}^{\frac{n}{2}}(\mu,\Lambda,g,\xi)\Gamma^{(n)}(\Lambda,g,\xi) \tag{1.5.8}$$

gives

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\overline{g},\overline{\xi})\frac{\partial}{\partial\overline{g}}+\delta(\overline{g},\overline{\xi})\frac{\partial}{\partial\overline{\xi}}-\frac{n}{2}\mu\frac{\partial}{\partial\mu}\ln Z\bigg|_{g,\xi,\Lambda\text{fixed}}\right]\overline{\Gamma}^{n}(\mu,\overline{g},\overline{\xi})=0 \tag{1.5.9}$$

where the expression $Z^{\frac{n}{2}}=\exp\{\frac{n}{2}\ln Z\}$ has been used. This is written as

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\overline{g},\overline{\xi})\frac{\partial}{\partial\overline{g}}+\delta(\overline{g},\overline{\xi})\frac{\partial}{\partial\overline{\xi}}-n\gamma(\overline{g},\overline{\xi})\right]\overline{\Gamma}^{n}(\mu,\overline{g},\overline{\xi})=0. \tag{1.5.10}$$

[7] This is a matter of convention – later in the thesis, the Z will be written in terms of the renormalised $\overline{g}$ and $\overline{\xi}$. The Z are only defined to remove the divergence and as long as one is clear which convention one uses, there is no ambiguity.

This equation is the Callan-Symanzik equation [22]. The functions β, δ and γ are characteristic to the theory and through this equation show how a change in μ can be reabsorbed into the definitions of the different parameters (such as the coupling) such that observable physics is unchanged.

1.6 The Need for Non-perturbative Methods and Schwinger-Dyson Studies

As has been stated before, the low-coupling limit of the theory gives rise to a natural expansion. However, the renormalisation group implies that the coupling is not a constant. It is observed that at low momentum scales (the infrared (IR) region), the coupling rises (see for example [5]) and so perturbation theory is not applicable. Thus, if one is to understand the IR content of QCD, other techniques must be used. If QCD in it's standard formulation is to describe the theory of strong interactions, then it must be able to account for confinement and the masses of the observable hadrons. The observed masses of the quarks are not simply related to the mass terms in the Lagrange density – the lightest quarks have masses measured at $\sim 5 - 10$MeV whereas the pion masses (two lightest quarks in a bound state) are ~ 140MeV and the proton mass (three lightest quarks in a bound state) is ~ 1GeV. The energy scales associated with the bound states are in the IR and so one is led to the idea of dynamical mass generation.

QCD as a fundamental theory must be able to explain confinement and mass generation in terms of basic parameters. These phenomena necessarily involve a mass scale other than the masses of the quarks and the only other scale in the theory is the renormalisation scale μ. This mass scale would be a gauge and renormalisation group invariant, so should satisfy (following [21])

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\overline{g})\frac{\partial}{\partial\overline{g}}\right] M(\mu,\overline{g}) = 0. \tag{1.6.1}$$

Since μ is the only scale involved[8], $M(\mu,\overline{g}) = \mu m(\overline{g})$ so the above equation has the solution

$$M(\mu,\overline{g}) = \mu \exp\left\{-\int^{\overline{g}} \frac{dg}{\beta(g)}\right\}. \tag{1.6.2}$$

Now, the perturbative expression for $\beta(\overline{g})$ goes like [23]

$$\beta(\overline{g}) = -\frac{1}{2}b_0\overline{g}^3 + O(\overline{g}^5) = -\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{\overline{g}^3}{16\pi^2} \tag{1.6.3}$$

[8]This is an assumption – there could be other mass scales in QCD, but there is no evidence for this.

where N_f is the number of quark flavours and $b_0 > 0$. Thus as $\overline{g} \to 0$

$$M(\mu, \overline{g}) = \mu \exp\left\{-\frac{1}{b_0\overline{g}^2}\right\} \tag{1.6.4}$$

which has an essential singularity for $\overline{g}^2 < 0$ and vanishes for $\overline{g}^2 > 0$. It is clear then that $\beta(\overline{g})$ must be derived non-perturbatively if one is to account for confinement and mass generation.

Given that one needs a non-perturbative description of QCD the question arises of how to go about this. One such method is to discreetize spacetime and use massive computation to extract information – the so-called lattice approach. This will not be discussed here.

Another approach is to study the Schwinger-Dyson equations directly. This has its advantages and disadvantages. On the plus side, the method is based on a continuous spacetime from which divergences can be naturally dealt with. These divergences may be of UV origin (and connected to the renormalisation) or of IR origin (ie. the region where an external momentum scale vanishes). Also, although the equations are obtained by expanding around the free field vacuum (being based on the generating functional), they make no reference to the vacuum or perturbations around it [21]. It is therefore hoped that even in a confining regime (where the quarks are not the observed states, rather it is the hadrons that are seen), the Schwinger-Dyson equations hold and can give the correct non-perturbative description of QCD.

However, there is no well-defined procedure for extracting reliable information from the Schwinger-Dyson equations other than perturbation theory. The Schwinger-Dyson equations are an infinite set of coupled non-linear integral equations and in order to solve the theory it would in principle be necessary to solve the whole set. The equations for the 2-point functions (see the next section) involve integrals containing the 3 and 4-point functions, the 3-point functions involve terms with the 4 and 5-point functions and so on. The technique is to find a way to reduce these equations down to a small subset of simpler equations and then solve these. This is called truncating the system. In general however, apart from perturbation theory, there is no method for truncating the system in a rigorously systematic manner. One is reduced to making ansatz for the unknown higher-point functions, trying to include as much physical input as possible (for example using the Slavnov-Taylor identity) and demanding that the solutions obtained are self-consistent. The results are then compared with observation. This may seem to be a pointless exercise at first, but it does provide insight into QCD which would not otherwise be possible. There are pieces of information that can be derived in spite of the technical problems encountered.

1.7 Feynman Rules, Notations and Conventions

Throughout this thesis, the Feynman rules for the fully dressed one-particle-irreducible quantities used are shown in table 1.1. Note that for convenience, the fermionic sector of the theory will be restricted to N_f massless quarks and for the most part they will only be included as corrections to the gluon self-energy and triple-gluon vertices. The transverse and longitudinal projectors are

$$\begin{aligned} t_{\mu\nu}(p) &= g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \\ l_{\mu\nu}(p) &= \frac{p_\mu p_\nu}{p^2}. \end{aligned} \tag{1.7.1}$$

At tree-level, the dressing functions reduce to

$$\begin{aligned} J_p^{-1} &= G_p = F_p = 1 \\ \widetilde{\Gamma}_\mu(p_1,p_2,p_3) &= p_{1\mu} \\ \Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) &= g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + \text{cyclic permutations (c.p.)} \\ \Gamma_\mu(p_1,p_2;p_3) &= \gamma_\mu \\ \Gamma^{a\ldots d}_{\mu_1\ldots\mu_4} &= -g^2\left\{g_{\mu_1\mu_2}g_{\mu_3\mu_4}\left[f^{ac,bd}-f^{ad,cb}\right] + g_{\mu_1\mu_3}g_{\mu_2\mu_4}\left[f^{ab,cd}f^{ad,bc}\right]\right. \\ &\left. +g_{\mu_1\mu_4}g_{\mu_2\mu_3}\left[f^{ac,db}-f^{ab,cd}\right]\right\} \end{aligned} \tag{1.7.2}$$

where $f^{ab,cd} = f^{abe}f^{cde}$. The SU(N) generators are denoted $T^a(a=1,\ldots,N^2-1)$, and obey

$$\begin{aligned} \left[T^a,T^b\right] &= \imath f^{abc}T^c \\ Tr\left[T^aT^b\right] &= \frac{1}{2}\delta^{ab} = T^a_{ij}T^b_{ji} \end{aligned} \tag{1.7.3}$$

with the structure constants having the following identities

$$\begin{aligned} f^{abc}f^{dbc} &= \delta^{ad}C_A \\ f^{ade}f^{bef}f^{cfd} &= \frac{1}{2}C_Af^{abc}. \end{aligned} \tag{1.7.4}$$

The loop integration measure is $-\imath \bar{d}^d\omega = -\imath d^d\omega/(2\pi)^d$.

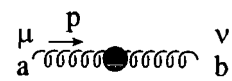
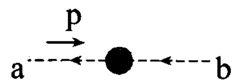
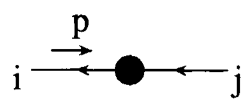
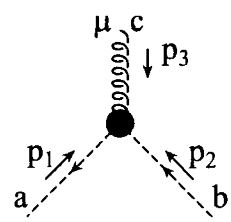
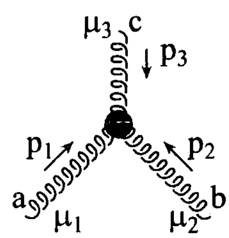
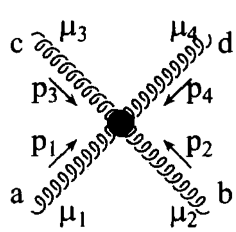
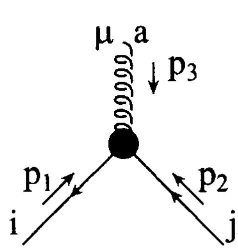
gluon propagator		$D^{ab}_{\mu\nu}(p) \equiv \delta^{ab}\frac{1}{p^2}\left[t_{\mu\nu}(p)J_p^{-1} + \xi l_{\mu\nu}(p)\right]$
ghost propagator		$D^{ab}_G(p) \equiv \delta^{ab}\frac{1}{p^2}G_p$
quark propagator		$S^F_{ij}(p) \equiv \delta_{ij}\frac{1}{\not p}F_p$
ghost-gluon vertex		$\tilde{\Gamma}^{abc}_{\mu}(p_1,p_2,p_3) \equiv -\imath g f^{abc}\tilde{\Gamma}_\mu(p_1,p_2;p_3)$
triple-gluon vertex		$\Gamma^{abc}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) \equiv -\imath g f^{abc}\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$
four-gluon vertex		$\Gamma^{a\ldots d}_{\mu_1\ldots\mu_4}(p_1,\ldots,p_4)$
quark-gluon vertex		$\Gamma^{Fa}_{\mu ij}(p_1,p_2;p_3) \equiv -gT^a_{ij}\Gamma_\mu(p_1,p_2;p_3)$

Table 1.1: Feynman rules used in this chapter. A blob indicates a fully dressed one-particle-irreducible quantity. The subscripts 'p' denote the argument of the function.

Further conventions used at various points in the thesis are as follows. The ghost-gluon vertex function can be decomposed as follows

$$\widetilde{\Gamma}_{\mu}(p_1, p_2, p_3) = p_1^{\nu}\widetilde{\Gamma}_{\nu\mu}(p_1, p_2, p_3) \tag{1.7.5}$$

by virtue of the rather peculiar form for the tree-level ghost-gluon vertex. The two-index object will be referred to as the ghost-gluon scattering-like kernel. Lastly, the triple-gluon vertex function obeys the following Slavnov-Taylor identity (written in the notation of [24])

$$p_3^{\rho}\Gamma_{\mu\nu\rho}(p_1, p_2, p_3)G_3^{-1} = J_2 p_2^2 t_{\nu}^{\lambda}(p_2)\widetilde{\Gamma}_{\lambda\mu}(p_2, p_3; p_1) - J_1 p_1^2 t_{\mu}^{\lambda}(p_1)\widetilde{\Gamma}_{\lambda\nu}(p_1, p_3; p_2). \tag{1.7.6}$$

The Schwinger-Dyson equations for the two-point Green's functions (inverse propagators) are presented in their diagrammatic form in fig. 1.1.

The renormalisation equations will be presented as required. All renormalised quantities will be denoted with an overbar, except where no confusion arises (in particular, the last chapter). Unrenormalised quantities will either be left as they are or for clarity denoted with a subscript $_b$.

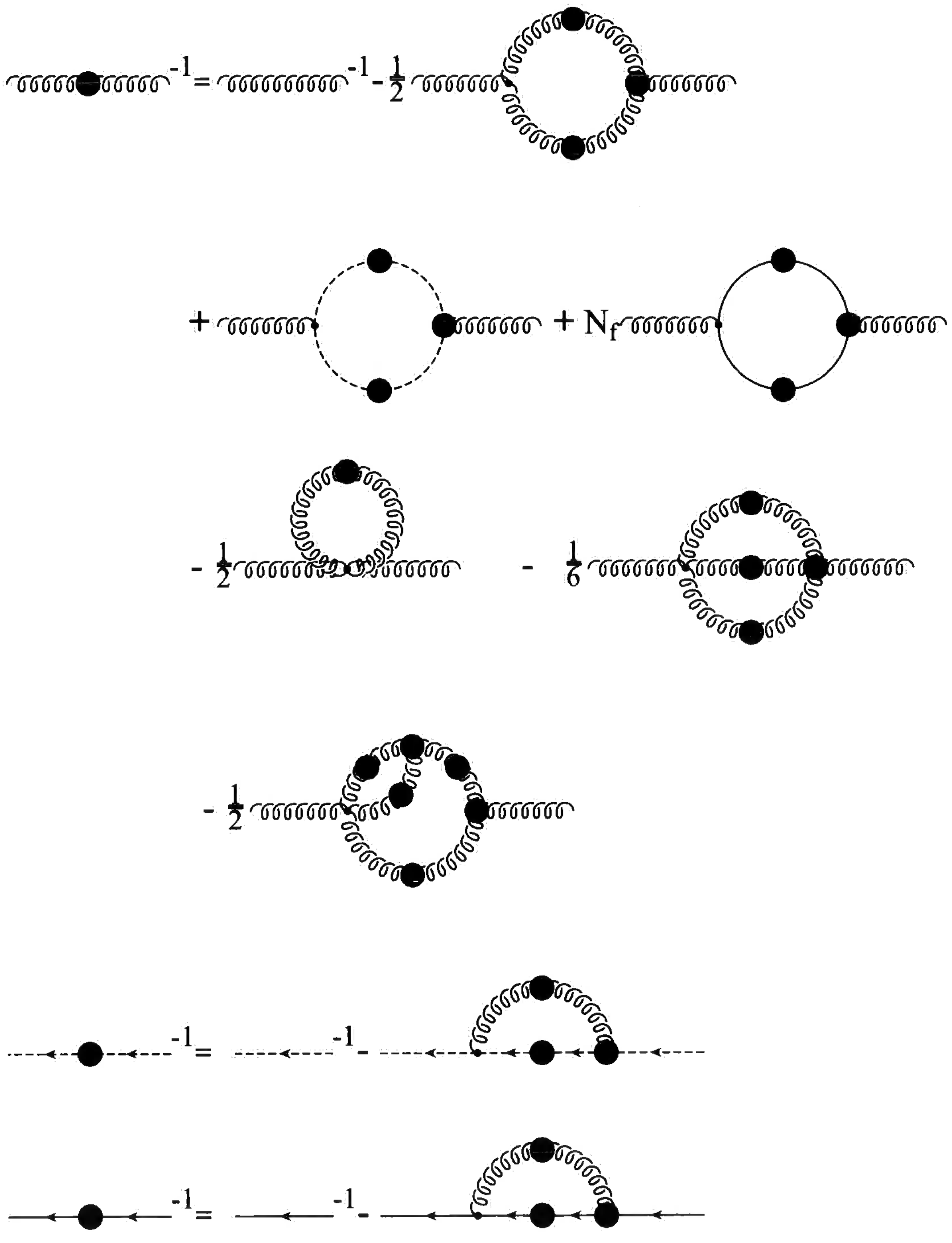


Figure 1.1: The full 2-point Schwinger-Dyson equations for QCD in covariant gauges.

Chapter 2

The Effect of Ghosts in Landau Gauge

The purpose of this chapter is to show how the inclusion of ghosts can affect Landau gauge Schwinger-Dyson studies of QCD. Perturbatively it is widely known [25, 26] that at the one-loop level the numerical significance of ghosts is negligible, although their importance to such issues as the transversality of the gluon propagator is beyond doubt. However, as will be demonstrated, they may play a crucial role in determining the infrared (IR) behaviour of the gluon propagator, which itself is important for confinement.

It is worth pointing out at this stage that the examples given in this chapter are not a complete survey of Schwinger-Dyson studies of QCD. There are many more schemes proposed in axial gauges [27, 28, 29, 30] and different techniques for looking at covariant gauges (for example [31, 32]). However, these examples do show that the inclusion of ghosts into an approximation scheme in covariant gauges is a delicate issue.

Besides explicit contributions to the Schwinger-Dyson equations themselves, the ghost propagator dressing function and ghost-gluon vertex form an integral part of the Slavnov-Taylor identity for the triple-gluon vertex and hence the vertex itself. The first part of the chapter focuses on this and shows how a deep knowledge of these ghost inclusions is required if one is to proceed in a consistent way.

The second section looks at an approximation scheme proposed originally by Mandelstam [25]. This study is based on the assumption that the perturbative numerical insignificance of the ghosts persists in the IR. By neglecting ghosts and making one further assumption about the triple-gluon vertex one arrives at a much simplified expression for the gluon polarisation. It is possible to infer a propagator which goes as $1/p^4$ in the limit $p^2 \to 0$ (and a running coupling which increases without bound).

The third section then steps out of chronological order and looks at a different scheme proposed by Atkinson and Bloch [33, 34]. Here, both the triple-gluon and ghost-gluon vertices are taken to be bare. Although this explicitly does not satisfy the Slavnov-Taylor identity, it does however lead to a simple system of equations. These can be dealt with in a particularly elegant way to get a gluon propagator that *vanishes* and a running coupling which attains a fixed point in the IR.

The final section deals with a more sophisticated attempt to include ghosts put forward by von Smekal *et al.* [1, 2, 3]. This in fact was the first piece of work to point out that the ghost sector could be crucial to IR QCD and was the motivation for the subject of this thesis. By neglecting irreducible ghost-ghost scattering (and all other four-point interactions) they obtain a particular form for the ghost-gluon and triple-gluon vertices which implements their respective Slavnov-Taylor identities. The analysis of the subsequent set of truncated Schwinger-Dyson equations leads to conclusions qualitatively the same as those of Atkinson and Bloch [33, 34].

2.1 Ghosts and the Triple-Gluon Vertex

One of the first stages in any Schwinger-Dyson propagator study is to make an ansatz for the unknown three and four-point functions that enter the equations. It is usually taken that for the purposes of confinement, the triple-gluon vertex is sufficient to give the basic features of the model. The four-gluon interaction is then neglected, partly under the assumption that it will not have a significant effect and partly because of the technical complications involved[1].

For the triple-gluon vertex ansatz, the Slavnov-Taylor identity is of great use. This identity can be solved to find a unique form for part of the vertex – the 'longitudinal' part. The rest of the vertex (the 'transverse' part) is left undetermined. Because the Slavnov-Taylor identity implements gauge invariance, the longitudinal vertex is sufficient to guarantee the correct gauge dependence of its contribution to the boson propagator. One such example is in QED, where the Ball-Chiu vertex [35] can be shown to result in the correct transverse structure of the photon propagator. To see this, one contracts the Schwinger-Dyson equation for the photon polarisation tensor with the external momen-

[1]One can also argue that the non-Abelian nature of the theory is already manifest with inclusion of the triple-gluon interaction, the four-gluon interaction only affecting matters quantitatively.

tum,

$$p^\nu \Pi_{\mu\nu}(p) = p^\nu \Pi^{(0)}_{\mu\nu}(p) + \int (-\imath)\, d\!\!^{-} {}^d k (g\gamma_\mu) S_F(k) S_F(k-p) p^\nu \Gamma_\nu(k, p-k; -p). \tag{2.1.1}$$

In the above, $\Pi_{\mu\nu}(p)$ is the photon polarisation (the inverse of the propagator), $S_F(k)$ is the fermion propagator and $\Gamma_\nu(k, p-k; -p)$ is the fermion-photon vertex. The Ward-Takahashi identity (the Abelian case of the Slavnov-Taylor identity) is

$$p^\nu \Gamma_\nu(k, p-k; -p) = S_F^{-1}(k) - S_F^{-1}(k-p) \tag{2.1.2}$$

so that any solution (ie the Ball-Chiu vertex) gives, under the condition that the integrals are finite or regularised in a translationally invariant way (eg dimensional regularisation)

$$\begin{aligned} p^\nu \Pi_{\mu\nu}(p) &= p^\nu \Pi^{(0)}_{\mu\nu}(p) + \int (-\imath) g\, d\!\!^{-} {}^d k \gamma_\mu \left\{ S_F^{-1}(k-p) - S_F^{-1}(k) \right\} \\ &= p^\nu \Pi^{(0)}_{\mu\nu}(p) \\ &= \xi \frac{p_\mu}{p^2} \end{aligned} \tag{2.1.3}$$

which is the correct gauge dependence of the photon polarisation [5]. The transverse part of the vertex, defined by

$$p^\nu \Gamma^t_\nu(k, p-k; -p) = 0 \tag{2.1.4}$$

can play no role in the transversality of the propagator, but is crucial to the multiplicative renormalisability of the equation [36].

Thus, one can see that it is desirable to have at least the longitudinal vertex from the point of view of the gauge dependence. Let us now see what this entails in the case of the triple-gluon vertex.

In order to solve the Slavnov-Taylor identity, the first step is to de-construct the general vertex into a form that makes the symmetry, colour and tensor properties explicit. This was done by Ball and Chiu [37] (also by Kim and Baker [38]) to give an expression involving six unknown functions, four of which are determined by the Slavnov-Taylor identity and two which are transverse to all three momenta (and therefore undetermined). A brief outline of their work is as follows. The full vertex is written

$$\Gamma^{abc}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) \equiv -\imath g f^{abc} \Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) \tag{2.1.5}$$

where the colour structure has been extracted. Bose symmetry then requires that Γ be antisymmetric under interchange of any two vectors and their respective Lorentz indices,

$(p_i, \mu_i) \leftrightarrow (p_j, \mu_j)$. This is done by constructing tensors antisymmetric under the interchange $(p_1, \mu_1) \leftrightarrow (p_2, \mu_2)$ and then demanding invariance under cyclic permutations. The vertex is conventionally written in the following way

$$\begin{aligned}\Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = & \; g_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} A_{123} + g_{\mu_1\mu_2}(p_1 + p_2)_{\mu_3} B_{123} \\ & -(p_1 \cdot p_2 g_{\mu_1\mu_2} - p_{1\mu_2} p_{2\mu_1})(p_1 - p_2)_{\mu_3} C_{123} \\ & +\frac{1}{3}(p_{1\mu_3} p_{2\mu_1} p_{3\mu_2} + p_{1\mu_2} p_{2\mu_3} p_{3\mu_1}) S_{123} \\ & +(p_1 \cdot p_2 g_{\mu_1\mu_2} - p_{1\mu_2} p_{2\mu_1})(p_2 \cdot p_3 p_{1\mu_3} - p_1 \cdot p_3 p_{2\mu_3}) F_{123} \\ & + \left(\frac{1}{3}(p_{1\mu_3} p_{2\mu_1} p_{3\mu_2} + p_{1\mu_2} p_{2\mu_3} p_{3\mu_1}) - g_{\mu_1\mu_2}(p_2 \cdot p_3 p_{1\mu_3} - p_1 \cdot p_3 p_{2\mu_3})\right) H_{123} \\ & + \text{cyclic permutations} \end{aligned} \quad (2.1.6)$$

where $A_{123} \equiv A(p_1^2, p_2^2, p_3^2)$ etc. The functions $(B), A, C$ and F are (anti)symmetric in their first two arguments, H is completely symmetric and S is completely antisymmetric. The functions F and H are multiplied by tensors that vanish under contraction with any momenta with its respective index and so make up the transverse vertex. The other four functions define the longitudinal vertex.

One further quantity to introduce is the ghost-gluon scattering-like kernel $\widetilde{\Gamma}_{\nu\mu}$ which is related to the ghost-gluon vertex in the following way

$$\widetilde{\Gamma}_{\mu}^{abc}(p_1, p_2; p_3) \equiv -\imath g f^{abc} p_1^{\nu} \widetilde{\Gamma}_{\nu\mu}(p_1, p_2; p_3) \quad (2.1.7)$$

This quantity has no immediate symmetry, two independent momenta with two free indices and hence can be de-constructed into five tensor components, conventionally written as

$$\widetilde{\Gamma}_{\nu\mu}(p_1, p_2; p_3) = g_{\nu\mu} a'_{321} - p_{3\nu} p_{2\mu} b'_{321} + p_{1\nu} p_{3\mu} c'_{321} + p_{3\nu} p_{1\mu} d'_{321} + p_{1\nu} p_{1\mu} e'_{321} \quad (2.1.8)$$

(again $a'_{321} \equiv a'(p_3^2, p_2^2, p_1^2)$).

Now, the Slavnov-Taylor identity for the triple-gluon vertex is

$$p_3^{\mu_3} \Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = G_3 J_2 p_2^2 t_{\mu_2}^{\nu}(p_2) \widetilde{\Gamma}_{\nu\mu_1}(p_2, p_3; p_1) - G_3 J_1 p_1^2 t_{\mu_1}^{\nu}(p_1) \widetilde{\Gamma}_{\nu\mu_2}(p_1, p_3; p_2) \quad (2.1.9)$$

(where $G_3 = G(-p_3^2)$) plus cyclic permutations (c.p.). The identity thus gives rise to a set of 15 linear equations in the coefficients of the five independent tensors. Of these, 3 can immediately be extracted by further contraction of the identity with $p_1^{\mu_1}$ and $p_2^{\mu_2}$ (and c.p.)

to give relations between the cyclic permutations of A, B, C and the fully antisymmetric S. This is because under the further contraction, the right-hand side of (2.1.9) vanishes and F, G are transverse. One is left with 9 unknown functions and 12 equations. Thus, A, B, C and S can be unambiguously expressed in terms of $a', \ldots, e', G, J$ and there are 3 equations left over which relate $a', \ldots, e'$. Writing $a_{123} = a'_{123}G_2J_3$, the solution can be expressed as

$$\begin{aligned}
4A_{123} &= 2(a_{321} + a_{312}) + p_3^2(b_{123} + b_{213}) + 2p_1\cdot p_3 d_{321} + 2p_2\cdot p_3 d_{312} \\
&\quad +(p_1^2 - p_2^2)(b_{231} - b_{132} + b_{312} - b_{321}) \qquad (2.1.10) \\
4B_{123} &= 2(a_{321} - a_{312}) - p_3^2(b_{231} - b_{132} + b_{312} - b_{321}) + 2p_1\cdot p_3 d_{321} - 2p_2\cdot p_3 d_{312} \\
&\quad -(p_1^2 - p_2^2)(b_{123} + b_{213}) \qquad (2.1.11) \\
(p_1^2 - p_2^2)C_{123} &= a_{231} - a_{132} + p_2\cdot p_3 d_{132} - p_1\cdot p_3 d_{231} \qquad (2.1.12) \\
-\frac{1}{2}S_{123} &= b_{123} + b_{231} + b_{312} - b_{132} - b_{321} - b_{213} \qquad (2.1.13)
\end{aligned}$$

and

$$a_{123} - a_{213} - p_1\cdot p_2(b_{123} - b_{213}) + p_1\cdot p_3 d_{123} - p_2\cdot p_3 d_{213} = 0 \qquad (2.1.14)$$

plus cyclic permutations.

Now one can clearly see the impact of the ghosts in determining the longitudinal part of the triple-gluon vertex. Aside from explicit contributions to the Schwinger-Dyson equations, in order to maintain the correct gauge dependence of the pure gluon loop via the longitudinal vertex it is necessary to know the non-perturbative forms for three of the five functions that define the kernel $\widetilde{\Gamma}_{\nu\mu}$ – ie, it is necessary to know the full ghost-gluon vertex.

2.2 The Mandelstam Approximation

Here, we shall discuss an approximation scheme originally put forward by Mandelstam [25]. This scheme has been studied by various groups [26, 39, 40] with the same conclusion that the gluon propagator diverges like $1/p^4$ as $p^2 \to 0$. The 'Mandelstam approximation' can be constructed in the following way. The first step is to neglect all four-gluon vertices and fermionic contributions. One then works in the Landau gauge. It is observed that the ghost-gluon scattering-like kernels $\widetilde{\Gamma}_{\nu\mu}$ occurring in the Slavnov-Taylor identity (2.1.9) reduce to their bare values as the in-ghost momentum (the second argument) tends to zero in the Landau gauge [21]. This leaves

$$p_3^{\mu_3}\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = G_3 J_2 p_2^2 t_{\mu_1\mu_2}(p_2) - G_3 J_1 p_1^2 t_{\mu_1\mu_2}(p_1) \tag{2.2.1}$$

The next step is to set the ghost dressing function to unity. This is necessary to ensure that condition (2.1.14) is met. The explicit ghost-loop contribution in the gluon propagator Schwinger-Dyson equation is now just the one-loop expression which numerically is small compared to the purely gluonic loop. The justification for dropping the ghost contributions is the postulate that the one-loop perturbative numerical insignificance can be extrapolated to the infrared [25].

The Schwinger-Dyson equation for the gluon propagator can then be written as

$$\Pi_{\mu\nu}^{ad}(p) = p^{\nu}\Pi_{\mu\nu}^{(0)ad}(p) - \frac{1}{2}\int(-\imath)\, \bar{d}^4k(-\imath g f^{abc})(-\imath g f^{bdc}) \times \Gamma^{(0)}_{\mu\rho\sigma}(p,-k,k-p)D^{\rho\alpha}(k)D^{\sigma\beta}(p-k)\Gamma_{\beta\alpha\nu}(p-k,k,-p). \tag{2.2.2}$$

where $\Pi_{\mu\nu}^{ad}(p)$ is the full gluon polarisation tensor. The triple-gluon vertex function above must now be dealt with. Taking the general solution for the longitudinal vertex (dropping the unknown transverse vertex), setting $a' = 1, b' = \ldots = e' = 0$ and $G = 1$, one gets

$$A_{123} = \frac{1}{2}(J_1 + J_2),\ B_{123} = \frac{1}{2}(J_1 - J_2),\ C_{123} = \frac{1}{(p_1^2 - p_2^2)}(J_1 - J_2) \tag{2.2.3}$$

Now it is assumed that J is a slowly varying function of its argument such that $J(p-k) \simeq J(k) \simeq J(p)$. The vertex is thus written

$$\Gamma_{\beta\alpha\nu}(p-k,k,-p) = J(p-k)\Gamma^{(0)}_{\mu\rho\sigma}(p,-k,k-p) \tag{2.2.4}$$

This simplifies the gluon Schwinger-Dyson equation (even more than using the bare vertex) to

$$\Pi_{\mu\nu}^{ad}(p) = p^{\nu}\Pi_{\mu\nu}^{(0)ad}(p) + \frac{\imath}{2}g^2 C_A \delta^{ad}\int \bar{d}^4k\Gamma^{(0)}_{\mu\rho\sigma}(p,-k,k-p)D^{\rho\alpha}(k)D^{\sigma\beta}(p-k)J(p-k)\Gamma^{(0)}_{\beta\alpha\nu}(p-k,k,-p)$$

$$t_{\mu\nu}(p)(J_p - 1) = \frac{\imath}{2}g^2 C_A\int\frac{\bar{d}^4k J_k^{-1}}{p^2k^2(p-k)^2}\Gamma^{(0)}_{\mu\rho\sigma}(p,-k,k-p)t^{\rho\alpha}(k)t^{\sigma\beta}(p-k)\Gamma^{(0)}_{\beta\alpha\nu}(p-k,k,-p) \tag{2.2.5}$$

Since a UV-cutoff will be used to regulate the integrals it is important to avoid quadratic divergences. Brown and Pennington [26] noted that these divergences can in general only occur in the part of the loop proportional to $g_{\mu\nu}$. The equation is thus contracted with[2]

$$R^{\mu\nu} = \frac{1}{3p^2}(4p^\mu p^\nu - g^{\mu\nu}p^2) \tag{2.2.6}$$

to give

$$J_p^{-1} = 1 - \frac{\imath}{2}g^2 C_A \int \frac{d^4k J_k^{-1}}{p^2k^2(p-k)^2} R^{\mu\nu}\Gamma^{(0)}_{\mu\rho\sigma}(p,-k,k-p)t^{\rho\alpha}(k)t^{\sigma\beta}(p-k)\Gamma^{(0)}_{\beta\alpha\nu}(p-k,k,-p) \tag{2.2.7}$$

Using a Wick rotation to Euclidean space ($g_{\mu\nu} \to -\delta_{\mu\nu}, \int d^4k \to \imath \int_E d^4k$) does not alter the integrand but the measure can now be expressed as

$$\int_E d^4k = \frac{1}{(2\pi)^3}\int_0^\pi d\theta \sin^2\theta \int_0^{\Lambda^2} dk^2 k^2, \; k\cdot p = |k||p|\cos\theta \tag{2.2.8}$$

Performing the tensor contractions in the integrand and writing $q = k - p$ gives

$$\begin{aligned} J_p^{-1} - 1 \;=\; & \frac{g^2 C_A}{16\pi^3}\int_0^{\Lambda^2}\frac{dk^2}{p^2}J_k^{-1}\int_0^\pi d\theta\sin^2\theta \;\times \\ & \left\{\frac{1}{q^4}\left[\frac{1}{12}\frac{p^6}{k^2} + \frac{2}{3}p^4 - \frac{5}{4}p^2k^2 + \frac{1}{3}\frac{k^6}{p^2} + \frac{1}{6}k^4\right]\right. \\ & + \frac{1}{q^2}\left[\frac{2}{3}\frac{p^4}{k^2} - \frac{19}{6}p^2 + \frac{8}{3}\frac{k^4}{p^2} - \frac{13}{6}k^2\right] \\ & \left. + \left[-\frac{13}{6} - \frac{5}{4}\frac{p^2}{k^2} - 6\frac{k^2}{p^2}\right] + q^2\left[\frac{8}{3}\frac{1}{p^2} + \frac{1}{6}\frac{1}{k^2}\right] + \frac{q^4}{3p^2k^2}\right\} \end{aligned} \tag{2.2.9}$$

To evaluate the angular integrals let us use a convenient formalism utilised by Atkinson and Bloch [34] which will be used extensively in the next section. Setting $x = p^2, y = k^2, z = q^2$ and defining

$$y_> = \begin{cases} x & y < x \\ y & y > x \end{cases}, \quad \frac{y_<}{y_>} = \begin{cases} y/x & y < x \\ x/y & y > x \end{cases} \tag{2.2.10}$$

the general angular integral can be written as

$$\int_0^\pi d\theta \sin^{2r}(\theta) z^n = \beta\left(r+\frac{1}{2},\frac{1}{2}\right) y_>^n {}_2F_1\left(-n,-n-r;r+1;\frac{y_<}{y_>}\right), \; r \geq 1 \tag{2.2.11}$$

where

$$\beta\left(r+\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(r+1)} \tag{2.2.12}$$

[2] This contraction also removes the tadpole contribution which is proportional to $g_{\mu\nu}$.

and

$$ {}_2F_1(a,b;c;t) = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\Gamma(b+r)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+r)} \frac{t^r}{r!} = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} \frac{t^r}{r!} \tag{2.2.13} $$

is a hypergeometric function. It is to be noted that in general, one cannot do the angular integrals exactly in a Schwinger-Dyson equation due to the usual occurrence of unknown functions G_z. However, the Mandelstam approximation does allow the integrals to be performed. On substituting the integrals into expression (2.2.11) and noting that the hypergeometric functions simplify, one gets[3]

$$ J_x - 1 = \frac{g^2 C_A}{32\pi^2} \left\{ \int_0^x \frac{dy}{x} J_y^{-1} \left[\frac{3}{4} - \frac{17}{3}\frac{y}{x} + \frac{7}{3}\frac{y^2}{x^2} \right] + \int_x^\Lambda \frac{dy}{x} J_y^{-1} \left[-\frac{14}{3}\frac{x}{y} + \frac{7}{12}\frac{x^2}{y^2} \right] \right\} \tag{2.2.14} $$

It is now pertinent to discuss the renormalisation of the equation. The renormalisation coefficients are defined as follows

$$ J_{x,\Lambda}^{-1} = Z_3(\mu,\Lambda)\overline{J}_{x,\mu}^{-1}, \; g = Z_g(\mu,\Lambda)\overline{g}(\mu) \tag{2.2.15} $$

where the bar denotes the renormalised quantity. This gives

$$ \overline{J}_{x,\mu} - Z_3 = Z_3^2 Z_g^2 \frac{\overline{g}^2(\mu) C_A}{16\pi^2} \left\{ \int_0^x \frac{dy}{x} \overline{J}_{y,\mu}^{-1} \left[-\frac{3}{8} - \frac{17}{6}\frac{y}{x} + \frac{7}{6}\frac{y^2}{x^2} \right] + \int_x^\Lambda \frac{dy}{x} \overline{J}_{y,\mu}^{-1} \left[-\frac{7}{3}\frac{x}{y} + \frac{7}{24}\frac{x^2}{y^2} \right] \right\} \tag{2.2.16} $$

Given the divergence structure of this equation, it is clear that in the Mandelstam approximation $Z_3^2 Z_g^2 = 1$. With a momentum subtraction scheme that requires the gluon self-energy to vanish at the renormalisation point ($\overline{J}_{\mu,\mu}^{-1} = 1$)

$$ \begin{aligned} Z_3(\mu,\Lambda) &= 1 - \frac{\overline{g}_\mu^2 C_A}{16\pi^2} \left\{ \int_0^\mu \frac{dy}{\mu} \overline{J}_{y,\mu}^{-1} \left[-\frac{3}{8} - \frac{17}{6}\frac{y}{\mu} + \frac{7}{6}\frac{y^2}{\mu^2} \right] + \int_\mu^\Lambda \frac{dy}{\mu} \overline{J}_{y,\mu}^{-1} \left[-\frac{7}{3}\frac{\mu}{y} + \frac{7}{24}\frac{\mu^2}{y^2} \right] \right\} \\ &= 1 - \frac{\alpha_s(\mu) C_A}{4\pi} I[\mu] \qquad (2.2.17) \\ \Rightarrow \overline{J}_{x,\mu} &= 1 + \frac{\alpha_s(\mu) C_A}{4\pi} \{ I[x] - I[\mu] \} \qquad (2.2.18) \end{aligned} $$

The infrared solution to this equation can be found by ansatz

I $\overline{J}_{x,\mu}^{-1} =$ constant. The term $\int_x dy\, y^{-1} \overline{J}_{y,\mu}^{-1} \sim \ln(x)$, but as $x \to 0$ this gives an infrared singularity that cannot be dealt with.

II $\overline{J}_{x,\mu}^{-1} \sim x$. Now the term $\int_x dy\, x y^{-2} \overline{J}_{y,\mu}^{-1} \sim \ln(x)$, again infrared singular.

[3]Note that after angular integration, all variables now refer to quantities with dimension [momentum]^2.

III $\overline{J}^{-1}_{x,\mu} \sim 1/x$. Putting this into $I[x]$ gives

$$I[x] = -\frac{3}{8x}\ln(y)|_0^x - \frac{35}{12x} \qquad (2.2.19)$$

In case 3, the terms exclusively give rise to gluon mass, ie the pole of the propagator has been shifted

$$\lim_{x\to 0}\Pi_{\mu\nu}(x) \neq 0. \qquad (2.2.20)$$

Such a condition is not allowed since it violates gauge invariance even in the present approximation scheme [40]. Therefore, these terms are subtracted in analogy with the UV-divergent terms [26]. The ansatz is then modified to

$$\overline{J}^{-1}_{x,\mu} = \frac{b}{x} + F(x) \qquad (2.2.21)$$

with the understanding that the terms b/x in the integrand are subtracted. Following the work of Hauck *et al.* [40] the equation is written as

$$\frac{b}{x} + F(x) = [A + Bx + C(x)]^{-1} \qquad (2.2.22)$$

where

$$\begin{aligned} A &= 1 - \frac{\alpha_s(\mu)C_A}{4\pi}\left[\frac{7}{3}\int_0^\Lambda \frac{dy}{y}F(y) + I[\mu]\right] \\ B &= \frac{7}{24}\frac{\alpha_s(\mu)C_A}{4\pi}\int_0^\infty \frac{dy}{y^2}F(y) \\ C(x) &= \frac{\alpha_s(\mu)C_A}{4\pi}\int_0^x dyF(y)\left[-\frac{9}{8}\frac{1}{x} - \frac{17}{6}\frac{y}{x^2} + \frac{7}{6}\frac{y^2}{x^3} + \frac{7}{3}\frac{1}{y} - \frac{7}{24}\frac{x}{y^2}\right] \end{aligned} \qquad (2.2.23)$$

since $\overline{J}^{-1}$ cannot be a constant as $x \to 0, A = 0$. Also, $b = B^{-1}$. To make the equation dimensionless and scale-independent the following changes of variable are made

$$\tilde{x} = \frac{x}{b}\sqrt{\frac{4\pi}{\alpha_s(\mu)}} \equiv \frac{x}{\lambda}, \quad \tilde{J}^{-1}_x = \sqrt{\frac{\alpha_s(\mu)}{4\pi}}\overline{J}^{-1}_{x,\mu} = \frac{b}{\tilde{x}} + \tilde{F}(\tilde{x}) \qquad (2.2.24)$$

This gives the full equation

$$\frac{\tilde{x}^2\tilde{F}(\tilde{x})}{1+\tilde{x}\tilde{F}(\tilde{x})} = -C_A\int_0^{\tilde{x}} d\tilde{y}\tilde{F}(\tilde{y})\left[-\frac{9}{8}\frac{1}{\tilde{x}} - \frac{17}{6}\frac{\tilde{y}}{\tilde{x}^2} + \frac{7}{6}\frac{\tilde{y}^2}{\tilde{x}^3} + \frac{7}{3}\frac{1}{\tilde{y}} - \frac{7}{24}\frac{\tilde{x}}{\tilde{y}^2}\right] \qquad (2.2.25)$$

with the constraint

$$\frac{7}{24}C_A\int_0^\infty \frac{d\tilde{y}}{\tilde{y}^2}\tilde{F}(\tilde{y}) = 1 \qquad (2.2.26)$$

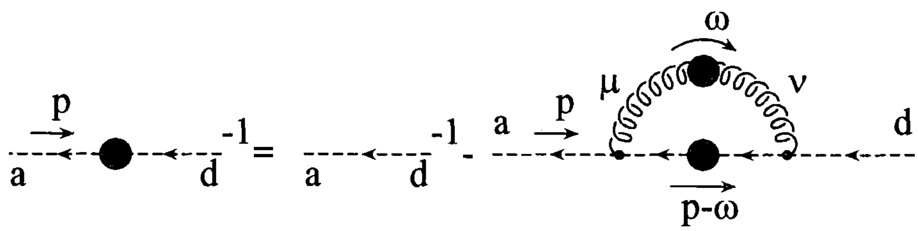


Figure 2.1: The ghost propagator Schwinger-Dyson equation with bare vertices. The blobs represent full one-particle-irreducible Green's functions. Shown here is the momentum routing with integration over the gluon momentum.

which fixes the overall scale of the solution. This was solved numerically [40] to obtain a consistent IR solution for $\widetilde{F}$. The details of the solution are not important for the purposes of this chapter, rather it the existence of the solution which has the lowest power $\widetilde{F}(\widetilde{x}) \sim \widetilde{x}^{\gamma_0}, \gamma_0 > 1$.

In summary, under the Mandelstam approximation, a consistent solution for the infrared behaviour of the gluon propagator is found from the Schwinger-Dyson equation. The propagator goes like $1/p^4$ as $p^2 \to 0$. This behaviour leads to a running coupling that increases without bound in the IR and a linearly rising potential that implies a simple picture of confinement [40]. Although it is not within the scope of this chapter it is worth pointing out that there is strong support for this result based on the phenomenological treatment of confinement and dynamical chiral symmetry breaking [41, 42].

2.3 The Bare Vertex Approximation in the Landau Gauge

In this section, results obtained by using an approximation scheme where all vertices are bare will be presented [33, 34]. It will be shown that the previously neglected ghost-gluon vertex may in fact be more important to the IR behaviour of the gluon than the triple-gluon (or four-gluon) vertex. The approximation is not physically motivated but is able to highlight some important points due to the fact that the integrals are far simpler than in the full case (although in the Mandelstam approximation they are even simpler).

With bare vertices, the ghost Schwinger-Dyson equation can be represented by fig. 2.1. There are two momentum routings for the loop and it will be important to have both.

With integration over the ghost momentum

$$\begin{aligned}\delta^{ad}p^2G_p^{-1} &= \delta^{ad}p^2 - \int(-\imath)\frac{d\!\!{}^{-}{}^4\omega\, G_\omega J_{p-\omega}^{-1}}{\omega^2(p-\omega)^2}(-\imath g f^{abc})(-\imath g f^{bdc})p_\alpha\omega_\beta t^{\alpha\beta}(p-w)\\ G_p^{-1} &= 1+\imath g^2C_A\int\frac{d\!\!{}^{-}{}^4\omega G_\omega J_{p-\omega}^{-1}}{p^2\omega^2(p-\omega)^2}p_\alpha\omega_\beta t^{\alpha\beta}(p-w) \qquad (2.3.1)\end{aligned}$$

Using a Wick rotation

$$G_p^{-1} = 1-\frac{g^2C_A}{16\pi^3}\int_0^\Lambda\frac{d\omega^2}{p^2}G_\omega\int_0^\pi d\theta\sin^2\theta\,\frac{J_{p-\omega}^{-1}}{(p-\omega)^2}p_\alpha\omega_\beta t^{\alpha\beta}(p-w) \qquad (2.3.2)$$

where $p\cdot\omega = |p||\omega|\cos\theta$.

$$G_p^{-1} = 1-\frac{g^2C_A}{8\pi^3}\int_0^\Lambda d\omega^2\omega^2G_\omega\int_0^\pi d\theta\sin^4\theta\,\frac{J_{p-\omega}^{-1}}{(p-\omega)^4} \qquad (2.3.3)$$

With integration over the gluon momentum, one gets

$$G_p^{-1} = 1-\frac{g^2C_A}{8\pi^3}\int_0^\Lambda d\omega^2J_\omega^{-1}\int_0^\pi d\theta\sin^4\theta\,\frac{G_{p-\omega}^{-1}}{(p-\omega)^2} \qquad (2.3.4)$$

Setting $x=p^2, y=\omega^2, z=x+y-2\sqrt{xy}\cos\theta$ gives the equations

$$\begin{aligned}G_x^{-1} &= 1-\frac{g^2C_A}{8\pi^3}\int_0^\Lambda dyyG_y\int_0^\pi d\theta\sin^4\theta\,\frac{J_z^{-1}}{z^2} \qquad (2.3.5)\\ G_x^{-1} &= 1-\frac{g^2C_A}{8\pi^3}\int_0^\Lambda dyJ_y^{-1}\int_0^\pi d\theta\sin^4\theta\,\frac{G_z}{z} \qquad (2.3.6)\end{aligned}$$

Applying multiplicative renormalisation and introducing the appropriate renormalisation coefficients (which will be used from now on)

$$G_{x,\Lambda} = \tilde{Z}_3(\mu,\Lambda)\overline{G}_{x,\mu},\ J_{x,\Lambda}^{-1} = Z_3(\mu,\Lambda)\overline{J}_{x,\mu}^{-1},\ g = \frac{\tilde{Z}_1}{Z_3^{1/2}\tilde{Z}_3}\overline{g}(\mu,\Lambda) \qquad (2.3.7)$$

where the bar denotes the renormalised quantity[4], one gets

$$\begin{aligned}\overline{G}_{x,\mu}^{-1} &= \tilde{Z}_3(\mu,\Lambda)-\tilde{Z}_1(\mu,\Lambda)\frac{\overline{g}(\mu,\Lambda)^2C_A}{8\pi^3}\int_0^\Lambda dyy\overline{G}_{y,\mu}\int_0^\pi d\theta\sin^4\theta\,\frac{\overline{J}_{z,\mu}^{-1}}{z^2} \qquad (2.3.8)\\ \overline{G}_{x,\mu}^{-1} &= \tilde{Z}_3(\mu,\Lambda)-\tilde{Z}_1(\mu,\Lambda)\frac{\overline{g}(\mu,\Lambda)^2C_A}{8\pi^3}\int_0^\Lambda dy\overline{J}_{y,\mu}^{-1}\int_0^\pi d\theta\sin^4\theta\,\frac{\overline{G}_{z,\mu}}{z} \qquad (2.3.9)\end{aligned}$$

It is observed that for the Landau gauge $\tilde{Z}_1(\mu,\Lambda)=1$ [21, 43]. The coefficient $\tilde{Z}_3(\mu,\Lambda)$ can be eliminated by subtracting the same equation at some finite IR scale σ. Concentrating on (2.3.9) and dropping the now redundant bar on the renormalised quantities

$$G_{x,\mu}^{-1}-G_{\sigma,\mu}^{-1} = \frac{g(\mu,\Lambda)^2C_A}{8\pi^3}\int_0^\Lambda dyyG_{y,\mu}\left\{\int_0^\pi d\theta\sin^4\theta\,\frac{J_{z',\mu}^{-1}}{z'^2}-\int_0^\pi d\theta\sin^4\theta\,\frac{J_{z,\mu}^{-1}}{z^2}\right\} \qquad (2.3.10)$$

[4]Note that the renormalised coupling depends on both the renormalisation scale μ and the cutoff scale Λ

where $z' = \sigma + y - 2\sqrt{\sigma y}\cos\theta$. To find a relationship between G, J one assumes a leading power law behaviour for both in the infrared[5]. Since the only scale to match the dimensions of x is μ, it is appropriate to consider the ratio x/μ.

$$J_{x,\mu}^{-1} = A\left(\frac{x}{\mu}\right)^{\alpha}, \quad G_{x,\mu} = B\left(\frac{x}{\mu}\right)^{\beta} \tag{2.3.11}$$

Putting this into the equation

$$\left(\frac{x}{\mu}\right)^{-\beta} - \left(\frac{\sigma}{\mu}\right)^{-\beta} = \frac{g(\mu,\Lambda)^2 C_A}{8\pi^3} AB^2 \int_0^{\Lambda} dyy \left(\frac{y}{\mu}\right)^{\beta} \int_0^{\pi} d\theta \sin^4\theta \left\{\frac{1}{z'^2}\left(\frac{z'}{\mu}\right)^{\alpha} - \frac{1}{z^2}\left(\frac{z}{\mu}\right)^{\alpha}\right\} \tag{2.3.12}$$

Now, it will be shown that $\alpha = -2\beta = 2\kappa$ is a consistent solution to both the ghost and the gluon Schwinger-Dyson equations. The first point to note is that this leading infrared power behaviour gives a renormalised coupling that is constant. This can be seen by considering the μ dependence of the renormalisation coefficients. Taking the gluon propagator function, recall that

$$J_{x,\Lambda}^{-1} = Z_3(\mu,\Lambda)\overline{J}_{x,\mu}^{-1}. \tag{2.3.13}$$

Plugging in the power law ansatz for the renormalised function and demanding that the powers match gives

$$A\left(\frac{x}{\Lambda}\right)^{\alpha} = \left(\frac{\mu}{\Lambda}\right)^{\alpha} A\left(\frac{x}{\mu}\right)^{\alpha}. \tag{2.3.14}$$

The renormalisation coefficient Z_3 can now be identified as $Z_3(\mu,\Lambda) = \left(\frac{\mu}{\Lambda}\right)^{\alpha}$. In a similar fashion, $\tilde{Z}_3(\mu,\Lambda) = \left(\frac{\mu}{\Lambda}\right)^{\beta}$. Consequently, the renormalised coupling will go like

$$\overline{g}(\mu,\Lambda)^2 = Z_3\tilde{Z}_3^2 g^2 = \left(\frac{\mu}{\Lambda}\right)^{\alpha+2\beta} g^2 = g^2. \tag{2.3.15}$$

With the behaviour of the renormalised coupling determined, the above equation becomes

$$\left(\frac{x}{\mu}\right)^{-\beta} - \left(\frac{\sigma}{\mu}\right)^{-\beta} = \frac{g^2 C_A}{8\pi^3} AB^2 \int_0^{\Lambda} dyy \left(\frac{y}{\mu}\right)^{\beta} \int_0^{\pi} d\theta \sin^4\theta \left\{\frac{1}{z'^2}\left(\frac{z'}{\mu}\right)^{\alpha} - \frac{1}{z^2}\left(\frac{z}{\mu}\right)^{\alpha}\right\}. \tag{2.3.16}$$

Since μ is the arbitrary renormalisation scale it is clear that $\alpha = -2\beta = 2\kappa$ is consistent such that the μ-dependence matches on both sides of the equation. Using the formula for

[5]A complete power expansion for both functions can be envisaged such that either side of the equation becomes a series in powers of x and σ.

the angular integrals (2.2.11) and simplifying

$$\begin{aligned} x^\kappa - \sigma^\kappa &= \frac{g^2 C_A}{16\pi^2}\frac{3}{4}AB^2\left\{\int_0^\sigma dy y^{1-\kappa}\sigma^{2\kappa-2}{}_2F_1(2-2\kappa,-2\kappa;3;\frac{y}{\sigma})\right. \\ &- \int_0^x dy y^{1-\kappa}x^{2\kappa-2}{}_2F_1(2-2\kappa,-2\kappa;3;\frac{y}{x}) + \int_\sigma^\Lambda dy y^{\kappa-1}{}_2F_1(2-2\kappa,-2\kappa;3;\frac{\sigma}{y}) \\ &\left. - \int_x^\Lambda dy y^{\kappa-1}{}_2F_1(2-2\kappa,-2\kappa;3;\frac{x}{y})\right\} \end{aligned} \quad (2.3.17)$$

The integration variables are scaled so that the limits are $0 \rightarrow 1$, the UV-cutoff being taken to infinity (there are no divergences in the equation due to the subtraction). Eliminating the overall factor $(x^\kappa - \sigma^\kappa)$

$$1 = -\frac{g^2 C_A}{16\pi^2}\frac{3}{4}AB^2\int_0^1 dt\left(t^{1-\kappa}+t^{-1-\kappa}\right){}_2F_1(2-2\kappa,-2\kappa;3;t) \quad (2.3.18)$$

This allows the use of the formula

$$\int_0^1 dt\, t^\nu {}_2F_1(a,b;c;t) = \frac{1}{\nu+1}\ {}_3F_2(a,b,\nu+1;c,\nu+2;1) \quad (2.3.19)$$

which gives

$$\begin{aligned} 1 &= -\frac{g^2 C_A}{16\pi^2}\frac{3}{4}AB^2\left\{\frac{1}{2-\kappa}\ {}_3F_2(2-2\kappa,-2\kappa,2-\kappa;3,3-\kappa;1)\right. \\ &\left. -\frac{1}{\kappa}\ {}_3F_2(2-2\kappa,-2\kappa,-\kappa;3,1-\kappa;1)\right\} \end{aligned} \quad (2.3.20)$$

Doing the same analysis on the other momentum routing (2.3.9) one gets

$$\begin{aligned} 1 &= -\frac{g^2 C_A}{16\pi^2}\frac{3}{4}AB^2\left\{\frac{1}{2\kappa+1}\ {}_3F_2(\kappa+1,\kappa-1,2\kappa+1;3,2\kappa+2;1)\right. \\ &\left. -\frac{1}{\kappa}\ {}_3F_2(\kappa+1,\kappa-1,-\kappa;3,1-\kappa;1)\right\} \end{aligned} \quad (2.3.21)$$

These last two expressions are in fact numerically equivalent for a range of κ [34]. It should be noted that in both equations, the pure power law ansatz for the functions has not given rise to higher powers.

To constrain κ further, it is necessary to study the Schwinger-Dyson equation for the gluon propagator. Atkinson and Bloch [33] found that a consistent IR solution can be derived by omitting the purely gluonic loop (as well as the two-loop terms and fermionic contributions). It shall now be demonstrated that all but the ghost-loop can be omitted in the IR, if one considers all vertices to be bare. The renormalised gluon Schwinger-Dyson equation contracted with $R^{\mu\nu}$ will have the generic form[6]

$$\begin{aligned} J_{x,\mu} \sim\; & Z_3(\mu,\Lambda) - \frac{1}{2}Z_1(\mu,\Lambda)\int_\omega \Gamma_3^{(0)}D^{(0)}D^{(0)}\Gamma_3^{(0)}J_{\omega,\mu}^{-2} + \tilde{Z}_1(\mu,\Lambda)\int_\omega \tilde{\Gamma}^{(0)}D_G^{(0)}D_G^{(0)}\tilde{\Gamma}^{(0)}G_{\omega,\mu}^2 \\ & -\frac{1}{6}Z_4(\mu,\Lambda)\int_\omega \Gamma_4^{(0)}D^{(0)}D^{(0)}D^{(0)}\Gamma_4^{(0)}J_{\omega,\mu}^{-3} \\ & -\frac{1}{2}Z_5(\mu,\Lambda)\int_\omega \Gamma_4^{(0)}D^{(0)}D^{(0)}D^{(0)}D^{(0)}\Gamma_3^{(0)}\Gamma_3^{(0)}J_{\omega,\mu}^{-4}. \end{aligned} \tag{2.3.22}$$

In this equation, it is possible to extract quite generally the μ-dependence of each term. The renormalisation coefficients Z_i will have a characteristic power of μ. Also, in each integral, given that the vertices are bare, the only μ-dependence comes from the propagator factors. Concentrating on the finite terms only, the only scale to balance the μ-dependence (to give a dimensionless gluon propagator function) is the external scale x.

Now, the vertices are bare and consequently will require no renormalisation. Thus the renormalisation coefficients will become

$$\tilde{Z}_1 = Z_1 = Z_4 = 1, \qquad Z_5 = \frac{1}{Z_3} = \left(\frac{\mu}{\Lambda}\right)^{-2\kappa}. \tag{2.3.23}$$

It is immediately apparent that this approximation scheme explicitly violates the Slavnov-Taylor identity (and consequently gauge invariance). The Slavnov-Taylor identity, which ensures that the physical coupling in QCD is uniquely defined via any of the vertices, can be expressed as [21, 43]

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_4}{Z_1} \tag{2.3.24}$$

which obviously cannot hold here. However, for heuristic purposes, the issue of gauge invariance will be neglected.

Returning to the gluon equation – by using the IR power law ansatz for the unknown functions, considering the μ-dependence of each term, restoring the dimensions with the external scale x and keeping only the finite terms, one gets the following

$$\begin{aligned} \left(\frac{x}{\mu}\right)^{-2\kappa} \sim\; & -\frac{1}{2}I_{gluon}\left(\frac{x}{\mu}\right)^{4\kappa} + I_{ghost}\left(\frac{x}{\mu}\right)^{-2\kappa} \\ & -\frac{1}{6}I_{sunset}\left(\frac{x}{\mu}\right)^{6\kappa} - \frac{1}{2}I_{2-loop}\left(\frac{x}{\mu}\right)^{6\kappa} \end{aligned} \tag{2.3.25}$$

From this it is clear that for the equation to be satisfied, unless $\kappa = 0$, the only term that contributes as $x \to 0$ is the ghost loop and that $\kappa > 0$. This tacitly assumes that the

[6] Note that the tadpole diagram vanishes under this contraction. Also, the renormalised coupling is omitted since it is simply a constant. Z_4 is the renormalisation coefficient for the four-gluon vertex. $Z_5 = \frac{Z_1^2}{Z_3}$.

ghost-loop contribution does not vanish and that unlike the previous section there is no subtraction of infrared divergent terms.

With this in mind the renormalised gluon Schwinger-Dyson equation becomes

$$\begin{aligned} J_{x,\mu} &= Z_3(\mu,\Lambda) \\ &+\frac{g^2C_A}{8\pi^3}\frac{1}{3x}\int_0^\Lambda dy G_{y,\mu}\int_0^\pi d\theta\sin^2\theta\, G_{z,\mu}\left\{\frac{1}{z}\left(\frac{x+y}{2}-\frac{y^2}{x}\right)+\frac{1}{2}+2\frac{y}{x}-\frac{z}{x}\right\} \end{aligned} \tag{2.3.26}$$

Again subtracting at some scale σ and putting in the power law ansatz for the functions

$$\begin{aligned} x^{-2\kappa}-\sigma^{-2\kappa} &= \frac{g^2C_A}{8\pi^3}\frac{1}{3x}AB^2\int_0^\Lambda dy G_{y,\mu}\int_0^\pi d\theta\sin^2\theta\,\times \\ &\left\{z^{-\kappa-1}\left(\frac{x+y}{2}-\frac{y^2}{x}\right)+z^{-\kappa}\left(\frac{1}{2}+2\frac{y}{x}\right)-z^{1-\kappa}\frac{1}{x}\right\} \\ &-(x\to\sigma) \end{aligned} \tag{2.3.27}$$

This leads directly to the form

$$\begin{aligned} 1=\frac{g^2C_A}{16\pi^2}\frac{1}{3}AB^2\int_0^1 dt\Big\{ &{}_2F_1(\kappa+1,\kappa;2;t)\left[\frac{1}{2}\left(t^{-\kappa}+t^{1-\kappa}+t^{2\kappa-1}+t^{2\kappa-2}\right)-\left(t^{2-\kappa}+t^{2\kappa-3}\right)\right] \\ &+{}_2F_1(\kappa,\kappa-1;2;t)\left[\frac{1}{2}\left(t^{-\kappa}+t^{2\kappa-2}\right)+2\left(t^{1-\kappa}+t^{2\kappa-3}\right)\right] \\ &-{}_2F_1(\kappa-1,\kappa-2;2;t)\left[t^{-\kappa}+t^{2\kappa-3}\right]\Big\} \end{aligned} \tag{2.3.28}$$

which is integrated to get

$$\begin{aligned} 1=&\frac{g^2C_A}{16\pi^2}\frac{1}{3}AB^2\times \\ &\left\{\frac{1}{2(1-\kappa)}\;{}_3F_2(\kappa+1,\kappa,1-\kappa;2,2-\kappa;1)+\frac{1}{2(2-\kappa)}\;{}_3F_2(\kappa+1,\kappa,2-\kappa;2,3-\kappa;1)\right. \\ &+\frac{1}{4\kappa}\;{}_3F_2(\kappa+1,\kappa,2\kappa;2,2\kappa+1;1)+\frac{1}{2(2\kappa-1)}\;{}_3F_2(\kappa+1,\kappa,2\kappa-1;2,2\kappa;1) \\ &+\frac{1}{(\kappa-3)}\;{}_3F_2(\kappa+1,\kappa,3-\kappa;2,4-\kappa;1)+\frac{1}{2(1-\kappa)}\;{}_3F_2(\kappa+1,\kappa,2\kappa-2;2,2\kappa-1;1) \\ &+\frac{1}{2(1-\kappa)}\;{}_3F_2(\kappa,\kappa-1,1-\kappa;2,2-\kappa;1)+\frac{1}{2(2\kappa-1)}\;{}_3F_2(\kappa,\kappa-1,2\kappa-1;2,2\kappa;1) \\ &+\frac{2}{(2-\kappa)}\;{}_3F_2(\kappa,\kappa-1,2-\kappa;2,3-\kappa;1)+\frac{1}{(\kappa-1)}\;{}_3F_2(\kappa,\kappa-1,2\kappa-2;2,2\kappa-1;1) \\ &+\frac{1}{(\kappa-1)}\;{}_3F_2(\kappa-1,\kappa-2,1-\kappa;2,2-\kappa;1) \\ &\left.+\frac{1}{2(1-\kappa)}\;{}_3F_2(\kappa-1,\kappa-2,2\kappa-2;2,2\kappa-1;1)\right\} \end{aligned} \tag{2.3.29}$$

This expression can be compared with either (2.3.20) or (2.3.21) coming from the ghost Schwinger-Dyson equation and it is found that they are equivalent in the limit $\kappa \to 1$[7] [34]. This can be seen analytically by setting $\kappa = 1 - \delta$ and considering firstly (2.3.20)

$$\begin{aligned}\left[\frac{g^2 C_A}{16\pi^2}\frac{1}{3}AB^2\right]^{-1} &= -\frac{9}{4}\left\{\frac{1}{1+\delta}\ {}_3F_2(2\delta, 2\delta-2, 1+\delta; 3, 2+\delta; 1)\right. \\ &\left. -\frac{1}{1-\delta}\ {}_3F_2(2\delta, 2\delta-2, \delta-1; 3, \delta; 1)\right\} \end{aligned} \tag{2.3.30}$$

Expanding the hypergeometric functions in terms of their series gives

$$\left[\frac{g^2 C_A}{16\pi^2}\frac{1}{3}AB^2\right]^{-1} = 3 \tag{2.3.31}$$

For the gluon equation (2.3.29) it is necessary to change the argument of the hypergeometric functions from 1 to the limit $\tau \to 1$ in order to avoid convergence problems

$$\begin{aligned}\left[\frac{g^2 C_A}{16\pi^2}\frac{1}{3}AB^2\right]^{-1} &= \lim_{\tau\to 1}\lim_{\delta\to 0}\left\{\ {}_3F_2(2,1,1;2,2;\tau) - \frac{1}{4}\ {}_3F_2(2,1,2;2,3;\tau)\right. \\ &+\frac{5}{2}\ {}_3F_2(1,0,1;2,2;\tau) + \frac{1}{2\delta}\left[\ {}_3F_2(2,1,\delta;2,1;\tau)\right. \\ &+\ {}_3F_2(2,1,-2\delta;2,1;\tau) + \ {}_3F_2(1,-\delta,\delta;2,1;\tau) \\ &-2\ {}_3F_2(1,-\delta,-2\delta;2,1;\tau) - 2\ {}_3F_2(-\delta,-1,\delta;2,1;\tau) \\ &\left.\left.+\ {}_3F_2(-\delta,-1,-2\delta;2,1;\tau)\right]\right\} \end{aligned} \tag{2.3.32}$$

Expanding those functions that have δ-dependence in terms of Pochhammer symbols

$$\begin{aligned}\left[\frac{g^2 C_A}{16\pi^2}\frac{1}{3}AB^2\right]^{-1} &= \lim_{\tau\to 1}\left\{\frac{5}{2} + {}_2F_1(1,1;2;\tau) - \frac{1}{4}{}_2F_1(1,2;3;\tau)\right. \\ &+\lim_{\delta\to 0}\sum_{r=0}^{\infty}\frac{\tau^r}{2r!\delta}\left[(\delta)_r + (-2\delta)_r + \frac{(-\delta)_r(\delta)_r}{(2)_r}\right. \\ &\left.\left.-2\frac{(-\delta)_r(-2\delta)_r}{(2)_r} - 2\frac{(-\delta)_r(-1)_r(\delta)_r}{(2)_r(1)_r} + \frac{(-\delta)_r(-1)_r(-2\delta)_r}{(2)_r(1)_r}\right]\right\} \end{aligned} \tag{2.3.33}$$

When $r = 0, (a)_r = 1$ so these terms in the series cancel (ensuring that there is no $1/\delta$ divergence). For $r = 1, (a)_r = a$ which conspires to give terms of $O(\delta)$ which drop out.

[7]This limit is actually very important. Naively setting $\kappa = 1$ immediately gives a divergent result. However, the hypergeometric functions have a complicated structure, which is such that these divergences do not in fact occur after integration.

When $r = 2, (a)_r = a(a+1)$ so the terms $(-1)_r$ are zero for $r \geq 2$. This leaves

$$\begin{aligned}
\left[\frac{g^2 C_A}{16\pi^2}\frac{1}{3}AB^2\right]^{-1} &= \lim_{\tau\to1}\lim_{\delta\to0}\left\{\frac{13}{4}+\sum_{r=1}^{\infty}\tau^r\left[\frac{1}{1+r}-\frac{1}{2(2+r)}+\frac{1}{2r!\delta}\left((\delta)_r+(-2\delta)_r\right)\right]\right\} \\
&= \frac{13}{4}+\lim_{\tau\to1}\sum_{r=1}^{\infty}\tau^r\left[\frac{1}{1+r}-\frac{1}{2(2+r)}+\frac{1}{2r!}\left((r-1)!-2(r-1)!\right)\right] \\
&= \frac{13}{4}+\lim_{\tau\to1}\sum_{r=1}^{\infty}\tau^r\left[\frac{1}{1+r}-\frac{1}{2(2+r)}-\frac{1}{2r}\right] \\
&= \frac{13}{4}+\lim_{\tau\to1}\left\{-\sum_{r=1}^{\infty}\tau^r\left(\frac{1}{2r}\right)+\sum_{r=2}^{\infty}\tau^r\left(\frac{1}{r}\right)-\sum_{r=3}^{\infty}\tau^r\left(\frac{1}{2r}\right)\right\} \\
&= 3+\lim_{\tau\to1}\sum_{r=3}^{\infty}\tau^r\left[-\frac{1}{2r}+\frac{1}{r}-\frac{1}{2r}\right] \\
&= 3
\end{aligned} \tag{2.3.34}$$

Thus it is seen that in the limit $\kappa\to1$ the coupled Schwinger-Dyson equations under this approximation scheme are consistently solved in the IR to get

$$G_x \sim x^{-1}, \; J_x^{-1} \sim x^2 \tag{2.3.35}$$

This corresponds to an IR vanishing gluon propagator and an enhanced ghost propagator in stark contrast to the results of the previous section. In addition, it is relatively straightforward to find the behaviour of the running coupling [33]. Replacing the bars on the renormalised quantities

$$\begin{aligned}
\overline{g}(\mu,\Lambda)^2 &= Z_3\tilde{Z}_3^2 g^2 \\
\overline{g}(\mu,\Lambda)^2 Z_3^{-1}(\mu,\Lambda)\tilde{Z}_3^{-2}(\mu,\Lambda) &= \hat{g}(\nu,\Lambda)^2 Z_3^{-1}(\nu,\Lambda)\tilde{Z}_3^{-2}(\nu,\Lambda) \\
\overline{g}(\mu,\Lambda)^2\overline{J}_{x,\mu}^{-1}\overline{G}_{x,\mu}^2 &= \hat{g}(\nu,\Lambda)^2\hat{J}_{x,\nu}^{-1}\hat{G}_{x,\nu}^2
\end{aligned} \tag{2.3.36}$$

The running coupling can be defined as $\alpha(x) = \frac{\overline{g}^2}{4\pi}\overline{J}_x^{-1}\overline{G}_x^2$ (independent of the prescription). Thus as $x\to0, \alpha(x)\to\frac{4\pi}{C_A}$, ie the running coupling approaches a constant in the IR.

2.4 Adding the Slavnov-Taylor identity for the Ghost-Gluon Vertex

In this final section, a more sophisticated attempt to include ghosts put forward by von Smekal *et al.* [1, 2, 3] will be reviewed. This truncation scheme as usual neglects all four-point interactions and fermionic contributions. The terms dismissed from the gluon

Schwinger-Dyson equation are the tadpole term (which vanishes under contraction with $R^{\mu\nu}$) and the explicit two-loop terms. In order to find a suitable form for the kernel $\widetilde{\Gamma}_{\nu\mu}$, a Slavnov-Taylor identity is derived under the truncation scheme. From this it is possible to deduce at least those parts of both Γ_3 and $\widetilde{\Gamma}$ that implement gauge invariance. It is hoped that these are sufficient to give rise to sensible physical results.

By utilising the BRS invariance of the full pure Yang-Mills theory, the following functional identity for full reducible ghost correlation functions was found [2]

$$\langle C^c(z)\overline{C}^b(y)\partial A^a(x)\rangle - \langle C^c(z)\overline{C}^a(x)\partial A^b(y)\rangle \sim \langle C^d(z)C^e(z)\overline{C}^a(x)\overline{C}^b(y)\rangle f^{cde} \tag{2.4.1}$$

where C and A are the ghost and gauge fields in configuration space. The rhs can be decomposed into connected and disconnected parts

$$\langle C^c C^a \overline{C}^b \overline{C}^d\rangle = \langle C^a\overline{C}^b\rangle\langle C^c\overline{C}^d\rangle - \langle C^a\overline{C}^d\rangle\langle C^c\overline{C}^b\rangle + \text{connected} \tag{2.4.2}$$

Under the truncation scheme, the connected ghost-ghost scattering contributions are neglected. This leads to the identity (true in all gauges)

$$G_3^{-1}p_3^\mu\widetilde{\Gamma}_\mu(p_1,p_2;p_3) + G_2^{-1}p_2^\mu\widetilde{\Gamma}_\mu(p_1,p_3;p_2) + p_1^2G_1^{-1} = 0 \tag{2.4.3}$$

One can notice immediately that as either $p_2 \to 0$ or $p_3 \to 0$, this equation (assuming that there are no singularities) reduces to

$$p^\mu\widetilde{\Gamma}_\mu(p,0;-p) = p^2 \tag{2.4.4}$$

from which one can infer that the vertex in this limit remains bare to all orders. This is known to be true non-perturbatively in the Landau gauge [21] but not in general gauges. This limit will be discussed in later chapters.

The identity (2.4.3) can be solved by making the following ansatz

$$\widetilde{\Gamma}_\mu(p_1,p_2;p_3) = (p_1 - p_2)_\mu X_{123} + p_{3\mu}Y_{123} \tag{2.4.5}$$

This choice is suitable because in general covariant gauges the symmetry requires $X_{123} = X_{213}$ [2]. The identity then becomes

$$(p_2^2 - p_1^2)G_3^{-1}X_{123} + (p_3^2 - p_1^2)G_2^{-1}X_{132} + p_3^2G_3^{-1}Y_{123} + p_2^2G_2^{-1}Y_{132} + p_1^2G_1^{-1} = 0 \tag{2.4.6}$$

Assuming that there are no kinematical factors in the unknown functions and looking at the coefficient of p_1^2

$$G_3^{-1}X_{123} + G_2^{-1}X_{132} = G_1^{-1} \tag{2.4.7}$$

This is most easily solved by setting

$$X_{123} = \frac{1}{2}\left(\frac{G_3}{G_1} + \frac{G_3}{G_2} - 1\right) \tag{2.4.8}$$

where the last two terms guarantee the symmetry. The function Y can then be derived by looking at the coefficients of either p_2^2 or p_3^2

$$Y_{123} = -\frac{1}{2}\left(\frac{G_3}{G_1} - \frac{G_3}{G_2} + 1\right) \tag{2.4.9}$$

$$\Rightarrow \widetilde{\Gamma}_\mu(p_1, p_2; p_3) = \frac{1}{2}(p_1 - p_2)_\mu \left(\frac{G_3}{G_1} + \frac{G_3}{G_2} - 1\right) - \frac{1}{2}p_{3\mu}\left(\frac{G_3}{G_1} - \frac{G_3}{G_2} + 1\right) \tag{2.4.10}$$

This vertex reduces to the bare form not only when the function G is taken to be bare, but also as $p_2 \to 0$ assuming that $p_{2\mu}G_2^{-1}$ vanishes. This is an indication that the function G should be singular in the IR to ensure consistency. Eliminating p_3 in the vertex

$$\widetilde{\Gamma}_\mu(p_1, p_2; p_3) = p_{1\mu}\left(\frac{G_3}{G_1}\right) + p_{2\mu}\left(1 - \frac{G_3}{G_2}\right) \tag{2.4.11}$$

and recalling the definition of the kernel $\widetilde{\Gamma}_{\nu\mu}$ leads to

$$\widetilde{\Gamma}_{\nu\mu}(p_1, p_2; p_3) = g_{\nu\mu}\left(\frac{G_3}{G_1}\right) + \frac{p_{1\nu}p_{2\mu}}{p_1^2}\left(1 - \frac{G_3}{G_2}\right) \tag{2.4.12}$$

or

$$a'_{321} = \frac{G_3}{G_1},\ c'_{321} = e'_{321} = -\frac{1}{p_1^2}\left(1 - \frac{G_3}{G_2}\right),\ b' = d' = 0 \tag{2.4.13}$$

Note that this form is only defined up to terms transverse in p_1^ν. However, as this is an ansatz, the simplest form consistent with all the physical requirements is from a practical point of view the best place to start. With the kernel above, the triple-gluon Slavnov-Taylor identity becomes

$$p_3^{\mu_3}\Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = \frac{G_3G_1}{G_2} J_2\, p_2^2\, t_{\mu_2\mu_1}(p_2) - \frac{G_3G_2}{G_1} J_1\, p_1^2\, t_{\mu_1\mu_2}(p_1) \tag{2.4.14}$$

and the longitudinal part of the triple-gluon vertex is given by the functions

$$A_{123} = \frac{1}{2}G_3\left(\frac{G_1}{G_2}J_2 + \frac{G_2}{G_1}J_1\right) \tag{2.4.15}$$

$$B_{123} = \frac{1}{2}G_3\left(\frac{G_2}{G_1}J_1 - \frac{G_1}{G_2}J_2\right) \tag{2.4.16}$$

$$C_{123} = \frac{1}{(p_1^2 - p_2^2)}G_3\left(\frac{G_2}{G_1}J_1 - \frac{G_1}{G_2}J_2\right) \tag{2.4.17}$$

$$S_{123} = 0 \tag{2.4.18}$$

with the consistency equation (2.1.14) satisfied.

Looking firstly at the ghost Schwinger-Dyson equation, substituting in for the vertex and noting the transversality of the gluon propagator in the Landau gauge one gets

$$\begin{aligned}
\delta^{ad}p^2G_p^{-1} &= \delta^{ad}p^2 - \int(-\imath)\frac{\bar{d}^4\omega G_\omega J_{p-\omega}^{-1}}{\omega^2(p-\omega)^2}(-\imath gf^{abc})(-\imath gf^{bdc})p_\alpha\widetilde{\Gamma}_\beta(\omega,-p;p-\omega)t^{\alpha\beta}(p-w) \\
G_p^{-1} &= 1+\imath g^2C_A\int\frac{\bar{d}^4\omega G_\omega J_{p-\omega}^{-1}}{p^2\omega^2(p-\omega)^2}p_\alpha\omega_\beta t^{\alpha\beta}(p-w)\left(\frac{G_{p-\omega}}{G_\omega}+\frac{G_{p-\omega}}{G_p}-1\right) \qquad (2.4.19)
\end{aligned}$$

Under a Wick rotation this becomes

$$G_p^{-1} = 1 - \frac{g^2C_A}{8\pi^3}\int_0^\Lambda\frac{d\omega^2}{p^2}\int_0^\pi d\theta\sin^2\theta\,\frac{J_{p-\omega}^{-1}}{(p-\omega)^2}p_\alpha\omega_\beta t^{\alpha\beta}(p-w)\left(G_{p-\omega}-G_\omega+\frac{G_\omega G_{p-\omega}}{G_p}\right) \qquad (2.4.20)$$

where $p{\cdot}\omega = |p||\omega|\cos\theta$. Now, in order to proceed it is necessary to make the following angular approximations. For $\omega^2 < p^2$ the argument of the functions $J_{p-\omega}^{-1}$ and $G_{p-\omega}$ become J_p^{-1} and G_p. This preserves the limit $\omega^2 \to 0$ of the integrand in Euclidean space. For $\omega^2 > p^2$ all arguments of the functions are replaced with ω^2. This assumes that in the UV the functions are slowly varying, which is borne out by perturbation theory. Thus

$$G_p^{-1} - 1 = -\frac{g^2 C_A}{8\pi^3}\left\{\int_0^{p^2}\frac{d\omega^2}{p^2}\int_0^{\pi} d\theta \sin^2\theta \frac{p_\alpha \omega_\beta t^{\alpha\beta}(p-w)}{(p-\omega)^2} G_p J_p^{-1} + \int_{p^2}^{\Lambda}\frac{d\omega^2}{p^2}\int_0^{\pi} d\theta \sin^2\theta \frac{p_\alpha \omega_\beta t^{\alpha\beta}(p-w)}{(p-\omega)^2} G_\omega J_\omega^{-1}\right\} \tag{2.4.21}$$

Doing the tensor contraction and setting $x = p^2, y = \omega^2, z = x + y - 2\sqrt{xy}\cos\theta$ as before gives

$$g_x^{-1} - 1 = -\frac{g^2 C_A}{8\pi^3}\left\{\int_0^x \frac{dy}{y}\int_0^\pi d\theta \sin^4\theta \frac{1}{z^2} G_x J_x^{-1} + \int_x^\Lambda \frac{dy}{y}\int_0^\pi d\theta \sin^4\theta \frac{1}{z^2} G_y J_y^{-1}\right\} \tag{2.4.22}$$

Substituting in the angular integrals

$$g_x^{-1} - 1 = -\frac{g^2 C_A}{16\pi^2}\frac{3}{4}\left\{\frac{1}{2} G_x J_x^{-1} + \int_x^\Lambda \frac{dy}{y} G_y J_y^{-1}\right\} \tag{2.4.23}$$

and renormalising as before (with $\tilde{Z}_1 = 1$) gives

$$\overline{G}_{x,\mu}^{-1} = \tilde{Z}_3(\mu,\Lambda) - \frac{\overline{g}^2(\mu,\Lambda) C_A}{16\pi^2}\frac{3}{4}\left\{\frac{1}{2}\overline{G}_{x,\mu}\overline{J}_{x,\mu}^{-1} + \int_x^\Lambda \frac{dy}{y}\overline{G}_{y,\mu}\overline{J}_{y,\mu}^{-1}\right\} \tag{2.4.24}$$

The bar will again be dropped since all quantities are renormalised. Notice again that the renormalised coupling depends on both the renormalisation scale μ and the cutoff scale Λ.

It is possible to discuss the IR behaviour of this equation. One makes the ansatz that as $x \to 0, G_{x,\mu} J_{x,\mu}^{-1} \sim (x/\mu)^\kappa$ for $\kappa \neq 0$, giving

$$\overline{G}_{x,\mu} \sim \left(\frac{x}{\mu}\right)^{-\kappa}, \quad \overline{J}_{x,\mu}^{-1} \sim \left(\frac{x}{\mu}\right)^{2\kappa}. \tag{2.4.25}$$

This ensures that the terms on the right-hand side that are integrated out under the angular approximations are consistent. The power law derived here is identical to that of the previous section[8]. Recall that in the previous section, assuming this power law led to a constant renormalised coupling – this behaviour holds here too such that $g(\mu, \Lambda) = g$. Putting this into the equations above and demanding that $x \to 0$

$$\left(\frac{x}{\mu}\right)^\kappa \sim \tilde{Z}_3(\mu,\Lambda) - \frac{g^2 C_A}{16\pi^2}\frac{3}{4}\left\{\frac{1}{2}\left(\frac{x}{\mu}\right)^\kappa + \frac{1}{\kappa}\left(\frac{\Lambda}{\mu}\right)^\kappa - \frac{1}{\kappa}\left(\frac{x}{\mu}\right)^\kappa\right\} \tag{2.4.26}$$

To obtain a positive definite function $G_{x,\mu}$ from a positive definite $J_{x,\mu}^{-1}$ entails that

$$0 < \kappa < 2 \tag{2.4.27}$$

[8]Chronologically speaking, it also came first.

Now consider the gluon Schwinger-Dyson equation. The renormalised equation will have the form[9]

$$\begin{aligned}\overline{J}_{x,\mu} \sim Z_3(\mu,\Lambda) & -\frac{1}{2}Z_1(\mu,\Lambda)\int_\omega \Gamma_3^{(0)}D^{(0)}D^{(0)}\Gamma_3^{(0)}\overline{J}_{\omega,\mu}^{-1}\overline{G}_{\omega,\mu} \\ & +\tilde{Z}_1(\mu,\Lambda)\int_\omega \tilde{\Gamma}^{(0)}D_G^{(0)}D_G^{(0)}\tilde{\Gamma}^{(0)}\overline{G}_{\omega,\mu}^2. \end{aligned} \tag{2.4.28}$$

Just as in the previous section, it is possible to isolate the μ-dependence of each term. As before, $\tilde{Z}_1 = 1$ but now, $Z_1 = \frac{\tilde{Z}_1 Z_3}{\tilde{Z}_3} = \left(\frac{\Lambda}{\mu}\right)^{-3\kappa}$. Putting this into the equation gives

$$\left(\frac{x}{\mu}\right)^{-2\kappa} \sim Z_3(\mu,\Lambda) - \frac{1}{2}\left(\frac{x}{\mu}\right)^{-2\kappa}\left(\frac{x}{\Lambda}\right)^{3\kappa} I_{gluon} + I_{ghost}\left(\frac{x}{\mu}\right)^{-2\kappa}. \tag{2.4.29}$$

In the previous section, it was possible to demand that only the terms independent of the cutoff scale Λ could contribute to the left-hand side, leaving a clear dependence on μ that could not be altered. However, in this case, it is seen that the μ dependence is the same for each of the terms. The integral I_{gluon} will actually be a function of x and Λ so if there is a contribution that cancels the Λ dependence, then the gluon loop will become important. Since $\kappa > 0$, this becomes a question of how singular the integral is.

Now, the important part of this integral is the region where the radial integration reaches the UV cutoff. In this region, the y-max approximation holds and so any function of z can be approximated by a function of y. It is also known that the bare tensor structure in the integrand gives rise to the most singular behaviour in perturbation theory which is at most quadratic. However, under the correct contraction of the equation, the most singular behaviour is logarithmic, which corresponds to the case where the power law ansatz is zero (and not allowed for consistency reasons). In this naive argument we shall however simply say that the power of the divergence coming from the bare factors is zero. Thus, the divergence all rests on the characteristic powers of the propagator functions in the integrand. The maximum possible power of the divergence corresponds to replacing the argument of all the propagator functions in the integrand with Λ and reading off the greatest power. With the triple-gluon vertex function ansatz outlined earlier, the integral behaves like

$$I_{gluon} \sim \int^\Lambda \frac{dy}{y} J^{-2}(y)\frac{J_1}{G_1}G_2G_3 \tag{2.4.30}$$

[9] Recall that under the present truncation scheme, all four-point interactions are neglected. The generic dependence of the vertices on the two-point functions has been made explicit.

where the two propagator factors in the integrand have the argument y as $y \to \Lambda$ and the subscripts in the vertex function denote the different orderings of the arguments due to the Bose symmetry. There are thus three cases to consider.

1. $G_2 = G_3 = G(y),\ G_1 = G(x)$

$$I_{gluon} \sim \int^{\Lambda} \frac{dy}{y} J^{-2}(y) \frac{J(x)}{G(x)} G^2(y) \sim \left(\frac{\Lambda}{x}\right)^{2\kappa} \tag{2.4.31}$$

2. $G_1 = G_3 = G(y),\ G_2 = G(x)$

$$I_{gluon} \sim \int^{\Lambda} \frac{dy}{y} J^{-2}(y) \frac{J(y)}{G(y)} G(x) G(y) \sim \left(\frac{\Lambda}{x}\right)^{2\kappa} \tag{2.4.32}$$

3. $G_1 = G_2 = G(y),\ G_3 = G(x)$

$$I_{gluon} \sim \int^{\Lambda} \frac{dy}{y} J^{-2}(y) \frac{J(y)}{G(y)} G(y) G(x) \sim \left(\frac{\Lambda}{x}\right)^{2\kappa} \tag{2.4.33}$$

In all three cases, the maximum divergence possible is therefore $\Lambda^{2\kappa}$.

Returning now to the gluon Schwinger-Dyson equation, and substituting in the most singular part of I_{gluon} one obtains

$$\left(\frac{x}{\mu}\right)^{-2\kappa} \sim Z_3(\mu,\Lambda) - \frac{1}{2}\left(\frac{x}{\mu}\right)^{-2\kappa}\left(\frac{x}{\Lambda}\right)^{3\kappa}\left(\frac{\Lambda}{x}\right)^{2\kappa} + I_{ghost}\left(\frac{x}{\mu}\right)^{-2\kappa}. \tag{2.4.34}$$

It is now clear that even the most singular part of I_{gluon} cannot give rise to leading x contributions in the infrared limit and so the gluon loop term can be neglected[10]. This is demonstrated explicitly in the work of von Smekal *et al.* [1, 2, 3] where in fact the greatest divergence of I_{gluon} is only Λ^{κ}. With this, it is also clear that the divergent part cancels the x-dependence and therefore contributes only to the renormalisation coefficient.

Thus, taking only the ghost loop term and employing the angular approximations outlined earlier, the gluon Schwinger-Dyson equation becomes

$$\overline{J}_{x,\mu} = Z_3(\mu,\Lambda) + \frac{\overline{g}^2(\mu) C_A}{16\pi^2}\frac{1}{3}\left\{-\frac{1}{3}\overline{G}^2_{x,\mu} + \frac{3}{2}\int_0^x \frac{dy\, y}{x^2}\overline{G}_{x,\mu}\overline{G}_{y,\mu} + \frac{1}{2}\int_x^{\Lambda}\frac{dy}{y}\overline{G}^2_{y,\mu}\right\}. \tag{2.4.35}$$

[10]This argument has been applied to the case when one is considering only pure infrared power law behaviour of the propagator functions. One may wonder if this argument can hold if one considers the propagator functions to be power series. However, in principle there is no problem since the UV behaviour of the propagator functions is known to be given by the perturbative results which would not generate such singularities.

Returning to the ghost equation (2.4.26) and writing

$$\overline{G}_{x,\mu}\overline{J}^{-1}_{x,\mu} = C\left(\frac{x}{\mu}\right)^{\kappa},\ \overline{G}_{x,\mu} = a\left(\frac{x}{\mu}\right)^{-\kappa},\ \gamma_0^G = \frac{C_A}{16\pi^2}\frac{3}{4} \tag{2.4.36}$$

(γ_0^G is the leading perturbative term for the anomalous ghost dimension) one obtains

$$a^{-1} = \overline{g}^2(\mu)\gamma_0^G\left(\frac{1}{\kappa}-\frac{1}{2}\right)C \tag{2.4.37}$$

Putting this into the gluon equation (2.4.35) gives

$$\left(\frac{1}{\kappa}-\frac{1}{2}\right)^{-1} = \frac{4}{9}\left(\frac{1}{\kappa}-\frac{1}{2}\right)^{-2}\left\{-\frac{1}{3}+\frac{3}{2}\frac{1}{(2-\kappa)}+\frac{1}{4\kappa}\right\} \tag{2.4.38}$$

Solving for κ and applying $0<\kappa<2$ readily gives $\kappa \simeq 0.92$. This result is very similar to the one derived by Atkinson and Bloch. The running coupling can be defined as before but this time reaches the fixed point $\alpha \simeq 9.45$ in the IR.

2.5 Summary

This chapter has demonstrated that the inclusion of ghosts into Landau gauge Schwinger-Dyson studies is an important issue. That their omission leads to an IR enhanced gluon propagator, whereas simple forms for their inclusion lead to the opposite is a striking observation. However, this must be tempered with the realisation that the work done so far is necessarily incomplete in nature. Various methods of finding a solution have been outlined, each with its own set of problems. For example, the angular approximations used in both the Mandelstam approach and the work of von Smekal *et al.* are not fully understood and simply using bare vertices is not physical from the point of view of gauge invariance and the Slavnov-Taylor identities. Also, the issue of the masslessness of the gluon has not been addressed. The self-consistent solution to the Schwinger-Dyson equations of the last two sections gave rise to an inverse propagator that diverges in the infrared; for the Mandelstam approximation, the subtraction of IR divergent terms was introduced *ad hoc*. Nonetheless, this work is crucial in pointing out the importance of the ghost contributions to QCD in the infrared.

Chapter 3

Perturbative Results and their Renormalisation

In this short chapter, the one-loop, arbitrary gauge results for the ghost and gluon propagators and the ghost-gluon vertex will be presented. Their renormalisation coefficients will also be derived. It will be seen that these results are necessary to later chapters.

The chapter is organised as follows. The first section sets all the notation and conventions that will be used in this and subsequent perturbative calculations. The second and third sections present the calculations for the ghost and gluon propagators respectively. The fourth section deals with the ghost-gluon vertex. Finally, the perturbative renormalisation of these quantities will be discussed. The results found will not necessarily be calculated or presented in the same way each time but rather, it will be convenient to express the functions in the manner in which they will be used later on.

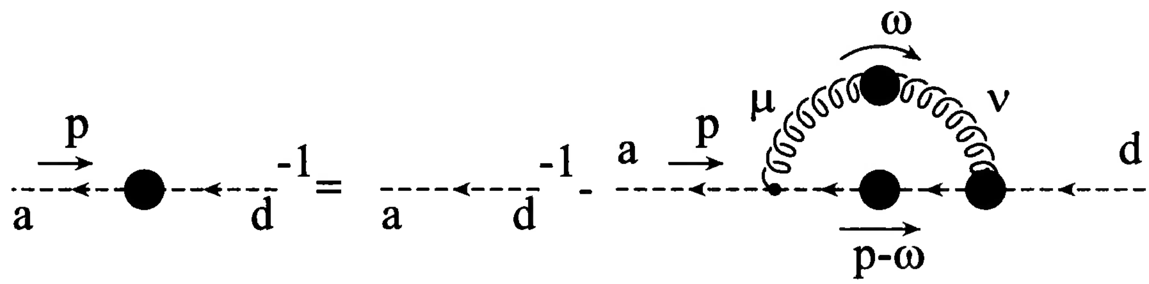


Figure 3.1: The ghost propagator Schwinger-Dyson equation.

3.1 Notation and Conventions

Throughout this chapter, the various unrenormalised dressing functions that occur in table 1.1 are expanded to the appropriate order in the coupling g as follows

$$\begin{aligned} J_p^{-1} &= 1 + g^2 \tilde{J}_p^{(1)} \\ G_p &= 1 + (-\frac{\imath}{2} g^2 C_A)\tilde{G}_p^{(1)} + (-\imath g^4 C_A)\hat{G}_p + (-\frac{\imath}{2} g^2 C_A)^2 \tilde{G}_p^{(2)} \\ \tilde{\Gamma}_\mu(p_1,p_2,p_3) &= p_{1\mu} + (-\frac{\imath}{2} g^2 C_A)\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3) + (-\imath g^4 C_A)\hat{\Gamma}_\mu(p_1,p_2,p_3) \\ &\quad + (-\frac{\imath}{2} g^2 C_A)^2 \tilde{\Gamma}_\mu^{(2)}(p_1,p_2,p_3). \end{aligned} \tag{3.1.1}$$

These functions will be calculated in later sections, using the various Schwinger-Dyson equations.

3.2 One Loop Expression for the Ghost Propagator

In this section, the general gauge result for $\tilde{G}_p^{(1)}$, the one-loop ghost function will be calculated. Consider the full Schwinger-Dyson equation for the ghost propagator (fig. 3.1)

$$\delta^{ad} p^2 G_p^{-1} = \delta^{ad} p^2 - \int (-\imath)\, đ^d\omega (-\imath g f^{abc}) p_\mu D^{bf}(p-\omega) \tilde{\Gamma}_\nu^{fdg}(p-\omega, -p, \omega) D^{cg\mu\nu}(\omega) \tag{3.2.1}$$

To obtain the one-loop expression for $\tilde{G}_p^{(1)}$, one simply replaces the dressing functions on the right-hand side (rhs) with their tree-level values to get

$$\delta^{ad} p^2 G_p^{-1} = \delta^{ad} p^2 - (-\imath)(-\imath g f^{abc})(-\imath g f^{bdc}) \int \frac{đ^d\omega}{\omega^2 (p-\omega)^2} p_\mu (p-\omega)_\nu \left[g_{\mu\nu} + (\xi - 1)\frac{\omega_\mu \omega_\nu}{\omega^2} \right]. \tag{3.2.2}$$

Doing the colour algebra gives

$$\delta^{ad}p^2(-\frac{\imath}{2}g^2C_A)\tilde{G}_p^{(1)} = -\imath g^2 C_A\delta^{ad}\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{p\cdot(p-w)+\frac{(\xi-1)}{\omega^2}p\cdot\omega(p-\omega)\cdot\omega\right\}. \tag{3.2.3}$$

Expanding the scalar products and cancelling factors

$$p^2\tilde{G}_p^{(1)} = \int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{p^2+(p-\omega)^2-\omega^2+\frac{(\xi-1)}{2\omega^2}\left[p^4-2p^2(p-\omega)^2-2p\cdot\omega(p-\omega)^2\right.\right.$$
$$\left.\left.+\omega^2(p-\omega)^2-w^4\right]\right\}.$$

Under the framework of dimensional regularisation, those integrals with no external scale (including those related by a translation of the integration variable) are zero, ie

$$\int\frac{\bar{d}^d\omega}{\omega^2}=\int\frac{\bar{d}^d\omega}{\omega^4}=\int\frac{\bar{d}^d\omega\omega_\mu}{\omega^4}=\int\frac{\bar{d}^d\omega}{(p-\omega)^2}=0. \tag{3.2.4}$$

This leaves

$$\tilde{G}_p^{(1)}=\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}+\frac{(\xi-1)}{2}p^2\int\frac{\bar{d}^d\omega}{\omega^4(p-\omega)^2} \tag{3.2.5}$$

Using the integration by parts technique outlined in appendix B.1, one then arrives at the final answer

$$\tilde{G}_p^{(1)}=\left\{1-(1-2\varepsilon)\frac{(\xi-1)}{2}\right\}\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2} \tag{3.2.6}$$

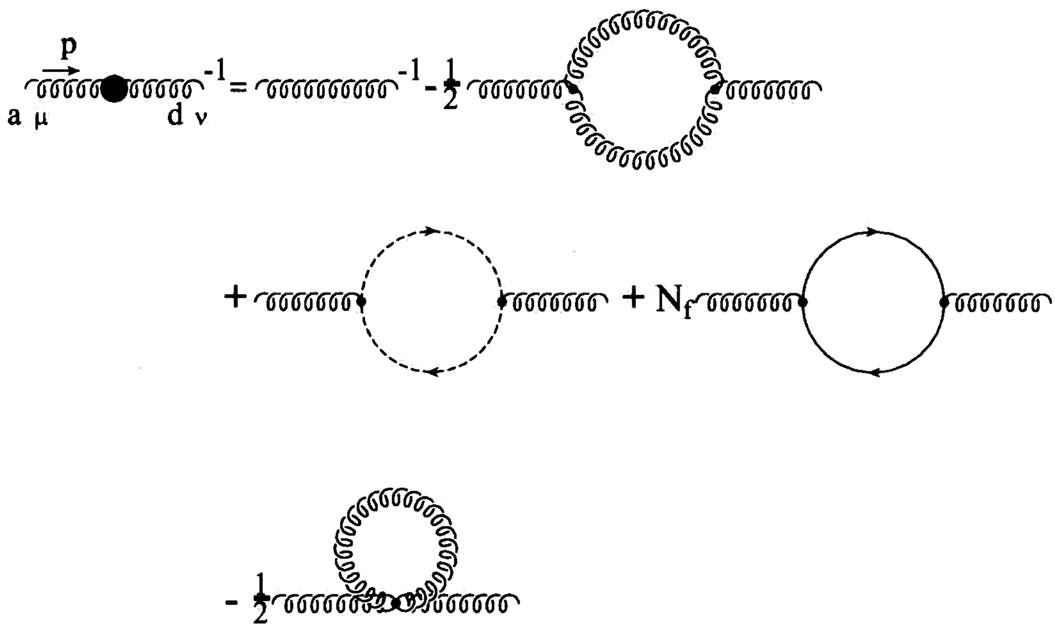


Figure 3.2: The inverse gluon propagator at one-loop. The relative signs and symmetry factors are included. Not shown are the momentum routing (for the bubble graphs see fig. A.1) and the contracted indices.

3.3 One Loop Expression for the Gluon Propagator

In order to calculate $\hat{J}_p^{(1)}$, the one-loop gluon function, consider fig. 3.2 contracted with $g^{\mu\nu}$.

$$
\begin{aligned}
\delta^{ad}p^2g^{\mu\nu}\left[t_{\mu\nu}(p)J_p+\frac{1}{\xi}l_{\mu\nu}(p)\right] &= \delta^{ad}p^2g^{\mu\nu}\left[t_{\mu\nu}(p)+\frac{1}{\xi}l_{\mu\nu}(p)\right] \\
&-\frac{1}{2}(-\imath)(-\imath g f^{abc})(-\imath g f^{dcb})\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{g^{\mu\nu}\Gamma^{tl}_{\mu\alpha\rho}(p,-\omega,\omega-p)\Gamma^{tl}_{\nu\sigma\beta}(-p,p-\omega,\omega)\times\right. \\
&\left.\left[t^{\alpha\beta}(\omega)+\xi l^{\alpha\beta}(\omega)\right]\left[t^{\rho\sigma}(p-\omega)+\xi l^{\rho\sigma}(p-\omega)\right]\right\} \\
&+(-\imath)(-\imath g f^{abc})(-\imath g f^{dcb})\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}g^{\mu\nu}(\omega-p)_\mu\omega_\nu \\
&+(-\imath)(-g)^2N_f\mathrm{tr}\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}g^{\mu\nu}\gamma_\mu T^a_{il}\delta_{ij}\not{\omega}\gamma_\nu T^d_{kj}\delta_{kl}(\not{\omega}-\not{p}) \\
&-\frac{1}{2}(-\imath)(-g^2)\int\frac{\bar{d}^d\omega}{\omega^2}\left[t^{\alpha\beta}(\omega)+\xi l^{\alpha\beta}(\omega)\right]g^{\mu\nu}\Gamma^{adbb}_{\mu\nu\alpha\beta}
\end{aligned}
\tag{3.3.1}
$$

where in the fermion loop term (second last line) the trace is over the γ - matrices. Extracting the colour structure, pre-factors and expanding the function J

$$
\begin{aligned}
&-\delta^{ad}p^2(d-1)\hat{J}_p^{(1)} = \\
&-1/2(-\imath C_A)\delta^{ad}\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{g^{\mu\nu}\Gamma^{tl}_{\mu\alpha\rho}(p,-\omega,\omega-p)\Gamma^{tl}_{\nu\sigma\beta}(-p,p-\omega,\omega)\times\right. \\
&\left.\left[t^{\alpha\beta}(\omega)+\xi l^{\alpha\beta}(\omega)\right]\left[t^{\rho\sigma}(p-\omega)+\xi l^{\rho\sigma}(p-\omega)\right]+2\omega\cdot(p-\omega)\right\} \\
&-\imath\frac{1}{2}N_f\delta^{ad}\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\mathrm{tr}\gamma^\mu\not{\omega}\gamma_\mu(\not{\omega}-\not{p}) \\
&-\imath\frac{1}{2}\int\frac{\bar{d}^d\omega}{\omega^2}\left[t^{\alpha\beta}(\omega)+\xi l^{\alpha\beta}(\omega)\right]g^{\mu\nu}\Gamma^{adbb}_{\mu\nu\alpha\beta}.
\end{aligned}
\tag{3.3.2}
$$

The first thing to notice is that the last term (the tadpole) vanishes. This is because the four-gluon vertex is bare and does not contain any momentum dependence, leaving the whole integral with no external scale which under dimensional regularisation is zero. The trace over the $\gamma-$ matrices is straightforward:

$$
\begin{aligned}
\mathrm{tr}\gamma^\mu\not{\omega}\gamma_\mu(\not{\omega}-\not{p}) &= \omega^{al}(\omega-p)^\beta\mathrm{tr}\gamma^\mu\gamma_\alpha\gamma_\mu\gamma_\beta \\
&= -(d-2)\omega^{al}(\omega-p)^\beta\mathrm{tr}\gamma_\alpha\gamma_\beta \\
&= 4(d-2)\omega\cdot(p-\omega).
\end{aligned}
\tag{3.3.3}
$$

The tensor contraction of the pure gluon loop is lengthy but simple and it is easier to evaluate this using FORM [44]. The result leads directly to:

$$
\begin{aligned}
(d-1)\hat{J}_p^{(1)} &= \frac{1}{2}(-\imath C_A)\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{\left[3d-\frac{7}{2}+\xi+\frac{1}{2}\xi^2\right]\right. \\
&\left.+\frac{p^2}{\omega^2}\left[4+2d(\xi-1)+\xi^2-5\xi\right]-\frac{1}{4}\frac{p^4}{\omega^2(p-\omega)^2}(1-\xi)^2\right\} \\
&+\imath N_f(d-2)\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}
\end{aligned}
\tag{3.3.4}
$$

Using the general formula (A.1.8) for the two-point integrals, one gets

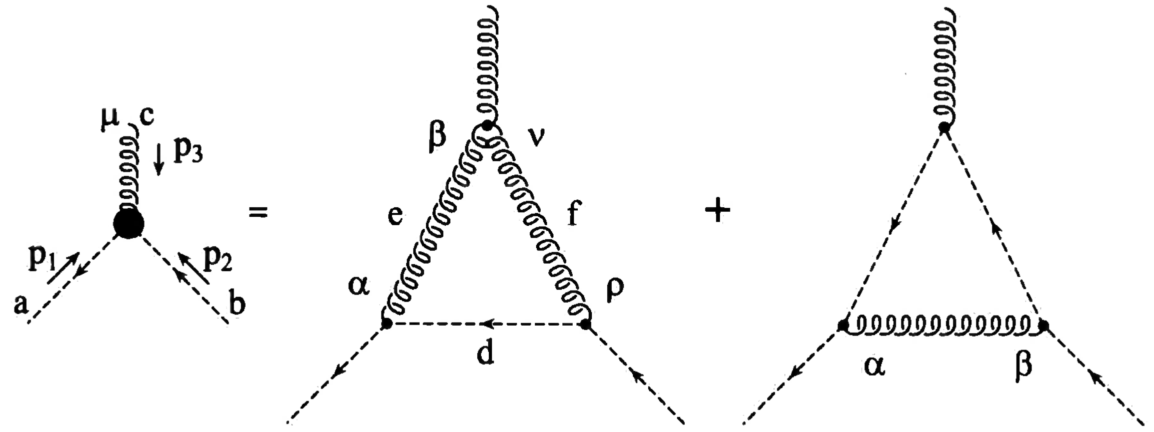


Figure 3.3: The ghost-gluon vertex at one-loop. Not shown is the momentum routing – see fig. A.2

$$
\begin{aligned}
\hat{J}_p^{(1)} = & \frac{1}{(d-1)}(4\pi)^{-d/2}(-p^2)^{-\varepsilon}\left\{\left(\frac{1}{2}C_A\left[3d-\frac{7}{2}+\xi+\frac{1}{2}\xi^2\right]-(d-2)N_f\right)\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}\right. \\
& +\frac{1}{2}C_A\left[4+2d(\xi-1)+\xi^2-5\xi\right]\frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)} \\
& \left.-\frac{1}{8}C_A(1-\xi)^2\frac{\Gamma(2+\varepsilon)\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)}\right\}. \qquad (3.3.5)
\end{aligned}
$$

Expanding this result in ε gives

$$
\begin{aligned}
\hat{J}_p^{(1)} = & (4\pi)^{-d/2}(-p^2)^{-\varepsilon}\left\{\frac{1}{\varepsilon}\left[\left(\frac{13}{6}-\frac{1}{2}\xi\right)C_A-\frac{2}{3}N_f\right]\right. \\
& \left.+\left[\left(-\frac{13}{6}\gamma+\frac{1}{2}\xi+\frac{1}{4}\xi^2+\frac{1}{2}\gamma\xi+\frac{97}{36}\right)C_A+\left(\frac{2}{3}\gamma-\frac{10}{9}\right)N_f\right]\right\} \qquad (3.3.6)
\end{aligned}
$$

where γ is the Euler constant.

3.4 One Loop Expression for the Ghost-Gluon Vertex

The one-loop expression for the ghost-gluon vertex can be immediately obtained from the Feynman diagrams (see fig. 3.3) and can be written

$$
\begin{aligned}
&-\imath g f^{abc}(-\frac{\imath}{2}g^2 C_A)\widetilde{\Gamma}^{(1)}_{\mu}(p_1,p_2,p_3) = \\
&\quad (-\imath)(-\imath g f^{ade})(-\imath g f^{dbf})(-\imath g f^{cef}) \int \frac{đ^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} \times \\
&\quad \Big\{ p_{1\alpha}(p_1-\omega)_\rho \left[t^{\alpha\beta}(\omega) + \xi l^{\alpha\beta}(\omega)\right] \left[t^{\rho\nu}(p_3+\omega) + \xi l^{\rho\nu}(p_3+\omega)\right] \Gamma^{tl}_{\beta\nu\mu}(\omega,-p_3-\omega,p_3) \\
&\quad + p_{1\alpha}\omega_\mu (p_3+\omega)_\beta \left[t^{\alpha\beta}(p_1-\omega) + \xi l^{\alpha\beta}(p_1-\omega)\right]\Big\}. \qquad (3.4.1)
\end{aligned}
$$

Extracting the colour factors and cancelling the pre-factors (arranged with this cancellation in mind) gives

$$
\begin{aligned}
&\widetilde{\Gamma}^{(1)}_{\mu}(p_1,p_2,p_3) = \int \frac{đ^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} \times \\
&\quad \Big\{ p_{1\alpha}(p_1-\omega)_\rho \left[t^{\alpha\beta}(\omega) + \xi l^{\alpha\beta}(\omega)\right] \left[t^{\rho\nu}(p_3+\omega) + \xi l^{\rho\nu}(p_3+\omega)\right] \Gamma^{tl}_{\beta\nu\mu}(\omega,-p_3-\omega,p_3) \\
&\quad + p_{1\alpha}\omega_\mu (p_3+\omega)_\beta \left[t^{\alpha\beta}(p_1-\omega) + \xi l^{\alpha\beta}(p_1-\omega)\right]\Big\}. \qquad (3.4.2)
\end{aligned}
$$

One can clearly that the tensor algebra is not difficult but it is very long. Also, there will be several vector and tensor integrals to deal with. Actually, it is possible to reduce the number of different integrals down to four by using the techniques of appendices A.2,A.4 and B. Using FORM, the first step is to do the tensor contraction. The next step is to reduce the vector triangle integrals to their respective combinations of scalar integrals (see section A.4). Then, one can deal in exactly the same way with the vector and tensor two-point integrals (see section A.2). This leaves a (large) set of scalar two and three-point integrals with various powers of the denominator factors and the Gram determinant

$$\Delta = p_1^2 p_3^2 - (p_1 \cdot p_3)^2. \qquad (3.4.3)$$

Using the technique of integration by parts outlined in appendix B, it is then straightforward to express the integrals with higher denominator factor recursively in terms of simpler integrals. Eventually, there are only four basic integrals which for clarity we will denote as

$$
\begin{aligned}
\int \frac{đ^d\omega}{\omega^2(p_1-\omega)^2} &\equiv I_1 \\
\int \frac{đ^d\omega}{\omega^2(p_3+\omega)^2} &\equiv I_3 \\
\int \frac{đ^d\omega}{(p_1-\omega)^2(p_3+\omega)^2} &= \int \frac{đ^d\omega}{\omega^2(p_2-\omega)^2} \equiv I_2 \\
\int \frac{đ^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} &\equiv \Phi
\end{aligned}
$$

The vertex function has one Lorentz index and two independent momenta and so can be written

$$\tilde{\Gamma}^{(1)}_{\mu}(p_1,p_2,p_3) = p_{1\mu}V_1(p_1,p_2,p_3) + p_{3\mu}V_2(p_1,p_2,p_3). \tag{3.4.4}$$

The result for the one-loop calculation of these two functions is:

$$\begin{aligned}
V_1(p_1,p_2,p_3) = \frac{1}{\Delta}\Bigg\{ \\
\Phi \bigg[& \frac{1}{4}p_1^4p_3^2\xi_j^2\varepsilon - p_1^2p_1\cdot p_2p_3^2 - \frac{1}{2}p_1^2p_2^2p_3^2\xi_j^2\varepsilon^2 + \frac{1}{4}p_1\cdot p_2p_3^4\xi_j^2 + \frac{1}{4}p_1\cdot p_3p_3^4\xi_j^2\varepsilon \\
& + p_1^2p_2\cdot p_3p_3^2\left(-\frac{3}{2} - \xi_j^2\varepsilon + \frac{1}{4}\xi_j^2 + \frac{1}{2}\xi_j\varepsilon + \frac{1}{4}\xi_j\right) + p_1\cdot p_2\Delta\left(1 - \frac{1}{2}\xi_j^2\varepsilon^2 + \frac{1}{2}\xi_j\varepsilon\right) \\
& + p_3^2\Delta\left(-1 + \frac{1}{2}\xi_j^2\varepsilon^2 - \frac{1}{2}\xi_j^2\varepsilon + \frac{1}{4}\xi_j^2 - \frac{1}{2}\xi_j\varepsilon + \frac{1}{2}\xi_j\right)\bigg] \\
+ I_1 \bigg[& -\frac{1}{2}p_1^2p_2\cdot p_3\xi_j^2\varepsilon^2 - \frac{1}{4}p_1\cdot p_2p_3^2\xi_j^2 - \frac{1}{4}p_1\cdot p_3p_3^2\xi_j^2\varepsilon + p_1^2p_1\cdot p_3\left(-1 - \frac{1}{4}\xi_j^2\varepsilon\right) \\
& + p_1^2p_3^2\left(-\frac{3}{2} - \frac{3}{4}\xi_j^2\varepsilon + \frac{1}{2}\xi_j\varepsilon + \frac{1}{4}\xi_j\right) + \Delta\left(1 - \frac{1}{2}\xi_j^2\varepsilon^2 + \frac{1}{4}\xi_j^2\varepsilon + \frac{1}{2}\xi_j\varepsilon - \frac{1}{2}\xi_j\right)\bigg] \\
+ I_2 \bigg[& \frac{1}{4}p_1^2p_1\cdot p_3\xi_j^2\varepsilon - \frac{1}{4}p_2^2p_3^2\xi_j^2 + \frac{1}{4}p_3^4\xi_j^2\varepsilon + p_1\cdot p_2p_3^2\left(-\frac{3}{2} + \frac{1}{2}\xi_j^2\varepsilon^2 - \xi_j^2\varepsilon + \frac{1}{2}\xi_j\varepsilon + \frac{1}{4}\xi_j\right) \\
& + p_1^2p_2\cdot p_3\left(-1 + \frac{1}{2}\xi_j^2\varepsilon^2\right) + \Delta\left(-1 + \frac{1}{2}\xi_j^2\varepsilon^2 - \frac{1}{4}\xi_j^2\varepsilon + \frac{1}{2}\xi_j\varepsilon - \frac{1}{2}\xi_j\right)\bigg] \\
+ I_3 \bigg[& -\frac{1}{2}p_1\cdot p_2p_3^2\xi_j^2\varepsilon^2 - \frac{1}{4}p_2\cdot p_3p_3^2\xi_j^2 - \frac{1}{4}p_3^4\xi_j^2\varepsilon + p_1^2p_3^2\left(-1 - \frac{1}{4}\xi_j^2\varepsilon\right) \\
& + p_1\cdot p_3p_3^2\left(-\frac{3}{2} - \frac{3}{4}\xi_j^2\varepsilon + \frac{1}{2}\xi_j\varepsilon + \frac{1}{4}\xi_j\right) + \Delta\left(1 - \frac{1}{2}\xi_j^2\varepsilon^2 + \frac{1}{2}\xi_j\varepsilon\right)\bigg] \Bigg\}
\end{aligned} \tag{3.4.5}$$

$$
\begin{aligned}
V_2(p_1,p_2,p_3) = \frac{1}{\Delta}\Bigg\{ & \\
\Phi\Bigg[& -\frac{1}{4}p_1^4 p_1\cdot p_3 \xi_j^2\varepsilon + \frac{1}{4}p_1^2 p_2^2 p_3^2 \xi_j^2 - \frac{1}{4}p_1^2 p_3^4 \xi_j^2 \varepsilon + 2\Delta p_1^2 + p_1^4 p_2\cdot p_3\left(1-\frac{1}{2}\xi_j^2\varepsilon^2\right) \\
& + p_1^2 p_1\cdot p_2 p_3^2\left(\frac{3}{2} - \frac{1}{2}\xi_j^2\varepsilon^2 + \xi_j^2\varepsilon - \frac{1}{2}\xi_j\varepsilon - \frac{1}{4}\xi_j\right) + \Delta p_1\cdot p_2\left(\xi_j^2\varepsilon^2 - \xi_j^2\varepsilon + \frac{1}{4}\xi_j^2 + \frac{1}{2}\xi_j\right) \\
& + \Delta\frac{p_1\cdot p_2 p_1\cdot p_3}{p_3^2}\left(\frac{1}{2}\xi_j^2\varepsilon^2 - \xi_j\varepsilon\right) + \Delta p_1\cdot p_3\left(\frac{1}{2}\xi_j^2\varepsilon^2 - \frac{1}{2}\xi_j^2\varepsilon + \xi_j\varepsilon\right) + \frac{1}{4}\Delta p_3^2\xi_j^2(\varepsilon-1)\Bigg] \\
+I_1\Bigg[& +\frac{1}{2}p_1^2 p_1\cdot p_2 \xi_j^2\varepsilon^2 + \frac{1}{4}p_1^2 p_2\cdot p_3 \xi_j^2 + \frac{1}{4}p_1^2 p_3^2 \xi_j^2\varepsilon + \frac{1}{4}\Delta\xi_j^2(1-\varepsilon) + p_1^4\left(1+\frac{1}{4}\xi_j^2\varepsilon\right) \\
& + p_1^2 p_1\cdot p_3\left(\frac{3}{2} + \frac{3}{4}\xi_j^2\varepsilon - \frac{1}{2}\xi_j\varepsilon - \frac{1}{4}\xi_j\right) + \Delta\frac{p_1\cdot p_2}{p_3^2}\left(-\frac{1}{2}\xi_j^2\varepsilon^2 + \frac{1}{4}\xi_j^2\varepsilon + \xi_j\varepsilon - \frac{1}{2}\xi_j\right)\Bigg] \\
+I_2\Bigg[& -\frac{1}{4}p_1^4\xi_j^2\varepsilon + p_1^2 p_1\cdot p_2 + \frac{1}{2}p_1^2 p_2^2\xi_j^2\varepsilon^2 - \frac{1}{4}p_1\cdot p_2 p_3^2\xi_j^2 - \frac{1}{4}p_1\cdot p_3 p_3^2\xi_j^2\varepsilon \\
& + p_1^2 p_2\cdot p_3\left(\frac{3}{2} + \xi_j^2\varepsilon - \frac{1}{4}\xi_j^2 - \frac{1}{2}\xi_j\varepsilon - \frac{1}{4}\xi_j\right) + \Delta\frac{p_1\cdot p_2}{p_3^2}\left(\frac{1}{2}\xi_j^2\varepsilon^2 - \frac{1}{4}\xi_j^2\varepsilon - \xi_j\varepsilon + \frac{1}{2}\xi_j\right) \\
& + \Delta\left(1 - \frac{1}{2}\xi_j^2\varepsilon^2 + \xi_j^2\varepsilon - \frac{1}{2}\xi_j^2 - \frac{1}{2}\xi_j\right)\Bigg] \\
+I_3\Bigg[& +\frac{1}{2}p_1^2 p_2\cdot p_3\xi_j^2\varepsilon^2 + \frac{1}{4}p_1\cdot p_2 p_3^2\xi_j^2 + \frac{1}{4}p_1\cdot p_3 p_3^2\xi_j^2\varepsilon - \Delta\frac{p_1\cdot p_3}{p_3^2}\xi_j\varepsilon - \frac{1}{2}\Delta\frac{p_2\cdot p_3}{p_3^2}\xi_j^2\varepsilon^2 \\
& + p_1^2 p_1\cdot p_3\left(1 + \frac{1}{4}\xi_j^2\varepsilon\right) + p_1^2 p_3^2\left(\frac{3}{2} + \frac{3}{4}\xi_j^2\varepsilon - \frac{1}{2}\xi_j\varepsilon - \frac{1}{4}\xi_j\right) \\
& + \Delta\left(-1 - \frac{3}{4}\xi_j^2\varepsilon + \frac{1}{4}\xi_j^2 + \frac{1}{2}\xi_j\right)\Bigg]\Bigg\}
\end{aligned}
\tag{3.4.6}
$$

where $\xi_j = 1-\xi$. This result agrees explicitly with [24].

3.5 Perturbative Renormalisation of the One Loop Expressions

All the quantities calculated in this chapter are ultraviolet divergent in four dimensions, where under dimensional regularisation the divergence is characterised by a simple pole in ε as $\varepsilon \to 0$. In order to make them finite, the theory must be renormalised. In practice, this can be achieved by introducing the coefficients Z. The idea is to isolate the divergence of the perturbative function and cancel it order by order in the coupling. One can define the renormalised functions and the coefficients Z in the following way[1]

$$\begin{aligned}
\overline{J}^{-1}(p^2|\overline{\xi},\overline{g}^2) &= Z_3^{-1}(1/\varepsilon,\overline{\xi},\overline{g}^2)J^{-1}(p^2|\xi_b,g_b^2,\varepsilon) \\
\overline{G}(p^2|\overline{\xi},\overline{g}^2) &= \tilde{Z}_3^{-1}(1/\varepsilon,\overline{\xi},\overline{g}^2)G(p^2|\xi_b,g_b^2,\varepsilon) \\
\overline{\tilde{\Gamma}}_\mu(p_1,p_2,p_3|\overline{\xi},\overline{g}^2) &= \tilde{Z}_1(1/\varepsilon,\overline{\xi},\overline{g}^2)\tilde{\Gamma}_\mu(p_1,p_2,p_3|\xi_b,g_b^2,\varepsilon)
\end{aligned} \tag{3.5.1}$$

where the overbar denotes the renormalised quantity, and the subscript b denotes the bare value of the parameter. It is understood that the bare coupling and gauge parameters (g_b^2 and ξ_b) occurring in the perturbative expansion of the unrenormalised quantities be replaced by their renormalised counterparts multiplied by the appropriate renormalisation coefficients, ie

$$g_b^2 = \frac{\tilde{Z}_1^2}{Z_3\tilde{Z}_3^2}\overline{g}^2, \quad \xi_b = Z_3\overline{\xi}. \tag{3.5.2}$$

This process can be seen easily in practice. Consider the unrenormalised perturbative expression for the ghost propagator function (3.2.6).

$$\begin{aligned}
G_p &= 1 + (-\frac{\imath}{2}g_b^2C_A)\tilde{G}_p^{(1)} + O(g_b^4) \\
&= 1 + (-\frac{\imath}{2}g_b^2C_A)\left\{1-(1-2\varepsilon)\frac{(\xi_b-1)}{2}\right\}\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2} + O(g_b^4)
\end{aligned} \tag{3.5.3}$$

This is expanded in ε (see appendix A.1) to obtain

$$G_p = 1 + \frac{g_b^2C_A}{(4\pi)^2}\left\{\left(\frac{3}{4}-\frac{1}{4}\xi_b\right)\left(\frac{1}{\varepsilon}-\gamma+\ln(4\pi)-\ln(-p^2)\right)+1\right\}+O(g_b^4) \tag{3.5.4}$$

Now consider the renormalisation of this. The first step is to expand the coefficients Z in the renormalised coupling $\overline{g}^2$. Actually, it is easier to expand in a new variable $h = \frac{\overline{g}^2}{(4\pi)^2}$ (no overbar on h since there will be no confusion), so

$$Z_i(1/\varepsilon,\overline{\xi},\overline{g}^2) = 1 + hZ_i^{(1)}(1/\varepsilon,\overline{\xi}) + h^2Z_i^{(2)}(1/\varepsilon,\overline{\xi}) + \ldots \tag{3.5.5}$$

It is clear to see that at lowest order, $g_b^2 = \overline{g}^2$ and $\xi_b = \overline{\xi}$. However, at the next order, there will be cross-terms coming from the expansion of the bare parameters. This gives

[1]Strictly speaking, one should redefine the Lagrangian in terms of renormalised fields multiplied by divergent coefficients in arbitrary dimension. This leads to the set of equations relating the renormalised quantities to the bare ones. In the process, the bare coupling takes on a dimension μ^ε. Subsequently, in deriving the renormalisation coefficients Z, one should add to the prescription (see later) the factor μ^ε such that the renormalised coupling is left dimensionless. That this mass scale μ is arbitrary leads to the concept of the renormalisation group. Here, the factor μ^ε will not be necessary and shall be omitted.

for the renormalised function $\overline{G}$

$$\begin{aligned}\overline{G}(p^2|\overline{\xi},\overline{g}^2) = &\left(1 - h\tilde{Z}_3^{(1)}(1/\varepsilon,\overline{\xi})\right) \times \\ &\left(1 + hC_A\left\{\left(\frac{3}{4} - \frac{1}{4}\overline{\xi}\right)\left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \ln(-p^2)\right) + 1\right\}\right) + O(h^2). \quad (3.5.6)\end{aligned}$$

Now, the whole idea here is to remove the divergence so it can easily be seen that $\tilde{Z}_3^{(1)}$ must be defined so as to cancel the $1/\varepsilon$ divergence. In fact, there is a complete freedom to cancel any finite (but not momentum dependent) part of the unrenormalised quantity – this gives rise to different renormalisation schemes and the concept of renormalisation scheme dependence. The most common scheme is the $\overline{\text{MS}}$ scheme and this shall be used. In the $\overline{\text{MS}}$ scheme, the Euler constant γ and the factor $\ln(4\pi)$ are removed since they always occur in the same combination alongside the $1/\varepsilon$ pole. Thus, to make the ghost propagator finite, the coefficient $\tilde{Z}_3^{(1)}$ is found to be

$$\tilde{Z}_3^{(1)}(1/\varepsilon,\overline{\xi}) = C_A\left(\frac{3}{4} - \frac{1}{4}\overline{\xi}\right)\left(\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right) \quad (3.5.7)$$

which leaves the finite ghost propagator function

$$\overline{G}(p^2|\overline{\xi},\overline{g}^2) = 1 + hC_A\left\{-\ln(-p^2)\left(\frac{3}{4} - \frac{1}{4}\overline{\xi}\right) + 1\right\} + O(h^2). \quad (3.5.8)$$

This procedure can be repeated for the gluon propagator function in exactly the same way. Using (3.3.6)

$$\begin{aligned}\overline{J}^{-1}(p^2|\overline{\xi},\overline{g}^2) = &\left(1 - hZ_3^{(1)}(1/\varepsilon,\overline{\xi})\right) \times \\ &\left(1 + h\left\{\left[\left(\frac{13}{6} - \frac{1}{2}\overline{\xi}\right)C_A - \frac{2}{3}N_f\right]\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \ln(-p^2)\right]\right.\right. \\ &\left.\left. + \left(\frac{1}{2}\overline{\xi} + \frac{1}{4}\overline{\xi}^2 + \frac{97}{36}\right)C_A - \frac{10}{9}N_f\right\}\right) + O(h^2) \quad (3.5.9)\end{aligned}$$

which means that in order for $\overline{J}^{-1}$ to be finite

$$Z_3^{(1)}(1/\varepsilon,\overline{\xi}) = \left[\left(\frac{13}{6} - \frac{1}{2}\overline{\xi}\right)C_A - \frac{2}{3}N_f\right]\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right]. \quad (3.5.10)$$

This leaves

$$\begin{aligned}&\overline{J}^{-1}(p^2|\overline{\xi},\overline{g}^2) = \\ &\quad 1 + h\left\{-\ln(-p^2)\left[\left(\frac{13}{6} - \frac{1}{2}\overline{\xi}\right)C_A - \frac{2}{3}N_f\right] + \left(\frac{1}{2}\overline{\xi} + \frac{1}{4}\overline{\xi}^2 + \frac{97}{36}\right)C_A - \frac{10}{9}N_f\right\} + O(h^2) \\ &\hfill (3.5.11)\end{aligned}$$

Now consider the ghost-gluon vertex function. Since the expression (3.4.6) is (to put it mildly) rather large, for clarity only the divergent parts are needed. By expanding the functions V_i in ε, one finds that the divergent part of the vertex function (retaining the factors γ and $\ln(4\pi)$) after replacing the bare coupling and gauge parameters with their renormalised counterparts can be written

$$\tilde{\Gamma}_\mu(p_1, p_2, p_3) = p_{1\mu}\left\{1 + hC_A\frac{\bar{\xi}}{2}\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right]\right\}. \tag{3.5.12}$$

This means that

$$\bar{\tilde{\Gamma}}_\mu(p_1, p_2, p_3)|\bar{\xi}, \bar{g}^2) = \left[1 + h\tilde{Z}_1^{(1)}(1/\varepsilon, \bar{\xi})\right] p_{1\mu}\left\{1 + hC_A\frac{\bar{\xi}}{2}\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right]\right\} + O(h^2) \tag{3.5.13}$$

which leads directly to the expression for $\tilde{Z}_1^{(1)}$

$$\tilde{Z}_1^{(1)}(1/\varepsilon, \bar{\xi}) = -C_A\frac{\bar{\xi}}{2}\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right] \tag{3.5.14}$$

Notice that in the Landau gauge, $\tilde{Z}_1^{(1)}$ vanishes. This very important property can be shown to be true at all orders [21, 43] and will be crucial to later calculations. Also, given that $\tilde{Z}_1^{(1)}$ has no Lorentz structure, there are restrictions placed on the divergence. Clearly, it would be inconsistent to have a divergence proportional to $p_{3\mu}$ at this order in the coupling. This can be extended to all orders – the divergence of the vertex part proportional to $p_{3\mu}$ must be less than that of the part proportional to $p_{1\mu}$.

Chapter 4

A Possible Identity for the Ghost-Gluon Vertex

In this chapter, the possibility of deriving an identity relating the ghost-gluon vertex to two-point functions is investigated. Earlier discussion focussed on the necessity for any vertex ansatz to obey the respective Slavnov-Taylor identity and that the inclusion of ghosts into IR Schwinger-Dyson studies of QCD was also needed. The desirability of a relation between the ghost-gluon vertex and the two-point functions is self-evident in Schwinger-Dyson studies, since this allows at least a first method of truncating the set of equations, as the work of Mandelstam [25], von Smekal *et al.* [1, 2, 3] and others clearly demonstrates. That this relation should also implement gauge invariance is a further crucial property.

In the work of von Smekal *et al.* [1, 2, 3], a functional form of the Slavnov-Taylor identity relating the ghost-gluon vertex to a four-point correlation function is put forward (2.4.1). This equation refers to full, reducible correlation functions and as such is not particularly suited to direct application to Schwinger-Dyson techniques. The more usual form of Slavnov-Taylor identities relates one-particle irreducible Green's functions and their renormalisation coefficients.

However, the equation (2.4.1) does contain a lot of useful information. The first point to note is that the Slavnov-Taylor identity involves a combination of contracted ghost-gluon vertices. On closer inspection, this is not so surprising given the tree-level form of the vertex itself. The tree-level vertex is proportional to the out-ghost momentum and the most likely contraction is with the gluon momentum, giving a scalar product that does not lend itself to a simple decomposition. There are three simple candidates for a combination of two tree-level contracted vertices

$$\begin{aligned}
p_3^\mu\widetilde{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu\widetilde{\Gamma}_\mu(p_3,p_2,p_1) &= 0 + O(g^2) \\
p_3^\mu\widetilde{\Gamma}_\mu(p_1,p_2,p_3) + p_2^\mu\widetilde{\Gamma}_\mu(p_1,p_3,p_2) &= -p_1^2 + O(g^2) \\
p_3^\mu\widetilde{\Gamma}_\mu(p_1,p_2,p_3) + p_3^\mu\widetilde{\Gamma}_\mu(p_2,p_1,p_3) &= -p_3^2 + O(g^2) \qquad (4.0.1)
\end{aligned}$$

The second of these possibilities corresponds to the case used by von Smekal *et al.* [2].

The second point to note about (2.4.1) is the explicit inclusion of the full reducible four-point ghost correlation function. This is important when one considers it's decomposition into one-particle irreducible Green's functions, connected reducible parts and disconnected scattering. This part of the equation is largely neglected in the work of von Smekal *et al.* who only retain the disconnected parts. The decomposition of the four-point function is not understood and this throws some doubt on the matter of which (if any) of the tree-level forms above is to be taken as a starting point[1]. It is entirely possible (and even likely) that the reducible scattering parts comprise some form of ghost-gluon vertex contribution.

With this in mind, the purpose of this chapter is to concentrate on trying to constrain perturbatively an identity relating the ghost-gluon vertex to some combination of two-point functions. That only two-point functions are involved is the most desirable property of such identities when dealing with Schwinger-Dyson studies since it allows the possibility of truncating the set of Schwinger-Dyson equations. The first section shows the one-loop derivation of such an identity, true in *all* gauges. This identity corresponds to the first of the tree-level forms above in contrast with the case used by von Smekal *et al.* [1, 2, 3]. This equation is remarkable in it's simplicity and that it is true in all gauges gives confidence that it may be the one-loop form of the Slavnov-Taylor identity.

The second section then goes on to show that the one-loop identity is also satisfied when one includes gluon self-energy corrections at the next order. This is important because it introduces the first fermionic contributions to the identity.

The third section uses the renormalisation properties of the Green's functions to constrain the form of the full identity. It shows that in order for the divergent parts of the equation to be consistent, there must be definite constraints satisfied by the two-loop parts of the equation. The technique is to assume a general form for the identity in terms

[1] This is *not* a criticism of the work done by von Smekal *et al.* – their motivation was to construct a truncation scheme where all connected four-point functions were consistently neglected.

of unrenormalised Green's functions. These functions can be re-expressed as products of the finite renormalised functions and the divergent renormalisation coefficients. By isolating order by order in the coupling the divergent parts of both sides of the general equation and demanding equivalence, it is possible to derive a set of constraints on the finite parts at lower orders. These constraint equations then need to be verified.

In the fourth section, the two-loop ghost propagator function is derived in the Feynman gauge. This calculation will not only be necessary to later sections, but is useful in introducing a compact notation and new techniques.

The fifth and sixth sections deal with the two-loop constraints on the equation in Feynman gauge and with a certain momentum configuration (in order to simplify matters). The vertex contributions are calculated in the fifth section and these are used in the sixth section where all the parts of the constraint equations are pieced together. It is found that the full identity (even with the restricted gauge and momentum configuration) cannot be satisfied. However, it is noted that with the omission of the explicit four-gluon interaction the identity may still hold.

4.1 The One Loop Identity

In this section, the derivation of an identity relating the difference of two contracted ghost-gluon vertices to a combination of two-point ghost propagator functions in general gauges is presented. Consider the one-loop expression for the ghost-gluon vertex contracted with the gluon momentum. Using (3.4.6), one finds that

$$
\begin{aligned}
&p_3^\mu \widetilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3) = \\
&\quad \Phi\left[p_1^2 p_3^2\left(-\frac{1}{2}+\frac{1}{2}\xi_j\varepsilon-\frac{1}{4}\xi_j\right)+\left((p_1\cdot p_3)^2+p_1\cdot p_3 p_1^2+p_1\cdot p_3 p_3^2\right)\left(-1+\frac{1}{2}\xi_j\varepsilon\right)\right] \\
&\quad +I_1\left[p_1^2\left(1-\xi_j\varepsilon+\frac{1}{2}\xi_j\right)+p_1\cdot p_3\left(1-\frac{1}{2}\xi_j\varepsilon\right)\right] \\
&\quad +I_2\left[p_1^2\left(-1+\xi_j\varepsilon-\frac{1}{2}\xi_j\right)+p_3^2\left(-\frac{1}{2}+\frac{1}{2}\xi_j\varepsilon-\frac{1}{4}\xi_j\right)+p_1\cdot p_3\left(-1+\frac{3}{2}\xi_j\varepsilon-\xi_j\right)\right] \\
&\quad +I_3\left[p_3^2\left(\frac{1}{2}-\frac{1}{2}\xi_j\varepsilon+\frac{1}{4}\xi_j\right)+p_1\cdot p_3\left(1-\frac{1}{2}\xi_j\varepsilon\right)\right]. \qquad (4.1.1)
\end{aligned}
$$

It is immediately clear that the terms proportional to Φ (the massless triangle integral) are symmetric under interchange of p_1 and p_3, since Φ itself is symmetric in the two. This integral is not present in the one-loop form of either the ghost or gluon propagators. Thus, in order to construct an identity relating the combination of contracted vertices to some combination of two-point functions, the simplest possibility at one-loop is the first equation of (4.0.1). By considering all possible contractions of the ghost-gluon vertex (using FORM), this turns out to be the *only* way of eliminating the triangle integral Φ using only two contracted vertices.

Taking then the difference of the two contracted vertices with p_1 and p_3 interchanged gives

$$
p_3^\mu \widetilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3) - p_1^\mu \widetilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1) = \frac{1}{2}\left(1-\xi_j\varepsilon+\frac{1}{2}\xi_j\right)\left[p_1^2 I_1 - p_1^2 I_2 - p_3^2 I_3 + p_3^2 I_2\right] \qquad (4.1.2)
$$

where as before $\xi_j = 1-\xi$. Comparison of this with the one-loop form of the ghost propagator function (3.2.6) and replacing the I_i with their integral forms gives

$$
p_3^\mu \widetilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3) - p_1^\mu \widetilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1) = \frac{1}{2}p_1^2\left(\tilde{G}_1^{(1)}-\tilde{G}_2^{(1)}\right) - \frac{1}{2}p_3^2\left(\tilde{G}_3^{(1)}-\tilde{G}_2^{(1)}\right). \qquad (4.1.3)
$$

This is the one-loop form of the identity, true in all gauges (and arbitrary dimension). It is this equation that all the fuss is about. The equation relates in a simple way the ghost-gluon vertex with the ghost propagator function, which was precisely what was set out to be achieved.

If this equation were to hold beyond the one-loop approximation (and there is as yet no reason why it should) then there are several possible forms that are admitted

$$
p_3^\mu \widetilde{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu \widetilde{\Gamma}_\mu(p_3,p_2,p_1) = \frac{1}{2}p_1^2\begin{pmatrix} G_1-G_2 \\ G_2^{-1}-G_1^{-1} \\ 1-G_2/G_1 \\ G_1/G_2-1 \end{pmatrix} - \frac{1}{2}p_3^2\begin{pmatrix} G_3-G_2 \\ G_2^{-1}-G_3^{-1} \\ 1-G_2/G_3 \\ G_3/G_2-1 \end{pmatrix}. \qquad (4.1.4)
$$

In addition, there could also be forms like

$$\begin{aligned} G_1 p_3^\mu \tilde{\Gamma}_\mu(p_1,p_2,p_3) - G_3 p_1^\mu \tilde{\Gamma}_\mu(p_3,p_2,p_1) = \\ \frac{1}{2}p_3^2\left(G_2-G_1\right)+\frac{1}{2}p_1^2\left(G_3-G_2\right)+\frac{1}{2}p_2^2\left(G_1-G_3\right). \end{aligned} \tag{4.1.5}$$

This last form is rather interesting due to the cyclic symmetry. The only way to distinguish between these possibilities is to go beyond one-loop.

4.2 Beyond the One Loop Identity

It is possible to extend the one-loop identity (4.1.3) to include gluon self-energy corrections. This is important since the gluon self-energy contains the first part of the fermionic contributions. To proceed, consider the ghost propagator function Schwinger-Dyson equation (see fig. 3.1). In order to include the next order gluon self-energy part, one only needs to modify the full expression by setting the ghost propagator and ghost-gluon vertex in the integrand to their respective bare values and setting the gluon propagator function to it's one-loop counterpart (explicitly calculated in the previous chapter). The pre-factors remain the same so it is simple to write

$$p^2(-\imath g^4 C_A)\hat{G}_p = (-\imath g^4 C_A)\int \frac{d\!\!^{-}{}^d\omega}{\omega^2(p-\omega)^2} p_\mu (p-\omega)_\nu t^{\mu\nu}(\omega)\hat{J}_\omega^{(1)} \tag{4.2.1}$$

After the tensor contraction, one finds that

$$\hat{G}_p = \int \frac{d\!\!^{-}{}^d\omega \hat{J}_\omega^{(1)}}{\omega^2(p-\omega)^2}\left[1-\frac{p\cdot\omega^2}{p^2\omega^2}\right]. \tag{4.2.2}$$

It will not be necessary to evaluate this integral explicitly.

Now consider the contracted one-loop ghost-gluon vertex function with the one-loop gluon propagator function inserted. Just as in the above, the expression can be written down almost immediately from consideration of fig. 3.3 and the one-loop expression. It is important to note though that in the first graph of fig. 3.3, the contraction of the external gluon momentum triggers the Slavnov-Taylor identity (1.7.6) for which there are gluon propagator function contributions. The contracted vertex is thus written

$$\begin{aligned} (-\imath g^4 C_A)p_3^\mu \hat{\Gamma}_\mu(p_1,p_2,p_3) = (-\frac{\imath}{2}g^4 C_A)\int \frac{d\!\!^{-}{}^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}\times \\ \Big\{p_{1\alpha}(p_1-\omega)_\rho\left[(p_3+\omega)^2 t_{\beta\nu}(p_3+\omega)-\omega^2 t_{\beta\nu}(\omega)\right] t^{\alpha\beta}(\omega)\hat{J}_\omega^{(1)}\left[t^{\rho\nu}(p_3+\omega)+\xi l^{\rho\nu}(p_3+\omega)\right] \\ +p_{1\alpha}(p_1-\omega)_\rho\left[(p_3+\omega)^2 t_{\beta\nu}(p_3+\omega)-\omega^2 t_{\beta\nu}(\omega)\right]\left[t^{\alpha\beta}(\omega)+\xi l^{\alpha\beta}(\omega)\right] t^{\rho\nu}(p_3+\omega)\hat{J}_{p_3+\omega}^{(1)} \end{aligned}$$

$$+p_{1\alpha}(p_1-\omega)_\rho\left[\omega^2 t_{\beta\nu}(\omega)\hat{J}^{(1)}_\omega-(p_3+\omega)^2 t_{\beta\nu}(p_3+\omega)\hat{J}^{(1)}_{p_3+\omega}\right]\times$$
$$\left[t^{\alpha\beta}(\omega)+\xi l^{\alpha\beta}(\omega)\right]\left[t^{\rho\nu}(p_3+\omega)+\xi l^{\rho\nu}(p_3+\omega)\right]$$
$$+p_{1\alpha}p3\cdot\omega(p_3+\omega)_\beta t^{\alpha\beta}(p_1-\omega)\hat{J}^{(1)}_{p_1-\omega}\Big\} \tag{4.2.3}$$

Eliminating those parts that cancel, one finds that the result is explicitly independent of the gauge parameter and is

$$p_3^\mu\hat{\Gamma}_\mu(p_1,p_2,p_3)=\frac{1}{2}\int\frac{đ^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}\times$$
$$\Big\{p_{1\alpha}(p_1-\omega)_\rho\left[\hat{J}^{(1)}_\omega(p_3+\omega)^2 t^{\beta\rho}(p_3+\omega)t^\alpha_\beta(\omega)-\hat{J}^{(1)}_{p_3+\omega}\omega^2 t^{\alpha\nu}(\omega)t^\rho_\nu(p_3+\omega)\right]$$
$$+p_{1\alpha}p3\cdot\omega(p_3+\omega)_\beta t^{\alpha\beta}(p_1-\omega)\hat{J}^{(1)}_{p_1-\omega}\Big\} \tag{4.2.4}$$

The tensor contraction is straightforward and so skipping a step, the result when the $p_1\leftrightarrow p_3$ interchanged result is subtracted is

$$p_3^\mu\hat{\Gamma}_\mu(p_1,p_2,p_3)-p_1^\mu\hat{\Gamma}_\mu(p_3,p_2,p_1)=\frac{1}{2}\int\frac{đ^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}\times$$
$$\left\{\hat{J}^{(1)}_\omega(p_3+\omega)^2\left[p_1^2-\frac{p_1\cdot\omega^2}{\omega^2}\right]-\hat{J}^{(1)}_\omega(p_1-\omega)^2\left[p_3^2-\frac{p_3\cdot\omega^2}{\omega^2}\right]\right.$$
$$+\omega^2\hat{J}^{(1)}_{p_1-\omega}\left[p_3^2+p_3\cdot\omega-\frac{(p_1-\omega)\cdot(p_3+\omega)}{(p_1-\omega)^2}(p_1\cdot p_3-p_3\cdot\omega)\right]$$
$$\left.-\omega^2\hat{J}_{p_3+\omega}\left[p_1^2-p_1\cdot\omega-\frac{(p_1-\omega)\cdot(p_3+\omega)}{(p_3+\omega)^2}(p_1\cdot p_3+p_1\cdot\omega)\right]\right\} \tag{4.2.5}$$

Changing variables on the last two lines such that the argument of $\hat{J}$ is ω gives

$$
\begin{aligned}
&p_3^\mu \hat{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu \hat{\Gamma}_\mu(p_3,p_2,p_1) = \\
&\quad \frac{1}{2}\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}\left\{(p_3+\omega)^2\left[p_1^2 - \frac{p_1\cdot\omega^2}{\omega^2}\right] - (p_1-\omega)^2\left[p_3^2 - \frac{p_3\cdot\omega^2}{\omega^2}\right]\right\} \\
&\quad +\frac{1}{2}\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega}{\omega^2(p_2-\omega)^2}\left\{p_3^2 - p_1^2 + \frac{p_2\cdot\omega}{\omega^2}(p_3-p_1)\cdot\omega\right\}.
\end{aligned} \tag{4.2.6}
$$

Using the method outlined in appendix A.2, it is clear that

$$
\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega p_2\cdot\omega}{\omega^4(p_2-\omega)^2}(p_3-p_1)\cdot\omega = \frac{1}{2}(p_1^2-p_3^2)\left\{\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega p_2\cdot\omega}{\omega^4(p_2-\omega)^2} + \int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega p_2\cdot\omega}{p_2^2\omega^2(p_2-\omega)^2}\right\}, \tag{4.2.7}
$$

but

$$
\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega p_2\cdot\omega^2}{\omega^4(p_2-\omega)^2} = p_2^2\frac{1}{2}\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega p_2\cdot\omega}{\omega^4(p_2-\omega)^2} + \frac{1}{2}\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega p_2\cdot\omega}{\omega^2(p_2-\omega)^2}. \tag{4.2.8}
$$

Comparing the last two equations one can see that

$$
\begin{aligned}
&p_3^\mu \hat{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu \hat{\Gamma}_\mu(p_3,p_2,p_1) = \\
&\quad \frac{1}{2}\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}\left\{(p_3+\omega)^2\left[p_1^2 - \frac{p_1\cdot\omega^2}{\omega^2}\right] - (p_1-\omega)^2\left[p_3^2 - \frac{p_3\cdot\omega^2}{\omega^2}\right]\right\} \\
&\quad +\frac{1}{2}(p_3^2-p_1^2)\int \frac{\bar{d}^d\omega \hat{J}^{(1)}_\omega}{\omega^2(p_2-\omega)^2}\left[1 - \frac{p_2\cdot\omega^2}{p_2^2\omega^2}\right].
\end{aligned} \tag{4.2.9}
$$

Putting the result (4.2.2) into the above immediately gives

$$
p_3^\mu \hat{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu \hat{\Gamma}_\mu(p_3,p_2,p_1) = \frac{1}{2}p_1^2\left(\hat{G}_1^{(1)} - \hat{G}_2^{(1)}\right) - \frac{1}{2}p_3^2\left(\hat{G}_3^{(1)} - \hat{G}_2^{(1)}\right) \tag{4.2.10}
$$

which is identical in form to the one-loop equation (4.1.3). This equation is encouraging since it shows that the one-loop identity may hold at higher orders.

4.3 Renormalisation and the Identity

It is widely known that the Slavnov-Taylor identity for the triple-gluon vertex (1.7.6) can be used to constrain the renormalisation coefficients of the theory. Just as the Ward identity in QED leads to the famous equality $Z_1 = Z_2$, such that the fermion-boson coupling is universal (see for example [5]), there is an analogous relation for QCD:

$$
\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} \tag{4.3.1}
$$

which implies that the ratio of bare to renormalised couplings is independent of whether the triple-gluon or ghost-gluon vertices are used to define the renormalised coupling [21].

It is pertinent to ask whether or not the identity (4.1.3) also implies some sort of relationship between renormalisation coefficients. Closer inspection of the identity shows that there will only be a relation between $\tilde{Z}_1$ and $\tilde{Z}_3$. However, this relation cannot be direct, since both sides of the one-loop identity are finite. Moreover, $\tilde{Z}_1$ and $\tilde{Z}_3$ are known to two-loops in perturbation theory (see for example [45]) and there is no obvious connection between them.

The underlying idea of this section is that the renormalisation coefficients give information about the divergence structure of the equation. By demanding that the divergent parts must be consistent, it is possible to narrow down the different possibilities allowed by the one-loop form of the identity (4.1.3). The general form of the full identity (assuming that it exists) is

$$G_x p_3^\mu \tilde{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu \tilde{\Gamma}_\mu(p_3,p_2,p_1) = G_y \tag{4.3.2}$$

where G_x and G_y are combinations of ghost propagator functions only. In principle, there may be vertex contributions multiplied by higher order terms too but this possibility will not be considered here. One may expect that the second vertex function above would be multiplied by some combination of propagator functions as well, but it is always possible to divide the whole equation by this combination. Each of the factors in the equation can be expressed in terms of renormalised quantities multiplied by renormalisation coefficients, using the definition of the renormalisation coefficients as follows[2]

$$\begin{aligned} G_x &= Z_x \overline{G}_x \\ G_y &= Z_y \overline{G}_y \\ \tilde{\Gamma}_\mu(p_1,p_2,p_3) &= \tilde{Z}_1^{-1} \overline{\tilde{\Gamma}}_\mu(p_1,p_2,p_3). \end{aligned} \tag{4.3.3}$$

Now, the renormalised functions are finite and so can be expanded as series in powers of the renormalised coupling, each of the terms in the series being finite. These series will be written as

$$\begin{aligned} p_3^\mu \overline{\tilde{\Gamma}}_\mu(p_1,p_2,p_3) &= p_1 \cdot p_3 + h\langle f_{123}^{(1)}\rangle + h^2\langle f_{123}^{(2)}\rangle + h^3\langle f_{123}^{(3)}\rangle \\ \overline{G}_x &= 1 + hG_x^{(1)} + h^2 G_x^{(2)} + h^3 G_x^{(3)} \end{aligned} \tag{4.3.4}$$

[2]It is understood that in the perturbative expansion of the unrenormalised quantities, the renormalised coupling is still used.

with a similar expansion for $\overline{G}_y$. The renormalisation coefficients are also expanded as a series in powers of the renormalised coupling, but this time, the terms in the series are explicitly divergent. Generically,

$$Z_i = 1 + hz_i^{(1)} + h^2 z_i^{(2)} + h^3 z_i^{(3)}. \tag{4.3.5}$$

It is now possible to write down the unrenormalised form of the identity (4.3.2) such that the divergent parts are made explicit.

$$Z_x \overline{G}_x \tilde{Z}_1^{-1} p_3^\mu \overline{\overline{\Gamma}}_\mu(p_1,p_2,p_3) - \tilde{Z}_1^{-1} p_1^\mu \overline{\overline{\Gamma}}_\mu(p_3,p_2,p_1) = Z_y \overline{G}_y \tag{4.3.6}$$

Both sides of this equation can be expanded in powers of the renormalised coupling. Up to $O(h^3)$, the left-hand side (lhs) will go like

$$\begin{aligned} lhs \;=\;& h\left\{\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle + p_1\cdot p_3\left(G_x^{(1)} + z_x^{(1)}\right)\right\} \\ &+h^2\left\{\langle f_{123}^{(2)}\rangle - \langle f_{321}^{(2)}\rangle + \langle f_{123}^{(1)}\rangle\left(G_x^{(1)} + z_x^{(1)}\right) + p_1\cdot p_3\left(G_x^{(2)} + G_x^{(1)}z_x^{(1)} + z_x^{(2)}\right)\right\} \\ &-h^2\tilde{z}_1^{(1)}\left\{\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle + p_1\cdot p_3\left(G_x^{(1)} + z_x^{(1)}\right)\right\} \\ &+h^3\left\{\langle f_{123}^{(3)}\rangle - \langle f_{321}^{(3)}\rangle + \langle f_{123}^{(2)}\rangle\left(G_x^{(1)} + z_x^{(1)}\right) + \langle f_{123}^{(1)}\rangle\left(G_x^{(2)} + G_x^{(1)}z_x^{(1)} + z_x^{(2)}\right)\right. \\ &\qquad\left. +p_1\cdot p_3\left(G_x^{(3)} + G_x^{(2)}z_x^{(1)} + G_x^{(1)}z_x^{(2)} + z_x^{(3)}\right)\right\} \\ &-h^3\tilde{z}_1^{(1)}\left\{\langle f_{123}^{(2)}\rangle - \langle f_{321}^{(2)}\rangle + \langle f_{123}^{(1)}\rangle\left(G_x^{(1)} + z_x^{(1)}\right) + p_1\cdot p_3\left(G_x^{(2)} + G_x^{(1)}z_x^{(1)} + z_x^{(2)}\right)\right\} \\ &+h^3(\tilde{z}_1^{(1)})^2\left\{\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle + p_1\cdot p_3\left(G_x^{(1)} + z_x^{(1)}\right)\right\} \\ &-h^3\tilde{z}_1^{(2)}\left\{\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle + p_1\cdot p_3\left(G_x^{(1)} + z_x^{(1)}\right)\right\}. \end{aligned} \tag{4.3.7}$$

The right-hand side can also be expanded in the same way. However, under the restriction that $\overline{G}_y$ is a combination of only ghost propagator functions, it must be independent of $\tilde{Z}_1$, otherwise there would be some relationship between this and $\tilde{Z}_3$. By demanding that the lhs is independent of $\tilde{Z}_1$, it is possible to gain a consistency requirement for the equation that is true for all momenta and gauges.

Consider then expression (4.3.7). At $O(h^2)$, there is a part that depends on $\tilde{z}_1^{(1)}$ which must vanish. In other words

$$\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle + p_1\cdot p_3\left(G_x^{(1)} + z_x^{(1)}\right) = 0. \tag{4.3.8}$$

This can be further split up into divergent and finite parts

$$\begin{aligned} z_x^{(1)} &= 0 \\ \langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle + p_1\cdot p_3 G_x^{(1)} &= 0 \end{aligned} \tag{4.3.9}$$

which now gives an explicit form for the (renormalised) one-loop factor $G_x^{(1)}$ and its renormalisation coefficient $z_x^{(1)}$. Substituting these back into the expression (4.3.7) gives

$$
\begin{aligned}
lhs &= h^2\left\{\langle f_{123}^{(2)}\rangle - \langle f_{321}^{(2)}\rangle + p_1\cdot p_3\left(G_x^{(2)} + z_x^{(2)}\right) - \frac{1}{p_1\cdot p_3}\langle f_{123}^{(1)}\rangle\left(\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle\right)\right\} \\
&+h^3\left\{\langle f_{123}^{(3)}\rangle - \langle f_{321}^{(3)}\rangle + p_1\cdot p_3\left(G_x^{(3)} + z_x^{(3)}\right) + \langle f_{123}^{(1)}\rangle G_x^{(2)} + \langle f_{321}^{(1)}\rangle z_x^{(2)}\right. \\
&\qquad\left. - \frac{1}{p_1\cdot p_3}\langle f_{123}^{(2)}\rangle\left(\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle\right)\right\} \\
&-h^3\tilde{z}_1\left\{\langle f_{123}^{(2)}\rangle - \langle f_{321}^{(2)}\rangle + p_1\cdot p_3\left(G_x^{(2)} + z_x^{(2)}\right) - \frac{1}{p_1\cdot p_3}\langle f_{123}^{(1)}\rangle\left(\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle\right)\right\}.
\end{aligned}
\tag{4.3.10}
$$

Again there is a part dependent on $\tilde{z}_1^{(1)}$ which must vanish. As before, this gives rise to the following constraints

$$
\begin{aligned}
z_x^{(2)} &= 0 \\
\langle f_{123}^{(2)}\rangle - \langle f_{321}^{(2)}\rangle + p_1\cdot p_3 G_x^{(2)} - \frac{1}{p_1\cdot p_3}\langle f_{123}^{(1)}\rangle\left(\langle f_{123}^{(1)}\rangle - \langle f_{321}^{(1)}\rangle\right) &= 0.
\end{aligned}
\tag{4.3.11}
$$

Notice that this actually makes the $O(h^2)$ part of the previous expression vanish. This means that up to $O(h^2)$ (two-loops), $\overline{G}_y = 0$.

In principle, this process of eliminating the $\tilde{Z}_1$-dependence of the lhs can be carried out to arbitrary order in the renormalised coupling h. However, in practice, it will only be possible to consider the $O(h^3)$ constraints due to the technical difficulties of higher-order loop corrections in perturbation theory.

All the preceding arguments have been carried out using the fact that the unrenormalised functions can be expressed as specific finite and divergent parts of a series in the renormalised coupling h. In practice though, when one calculates perturbative expressions, the expansion is done in the bare coupling (and with the bare gauge parameter). The process of renormalisation is done afterwards. Thus, for the constraint equations (4.3.9) and (4.3.11), the quantities appearing should be translated into the proper unrenormalised perturbative expressions. To see how this is done, consider a general Green's function F. Denote all the unrenormalised quantities with the subscript b, the renormalisation coefficient for F as Z, and the renormalisation coefficients for the coupling and gauge parameters as Z_g and Z_ξ respectively such that

$$
\overline{F} = ZF_b,\quad h_b = Z_g h,\quad \xi_b = Z_\xi\overline{\xi}
\tag{4.3.12}
$$

with the expansions

$$\begin{aligned}\overline{F} &= 1 + hF^{(1)}(\overline{\xi}) + h^2F^{(2)}(\overline{\xi}) + \ldots \\ Z &= 1 + hz^{(1)} + h^2z^{(2)} + \ldots \\ F_b &= 1 + h_bF_b^{(1)}(\xi_b) + h_b^2F_b^{(2)}(\xi_b) + \ldots\end{aligned}$$

The quantity that one derives in perturbation theory is F_b as an expansion in h_b and with the parameter ξ_b. To renormalise, the first step is to rewrite these bare parameters in terms of their renormalised counterparts multiplied by the appropriate renormalisation coefficients. This gives

$$F_b = 1 + Z_g hF_b^{(1)}(Z_\xi\overline{\xi}) + Z_g^2h^2F_b^{(2)}(Z_\xi\overline{\xi}) + \ldots \tag{4.3.13}$$

The renormalisation coefficients are then expanded and so

$$F_b = 1 + hF_b^{(1)}(\overline{\xi}) + h^2\left\{F_b^{(2)}(\overline{\xi}) + z_g^{(1)}F_b^{(1)}(\overline{\xi}) + F_b^{(1)}(z_\xi^{(1)}\overline{\xi})\right\} + \ldots \tag{4.3.14}$$

where the last term arises from the expansion of the gauge parameter. The next step in the renormalisation process is to then write the expansion for $\overline{F}$. It is

$$\overline{F} = 1+h\left\{F_b^{(1)}(\overline{\xi}) + z^{(1)}\right\}+h^2\left\{F_b^{(2)}(\overline{\xi}) + z_g^{(1)}F_b^{(1)}(\overline{\xi}) + F_b^{(1)}(z_\xi^{(1)}\overline{\xi}) + z^{(1)}F_b^{(1)}(\overline{\xi}) + z^{(2)}\right\}+\ldots \tag{4.3.15}$$

Finally, one identifies the coefficients z as those functions that remove the divergences (Z_g is some known combination of renormalisation functions so can be calculated as well). This then leaves the finite $F^{(n)}(\overline{\xi})$. Even without identifying the z's, one can now write

$$\begin{aligned}F^{(1)}(\overline{\xi}) &= F_b^{(1)}(\overline{\xi}) + z^{(1)} \\ F^{(2)}(\overline{\xi}) &= F_b^{(2)}(\overline{\xi}) + z_g^{(1)}F_b^{(1)}(\overline{\xi}) + F_b^{(1)}(z_\xi^{(1)}\overline{\xi}) + z^{(1)}F_b^{(1)}(\overline{\xi}) + z^{(2)}.\end{aligned} \tag{4.3.16}$$

It may appear that there is a problem with the gauge dependence of these functions. What is needed is to be able to write the renormalised functions at the appropriate order in terms of unrenormalised quantities. Although the dependence of the coupling in the $F^{(n)}(\overline{\xi})$ has been taken care of, they are still dependent on the renormalised gauge parameter $\overline{\xi}$. However, consider the meaning of each of the terms $F_b^{(n)}$. Where the argument is $\overline{\xi}$, this indicates that the function has simply had ξ_b replaced by $\overline{\xi}$. The only other remaining function is $F_b^{(1)}(z_\xi^{(1)}\overline{\xi})$. This is a one-loop term where only the parts dependent on the gauge parameter have been kept, and ξ_b has been replaced with $z_\xi^{(1)}\overline{\xi}$.

This term can easily be evaluated. Thus, the way to relate the functions appearing in the renormalised Green's function to those quantities which would be calculated when doing unrenormalised perturbation theory has been made explicit.

Now consider the first of the constraint equations (4.3.9). The first vertex term is translated in the following way[3] (the gauge dependence will be made explicit only where necessary from now on)

$$\langle f_{123}^{(1)} \rangle = \left(-\frac{\imath}{2} C_A\right) p_3^\mu \tilde{\Gamma}_\mu^{(1)}(p_1, p_2, p_3) + p_1 \cdot p_3 \tilde{z}_1^{(1)}. \quad (4.3.17)$$

and similarly for $\langle f_{321}^{(1)} \rangle$. The finite part of the combination of two-point functions becomes

$$G_x^{(1)} = G_{b,x}^{(1)} - z_x^{(1)} \quad (4.3.18)$$

with the minus sign coming from the different definition of the renormalisation coefficient. Thus, (4.3.9) is rewritten as

$$\left(-\frac{\imath}{2} C_A\right) p_3^\mu \tilde{\Gamma}_\mu^{(1)}(p_1, p_2, p_3) - \left(-\frac{\imath}{2} C_A\right) p_1^\mu \tilde{\Gamma}_\mu^{(1)}(p_3, p_2, p_1) + p_1 \cdot p_3 \left\{ G_{b,x}^{(1)} - z_x^{(1)} \right\} = 0. \quad (4.3.19)$$

However, it is already known that $z_x^{(1)} = 0$ so now

$$G_{b,x}^{(1)} = -\left(-\frac{\imath}{2} C_A\right) \frac{1}{p_1 \cdot p_3} \left\{ p_3^\mu \tilde{\Gamma}_\mu^{(1)}(p_1, p_2, p_3) - p_1^\mu \tilde{\Gamma}_\mu^{(1)}(p_3, p_2, p_1) \right\}. \quad (4.3.20)$$

The right-hand side of this equation is nothing but the one-loop identity already derived (4.1.3). Thus

$$G_{b,x}^{(1)} = -\left(-\frac{\imath}{2} C_A\right) \frac{1}{2 p_1 \cdot p_3} \left\{ p_1^2 \left(\tilde{G}_1^{(1)} - \tilde{G}_2^{(1)}\right) - p_3^2 \left(\tilde{G}_3^{(1)} - \tilde{G}_2^{(1)}\right) \right\}. \quad (4.3.21)$$

The second constraint equation (4.3.11) can be dealt with in exactly the same way. The finite two-loop vertex function will become

$$\begin{aligned} \langle f_{123}^{(2)} \rangle = & \ (-\imath C_A) p_3^\mu \hat{\Gamma}_\mu(p_1, p_2, p_3) + \left(-\frac{\imath}{2} C_A\right)^2 p_3^\mu \tilde{\Gamma}_\mu^{(2)}(p_1, p_2, p_3) \\ & + \left(z_g^{(1)} + \tilde{z}_1^{(1)}\right) \left(-\frac{\imath}{2} C_A\right) p_3^\mu \tilde{\Gamma}_\mu^{(1)}(p_1, p_2, p_3) + \left(-\frac{\imath}{2} C_A\right) p_3^\mu \tilde{\Gamma}_\mu^{(1)}(p_1, p_2, p_3 | z_\xi^{(1)} \xi) \\ & + p_1 \cdot p_3 \tilde{z}_1^{(2)}. \end{aligned} \quad (4.3.22)$$

The finite two-loop unknown combination of two-point functions becomes

$$G_x^{(2)} = G_{b,x}^{(2)} + \left(z_g^{(1)} - \tilde{z}_x^{(1)}\right) G_{b,x}^{(1)} + G_{b,x}^{(1)}(z_\xi^{(1)} \xi) - z_x^{(2)} + (z_x^{(1)})^2. \quad (4.3.23)$$

[3]Recall the earlier definitions. There is an overall factor of $(4\pi)^2$ that shall be omitted for clarity.

Putting these expressions into the equation (4.3.11), using (4.3.20) and cancelling any appropriate terms then gives

$$\begin{aligned} p_1\cdot p_3 G^{(2)}_{b,x} = &-(-\imath C_A)\left\{p_3^\mu \hat{\Gamma}_\mu(p_1,p_2,p_3) - p_1^\mu \hat{\Gamma}_\mu(p_3,p_2,p_1)\right\} \\ &-\left(-\frac{\imath}{2}C_A\right)^2\left\{p_3^\mu \tilde{\Gamma}^{(2)}_\mu(p_1,p_2,p_3) - p_1^\mu \tilde{\Gamma}^{(2)}_\mu(p_3,p_2,p_1)\right\} \\ &+\frac{1}{p_1\cdot p_3}\left(-\frac{\imath}{2}C_A\right)^2\left\{\left(p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1,p_2,p_3)\right)^2 - p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1,p_2,p_3)p_1^\mu \tilde{\Gamma}^{(1)}_\mu(p_3,p_2,p_1)\right\}. \end{aligned} \tag{4.3.24}$$

It is immediately apparent that the first two terms involving $\hat{\Gamma}$ are just the gluon self-energy insertions to the one-loop identity considered in the previous section. This part of the identity was explicitly shown to be satisfied so we can write

$$\begin{aligned} p_1\cdot p_3 \tilde{G}^{(2)}_{b,x} = &-\left(-\frac{\imath}{2}C_A\right)^2\left\{p_3^\mu \tilde{\Gamma}^{(2)}_\mu(p_1,p_2,p_3) - p_1^\mu \tilde{\Gamma}^{(2)}_\mu(p_3,p_2,p_1)\right\} \\ &+\frac{1}{p_1\cdot p_3}\left(-\frac{\imath}{2}C_A\right)^2\left\{\left(p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1,p_2,p_3)\right)^2 - p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1,p_2,p_3)p_1^\mu \tilde{\Gamma}^{(1)}_\mu(p_3,p_2,p_1)\right\}. \end{aligned} \tag{4.3.25}$$

where now $\tilde{G}^{(2)}_{b,x}$ is some combination of ghost propagator functions with the gluon self-energy insertions omitted.

The rest of this chapter will be devoted to the task of finding out whether or not $\tilde{G}^{(2)}_{b,x}$ can be expressed in terms of ghost propagator functions alone. If it cannot, then the identity cannot be true in this form.

4.4 The Two-Loop Ghost Propagator in Feynman Gauge

In order to proceed, it is now necessary to evaluate the two-loop ghost propagator explicitly. From now onwards, attention will be focussed on the Feynman gauge. Whilst it would be desirable to do an arbitrary gauge calculation, the technical details are prohibitively complex. Recalling the expansion of the ghost propagator function,

$$G_p = 1 + (-\frac{\imath}{2}g^2C_A)\tilde{G}^{(1)}_p + (-\imath g^4 C_A)\hat{G}_p + (-\frac{\imath}{2}g^2C_A)^2\tilde{G}^{(2)}_p \tag{4.4.1}$$

it easy to calculate the inverse function

$$G_p^{-1} = 1 - (-\frac{\imath}{2}g^2C_A)\tilde{G}^{(1)}_p - (-\imath g^4 C_A)\hat{G}_p - (-\frac{\imath}{2}g^2C_A)^2\left[\tilde{G}^{(2)}_p - \left(\tilde{G}^{(1)}_p\right)^2\right]. \tag{4.4.2}$$

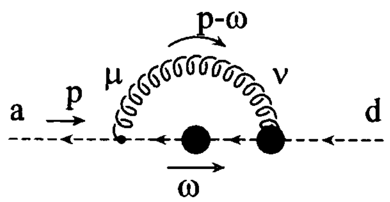


Figure 4.1: The relevant parts of the loop integral of the ghost propagator Schwinger-Dyson equation. The blobs represent one-loop insertions.

Earlier, the gluon self-energy insertions to the one-loop identity were explicitly dealt with. Now the interest lies in the remainder of the two-loop propagator function, $\tilde{G}_p^{(2)}$. This is calculated from the Schwinger-Dyson equation just as before. The relevant terms in the equation are shown in fig. 4.1. From this, and recalling the forms of the full equation (3.2.1) and one-loop insertions, it is simple to write down the appropriate expression

$$-\delta^{ad}p^2(-\frac{\imath}{2}g^2C_A)^2\left[\tilde{G}_p^{(2)}-\left(\tilde{G}_p^{(1)}\right)^2\right]=$$
$$-(-\imath)(-\imath gf^{abc})(-\imath gf^{bdc})(-\frac{\imath}{2}g^2C_A)\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{p{\cdot}\omega\tilde{G}_\omega^{(1)}+p^\nu\tilde{\Gamma}_\nu^{(1)}(\omega,-p,p-\omega)\right\}. \tag{4.4.3}$$

Doing the colour algebra and cancelling factors leaves

$$p^2\left[\tilde{G}_p^{(2)}-\left(\tilde{G}_p^{(1)}\right)^2\right]=2\int\frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2}\left\{p{\cdot}\omega\tilde{G}_\omega^{(1)}+p^\nu\tilde{\Gamma}_\nu^{(1)}(\omega,-p,p-\omega)\right\}. \tag{4.4.4}$$

One may imagine that for the one-loop vertex corrections, it would be possible to insert the explicit expressions for the two parts of the vertex function (3.4.6) and then integrate over ω. However, the appearance of kinematical factors, especially the Gram determinant Δ means that the integrals cannot be done in a simple manner. It is better to derive a form for the one-loop vertex which does not evaluate the integrals. This will then give for the above expression a set of two-loop integrals which can be evaluated in a reasonably straightforward way. Thus consider the expression (3.4.2) in Feynman gauge,

$$\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)=\int\frac{\bar{d}^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}\times$$
$$\left\{p_{1\alpha}(p_1-\omega)_\rho g^{\alpha\beta}g^{\rho\nu}\Gamma^{tl}_{\beta\nu\mu}(\omega,-p_3-\omega,p_3)+p_{1\alpha}\omega_\mu(p_3+\omega)_\beta g^{\alpha\beta}\right\}. \tag{4.4.5}$$

Now, in the next section the one-loop ghost-gluon scattering-like kernel $\tilde{\Gamma}_{\alpha\mu}^{(1)}(p_1,p_2,p_3)$ will be needed. This quantity is just the same as the vertex function above but with the bare vertex term connected to the out-ghost line omitted. The reason that this can be done is that this factor is simply the external momentum – in this case just $p_{1\alpha}$. The omission of this factor gives

$$\widetilde{\Gamma}^{(1)}_{\alpha\mu}(p_1, p_2, p_3) = \int \frac{đ^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} \left\{(p_1-\omega)^\nu \Gamma^{tl}_{\alpha\nu\mu}(\omega, -p_3-\omega, p_3) + \omega_\mu(p_3+\omega)_\alpha\right\}. \tag{4.4.6}$$

Expanding out the tree-level triple-gluon vertex function $\Gamma^{tl}_{\alpha\nu\mu}$ and collecting terms together leads to

$$\tilde{\Gamma}^{(1)}_{\alpha\mu}(p_1,p_2,p_3) = \int \frac{\bar{d}^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} \left\{ \frac{1}{2} g_{\alpha\mu} \left[p_2^2 - 2p_1^2 + (p_1-\omega)^2 - (p_3+\omega)^2 + 2\omega^2 \right] \right.$$
$$\left. + p_{1\alpha} p_{3\mu} - 2p_{3\alpha} p_{1\mu} + (p_3 - 2p_2)_\alpha \omega_\mu + p_{2\mu}\omega_\alpha \right\}. \tag{4.4.7}$$

It is now useful to adopt the notation of appendices A.3 and A.4 for the scalar and vector triangle integrals, and the notation of section 3.4 for the two-point integrals. The above expression becomes

$$\tilde{\Gamma}^{(1)}_{\alpha\mu}(p_1,p_2,p_3) = \frac{1}{2} g_{\alpha\mu} \left[I_{p_3} + 2I_{p_2} - I_{p_1} \right] + I(p_1,p_2,p_3) \left[\frac{1}{2} g_{\alpha\mu}(2p_2^2 - p_1^2) + p_{1\alpha}p_{3\mu} - 2p_{3\alpha}p_{1\mu} \right]$$
$$+ I(\mu;p_1,p_2,p_3)(p_3 - 2p_2)_\alpha + I(\alpha;p_1,p_2,p_3)p_{2\mu}. \tag{4.4.8}$$

The vertex function $\tilde{\Gamma}^{(1)}_{\mu}(p_1,p_2,p_3)$ can be derived from $\tilde{\Gamma}^{(1)}_{\alpha\mu}(p_1,p_2,p_3)$ by contraction with $p_{1\alpha}$. The only complication for this is the appearance of the contracted integral $p_{1\alpha} I(\alpha;p_1,p_2,p_3)$. However, this can be expanded quite easily by considering the following

$$\begin{aligned} p_{1\alpha} I(\alpha;p_1,p_2,p_3) &= \int \frac{\bar{d}^d v \; p_1 \cdot v}{v^2(p_1-v)^2(p_3+v)^2} \\ &= -\frac{1}{2} \int \frac{\bar{d}^d v}{v^2(p_1-v)^2(p_3+v)^2} \left[(p_1-v)^2 - p_1^2 - v^2 \right] \\ &= -\frac{1}{2} \left[I_{p_3} - p_1^2 I(p_1,p_2,p_3) - I_{p_2} \right]. \end{aligned} \tag{4.4.9}$$

Using this, the contraction becomes

$$\begin{aligned} \tilde{\Gamma}^{(1)}_{\mu}(p_1,p_2,p_3) &= \frac{1}{2} p_{1\mu} \left[I_{p_3} + 2I_{p_2} - I_{p_1} \right] \\ &+ \frac{1}{2} p_{2\mu} \left[I_{p_2} - I_{p_3} \right] + \frac{1}{2} I(p_1,p_2,p_3) \left[p_{1\mu}(2p_3^2 - p_1^2 - p_2^2) + p_1^2 p_{3\mu} \right] \\ &+ \frac{1}{2} I(\mu;p_1,p_2,p_3) \left[p_1^2 + 3p_2^2 - 3p_3^2 \right] \end{aligned} \tag{4.4.10}$$

which is precisely the form that is required for this section.

Inserting the one-loop vertex function and identifying the one-loop Feynman gauge propagator using (3.2.6) as

$$\tilde{G}^{(1)}_{\omega} = I_\omega \tag{4.4.11}$$

gives for the two-loop part of the inverse ghost propagator

$$p^2\left[\tilde{G}_p^{(2)} - \left(\tilde{G}_p^{(1)}\right)^2\right] = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2}\Big\{p^\nu I(\nu;\omega,-p,p-\omega)(3p^2-3(p-\omega)^2+\omega^2)$$
$$+I(\omega,-p,p-\omega)\left(\omega^2(p^2-2p\cdot\omega)+2p\cdot\omega(p-\omega)^2-p^2p\cdot\omega\right)$$
$$+I_p(2p\cdot\omega-p^2)+I_\omega p\cdot\omega+I_{p-\omega}(p^2+p\cdot\omega)\Big\}. \quad (4.4.12)$$

Using the integral transforms of appendices A.3 and A.4, it is possible to permute the order of the arguments occurring in the one-loop triangle integrals. This then allows the expansion of the contracted vector integral just as before.

$$\begin{aligned} I(\omega,-p,p-\omega) &= I(\omega,p-\omega,-p) \\ p^\nu I(\nu;\omega,-p,p-\omega) &= p^\nu\left(\omega_\nu I(\omega,p-\omega,-p)-I(\nu;\omega,p-\omega,-p)\right) \\ &= p\cdot\omega I(\omega,p-\omega,-p)-\int\frac{d\!\!{}^-{}^d v\; p\cdot v}{v^2(\omega-v)^2(p-v)^2} \\ &= \left(p\cdot\omega-\frac{1}{2}p^2\right)I(\omega,p-\omega,-p)+\frac{1}{2}I_\omega-\frac{1}{2}I_{p-\omega} \quad (4.4.13)\end{aligned}$$

Inserting the above into (4.4.12) gives an expression involving both I_ω and $I_{p-\omega}$. It is trivial to change variables $\omega \to p-\omega$ on those terms involving $I_{p-\omega}$. Doing this and expanding the scalar product $p\cdot\omega$ then leads to

$$p^2\left[\tilde{G}_p^{(2)} - \left(\tilde{G}_p^{(1)}\right)^2\right] = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2}\Big\{I(\omega,p-\omega,-p)\left(-\frac{1}{2}\omega^4+p^2\omega^2+\frac{1}{2}(p-\omega)^4-\frac{1}{2}p^4\right)$$
$$+I_p\left(\omega^2-(p-\omega)^2\right)+I_\omega\left(2\omega^2-2(p-\omega)^2+2p^2\right)\Big\}. \quad (4.4.14)$$

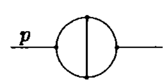
Notice that changing variable $\omega \to p-\omega$ on the term involving $(p-\omega)^4$ gives a cancellation with the term involving ω^4 since the scalar triangle integral is invariant. Also, under the framework of dimensional regularisation , the terms involving both I_p and $I_\omega(p-\omega)^2$ vanish. Thus

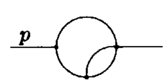
$$p^2\left[\tilde{G}_p^{(2)} - \left(\tilde{G}_p^{(1)}\right)^2\right] = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2}\Big\{I(\omega,p-\omega,-p)\left(p^2\omega^2-\frac{1}{2}p^4\right)$$
$$+I_\omega\left(2\omega^2+2p^2\right)\Big\}. \quad (4.4.15)$$

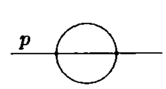
Writing out the integrals explicitly

$$p^2\left[\tilde{G}_p^{(2)} - \left(\tilde{G}_p^{(1)}\right)^2\right] = -\frac{1}{2}p^4\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{\omega^2(p-\omega)^2v^2(p-v)^2(\omega-v)^2}$$
$$+p^2\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{(p-\omega)^2v^2(p-v)^2(\omega-v)^2}$$
$$+2\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{(p-\omega)^2v^2(\omega-v)^2} + 2p^2\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{\omega^2(p-\omega)^2v^2(\omega-v)^2}. \tag{4.4.16}$$

The second and fourth of these integrals are in fact the same (related on by a change of variables). These two-loop propagator integrals are presented in appendix C. It is useful to introduce a diagrammatic notation for the integrals because in the next section the explicit forms will simply become too big to be read easily.

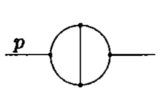
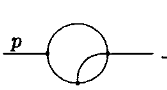
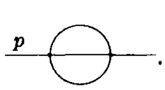
$$\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{\omega^2(p-\omega)^2v^2(p-v)^2(\omega-v)^2} =$$


$$\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{\omega^2(p-\omega)^2v^2(\omega-v)^2} =$$


$$\int\frac{d\!\!{}^-{}^d\omega\, d\!\!{}^-{}^d v}{(p-\omega)^2v^2(\omega-v)^2} =$$


(4.4.17)

This gives the expression for the two-loop function $\tilde{G}_p^{(2)}$

$$\tilde{G}_p^{(2)} = \left(\tilde{G}_p^{(1)}\right)^2 - \frac{1}{2}p^2$$

$$+\,3$$

$$+\,\frac{2}{p^2}$$


(4.4.18)

This type of expression will be used heavily in the next section. Expanding out the integrals in powers of ε and setting $\varepsilon=0$ gives

$$\begin{aligned}\tilde{G}_p^{(2)} &= \left(\tilde{G}_p^{(1)}\right)^2 \\ &\quad +(4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{-\frac{3}{2\varepsilon^2}+(3\gamma-7)\frac{1}{\varepsilon}-\frac{101}{4}+14\gamma-3\gamma^2+\frac{\pi^2}{4}+3\zeta_3+O(\varepsilon)\right\} \\ &= (4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{-\frac{5}{2\varepsilon^2}+(5\gamma-11)\frac{1}{\varepsilon}-\frac{149}{4}+22\gamma-5\gamma^2+\frac{5\pi^2}{12}+3\zeta_3+O(\varepsilon)\right\}.\end{aligned} \tag{4.4.19}$$

4.5 Two-Loop Vertex Contributions to the Identity in Feynman Gauge

In this section will be presented the calculation of the two-loop quantity $p_3^\mu\tilde{\Gamma}_\mu^{(2)}(p_1,p_2,p_3) - p_1^\mu\tilde{\Gamma}_\mu^{(2)}(p_3,p_2,p_1)$. Due to the highly complicated nature of such a two-loop calculation, it

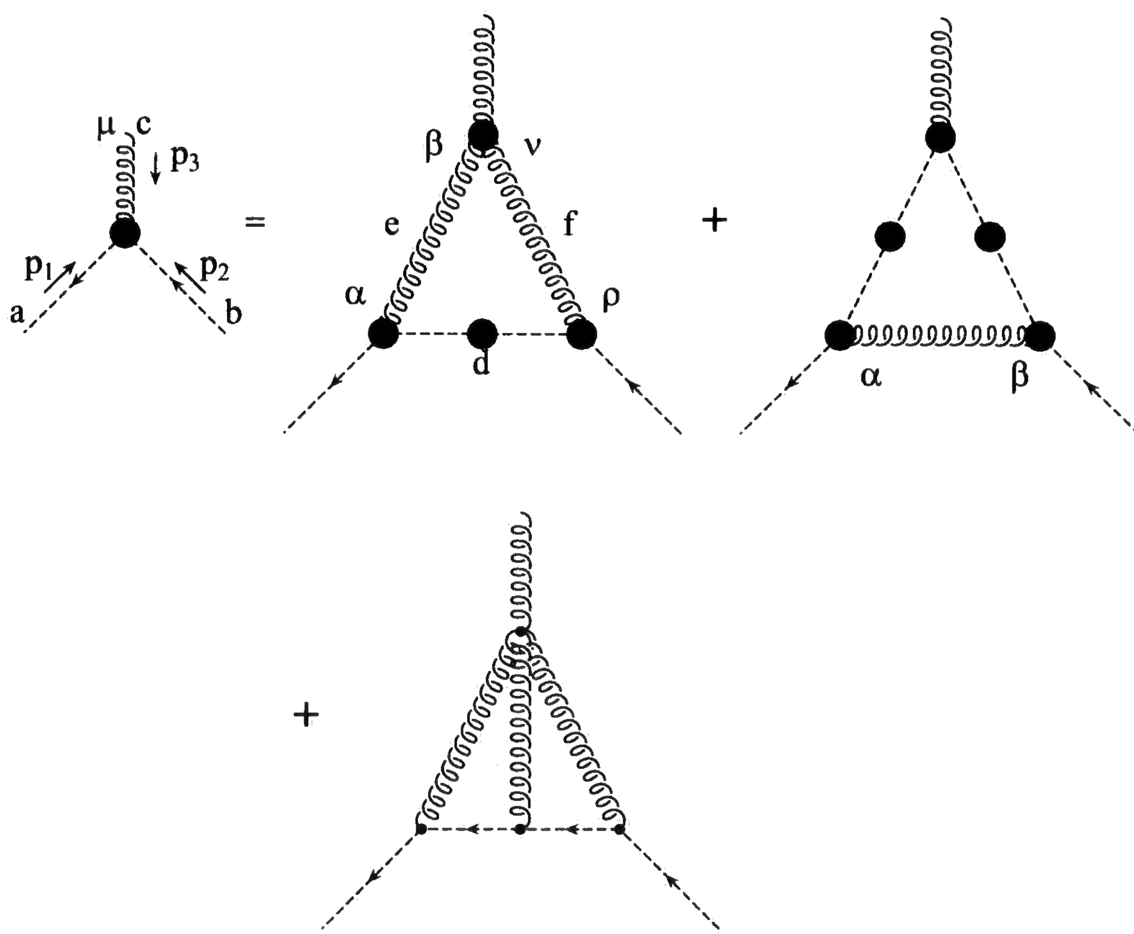


Figure 4.2: The ghost-gluon vertex at two-loops. The blobs represent the insertion of one-loop Green's functions but with the gluon self-energy parts omitted. The momentum routing for the first two graphs is the same as in the one-loop case. The colour factors, Lorentz indices and momentum routing of the third graph will be presented later.

is necessary to restrict to Feynman gauge and the momentum configuration $p_1 = p_2 = p$. This means that there is only one momentum scale and the integrals simplify greatly. In order to do this, consider the Feynman graphs of fig. 4.2. Note that non-planar graphs do not contribute because their colour factor is zero [46]. The quantity shown is the two-loop part of the vertex $\widetilde{\Gamma}^{abd}_\mu(p_1, p_2, p_3)$. It is simpler to split this up into smaller factors as follows

$$\begin{aligned} p_3^\mu \widetilde{\Gamma}^{abd}_\mu(p_1, p_2; p_3) &\rightarrow -\imath g f^{abc}\left(-\frac{\imath}{2}g^2 C_A\right)^2 p_3^\mu \widetilde{\Gamma}^{(2)}_\mu(p_1, p_2; p_3) \\ &= -\imath g f^{abc}\left(-\frac{\imath}{2}g^2 C_A\right)^2 (V_1 + V_2 + V_3) \end{aligned} \tag{4.5.1}$$

with the V_i denoting the contributions from each graph of fig. 4.2 in turn. Due to the similarity in form of the first two graphs, they will be considered together. The third graph (V_3) will be considered at the end. The algebra for this calculation has been done using FORM.

Consider then the contributions of V_1 and V_2. Writing out the expressions and doing the colour algebra (which is identical to that of the one-loop case) immediately gives

$$\begin{aligned} V_1 + V_2 = \int & \frac{\bar{d}^d\omega}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} \times \\ & \Big\{ p_3^\mu \Gamma^{tl}_{\beta\nu\mu}(\omega, -p_3-\omega, p_3)\Big[(p_1-\omega)^\nu \widetilde{\Gamma}^{(1)\beta}(p_1, \omega - p_1; -\omega) + p_1^\beta \widetilde{\Gamma}^{(1)\nu}(p_1-\omega, p_2, p_3+\omega) \\ & \qquad + p_1^\beta (p_1-\omega)^\nu \left(\tilde{G}^{(1)}_{p_1-\omega} + \tilde{G}^{(1)}_{p_3}\right)\Big] \\ & + p_3\cdot\omega \Big[(p_3+\omega)^\alpha \widetilde{\Gamma}^{(1)}_\alpha(p_1, -\omega; \omega - p_1) + p_1^\beta \widetilde{\Gamma}^{(1)}_\beta(p_3+\omega, p_2, p_1-\omega) \\ & \qquad + p_1\cdot(p_3+\omega)\left(\tilde{G}^{(1)}_{p_3+\omega} + \tilde{G}^{(1)}_\omega\right)\Big] \\ & + p_1\cdot(p_3+\omega) p_3^\nu \widetilde{\Gamma}^{(1)}_\mu(\omega, -p_3-\omega, p_3) \\ & + (p_3+\omega)^2 (p_1-\omega)^\nu t^\lambda_\nu(p_3+\omega) p_1^\beta \widetilde{\Gamma}_{\lambda\beta}(-p_3-\omega, p_3, \omega) \\ & - \omega^2 p_1^\beta t^\lambda_\beta(\omega)(p_1-\omega)^\nu \widetilde{\Gamma}_{\lambda\nu}(\omega, p_3, -p_3-\omega)\Big\}. \end{aligned} \tag{4.5.2}$$

Notice the occurrence of the function $\widetilde{\Gamma}_{\nu\mu}$ which comes from the use of the triple-gluon vertex Slavnov-Taylor identity (1.7.6).

Now consider the second two-loop vertex function. Denoting

$$p_1^\mu \widetilde{\Gamma}^{(2)}_\mu(p_3, p_2, p_1) = V_1' + V_2' + V_3' \tag{4.5.3}$$

it is simple to write down $V_1 + V_2 - V_1' - V_2'$ since V_1' and V_2' are just related to V_1 and V_2 by the interchange of $p_1 \leftrightarrow p_3$ and $\omega \rightarrow -\omega$. The next step is to set $p_1 = p_2 = p$, $p_3 = -2p$.

The result is[4]

$$
\begin{aligned}
&[V_1 + V_2 - V_1' - V_2']_{p_1=p_2=p} = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2(2p-\omega)^2} \times \\
&\Big\{-(2p-\omega)^2 p^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(2p-\omega,-2p;\omega) - 2(p-\omega)^2 p^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(\omega-p,p;-\omega) \\
&-w^2(p-\omega)^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(\omega,-2p;2p-\omega) + 2w^2(2p-\omega)^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(-\omega,p;\omega-p) \\
&+p\cdot(p+\omega)\omega^\mu \widetilde{\Gamma}^{(1)}_\mu(-2p,2p-\omega;\omega) + 2p^2\omega^\mu \widetilde{\Gamma}^{(1)}_\mu(p-\omega,p;\omega-2p) \\
&+2p^2\omega^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega-2p,p;p-\omega) - 2p\cdot(p-\omega)\omega^\mu \widetilde{\Gamma}^{(1)}_\mu(p,\omega-p;-\omega) \\
&-p\cdot\omega(\omega-2p)^\mu \widetilde{\Gamma}^{(1)}_\mu(-2p,\omega;2p-\omega) - p\cdot\omega(\omega-2p)^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega,-2p;2p-\omega) \\
&+2p\cdot\omega(p-\omega)^\mu \widetilde{\Gamma}^{(1)}_\mu(-\omega,p;\omega-p) + 2p\cdot\omega(p-\omega)^\mu \widetilde{\Gamma}^{(1)}_\mu(p,-\omega;\omega-p) \\
&-p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(-2p,\omega;2p-\omega) - \omega\cdot(p+\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(-2p,2p-\omega;\omega) \\
&+2p\cdot(p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(-\omega,\omega-p;p) + 2p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(-\omega,p;\omega-p) \\
&-p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega,-2p;2p-\omega) + 2p\cdot(2p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega,2p-\omega;-2p) \\
&-4p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(p-\omega,p;\omega-2p) + p\cdot(2p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(2p-\omega,-2p;\omega) \\
&-4p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega-2p,p;p-\omega) - 2p\cdot(p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega-p,p;-\omega) \\
&+2p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(p,-\omega;\omega-p) + 2\omega\cdot(p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(p,\omega-p;-\omega) \\
&+2p^2 p\cdot\omega \tilde{G}^{(1)}_\omega + \left(4p\cdot\omega^2 - 2p^2\omega^2 - 2p^2 p\cdot\omega\right)\tilde{G}^{(1)}_{p-\omega} + \left(-4p\cdot\omega^2 + 2p^2\omega^2 + 4p^2 p\cdot\omega\right)\tilde{G}^{(1)}_{2p-\omega} \\
&+2\left(p^2\omega^2 - p\cdot\omega^2\right)\left(\tilde{G}_p - \tilde{G}_{2p}\right)\Big\}. \qquad (4.5.4)
\end{aligned}
$$

The first thing to notice about this expression is the invariance of the denominator factors under the change of variable $\omega \to 2p-\omega$. This means that it is possible to cancel certain terms in the numerator. This is done to all the one-loop vertex terms that have $2p-\omega$ as the last argument except the term

$$p\cdot\omega(\omega-2p)^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega,-2p;2p-\omega) \qquad (4.5.5)$$

which will be dealt with later. The change of variable is also used on the two following terms

$$
\begin{gathered}
p^2\omega^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega-2p,p;p-\omega) \\
p\cdot(2p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(2p-\omega,-2p;\omega)
\end{gathered}
$$

for reasons that will become apparent.

[4]One can now appreciate the magnitude of the calculation!

The expression is now

$$
\begin{aligned}
&[V_1 + V_2 - V_1' - V_2']_{p_1=p_2=p} = \int \frac{đ^d\omega}{\omega^2(p-\omega)^2(2p-\omega)^2}\times \\
&\Big\{3p^2\omega^\mu\tilde{\Gamma}^{(1)}_\mu(-2p,2p-\omega;\omega) + 2p\cdot(2p-\omega)p^\mu\tilde{\Gamma}^{(1)}_\mu(\omega,2p-\omega;-2p) \\
&-2p^2\omega^\mu\tilde{\Gamma}^{(1)}_\mu(\omega-p,p;-\omega) + 2p\cdot(p+\omega)(p-\omega)^\mu\tilde{\Gamma}^{(1)}_\mu(-\omega,p;\omega-p) \\
&+2p\cdot(p-\omega)p^\mu\tilde{\Gamma}^{(1)}_\mu(-\omega,\omega-p;p) - 2p\cdot(p-\omega)\omega^\mu\tilde{\Gamma}^{(1)}_\mu(p,\omega-p;-\omega) \\
&-2(p-\omega)^2p^\mu p^\nu\tilde{\Gamma}^{(1)}_{\nu\mu}(\omega-p,p;-\omega) - (2p-\omega)^2\omega^\mu p^\nu\tilde{\Gamma}^{(1)}_{\nu\mu}(2p-\omega,-2p;\omega) \\
&+2w^2(2p-\omega)^\mu p^\nu\tilde{\Gamma}^{(1)}_{\nu\mu}(-\omega,p;\omega-p) - p\cdot\omega(\omega-2p)^\mu\tilde{\Gamma}^{(1)}_\mu(\omega,-2p;2p-\omega) \\
&+2p\cdot\omega(p-\omega)^\mu\tilde{\Gamma}^{(1)}_\mu(p,-\omega;\omega-p) - (2p^2+\omega^2)p^\mu\tilde{\Gamma}^{(1)}_\mu(-2p,2p-\omega;\omega) \\
&-6p\cdot(p-\omega)p^\mu\tilde{\Gamma}^{(1)}_\mu(-\omega,p;\omega-p) - 6p\cdot(p-\omega)p^\mu\tilde{\Gamma}^{(1)}_\mu(\omega-p,p;-\omega) \\
&+2p\cdot\omega p^\mu\tilde{\Gamma}^{(1)}_\mu(p,-\omega;\omega-p) + 2\omega\cdot(p-\omega)p^\mu\tilde{\Gamma}^{(1)}_\mu(p,\omega-p;-\omega) \\
&+2p^2p\cdot\omega\tilde{G}^{(1)}_\omega + \left(4p\cdot\omega^2 - 2p^2\omega^2 - 2p^2p\cdot\omega\right)\tilde{G}^{(1)}_{p-\omega} + \left(-4p\cdot\omega^2 + 2p^2\omega^2 + 4p^2p\cdot\omega\right)\tilde{G}^{(1)}_{2p-\omega} \\
&+2\left(p^2\omega^2 - p\cdot\omega^2\right)\left(\tilde{G}_p - \tilde{G}_{2p}\right)\Big\}. \qquad (4.5.6)
\end{aligned}
$$

Notice that in the first three lines of the above, the vertices have their first and last arguments interchanged. This means that the one-loop identity (4.1.3) can be used to get rid of some of the vertex functions in favour of two-point functions. This is done to obtain yet another large expression. However, by again using the invariance of the denominator under $\omega \to 2p-\omega$, it is possible to eliminate the function $\tilde{G}^{(1)}_{2p-\omega}$. The (slightly smaller!) result is

$$[V_1 + V_2 - V_1' - V_2']_{p_1=p_2=p} = \int \frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2(2p-\omega)^2} \times$$
$$\Big\{-2(p-\omega)^2 p^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(\omega-p,p;-\omega) - (2p-\omega)^2\omega^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(2p-\omega,-2p;\omega)$$
$$+2w^2(2p-\omega)^\mu p^\nu \widetilde{\Gamma}^{(1)}_{\nu\mu}(-\omega,p;\omega-p) - p\cdot\omega(\omega-2p)^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega,-2p;2p-\omega)$$
$$+2p\cdot\omega(p-\omega)^\mu \widetilde{\Gamma}^{(1)}_\mu(p,-\omega;\omega-p) - (2p^2+\omega^2)p^\mu \widetilde{\Gamma}^{(1)}_\mu(-2p,2p-\omega;\omega)$$
$$-6p\cdot(p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(-\omega,p;\omega-p) - 6p\cdot(p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(\omega-p,p;-\omega)$$
$$+2p\cdot\omega p^\mu \widetilde{\Gamma}^{(1)}_\mu(p,-\omega;\omega-p) + 2\omega\cdot(p-\omega)p^\mu \widetilde{\Gamma}^{(1)}_\mu(p,\omega-p;-\omega)$$
$$\left(-4p\cdot\omega^2 + 2p^2\omega^2 - \omega^2 p\cdot\omega\right)\tilde{G}^{(1)}_\omega + \left(2p^4 + 4p\cdot\omega^2 - 2p^2\omega^2 - 5p^2 p\cdot\omega + \omega^2 p\cdot\omega\right)\tilde{G}^{(1)}_{p-\omega}$$
$$+\left(-2p^4 - 2p\cdot\omega^2 + 2p^2\omega^2 + 3p^2 p\cdot\omega\right)\tilde{G}^{(1)}_p + \left(6p^4 + 2p\cdot\omega^2 - 2p^2\omega^2\right)\tilde{G}^{(1)}_{2p}\Big\}. \quad (4.5.7)$$

It is now necessary to expand the one-loop ghost-gluon scattering-like kernel and vertex functions into their integral form. This is done using the expressions (4.4.8) and (4.4.10) from the last section[5]. The two-point function $\tilde{G}^{(1)}_x$ is again identified as the integral I_x The result of this is a plethora of terms of the form

$$\int \frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2(2p-\omega)^2} I(?)(?) \quad (4.5.8)$$

where the question mark denotes some combination of Lorentz indices and momentum arguments. In order to deal with the various combinations, the order of the arguments in the triangle integrals $I(p_1, p_2, p_3)$ (both scalar and vector) are transformed into two distinct permutations using the techniques of appendices A.3 and A.4. These permutations are, in the scalar case:

$$\begin{aligned} I(p,\omega-p,-\omega) &= \int \frac{\bar{d}^d v}{v^2(p-v)^2(\omega-v)^2} \\ I(\omega,2p-\omega,-2p) &= \int \frac{\bar{d}^d v}{v^2(\omega-v)^2(2p-v)^2} \end{aligned} \quad (4.5.9)$$

and similarly for the vector case. The vector triangle integrals are contracted with either p or ω and this is dealt with in exactly the same way as (4.4.9) by expanding out the scalar product in the numerator. This then leaves a manageable number of terms in the overall expression. This is again minimised by exploiting the invariance of the denominator factors under the change of variable $\omega \to 2p-\omega$ to eliminate $I_{2p-\omega}$. The result of all these

[5]Because of the very large nature of the expressions involved, they will not be presented in full.

operations is:

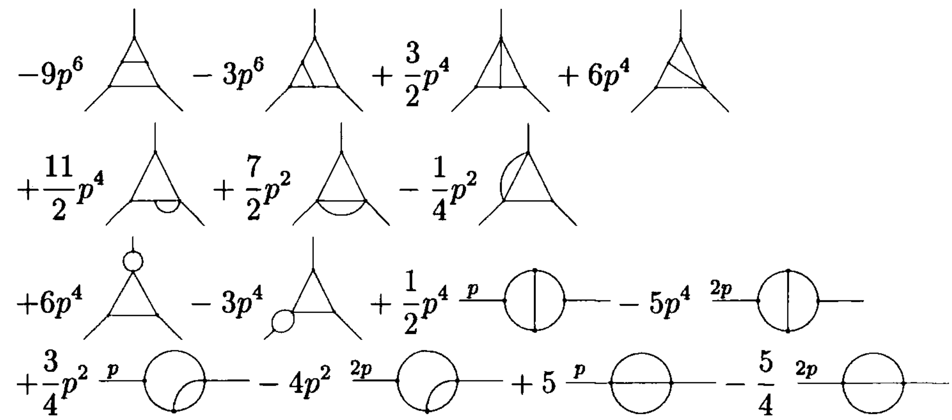
$$[V_1 + V_2 - V_1' - V_2']_{p_1=p_2=p} = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2(2p-\omega)^2} \times$$
$$\Big\{I(\omega, 2p-\omega, -2p)p^2\left(4p^2\omega^2 - 4p\cdot\omega^2 - 4\omega^2 p\cdot\omega + 2\omega^4\right)$$
$$+I(p, \omega - p, -\omega)p^2\left(4p^2\omega^2 + 2p\cdot\omega^2 + \omega^2 p\cdot\omega - 2\omega^4 - 5p^2 p\cdot\omega\right)$$
$$+I_\omega\left(3p^2\omega^2 + 15p^2 p\cdot\omega - 10p\cdot\omega^2\right) + I_{p-\omega}\left(4p^4 + 4p\cdot\omega^2 - 6p^2\omega^2 + 3\omega^2 p\cdot\omega - 5p^2 p\cdot\omega\right)$$
$$+I_p p^2\left(2\omega^2 - 2p\cdot\omega\right) + I_{2p}\left(6p^4 + 4p\cdot\omega^2 - 2\omega^2 p\cdot\omega - 6p^{2p\omega}\right)\Big\}. \quad (4.5.10)$$

The next step is to eliminate the numerator structure of the expression. This done by cancelling firstly ω^2 factors in the numerator and denominator, then recursively applying the identities

$$\begin{aligned} p\cdot\omega &= \frac{1}{2}\left((p-\omega)^2 - (2p-\omega)^2 + 3p^2\right) \\ \omega^2 &= 2p^2 + 2(p-\omega)^2 - (2p-\omega)^2 \end{aligned} \quad (4.5.11)$$

and cancelling again until there are no factors involving ω in the numerator. Those integrals with no external scale (and those related to them by some change of variable) vanish under the framework of dimensional regularisation and are dropped. At this point the useful notation I for the two-point and vector (sub-)integrals is dropped and the integrals written out explicitly. The resulting (still large) expression is a set of two-loop integrals. These integrals can be written in more than one way, although the different forms are related by changes of variable. It is thus better to express the integral in a diagrammatic fashion as was the case in the last section. All the two-loop propagator type integrals are presented in appendix C, the two-loop vertex type integrals in appendix D. The expression can be written as

$$[V_1 + V_2 - V_1' - V_2']_{p_1=p_2=p} =$$



$$+2p^2 \left(I_{2p}\right)^2 + 3p^2 \left(I_p\right)^2 - 5p^2 I_p I_{2p}. \tag{4.5.12}$$

It is not difficult to substitute in the value of each of the integrals as an expansion in ε, using the results of appendices A.1, C and D. Where there is a factor like $(-4p^2)^{-2\varepsilon}$, one uses the following:

$$(-4p^2)^{-2\varepsilon} = (-p^2)^{-\varepsilon}\left[1 - 4\varepsilon \ln 2 + 8\varepsilon^2 \ln^2 2 + \ldots\right]. \tag{4.5.13}$$

Doing this immediately gives the final result for this part of the calculation.

$$[V_1 + V_2 - V_1' - V_2']_{p_1=p_2=p} = (4\pi)^{-d}(-p^2)^{-2\varepsilon} p^2 \ln 2 \left\{-\frac{20}{\varepsilon} + 40\gamma + 44 \ln 2 - 84 + O(\varepsilon)\right\}. \tag{4.5.14}$$

Now consider the third graph for the two-loop ghost-gluon vertex in Feynman gauge. The expression for this can be written down by considering fig. 4.3. It is

$$-\imath g f^{abc}\left(-\frac{\imath}{2} g^2 C_A\right)^2 V_3 = (-\imath g f^{adh_1})(-\imath g f^{deh_2})(-\imath g f^{ebh_3})(-g^2)\times$$
$$\int \frac{(-\imath)^2\, đ^d\omega\, đ^d v}{\omega^2 (p_1-\omega)^2 (\omega - v)^2 (p_1 - v)^2 (p_3 + v)^2} p_1^{\mu_1} p_3^{\mu} (p_1 - \omega)^{\mu_2} (p_1 - v)^{\mu_3} \Gamma^{(0) h_1 h_2 h_3 c}_{\mu_1 \mu_2 \mu_3 \mu}. \tag{4.5.15}$$

Collecting together factors and expanding the tree-level four-gluon vertex using (1.7.2) gives

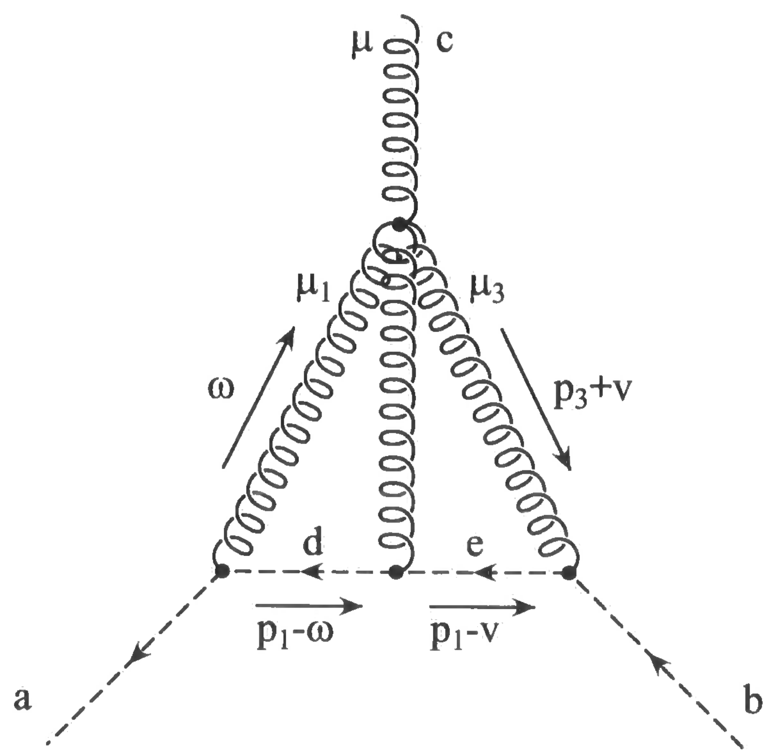


Figure 4.3: The third graph for the two-loop ghost-gluon vertex in Feynman gauge. The internal gluon lines have the Lorentz indices μ_1-μ_3 (contracted straight into the bare four-gluon vertex) and respective colour indices h_1-h_3.

$$-ig f^{abc}\left(-\frac{i}{2}g^2C_A\right)^2 V_3 = ig^5\int \frac{đ^d\omega\, đ^d v\; p_1^{\mu_1} p_3^{\mu}(p_1-\omega)^{\mu_2}(p_1-v)^{\mu_3}}{\omega^2(p_1-\omega)^2(\omega-v)^2(p_1-v)^2(p_3+v)^2} f^{adh_1}f^{deh_2}f^{ebh_3}\times$$
$$\left\{f^{h_1h_2,h_3c}\left(g_{\mu_1\mu_3}g_{\mu_2\mu}-g_{\mu_1\mu}g_{\mu_2\mu_3}\right)+f^{h_1h_3,h_2c}\left(g_{\mu_1\mu_2}g_{\mu_3\mu}-g_{\mu_1\mu}g_{\mu_2\mu_3}\right)\right.$$
$$\left.+f^{h_1c,h_2h_3}\left(g_{\mu_1\mu_2}g_{\mu_3\mu}-g_{\mu_1\mu_3}g_{\mu_2\mu}\right)\right\}. \qquad (4.5.16)$$

The colour algebra is done using the identity

$$f^{ade}f^{bef}f^{cfd} = \frac{1}{2}C_A f^{abc}, \qquad (4.5.17)$$

which gives for the first and last terms

$$\begin{aligned} f^{adh_1}f^{deh_2}f^{ebh_3}f^{h_1h_2,h_3c} &= \frac{1}{4}f^{abc}C_A^2 \\ f^{adh_1}f^{deh_2}f^{ebh_3}f^{h_1c,h_2h_3} &= -\frac{1}{4}f^{abc}C_A^2. \end{aligned} \qquad (4.5.18)$$

The colour factor for the middle term corresponds to an non-planar configuration and is known to be zero [46]. Cancelling then the pre-factors

$$V_3 = \int \frac{đ^d\omega\, đ^d v\; p_1^{\mu_1}p_3^{\mu}(p_1-\omega)^{\mu_2}(p_1-v)^{\mu_3}}{\omega^2(p_1-\omega)^2(\omega-v)^2(p_1-v)^2(p_3+v)^2}\left\{2g_{\mu_1\mu_3}g_{\mu_2\mu}-g_{\mu_1\mu}g_{\mu_2\mu_3}-g_{\mu_1\mu_2}g_{\mu_3\mu}\right\}. \qquad (4.5.19)$$

It is now possible to write down the expression for $V_3 - V_3'$,

$$\begin{aligned} V_3 - V_3' &= \int \frac{đ^d\omega\, đ^d v\; \left\{2g_{\mu_1\mu_3}g_{\mu_2\mu}-g_{\mu_1\mu}g_{\mu_2\mu_3}-g_{\mu_1\mu_2}g_{\mu_3\mu}\right\}}{\omega^2(p_1-\omega)^2(p_3+\omega)^2(\omega-v)^2(p_1-v)^2(p_3+v)^2}\times \\ &\left\{(p_3+\omega)^2 p_1^{\mu_1}p_3^{\mu}(p_1-\omega)^{\mu_2}(p_1-v)^{\mu_3}-(p_1-\omega)^2 p_3^{\mu_1}p_1^{\mu}(p_3+\omega)^{\mu_2}(p_3+v)^{\mu_3}\right\}. \end{aligned} \qquad (4.5.20)$$

Setting $p_1 = p_2 = p$ and performing the tensor contraction, this simplifies to

$$[V_3 - V_3']_{p_1=p_2=p} = 2\int \frac{đ^d\omega\, đ^d v\; \left\{(p-\omega)^2-(2p-\omega)^2\right\}}{\omega^2(p-\omega)^2(2p-\omega)^2(\omega-v)^2(p-v)^2(2p-v)^2}\left(p\cdot\omega p\cdot v - p^2\omega\cdot v\right). \qquad (4.5.21)$$

The scalar products in the numerator are expanded using

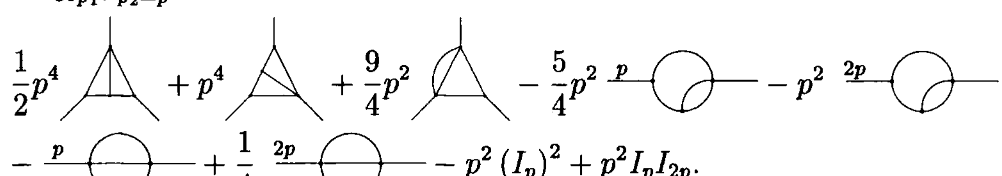
$$\begin{aligned} p\cdot v &= \frac{1}{2}\left[(p-v)^2-(2p-v)^2+3p^2\right] \\ p\cdot\omega &= -\frac{1}{2}\left[(p-\omega)^2-\omega^2-p^2\right] \\ &= -\frac{1}{4}\left[(2p-\omega)^2-\omega^2-4p^2\right] \\ \omega\cdot v &= -\frac{1}{2}\left[(\omega-v)^2-\omega^2-v^2\right] \end{aligned} \qquad (4.5.22)$$

where appropriate. In order to eliminate irreducible numerator factors, the following further identity is used

$$v^2 = 2p^2 + 2(p-v)^2 - (2p-v)^2. \tag{4.5.23}$$

The result is a set of two-loop scalar integrals. Writing this out using the diagrammatic notation for the integrals,

$$[V_3 - V_3']_{p_1=p_2=p} =$$



(4.5.24)

All the integrals have been done (see appendices A.1,C and C) and substituting in the results gives the final expression

$$[V_3 - V_3']_{p_1=p_2=p} = (4\pi)^{-d}(-p^2)^{-2\varepsilon}p^2 \left\{ 6\ln 2 + 2\ln^2 2 - \frac{39}{8}\zeta_3 + O(\varepsilon) \right\}. \tag{4.5.25}$$

Thus, the combination of two-loop ghost-gluon vertices (without gluon self-energy insertions) has been calculated in Feynman gauge with the momentum configuration $p_1 = p_2 = p$. The result is the sum of (4.5.14) and (4.5.25):

$$\begin{aligned} & p_3^\mu \tilde{\Gamma}^{(2)}_\mu(p_1,p_2,p_3) - p_1^\mu \tilde{\Gamma}^{(2)}_\mu(p_3,p_2,p_1) \\ & \quad \to [V_1 + V_2 + V_3 - V_1' - V_2' - V_3']_{p_1=p_2=p} \\ & \quad = (4\pi)^{-d}(-p^2)^{-2\varepsilon}p^2 \left\{ \ln 2 \left[-\frac{20}{\varepsilon} + 40\gamma + 46\ln 2 - 78 \right] - \frac{39}{8}\zeta_3 + O(\varepsilon) \right\}. \end{aligned} \tag{4.5.26}$$

This concludes the section.

4.6 The Two-Loop Identity

In this section, the equation (4.3.25) will be investigated. To recap, this equation was derived from the insistence that the full identity should have a renormalisable form. This meant that order by order, contributions dependent on the renormalisation coefficient for the ghost-gluon vertex ($\tilde{Z}_1$) had to vanish. This gave rise to the function G_x, which had the following form:

$$
\begin{aligned}
G_{b,x}^{(0)} &= 1 \\
G_{b,x}^{(1)} &= -\left(-\frac{\imath}{2}C_A\right)\frac{1}{2p_1\cdot p_3}\left\{p_1^2\left(\tilde{G}_1^{(1)}-\tilde{G}_2^{(1)}\right)-p_3^2\left(\tilde{G}_3^{(1)}-\tilde{G}_2^{(1)}\right)\right\} \\
z_x^{(1)} &= 0 \\
p_1\cdot p_3\tilde{G}_{b,x}^{(2)} &= -\left(-\frac{\imath}{2}C_A\right)^2\left\{p_3^\mu\tilde{\Gamma}_\mu^{(2)}(p_1,p_2,p_3)-p_1^\mu\tilde{\Gamma}_\mu^{(2)}(p_3,p_2,p_1)\right\} \\
&\quad+\frac{1}{p_1\cdot p_3}\left(-\frac{\imath}{2}C_A\right)^2\left\{\left(p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)\right)^2-p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)p_1^\mu\tilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1)\right\} \\
z_x^{(2)} &= 0.
\end{aligned}
\tag{4.6.1}
$$

It was originally assumed that the function G_x was some combination of ghost propagator functions. Thus, it is necessary to test whether or not $\tilde{G}_{b,x}^{(2)}$ can be decomposed into such a form. So far, the two-loop ghost propagator function and the ghost-gluon vertex function have been evaluated in the Feynman gauge and with a certain momentum configuration. All that is left is to evaluate $\left(p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)\right)^2-p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)p_1^\mu\tilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1)$ under these constraints. To do this consider

$$
\begin{aligned}
&\left(p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)\right)^2-p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)p_1^\mu\tilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1)= \\
&\qquad p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)\left[p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)-p_1^\mu\tilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1)\right].
\end{aligned}
\tag{4.6.2}
$$

The quantity in square brackets is nothing but the one-loop identity, which can be written as a combination of ghost propagator functions. Thus

$$
\begin{aligned}
&\left(p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)\right)^2-p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)p_1^\mu\tilde{\Gamma}_\mu^{(1)}(p_3,p_2,p_1)= \\
&\qquad \frac{1}{2}p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)\left\{p_1^2\left(\tilde{G}_1^{(1)}-\tilde{G}_2^{(1)}\right)-p_3^2\left(\tilde{G}_3^{(1)}-\tilde{G}_2^{(1)}\right)\right\}.
\end{aligned}
\tag{4.6.3}
$$

The function $p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3)$ has already been evaluated for arbitrary gauge and general momenta (4.1.1). Restricting to Feynman gauge, the result is

$$
\begin{aligned}
p_3^\mu\tilde{\Gamma}_\mu^{(1)}(p_1,p_2,p_3) &= \Phi\left[-\frac{1}{2}p_1^2p_3^2-\left((p_1\cdot p_3)^2+p_1\cdot p_3p_1^2+p_1\cdot p_3p_3^2\right)\right] \\
&\quad+I_1\left[p_1^2+p_1\cdot p_3\right]+I_2\left[-p_1^2-\frac{1}{2}p_3^2-p_1\cdot p_3\right]+I_3\left[\frac{1}{2}p_3^2+p_1\cdot p_3\right].
\end{aligned}
\tag{4.6.4}
$$

where the function Φ is nothing but the triangle integral $I(p_1, p_2, p_3)$. Setting $p_1 = p_2 = p$, Φ is expressed in the form (A.3.10)

$$\Phi = I(p, p, -2p) = \frac{1}{p^2}\left[I_p - I_{2p}\right]. \tag{4.6.5}$$

Thus

$$p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1, p_2, p_3)_{p_1=p_2=p} = 2p^2\left[I_p - 2I_{2p}\right]. \tag{4.6.6}$$

This gives

$$\left[\left(p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1, p_2, p_3)\right)^2 - p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1, p_2, p_3) p_1^\mu \tilde{\Gamma}^{(1)}_\mu(p_3, p_2, p_1)\right]_{p_1=p_2=p} = 4p^4\left[I_p - 2I_{2p}\right]\left[I_p - I_{2p}\right]. \tag{4.6.7}$$

Evaluating these integrals (see appendix A.1) and expanding in ε gives

$$\left[\left(p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1, p_2, p_3)\right)^2 - p_3^\mu \tilde{\Gamma}^{(1)}_\mu(p_1, p_2, p_3) p_1^\mu \tilde{\Gamma}^{(1)}_\mu(p_3, p_2, p_1)\right]_{p_1=p_2=p} = (4\pi)^{-d}(-p^2)^{-2\varepsilon} p^4 \ln 2 \left\{\frac{8}{\varepsilon} + 32 - 16\gamma - 40\ln 2 + O(\varepsilon)\right\}. \tag{4.6.8}$$

After all this, the equation for $\tilde{G}^{(2)}_{b,x}$ can now be written as an expansion in ε, albeit in Feynman gauge and with $p_1 = p_2 = p$. Putting the pieces together, one obtains

$$\tilde{G}^{(2)}_{b,x}|_{p_1=p_2=p} = \left(-\frac{\imath}{2}C_A\right)^2 (4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\ln 2\left[-\frac{8}{\varepsilon} - 31 + 16\gamma + 13\ln 2\right] - \frac{39}{16}\zeta_3 + O(\varepsilon)\right\}. \tag{4.6.9}$$

Now consider the possible forms that $\tilde{G}^{(2)}_{b,x}$ could have. Since the renormalisation coefficient z_x is unity up to two-loops, this means that there can be no leading divergence (ie $(g^2)^n\varepsilon^{-n}$ terms). Therefore, $\tilde{G}^{(2)}_{b,x}$ can only depend on either the ratio or difference of ghost propagator functions. However, if one considers the term proportional to ζ_3, this occurs in the finite part of the propagator function at two-loops. If $\tilde{G}^{(2)}_{b,x}$ were to be either the ratio or difference of propagator functions then these terms *must vanish.* Thus (4.1.3) cannot be expressed as a combination of ghost propagator functions and *the proposed form of the identity cannot be true.*

However, it is noted that the ζ_3-dependence comes solely from the $V_3 - V_3'$ part of the two-loop vertex. This type of graph was not present at one-loop. This leads to the question of whether or not $\tilde{G}^{(2)}_{b,x}$ can be constructed from ghost propagator functions alone if this graph is omitted. Without this graph,

$$\tilde{G}^{(2)}_{b,x}|_{p_1=p_2=p} = \left(-\frac{\imath}{2}C_A\right)^2 (4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\ln 2\left[-\frac{8}{\varepsilon} - 34 + 16\gamma + 12\ln 2\right] + O(\varepsilon)\right\}. \tag{4.6.10}$$

Since $\tilde{G}^{(2)}_{b,x}$ must be constructed from the difference or ratio of ghost propagator functions and the momenta are restricted to $p_1 = p_2 = p$, there are three two-loop configurations that may contribute. These are:

$$\begin{aligned}
\tilde{G}^{(2)}_{2p} - \tilde{G}^{(2)}_{p} &= (4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\ln 2\left[\frac{10}{\varepsilon} + 44 - 20\gamma - 20\ln 2\right] + O(\varepsilon)\right\} \\
\tilde{G}^{(1)}_{p}\left(\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right) &= (4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\ln 2\left[-\frac{2}{\varepsilon} - 8 + 4\gamma + 2\ln 2\right] + O(\varepsilon)\right\} \\
\tilde{G}^{(1)}_{2p}\left(\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right) &= (4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\ln 2\left[-\frac{2}{\varepsilon} - 8 + 4\gamma + 6\ln 2\right] + O(\varepsilon)\right\}.
\end{aligned} \tag{4.6.11}$$

It is found that $\tilde{G}^{(2)}_{b,x}$ can be constructed from these three combinations. The result is

$$\tilde{G}^{(2)}_{b,x}|_{p_1=p_2=p} = \left(-\frac{\imath}{2}C_A\right)^2 \left\{-\frac{1}{2}\left[\tilde{G}^{(2)}_{2p} - \tilde{G}^{(2)}_{p}\right] + \frac{7}{4}\tilde{G}^{(1)}_{p}\left[\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right] - \frac{1}{4}\tilde{G}^{(1)}_{2p}\left[\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right]\right\}. \tag{4.6.12}$$

which can be verified by inspection. The function $\tilde{G}_{b,x}$ can now be written in four dimensions as

$$\begin{aligned}
\tilde{G}_{b,x}|_{p_1=p_2=p} &= 1 + \left(-\frac{\imath}{2}g^2C_A\right)\left[\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right] \\
&\quad + \left(-\frac{\imath}{2}g^2C_A\right)^2\left\{\frac{1}{2}\left[\tilde{G}^{(2)}_{p} - \tilde{G}^{(2)}_{2p}\right] + \frac{1}{4}\left[\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right]^2 + \frac{3}{2}\tilde{G}^{(1)}_{p}\left[\tilde{G}^{(1)}_{p} - \tilde{G}^{(1)}_{2p}\right]\right\} \\
&\quad + O(g^6).
\end{aligned} \tag{4.6.13}$$

4.7 Summary

In this chapter, the possibility of relating two contracted ghost-gluon vertices with some combination of two-point functions has been investigated using perturbation theory. It was found that there exists a unique identity, true in all gauges and dimensions at the one-loop level.

The one-loop identity found (4.1.3) was derived by demanding that a simple combination of contracted ghost-gluon vertices should be independent of the massless triangle integral Φ which does not occur in the expressions for two-point functions. This was achieved uniquely and it was discovered that in doing so, the two-point integrals decomposed immediately into exactly the right combinations as to be identified with the ghost propagator function. The one-loop identity admits several extrapolated forms for a proposed identity true to all orders. It was then found that the one-loop identity (4.1.3) could be extended to include the first level of gluon self-energy corrections. These corrections contain the first fermionic contributions to the equation.

The renormalisation properties of QCD were then used to constrain the form of the identity. By demanding that there was no connection between the different renormalisation coefficients, it was possible to restrict the possibilities. This led to a single equation at two-loop perturbative order that had to be satisfied if the identity were to be true.

The two-loop perturbative calculations were done in the Feynman gauge and with a certain momentum configuration. This simplified the integrals greatly as there were only single powers of denominator factors and all the integrals had only one external scale.

Finally, the components of the two-loop renormalisation constraint equation were put together. It was found that the identity, even in Feynman gauge and one particular momentum configuration, could not be satisfied. However, it was seen that by omitting a certain graph containing an explicit four-gluon interaction that the constraint equation could be satisfied under these conditions.

Chapter 5

The Identity for the Ghost-Gluon Vertex

In the previous chapter, an attempt was made to find an identity relating the ghost-gluon vertex to some combination of two-point functions. The starting point for this was the functional identity (2.4.1) which involved connected and disconnected ghost-ghost scattering. As such, this was not suitable for application directly into a useful form and so, perturbation theory was used in order to see whether or not an identity existed that would not depend on the four-point functions. Unfortunately, no such identity was found.

In this short chapter, the equation (2.4.1) is studied more directly. By decomposing the Green's functions and Fourier transforming into momentum space, an identity relating the two, three and four-point one-particle-reducible and irreducible functions is found. This identity is not suitable for Schwinger-Dyson purposes since the four-point functions cannot be dealt with, but nonetheless some information can be obtained.

In the first section, the identity is derived in momentum space. In the second section, it is checked to one-loop order in perturbation theory and then with one-loop gluon insertions. This is interesting because the identity involves terms that are reducible as well as the more usual one-particle-irreducible terms. In the final section, it is shown that some information about the infrared behaviour of the ghost-gluon vertex can be extracted which will be of use later on.

5.1 The Momentum Space Identity

In section 2.4, a functional identity relating the ghost-gluon vertex to some combination of connected and disconnected ghost-ghost scattering was presented. This equation, derived by von Smekal *et al.* [2], follows directly from the BRS invariance of the full pure Yang-Mills theory and has the following form in Euclidean space:

$$\frac{1}{\xi}\langle C^c(z)\overline{C}^b(y)\partial A^a(x)\rangle - \frac{1}{\xi}\langle C^c(z)\overline{C}^a(x)\partial A^b(y)\rangle = -\frac{g}{2}f^{cde}\langle C^d(z)C^e(z)\overline{C}^a(x)\overline{C}^b(y)\rangle. \tag{5.1.1}$$

Note that these are full, reducible correlation functions and so, the four-point function on the right-hand side can be decomposed into connected and disconnected parts as follows

$$\langle C^c C^a \overline{C}^b \overline{C}^d\rangle = \langle C^a\overline{C}^b\rangle\langle C^c\overline{C}^d\rangle - \langle C^a\overline{C}^d\rangle\langle C^c\overline{C}^b\rangle + \langle C^c C^a\overline{C}^b\overline{C}^d\rangle_c \tag{5.1.2}$$

where the minus sign is by virtue of the Grassmannian nature of the ghost fields and the subscript c denotes the connected four-point function. The correlation functions in (5.1.1) are not truncated and so can be further decomposed to make the external legs explicit.

$$\begin{aligned}
&\frac{1}{\xi}\int du\,dv\,dw\,\widetilde{\Gamma}_\nu^{cba}(v,w;u)\partial_\mu^x D^{\mu\nu}(x,u)D(z,v)D(w,y)\\
&-\frac{1}{\xi}\int du\,dv\,dw\,\widetilde{\Gamma}_\nu^{cab}(v,u;w)\partial_\mu^y D^{\mu\nu}(y,w)D(z,v)D(u,x)\\
&\quad= -gf^{cba}D(z,x)D(z,y)\\
&\quad-\frac{g}{2}f^{cde}\int du\,dv\,dw\,dt\,\Gamma_4^{deab}(v,t;u,w)D(z,v)D(z,t)D(u,x)D(w,y)
\end{aligned} \tag{5.1.3}$$

where now, the ghost gluon vertex and ghost propagator functions (in configuration space) are one-particle irreducible, whilst the four-point ghost function is still reducible. Both sides of the equation can be expressed in the form $F(x,y,z)$ and it is now useful to do a Fourier transform to momentum space in the following way[1]

$$F(p_1,p_2,p_3) = \int dx\,dy\,dz F(x,y,z)\exp\{-\imath[p_1z+p_2y+p_3x]\}. \tag{5.1.4}$$

For clarity, it is better to consider each term in turn. The first term on the left-hand side of (5.1.3) becomes under the transform

$$\int dx\,dy\,dz\,du\,dv\,dw\,\widetilde{\Gamma}_\nu^{cba}(v,w;u)\frac{1}{\xi}\partial_\mu^x D^{\mu\nu}(x,u)D(z,v)D(w,y)\exp\{-\imath[p_1z+p_2y+p_3x]\}. \tag{5.1.5}$$

Now it is possible to make the following identification for the two-point functions

$$\begin{aligned}
D(z,v) &= \frac{1}{(2\pi)^{2d}}\int dq_1\,dq_2\,D(q_1,q_2)\exp\{\imath[q_1z+q_2v]\}\\
\partial_\mu^x D^{\mu\nu}(x,u) &= \frac{1}{(2\pi)^{2d}}\int dq_1\,dq_2\,\imath q_{1\mu}D(q_1,q_2)\exp\{\imath[q_1x+q_2u]\}.
\end{aligned} \tag{5.1.6}$$

[1]In what follows, the integral measure d^dx or d^dq will be abbreviated to dx or dq to keep the equations from getting too cluttered.

Putting these into the term then gives

$$\int dx\,dy\,dz\,du\,dv\,dw\,\tilde{\Gamma}_\nu^{cba}(v,w;u)\frac{1}{(2\pi)^{6d}}\int dq_1\ldots dq_6\frac{1}{\xi}\imath q_{1\mu}D^{\mu\nu}(q_1,q_4)\times$$
$$D(q_2,q_5)D(q_3,q_6)\exp\{\imath[(q_1-p_3)x+(q_2-p_1)z+(q_3-p_2)y+q_4u+q_5v+q_6w]\}. \tag{5.1.7}$$

Recognising the Fourier transform of the momentum space ghost-gluon vertex

$$\tilde{\Gamma}_\nu^{cba}(-q_5,-q_6;-q_4)=\int du\,dv\,dw\,\tilde{\Gamma}_\nu^{cba}(v,w;u)\exp\{\imath[q_4u+q_5v+q_6w]\} \tag{5.1.8}$$

then leads to

$$\int dx\,dy\,dz\,\frac{1}{(2\pi)^{6d}}\int dq_1\ldots dq_6\tilde{\Gamma}_\nu^{cba}(-q_5,-q_6;-q_4)\frac{1}{\xi}\imath q_{1\mu}D^{\mu\nu}(q_1,q_4)\times$$
$$D(q_2,q_5)D(q_3,q_6)\exp\{\imath[(q_2-p_1)z+(q_3-p_2)y+(q_1-p_3)x]\}. \tag{5.1.9}$$

The next step is to realise that since the propagator is invariant under a translation in configuration space, it can only depend on one momentum. In fact, it can be shown that

$$D(q_1,q_2)=(2\pi)^d\delta^d(q_1+q_2)D(q_1). \tag{5.1.10}$$

Substituting in for the propagator functions and integrating out the delta-functions gives

$$\int dx\,dy\,dz\,\frac{1}{(2\pi)^{3d}}\int dq_1\ldots dq_3\tilde{\Gamma}_\nu^{cba}(q_2,q_3;q_1)\frac{1}{\xi}\imath q_{1\mu}D^{\mu\nu}(q_1)D(q_2)D(q_3)\times$$
$$\exp\{\imath[(q_2-p_1)z+(q_3-p_2)y+(q_1-p_3)x]\}. \tag{5.1.11}$$

Now, the remaining integrals over x, y and z can be expressed as delta-functions since

$$\int dx\,\exp\{\imath(q_1+q_2)x\}=(2\pi)^d\delta^d(q_1+q_2). \tag{5.1.12}$$

This, coupled with the recognition of the Slavnov-Taylor identity for the gluon propagator

$$q_{1\mu}D^{\mu\nu}(q_1)=\xi\frac{q_1^\nu}{q^2} \tag{5.1.13}$$

gives the final form for the first term in (5.1.3)

$$\imath\frac{p_3^\nu}{p_3^2}\tilde{\Gamma}_\nu^{cba}(p_1,p_2;p_3)D(p_1)D(p_2). \tag{5.1.14}$$

The second term in (5.1.3) is identical to the first but with (x,a) interchanged with (y,b). This is simply an exchange of the labels associated with the gluon and in-ghost legs and thus, the second term can be written down immediately as

$$\imath\frac{p_2^\nu}{p_2^2}\tilde{\Gamma}_\nu^{cab}(p_1,p_3;p_2)D(p_1)D(p_3). \tag{5.1.15}$$

The first term on the right-hand side of (5.1.3) corresponds to the disconnected scattering of two ghost fields. Taking the Fourier transform in exactly the same way as for the previous expressions gives[2]

$$\int dx\, dy\, dz\, D(z,x)D(z,y) \exp\{-\imath[p_1 z + p_2 y + p_3 x]\}. \tag{5.1.16}$$

Using (5.1.6), immediately leads to

$$\int dx\, dy\, dz \frac{1}{(2\pi)^{4d}} \int dq_1 \ldots dq_4 D(q_1,q_3)D(q_2,q_4)\times \\ \exp\{\imath[(q_3-p_3)x + (q_1+q_2-p_1)z + (q_4-p_2)y]\}. \tag{5.1.17}$$

Again using the scale invariance of the propagator, the expression becomes

$$\int dx\, dy\, dz \frac{1}{(2\pi)^{2d}} \int dq_1\, dq_2 D(q_1)D(q_2)\times \\ \exp\{\imath[(-q_1-p_3)x + (q_1+q_2-p_1)z + (-q_2-p_2)y]\}. \tag{5.1.18}$$

The integrals over x and y are delta-functions and integrating these out gives

$$D(p_3)D(p_2)\int dz\, \exp\{\imath(-p_3-p_2-p_1)z\}. \tag{5.1.19}$$

The final integral over z is again a delta-function but this time remains as an external factor, giving the final expression for the term

$$D(p_3)D(p_2)(2\pi)^d\delta^d(p_1+p_2+p_3). \tag{5.1.20}$$

So far, the expressions presented are no more than those contained in (2.4.3). However, the final term on the right-hand side of (5.1.3) is new. Under the truncation scheme of von Smekal *et al.* [2], this term was neglected. The term corresponds to connected but not one-particle irreducible ghost-ghost scattering. In the convention adopted here, the ordering of the legs is as follows: the legs are defined in an anti-clockwise way, with the first two arguments referring to out-ghost fields (just as the first argument of the ghost-gluon vertex refers to the out-ghost leg). As far as the Fourier transform is concerned though, the details of the function are not important. Thus, it is possible to proceed as before and the transform of this final term is

$$\int dx\, dy\, dz\, du\, dv\, dw\, dt\, \Gamma_4^{deab}(v,t;u,w)D(z,v)D(z,t)D(u,x)D(w,y)\times \\ \exp\{-\imath[p_1 z + p_2 y + p_3 x]\}. \tag{5.1.21}$$

[2] Omitting the pre-factors for now.

Again, the first thing to do is to replace the configuration space propagators with their Fourier transformed momentum space counterparts as in (5.1.6), giving

$$\int dx\,dy\,dz\,du\,dv\,dw\,dt\,\Gamma_4^{deab}(v,t;u,w)\frac{1}{(2\pi)^{8d}}\int dq_1\dots dq_8 D(q_1,q_5)D(q_2,q_6)D(q_3,q_7)\times$$
$$D(q_4,q_8)\exp\{\imath[(q_1+q_2-p_1)z+(q_4-p_2)y+(q_3-p_3)x+q_5v+q_6t+q_7u+q_8w]\}. \tag{5.1.22}$$

Now one can identify the Fourier transform of the four-point function

$$\Gamma_4^{deab}(-q_5,-q_6;-q_7,-q_8)=\int du\,dv\,dw\,dt\,\Gamma_4^{deab}(v,t;u,w)\exp\{\imath[q_5v+q_6t+q_7u+q_8w]\} \tag{5.1.23}$$

which leaves

$$\int dx\,dy\,dz\,\frac{1}{(2\pi)^{8d}}\int dq_1\dots dq_8\Gamma_4^{deab}(-q_5,-q_6;-q_7,-q_8)D(q_1,q_5)D(q_2,q_6)D(q_3,q_7)\times$$
$$D(q_4,q_8)\exp\{\imath[(q_1+q_2-p_1)z+(q_4-p_2)y+(q_3-p_3)x]\}. \tag{5.1.24}$$

Using the translational invariance gives

$$\int dx\,dy\,dz\,\frac{1}{(2\pi)^{4d}}\int dq_1\dots dq_4\Gamma_4^{deab}(q_1,q_2;q_3,q_4)D(q_1)D(q_2)D(q_3)D(q_4)\times$$
$$\exp\{\imath[(q_1+q_2-p_1)z+(q_4-p_2)y+(q_3-p_3)x]\}. \tag{5.1.25}$$

The integrals over x, y and z are all now delta-functions which when written out explicitly give the form

$$\frac{1}{(2\pi)^d}\int dq_1\dots dq_4\delta^d(q_3-p_3)\delta^d(q_4-p_2)\delta^d(q_1+q_2-p_1)\Gamma_4^{deab}(q_1,q_2;q_3,q_4)\times$$
$$D(q_1)D(q_2)D(q_3)D(q_4). \tag{5.1.26}$$

Integrating over q_2, q_3 and q_4 then gives

$$\frac{1}{(2\pi)^d}\int dq_1\Gamma_4^{deab}(q_1,p_1-q_1;p_3,p_2)D(q_1)D(p_1-q_1)D(p_3)D(p_2). \tag{5.1.27}$$

This is the final result for the last term in (5.1.3). It is interesting to note that it carries precisely the right integral measure as to be indistinguishable from a normal loop integral.

Putting the equation (5.1.3) back together gives

$$\begin{aligned}
&\imath\frac{p_3^\nu}{p_3^2}\tilde{\Gamma}_\nu^{cba}(p_1,p_2;p_3)D(p_1)D(p_2)-\imath\frac{p_2^\nu}{p_2^2}\tilde{\Gamma}_\nu^{cab}(p_1,p_3;p_2)D(p_1)D(p_3)=\\
&\quad -gf^{cba}D(p_3)D(p_2)(2\pi)^d\delta^d(p_1+p_2+p_3)\\
&\quad -\frac{g}{2}f^{cde}\frac{1}{(2\pi)^d}\int dq_1\Gamma_4^{deab}(q_1,p_1-q_1;p_3,p_2)D(q_1)D(p_1-q_1)D(p_3)D(p_2).
\end{aligned} \tag{5.1.28}$$

This equation now involves momentum space Green's functions alone. In order to proceed, it is necessary to extract the more usual propagator and vertex functions that appear in perturbation theory. It is possible to eliminate the explicit delta-function by using the translational invariance of both the three and four-point functions in configuration space, just as in the case of the two-point functions to give

$$\begin{aligned}
&\imath\frac{p_3^\nu}{p_3^2}\tilde{\Gamma}_\nu^{cba}(p_1,p_2;p_3)D(p_1)D(p_2)-\imath\frac{p_2^\nu}{p_2^2}\tilde{\Gamma}_\nu^{cab}(p_1,p_3;p_2)D(p_1)D(p_3)=-gf^{cba}D(p_3)D(p_2)\\
&\quad -\frac{g}{2}f^{cde}\frac{1}{(2\pi)^d}\int dq_1\Gamma_4^{deab}(q_1,p_1-q_1;p_3,p_2)D(q_1)D(p_1-q_1)D(p_3)D(p_2)
\end{aligned} \tag{5.1.29}$$

where now, the momentum conservation is included implicitly in the definition of the Green's functions. The next step is to rearrange this slightly, dividing through by the propagators to obtain

$$\begin{aligned}
&D(p_3)^{-1}\imath\frac{p_3^\nu}{p_3^2}\tilde{\Gamma}_\nu^{cba}(p_1,p_2;p_3)-D(p_2)^{-1}\imath\frac{p_2^\nu}{p_2^2}\tilde{\Gamma}_\nu^{cab}(p_1,p_3;p_2)=-gf^{cba}D(p_1)^{-1}\\
&\quad -\frac{g}{2}f^{cde}D(p_1)^{-1}\frac{1}{(2\pi)^d}\int dq_1\Gamma_4^{deab}(q_1,p_1-q_1;p_3,p_2)D(q_1)D(p_1-q_1).
\end{aligned} \tag{5.1.30}$$

Now, this equation is written in Euclidean space. Performing a Wick rotation into Minkowski space alters only the loop integration measure dq_1 since the whole equation has the same dimension throughout

$$\begin{aligned}
&D(p_3)^{-1}\imath\frac{p_3^\nu}{p_3^2}\tilde{\Gamma}_\nu^{cba}(p_1,p_2;p_3)-D(p_2)^{-1}\imath\frac{p_2^\nu}{p_2^2}\tilde{\Gamma}_\nu^{cab}(p_1,p_3;p_2)=-gf^{cba}D(p_1)^{-1}\\
&\quad -\frac{g}{2}f^{cde}D(p_1)^{-1}\frac{1}{(2\pi)^d}\int(-\imath)dq_1\Gamma_4^{deab}(q_1,p_1-q_1;p_3,p_2)D(q_1)D(p_1-q_1).
\end{aligned} \tag{5.1.31}$$

Finally, replacing the Green's functions above with the Feynman rules outlined in 1.1 leads to

$$g f^{cba}\left[G_3^{-1}p_3^{\nu}\widetilde{\Gamma}_{\nu}(p_1,p_2,p_3)+G_2^{-1}p_2^{\nu}\widetilde{\Gamma}_{\nu}(p_1,p_3,p_2)+p_1^2G_1^{-1}\right]=$$
$$-\frac{g}{2}f^{cde}p_1^2G_1^{-1}\int\frac{(-\imath)\,\bar{d}^d\omega}{\omega^2(p_1-\omega)^2}G_{\omega}G_{p_1-\omega}\Gamma_4^{deab}(\omega,p_1-\omega;p_3,p_2). \qquad (5.1.32)$$

The last term in the above equation refers to connected ghost-ghost scattering. It is not usual for such an identity to contain one-particle-reducible parts (ie graphs that become disconnected when one internal propagator is cut) and further, the one-particle-irreducible graphs do not occur at the one-loop perturbative level. It is pertinent to verify the equation at the one-loop level, if only to check the pre-factors and ordering of external legs.

Notice the symmetry of the equation. Since the four-point function on the right-hand side involves combinations of identical Grassmann fields, then under interchange there will be a relative minus sign. For example, consider the interchange of the two in-ghost fields, ie $(p_2,b)\leftrightarrow(p_3,a)$. The right-hand side gains a minus sign. On the left-hand side, there is an explicit symmetry between p_2 and p_3 but the structure constant f^{cba} automatically changes sign under interchange of a and b. Now consider the one-loop identity of the previous chapter (4.1.3). Neglecting colour pre-factors, this equation was *antisymmetric* under interchange of p_2 and p_3. This indicates that (5.1.32) is not the same as the form under consideration in the previous chapter. However, (5.1.32) still contains reducible four-point interactions and the whole purpose of the last chapter was to see if an identity independent of such contributions existed.

5.2 The One-Loop Identity and Gluon Self-Energy Insertions

It is relatively straightforward to check the identity (5.1.32) at the one-loop level. In fact, since such an exercise has already been done before, it will be no more complicated to include gluon self-energy corrections at the same time. The reason that this is no more complicated is that in general gauge, the gluon propagator splits naturally into the gauge dependent longitudinal part and the transverse part which is multiplied only by a dimensionless function (the self-energy) of one argument. At tree-level, this function is

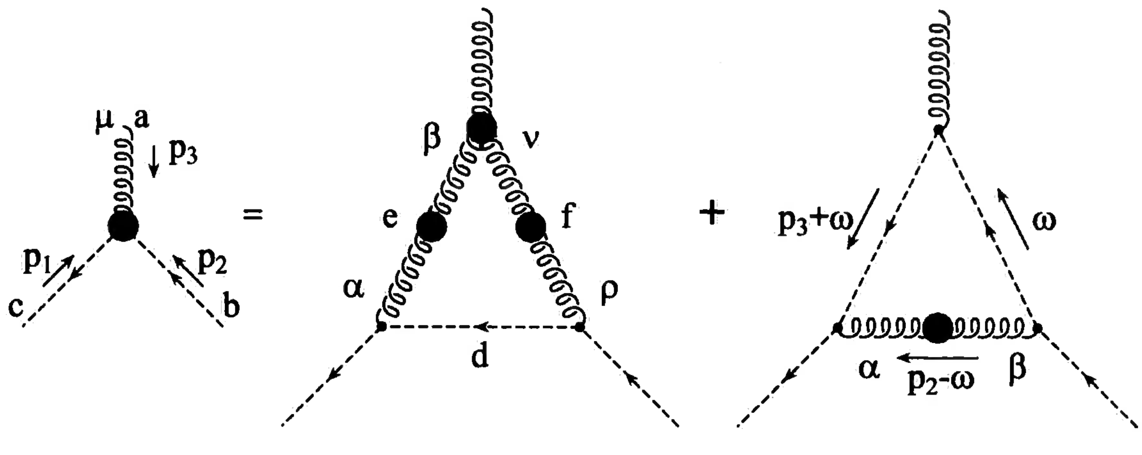


Figure 5.1: The ghost-gluon vertex at one-loop. The blobs represent the insertion of the gluon self-energy.

unity, but for the purposes here, all one needs to do is retain this function and it is not necessary to evaluate it.

To start the calculation, consider the one-loop form of the contracted ghost-gluon vertex with the bare gluon propagators replaced with the full forms (see fig. 5.1). This can be written as

$$-\imath g f^{cba} p_3^\nu \tilde{\Gamma}_\nu(p_1,p_2,p_3) = -\imath g f^{cba} p_1\cdot p_3 + (-\imath)^4 f^{cde} f^{aef} f^{bfd} \int \frac{\bar{d}^d\omega\, p_{1\alpha}}{\omega^2(p_2-\omega)^2(p_3+\omega)^2} \times \left\{(\omega-p_2)_\rho A^{\alpha\beta}(p_3+w)A^{\rho\nu}(\omega)p_3^\mu \Gamma_{\beta\nu\mu}(-p_3-\omega,\omega,p_3) + p_3\cdot(p_3+\omega)\omega_\beta A^{\alpha\beta}(p_2-\omega)\right\} \tag{5.2.1}$$

where

$$A^{\rho\nu}(\omega) = t^{\rho\nu}(\omega)J_w^{-1} + \xi l^{\rho\nu}(\omega). \tag{5.2.2}$$

Doing the colour algebra, using the Slavnov-Taylor identity (1.7.6) for the triple-gluon vertex (retaining the full gluon propagator functions) and combining terms leads to

$$p_3^\nu \tilde{\Gamma}_\nu(p_1,p_2,p_3) = p_1\cdot p_3 + \left(\frac{-\imath}{2}g^2 C_A\right)\int \frac{\bar{d}^d\omega\, p_{1\alpha}}{\omega^2(p_2-\omega)^2(p_3+\omega)^2}\times \left\{-p_{2\rho}\omega^2 t_\beta^\rho(\omega)A^{\alpha\beta}(p_3+w) + (p_2-\omega)_\rho(p_3+\omega)^2 t_\nu^\alpha(p_3+\omega)A^{\rho\nu}(\omega) + p_3\cdot(p_3+\omega)\omega_\beta A^{\alpha\beta}(p_2-\omega)\right\}. \tag{5.2.3}$$

Now, the two-point ghost function G can be constructed in exactly the same way by taking the one-loop form (see section 3.2) and putting back the gluon self-energy function. It can be written as

$$G_3^{-1} = 1 - \left(\frac{-\imath}{2} g^2 C_A\right) \frac{2}{p_3^2} \int \frac{\bar{d}^d\omega}{\omega^2(p_3+\omega)^2} p_{3\mu}(p_3+\omega)_\nu A^{\mu\nu}(\omega). \tag{5.2.4}$$

The term $p_1^2 G_1^{-1}$ is written in such a way as the integrals are expressly symmetric under the interchange $p_2 \leftrightarrow p_3$ by suitable changes of variable. The expression is

$$\begin{aligned} p_1^2 G_1^{-1} = p_1^2 \; - \; & \left(\frac{-\imath}{2} g^2 C_A\right) \int \frac{\bar{d}^d\omega}{(p_2-\omega)^2(p_3+\omega)^2} (-p_{1\mu}) \times \\ & [(p_3+\omega)_\nu A^{\mu\nu}(p_2-\omega) + (p_2-\omega)_\nu A^{\mu\nu}(p_3+\omega)] . \end{aligned} \tag{5.2.5}$$

It is now possible to construct the left-hand side of (5.1.32). Writing for convenience

$$gf^{cba}[\text{lhs}] = gf^{cba}\left[G_3^{-1} p_3^\nu \widetilde{\Gamma}_\nu(p_1,p_2,p_3) + G_2^{-1} p_2^\nu \widetilde{\Gamma}_\nu(p_1,p_3,p_2) + p_1^2 G_1^{-1}\right] \tag{5.2.6}$$

then after expanding the transverse projectors and rearranging the tensors one obtains

$$\begin{aligned} [\text{lhs}] = & \left(\frac{-\imath}{2} g^2 C_A\right) \int \frac{\bar{d}^d\omega}{\omega^2(p_2-\omega)^2(p_3+\omega)^2} \times \\ & \Big\{ -p_{1\mu} A^{\mu\nu}(p_2-\omega)\left[2\omega^2(p_3+\omega)_\nu - (p_3+\omega)^2\omega_\nu\right] \\ & -p_{1\mu} A^{\mu\nu}(p_3+\omega)\left[2\omega^2(p_2-\omega)_\nu + (p_2-\omega)^2\omega_\nu\right] \\ & +A^{\mu\nu}(\omega)\Big[(p_2-\omega)^2(p_3+\omega)_\mu \left(p_{1\nu} - \frac{2p_1\cdot p_3}{p_3^2} p_{3\nu}\right) \\ & \qquad + (p_3+\omega)^2(p_2-\omega)_\mu \left(p_{1\nu} - \frac{2p_1\cdot p_2}{p_2^2} p_{2\nu}\right) \\ & \qquad + p_1^2(p_3+\omega)_\mu(p_2-\omega)_\nu\Big]\Big\} . \end{aligned} \tag{5.2.7}$$

Changing variables such that the argument of A is always ω immediately gives

$$\begin{aligned} [\text{lhs}] = & \left(\frac{-\imath}{2} g^2 C_A\right) \int \frac{\bar{d}^d\omega}{\omega^2(p_2-\omega)^2(p_3+\omega)^2} A^{\mu\nu}(\omega) \Big\{ p_1^2(p_3+\omega)_\mu(p_2-\omega)_\nu \\ & +2(p_2-\omega)^2(p_3+\omega)_\mu \left(p_{1\nu} - \frac{p_1\cdot p_3}{p_3^2} p_{3\nu}\right) + 2(p_3+\omega)^2(p_2-\omega)_\mu \left(p_{1\nu} - \frac{p_1\cdot p_2}{p_2^2} p_{2\nu}\right)\Big\} . \end{aligned} \tag{5.2.8}$$

Now consider the integral

$$\int \frac{\bar{d}^d\omega}{\omega^2(p_3+\omega)^2} A^{\mu\nu}(\omega)(p_3+\omega)_\mu = p_3^\nu I. \tag{5.2.9}$$

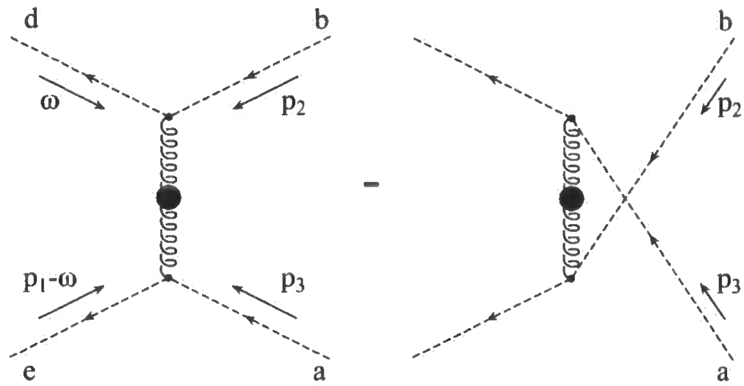


Figure 5.2: The function $\Gamma_4^{deab}(\omega, p_1-\omega; p_3, p_2)$ at $O(g^2)$. The blobs represent the insertion of the gluon self-energy. Note the relative minus sign due to the interchange of two identical Grassmann fields.

This means that

$$\begin{aligned} \int \frac{\bar{d}^d\omega}{\omega^2(p_3+\omega)^2} A^{\mu\nu}(\omega)(p_3+\omega)_\mu \left(p_{1\nu} - \frac{p_1 \cdot p_3}{p_3^2} p_{3\nu}\right) &= 0 \\ \int \frac{\bar{d}^d\omega}{\omega^2(p_2-\omega)^2} A^{\mu\nu}(\omega)(p_2-\omega)_\mu \left(p_{1\nu} - \frac{p_1 \cdot p_2}{p_2^2} p_{2\nu}\right) &= 0 \end{aligned} \tag{5.2.10}$$

such that finally

$$[\text{lhs}] = \left(\frac{-\imath}{2} g^2 C_A\right) \int \frac{\bar{d}^d\omega}{\omega^2(p_2-\omega)^2(p_3+\omega)^2} A^{\mu\nu}(\omega) p_1^2 (p_3+\omega)_\mu (p_2-\omega)_\nu \tag{5.2.11}$$

The right-hand side of (5.1.32) at the one-loop level involves only the one-particle-reducible terms (see fig. 5.2). It is convenient to write

$$g f^{cba}[\text{rhs}] = -\frac{g}{2} f^{cde} p_1^2 G_1^{-1} \int \frac{(-\imath)\,\bar{d}^d\omega}{\omega^2(p_1-\omega)^2} G_\omega G_{p_1-\omega} \Gamma_4^{deab}(\omega, p_1-\omega; p_3, p_2). \tag{5.2.12}$$

Now, with the full gluon propagators

$$\begin{aligned} f^{cde}\Gamma_4^{deab}(\omega, p_1-\omega; p_3, p_2) &= g^2 f^{cde} f^{aef} f^{bfd} \frac{\omega_\mu (p_1-\omega)_\nu}{(p_2+\omega)^2} A^{\mu\nu}(p_2+\omega) \\ &\quad - g^2 f^{cde} f^{bef} f^{afd} \frac{\omega_\mu (p_1-\omega)_\nu}{(p_3+\omega)^2} A^{\mu\nu}(p_3+\omega). \end{aligned} \tag{5.2.13}$$

Inserting this into the expression for the right-hand side at the appropriate order in the coupling gives

$$\begin{aligned} g f^{cba}[\text{rhs}] &= \frac{1}{2} g f^{cba} \left(\frac{-\imath}{2} g^2 C_A\right) p_1^2 \times \\ &\left\{ \int \frac{\bar{d}^d\omega\, \omega_\mu (p_1-\omega)_\nu}{\omega^2(p_1-\omega)^2(p_2+\omega)^2} A^{\mu\nu}(p_2+\omega) + \int \frac{\bar{d}^d\omega\, \omega_\mu (p_1-\omega)_\nu}{\omega^2(p_1-\omega)^2(p_3+\omega)^2} A^{\mu\nu}(p_3+\omega) \right\}. \end{aligned} \tag{5.2.14}$$

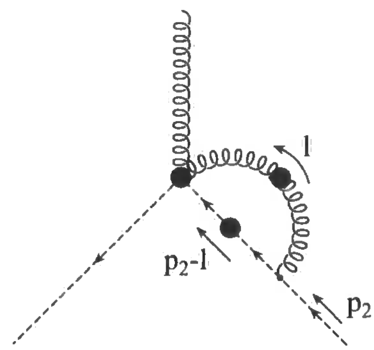


Figure 5.3: Diagram showing the decomposition of the ghost-gluon vertex.

Changing variables such that again the argument of A is always ω immediately leads to

$$[\text{rhs}] = p_1^2 \left(\frac{-\imath}{2} g^2 C_A\right) \int \frac{\bar{d}^d\omega}{\omega^2 (p_2-\omega)^2 (p_3+\omega)^2} A^{\mu\nu}(\omega)(p_3+\omega)_\mu (p_2-\omega)_\nu \tag{5.2.15}$$

which by inspection is identical to the form found for [lhs]. Thus, the equation (5.1.32) has been verified at the one-loop order with gluon self-energy corrections added in. The pre-factors agree and the ordering of the external legs is now apparent.

5.3 Extracting Information from the Identity

One of the useful properties of higher-point Green's functions when working in Landau gauge is that as the momentum of an in-ghost vanishes, all but the tree-level graphs vanish. This property is intimately related to the observation that $\tilde{Z}_1$ is unity in the Landau gauge ([21, 43] and the end of this section). In order to see this reduction, consider the decomposition of the ghost-gluon vertex into the tree-level part and the form shown in fig. 5.3. This graph has isolated the bare ghost-gluon vertex connected directly to the in-ghost line by virtue of the fact that this is the *only* interaction term involving the ghost field. Thus, for any graph contributing to fig. 5.3, there will be the term

$$(p_2 - l)_\nu t^{\mu\nu}(l) = p_{2\nu} t^{\mu\nu}(l). \tag{5.3.1}$$

This factor vanishes as $p_2 \to 0$, and since the integrand vanishes, so must the integral. Thus the full ghost-gluon vertex reduces to the bare vertex in Landau gauge as the momentum of the in-ghost vanishes.

One would imagine that this argument could be applied to all other Green's functions involving ghost fields. This is the case for four-point and higher functions but not for the two-point function. For a one-particle-irreducible four-point function (or higher), since there is no tree-level expression, the whole function must vanish. In the case of the ghost self-energy, when one considers the Schwinger-Dyson equation, it is immediately apparent that the whole equation is proportional to the external momentum squared, ie it has a non-zero dimension. Whilst the integrand of the loop term does indeed contain the factor above, when the factor p^2 is divided through to get an expression for the dressing function, this factor becomes

$$\frac{p_\nu}{p^2} t^{\mu\nu}(l) \tag{5.3.2}$$

which does not vanish as $p \to 0$. The ghost Schwinger-Dyson equation will be studied in detail in the next chapter.

Consider now the reducible terms occurring in the function $\Gamma_4^{deab}(\omega, p_1 - \omega; p_3, p_2)$. These terms are effectively those shown in fig. 5.2 but with full vertices. If either p_2 or p_3 vanishes in either graph, then the vertex connected to that line will become bare. Now since this vertex is bare, the graph is simply proportional to the out-ghost momentum contracted with the (transverse in the Landau gauge) gluon propagator, which by momentum conservation will have the same momentum. Thus both the reducible graphs vanish.

It is now possible to study the identity (5.1.32) as $p_2 \to 0$ (or equivalently $p_3 \to 0$). Recall that the identity is

$$gf^{cba}\left[G_3^{-1}p_3^{\nu}\tilde{\Gamma}_{\nu}(p_1,p_2,p_3)+G_2^{-1}p_2^{\nu}\tilde{\Gamma}_{\nu}(p_1,p_3,p_2)+p_1^2G_1^{-1}\right]=$$
$$-\frac{g}{2}f^{cde}p_1^2G_1^{-1}\int\frac{(-\imath)\,\bar{d}^d\omega}{\omega^2(p_1-\omega)^2}G_{\omega}G_{p_1-\omega}\Gamma_4^{deab}(\omega,p_1-\omega;p_3,p_2). \quad (5.3.3)$$

By the above arguments, the right-hand side vanishes since both the reducible and irreducible parts of the four-point function in the integrand vanish. The ghost-gluon vertex of the first term becomes bare but since as $p_2\to 0$, $p_3\to -p_1$, then this cancels the last term of the left-hand side. This leaves

$$G_2^{-1}p_2^{\nu}\tilde{\Gamma}_{\nu}(p_1,p_3,p_2)|_{p_2\to 0}=0. \quad (5.3.4)$$

Finally, it is now pertinent to show that $\tilde{Z}_1=1$ in Landau gauge [21, 43]. As before, the vertex is renormalised as follows

$$\overline{\tilde{\Gamma}}_{\mu}(p_i|\overline{\xi},\overline{g},\mu)=\tilde{Z}_1(\overline{\xi},\overline{g},\mu,\Lambda)\tilde{\Gamma}_{\mu}(p_i|\xi_b,g_b,\Lambda). \quad (5.3.5)$$

where μ is the renormalisation scale[3] and the divergence is simply expressed in terms of some quantity Λ[4]. The momentum arguments have been condensed to $(p_1,p_2,p_3)=p_i$. Now, the unrenormalised vertex function $\tilde{\Gamma}_{\mu}(p_i|\xi_b,g_b^2,\Lambda)$ in general gauge is singular. After the bare coupling and gauge parameters have been replaced by their renormalised counterparts

$$g_b^2=\frac{\tilde{Z}_1^2}{Z_3\tilde{Z}_3^2}\overline{g}^2,\quad \xi_b=Z_3\overline{\xi}, \quad (5.3.6)$$

the vertex function can be expressed (expanding out the Lorentz structure) as

$$\tilde{\Gamma}_{\mu}(p_i|\overline{\xi},\overline{g},\Lambda)=p_{1\mu}X(p_i|\overline{\xi},\overline{g},\Lambda)+p_{3\mu}Y(p_i|\overline{\xi},\overline{g},\Lambda). \quad (5.3.7)$$

The renormalisation equation above can thus be written

$$\overline{\tilde{\Gamma}}_{\mu}(p_i|\overline{\xi},\overline{g},\mu)=\tilde{Z}_1(\overline{\xi},\overline{g},\mu,\Lambda)\left\{p_{1\mu}X(p_i|\overline{\xi},\overline{g},\Lambda)+p_{3\mu}Y(p_i|\overline{\xi},\overline{g},\Lambda)\right\}. \quad (5.3.8)$$

Now, in the limit $p_2\to 0$ in Landau gauge, the unrenormalised vertex function reduces to it's tree-level form so

$$\left.\overline{\tilde{\Gamma}}_{\mu}(p_i|\overline{\xi},\overline{g},\mu)\right|_{\overline{\xi}=0,p_2\to 0}=\tilde{Z}_1(0,\overline{g},\mu,\Lambda)p_{1\mu}. \quad (5.3.9)$$

[3]Not to be confused with the Lorentz index!

[4]The specific regularisation procedure is not important – it is assumed that there exists some 'perfect' method

Now, since the tree-level vertex is explicitly finite, the renormalisation coefficient cannot be dependent on the regularisation parameter. Also, $\tilde{Z}_1$ is a dimensionless function so it cannot depend on the dimensionful renormalisation scale (the renormalised coupling is dimensionless). Furthermore, it cannot depend on the coupling since this has its own dependence on Λ and μ[5] so

$$\overline{\overline{\Gamma}}_\mu(p_i|\overline{\xi},\overline{g},\mu)\Big|_{\overline{\xi}=0,p_2\to 0} = \tilde{z}_1 p_{1\mu} \tag{5.3.10}$$

where $\tilde{z}_1$ is simply a constant determined by the renormalisation prescription. In perturbation theory, where one conventionally uses a subtraction scheme such as $\overline{MS}$, this constant would simply be the lowest order part of Z, ie. unity. The only scheme in which this constant is not unity is the momentum subtraction scheme where $\tilde{z}_1$ depends on how one defines the physical value of the vertex. However, in this thesis, we shall not be using the momentum subtraction scheme. Thus

$$\tilde{Z}_1 = 1. \tag{5.3.11}$$

The renormalisation coefficient is independent of the external momentum, and so, no matter what the external momentum scales are, the vertex needs no renormalisation. This means that the unrenormalised vertex (in Landau gauge) is explicitly finite[6] and is completely independent of the regularisation. Also the renormalised vertex is independent of the renormalisation scale.

The reduction of the full ghost-gluon vertex to it's bare form as the in-ghost momentum vanishes and the simple form of it's renormalisation coefficient will be used heavily in the next chapter. These two observations simplify the Schwinger-Dyson equations sufficiently to give at least a promising starting point for extracting useful information.

[5] Recall that Z can be defined in terms of either the renormalised of unrenormalised coupling.

[6] Remember that the unrenormalised vertex is expressed in terms of the renormalised coupling.

Chapter 6

Powerlaw Behaviour in Landau Gauge

In this chapter, the infrared behaviour of the QCD Schwinger-Dyson equations in Landau Gauge will be discussed. In previous chapters, the infrared behaviour of the truncated equations was extracted qualitatively from the dimensional consideration of the renormalisation scale dependence of each of the terms. Here, this argument will be refined and applied to the Yang-Mills sector of the theory. It will be seen that there is only one function necessary to completely describe the infrared behaviour of the propagators. Unfortunately, it is found that this function cannot be constrained completely, and so only a qualitative analysis can be done without introducing some form of ansatz.

In sections 2.3 and 2.4, it was the truncation of the Schwinger-Dyson equations that allowed the analysis to be done. In the case of section 2.3, the bare vertices provided an excellent means of introducing the idea behind the dimensional arguments. However, with bare vertices, the gauge invariance of the theory cannot hold. In the case of section 2.4, the angular approximations played a crucial role in the derivation of the results. In the light of the failure to find a simple identity relating the ghost-gluon vertex (and hence the triple-gluon vertex) to some combination of two-point functions (the previous two chapters), there is no natural truncation of the system and so one may be greatly discouraged. However, the dimensional technique for treating the IR region may be able to provide more information even without a specific truncation scheme. In order to do this, it will be necessary to refine the IR powerlaw arguments.

Consider a general two-point dressing function F in a massless gauge theory. It is a dimensionless function of a single variable. Now, in the renormalised theory, there are only two scales (with dimension momentum squared), the external scale p^2 and the renormalisation scale μ which this dressing function can depend on, anything else being simply a constant (for example the number of quark flavours or number of colours). Thus, the argument of the function must be the ratio of the two scales in order to be dimensionless. As this ratio tends to zero, the function can be written as a Laurent expansion and this expansion will have a unique lowest power which is a constant. Thus, one can write to lowest order in the expansion

$$F(p^2,\mu) = F\left(\frac{p^2}{\mu}\right) \stackrel{p^2\to 0}{\sim} \left(\frac{p^2}{\mu}\right)^{\alpha}. \tag{6.0.1}$$

One may imagine that a complete expansion for F could be made in this manner. However, in practice this cannot be done since the general solution for F implied by the renormalisation group involves dependence on the running coupling of the theory, which itself has an implicit dependence on the scale. It is only the lowest power of the expansion that can be written in a simple way.

This principle will be applied to the two-point functions of QCD. These functions obey a coupled set of Schwinger-Dyson equations which are in practice insoluble, but by using the IR powerlaw behaviour noted above and by demanding that the equations reproduce consistent behaviour, it is possible to at least get a feel for the IR properties of the theory. In the first part of this chapter, the general notations will be presented. The second part looks at a toy model of a coupled set of Schwinger-Dyson equations that involves most of the features present in the analysis. These features include vertices that respect qualitatively the ideas of renormalisability, the improved angular approximations necessary and the interplay between the IR and UV parts of the integrals that will be shown to be inherent in the study of powerlaw behaviour. The next part then looks in more detail at the ghost-gluon system without quarks. By using a quite general ansatz for the ghost-gluon vertex, the important features of the system that will be seen in practical studies are highlighted.

One issue to be addressed immediately is the possible existence of an IR scale (κ) which is associated with the transition from the perturbative to the non-perturbative regions. This scale could for instance be a physical scale such as the proton mass squared. The view taken throughout this work is that although this scale may exist, the Schwinger-Dyson equations do not explicitly contain information about it from the outset and so, this scale is not used in determining their solution. It is hoped that the equations themselves will give rise to a solution that manifests this scale somehow. The most likely scenario is that this scale would be associated with the transition region where powerlaw behaviour is replaced with logarithmic behaviour. Since the idea is to work with only the lowest power, then this scale will not interfere with the discussion.[1]

6.1 Notation and Conventions

In this chapter, the renormalisation constants of the theory are defined in the following way

$$\begin{aligned}
\overline{J}^{-1}(p^2|\mu) &= Z_3^{-1}(\mu,\Lambda)\, J^{-1}(p^2|\Lambda) \\
\overline{G}(p^2|\mu) &= \tilde{Z}_3^{-1}(\mu,\Lambda)\, G(p^2|\Lambda) \\
\overline{F}(p^2|\mu) &= Z_2^{-1}(\mu,\Lambda)\, F(p^2|\Lambda) \\
\overline{\tilde{\Gamma}}_\mu(p_1,p_2,p_3|\mu) &= \tilde{Z}_1(\mu,\Lambda)\, \tilde{\Gamma}_\mu(p_1,p_2,p_3|\Lambda) \\
\overline{\Gamma}_{\mu\nu\rho}(p_1,p_2,p_3|\mu) &= Z_1(\mu,\Lambda)\, \Gamma_{\mu\nu\rho}(p_1,p_2,p_3|\Lambda) \\
\overline{\Gamma}^F_\mu(p_1,p_2,p_3|\mu) &= Z_{1F}(\mu,\Lambda)\, \Gamma^F_\mu(p_1,p_2,p_3|\Lambda)
\end{aligned} \tag{6.1.1}$$

In Landau gauge, $\tilde{Z}_1(\mu,\Lambda)=1$. The renormalised coupling is written as

$$g^2 = \frac{\tilde{Z}_1^2}{Z_3\tilde{Z}_3^2}\,\overline{g}^2(\mu,\Lambda) \tag{6.1.2}$$

and the Slavnov-Taylor identity (which ensures that the renormalised coupling is independent of the vertex used to define it) is written here in it's renormalisation coefficient form [21, 43]:

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} = \frac{Z_2}{Z_{1F}}. \tag{6.1.3}$$

In the IR the renormalised two-point functions reduce to their powerlaw forms which are written as

$$\begin{aligned}
\overline{J}^{-1}(p^2|\mu) &\overset{p^2\to 0}{=} a\left(\frac{p^2}{\mu}\right)^{\alpha} \\
\overline{G}(p^2|\mu) &\overset{p^2\to 0}{=} b\left(\frac{p^2}{\mu}\right)^{\beta} \\
\overline{F}(p^2|\mu) &\overset{p^2\to 0}{=} f\left(\frac{p^2}{\mu}\right)^{\gamma}.
\end{aligned} \tag{6.1.4}$$

[1] One may also wonder about the perturbatively generated Λ_{QCD}. However, this scale is used to give a physical reference scale (instead of the arbitrary renormalisation scale) from which the renormalisation group can be used to find results at some other external scale, so there are still only two scales with which to work with.

6.2 A Toy Model

In this section, a toy model of Landau gauge QCD will be studied in order to introduce certain technical issues relating to subsequent work. The model incorporates many of the salient features of the Schwinger-Dyson equations necessary to investigate the IR powerlaw behaviour of the theory and is intended only to provide examples of these such that subsequent work will be made more transparent. The gluon self-interaction vertices are completely neglected for heuristic purposes. The ghost-gluon vertex is taken to be bare and the quark-gluon vertex is reduced to a minimal form which respects the Slavnov-Taylor identity in the form written in the previous section. These vertices will be seen to have the right features necessary for a consistent discussion. It is understood that the gluon propagator will of course not be transverse, but that is not the issue here. The unrenormalised vertices are written as

$$\begin{aligned} \widetilde{\Gamma}_\mu(p_1,p_2,p_3|\Lambda) &= p_{1\mu} \\ \Gamma^F_\mu(p_1,p_2,p_3|\Lambda) &= \frac{1}{2}\gamma_\mu G_3\left[F_1^{-1}+F_2^{-1}\right]. \end{aligned} \tag{6.2.1}$$

6.2.1 The Schwinger-Dyson Equations

It is now necessary to write down the Schwinger-Dyson equations (Minkowski space). In order to make this chapter more self-contained and to highlight certain points such as the contraction of the gluon equation and the different vertex forms, the equations will be derived in full although certain parts will be identical to previous discussion. Using the Feynman rules of section 1.7, the ghost equation with a bare vertex is written as

$$\delta^{ad}p^2G_p^{-1} = \delta^{ad}p^2 - \int(-\imath)\frac{\bar{d}^4\omega\, G_{p-\omega}J_\omega^{-1}}{\omega^2(p-\omega)^2}(-\imath gf^{abc})(-\imath gf^{bdc})p_\alpha(p-\omega)_\beta t^{\alpha\beta}(\omega), \tag{6.2.2}$$

so that

$$G_p^{-1} = 1+\imath g^2C_A\int\frac{\bar{d}^4\omega G_{p-\omega}J_\omega^{-1}}{p^2\omega^2(p-\omega)^2}p_\alpha p_\beta t^{\alpha\beta}(\omega). \tag{6.2.3}$$

The quark equation is

$$\begin{aligned} \delta_{ij}\not{p}F_p^{-1} &= \delta_{ij}\not{p} - \int\frac{(-\imath)\,\bar{d}^4\omega\, F_\omega J_{p-\omega}^{-1}}{\omega^2(p-\omega)^2}(-g\gamma_\mu T^a_{ik})\not{\omega}(-gT^a_{kj})\Gamma^F_\nu(\omega,-p;p-\omega)t^{\mu\nu}(p-\omega) \\ &= \delta_{ij}\not{p} + \delta_{ij}\imath g^2\frac{C_A^2-1}{2C_A}\int\frac{\bar{d}^4\omega\, F_\omega J_{p-\omega}^{-1}G_{p-\omega}}{\omega^2(p-\omega)^2}\gamma_\mu\not{\omega}\gamma_\nu t^{\mu\nu}(p-\omega)\frac{1}{2}\left[F_\omega^{-1}+F_p^{-1}\right]. \end{aligned} \tag{6.2.4}$$

In order to turn this into a scalar equation, one pre-multiplies by $\not p$ and takes the trace over the Dirac gamma matrices. This gives

$$4p^2F_p^{-1} = 4p^2 + \imath g^2\frac{C_A^2-1}{2C_A}\int \frac{d\!\!{}^-{}^4\omega\; F_\omega J_{p-\omega}^{-1} G_{p-\omega}}{\omega^2(p-\omega)^2}\frac{1}{2}\left[F_\omega^{-1}+F_p^{-1}\right]t^{\mu\nu}(p-\omega)Tr\left\{\not p\gamma_\mu\not\omega\gamma_\nu\right\}. \tag{6.2.5}$$

Explicitly evaluating the trace and doing the tensor algebra

$$F_p^{-1} = 1 + \imath g^2\frac{C_A^2-1}{2C_A}\int \frac{d\!\!{}^-{}^4\omega\; F_\omega J_{p-\omega}^{-1} G_{p-\omega}}{\omega^2(p-\omega)^2}\frac{1}{2}\left[F_\omega^{-1}+F_p^{-1}\right]\times$$
$$\left[2\frac{\omega^2}{(p-\omega)^2}\left(1-\frac{p\cdot\omega^2}{p^2\omega^2}\right)+\frac{3}{2}\left(\frac{(p-\omega^2)}{p^2}-1-\frac{\omega^2}{p^2}\right)\right]. \tag{6.2.6}$$

The gluon equation is written (with a single quark flavour) as

$$\begin{aligned}\delta^{ad}p^2t^{\mu\nu}(p)J_p &= \delta^{ad}p^2t^{\mu\nu}(p) + \int(-\imath)\frac{d\!\!{}^-{}^4\omega\, G_{p-\omega}G_\omega}{\omega^2(p-\omega)^2}(-\imath g f^{abc})(-\imath g f^{bdc})(-\omega)^\mu(p-\omega)^\nu \\ &+\int(-\imath)\frac{d\!\!{}^-{}^4\omega\, G_pF_{p-\omega}F_\omega}{\omega^2(p-\omega)^2}(-gT^a_{ij})(-gT^d_{ji})\frac{1}{2}\left[F_{p-\omega}^{-1}+F_\omega^{-1}\right]Tr\left\{\gamma^\mu\not\omega\gamma^\nu(\not\omega-\not p)\right\} \\ &= \delta^{ad}p^2t^{\mu\nu}(p) + \imath g^2C_A\delta^{ad}\int\frac{d\!\!{}^-{}^4\omega\, G_{p-\omega}G_\omega}{\omega^2(p-\omega)^2}\omega^\mu(p-\omega)^\nu \\ &-\imath g^2\delta^{ad}\frac{1}{4}\int\frac{d\!\!{}^-{}^4\omega\, G_pF_{p-\omega}F_\omega}{\omega^2(p-\omega)^2}\left[F_{p-\omega}^{-1}+F_\omega^{-1}\right]Tr\left\{\gamma^\mu\not\omega\gamma^\nu(\not\omega-\not p)\right\}.\end{aligned} \tag{6.2.7}$$

Evaluating the trace

$$p^2t^{\mu\nu}(p)J_p = p^2t^{\mu\nu}(p) + \imath g^2C_A\int\frac{d\!\!{}^-{}^4\omega\, G_{p-\omega}G_\omega}{\omega^2(p-\omega)^2}\omega^\mu(p-\omega)^\nu$$
$$-\imath g^2\int\frac{d\!\!{}^-{}^4\omega\, G_pF_{p-\omega}F_\omega}{\omega^2(p-\omega)^2}\left[F_{p-\omega}^{-1}+F_\omega^{-1}\right]\left[\omega^\mu(\omega-p)^\nu+\omega^\mu(\omega-p)^\nu-g^{\mu\nu}\omega\cdot(\omega-p)\right]. \tag{6.2.8}$$

Now, this is a tensor equation and for convenience it is suitable to contract with $R_{\mu\nu} = \left(g_{\mu\nu}-4\frac{p_\mu p_\nu}{p^2}\right)/(3p^2)$. This contraction will be discussed further in the next section, because there are some considerably important points attached with it. That aside, the gluon equation simplifies to

$$J_p = 1+\imath g^2\int\frac{d\!\!{}^-{}^4\omega}{p^2\omega^2(p-\omega)^2}R_{\mu\nu}\omega^\mu(p-\omega)^\nu\left[C_AG_{p-\omega}G_\omega+2G_pF_{p-\omega}F_\omega\left(F_{p-\omega}^{-1}+F_\omega^{-1}\right)\right]. \tag{6.2.9}$$

The three equations (6.2.3,6.2.6,6.2.9) must now be studied. In order to extract any kind of information, one Wick rotates into Euclidean space. This of course raises many issues related to the validity of such a procedure but from our point of view, the whole theory could have been defined from the outset in Euclidean space and so these concerns are not important. The next step is to regulate the integrals with an ultraviolet cutoff scale Λ. Again there are many questions raised about the validity of the approach – these will become especially important in the next section.

After the Wick rotation, our toy Schwinger-Dyson equations become

$$\begin{aligned} G_p^{-1} &= 1 - g^2 C_A \int \frac{d\!\!{}^-{}^4\omega G_{p-\omega} J_\omega^{-1}}{\omega^2 (p-\omega)^2} \sin^2\theta, \\ F_p^{-1} &= 1 - g^2 \frac{C_A^2 - 1}{2C_A} \int \frac{d\!\!{}^-{}^4\omega\, F_\omega J_{p-\omega}^{-1} G_{p-\omega}}{\omega^2 (p-\omega)^2} \frac{1}{2}\left[F_\omega^{-1} + F_p^{-1}\right] \times \\ &\left[2\frac{\omega^2}{(p-\omega)^2}\sin^2\theta + \frac{3}{2}\left(\frac{(p-\omega^2)}{p^2} - 1 - \frac{\omega^2}{p^2}\right)\right], \\ J_p &= 1 - g^2 \int \frac{d\!\!{}^-{}^4\omega}{p^2\omega^2(p-\omega)^2}\left[-\frac{4}{3}\omega^2\sin^2\theta + \frac{1}{2}(p-\omega)^2 + \frac{1}{2}\omega^2 - \frac{1}{2}p^2\right] \times \\ &\left[C_A G_{p-\omega} G_\omega + 2G_p F_{p-\omega} F_w \left(F_{p-\omega}^{-1} + F_\omega^{-1}\right)\right], \end{aligned} \tag{6.2.10}$$

where θ is the angle between vectors p and ω. The integration measure can be split into radial and angular parts and it is convenient to rename the momenta such that $p^2 = x$, $\omega^2 = y$, $(p-\omega)^2 = x + y - 2\sqrt{xy}\cos\theta = z$. The equations thus become

$$\begin{aligned} G_x^{-1} &= 1 - \frac{g^2}{16\pi^2} C_A \int_0^\Lambda dy\; J_y^{-1} \frac{2}{\pi} \int_0^\pi d\theta\; \sin^4\theta \frac{G_z}{z}, \\ F_x^{-1} &= 1 \\ &- \frac{g^2}{16\pi^2}\frac{C_A^2-1}{2C_A} \int_0^\Lambda dy\; \frac{1}{2}\left[1 + \frac{F_y}{F_x}\right] \frac{2}{\pi}\int_0^\pi d\theta\; \sin^2\theta \frac{J_z^{-1} G_z}{z}\left[2\frac{y}{z}\sin^2\theta + \frac{3}{2}\left(\frac{z}{x} - 1 - \frac{y}{x}\right)\right], \\ J_x &= 1 \\ &- \frac{g^2}{16\pi^2}\int_0^\Lambda dy\; \frac{2}{\pi}\int_0^\pi d\theta\; \sin^2\theta \frac{1}{z}\left[-\frac{4y}{3x}\sin^2\theta + \frac{z}{2x} + \frac{y}{2x} - \frac{1}{2}\right]\left[C_A G_z G_y + 2G_x\left(F_y + F_z\right)\right]. \end{aligned} \tag{6.2.11}$$

It is now appropriate to re-express these equations in terms of renormalised quantities. This is done using the earlier definitions of the renormalisation coefficients. Since all quantities in the equations are renormalised, it is possible to omit the bars. The equations are

$$G_{x,\mu}^{-1} = \tilde{Z}_3 - \lambda C_A \int_0^\Lambda dy\; J_{y,\mu}^{-1} \frac{2}{\pi}\int_0^\pi d\theta\; \sin^4\theta \frac{G_{z,\mu}}{z}, \tag{6.2.12}$$

$$F^{-1}_{x,\mu} = Z_2 - Z_{1F}\lambda\frac{C_A^2-1}{2C_A}\int_0^\Lambda dy\,\frac{1}{2}\left[1+\frac{F_{y,\mu}}{F_{x,\mu}}\right]\frac{2}{\pi}\int_0^\pi d\theta\,\sin^2\theta\frac{J^{-1}_{z,\mu}G_{z,\mu}}{z}\times$$
$$\left[2\frac{y}{z}\sin^2\theta+\frac{3}{2}\left(\frac{z}{x}-1-\frac{y}{x}\right)\right], \qquad (6.2.13)$$
$$J_{x,\mu} = Z_3 - \lambda\int_0^\Lambda dy\,\frac{2}{\pi}\int_0^\pi d\theta\,\sin^2\theta\frac{1}{z}\left[-\frac{4y}{3x}\sin^2\theta+\frac{z}{2x}+\frac{y}{2x}-\frac{1}{2}\right]\times$$
$$\left[C_A G_{z,\mu}G_{y,\mu}+2Z_{1F}G_{x,\mu}\left(F_{y,\mu}+F_{z,\mu}\right)\right], \qquad (6.2.14)$$

where we have used $\tilde{Z}_1=1$ and $\lambda = g^2(\mu,\Lambda)/(16\pi^2)$.

6.2.2 A Simple Integral and the Powerlaw Approach

In order to make the subsequent discussion clearer, a simple (though unphysical) integral will be studied here. This integral shows that whilst it is relevant to expand a two-point dressing function as a Laurent expansion in x/μ, it is not the same as expanding the Schwinger-Dyson integral due to the presence of the additional scale Λ. However, it will be shown that from the point of view of the power of the lowest term in the Laurent expansion, there is no difference.

A general renormalised Schwinger-Dyson equation has the form

$$G^{-1}(x,\mu) = Z(\Lambda,\mu) - I \qquad (6.2.15)$$

where I is some integral. Suppose that the (complete) integral term of a Schwinger-Dyson equation was

$$I = \int_x^\Lambda \frac{dy}{\mu}\ln\left(1+\frac{y}{\mu}+\frac{x}{\mu}\right) \qquad (6.2.16)$$

where x is the IR external scale, μ is the renormalisation scale and Λ is the arbitrarily large cutoff scale. The result of this integral is:

$$I = \left(1+\frac{\Lambda}{\mu}+\frac{x}{\mu}\right)\ln\left(1+\frac{\Lambda}{\mu}+\frac{x}{\mu}\right)+\frac{x}{\mu}-\frac{\Lambda}{\mu}-\left(1+2\frac{x}{\mu}\right)\ln\left(1+2\frac{x}{\mu}\right). \qquad (6.2.17)$$

As $x\to 0$, this becomes

$$I = \left(1+\frac{\Lambda}{\mu}\right)\ln\left(1+\frac{\Lambda}{\mu}\right)-\frac{\Lambda}{\mu}+\frac{x}{\mu}\ln\left(1+\frac{\Lambda}{\mu}\right)-\frac{1}{2}\frac{x^2}{\mu^2}\left[4-\frac{1}{1+\frac{\Lambda}{\mu}}\right]+O(x^3). \qquad (6.2.18)$$

It is clear that the terms independent of x should be identified with the renormalisation coefficient Z and these would be subtracted away. The rest of the integral is more complicated due to the mixing of terms dependent on x and terms dependent on Λ. This is the general behaviour of such integrals. However, it is also immediately apparent that this integral is not physical because it cannot be renormalised properly. This is the principle that will be used – it will be assumed that the integrals in the Schwinger-Dyson equations have a renormalisable form such that the dependence on x and the dependence on Λ can be separated. What this means in practice is that the equation will be expanded in powers of $1/\mu$. If the integral is renormalisable, then it is in principle possible to extract the lowest (but only the lowest) power of x/μ from the integral. This will become clear in the following discussion.

The idea is as follows – all 2-point functions will be replaced by their lowest powerlaw term. On the left-hand side of the equation, this is clearly valid in the regime where the external momentum scale is vanishingly small. In a loop, as the argument of any internal propagator factor vanishes then this is also valid. Now, as the argument of any unrenormalised internal propagator approaches the UV, the function must become unity. This is achieved by noting the following

$$\begin{aligned}\overline{G}(p^2|\mu)\tilde{Z}_3(\mu,\Lambda) &= G(p^2|\Lambda)\\ &\sim \left(\frac{p^2}{\mu}\right)^{\beta}\tilde{Z}_3(\mu,\Lambda).\end{aligned}$$

The renormalisation coefficient $\tilde{Z}_3$ must be identified as that function that removes the μ-dependence of the renormalised function such that the unrenormalised function is independent of the renormalisation scale μ. Thus

$$\tilde{Z}_3(\mu,\Lambda) \sim \left(\frac{\Lambda}{\mu}\right)^{-\beta}, \tag{6.2.19}$$

so that

$$G(p^2|\Lambda) \sim \left(\frac{p^2}{\Lambda}\right)^{\beta} \stackrel{p^2\to\Lambda}{\sim} 1. \tag{6.2.20}$$

The renormalised Schwinger-Dyson equation is in fact only just the unrenormalised equation with all unrenormalised quantities replaced by their appropriate renormalisation coefficients multiplied by the renormalised function. Thus, the UV properties of the internal propagators will be consistent as long as the renormalisation coefficients have been identified correctly. Since only a simple power can satisfy the multiplicative renormalisability condition above, then only a single power can be used.

The vertices have their complete μ-dependence specified via the Slavnov-Taylor identity. In its renormalisation coefficient form, this is written as

$$\frac{Z_3}{Z_1} = \frac{\tilde{Z}_3}{\tilde{Z}_1} = \frac{Z_2}{Z_{1F}}. \tag{6.2.21}$$

In Landau gauge, $\tilde{Z}_1 = 1$ and so it can be seen that all the μ-dependence of the remaining vertices can be expressed in terms of the propagator renormalisation coefficients. So again, if one can identify the renormalisation coefficients, then the full μ-dependence of the vertices can be extracted at the lowest power in $1/\mu$. The problem of the vertices will be seen to be one of determining the behaviour as two of the momenta approach the cutoff Λ and the third (external) momentum vanishes such that the proper x and Λ-dependence of the integral is extracted (the μ-dependence should already be correct). One issue connected to this regards the major advantage of the powerlaw approach. The UV behaviour of the integral is *not* the same as the perturbative expression due to the presence of the explicitly non-perturbative scale x. The integral must give rise to the lowest powerlaw term on the left-hand side of the equation and the only candidates for this must have the lowest power of $1/\mu$. The Λ-dependence coming from the vertices is a secondary concern and can be subsequently derived in order to give consistency. The powerlaw approach gives the natural framework from which the x-dependence of the integral can in principle be extracted. With a suitable framework it is possible to see what parts of the various vertices are then important and concentrate on only these. This will be seen in practice in later sections when using a general form for the ghost-gluon vertex.

Lastly, the renormalised coupling has it's μ-dependence given directly by the renormalisation coefficients in the following way

$$\overline{g}^2(\mu, \Lambda) = \frac{Z_3 \tilde{Z}_3^2}{\tilde{Z}_1^2} g^2. \tag{6.2.22}$$

In summary, the tactic is to try and identify the renormalisation coefficients for the propagators at the lowest order in an expansion of $1/\mu$. These coefficients depend on μ and Λ. Clearly, an expansion in powers of Λ/μ cannot be convergent, since Λ is arbitrarily large. However, this does not matter. What is required is simply that these coefficients remove the μ-dependence of the renormalised functions. The renormalisation coefficients thus need only the lowest $1/\mu$ term and the rest is irrelevant. The form of the Schwinger-Dyson equations will automatically give this term. Since in the integral, all lowest factors of $1/\mu$ coming from the propagators, vertices, renormalisation coefficients and the coupling have been isolated, then the integral can only give the terms with the lowest power of $1/\mu$. The propagator renormalisation coefficient is then that part of the integral that is explicitly dependent on Λ and independent of x. For consistency, this must have the same power of $1/\mu$ as the renormalised function on the left-hand side of the equation in order to satisfy the multiplicative renormalisability condition.

After the renormalisation coefficients of the coupled equations have been identified and the UV divergences removed, one is left in principle with only those parts of the integral that are independent of Λ. However, in practice this will not be the case since the powers are not known. The resulting integral will indeed have only those terms with the identified lowest power of $1/\mu$ but there will be several such terms, each with a certain x and Λ-dependence at a certain power. It is then necessary to consider each of these terms in turn. By setting the unknown power of Λ to zero in one of the terms, (such that the term is definitely the lowest power of x/μ) and demanding that all the other terms then vanish due to positive powers of x/Λ, one can look for consistency in the coupled set of Schwinger-Dyson equations. If consistency can be found, then the unknown powers have been constrained. This is what we are after.

6.2.3 The Ghost Schwinger-Dyson Equation

Now consider a real Schwinger-Dyson equation. It is easiest to study the ghost equation (6.2.12) first since this is the simplest of all the Schwinger-Dyson equations in QCD. The arguments of the preceding section will be reiterated for clarity, since now they are in a definite context.

On the left-hand side (lhs) of (6.2.12) as $x \to 0$, the function will reduce to it's IR lowest powerlaw form. This means that it has the following μ-dependence:

$$\text{lhs} \sim \left(\frac{1}{\mu}\right)^{-\beta}. \tag{6.2.23}$$

The right-hand side (rhs) of the equation must also have this dependence. Now consider the form of the integral of (6.2.12). Since the vertex is bare and the integration knows nothing of the renormalisation scale μ, the only μ-dependence can come from the two propagator functions. Again, it is emphasized that in this IR analysis, one is looking for the lowest power of $1/\mu$, so it is valid to replace the unknown propagator functions with their respective IR powerlaw forms. This gives

$$\text{rhs} \sim \tilde{Z}_3 - \lambda \left(\frac{1}{\mu}\right)^{\alpha+\beta} . \tag{6.2.24}$$

Recall that the quantity λ has it's own μ-dependence given by the renormalisation coefficients Z_3 and $\tilde{Z}_3$. To ensure that multiplicative renormalisability (MR) holds (in the powerlaw sense), these coefficients must have a μ-dependence that cancels that of their respective renormalised propagator functions such that the unrenormalised functions do not depend on μ. In general, they will be complicated functions but they must contain such factors so as to ensure that the assumed IR powerlaw behaviour is multiplicatively renormalisable. Thus,

$$\tilde{Z}_3 \sim \left(\frac{\Lambda}{\mu}\right)^{-\beta}, \; Z_3 \sim \left(\frac{\Lambda}{\mu}\right)^{-\alpha} \tag{6.2.25}$$

which (recalling the earlier definition) gives rise to

$$\lambda \sim Z_3 \tilde{Z}_3^2 \sim \left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} . \tag{6.2.26}$$

One can then see that the rhs of the ghost equation naturally has the same μ-dependence as the lhs. This argument could have just as easily been used in reverse. That the rhs must have this dependence for consistency means that λ and $\tilde{Z}_3$ must have the above forms and MR is then seen to hold (for this IR powerlaw analysis)[2].

It is pertinent to discuss here the main difference between this and previous studies of the same equation [1, 2, 3, 33, 34]. That in the renormalised equation, the coupling is allowed to also be more than simply a constant has not been emphasized explicitly before. If one were to take λ to be simply a constant then automatically $\alpha = -2\beta$. That previous authors found that this gives rise to wonderfully consistent results gives credence to the original assertion that λ is indeed a constant. The ethos of this work is to show that this is the only consistent scenario in the IR.

The next point to be raised regards the renormalisation scheme being used. The often used momentum subtraction scheme is not used here, since this violates one of the greatest

[2]Again, note that this is only true for the single lowest power. In general, there is no simple power series that can keep multiplicative renormalisability. Studying the dimensionality of the equations that define the renormalisation coefficients one is led to the conclusion that naively, there can be only a single term for each function with a unique power for each equation. However, this is not what the renormalisation group implies. The complete solution certainly involves more than a simple power law. The resolution to this is that the characteristic powers and coefficients associated with each equation are actually dependent on the coupling, which itself has a dependence on both the renormalisation scale and the cutoff. It is only in this IR analysis that the simple power behaviour may hold.

simplifications of QCD in the Landau gauge – namely that for the full theory $\tilde{Z}_1 = 1$. To recap, in the Schwinger-Dyson equations, a slightly different scheme must be employed. Put quite simply, the renormalisation coefficients will be defined as a powerlaw term that removes any terms dependent on Λ and independent of x. If the power is zero (ie the term is a constant) then there is necessarily a confusion as to whether this constant is part of the renormalisation coefficient or part of the renormalised function. This will be seen in practice later. The use of this scheme allows one to see that naively, there is an interplay between the IR and the UV terms. Such interplay will provide many interesting points.

Having introduced the powerlaw nature of the equation, it is now possible to proceed. It is necessary to define precisely the form of the renormalisation coefficients and the quantity λ:

$$\tilde{Z}_3 = \tilde{z}_3 \left(\frac{\Lambda}{\mu}\right)^{-\beta}, \; Z_3 = z_3 \left(\frac{\Lambda}{\mu}\right)^{-\alpha}, \tag{6.2.27}$$

$$\lambda = z_3 \tilde{z}_3^2 \frac{g^2}{16\pi^2} \left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} = \lambda' \left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}. \tag{6.2.28}$$

The ghost Schwinger-Dyson equation (6.2.12) can now be written as

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3 \left(\frac{\Lambda}{\mu}\right)^{-\beta} - ab^2 C_A \lambda' \left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} \left(\frac{x}{\mu}\right)^{\alpha+\beta} \int_0^\Lambda \frac{dy}{x} \left(\frac{y}{x}\right)^{\alpha} \frac{2}{\pi} \int_0^\pi d\theta \, \sin^4\theta \left(\frac{z}{x}\right)^{\beta-1}. \tag{6.2.29}$$

It is now necessary to do the angular integration. If one were to do the angular integrals exactly, the result would be a combination of hypergeometric functions. The general angular integral is actually

$$\int_0^\pi d\theta \, \sin^{2r}(\theta) \left(\frac{z}{x}\right)^n = \beta\left(r+\frac{1}{2},\frac{1}{2}\right) \left(\frac{y_>}{x}\right)^n {}_2F_1\!\left(-n, -n-r; r+1; \frac{y_<}{y_>}\right), \; r \geq 1 \tag{6.2.30}$$

where

$$y_> = \begin{cases} x & y < x \\ y & y > x \end{cases}, \; \frac{y_<}{y_>} = \begin{cases} y/x & y < x \\ x/y & y > x \end{cases} \tag{6.2.31}$$

and

$$\beta\left(r+\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(r+1)}. \tag{6.2.32}$$

In order to make the subsequent discussion tractable, it is necessary to make the following approximation. The hypergeometric function will be replaced by the first two non-zero terms in it's expansion with respect to the argument $y_</y_>$. For the angular integral in the ghost equation, this becomes

$$\frac{2}{\pi}\int_0^\pi d\theta\ \sin^4\theta\left(\frac{z}{x}\right)^{\beta-1} = \frac{3}{4}\left(\frac{y_>}{x}\right)^{\beta-1} {}_2F_1\left(1-\beta,-1-\beta;3;\frac{y_<}{y_>}\right)$$
$$\approx \frac{3}{4}\begin{cases}\left[1+\frac{(\beta^2-1)}{3}\frac{y}{x}\right] & y<x \\ \left(\frac{y}{x}\right)^{\beta-1}\left[1+\frac{(\beta^2-1)}{3}\frac{x}{y}\right] & y>x\end{cases} \qquad (6.2.33)$$

The validity of this approximation can be tested directly using MAPLE to plot the difference between the exact and approximated forms for a range of β and $y_</y_>$. Typically, this approximation leads to an error of around $5-10\%$. However, the justification for this approximation does not lie in it's numerical accuracy, but rather will be seen to be the necessity for a way of looking at the UV part of the radial integrals. It is to be noted that this approximation is vastly more accurate than the widely used y-max approximation (which in fact only retains the first term of the expansion) because it includes at least in a crude way the dependence of the integral on the unknown power β. This allows the approximation to follow the shape of the hypergeometric function.

Substituting the angular integrals into (6.2.29) and using appropriate changes of variable gives

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}\times$$
$$\left\{\int_0^1 dt\left[t^\alpha+\frac{(\beta^2-1)}{3}t^{\alpha+1}\right]+\int_1^{\Lambda/x}dt\left[t^{\alpha+\beta-1}+\frac{(\beta^2-1)}{3}t^{\alpha+\beta-2}\right]\right\}. \qquad (6.2.34)$$

It is immediately apparent that the lower integral and lower limit of the upper integral are simply numbers. This is the case for all such integrals. On important point to note is that in this lower integral $\alpha > -1$ must be true. This ensures that there is no IR regularisation needed. If one were to evaluate this integral with integration over the ghost momentum (rather than the gluon momentum), then one would obtain the bound $\beta > -2$. In the upper integral, there are similar restrictions. Because this whole analysis is based on a powerlaw approach, there can be no logarithms mixing with the lowest power. This means that $\alpha+\beta \neq 0, 1$. This will prove to be a very useful restriction. Proceeding with the integration, one gets

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\times$$
$$\left\{\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}I_{gh}(\alpha,\beta)+\frac{1}{(\alpha+\beta)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}+\frac{(\beta^2-1)}{3(\alpha+\beta-1)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\frac{x}{\Lambda}\right\}. \qquad (6.2.35)$$

where $I_{gh}(\alpha,\beta)$ denotes the combination of the lower integral and the lower limit of the upper integral, which is a pure number.

There are several things to discuss at this stage. The first is the final form of the upper integral. If one were to try and do the angular integrals exactly, the resulting upper radial integral does not have the convenient limits $0-1$ which allow the radial integral to be done nicely. Instead, there is the more awkward Λ/x. Thus, the exact integral would be some incomplete hypergeometric function. This can be expanded as a power series in x/Λ, but the fact remains that one would end up with a similar form to the above (albeit with different numerator coefficients). The result would still be terms with the same powers of Λ. The next term in the series would have the factor x^2/Λ^2. However, recalling the IR bound that the lowest power of the overall equation, $\beta > -2$, this term must definitely be subleading and can play no part. This is the justification for the angular approximation - only the first two terms in the series will contribute and whilst it is known that the numerator coefficient will in general not be correct[3], it will become clear that this is of no consequence. It is only the fact that the powers of x and the denominators (whose zeroes indicate the logarithmic bounds) will be as above that is important.

The second point to discuss is the retention of the upper part of the integral. In previous studies, this part of the integral has been assumed to only contribute to the renormalisation and has been removed along with $\tilde{Z}_3$. The central theme to this work is that this procedure is not valid. It is asserted that until proven otherwise, it is entirely possible for the UV part of the integral to give rise to terms that may be contributing to the lowest IR power or will at the very least restrict the IR behaviour. In the above equation, this corresponds to the possibilities that $\beta = 0$ or $\beta = -1$. The point is that these cases must be considered in detail.

The third point is that this UV term proportional to x/Λ gives a lower bound for β. In the equation, one is looking for the most singular term that gives a consistent value for β. There are three possibilities here: $-\beta = 1, 0$ or $-\beta = \alpha + \beta$. If it is shown that $\beta \neq 0$, then it will take on the lower of the remaining two values. So if $-\beta = \alpha+\beta$ then $\beta > -1$ (it cannot be equal to -1, since this would give rise to a logarithm, ie. $\alpha+\beta = 1$), on the other hand if $-\beta = 1$ then $\alpha + \beta > 1$ - either way $\beta \geq -1$. One may be tempted to say that the numerator of the x/Λ term vanishes in the above equation, but then one must

[3]This is clearly the case when one considers the other momentum routing of the equation. The numerator coefficient is $\alpha(\alpha-2)$ instead of β^2-1.

remember that the value of this numerator is due to the angular approximation.

The final point is that the three possibilities mentioned above are mutually exclusive. This will be important when one considers the coupled system.

6.2.4 The Quark Schwinger-Dyson Equation

Now consider the quark Schwinger-Dyson equation (6.2.13). This can be treated in exactly the same way as the ghost equation but now one encounters an extra renormalisation coefficient Z_{1F}. This coefficient is given by the Slavnov-Taylor identity (2.1.9). In the notation employed here

$$Z_{1F} = z_2\tilde{z}_3^{-1}\left(\frac{\Lambda}{\mu}\right)^{\beta-\gamma}. \tag{6.2.36}$$

Using the powerlaw description, (6.2.13) can be rewritten as

$$\left(\frac{x}{\mu}\right)^{-\gamma} = fz_2\left(\frac{\Lambda}{\mu}\right)^{-\gamma} - abf\lambda'\frac{C_A^2-1}{2C_A}z_2\tilde{z}_3^{-1}\left(\frac{\Lambda}{\mu}\right)^{-\alpha-\beta-\gamma}\left(\frac{x}{\mu}\right)^{\alpha+\beta}\int_0^\Lambda\frac{dy}{x}\frac{1}{2}\left[1+\left(\frac{y}{x}\right)^\gamma\right]\times$$
$$\frac{2}{\pi}\int_0^\pi d\theta\,\sin^2\theta\left(\frac{z}{x}\right)^{\alpha+\beta-1}\left[2\frac{y}{z}\sin^2\theta+\frac{3}{2}\left(\frac{z}{x}-1-\frac{y}{x}\right)\right]. \tag{6.2.37}$$

Again there is the powerlaw consistency with the powers of μ, indicating that at least qualitatively MR holds. The particular form of the vertex used here has been chosen in such a way that this would automatically be the case. Using the angular approximation technique and appropriate changes of variable, one finds

$$\begin{aligned}\left(\frac{x}{\mu}\right)^{-\gamma} &= fz_2\left(\frac{\Lambda}{\mu}\right)^{-\gamma} - abf\lambda'\frac{C_A^2-1}{2C_A}z_2\tilde{z}_3^{-1}\left(\frac{\Lambda}{\mu}\right)^{-\alpha-\beta-\gamma}\left(\frac{x}{\mu}\right)^{\alpha+\beta}\times\\ &\left\{\int_0^1 dt\,\frac{1}{2}\left[1+t^\gamma\right]\left[(\alpha+\beta)t+\frac{1}{3}(\alpha+\beta)^2(\alpha+\beta-2)t^2\right]\right.\\ &\left.+\int_1^{\Lambda/x}dt\,\frac{1}{2}\left[1+t^\gamma\right]t^{\alpha+\beta}\left[(\alpha+\beta)t^{-1}+\frac{1}{3}(\alpha+\beta)^2(\alpha+\beta-2)t^{-2}\right]\right\}.\end{aligned} \tag{6.2.38}$$

Again it is seen that the lower parts of the integral are just a number. Also, there is the IR restriction that $\gamma > -2$. There are also the UV restrictions that in the lowest contributing power, there can be no logarithms so $\alpha+\beta \neq 0,1$ and $\alpha+\beta+\gamma \neq 0,1$. It is to be noted that these restrictions are dependent on the vertex ansatz. This is clearly evident if one were to omit the second term of the vertex which gives rise to the terms involving powers of γ – the second set of restrictions depend on the inclusion of this term. It is entirely possible that a complete vertex could completely change the UV integrals such that the restrictions are different. This is a possibility investigated later with the ghost-gluon vertex.

However, simply doing the integrals, one arrives at the equation

$$\begin{aligned}
\left(\frac{x}{\mu}\right)^{-\gamma} &= f z_2 \left(\frac{\Lambda}{\mu}\right)^{-\gamma} - abf\lambda' \frac{C_A^2-1}{2C_A} z_2 \tilde{z}_3^{-1} \times \\
&\left\{ I_q(\alpha,\beta,\gamma) \left(\frac{\Lambda}{\mu}\right)^{-\gamma} \left(\frac{x}{\Lambda}\right)^{\alpha+\beta} + \left(\frac{\Lambda}{\mu}\right)^{-\gamma} + \frac{(\alpha+\beta)^2(\alpha+\beta-2)}{3(\alpha+\beta-1)} \left(\frac{\Lambda}{\mu}\right)^{-\gamma} \frac{x}{\Lambda} \right. \\
&\left. + \frac{(\alpha+\beta)}{(\alpha+\beta+\gamma)} \left(\frac{x}{\mu}\right)^{-\gamma} + \frac{(\alpha+\beta)^2(\alpha+\beta-2)}{3(\alpha+\beta+\gamma-1)} \left(\frac{x}{\mu}\right)^{-\gamma} \frac{x}{\Lambda} \right\} \qquad (6.2.39)
\end{aligned}$$

where $I_q(\alpha,\beta,\gamma)$ denotes the pure number coming from the lower parts of the integral. The first thing to note is that the term in the vertex proportional to the external momentum scale (the term mentioned in the previous paragraph) has given rise to a term in the UV part of the integral with exactly the same x/μ dependence as the left-hand side of the equation. This term is very interesting because it tells us that the IR powerlaw cannot be given by the IR part of the integral but rather by the UV part. To see this, consider the case where this integral goes logarithmic $\alpha+\beta+\gamma=0$ (the logarithm is of Λ/x). This log is multiplied by the external factor

$$\left(\frac{\Lambda}{\mu}\right)^{-\alpha-\beta-\gamma} \left(\frac{x}{\mu}\right)^{\alpha+\beta} = \left(\frac{x}{\mu}\right)^{-\gamma} \qquad (6.2.40)$$

and so one is faced with the inconsistent term[4]

$$\ln\frac{\Lambda}{x}\left(\frac{x}{\mu}\right)^{-\gamma}. \qquad (6.2.41)$$

Thus, the vertex term has created a term in the UV part of the integral that explicitly tells us that $\alpha+\beta+\gamma \neq 0$. This powerlaw is what one would have concluded if the UV part of the integral had been assumed to be simply part of the renormalisation coefficient, showing that the IR powerlaw analysis must be very careful about which terms contribute. As a direct result of this inequality, the IR number term must be subleading as $x \to 0$ and so

$$\alpha+\beta > -\gamma. \qquad (6.2.42)$$

[4]The inconsistency is a consequence of the assumption that the IR behaviour is a pure powerlaw. Higher order terms in the IR expansion may indeed contain both logarithmic and powerlaw factors but it is demanded that for the lowest power, such terms cannot contribute.

The second point to note in this equation is that the term simply proportional to $\left(\frac{\Lambda}{\mu}\right)^{-\gamma}\frac{x}{\Lambda}$, as in the case of the ghost equation, places a lower bound on γ. Just as before $\gamma \geq -1$.

6.2.5 The Gluon Schwinger-Dyson Equation

The last equation to consider is the gluon equation (6.2.14). Having introduced virtually all of the important concepts via the simpler ghost and quark equations, this will be analysed in the same way but with the details omitted. Inserting the powerlaw forms for all the functions

$$\left(\frac{x}{\mu}\right)^{-\alpha} = az_3\left(\frac{\Lambda}{\mu}\right)^{-\alpha} - a\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} \times \int_0^\Lambda \frac{dy}{x}\frac{2}{\pi}\int_0^\pi d\theta\, \sin^2\theta \frac{x}{z}\left[-\frac{4y}{3x}\sin^2\theta + \frac{z}{2x} + \frac{y}{2x} - \frac{1}{2}\right] \times \left[b^2C_A\left(\frac{x}{\mu}\right)^{2\beta}\left(\frac{y}{x}\right)^{\beta}\left(\frac{z}{x}\right)^{\beta} + 2bfz_2\tilde{z}_3^{-1}\left(\frac{\Lambda}{\mu}\right)^{\beta-\gamma}\left(\frac{x}{\mu}\right)^{\beta+\gamma}\left(\left(\frac{y}{x}\right)^{\gamma} + \left(\frac{z}{x}\right)^{\gamma}\right)\right] \quad (6.2.43)$$

Doing the angular integrals and using suitable changes of variable

$$\begin{aligned}\left(\frac{x}{\mu}\right)^{-\alpha} = az_3\left(\frac{\Lambda}{\mu}\right)^{-\alpha} - a\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} \times \\ \Bigg\{b^2C_A\left(\tfrac{x}{\mu}\right)^{2\beta} &\left\{\int_0^1 dt\,\left[\frac{1}{2}(\beta-1)t^{\beta+1} + \frac{1}{6}(\beta^2-1)(\beta-2)t^{\beta+2}\right]\right.\\ &\left.+\int_1^{\Lambda/x} dt\,\left[\frac{1}{6}(\beta^2-1)t^{2\beta-1} + \frac{1}{24}\beta(\beta^2-1)(\beta-2)t^{2\beta-2}\right]\right\}\\ +2bfz_2\tilde{z}_3^{-1}\left(\tfrac{\Lambda}{\mu}\right)^{\beta-\gamma}\left(\tfrac{x}{\mu}\right)^{\beta+\gamma} &\left\{\int_0^1 dt\,\left[-\frac{1}{2}t^{\gamma+1} + \frac{1}{3}t^{\gamma+2} + \frac{1}{2}(\gamma-1)t + \frac{1}{6}(\gamma^2-1)(\gamma-2)t^2\right]\right.\\ &\left.+\int_1^{\Lambda/x} dt\,\left[\frac{1}{6}(\gamma^2-2)t^{\gamma-1} + \frac{1}{24}\gamma(\gamma^2-1)(\gamma-2)t^{\gamma-2}\right]\right\}\Bigg\}.\end{aligned} \quad (6.2.44)$$

Again it is seen that for IR consistency $\beta > -2$ and $\gamma > -2$. The UV logarithmic consistency argument should also be noted. It is demanded that there be no logarithmic contributions in the leading order powerlaw term. This means that for example γ could be zero, as long as it was multiplied by a factor of x/μ that had a higher power than $-\alpha$. In fact, in this case we know that this is so – the factor multiplying the log would be $\sim x^{\beta+\gamma}$ and from the quark equation it is known that $-\alpha < \beta + \gamma$. This aside, the result for the integral can be written as

$$\left(\frac{x}{\mu}\right)^{-\alpha} = az_3\left(\frac{\Lambda}{\mu}\right)^{-\alpha} - a\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} \times$$
$$\left\{b^2C_A\left(\frac{x}{\mu}\right)^{2\beta}J_{gh}(\beta) + 2bfz_2\tilde{z}_3^{-1}\left(\frac{\Lambda}{\mu}\right)^{\beta-\gamma}\left(\frac{x}{\mu}\right)^{\beta+\gamma}J_q(\gamma)\right.$$
$$+b^2C_A\left[\frac{(\beta^2-1)}{12\beta}\left(\frac{\Lambda}{\mu}\right)^{2\beta} + \frac{\beta(\beta^2-1)(\beta-2)}{24(2\beta-1)}\left(\frac{\Lambda}{\mu}\right)^{2\beta}\frac{x}{\Lambda}\right]$$
$$\left.+2bfz_2\tilde{z}_3^{-1}\left[\frac{(\gamma^2-2)}{6\gamma}\left(\frac{\Lambda}{\mu}\right)^{\beta}\left(\frac{x}{\mu}\right)^{\beta} + \frac{1}{24}\gamma(\gamma+1)(\gamma-2)\left(\frac{\Lambda}{\mu}\right)^{\beta}\left(\frac{x}{\mu}\right)^{\beta}\frac{x}{\Lambda}\right]\right\} \tag{6.2.45}$$

where $J_{gh}(\beta)$ and $J_q(\gamma)$ represent the pure number content of the lower integrals.

6.2.6 The Coupled Schwinger-Dyson Equations

All three of the Schwinger-Dyson equations have been written in a suitable form (equations 6.2.35, 6.2.39 and 6.2.45), so it is now possible to couple the system and derive constraints between the unknown lowest powers α, β and γ.

The first such constraint is that $\beta \neq 0$. In the integral form for the gluon equation (6.2.44) this clearly gives rise to a single logarithmic term ($\sim \ln\Lambda/x$). The factor multiplying this log would be $(\Lambda/\mu)^{-\alpha}$ (if $\beta = 0$). This means that it would, in general, be part of the renormalisation (since $\alpha + \beta \neq 0$ is already known from the ghost equation constraints, it cannot be part of the left-hand side either). However, the renormalisation coefficient must be strictly independent of x and so by elimination $\beta \neq 0$.

This leaves only two possibilities for the powerlaw of the ghost equation (6.2.35) and it is easiest to investigate these next. Consider the scenario where $\beta = -1$. In the ghost equation, since $\alpha+\beta \neq 1$ and one is looking at the lowest power, this means that $\alpha > 2$. By inspection, the gluon equation (6.2.45) has only one term that is a candidate — the term proportional to $J_q(\gamma)$. However, from the quark equation, it was seen that $-\alpha < \beta+\gamma$, so this term cannot match the lowest powerlaw for any values of α, β or γ. Thus in the gluon equation, there is no term that can give rise to a consistent IR powerlaw with $\beta = -1$. Again by elimination, it is found that $\beta \neq -1$. The only remaining possibility is

$$\alpha = -2\beta, \qquad \beta > -1,\ \beta \neq 0. \tag{6.2.46}$$

It is thus possible to rewrite the three Schwinger-Dyson equations with this powerlaw substituted in. They become

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\left\{\left(\frac{x}{\mu}\right)^{-\beta} I_{gh}(\alpha,\beta) + \frac{1}{(\alpha+\beta)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\right\}, \tag{6.2.47}$$

$$\begin{aligned}\left(\frac{x}{\mu}\right)^{-\gamma} &= fz_2\left(\frac{\Lambda}{\mu}\right)^{-\gamma} - abf\lambda'\frac{C_A^2-1}{2C_A}z_2\tilde{z}_3^{-1}\\ &\left\{\left(\frac{\Lambda}{\mu}\right)^{-\gamma} + \frac{(\alpha+\beta)^2(\alpha+\beta-2)}{3(\alpha+\beta-1)}\left(\frac{\Lambda}{\mu}\right)^{-\gamma}\frac{x}{\Lambda}\right.\\ &\left.+\frac{(\alpha+\beta)}{(\alpha+\beta+\gamma)}\left(\frac{x}{\mu}\right)^{-\gamma} + \frac{(\alpha+\beta)^2(\alpha+\beta-2)}{3(\alpha+\beta+\gamma-1)}\left(\frac{x}{\mu}\right)^{-\gamma}\frac{x}{\Lambda}\right\}\end{aligned} \tag{6.2.48}$$

$$\begin{aligned}\left(\frac{x}{\mu}\right)^{2\beta} &= az_3\left(\frac{\Lambda}{\mu}\right)^{2\beta} - a\lambda'\\ &\left\{b^2C_A\left(\frac{x}{\mu}\right)^{2\beta}J_{gh}(\beta) + b^2C_A\left[\frac{(\beta^2-1)}{12\beta}\left(\frac{\Lambda}{\mu}\right)^{2\beta} + \frac{\beta(\beta^2-1)(\beta-2)}{24(2\beta-1)}\left(\frac{\Lambda}{\mu}\right)^{2\beta}\frac{x}{\Lambda}\right]\right.\\ &\left.+2bfz_2\tilde{z}_3^{-1}\left[\frac{(\gamma^2-2)}{6\gamma}\left(\frac{\Lambda}{\mu}\right)^{\beta}\left(\frac{x}{\mu}\right)^{\beta} + \frac{1}{24}\gamma(\gamma+1)(\gamma-2)\left(\frac{\Lambda}{\mu}\right)^{\beta}\left(\frac{x}{\mu}\right)^{\beta}\frac{x}{\Lambda}\right]\right\}\end{aligned} \tag{6.2.49}$$

where terms that clearly do not contribute to the leading power behaviour have been eliminated. The constraint $-\alpha < \beta + \gamma$ becomes $\gamma > \beta$.

In the gluon equation above (6.2.49), there is a term coming from the UV part of the quark loop with an x-dependence x^{β}. Since $\beta \neq 0$, this term must be subleading in the IR so $2\beta < \beta$, thus $\beta < 0$ and $\alpha > 0$. This relation allows one to dismiss in the gluon equation all but the first of the terms coming from the UV part of the integrals. The equation reduces to

$$\left(\frac{x}{\mu}\right)^{2\beta} = az_3\left(\frac{\Lambda}{\mu}\right)^{2\beta} - a\lambda'\left\{b^2C_A\left(\frac{x}{\mu}\right)^{2\beta}J_{gh}(\beta) + b^2C_A\frac{(\beta^2-1)}{12\beta}\left(\frac{\Lambda}{\mu}\right)^{2\beta}\right\}. \tag{6.2.50}$$

It is seen that the quark contributions are completely subleading.

In the quark equation (6.2.49), given that $\gamma > \beta$, then $\gamma > -1$ and so it is possible to eliminate most of the terms.

$$\left(\frac{x}{\mu}\right)^{-\gamma} = fz_2\left(\frac{\Lambda}{\mu}\right)^{-\gamma} - abf\lambda'\frac{C_A^2-1}{2C_A}z_2\tilde{z}_3^{-1}\left\{\left(\frac{\Lambda}{\mu}\right)^{-\gamma} + \frac{(\alpha+\beta)}{(\alpha+\beta+\gamma)}\left(\frac{x}{\mu}\right)^{-\gamma}\right\}. \tag{6.2.51}$$

In summary, the coupled Schwinger-Dyson equations have reduced to

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\left\{\left(\frac{x}{\mu}\right)^{-\beta} I_{gh}(\alpha,\beta) - \frac{1}{(\beta)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\right\}, \quad (6.2.52)$$

$$\left(\frac{x}{\mu}\right)^{-\gamma} = fz_2\left(\frac{\Lambda}{\mu}\right)^{-\gamma} - abf\lambda'\frac{C_A^2-1}{2C_A}z_2\tilde{z}_3^{-1}\left\{\left(\frac{\Lambda}{\mu}\right)^{-\gamma} - \frac{(\beta)}{(\gamma-\beta)}\left(\frac{x}{\mu}\right)^{-\gamma}\right\} \quad (6.2.53)$$

$$\left(\frac{x}{\mu}\right)^{2\beta} = az_3\left(\frac{\Lambda}{\mu}\right)^{2\beta} - a\lambda'\left\{b^2C_A\left(\frac{x}{\mu}\right)^{2\beta} J_{gh}(\beta) + b^2C_A\frac{(\beta^2-1)}{12\beta}\left(\frac{\Lambda}{\mu}\right)^{2\beta}\right\} \quad (6.2.54)$$

with the restriction that $-1 < \beta < 0$ and $\gamma > \beta$.

6.2.7 Summary of the Toy Model

In this section, a toy model of the Schwinger-Dyson equations has been studied in the IR. The purpose of this was not to find specific results, but rather to introduce a few of the concepts involved in an IR powerlaw analysis. The most important concept was the existence of a lowest power for each of the two-point functions. This led naturally onto the idea behind the renormalisation of the equations in this regime and the possibility that the renormalised coupling was also not a constant with respect to the renormalisation scale.

In order to proceed, an angular approximation was used. This approximation is superior to the oft used y-max approximation, but the real justification was the observation that even with exact angular integrals, the qualitative features of the integrals were preserved. This of course negates the possibility of a proper quantitative analysis.

The next point raised was the necessity to include the UV parts of the integral in a consistent way. If some kind of momentum subtraction renormalisation scheme had been used, vital information would have been lost. It was seen (in the quark equation) that specific terms in the vertex used could indeed give a UV integral that contributed to the IR behaviour of the equation. Moreover, these integrals provide bounds to the unknown powers. In order to ensure consistency in the lowest power terms, it was demanded that there be no mixing of the lowest power with logarithmic terms. This proved to be a very useful constraint.

Finally, it was shown how such a set of equations could be coupled. Although the complete solution was not found, a unique powerlaw relationship was unambiguously obtained and the value of the unknown powers was constrained. This is the information that was sought.

6.3 The Ghost-Gluon System

In the second part of this chapter, the quarks will be neglected. This leaves the pure Yang-Mills sector of the theory. It will be assumed (though this should be verified) that the four-gluon interaction can be safely dropped because it is purely subleading in the IR, or at the very least, does not produce any new features. The triple-gluon vertex will be shown explicitly not to interfere with the discussion. This will lead to the quite general result that the gluon propagator function does not diverge in the IR, whilst the ghost propagator function does not vanish.

The first section looks at a general form of the ghost-gluon vertex that is suitable for the type of analysis used. This draws heavily on the experience gained with the toy model considered previously. This vertex form leads to the full ghost Schwinger-Dyson equation in the IR which is seen to be not much more complicated than with the bare vertex. However, the arguments necessary to proceed with the derivation of the IR powerlaw will be seen to be more subtle than before.

The second section then looks at the gluon loop of the gluon Schwinger-Dyson equation. Using a completely general argument it will be shown that for the purposes of a practical study of the Schwinger-Dyson system, the gluon loop can be omitted as long as one remembers certain caveats. It will also be shown that for consistency, the gluon propagator function cannot diverge in the IR.

The third section goes on to study the remainder of the gluon equation. The issue of *practical* transversality of the propagator is discussed and this leads to the not inconsiderably contentious issue of the quadratic UV divergence of the bosonic equation. This phenomenon will occur quite naturally in the powerlaw analysis used here and so will be discussed in detail.

Finally, the system will be coupled. It will be seen that although no single conclusion can be made, the possibilities are clearly demarcated. They are shown to depend on a certain kinematical limit of the particular form of the vertex used.

6.3.1 The Generalised Ghost-Gluon Vertex and the Ghost Schwinger-Dyson Equation

Consider the ghost-gluon vertex. From the Lorentz structure, it is possible to write down the general form for the renormalised vertex function in terms of two dimensionless scalar functions[5]

$$\widetilde{\Gamma}_\mu(p_1,p_2;p_3) = p_{1\mu}X(p_1^2,p_2^2;p_3^2) + p_{3\mu}Y(p_1^2,p_2^2;p_3^2). \tag{6.3.1}$$

For simplicity, the arguments of the functions X and Y will be abbreviated to X_{123} and Y_{123}. In Landau gauge the vertex has the wonderful property (that in fact gives rise to all the simplifications of full QCD necessary for the powerlaw approach!) that as the second argument p_2 vanishes, the vertex becomes bare [21]. This means that

$$\begin{aligned}\widetilde{\Gamma}_\mu(p_1,p_2;p_3) &\overset{p_2\to 0}{=} p_{1\mu}\left(X_{123}-Y_{123}\right) - p_{2\mu}Y_{123} \\ &= p_{1\mu}.\end{aligned} \tag{6.3.2}$$

Thus, as $p_2 \to 0$, the functions X_{123} and Y_{123} cannot be singular.

Now consider the full ghost Schwinger-Dyson equation in powerlaw form (analogous to (6.2.29))[6]. It is

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - ab^2C_A\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}\int_0^\Lambda \frac{dy}{x}\left(\frac{y}{x}\right)^{\alpha}\frac{2}{\pi}\int_0^\pi d\theta\,\sin^4\theta\left(\frac{z}{x}\right)^{\beta-1}X_{zxy}. \tag{6.3.3}$$

Recall that all the Schwinger-Dyson equations analysed in this manner split into two parts after angular integration. The lower part of the integral is simply a number multiplied by the generic dependence on x, μ and Λ coming from the powers associated with the renormalisation coefficients, the renormalised coupling and the propagator functions of the integrand. The upper part of the integral, where the integration variable takes the value of the cutoff Λ depends specifically on the vertex ansatz used. Because the lower integral reduces to a number whose value will not be important to the powerlaw relations that are being investigated[7], it is possible to tailor the general vertex ansatz to be correct only in the UV part of the integral without loss of information. This allows one to utilise a general power expansion of the vertex in order to discuss the powerlaw behaviour of the equation.

The general form of the vertex suitable for the equation above is written

$$X_{zxy} = r_0 + r_1\left(\frac{x}{y}\right)^{\rho_1} + r_2\left(\frac{x}{z}\right)^{\rho_2} + r_3\left(\frac{y}{z}\right)^{\rho_3}. \tag{6.3.4}$$

[5]Recall that in Landau gauge the vertex is independent of the renormalisation scale μ.

[6]There is no need for a discussion of the justification of the powerlaw form here, since the equation is virtually identical to before.

[7]This tacitly assumes that this number is not zero. This case will not be considered because it is highly unlikely that a practical calculation could produce this, unless specifically engineered to do so.

This form requires some discussion. The coefficients r_n are simply constants and the $\rho_n \neq 0$. Since it is known that this function, as $x \to 0$ or equivalently (as is the case in the equation (6.3.3)) $y, z \to \Lambda$, is not divergent, this forces $\rho_1, \rho_2 > 0$. The next point to note is the inclusion of a dependence on z. It would appear that since $z = x+y-2\sqrt{xy}\cos\theta \to y$, as $x \to 0$ these terms are not necessary to the UV part of the integral. However, the effect of the angular integration cannot be foreseen, so these terms are included for completeness. It will be seen that the last term is definitely important. This is actually an 'indicator term' in the sense that when $x = 0$, all such terms will combine to become a single constant. Without a specific vertex ansatz though and not knowing what the effect of the angular integration could be, it would be necessary to include all such terms. Thus when considering factors proportional to r_3, one should be mindful that this actually indicates a complete set of factors.

One now substitutes this vertex into the equation (6.3.3). As in the previous discussion of the bare vertex case, it is possible to use the angular approximation. Because the result for the bare vertex is already known, one can anticipate the form of the resulting integrals and omit terms that are definitely subleading. The result is

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}\{I_{gh}(\alpha,\beta)+ \int^{\Lambda/x}dt\left[(r_0+r_3)t^{\alpha+\beta-1}+\frac{1}{3}\left(r_0(\beta^2-1)+r_3((\beta-\rho_3)^2-1)\right)t^{\alpha+\beta-2} +r_1t^{\alpha+\beta-\rho_1-1}+r_2t^{\alpha+\beta-\rho_2-1}\right]\Big\}. \tag{6.3.5}$$

Note that there will be logarithms generated if $\alpha+\beta = 0, 1, \rho_1$ or ρ_2. Proceeding with the integration for now,

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\left\{\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}I_{gh}(\alpha,\beta)+ \left[\frac{(r_0+r_3)}{\alpha+\beta}\left(\frac{\Lambda}{\mu}\right)^{-\beta}+\frac{(r_0(\beta^2-1)+r_3((\beta-\rho_3)^2-1))}{3(\alpha+\beta-1)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\frac{x}{\Lambda} +\frac{r_1}{(\alpha+\beta-\rho_1)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\left(\frac{x}{\Lambda}\right)^{\rho_1}+\frac{r_2}{(\alpha+\beta-\rho_2)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\left(\frac{x}{\Lambda}\right)^{\rho_2}\right]\right\}. \tag{6.3.6}$$

The terms dropped from this equation are terms that have a resulting power of x higher than one. It is assumed that the numerator of the coefficient multiplying the term $\sim x/\Lambda$ is non-vanishing[8]. This of course means that $\beta \geq -1$. It is seen that the two terms in

[8] The reader is reminded that in this term, the numerator coefficient is not actually known. In order

the vertex dependent on x (ie those with r_1 and r_2) give rise to exactly the same form of term. Thus they are combined into a single term denoted by r_i and ρ_i giving

to find it, one would not only have to do the angular integrals exactly (as in the case of the bare vertex) but would also have to know exactly the form of the vertex functions independent of x. This is because the term arises from the subleading parts of the angular integral and thus depends on the value of ρ_3, which represents the complete set of all such terms.

$$\left(\frac{x}{\mu}\right)^{-\beta} = b\tilde{z}_3\left(\frac{\Lambda}{\mu}\right)^{-\beta} - \frac{3}{4}ab^2C_A\lambda'\left\{\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}I_{gh}(\alpha,\beta) + \frac{(r_0+r_3)}{\alpha+\beta}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\right.$$
$$\left. + \frac{(r_0(\beta^2-1)+r_3((\beta-\rho_3)^2-1))}{3(\alpha+\beta-1)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\frac{x}{\Lambda} + \sum_{i=1,2}\frac{r_i}{(\alpha+\beta-\rho_i)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\left(\frac{x}{\Lambda}\right)^{\rho_i}\right\}. \tag{6.3.7}$$

This equation should now provide all the information content possible for a completely general vertex ansatz in the IR powerlaw regime.

6.3.2 The IR Part of the Gluon Loop in the Gluon Schwinger-Dyson Equation

Consider the gluon Schwinger-Dyson equation. It contains a loop term involving two gluon propagators and the full triple-gluon vertex. Now recall the generic form of this integral. It will have a lower part which is simply a number multiplied by some known factors involving x, μ and Λ and an upper part which is governed by the UV integral (which is dependent on the form of the vertex in the limit that the external momentum goes to zero and the other two momenta go to Λ).

The factors multiplying the lower integral number can easily be found by analogy with the integrals that have already been studied. These factors are the renormalised coupling, the triple-gluon vertex renormalisation coefficient, the two gluon propagator functions in the integrand and the factors inherent to the renormalised vertex function (which obeys it's Slavnov-Taylor identity). They are derived from their μ-dependence and are written respectively as follows

$$\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{\Lambda}{\mu}\right)^{-\alpha+\beta}\left(\frac{x}{\mu}\right)^{2\alpha}\left(\frac{x}{\mu}\right)^{\beta-\alpha} = \left(\frac{\Lambda}{\mu}\right)^{-2\alpha-\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}. \tag{6.3.8}$$

Note that the overall μ-dependence is equal to that of the inverse gluon propagator function appearing on the left-hand side of the full equation, showing that MR is again holding in the IR powerlaw sense.

Suppose that this term was responsible for generating the lowest power of the equation as $x \to 0$. This would mean that $\beta = -2\alpha$. Now consider the powers associated with the ghost equation (6.3.7) with this powerlaw substituted in:

$$2\alpha \sim -\alpha, 0, 1, \rho_i \tag{6.3.9}$$

The first of these powers (coming from the IR part of the integral) tells us that in order to be consistent $\alpha \leq 0$. The other three powers (and indeed all other powers) are all greater than zero so this would force $\alpha = \beta = 0$. However, it can already be foreseen that this possibility is contained within the ghost loop of the same equation[9]. This point is not trivial because it indicates that in this study, the IR part of the gluon loop can be entirely neglected. If the only way that the IR term of the gluon loop can contribute is indistinguishable from another term in the ghost loop, then the only change that occurs in it's omission is in the coefficient. However, this coefficient cannot be known without a specific vertex ansatz and that is not what is under consideration here – we are only concerned with the consistency of the powerlaw.

Given that the gluon loop IR term cannot give rise to a power lower than that of the lhs (unless a highly unlikely cancellation occurs) this means that $\alpha + \beta \geq -\alpha$ or $2\alpha \geq -\beta$. Again one considers the terms arising in the ghost equation. If the first (IR) term dominates then $\alpha = -2\beta$ and the gluon loop constraint becomes $\alpha \geq 0$ and $\beta \leq 0$. If any of the other terms in the ghost equation dominates then $\beta \leq 0$ and again $\alpha \geq 0$. Thus, neglecting the possibility of an extremely delicate cancellation in the IR part of the gluon loop,

$$\alpha \geq 0, \ -1 \leq \beta \leq 0 \tag{6.3.10}$$

and the IR part of the gluon loop can be neglected. This is a central result to the study.

6.3.3 Transversality and the Ghost Loop of the Gluon Equation

It is a well known fact that in Landau gauge, the gluon propagator must be transverse. The gluon Schwinger-Dyson equation is a tensor equation and in the powerlaw regime what this means is that the dominating IR contribution must be multiplied by the transverse projector $t_{\mu\nu}$. Now, it is not known a priori which term (or terms) give rise to the lowest power and so, a slightly esoteric approach to this issue must be taken. The approach is as follows. It is assumed that the only candidate for the lowest power term coming from the IR part of one of the loop integrals is that from the ghost loop. All other candidates have origins in the UV parts of their respective integrals. It will be demanded that the ghost loop is transverse in the IR on the basis that if the IR part of this integral dominates[10] then

[9]In fact, the possibility that the lowest IR power is zero is contained within any properly renormalised loop integral since it merely corresponds to the term independent of x, usually associated with the propagator renormalisation coefficient.

[10]This is actually quite likely. The consistency gained in this scenario is remarkable.

since it is the only such part, then it must be transverse. If on the other hand the IR part of the ghost loop does not dominate, then one of the UV parts of either the ghost or gluon loops must be doing the job. However, the UV parts of such a Schwinger-Dyson integral give rise to only certain powers (specific integers or $-\alpha$ itself) multiplied by coefficients that cannot be derived without knowing the vertex. These powers can arise in the ghost loop alone and so the gluon loop is neglected for now. When discussing the UV parts, it will always be assumed that more than one term of the Schwinger-Dyson equation may be contributing but this total combination must be transverse so the above condition on the ghost loop seems a reasonable place to start. The UV part of the gluon loop will be discussed later.

The gluon Schwinger-Dyson equation with only the ghost loop contribution can be written as

$$t^{\mu\nu}(p)J_p = t^{\mu\nu}(p) + \imath g^2 C_A \int \frac{\bar{d}^4\omega\, G_{p-w}G_\omega}{p^2\omega^2(p-\omega)^2}\omega^\mu\widetilde{\Gamma}^\nu(p-\omega,\omega;-p). \tag{6.3.11}$$

Writing the vertex function in the form of (6.3.1) gives

$$\begin{aligned} t^{\mu\nu}(p)J_p = t^{\mu\nu}(p) + \imath g^2 C_A \int \frac{\bar{d}^4\omega\, G_{p-w}G_\omega}{p^2\omega^2(p-\omega)^2}\times \\ \left\{-\omega^\mu\omega^\nu X(p-\omega,\omega;-p) + w^\mu p^\nu\left[X(p-\omega,\omega;-p) - Y(p-\omega,\omega;-p)\right]\right\}. \end{aligned} \tag{6.3.12}$$

Now, the tensor integrals can be rewritten as

$$\begin{aligned}
\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)\omega^\mu &= \frac{p^\mu}{p^2}\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)p{\cdot}\omega \\
\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)\omega^\mu\omega^\nu &= t^{\mu\nu}(p)\frac{1}{3}\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)\omega^2 \\
&\quad -\left(g^{\mu\nu}-4\frac{p^\mu p^\nu}{p^2}\right)\frac{1}{3}\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)\frac{p{\cdot}\omega^2}{p^2} \\
&= t^{\mu\nu}(p)\frac{1}{3}\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)\left[\omega^2-\frac{p{\cdot}\omega^2}{p^2}\right] \\
&\quad +\frac{p^\mu p^\nu}{p^2}\int d\!\!{}^-{}^4\omega\, f(p-\omega,\omega;-p)\frac{p{\cdot}\omega^2}{p^2}.
\end{aligned} \tag{6.3.13}$$

Using these, the gluon equation becomes

$$\begin{aligned}
&t^{\mu\nu}(p)J_p = t^{\mu\nu}(p) \\
&\quad -t^{\mu\nu}(p)\frac{1}{3}\imath g^2 C_A\int\frac{d\!\!{}^-{}^4\omega\, G_{p-w}G_\omega}{p^2(p-\omega)^2}X(p-\omega,\omega;-p)\left[1-\frac{p{\cdot}\omega^2}{p^2\omega^2}\right] \\
&\quad +\frac{p^\mu p^\nu}{p^2}\imath g^2 C_A\int\frac{d\!\!{}^-{}^4\omega\, G_{p-w}G_\omega}{p^2\omega^2(p-\omega)^2}\left[X(p-\omega,\omega;-p)\left(p{\cdot}\omega-\frac{p{\cdot}\omega^2}{p^2}\right)-Y(p-\omega,\omega;-p)p{\cdot}\omega\right].
\end{aligned} \tag{6.3.14}$$

It is now apparent that if only the ghost loop is to contribute to the IR behaviour of the equation then

$$\int\frac{d\!\!{}^-{}^4\omega\, G_{p-w}G_\omega}{p^2\omega^2(p-\omega)^2}\left[X(p-\omega,\omega;-p)\left(p{\cdot}\omega-\frac{p{\cdot}\omega^2}{p^2}\right)-Y(p-\omega,\omega;-p)p{\cdot}\omega\right]\stackrel{p^2\to 0}{=}0 \tag{6.3.15}$$

is a necessary condition to ensure transversality. It is to be noted that this condition holds only when $p^2\to 0$. The remainder of the equation is thus

$$J_p = 1-\frac{1}{3}\imath g^2 C_A\int\frac{d\!\!{}^-{}^4\omega\, G_{p-w}G_\omega}{p^2(p-\omega)^2}X(p-\omega,\omega;-p)\left[1-\frac{p{\cdot}\omega^2}{p^2\omega^2}\right]. \tag{6.3.16}$$

Performing a Wick rotation, renormalising and writing in the usual way, this equation is written as

$$J_{x,\mu} = Z_3+\frac{1}{3}\lambda C_A\int_0^\Lambda\frac{dy}{x}\frac{y}{x}G_{y,\mu}\frac{2}{\pi}\int_0^\pi d\theta\sin^4\theta\frac{x}{z}G_{z,\mu}X_{zyx}. \tag{6.3.17}$$

Again one can use the powerlaw form, suitable for the IR analysis. The equation becomes

$$\begin{aligned}
\left(\frac{x}{\mu}\right)^{-\alpha} &= az_3\left(\frac{\Lambda}{\mu}\right)^{-\alpha} \\
&\quad +\frac{1}{3}ab^2C_A\lambda'\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{2\beta}\int_0^\Lambda\frac{dy}{x}\left(\frac{y}{x}\right)^{\beta+1}\frac{2}{\pi}\int_0^\pi d\theta\,\sin^4\theta\left(\frac{z}{x}\right)^{\beta-1}A_{zyx}.
\end{aligned} \tag{6.3.18}$$

One now proceeds in exactly the same way as for the ghost equation. The vertex function X_{zyx} will be expanded in a form appropriate to match the UV behaviour of the integral. This form is written as[11]

$$X_{zyx} = s_0 + s_1 \left(\frac{x}{y}\right)^{\sigma_1} + s_2 \left(\frac{x}{z}\right)^{\sigma_2} + s_3 \left(\frac{y}{z}\right)^{\sigma_3}. \tag{6.3.19}$$

It is demanded that the vertex be free of kinematical singularities as $x \to 0$, so one may naturally think that $\sigma_1, \sigma_2 > 0$ must hold. However, the σ do not denote kinematical content solely – they denote the full powerlaw content of one part of the vertex (recall that there are two functions involved with the ghost-gluon vertex). There is no reason to expect these terms not to be singular, just as the two-point function may or may not be singular[12]. Doing the angular integrals as before and neglecting terms that are definitely subleading with respect to x (in this case any terms that are definitely vanishing as $x \to 0$) one can write

$$\left(\frac{x}{\mu}\right)^{-\alpha} = az_3 \left(\frac{\Lambda}{\mu}\right)^{-\alpha} + \frac{1}{4}ab^2 C_A \lambda' \left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta} \left(\frac{x}{\mu}\right)^{2\beta} \left\{ J(\beta) \right.$$
$$\left. + \int^{\Lambda/x} dt \left[(s_0 + s_3)t^{2\beta} + \frac{1}{3}\left(s_0(\beta^2 - 1) + s_3((\beta - \sigma_3)^2 - 1)\right) t^{2\beta-1} + s_1 t^{2\beta-\sigma_1} + s_2 t^{2\beta-\sigma_2} \right] \right\}. \tag{6.3.20}$$

It is to be noted that logarithms may occur if $2\beta = -1, 0, \sigma_1 - 1, \sigma_2 - 1$. Proceeding with the radial integration and again noting that there is no difference between σ_1 and σ_2 gives

[11] Although this is identical in form to the expansion used in the ghost equation, the terms have completely different meanings, since the order of the arguments is not the same. The term proportional to s_3 plays the same indicator role as that proportional to r_3 in the ghost equation.

[12] There is a mild restriction derived from the full identity for the ghost-gluon vertex but this refers to the full vertex, not just the function A considered here.

$$\left(\frac{x}{\mu}\right)^{-\alpha} = az_3\left(\frac{\Lambda}{\mu}\right)^{-\alpha} + \frac{1}{4}ab^2C_A\lambda'\left\{\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{2\beta}J(\beta) + \frac{(s_0+s_3)}{(2\beta+1)}\left(\frac{\Lambda}{\mu}\right)^{-\alpha}\frac{\Lambda}{x}\right.$$
$$\left. + \frac{(s_0(\beta^2-1)+s_3((\beta-\sigma_3)^2-1))}{6\beta}\left(\frac{\Lambda}{\mu}\right)^{-\alpha} + \sum_{i=1,2}\frac{s_i}{(2\beta-\sigma_1+1)}\left(\frac{\Lambda}{\mu}\right)^{-\alpha}\left(\frac{x}{\Lambda}\right)^{\sigma_i-1}\right\}. \tag{6.3.21}$$

We are now faced with a potentially major problem: the term that should naturally be giving rise to the renormalisation coefficient Z_3 is now what seems to be a quadratic divergence (characterised by the single power of Λ/x). To explain: as the external scale x increases, the term on the left-hand side of the equation when combined with its renormalisation coefficient should become unity – this means that effectively in this region, the power α should be zero. Putting α to zero in the right-hand side would therefore give a single factor Λ/x. In perturbation theory (where the external scale is large) such a term cannot be renormalised and so cannot occur. However, *we are not doing perturbation theory*! This term arises quite naturally from the tensor nature of the equation and in the powerlaw way of looking at things, the term is completely usual (indeed expected) and can be interpreted not as a nuisance for renormalisation but as part of the IR expansion.

If one were to look at the full gluon equation in the region where x is large then the quadratic divergence must cancel, because all the two-point functions in the integrand reduce to their bare forms (all momentum scales are large in this region of the integral) and one is left with the one-loop expression. However, when the scale x is non-perturbative (ie. our case) then the two-point functions do not reduce and the cancellation may not occur. In the full gluon equation where there are contributions from both the ghost-gluon and gluon self-interaction vertices, there will certainly be factors containing the two-point functions whose arguments are (the non-perturbative) x and it is this that removes the need for the absence of the quadratic type term. This term arises from the UV part of the integral and is dependent on the vertex (actually the combination of the various vertices). Thus, the most likely scenario is that if one were to set the σ_i to zero (mimicking the perturbative behaviour), then those terms (which are now purely quadratic too) would cancel against the rest of the quadratic type terms[13]. This is a consequence of the vertices reducing to their tree-level forms which automatically gives

[13] Assuming that the regularisation were ideal.

the one-loop form of the integrals and so cannot give rise to a quadratic divergence. Thus it is concluded that even if the UV cutoff were translationally invariant, the quadratic type terms in the pure Yang-Mills part of the IR gluon Schwinger-Dyson equation remain so that now $\alpha \geq 1$.

Recall that the discussion of the previous section centred on the idea that the IR part of the gluon loop could be omitted. Now consider the UV part of the gluon loop. This will have exactly the same form as the UV part of the ghost loop but with different coefficients and powers associated with the general expansion of the vertex. This means that one can include these terms into the equation above simply by saying that the coefficients are not known[14]. In fact, this is tautological, since these coefficients are not known for the ghost loop anyway due to the ignorance of the vertex function and angular approximations necessary. This means that as long as one bears in mind that the terms may have more than one origin, one only need consider a transverse ghost loop since this includes all the generic terms that will arise.

6.3.4 Coupling the Schwinger-Dyson Equations

Let us now consider what happens when one tries to couple the equations (6.3.7) and (6.3.21). Much emphasis has been placed on the inclusion of all the most general terms in the equation and it would seem that this negates the possibility of extracting definite information. However, the technique used in this section follows the pattern used with the toy model and tries to limit the options in a systematic way. There are four distinct possibilities allowed in the ghost equation (6.3.7), $\beta = 0, -1, -\alpha - \beta$, or $-\rho_i$.

Consider the case $\beta = 0$. In the gluon equation, this naturally gives rise to a logarithmic term in the ghost loop (the gluon loop may or may not have such a logarithm, but this does not matter). This is similar to the case with the toy model but now, the coefficient of this term is definitely not known, since it relies on the set of terms indicated by s_3 and σ_3. It is unreasonable to expect a practical calculation to provide a cancellation (which in any case could be spurious) and so $\beta \neq 0$.

Now consider the case $\beta = -1$. In order to avoid logarithmic behaviour and provide a consistent powerlaw in the ghost equation (6.3.7), $\alpha > 2$. In the gluon equation, the term

[14]For the case of the terms that could come from the gluon loop analogous to the σ_i term, it is noted that just as in the case for the ghost loop, the only way that these terms contribute is if their power is $-\alpha$. Thus again, the terms are on the same footing and can quite easily be included without loss of generality.

with σ_i is the only term capable of providing this and this would require $\sigma_i < -1$. This is a remote possibility, but the vertex function is dangerously singular if one remembers that there is a restriction placed on the full contracted vertex whose last argument vanishes to be less singular than the two-point function.

The next possibility is that $\alpha = -2\beta$ with $-1 < \beta < 0$. This is the most consistent case. In both equations, it is the IR term that contributes to the powerlaw. In order to avoid logarithmic problems, the other terms (in both equations) must be either subleading in the IR or vanishing, either way these terms cannot be present. This places restrictions on the form of the vertex. For example in the ghost equation $\rho_i \neq -\beta$ unless $r_i = 0$[15]. In the gluon equation similar restrictions apply to the σ_i terms. There is one special term and that is the quadratic term of the gluon equation. If $\beta = -\frac{1}{2}$ then this term must vanish because it would otherwise produce a logarithm. However if the term vanishes then there is no problem. The interesting case arises if the coefficient of the term does not vanish (and in practical cases it won't). This would necessarily mean that $\alpha > 1$ and $\beta < -\frac{1}{2}$.

The final possibility is that $\beta = -\rho_i$. To avoid logarithms in the ghost equation, $\alpha > -2\beta$ and so the IR term of the gluon equation must be subleading. Again, if the quadratic term is present in the gluon equation, then $\alpha \geq 1$. The only term that can give rise to $\alpha > 1$ is the vertex term σ_i.

In summary, we have the following. The equations give rise to the logarithmic constraints: $\alpha + \beta \neq 0, 1$ and $\beta \neq 0, -1$. The powers are restricted to the ranges

$$\alpha \geq 1, \quad -1 < \beta \leq -\frac{1}{2}, \quad \alpha \geq -2\beta. \tag{6.3.22}$$

Given this, it is possible to reduce the equations to

$$\left(\frac{x}{\mu}\right)^{-\beta} = -\frac{3}{4}ab^2C_A\lambda'\left\{\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{\alpha+\beta}I_{gh}(\alpha,\beta) + \sum_{i=1,2}\frac{r_i}{(\alpha+\beta-\rho_i)}\left(\frac{\Lambda}{\mu}\right)^{-\beta}\left(\frac{x}{\Lambda}\right)^{\rho_i}\right\}, \tag{6.3.23}$$

[15] This in fact does not depend on the regularisation scheme used. One could imagine doing the integral under dimensional regularisation and still getting the same result with the ghost equation. This is because when $\alpha = -2\beta$, the coupling is now just a constant. In the integral, any factors of x are still non-perturbative and have the powerlaw form so can be taken outside. The remaining integral multiplying this factor has a part in the UV where all functions behave perturbatively and so will mimic the one-loop form so giving a UV singularity in the form of an ε pole. The case $\rho_i = -\beta$ corresponds to a function G_x^{-1} in the vertex. Thus, one would have on the right-hand side of the equation a term with $G_x^{-1}\varepsilon^{-1}$ whilst on the left-hand side the simple term G_x^{-1}. One can see now that such terms in the vertex must cancel or the powerlaw cannot be valid such that the term is subleading.

$$\left(\frac{x}{\mu}\right)^{-\alpha} = \frac{1}{4}ab^2C_A\lambda'\left\{\left(\frac{\Lambda}{\mu}\right)^{-\alpha-2\beta}\left(\frac{x}{\mu}\right)^{2\beta}J(\beta)+\frac{(s_0+s_3)}{(2\beta+1)}\left(\frac{\Lambda}{\mu}\right)^{-\alpha}\frac{\Lambda}{x}\right.$$
$$\left.+\sum_{i=1,2}\frac{s_i}{(2\beta-\sigma_1+1)}\left(\frac{\Lambda}{\mu}\right)^{-\alpha}\left(\frac{x}{\Lambda}\right)^{\sigma_i-1}\right\}. \tag{6.3.24}$$

6.4 Conclusions

In this chapter, the powerlaw approach to studying the IR behaviour of two-point QCD Green's functions in Landau gauge has been refined. By expressing these functions as single powers in the IR, the consistency requirements on the Schwinger-Dyson equations and the multiplicatively renormalisable nature of the theory are shown to restrict the number of possibilities down to two. These are either that there exists a relationship between the different powers or that the specific form of the vertex ansatz used determines the value of the power. It is seen that in order to distinguish between the two, it is necessary in general to know a certain kinematical regime of the vertex functions involved.

The central idea in the study is the existence of a powerlaw regime of the theory. This was based on the observation that there are only two scales present in the renormalised theory (the external scale and the renormalisation scale). Due to this, all two-point functions could be expressed as powers of the ratio of the two scales such that as the external scale vanished (the IR regime) the function reduced to a single powerlaw term with a characteristic power. With this lowest power in mind, the Schwinger-Dyson equations had to reproduce consistent terms and the renormalisable nature of the theory had to be maintained. It is recognised that this simplistic view of the equations only holds for the lowest power since the variables involve implicit dependencies on each other.

The first parts involved a toy model with which important technical aspects of the approach could be introduced. Such issues were the notations, the integral approximations necessary (the UV cutoff Λ, and the angular approximations) and the way in which the consistency requirements of the coupled system could be employed in order to extract information. It was recognised that due to the necessarily approximate way of doing the integrals (even with definite vertices), the information gained would be at best qualitative. However, there were useful constraints found.

The next part then looked at the pure Yang-Mills sector of the theory in more detail. Using the experience gained with the toy model, it was possible to anticipate the form of the ghost equation and tailor a generalised vertex in order to analyse the behaviour of the complete equation. It was discovered that the most crucial piece of information that one needs in order to proceed is the limiting form of the vertex as the second argument (the in-ghost momentum) vanishes. This is not known presently but would allow one to work out whether the lowest power of the ghost and gluon propagator functions are simply related (in which case the IR part of the self-energy integral dominates the IR behaviour of the equation), or whether the important part of the integral is the UV part (and the equation is dominated by a vertex term dependent on the external momentum scale). The next sections looked at the gluon equation and the omission of the pure gluon loop was justified. This led to the general result that the gluon propagator function cannot diverge and the ghost propagator function cannot vanish in the IR.

In order to proceed, it is concluded that one must first derive the limit of the first ghost-gluon vertex function as the in-ghost momentum vanishes such that one may be able to distinguish between the two scenarios. It is also recognised that the use of the UV-cutoff leads to a lack of translational invariance in the integrals. The deficiencies of the cutoff regularisation must therefore also be overcome in order to further constrain the system (this is in progress [48]). Lastly, it would be necessary to include both the four-gluon interaction and the quark sector.

Chapter 7

Conclusions

In this thesis has been presented work on the IR properties of the Schwinger-Dyson equations of Landau gauge QCD. The main conclusions were that a suitable truncation scheme for the ghost sector could not be found due to the absence of an appropriate identity for the ghost-gluon vertex but that by using a different approach, the IR behaviour of the propagator functions could still be constrained.

The thesis started with a brief introduction to QCD, the ghost sector, renormalisation and the Schwinger-Dyson equations. Next was a short review of the recent work that has led to the belief that the ghost sector may be of crucial importance to IR QCD.

The next few chapters of the thesis were concerned with extracting information about the ghost-gluon vertex. In standard Schwinger-Dyson studies, the system of coupled equations is truncated with vertex ansatze so it is necessary to constrain the vertices as much as possible. After a preliminary chapter presenting necessary results, it was proposed to look for such an identity for the ghost-gluon vertex perturbatively. The justification for this approach was that any non-perturbative identity reduces in complexity when one considers the tree level and one-loop perturbative forms. It was hoped that this simplicity would give an indication of how to proceed. By demanding that the identity relate the vertex to some combination of propagator functions, it was immediately found that there does indeed exist such an identity at the one-loop level. This one-loop identity is true for all gauges and arbitrary dimension. It was extended to include the first fermionic contributions. By utilising the renormalisability of the theory, it was possible to constrain the higher order contributions to the identity. This was again based on the assumption that the full non-perturbative identity had a simple form that related vertices to propagators. From this was generated consistency equations for the two-loop perturbative expressions. By calculating the appropriate two-loop parts, it was found that these consistency equations could not be satisfied in Feynman gauge and with a specific momentum configuration. It is concluded therefore that the full non-perturbative identity sought does not simply relate vertices to propagators but involves other contributions. These could include the four-point ghost-ghost scattering. It was noted that omitting a certain graph explicitly containing the four-gluon interaction led to a possible agreement within the renormalisation inspired consistency equations. This was not followed up but may be a possible extension of the work.

The original non-perturbative identity put forward by von Smekal *et al.* [1, 2, 3] was then studied. Although not directly suitable for applying to a complete truncation scheme, this identity was shown to lead to a minor restriction on the ghost-gluon vertex function under a certain momentum configuration. Also shown were the important Landau gauge results that the ghost-gluon vertex reduces to it's tree-level form as one momentum argument vanishes and that it needs no renormalisation.

The last chapter concentrated on a different method of extracting information from the Schwinger-Dyson equations. Instead of relying on a specific vertex ansatz to truncate the system, the approach centred on the idea that as the argument of a two-point function vanishes, the function reduces to a single IR power. The multiplicative renormalisability of the theory was used to rewrite the Schwinger-Dyson equations in a consistent way that would allow these lowest IR powers to be extracted. It was found that for the pure Yang-Mills theory, in order to find more information about this lowest power it would be necessary to study the ghost-gluon vertex as one momentum argument vanishes. It was also found that in order to be consistent, the gluon propagator dressing function must vanish in the IR whilst the ghost propagator function must be singular.

There are many areas for further study that have not been touched on in this work. The first concerns the identity for the ghost-gluon vertex. That such a simple identity could be found at the one-loop perturbative level gives a tantalising glimpse of something, but quite what, is certainly not clear. One way to proceed would be to look at the two-loop quantities in Landau gauge. This would give more information on the possibilities, but is hampered by the obvious technical difficulties. Another possible area of study is the advancement of the powerlaw technique. The UV-cutoff regularisation has the problem that it is not translationally invariant, and this would be a natural place to start. Further, the quark sector has been neglected. Although work has been started along this path, the problem is compounded by the complexity of the quark-gluon vertex. Ultimately, the aim of this is to provide insight into colour confinement, the outstanding feature that separates QCD from QED.

Appendix A

One Loop Integrals

In this appendix, the one-loop integrals used in the various chapters are presented. These include the bubble integral with arbitrary powers of the denominators, the two-point vector integral and the scalar and vector triangle integrals with unit powers of the denominators. Dimensional regularisation is used throughout.

A.1 The Bubble Integral

The basic one-loop two-point scalar integral (the 'bubble') is written in Minkowski space as follows[1] (see fig. A.1)

$$I(\nu_1, \nu_2|p) = \int \frac{đ^d\omega}{[\omega^2]^{\nu_1}[(p-\omega)^2]^{\nu_2}}. \tag{A.1.1}$$

The integral is trivially symmetric under interchange of ω and $-\omega$ (which will be used often) and invariant under translation of ω (which will be pointed out when used). It can

[1]The arguments will be dropped for the most part, where no confusion arises.

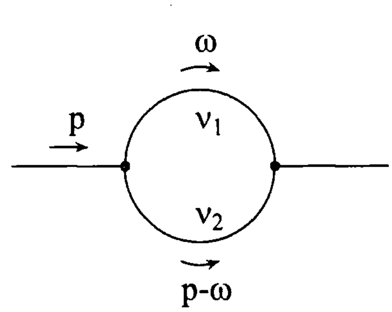


Figure A.1: The one-loop bubble integral with arbitrary powers of the denominators.

easily be evaluated in general dimension $d = 4 - 2\varepsilon$ using the following straightforward steps [5]. Step one is to introduce the standard Feynman parameterisation identity

$$\frac{1}{A_1^{m_1}\dots A_n^{m_n}} = \int_0^1 dx_1 \dots dx_n \delta(\sum x_i - 1) \frac{\Gamma(m_1 + \dots + m_n)}{\Gamma(m_1)\dots\Gamma(m_n)} \frac{\prod x_i^{m_i-1}}{[\sum x_i A_i]^{\sum m_i}}. \tag{A.1.2}$$

The integral then becomes

$$I = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 dx x^{\nu_1-1}(1-x)^{\nu_2-1} \int \frac{d^d\omega}{[x\omega^2 + (1-x)(p-\omega)^2]^{\nu_1+\nu_2}}. \tag{A.1.3}$$

The next step is to complete the square in the denominator by changing variables using $\omega = l + p(1-x)$ to get

$$I|_{w\to l+p(1-x)} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 dx x^{\nu_1-1}(1-x)^{\nu_2-1} \int \frac{d^d l}{[l^2 + p^2 x(1-x)]^{\nu_1+\nu_2}}. \tag{A.1.4}$$

The integral over l is standard and in $d = 4 - 2\varepsilon$ dimensions is

$$\int \frac{d^d l}{[l^2+\Delta]^n} = \imath(-1)^n (4\pi)^{-d/2} \frac{\Gamma(n-2+\varepsilon)}{\Gamma(n)} (-\Delta)^{-n+2-\varepsilon} \tag{A.1.5}$$

which immediately gives

$$I = \imath(-1)^{\nu_1+\nu_2}(4\pi)^{-d/2}(-p^2)^{-\nu_1-\nu_2+2-\varepsilon} \frac{\Gamma(\nu_1+\nu_2-2+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 dx x^{1-\nu_2-\varepsilon}(1-x)^{1-\nu_1-\varepsilon}. \tag{A.1.6}$$

The integral over x is just the Beta-function [49], which can be written generally as

$$\int_0^1 dx x^\alpha (1-x)^\beta = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{A.1.7}$$

giving the final result for the bubble integral

$$\begin{aligned} &I(\nu_1, \nu_2|p) \\ &= \imath(-1)^{\nu_1+\nu_2}(4\pi)^{-d/2}(-p^2)^{-\nu_1-\nu_2+2-\varepsilon} \frac{\Gamma(\nu_1+\nu_2-2+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(2-\nu_1-\varepsilon)\Gamma(2-\nu_2-\varepsilon)}{\Gamma(4-\nu_1-\nu_2-2\varepsilon)}. \end{aligned} \tag{A.1.8}$$

It is convenient to note that when the powers of the denominators are unity, the result simplifies to

$$I_p \equiv I(1,1|p) = \int \frac{d^d\omega}{\omega^2(p-\omega)^2} = \imath(4\pi)^{-d/2}(-p^2)^{-\varepsilon}\Gamma(\varepsilon)\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}. \tag{A.1.9}$$

This result is divergent as $\varepsilon \to 0$, with the divergence characterised by a simple pole in ε. Expanding the Gamma functions [49], one readily obtains

$$I_p = \imath(4\pi)^{-d/2}(-p^2)^{-\varepsilon}\left\{\frac{1}{\varepsilon} + (2-\gamma) + (2 - 2\gamma + \frac{1}{2}\gamma^2 - \frac{1}{12}\pi^2)\varepsilon + O(\varepsilon^2)\right\} \tag{A.1.10}$$

where γ is the Euler constant.

A.2 The Two-point Tensor Integrals

A natural extension of the bubble integral considered in the previous section is the two-point vector integral defined as[2]:

$$I(\mu|\nu_1,\nu_2|p) = \int \frac{đ^d\omega\omega_\mu}{[\omega^2]^{\nu_1}[(p-\omega)^2]^{\nu_2}}. \tag{A.2.1}$$

Notice that the symmetry properties in the bubble integral are no longer present. This integral can be calculated directly using the procedure of the last section but there is a simpler and more elegant way [50]. The integral has one external momentum scale p and one Lorentz index μ, so can be written

$$I_\mu = p_\mu J. \tag{A.2.2}$$

It is trivial to see that

$$J = \frac{1}{p^2}p^\mu I_\mu. \tag{A.2.3}$$

Now

$$p^\mu I_\mu = \int \frac{đ^d\omega p\cdot\omega}{[\omega^2]^{\nu_1}[(p-\omega)^2]^{\nu_2}} = \frac{1}{2}\int \frac{đ^d\omega}{[\omega^2]^{\nu_1}[(p-\omega)^2]^{\nu_2}}\left(p^2+\omega^2-(p-\omega)^2\right) \tag{A.2.4}$$

which immediately gives

$$I(\mu|\nu_1,\nu_2|p) = \frac{1}{2}p_\mu\left(I(\nu_1,\nu_2|p)+\frac{1}{p^2}I(\nu_1-1,\nu_2|p)-\frac{1}{p^2}I(\nu_1,\nu_2-1|p)\right). \tag{A.2.5}$$

This technique is actually a general method of calculating one-loop vector (and tensor) integrals in terms of simpler scalar integrals and introduces a kinematical quantity known as the Gram determinant (in this case just the factor $2p^2$) which can become quite complicated for higher-point integrals. Notice that in for the case when $\nu_1=\nu_2$, the last two-integrals cancel by virtue of the translational invariance of ω.

Now consider the tensor integral

$$I(\mu\alpha|\nu_1,\nu_2|p) = \int \frac{đ^d\omega\omega_\mu\omega_\alpha}{[\omega^2]^{\nu_1}[(p-\omega)^2]^{\nu_2}}. \tag{A.2.6}$$

This now has two Lorentz indices and so can be written

$$I_{\mu\alpha} = p_\mu p_\alpha J_1 + g_{\mu\alpha}p^2 J_2. \tag{A.2.7}$$

[2] Again the arguments will be dropped, apart from the Lorentz index.

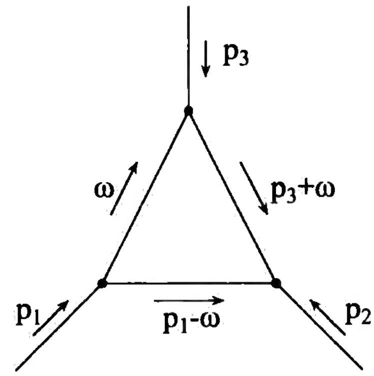


Figure A.2: Momentum routing for the one-loop triangle integral

Contracting with the transverse projector (1.7.1) and the metric then gives

$$J_2 = \frac{1}{(d-1)p^2} t^{\mu\alpha} I_{\mu\alpha} \tag{A.2.8}$$

$$J_1 = \frac{1}{p^2} g^{\mu\alpha} I_{\mu\alpha} - dJ_2. \tag{A.2.9}$$

The contractions involved in these equations are no more complicated than for the vector integral before and can be done to find an expression for $I(\mu\alpha|\nu_1, \nu_2|p)$ involving only the scalar integrals with differing powers of the denominator. In practice, it will not be necessary to know the explicit formulae for J_1 and J_2, but rather the definitions above since the tensor integrals that occur do so as part of lengthy calculations for which FORM programs are needed and for which, these expressions are ideally suited in terms of computer code.

A.3 The Triangle Integral

Now consider the one-loop scalar triangle integral with unit powers of the denominator factors. It can be written with all momenta incoming as follows (see also fig. A.2)

$$I(p_1, p_2, p_3) = \int \frac{\bar{d}^d\omega}{\omega^2 (p_1 - \omega)^2 (p_3 + \omega)^2}. \tag{A.3.1}$$

The first thing to notice is that the integral is symmetric under interchange of any of it's arguments, as can be seen by the following changes of variable

$$I(p_3, p_2, p_1) = \int \frac{\bar{d}^d\omega}{\omega^2 (p_3 - \omega)^2 (p_1 + \omega)^2}\Big|_{\omega \to -\omega} = I(p_1, p_2, p_3)$$

$$\begin{aligned} I(p_2,p_1,p_3) = & \int \frac{\bar{d}^d\omega}{\omega^2(p_2-\omega)^2(p_3+\omega)^2}\Big|_{\omega\to\omega-p_3} = \int \frac{\bar{d}^d\omega}{(p_3+\omega)^2(p_1-\omega)^2\omega^2} = I(p_1,p_2,p_3) \\ I(p_1,p_3,p_2) = & \int \frac{\bar{d}^d\omega}{\omega^2(p_1-\omega)^2(p_2+\omega)^2}\Big|_{\omega\to p_1-\omega} = \int \frac{\bar{d}^d\omega}{(p_1-\omega)^2\omega^2(p_3+\omega)^2} = I(p_1,p_2,p_3). \end{aligned} \quad (A.3.2)$$

The triangle integral is finite as $\varepsilon \to 0$ (as long as none of the momenta vanish) and in $d = 4$ dimensions can be calculated using various techniques (see for example [51]). The result is

$$I(p_1,p_2,p_3) = \imath(4\pi)^{-2}\frac{1}{p_3^2}\varphi^{(1)}(x,y) \quad (A.3.3)$$

where

$$\begin{aligned} \varphi^{(1)}(x,y) &= \frac{1}{\lambda}\left\{2Li_2(-\rho x)+2Li_2(-\rho y)+\ln\left(\frac{y}{x}\right)\ln\left(\frac{1+\rho y}{1+\rho x}\right)+\ln(\rho x)\ln(\rho y)+\frac{pi^2}{3}\right\} \\ \lambda(x,y) &= \sqrt{(1-x-y)^2-4xy}, \quad \text{(the Gram determinant)} \\ \rho(x,y) &= \frac{2}{1-x-y-\lambda} \end{aligned} \quad (A.3.4)$$

where

$$x = \frac{p_1^2}{p_3^2},\ y = \frac{p_2^2}{p_3^2} \quad (A.3.5)$$

and Li_2 is the dilogarithm.

It turns out that this integral need only be evaluated under the condition $p_1 = p_2 = p$, $p_3 = -2p$. There are two ways to do this. The first method is to calculate $\varphi^{(1)}(\frac{1}{4},\frac{1}{4})$ directly. Because the Gram determinant λ vanishes, the technique is to expand $\varphi^{(1)}$ in powers of λ first (using MAPLE [47]) and then set $x = y = \frac{1}{4}$ to get

$$I(p,p,-2p)|_{d=4} = \imath(4\pi)^{-2}\frac{1}{p^2}2\ln(2). \quad (A.3.6)$$

It is a well known result that there are no kinematical singularities in the integral when λ vanishes.

The second way to evaluate $I(p,p,-2p)$ is less straightforward and uses a trick involving the vector integral. Consider

$$I(p,p,-2p) = \int \frac{\bar{d}^d\omega}{\omega^2(p-\omega)^2(2p-\omega)^2}\Bigg|_{\omega\to p-\omega} = \int \frac{\bar{d}^d\omega}{(p-\omega)^2\omega^2(p+\omega)^2}. \quad (A.3.7)$$

Now,

$$\int \frac{\bar{d}^d\omega 2p\cdot\omega}{(p-\omega)^2\omega^2(p+\omega)^2}\Bigg|_{\omega\to-\omega} = -\int \frac{\bar{d}^d\omega 2p\cdot\omega}{(p-\omega)^2\omega^2(p+\omega)^2} = 0 \quad (A.3.8)$$

and expanding the scalar product

$$p^2 \int \frac{d\!\!{}^-{}^d\omega}{(p-\omega)^2\omega^2(p+\omega)^2} = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2} - \int \frac{d\!\!{}^-{}^d\omega}{(p-\omega)^2(p+\omega)^2}. \tag{A.3.9}$$

Changing variable $\omega \to \omega - p$ on the last integral gives

$$p^2 I(p,p,-2p) = \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2} - \int \frac{d\!\!{}^-{}^d\omega}{\omega^2(2p-\omega)^2}. \tag{A.3.10}$$

These integrals are nothing but the bubble integral evaluated at different momenta. Plugging in the result (A.1.9) immediately leads to

$$I(p,p,-2p) = \imath(4\pi)^{-d/2}\frac{1}{p^2}(-p^2)^{-\varepsilon}\Gamma(\varepsilon)\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}\left[1-(4)^{-\varepsilon}\right] \tag{A.3.11}$$

and restricting further to the expansion in ε,

$$I(p,p,-2p) = \imath(4\pi)^{-d/2}\frac{1}{p^2}(-p^2)^{-\varepsilon}\left[\frac{1}{\varepsilon}+O(\varepsilon^0)\right]\left[2\varepsilon\ln(2)+O(\varepsilon^2)\right]. \tag{A.3.12}$$

Setting $\varepsilon = 0$, one sees that the answer is finite and is

$$I(p,p,-2p)|_{d=4} = \imath(4\pi)^{-2}\frac{1}{p^2}2\ln(2) \tag{A.3.13}$$

which is the same as before. Note that now, the result can be expressed for any value of ε.

A.4 The Vector Triangle Integral

Now consider the one-loop vector triangle integral with unit powers of the denominator factors. It is written (with the same conventions as the scalar integral of the last section) as

$$I(\mu;p_1,p_2,p_3) = \int \frac{d\!\!{}^-{}^d\omega\,\omega_\mu}{\omega^2(p_1-\omega)^2(p_3+\omega)^2}. \tag{A.4.1}$$

It is possible to permute the order of the arguments by using changes of variable essentially identical to (A.3.2)

$$\begin{aligned}
I(\mu;p_3,p_2,p_1)|_{\omega\to-\omega} &= -I(\mu;p_1,p_2,p_3)\\
I(\mu;p_2,p_1,p_3)|_{\omega\to-p_3-\omega} &= -p_{3\mu}I(p_1,p_2,p_3) - I(\mu;p_1,p_2,p_3)\\
I(\mu;p_1,p_3,p_2)|_{\omega\to p_1-\omega} &= p_{1\mu}I(\mu;p_1,p_2,p_3) - I(\mu;p_1,p_2,p_3)
\end{aligned} \tag{A.4.2}$$

Lastly, consider the one-loop vector triangle integral with arbitrary powers of the denominator factors. This integral can be evaluated in terms of the scalar triangle and bubble integrals (and the appropriate Gram determinant) by noting that there are two independent momenta and one free Lorentz index, such that

$$\begin{aligned} I(\mu|\nu_1,\nu_2,\nu_3|p_1,p_2,p_3) &= \int \frac{đ^d\omega\omega_\mu}{[\omega^2]^{\nu_1}[(p_1-\omega)^2]^{\nu_2}[(p_3+\omega)^2]^{\nu_3}} \\ &= p_{1\mu}J_1 + p_{2\mu}J_2. \end{aligned} \quad (A.4.3)$$

By contraction with p_1 and p_2, one obtains two simultaneous equations in the two unknowns J_1 and J_2[3]

$$\begin{aligned} p_1^\mu I(\mu|\nu_1,\nu_2,\nu_3) &= p_1^2 J_1 + p_1{\cdot}p_3 J_2 \\ p_3^\mu I(\mu|\nu_1,\nu_2,\nu_3) &= p_1{\cdot}p_3 J_1 + p_3^2 J_2. \end{aligned} \quad (A.4.4)$$

Thus

$$\begin{aligned} J_1 &= \frac{1}{\Delta}\Big(p_3^2 p_1^\mu I(\mu|\nu_1,\nu_2,\nu_3) - p_1{\cdot}p_3 p_3^\mu I(\mu|\nu_1,\nu_2,\nu_3)\Big) \\ J_2 &= \frac{1}{\Delta}\Big(p_1^2 p_3^\mu I(\mu|\nu_1,\nu_2,\nu_3) - p_1{\cdot}p_3 p_1^\mu I(\mu|\nu_1,\nu_2,\nu_3)\Big) \end{aligned} \quad (A.4.5)$$

where

$$\Delta = p_1^2 p_3^2 - (p_1{\cdot}p_3)^2 \quad (A.4.6)$$

is the Gram determinant. Substituting the expressions for J_1 and J_2 back into the original and doing the contractions (cf the two-point functions), one obtains an expression for the vector integral in terms of scalar integrals. Again, the explicit formula is not needed since the contractions can be done *en masse* as part of the FORM code in an actual calculation.

[3]Here, only the momentum arguments can be omitted since there are no changes of variable.

Appendix B

Integration by Parts

In this appendix, a useful technique (integration by parts) for evaluating integrals with higher powers of the denominator factors will be presented [52]. This technique is based on the observation that the integral of a total derivative is zero. By expanding the derivative, one can obtain an expression relating integrals with denominator powers differing by unity.

B.1 The Two-point Integral

Consider the following integral of a total derivative

$$\int đ^d v \frac{\partial}{\partial v_\mu} \frac{q_\mu - v_\mu}{[v^2]^{\nu_1}[(p-v)^2]^{\nu_2}} = 0. \tag{B.1.1}$$

The product rule for differentiation applies, and the following formula are necessary

$$\begin{aligned} \frac{\partial}{\partial v_\mu} v_\mu &= d \\ \frac{\partial}{\partial v_\mu} \frac{1}{[(a-v)^2]^\nu} &= \frac{\partial[(a-v)^2]^{-\nu}}{\partial (a-v)^2} \frac{\partial (a-v)^2}{\partial v_\mu} \\ &= -\frac{\nu}{[(a-v)^2]^{\nu+1}} \frac{\partial}{\partial v_\mu} \left[a^2 - 2a^\alpha v_\alpha + v^\alpha v_\alpha\right] \\ &= 2\nu \frac{a_\mu - v_\mu}{[(a-v)^2]^{\nu+1}}. \end{aligned} \tag{B.1.2}$$

Thus (B.1.1) becomes

$$\int \frac{đ^d v}{[v^2]^{\nu_1}[(p-v)^2]^{\nu_2}} \left\{ -d - \frac{2\nu_1 (q-v)\cdot v}{v^2} + \frac{2\nu_2 (q-v)\cdot(p-v)}{(p-v)^2} \right\} = 0. \tag{B.1.3}$$

Expanding out the scalar products gives

$$\int \frac{đ^d v}{[v^2]^{\nu_1}[(p-v)^2]^{\nu_2}} \left\{ -d + \nu_1 + \nu_2 + (q-v)^2 \left[\frac{\nu_1}{v^2} + \frac{\nu_2}{(p-v)^2}\right] - \nu_1 \frac{q^2}{v^2} - \nu_2 \frac{(q-p)^2}{(p-v)^2} \right\} = 0. \tag{B.1.4}$$

Setting $q=0$ or $q=p$ and rearranging gives (using the notation established in the previous appendix)

$$I(\nu_1,\nu_2+1|p) = \frac{1}{\nu_2 p^2}\{-(d-2\nu_1-\nu_2)I(\nu_1,\nu_2|p)+\nu_1 I(\nu_1-1,\nu_2+1|p)\} \quad \text{(B.1.5)}$$

$$I(\nu_1+1,\nu_2|p) = \frac{1}{\nu_1 p^2}\{-(d-\nu_1-2\nu_2)I(\nu_1,\nu_2|p)+\nu_1 I(\nu_1+1,\nu_2-1|p)\}. \quad \text{(B.1.6)}$$

(B.1.6) can be applied recursively to reduce the denominator factors. Where either of the factors is zero, that integral vanishes under dimensional regularisation since there is no external scale (up to a translation of the integration variable).

B.2 The Three-point Integral

Consider

$$\int đ^d v \frac{\partial}{\partial v_\mu}\frac{q_\mu - v_\mu}{[v^2]^{\nu_1}[(p_1-v)^2]^{\nu_2}[(p_3+v)^2]^{\nu_3}} = 0. \quad \text{(B.2.1)}$$

Applying the product rule for differentiation, equations (B.1.2) and expanding the scalar products gives

$$\int \frac{đ^d v}{[v^2]^{\nu_1}[(p_1-v)^2]^{\nu_2}[(p_3+v)^2]^{\nu_3}} \left\{-d+\nu_1+\nu_2+\nu_3 \right.$$
$$+(q-v)^2\left[\frac{\nu_1}{v^2}+\frac{\nu_2}{(p_1-v)^2}+\frac{\nu_3}{(p_3+v)^2}\right]$$
$$\left. -\nu_1\frac{q^2}{v^2}-\nu_2\frac{(q-p_1)^2}{(p_1-v)^2}-\nu_3\frac{(q+p_3)^2}{(p_3+v)^2}\right\} = 0. \quad \text{(B.2.2)}$$

By setting the value of q to $0, p_1$ and p_3 in turn[1]

[1] Using the obvious notation where the arguments denote the powers of the denominators, cf the latter part of sect. A.4

$$
\begin{aligned}
&q = 0: \\
&\nu_2 p_1^2 I(\nu_1, \nu_2+1, \nu_3) + \nu_3 p_3^2 I(\nu_1, \nu_2, \nu_3+1) = \\
&(-d + 2\nu_1 + \nu_2 + \nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_2 I(\nu_1 - 1, \nu_2+1, \nu_3) + \nu_3 I(\nu_1 - 1, \nu_2, \nu_3+1) \\
&q = p_1: \\
&\nu_1 p_1^2 I(\nu_1+1, \nu_2, \nu_3) + \nu_3 p_2^2 I(\nu_1, \nu_2, \nu_3+1) = \\
&(-d + \nu_1 + 2\nu_2 + \nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2-1, \nu_3) + \nu_3 I(\nu_1, \nu_2-1, \nu_3+1) \\
&q = -p_3: \\
&\nu_1 p_3^2 I(\nu_1+1, \nu_2, \nu_3) + \nu_2 p_2^2 I(\nu_1, \nu_2+1, \nu_3) = \\
&(-d + \nu_1 + \nu_2 + 2\nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2, \nu_3-1) + \nu_2 I(\nu_1, \nu_2+1, \nu_3-1).
\end{aligned}
\tag{B.2.3}
$$

This can be rewritten in matrix form as

$$
\begin{pmatrix} \nu_1 p_3^2 & \nu_2 p_2^2 & 0 \\ \nu_1 p_1^2 & 0 & \nu_3 p_2^2 \\ 0 & \nu_2 p_1^2 & \nu_3 p_3^2 \end{pmatrix}
\begin{pmatrix} I(\nu_1+1, \nu_2, \nu_3) \\ I(\nu_1, \nu_2+1, \nu_3) \\ I(\nu_1, \nu_2, \nu_3+1) \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
\tag{B.2.4}
$$

where

$$
\begin{aligned}
b_1 &= (-d + \nu_1 + \nu_2 + 2\nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2, \nu_3-1) + \nu_2 I(\nu_1, \nu_2+1, \nu_3-1) \\
b_2 &= (-d + \nu_1 + 2\nu_2 + \nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_1 I(\nu_1 + 1, \nu_2-1, \nu_3) + \nu_3 I(\nu_1, \nu_2-1, \nu_3+1) \\
b_3 &= (-d + 2\nu_1 + \nu_2 + \nu_3) I(\nu_1, \nu_2, \nu_3) + \nu_2 I(\nu_1 - 1, \nu_2+1, \nu_3) + \nu_3 I(\nu_1 - 1, \nu_2, \nu_3+1)
\end{aligned}
\tag{B.2.5}
$$

Now our system of equations is in the form $A\boldsymbol{x} = \boldsymbol{b}$ where A is a matrix and $\boldsymbol{x}$ and $\boldsymbol{b}$ are column vectors. Cramer's rule tells us that the components of the vector $\boldsymbol{x}$ are given by

$$
x_i = \frac{\det(A_i)}{\det(A)} \tag{B.2.6}
$$

where A_i is the matrix A with the i$^{\text{th}}$ column replaced by the column vector $\boldsymbol{b}$. Now,

$$\begin{aligned}
\det(A) &= -2\nu_1\nu_2\nu_3 p_1^2 p_2^2 p_3^2 \\
\det(A_1) &= -\nu_2\nu_3 p_2^2 \left(p_1^2 b_1 + p_3^2 b_2 - p_2^2 b_3\right) \\
\det(A_2) &= -\nu_1\nu_3 p_3^2 \left(p_1^2 b_1 - p_3^2 b_2 + p_2^2 b_3\right) \\
\det(A_2) &= -\nu_1\nu_2 p_1^2 \left(-p_1^2 b_1 + p_3^2 b_2 + p_2^2 b_3\right)
\end{aligned} \tag{B.2.7}$$

giving

$$\begin{aligned}
I(\nu_1+1,\nu_2,\nu_3) &= \frac{1}{2\nu_1 p_1^2 p_3^2}\left(p_1^2 b_1 + p_3^2 b_2 - p_2^2 b_3\right) \\
I(\nu_1,\nu_2+1,\nu_3) &= \frac{1}{2\nu_2 p_1^2 p_2^2}\left(p_1^2 b_1 - p_3^2 b_2 + p_2^2 b_3\right) \\
I(\nu_1,\nu_2,\nu_3+1) &= \frac{1}{2\nu_3 p_2^2 p_3^2}\left(-p_1^2 b_1 + p_3^2 b_2 + p_2^2 b_3\right).
\end{aligned} \tag{B.2.8}$$

These formulae, with the b_i inserted give a recursive method for dealing with integrals with higher powers of the denominator factors. If any of the ν_i become zero, then that integral becomes a two-point integral, which has been dealt with in the previous section. For example

$$\begin{aligned}
I(0,1,1) &= \int \frac{\bar{d}^d v}{(p_1 - v)^2 (p_3 + v)^2} \\
&= \int \frac{\bar{d}^d v}{v^2 (p_2 - v)^2}
\end{aligned} \tag{B.2.9}$$

(in the second line, the change of variable $v \to p_1 + v$ was made). Thus, it is a reasonably simple matter to reduce the number of integrals actually performed to a minimal amount.

Appendix C

Two Loop Propagator Integrals

In this appendix, the two-loop propagator type integrals will be evaluated. There are four basic topologies to consider (see fig. C.1). It is only necessary to consider the integrals with unit powers of the denominators. Note that all these integrals can be expressed in a number of ways by translating the integration variables.

C.1 Graph a

Graph a (fig. C.1) is trivial since it is just the square of the bubble integral with unit powers of the denominators (A.1.8), ie

$$\begin{aligned} I_a &= \int \frac{đ^d\omega}{\omega^2(p-\omega)^2}\int \frac{đ^d v}{v^2(p-v)^2} \\ &= -(4\pi)^{-d}(-p^2)^{-2\varepsilon}\Gamma(\varepsilon)^2\frac{\Gamma(1-\varepsilon)^4}{\Gamma(2-2\varepsilon)^2}. \end{aligned} \tag{C.1.1}$$

Expanding the Gamma functions in powers of ε using MAPLE, one readily obtains

$$I_a = -(4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\frac{1}{\varepsilon^2}+(4-2\gamma)\frac{1}{\varepsilon}+12-8\gamma+2\gamma^2-\frac{\pi^2}{6}+O(\varepsilon)\right\}. \tag{C.1.2}$$

C.2 Graph b

Graph b (fig. C.1) is the so-called sunset diagram and is evaluated using the equation for the bubble integral of appendix a with arbitrary powers of the denominators (A.1.8). The crucial property is that each integral can be done in turn.

$$I_b = \int \frac{đ^d\omega}{(p-\omega)^2}\int \frac{đ^d v}{v^2(\omega-v)^2}$$

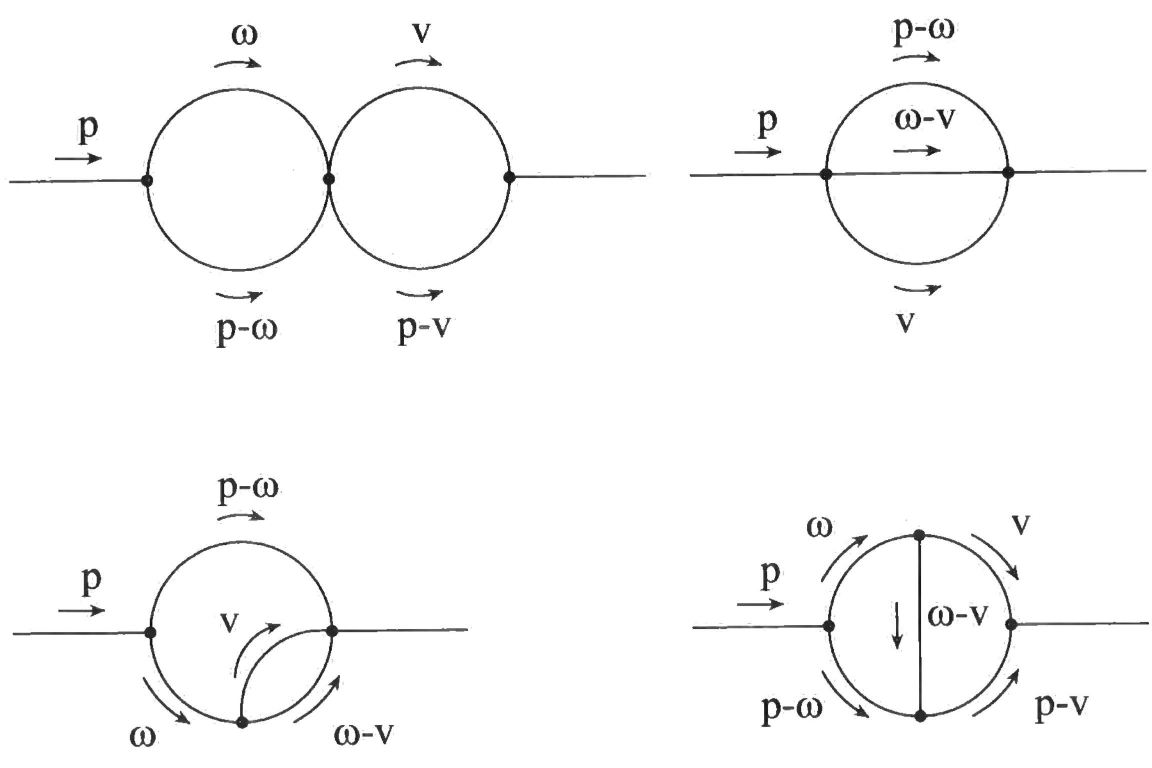


Figure C.1: (clockwise from top left) Graphs a–d, showing the momentum routing for each of the basic two-loop propagator integrals.

$$= \imath(4\pi)^{-d/2}\Gamma(\varepsilon)\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}(-1)^{-\varepsilon}\int\frac{d\!\!{}^-{}^d\omega}{[\omega^2]^{\varepsilon}(p-\omega)^2}$$

$$= (4\pi)^{-d}p^2(-p^2)^{-2\varepsilon}\frac{1}{(1-2\varepsilon)}\Gamma(2\varepsilon)\frac{\Gamma(1-\varepsilon)^3}{\Gamma(3-3\varepsilon)} \qquad \text{(C.2.1)}$$

Again expanding in powers of ε, one gets

$$I_b = (4\pi)^{-d}p^2(-p^2)^{-2\varepsilon}\left\{\frac{1}{4\varepsilon}+\frac{13}{8}-\frac{1}{2}\gamma+O(\varepsilon)\right\}. \qquad \text{(C.2.2)}$$

C.3 Graph c

Graph c (fig. C.1) is similar to graph b in its evaluation.

$$I_c = \int\frac{d\!\!{}^-{}^d\omega}{\omega^2(p-\omega)^2}\int\frac{d\!\!{}^-{}^d v}{v^2(\omega-v)^2}$$

$$= \imath(4\pi)^{-d/2}\Gamma(\varepsilon)\frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}(-1)^{-\varepsilon}\int\frac{d\!\!{}^-{}^d\omega}{[\omega^2]^{1+\varepsilon}(p-\omega)^2}$$

$$= -(4\pi)^{-d}(-p^2)^{-2\varepsilon}\frac{1}{\varepsilon}\Gamma(2\varepsilon)\frac{\Gamma(1-\varepsilon)^3}{\Gamma(1-2\varepsilon)\Gamma(2-3\varepsilon)} \qquad \text{(C.3.1)}$$

Again expanding in powers of ε, one gets

$$I_c = -(4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\frac{1}{2\varepsilon^2}+(\frac{5}{2}-\gamma)\frac{1}{\varepsilon}+\frac{19}{2}-5\gamma+\gamma^2-\frac{\pi^2}{12}+O(\varepsilon)\right\}. \qquad \text{(C.3.2)}$$

C.4 Graph d

Graph d (fig. C.1) is considerably more complicated than the rest to evaluate and to do so requires integration by parts. Consider the following

$$\int d\!\!{}^-{}^d v\frac{\partial}{\partial v_\mu}\frac{v_\mu-\omega_\mu}{(p-v)^2v^2(\omega-v)^2}=0. \qquad \text{(C.4.1)}$$

The product rule for differentiation applies, and using (B.1.2) leads to

$$\int\frac{d\!\!{}^-{}^d v}{(p-v)^2v^2(\omega-v)^2}\left\{d-2+\frac{2(p-v)\cdot(v-w)}{(p-v)^2}-\frac{2v\cdot(v-w)}{v^2}\right\}=0. \qquad \text{(C.4.2)}$$

Expanding the scalar products gives

$$\int\frac{d\!\!{}^-{}^d v}{(p-v)^2v^2(\omega-v)^2}\left\{d-4+\frac{(p-\omega)^2}{(p-v)^2}-\frac{(\omega-v)^2}{(p-v)^2}+\frac{\omega^2}{v^2}-\frac{(\omega-v)^2}{v^2}\right\}=0, \qquad \text{(C.4.3)}$$

and rearranging

$$(d-4)\int \frac{d\!\!{}^-{}^d v}{(p-v)^2 v^2 (\omega-v)^2} = -\int \frac{d\!\!{}^-{}^d v (p-\omega)^2}{(p-v)^4 v^2 (\omega-v)^2} - \int \frac{d\!\!{}^-{}^d v \omega^2}{(p-v)^2 v^4 (\omega-v)^2} + \int \frac{d\!\!{}^-{}^d v}{(p-v)^4 v^2} + \int \frac{d\!\!{}^-{}^d v}{(p-v)^2 v^4}. \quad (C.4.4)$$

Changing variables $v \to p-v$ in the penultimate integral then gives

$$(d-4)\int \frac{d\!\!{}^-{}^d v}{(p-v)^2 v^2 (\omega-v)^2} = -\int \frac{d\!\!{}^-{}^d v (p-\omega)^2}{(p-v)^4 v^2 (\omega-v)^2} - \int \frac{d\!\!{}^-{}^d v \omega^2}{(p-v)^2 v^4 (\omega-v)^2} + 2\int \frac{d\!\!{}^-{}^d v}{(p-v)^2 v^4}. \quad (C.4.5)$$

It is now possible to consider the integral in question, graph d of (fig. C.1)

$$\begin{aligned} I_d &= \int \frac{d\!\!{}^-{}^d \omega\, d\!\!{}^-{}^d v}{\omega^2 (p-\omega)^2 (p-v)^2 v^2 (\omega-v)^2} \\ &= \frac{1}{(d-4)} \left\{ 2\int \frac{d\!\!{}^-{}^d \omega}{\omega^2 (p-\omega)^2} \int \frac{d\!\!{}^-{}^d v}{(p-v)^2 v^4} \right. \\ &\quad \left. - \int \frac{d\!\!{}^-{}^d \omega\, d\!\!{}^-{}^d v}{\omega^2 (p-v)^4 v^2 (\omega-v)^2} - \int \frac{d\!\!{}^-{}^d \omega\, d\!\!{}^-{}^d v}{(p-\omega)^2 (p-v)^2 v^4 (\omega-v)^2} \right\}. \end{aligned} \quad (C.4.6)$$

In the second last integral, the variables ω and v are swapped. For the last integral, one makes the changes of variable $v \to p-\omega$, $\omega \to p-v$. This gives

$$I_d = \frac{2}{d-4} \left\{ 2\int \frac{d\!\!{}^-{}^d \omega}{\omega^2 (p-\omega)^2} \int \frac{d\!\!{}^-{}^d v}{(p-v)^2 v^4} - 2\int \frac{d\!\!{}^-{}^d \omega\, d\!\!{}^-{}^d v}{\omega^2 (p-\omega)^4 v^2 (\omega-v)^2} \right\}. \quad (C.4.7)$$

It is possible to proceed in two ways. One method is to use integration by parts for each of the integrals. This is not particularly complicated but in this this case, the result (A.1.8) for the bubble integral can simply be plugged in.

$$\begin{aligned} I_d &= \frac{2}{d-4} \imath (4\pi)^{-d/2} \Gamma(\varepsilon) \frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} \\ &\quad \left\{ (-p^2)^{-\varepsilon} \int \frac{d^d v}{(p-v)^2 v^4} - (-1)^{-\varepsilon} \int \frac{d^d \omega}{[\omega^2]^{1+\varepsilon} (p-\omega)^4} \right\} \\ &= -\frac{1}{\varepsilon} (4\pi)^{-d} (-p^2)^{-1-2\varepsilon} \Gamma(\varepsilon) \Gamma(-\varepsilon) \frac{\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)} \\ &\quad \left\{ \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} - \frac{\Gamma(1+2\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)} \right\} \end{aligned} \tag{C.4.8}$$

Expanding in powers of ε gives

$$I_d = -(4\pi)^{-d} \frac{1}{p^2} (-p^2)^{-2\varepsilon} \{6\zeta_3 + O(\varepsilon)\} \tag{C.4.9}$$

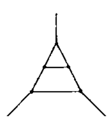
where $\zeta_3 \approx 1.202$ is the Riemann Zeta function. ζ_3 arises naturally from the higher order terms in the expansion of the Gamma functions, the lower order parts cancelling in this expression to give a startlingly simple result.

Appendix D

Two Loop Vertex Integrals

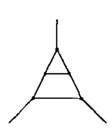
In this appendix, the two-loop vertex type integrals used will be derived. There are nine such integrals. The integrals are evaluated with a certain configuration of external momenta ($p_1 = p_2 = p$). The generic integral will be represented as a diagram, with the external legs having the configuration of fig. D.1. Because of the external momentum configuration, there is a reflection symmetry about the vertical line cutting the middle of the graph. This symmetry will be exploited wherever possible.

Consider then the integral



$$= \int \frac{\bar{d}^d\omega\, \bar{d}^d v}{\omega^2(p-\omega)^2(2p-\omega)^2 v^2(2p-v)^2(\omega-v)^2}. \tag{D.0.1}$$

This has been explicitly evaluated by Davydychev and others [53, 54] to be (in $d = 4$ dimensions)



$$= (2\pi)^{-2d}\left(\frac{\imath\pi^2}{4p^2}\right)^2 \varphi^{(2)}\left(\frac{1}{4},\frac{1}{4}\right) \tag{D.0.2}$$

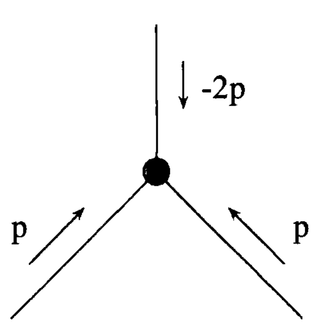


Figure D.1: Momentum configuration of the external legs for the generic two-loop vertex type integral.

where

$$\begin{aligned}\varphi^{(2)}(x,y) &= \frac{1}{\lambda}\Big\{6\left(Li_4(-\rho x)+Li_4(-\rho y)\right)+3\ln\frac{y}{x}\left(Li_3(-\rho x)-Li_3(-\rho y)\right)\\ &+\frac{1}{2}\ln^2\frac{y}{x}\left(Li_2(-\rho x)+Li_2(-\rho y)\right)+\frac{1}{4}\ln^2(\rho x)\ln^2(\rho y)\\ &+\frac{\pi^2}{2}\ln(\rho x)\ln(\rho y)+\frac{\pi^2}{12}\ln^2\frac{y}{x}+\frac{7\pi^4}{60}\Big\}\end{aligned} \tag{D.0.3}$$

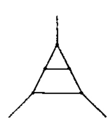
with x, y, λ and ρ defined as in appendix A.3 and the polylogarithms $Li_n(z)$ defined as

$$Li_n(z) = \frac{(-1)^n}{(n-1)!}\int_0^1 dt\frac{\ln^{n-1}t}{t-z^{-1}}. \tag{D.0.4}$$

The function $\varphi^{(2)}$ can be explicitly evaluated just like the one-loop triangle integral of appendix A.3 using MAPLE and the result is

$$\varphi^{(2)}\left(\frac{1}{4},\frac{1}{4}\right) = 18\zeta_3. \tag{D.0.5}$$

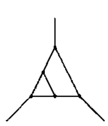
This result is finite and this allows the integral to be written with the same prefactors as the two-loop propagator type integrals of appendix C in the following way



$$= -\frac{1}{p^4}(4\pi)^{-d}(-p^2)^{-2\varepsilon}\frac{9}{8}\zeta_3. \tag{D.0.6}$$

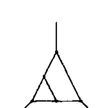
since the expansion in ε will only produce terms of $O(\varepsilon)$.

The second integral to consider is



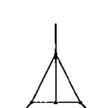
$$= \int \frac{\bar{d}^d\omega\,\bar{d}^dv}{\omega^2(p-\omega)^2(2p-\omega)^2v^2(p-v)^2(\omega-v)^2}. \tag{D.0.7}$$

This has the same form as the previous integral, but with different arguments. It is evaluated in exactly the same way.



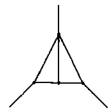
$$= \left(\frac{i\pi^2}{p^2}\right)^2\varphi^{(2)}(4,1) = -\frac{1}{p^4}(4\pi)^{-d}(-p^2)^{-2\varepsilon}\frac{21}{8}\zeta_3. \tag{D.0.8}$$

Consider now the third integral.

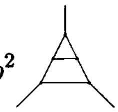


$$= \int \frac{\bar{d}^d\omega\,\bar{d}^dv}{\omega^2(p-\omega)^2(p-v)^2(2p-v)^2(\omega-v)^2} \tag{D.0.9}$$

This is evaluated using the so-called uniqueness relations [53, 54]. It turns out that

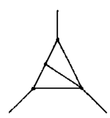


$$= 4p^2$$



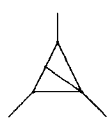
$$= -\frac{1}{p^2}(4\pi)^{-d}(-p^2)^{-2\varepsilon}\frac{9}{2}\zeta_3. \tag{D.0.10}$$

The fourth integral is expressed as

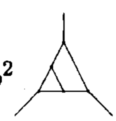


$$= \int \frac{đ^d\omega\, đ^d v}{\omega^2(p-\omega)^2 v^2(2p-v)^2(\omega-v)^2}. \tag{D.0.11}$$

This can be evaluated in two ways. Again using the uniqueness relations [53, 54] leads immediately to



$$= p^2$$

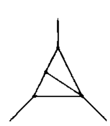


$$= -\frac{1}{p^2}(4\pi)^{-d}(-p^2)^{-2\varepsilon}\frac{21}{8}\zeta_3. \tag{D.0.12}$$

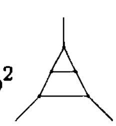
The second way to calculate this uses the same vector integral trick as appendix A.3. Consider the following

$$\int \frac{đ^d\omega\, đ^d v\; 2p\cdot\omega}{(p-\omega)^2\omega^2(p+\omega)^2(p-v)^2(\omega-v)^2(p+v)^2} = 0 \tag{D.0.13}$$

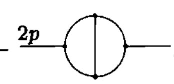
which comes about due to the antisymmetry of the numerator. Expanding the scalar product leads to the relation



$$= p^2$$

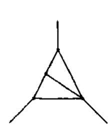


$$+$$



$$. \tag{D.0.14}$$

Plugging in the results (D.0.6) and (C.4.9) gives again

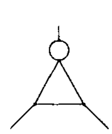


$$= -\frac{1}{p^2}(4\pi)^{-d}(-p^2)^{-2\varepsilon}\frac{21}{8}\zeta_3. \tag{D.0.15}$$

The next two two-loop vertex integrals involve a bubble integral multiplying the one-loop triangle integral. The one-loop triangle integral was calculated in appendix A.3. Rewriting the result (A.3.10)

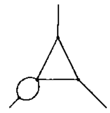
$$p^2 I(p,p,-2p) = \int \frac{đ^d\omega}{\omega^2(p-\omega)^2} - \int \frac{đ^d\omega}{\omega^2(2p-\omega)^2} = I_p - I_{2p} \tag{D.0.16}$$

gives immediately



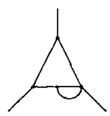
$$= \frac{1}{p^2} I_{2p}\left[I_p - I_{2p}\right] \tag{D.0.17}$$

and

 $$= \frac{1}{p^2} I_p \left[I_p - I_{2p}\right]. \qquad \text{(D.0.18)}$$

It is not necessary to expand these integrals as power series in ε at this point because the two-point integrals will be evaluated after the vertex-type integrals.

Consider now the integral

 $$= \int \frac{\bar{d}^d\omega\, \bar{d}^d v}{\omega^2(p-\omega)^2(2p-\omega)^2(p-v)^2(\omega-v)^2}. \qquad \text{(D.0.19)}$$

In order to calculate this, it is necessary to use the integration by parts technique. The starting point is the relation (B.1.6)

$$-(d-3)I_p = p^2 \int \frac{\bar{d}^d v}{v^2(p-v)^4}. \qquad \text{(D.0.20)}$$

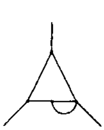
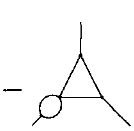
This can be inserted into the following two-loop integral

$$-(d-3)\int \frac{\bar{d}^d\omega\, \bar{d}^d v}{\omega^2(p-\omega)^2(2p-\omega)^2 v^2(\omega-v)^2} = \int \frac{\bar{d}^d\omega\, \bar{d}^d v}{(p-\omega)^2(2p-\omega)^2 v^2(\omega-v)^4}. \qquad \text{(D.0.21)}$$

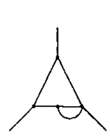
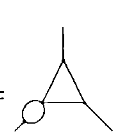
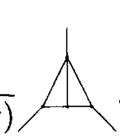
Notice now that the integral over ω can be considered as a one-loop triangle with one of the denominator factors squared. To see this, write the previous equation as

$$-(d-3)\int \frac{\bar{d}^d\omega\, \bar{d}^d v}{\omega^2(p-\omega)^2(2p-\omega)^2 v^2(\omega-v)^2} =$$
$$\int \frac{\bar{d}^d v(p-v)^2(2p-v)^2}{v^2(p-v)^2(2p-v)^2} \int \frac{\bar{d}^d\omega}{(p-\omega)^2(2p-\omega)^2(\omega-v)^4}. \qquad \text{(D.0.22)}$$

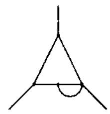
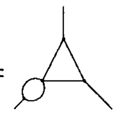
In appendix B.2, the technique for expressing the one-loop triangle integral with higher denominators was presented. Using this technique and cancelling factors that are related by the vertical reflection symmetry about the centre, it is possible to show that

$$0 = -\frac{(d-4)}{2}$$  $$-(d-3)\Big\{$$  $$-$$  $$\Big\}. \qquad \text{(D.0.23)}$$

The original two-loop vertex integral has dropped out of the equation, leaving

 $$=$$  $$+\frac{\varepsilon}{(1-2\varepsilon)}$$  $$. \qquad \text{(D.0.24)}$$

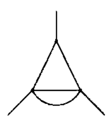
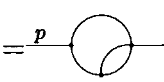
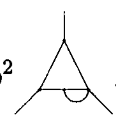
Now the second integral on the right-hand side was evaluated before (D.0.10) and is finite. This means that in $d=4$ dimensions

 $=$  $= \frac{1}{p^2} I_p \left[I_p - I_{2p} \right].$ (D.0.25)

In order to evaluate the next integral, it is necessary to use the vector triangle trick again. Consider

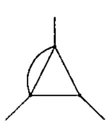
$$\int \frac{đ^d\omega\, đ^d v\; 2p\cdot\omega}{(p-\omega)^2\omega^2(p+\omega)^2 v^2(\omega-v)^2} = 0. \tag{D.0.26}$$

Expanding the scalar product, one readily obtains

 $=$  $- p^2$  . (D.0.27)

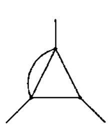
This result need not be evaluated explicitly.

The last two-loop vertex integral is evaluated using the results of Ussyukina and Davydychev [53]. The integral is expressed as



$$= -(4\pi)^{-d}(-p^2)^{-2\varepsilon}\Gamma(\varepsilon)\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{2(1-3\varepsilon)\Gamma(2-2\varepsilon)}\left\{\frac{1}{\varepsilon} - 4\varepsilon\varphi^{(1)}(1,4) - \varepsilon\frac{\pi^2}{6} + O(\varepsilon^2)\right\}. \tag{D.0.28}$$

Putting in the expression for $\varphi^{(1)}(1,4)$ derived in appendix A.3 gives the final result expanded in ε



$$= -(4\pi)^{-d}(-p^2)^{-2\varepsilon}\left\{\frac{1}{2\varepsilon^2} + \left(\frac{5}{2}-\gamma\right)\frac{1}{\varepsilon} - 4\ln 2 + \frac{19}{2} - 5\gamma + \gamma^2 - \frac{\pi^2}{12} + O(\varepsilon)\right\}. \tag{D.0.29}$$

Bibliography

[1] L.v. Smekal, A. Hauck, and R. Alkofer, Phys. Rev. Lett. **79** (1997) 3591.

[2] L.v. Smekal, A. Hauck, and R. Alkofer, Ann. Phys. **267** (1998) 1; Erratum-ibid. **269** (1998) 182.

[3] A. Hauck, L.v. Smekal, and R. Alkofer, Comput. Phys. Commun. **112** (1998) 166.

[4] F. Halzen, and A.D. Martin, "Quarks and Leptons: An Introductory Course in Modern Particle Physics", John Wiley and Sons (1984).

[5] M.E. Peskin, and D.V. Schroeder, "An Introduction to Quantum Field Theory", Addison-Wesley Publishing Company (1995).

[6] T. Muta, "Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories", World Scientific Publishing Co. Pte. Ltd. (1987).

[7] C. Itzykson, and J.-B. Zuber, "Quantum Field Theory", McGraw-Hill Book Co. (1985).

[8] E.S. Abers, and B.W. Lee, Phys. Rep. **9C** (1973) 1.

[9] R.P. Feynman, Rev. Mod. Phys. **20** (1948) 367.

[10] R.P. Feynman, and A.R. Hibbs, "Quantum Mechanics and Path Integrals", McGraw-Hill Book Co. (1965).

[11] L.H. Ryder, "Quantum Field Theory", Cambridge University Press (1996).

[12] R.J. Rivers, "Path Integral Methods in Quantum Field Theory", Cambridge University Press (1988).

[13] I.D. Lawrie, "A Unified Grand Tour of Theoretical Physics", IOP Publishing Ltd (1990).

[14] L.S. Faddeev, and U.N. Popov, "Perturbation Theory for Gauge Invariant Fields", Kiev ITP Report (unpublished); Phys, Lett. **25B** (1967) 29.

[15] V.N. Gribov, Nuc. Phys. **B139** (1978) 1.

[16] J.S. Schwinger, Proc. Nat. Acad. Sc. **37** (1951) 452, 455.

[17] F.J. Dyson, Phys. Rev. **75** (1949) 1736.

[18] C. Becchi, A. Rouet, and R. Stora, Comm. Math. Phys. **42** (1975) 127; Ann. Phys. **98** (1976) 287.

[19] A.A. Slavnov, Theor. Math. Phys. **10** (1972) 99; J.C. Taylor, Nuc. Phys. **B33** (1971) 436.

[20] G. 't Hooft, Nuc. Phys. **B33** (1971) 173.

[21] W. Marciano, and H. Pagels, Phys. Rep. **36C** (1978) 137.

[22] C.K. Callan, Phys. Rev. **D2** (1970) 1541; K. Symanzik, Commun. Math. Phys. **18** (1970) 227.

[23] M.R. Pennington, Rep. Prog. Phys. **46** (1983) 393.

[24] A.I. Davydychev, P. Osland, and O.V. Tarasov, Phys. Rev. **D54** (1996) 4087.

[25] S. Mandelstam, Phys. Rev. **D20** (1979) 3223.

[26] N. Brown, and M.R. Pennington, Phys. Rev. **D39** (1989) 2723.

[27] M. Baker, J.S. Ball, and F. Zachariasen, Nuc. Phys. **B186** (1981) 531.

[28] G.B. West, Phys. Rev. **D27** (1983) 1878.

[29] W.J. Schoenmaker, Nuc. Phys. **B194** (1982) 535.

[30] K. Buttner, M.R. Pennington, Phys. Rev. **D52** (1995) 5220.

[31] U. Bar-Gadda, Nucl. Phys **B163** (1980) 312.

[32] M. Stingl, Z. Phys. **A253** (1996) 423; Eur. Phys. J. **A4** (1999) 381; Eur. Phys. J. **A4** (1999) 401.

[33] D. Atkinson, and J.C.R. Bloch, Phys. Rev. **D58** (1998) 094036.

[34] D. Atkinson, and J.C.R. Bloch, Mod.Phys.Lett. **A13** (1998) 1055.

[35] J.S. Ball, and T-W. Chiu, Phys. Rev. **D22** (1980) 2542.

[36] D.C. Curtis, and M.R. Pennington, Phys. Rev. **D42** (1990) 4165.

[37] J.S. Ball, and T-W. Chiu, Phys. Rev. **D22** (1980) 2550.

[38] S.K. Kim, and M. Baker, Nuc. Phys. **B164** (1980) 152.

[39] D. Atkinson, J.K. Drohm, P.W. Johnson, and K. Stam, J. Math. Phys. **22** (1981) 2704.

[40] A. Hauck, L.v. Smekal, and R. Alkofer, Comp. Phys. Comm. **112** (1998) 149.

[41] F.T. Hawes, P. Maris, and C.D. Roberts, Phys. Lett. **B440** (1998) 353.

[42] C.D. Roberts, Report No. nucl-th/9807026.

[43] J.C. Taylor, Nucl. Phys. **B33** (1971) 436.

[44] J.A.M. Vermaseren, "Symbolic Manipulation with FORM", CAN, Amsterdam (1991).

[45] A.I. Davydychev, P. Osland, and O.V. Tarasov, Phys. Rev. **D58** (1998) 036007.

[46] P. Cvitanovic, Phys. Rev. **D14** (1976) 1536.

[47] D. Redfern, "The MAPLE Handbook", Springer-Verlag New York, Inc. (1994).

[48] Ayse Kizilersu, Tom Sizer, and Anthony G. Williams, Report No. hep-ph/0001147.

[49] I.S. Gradshteyn, and I.M. Ryzhik, "Table of Integrals, Series, and Products", Academic Press (1980).

[50] L.M. Brown, and R.P. Feynman, Phys. Rev. **85** (1952) 231;
G. Passarino, and M. Veltman, Nuc. Phys. **B160** (1979) 151.

[51] J.M. Campbell, E.W.N. Glover, and D.J. Miller, Nucl. Phys. **B498** (1997) 397.

[52] F.V. Tkachov, Phys. Lett. **B100** (1981) 65;
K.G. Chetyrkin, and F.V. Tkachov, Nuc. Phys. **B192** (1981) 159.

[53] N.I. Ussyukina, and A.I. Davydychev, Report No. hep-ph/9402223.

[54] N.I. Ussyukina, and A.I. Davydychev, Phys. Lett. **B298** (1993) 363.