Harmonic maps, SU (N) skyrme models and yang-mills theories

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Harmonic Maps, $SU(N)$ Skyrme Models and Yang-Mills Theories

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Abstract

This thesis examines the construction of static solutions of (3+1)-dimensional $SU(N)$ Skyrme models, usual and alternative, and pure massive $SU(N)$ Yang-Mills theories. In particular, the application of harmonic maps from S^2 into the subspace of fiélds configuration space $\mathcal{M}$. Here, the harmonic maps are used as an ansatz to factoring out the angular dependence part of the solutions from the field equations. In this thesis, we consider the harmonic maps $S^2 \to Gr(n,\, N)$, where $Gr(n,\, N)$ is the Grassmann manifold of n-dimensional planes passing through the origin in C^N.

Using the harmonic map ansatz of $S^2 \to Gr(2,\, N)$ to study the usual $SU(N)$ Skyrme models, we have found that our approximate solutions have marginally higher energies in comparison to the corresponding results previously obtained using CP^{N-1} as target space $\mathcal{M}$. For exact spherically symmetric solutions, we present arguments which suggest that the only solutions obtained this way are embeddings.

For the alternative $SU(N)$ Skyrme models, using the harmonic map ansatz of $S^2 \to CP^{N-1}$, we have found that our results for the energies of the exact spherically symmetric solutions are higher than in the usual models.

When considering the pure massive $SU(N)$ Yang-Mills theories, we have shown that by choosing the gauge potential to be of almost pure gauge form, the theories reduce to the usual $SU(N)$ Skyrme models. This observation has suggested to us to consider the harmonic map ansatz of $S^2 \to CP^{N-1}$ previously applied to monopole theories. Using this ansatz, we have constructed some bounded spherically symmetric solutions of the theories having finite energies.

Declaration

This thesis is based on research carried out by the author during the period 2000-2002 at the University of Durham, England, under the supervision of Professor Wojtek Zakrzewski. No part of it has been submitted for any degree, either at the university or anywhere else.

With the exception of chapters 1, 2, 3 (except for section 3.2.3) and wherever a reference is given, this work is believed to be original. Section 3.2.3 of chapter 3 and chapter 4 are based on two published papers in collaboration with Wojtek Zakrzewski [17, 18], while chapter 5 is based on an unpublished work [19].

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Chapter 1

Introduction

In the course of trying to understand nucleon charge radius, which has a size of roughly 1 fermi [1], T. H. R Skyrme in 1962 put forward the idea that strongly interacting particles (hadrons) are locally concentrated static solutions of classical extended nonlinear sigma (chiral) model field theories [2]. This is a very interesting and challenging proposal as it was against the established understanding of the mainstream physics at that time. "Elementary particles are quantum mechanical objects and that their intractions are described by perturbed quantum field theories" [3]. Skyrme proposal was thus left aside and was forgotten.

A breakthrough which shed light along this line was then made by Gardner *et. al* in 1967 with the invention of *inverse scattering* method revealing the existence of *multi-solitons* in the $(1+1)D$ (dimensional) Korteweg-de Vries (KdV) equation [4], which describes shallow water waves phenomena [5]. These solitons are locally concentrated classical solutions in one spatial dimension exhibiting particle-like behaviour in interactions, the analog of $3D$ entities envisaged by Skyrme. However, their dynamics is very simple, for instance, their collisions occur elastically and they undergo no more than a phase shift. This peculiar property is believed to have relation to the fact uncovered by Gardner *et. al* that KdV equation possesses an infinite number of globally conserved quantities of dynamical origin. These conserved quantities was then found to exist in a large class of $(1+1)D$ nonlinear classical field equations classified as *integrable systems* [5, 6]. Even though these $1D$ soliton picture is outside the realm of $3D$ particles, which could interact nonelastically and annihilate each other, the revealed basic phenomena is very interesting and tantalising.

The $3D$ solitons of the Skyrme models, also known as *skyrmions*, possess a conserved quantity that is purely topological in origin and so they are classified as *topological solitons* [7]. This conserved quantity, generally called *topological charge*, is related to the global aspects, *i.e., homotopy,* of static field configurations as maps from compactified $3D$ space: $R^3 \cup \{\infty\} \simeq S^3$ into target space of fields, *i.e.* the $SU(N)$ group manifold. For the original Skyrme model, $N = 2$, the target space is isomorphic to the 3-sphere S^3. Skyrme main proposal is to identify this integer valued winding number, *i.e.,* the topological charge, with *baryon number* B of hadrons.

With the advent of renormaliseable quantised nonabelian gauge field theories in the early '70s, an alternative theory of strong interactions was introduced around 1973-74 in the form of $SU(3)$ *quantum chromodynamics* (QCD) [3]. This theory explains that quarks, the building blocks of hadrons, and gluons, the quantised mediating coloured gauge fields between quarks, must be *confined* within a region of about 1 fermi radius. This seems to imply that the original Skyrme motivation, seeking classical explanation for nucleon charge radius, was no longer relevant that added further discouragement to the exploration of Skyrme's proposal.

However, in the late '70s E. Witten [8] resurrected Skyrme's idea by showing that the Skyrme model (sigma model part) is deriveable from $SU(N_c)$ QCD in the large limit of the colour number N_c. Since then, Skyrme models has been used to study various properties of low energy mesons and baryons [9, 10]. Especially, for the $N = 2$ case, it has been argued [2, 9] that they describe, at a classical level, low energy states of nucleons and light nuclei.

The classical static solutions of Skyrme models with global $SU(N)$ symmetry is an interesting topic in its own right and is a challenging problem in the 3-dimensional nonlinear field theories. Especially, the search for bounded solutions having finite energies (or mass), which is one of the main problem that we are concerned in this thesis.

The first exact solution, in the $SU(2)$ case, was found by Skyrme [2]. This solution, also known as the hedgehog solution, describes a spherically symmetric energy lump and has baryon number $B = 1$. Since Witten's suggestion that the Skyrme model arises in the large N_c limit of QCD [8], most of the studies involving $SU(N)$ for $N > 2$ considered configurations which were embeddings of the $SU(2)$ fields [10]. The first non-embedding solution, for $N = 3$, was presented by Balachandran *et. al.* [7] who found an $SO(3)$ subgroup soliton which has baryon number $B = 2$. Another configuration, which has a large strangeness content was then found by Kopeliovich *et. al.* [11].

Until very recently very little attention has been paid to field configurations describing many skyrmions in $SU(N)$ models which were not embeddings of $SU(2)$ skyrmions. Although some work has been done earlier [7, 11] the real progress has only been made since Houghton, Manton and Sutcliffe had produced their *harmonic map ansatz* [12], which revealed the connection between $3D$ skyrmions and $2D$ harmonic maps. Here, the harmonic map ansatz is used to describe the angular dependence of field configurations. In their seminal work [12] Houghton *et. al.* have shown how to use rational maps of $S^2 \to S^2$ to construct field configurations for the $SU(2)$ Skyrme model which have arbitrary baryon number B and, for low values of B, are close to the exact solutions of the model. In Ref. [13] Ioannidou *et. al.* took the $SU(2)$ ansatz of Houghton *et. al.* and rewrote it in terms of a projector of S^2 into complex projective space CP^1 and then generalised it to more general projectors of $S^2 \to CP^{N-1}$ of rank-1 [14]. This method enabled the construction of new static spherically symmetric solutions for any N. Moreover, it also presents field configurations, which though not solutions of the equations, are close to them - thus providing us with good approximants to other solutions [13].

The main aim of this thesis is twofolds. Firstly, to seek other possible static spherically symmetric solutions of the $SU(N)$ Skyrme models having lower energy by using a further generalisation of Ioannidou *et. al*'s harmonic map ansatz method. Secondly, to apply harmonic map ansatz method to *alternative* $SU(N)$ Skyrme models and Yang-Mills theories.

The organisation of this thesis is as follows. Chapters 2 and 3 contain the necessary background materials for the rest of discussions. In chapter 2, we start by briefly reviewing basic concepts of harmonic mapping theories (also known as sigma models) as maps $\mathcal{M}_0 \to \mathcal{M}$ satisfying sigma model equations. This is followed by extensive discussions on group and geometrical formulations of sigma models field configurations space $\mathcal{M}$ as coset group manifolds. In chapter 3, we concentrate on geometrical formulation of Grassmannian sigma model $Gr(n,\, N)$, $n < N$, in terms of $N \times N$ matrix projector fields of rank-n [15]. Here, $Gr(n,\, N)$ is the Grassmann manifold of n-dimensional planes passing through the origin in C^N, for which $CP^{N-1} = Gr(1,\, N)$. Our main stress is on the constructions of *full solutions* and the related *topological invariants* of the $2D$ Grassmannian sigma model, using projectors of higher rank. These full solutions, first constructed by A. Din and W. J. Zakrzewski [16], are backbones of the $2D$ harmonic maps, as they constitute mutually orthogonal $(N \times n)$ matrix fields in C^N, which are *complete* for the CP^{N-1} case. Of particular importance is the *Veronese sequence* of N mutually orthogonal vector fields in CP^{N-1}, which play an important role in the construction of exact spherically symmetric solutions of $SU(N)$ Skyrme models and Yang-Mills field theories.

The $SU(N)$ Skyrme models are then considered in chapter 4 where in sections 4.1-4.7, which are taken from Ref. [17], we generalise the method of Ioannidou *et. al.* further by considering projectors of S^2 into the Grassman manifold $Gr(2, N)$, *i.e.,* using rank-2 projectors. We find that, in contrast to the rank-1 case, in which exact spherically symmetric solutions can be found (numerically) by using the Veronese sequence, such a construction is now more involved. In section 4.7, in particular, we formulate a condition on the sequence of $\lambda < N$ mutually orthogonal $(N \times n)$ matrix fields in $Gr(n, N)$ which would give exact spherically symmetric solutions of the $SU(N)$ Skyrme models. After analysing this condition further and considering some special configurations, we present arguments suggesting that its only solutions are embeddings. In sections 4.8-4.10, which are taken from Ref. [18], we consider alternative $SU(N)$ Skyrme models which possess a modified Skyrme term, which deviates from the usual one when $N > 2$. We then show that all the ideas involving harmonic maps (where we concentrate our attention to rank-1 projectors) work as in the usual models [13].

Finally, in chapter 5, we apply the harmonic map ansatz method to pure $SU(N)$ Yang-Mills field theories where we concentrate on the massive case only [19] . First, we show that its action reduces to the $SU(N)$ Skyrme models in the special case when the gauge potential is chosen to be of *almost pure gauge* form, which suggests the existence of stable static solutions having finite energies. Then, using Ioannidou-Sutcliffe [65] harmonic map ansatz (with rank-1 projectors) for static spherically symmetric gauge potential, we finally found that some bounded solutions with finite energies can be constructed as we have expected.

To keep our presentation self-contained, as far as possible, we have added 9 appendices, $A - I$, containing details of the derivations of some formula, concrete examples of abstract constructions, proofs of propositions, etc.

The thesis is concluded with a summary of the results obtained and an outlook for further research.

Chapter 2

Harmonic Maps and Nonlinear Sigma Models

The concept of harmonic map is in fact a generalisation of the concept of geodesic in differential geometry. Basically, harmonic map is a map between Riemannian manifolds which extremises a certain functional called "energy" integral in mathematics literature. In physics, this functional is nothing but the static energy of nonlinear *sigma* model field theories. These containing several models of special interest such as chiral models, $O(N)$ and Grassmannian (which includes CP^{N-1}) sigma models. Thus, harmonic maps are (static) solutions of nonlinear sigma model field equations.

The theory of harmonic maps was introduced in mathematics in 1954 by B. F. Fuller [20] and the general theory was laid by J. Eeels and J. M. Sampson 10 years later [21]. Its role in physics was first emphasized by C. W. Misner [22] in 1978, who formulated nonlinear sigma model field theories in a more direct geometrical description. The nonlinear sigma models in 2-dimensions are of special interest because they bear much resemblance to 4-dimensional nonabelian gauge theories and that they have the property of being an "integrable" system. This generally means that the corresponding solutions or harmonic maps can be constructed explicitly and that they bear soliton-like structures, *i.e.* they are stable and having finite energy integral.

In this chapter, we first introduce harmonic map theory, and then we discuss group and geometrical formulations of the nonlinear sigma models.

2.1 Theory of Harmonic Maps

Let us start by briefly recalling some basic definitions of harmonic map theory [23–25]. Let $\mathcal{M}_0$ be a Riemannian manifold with local coordinates x^μ, $\mu = 1, 2, \ldots,$ $dim[\mathcal{M}_0]$, with metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

and let $\mathcal{M}$ be another Riemannian manifold with local coordinates f^A, $A = 1, 2, ...,$ $dim[\mathcal{M}]$, with metric

$$d\sigma^2 = h_{AB} df^A df^B, \tag{2.2}$$

where $g_{\mu\nu}$ and h_{AB} are the corresponding metric tensors. Here summation convention on repeated indices is understood.

Then the map:

$$f: \quad \mathcal{M}_0 \to \mathcal{M}, \qquad f = (f^A(x)),$$

is called a *harmonic map* if it extremises the action

$$S(f) = \int_{\mathcal{M}_0} d^{n_0}x \sqrt{g} \mathcal{L}(x), \tag{2.3}$$

where $n_0 = dim(\mathcal{M}_0)$, $g = |det(g_{\mu\nu})|$ and where

$$\mathcal{L}(x) = \frac{1}{2} h_{AB} \frac{\partial f^A}{\partial x^\mu} \frac{\partial f^B}{\partial x^\nu} g^{\mu\nu}, \tag{2.4}$$

is the Lagrangian density.

Note that, we do not use the terminologies "energy" and "energy density" for (2.3) and (2.4), respectively, in order to avoid confusion with the corresponding definitions in physics. In fact, for the static case, where $\mathcal{M}_0$ is the spatial submanifold of the (2+1) or (3+1)-dimensional spacetime manifold, both terminologies coincide. In this thesis, we will assume $\mathcal{M}_0$ to be general, *i.e.* the metric (2.1) needs not to be positive definite.

Thus f is a harmonic map if and only if it satisfies the Euler-Lagrange equation of the action $S(f)$, *i.e.*

$$\Box f^A + \Gamma^A_{BC}(f) \frac{\partial f^B}{\partial x^\mu} \frac{\partial f^C}{\partial x^\nu} g^{\mu\nu} = 0, \tag{2.5}$$

where

$$\Box f^A \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial f^A}{\partial x^\nu} \right), \tag{2.6}$$

is the covariant Laplacian of f^A and where

$$\Gamma^A_{BC}(f) = \frac{1}{2}h^{AD}\left(\partial_C h_{BD} + \partial_B h_{CD} - \partial_D h_{BC}\right), \qquad \partial_A = \partial/\partial f^A, \tag{2.7}$$

are the components of the Christoffel symbol of $\mathcal{M}$. Equation (2.5) is a second order nonlinear partial differential equations whose type depends on the signature $s(g)$ = sign of diag$(g_{\mu\nu})$ of the metric tensor of the base manifold $\mathcal{M}_0$. If $s(g) = (+, ..., +)$ then it is elliptic, otherwise it is hyperbolic. The manifolds $\mathcal{M}_0$ and $\mathcal{M}$ are called "source" (or base) manifold and "target" (or field configuration) manifold, respectively.

Classical examples of harmonic maps appear for the following cases. When the target manifold $\mathcal{M}$ is a real line, *i.e.* $dim(\mathcal{M}) = 1$, then the Euler-Lagrange equation (2.5) reduces to harmonic equation:

$$\Box f = 0. \tag{2.8}$$

In this case, a harmonic map f is nothing else but a harmonic function $f = f(x)$ in $\mathcal{M}_0$. When the source manifold $\mathcal{M}_0$ is a real line, *i.e.* $dim(\mathcal{M}_0) = 1$, then (2.5) becomes

$$\frac{d^2 f^A}{ds^2} + \Gamma^A_{BC}(f)\frac{df^B}{ds}\frac{df^C}{ds} = 0. \tag{2.9}$$

Thus, the harmonic maps $f = (f^A(s))$ in this case are geodesics of $\mathcal{M}$ parametrised proportionally to arc length s. The corresponding map is called a *geodesic map*.

Sometimes, it is useful to consider the composition of two harmonic maps:

$$\eta : \mathcal{M}_0 \to \mathcal{M}', \qquad f : \mathcal{M}' \to \mathcal{M}. \tag{2.10}$$

Then we have the following.

Proposition 2.1 If η: $\mathcal{M}_0 \to \mathcal{M}'$, and f: $\mathcal{M}' \to \mathcal{M}$ being two harmonic maps then if $dim[\mathcal{M}'] = 1$, the composite map: $fo\eta(x)$: $\mathcal{M}_0 \to \mathcal{M}$ is a harmonic map.

Proof: As $\dim[\mathcal{M}'] = 1$, η is a scalar function in $\mathcal{M}_0$, *i.e.* $\eta = \eta(x)$. By considering the composition $fo\eta(x) = f(\eta(x))$ and using the chain rule in (2.5) we obtain

$$\frac{df^A}{d\eta}\Box\eta + g^{\mu\nu}\frac{\partial\eta}{\partial x^\mu}\frac{\partial\eta}{\partial x^\nu}\left[\frac{d^2 f^A}{d\eta^2} + \Gamma^A_{BC}(f)\frac{df^B}{d\eta}\frac{df^C}{d\eta}\right] = 0. \tag{2.11}$$

As η: $\mathcal{M}_0 \to \mathcal{M}'$ is a harmonic map, $\eta(x)$ satisfies the harmonic equation (2.8). Hence, if $g^{\mu\nu}\frac{\partial\eta}{\partial x^\mu}\frac{\partial\eta}{\partial x^\nu} \neq 0$, then $fo\eta(x) = f(\eta(x))$ is a geodesic map parametrised proportionally to the harmonic function $\eta(x)$. Hence, $fo\sigma(x)$ is a harmonic map [26]. □

This proposition then implies the following.

Corollary 2.1 The Lagrangian density of the harmonic map $fo\eta(x)$ in proposition 1 is proportional to a free scalar field Lagrangian density.

Proof: As $fo\eta(x)$ in the previous proposition is a geodesic map, so using the chain rule, the Lagrangian density (2.4) becomes

$$\mathcal{L}(x) = \frac{1}{2}h_{AB}\frac{df^A}{d\eta}\frac{df^B}{d\eta}\frac{\partial\eta}{\partial x^\mu}\frac{\partial\eta}{\partial x^\nu}g^{\mu\nu}. \tag{2.12}$$

Thus, if we choose η to be proportional to the arc length of the geodesic in $\mathcal{M}$, *i.e.*

$$d\eta^2 = \frac{1}{\lambda}d\sigma^2 = \frac{1}{\lambda}h_{AB}df^A df^B, \tag{2.13}$$

where λ is an arbitrary nonzero constant parameter, then the Lagrangian density (2.12) reduces to the free scalar field Lagrangian density

$$\mathcal{L}(x) = \frac{\lambda}{2}\frac{\partial\eta}{\partial x^\mu}\frac{\partial\eta}{\partial x^\nu}g^{\mu\nu}, \tag{2.14}$$

as required. □

Given a map $f : \mathcal{M}_0 \to \mathcal{M}$, not necessarily harmonic, then its *second fundamental form* is defined by [21,25]

$$\alpha^A_{\mu\nu}(f) \equiv \frac{\partial^2 f^A}{\partial x^\mu \partial x^\nu} - \gamma^\rho_{\mu\nu}\frac{\partial f^A}{\partial x^\rho} + \Gamma^A_{BC}\frac{\partial f^B}{\partial x^\mu}\frac{\partial f^B}{\partial x^\mu}, \tag{2.15}$$

where $\gamma^\rho_{\mu\nu}$ are the components of the Christoffel symbol of g on $\mathcal{M}_0$.

If

$$\alpha^A_{\mu\nu}(f) = 0, \tag{2.16}$$

then the map f is called a *totally geodesic map*. This map is unique as it maps geodesics of $\mathcal{M}_0$ into geodesics of $\mathcal{M}$ linearly.

This can be seen as follows.

Let $x^\mu = x^\mu(s)$ be an arbitrary twice differentiable curve in $\mathcal{M}_0$ and so $f^A = f^A(x^\mu(s)) = f^A(s)$. Then from (2.15), we derive that

$$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\alpha^A_{\mu\nu}(f) = \left[\frac{d^2 f^A}{ds^2} + \Gamma^A_{BC}\frac{df^B}{ds}\frac{df^C}{ds}\right] - \left[\frac{d^2x^\mu}{ds^2} + \gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds}\right]\frac{\partial f^A}{\partial x^\mu}. \quad (2.17)$$

Thus, if $x^\mu(s)$ is a geodesic in $\mathcal{M}_0$ then the second term in the right hand side of (2.17) vanishes, and with $\alpha^A_{\mu\nu}(f) = 0$, it follows that $f^A = f^A(x^\mu(s))$ satisfies the geodesic equation in $\mathcal{M}$.

This fact leads to the following.

Proposition 2.2 Any totally geodesic map $f : \mathcal{M}_0 \to \mathcal{M}$ is also a harmonic map.

Proof: Let us consider $\Box f$ in (2.8). Using

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^\mu} = \frac{1}{2}g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}, \quad (2.18)$$

and

$$\frac{\partial g^{\mu\nu}}{\partial x^\mu} = -g^{\mu\rho}\frac{\partial g_{\rho\sigma}}{\partial x^\mu}g^{\sigma\nu}, \quad (2.19)$$

then

$$\Box f = g^{\mu\nu}\left[\frac{\partial^2 f^A}{\partial x^\mu \partial x^\nu} - \gamma^\rho_{\mu\nu}\frac{\partial f^A}{\partial x^\rho}\right]. \quad (2.20)$$

Substituting (2.20) into the harmonic map equation (2.5) yields the equivalent equation

$$g^{\mu\nu}\alpha^A_{\mu\nu}(f) = 0. \quad (2.21)$$

Clearly, $\alpha^A_{\mu\nu} = 0$ satisfies (2.21) which completes our proof. □

The functional

$$\tau(f) = g^{\mu\nu}\alpha^A_{\mu\nu}(f), \quad (2.22)$$

is called the *tension* of the map $f : \mathcal{M}_0 \to \mathcal{M}$ which was first introduced by J. Eeels and J. M. Sampson [21]. Thus, f is a harmonic map if its tension vanishes.

2.2 Nonlinear Sigma Models

As mentioned previously, the theory of harmonic map is what in physics literature called *nonlinear sigma model*. Here, by a nonlinear sigma model we mean a field theory with the following properties [7]:

(1) The fields $\phi(x)$ of the model are subjected to nonlinear constraints for all points $x \in \mathcal{M}_0$.

(2) The constraints and the Lagrangian density are invariant under the action of a global symmetry group G on $\phi(x)$.

To illustrate these models, in the following subsections we discuss two examples: the $O(N)$ and its descendant the RP^{N-1} nonlinear sigma models. From now on, we use the Greek letter σ to abbreviate the adjective "sigma" in the name of these models.

2.2.1 $O(N)$ σ model

The simplest example of these models is the $O(N)$ nonlinear σ model which consist of N-real scalar fields, $\phi^{\hat{A}}$, $\hat{A} = 1, ..., N$, having the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\frac{\partial\phi^{\hat{A}}}{\partial x^{\mu}}\frac{\partial\phi^{\hat{A}}}{\partial x^{\nu}}g^{\mu\nu}, \tag{2.23}$$

where the scalar fields $\phi^{\hat{A}}$ satisfy the constraint:

$$\phi^{\hat{A}}\phi^{\hat{A}} = 1. \tag{2.24}$$

The Lagrangian density (2.23) is obviously invariant under the global (or space independent) orthogonal transformations $O(N)$, *i.e.* the group of N-dimensional rotations:

$$\phi^{\hat{A}} \to \phi'^{\hat{A}} = O^{\hat{A}}_{\hat{B}}\phi^{\hat{B}}. \tag{2.25}$$

In the following, we shall derive Euler-Lagrange equations for the Lagrangian density (2.23) taken together with the constraint (2.24). We do this by applying Lagrange multiplier method in which, instead of (2.23), we consider the extended Lagrangian density:

$$\mathcal{L}_{\eta} = \frac{1}{2}\frac{\partial\phi^{\hat{A}}}{\partial x^{\mu}}\frac{\partial\phi^{\hat{A}}}{\partial x^{\nu}}g^{\mu\nu} + \frac{1}{2}\eta\left(\phi^{\hat{A}}\phi^{\hat{A}} - 1\right), \tag{2.26}$$

where $\eta = \eta(x)$ is a $\phi^{\hat{A}}$-independent multiplier. As the fields $\phi^{\hat{A}}$ are now being independent, *i.e.* unconstrained, the variation of the action $S_{\eta} = \int_{\mathcal{M}_0} d^{n_0}x\sqrt{g}\mathcal{L}_{\eta}(x)$, with respect to the field variations $\delta\phi^{\hat{A}}$ and $\delta\eta$ is

$$\begin{aligned}\delta S_\eta &= \int_{\mathcal{M}_0} d^{n_0}x\sqrt{g}\Big[\frac{\partial(\delta\phi^{\hat{A}})}{\partial x^\mu}\frac{\partial\phi^{\hat{A}}}{\partial x^\nu}g^{\mu\nu} + \eta(\delta\phi^{\hat{A}})\phi^{\hat{A}} + \frac{1}{2}\delta\eta\left(\phi^{\hat{A}}\phi^{\hat{A}} - 1\right)\Big] \\ &= \int_{\mathcal{M}_0} d^{n_0}x\left[\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\delta\phi^{\hat{A}}\frac{\partial\phi^{\hat{A}}}{\partial x^\nu}g^{\mu\nu}\right) - \delta\phi^{\hat{A}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\frac{\partial\phi^{\hat{A}}}{\partial x^\nu}g^{\mu\nu}\right)\right. \\ &\qquad \left. + \delta\phi^{\hat{A}}\left(\sqrt{g}\eta\phi^{\hat{A}}\right) + \frac{1}{2}\delta\eta\left(\phi^{\hat{A}}\phi^{\hat{A}} - 1\right)\right]. \end{aligned} \tag{2.27}$$

The first term is a total divergence term, so it vanishes after being transformed into a surface integral due to the boundary conditions: $\delta\phi^{\hat{A}} = 0$ and $\delta\eta = 0$ on the boundary $\partial\mathcal{M}_0$.

Thus, from the least action principle: $\delta S_\eta = 0$, we derive

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}g^{\mu\nu}\frac{\partial\phi^{\hat{A}}}{\partial x^\nu}\right) - \eta\phi^{\hat{A}} = 0, \tag{2.28}$$

$$\phi^{\hat{A}}\phi^{\hat{A}} = 1. \tag{2.29}$$

Eliminating η from (2.28) by multipling (2.28) with $\phi^{\hat{A}}$ and summing over $\hat{A}$ then after using (2.29) we obtain

$$\eta = -g^{\mu\nu}\frac{\partial\phi^{\hat{A}}}{\partial x^\mu}\frac{\partial\phi^{\hat{A}}}{\partial x^\nu}, \tag{2.30}$$

where we have used: $\phi^{\hat{A}}\partial_\nu\phi^{\hat{A}} = 0$.

Substituting (2.30) into (2.28) then yields the required Euler-Lagrange equations:

$$\Box\phi^{\hat{A}} + g^{\mu\nu}\frac{\partial\phi^{\hat{B}}}{\partial x^\mu}\frac{\partial\phi^{\hat{B}}}{\partial x^\nu}\phi^{\hat{A}} = 0. \tag{2.31}$$

Geometrically, the constraint (2.24) defines an (N-1)-dimensional sphere S^{N-1} in the N-dimensional Euclidean space R^N of field manifolds $\phi^{\hat{A}}$. This constraint can be solved, for example by introducing the parametrisation:

$$\phi^A = f^A, \quad \phi^N = \pm\sqrt{(1-|f|^2)}, \qquad A = 1, 2, ..., (N-1), \tag{2.32}$$

where $|f|^2 = f^A f^A$ and the range of $|f|$ is restricted to $-1 \leq |f| \leq 1$. The sign choice determines we parametrise either the upper or the lower hemisphere of S^{N-1}. Using either of the two possible solutions of (2.32) allows us to write

$$\mathcal{L} = \frac{1}{2}h_{AB}\frac{\partial f^A}{\partial x^\mu}\frac{\partial f^B}{\partial x^\nu}g^{\mu\nu}, \tag{2.33}$$

where the metric tensor of the target space S^{N-1} is given by

$$h_{AB}(f) = \delta_{AB} + \frac{f^A f^B}{(1-|f|^2)}, \tag{2.34}$$

and where δ_{AB} is the Kronecker's delta.

In terms of the non-constraint field $f = (f^A(x))$, the $O(N)$ σ model equations (2.31) are

$$\Box f^A + f^A h_{BC} \frac{\partial f^B}{\partial x^\mu}\frac{\partial f^C}{\partial x^\nu} g^{\mu\nu} = 0, \tag{2.35}$$

which coincide with the general harmonic map equations (2.5) as the Christoffel symbol (2.7) of the metric tensor h_{AB} in (2.34) is

$$\Gamma^A_{BC}(f) = f^A h_{BC}. \tag{2.36}$$

Thus, the non-constraint field $f = (f^A(x))$ which solves the $O(N)$ σ model field equations (2.35), describes the harmonic map:

$$f: \quad \mathcal{M}_0 \to S^{N-1}.$$

There are, of course, infinitely many parametrisations of the field $\phi^{\hat{A}}$, besides (2.32), that can be introduced to solve the constraint (2.24). To prepare the geometrical background for our discussion in the next chapters, in the following two subsections we discuss two other special parametrisations of the sphere S^{N-1}:

(1) the *stereographic projection*

$$\phi^A = \frac{2f^A}{1+|f|^2}, \quad \phi^N = \frac{1-|f|^2}{1+|f|^2}, \tag{2.37}$$

(2) the *projective coordinates projection*

$$\phi^A = \frac{f^A}{\sqrt{1+|f|^2}}, \quad \phi^N = \frac{1}{\sqrt{1+|f|^2}}. \tag{2.38}$$

These two parametrisations are unique as we shall see below that they provide conformal and projective perspectives of the sphere S^{N-1}, respectively.

2.2.2 Compactified R^{N-1}

For the stereographic projection, as from (2.37)

$$f^A = \frac{\phi^A}{1+\phi^N}, \tag{2.39}$$

the "south" pole $\phi^{\hat{A}}_{(S)} = (0, \dots, -1)$ acts as the centre of projection. A point $(\phi^{\hat{A}})$ on the sphere S^{N-1} is projected by a "ray" emerging from $\phi^{\hat{A}}_S$ to a point (f^A) on the $(N-1)$-dimensional "equatorial plane" R^{N-1}, $\phi^N = 0$. As (f^A) is the intersection point of the ray with the equatorial plane, so points on the northern (resp. southern) hemisphere have $|f| < 1$ (resp. $|f| > 1$).

The projection (2.39) is, unfortunately, pathological as $\phi^{\hat{A}}_{(S)}$ goes to points at ∞ which are not points of R^{N-1} at all. To get rid of this, we change ϕ^N to -ϕ^N in (2.39) which reverses the roles of north and south poles. Thus, we need two coordinate charts: $U_S = S^{N-1}/\{\phi^{\hat{A}}_{(S)}\}$ and $U_N = S^{N-1}/\{\phi^{\hat{A}}_{(N)}\}$ which form the atlas $\{U_N, U_S\}$ to cover S^{N-1}. Hence, f^A in (2.39) are the *stereographic coordinates* of the sphere S^{N-1} in the chart U_S. Relations between these two coordinates: $f^A_{(S)} \in U_S$ and $f^A_{(N)} \in U_N$ for each point in the overlap region $U_N \bigcap U_S$ is given by the *transition functions*:

$$f^A_{(N)} = \frac{f^A_{(S)}}{|f_{(S)}|^2}, \qquad f^A_{(S)} = \frac{f^A_{(N)}}{|f_{(N)}|^2}. \tag{2.40}$$

The corresponding metric tensor, resulting from (2.37) is

$$h_{AB}(f) = \frac{4\delta_{AB}}{(1+|f|^2)^2}, \tag{2.41}$$

which is conformally equivalent to the flat metric δ_{AB} of R^{N-1} with the conformal factor $\Lambda = 4(1+|f|^2)^{-2}$. Hence, by including points at ∞, the "plane" R^{N-1} is compactified to $R^{N-1} \bigcup \{\infty\} \simeq S^{N-1}$. (Note that, this compactified sphere has infinite radius rather than unit radius).

As an illustration, let us consider the S^2 case:

$$(x, y, z) \in S^2 \to (\xi^1, \xi^2) \in R^2.$$

Here, it is convenient to use the complex coordinates: $\xi = (\xi^1 + i\xi^2) \in \mathbb{C}$, $(i^2 = -1)$, on the plane R^2. Then, in terms of the spherical polar angle coordinates $(0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi)$, of S^2 *i.e.* $(x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta)$, the stereographic coordinate (2.37) in U_S is

$$\xi = \xi^1_{(S)} + i\xi^2_{(S)} = \frac{x+iy}{1+z} = \tan(\theta/2)e^{i\phi}, \tag{2.42}$$

and in U_N is

$$\zeta = \mathrm{cotan}(\theta/2)e^{-i\phi}. \tag{2.43}$$

Note that we have chosen the phase of ζ to be the conjugate of the phase of ξ, in order that in the overlap region $U_N \bigcap U_S$ the transition function is *holomorphic* [27, 28], *i.e.*

$$\xi = \frac{1}{\zeta}. \tag{2.44}$$

For this case, the compactified complex plane $C \bigcup \{\infty\} \simeq S^2$ is known as a *Riemann sphere.* Its metric is

$$d\sigma^2 = \frac{4d\bar{\xi}d\xi}{(1+|\xi|^2)^2}. \tag{2.45}$$

2.2.3 RP^{N-1} σ Model

For the projective coordinates projection, as from (2.38)

$$f^A = \frac{\phi^A}{\phi^N}, \tag{2.46}$$

so here the centre of projection is the *centre* $\phi_O^{\hat{A}} = (0, \dots, 0)$ of S^{N-1}. A point (ϕ^A) on the northern (resp. southern) hemisphere $\phi^N > 0$ (resp. $\phi^N < 0$) is projected by a ray emerging from $\phi_O^{\hat{A}}$ to a point (f^A) on the $(N-1)$-dimensional plane $\phi^N = 1$ (resp. $\phi^N = -1$). From (2.46) we see that $(\phi^{\hat{A}})$ and $(-\phi^{\hat{A}})$ have the same value of f^A, but the corresponding projective points lying on different projective planes. This fact suggests us to identify antipodal (or diametrically opposite) points on the sphere S^{N-1} and by doing this, we obtain the so called *real projective space* $RP^{N-1} \simeq S^{N-1}/\{1, -1\}$.

The division notation "$S^{N-1}/\{1, -1\}$" means that two points with coordinates $\phi^{\hat{A}}$, $\tilde{\phi}^{\hat{A}} \in S^{N-1}$ that satisfy $\tilde{\phi}^{\hat{A}} = \lambda\phi^{\hat{A}}$ with $\lambda^2 = 1$ (or $\lambda = \pm 1$) are considered equivalent which defines the equivalence classes $\{\phi^{\hat{A}}\} \sim \{\lambda\phi^{\hat{A}}\}$. As $\lambda^2 = 1$ is also considered as the equation for the zeroth sphere S^0, so $RP^{N-1} \simeq S^{N-1}/S^0$.

Thus RP^{N-1} may be thought of as a hemisphere of S^{N-1} with antipodal points on the "equator" identified. Geometrically, with no reference to S^{N-1}, RP^{N-1} is defined as an (N-1)-dimensional manifold of straightlines passing through the origin in R^N. Thus a "point" of RP^{N-1} is represented by a line in R^N whereas a point on S^{N-1} represents an equivalence class in R^N [29].

The extension of RP^{N-1} to complex manifold C^N results in *the complex projective space* CP^{N-1} and its generalisation, *Grassmann manifold*, which we shall be discussing separately in Chapter 3.

Note that, under the $O(N)$ transformation (2.25) on $\phi^{\hat{A}}$, by virtue of (2.38), f^A transforms as

$$f^A = \frac{O^A_B f^B + O^A_N}{O^N_B f^B + O^N_N}, \tag{2.47}$$

which is nothing but the fractional or homographic map of RP^{N-1} onto itself. This suggests the projective interpretation of the fields $\phi^{\hat{A}}$ and f^A in (2.38) as homogeneous and inhomogeneous (or *affine*) coordinates of RP^{N-1}, respectively. With no reference to S^{N-1}, the projection (2.38) is generalised to

$$f^A_{(\hat{C})} = \frac{\phi^{\hat{A}}}{\phi^{(\hat{C})}}, \qquad \hat{A} \neq \hat{C}, \tag{2.48}$$

where $(\hat{C})$ denotes a fixed index value $\hat{C}$ for which $\phi^{\hat{C}} \neq 0$. Hence, $f^A_{(\hat{C})}$ are the inhomogeneous coordinates of RP^{N-1} in the coordinate chart $U_{(\hat{C})} = R^N/\{\phi^{\hat{C}}\}$. In general, RP^{N-1} is covered by N-coordinate charts $U_{(\hat{C})}$, $\hat{C} = 1, \ldots, N$ and that the transition function in the overlap region $U_{(\hat{C})} \bigcap U_{(\hat{D})}$ is

$$f^A_{(\hat{C})} = \left(\frac{\phi^{\hat{D}}}{\phi^{\hat{C}}}\right) f^A_{(\hat{D})}. \tag{2.49}$$

In fact, the corresponding metric tensor, resulting from (2.38) is

$$h_{AB}(f) = \frac{(1+|f|^2)\delta_{AB} - f^A f^B}{(1+|f|^2)^2}, \tag{2.50}$$

which is the standard RP^{N-1} metric tensor. As "points" of RP^{N-1} are lines in R^N, the metric ds here measures the smaller angle between two lines.

Hence, the non-constraint field $f = (f^A(x))$ in (2.46), desribes the harmonic map:

$$f: \quad \mathcal{M}_0 \to RP^{N-1}.$$

In the rest of this thesis, by the term "σ model", we will always mean for "nonlinear σ model". The adjective "sigma" in these models, historically, was introduced in relation to scalar field σ in M. Gell Mann and M. Levy linear σ model field theory of current algebra [30] where its nonlinear extension was originated by S. Weinberg [31–33].

2.3 Group Theoretical Formulation of σ Models

There is another way of formulating σ models using group theoretical method which brings out group and coset space nature of the target manifold $\mathcal{M}$. Here, by a group G we mean a matrix Lie group and in this thesis we assume G to be compact. This formulation, which was first systematically formulated by F. Eichenher and M. Forger [34], is interesting in the sense that it shows an intimate relationship between differential geometry and gauge theories.

The idea is based on the observation that, taking the above $O(N)\,\sigma$ model as an example, the global invariance group G acts *transitively* on the target or field manifold $\mathcal{M}$, which means that the action of G over a given field $\phi_p \in \mathcal{M}$ produces the whole field manifold $\mathcal{M}$. More specifically, for any $\phi_q \neq \phi_p \in \mathcal{M}$, there is at least one element $g \in G$ such that $\phi_q = g\phi_p$. The set of fields ϕ that can be reached from ϕ_p by applying elements of G is called the *orbit* of G at ϕ_p. The field manifold $\mathcal{M}$ with this property is called a *homogeneous space* of G; geometrically, this means that any point of $\mathcal{M}$ is like any other point under the action of G.

But, if $H \subset G$ is the *stability* or *isotropy* group of a field value $\phi_p \in \mathcal{M}$, *i.e.*

$$H = \{h \in G \,|\, h\phi_p = \phi_p\}\,, \tag{2.51}$$

then the set of group elements gH acting on ϕ_p will produce the same field value $\phi_q \neq \phi_p$ as acting by g alone. Therefore, in order to have a one to one correspondence between points of $\mathcal{M}$ and elements of G, we need to "divide out" H from G, *i.e.* by identifying all the elements of G of the form gh, for a given g and arbitrary $h \in H$. Formally, we do this by performing the quotient

$$G/H = \{gh|g \in G\}\,, \tag{2.52}$$

which is the *coset space of* equivalence classes of the group G modulo the subgroup H, $G \sim GH$, called *left cosets* (of G relative to H or of H in G).

Then, any $g \in G$ can be written as

$$g = g(\theta)h, \qquad g(\theta) \in G/H, \tag{2.53}$$

and so any $\phi_q \in \mathcal{M}$ can be obtained from ϕ_p as

$$\phi_q = g(\theta)h\phi_p = g(\theta)\phi_p. \tag{2.54}$$

Here, $g(\theta)$ is labeled by $dim[G/H] = (dim[G] - dim[H])$ parameters (θ). Hence, the element $g(\theta)$ of the cosets G/H, also called *coset representative*, yields a parametrisation of $\mathcal{M}$ where $dim[\mathcal{M}] = dim[G/H]$. Therefore, we can identify $\mathcal{M}$ with the cosets G/H. However, if the identity group Id_G is the only subgroup of G which leaves every field $\phi \in \mathcal{M}$ invariant, *i.e.* $H = Id_G$ then G is said to act *effectively* on $\mathcal{M}$. Hence, $\mathcal{M} = G/H = G$, the manifold of the group G. In this case, the corresponding σ model is called *chiral model.*

It is interesting to note that this coset construction fit naturally into the *fibre bundle* setting (E, B, F, p) [27–29]. Here G is the *total space* E, $G/H = \mathcal{M}$ the *base manifold* B with $p: G \to G/H$ the *projection*, and $p^{-1}(G/H) = H$ the typical *fibre* F. In fact, this is a *principal bundle* with *structure group* H. If we let $\rho_V : H \to GL(V)$ be a fixed representation of H on some k-dimensional vector space V, then the bundle $((G,\, G/H,\, H,\, p)) \otimes V)/H$ defines a *vector bundle* of rank-k associated to $(G,\, G/H,\, H,\, p)$. For $k = 1$, it is called a *line bundle.*

In the following section, we shall discuss the chiral model first which form the foundations of our discussion of σ models on coset spaces in the next section.

2.4 Chiral Models

By definition, a *chiral* model with a global symmetry group G, which in this thesis we call G-*chiral* model, is a σ model where the fields ϕ of the model take values in the manifold of the group G. To have an idea about this statement, let us consider the $O(4)$ σ model. Here, the field variables $\phi^{\hat{A}}$, $(\hat{A} = 1, \ldots, 4)$, can be identified with the elements of the matrices $G \in SU(2)$, in the fundamental (or spinor) representation, as

$$G = \phi^4 \sigma_4 + i\phi^A \sigma_A, \tag{2.55}$$

where σ_A $(A = 1,2,3)$ are the Pauli matrices and $\sigma_4 = I_2$ a (2×2) unit matrix. We see that, this identification is consistent with the $O(4)$ σ model constraint (2.24) which follows directly from the unitarity condition of the group $SU(2)$, *i.e.* $G^\dagger G = I_2$. The condition $det\, G = 1$ follows automatically. Note that the northern (resp. southern) hemisphere is represented by G (resp. -G). Hence, the elements of $SU(2)$

can be parametrised by the coordinates of the target manifold $\mathcal{M}$ and that the origin $\phi^A = 0$ of $\mathcal{M}$ corresponds to I_2, the identity element of $SU(2)$.

2.4.1 Group and Geometrical Formulations

In general, as mentioned previously, the target space $\mathcal{M}$ of the G-chiral model is the group manifold G. Thus, elements of G correspond to points on $\mathcal{M}$ and may be parametrised in terms of the real coordinates of $\mathcal{M}$, *i.e.* f^A, $(A = 1, \ldots, n = dim(G))$, where the origin $f^A = 0$ determines the unit element $Id_G \in G$. Hence, the target space $\mathcal{M}$ is equipped with a group structure.

Let Φ be the composition function which determines the group multiplication [35–37]. Then if f_1^A and f_2^A be the parameters corresponding to g_1, $g_2 \in G$, *i.e.* $g_1 = g(f_1)$ and $g_2 = g(f_2)$, the group product $g = g_1 g_2$ corresponds to the equations

$$f^A = \Phi^A(g_1, g_2) = \Phi^A(f_1, f_2). \tag{2.56}$$

Let $G(f)$ be a matrix representation of G, satisfying

$$\left.\frac{\partial G(f)}{\partial f^A}\right|_{f^A=0} = -iT_{(A)}, \tag{2.57}$$

where $T_{(A)}$ are the generators of the Lie algebra $\mathcal{G}$ of the group G in the chosen representation. These generators are normalised to $\mathrm{Tr}(T_{(A)}T_{(B)}) = \frac{1}{2}\delta_{(A)(B)}$, where Tr is for *trace*, and that they satisfy the commutation relations

$$[T_{(A)}, T_{(B)}] = i f_{(A)(B)}^{(C)} T_{(C)}, \tag{2.58}$$

where $f_{(A)(B)}^{(C)}$ are the corresponding structure constants of $\mathcal{G}$. Note that, we have introduced bracketed indices, like (A), for labelling the components of a Lie algebra element with respect to the generator $T_{(A)}$.

Then (2.56) implies that

$$G\Big(\Phi^A(f_1, f_2)\Big) = G(f_1)G(f_2), \tag{2.59}$$

and so by taking the derivative of (2.59) with respect to f_2 evaluated at $f_2^A = 0$, yield

$$\left.\frac{\partial \Phi^B}{\partial f_2^A}\frac{\partial G}{\partial f^B}\right|_{f_2^A=0} = G(f_1)\left.\frac{\partial G}{\partial f_2^A}\right|_{f_2^A=0}. \tag{2.60}$$

The quantity

$$\rho^{B}_{(A)}(f) = \left.\frac{\partial \Phi^B(f, f_2)}{\partial f_2^A}\right|_{f_2^A=0} \tag{2.61}$$

is defined as the *left auxiliary* coordinate dependent matrix. Equation (2.60) implies that

$$\mathcal{R}_{(A)} = \rho^{B}_{(A)}\partial_B, \tag{2.62}$$

are the infinitesimal generators of transformations via group multiplication from the right: $G \to GG_r$, which form a basis of the vector space $T_f(\mathcal{M})$ of tangent vectors in $\mathcal{M}$.

Substituting (2.57) and (2.61) into (2.60) gives the relation

$$\rho^{B}_{(A)}\partial_B G = -iGT_{(A)}, \tag{2.63}$$

or

$$\lambda^{(A)}_B T_{(A)} = iG^{-1}\partial_B G \tag{2.64}$$

where λ is the inverse of ρ, *i.e.*

$$\lambda^{(A)}_B \rho^{B}_{(C)} = \delta^{(A)}_{(C)}. \tag{2.65}$$

In terms of the exterior derivative d, the relation (2.64) in differential form expression is [27, 29]

$$\lambda^{(A)}_B T_{(A)} df^B = iG^{-1}dG, \tag{2.66}$$

which shows that the matrix 1-form $G^{-1}dG$ takes values in the Lie algebra $\mathcal{G}$ of G. This matrix 1-form $G^{-1}dG$ is obviously invariant under the global transformation from the left: $G \to G_l G$, and is called *Maurer-Cartan left-invariant* 1-form. We will see that it plays an important role in the geometrical formulation of Lie group. Note that, when in (2.59) we take the derivative with respect to the first argument, then the roles of left and right are reversed.

Taking the exterior derivative of (2.66), then using $dG^{-1} = -G^{-1}dGG^{-1}$ and Poincaré lemma $d^2G = 0$, we find that the the Maurer-Cartan 1-form components:

$$\lambda^{(A)} = \lambda^{(A)}_B df^B, \tag{2.67}$$

satisfy

$$d\lambda^{(A)} = -\frac{1}{2} f^{(A)}_{(B)(C)} \lambda^{(B)} \wedge \lambda^{(C)}, \tag{2.68}$$

the celebrated *Maurer-Cartan equations.* Poincaré lemma $d^2\lambda^{(A)} = 0$, then yields

$$f^{(C)}_{(A)(B)} f^{(E)}_{(C)(D)} + f^{(C)}_{(D)(A)} f^{(E)}_{(C)(B)} + f^{(C)}_{(B)(D)} f^{(E)}_{(C)(A)} = 0, \tag{2.69}$$

the *Jacobi identity* for the structure constants $f^{(C)}_{(A)(B)}$.

Using the Maurer-Cartan 1-form components $\lambda^{(A)}$ as "building blocks", we can now apply *Cartan's moving frames* method to extract geometrical objects of the curved target manifold $\mathcal{M}$ that we need to construct Lagrangian density of the G-chiral model. Within this approach, the 1-forms $\lambda^{(A)}$ are taken to be the basis of the dual vector space $T^*_f(\mathcal{M})$ which is a space of 1-forms on $\mathcal{M}$ at point f [27].

Hence, the components $\lambda^{(A)}_B$ can be taken as *vielbein* fields of $\mathcal{M}$, and so the natural *metric tensor* on the target manifold $\mathcal{M}$, known as *Cartan-Killing* metric tensor, is

$$h_{AB} = \lambda^{(C)}_A \lambda^{(C)}_B \ = \ 2\mathrm{Tr}\left(\lambda_A \lambda_B\right), \tag{2.70}$$

where

$$\lambda_A \ = \ \lambda^{(C)}_A T_{(C)}. \tag{2.71}$$

The inverse metric tensor then reads

$$h^{AB} = \rho^A_{(C)} \rho^B_{(C)}. \tag{2.72}$$

For a given representation $G(f)$, the metric tensor can also be written as

$$h_{AB} = -2\mathrm{Tr}\left[G^{-1}\left(\partial_A G\right) G^{-1}\left(\partial_B G\right)\right], \tag{2.73}$$

which evidently invariant under the global left and right actions: $G \rightarrow G_l G$ and $G \rightarrow G G_r$, respectively.

In terms of the Maurer-Cartan 1-form $\lambda^{(A)}$, the *group volume* is given by

$$\begin{aligned} V_G \ &= \ \int_{\mathcal{M}} \left(\lambda^{(1)} \wedge \lambda^{(2)} \wedge \ldots \wedge \lambda^{(n)}\right) \\ &= \ \int_{\mathcal{M}} \left(\lambda^{(1)}_{A_1} \lambda^{(2)}_{A_2} \ldots \lambda^{(n)}_{A_n}\right) \epsilon^{A_1 A_2 \ldots A_n}\, df^1 df^2 \ldots df^n, \end{aligned} \tag{2.74}$$

where

$$\epsilon^{A_1 A_2 \ldots A_n} = \delta^{A_1 A_2 \ldots A_n}_{1\ 2\ \ldots n} = det \begin{bmatrix} \delta^{A_1}_1 & \ldots & \delta^{A_n}_1 \\ \vdots & \times & \vdots \\ \delta^{A_1}_n & \ldots & \delta^{A_n}_n \end{bmatrix}, \tag{2.75}$$

is the Levi-Civita antisymmetric symbol and where *det* is for determinant.

Considering $\lambda^{(A)}_B$ as elements of the $(n \times n)$ matrix $(\lambda^{(A)}_B)$, (2.74) simplified to

$$V_G = \int_{\mathcal{M}} \left[det\left(\lambda^{(A)}_B\right)\right] df^1 df^2 \ldots df^n, \tag{2.76}$$

where we have used the definition of determinant $det\left(\lambda^{(A)}_B\right)$ given by the integrand in (2.74). Furthermore, the Cartan-Killing metric tensor (2.70) can also be interpreted as matrix multiplication

$$(h_{AB}) = \left(\lambda^{(A)}_B\right)^2, \tag{2.77}$$

and so (2.76) becomes

$$V_G = \int_{\mathcal{M}} \left[\sqrt{det\,(h_{AB})}\right] df^1 df^2 \ldots df^n, \tag{2.78}$$

the standard volume integral in general coordinates $(f^1, f^2, \ldots, f^n)$.

To have an explicit example about this metric tensor, let us consider the $SU(2)$-chiral model. As here $G^{-1} = G^\dagger$ so (2.70) becomes $h_{AB} = 2\mathrm{Tr}\big(\partial_A G^\dagger \partial_B G\big)$ and using the representation (2.55) for G, we obtain

$$h_{AB}(f) = 4\Big[\delta_{AB} + \frac{f^A f^B}{(1 - |f|^2)}\Big], \tag{2.79}$$

which coincides with the metric tensor (2.34) for S^3 up to the overall factor 4.

Note that, the Lie algebra $\mathcal{G}$ is a metric space as well with the *Killing form*

$$\tilde{g}_{(A)(B)} = \mathrm{Tr}\left(adT_{(A)} adT_{(B)}\right) = f^{(C)}_{(A)(D)} f^{(D)}_{(B)(C)}, \tag{2.80}$$

plays the role of metric tensor. Here $adT_{(A)}$ is the *adjoint* representation matrix of $T_{(A)}$ defined by

$$\left(adT_{(A)}\right) T_{(B)} = \left[T_{(A)}, T_{(B)}\right], \tag{2.81}$$

where $\left(adT_{(A)}\right) T_{(B)} = T_{(C)} \left(adT_{(A)}\right)_{(C)(B)}$ which implies that $\left(adT_{(A)}\right)_{(C)(B)} = f^{(C)}_{(A)(B)}$. Thus, for example,

$$f_{(C)(A)(B)} = \tilde{g}_{(C)(D)} f^{(D)}_{(A)(B)}. \tag{2.82}$$

In fact, by multiplying the Jacobi identity (2.69) by $f^{(D)}_{(E)(F)}$, we deduce that $f_{(C)(D)(F)}$ is antisymmetric in (C) and (D). Thus, $f_{(A)(B)(C)}$ is antisymmetric in all its indices. If $det(\tilde{g}_{(A)(B)}) \neq 0$, then we can define the inverse metric tensor $\tilde{g}^{(A)(B)}$, *i.e.*

$\tilde{g}^{(A)(B)}\tilde{g}_{(B)(C)} = \delta^{(A)}_{(C)}$. Note that, the condition $det(\tilde{g}_{(A)(B)}) \neq 0$ is Cartan's criteria for the group G to be *semi-simple*, *i.e.* G has no abelian subgroup [36].

Next, let us return to the target manifold $\mathcal{M}$ and assume that it is equipped with an *affine connection* 1-form $\omega^{(A)}_{\ (B)}$. Then we have the *Cartan structural equations* for $\mathcal{M}$ [27,37,40]:

$$\mathcal{T}^{(A)} = d\lambda^{(A)} + \omega^{(A)}_{\ (B)} \wedge \lambda^{(B)}, \tag{2.83}$$

$$R^{(A)}_{\ (B)} = d\omega^{(A)}_{\ (B)} + \omega^{(A)}_{\ (C)} \wedge \omega^{(C)}_{\ (B)}, \tag{2.84}$$

where $\mathcal{T}^{(A)}$ and $R^{(A)}_{\ (B)}$ are the torsion 2-form and the curvature 2-form of $\mathcal{M}$, respectively.

Now a question arises. What is the affine connection 1-form $\omega^{(A)}_{\ (B)}$ of $\mathcal{M}$? In fact, the answer is already provided by the Maurer-Cartan equations (2.68) that *can be regarded* as the Cartan's first structural equation (2.83). With this interpretation, we conclude that $\mathcal{M}$ is *torsion-less*, *i.e.* $\mathcal{T}^{(A)} = 0$, and that the corresponding affine connection 1-form is given by

$$\omega^{(A)}_{\ (C)} = \frac{1}{2} f^{(A)}_{\ (B)(C)} \lambda^{(B)}. \tag{2.85}$$

Armed with this connection 1-form, we can now determine the value of the curvature of $\mathcal{M}$ that proceeds as follows.

Firstly, we take the exterior derivative of $\omega^{(A)}_{\ (B)}$ in (2.85) and after using again the Maurer-Cartan equations (2.68), we obtain

$$d\omega^{(A)}_{\ (B)} = -\frac{1}{4} f^{(A)}_{\ (B)(C)} f^{(C)}_{\ (D)(E)} \lambda^{(D)} \wedge \lambda^{(E)}. \tag{2.86}$$

Next, introducing (2.85) and (2.86) in the Cartan's second structural equation (2.84), and then using the Jacobi identity (2.69), we find that the curvature 2-form $R^{(A)}_{\ (B)}$ is

$$R^{(A)}_{\ (B)} = \frac{1}{8} f^{(A)}_{\ (B)(C)} f^{(C)}_{\ (D)(E)} \lambda^{(D)} \wedge \lambda^{(E)} \equiv \frac{1}{2} R^{(A)}_{\ (B)FG} df^F \wedge df^G, \tag{2.87}$$

where $R^{(A)}_{\ (B)FG}$ are its local components. The *Riemann curvature tensor* is defined by

$$R^K_{\ LFG} = \rho^K_{(A)} \lambda^{(B)}_L R^{(A)}_{\ (B)FG}. \tag{2.88}$$

Hence, the scalar curvature of $\mathcal{M}$ is

$$R = G^{LG} R^K_{\ LKG} = \frac{1}{4} f^{(C)}_{\ (A)(B)} f^{(B)}_{\ (C)(A)} = -\frac{1}{4} \tilde{g}_{(A)(A)}. \tag{2.89}$$

For the $SU(2)$-chiral model, $f^{(A)}_{(B)(C)} = \epsilon^{(A)}_{(B)(C)}$, so $\tilde{g}_{(A)(A)} = -6$, hence $R = \frac{3}{2}$. (Here $\epsilon_{(A)(B)(C)}$ is the Levi-Civita antisymmetric symbol given by (2.75)). As R in (2.89) is determined solely by the Killing form, *i.e.* the structure constants, of $\mathcal{G}$, we conclude that the target space $\mathcal{M}$ of the G-chiral model is a Riemannian manifold of *constant curvature.*

2.4.2 G-Chiral Lagrangian Density

Now, we are ready to construct the Lagrangian density for the G-chiral model. Substituting the metric tensor (2.73) into the Lagrangian density of the harmonic maps (2.4), yields

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{2} h_{AB} \frac{\partial f^A}{\partial x^\mu} \frac{\partial f^B}{\partial x^\nu} g^{\mu\nu} \\ &= -\mathrm{Tr}\left[G^{-1}\left(\partial_\mu G\right) G^{-1}\left(\partial_\nu G\right)\right] g^{\mu\nu}, \end{aligned} \tag{2.90}$$

where $G^{-1}(\partial_\mu G)$ is the pull back of the Maurer-Cartan 1-form onto the base manifold $\mathcal{M}_0$, *i.e.*

$$G^{-1}\left(\partial_\mu G\right) = G^{-1}\left(\partial_A G\right) \partial_\mu f^A. \tag{2.91}$$

Equation (2.90) is the basic expression that is taken as the definition of the *G-chiral model Lagrangian density.* The Euler-Lagrange equation that follows from (2.90) is

$$\partial^2 G - \partial^\mu G G^{-1} \partial_\mu G = 0, \tag{2.92}$$

which is the G-chiral model field equation. If we define

$$L_\mu = G^{-1} \partial_\mu G, \tag{2.93}$$

then in terms of it, the G-chiral model Lagrangian density (2.90) becomes

$$\mathcal{L} = -\mathrm{Tr}\left(L^\mu L_\mu\right), \tag{2.94}$$

and the Euler-Lagrange equations (2.92) simplify to

$$\partial^\mu L_\mu = 0. \tag{2.95}$$

Equation (2.95) is a conservation law equation for L_μ which suggests the name *left current* for L_μ. An equivalent formulation is given in terms of the *right current:* $R_\mu = \partial_\mu G G^{-1}$.

As the Lagrangian density (2.90) is invariant under the global left-right action:

$$G \to G' = G_l G G_r, \tag{2.96}$$

the real invariance of the G-chiral model is $G_l \times G_r$. In fact, the adjective *chiral* (derived from the name "chira" which means a hand in Greek) in the name of the models is introduced in relation to this left-right invariance.

2.5 Coset σ Models

We now possess basic group and geometrical knowledge necessary to formulate σ models with coset space as the target space. These include the $O(N)$, RP^{N-1} and its generalisation Grassmannian σ model. In this section, we discuss a coset formulation of these models, in general, while in Chapter 3, we shall deal mainly with the Grassmannian σ model [7,34,36,38,39]. For further reference, we call them *coset σ models.*

2.5.1 Coset Formulation

To start our discussion on coset formulation of these models, let us take the $O(N)$ σ model as an example [36, 39]. Here, we identify the fields $\phi^{\hat{A}}$ as the elements in the last column of the matrix $(G^{\hat{A}\hat{B}}) \in O(N)$, in the fundamental or vector representation, *i.e.*

$$\phi^{\hat{A}} = G^{\hat{A}N}. \tag{2.97}$$

This identification partitions the $(N \times N)$ matrix G into

$$G = \begin{bmatrix} \Phi^{\hat{A}B} & \phi^{\hat{A}} \end{bmatrix}, \tag{2.98}$$

where $\Phi^{\hat{A}B}$ is an $(N \times (N-1))$ matrix whereas $\phi^{\hat{A}}$ is an N-component column vector. The constraint (2.24) then follows from the orthogonality condition of O, *i.e.* $G^T G = I_N$, where the superscript T is for transposition and I_N the $(N \times N)$ unit matrix. Explicitly, as

$$G^T G = \begin{bmatrix} (\Phi^T)^{B\hat{A}}\Phi^{\hat{A}C} & (\Phi^T)^{B\hat{A}}\phi^{\hat{A}} \\ & \\ (\phi^T)^{\hat{A}}\Phi^{\hat{A}C} & (\phi^T)^{\hat{A}}\phi^{\hat{A}} \end{bmatrix}, \tag{2.99}$$

Φ and ϕ satisfy the constraints:

$$\Phi^T \Phi = I_{N-1}, \qquad \Phi^T \phi = 0, \qquad \phi^T \phi = 1. \tag{2.100}$$

Now, under the right action of the group $H \in O(N)$ on G, *i.e.* $G \to GH$, $\phi^{\hat{A}}$ transforms according to [7]

$$\phi^{\hat{A}} \to \phi^{\hat{A}} H^{NN} + \Phi^{\hat{A}B} H^{BN}. \tag{2.101}$$

Thus, $\phi^{\hat{A}}$ is left invariant under this transformation if we set $H^{\hat{A}N} = \delta^{\hat{A}N}$, *i.e.*

$$H = \begin{bmatrix} \times & \dots & \times & 0 \\ \vdots & \times & \vdots & \vdots \\ \times & \dots & \times & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}. \tag{2.102}$$

With this definition, H defines an embedding of the group $O(N-1)$ in $O(N)$. Thus, $O(N-1)$ is the isotropy group (or in physics terminology, the little group) of the field manifold $\phi^{\hat{A}}$.

Therefore, if a transformation $G_1 \in O(N)$ maps a certain field value $\phi_p \in S^{N-1}$, for example the "north pole" $(0, \dots, 1)$, into another point $\phi_q \in S^{N-1}$, then the transformation $G_2 = G_1 H$ on S^{N-1} will produce the same field value as well, *i.e.* $\phi_q = G_2 \phi_p$. Hence, points on the sphere S^{N-1} can be associated with the left cosets of $O(N)$ relative to $O(N-1)$, *i.e.* $S^N = O(N)/O(N-1)$.

In general, as mentioned previously, the elements of G/H are equivalence classes of the form GH and that $dim[G/H] = (dim[G] - dim[H])$. For our case,

$$dim[O(N)/O(N-1)] = \left[\frac{1}{2}N(N-1)\right] - \left[\frac{1}{2}(N-1)(N-2)\right] = (N-1), \tag{2.103}$$

as expected for the dimension of S^{N-1}.

Giving a parametrisation $\phi^{\hat{A}} = \phi^{\hat{A}}(f^1, \dots, f^{N-1})$, as in sections 2.2.1-2.2.3, is equivalent to take a particular coset representantive from the cosets $O(N)/O(N-1)$ which assigns a single $O(N)$ element to every coset [36, 39]. To find such a parametrisation within this coset formalism, we observe that every element $G \in O(N)$ can be decomposed as the product:

$$G = G(\theta) H, \tag{2.104}$$

where $H \in O(N-1)$, and $G(\theta)$, $(\theta^1, \dots, \theta^{(N-1)}) \in R^{N-1}$, is the coset space element which we choose as a representative element of each H-equivalence class.

Here $G(\theta)$ can be locally obtained by exponentiating the component of the Lie algebra of $O(N)$ orthogonal to the Lie algebra of $O(N-1)$. Let $\mathcal{O}(M)$ be the Lie algebra of the orthogonal group $O(M)$ then

$$\mathcal{O}(N) = \mathcal{O}(N-1) \oplus \mathcal{K}, \tag{2.105}$$

where $\mathcal{K}$ is the complement to $\mathcal{O}(N-1)$ in $\mathcal{O}(N)$. Thus, $\mathcal{K}$ consists of matrices of the form [36,39]

$$\mathcal{K}(\theta) = \begin{bmatrix} 0 & \dots & 0 & \theta^1 \\ \vdots & 0 & \vdots & \vdots \\ 0 & \dots & 0 & \theta^{(N-1)} \\ -\theta^1 & \dots & -\theta^{(N-1)} & 0 \end{bmatrix}, \tag{2.106}$$

Hence, the coset representative $G(\theta)$ is given by

$$G(\theta) = \mathrm{Exp}\Big(\mathcal{K}(\theta)\Big) = \begin{bmatrix} \delta^{AB} + \theta^A\theta^B \frac{(\cos|\theta|-1)}{|\theta|^2} & \theta^A \frac{\sin|\theta|}{|\theta|} \\ & \\ -\theta^B \frac{\sin|\theta|}{|\theta|} & \cos|\theta| \end{bmatrix}, \tag{2.107}$$

as evaluated in appendix A, where $|\theta| = \sqrt{\theta^A\theta^A}$, $0 \leq |\theta| < \pi$. Setting,

$$f^A = \theta^A \frac{\sin|\theta|}{|\theta|}, \tag{2.108}$$

then the coset representative (2.107) reads

$$G(f) = \begin{bmatrix} \delta^{AB} + f^A f^B \frac{(\pm\sqrt{(1-|f|^2)}-1)}{|f|^2} & f^A \\ & \\ -f^B & \pm\sqrt{(1-|f|^2)}. \end{bmatrix}. \tag{2.109}$$

Of course, one may choose a different representative of the cosets G/H. This is due to the fact that all different representatives $G(f)$ of the cosets are related by f-dependent H transformation from the right on $G(f)$, *i.e.*

$$G(f) \to G(f)h(f), \tag{2.110}$$

for $h(f) \in H$.

Since $G(f) \in G$, we may examine the effect of a G transformation acting on $G(f)$ from the left. To see this explicitly, let us consider the following infinitesimal $g \in O(N)$ transformation [39]

$$g \approx I + \begin{bmatrix} 0 & \dots & 0 & \epsilon^1 \\ \vdots & 0 & \vdots & \vdots \\ 0 & \dots & 0 & \epsilon^{N-1} \\ -\epsilon^1 & \dots & -\epsilon^{N-1} & 0 \end{bmatrix}, \tag{2.111}$$

where $(\epsilon^1, \dots, \epsilon^{N-1})$ are infinitesimal parameters. Acting on $G(f)$ in (2.109) this leads to

$$\begin{aligned} gG(f) &\approx G(f) \\ &+ \begin{bmatrix} -\epsilon^A f^B & \pm\epsilon^A\sqrt{(1-|f|^2)} \\ -\epsilon^B - f^B(\epsilon^C f^C)\frac{(\pm\sqrt{(1-|f|^2)}-1)}{|f|^2} & -(\epsilon^C f^C) \end{bmatrix}. \end{aligned} \tag{2.112}$$

We see that $gG(f)$ is no longer compatible with the coset representative $G(f)$. In fact, we can interpret it as having the effect of inducing a coordinate transformation on the coset space S^{N-1}, *i.e.*

$$f^A \to f'^A \approx f^A \pm \epsilon^A\sqrt{1-|f|^2}. \tag{2.113}$$

In general, the action of a transformation $g \in G$ on a coset representative $G(f)$ will give another element $g' \in G$. Since g' can be written uniquely in a coset decomposition as in (2.104), the transformation $gG(f)$ could also be interpreted as having the effect of transforming $G(f)$ into a different equivalence class in G/H, whose representative element we denote by $G(f')$, *i.e.*

$$gG(f) = G(f')h(f,g), \qquad h \in H. \tag{2.114}$$

This equation determines f' and h as functions of both f and g [36,38,39] .

The abstract relation (2.114) is given concretely, for our case $O(N)/O(N-1)$, by the matrix multiplication [36]

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix} \begin{bmatrix} \Phi & f \\ -f^T & \phi^N \end{bmatrix} = \begin{bmatrix} \Phi' & f' \\ -f'^T & \phi'^N \end{bmatrix} \begin{bmatrix} \widetilde{H} & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.115}$$

where A, $\widetilde{\Phi}$ and $\widetilde{H}$ are $(N-1)\times(N-1)$ matrices, whereas b, c and f are $(N-1)$-component column vectors. For example, to determine f' in terms of $g=(A,\, b,\, c,\, d)$ and f, we need to solve the equation

$$\begin{bmatrix} \times & (Af+b\phi^N) \\ \\ \times & (c^T f + d\phi^N) \end{bmatrix} = \begin{bmatrix} \times & f'(1) \\ \\ \times & \phi'^N(1) \end{bmatrix}. \tag{2.116}$$

The effect is to change the coordinate $f \to f'$ in a way that it satisfies the group multiplication law.

We notice that the "north pole" $\phi_p = (0,\ldots,1)^T \in S^{N-1}$ is mapped onto any point of S^{N-1} by the coset representative $G(\theta)$, *i.e.*

$$G(\theta)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \phi^1 \\ \vdots \\ \phi^{N-1} \\ \phi^N \end{bmatrix}. \tag{2.117}$$

In fact, using (2.107) for $G(\theta)$) in the left hand side of (2.117) gives us

$$\begin{aligned} \phi^1 &= \frac{\theta^1 \sin|\theta|}{|\theta|} = \sin\theta \sin\varphi_1 \ldots \sin\varphi_{N-2}, \\ &\vdots \\ \phi^{\tilde{A}} &= \frac{\theta^{\tilde{A}} \sin|\theta|}{|\theta|} = \sin\theta \left(\prod_{k=1}^{N-\tilde{A}-1} \sin\varphi_k\right) \cos\varphi_{\tilde{A}}, \quad (2 \le \tilde{A} \le N-2), \\ &\vdots \\ \phi^{N-1} &= \frac{\theta^{N-1}\sin|\theta|}{|\theta|} = \sin\theta\cos\varphi_1, \\ \phi^N &= \frac{\theta^N \sin|\theta|}{|\theta|} = \cos\theta, \end{aligned} \tag{2.118}$$

the parametrisation of $(\phi^1,\, \ldots,\, \phi^N)$ in terms of the spherical polar coordinates of S^{N-1}, $N \geq 3$, where $0 \leq \theta,\ \varphi_{\tilde{A}-1} < \pi$ and $0 \leq \phi_{N-2} < 2\pi$. Using $G(f)$ from (2.109), instead of $G(\theta)$, in (2.117) gives us $\phi^A = (f^A,\, \pm\sqrt{1-|f|^2})$.

Thus, we see that $S^{N-1} \subset R^N$ is the *orbit* of ϕ_p under the action of $O(N)$. In fact, $\mathrm{Exp}(t\mathcal{K}(\theta))\phi_p$ with $t \in \mathbb{R}$ is a *geodesic* starting from ϕ_p; conversely, every geodesic from ϕ_p is of this form [36].

2.5.2 Lie Algebra Decomposition

In the above construction, we have seen that the coset manifold G/H is generated through the exponential map Exp($\mathcal{K}$) where $\mathcal{K}$ is the matrix vector space complement of the Lie algebra $\mathcal{H}$ of H in the Lie algebra $\mathcal{G}$ of G. Thus, geometrically, $\mathcal{K}$ plays the role as tangent space to the coset manifold.

To prepare the background for our discussion in the next subsection about geometric structures of this coset manifold, here we consider the algebraic structure of the Lie algebra decomposition:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{K}. \tag{2.119}$$

Let $T_{\hat{a}}$ ($\hat{a} = 1, \ldots, dim(\mathcal{G})$) be the generators of $\mathcal{G}$ satisfying the commutation relations:

$$[T_{\hat{a}}, T_{\hat{b}}] = f_{\hat{a}\hat{b}}^{\hat{c}} T_{\hat{c}}, \tag{2.120}$$

where $f_{\hat{a}\hat{b}}^{\hat{c}}$ are the corresponding structure constants of $\mathcal{G}$. Note that, here we have deleted the i factor from the right hand side of (2.120) as this amounts to multiplying the generators $T_{\hat{a}}$ by i and the corresponding group parameters by $(-i)$ which changes nothing. Thus, here the generators are normalised to $\mathrm{Tr}(T_{\hat{a}}T_{\hat{b}}) = -\frac{1}{2}\delta_{\hat{a}\hat{b}}$.

If we let T_a ($a = 1, \ldots, dim(\mathcal{G}) - dim(\mathcal{H})$) and $T_{\tilde{a}}$ ($\tilde{a} = dim(\mathcal{G}) - dim(\mathcal{H}) + 1, \ldots, dim(\mathcal{G})$) be the generators of $\mathcal{K}$ and $\mathcal{H}$, respectively, then from (2.120) we have [7]

$$[T_{\tilde{a}}, T_{\tilde{b}}] = f_{\tilde{a}\tilde{b}}^{\tilde{c}} T_{\tilde{c}}, \tag{2.121}$$

$$[T_{\tilde{a}}, T_b] = f_{\tilde{a}b}^{c} T_c, \tag{2.122}$$

$$[T_a, T_b] = f_{ab}^{\tilde{c}} T_{\tilde{c}} + f_{ab}^{c} T_c. \tag{2.123}$$

The first commutation relation (2.121) states that $\mathcal{H}$ is a *subalgebra* which follows from the fact that H is a subgroup of G, whereas (2.122) implies that the generators of $\mathcal{K}$ form a representation of $\mathcal{H}$. The absence of $T_{\tilde{c}}$ in the right hand side of (2.122) is due to

$$\mathrm{Tr}\left(T_{\tilde{d}}\left[T_{\tilde{a}}, T_b\right]\right) = \mathrm{Tr}\left(T_b\left[T_{\tilde{d}}, T_{\tilde{a}}\right]\right) = 0. \tag{2.124}$$

Thus, schematically, we have

$$[\mathcal{H}, \mathcal{H}] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{K}] = \mathcal{K}, \quad [\mathcal{K}, \mathcal{K}] = \mathcal{H} \oplus \mathcal{K}. \tag{2.125}$$

If for a specific coset G/H, the corresponding subgroup H is such that the third commutator in (2.125) satisfies

$$[\mathcal{K},\mathcal{K}] = \mathcal{H}, \tag{2.126}$$

i.e. $f^{c}_{ab} = 0$ in (2.123), then the coset G/H is a *symmetric space.* A detailed group theoretical formulation of symmetric space will be given in Chapter 3.

As an illustration of this construction, let us consider the Lie algebra $\mathcal{O}(N)$. Here the corresponding $\frac{1}{2}N(N-1)$ generators, in two indices notation, are $M_{[\hat{A}\hat{B}]} = -M_{[\hat{B}\hat{A}]}$, which satisfy [35]

$$\left[M_{[\hat{A}\hat{B}]}, M_{[\hat{C}\hat{D}]}\right] = \delta_{\hat{A}\hat{C}}M_{[\hat{B}\hat{D}]} - \delta_{\hat{A}\hat{D}}M_{[\hat{B}\hat{C}]} + \delta_{\hat{B}\hat{D}}M_{[\hat{A}\hat{C}]} - \delta_{\hat{B}\hat{C}}M_{[\hat{A}\hat{D}]}. \tag{2.127}$$

Let (AB) stand for the indices in a fixed order $A < B$; then the identifications:

$$T_{\tilde{a}} \equiv M_{(AB)}, \quad T_a \equiv M_{(AN)}, \tag{2.128}$$

constitute the decomposition $\mathcal{O}(N) = \mathcal{O}(N-1) \oplus \mathcal{K}$ with nonzero structure constants:

$$f^{\tilde{c}}_{\tilde{a}\tilde{b}} \equiv f^{(EF)}_{(AB)(CD)} = \frac{1}{2}\left[\delta_{AC}\delta^{EF}_{BD} - \delta_{AD}\delta^{EF}_{BC} + \delta_{BD}\delta^{EF}_{AC} - \delta_{BC}\delta^{EF}_{AD}\right], \tag{2.129}$$

$$f^{d}_{\tilde{a}c} \equiv f^{d}_{(AB)c} = \delta_{Ac}\delta^{d}_{B} - \delta_{Bc}\delta^{d}_{A}, \tag{2.130}$$

$$f^{\tilde{c}}_{ab} \equiv f^{(CD)}_{ab} = \frac{1}{2}\delta^{CD}_{ab}, \tag{2.131}$$

where δ^{UV}_{XY} is given by (2.75). We see that, as $f^{c}_{\tilde{a}\tilde{b}} = f^{\tilde{c}}_{\tilde{a}b} = f^{c}_{ab} = 0$, S^{N-1} is a symmetric space.

In the vector representation ρ_V, as we have used above for the coset construction of the $O(N)$ σ model, the matrix elements of the generators $M_{[\hat{A}\hat{B}]}$ are

$$\left(M_{[\hat{A}\hat{B}]}\right)_{(\hat{C})(\hat{D})} = \delta_{\hat{A}(\hat{C})}\delta_{\hat{B}(\hat{D})} - \delta_{\hat{B}(\hat{C})}\delta_{\hat{A}(\hat{D})}. \tag{2.132}$$

Here the bracketed indices, $(\hat{A})$ for example, label the components of the representation vector, $\phi^{\hat{A}} \equiv |(\hat{A})>$, where the matrix acts. We see that $\mathcal{K}(\theta)_{(\hat{C})(\hat{D})} = \theta^A \left(M_{[AN]}\right)_{(\hat{C})(\hat{D})}$ coincides with (2.106).

The case $N = 2n$ is of particular interest, as here $O(2n)$ has *spinor* representation ρ_S for which

$$M_{[\hat{A}\hat{B}]} = \frac{1}{4}[\Gamma_{\hat{A}}, \Gamma_{\hat{B}}], \tag{2.133}$$

where $\Gamma_{\hat{A}}$ are the generalised $(2^n \times 2^n)$ Dirac gamma matrices satisfying the Clifford algebra [27, 37]:

$$\Gamma_{\hat{A}}\Gamma_{\hat{B}} + \Gamma_{\hat{B}}\Gamma_{\hat{A}} = 2\delta_{\hat{A}\hat{B}}. \tag{2.134}$$

For this case, the coset representative is [39]

$$\begin{aligned} G(\theta) &= \text{Exp}\left[\frac{1}{2}\theta^A\left(\Gamma_A\Gamma_N\right)\right] \\ &= \cos(|\theta|/2) + \frac{\sin(|\theta|/2)}{|\theta|}\theta^A\left(\Gamma_A\Gamma_N\right). \end{aligned} \tag{2.135}$$

Applying this on a 2^n-components spinor: $\psi^T = (0, 0, \ldots, 0, 1)$ (the analog of (2.117)), we obtain a spinor on S^{N-1}.

As the coset space G/H plays an important role as the manifold for the σ-model field configurations, we shall also use geometric terminology for these models, such as "S^{N-1} σ model" for $O(N)$ σ model.

2.5.3 Geometric Formulation

From the above discussion, we see that within this coset formalism, the nonlinear σ model fields take value in the coset representative $G(f(x)) \in G$ which is defined up to the action of global $g \in G$ from the left and the local invariant subgroup $h(f) \in H$ (or gauge transformation) from the right, *i.e.*

$$G(f) \sim gG(f)h(f). \tag{2.136}$$

Due to this gauge arbitrariness, the parametrisation $G(f)$ contains gauge degrees of freedom. Therefore, the true dynamical (physical) fields (also known as Goldstone bosons in physics literature) are the equivalence classes, *i.e.* have values in the coset space $\mathcal{M} = G/H$.

Thus, the invariant Lagrangian density for the G-valued fields $G(f)$ should be constructed from the geometrical quantities of the coset space G/H. We have seen in the previous section that the basic geometrical object in this construction is the Maurer-Cartan left-invariant 1-form $G^{-1}dG$. As it takes value in the Lie algebra $\mathcal{G}$ of G, it can be decomposed into the generators of $\mathcal{H}$ and $\mathcal{K}$, as follows:

$$G^{-1}dG = \Omega + V, \tag{2.137}$$

where [7,38,39]

$$\Omega \equiv \left(G^{-1}dG\right)_{\mathcal{H}} = -2T_{\bar{a}}\mathrm{Tr}\left(T_{\bar{a}}G^{-1}dG\right), \tag{2.138}$$

$$V \equiv \left(G^{-1}dG\right)_{\mathcal{K}} = -2T_{a}\mathrm{Tr}\left(T_{a}G^{-1}dG\right). \tag{2.139}$$

Hence,

$$d\Omega + \Omega \wedge \Omega = -\left(V \wedge V\right)_{\mathcal{H}}, \tag{2.140}$$

$$dV + \Omega \wedge V + V \wedge \Omega = -\left(V \wedge V\right)_{\mathcal{K}}. \tag{2.141}$$

Under the *H-gauge transformation*:

$$G(f) \to G(f)h(f), \tag{2.142}$$

$h(f) \in H$, each part of the 1-form $G^{-1}dG$ transforms as

$$\Omega(Gh) \to h^{-1}\Omega(G)h + h^{-1}dh, \tag{2.143}$$

$$V(Gh) \to h^{-1}V(G)h. \tag{2.144}$$

Thus, Ω can be interpreted as a *gauge potential* or a *connection* 1-form matrix for H whereas V transforms *covariantly.* Their explicit expressions with respect to the local coordinates f^A and the generators $T^{\hat{a}}$ are

$$\Omega = \Omega^{\bar{a}}T_{\bar{a}} = \Omega^{\bar{a}}_{A}T_{\bar{a}}df^{A}, \tag{2.145}$$

$$V = V^{a}T_{a} = V^{a}_{A}T_{a}df^{A}. \tag{2.146}$$

The 1-forms V^a can be regarded as the basis of $T^*_f(\mathcal{M})$ of 1-forms on $\mathcal{M}$ at point f, whereas $\Omega^{\bar{a}}$ defines the "spin" connection 1-form with the tangent space "rotation" on $\mathcal{M}$ given by the isotropy group H [38,39].

Hence, as in the previous section, the V^a_A can be taken as the *vielbein* fields on $\mathcal{M}$, and so the natural metric tensor on $\mathcal{M}$ is

$$h_{AB} = V^{a}_{A}V^{a}_{B} = -2\mathrm{Tr}\left(V_{A}V_{B}\right), \tag{2.147}$$

where

$$V_A = V^{a}_{A}T_{a}. \tag{2.148}$$

The metric tensor h_{AB} in (2.147) is obviously H-gauge invariant, as V_A is H-covariant.

In terms of the 1-form basis: $V^a = V^a_A df^A$, the *coset space volume integral* is also given by (2.74), with the replacement: $\lambda^{(A)} \to V^a$, which results in the standard general coordinates formula (2.78) as well, where the metric tensor h_{AB} is now given by (2.147).

To have an explicit example about the coset space metric tensor (2.147), let us consider the coset space $S^{N-1} = O(N)/O(N-1)$. Here, the matrix G in the fundamental representation of $O(N)$ is given by the partitions (2.98). Hence, the generic $(N \times N)$ matrix representation of (2.137) is

$$G(f)^{-1}dG(f) = \begin{bmatrix} \Omega^{BC}_D & V^B_D \\ \\ -(V^T)^C_D & 0 \end{bmatrix} df^D, \tag{2.149}$$

where

$$\Omega^{BC}_D = \left(\Phi^T\right)^{B\hat{A}} \partial_D \Phi^{\hat{A}C}, \qquad V^B_D = \left(\Phi^T\right)^{B\hat{A}} \partial_D \phi^{\hat{A}}, \tag{2.150}$$

and where we have used the properties (2.100) so that $\phi^T d\phi = 0$, and $(d\Phi^T)\phi = -\Phi^T d\phi$. These properties also yield: $\Omega^{BC}_D = -\Omega^{CB}_D$.

Using the explicit coset representative form (2.109) we derive [39]

$$\Omega^{BC}_D = \left(f^B \delta^C_D - f^C \delta^B_D\right) \frac{\left(\pm\sqrt{1-|f|^2} - 1\right)}{|f|^2}, \tag{2.151}$$

$$V^B_D = \delta^B_D + \frac{f^B f^D}{f^2}\left(\pm\frac{1}{\sqrt{1-|f|^2}} - 1\right). \tag{2.152}$$

Thus, according to (2.147) the corresponding metric tensor is

$$h_{AB} = \delta_{AB} + \frac{f^A f^B}{(1-|f|^2)}, \tag{2.153}$$

which coincides with (2.34).

Now let us proceed to consider the effect of the left action of the global symmetry group $g \in G$, *i.e.* $G \to G' = gG$ on the metric tensor h_{AB}. In the previous subsection we have seen that it has two effects on our coset formalism:

(1). It transforms a particular coset representative $G(f)$ into $G(f')$ of a different equivalence class, and so from the construction (2.147), the dependence of the metric tensor on f changes accordingly into

$$h_{AB}(f) \to h_{AB}(f'). \tag{2.154}$$

(2). It induces a coordinate transformation on the coset space $\mathcal{M}$: $f \to f'(f)$. Hence, the metric tensor transforms into

$$h'_{AB}(f') = \frac{\partial f'^C}{\partial f^A}\frac{\partial f'^D}{\partial f^B} h_{CD}(f). \tag{2.155}$$

However, as the Maurer-Cartan 1-form $G^{-1}dG$ is left-invariant, so

$$h_{AB}(f') = h'_{AB}(f'), \tag{2.156}$$

which means that g is an *isometry*. In fact, if we consider the infinitesimal transformation:

$$f'^A \approx f^A + \xi^A(f), \tag{2.157}$$

then by virtue of the isometry condition (2.156), the tangent vector

$$X = \xi^A \frac{\partial}{\partial f^A} \tag{2.158}$$

is a *Killing vector*, *i.e.* its components ξ^A satisfy the *Killing equations*:

$$\xi_{A,B} + \xi_{B,A} - 2\Gamma^C_{AB}\xi_C = 0, \tag{2.159}$$

where Γ^C_{AB} is the Cristoffel symbol (2.7), and where $\xi_A = h_{AB}\xi^B$. As G is the isometry group of $\mathcal{M}$ so the number of Killing vectors admitted by $\mathcal{M}$ is equal to $dim(G)$, *i.e.* $\xi^A = \epsilon^{\hat{a}}\xi^{\hat{a}A}$, where $\epsilon^{\hat{a}}$, ($\hat{a} = 1, \ldots, dim(G)$), are constant parameters.

One can check that the infinitesimal part of the transformation (2.113), *i.e.* $\xi^A = \pm\,\epsilon^A\sqrt{1-f^2}$ do satisfy the Killing equation (2.159), which is consistent with the fact that it is a G-transformation.

Let us now compute the curvature of the coset space $\mathcal{M} = G/H$ using Cartan's moving frames method. Here, the basic equations to start with are equations (2.140)-(2.141) that could be interpreted as *embedding equations* of the coset space $\mathcal{M}$ into the group manifold G. In terms of the Lie algebra components of Ω and V in (2.145) and (2.146), respectively, the embedding equations (2.140)-(2.141) become

$$d\Omega^{\tilde{a}} + \frac{1}{2} f^{\tilde{a}}_{\tilde{b}\tilde{c}}\Omega^{\tilde{b}} \wedge \Omega^{\tilde{c}} = -\frac{1}{2} f^{\tilde{a}}_{bc} V^b \wedge V^c, \tag{2.160}$$

$$dV^a + f^a_{\tilde{b}c}\Omega^{\tilde{b}} \wedge V^c = -\frac{1}{2} f^a_{bc} V^b \wedge V^c. \tag{2.161}$$

As the matrix curvature 2-form of the matrix H-connection 1-form Ω is defined by

$$R(\Omega) = d\Omega + \Omega \wedge \Omega, \tag{2.162}$$

the H-connection curvature 2-form that we read-off from (2.160) is

$$R^{\tilde{a}} = -\frac{1}{2} f^{\tilde{a}}_{bc} V^b \wedge V^c \equiv \frac{1}{2} R^{\tilde{a}}{}_{BC} df^B \wedge df^C. \tag{2.163}$$

As in section 2.4.1, we regard equation (2.161) as the Cartan's first structural equation (2.83) from which we shall define the *affine connection* 1-form $\omega^a_{\ c}$ of the coset space $\mathcal{M}$. In fact, there are two ways to define this connection [38].

The simplest one is to choose

$$\omega^a_{\ c} = f^a_{\tilde{b}c} \Omega^{\tilde{b}}, \tag{2.164}$$

which has *non*-zero torsion 2-form

$$\mathcal{T}^a = -\frac{1}{2} f^a_{bc} V^b \wedge V^c \equiv \frac{1}{2} \mathcal{T}^a_{BC} df^B \wedge df^C. \tag{2.165}$$

The corresponding curvature 2-form of $\mathcal{M}$ is then obtained by substituting the affine connection (2.164) into the Cartan's second structural equation (2.84), which yields

$$R^a_{\ c} = f^a_{\tilde{b}c} d\Omega^{\tilde{b}} + f^a_{\tilde{d}b} f^b_{\tilde{e}c} \Omega^{\tilde{d}} \wedge \Omega^{\tilde{e}}. \tag{2.166}$$

The second choice is to choose

$$\omega^a_{\ c} = f^a_{\tilde{b}c} \Omega^{\tilde{b}} + \frac{1}{2} f^a_{bc} V^b, \tag{2.167}$$

which has a *vanishing* torsion 2-form, *i.e.* $\mathcal{T} = 0$. The corresponding 2-form curvature $R^a_{\ c}$ is then obtained by substituting (2.167) into (2.84).

In the following discussion, we consider $\mathcal{M}$ to be a *symmetric space*, which means: $f^c_{ab} = 0$, so the corresponding torsion 2-form $\mathcal{T}^a$ in (2.165) *vanishes* and the above two assignments become identical.

Substituting (2.160) into (2.166) and using the Jacobi identity (2.69) for the special structure constants $f^{\hat{c}}_{\hat{a}\hat{b}}$ of $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$, we finally obtain

$$R^a_{\ b} = f^a_{\tilde{c}b} R^{\tilde{c}} = -\frac{1}{2} f^a_{\tilde{c}b} f^{\tilde{c}}_{de} V^d \wedge V^e \equiv \frac{1}{2} R^a_{\ bCD} df^C \wedge df^D. \tag{2.168}$$

Thus, the Riemann curvature tensor of $\mathcal{M}$ is

$$R_{ABCD} = h_{AE}R^E{}_{BCD} = V_A^f V_E^f \left(\rho_a^E V_B^b R^a{}_{bCD}\right) = f_{b\bar{c}}^a f_{de}^{\bar{c}} V_A^a V_B^b V_C^d V_D^e, \tag{2.169}$$

and so the corresponding scalar curvature is

$$R \equiv h^{BD} R^A{}_{BAD} = -\tilde{g}_{bb}, \tag{2.170}$$

where $\tilde{g}_{bb} = -f_{b\bar{c}}^a f_{ba}^{\bar{c}}$ is the Killing form of $\mathcal{G}$.

For the S^{N-1} σ model case, the structure constants (2.130)-(2.131) yield

$$f_{b\bar{c}}^a f_{de}^{\bar{c}} = f_{b(PQ)}^a f_{de}^{(PQ)} = \delta_d^a \delta_{be} - \delta_e^a \delta_{bd}. \tag{2.171}$$

Hence, from (2.169) we obtain

$$R_{ABCD} = h_{AC}h_{BD} - h_{AD}h_{BC}, \tag{2.172}$$

the standard curvature tensor of the sphere, and so the scalar curvature of S^{N-1} is

$$R = (N-1)(N-2). \tag{2.173}$$

Thus, we conclude that the coset σ model target space $\mathcal{M} = G/H$ is a Riemannian manifold with constant curvature if $\mathcal{M}$ is a symmetric space.

2.5.4 Coset σ Models Lagrangian Density

Now we have discussed all the basic group and geometrical formalisms we need to construct the Lagrangian density for the coset σ model. First, we note that, by rewriting the decomposition (2.137) as

$$dG - G\Omega = GV, \tag{2.174}$$

the extended matrix exterior derivative on the left hand side, *i.e.*

$$DG \equiv dG - G\Omega, \tag{2.175}$$

transforms covariantly under the H-gauge transformation (2.144). Hence, D plays a role as *H-gauge covariant exterior derivative* in $\mathcal{M}$.

In terms of D, the matrix vielbein field $V_A = V_A^a T_a$, according to the decomposition (2.137) is given by [34]

$$V_A = G^{-1} D_A G, \tag{2.176}$$

and the metric tensor (2.147) reads

$$h_{AB} = -2\mathrm{Tr}\left(V_A V_B\right) = -2\mathrm{Tr}\left[G^{-1}\left(D_A G\right) G^{-1}\left(D_B G\right)\right], \tag{2.177}$$

which is H-gauge invariant.

Now we are ready to construct the Lagrangian density for the coset σ model. Substituting the metric tensor h_{AB} in (2.177) into the Lagrangian density of the harmonic maps (2.4), yields

$$\mathcal{L}(x) = -\mathrm{Tr}\left(V^\mu V_\mu\right) = -\mathrm{Tr}\left[G^{-1}\left(D^\mu G\right) G^{-1}\left(D_\mu G\right)\right], \tag{2.178}$$

where

$$D_\mu G \equiv \partial_\mu G - G\Omega_\mu, \tag{2.179}$$

is the pull back of D_A in $\mathcal{M}$ onto the base manifold $\mathcal{M}_0$, *i.e.*

$$\partial_\mu G = \partial_A G \partial_\mu f^A, \tag{2.180}$$

whereas the pullback of Ω_A and V_A are

$$\Omega_\mu = \Omega_A^{\bar{a}} T_{\bar{a}} \partial_\mu f^A, \qquad V_\mu = V_A^a T_a \partial_\mu f^A, \tag{2.181}$$

respectively. Equation (2.178) defines the Lagrangian density of the coset σ models.

At this stage one still have the full gauge invariance with respect to local H transformations and we can impose a gauge restricting G to the form of a particular coset representative. For example, let us reconsider the coset representative of S^{N-1} σ model as given in (2.98). From the results (2.149)-(2.150), we deduce that the matrix partitions of the pullback of Ω_μ and V_μ onto the base space $\mathcal{M}_0$ are

$$\Omega_\mu = \begin{bmatrix} \Phi^T \partial_\mu \Phi & 0 \\ 0 & 0 \end{bmatrix}, \qquad V_\mu = \begin{bmatrix} 0 & \Phi^T \partial_\mu \phi \\ -\left(\Phi^T \partial_\mu \phi\right)^T & 0 \end{bmatrix}, \tag{2.182}$$

respectively, and so

$$D_\mu G = \left[D_\mu \Phi \quad \partial_\mu \phi\right], \tag{2.183}$$

where

$$D_\mu \Phi = \partial_\mu \Phi - \Phi \Phi^T \partial_\mu \Phi. \tag{2.184}$$

Substituting V_μ from (2.182) and $D_\mu G$ from (2.184) into the first and second equality of the Lagrangian density (2.178), respectively, yield the explicit expressions:

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi^{\hat{A}} \partial_\mu \phi^{\hat{A}} = \mathrm{Tr}\, (D^\mu \Phi)^T D_\mu \Phi. \tag{2.185}$$

The field equations of the first and second Lagrangian in (2.185), taken together with the constraints (2.100) are

$$\Box \phi^{\hat{A}} + \partial^\mu \phi^{\hat{B}} \partial_\mu \phi^{\hat{B}} \phi^{\hat{A}} = 0, \tag{2.186}$$

and

$$D^\mu D_\mu \Phi + \Phi \, (D^\mu \Phi)^T D_\mu \Phi = 0, \tag{2.187}$$

respectively.

We note that we have obtained two different formulations of the S^{N-1} model: the first one involves the $O(N)$ vector field ϕ which is the usual one, whereas the second one involves the matrix field Φ and for $N > 3$ has a non-Abelian $O(N-1)$ gauge symmetry. Classically, this simply amounts to different parametrisations of S^{N-1} σ model [34].

Chapter 3

Grassmannian σ Model and Their 2D Solutions

In our application of harmonic map theory to construct solutions of static $SU(N)$ Skyrme models in Chapter 4, and Yang-Mills theories in Chapter 5, we will be using harmonic maps from S^2 into complex projective space $CP^{(N-1)}$ and Grassmann manifold, $Gr(2, N)$. Geometrically, the Grassmann manifold $Gr(n, N)$, $1 \leq n < N$ is the manifold of n-dimensional planes passing through the origin in the N-dimensional complex space C^N, for which $CP^{(N-1)} = Gr(1, N)$.

In the first section of this chapter we shall introduce the $Gr(n, N)$ σ model and then discuss its coset formulation which we use to construct the corresponding local $U(n)$ gauge invariant Lagrangian density. Next, after showing that $Gr(n, N)$ is a symmetric space, we proceed to reformulate the $Gr(n, N)$ σ model in terms of rank-n projector matrices. Using this projector formalism we discuss the method of constructing *full* harmonic maps: $R^2 \to Gr(n, N)$, starting from a given *instanton* solution which was originally introduced by A. Din and W. J. Zakrzewski [16]. We then discuss the Veronese sequence as an example of the full harmonic maps: $S^2 \to CP^{N-1}$, which play an important role in the construction of exact spherically symmetric solutions of the $SU(N)$ Skyrme models and Yang-Mills theories.

In the final section, after discussing topological meanings of the 2D solutions, we discuss the Hobart-Derrick scale stability argument for the existence of soliton-like solutions in higher dimensions.

3.1 Grassmannian σ Model

Let us start by recalling some basic definitions of the model [15]. The complex Grassmannian or $Gr(n,N)$ σ model, consists of $(N \times n)$, $1 \leq n < N$, complex matrix fields $Z = (Z^{\hat{A}a}(x))$ with $\hat{A} = 1,\ldots,N$; $a = 1,\ldots,n$ (the analogue of the vector $\phi^{\hat{A}}$ of the S^{N-1} σ model) which satisfy the constraint:

$$(Z^\dagger)^{a\hat{A}} Z^{\hat{A}b} = \delta^{ab}, \qquad \text{or} \qquad Z^\dagger Z = I_n, \tag{3.1}$$

where † denotes Hermitian conjugation, and I_n the $(n \times n)$ unit matrix.

The Lagrangian density of the model is required to be invariant under the global unitary transformations $G \in U(N)$ acting from the left on Z,

$$Z \to Z' = GZ, \qquad G \in U(N), \tag{3.2}$$

and under the local $H(x) \in U(n)$ transformations from the right,

$$Z \to Z' = ZH(x), \tag{3.3}$$

and is given by

$$\mathcal{L} = (D^\mu Z)^\dagger (D_\mu Z), \tag{3.4}$$

where

$$D_\mu Z = \partial_\mu Z - ZZ^\dagger \partial_\mu Z. \tag{3.5}$$

Note that $ZZ^\dagger$ in (3.5) is an $N \times N$ matrix, *i.e.* $(ZZ^\dagger)^{\hat{A}\hat{B}} = Z^{\hat{A}a}(Z^\dagger)^{a\hat{B}}$. Taking into account the constraint (3.1), the Euler-Lagrange equation for the Lagrangian density (3.4) is

$$D^\mu D_\mu Z + Z\,(D^\mu Z)^\dagger D_\mu Z = 0. \tag{3.6}$$

Note that, the special case $n = 1$ corresponds to CP^{N-1} σ model.

3.1.1 Coset Formulation

In this subsection we discuss group and geometrical formulations of the $Gr(n,N)$ σ model, by formulating its target space as coset space. This we do by identifying the $Gr(n,N)$ σ model matrix fields Z in terms of $U(N)$-valued field, as we did for the S^{N-1} model in chapter 2.

First, we note that if we let $\mathbf{w}^a$ $(a = 1, \dots, n)$ to be a set of N-components orthonormal vectors in C^N, *i.e.* $\mathbf{w}^a.\mathbf{w}^b = \left((w^\dagger)^{a\hat{A}} w^{\hat{A}b}\right) = \delta^{ab}$, $(\hat{A}, = 1, \dots, N)$, then the matrix field Z can be represented as

$$Z = (\mathbf{w}^1, \dots, \mathbf{w}^n) = (Z^{\hat{A}a}), \qquad n < N. \tag{3.7}$$

Thus, with this representation, Z defines an orthonormal *n-frame* in C^N. Now, let us choose $Z^{\hat{A}a}$ as the last n-columns of the matrices $G^{\hat{A}\hat{B}}$ in the fundamental representation of $U(N)$, *i.e.*

$$Z^{\hat{A}a} = U^{\hat{A}(N-n+a)}, \tag{3.8}$$

which partitioning $G^{\hat{A}}_{\hat{B}}$ into

$$G = \begin{bmatrix} Y^{\hat{A}B} & Z^{\hat{A}b} \end{bmatrix}, \qquad B = 1, \dots, (N-n), \tag{3.9}$$

where Y is an $(N \times (N-n))$ matrix and Z an $(N \times n)$ matrix. Since

$$G^\dagger G = \begin{bmatrix} Y^\dagger Y & Y^\dagger Z \\ & \\ Z^\dagger Y & Z^\dagger Z \end{bmatrix}, \qquad GG^\dagger = \begin{bmatrix} YY^\dagger + ZZ^\dagger \end{bmatrix}, \tag{3.10}$$

the unitarity condition, $G^\dagger G = GG^\dagger = I_N$ implies that

$$Y^\dagger Y = I_{N-n}, \qquad Y^\dagger Z = 0, \qquad Z^\dagger Z = I_n, \qquad YY^\dagger + ZZ^\dagger = I_N. \tag{3.11}$$

Thus Z satisfies the constraint (3.1) as required.

Under the right action of $H \in U(N)$ on G,

$$Z^{\hat{A}a} \to Z'^{\hat{A}a} = Z^{\hat{A}b} H^{(N-n+b)(N-n+a)} + Y^{\hat{A}B} H^{B(N-n+a)}, \tag{3.12}$$

and we see that $Z^{\hat{A}a}$ is left invariant if

$$H^{\hat{A}(N-n+a)} = \delta^{\hat{A}(N-n+a)} \qquad \text{or} \qquad H = \begin{bmatrix} \times & \dots & \times & \hat{0}_n^\dagger \\ \vdots & \times & \vdots & \vdots \\ \times & \dots & \times & \hat{0}_n^\dagger \\ \hat{0}_n & \dots & \hat{0}_n & I_n \end{bmatrix}, \tag{3.13}$$

where $\hat{0}_n$ is a zero column vector with n-components. With this definition, H defines an embedding of the group $U(N-n)$ in $U(N)$. Thus, $U(N-n)$ is the isotropy group of the matrix fields Z, and so the corresponding target space is the coset space

$$V(n,\, N) \;=\; U(N)/U(N-n). \tag{3.14}$$

This coset space is known as the *Stiefel manifold* of orthonormal n-frames in C^N, *i.e.*

$$V(n,\, N) \;=\; \left\{Z \in M(N,n) | Z^\dagger Z \;=\; I_n\right\}, \tag{3.15}$$

where $M(N,n)$ is the vector space of $(N \times n)$-complex matrices. Since the condition $Z^\dagger Z \;=\; I_n$ gives n^2 real equations so the real dimension of $V(n,\, N)$ is,

$$dim[V(n,\, N)] \;=\; 2Nn \;-\; n^2 \;=\; 2n(N-n) + n^2, \tag{3.16}$$

which coincides with the dimension of the coset $U(N)/U(N-n)$ as it should be. Geometrically, "a point" of $V(n,\, N)$ is represented by an orthonormal n-frame.

The Grassmann manifold $Gr(n,\, N)$, as we have mentioned previously, is the manifold of n-dimensional planes L^n passing through the origin in C^N. Thus, geometrically, a particular plane L^n represents "a point" of $Gr(n,\, N)$, and so to construct the coset space representation of $Gr(n,\, N)$ we need to find the isotropy group of L^n. We observe that if we let $(\mathbf{w}^1, \ldots, \mathbf{w}^n)$ being a basis for L^n then we can associate the exterior vector:

$$\begin{aligned} \zeta \;&=\; \mathbf{w}^1 \wedge \ldots \wedge \mathbf{w}^n \\ &=\; \frac{1}{n!}\zeta^{[\hat{a}_1 \ldots \hat{a}_n]}\mathbf{e}_{\hat{a}_1} \wedge \ldots \wedge \mathbf{e}_{\hat{a}_n}, \end{aligned} \tag{3.17}$$

where $(\mathbf{e}_{\hat{a}})$ are orthonormal vector basis in C^N and where $\zeta^{[\hat{a}_1 \ldots \hat{a}_n]}$ is antisymmetric in its indices. The set of $\binom{N}{n}$ complex numbers $\zeta^{(\hat{a}_1 \ldots \hat{a}_n)}$, $1 \leq \hat{a}_1 < \ldots \hat{a}_n \leq N$, form the so called *Plücker-Grassmann coordinates* of L^n in $Gr(n, N)$. From the property of exterior product, $\zeta \wedge \zeta \;=\; 0$, the coordinates $\zeta^{(\hat{a}_1 \ldots \hat{a}_n)}$ satisfy the *Plücker relation:*

$$\epsilon_{\hat{a}_1 \ldots \hat{a}_n \hat{b}_1 \ldots \hat{b}_n}\zeta^{[\hat{a}_1 \ldots \hat{a}_n]}\zeta^{[\hat{b}_1 \ldots \hat{b}_n]} \;=\; 0, \tag{3.18}$$

where $\epsilon_{\hat{a}_1 \ldots \hat{a}_n \hat{b}_1 \ldots \hat{b}_n}$ is given by (2.75). Thus $\zeta^{(\hat{a}_1 \ldots \hat{a}_n)}$ could be considered as a generalisation of homogeneous coordinates in projective space $CP^{\binom{N}{n}-1}$ into which $Gr(n,\, N)$ embeds.

For example, let us consider $Gr(2,4)$. Here we have

$$\mathbf{w^1} = p^{\hat{a}}\mathbf{e}_{\hat{a}}, \qquad \mathbf{w^2} = q^{\hat{a}}\mathbf{e}_{\hat{a}}, \qquad \hat{a} = 1, \ldots, 4, \tag{3.19}$$

and so

$$\zeta = \mathbf{w^1} \wedge \mathbf{w^2} = \frac{1}{2!}\zeta^{[\hat{a}\hat{b}]}\mathbf{e}_{\hat{a}} \wedge \mathbf{e}_{\hat{b}}, \tag{3.20}$$

where

$$\zeta^{[\hat{a}\hat{b}]} = p^{\hat{a}}q^{\hat{b}} - q^{\hat{a}}p^{\hat{b}}. \tag{3.21}$$

By considering $p^{\hat{a}}$ and $q^{\hat{b}}$ as homogeneous coordinates in CP^3 then $\zeta^{(\hat{a}\hat{b})}$ is uniquely determined by the line $\overrightarrow{w^1w^2}$ through the points $\overrightarrow{w^1}$ and $\overrightarrow{w^2}$ (not on the chosen points). The six complex numbers

$$(\zeta^{12}, \zeta^{13}, \zeta^{14}, \zeta^{23}, \zeta^{24}, \zeta^{34}), \tag{3.22}$$

are precisely the Plücker-Grassmann coordinates in $Gr(2,4)$ which can be considered as homogeneous coordinates in CP^5. The corresponding Plücker relation (3.18) is

$$\zeta^{12}\zeta^{34} - \zeta^{13}\zeta^{24} + \zeta^{14}\zeta^{23} = 0. \tag{3.23}$$

By the identification

$$\begin{aligned} z_1 &= \zeta^{12} + \zeta^{34}, & z_3 &= \zeta^{24} + \zeta^{13}, & z_5 &= \zeta^{14} - \zeta^{23}, \\ z_2 &= \zeta^{12} - \zeta^{34}, & z_4 &= i(\zeta^{24} - \zeta^{13}), & z_6 &= i(\zeta^{14} + \zeta^{23}). \end{aligned} \tag{3.24}$$

equation (3.23) reads

$$z_1^2 = z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2, \tag{3.25}$$

which defines a complex *quadric* Q_4 in CP^5. Thus we see that the space of lines in CP^3 may be thought of as a quadric hypersurface Q_4 in CP^5. This is known as *Klein representation* of lines in CP^3 which is the base of Penrose's twistor formulation [28]. If we restrict to the real projective subspace $RP^5 \subset CP^5$ then we obtain the identification

$$S^4 = Q_4 \bigcap \left\{ z \in CP^5 | z = \bar{z} \right\}, \tag{3.26}$$

and for $z \in S^4$ automatically $z_1 \neq 0$.

If the basis $\mathbf{w}^a$ of L^n transforms by a matrix A, $\mathbf{w}'^a = \mathbf{w}'^b A^a_b$ then from the property of the exterior product, $\zeta' = det(A)\zeta$, which describes the same plane L^n up to an orientation. Thus, under the right action of

$$H = U(N-n) \times U(n) = \begin{bmatrix} U(N-n) & 0 \\ 0 & U(n) \end{bmatrix}, \tag{3.27}$$

ζ transforms into $det(U(n))\zeta$, and so L^n is mapped onto itself (up to the complex number $det(U(n))$. Thus, H is the isotropy group of L^n and so we have

$$Gr(n,\, N) \;=\; U(N)/(U(N-n) \times U(n)). \tag{3.28}$$

Therefore, the real dimension of $Gr(n, N)$ is

$$dim\,[Gr(n,\, N)] \;=\; dim\,[U(N)] \;-\; \Big(dim\,[U(N-n)] \;+\; dim\,[U(n)]\Big) \;=\; 2n(N-n). \tag{3.29}$$

Note that this dimension is n^2 less than dim$[V(n, N)]$, as two frames Z and Z' in $Gr(n, N)$ are considered equivalent if they are related by an $U(n)$ transformation acting from the right, *i.e.* $Z \sim Z'$ if $Z' = ZU(n)$ where $U(n)$ has n^2 parameters. If the orientation of C^N is taken into account, we obtain the manifold of oriented n-planes given by

$$\tilde{Gr}(n,\, N) \;=\; SU(N)/S(U(N-n) \times U(n)), \tag{3.30}$$

which has the same dimension as $Gr(n, N)$.

The corresponding Lie algebra decomposition of $\mathcal{G} = \mathcal{U}(N)$ is

$$\mathcal{U}(N) \;=\; \mathcal{H} \oplus \mathcal{K}, \tag{3.31}$$

where

$$\mathcal{H} \;=\; \begin{bmatrix} \mathcal{U}(N-n) & 0 \\ 0 & \mathcal{U}(n) \end{bmatrix}, \qquad \mathcal{K} \;=\; \begin{bmatrix} 0 & X \\ -X^\dagger & 0 \end{bmatrix}, \tag{3.32}$$

and where $X \in M(N-n,\, n)$. Now, as here $[\mathcal{K}, \mathcal{K}] = \mathcal{H}$, so according to the definition in section 2.5.2, $Gr(n, N)$ is a *symmetric space.*

For the Stiefel manifold, $V(n, N)$, we have

$$\mathcal{H}' \;=\; \begin{bmatrix} \mathcal{U}(N-n) & 0 \\ 0 & I_n \end{bmatrix}, \tag{3.33}$$

and $\mathcal{K}$ as in (3.32). As here, for $2 \leq n < N$: $[\mathcal{K}, \mathcal{K}] = \mathcal{H}' \oplus \mathcal{K}$, so the corresponding Stiefel manifold $V(n,\ N)$ is *not* a symmetric space. However, $V(1, N)$ is a symmetric space, as $V(1, N) = S^{2N-1}$.

In fact, the coset definition (3.30) for $Gr(n,\ N)$, in relation to $V(n,\ N)$ in (3.14), shows that

$$Gr(n,\ N) = V(n,\ N)/U(n). \tag{3.34}$$

This suggests the principle bundle picture: $(V(n,\ N),\ Gr(r, N),\ U(n),\ p)$, whose base space is the Grassmann manifold $Gr(n,\ N)$, and the Stiefel manifold $V(n,\ N)$ is the total space where the fibre $p^{-1}(Gr(n,\ N)) = U(n)$ acts. Here, the projection $p : V(n,\ N) \to Gr(r, N)$ takes an n-frame to the n-plane it spans. In particular, $CP^{N-1} = V(1, N)/U(1) \simeq S^{2N-1}/S^1$, which is an immediate extension of the real case: $RP^{N-1} = S^{N-1}/S^0$ that we have considered in section 2.2.3.

3.1.2 $U(n)$ Gauge Invariant Lagrangian Density

To construct the invariant Lagrangian density of the model, let us choose the G-invariant fields as given by the coset representative (3.9). Then, from the corresponding Maurer-Cartan one-forms $G^{-1}dG$, we find that the pull-back onto the base manifold $\mathcal{M}_0$ of the $H = (U(N-n) \times U(n))$ connection part Ω_μ and the orthonormal basis part V_μ, as described in section 2.5.4, are given by

$$\Omega_\mu = \begin{bmatrix} Y^\dagger \partial_\mu Y & 0 \\ 0 & Z^\dagger \partial_\mu Z \end{bmatrix}, \quad V_\mu = \begin{bmatrix} 0 & Y^\dagger \partial_\mu Z \\ -\left(Y^\dagger \partial_\mu Z\right)^T & 0 \end{bmatrix}, \tag{3.35}$$

respectively. Thus the covariant derivative on a G-valued field, according to (2.179), is given by

$$D_\mu G = [D_\mu Y \quad D_\mu Z], \tag{3.36}$$

where

$$D_\mu Y = \partial_\mu Y - Y Y^\dagger \partial_\mu Y, \qquad D_\mu Z = \partial_\mu Z - Z Z^\dagger \partial_\mu Z, \tag{3.37}$$

We see that, $D_\mu Z$ in (3.37) coincides with (3.5) and that

$$A_\mu = Z^\dagger \partial_\mu Z, \qquad B_\mu = Y^\dagger \partial_\mu Y, \tag{3.38}$$

play the role as gauge field potentials or connections for the local $U(n)$ and $U(N-n)$ gauge symmetry, respectively.

As $Y^\dagger Z = 0$, so the submatrix $Y^\dagger \partial_\mu Z$ of V_μ in (3.35) is in fact $Y^\dagger D_\mu Z$. Bearing this in mind, then by substituting V_μ and Ω_μ from (3.35) into the Lagrangian density (2.178), we obtain

$$\mathcal{L} = \mathrm{Tr}\,(D^\mu Y)^\dagger\,(D_\mu Y) = \mathrm{Tr}\,(D^\mu Z)^\dagger\,D_\mu Z, \tag{3.39}$$

and the corresponding field equations are

$$D^\mu D_\mu Y + Y\,(D^\mu Y)^\dagger\,D_\mu Y = 0, \tag{3.40}$$

and

$$D^\mu D_\mu Z + Z\,(D^\mu Z)^\dagger\,D_\mu Z = 0, \tag{3.41}$$

respectively.

We see that the formulation in terms of Z coincides with those presented in the beginning of this section. Note that, the number of independent fields of Z, due to the constraint (3.1) is $n(2N-n)$. However, due to the $U(n)$ gauge invariance of the Lagrangian density (3.39), this number is reduced further to become $2n(N-n)$ which coincides with the real dimension of the Grassmann manifold $Gr(n,\,N)$.

3.1.3 Projection Matrix Formulation

In this subsection, we discuss group and geometrical formulations of the Grassmannian σ model target space $\mathcal{M}$ as a symmetric space, and reformulating it in terms of rank-n matrix projector $P \in C^{(N\times N)}$ [34]. We shall see that this formalism enables us to derive a concise equivalent invariant Lagrangian density and offers us a simple but much structured approach to construct solutions of the models.

To start our discussion, we need to introduce additional symmetry group σ called *involutive automorphism* of G [36, 40]. This term simply means that

$$\sigma^2 = Id_G, \qquad \sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2), \tag{3.42}$$

for $g_1,\, g_2 \in G$ where Id_G is the identity element of G. The corresponding involutive automorphism of the Lie algebra $\mathcal{G}$ is

$$\dot{\sigma}^2 = Id_{\mathcal{G}}, \qquad \dot{\sigma}\left([\mathcal{I}_1,\,\mathcal{I}_2]\right) = \left[\dot{\sigma}(\mathcal{I}_1),\,\dot{\sigma}(\mathcal{I}_2)\right], \tag{3.43}$$

for $\mathcal{I}_1, \mathcal{I}_2 \in \mathcal{G}$.

We recall that if $\dot{\sigma}$ is an automorphism of order N, *i.e.*

$$\dot{\sigma}^N = Id_{\mathcal{G}}, \tag{3.44}$$

then as a vector space the Lie algebra $\mathcal{G}$ splits into the direct sum

$$\mathcal{G} = \mathcal{G}_{(0)} \oplus \mathcal{G}_{(1)} \oplus \ldots \oplus \mathcal{G}_{(N)}, \tag{3.45}$$

of eigenspaces of $\dot{\sigma}$, with

$$\mathcal{G}_{(\alpha)} = \left\{ \mathcal{I} \in \mathcal{G} \middle| \dot{\sigma}(\mathcal{I}) = e^{2\pi i\alpha/N}\mathcal{I} \right\}. \tag{3.46}$$

The automorphism property of $\dot{\sigma}$ implies that

$$\left[\mathcal{G}_{(\alpha)}, \mathcal{G}_{(\beta)}\right] \subseteq \mathcal{G}_{(\alpha+\beta \; mod \, N)}. \tag{3.47}$$

Thus we see that the eigenspace $\mathcal{G}_0$ with eigenvalue $+1$ is a subalgebra.

As involutive automorphism is an automorphism of order $N = 2$, so the Lie algebra $\mathcal{G}$ decomposes into

$$\mathcal{G} = \mathcal{G}_{(0)} \oplus \mathcal{G}_{(1)}, \tag{3.48}$$

and from the decomposition (3.31), we can identify $\mathcal{G}_{(0)} = \mathcal{H}$ and $\mathcal{G}_{(1)} = \mathcal{K}$. Hence, $\dot{\sigma}(\mathcal{H}) = \mathcal{H}$ and $\dot{\sigma}(\mathcal{K}) = e^{i\pi}\mathcal{K} = -\mathcal{K}$.

Geometrically, σ defines the symmetry σ_p at a distinguished point $p \in \mathcal{M}$ which is a diffeomorphism of a neighbourhood of p onto itself. As $\mathcal{K}$ can be considered as the tangent space of $\mathcal{M}$, σ_p maps $\text{Exp}_p(T)$ into $\text{Exp}_p(-T)$ where $T \in T_p(\mathcal{M})$, the tangent plane at p. Hence σ_p^2, which is an identitity transformation of G, is an involutive diffeomorphism. If $(\theta^1, \ldots, \theta^{dim(M)})$ is the local coordinates of p then σ_p sends $(\theta^1, \ldots, \theta^{dim(M)})$ to $(-\theta^1, \ldots, -\theta^{dim(M)})$ and so the differential $\dot{\sigma}_p$ of σ_p at p is $-Id_p$ where Id_p is the identity transformation at $T_p(M)$.

For $Gr(n, N)$, the involutive automorphism σ of $U(N)$ is defined as

$$\sigma(g) = \eta_n g \eta_n^{-1}, \qquad \eta_n = \begin{bmatrix} I_{N-n} & 0 \\ & \\ 0 & -I_n \end{bmatrix}, \tag{3.49}$$

for $g \in U(N)$. Thus,

$$\sigma(H) = \eta_n H \eta_n^{-1} = H, \tag{3.50}$$

which means that $H \in U(n)$ are fixed points of $Gr(n, N)$ involutive automorphism σ. For the coresponding Lie algebra decomposition, $\mathcal{H}$ and $\mathcal{K}$, we have $\dot{\sigma}(\mathcal{H}) = \mathcal{H}$ and $\dot{\sigma}(\mathcal{K}) = -\mathcal{K}$ as required.

With this brief introduction, we are now ready to give the group theoretical definition of the symmetric space $M = G/H$. A *symmetric space* is defined by the triple (G, H, σ) where G is a connected Lie group, and H a closed subgroup of G such that [34, 41, 42]

$$(G_\sigma)_o \subset H \subset G_\sigma, \tag{3.51}$$

with G_σ be the set of fixed points of σ, *i.e.* $G_\sigma = \{g \in G, \sigma(g) = g\}$ and $(G_\sigma)_o$ its identity component. For $Gr(n, N)$, we have $(G_\sigma)_o = H = G_\sigma$ which implies that $Gr(n, N)$ is a *symmetric space.*

Let $\tilde{\Phi} : G \to G/H$ be the natural projection which assigns to an element $g \in G$ the coset gH. Then, it is a well known result in differential geometry that the map

$$\begin{aligned} \tilde{\Phi}^{-1} : \quad G/H &\to G \\ gH &\to \tilde{\Phi}^{-1}(gH) = \sigma(g)g^{-1}, \end{aligned} \tag{3.52}$$

is a diffeomorphism of G/H onto the closed *totally geodesic* submanifold

$$\mathcal{M}_\sigma = \left\{g \in G | \sigma(g)g = Id_G\right\}, \tag{3.53}$$

of G. This is called *Cartan immersion* of G/H in G. We recall that a submanifold $\mathcal{N}'$ of $\mathcal{N}$ is called totally geodesic if any geodesic $\gamma(s)$ in $\mathcal{N}'$ is a geodesic in $\mathcal{N}$. Thus, according to section 2.1, the harmonic map $f : \mathcal{N}' \to \mathcal{N}$ is a totally geodesic map.

As H is a fixed element of σ, *i.e.* $\sigma(H) = H$, so the G-valued field $\tilde{\Phi}^{-1}$ is gauge invariant and satisfies the constraint

$$\tilde{\Phi}^{-1}\sigma(\tilde{\Phi}^{-1}) = I_N. \tag{3.54}$$

Let us return to $Gr(n, N)$ σ model and define the field

$$\Phi = \eta_n \tilde{\Phi}^{-1} = g\eta_n^{-1}g^{-1}. \tag{3.55}$$

This field has the properties:

$$\Phi^2 = I_N, \qquad \Phi^\dagger = \Phi. \tag{3.56}$$

Introducing the partition $g = (Y, Z) \in U(N)$, as in (3.9), then, since $g\eta_k^{-1} = (Y, -Z)$,

$$\Phi = \begin{bmatrix} Y & -Z \end{bmatrix} \begin{bmatrix} Y^\dagger \\ Z^\dagger \end{bmatrix} = \left\{ [Y \quad Z] - [0 \quad 2Z] \right\} \begin{bmatrix} Y^\dagger & Z^\dagger \end{bmatrix}. \tag{3.57}$$

Using the third formula in (3.11) this reduces to

$$\Phi = (I - 2P), \tag{3.58}$$

where

$$P = ZZ^\dagger, \tag{3.59}$$

is an $(N \times N)$ matrix *projector* field which projects vectors in C^N into the n-dimensional plane L^n spanned by the column vectors $\mathbf{w}^a$, $(a = 1, \ldots, n)$, of Z as defined in (3.7). In terms of these colum vectors,

$$P = \sum_{a=1}^{n} \mathbf{w}^a \mathbf{w}^{a\dagger}, \tag{3.60}$$

which shows that P has maximal rank-n.

Note that, the projector P satisfies the properties,

$$P^2 = P = P^\dagger, \qquad \mathrm{Tr}P = n, \tag{3.61}$$

where the trace property means that P is a rank-n projector, which is consistent with the rank definition in (3.60).

Using the decomposition (3.9) then for any $U \in U(N)$, we have

$$P = UQU^\dagger, \qquad Q = \begin{bmatrix} 0_{N-n} & 0_n \\ & \\ 0_n^\dagger & I_n \end{bmatrix}. \tag{3.62}$$

Furthermore, under the action of the subgroup $H = (U(N - n) \times U(n))$ on Z from the right: $P' = Z'Z'^\dagger = ZU(n)U(n)^\dagger Z^\dagger$. As $U(n)U(n)^\dagger = I_n$, we see that

$H = (U(N-n) \times U(n))$ is the isotropy group of the projector P. In this way, the Grassmann manifold $Gr(n, N)$ is also defined as

$$Gr(n,N) = \left\{P \in M(N,N) | P^2 = P = P^\dagger,\ \mathrm{Tr}P = n\right\}. \tag{3.63}$$

Since Φ is H-gauge invariant, and so is with P, the invariant Lagrangian density for the $Gr(n, N)$ σ model is simply given by [15]

$$\mathcal{L} = \frac{1}{8}\mathrm{Tr}\partial^\mu\Phi\partial_\mu\Phi = \frac{1}{2}\mathrm{Tr}\partial^\mu P\partial_\mu P. \tag{3.64}$$

which coincides with (3.39). Taking into account the constraint (3.61), the corresponding Euler-Lagrange equations for the projector P is

$$[P, \Box P] = 0. \tag{3.65}$$

As (3.56) constraining $\Phi \in U(N)$, we conclude that the solution space of the $Gr(n, N)$ σ model is a subspace of the $U(N)$-chiral models solution space.

3.2 Harmonic Maps $R^2 \to Gr(n, N)$

In this section we shall discuss the construction of solutions of the $Gr(n, N)$ σ-model field equations (3.65) in 2-dimensional Euclidean space R^2 or the complex plane C which we will be using in chapters 4 and 5. Later we compactify R^2 by including points at ∞ to obtain the Riemann sphere $S^2 = R^2 \cup \{\infty\}$ and consider the harmonic maps: $S^2 \to Gr(n, N)$. In the following, by a $Gr(n, N)$ σ model we always mean that the base manifold is a 2-dimensional Euclidean space.

In searching for these solutions, we shall use the complex coordinates $(\xi, \bar{\xi})$ where $\xi \in \mathbb{C}$, in terms of which the $Gr(n, N)$ σ model field equations (3.65) becomes

$$\left[P, \partial_\xi\partial_{\bar{\xi}}P\right] = 0. \tag{3.66}$$

This can also be written in a conservation law equation form

$$\partial_\xi\left[P, \partial_{\bar{\xi}}P\right] + \partial_{\bar{\xi}}\left[P, \partial_\xi P\right] = 0. \tag{3.67}$$

3.2.1 Instanton Solutions

Using the projector property, $P^2 = P$, we see that two special classes of solutions for equation (3.67) are given by the following simpler equations:

$$P\partial_\xi P = 0, \qquad \text{and} \qquad P\partial_{\bar{\xi}} P = 0, \tag{3.68}$$

which are called *selfdual* and *anti-selfdual* equations, respectively.

Since both equations in (3.68) have similar structure, in the following we discuss the construction of the solutions for self-dual equation only. For this purpose, let us consider an *un*-normalised $(N \times n)$ matrix field $M = M(\xi, \bar{\xi})$ for which the $(n \times n)$ matrix $|M|^2 = M^\dagger M$ is assumed to be non-singular. As $|M|^2$ is hermitian, its eigenvalues are real and so there exist an unitary matrix U such that

$$|M|^2 = U^\dagger \Lambda^2 U, \tag{3.69}$$

where Λ^2 is a diagonal matrix with eigenvalues $(\lambda_1^2, \ldots, \lambda_N^2)$. Let Λ be the square root matrix of Λ^2, *i.e.* $(\Lambda^2)^{\frac{1}{2}} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_N)$. Then, as $|M|^2 = \left(U^\dagger \Lambda U\right)\left(U^\dagger \Lambda U\right)$, so

$$|M| = \left(|M|^2\right)^{\frac{1}{2}} = U^\dagger \Lambda U. \tag{3.70}$$

Hence, in terms of M, the $Gr(n, N)$ σ model matrix field Z is given by

$$Z = M|M|^{-1}, \tag{3.71}$$

and so the $(N \times N)$ matrix projector field P is

$$P = ZZ^\dagger = M|M|^{-2}M^\dagger. \tag{3.72}$$

As, $(I - P)M = 0$, the first equation in (3.68) is equivalent to

$$M|M|^{-2}\left(\partial_{\bar{\xi}} M\right)^\dagger [I - P] = 0. \tag{3.73}$$

We see that the obvious solution is given by the matrix field M which satisfies

$$\partial_{\bar{\xi}} M = 0, \tag{3.74}$$

i.e. $M = M_0(\xi)$ is a *holomorphic* matrix field. Analogously, the solution of the anti-selfdual equation is given by an *anti-holomorphic* field $M = \bar{M}_0(\bar{\xi})$. The first class of solution is called *instanton* solutions while the second, *anti-instanton.*

3.2.2 Full Solutions

In this subsection, we proceed to discuss the method of generating more general exact solutions of the $2D$ $Gr(n, N)$ σ model field equations (3.66) starting from an instanton solution. The method is very powerful and it was originally introduced by A. Din and W. J. Zakrzewski [15, 16]. From now on, we abandon summation convention on repeated lower case *latin* indices.

Let $M_k = M_k(\xi, \bar{\xi})$, $k = 0, 1, ..., \lambda$ where $\lambda \leq (N-1)$, be a set of $(\lambda + 1)$ mutually orthogonal $(N \times n)$ matrices $(n < N)$, *i.e.*

$$M_k^\dagger M_l = |M_k|^2 \delta_{kl}, \tag{3.75}$$

where

$$|M_k|^2 = M_k^\dagger M_k, \tag{3.76}$$

are $(n \times n)$ nonsingular matrices. Then the corresponding projector $P_k(n)$ onto each matrix M_k is given by

$$P_k(n) = M_k |M_k|^{-2} M_k^\dagger. \tag{3.77}$$

Clearly, $\mathrm{Tr}\, P_k(n) = \mathrm{Tr}\, I_n = n$ where I_n is an $(n \times n)$ unit matrix, which means that each projector $P_k(n)$ has rank-n. From (3.77), we see that the projectors $P_k(n)$ are mutually orthogonal, *i.e.* $P_k(n) P_l(n) = \delta_{kl} P_l(n)$, and are Hermitian, *i.e.* $P_k(n)^\dagger = P_k(n)$, as by definition M_k are mutually orthogonal and, by construction (3.76), $|M_k|^2$ are Hermitian.

In the following we want to present a generalised harmonic map ansatz. To do this we use a sequence of mutually orthogonal matrices $(M_0, M_1, ..., M_\lambda)$ obtained from a sequence of holomorphic (analytic) matrices $(M, \partial_\xi M, ..., \partial_\xi^\lambda M)$, $\partial_{\bar{\xi}} M = 0$, via the Gram-Schmidt orthogonalisation process.

We can do this using the operator P_+ which is defined by its action on any matrix $M \in C^{N \times n}$ as [15, 16]

$$P_+ M = \partial_\xi M - M |M|^{-2} (M^\dagger \partial_\xi M). \tag{3.78}$$

Then we have

$$M_0 = M, \quad M_1 = P_+ M, \quad ..., \quad M_k = P_+^k M = P_+(P_+^{k-1} M), \quad ..., \quad M_\lambda = P_+^\lambda M,$$

or simply

$$M_0 = M, \quad M_k = (I - P_{k-1})\partial_\xi M_{k-1}, \quad k = 1, \dots, \lambda, \tag{3.79}$$

where P_{k-1} is the projector (3.77).

An equivalent formula for the sequence M_k, in terms of the projectors P_k, is given by

$$M_k = (I - P_0 - \dots - P_{k-1})\, \partial_\xi^k M_0. \tag{3.80}$$

With either one of these constructions the following properties of the matrices M_k hold when M_0 is holomorphic [15]:

$$\partial_{\bar\xi} M_k = -M_{k-1}|M_{k-1}|^{-2}|M_k|^2, \tag{3.81}$$

$$\partial_\xi (M_k|M_k|^{-2}) = M_{k+1}|M_k|^{-2}, \tag{3.82}$$

as derived in appendix B.

For rank-2 projectors, the matrix M_k is given by

$$M_k = (M_{k1}, M_{k2}), \tag{3.83}$$

where M_{k1} and M_{k2} are two N-component column vector fields and $|M_k|^2$ is a (2×2) matrix. Using this column vector notation the entries of the projector $P_k(2)$ are given by

$$\begin{aligned}(P_k(2))_{ab} = \frac{1}{D_k}\Big[& |M_k|^2_{22}(M_{k1})_a(\bar{M}_{k1})_b + |M_k|^2_{11}(M_{k2})_a(\bar{M}_{k2})_b \\ & - |M_k|^2_{12}(M_{k1})_a(\bar{M}_{k2})_b - |M_k|^2_{21}(M_{k2})_a(\bar{M}_{k1})_b\Big],\end{aligned} \tag{3.84}$$

where $a, b = 1, 2, \dots, N$, $(\bar{M}_{kj})_a$, $j = 1, 2$ is the complex conjugate of $(M_{kj})_a$, and where

$$D_k = Det\, |M_k|^2. \tag{3.85}$$

Clearly, $\mathrm{Tr} P_k(2) = 2$. Furthermore, if we let

$$P_{kj}(1) = M_{kj}|M_{kj}|^{-2}M_{kj}^\dagger, \tag{3.86}$$

then from (3.84) it follows that

$$P_k(2) = P_{k1}(1) + \widetilde{P}_{k2}(1), \tag{3.87}$$

where

$$\widetilde{P}_{k2}(1) = \widetilde{M}_{k2}|\widetilde{M}_{k2}|^{-2}\widetilde{M}_{k2}^{\dagger}, \tag{3.88}$$

with

$$\widetilde{M}_{k2} = [I - P_{k1}(1)]M_{k2}, \tag{3.89}$$

which is orthogonal to M_{k1}, *i.e.* $M_{k1}^{\dagger}\widetilde{M}_{k2} = 0$. Thus, we see from (3.87) that each projector of $Gr(2, N)$ is really a sum of two mutually orthogonal rank-1 projectors.

Note that for some cases this construction does not work. To see this take the case where the initial $(N \times 2)$ matrix M_0 is chosen to be given by,

$$M_0 = (M_{01}, \partial_\xi M_{01}), \tag{3.90}$$

i.e. M_{02} is a derivative of M_{01}. Then it follows from (3.79) that

$$M_1 = (0, M_{12}). \tag{3.91}$$

Thus, in this special case, $|M_1|^2$ is singular, and so the projector $P_1(2)$ does not exist.

Notice also that, for the $CP^{(N-1)}$ case, the projectors P_k, $k = 0, \ldots, (N-1)$ are *complete*, *i.e.*

$$P_0 + P_1 + \ldots + P_{N-1} = I, \tag{3.92}$$

and so according to the construction (3.80)

$$M_N = 0. \tag{3.93}$$

With the projectors P_k that we have constructed above, we have the following result that was originally proved by A. Din and W. J. Zakrzewski [16] using the Z fields formalism.

Theorem 3.1 Each $(N \times N)$ projector $P_k(n) = M_k|M_k|^{-2}M_k^{\dagger}$, $(k = 0, 1, ..., \lambda)$, where $M_k = P_+^k M_0$, with $M_0 = M_0(\xi)$ a holomorphic $(N \times n)$ matrix field, solves the $Gr(n, N)$ σ-model $Gr(n, N)$ field equation (3.66).

Proof: Here we shall apply Sasaki's method [43], where we shall be using the projector formalism in terms of M_k, which proceeds as follows.

From the properties (3.81) and (3.82) we derive

$$\begin{aligned} \partial_\xi P_0 &= M_1 |M_0|^{-2} M_0^\dagger, \\ \partial_\xi P_k &= M_{k+1} |M_k|^{-2} M_k^\dagger - M_k |M_{k-1}|^{-2} M_{k-1}^\dagger \qquad k = 1, ..., \lambda. \end{aligned} \tag{3.94}$$

Define

$$Q_k = \sum_{l=0}^{k-1} P_l, \qquad Q_0 = 0, \tag{3.95}$$

then (3.94) imply that

$$\partial_\xi Q_k = M_k |M_{k-1}|^{-2} M_{k-1}^\dagger. \tag{3.96}$$

Therefore, the orthogonality of M_k's implies that Q_k satisfy the self-dual equation

$$Q_k \partial_\xi Q_k = 0, \tag{3.97}$$

whereas P_k satisfy

$$Q_k \partial_\xi P_k = 0, \qquad P_k \partial_\xi Q_k = \partial_\xi Q_k. \tag{3.98}$$

Thus, the following quantities

$$R_k = Q_k + P_k, \tag{3.99}$$

satisfies the self-dual equation:

$$R_k \partial_\xi R_k = 0. \tag{3.100}$$

Using (3.97) - (3.98), then (3.100) reduces to

$$P_k \partial_\xi P_k + \partial_\xi Q_k = 0. \tag{3.101}$$

Taking the Hermitian conjugate of (3.101) gives

$$\left(\partial_{\bar\xi} P_k\right) P_k + \partial_{\bar\xi} Q_k = 0. \tag{3.102}$$

Now, from the integrability condition: $\partial_\xi \left(\partial_{\bar\xi} Q\right) = \partial_{\bar\xi} \left(\partial_\xi Q\right)$, we obtain from (3.101) and (3.102) that the projectors P_k satisfy

$$\left[P_k, \partial_\xi \partial_{\bar\xi} P_k\right] = 0, \tag{3.103}$$

which completes the proof. □

3.2.3 The Action of Full Solutions and Nonabelian Toda Equations

As each projector $P_k(n)$ solves the $Gr(n, N)$ σ model equations as stated in the previous theorem, each projector $P_k(n)$ describes a specific field configuration having action or "energy"

$$S_k = i\int d\xi\, d\bar{\xi}\, \mathrm{Tr}\left(\partial_{\bar{\xi}}P_k\partial_{\xi}P_k\right), \tag{3.104}$$

where S_0 corresponds to instanton (or anti-instanton) configuration.

Using relation (3.94) for $\partial_{\xi}P_k$ and the fact that the matrices M_k are mutually orthogonal, then in terms of M_k, the action S_k in (3.104) becomes

$$S_k = 2\pi\left(\mathcal{N}_k + \mathcal{N}_{k-1}\right), \tag{3.105}$$

where

$$\mathcal{N}_k = \frac{i}{2\pi}\int d\xi\, d\bar{\xi}\, \mathrm{Tr}\left(|M_{k+1}|^2|M_k|^{-2}\right). \tag{3.106}$$

and where by definition $\mathcal{N}_{-1} = 0$ (as $\partial_{\bar{\xi}}M_0 = 0$).

In the following we derive recurrence relations for $\mathrm{Tr}\left(|M_k|^2|M_{k-1}|^{-2}\right)$ appearing in the integral $\mathcal{N}_k$ of (3.106). To do this we rewrite the definition of M_{k+1} in (3.79) as follows:

$$\partial_{\xi}M_k = M_{k+1} + P_k\partial_{\xi}M_k. \tag{3.107}$$

As $\partial_{\bar{\xi}}M_k$ is given by (3.81) so from the integrability condition: $\partial_{\xi}\partial_{\bar{\xi}}M_k = \partial_{\bar{\xi}}\partial_{\xi}M_k$, we derive the recurrence relations [17]:

$$\partial_{\xi}\left[\left(\partial_{\bar{\xi}}|M_k|^2\right)|M_k|^{-2}\right] = |M_{k+1}|^2|M_k|^{-2} - |M_k|^2|M_{k-1}|^{-2}. \tag{3.108}$$

We note that for the $n = 1$ case, *i.e.* when M_k are a sequence of N-component vector fields, equation (3.108) gives the celebrated *Toda equation* [44]. Thus, for $n \neq 1$ our equation (3.108) could be considered as its generalisation to the *nonabelian* case. In fact, it coincides with the *nonabelian Toda equation* considered in Refs. [45, 46].

Furthermore, taking the trace of (3.108) we obtain

$$\partial_{\xi}\partial_{\bar{\xi}}\left[log\, Det\, |M_k|^2\right] = \mathrm{Tr}(|M_{k+1}|^2|M_k|^{-2}) - \mathrm{Tr}(|M_k|^2|M_{k-1}|^{-2}), \tag{3.109}$$

i.e. our recurrence relations for $\mathrm{Tr}\left(|M_k|^2|M_{k-1}|^{-2}\right)$. Note that, for $n = 1$, *i.e.* the $Gr(1, N) = CP^{(N-1)}$ case, equations (3.108) and (3.109) are equivalent.

By virtue of (3.109), we have the following [17]:

Proposition 3.1 If $D_k = Det\,|M_k|^2 \neq 0$ in the whole complex plane C and

$$lim_{|\xi|\to\infty} D_k \to |\xi|^{2\omega_k}, \tag{3.110}$$

i.e. ω_k is the highest degree of $|\xi|^2$ in D_k, then

$$\omega_k = \mathcal{N}_k - \mathcal{N}_{k-1}, \tag{3.111}$$

where $\mathcal{N}_k$, $(k = 0, \ldots, \lambda - 1)$, are given by (3.106).

Proof: As we have assumed that $D_k \neq 0$ in the whole complex plane C so by applying Stokes' theorem in the plane to (3.109), *i.e.*

$$\int d\xi\, d\bar{\xi} (\psi)_{\bar{\xi}\xi} = \frac{1}{2} \oint_{|\xi|\to\infty} \left[d\bar{\xi}\psi_{\bar{\xi}} - d\xi\psi_{\xi} \right], \tag{3.112}$$

we obtain

$$\frac{i}{4\pi} \oint_{|\xi|\to\infty} \left[d\bar{\xi}\partial_{\bar{\xi}}(logD_k) - d\xi\partial_{\xi}(logD_k) \right] = \mathcal{N}_k - \mathcal{N}_{k-1}, \tag{3.113}$$

where we have used (3.106) in the right hand side.

From (3.110), it follows that $\partial_{\xi}(logD_k) \to \omega_k/\xi$. Using polar coordinate $\xi = re^{i\phi}$, then on the circle $|\xi| = R_0 \to \infty$, $d\xi = i\xi d\phi$, so the left hand side of (3.113) becomes

$$\frac{i}{4\pi} \oint_{|\xi|\to\infty} \left[d\bar{\xi}\partial_{\bar{\xi}}(logD_k) - d\xi\partial_{\xi}(logD_k) \right] = \frac{i}{4\pi}(-2i\omega_k) \int_0^{2\pi} d\phi = \omega_k, \tag{3.114}$$

which yields (3.111) as required. □

Note that, if $D_k = 0$ at some points, then this proposition cease to hold. In this case, if the singularities of D_k^{-1} are poles, then they must be substracted from (3.112), *i.e.* using the residue theorem. As we have assumed that D_k is analytic in C, so $\omega_k \in \mathbf{Z}$. Hence, we have the following [16]:

Corollary 3.1 If the elements of the initial matrix M_0 are *polynomials* in ξ such that the conditions of Proposition 3.1 hold then each solution M_k has finite action.

Proof: See the preceding outline of this construction. □

3.2.4 Topological Lowest Bound for the Action

In this subsection we consider the problem of finding the lowest bound finite action solutions of the $2D$ $Gr(n, N)$ σ model field equations (3.66). First, we observe that the $2D$ Lagrangian density (3.64), in terms of the real coordinates $x^\mu = (x, y)$, can be written in the form of perfect square plus "something", as follows

$$\mathcal{L} = \frac{1}{2}\text{Tr}\left[(\partial_\mu P \pm \epsilon_{\mu\nu} P \partial_\nu P)^2 \mp 2\epsilon_{\mu\nu} P \partial_\mu P \partial_\nu P \right], \quad \mu, \nu = 1, 2, \tag{3.115}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ with $\epsilon_{12} = 1$. Here, P is the generic expression for P_k. Hence, the corresponding action is

$$S = \frac{1}{2}\int d^2x \text{Tr}\Big[(\partial_\mu P \pm \epsilon_{\mu\nu} P \partial_\nu P)^2 \Big] \mp 2\pi \mathcal{Q}, \tag{3.116}$$

where

$$\mathcal{Q} = \frac{1}{2\pi}\int d^2x \text{Tr}\left[\epsilon_{\mu\nu} P \partial_\mu P \partial_\nu P\right]. \tag{3.117}$$

Note that the integrand in (3.117) is metric independent. In section 3.3.1, we shall show that $\mathcal{Q}$ is a topological quantity; in fact, it is the *topological charge* of the $Gr(n, N)$ σ model.

As the first term in (3.116) is positive definite so we have the topological lowest bound on the action:

$$S \geq 2\pi |\mathcal{Q}|. \tag{3.118}$$

This lowest bound, known as Bogomol'nyi bound [48], would be saturated if the perfect square term in (3.116) vanishes, *i.e.*

$$\partial_\mu P \pm \epsilon_{\mu\nu} P \partial_\nu P = 0, \tag{3.119}$$

which is nothing but the (anti-)selfdual equations (3.68). This means that all (anti-) instanton solutions are the lowest finite action solutions with value: $S_0 = 2\pi|\mathcal{Q}_0|$. For these special class of solutions, $\mathcal{Q}_0$ is also called (*anti*)-*instanton number.*

Let us now have a closer look on the topological charge $\mathcal{Q}_k$ of the solution P_k, by rewriting (3.117) in the complex coordinate $\xi = x + iy$, which takes the form

$$\mathcal{Q}_k = \frac{i}{2\pi}\int d\bar{\xi} d\xi \text{Tr}\left[P_k \left(P_{k\xi} P_{k\bar{\xi}} - P_{k\bar{\xi}} P_{k\xi}\right)\right]. \tag{3.120}$$

Using the relation (3.94) for $\partial_\xi P_k$ and noting that the matrices M_k are mutually orthogonal, then in terms of M_k, we find that

$$\mathcal{Q}_k = \mathcal{N}_k - \mathcal{N}_{k-1}, \tag{3.121}$$

where $\mathcal{N}_k$ is given in (3.106). Thus, by virtue of Proposition 3.1,

$$\mathcal{Q}_k = \omega_k, \qquad \omega_k \in \mathbb{Z}. \tag{3.122}$$

3.2.5 Veronese Map

In this subsection, we shall illustrate explicit construction of the full solutions of the $2D$-Grassmannian σ model as we have described in the previous subsections. Here, for simplicity, we shall consider the $Gr(1,N) = CP^{(N-1)}$ case for which $M_0 \in C^{N\times 1}$ is a vector field, *i.e.*

$$\begin{aligned} M_0 \ &: \ C \to C^N, \\ &\xi \to M_0 = (f_0, \ldots, f_p, \ldots, f_{N-1})^T. \end{aligned} \tag{3.123}$$

In particular, to prepare analytical background for our discussions in chapters 4 and 5, we shall discuss *Veronese* map or embedding [47], in which each component f_p is a monomial in ξ of order p in such a way that

$$|M_0|^2 = \left(1 + |\xi|^2\right)^{N-1}. \tag{3.124}$$

This constraint restricts the components f_p to have the form

$$f_p = \sqrt{C_p^{N-1}}\xi^p, \tag{3.125}$$

where C_p^{N-1} is the combinatorial factor and so the corresponding $CP^{(N-1)}$ field is

$$Z_0 = \frac{M_0}{|M_0|} = \frac{\left(1, \ldots, \sqrt{C_p^{N-1}}\xi^p, \ldots, \xi^{N-1}\right)^T}{\sqrt{(1+|\xi|^2)^{N-1}}}. \tag{3.126}$$

To construct full solutions of the $CP^{(N-1)}$ field equation (3.66) generating from this special Veronese map, explicitly, we need to choose a specific low N, in order to simplify the task.

For $N = 4$, we find, after using the construction (3.79) or (3.80), that

$$M_0 = \left(1,\, \sqrt{3}\xi,\, \sqrt{3}\xi^2,\, \xi^3\right)^T, \tag{3.127}$$

$$M_1 = \frac{\left(-3\bar{\xi},\, \sqrt{3}\,(1-2|\xi|^2)\,,\, \sqrt{3}\xi\,(2-|\xi|^2)\,,\, 3\xi^2\right)^T}{(1+|\xi|^2)}, \tag{3.128}$$

$$M_2 = \frac{2\left(3\bar{\xi}^2,\, -\sqrt{3}\bar{\xi}\,(2-|\xi|^2)\,,\, \sqrt{3}\,(1-2|\xi|^2)\,,\, 3\xi\right)^T}{(1+|\xi|^2)^2}, \tag{3.129}$$

$$M_3 = \frac{6\left(-\bar{\xi}^3,\, \sqrt{3}\bar{\xi},\, -\sqrt{3}\bar{\xi},\, 1\right)^T}{(1+|\xi|^2)^3}, \tag{3.130}$$

whereas $M_4 = 0$, according to (3.93), and so

$$\begin{aligned} |M_0|^2 &= \left(1+|\xi|^2\right)^3, \\ |M_1|^2 &= 3\left(1+|\xi|^2\right), \\ |M_2|^2 &= 12\left(1+|\xi|^2\right)^{-1}, \\ |M_3|^2 &= 36\left(1+|\xi|^2\right)^{-3}. \end{aligned} \tag{3.131}$$

For completeness, in the appendix C, we present the complete set of the corresponding (4×4) matrix projectors $P_k = M_k|M_k|^{-2}M^{\dagger}$, $(k = 0, \ldots, 3)$.

As for the $CP^{(N-1)}$ case, $D_k = |M_k|^2$, so we conclude that

$$\omega_0 = 3, \qquad \omega_1 = 1, \qquad \omega_2 = -1, \qquad \omega_3 = -3, \tag{3.132}$$

which are the topological charges $\mathcal{Q}_k$ for the configurations M_0, M_1, M_2 and M_3, consecutively.

Putting the results for $|M_k|^2$, $k = 0, \ldots, 4$ from (3.131) into the integrand of $\mathcal{N}_k$ in (3.106) and using the integral formula

$$i \int \frac{d\bar{\xi} d\xi}{(1+|\xi|^2)^2} = 2\pi, \tag{3.133}$$

we obtain

$$\mathcal{N}_0 = 3, \qquad \mathcal{N}_1 = 4, \qquad \mathcal{N}_2 = 3, \qquad \mathcal{N}_3 = 0. \tag{3.134}$$

With the above results for $|M_k|^2$, $\mathcal{N}_k$ and ω_k, $(k = 0, \ldots, 3)$, we see that the recurrence relations (3.108) or (3.109) and (3.111) are satisfied, explicitly.

Let us now return to the general case N and proof the following result that was first given by Ioannidou *et. al* [14]:

Proposition 3.2 For the Veronese map M_0 in (3.126), the sequence M_k, $(k = 0, \ldots, N-1)$, constructed by the scheme (3.79) or (3.80), called *Veronese sequence* [47], satisfy

$$\frac{|M_{k+1}|^2}{|M_k|^2} = \frac{(k+1)(N-k-1)}{(1+|\xi|^2)^2}. \tag{3.135}$$

Proof. Here, we shall make use of the recurrence relations (3.108) or (3.109). As $|M_0|^2 = (1+|\xi|^2)^{(N-1)}$,

$$\partial_{\bar{\xi}}\partial_{\xi}(log|M_0|^2) = (N-1)\left(1+|\xi|^2\right)^{-2}. \tag{3.136}$$

Thus, from (3.108) or (3.109), we obtain

$$\frac{|M_1|^2}{|M_0|^2} = \frac{(N-1)}{(1+|\xi|^2)^2}, \tag{3.137}$$

as by definition $M_{-1} = 0$.

For general k, we shall use inductive proof, by assuming (3.135) holds up to $1 < k < (N-1)$, *i.e.*

$$\frac{|M_k|^2}{|M_{k-1}|^2} = \frac{k(N-k)}{(1+|\xi|^2)^2}, \tag{3.138}$$

which is already true for $k = 1$, as we have seen in (3.137). Equation (3.138) then implies that

$$\begin{aligned} |M_k|^2 &= \frac{|M_k|^2}{|M_{k-1}|^2}\frac{|M_{k-1}|^2}{|M_{k-2}|^2}\cdots\frac{|M_1|^2}{|M_0|^2}|M_0|^2 \\ &= \frac{k!(N-1)!}{(N-k-1)!}(1+|\xi|^2)^{(N-2k-1)}, \end{aligned} \tag{3.139}$$

and so

$$\partial_{\bar{\xi}}\partial_{\xi}(log|M_k|^2) = \frac{(N-2k-1)}{(1+|\xi|^2)^2}. \tag{3.140}$$

Substituting (3.138) and (3.140) into (3.108) or (3.109) gives

$$\frac{|M_{k+1}|^2}{|M_k|^2} = \frac{(k+1)(N-k-1)}{(1+|\xi|^2)^2}, \tag{3.141}$$

which completes our inductive proof. □

From this proposition, follows:

Corollary 3.2 Each configuration M_k, $(k = 0, \ldots, N-1)$, of the Veronese map (3.126), has

$$\mathcal{N}_k = (k+1)(N-k-1), \tag{3.142}$$

and topological charge

$$Q_k = (N - 2k - 1). \tag{3.143}$$

Proof: Equation (3.142) follows after substituting (3.135) and using the integral formula (3.133).

Since for CP^{N-1}, $D_k = |M_k|^2$, where $|M_k|^2$ is given by (3.139),

$$lim_{|\xi|\to\infty} D_k \to |\xi|^{2(N-2k-1)}. \tag{3.144}$$

Thus, according to (3.110), $\omega_k = (N - 2k - 1)$ and so, by virtue of (3.122), $Q_k = (N - 2k - 1)$ as required, which is consistent with (3.122). □

3.3 Topological Consideration

We now turn to consider the topological meanings of the topological charges Q_k and the solutions $Z_k = M_k|M_k|^{-1}$ that we have constructed in section 3.2.2. This needs some algebraic topological backgrounds, in particular, de Rham cohomology and homotopy groups theories, which we shall discuss in this section. Here we shall only discuss the common framework relevant to the case under consideration. In fact, we do also need these topological theories background as we shall encounter similar topological quantities in chapters 4 and 5, which deal with the $3D$ cases.

3.3.1 Topological Charge: de Rham Cohomology

First, let us consider the topological meaning of the topological charge Q_k. By a little inspection on the integral (3.117) we note that the integrand is in fact the pull-back of the celebrated *1-st Chern form* in $Gr(n, N)$ [27,53] into the base manifold R^2, *i.e.*

$$c_1(Gr(n, N)) = \mathrm{Tr}(F) = \mathrm{Tr}\left[PdP \wedge dP\right]. \tag{3.145}$$

Here

$$F = Z^\dagger dP \wedge dPZ, \tag{3.146}$$

is the curvature 2-form of the local $U(n)$ gauge transformation having connection 1-form

$$A = Z^\dagger dZ, \tag{3.147}$$

with local components in R^2 as given in (3.38). Using (3.59) for the projector P, we find that

$$dc_1 = 0, \tag{3.148}$$

which is consistent with the fact that c_1 is the generator of $H^2_{dR}(Gr(n,\, N),\, \mathbb{Z})$, the second *de Rham cohomology group* of $Gr(n,\, N)$ with coefficients in the integers $\mathbb{Z}$ [28,29,49,51,52].

We recall that if $\mathcal{F}^p(\mathcal{M})$ denotes the set of p-forms on $\mathcal{M}$, then $\alpha \in \mathcal{F}^p(\mathcal{M})$ is called *closed* if $d\alpha = 0$, and *exact* if $\alpha = d\beta$ for $\beta \in \mathcal{F}^{p-1}(\mathcal{M})$. Two closed p-forms are called *cohomologous*, denoted $\alpha \sim \alpha'$ if $\alpha - \alpha'$ is exact. The set of the equivalence relation $\sim$ is called the *p-th de Rham cohomology group* of $\mathcal{M}$ and is denoted by $H^p_{dR}(\mathcal{M})$. Its elements $[\alpha]$, called *cohomology classes*, form an additive group and even a vector space structure, *i.e.* if $\alpha_1 \sim \alpha_1'$ and $\alpha_2 \sim \alpha_2'$ then $(\alpha_1 + \alpha_2) \simeq (\alpha_1 + \alpha_2)$, $c\alpha_1 \simeq c\alpha_1'$, for $c \in \mathbb{R}$. Since any exact form is closed ($d^2 = 0$), so we can rephrase the definition of $H^p(\mathcal{M})$ by saying that it is a quotient vector space

$$H^p_{dR}(\mathcal{M}) = \frac{\text{closed } p-forms}{\text{exact } p-forms}. \tag{3.149}$$

On a compact manifold $\mathcal{M}$, $H^p_{dR}(\mathcal{M})$ is finite dimensional and its dimension b^p_{dR} is called the p-th *Betti number* which is a topological invariant. More precisely, $b^p_{dR} = n$ if there exist n classes of closed p-forms $\alpha_{(k)}$, $k = 1, \ldots, n$, each representing $[\alpha_{(k)}]$, such that any element α is given by the finite sum of the form

$$\alpha = \sum_{k=1}^{n} r_k \alpha_{(k)}, \tag{3.150}$$

where the coefficient r_k is an element of a ring R. The notation $H^p(\mathcal{M}, R)$ is then used to denote the ring type of this coefficient. As $\alpha = 0$ for $p > dim[\mathcal{M}]$, the p-th *Betti number* b^p_{dR} is always finite.

For our case here, we have $H^2_{dR}(Gr(n,\, N),\, \mathbb{Z}) = \mathbb{Z}$. Here the ring of integers $\mathbb{Z}$ is considered as an infinite cyclic group with one independent generator which can be chosen to be $+1$. The corresponding cohomology class of the 1-st Chern form c_1, *i.e.* $[c_1]$, is called the *1-st Chern class* c_1. In relation to this, the topological charge Q is also called the *1-st Chern number.*

As the finiteness of the action (3.104) requires the boundary condition $D_k \to |\xi|^{2\omega_k}$ at $|\xi| \to \infty$, we may identify all points at infinity to a single point and view

$$Z_k : \ R^2 \cup \{\infty\} \simeq S^2 \to Gr(n,\, N). \tag{3.151}$$

Since the 1-st Chern number measures the number of times S^2 is covered by the mapping, the integers ω_k has geometric interpretation as the *degree of maps* $Z_k : S^2 \to Gr(n,\, N)$. We refer to Refs. [53, 54] for more details.

3.3.2 Discrete Solutions: Homotopy

The solutions $Z_k = M_k|M_k|^{-1}$ by themselves do also provide another topological meaning. We have seen that each solution Z_k attains discrete topological charge value $\omega_k \in \mathbb{Z}$. This implies that solutions with different ω_k fall into disjoint classes of field configurations which can not be transformed into one another by a continuous or topological deformations. In this case, the field configurations Z_k and Z_l with $k \neq l$ are said to be *nonhomotopic.*

We recall that if $F(x,\, \tau)$, $x \in S^D$ denotes a continuous function from $S^D \times [0, 1] \to \mathcal{M}$ then two maps f and g are said *homotopic* to each other, denoted $f \sim g$, if $F(x,\, \tau)$ compatible with the boundary conditions (imposed on f and g) and fulfills $F(x,\, 0) = f$, $F(x,\, 1) = g$. If $F(x,\, \tau)$ exist then it is called a *homotopy* from f to g. The set of these equivalence classes can be provided with a group structure $\Pi_D(\mathcal{M})$ called D-dimensional *homotopy group* of manifold $\mathcal{M}$. Here, the notation $\Pi_D(\mathcal{M})$ means that we map the manifold of the D-sphere S^D into $\mathcal{M}$ [49, 51, 52].

A brief review on this homotopy group and their computations are given in the appendix E, where we have shown that

$$\Pi_2(Gr(n, N)) = \mathbb{Z}. \tag{3.152}$$

Homotopically, the result (3.152) means that there is a 2-dimensional submanifold $\widetilde{S}^2 \subset Gr(n,\, N) \simeq S^2$ which is nondeformable to one point. As we have seen in the previous subsection that $H^2(Gr(n,\, N)) = \mathbb{Z}$, we have the isomorphism $H^2(Gr(n,\, N)) \simeq \Pi_2(Gr(n,\, N))$. This is consistent with the celebrated *Hurewicz theorem* [51, 52] which asserts that: if $\Pi_k(\mathcal{M}) = 0$ for $k = 1, 2, \ldots, (D-1)$, and $\Pi_D(\mathcal{M}) \neq 0$ then $H^D(\mathcal{M}) \simeq \Pi_D(\mathcal{M})$.

3.4 Scale Stability and Multidimensional Solutions

From the above discussions, we see that the solutions of the $2D$ $Gr(n, N)$ σ models satisfy the properties of being nonsingular, having finite energies, and topologically stable. Thus, they belong to a class of soliton solutions called *topological solitons* [7].

Their stability with respect to small initial perturbations, *i.e.* topological deformations, as was claimed in the section 3.3.2, is a neccessary condition for the applicability of the classical solutions to the construction of extended quantum mechanical objects such as particles and nuclei which we shall consider in chapters 4 and 5. This then brought into focus the problem of analytical investigations to the existence of topological solitons in higher dimensions which we shall carry out in this section.

The simplest stability criterion for multidimensional field theory classical solutions is given by a scale argument that was first put forward by Hobart and Derrick [56, 57], which considers energy variation under scale perturbations. This argument is stated as follows:

Theorem 3.2 Let

$$E[f] = \int d^D x\, \mathcal{E}(f(x)), \tag{3.153}$$

be the static energy functional of field theories in a $(D+1)$-dimensional spacetime. If under the scaling:

$$x^\mu \to \lambda x^\mu, \tag{3.154}$$

where λ is a scale parameter, the energy functional varies as

$$E[f] \to E[f(\lambda)], \tag{3.155}$$

then a necessary but not sufficient conditions for stability of the classical solutions $f(x)$ are given by

$$\left.\frac{dE[f(\lambda)]}{d\lambda}\right|_{\lambda=1} = 0, \tag{3.156}$$

$$\left.\frac{d^2E[f(\lambda)]}{d\lambda^2}\right|_{\lambda=1} \geq 0. \tag{3.157}$$

Proof. Consider the energy functional $E[f(\lambda)]$ as an ordinary function of one variable $E(\lambda)$ and apply calculus analysis. □

Let us study the application of the above theorem by starting from the σ model energy functional (2.3) in arbitrary dimension D:

$$E_\sigma[f] = \frac{1}{2}\int d^D x\, h_{AB}\frac{\partial f^A}{\partial x^k}\frac{\partial f^B}{\partial x^k}, \tag{3.158}$$

where we have chosen the base space $\mathcal{M}_0$ to be Euclidean and that that the summation convention on repeated indices is understood. Under the scale transformation (3.154),

$$f^A(x) \to f^A(\lambda x), \tag{3.159}$$

and the energy functional (3.158) changes to

$$\begin{aligned} E_\sigma[f(\lambda)] &= \frac{1}{2}\int d^D x\, h_{AB}(f(\lambda x))\frac{\partial f^A(\lambda x)}{\partial x^k}\frac{\partial f^B(\lambda x)}{\partial x^k} \\ &= \frac{1}{2}\int \frac{d^D(\lambda x)}{\lambda^D}\, h_{AB}(f(\lambda x))\,(\lambda^2)\,\frac{\partial f^A(\lambda x)}{\partial(\lambda x^k)}\frac{\partial f^B(\lambda x)}{\partial(\lambda x^k)} \\ &= \lambda^{(2-D)}E_\sigma[f]. \end{aligned} \tag{3.160}$$

From (3.160) it follows that

$$\left.\frac{dE_\sigma[f(\lambda)]}{d\lambda}\right|_{\lambda=1} = (2-D)E_\sigma, \tag{3.161}$$

$$\left.\frac{d^2E_\sigma[f(\lambda)]}{d\lambda^2}\right|_{\lambda=1} = (2-D)(1-D)E_\sigma. \tag{3.162}$$

We see from (3.160) that for $D = 2$, $E_\sigma[f]$ is scale invariant and that the stability conditions (3.156) and (3.157) are satisfied identically by (3.161) and (3.162). However, for $D \geq 3$, since $E_\sigma[f] \geq 0$, the only stable solution is the trivial one, $f = \textit{constant}$. Other nontrivial solutions, for example the composite geodesic map solutions in Corollary 2.1, have $E_\sigma[f] \to \infty$.

Let us see whether the addition of arbitrary nonderivative potential energy

$$V[f] = \int d^D x \mathcal{V}(f(x)), \tag{3.163}$$

could improve this result. As under the scale transformation (3.154)

$$V[f] \to V[f(\lambda)] = \lambda^{-D}V[f], \tag{3.164}$$

the total energy: $E = E_\sigma + V$ scales as

$$E[f(\lambda)] = \lambda^{(2-D)}E_\sigma[f] + \lambda^{-D}V[f], \tag{3.165}$$

from which it follows that

$$dE[f(\lambda)]/d\lambda\Big|_{\lambda=1} = (2-D)E_\sigma - DV, \tag{3.166}$$

$$d^2E[f(\lambda)]/d\lambda^2\Big|_{\lambda=1} = (2-D)(1-D)E_\sigma + D(D+1)V. \tag{3.167}$$

The extremum condition (3.156) then implies the virial type relation: $V = \frac{(2-D)}{D}E_\sigma$ from (3.166), which after we put this relation into (3.167) yields

$$d^2E_\sigma[f(\lambda)]d\lambda^2\Big|_{\lambda=1} = 2(2-D)E_\sigma. \tag{3.168}$$

Hence, we conclude that all static solutions in $D \geq 3$ should be unstable.

In chapter 4 when we discuss the $SU(N)$ Skyrme models, we will see how to evade the above no-go theorem by the addition of terms of higher power in field derivatives to E_σ in (3.158).

Chapter 4

$SU(N)$ Skyrme Models and Harmonic Maps

In this chapter we discuss the $SU(N)$ Skyrme models and the alternative models, which are minimal generalisations of the $SU(N)$ chiral models in (3+1)-dimensional spacetime that possess static finite energy solutions called multiskyrmions. They are examples of topological solitons in 3 spatial dimensions that evade the Hobart-Derrick no-go theorem, which we have discussed in chapter 3. This is achieved by the addition of terms of higher power in field derivatives to the σ-model action.

First, we introduce formulation of the $SU(N)$ Skyrme models and the related $3D$ topological charge quantity. Then we discuss the application of harmonic map ansatz method to construct approximate and exact solutions of static field equations. This ansatz is used to factoring out the angular dependence parts of the solutions from the equations which leaves us with radial equations for the profile functions g_k. Here, we generalise the harmonic map ansatz method of Ioannidou *et. al* [14] by considering rank-2 projectors of $S^2 \to Gr(2, N)$. When comparing our results for g_k with those of rank-1 case, we found that they are very close but having marginally higher energies and that exact solutions are just embeddings.

In section 4.8 and the rest, we consider alternative $SU(N)$ Skyrme models and show that the harmonic map ansatz methods work as in the usual models. Here, we use instead the rank-1 projectors of $S^2 \to CP^{N-1}$ in order to compare exact results of both models. We found that the alternative models have higher energies.

4.1 $SU(N)$ Skyrme Models

The $SU(N)$ Skyrme models are described by $SU(N)$ group valued functions U of $(3+1)$-D spacetime coordinates $x^\mu = (x^0, \vec{x} = (x^a))$. Their dynamics is determined by the action [13]

$$S = \int d^4x \mathcal{L}, \tag{4.1}$$

where

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{F^2}{16} L_\mu L^\mu + \frac{1}{32a^2}[L_\mu, L_\nu][L^\mu, L^\nu] + \frac{F^2}{16} M_\pi^2 \left(U^{-1} + U - 2I\right) \Big], \tag{4.2}$$

is the corresponding Lagrangian density and where

$$L_\mu = U^{-1}\partial_\mu U, \tag{4.3}$$

are the left chiral currents with values in the Lie algebra $su(N)$, $F \cong 189$ MeV is the pion decay constant and a is a dimensionless constant. Notice that, our convention for the spacetime metric is: $ds^2 = (dx^0)^2 - (dx^a)^2$.

The first term in (4.2) is the $SU(N)$ chiral models Lagrangian density (2.94) whereas the second term is the celebrated *Skyrme term* that is responsible for stabilising the would be solitonic solutions. In terms of U the Skyrme term is of order four in field derivatives. The last term in (4.2) describes the mass term where M_π is the pion (meson) mass. The action (4.1) has a global $SU(N)/\mathbb{Z}_2$ symmetry, as it is invariant under the conjugation: $U \to \Omega U \Omega^\dagger$, where $\Omega \in SU(N)$ is a constant matrix.

To derive the Euler-Lagrange equations of the models, we start by considering variation of the action (4.1) with respect to the field variation δU, *i.e.*

$$\delta S = \int d^4x \delta\mathcal{L}, \tag{4.4}$$

where

$$\delta\mathcal{L} = \mathrm{Tr}\Big[-\frac{F^2}{8}(\delta L_\mu)L^\mu + \frac{1}{16a^2}(\delta[L_\mu, L_\nu])[L^\mu, L^\nu] + \frac{F^2}{16} M_\pi^2 \left(\delta U^{-1} + \delta U\right) \Big]. \tag{4.5}$$

As $\delta U^{-1} = -U^{-1}(\delta U)U^{-1}$, we have

$$\delta L_\mu = -U^{-1}(\delta U)L_\mu + L_\mu U^{-1}(\delta U) + \partial_\mu \left(U^{-1}\delta U\right), \tag{4.6}$$

and so

$$\mathrm{Tr}\left((\delta L_\mu)L^\mu\right) = -\mathrm{Tr}\left[(\partial_\mu L^\mu)\, U^{-1}\delta U\right] \tag{4.7}$$

plus a total divergence term.

Next, using the fact that L_μ is the pull-back of the Maurer-Cartan 1-form which satisfies the Maurer-Cartan equation:

$$\partial_\mu L_\nu - \partial_\nu L_\mu = -[L_\mu, L_\nu], \tag{4.8}$$

then

$$\delta[L_\mu, L_\nu] = -\partial_\mu \delta L_\nu + \partial_\nu \delta L_\mu, \tag{4.9}$$

from which we derive that

$$\mathrm{Tr}\left((\delta[L_\mu, L_\nu])[L^\mu, L^\nu]\right) \;=\; 2\mathrm{Tr}\left((\partial_\mu[L_\nu, [L^\nu, L^\mu]])\, U^{-1}\delta U\right) \tag{4.10}$$

plus a total divergence term.

Substituting (4.7) and (4.10) in (4.4) and throwing away the total divergence terms, which are transformed into surface integral terms, due to the vanishing of the variation δU on the boundary, we arrive at

$$\delta S = \int d^4x \mathrm{Tr}\Big[\Big(\frac{F^2}{8}\partial_\mu L^\mu \;+\; \frac{1}{8a^2}\partial_\mu[L_\nu, [L^\nu, L^\mu]] \;+\; \frac{F^2}{16}M_\pi^2(U - U^{-1})\Big)\left(U^{-1}\delta U\right)\Big]. \tag{4.11}$$

Hence, the equations of motion, in matrix form, that we read-off from (4.11) are

$$\partial_\mu \left(L^\mu \;-\; \frac{1}{a^2F^2}[L_\nu, [L^\mu, L^\nu]]\right) \;+\; \frac{1}{2}M_\pi^2\left(U - U^{-1}\right) \;=\; 0. \tag{4.12}$$

4.2 Static Energy and Topological Charge

In the following discussions we shall concentrate on the static case only, for which $L_0 = 0$.

4.2.1 Static Energy and Static Field Equations

The static energy of the $SU(N)$ Skyrme models, as we have derived in the appendix H.1, is given by $E = E_{stat}$ in (H.1.9), *i.e.*

$$E = -\int d^3x \mathrm{Tr}\Big[\frac{F^2}{16}L_a^2 \;+\; \frac{1}{32a^2}[L_a, L_b]^2 \;+\; \frac{F^2}{16}M_\pi^2\left(U^{-1} + U - 2I\right)\Big], \tag{4.13}$$

We see that the energy density $\mathcal{E}$ of (4.13) coincides with the negative of the Lagrangian density (4.2), *i.e.* $\mathcal{E} = -\mathcal{L}_{stat}$ as derived from the canonical method.

To study solitonic properties of the $SU(N)$ Skyrme models from the static energy E in (4.13), it is convenient to scale the spatial coordinates $\vec{x}$ by setting $\vec{x} \to 2\vec{x}/aF$ and give the energy in units of $F/4a$, *i.e.* by taking $F/4a = 1/(12\pi^2)$. In this unit the energy (4.13) reads

$$E = \frac{1}{12\pi^2}\int d^3x \left(-\frac{1}{2}\right) \mathrm{Tr}\left[L_a^2 + \frac{1}{8}\left([L_a, L_b]^2\right) + m_\pi^2\left(U^{-1} + U - 2I\right)\right], \quad (4.14)$$

where $m_\pi = 2M_\pi/aF$ and the equations of motion (4.12), in the static case, read

$$\partial_a\left(L_a - \frac{1}{4}[L_b,[L_a,L_b]]\right) - \frac{m_\pi^2}{2}\left(U - U^{-1}\right) = 0. \quad (4.15)$$

which coincides with stationary points (minima or saddle points) of the static energy (4.14).

4.2.2 Scale Stability

Let us now examine scale stability of the static energy (4.14). As under the scale transformation: $\vec{x} \to \lambda\vec{x}$, the currents L_a scale as

$$L_a(\vec{x}) \to U^{-1}(\lambda\vec{x})\frac{\partial U(\lambda\vec{x})}{\partial x^a} = \lambda L_a(\lambda\vec{x}), \quad (4.16)$$

the energy (4.14) changes to

$$\begin{aligned} E[(\lambda)] &= \frac{1}{12\pi^2}\int \frac{d^3(\lambda\vec{x})}{\lambda^3}\mathrm{Tr}\left[-\frac{1}{2}\lambda^2 L_a(\lambda\vec{x})^2 - \frac{1}{16}\lambda^4[L_a(\lambda\vec{x}),\, L_b(\lambda\vec{x})]^2\right] \\ &= \frac{1}{\lambda}E_\sigma + \lambda E_{Skr}, \end{aligned} \quad (4.17)$$

where E_σ and E_{Skr} are the static chiral energy term and the Skyrme term, respectively.

From (4.17) it follows that

$$\left.\frac{dE[\lambda]}{d\lambda}\right|_{\lambda=1} = \left.-\frac{1}{\lambda^2}E_\sigma + E_{Skr}\right|_{\lambda=1} = -E_\sigma + E_{Skr}, \quad (4.18)$$

$$\left.\frac{d^2E[\lambda]}{d\lambda^2}\right|_{\lambda=1} = \left.\frac{2}{\lambda^3}E_\sigma\right|_{\lambda=1} = 2E_\sigma. \quad (4.19)$$

The extremum condition $dE[\lambda]/d\lambda\Big|_{\lambda=1} = 0$ implies that $E_\sigma = E_{Skr}$, and so, if $E_\sigma > 0$, (4.19) implies that the static energy (4.14) is stable against scale perturbations. Thus, we conclude that the $SU(N)$ Skyrme models admit the existence of stable static finite energy solutions. We see that it is the additional Skyrme term that is responsible for stabilising the would be solitonic solutions.

4.2.3 Topological Charge

We observe that the energy (4.14), in the massless case $m_\pi = 0$, can be written in the form of a perfect square plus "something" as follows

$$E = \frac{1}{12\pi^2}\int d^3x \left(-\frac{1}{2}\right) \mathrm{Tr}\left[\left(L_a \pm \frac{1}{4}\epsilon_{abc}[L_b, L_c]\right)^2\right] \pm \frac{1}{24\pi^2}\int d^3x \epsilon_{abc}\, \mathrm{Tr}\,(L_a L_b L_c)\,. \tag{4.20}$$

where ϵ_{abc} is the Levi-Civita symbol, *i.e.* $\epsilon_{abc} = \delta^{abc}_{123}$, where the right hand side is defined in (2.75). The first integral is positive definite and thus we arrive at the energy lower bound

$$E \geq B, \tag{4.21}$$

where

$$B = \frac{1}{24\pi^2}\int d^3x \epsilon_{abc}\, \mathrm{Tr}\,(L_a L_b L_c)\,. \tag{4.22}$$

This bound is known as Faddeev bound that would be saturated if

$$L_a \mp \frac{1}{4}\epsilon_{abc}[L_b, L_c] \;=\; 0. \tag{4.23}$$

It turns out that the only solution to this equation is the trivial one, namely $U =$ *constant* and so $L_a = 0$. We note that B is metric independent and we shall show that it is a topological quantity. In fact, B is the *topological charge* of the $SU(N)$ Skyrme models. Thus, equation (4.21) shows that the energy E is measured in the *topological charge unit.*

In order that a given configuration corresponds to a finite energy lump, we must impose the boundary condition that the field $U(\vec{x})$ goes to a constant matrix U_0 at spatial infinity. As by a global $SU(N)$ transformation, this U_0 can be brought to the identity matrix I, so without the loss of generality we can impose the following boundary condition on U: $U \to I$ as $|\vec{x}| \to \infty$.

As $L_a = U^{-1}\partial_a U$, this boundary condition implies that $U(\vec{x})$ is effectively a mapping:

$$U: \; R^3 \cup \{\infty\} \simeq S^3 \to SU(N). \tag{4.24}$$

In chapter 3, we have seen that such maps fall into homotopy classes and are simply the elements of the 3th-homotopy group $\Pi_3(SU(N)) = \mathbb{Z}$ as computed in the appendix E.

Let us now have a look at the topological meaning of the topological charge B in (4.22). We notice that $L = U^{-1}dU$ is the Maurer-Cartan left invariant 1-form of the $SU(N)$ group manifold and so the corresponding integrand is the pull-back of the 3-form

$$\Omega = \mathrm{Tr}\,(L \wedge L \wedge L)\,, \tag{4.25}$$

into the base space R^3. As L satisfies the Maurer-Cartan equation $dL = -L \wedge L$,

$$\Omega = -\mathrm{Tr}\,(dL \wedge L)\,, \tag{4.26}$$

and we derive that $d\Omega = 0$. This result is obtained by using the Lie algebra component form $L = U^{-1}dU = \lambda^{(A)}T_{(A)}$ where $T_{(A)}$ are the $SU(N)$ generators and $\lambda^{(A)}$ the vielbein 1-forms of the $SU(N)$ group manifold. Thus, $dL = d\lambda^{(A)}T_{(A)}$ where $d\lambda^{(A)}$ is given by (2.68). Assuming that the generators has been chosen so that the Cartan-Killing form is simply the Kronecker delta, then the vanishing of $d\Omega$ follows from the Jacobi identities (2.69) [58].

Thus Ω generates the $3th$-de Rham cohomology group $H^3_{dR}(SU(N))$. As we have shown in appendix E that $\Pi_1(SU(N)) = \Pi_2(SU(N)) = 0$, so according to Hurewicz theorem:

$$H^3_{dR}(SU(N)) = \Pi_3(SU(N)) = \mathbb{Z}. \tag{4.27}$$

Hence, B is the integer valued winding number of the map U: $S^3 \to SU(N)$.

In relation to (static) solutions of the $SU(N)$ Skyrme models, B classifies the solitonic sectors of the models. Following Skyrme [2] and Witten [8], B is identified with *baryon number* of the finite energy lump configurations called multiskyrmions. Thus, multiskyrmions are stationary points (minima or saddle points) of the static energy functional (4.13).

4.3 Generalised Harmonic Map Ansatz

In this section, we first rewrite the energy (4.14) and the topological charge (4.22) in the spherical polar coordinates $(r,\ \theta,\ \varphi)$. Later we introduce the harmonic map ansatz - so it is convenient to replace the spherical angular coordinates by the complex (or holomorphic) stereographic coordinates $(\xi,\ \bar{\xi})$ where ξ is related to the θ, φ, via $\xi = \tan\frac{\theta}{2}e^{i\varphi}$ as given in (2.42).

The spatial metric in the spherical polar coordinates $(r,\ \theta,\ \varphi)$:

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.28}$$

then reads

$$ds^2 = dr^2 + r^2\left[\frac{4d\bar{\xi}d\xi}{(1+|\xi|^2)^2}\right], \tag{4.29}$$

in the $(r,\ \xi,\ \bar{\xi})$ coordinates, whereas for the volume element:

$$d^3x = r^2 dr\left[\sin\theta d\theta d\phi\right] = r^2 dr\left[\frac{2id\bar{\xi}d\xi}{(1+|\xi|^2)^2}\right]. \tag{4.30}$$

We also note that in $(\xi,\ \bar{\xi})$ coordinates, the angular derivatives ∂_θ and ∂_ϕ read:

$$\partial_\theta = \frac{(1+|\xi|^2)}{2|\xi|}\left(\xi\partial_\xi + \bar{\xi}\partial_{\bar{\xi}}\right), \tag{4.31}$$

$$\partial_\phi = i\left(\xi\partial_\xi - \bar{\xi}\partial_{\bar{\xi}}\right), \tag{4.32}$$

respectively, where $|\xi| = \sqrt{|\xi|^2}$.

Taking the sequence of rank-n projectors $P_k(n) = P_k$ in section 3.3.2, we can now formulate our *generalised* harmonic map ansatz for the $SU(N)$ Skyrme fields in 3 dimensions. Namely, we take the matrix $U \in SU(N)$ of the form

$$\begin{aligned} U &= \exp\left[i\sum_{k=0}^{\lambda-1} g_k\left(P_k - \frac{nI}{N}\right)\right] \\ &= e^{-ni\sum_{k=0}^{\lambda-1} g_k/N}(I + \sum_{l=0}^{\lambda-1} A_l P_l), \end{aligned} \tag{4.33}$$

where $g_k = g_k(r)$ for $k = 0, ..., \lambda-1$ are the profile functions and

$$A_k = (e^{ig_k} - 1). \tag{4.34}$$

The profile functions $g_k(r)$ are required to satisfy the boundary conditions: $g_k(\infty) = 0$ and $g_k(0) = 2\alpha\pi$, where $\alpha = 0$ or ± 1.

4.3.1 Energy

The energy (4.14), when writen in the spherical polar coordinates $(r,\ \theta,\ \phi)$ takes the form

$$E = \frac{1}{12\pi^2}\int drd\theta d\phi(r^2\sin\theta)\left(-\frac{1}{2}\right)\mathrm{Tr}\left[L_r^2+\frac{1}{r^2}L_\theta^2+\frac{1}{r^2\sin^2\theta}L_\phi^2\right.$$
$$\left.+\frac{1}{4r^2}\left([L_r,\ L_\theta]^2+\frac{1}{\sin^2\theta}[L_r,\ L_\phi]^2+\frac{1}{r^2\sin^2\theta}[L_\theta,\ L_\phi]^2\right)\right], \quad (4.35)$$

where, for simplicity, we have put the pion (meson) mass $m_\pi = 0$. In the spherical (holomorphic) coordinates $(r,\ \xi,\ \bar{\xi})$, using (4.30)-(4.32), (4.35) takes the form:

$$E = -\frac{i}{12\pi^2}\int d\xi\, d\bar{\xi}\, dr\, r^2\mathrm{Tr}\left[\frac{1}{(1+|\xi|^2)^2}L_r^2 + \frac{1}{r^2}L_\xi L_{\bar{\xi}} + \frac{1}{4r^2}\left[L_r, L_\xi\right]\left[L_r, L_{\bar{\xi}}\right]\right.$$
$$\left.-\frac{(1+|\xi|^2)^2}{16r^4}\left[L_\xi, L_{\bar{\xi}}\right]^2\right], \quad (4.36)$$

With the matrix U given by the harmonic map ansatz (4.33), the currents L_a's in (4.3) take the following forms:

$$L_r = i\sum_{k=0}^{\lambda-1}\dot{g}_k\left(P_k-\frac{nI}{N}\right), \quad (4.37)$$

where $\dot{g}_k = \frac{dg_k}{dr}$ and

$$L_\xi = \left[I+\sum_{k=0}^{\lambda-1}(e^{-ig_k}-1)P_k\right]\left[\sum_{l=0}^{\lambda-1}(e^{ig_l}-1)P_{l\xi}\right]$$
$$= \sum_{k=0}^{\lambda-1}\left[e^{i(g_k-g_{k+1})}-1\right]\left(M_{k+1}|M_k|^{-2}M_k^\dagger\right), \quad (4.38)$$

where by definition $g_\lambda = 0$, and that $L_{\bar{\xi}} = -(L_\xi)^\dagger$.

Using the expressions for L_r and L_ξ in (4.37) and (4.38), respectively, we find that the traces in the energy functional (4.36) become

$$\mathrm{Tr}(L_r^2) = \frac{n^2}{N}\left(\sum_{k=0}^{\lambda-1}\dot{g}_k\right)^2 - n\sum_{k=0}^{\lambda-1}\dot{g}_k^2, \quad (4.39)$$

$$\mathrm{Tr}(L_\xi L_{\bar{\xi}}) = -2\sum_{k=0}^{\lambda-1}\mathcal{S}_k\,\mathrm{Tr}\left(|M_{k+1}|^2|M_k|^{-2}\right), \quad (4.40)$$

$$\mathrm{Tr}\left([L_r, L_\xi]\left[L_r, L_{\bar{\xi}}\right]\right) = -2\sum_{k=0}^{\lambda-1}(\dot{g}_k-\dot{g}_{k+1})^2\,\mathcal{S}_k\,\mathrm{Tr}\left(|M_{k+1}|^2|M_k|^{-2}\right), \quad (4.41)$$

$$\mathrm{Tr}\left([L_\xi, L_{\bar\xi}]^2\right) = 8\sum_{k=0}^{\lambda-1}\Big[\mathcal{S}_k^2\,\mathrm{Tr}\left([|M_{k+1}|^2|M_k|^{-2}]^2\right) - \mathcal{S}_k\mathcal{S}_{k-1}\,\mathrm{Tr}\left(|M_{k+1}|^2|M_{k-1}|^{-2}\right)\Big], \tag{4.42}$$

where by definition $\mathcal{S}_{-1} = 0$ and for $k = 0, ..., \lambda - 1$:

$$\mathcal{S}_k = [1 - \cos(g_k - g_{k+1})]. \tag{4.43}$$

4.3.2 Topological Charge

The topological charge (4.22), in covariant form, is

$$B = \frac{1}{24\pi^2}\int_{R^3}\sqrt{g}\,d^3x\left(\frac{\epsilon^{abc}}{\sqrt{g}}\right)\mathrm{Tr}\left(L_aL_bL_c\right), \tag{4.44}$$

where $g = det(g_{ab})$ and ϵ^{abc} the Levi-Civita symbol. In the spherical polar coordinates $(r,\,\theta,\,\phi)$, with $\epsilon^{abc} = \delta^{abc}_{r\theta\phi}$:

$$B = \frac{1}{8\pi^2}\int(dr\,d\theta\,d\phi)\,\mathrm{Tr}\left(L_r[L_\theta,\,L_\phi]\right). \tag{4.45}$$

This takes the form:

$$B = \frac{1}{8\pi^2}\int drd\xi d\bar\xi\,\mathrm{Tr}\left(L_r[L_\xi, L_{\bar\xi}]\right) \tag{4.46}$$

in the $(r,\,\xi,\,\bar\xi)$ coordinates.

Using the expression for L_ξ in (4.38) we find that

$$[L_\xi,\,L_{\bar\xi}] = 2\sum_{k=0}^{\lambda-1}\mathcal{S}_k\left(M_k|M_k|^{-2}|M_{k+1}|^2|M_k|^{-2}M_k^\dagger - M_{k+1}|M_k|^{-2}M_{k+1}^\dagger\right), \tag{4.47}$$

and so, using this commutator and L_r as given in (4.37), the topological charge (4.46) becomes

$$B = -\frac{1}{2\pi}\int dr\sum_{k=0}^{\lambda-1}(\dot g_k - \dot g_{k+1})\,(1-\cos(g_k - g_{k+1}))\mathcal{N}_k, \tag{4.48}$$

where $\mathcal{N}_k$ is given in (3.106).

As $g_k(\infty) = 0$, we finally obtain

$$B = \frac{1}{2\pi}\sum_{k=0}^{\lambda-1}\mathcal{N}_k\left(g_k(0) - g_{k+1}(0)\right), \tag{4.49}$$

and so we see that the only contributions to the topological charge comes from $g_k(0)$.

According to proposition 3.1, if $D_k = Det|M_k|^2$ is nonsingular in the whole complex ξ plane then $\mathcal{N}_k$ obey the recurrence relations (3.111). Thus if we know ω_k, the highest degree of $|\xi|^2$ in D_k, as determined from (3.110), then we can determine $\mathcal{N}_k$. In fact, if

$$H_0 = M_{01} \wedge M_{02} \wedge \ldots \wedge M_{0n}, \tag{4.50}$$

is the exterior product of the column vectors of M_0 which form the $n-$dimensional subspace of C^N, then

$$\omega_0 = deg(H_0), \tag{4.51}$$

i.e. is the highest degree of ξ in H_0.

4.4 Approximate Formulations

In this section, we derive field equations for the profile functions g_k from the energy (4.36) into which we have inserted the expressions of the generalised harmonic map ansatz.

To do this we, first of all, take a holomorphic matrix $M_0 = M(\xi)$ and then evaluate the sequence $(M_1, M_2, ..., M_\lambda)$ using the formulation of section 3.2.2. Then we compute the angular integrals $\mathcal{N}_k$ in (3.106),

$$\mathcal{I}_k = \frac{i}{2\pi}\int d\xi\, d\bar{\xi}\,(1+|\xi|^2)^2\,\mathrm{Tr}\left(\left[|M_{k+1}|^2|M_k|^{-2}\right]^2\right), \tag{4.52}$$

and

$$\mathcal{H}_k = \frac{i}{2\pi}\int d\xi\, d\bar{\xi}\,(1+|\xi|^2)^2\,\mathrm{Tr}\left(|M_{k+1}|^2|M_{k-1}|^{-2}\right), \tag{4.53}$$

for $k = 0, ..., \lambda - 1$, where by definition $\mathcal{H}_0 = 0$.

In terms of $\mathcal{N}_k$, $\mathcal{I}_k$ and $\mathcal{H}_k$, the energy (4.36), for our ansatz (22), reduces to

$$E = \frac{1}{6\pi}\int dr \left(r^2 \left[-\frac{n^2}{N}\left(\sum_{k=0}^{\lambda-1} \dot{g}_k\right)^2 + n\sum_{k=0}^{\lambda-1} \dot{g}_k^2 \right] \right.$$

$$\left. + 2\sum_{k=0}^{\lambda-1} \mathcal{N}_k \left[1 + \frac{1}{4}\left(\dot{g}_k - \dot{g}_{k+1}\right)^2\right] \mathcal{S}_k + \frac{1}{2r^2}\sum_{k=0}^{\lambda-1}\left[\mathcal{I}_k\mathcal{S}_k^2 - \mathcal{H}_k\mathcal{S}_k\mathcal{S}_{k-1}\right] \right). \tag{4.54}$$

Introducing

$$F_k = g_k - g_{k+1}, \tag{4.55}$$

with $g_\lambda = 0$ we find that, in terms of F_k, the energy integral (4.54) becomes

$$E = \frac{1}{6\pi}\int dr \left(r^2 \left[-\frac{n^2}{N}\left(\sum_{k=0}^{\lambda-1}(k+1)\dot{F}_k\right)^2 + n\sum_{k=0}^{\lambda-1}\left(\sum_{l=k}^{\lambda-1}\dot{F}_l\right)^2\right] \right.$$

$$+\frac{1}{2}\sum_{k=0}^{\lambda-1}\mathcal{N}_k\left(4+\dot{F}_k^2\right)(1-\cos F_k)$$

$$\left. +\frac{1}{2r^2}\sum_{k=0}^{\lambda-1}\left[\mathcal{I}_k(1-\cos F_k)^2 - \mathcal{H}_k(1-\cos F_k)(1-\cos F_{k-1})\right]\right), \tag{4.56}$$

and the topological charge (4.49) becomes

$$B = \frac{1}{2\pi}\sum_{k=0}^{\lambda-1}\mathcal{N}_k F_k(0). \tag{4.57}$$

To derive the equations for the profile functions g_k from (4.56) we note that

$$\frac{\partial\mathcal{E}}{\partial\dot{F}_l} = r^2\left[-\frac{2n^2(l+1)}{N}\sum_{i=0}^{\lambda-1}(i+1)\dot{F}_i + 2n\sum_{i=0}^{l}\left(\sum_{j=i}^{\lambda-1}\dot{F}_j\right)\right]+\mathcal{N}_l\dot{F}_l(1-\cos F_l), \tag{4.58}$$

where $\mathcal{E}$ denotes the integrand of E.

Thus our field equations for the functions F_i and so also for g_i are given by

$$\left[-\frac{2n^2(l+1)}{N}\sum_{i=0}^{\lambda-1}(i+1)\ddot{F}_i + 2n\sum_{i=0}^{l}\sum_{j=i}^{\lambda-1}\ddot{F}_j\right] + \frac{1}{r^2}\mathcal{N}_l\ddot{F}_l(1-\cos F_l)$$

$$+\frac{2}{r}\left[-\frac{2n^2(l+1)}{N}\sum_{i=0}^{\lambda-1}(i+1)\dot{F}_i + 2n\sum_{i=0}^{l}\sum_{j=i}^{\lambda-1}\dot{F}_j\right] + \frac{\sin F_l}{2r^2}\left[\mathcal{N}_l(\dot{F}_l^2-4)\right.$$

$$\left. -\frac{2\mathcal{I}_l(1-\cos F_l)}{r^2} + \frac{\mathcal{H}_l(1-\cos F_{l-1})}{r^2} + \frac{\mathcal{H}_{l+1}(1-\cos F_{l+1})}{r^2}\right] = 0. \tag{4.59}$$

Now a question arises: what is the best choice of the initial matrix M_0 that would yield low energy field configurations which, hopefully, are close to the exact solutions of the full equations of the model, *i.e.* equations (4.15). To answer this question we note that each $\mathcal{N}_k$ is, in fact, the energy of the Grassmannian $Gr(n, N)$ models. Thus in order to have minimal $\mathcal{N}_k$, and so also energy to be close to the exact multiskyrmion energy, the entries of the matrix M_0 must be *polynomials* in ξ as stated in Corollary 3.2 [15, 16].

In the following sections we make this choice for M_0 and consider various fields for the $SU(3)$, $SU(4)$ and $SU(5)$ cases with 1 and 2 rank-2 projector approximations. Our choices are dictated by simplicity and they lead to energy density distributions which are spherically symmetric. Moreover, looking at the general case we see that they also correspond to setting some g_i functions in (4.33) equal to zero.

4.5 One Projector Approximations

In this case, we take only one profile function $g_0 = F_0 = F$, *i.e.* $\lambda = 1$, and so the approximate energy (4.56) with the mass term reduces to

$$E = \frac{1}{6\pi}\int dr\Bigg(\frac{1}{2}\dot{F}^2\left[A_N(n)r^2 + \mathcal{N}_0(1-\cos F)\right] + 2\mathcal{N}_0(1-\cos F) + \frac{\mathcal{I}_0(1-\cos F)^2}{2r^2} + m_\pi^2\left[(N-n)\left(1-\cos\left[\frac{nF}{N}\right]\right) + n\left(1-\cos\left[\frac{(N-n)F}{N}\right]\right)\right]\Bigg), \tag{4.60}$$

where

$$A_N(n) = \frac{2n(N-n)}{N}, \tag{4.61}$$

and the field equation for the approximate function F becomes

$$\ddot{F}\left[A_N(n) + \frac{\mathcal{N}_0(1-\cos F)}{r^2}\right] + \frac{2A_N(n)}{r}\dot{F} + \frac{\sin F}{2r^2}\left[\mathcal{N}_0(\dot{F}^2-4) - \frac{2\mathcal{I}_0(1-\cos F)}{r^2}\right] - m_\pi^2 A_N(n)\left[\sin\left(\frac{nF}{N}\right) + \sin\left(\frac{(N-n)F}{N}\right)\right] = 0. \tag{4.62}$$

In the following we restrict our attention to the rank-2 case only, *i.e.* $n = 2$. To solve (4.62) we impose the boundary conditions: $F(0) = 2\pi$ and $F(\infty) = 0$. Thus, the baryon number of these configurations is $B = \mathcal{N}_0$. Finally, we compare the approximate energies of each subcase with the corresponding energies of the one rank-1 projector approximations [14].

As a first attempt to solve (4.62), let us take the initial matrix $M_0 = (M_{01}, M_{02})$, where both column vectors are given by the following Veronese type form:

$$M_{01} = \left(1, \sqrt{C_1^{N-1}}\xi, \ldots, \sqrt{C_k^{N-1}}\xi^k, \ldots, \xi^{N-1}\right)^T, \quad M_{02} = \partial_\xi M_{01}. \tag{4.63}$$

This special form of M_0 enables us to express the determinant D_0 of $|M_0|^2$ in the following closed form,

$$D_0 = (N-1)(1+|\xi|^2)^{2(N-2)}. \tag{4.64}$$

Then from (3.110) we conclude that $\mathcal{N}_0 = 2(N-2)$, which is consistent with (4.51). This result can be verified explicitly using definition (3.106) with the help of the recurrence relations (3.109) which yields

$$\mathrm{Tr}\left(|M_1|^2|M_0|^{-2}\right) = \frac{2(N-2)}{(1+|\xi|^2)^2}. \tag{4.65}$$

For this case, according to (3.91), $D_1 = 0$, and so according to formula (F.0.3), $\mathcal{I}_0 = \mathcal{N}_0^2 = 4(N-2)^2$.

In the $SU(3)$ case, we find that $A_3(2) = \frac{4}{3}$, $\mathcal{N}_0 = 2$, and so $\mathcal{I}_0 = 4$, which all coincide with the values of the corresponding quantities in the rank-1 projector approximation [13]. Thus, we conclude that their energies also coincide, *i.e.* $E = 2.44404$ ($m_\pi = 0$). This equality holds for $m_\pi \neq 0$ as well. This is to be expected as in this case our rank-2 projector is really a sum of two rank-1 projectors constructed from the first two vectors of the Veronese sequence.

It is clear that, for $SU(N)$ with $N > 3$ we have: $A_N(2) > A_N(1)$, $\mathcal{N}_0 > (N-1)$, and $\mathcal{I}_0 > (N-1)^2$. Thus, this Veronese type configuration for $N > 3$ will lead to energies higher in comparison with those for the rank-1 projector approximations [14].

So, in the following, we look only at M_0 with $\mathcal{N}_0 = (N-1)$ and $\mathcal{I}_0 < (N-1)^2$ in order to compensate $A_N(2) > A_N(1)$ in the energy integral (4.60). More specifically, we look at the following 2 subcases for which the determinant of $|M_0|^2$ is of the form:

$$D_0 = c(1+|\xi|^2)^{N-1}, \tag{4.66}$$

or

$$D_0 = c(1+\sigma|\xi|^2)^{N-3}(1+|\xi|^2)^2, \tag{4.67}$$

where c and σ are some constants. For the subcase (4.67) we choose the column vector M_{01} to have the Veronese type form (4.63) while $M_{02} = \partial_\xi^{N-2} M_{01}$.

4.5.1 $SU(3)$

As here we have $N = 3$, so subcase (4.67) is the same with the Veronese type subcase that we have discussed previously. Thus in this section we consider only the subcase (4.66).

In this subcase, we can choose the initial matrix M_0 to be given by

$$M_0 = \begin{bmatrix} 1 & \sqrt{a}\xi & \sqrt{b}\xi^2 \\ \\ 0 & 1 & \sqrt{c}\xi \end{bmatrix}^T . \tag{4.68}$$

If we now require that $D_0 = (1 + |\xi|^2)^2$, then $c = 2$ while there is an infinite number of solutions for a and b as here we have only one equation for a and b. In the following we restrict our attention to the solution: $a = 0$ and $b = 1$.

Starting from the corresponding initial matrix M_0 we find that

$$M_1 = \frac{\sqrt{2}}{D_0} \begin{bmatrix} \sqrt{2}\xi(-\bar{\xi}^2 & -\sqrt{2}\bar{\xi} & 1) \\ \\ -\bar{\xi}^2 & -\sqrt{2}\bar{\xi} & 1 \end{bmatrix}^T , \tag{4.69}$$

which is orthogonal to M_0, *i.e.* $M_0^\dagger M_1 = 0$. Note that $|M_1|^2$ is singular, *i.e.* $D_1 = 0$.

Using these two basis matrices, M_0 and M_1, we find from (3.106) that $\mathcal{N}_0 = 2$, and from (4.52) $\mathcal{I}_0 = 4$. These results coincide with the corresponding numbers in the one rank-1 projector approximation of the $SU(3)$ case described by the initial vector field [14]

$$f_0 = (1, \sqrt{2}\xi, \xi^2)^T. \tag{4.70}$$

As $A_3(2) = \frac{2}{3} = A_3(1)$ both rank-1 and rank-2 projector approximations have equal energy, *i.e.* $E = 2.44404$ ($m_\pi = 0$). This equality holds for $m_\pi \neq 0$ as well.

The result $D_1 = 0$ that we have encountered previously is in fact a general property for $SU(3)$ case with rank-2 projectors ansatz, which can be seen as follows.

Using the splitting relations (3.87) we see that

$$M_1 = \Big([I - P_{01}(1) - \widetilde{P}_{02}(1)]\partial_\xi M_{01},\, [I - P_{01}(1) - \widetilde{P}_{02}(1)]\partial_\xi M_{02}\Big)\,. \tag{4.71}$$

As the case $\partial_\xi M_{02} = 0$ is trivial so we will only consider the nonzero case. We observe that

$$\begin{aligned} V_0 &= M_{01}, \\ V_1 &= [I - P_{01}(1)]M_{02}, \\ V_2 &= [I - P_{01}(1) - \widetilde{P}_{02}(1)]\partial_\xi M_{02}, \end{aligned} \tag{4.72}$$

are mutually orthogonal and so they span C^3. Thus,

$$\partial_\xi M_{01} = \alpha V_0 + \beta V_1 + \gamma V_2, \tag{4.73}$$

where α, β and γ are expansion coefficients which depend on ξ and $\bar{\xi}$. As

$$[I - P_{01}(1) - \widetilde{P}_{02}(1)][\alpha V_0 + \beta V_1 + \gamma V_2] = \gamma V_2, \tag{4.74}$$

so

$$M_1 = (\gamma V_2, V_2)\,, \tag{4.75}$$

and it clearly follows that $|M_1|^2$ has a vanishing determinant.

4.5.2 $SU(4)$

$D_0 = c(1 + |\xi|^2)^3$

In this subcase we can choose the initial matrix M_0 to be given by

$$M_0 = \begin{bmatrix} 1 & 0 & \sqrt{a}\xi & \sqrt{b}\xi^2 \\ & & & \\ 0 & 1 & \sqrt{c}\xi & \sqrt{d}\xi^2 \end{bmatrix}^T . \tag{4.76}$$

If we require $D_0 = (1 + |\xi|^2)^3$, then there is an infinite number of solutions for a, b, c and d as here we have only 3 equations for the 4 parameters a, b, c, and d. In the following we consider the solution: $a = 2$, $b = 1$, $c = 1$, $d = 2$.

Then, starting from M_0 in (4.76) we have computed the corresponding mutually orthogonal matrix M_1 and by using these two basis matrices, M_0 and M_1, we have

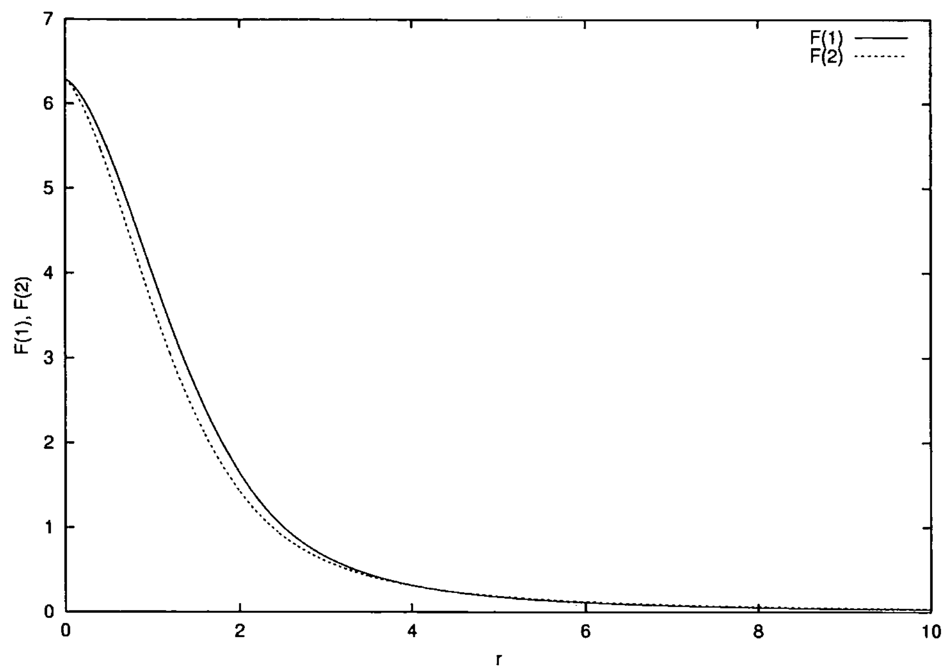

Figure 4.1: Approximate $SU(4)$ profile functions for rank-1 and 2 cases.

computed explicitly the integrals $\mathcal{N}_0$ and $\mathcal{I}_0$ in expressions (3.106) and (4.52), respectively and we have found that $\mathcal{N}_0 = 3$ and $\mathcal{I}_0 = \frac{23}{3}$. We note that, in the one projector rank-1 case with initial vector field

$$f_0 = (1, \sqrt{3}\xi, \sqrt{3}\xi^2, \xi^3)^T, \tag{4.77}$$

$\mathcal{I}_0 = 9$ which is larger than the above result. However, as $A_4(2) = 2$ while $A_4(1) = \frac{3}{4}$ it is not clear which energy is larger.

To assess this we have solved numerically equation (4.62) for F, and we found that it is very close to that found in [14] using rank-1 projector ansatz. In fig. 4.1, we compare the graphs of the solution F, denotes by $F(2)$, with the approximate profile functions using one rank-1 projector, $F(1)$, whereas the comparison of the corresponding energy densities (*i.e.* radial energy distributions) is presented in fig. 4.2.

In table 4.1 we present our results for the energies $E(2)$ and compare them with the results using one rank-1 projector, $E(1)$ [14], for different values of the mass m_π. We see that for all the masses (at least to $m_\pi = 30.0$) we always have $E(2) > E(1)$.

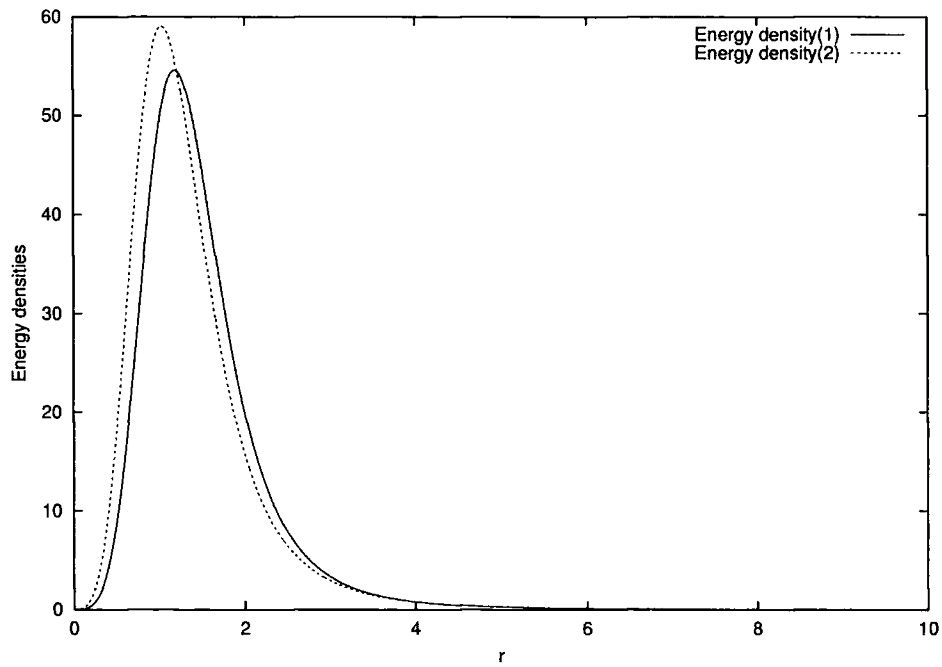

Figure 4.2: Approximate $SU(4)$ energy densities for rank-1 and 2 cases.

m_π	$E(1)$	$E(2)$
0.0	3.64410	3.81387
0.2	3.68291	3.86158
1.0	4.17241	4.42030
2.23	5.00186	5.33405
7.0	7.47187	8.02336
30.0	14.3393	15.4287

Table 4.1: Approximate energies of the $SU(4)$ Skyrme model.

$D_0 = c(1+\sigma|\xi|^2)(1+|\xi|^2)^2$

In this subcase, we choose the initial matrix M_0 to be given by the following Veronese type form

$$M_0 = \begin{bmatrix} 1 & \sqrt{3}\xi & \sqrt{3}\xi^2 & \xi^3 \\ \\ 0 & 0 & 2\sqrt{3} & 6\xi \end{bmatrix}^T, \quad (4.78)$$

which has $D_0 = 12(1+4|\xi|^2)(1+|\xi|^2)^2$. For this configuration, we have found that it has $\mathcal{N}_0 = 3$ but $\mathcal{I}_0 = 7.14357$ and its energy is $E = 3.76929$ ($m_\pi = 0$), which is a little lower than the energy of the previous subcase, but it is still higher than the energy of the one rank-1 projector approximation, *i.e.* $E = 3.64410$.

4.5.3 $SU(5)$

$D_0 = c(1+|\xi|^2)^4$

In this subcase we can choose the initial matrix M_0 to be

$$M_0 = \begin{bmatrix} 1 & \sqrt{a}\xi & \sqrt{b}\xi^2 & \sqrt{c}\xi^3 & \sqrt{d}\xi^4 \\ \\ 0 & 0 & 0 & 1 & \sqrt{e}\xi \end{bmatrix}^T. \quad (4.79)$$

If we require $D_0 = (1+|\xi|^2)^4$, then there is an infinite number of solutions for a, b, c, d and e as here we have only 4 equations for these 5 parameters. Here we consider only the solution: $a = b = c = e = \sqrt{2}$ and $d = 1$.

For this configuration, we have found that $\mathcal{N}_0 = 4$ but $\mathcal{I}_0 = 12.2667$ which is much lower than 16. However, as $A_5(2) = \frac{12}{5}$, we see that its energy is $E = 5.10580$ ($m_\pi = 0$) which is still higher than $E = 4.83792$ in the rank-1 projector approximation [18]. When we have solved the equations for the approximate profile function $F = g_0$ for each rank (1 and 2), we found that their difference is very small.

$$D_0 = c(1+\sigma|\xi|^2)^2(1+|\xi|^2)^2$$

In this subcase we choose the initial matrix M_0 to have the following Veronese type form

$$M_0 = \begin{bmatrix} 1 & 2\xi & \sqrt{6}\xi^2 & 2\xi^3 & \xi^4 \\ \\ 0 & 0 & 0 & 12 & 24\xi \end{bmatrix}^T, \tag{4.80}$$

which has $D_0 = 144(1+3|\xi|^2)^2(1+|\xi|^2)^2$.

We have found that this configuration has $\mathcal{N}_0 = 4$ but $\mathcal{I}_0 = 12.4444$ and its energy is $E = 5.11875$ ($m_\pi = 0$) which is higher than the energy of the previous subcase.

4.6 Two Projector Approximations

Now we consider the case of two projectors. Here we have 2 profile functions: F_0 and F_1, *i.e.* $\lambda = 2$, and so the energy integral (4.56) becomes

$$E = \frac{1}{12\pi}\int dr \Bigg(r^2\left[A_N(n)\dot{F}_0^2 + A_N(2n)\dot{F}_0\dot{F}_1 + A_N(2n)\dot{F}_1^2\right]$$
$$+\mathcal{N}_0(4+\dot{F}_0^2)(1-\cos F_0) + \mathcal{N}_1(4+\dot{F}_1^2)(1-\cos F_1)$$
$$+\frac{1}{r^2}\left[\mathcal{I}_0(1-\cos F_0)^2 - \mathcal{H}_1(1-\cos F_0)(1-\cos F_1) + \mathcal{I}_1(1-\cos F_1)^2\right]\Bigg), \tag{4.81}$$

where now, for simplicity, we have set $m_\pi = 0$.

The field equations for F_0 and F_1 are

$$\ddot{F}_0\left[A_N(n) + \frac{\mathcal{N}_0(1-\cos F_0)}{r^2}\right] + \frac{1}{2}A_N(2n)\ddot{F}_1 + \frac{2}{r}\left[A_N(n)\dot{F}_0 + \frac{1}{2}A_N(2n)\dot{F}_1\right]$$
$$+\frac{\sin F_0}{2r^2}\left[\mathcal{N}_0(\dot{F}_0^2-4) - \frac{2\mathcal{I}_0(1-\cos F_0)}{r^2} + \frac{\mathcal{H}_1(1-\cos F_1)}{r^2}\right] = 0, \tag{4.82}$$

$$\frac{1}{2}A_N(2n)\ddot{F}_0 + \ddot{F}_1\left[A_N(2n) + \frac{\mathcal{N}_1(1-\cos F_1)}{r^2}\right] + \frac{2}{r}\left[\frac{1}{2}A_N(2n)\dot{F}_0 + A_N(2n)\dot{F}_1\right]$$
$$+\frac{\sin F_1}{2r^2}\left[\mathcal{N}_1(\dot{F}_1^2-4) - \frac{2\mathcal{I}_1(1-\cos F_1)}{r^2} + \frac{\mathcal{H}_1(1-\cos F_0)}{r^2}\right] = 0. \tag{4.83}$$

In the following, we take $n = 2$ and solve these equations (numerically) by imposing the boundary conditions: $F_0(0) = 2\pi$, $F_1(0) = 0$, and $F_0(\infty) = F_1(\infty) = 0$. Thus, the baryon number of these configurations is $B = \mathcal{N}_0$. We then compare the approximate energies of each of these cases with the corresponding energies of the one rank-1 projector approximations [14].

For the $SU(3)$ configurations, we have shown that $|M_1|^2$ is singular. Thus for this case the projector P_1 does not exist. As for the $SU(4)$ and $SU(5)$ cases that we have considered previously $|M_1|^2$ is nonsingular, so in this section we consider these two cases only.

4.6.1 $SU(4)$

Starting from the initial matrix M_0 in (4.76), we find that

$$M_2 = 0, \tag{4.84}$$

so from (3.106), (4.52) and (4.53) we have $\mathcal{N}_1 = 0$, $\mathcal{I}_1 = 0$, and $\mathcal{H}_1 = 0$. As for this case, $N = 4$ and $n = 2$, the energy integral (4.81) reduces to the energy integral (4.60) for the corresponding one projector of rank-2 projector approximation.

We note that $M_2 = 0$ is in fact a general property for $SU(4)$ with rank-2 projectors ansatz. To prove this it is convenient to use the construction (3.80). Then using the splitting relations (3.87) in (3.80) for M_2 gives

$$\begin{aligned} M_2 &= [I - P_0(2) - P_1(2)]\partial_\xi^2 M_0, \\ &= [I - (P_{01}(1) + \widetilde{P}_{02}(1)) - (P_{11}(1) + \widetilde{P}_{12}(1))]\partial_\xi^2 M_0. \end{aligned} \tag{4.85}$$

Now, $M_2 = 0$ follows from the completeness relation for the rank-1 projectors in C^4, *i.e.*

$$P_{01}(1) + \widetilde{P}_{02}(1) + P_{11}(1) + \widetilde{P}_{12}(1) = I. \tag{4.86}$$

4.6.2 $SU(5)$

As the subcase $D_0 = c(1+|\xi|^2)^4$ in the one projector approximation has lower energy, we restrict our attention to this subcase. Starting from the matrix M_0 in (4.80), we have computed the corresponding matrices M_1 and M_2, and we have found that $\mathcal{N}_1 = 2.0$, $\mathcal{I}_1 = 4.28989$, and $\mathcal{H}_1 = 3.57357$.

Solving equations (4.82)-(4.83) with the correct values for $\mathcal{N}_k$, $\mathcal{I}_k$, and $\mathcal{H}_k$, $k = 0$ and 1, we have found that, as $A_5(2) = \frac{12}{5}$ and $A_5(4) = \frac{8}{5}$, this configuration has energy: $E = 5.02469$. This is higher than the energy of the one rank-1 projector approximation case, *i.e.* $E = 4.83792$; however, they are marginally higher than the exact energy of the $SU(2)$ case with $B = 4$, *i.e.* $E = 4.464$ [59].

4.7 Exact Spherically Symmetric Solutions

In this section we consider whether one could also construct exact spherically symmetric solutions of the $SU(N)$ Skyrme field equations (4.15) using the general harmonic maps ansatz (4.33) with λ rank-2 projectors.

4.7.1 Condition for an Exact Spherically Symmetric Solution

When written in the spherical polar coordinates $(r,\, \theta,\, \phi)$, equations (4.15) without the mass term become

$$\partial_r\left(r^2 L_r\right) + \frac{1}{\sin\theta}\left[\partial_\theta\left(\sin\theta L_\theta\right) + \frac{1}{\sin\theta}\partial_\phi L_\phi\right]$$
$$-\frac{1}{4}\left[\partial_r\left(r^2 C_r\right) + \frac{1}{\sin\theta}\left\{\partial_\theta\left(\sin\theta C_\theta\right) + \frac{1}{\sin\theta}\partial_\phi C_\phi\right\}\right] = 0, \qquad (4.87)$$

where

$$C_a = [L_r,\, [L_r,\, L_a]] + \frac{1}{r^2}[L_\theta,\, [L_\theta,\, L_a]] + \frac{1}{r^2\sin^2\theta}[L_\phi,\, [L_\phi,\, L_a]], \qquad (4.88)$$

for $a = (r,\, \theta,\, \phi)$.

In the spherical holomorphic coordinates $(r,\, \xi,\, \bar{\xi})$, using (4.31)-(4.32), equations (4.87) become

$$\partial_r \left[r^2 L_r + \frac{(1+|\xi|^2)^2}{8} ([L_{\bar{\xi}}, [L_r, L_\xi]] + [L_\xi, [L_r, L_{\bar{\xi}}]]) \right] + \frac{(1+|\xi|^2)^2}{2} \left(\partial_{\bar{\xi}} L_\xi + \partial_\xi L_{\bar{\xi}} \right) + \frac{(1+|\xi|^2)^2}{8} \left(\partial_\xi \left([L_r, [L_{\bar{\xi}}, L_r]]\right) + \partial_{\bar{\xi}} \left([L_r, [L_\xi, L_r]]\right) \right) + \frac{(1+|\xi|^2)^2}{16 r^2} \left[\partial_{\bar{\xi}} \Big((1+|\xi|^2)^2 [L_\xi, [L_\xi, L_{\bar{\xi}}]] \Big) - \partial_\xi \Big((1+|\xi|^2)^2 [L_{\bar{\xi}}, [L_\xi, L_{\bar{\xi}}]] \Big) \right] = 0. \tag{4.89}$$

Now, let us look in detail at all the terms in these equations in our generalised harmonic map ansatz (4.33) case. We find that

$$\partial_{\bar{\xi}} L_\xi + \partial_\xi L_{\bar{\xi}} = 2i \sum_{l=0}^{\lambda-1} \sin F_l \Big[M_{l+1} |M_l|^{-2} M_{l+1}^\dagger - M_l |M_l|^{-2} |M_{l+1}|^2 |M_l|^{-2} M_l^\dagger \Big], \tag{4.90}$$

$$[L_{\bar{\xi}}, [L_r, L_\xi]] = -2i \sum_{l=0}^{\lambda-1} \mathcal{S}_l \dot{F}_l \Big[M_{l+1} |M_l|^{-2} M_{l+1}^\dagger - M_l |M_l|^{-2} |M_{l+1}|^2 |M_l|^{-2} M_l^\dagger \Big], \tag{4.91}$$

$$[L_r, [L_\xi, L_r]] = \sum_{l=0}^{\lambda-1} \dot{F}_l^2 b_l M_{l+1} |M_l|^{-2} M_l^\dagger, \tag{4.92}$$

$$[L_\xi, [L_\xi, L_{\bar{\xi}}]] = 2 \sum_{l=0}^{\lambda-1} \Bigg[2 \mathcal{S}_l b_l M_{l+1} |M_l|^{-2} |M_{l+1}|^2 |M_l|^{-2} |M_l^\dagger - \mathcal{S}_l b_{l+1} M_{l+2} |M_l|^{-2} M_{l+1}^\dagger - \mathcal{S}_{l+1} b_l M_{l+1} |M_{l+1}|^{-2} |M_{l+2}|^2 |M_l|^{-2} M_l^\dagger \Bigg], \tag{4.93}$$

where $\mathcal{S}_l$ is given by (4.43) and where

$$b_l = \left(e^{iF_l} - 1 \right). \tag{4.94}$$

We have checked that none of the configurations that we have considered so far in this paper is an exact solution of (4.89). This can be seen as follows. First, we multiply (4.89) by M_l from the right, which results in a set of equations for M_l. As the terms $\partial_r(r^2 L_r) M_l$ are proportional to M_l while the others are not, the contracted equations have the following general structure

$$a_l M_l + \sum_{k=0}^{\lambda-1} M_k A_{kl} = 0, \tag{4.95}$$

where a_l depend only on N, n, $\dot{g}_i$, $\ddot{g}_i$, and r while the $(n \times n)$ matrices A_{kl} depend on g_i, ξ and $\bar{\xi}$ as well. Clearly, these equations are inconsistent, unless each matrix A_{kl} is proportional to the $(n \times n)$ unit matrix I_n, *i.e.* $A_{kl} = b_{kl} I_n$ where each b_{kl} is independent of ξ and $\bar{\xi}$.

Armed with this observation we can now ask the following question: for which forms of the matrices M_k do the $SU(N)$ Skyrme field equations (4.89) have exact solutions? In appendix G, we have looked at the $SU(N)$ chiral models and we found that, if we are able to find the matrices M_k which satisfy the condition

$$|M_{k+1}|^2 |M_k|^{-2} = \mathcal{K}_k (1 + |\xi|^2)^{-2} I_n, \tag{4.96}$$

where $\mathcal{K}_k$ are some constants which depend on N, n and k then this configuration could possibly give exact solutions for the profile functions g_k. Note that, for rank-1 projectors case, this condition is satisfied by the Veronese sequence M_k, $k = 1, \ldots, (N-1)$, where in Ref. [14] it had been shown that the corresponding field configurations give exact solutions for g_k.

To see how this may work in our case we have put the condition (4.96) into (4.90)- (4.93), which has turned the field equations (4.89) into the following reduced set

$$\sum_{k=0}^{\lambda-1} \left[(P_k - \frac{nI}{N})\alpha_k + (P_{k+1} - P_k)\beta_k + (P_{k+2} - P_{k+1})\gamma_k \right] = 0, \tag{4.97}$$

where

$$\alpha_k = \partial_r \left(r^2 \dot{g}_k \right), \tag{4.98}$$

$$\beta_k = -\frac{1}{2}\mathcal{K}_k \dot{F}_k + \mathcal{K}_k \left(1 + \frac{1}{4}\dot{F}_k^2 \right) \sin F_k + \frac{1}{4r^2} \left[2\mathcal{K}_k^2 \mathcal{S}_k - \mathcal{K}_k \mathcal{K}_{k+1} \mathcal{S}_{k+1} \right] \sin F_k, \tag{4.99}$$

$$\gamma_k = -\frac{1}{4r^2} \left[\mathcal{K}_k \mathcal{K}_{k+1} \mathcal{S}_k \sin F_{k+1} \right]. \tag{4.100}$$

Next, we multiply (4.97) from the right by M_l and follow the procedure of appendix G, which yields

$$\partial_r \left[r^2 \left(\sum_{p=0}^{l} \sum_{q=p} \dot{F}_q - \frac{n(l+1)}{N} \sum_{p=0}^{N-2} (p+1)\dot{F}_p \right) + \frac{1}{2}\mathcal{K}_l \mathcal{S}_l \dot{F}_l \right]$$

$$-\frac{1}{4}\sin F_l \left[\mathcal{K}_l \left(4 + \dot{F}_l\right)^2 + \frac{1}{r^2}\left\{2\mathcal{K}_l^2 \mathcal{S}_l - \mathcal{K}_{l-1}\mathcal{K}_l \mathcal{S}_{l-1} + \mathcal{K}_l \mathcal{K}_{l+1}\mathcal{S}_{l+1}\right\}\right] = 0. \quad (4.101)$$

Substituting condition (4.96) into (3.106), (4.52) and (4.53) yields the relations:

$$\mathcal{N}_k = n\mathcal{K}_k, \quad (4.102)$$

$$\mathcal{I}_k = n\mathcal{K}_k^2, \quad (4.103)$$

$$\mathcal{H}_k = n\mathcal{K}_k \mathcal{K}_{k-1}. \quad (4.104)$$

Using relations (4.102) - (4.104) in (4.101), we found that the resulting equations for the profile functions g_k do, indeed, coincide with (4.144). This also means that the energy integral (4.36), in this case, is exact (*i.e.* not an approximation). As the corresponding energy density is a function of the radial coordinate r only, solutions of the field equations (4.87) are spherically symmetric.

4.7.2 Further Analysis of the Condition (4.96)

As the matrices M_k for $k \neq 0$ are generated from the initial matrix M_0, so in this section, we derive the conditions that M_0 should satisfy in order for condition (4.96) to hold. To do so, we put (4.96) into (3.108) and (3.109) with $k = 0$, and obtain

$$\partial_\xi \left[\left(\partial_{\bar{\xi}} |M_0|^2\right) |M_0|^{-2} \right] = \mathcal{K}_0 (1 + |\xi|^2)^{-2} I_n, \quad (4.105)$$

and

$$\partial_\xi \partial_{\bar{\xi}} \left[log\, D_0 \right] = n\mathcal{K}_0 (1 + |\xi|^2)^{-2}, \quad (4.106)$$

respectively.

The general solution of equation (4.105) is

$$|M_0|^2 = (1 + |\xi|^2)^{\mathcal{K}_0} G(\bar{\xi}, \xi), \quad (4.107)$$

where $G(\bar{\xi}, \xi)$ is an $(n \times n)$ Hermitian matrix satisfying

$$\partial_\xi \left[\left(\partial_{\bar{\xi}} G\right) G^{-1} \right] = 0. \quad (4.108)$$

Equation (4.108) has the general solution [60]

$$G(\bar{\xi},\xi) = \bar{H}(\bar{\xi})H(\xi), \tag{4.109}$$

where H and $\bar{H} = H^{\dagger}$ are arbitrary $(n \times n)$ matrices of one variable.

Thus, with $G(\bar{\xi},\xi)$ given by (4.109), solution (4.107) becomes

$$|M_0|^2 = (1+|\xi|^2)^{\mathcal{K}_0}\bar{H}(\bar{\xi})H(\xi), \tag{4.110}$$

and from condition (4.96) it follows that (for $l \geq 1$)

$$|M_l|^2 = \mathcal{K}_0\mathcal{K}_1 \dots \mathcal{K}_{l-1}(1+|\xi|^2)^{\mathcal{K}_0-2l}\bar{H}(\bar{\xi})H(\xi). \tag{4.111}$$

Furthermore, from (4.107) it follows that

$$D_0 = (1+|\xi|^2)^{n\mathcal{K}_0}Det[\bar{H}(\bar{\xi})]Det[H(\xi)], \tag{4.112}$$

is a solution of (4.106). By putting D_0 into (3.113) we have found that $\mathcal{N}_0 = n\mathcal{K}_0$, as we have assumed that $Det[H(\xi)]$ is holomorphic (and $Det[\bar{H}(\bar{\xi})]$ is antiholomorphic) with the only singularity at $|\xi| \to \infty$, which is consistent with (4.102). Thus, we can choose $Det[G(\bar{\xi},\xi)] = 1$, and so from (4.111) it follows that

$$D_l = (\mathcal{K}_0\mathcal{K}_1 \dots \mathcal{K}_{l-1})^n\,(1+|\xi|^2)^{n(\mathcal{K}_0-2l)}, \tag{4.113}$$

from which, according to (3.110),

$$\omega_l = n(\mathcal{K}_0 - 2l). \tag{4.114}$$

Using the recurrence relations (3.111), with ω_l given by (4.114), we derive

$$\mathcal{N}_l = (l+1)\,(\mathcal{N}_0 - nl)\,. \tag{4.115}$$

Now, with $|M_0|^2$ given by (4.110), the projector $P_0(n) = M_0|M_0|^{-2}M_0^{\dagger}$ becomes

$$P_0(n) = (1+|\xi|^2)^{-\mathcal{K}_0}\tilde{M}_0\tilde{M}_0^{\dagger}, \tag{4.116}$$

where $\tilde{M}_0 = M_0H^{-1}$. Using (4.110) then it follows that the matrix $\tilde{M}_0$ satisfies

$$|\tilde{M}_0|^2 = \bar{H}^{-1}|M_0|^2H^{-1} = (1+|\xi|^2)^{\mathcal{K}_0}I_n. \tag{4.117}$$

Equation (4.117) implies that the column vectors of the $(N \times n)$ matrix $\tilde{M}_0 = \left(\tilde{M}_{01}, \ldots, \tilde{M}_{0n}\right)$ are orthogonal. Thus, using the $SU(N)$ global symmetry, we can bring the column vectors $\tilde{M}_{0j}$, $(j = 1, 2, .., n)$ to live in n-disjoint subspaces. In this case, the projector $P_0(n)$ has a block diagonal form. For example, in the rank-2 projector ansatz, *i.e.* $n = 2$, for $N = even$, we have

$$P_0(2) = \begin{bmatrix} P_{01}(1) & 0 \\ & \\ 0 & P_{02}(1) \end{bmatrix}, \tag{4.118}$$

where

$$P_{0s}(1) = \frac{\tilde{M}_0 \tilde{M}_0^\dagger}{|\tilde{M}_0|^2}, \tag{4.119}$$

$s = 0, 1$, are rank-1 projectors. For $N = odd$, still in the rank-2 projector ansatz, equation (4.117) requires that one of the entries of $\tilde{M}_{0j}$ should be zero, so if we choose $\left(\tilde{M}_{0j}\right)_N = 0$, then the $((N-1) \times (N-1))$ submatrix of $P_0(2)$ has the block form (4.118) while $P_0(2)_{NN} = 0$.

Thus, as far as condition (4.96) is concerned, it seems that the only exact spherically symmetric solutions of the $SU(N)$ Skyrme models using projectors of rank-2 are embeddings of a pair of $SU([N]/2)$ solutions of rank-1 projector ansatz, where $[N] = N$, or $(N-1)$ for N even or odd, respectively.

4.7.3 Some Specific Configurations

In this final subsection we consider only the rank-2 projector ansatz, *i.e.* $n = 2$. Thus, here, the $SU(3)$ case should be excluded, as from our analysis in section 4.5.1, $|M_1|^2$ is singular so it is automatically not proportional to the nonsingular matrix $|M_0|^2$ as required by (4.96). For the same reason, we also exclude the case $M_{02} = \partial_\xi M_{01}$, *i.e.* equation (3.90). Next, we have a look at some specific forms of the initial matrix M_0, for the $N = 4$ and 6 cases.

SU(4)

In this case, we can choose the initial matrix M_0 to be given by

$$M_0 = \begin{bmatrix} 1 & \xi & f(\xi) & \xi f(\xi) \\ \\ 0 & 0 & 1 & \xi \end{bmatrix}^T, \tag{4.120}$$

where $f(\xi)$ is an arbitrary polynomial function of only ξ. Then

$$|M_0|^2 = (1+|\xi|^2) \begin{bmatrix} 1 & \bar{f}(\bar{\xi}) \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ f(\xi) & 1 \end{bmatrix}, \tag{4.121}$$

which is of the form (4.110).

For the corresponding projector $P_0(2)$, we have found by direct evaluation that it has the block form (4.118) with

$$P_{0s}(1) = \frac{1}{(1+|\xi|^2)} \begin{bmatrix} 1 & \bar{\xi} \\ \\ \xi & |\xi|^2 \end{bmatrix}, \tag{4.122}$$

$s = 0, 1$, which clearly are the rank-1 projectors of a one skyrmion $SU(2)$ solution. As $D_0 = (1+|\xi|^2)^2$, this configuration has $\mathcal{N}_0 = 2$, or $\mathcal{K}_0 = 1$, and so according to (4.103), $\mathcal{I}_0 = 2$. Then from (4.115) we obtained $\mathcal{N}_1 = 0$ or $\mathcal{K}_1 = 0$ (according to (4.102)) so from (4.111) we conclude that $M_l = 0$, for $l \geq 2$ which is consistent with our general result in subsection VI.A. Thus, this configuration has only one projector, *i.e.* $P_0(2)$.

Substituting $\mathcal{N}_0 = \mathcal{I}_0 = A_4(2) = 2$ in the 1-projector energy integral (4.60) we found that it has energy: $E = 2E_{SU(2)}$, where $E_{SU(2)}$ is the energy of the $SU(2)$ Skyrme model [14], as we would have been expected.

SU(6)

In this case, we can choose the initial matrix M_0 to be given by

$$M_0 = \begin{bmatrix} 1 & \sqrt{2}\xi & \xi^2 & f(\xi) & \sqrt{2}\xi f(\xi) & \xi^2 f(\xi) \\ \\ 0 & 0 & 0 & 1 & \sqrt{2}\xi & \xi^2 \end{bmatrix}^T, \tag{4.123}$$

from which it follows that $|M_0|^2$ is of the form (4.110) as well with $\mathcal{K}_0 = 2$.

We have also found that the corresponding projector $P_0(2)$, has the block form (4.118) with $P_{0s}(1)$, $s = 0, 1$, are the rank-1 projectors of the $SU(3)$ solution [14]. As $D_0 = (1 + |\xi|^2)^4$, this configuration has $\mathcal{N}_0 = 4$, from which we derived that $\mathcal{I}_0 = 8$, $\mathcal{N}_1 = 4$, $\mathcal{I}_1 = 8$, $\mathcal{H}_1 = 8$, but $\mathcal{N}_2 = 0$ or $\mathcal{K}_2 = 0$ and so $M_l = 0$, for $l \geq 3$. Thus, this configuration has only 2 projectors, *i.e.* $P_0(2)$ and $P_1(2)$.

Substituting the correct values for $\mathcal{N}_k$, $\mathcal{I}_k$, and $\mathcal{H}_k$, $k = 0, 1$ into the 2 projectors energy integral (4.81) we have found that, as $A_6(2) = A_6(4) = \frac{8}{3}$, this configuration has energy: $E = 2E_{SU(3)}$, where $E_{SU(3)}$ is the energy of the $SU(3)$ Skyrme model [14].

4.8 Alternative $SU(N)$ Skyrme Models

Until very recently most of the studies involving the Skyrme models have concentrated on the $SU(2)$ version of the model. However, when one considers $SU(N)$, for $N \geq 3$, one has to bear in mind that the Skyrme model is not unique. In fact, there are two possible versions of the fourth order Skyrme term [61] and so in this section we study the model based on the alternative form of the fourth-order Skyrme term.

Thus instead of looking at

$$\frac{1}{32a^2}\mathrm{Tr}\,[L_\mu,\, L_\nu]^2\,, \tag{4.124}$$

in (4.1) we consider

$$-\frac{1}{16a^2}\left[(\mathrm{Tr}\,(L_\mu L^\mu))^2 - (\mathrm{Tr}\,(L_\mu L_\nu))^2\right]. \tag{4.125}$$

Note that when $N = 2$ the two expressions are the same. To see this we introduce

$$L_\mu = U^{-1}\,\partial_\mu U \;=\; iL_\mu^{(k)}\sigma^{(k)}, \tag{4.126}$$

where $\sigma^{(k)}$, $k = 1, 2, 3$, are Pauli matrices. Then, using the properties of Pauli matrices, we find that

$$\mathrm{Tr}\,(L_\mu L_\nu) \;=\; -2\left(L_\mu^{(k)}L_\nu^{(k)}\right), \tag{4.127}$$

and so

$$\begin{aligned} \mathrm{Tr}\,[L_\mu, L_\nu]^2 &= -8\left[L_\mu^{(k)}L^{(k)\mu}L_\nu^{(l)}L^{(l)\nu} - L_\mu^{(k)}L_\nu^{(k)}L^{(l)\mu}L^{(l)\nu}\right] \\ &= -2\left[(\mathrm{Tr}\,(L_\mu L^\mu))^2 - (\mathrm{Tr}\,(L_\mu L_\nu))^2\right]. \end{aligned} \tag{4.128}$$

However, for $N > 2$ the two expressions are different.

Thus, the alternative $SU(N)$ Skyrme models are determined by the action (4.1) with the Skyrme term (4.124) is replaced by the alternative term (4.125), *i.e.*

$$S = \int d^4x \left(-\frac{F^2}{16}\mathrm{Tr}\,(L_\mu L^\mu) - \frac{1}{16a^2}\left[(\mathrm{Tr}\,(L_\mu L^\mu))^2 - (\mathrm{Tr}\,(L_\mu L_\nu))^2\right]\right), \tag{4.129}$$

where here we have put $M_\pi = 0$. The corresponding field equations are

$$\partial_\mu \left(L^\mu + \frac{2}{a^2F^2}\left[\left(\mathrm{Tr}\,(L_\nu)^2\right)L^\mu - \left(\mathrm{Tr}\,(L^\mu L_\nu)\right)L^\nu\right]\right) = 0. \tag{4.130}$$

In the following two sections, *i.e.* 4.9-4.10, which are taken from Ref. [18], we apply the same ideas of harmonic map ansatz to these models.

4.9 Spherically Symmetric Solutions - General Discussion

In this section, we shall consider general discussion on static spherically symmetric energy and field equations of the alternative $SU(N)$ Skyrme models by applying harmonic map ansatz (4.33) involving $(N-1)$ rank-1 projectors P_k, $k = 0, \ldots, (N-2)$. As for the ususal $SU(N)$ Skyrme models [14], we show that this leads to *exact* solutions for the alternative models as well.

4.9.1 Static Energy and Equations for the Profile Functions

From the action (4.129) we derive that the corresponding static energy and field equations in the spherical holomorphic coordinates $(r, \xi, \bar{\xi})$, in topological charge unit, are

$$E = -\frac{i}{12\pi^2}\int \frac{d\xi d\bar{\xi}}{(1+|\xi|^2)^2} r^2 dr \left[\mathrm{Tr}(L_r^2) + \frac{1}{r^2}Q\left(1 - \frac{1}{2}\mathrm{Tr}(L_r^2) - \frac{1}{8r^2}Q\right)\right], \tag{4.131}$$

and

$$\partial_r\left(r^2L_r\right)+\frac{(1+|\xi|^2)^2}{2}\left(\partial_\xi L_{\bar\xi}+\partial_{\bar\xi}L_\xi\right)$$
$$-\frac{(1+|\xi|^2)^2}{4}\partial_r\left[\{\mathrm{Tr}\left(L_\xi L_{\bar\xi}\right)L_r-\mathrm{Tr}\left(L_rL_\xi\right)L_{\bar\xi}\}+(\xi\leftrightarrow\bar\xi)\right]$$
$$-\frac{(1+|\xi|^2)^2}{4}\left[\partial_\xi\{\mathrm{Tr}\left(L_r^2\right)L_{\bar\xi}-\mathrm{Tr}\left(L_{\bar\xi}L_r\right)L_r\}+(\xi\leftrightarrow\bar\xi)\right]$$
$$+\frac{(1+|\xi|^2)^2}{4r^2}\left[\partial_\xi\{\frac{(1+|\xi|^2)^2}{2}\left(\mathrm{Tr}\left(L_{\bar\xi}^2\right)L_\xi-\mathrm{Tr}\left(L_\xi L_{\bar\xi}\right)L_{\bar\xi}\right)\}+(\xi\leftrightarrow\bar\xi)\right]=0, \tag{4.132}$$

respectively, where L_r is given by (4.37), L_ξ by (4.38) and

$$Q=Q(r,\xi,\bar\xi)=(1+|\xi|^2)^2\mathrm{Tr}(L_\xi L_{\bar\xi}). \tag{4.133}$$

Let us simplify the field equations (4.132). First, we note that the orthogonality of the vectors M_k yields

$$\mathrm{Tr}(L_rL_\xi)=\mathrm{Tr}(L_\xi^2)=\mathrm{Tr}(L_{\bar\xi}^2)=0. \tag{4.134}$$

Next, using these results and the fact that $\mathrm{Tr}\,(L_r^2)$, given in (4.39), with $n=1$ and $\lambda=(N-1)$, is independent of $(\xi,\bar\xi)$ we see that (4.132) reduces to

$$\partial_r\left[\left(r^2-\frac{1}{2}Q\right)L_r\right]$$
$$+\frac{(1+|\xi|^2)^2}{2}\left[\left(\partial_\xi L_{\bar\xi}+\partial_{\bar\xi}L_\xi\right)\left\{1-\frac{1}{2}\mathrm{Tr}(L_r^2)\right\}-\frac{1}{4r^2}\{\partial_\xi\left(QL_{\bar\xi}\right)+\partial_{\bar\xi}\left(QL_\xi\right)\}\right]=0. \tag{4.135}$$

Let us choose the vectors M_k $(k=0,\ldots,(N-2))$ to be given by the Veronese sequence as described in proposition 3.2, so that the recursive quotient $(|M_{k+1}|^2/|M_k|^2)$ is given by (3.135). Thus from now on, we restrict to the $n=1$, $\lambda=(N-1)$ case of the general formula we derive in section 4.3.1.

Then we find that Q in (4.133), with $\mathrm{Tr}(L_\xi L_{\bar\xi})$ given by (4.40), reduces to

$$Q=-2\sum_{k=0}^{N-2}\mathcal{D}_k, \tag{4.136}$$

where

$$\mathcal{D}_k=(k+1)(N-k-1)\mathcal{S}_k, \tag{4.137}$$

and where $\mathcal{S}_k$ is given by (4.43).

Let us reconsider the energy (4.131) using this special choice of the vectors M_k. Substituting (4.136), $\mathrm{Tr}(L_r^2)$ as given by (4.39), and (4.136) into (4.131) and then carried out the integration with respect to the holomorphic coordinates $(\xi, \bar{\xi})$, gives us

$$E = -\frac{1}{6\pi}\int r^2 dr\Bigg[\left\{\frac{1}{N}\left(\sum_{k=0}^{N-2}(k+1)\dot{F}_k\right)^2 - \sum_{k=0}^{N-2}\left(\sum_{l=k}^{N-2}\dot{F}_l\right)^2\right\}\left\{1 + \frac{1}{r^2}\sum_{l=0}^{N-2}\mathcal{D}_l\right\}$$
$$- \frac{2}{r^2}\sum_{k=0}^{N-2}\mathcal{D}_k\left(1 + \frac{1}{4r^2}\sum_{l=0}^{N-2}\mathcal{D}_l\right)\Bigg], \tag{4.138}$$

where $F_k = (g_k - g_{k+1})$, $k = 0, \ldots, (N-2)$, with $g_{N-1} = 0$.

Let us now consider the field equations (4.135) to derive the equations for the profile functions g_k. To do this we look in detail at all the terms in these equations. From (4.90), with M_k given by the Veronese sequence, we find that

$$\partial_\xi L_{\bar{\xi}} + \partial_{\bar{\xi}} L_\xi = \frac{2i}{(1+|\xi|^2)^2}\sum_{k=0}^{N-2}(k+1)(N-k-1)\sin F_k(P_{k+1} - P_k). \tag{4.139}$$

The term L_r is given in (4.37), while $\mathrm{Tr}(L_r^2)$ is given in (4.39).

Using these results we find that (4.135) reduce to the the following factorised form

$$\sum_{k=0}^{N-2}\left[(P_k - \frac{1}{N})\widetilde{\alpha}_k + (P_{k+1} - P_k)\widetilde{\beta}_k\right] = 0, \tag{4.140}$$

where

$$\widetilde{\alpha}_k = \partial_r\left[(r^2 - \frac{1}{2}Q)\dot{g}_k\right], \tag{4.141}$$

and

$$\widetilde{\beta}_k = (k+1)(N-k-1)\sin F_k\left[1 - \frac{1}{2}\mathrm{Tr}(R_r^2) - \frac{1}{4r^2}Q\right]. \tag{4.142}$$

Next, we multiply (4.140) from the right by the vectors M_m $(m = 0, 1, \ldots, N-2)$ and follow the procedure of appendix G. This yields

$$\sum_{k=0}^{l}\widetilde{\alpha}_k - \frac{(l+1)}{N}\sum_{n=0}^{N-2}\widetilde{\alpha}_n - \widetilde{\beta}_l = 0, \tag{4.143}$$

which gives the required equations for the profile functions F_k or g_k. They are

$$\left[-\frac{l+1}{N}\sum_{i=0}^{N-2}(i+1)\ddot{F}_i + \sum_{i=0}^{l}\sum_{j=i}^{N-2}\ddot{F}_j\right] A_{N2} +$$
$$\frac{2}{r}\left[-\frac{l+1}{N}\sum_{i=0}^{N-2}(i+1)\dot{F}_i + \sum_{i=0}^{l}\sum_{j=i}^{N-2}\dot{F}_j\right] A_{N1}$$
$$-\frac{1}{r^2}\left[(l+1)(N-l-1)\sin F_l\right] A_{N0} = 0, \tag{4.144}$$

where

$$A_{N2} = 1 + \frac{1}{r^2}\sum_{k=0}^{N-2}\mathcal{D}_k, \tag{4.145}$$

$$A_{N1} = 1 + \frac{1}{2r}\sum_{k=0}^{N-2}\dot{\mathcal{D}}_k, \tag{4.146}$$

$$A_{N0} = -\frac{1}{2N}\left(\sum_{i=0}^{N-2}(i+1)\dot{F}_i\right)^2 + \frac{1}{2}\sum_{i=0}^{N-2}\left(\sum_{j=i}^{N-2}\dot{F}_j\right)^2 + 1 + \frac{1}{2r^2}\sum_{k=0}^{N-2}\mathcal{D}_k. \tag{4.147}$$

Equation (4.144) coincides with stationary points (minima or saddle points) of the static energy (4.138) [18]. This means that, with the vectors M_k given by the Veronese sequence, the energy integral (4.138) is exact (*i.e.* not an approximation).

In the next section, to compare the obtained results with the results discussed in [13], we look in detail at the simplest cases of $N = 3$, 4 and 5.

4.9.2 Symmetries

Our $(N-1)$ equations (4.144) for functions F_k, $k = 0, ..., N-2$ and so g_k have many symmetries. These symmetries allow us to find special solutions which involve only a smaller number of functions. So, before looking at special cases, let us mention some of these symmetries.

The main symmetry, which is relatively easy to spot is the symmetry under the independent interchanges

$$F_0 \leftrightarrow F_{N-2}, \qquad F_1 \leftrightarrow F_{N-3}, \qquad F_k \leftrightarrow F_{N-k-2}, \qquad \dots \tag{4.148}$$

To see this symmetry we look at the expression for the energy and note that as

$$\mathcal{D}_k = (k+1)(N-k-1)(1-\cos F_k)$$

$\mathcal{D}_k$ have this symmetry; *i.e.* $\mathcal{D}_k \leftrightarrow \mathcal{D}_{N-k}$, when $F_k \leftrightarrow F_{N-k}$.

This symmetry is evident in all terms involving $\mathcal{D}_k$'s and so we are left with having to look at the terms involving $\dot{F}_k$'s. For the terms

$$-\frac{1}{N}\left(\sum_{k=0}^{N-2}(k+1)\dot{F}_k\right)^2 + \sum_{k=0}^{N-2}\left(\sum_{l=k}^{N-2}\dot{F}_l\right)^2 \tag{4.149}$$

we note that the coefficient of $\dot{F}_k^2$ is given by

$$-\frac{1}{N}(k+1)^2 + (k+1) = (k+1)\frac{(N-k-1)}{N}, \tag{4.150}$$

which is also the coefficient of $\dot{F}_{N-k-2}^2$. Moreover, as the coefficient of $\dot{F}_k\dot{F}_l$ (when $k > l$) is

$$2k - \frac{2}{N}(k+1)(l+1), \tag{4.151}$$

which is also the coefficient of $\dot{F}_{N-k-2}\dot{F}_{N-l-2}$ we see that we have demonstrated the validity of our symmetry.

4.10 Special Cases

Now we look at the cases of low N.

4.10.1 $SU(3)$

For this case we have two functions: F_0 and F_1. The radial energy density is given by

$$\mathcal{E} = \frac{2}{3}(\dot{F}_0^2 + \dot{F}_1^2 + \dot{F}_0\dot{F}_1)\left[r^2 + 2(1-\cos F_0) + 2(1-\cos F_1)\right]$$
$$+4\left[(1-\cos F_0) + (1-\cos F_1)\right]\left[1 + \frac{1}{2r^2}(1-\cos F_0) + \frac{1}{2r^2}(1-\cos F_1)\right]. \tag{4.152}$$

The equations for F_0 and F_1 are

$$(2\ddot{F}_0 + \ddot{F}_1)A_{32} + \frac{2}{r}(2\dot{F}_0 + \dot{F}_1)A_{31} - \frac{6}{r^2}(\sin F_0)A_{30} = 0,$$

$$(\ddot{F}_0 + 2\ddot{F}_1)A_{32} + \frac{2}{r}(\dot{F}_0 + 2\dot{F}_1)A_{31} - \frac{6}{r^2}(\sin F_1)A_{30} = 0, \tag{4.153}$$

where

$$\begin{aligned} A_{32} &= 1+\frac{2}{r^2}(1-\cos F_0)+\frac{2}{r^2}(1-\cos F_1), \\ A_{31} &= 1+\frac{1}{r}\dot{F}_0\sin F_0+\frac{1}{r}\dot{F}_1\sin F_1, \\ A_{30} &= \frac{1}{3}(\dot{F}_0^2+\dot{F}_1^2+\dot{F}_0\dot{F}_1)+1+\frac{1}{r^2}(1-\cos F_0)+\frac{1}{r^2}(1-\cos F_1). \quad (4.154) \end{aligned}$$

The equations (72) can be solved for the two functions F_0 and F_1. Clearly, we cannot put either of them to zero but, due to the symmetry, we can take $F_0 = F_1 = F$ in which case both equations reduce to

$$\ddot{F}\left[1+\frac{4}{r^2}(1-\cos F)\right]+\frac{2}{r}\dot{F}+\frac{2}{r^2}\sin F\left[\dot{F}^2-1-\frac{2}{r^2}(1-\cos F)\right] = 0. \quad (4.155)$$

This equation coincides with the equation of the usual $SU(2)$ Skyrme model after rescaling the coordinate $r = 2\tilde{r}$. Performing this coordinate rescaling in the corresponding energy integral, we find that its energy is $E = 8 \times 1.232$, *i.e.* is exactly 8 times the energy of one $SU(2)$ skyrmion (taking $F(0) = 2\pi$). This agrees with our numerical result 9.85242 obtained from (74) (within our numerical accuracy the energy of one $SU(2)$ skyrmion is 1.23146). The topological charge of this configuration is clearly $B = 4$, so energy per baryon is 2 times the energy of one $SU(2)$ skyrmion.

In addition, there is a further symmetry; we can put $F_0 = -F_1 = G$. In this case both equations reduce to

$$\ddot{G}\left[1+\frac{4}{r^2}(1-\cos G)\right]+\frac{2}{r}\dot{G}+\frac{2}{r^2}\sin G\left[\dot{G}^2-3-\frac{6}{r^2}(1-\cos G)\right] = 0. \quad (4.156)$$

This case, as $F_0 = g_0 - g_1$ and $F_1 = g_1$, corresponds to the case of $g_0 = 0$ and so our solution involves only one projector, namely P_1. Its topological charge is $B = 2 - 2 = 0$ and its energy is 5.11338. A similar solution was discussed, in the usual Skyrme model case, in [14] (there its energy is 3.861).

In general, however, our solutions depend on two functions F_0 and F_1. For example, by imposing the boundary conditions: $F_0(0) = 2\pi, F_0(\infty) = 0$ and $F_1(0) = 0, F_1(\infty) = 0$ in (4.153) we found that the energy of this solution is $E = 2.61503$ and its baryon number is $B = 2$. (In the usual Skyrme model, a similar solution has energy 2.3764).

4.10.2 $SU(4)$

In this case we have three functions F_0, F_1 and F_2. The energy density becomes

$$\begin{aligned}\mathcal{E} &= \frac{1}{4}\left[3\dot{F}_0^2 + 4\dot{F}_1^2 + 3\dot{F}_2^2 + 4\dot{F}_0\dot{F}_1 + 2\dot{F}_0\dot{F}_2 + 4\dot{F}_1\dot{F}_2\right] \\ &\quad \times [r^2 + 3(1-\cos F_0) + 4(1-\cos F_1) + 3(1-\cos F_2)] \\ &\quad + 2\,[3(1-\cos F_0) + 4(1-\cos F_1) + 3(1-\cos F_2)] \\ &\quad \times \left[1 + \frac{3}{4r^2}(1-\cos F_0) + \frac{1}{r^2}(1-\cos F_1) + \frac{3}{4r^2}(1-\cos F_2)\right]. \quad (4.157)\end{aligned}$$

The equations for F_0, F_1 and F_2 are very complicated. They read

$$\begin{aligned}(3\ddot{F}_0 + 2\ddot{F}_1 + \ddot{F}_2)A_{42} + \frac{2}{r}(3\dot{F}_0 + 2\dot{F}_1 + \dot{F}_2)A_{41} - \frac{12}{r^2}(\sin F_0)A_{40} &= 0, \\ (\ddot{F}_0 + 2\ddot{F}_1 + \ddot{F}_2)A_{42} + \frac{2}{r}(\dot{F}_0 + 2\dot{F}_1 + \dot{F}_2)A_{41} - \frac{8}{r^2}(\sin F_1)A_{40} &= 0, \\ (\ddot{F}_0 + 2\ddot{F}_1 + 3\ddot{F}_2)A_{42} + \frac{2}{r}(\dot{F}_0 + 2\dot{F}_1 + 3\dot{F}_2)A_{41} - \frac{12}{r^2}(\sin F_2)A_{40} &= 0, \quad (4.158)\end{aligned}$$

where

$$\begin{aligned}A_{42} &= 1 + \frac{3}{r^2}(1-\cos F_0) + \frac{4}{r^2}(1-\cos F_1) + \frac{3}{r^2}(1-\cos F_2), \\ A_{41} &= 1 + \frac{3}{2r}\dot{F}_0\sin F_0 + \frac{2}{r}\dot{F}_1\sin F_1 + \frac{3}{2r}\dot{F}_2\sin F_2, \\ A_{40} &= \frac{3}{8}(\dot{F}_0^2 + \frac{4}{3}\dot{F}_1^2 + \dot{F}_2^2 + \frac{4}{3}\dot{F}_0\dot{F}_1 + \frac{2}{3}\dot{F}_0\dot{F}_2 + \frac{4}{3}\dot{F}_1\dot{F}_2) + \\ &\quad 1 + \frac{3}{2r^2}(1-\cos F_0) + \frac{2}{r^2}(1-\cos F_1) + \frac{3}{2r^2}(1-\cos F_2). \quad (4.159)\end{aligned}$$

These equations have the previously mentioned symmetry $F_0 \leftrightarrow F_2$ which allows us to set $F_0 = F_2 = F$ while keeping F_1 arbitrary. In this case the above equations reduce to the following two coupled equations:

$$\begin{aligned}(2\ddot{F} + \ddot{F}_1)\tilde{A}_{42} + \frac{2}{r}(2\dot{F} + \dot{F}_1)\tilde{A}_{41} - \frac{6}{r^2}(\sin F)\tilde{A}_{40} &= 0, \\ (\ddot{F} + \ddot{F}_1)\tilde{A}_{42} + \frac{2}{r}(\dot{F} + \dot{F}_1)\tilde{A}_{41} - \frac{4}{r^2}(\sin F_1)\tilde{A}_{40} &= 0, \quad (4.160)\end{aligned}$$

where

$$\begin{aligned}\tilde{A}_{42} &= 1 + \frac{6}{r^2}(1-\cos F) + \frac{4}{r^2}(1-\cos F_1), \\ \tilde{A}_{41} &= 1 + \frac{3}{r}\dot{F}\sin F + \frac{2}{r}\dot{F}_1\sin F_1, \\ \tilde{A}_{40} &= (\dot{F}^2 + \frac{1}{2}\dot{F}_1^2 + \dot{F}\dot{F}_1) + 1 + \frac{3}{r^2}(1-\cos F) + \frac{2}{r^2}(1-\cos F_1). \quad (4.161)\end{aligned}$$

By imposing: $F(0) = 2\pi, F(\infty) = 0$ and $F_1(0) = 0, F_1(\infty) = 0$ the corresponding solution is found to have energy 13.2006 and its baryon number is 6.

If we further set $F_1 = F = G$ then the above coupled equations reduce to

$$\ddot{G}\left[1+\frac{10}{r^2}(1-\cos G)\right]+\frac{2}{r}\dot{G}+\frac{1}{r^2}\sin G\left[5\dot{G}^2-2-\frac{10}{r^2}(1-\cos G)\right]=0. \quad (4.162)$$

This equation coincides with the usual $SU(2)$ Skyrme model equation, after rescaling the coordinate $r=\sqrt{10}\tilde{r}$. Performing this coordinate rescaling in the corresponding energy integral we find that this configuration has energy $E = 10\sqrt{10} \times 1.232$. Our numerical result for the energy obtained from (81) is 38.9551 which is in good agreement with the above exact result. Note that the topological charge of this solution is 10, so energy per baryon is $\sqrt{10}$ times the energy of one $SU(2)$ skyrmion.

Another solution can be found by setting $F_0 = -F_2 = Z$. Then the equations have a solution if $F_1 = 0$. This case corresponds to $g_0 = 0$ and $g_1 = g_2$ and so, effectively, the field configuration is described by a one projector of rank two; namely, $P_1 + P_2$. The corresponding equation for Z is

$$\ddot{Z}\left[1+\frac{6}{r^2}(1-\cos Z)\right]+\frac{2}{r}\dot{Z}+\frac{3}{r^2}\sin Z\left[\dot{Z}^2-2-\frac{6}{r^2}(1-\cos Z)\right]=0. \quad (4.163)$$

This solution has energy 9.39388 and its charge is $B = 3 - 3 = 0$.

When we use all 3 functions we get results which depend on $F_i(0)$. In Table 2 we present our results for E_a and compare them with the similar results for E_u derived in the usual Skyrme models [14].

We see that our energies are higher (especially for larger values of B).

4.10.3 $SU(5)$

This time we have four functions F_0, F_1, F_2 and F_3. The energy density becomes

$$\begin{aligned}\mathcal{E} = \frac{2}{5}&\left[2\dot{F}_0^2+3\dot{F}_1^2+3\dot{F}_2^2+2\dot{F}_3^2+3\dot{F}_0\dot{F}_1+2\dot{F}_0\dot{F}_2+\dot{F}_0\dot{F}_3+4\dot{F}_1\dot{F}_2+2\dot{F}_1\dot{F}_3+3\dot{F}_2\dot{F}_3\right]\\ &\times\left[r^2+2\left\{2(1-\cos F_0)+3(1-\cos F_1)+3(1-\cos F_2)+2(1-\cos F_3)\right\}\right]\\ &+4\left[2(1-\cos F_0)+3(1-\cos F_1)+3(1-\cos F_2)+2(1-\cos F_3)\right]\\ &\times\left[1+\frac{1}{2r^2}\left\{2(1-\cos F_0)+3(1-\cos F_1)+3(1-\cos F_2)+2(1-\cos F_3)\right\}\right].\end{aligned} \quad (4.164)$$

$F_0(0)$	$F_1(0)$	$F_2(0)$	B	E_a	E_u
2π	0	0	3	3.96601	3.518
0	2π	0	4	5.87187	4.788
2π	0	2π	6	13.2006	7.22553
2π	2π	0	7	18.9833	8.45219
2π	2π	2π	10	38.9551	12.32
2π	-2π	2π	6-4	14.3419	8.852
2π	2π	-2π	7-3	20.5668	9.896
2π	0	-2π	3-3	9.39388	6.63422
-2π	2π	0	4-3	9.07753	6.61478

Table 4.2: Energies of the alternative and the usual $SU(4)$ Skyrme models.

The equations for F_0, F_1, F_2 and F_3 are now given by

$$\begin{aligned}
(4\ddot{F}_0 + 3\ddot{F}_1 + 2\ddot{F}_2 + \ddot{F}_3)A_{52} + \frac{2}{r}(4\dot{F}_0 + 3\dot{F}_1 + 2\dot{F}_2 + \dot{F}_3)A_{51} - \frac{20}{r^2}(\sin F_0)A_{50} &= 0, \\
(3\ddot{F}_0 + 6\ddot{F}_1 + 4\ddot{F}_2 + 2\ddot{F}_3)A_{52} + \frac{2}{r}(3\dot{F}_0 + 6\dot{F}_1 + 4\dot{F}_2 + 2\dot{F}_3)A_{51} - \frac{30}{r^2}(\sin F_1)A_{50} &= 0, \\
(2\ddot{F}_0 + 4\ddot{F}_1 + 6\ddot{F}_2 + 3\ddot{F}_3)A_{52} + \frac{2}{r}(2\dot{F}_0 + 4\dot{F}_1 + 6\dot{F}_2 + 3\dot{F}_3)A_{51} - \frac{30}{r^2}(\sin F_2)A_{50} &= 0, \\
(\ddot{F}_0 + 2\ddot{F}_1 + 3\ddot{F}_2 + 4\ddot{F}_3)A_{52} + \frac{2}{r}(\dot{F}_0 + 2\dot{F}_1 + 3\dot{F}_2 + 4\dot{F}_3)A_{51} - \frac{20}{r^2}(\sin F_3)A_{50} &= 0, \quad (4.165)
\end{aligned}$$

where

$$\begin{aligned}
A_{52} &= 1 + \frac{2}{r^2}\left[2(1-\cos F_0) + 3(1-\cos F_1) + 3(1-\cos F_2) + 2(1-\cos F_3)\right], \\
A_{51} &= 1 + \frac{1}{r}\left[2\dot{F}_0 \sin F_0 + 3\dot{F}_1 \sin F_1 + 3\dot{F}_2 \sin F_2 + 2\dot{F}_3 \sin F_3\right], \\
A_{50} &= \frac{1}{5}[2\dot{F}_0^2 + 3\dot{F}_1^2 + 3\dot{F}_2^2 + 2\dot{F}_3^2 + 3\dot{F}_0\dot{F}_1 + 2\dot{F}_0\dot{F}_2 + \dot{F}_0\dot{F}_3 + 4\dot{F}_1\dot{F}_2 \\
&\quad + 2\dot{F}_1\dot{F}_3 + 3\dot{F}_2\dot{F}_3] + 1 + \frac{1}{r^2}[2(1-\cos F_0) + 3(1-\cos F_1) \\
&\quad + 3(1-\cos F_2) + 2(1-\cos F_3)]. \quad (4.166)
\end{aligned}$$

It is easy to spot, as we have mentioned before, that these expressions have symmetries $F_0 \leftrightarrow F_3$ and, independently, $F_1 \leftrightarrow F_2$.

So we can seek solutions involving only two functions $F_0 = F_3 = F$ and $F_1 = F_2 = G$. If we impose this condition our equations reduce to the following two coupled equations:

$$(\ddot{F}+\ddot{G})\tilde{A}_{52} + \frac{2}{r}(\dot{F}+\dot{G})\tilde{A}_{51} - \frac{4}{r^2}(\sin F)\tilde{A}_{50} = 0,$$

$$(\ddot{F}+2\ddot{G})\tilde{A}_{52} + \frac{2}{r}(\dot{F}+2\dot{G})\tilde{A}_{51} - \frac{6}{r^2}(\sin G)\tilde{A}_{50} = 0, \qquad (4.167)$$

where

$$\begin{aligned}
\tilde{A}_{52} &= 1+\frac{4}{r^2}\left[2(1-\cos F)+3(1-\cos G)\right],\\
\tilde{A}_{51} &= 1+\frac{2}{r}\left[2\dot{F}\sin F+3\dot{G}\sin G\right],\\
\tilde{A}_{50} &= \left[\dot{F}^2+2\dot{G}^2+2\dot{F}\dot{G}\right]\\
&\quad +1+\frac{2}{r^2}\left[2(1-\cos F)+3(1-\cos G)\right]. \qquad (4.168)
\end{aligned}$$

By imposing: $F(0)=0, F(\infty)=0$ and $G(0)=2\pi, G(\infty)=0$, the corresponding solution is found to have energy 37.3436 and its baryon number is 12. We observe that equations (4.167) coincide with equations (4.160) in the $SU(4)$ case, after rescaling the coordinate $r=\sqrt{2}\tilde{r}$. Performing this coordinate rescaling in the corresponding energy integral of (4.167), we find that its energy is $2\sqrt{2}$ times the energy of (4.160) which agrees with our numerical results above.

Note that if in addition, we further let $G=F$ then the above coupled equations reduce to

$$\ddot{F}\left[1+\frac{20}{r^2}(1-\cos F)\right] + \frac{2}{r}\dot{F} + \frac{2}{r^2}\sin F\left[5\dot{F}^2-1-\frac{10}{r^2}(1-\cos F)\right] = 0. \quad (4.169)$$

This equation, again, coincides with the usual $SU(2)$ Skyrme model equation after rescaling the coordinate $r=\sqrt{20}\tilde{r}$ and from the corresponding energy integral we find that its energy is $E=20\sqrt{20}\times 1.232$. Our numerical result for the energy obtained from (88) is 110.251 which is also in good agreement with the above exact result. As its topological charge is 20 we see that the energy per baryon of this solution is $\sqrt{20}$ times the energy of one $SU(2)$ skyrmion.

There is still a further symmetry, which we could exploit, and which allows us to put $F_0=-F_3=Y$ and $F_1=-F_2=Z$. The corresponding equations for Y and Z are

$$(3\ddot{Y}+\ddot{Z})\widetilde{A}_{52}+\frac{2}{r}(3\dot{Y}+\dot{Z})\widetilde{A}_{51}-\frac{20}{r^2}(\sin Y)\widetilde{A}_{50}=0,$$
$$(\ddot{Y}+2\ddot{Z})\widetilde{A}_{52}+\frac{2}{r}(\dot{Y}+2\dot{Z})\widetilde{A}_{51}-\frac{30}{r^2}(\sin Z)\widetilde{A}_{50}=0, \quad (4.170)$$

where

$$\begin{aligned}\widetilde{A}_{52} &= 1+\frac{4}{r^2}\left[2(1-\cos Y)+3(1-\cos Z)\right],\\ \widetilde{A}_{51} &= 1+\frac{2}{r}\left[2\dot{Y}\sin Y+3\dot{Z}\sin Z\right],\\ \widetilde{A}_{50} &= \frac{1}{5}\left[3\dot{Y}^2+2\dot{Z}^2+2\dot{Y}\dot{Z}\right]\\ &\quad +1+\frac{2}{r^2}\left[2(1-\cos Y)+3(1-\cos Z)\right]. \quad (4.171)\end{aligned}$$

This case corresponds to $g_0=0$ and $g_1=g_3$ and so the corresponding field configurations are described by two projectors - namely P_2 and P_1+P_3. By imposing: $Y(0)=2\pi, Y(\infty)=0$ and $Z(0)=0, Z(\infty)=0$, we found that this configuration has energy 13.4618. Its charge is $B=4-4=0$.

More general solutions, however, depend on all four functions.

Finally, let us note that for $SU(N)$ with $N>2$, when all profile functions F_k are the same, *i.e.* $F_0=F_1=\ldots=F_{N-2}=F$ equations (4.144) reduce to single equation

$$\ddot{F}\left[1+\frac{\tilde{B}}{r^2}(1-\cos F)\right]+\frac{2}{r}\dot{F}+\frac{2}{r^2}\sin F\left[\frac{\tilde{B}}{4}\dot{F}^2-1-\frac{\tilde{B}}{2r^2}(1-\cos F)\right]=0, \quad (4.172)$$

and the corresponding energy integral (4.138) reduces to

$$E=\frac{1}{6\pi}\int r^2dr\left[\frac{\tilde{B}}{2}\dot{F}^2\left(1+\frac{\tilde{B}}{r^2}(1-\cos F)\right)+2\frac{\tilde{B}}{r^2}(1-\cos F)+\frac{\tilde{B}^2}{2r^4}(1-\cos F)^2\right], \quad (4.173)$$

where

$$\tilde{B}=\frac{(N-1)N(N+1)}{6}, \quad (4.174)$$

and $\tilde{B}$ is the baryon number of our configuration.

We observe that by rescaling the radial coordinate r to

$$r=\tilde{r}\sqrt{\tilde{B}}, \quad (4.175)$$

equation (4.172) reduces to the usual $SU(2)$ Skyrme model equation (in the coordinate $\tilde{r}$) and the energy integral (4.173) becomes

$$E = \tilde{B}\sqrt{\tilde{B}}\, E_{SU(2)}, \tag{4.176}$$

where $E_{SU(2)}$ is the energy integral of the usual $SU(2)$ Skyrme model (in the coordinate $\tilde{r}$). Thus this configuration has energy $E = \tilde{B}\sqrt{\tilde{B}} \times 1.232$ (taking $F(0) = 2\pi$), which agrees with our results for the cases: $N = 3, 4$, and 5 above. As the baryon number of this configuration is $\tilde{B}$, the energy per baryon is $\sqrt{\tilde{B}}$ times the energy of one $SU(2)$ skyrmion.

Chapter 5

$SU(N)$ Yang-Mills Theories and Harmonic Maps

In this chapter we apply the harmonic mapping method to pure $SU(N)$ Yang-Mills theories where we concentrate on the massive case only as it has been known that the massless case does not admit classical particle-like solutions [62, 63]. The pure massive case is of course pathological, as its action is nongauge invariant. In order to get rid of this nongauge invariancy, here we consider a Stückelberg type gauge invariant formalism [64], where $SU(N)$ chiral currents $\widetilde{U}^{-1}\partial_\mu\widetilde{U}$ with $\widetilde{U} \in SU(N)$ are added to render the massive terms gauge invariant. The pure massive case now corresponds to choosing the special gauge $\widetilde{U} = I$. As the pure massive $SU(N)$ Yang-Mills fields action is *not scale* invariant, this raises the expectation that solitonic solutions might exist in these theories. In fact, we observe that if we choose the gauge potential to be of *almost* pure gauge form, we recover the $SU(N)$ Skyrme models action, which adds strong support to our previous expectation.

Armed with this observation, we turn to consider a static magnetic type case by using the harmonic map ansatz for the gauge potential that was introduced by Ioannidou and Sutcliffe in their study of non-Bogomolnyi BPS monopoles [65]. When we studied spherically symmetric solutions of the equations (numerically) for lower N cases (2, 3 and 4), we found that by letting the profile functions $g_l(r)$ vanish at $r \to \infty$ and appropriately choosing the boundary conditions to be imposed at the origin $r = 0$, some bounded solutions with finite energies can be constructed.

5.1 Massive $SU(N)$ Yang-Mills Theories and Skyrme Models

Let $\widetilde{U}(x)$ be an $SU(N)$ group valued function of spacetime coordinates. Then a Stückelberg type formalism of massive $SU(N)$ Yang-Mills theories is given by the action [64]

$$S = \int d^4x \, \mathrm{Tr}\left[-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + M^2\left(A^\mu - \widetilde{L}^\mu\right)\left(A_\mu - \widetilde{L}_\mu\right)\right], \tag{5.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu], \tag{5.2}$$

$$\widetilde{L}_\mu = -i\widetilde{U}^{-1}\partial_\mu \widetilde{U}, \tag{5.3}$$

with M a mass parameter and where we have chosen arbitrarily the gauge coupling $e = 1$. The gauge potential A_μ and the left chiral current $\widetilde{L}_\mu$ have values in the Lie algebra $su(N)$ and here we have chosen them to be Hermitian, *i.e.* $A_\mu^\dagger = A_\mu$ and $\widetilde{L}_\mu^\dagger = \widetilde{L}_\mu$, respectively.

We note that the action (5.1) is *invariant* under the gauge transformations:

$$A'_\mu = \Omega^{-1}A_\mu\Omega - i\Omega^{-1}\partial_\mu\Omega, \tag{5.4}$$

$$\widetilde{U}' = \widetilde{U}\Omega, \tag{5.5}$$

where $\Omega \in SU(N)$. On the other hand, as under the scale transformation $x^\mu \to \lambda x^\mu$, the gauge potential and the chiral current scale as $\lambda A_\mu(\lambda x)$ and $\lambda\widetilde{L}_\mu(\lambda x)$, respectively, we see that the mass term breaks the scale invariancy of the action (5.1).

To derive the Euler-Lagrange equations of the action (5.1), let us consider its variation under the variations of field variables δA_μ and $\delta\widetilde{U}$. We find

$$\delta S = \int d^4x \, \mathrm{Tr}\left[-F^{\mu\nu}\delta F_{\mu\nu} + 2M^2\left(A^\mu - \widetilde{L}^\mu\right)\left(\delta A_\mu - \delta\widetilde{L}_\mu\right)\right]. \tag{5.6}$$

As

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu + i\,[\delta A_\mu, A_\nu] - (\mu \leftrightarrow \nu), \tag{5.7}$$

where $\delta\widetilde{L}_\mu$ is given by (4.6), so up to the total divergence terms, which vanish due to the boundary conditions: $\delta A_\mu = 0$ and $\delta\widetilde{U} = 0$ on the boundary, we obtain

$$\delta S = -2 \int d^4x \, \mathrm{Tr}\Big[\Big\{ D_\mu F^{\mu\nu} - M^2 \left(A^\nu - \widetilde{L}^\nu \right) \Big\} \delta A_\mu - M^2 \left(\partial_\mu A^\mu - D_\mu \widetilde{L}^\mu \right) \widetilde{U}^{-1} \delta \widetilde{U} \Big], \tag{5.8}$$

where D_μ is the covariant derivative

$$D_\mu G \equiv \partial_\mu G + i[A_\mu, G] \tag{5.9}$$

for any $G \in su(N)$.

Thus, the corresponding Euler-Lagrange equations are

$$D_\mu F^{\mu\nu} - M^2 \left(A^\nu - \widetilde{L}^\nu \right) = 0, \tag{5.10}$$

$$\partial_\mu A^\mu - D_\mu \widetilde{L}^\mu = 0. \tag{5.11}$$

Taking $\Omega = \widetilde{U}^{-1}$ in (5.4) and (5.5), $\widetilde{U}$ becomes the unit element I and the action (5.1) reduces to

$$S = \int d^4x \, \mathrm{Tr} \left[-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + M^2 A^\mu A_\mu \right], \tag{5.12}$$

i.e. it is the *pure massive* $SU(N)$ Yang-Mills action. The field equations (5.10) and (5.11) then reduce to

$$D_\mu F^{\mu\nu} - M^2 A^\nu = 0, \tag{5.13}$$

$$\partial_\mu A^\mu = 0, \tag{5.14}$$

respectively.

Equation (5.14) means that the gauge potential A_μ satisfies the *Lorentz condition.* Thus, we may interpret the action (5.12) as a gauge fixed version of the gauge invariant action (5.1). As the field equations (5.13) and (5.14) imply 3 degrees of freedom (polarisation states) for each vector field A^a_μ where a is the Lie algebra index, we call the gauge $\widetilde{U} = I$ the *physical* or *unitary gauge.* In the following, by massive $SU(N)$ Yang-Mills theories we mean the pure case (5.12).

Now, let us make the following observation by choosing the gauge potential A_μ in the action (5.12) to be of *almost* pure-gauge form, *i.e.*

$$A_\mu = iqU^{-1}\partial_\mu U, \tag{5.15}$$

where U takes value in the group $SU(N)$, and q is a spacetime independent free parameter. In terms of U, the gauge field strength $F_{\mu\nu}$, according to (5.2), becomes

$$F_{\mu\nu} = -iq(1+q)\,[L_\mu, L_\nu]\,, \tag{5.16}$$

where

$$L_\mu = U^{-1}\partial_\mu U. \tag{5.17}$$

Thus we see that, if we choose $q = -1$ then $F_{\mu\nu} = 0$ and so the corresponding gauge potential A_μ is a pure-gauge.

Putting (5.15) and (5.16) into the action (5.12) yields a new action

$$S = \int d^4x\,\mathrm{Tr}\left(\frac{1}{2}q^2(1+q)^2[L^\mu, L^\nu][L_\mu, L_\nu] - q^2M^2L^\mu L_\mu\right), \tag{5.18}$$

which coincides with the $SU(N)$ Skyrme models action (4.1) for $M_\pi = 0$, with the parameters identification

$$F = 4qM, \qquad a = \frac{1}{4q(1+q)}, \tag{5.19}$$

where F is the pion decay constant, and a the Skyrme models dimensionless constant.

As the $SU(N)$ Skyrme models have solitonic solutions (*i.e.* the skyrmions), this coincidence gives us a strong basis to expect that massive $SU(N)$ Yang-Mills theories might admit solitonic solutions as well. To check this further, in the following sections 5.2-5.4, we choose to study the static magnetic type case.

5.2 Static Magnetic type Energy and B-integral

As we are interested in the static magnetic type case, we exclude the electric type fields by imposing

$$A_0 = 0, \qquad \partial_0 A_a = 0, \tag{5.20}$$

from which it follows that

$$F_{0a} = 0, \tag{5.21}$$

where $a = 1, 2, 3$. Then, from the action (5.12), the static magnetic type energy of the massive $SU(N)$ Yang-Mills fields, as derived in appendix H.2, is

$$E = \int d^3x\,\mathrm{Tr}\left[\frac{1}{2}F_{ab}^2 + M^2A_a^2\right]. \tag{5.22}$$

In Ref. [66], Sheng had shown that the static magnetic type massive $SU(N)$ Yang-Mills fields in $(n+1)$-D spacetime with $n \neq 3$ do not allow for the existence of finite energy static solutions. Thus, the $n=3$ case that we are considering here evades Sheng non-existence theorems which was left as an open problem in Ref. [66].

To examine this $n=3$ case explicitly, let us study the scale stability of the energy integral (5.22) under the scale transformation: $\vec{x} \to \lambda\vec{x}$. As the gauge potentials A_μ scale as

$$A_\mu(\vec{x}) \to \lambda A_\mu(\lambda\vec{x}), \tag{5.23}$$

from which

$$F_{\mu\nu}(\vec{x}) \to \lambda^2 F_{\mu\nu}(\lambda\vec{x}), \tag{5.24}$$

the energy integral (5.22) scales as

$$\begin{aligned} E \to E[\lambda] &= \int \frac{d^3(\lambda x)}{\lambda^3} \mathrm{Tr}\left[\frac{1}{2}\lambda^4 F_{ab}(\lambda\vec{x})^2 + M^2\lambda^2 A_a(\lambda\vec{x})^2\right] \\ &= \lambda E_F + \frac{1}{\lambda}E_M, \end{aligned} \tag{5.25}$$

where E_F and E_M are the F-term and the massive term, respectively. Thus we see that the pure massive $SU(N)$ Yang-Mills static energy (5.22) scales in the same fashion as the static energy of the $SU(N)$ Skyrme models (4.17).

As

$$\left.\frac{dE[\lambda]}{d\lambda}\right|_{\lambda=1} = E_F - E_M, \tag{5.26}$$

$$\left.\frac{d^2E[\lambda]}{d\lambda^2}\right|_{\lambda=1} = 2E_M, \tag{5.27}$$

the extremum condition guarantees that $E_M = E_F$. Thus we see that the stability condition could be satisfied, and so we conclude that static massive $SU(N)$ Yang-Mills field theories admit the existence of solitonic solutions as we expected.

Next, we note that the energy integral (5.22) can be expressed in the form of a perfect square term plus "something" as follows

$$E = \int d^3x\, \mathrm{Tr}\left[\left(\frac{1}{2}\epsilon_{abc}F_{ab} - MA_c\right)^2 + M\epsilon_{abc}F_{ab}A_c\right]. \tag{5.28}$$

The second term in (5.28) has structure which is independent of the metric tensor, which raises the expectation that it is a topological quantity. As it is proportional to the baryon number B of the $SU(N)$ Skyrme models if A_μ is of almost pure gauge form, so for convenience we keep the same letter B for it, and define

$$B = \frac{1}{16\pi}\int d^3x \,\mathrm{Tr}\left[\epsilon_{abc}F_{ab}A_c\right]. \tag{5.29}$$

For later reference, we call it by the name *B-integral.*

From (5.28) it is obvious that, for finite B,

$$E \geq 16\pi M B, \tag{5.30}$$

and so the lower bound of the energy would be saturated if

$$\epsilon_{abc}F_{ab} - 2MA_c = 0, \tag{5.31}$$

which is analogous to Bogomolnyi's bound in Yang-Mills-Higgs monopole theories [65]. Note that (5.31) is consistent with the Lorentz condition (5.14) in this static magnetic type case.

In terms of the spherical polar coordinates (r, θ, ϕ), the energy (5.22) becomes

$$E = \int dr(\sin\theta d\theta d\phi)\,\mathrm{Tr}\left[F_{r\theta}^2 + \frac{1}{\sin^2\theta}F_{r\phi}^2 + \frac{1}{r^2\sin^2\theta}F_{\theta\phi}^2 \right.$$
$$\left. + M^2r^2A_r^2 + M^2A_\theta^2 + \frac{M^2}{\sin^2\theta}A_\phi^2\right]. \tag{5.32}$$

As we are going to apply the harmonic maps $S^2 \to CP^{N-1}$ to this problem, in the following we choose to use the holomorphic coordinates (ξ, $\bar{\xi}$) for S^2. The transformation relations between the angular components of the gauge potential and field strength in the spherical polar and holomorphic coordinates are

$$A_\theta = \frac{(1+|\xi|^2)}{2|\xi|}\left(\xi A_\xi + \bar{\xi}A_{\bar{\xi}}\right), \tag{5.33}$$
$$A_\phi = i\left(\xi A_\xi - \bar{\xi}A_{\bar{\xi}}\right), \tag{5.34}$$
$$F_{r\theta} = \frac{(1+|\xi|^2)}{2|\xi|}\left(\xi F_{r\xi} + \bar{\xi}F_{r\bar{\xi}}\right), \tag{5.35}$$
$$F_{r\phi} = i\left(\xi F_{r\xi} - \bar{\xi}F_{r\bar{\xi}}\right), \tag{5.36}$$
$$F_{\theta\phi} = i|\xi|\left(1+|\xi|^2\right)F_{\bar{\xi}\xi}. \tag{5.37}$$

In terms of the complex quantities on the right hand side of (5.33) - (5.37), the energy (5.32) in the spherical holomorphic coordinates (r, ξ, $\bar{\xi}$) is

$$E = 2i\int drd\xi d\bar{\xi}\mathrm{Tr}\left[F_{r\bar{\xi}}F_{r\xi} - \frac{(1+|\xi|^2)^2}{4r^2}F_{\bar{\xi}\xi}F_{\bar{\xi}\xi} + \frac{M^2r^2}{(1+|\xi|^2)^2}A_rA_r + M^2A_{\bar{\xi}}A_\xi\right], \tag{5.38}$$

and the field equations (5.13) become

$$D_{\bar{\xi}}F_{\xi r} + D_{\xi}F_{\bar{\xi}r} - \frac{2Mr^2}{(1+|\xi|^2)^2}A_r = 0, \tag{5.39}$$

$$D_rF_{r\bar{\xi}} - D_{\bar{\xi}}\left[\frac{(1+|\xi|^2)^2}{2r^2}F_{\bar{\xi}\xi}\right] - M^2A_{\bar{\xi}} = 0, \tag{5.40}$$

while the Lorentz condition (5.14) becomes

$$\partial_r\left(r^2A_r\right) + \frac{(1+|\xi|^2)^2}{2}\left[\partial_{\xi}A_{\bar{\xi}} + \partial_{\bar{\xi}}A_{\xi}\right] = 0. \tag{5.41}$$

5.3 $SU(N)$ Harmonic Map Ansatz

In this section, we solve the massive $SU(N)$ Yang-Mills field equations (5.39) and (5.40) together with the constraint (5.41), numerically, by using harmonic maps $S^2 \to CP^{N-1}$. Explicitly, we choose to take the following harmonic map ansatz for the gauge potentials [65]:

$$A_r = 0, \qquad A_{\xi} = i\sum_{k=0}^{N-2} g_k\left[P_k,\, \partial_{\xi}P_k\right], \tag{5.42}$$

where $P_k = P_k(\xi,\bar{\xi})$ are rank-1 projector fields of the CP^{N-1} σ model, and the profile functions g_k are functions of the radial coordinate r only. Note that, $A_{\bar{\xi}} = (A_{\xi})^{\dagger}$. From now on, we abandon summation convention on repeated indices.

As the projectors satisfy the CP^{N-1} equation: $[P_k, \partial_{\bar{\xi}\xi}P_k] = 0$, it follows from the ansatz (5.42) that the gauge potentials A_{ξ} and $A_{\bar{\xi}}$, satisfy the identity

$$\partial_{\xi}A_{\bar{\xi}} + \partial_{\bar{\xi}}A_{\xi} = 0, \tag{5.43}$$

and so (5.41) is solved by this ansatz automatically.

Furthermore, we find that with the ansatz (5.42), equation (5.39) is also satisfied identically. Thus, the only nontrivial equation left is (5.40). To derive the explicit equations for the profile functions g_k from (5.40), we need to extract out the angular dependence in a consistent way. To carry out this manipulation directly is a formidable task due to the complexities in evaluating the derivatives of the projectors. In order to get rid of it, we reduce the dependencies on the holomorphic coordinates $(\xi, \bar{\xi})$ by multiplying (5.40) from the right by the mutually orthogonal vector fields M_l $(l = 0, \dots, (N-1))$, *i.e.*

$$\left(D_r F_{r\bar{\xi}} - D_{\bar{\xi}} \left[\frac{(1+|\xi|^2)^2}{2r^2} F_{\bar{\xi}\xi}\right] - M^2 A_{\bar{\xi}}\right) M_l = 0, \tag{5.44}$$

and then use the following properties of the projector operators $P_k = M_k |M_k|^{-2} M_k^\dagger$ and its derivatives applied to M_l:

$$P_k M_l = \delta_{kl} M_l, \tag{5.45}$$

$$(\partial_\xi P_k) M_l = (\delta_{kl} - \delta_{k,l+1}) M_{l+1}, \tag{5.46}$$

$$(\partial_{\bar{\xi}} P_k) M_l = \kappa_{l-1} (\delta_{k,l-1} - \delta_{kl}) M_{l-1}, \tag{5.47}$$

where

$$\kappa_l = \frac{|M_{l+1}|^2}{|M_l|^2}. \tag{5.48}$$

In deriving the above results we have taken M_0 to be holomorphic and we have used the derivative properties of M_l as given by equations (3.81)-(3.82). Note that, by definition: $\kappa_{-1} = 0$. For example, in deriving (5.47) we first write

$$(\partial_{\bar{\xi}} P_k) M_l = \partial_{\bar{\xi}}(P_k M_l) - P_k \partial_{\bar{\xi}} M_l. \tag{5.49}$$

Then, using (5.45) and (3.81), *i.e.* $\partial_{\bar{\xi}} M_l = -\kappa_{l-1} M_{l-1}$, gives (5.47).

Hence, the action of the gauge potential, as given by the ansatz (5.42), on the vectors M_l are

$$A_\xi M_l = -i G_l M_{l+1}, \tag{5.50}$$

$$A_{\bar{\xi}} M_l = i G_{l-1} \kappa_{l-1} M_{l-1}, \tag{5.51}$$

where

$$G_l = (g_l + g_{l+1}), \tag{5.52}$$

from which we derive that

$$F_{r\xi} M_l = -i \dot{G}_l M_{l+1}, \tag{5.53}$$

$$F_{r\bar{\xi}} M_l = i \dot{G}_{l-1} \kappa_{l-1} M_{l-1}, \tag{5.54}$$

$$F_{\bar{\xi}\xi} M_l = i\left(\widetilde{Q}_l \kappa_l - \widetilde{Q}_{l-1} \kappa_{l-1}\right) M_l, \tag{5.55}$$

where

$$\widetilde{Q}_l = G_l(2 - G_l). \tag{5.56}$$

Note that, by definition: $g_l = 0$ if $l \notin [0, 1, \ldots, (N-2)]$.

Let us now return to equation (5.44). We observe that, in order to have a compatible set of equations for the profile functions g_l, we have to choose the vectors M_l in such a way that each factor $(1 + |\xi|^2)^2 \kappa_l$ is equal to a constant, *i.e.*

$$\kappa_l = \frac{\mathcal{K}_l}{(1 + |\xi|^2)^2}, \tag{5.57}$$

where $\mathcal{K}_l$ are some constants depending on the index l. In fact, we found that the condition (5.57) is satisfied if we choose the initial vector M_0 to be given by the *Veronese map*, (3.123)-(3.125), *i.e.*

$$M_0 = \left[1, \sqrt{C_1^{N-1}}\xi, \ldots, \sqrt{C_k^{N-1}}\xi^k, \ldots, \xi^{N-1}\right]^T, \tag{5.58}$$

With this choice then from the construction of the vectors M_l as described in proposition 3.2, *i.e.* equations (3.135) and (3.142):

$$\mathcal{K}_l = (l+1)(N-l-1) = \mathcal{N}_l. \tag{5.59}$$

Using (5.50) - (5.54), and (5.57) for κ_l then for the first two terms of equation (5.40), we obtain

$$\left(D_r F_{r\bar{\xi}} - D_{\bar{\xi}} \left[\frac{(1 + |\xi|^2)^2}{2r^2} F_{\bar{\xi}\xi} \right] \right) M_l =$$

$$i \left(\ddot{G}_{l-1} + \frac{1}{2r^2} \left[\mathcal{N}_l \widetilde{Q}_l - 2\mathcal{N}_{l-1} \widetilde{Q}_{l-1} + \mathcal{N}_{l-2} \widetilde{Q}_{l-2} \right] (1 - G_{l-1}) \right) \kappa_{l-1} M_{l-1}. \tag{5.60}$$

Substituting (5.51) and (5.60) in equation (5.40), and noticing that M_l are independent vector fields, we find that the profile functions g_l satisfy the following second order nonlinear ordinary differential equations:

$$\ddot{G}_l + \frac{1}{2r^2} \left[\mathcal{N}_{l+1} \widetilde{Q}_{l+1} - 2\mathcal{N}_l \widetilde{Q}_l + \mathcal{N}_{l-1} \widetilde{Q}_{l-1} \right] (1 - G_l) - M^2 G_l = 0. \tag{5.61}$$

Thus we see that the harmonic map ansatz (5.42) with the initial vector M_0 given by the Veronese map (5.58) is an exact spherically symmetric solution of the massive Yang-Mills field equations (5.39) - (5.41).

Next, we want to express the static magnetic type energy (5.22) in terms of the profile functions g_l. To simplify the evaluation of the traces in (5.22), we choose to use the formula:

$$\mathrm{Tr}[R] = \sum_{k=0}^{N-1} \frac{1}{|M_k|^2} M_k^{\dagger}[R] M_k, \tag{5.62}$$

where R is a $(N \times N)$ nonsingular complex matrix which is diagonal in each basis vector M_k, $(k = 0, 1, \ldots, N-1)$. Note that the upper sum in (5.62) is $(N-1)$, instead of $(N-2)$, because here we have to sum over the whole *complete set* of basis vectors M_k in C^N.

Using (5.50) - (5.54) and (5.57) we obtain

$$\mathrm{Tr}\left(F_{r\bar{\xi}} F_{r\xi}\right) = \frac{1}{(1+|\xi|^2)^2} \sum_{k=0}^{N-2} \mathcal{N}_l \dot{G}_l^2, \tag{5.63}$$

$$\mathrm{Tr}\left(F_{\bar{\xi}\xi} F_{\bar{\xi}\xi}\right) = -\frac{1}{(1+|\xi|^2)^4} \sum_{k=0}^{N-1} \left(\mathcal{N}_l \widetilde{Q}_l - \mathcal{N}_{l-1} \widetilde{Q}_{l-1}\right)^2 \tag{5.64}$$

$$\mathrm{Tr}\left(A_{\bar{\xi}} A_{\xi}\right) = \frac{1}{(1+|\xi|^2)^2} \sum_{k=0}^{N-2} \mathcal{N}_l G_l^2, \tag{5.65}$$

where, by definition, $G_{N-1} = 0$ and $\mathcal{N}_{-1} = 0$.

With the above results for the traces, the static energy (5.38), written in a symmetrical form, becomes:

$$E = 4\pi \int_0^{\infty} dr \sum_{l=0}^{N-1} \left[\mathcal{N}_l \dot{G}_l^2 + \frac{1}{4r^2} \left(\mathcal{N}_l \widetilde{Q}_l - \mathcal{N}_{l-1} \widetilde{Q}_{l-1}\right)^2 + M^2 \mathcal{N}_l G_l^2 \right]. \tag{5.66}$$

We note that equations for the critical points of the energy (5.66) coincide with the equations for the profile functions g_l in (5.61) as we expected. From this expression we see that the energy is finite provided the profile functions G_l, for all $l = 0, 1 \ldots, (N-2)$, are bounded and that they approach zero at infinity as re^{-Mr} and at the origin the boundary conditions: $G_l = 0$ or 2, are imposed.

The B-integral in this coordinate system is

$$B = -\frac{2i}{16\pi} \int dr d\xi d\bar{\xi} \mathrm{Tr}\left[F_{r\xi} A_{\bar{\xi}} + F_{\bar{\xi} r} A_{\xi}\right], \tag{5.67}$$

and so using the formula (5.62) to express the traces, we obtain

$$B = -\frac{1}{2} \sum_{l=0}^{N-2} \mathcal{N}_l \int_0^{\infty} dr \left(\dot{G}_l G_l\right) = -\frac{1}{4} \sum_{l=0}^{N-2} \mathcal{N}_l \left[G_l^2(r)\right]_{r=0}^{\infty}, \tag{5.68}$$

where $\mathcal{N}_l$ is given by (5.59). Thus, by taking the boundary conditions $G_l(\infty) = 0$, we see that the B-integral is determined solely by the boundary conditions at the origin.

5.4 Spherically Symmetric Solutions

In this section, we have a look at the numerical solutions of the profile equations (5.61) for some different values of the mass M by imposing the boundary conditions: $G_l(\infty) = 0$ and $G_l(0) = 0$ or 2, as required for having finite energy solutions. In fact, we only consider the cases: $N = 2, 3$ and 4.

5.4.1 $SU(2)$

Here we have only one profile function G_0, with $\mathcal{N}_0 = 1$, so the profile equations (5.61) reduce to a single equation

$$\ddot{G}_0 - \frac{1}{r^2}\widetilde{Q}_0(1 - G_0) - M^2 G_0 = 0. \tag{5.69}$$

Solving (5.69) for some different values of the mass M with the boundary condition $G_0(0) = 2$ at the origin, we have found that each solution G_0 is bounded. In fig. 5.1 we show the solution for $M = 1$.

We have also computed the corresponding energy from (5.66), *i.e.*

$$E = 4\pi \int_0^\infty dr \left[\dot{G}_0^2 + \frac{1}{2r^2}\widetilde{Q}_0^2 + M^2 G_0^2\right], \tag{5.70}$$

and the results for 4 different mass parameters, *i.e.* $M = 1$, 5, 10 and 50, are summarised in Table 5.1.

$G_0(0)$	B	$E_{M=1}/4\pi$	$E_{M=5}/4\pi$	$E_{M=10}/4\pi$	$E_{M=50}/4\pi$
2	1	4.94629e+00	2.47320e+01	4.94670e+01	2.47755e+02

Table 5.1: Energies of the massive $SU(2)$ YM fields.

In fig. 5.2 we show the radial energy density for $M = 1$ and we see that it looks like a trough ball.

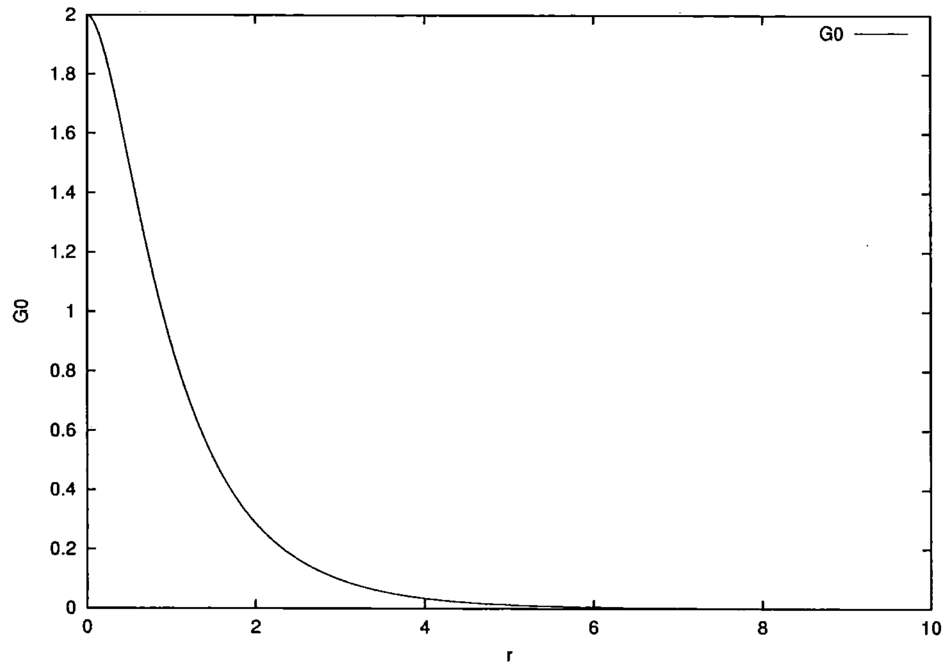

Figure 5.1: Profile function of the massive $SU(2)$ YM field for $M = 1$.

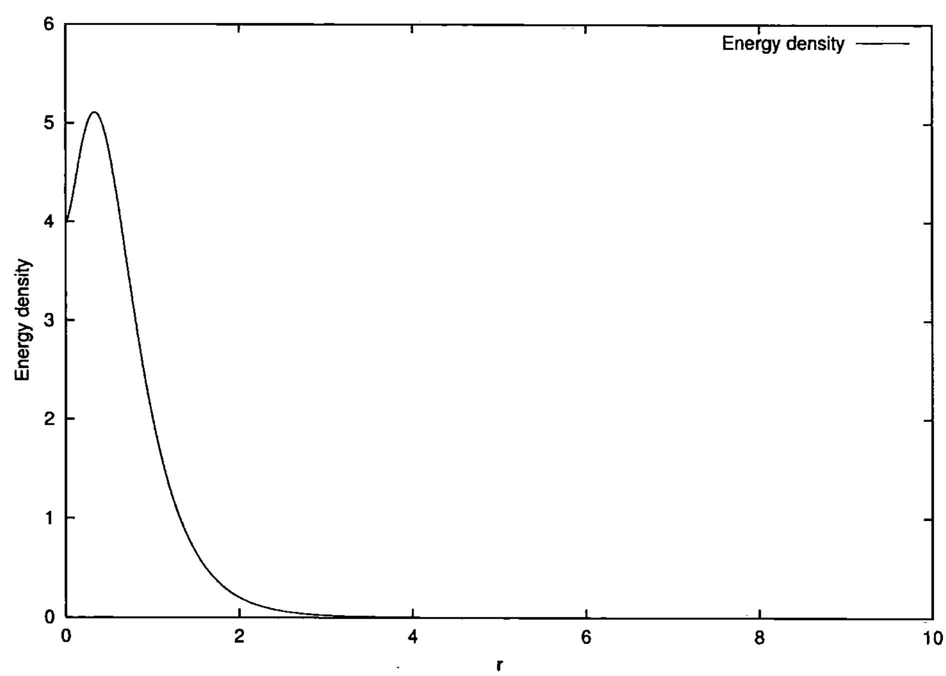

Figure 5.2: Energy density of the massive $SU(2)$ YM field for $M = 1$.

These solutions has B-integral equal to 1 and so, according to (5.30), the lower bound of the energy is $E/(4\pi) = 4M$. We see that each energy is about 25 precent higher than the corresponding lower bound value.

For the boundary condition $G_0(0) = 0$, we have found that equation (5.69) has trivial solution $G_0(r) = 0$.

5.4.2 $SU(3)$

For $N = 3$, we have two profile functions G_0 and G_1 which satisfy

$$\ddot{G}_0 - \frac{1}{r^2}\left[2\widetilde{Q}_0 - \widetilde{Q}_1\right](1 - G_0) - M^2 G_0 = 0, \tag{5.71}$$

$$\ddot{G}_1 - \frac{1}{r^2}\left[-\widetilde{Q}_0 + 2\widetilde{Q}_1\right](1 - G_1) - M^2 G_1 = 0, \tag{5.72}$$

as $\mathcal{N}_0 = \mathcal{N}_1 = 2$.

We have solved these equations for each of the following two different choices of boundary conditions imposing at the origin $(G_0(0), G_1(0)) = (2, 0)$ and $(2, 2)$, and we have also found that the corresponding solutions are bounded. In fig. 5.3, we show the graphs of these solutions for $M = 1$, and the corresponding radial energy distribution is presented in fig. 5.4. In table 5.2, we have summarised the result of energies computed from (5.66), *i.e.*

$$E = 4\pi(2)\int_0^\infty dr\Big(\dot{G}_0^2 + \dot{G}_1^2 + \frac{1}{r^2}\left[\widetilde{Q}_0^2 - \widetilde{Q}_0\widetilde{Q}_1 + \widetilde{Q}_1^2\right] + M^2G_0^2 + M^2G_1^2\Big), \tag{5.73}$$

for different values of mass M.

$G_0(0)$	$G_1(0)$	B	$E_{M=1}/4\pi$	$E_{M=5}/4\pi$	$E_{M=10}/4\pi$	$E_{M=50}/4\pi$
2	0	2	1.19080e+01	5.95409e+01	1.19087e+02	5.96217e+02
2	2	4	2.16498e+01	1.08251e+02	2.16516e+02	1.08440e+03

Table 5.2: Energies of the massive $SU(3)$ YM fields.

We notice that equations (5.71) and (5.72) are symmetric with respect to the interchange $G_0 \leftrightarrow G_1$. This allows us to set $G_0 = G_1$ which reduces the system to a single $SU(2)$ profile equation (5.69). This configuration has $B = 4$ (taking $G_0(0) = 2$) and having energy 4 times the energy of a $SU(2)$ configuration.

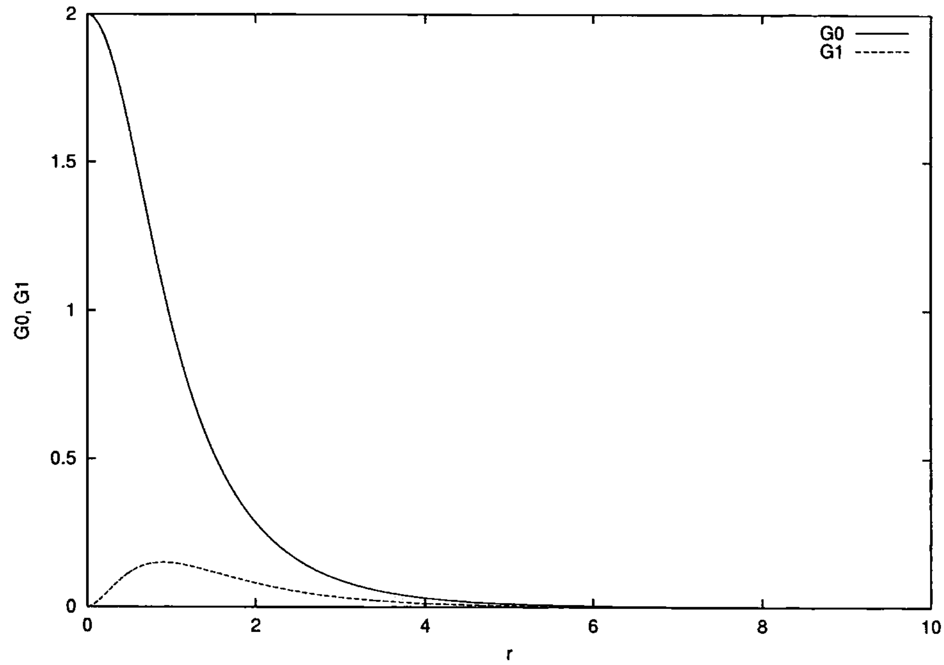

Figure 5.3: Profile functions of the massive $SU(3)$ YM fields for $M = 1$, $G_0(0) = 2$ and $G_1(0) = 0$.

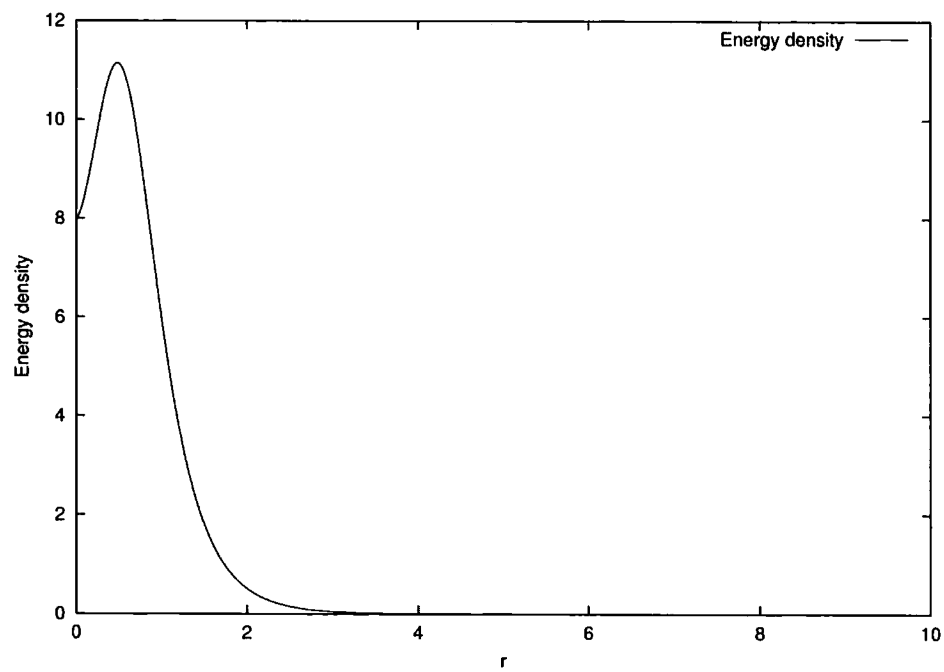

Figure 5.4: Energy density of the massive $SU(3)$ YM fields for $M = 1$, $G_0(0) = 2$ and $G_1(0) = 0$.

5.4.3 $SU(4)$

Here we have three profile functions G_0, G_1 and G_2, with $\mathcal{N}_0 = \mathcal{N}_2 = 3$ and $\mathcal{N}_1 = 4$, so the profile equations (5.61) reduce to

$$\ddot{G}_0 - \frac{1}{r^2}\left[3\widetilde{Q}_0 - 2\widetilde{Q}_1\right](1 - G_0) - M^2 G_0 = 0, \tag{5.74}$$

$$\ddot{G}_1 - \frac{1}{2r^2}\left[-3\widetilde{Q}_0 + 8\widetilde{Q}_1 - 3\widetilde{Q}_2\right](1 - G_1) - M^2 G_1 = 0, \tag{5.75}$$

$$\ddot{G}_2 - \frac{1}{r^2}\left[-2\widetilde{Q}_1 + 3\widetilde{Q}_2\right](1 - G_2) - M^2 G_2 = 0, \tag{5.76}$$

The corresponding energy (5.66) is

$$E = 4\pi \int_0^\infty dr\Big(3\dot{G}_0^2 + 4\dot{G}_1^2 + 3\dot{G}_2^2 + \frac{1}{2r^2}\left[9\widetilde{Q}_0^2 - 12\widetilde{Q}_0\widetilde{Q}_1 + 16\widetilde{Q}_1^2 - 12\widetilde{Q}_1\widetilde{Q}_2 + 9\widetilde{Q}_2^2\right]$$
$$+3M^2G_0^2 + 4M^2G_1^2 + 3M^2G_2^2\Big). \tag{5.77}$$

We observe that the system (5.74) - (5.76) has symmetry $G_0 \leftrightarrow G_2$, which allows us to set $G_0 = G_2 = G$ by keeping G_1 arbitrary. The energies for this configuration are summarised in table 5.3.

$G(0)$	$G_1(0)$	B	$E_{M=1}/4\pi$	$E_{M=5}/4\pi$	$E_{M=10}/4\pi$	$E_{M=50}/4\pi$
0	2	4	2.69092e+01	1.34550e+02	2.69120e+02	1.34817e+03
2	0	6	3.66547e+01	1.83276e+02	3.66567e+02	1.83496e+03

Table 5.3: Energies of the massive $SU(4)$ YM fields (reduced case).

In addition, letting $G_0 = G_1 = G_2 = G$ the above system of equations reduces to the $SU(2)$ profile equation (5.69). This configuration has $B = 10$ (taking $G(0) = 2$) and has energy 10 times the energy of a one $SU(2)$ configuration.

In table 5.4, we have summarised the values of energies for general configurations $G_0 \neq G_1 \neq G_2$ for 4 different combinations of boundary conditions at the origin. In fig. 5.5, we show the graphs of the solutions for $M = 1$, $G_0(0) = 0$, $G_1(0) = 2$ and $G_2(0) = 2$. The corresponding radial energy distribution is presented in fig. 5.6.

$G_0(0)$	$G_1(0)$	$G_2(0)$	B	$E_{M=1}/4\pi$	$E_{M=5}/4\pi$	$E_{M=10}/4\pi$	$E_{M=50}/4\pi$
0	0	2	3	1.91795e+01	9.58985e+01	1.91804e+02	9.60104e+02
0	2	0	4	2.77712e+01	1.38860e+02	2.77748e+02	1.39303e+03
2	0	2	6	3.66540e+01	1.83273e+02	3.66568e+02	1.83597e+03
0	2	2	7	4.13643e+01	2.06826e+02	4.13680e+02	2.07268e+03

Table 5.4: Energies of the massive $SU(4)$ YM fields.

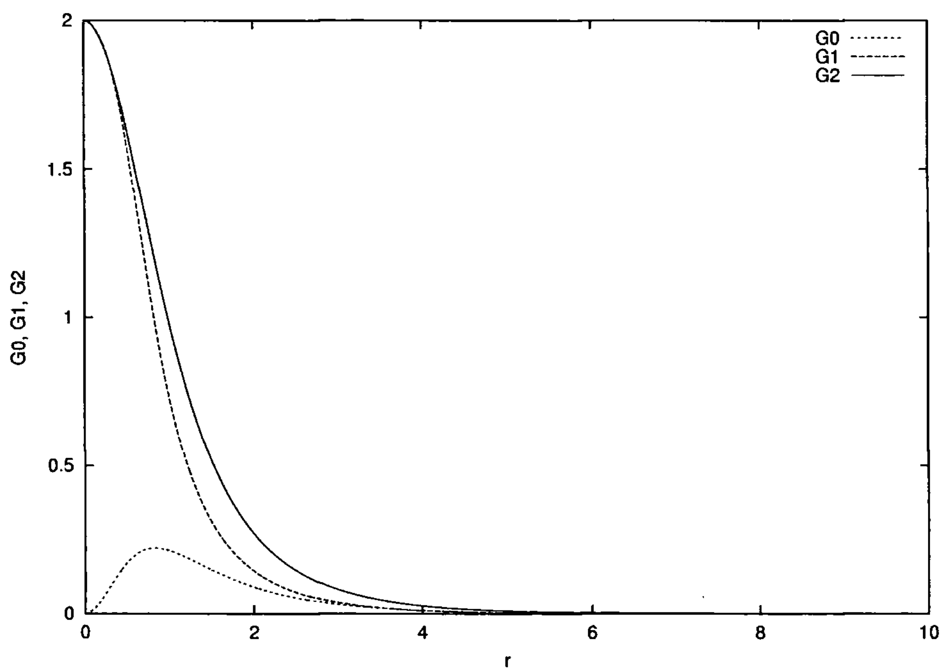

Figure 5.5: Profile functions of the massive $SU(4)$ YM fields for $M = 1$, $G_0(0) = 0$, $G_1(0) = 2$ and $G_2(0) = 2$.

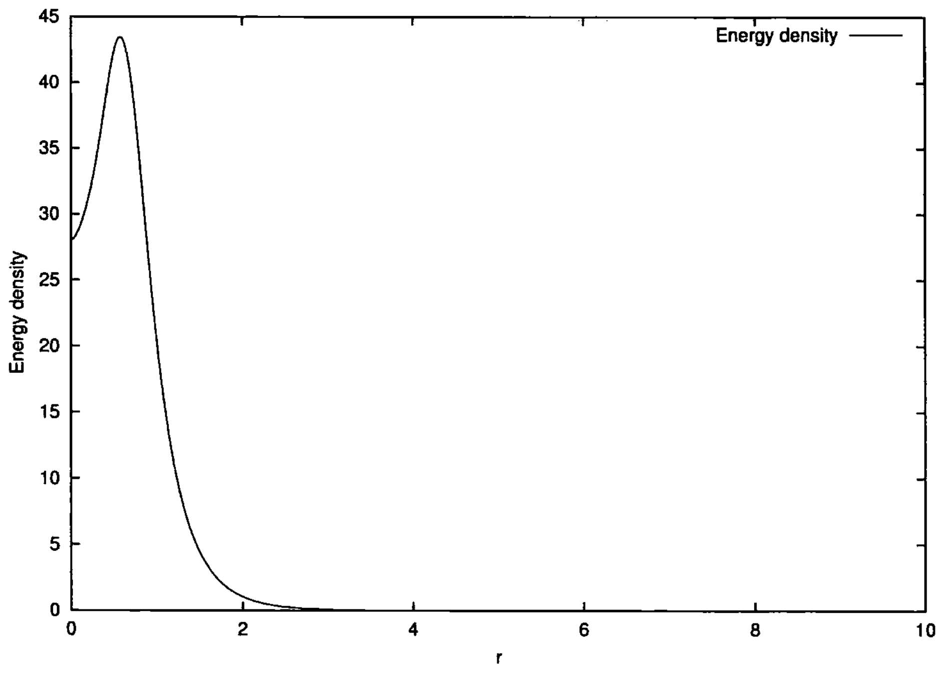

Figure 5.6: Energy density of the massive $SU(4)$ YM fields for $M = 1$, $G_0(0) = 0$, $G_1(0) = 2$ and $G_2(0) = 2$.

Chapter 6

Conclusions and Outlook

In this thesis we have discussed the applications of harmonic mapping methods to the construction of solutions of static 3-dimensional $SU(N)$ Skyrme models, usual and alternative, and pure massive $SU(N)$ Yang-Mills theories, which are based on the results of our research works [17–19].

In the following sections, 6.1-6.4, we present concluding remarks of each chapter, starting from chapter 2. Chapter 1 is omitted as it contained a general Introduction. Discussion of the outlook for further research is presented in the final section, *i.e.* 6.5.

6.1 Harmonic maps

In chapter 2 we briefly reviewed harmonic mapping theories as maps between two Riemannian manifolds $\mathcal{M}_0 \to \mathcal{M}$, generalising the concept of a geodesic in differential geometry. Based on the fact that solutions of the nonlinear sigma (or σ) models field equations coincide with harmonic maps, we reformulated the σ model field configurations space $\mathcal{M}$ as group and coset space manifolds revealing intimate relationship between differential geometry and gauge theories.

In chapter 3 we concentrated on the Grassmannian σ model, and reformulated the model in terms of rank-n projectors P_k of S^2 into Grassmann manifold $Gr(n, N) = SU(N)/S(U(N-n) \times U(n))$, which offers us a simple but much structured approach to construct full solutions of the models. There, we discussed explicitly the Veronese map or embedding: $S^2 \to CP^{N-1} = Gr(1, N)$ that plays an important role in the construction of exact spherically symmetric solutions of the $SU(N)$ Skyrme models and Yang-Mills theories in chapters 4 and 5, respectively.

In that chapter we also discussed some relevant algebraic topology concepts, in particular, de Rham cohomology and homotopy theories, which gives us topological understanding about the existence of topological charges and discrete solutions in sigma models, $SU(N)$ Skyrme models and massive Yang-Mills theories.

6.2 $SU(N)$ Skyrme models

In sections 4.1-4.7 of chapter 4, we studied the $SU(N)$ Skyrme models by constructing $SU(N)$ multiskyrmions fields using harmonic mappings method. We generalised the method of Ioannidou *et. al.* [14] by considering projectors of $S^2 \to Gr(n, N)$, *i.e.* using projectors of rank $n > 1$. In particular, we concentrated our attention only on the rank-2 projectors.

Using our construction we studied some approximate spherically symmetric configurations of $SU(N)$ Skyrme models. When we solved the equations for the profile functions for configurations with baryon number $B = (N-1)$ we found that they are very close to those for the rank-1 cases and that they have marginally higher energies. These results indicate that the rank-1 projector ansatz [13] is the best way to approximate energy minima of the $SU(N)$ Skyrme models.

We also discussed the possibility of generating exact spherically symmetric solutions using this construction. However, we found that, in contrast to the rank-1 projector ansatz in which exact spherically symmetric solutions can be found (numerically) by using the Veronese sequence of N mutually orthogonal vector fields in CP^{N-1}, such a construction is more involved in our case. In particular, we found that if the sequence of the $(N \times n)$ matrix fields M_k satisfy the condition (4.96) then it seems that the only possible exact solutions are embeddings. For example, for the rank-2 projector ansatz these are embeddings of a pair of $SU([N]/2)$ solutions of rank-1 projector ansatz, where $[N] = N$, or $(N-1)$ for N even or odd, respectively.

6.3 Alternative $SU(N)$ Skyrme models

In sections 4.8 and the rest of chapter 4, we discussed the alternative $SU(N)$ Skyrme models where we showed that to study these models we could use the harmonic maps ansatz [12] as has been applied to the usual Skyrme models [13].

We found that, as in the case of usual Skyrme models, the use of $(N-1)$ rank-1 projectors constructed from the Veronese sequence of vectors gives us spherically symmetric solutions of the alternative models. These solutions are characterised by the appropriate profile functions, which have to be determined numerically. In some cases we can exploit symmetries of the energy densities and reduce the number of functions.

Thus, almost everything has worked exactly like for the usual Skyrme models; only the equations for the profile functions have been a little modified. When we solved the equations for the profile functions we found that the solutions of the alternative models have energies higher than the corresponding solutions of the usual models. This can be traced to the extra terms in the expression for the energy density which give an additional positive contribution to the total energy.

6.4 Massive $SU(N)$ Yang-Mills Theories

In chapter 5 we considered the pure massive $SU(N)$ Yang-Mills theories where we first showed that for the case when the gauge potential is chosen to be almost pure gauge the theories reduce to the $SU(N)$ Skyrme models. When we studied the static magnetic type case we found that the theories do also admit the existence of a topological charge like quantity that we called by the name B-integral.

To solve the corresponding static equations, we used Ioannidou-Sutcliffe harmonic map ansatz that they introduced in their study of non-Bogomolnyi BPS monopoles [65]. This ansatz enabled us to construct some bounded solutions having finite energies. These solutions are very special in the sense that they depend very much on the chosen boundary conditions to be imposed on the profile functions g_k, *i.e.* $g_k = -2$, 0 or 2 at the origin and zero at infinity.

6.5 Outlook

We have seen that the generalised harmonic map ansatz (4.33) is compatible with the $SU(N)$ Skyrme equations if the sequence of $(N \times n)$ matrices $M_k(\xi)$ of the map $S^2 \to Gr(n, N)$ satisfy the condition (4.96). For the rank-1 projectors case, *i.e.* $n = 1$, as applied to the massive $SU(N)$ Yang-Mills equations as well, the solution of (4.96) for M_k are given by the Veronese sequence. For higher rank cases, however, our analysis suggests that the only solutions are embeddings of the rank-1 cases.

It would be interesting to see, whether there exists *other* conditions than (4.96) for the $(N \times n)$ matrix fields M_k that might lead to non-embedding solutions.

For the pure massive $SU(N)$ Yang-Mills theories, it would be interesting to see if there exist an extended version of the harmonic map ansatz (5.42) that can be applied to solve static *electric*-magnetic type equations.

As far as *spatial* symmetry is concerned, it is not clear, how to generalise the harmonic map ansatz (4.33) and (5.42) to derive an analytical or quasi-analytical form of non-spherically symmetric solutions for the Skyrme models and the Yang-Mills theories.

It would be interesting, as a first step in this direction, to see if the harmonic map ansatz method can be applied, at least, to the oblate or prolate spheroidal symmetric cases, where for the $SU(2)$ Skyrme model, a hedgehog-like ansatz has been applied to construct solutions having this deformed symmetry [69, 70].

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Appendix A

Evaluation of Exp($\mathcal{K}(\theta)$)

In this appendix we shall derive the closed matrix expression of the exponential matrix $G(\theta) = \text{Exp}\,(\mathcal{K}(\theta))$ in (2.107) from the series expansion:

$$\text{Exp}(M) = I + \sum_{p=1}^{\infty} \frac{1}{p!} M^p, \tag{A.0.1}$$

which is defined for any $(N \times N)$ matrix M [36].

For our case,

$$M = \mathcal{K}(\theta) \quad = \quad \begin{bmatrix} 0_{N-1} & \theta \\ & \\ -\theta^T & 0 \end{bmatrix}, \qquad \theta = (\theta^1, \ldots, \theta^{N-1})^T, \tag{A.0.2}$$

where 0_{N-1} is an $(N-1) \times (N-1)$ zero matrix. We observe that the powers of the matrix $\mathcal{K}(\theta)$ satisfy the following properties:

$$\begin{aligned} \mathcal{K}(\theta)^{2p} &= (-|\theta|^2)^{p-1}\mathcal{K}(\theta)^2, \\ \mathcal{K}(\theta)^{2p+1} &= (-|\theta|^2)^p \mathcal{K}(\theta), \end{aligned} \tag{A.0.3}$$

for $p = 1,\, 2,\, \ldots,$ where $|\theta|^2 = \theta^T\theta$,

$$\mathcal{K}(\theta)^2 \quad = \quad \begin{bmatrix} -\theta\theta^T & \hat{0}_{N-1} \\ & \\ \hat{0}^T_{N-1}, & -|\theta|^2 \end{bmatrix}, \tag{A.0.4}$$

and where $\hat{0}_{N-1}$ is a zero column vector with $(N-1)$ components.

Thus, the series expansion (A.0.1) for our case becomes

$$\mathrm{Exp}\Big(\mathcal{K}(\theta)\Big) = I + \sum_{p=0}^{\infty} \frac{(-|\theta|^2)^p}{(2p+1)!}\mathcal{K}(\theta) + \sum_{p=0}^{\infty} \frac{(-|\theta|^2)^{p-1}}{(2p)!}\mathcal{K}(\theta)^2. \tag{A.0.5}$$

Recall the Taylor series expansions of the sine and cosine functions:

$$\sin|\theta| = \sum_{p=0}^{\infty} \frac{(-)^p|\theta|^{2p+1}}{(2p+1)!}, \tag{A.0.6}$$

$$\cos|\theta| = \sum_{p=0}^{\infty} \frac{(-)^p|\theta|^{2p}}{(2p)!}, \tag{A.0.7}$$

the series (A.0.5) has the closed form

$$\mathrm{Exp}\Big(\mathcal{K}(\theta)\Big) = I + \frac{\sin|\theta|}{|\theta|}\mathcal{K}(\theta) + \frac{(1-\cos|\theta|)}{|\theta|^2}\mathcal{K}(\theta)^2. \tag{A.0.8}$$

With $\mathcal{K}(\theta)$ and $\mathcal{K}(\theta)^2$ given by (A.0.2) and (A.0.4), respectively, (A.0.8) has the matrix form

$$\mathrm{Exp}\Big(\mathcal{K}(\theta)\Big) = \begin{bmatrix} I_{N-1} + \theta\theta^T\frac{(\cos|\theta|-1)}{|\theta|^2} & \theta\frac{\sin|\theta|}{|\theta|} \\ -\theta^T\frac{\sin|\theta|}{|\theta|} & \cos|\theta| \end{bmatrix}. \tag{A.0.9}$$

Explicitly,

$$\mathrm{Exp}\Big(\mathcal{K}(\theta)\Big) = \begin{bmatrix} \delta^{AB} + \theta^A\theta^B\frac{(\cos|\theta|-1)}{|\theta|^2} & \theta^A\frac{\sin|\theta|}{|\theta|} \\ -\theta^B\frac{\sin|\theta|}{|\theta|} & \cos|\theta| \end{bmatrix}, \tag{A.0.10}$$

$(A, B = 1, \ldots, N-1)$, which completes our derivation of (2.107).

Appendix B

Derivations of the properties (3.81) and (3.82)

In this appendix we present the derivations of the properties (3.81) and (3.82) of the $(N \times n)$ mutually orthogonal matrices M_k, $(k = 0, \ldots, \lambda)$, that we constructed in section 3.2.2.

B.0.1 Derivation of (3.81)

Using the fact that the sequence M_k are independent, we have the expansion:

$$\partial_{\bar{\xi}} M_l = \sum_{k=0}^{\lambda} M_k a_{kl}, \quad l = 1, \ldots, \lambda, \tag{B.0.1}$$

where a_{kl} are $(n \times n)$ matrices. Note that, $\partial_{\bar{\xi}} M_0 = 0$, as we have assumed M_0 to be holomorphic.

Multiplying (B.0.1) from the left by $M_m^\dagger$ then yields

$$M_m^\dagger \partial_{\bar{\xi}} M_l = |M_m|^2 a_{ml}, \tag{B.0.2}$$

where we have used the orthogonality property (3.75).

Next, using the construction (3.79), the left hand side of (B.0.2) becomes

$$\begin{aligned} M_m^\dagger \partial_{\bar{\xi}} M_l &= \partial_{\bar{\xi}} \left(M_m^\dagger M_l \right) - \left(\partial_{\xi} M_m \right)^\dagger M_l \\ &= \delta_{lm} M_l^\dagger \partial_{\bar{\xi}} M_l - \delta_{l,m+1} |M_l|^2. \end{aligned} \tag{B.0.3}$$

Thus, using (B.0.2) in (B.0.3), we obtain

$$a_{ml} = \delta_{lm}|M_m|^{-2}M_l^\dagger \partial_{\bar{\xi}} M_l - \delta_{l,m+1}|M_m|^{-2}|M_l|^2, \tag{B.0.4}$$

and so the expansion (B.0.1) reduces to

$$\partial_{\bar{\xi}} M_l = M_l a_{ll} + M_{l-1} a_{l-1,l}. \tag{B.0.5}$$

In the following we shall show that

$$a_{ll} = 0, \quad l = 1, \dots, \lambda, \tag{B.0.6}$$

which is, according to (B.0.4), equivalent to

$$M_l^\dagger \partial_{\bar{\xi}} M_l = 0. \tag{B.0.7}$$

For the case $l = 1$, using the construction (3.79) or (3.80) and $\partial_{\bar{\xi}} M_0 = 0$, we obtain

$$\partial_{\bar{\xi}} M_1 = -\partial_{\bar{\xi}} P_0 \partial_\xi M_0 = -M_0 \partial_{\bar{\xi}} \left(|M_0|^{-2} M_0^\dagger\right) \partial_\xi M_0, \tag{B.0.8}$$

and so

$$M_1^\dagger \partial_{\bar{\xi}} M_1 = 0, \qquad i.e. \qquad a_{11} = 0. \tag{B.0.9}$$

For the general case: $1 < k < \lambda$, we use the inductive argument by assuming that

$$a_{kk} = 0, \tag{B.0.10}$$

i.e.

$$\partial_{\bar{\xi}} M_k = M_{k-1} a_{k-1,k}. \tag{B.0.11}$$

As

$$\begin{aligned} \partial_{\bar{\xi}} P_k &= \partial_{\bar{\xi}} M_k \left(|M_k|^{-2} M_k^\dagger\right) + M_k \partial_{\bar{\xi}} \left(|M_k|^{-2} M_k^\dagger\right) \\ &= \left[M_{k-1} a_{k-1,k} + M_k \partial_{\bar{\xi}}\right] \left(|M_k|^{-2} M_k^\dagger\right), \end{aligned} \tag{B.0.12}$$

using the construction (3.80) and the orthogonality property (3.75), it follows that

$$M_{k+1}^\dagger \partial_{\bar{\xi}} M_{k+1} = M_{k+1}^\dagger \left[-\sum_{l=0}^{k} \left(\partial_{\bar{\xi}} P_l\right) \partial_\xi^{k+1} M_0 \right] = 0, \tag{B.0.13}$$

i.e.

$$a_{k+1,k+1} = 0, \tag{B.0.14}$$

which completes our inductive proof of (B.0.6).

Thus, (B.0.5) becomes

$$\partial_{\xi} M_l = -M_{l-1}|M_{l-1}|^{-2}|M_l|^2, \tag{B.0.15}$$

i.e. the equation (3.81).

B.0.2 Derivation of (3.82)

As from (3.81),

$$\partial_{\xi} M_k^{\dagger} = \left(\partial_{\xi} M_k\right)^{\dagger} = \left(a_{k-1,k}\right)^{\dagger} M_{k-1}^{\dagger}, \tag{B.0.16}$$

it follows from the orthogonality property (3.75) that

$$\partial_{\xi}|M_k|^2 = M_k^{\dagger}\partial_{\xi} M_k. \tag{B.0.17}$$

Thus,

$$\begin{aligned} \partial_{\xi}\left(M_k|M_k|^{-2}\right) &= \left(\partial_{\xi} M_k\right)|M_k|^{-2} - M_k|M_k|^{-2}\left(\partial_{\xi}|M_k|^2\right)|M_k|^{-2} \\ &= \left[\left(I - P_k\right)\partial_{\xi} M_k\right]|M_k|^{-2}, \end{aligned} \tag{B.0.18}$$

and (3.82) follows by using the construction (3.79).

Appendix C

Projectors of $C \to CP^3$

In this appendix we present the complete (4×4) matrix projectors P_k, $(k = 0, \ldots, 3)$, of the Veronese map $M_0 : C \to CP^3$, *i.e.*

$$M_0 = \left(1,\ \sqrt{3}\xi,\ \sqrt{3}\xi^2,\ \xi^3\right)^T. \tag{C.0.1}$$

Using M_0 in (C.0.1) and the remaining mutually orthogonal vector fields M_k $(k = 1, 2, 3)$, that we have computed in section 3.2.5, we find that these (4×4) matrix projectors

$$P_k = M_k |M_k|^{-2} M_k^\dagger, \tag{C.0.2}$$

are

$$P_k = (1 + |\xi|^2)^{-3} \widetilde{P}_k, \qquad k = 0, \ldots 3, \tag{C.0.3}$$

where the diagonal and the upper diagonal entries of the (4×4) matrix $\widetilde{P}_k$ are given in table A.1. The other elements are obtained from the Hermiticity property: $\widetilde{P}_k = \widetilde{P}_k^\dagger$.

From table A.1, we see explicitly that the projectors P_k, $(k = 0, \ldots, 3)$, satisfy the rank-1 condition: $\mathrm{Tr}(P_k) = 1$, and the completeness relation: $P_0 + P_1 + P_2 + P_3 = I_4$, where I_4 is the (4×4) unit matrix.

Entries	$\widetilde{P}_0$	$\widetilde{P}_1$	$\widetilde{P}_2$	$\widetilde{P}_3$
11	1	$3\lvert\xi\rvert^2$	$3\lvert\xi\rvert^4$	$\lvert\xi\rvert^6$
12	$\sqrt{3}\bar{\xi}$	$-\sqrt{3}\bar{\xi}(1-2\lvert\xi\rvert^2)$	$-\sqrt{3}\bar{\xi}\lvert\xi\rvert^2(2-\lvert\xi\rvert^2)$	$-\sqrt{3}\bar{\xi}\lvert\xi\rvert^4$
13	$\sqrt{3}\bar{\xi}^2$	$-\sqrt{3}\bar{\xi}^2(2-\lvert\xi\rvert^2)$	$\sqrt{3}\bar{\xi}^2(1-2\lvert\xi\rvert^2)$	$\sqrt{3}\bar{\xi}^2\lvert\xi\rvert^4$
14	$\bar{\xi}^3$	$-3\bar{\xi}^3$	$3\bar{\xi}^3$	$-\bar{\xi}^3$
22	$3\lvert\xi\rvert^2$	$(1-2\lvert\xi\rvert^2)^2$	$\lvert\xi\rvert^2(2-\lvert\xi\rvert^2)^2$	$3\lvert\xi\rvert^4$
23	$3\bar{\xi}\lvert\xi\rvert^2$	$\bar{\xi}(1-2\lvert\xi\rvert^2)(2-\lvert\xi\rvert^2)$	$-\bar{\xi}(1-2\lvert\xi\rvert^2)(2-\lvert\xi\rvert^2)$	$-3\bar{\xi}\lvert\xi\rvert^2$
24	$\sqrt{3}\bar{\xi}^2\lvert\xi\rvert^2$	$\sqrt{3}\bar{\xi}^2(1-2\lvert\xi\rvert^2)$	$-\sqrt{3}\bar{\xi}^2(2-\lvert\xi\rvert^2)$	$\sqrt{3}\bar{\xi}^2$
33	$3\lvert\xi\rvert^4$	$\lvert\xi\rvert^2(2-\lvert\xi\rvert^2)^2$	$(1-2\lvert\xi\rvert^2)^2$	$3\lvert\xi\rvert^2$
34	$\sqrt{3}\bar{\xi}\lvert\xi\rvert^4$	$\sqrt{3}\bar{\xi}\lvert\xi\rvert^2(2-\lvert\xi\rvert^2)$	$\sqrt{3}\bar{\xi}(1-2\lvert\xi\rvert^2)$	$-\sqrt{3}\bar{\xi}$
44	$\lvert\xi\rvert^6$	$3\lvert\xi\rvert^4$	$3\lvert\xi\rvert^2$	1

Table C.1: Elements of the (4×4) matrices $\widetilde{P}_k = \widetilde{P}_k^\dagger$ $(k = 0, \ldots, 3)$.

Appendix D

Symmetry of the Veronese Sequence

In this appendix we shall consider the action of the coordinate transformation $\xi \to \xi'$, on the CP^{N-1} fields $Z_k = M_k|M_k|^{-1}$ where M_k are the Veronese sequence. Here, we assume $\xi \in S^2$, which is our main concern in chapters 4 and 5. We do not consider the complex plane C as base space, since for this case, the fields Z_k could take an arbitrary, but fixed, value at $|\xi| \to \infty$, which breaks the global $SU(N)$ symmetry.

Then we have the following result, first proved by Ioannidou *et. al.* [13].

Proposition C.1

Let $M_k(\xi) : \;\; S^2 \to CP^{N-1}$ be the Veronese sequence. Then the CP^{N-1} fields $Z_k = M_k|M_k|^{-1}$ transform covariantly under the Mobius transformation:

$$\xi \to \xi' = \frac{a\xi + b}{-\bar{b}\xi + \bar{a}}, \qquad |a|^2 + |b|^2 = 1, \tag{D.0.1}$$

i.e.

$$Z_k(\xi') = \Omega Z_k(\xi), \qquad (k = 0, \ldots, N-1), \tag{D.0.2}$$

where $\Omega \in SU(N)$ depends on a and b and their complex conjugates.

Proof

Our proof will consist of two parts: (1) $\Omega \in U(N)$, and (2) $det\,\Omega = 1$.

(1) $\Omega \in U(N)$

Here we shall use the construction (3.79). First, we consider the case $k = 0$. By acting the Mobius transformation (D.0.1) on the Veronese map M_0 in (3.123), with the components $[M_0(\xi)]_p = f_p(\xi)$ given by (3.125), we find that

$$M_0(\xi') = \frac{AM_0(\xi)}{(-\bar{b}\xi + \bar{a})^{N-1}}, \tag{D.0.3}$$

where A is an $(N \times N)$ matrix depending only on a and b and their complex conjugates. The elements A_{pq} of A are extracted from the expression

$$\begin{aligned}(-\bar{b}\xi + \bar{a})^{N-1}[M_0(\xi')]_p &= \sqrt{C_p^{N-1}}(a\xi + b)^p(-\bar{b}\xi + \bar{a})^{N-p-1} \\ &= \sum_{q=0}^{N-1} A_{pq}\left(\sqrt{C_q^{N-1}}\xi^q\right). \end{aligned} \tag{D.0.4}$$

Explicitly,

$$A_{pq} = \sqrt{C_p^{N-1}} \sum_{s=0}^{q} \frac{(-)^s C_{q-s}^p C_s^{N-p-1} a^{q-s} \bar{a}^{N-p-s-1} b^{p-q+s} \bar{b}^s}{\sqrt{C_q^{N-1}}}, \tag{D.0.5}$$

where $C_l^k = 0$, for $l > k$.

By virtue of (3.139),

$$|M_0(\xi')|^2 = (1 + |\xi'|^2)^{N-1} = \frac{|M_0(\xi)|^2}{(-\bar{b}\xi + \bar{a})^{N-1}}, \tag{D.0.6}$$

where we have used $|a|^2 + |b|^2 = 1$. Thus, from (D.0.3) and (D.0.6),

$$|AM_0(\xi)|^2 = |(-\bar{b}\xi + \bar{a})^{N-1} M_0(\xi')|^2 = (1 + |\xi|^2)^{N-1}, \tag{D.0.7}$$

which proves that $A \in U(N)$.

Let us now introduce the notations: $\tilde{M}_k = M_k(\xi')$ and $M_k = M_k(\xi)$ and consider the case $k = 1$. As

$$\partial_{\xi'} = \frac{\partial \xi}{\partial \xi'} \partial_\xi = \frac{(1 + |\xi|^2)}{(-\bar{b}\xi + \bar{a})^2} \partial_\xi, \tag{D.0.8}$$

and

$$\tilde{P}_0 = \tilde{M}_0 |\tilde{M}_0|^{-2} \tilde{M}_0^\dagger = AP_0A^\dagger, \tag{D.0.9}$$

which follows from (D.0.3), so, by the construction (3.79),

$$\begin{aligned}\tilde{M}_1 &= (I - \tilde{P}_0)\partial_{\xi'}\tilde{M}_0 \\ &= A(I - P_0)A^\dagger(-\bar{b}\xi + \bar{a})^2 \partial_\xi \left[\frac{AM_0}{(-\bar{b}\xi + \bar{a})^{N-1}}\right] \\ &= \frac{A(I - P_0)\partial_\xi M_0}{(-\bar{b}\xi + \bar{a})^{N-3}}, \end{aligned} \tag{D.0.10}$$

since $(I - P_0)M_0 = 0$. Thus,

$$\tilde{M}_1 = \frac{AM_1}{(-\bar{b}\xi + \bar{a})^{N-3}}, \tag{D.0.11}$$

and so $\tilde{P}_1 = \tilde{M}_1|\tilde{M}_1|^{-2}\tilde{M}_1^\dagger = AP_1A^\dagger$.

Now we consider the general case: $1 < k \leq (N-1)$, where we shall use inductive proof by assuming

$$\tilde{M}_k = \frac{AM_k}{(-\bar{b}\xi + \bar{a})^{N-2k-1}}, \tag{D.0.12}$$

and so $\tilde{P}_k = AP_kA^\dagger$, holds up to $k < (N-1)$.

Note that (D.0.12) is consistent with $A \in U(N)$, as from (3.139),

$$\begin{aligned} |\tilde{M}_k|^2 &= \frac{k!(N-1)!}{(N-k-1)!}(1+|\xi'|^2)^{N-2k-1} \\ &= \frac{|M_k|^2}{|(-\bar{b}\xi + \bar{a})^{N-2k-1}|^2}, \end{aligned} \tag{D.0.13}$$

and so

$$|AM_k|^2 = |(-\bar{b}\xi + \bar{a})^{N-2k-1}\tilde{M}_k|^2 = |M_k|^2. \tag{D.0.14}$$

The construction (3.79) then gives us

$$\begin{aligned} \tilde{M}_{k+1} &= (I - \tilde{P}_k)\partial_{\xi'}\tilde{M}_k \\ &= A(I-P_k)A^\dagger(-\bar{b}\xi + \bar{a})^2\partial_\xi\left[\frac{AM_k}{(-\bar{b}\xi + \bar{a})^{N-2k-1}}\right] \\ &= \frac{AM_{k+1}}{(-\bar{b}\xi + \bar{a})^{N-2(k+1)-1}}, \end{aligned} \tag{D.0.15}$$

since $(I - P_k)M_k = 0$, which completes our inductive proof of (D.0.12).

(2) $det\,\Omega = 1$.

Now we give a proof of $det\,A = 1$. For this purpose, it is enough to consider A around the identity, *i.e.*

$$A = e^X \simeq I + X, \qquad X^2 \simeq 0, \tag{D.0.16}$$

where X takes value in the Lie algebra $u(N)$.

Since $det\,A = e^{\mathrm{Tr}X}$, so to proof $det\,A = 1$ is equivalent to proof

$$\mathrm{Tr}\,X = 0. \tag{D.0.17}$$

Thus, according to (D.0.16), we need to consider the infinitesimal part of the diagonal elements A_{pp} of A only. For the finite expression, (D.0.5) gives us

$$A_{pp} = \sum_{s=0}^{q} (-)^s C_{p-s}^{p} C_s^{N-p-1} a^{p-s} \bar{a}^{N-p-s-1} |b|^{2s}. \tag{D.0.18}$$

Now a question arises: what is the infinitesimal expressions of the elements a and b? To answer this question, let us reconsider the Mobius transformation (D.0.1). In the homogeneous coordinates $[z_1, z_2]$, where $\xi = z_1/z_2$, $z_2 \neq 0$, the action of (D.0.1) is simply matrix multiplication [28, 42]:

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{D.0.19}$$

showing that a and b are elements of $SU(2)$. Thus, their infinitesimal forms are [36]

$$a \simeq 1 + i\theta, \qquad b \simeq i\lambda + \eta, \qquad \theta^2 \simeq \lambda^2 \simeq \eta^2 \simeq 0. \tag{D.0.20}$$

As $|b|^2 \simeq 0$, the infinitesimal form of the diagonal elements A_{pp} in (D.0.18) is nonzero for $s = 0$ only. Thus,

$$A_{pp} \simeq C_p^p C_0^{N-p-1} a^p \bar{a}^{N-p-1} \simeq 1 + i\theta(2p - N + 1), \tag{D.0.21}$$

and so the diagonal elements of X, according to (D.0.16), are

$$X_{pp} = i\theta(2p - N + 1). \tag{D.0.22}$$

As

$$\sum_{p=0}^{N-1} (2p - N + 1) = 2.\frac{1}{2}(N-1)N - N(N-1) = 0, \tag{D.0.23}$$

we see that (D.0.17) is satisfied, which completes our proof of $det\, A = 1$.

Thus, for the CP^{N-1} fields Z_k, by virtue of (D.0.12),

$$\tilde{Z}_k = \tilde{M}_k |\tilde{M}_k|^{-1} = A M_k |M_k|^{-1} = A Z_k, \qquad k = 0, 1, \ldots, (N-1), \tag{D.0.24}$$

which proves that (D.0.2) holds with $\Omega = A \in SU(N)$. □

Appendix E

Homotopy Groups

In this appendix we briefly review the definitions of the D-th homotopy groups, $\Pi_D(\mathcal{M})$, for $D = 0$, 1 and $D > 1$ and discuss their computations [49–52]. In fact, our main aim here is to proof that:

(1) $\Pi_2(Gr(n, N)) = \mathbb{Z}$, and

(2) $\Pi_3(SU(N)) = \mathbb{Z}$,

which are relevant to our discussion in sections 3.3.2 and 4.2.3, respectively.

E.1 $\Pi_0(\mathcal{M})$

Here, as S^0 consist of two points: -1 and +1 only, $\Pi_0(\mathcal{M})$ is not a group but just a set. This is explained as follows. Let $\alpha, \beta : \{-1, 1\} \simeq S^0 \to \mathcal{M}$ where we fix $\alpha(-1) = \beta(-1) = y_0 \in \mathcal{M}$. Then $\alpha(1)$ and $\beta(1)$ could be any two points in $\mathcal{M}$. If $F(s, \tau) : S^0 \times [0, 1] \to \mathcal{M}$ be a homotopy from α to β then $F(-1, \tau) = y_0$ for $\tau \in [0, 1]$ whereas $F(1, \tau)$ is a continuous function from $[0, 1] \to \mathcal{M}$ such that $F(1, 0) = \alpha(1)$ and $F(1, 1) = \beta(1)$. Thus $\Pi_0(\mathcal{M})$ corresponds to *set* of path-connected components of $\mathcal{M}$. If $\Pi_0(\mathcal{M})$ has only one component then $\mathcal{M}$ is *connected* and we write $\Pi_0(\mathcal{M}) = 0$. The disconnected case $\Pi_0(\mathcal{M}) \neq 0$ is the analog of topological charges for the maps $S^0 \to \mathcal{M}$ called *domain walls* in physics literature. Clearly, $\Pi_0(\mathbb{R}) = 0$, while $\Pi_0(S^0) = \mathbb{Z}_2$. Here, $\mathbb{Z}_2$ is considered as a *set* with just two elements $\{-1, 1\}$. For the integers $\mathbb{Z}$, which consists of infinitely countable disconnected elements, we have $\Pi_0(\mathbb{Z}) = \mathbb{Z}$.

E.2 $\Pi_1(\mathcal{M})$

The homotopy group $\Pi_1(\mathcal{M})$, called *fundamental group*, is of particular interest. Its elements $[\alpha]$ consist of closed paths or loops $\alpha(s)$, $s \in [0,\,1] \simeq S^1$, which begin and end at a definite point $y_0 \in \mathcal{M}$, *i.e.* $\alpha(0) = \alpha(1) = y_0$ called the *base point.* Let $[e]$ denotes class of loops homotopic to a point. Then $\Pi_1(\mathcal{M}) = 0 \simeq [e]$ means that any loops on $\mathcal{M}$ can be shrunk to a point, which implies that $\mathcal{M}$ is *simply connected.* Otherwise, $\mathcal{M}$ is *multiply connected.*

The *group multiplication* in $\Pi_1(\mathcal{M})$ is defined as follows. Let $[\alpha]$, $[\beta] \in \Pi_1(\mathcal{M})$, then we define

$$[\alpha][\beta] = [\alpha \cup \beta]. \tag{E.2.1}$$

The product loop $\gamma = \alpha \cup \beta$ is formally defined by

$$\gamma(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases} \tag{E.2.2}$$

Geometrically, γ corresponds to traversing α in the first half, then followed by β in the remaining half. Traversing α in the reverse direction defines the *inverse* loop

$$\alpha^{-1}(s) = \alpha(1-s). \tag{E.2.3}$$

One can show that this really gives a group structure, *i.e.* the multiplication is associative, it has inverse α^{-1} and an identity element $[e]$ [51,52]. The multiplication (E.2.2) could be non-commutative, in that case $\Pi_1(\mathcal{M})$ is nonabelian [7].

For illustration, let us consider the computation of $\Pi_1(S^k)$ for $k \geq 1$. The case $k = 1$ corresponds to the map $S^1_s \to S^1$ which can be specified by a complex function of unit modulus, $f(s) = e^{i\phi(s)}$, $s \in [0,\,1] \in S^1_s$. The single valuedness of the map then requires that $\phi(1) = \phi(0) + n(2\pi)$, $n \in \mathbb{Z}$, which is precisely the set of elements generated by the action of $\mathbb{Z}$. Intuitively, this means that a loop winding n times around the circle cannot be shrunk to a point without leaving it. Thus, maps with different n fall into different classes $[\alpha_n]$. Therefore, we have $\Pi_1(S^1) = \mathbb{Z}$. Since S^1 is isomorphic to the unitary group $U(1)$, we also have $\Pi_1(U(1)) = \mathbb{Z}$. For $k > 1$, S^1_s maps into arbitrary loops on S^k. As any curve on S^k can always be deformed into a single point, we have $\Pi_1(S^k) = 0$. Next, since $SU(2) \simeq S^3$, we also have

$\Pi_1(SU(2)) = 0$. In general, as $SU(N)$ is simply-connected, $\Pi_1(SU(N)) = 0$ as well. Let us consider $\Pi_1(RP^{N-1})$. As $RP^{N-1} \simeq S^{N-1}/[-1,\, 1]$, *i.e.* S^{N-1} covers RP^{N-1} twice, we have $\Pi_1(RP^{N-1}) = \mathbb{Z}_2$. Here, $\mathbb{Z}_2$ is considered as a cyclic group of order two which includes just two elements: -1 and 1.

E.3 $\Pi_D(\mathcal{M})$, $D > 1$

Let us now consider the higher homotopy groups $\Pi_D(\mathcal{M})$, $D > 1$. They are generated by homotopy classes of the higher dimensional generalisation of a loop *i.e.* D-loop which is homeomorphic to S^D. Here the interval $\mathcal{I} = [0,\, 1]$ is generalised to D-dimensional *interval* or *cube*: $[0,\, 1] \times \ldots \times [0,\, 1] = \mathcal{I}^D$, where the boundary $\partial\mathcal{I}^D$ is made of points: $s = (s_1, \ldots, s_D)$ such that at least one coordinate s_i is 0 or 1. Then a *D-loop* $\alpha(s)$, $s \in \mathcal{I}^D$, based at $y_0 \in \mathcal{M}$ is a map: $\mathcal{I}^D \to \mathcal{M}$ such that $\alpha(s) = y_0$ for all $s \in \partial\mathcal{I}^D$. Loops α and β are *homotopic* if there exist a continuous function $F(s,\, \tau) : \mathcal{I}^D \times [0,\, 1] \to \mathcal{M}$ such that

$$F(s,0) = \alpha(s), \qquad F(1,s) = \beta(s), \tag{E.3.4}$$

and

$$F(s,\tau) = y_0, \quad \text{if} \quad s \in \partial\mathcal{I}^D. \tag{E.3.5}$$

The homotopy class $[\alpha]$ is now the quotient of the space of the D-loops in $\mathcal{M}$ by the homotopy equivalence as defined above.

The group product of the homotopy classes $[\alpha]$ and $[\beta]$ is defined as in (E.2.1) where here the product loop $\gamma = \alpha \cup \beta$ is defined by

$$\gamma(s) = \begin{cases} \alpha(2s_1, s_2, \ldots, s_D), & 0 \le s \le \frac{1}{2} \\ \beta(2s_1 - 1, s_2, \ldots, s_D), & \frac{1}{2} \le s \le 1, \end{cases} \tag{E.3.6}$$

and the *invers* loop

$$\alpha^{-1}(s) = \alpha(1 - s_1, s_2, \ldots, s_n). \tag{E.3.7}$$

The set with elements $[\alpha]$ and group multiplication $[\alpha][\beta]$ as defined above form the D-th homotopy group $\Pi_D(\mathcal{M})$. If it is nontrivial, *i.e.* $\Pi_D(\mathcal{M}) \neq 0$, then there is a D-dimensional submanifold $\widetilde{S}^D$ in $\mathcal{M}$ which is topologically equivalent to S^D

that cannot be shrunk into one point. In fact, since S^D is a smooth manifold, $\Pi_D(S^N) = 0$ for $D < N$ whereas $\Pi_D(S^D) = \mathbb{Z}$.

E.4 Computation of $\Pi_2(Gr(n,\, N))$

We now come to the first aim of this appendix, the computation of the second homotopy group of the Grassmann manifold, $\Pi_2(Gr(n,\, N))$. This we need to add homotopy perspective to the topological meaning to the disrete solutions Z_k and the topological charge integral (3.117).

Here we shall follow the general scheme to compute the n-th homotopy groups $\Pi_D(\mathcal{M})$ which is set by presenting the manifold $\mathcal{M}$ as coset space G/H. Fitting this construction into the bundle picture $(E,\, B,\, F,\, p)$ where $E = G$, $B = G/H$, $F = H$ and $p : E \to B$ the projection then there exist a long *exact homotopy sequence* [49–52]:

$$\ldots \to \Pi_D(F) \overset{i_D}{\to} \Pi_D(E) \overset{p_D}{\to} \Pi_D(B) \overset{\partial_D}{\to} \Pi_{D-1}(F) \to \ldots \tag{E.4.8}$$

Here i_D is a group homomorphism of the inclusion map $i : F \to i(F) \subset E$, and p_D as well, as $p : i(F) \to B$, whereas $\partial : \{\mathcal{I}^{D-1} \times \mathcal{I} \to B\} \to \{\mathcal{I}^{D-1} \times \partial\mathcal{I} \to F\}$. Recall that the sequence:

$$\ldots \to \Phi_{k+1} \overset{f_{k+1}}{\to} \Phi_k \overset{f_k}{\to} \Phi_{k-1} \overset{f_{k-1}}{\to} \Phi_{k-2} \to \ldots \tag{E.4.9}$$

is called *exact* if the *kernel* of the mapping f_k, Ker$(f_k) \subset \Phi_k$, is the *image* of the mapping f_{k+1}, Im$(f_{k+1}) \subset \Phi_k$, where by definition: $f_k(\text{Ker}(f_k)) = 0$.

To summarise the relevant theorems related to the exact homotopy sequence (E.4.8), let us introduce the following notations. Let Φ_k, $k = 0,\, 1,\, \ldots$ be the k-th element in the sequence (E.4.8) counting from the right end. Thus, if $k = 3D + r$, with $r = 0,\, 1,\, 2$, then $\Phi_k = \Pi_D(X_r)$ where $X_0 = B$, $X_1 = E$ and $X_2 = F$. Using this notation, then the relevant theorems are summarised as follows:

(1) If $\Phi_{k+1} = 0$ and $\Phi_{k-1} = 0$ then $\Phi_k = 0$.

(2) If $\Phi_{k+1} = 0$ and $\Phi_{k-2} = 0$ then $\Phi_k = \Phi_{k-1}$, *i.e.* $f_k : \Phi_k \to \Phi_{k-1}$ is an isomorphism (onto).

(3) If $E = B \times F$ then $\Phi_k(E) = \Phi_k(B) \oplus \Phi_k(F)$.

For detail explanations and proofs we refer to Ref. [51,52].

As the Grassmann manifold $Gr(n,\, N) = V(n,\, N)/U(n)$ where $V(n,\, N)$ is the Stiefel manifold, *i.e.* $V(n,\, N) = U(N)/U(N-n)$, we shall first compute the *D-th* homotopy group of $U(N)$ and $V(n,\, N)$. This we do in steps by starting from the simpler manifold S^1 as follows.

E.4.1 $\Pi_D(S^1)$, $D \geq 2$

As $S^1 \simeq \mathbb{R}/\mathbb{Z}$, the corresponding exact homotopy sequence is

$$\ldots \rightarrow \Pi_{D+1}(S^1) \rightarrow \Pi_D(\mathbb{Z}) \rightarrow \Pi_D(\mathbb{R}) \rightarrow \Pi_D(S^1) \rightarrow \Pi_{D-1}(\mathbb{Z}) \rightarrow \ldots \tag{E.4.10}$$

Since $\Pi_D(\mathbb{R}) = 0$, $D \geq 0$, $\Pi_D(\mathbb{Z}) = 0$, $D \geq 1$, $\Pi_0(\mathbb{Z}) = \mathbb{Z}$, and $\Pi_0(S^1) = 0$, it follows from (E.4.10) that $\Pi_1(S^1) = \Pi_0(\mathbb{Z}) = \mathbb{Z}$, as we have found before, and

$$\Pi_D(S^1) = 0, \quad \text{for} \quad D \geq 2. \tag{E.4.11}$$

E.4.2 $\Pi_D(U(N))$

Here we use the coset relation $S^{2N+1} = V(1,\, N+1) = U(N+1)/U(N)$, for which the corresponding exact homotopy sequence is

$$\ldots \rightarrow \Pi_{D+1}(S^{2N+1}) \rightarrow \Pi_D(U(N)) \rightarrow \Pi_D(U(N+1)) \rightarrow \Pi_D(S^{2N+1})$$
$$\rightarrow \Pi_{D-1}(U(N)) \rightarrow \ldots \tag{E.4.12}$$

Since $\Pi_k(S^l) = 0$, for $k < l$, it follows from (E.4.12) that

$$\Pi_D(U(N)) = \Pi_D(U(N+1)), \quad \text{for} \quad D < 2N. \tag{E.4.13}$$

Using the result $\Pi_1(U(1)) = \Pi_1(S^1) = \mathbb{Z}$ in (E.4.13) then by induction we obtain

$$\Pi_1(U(N)) = \mathbb{Z}, \quad N \geq 1. \tag{E.4.14}$$

Next, since $\Pi_2(S^3) = 0$ $(2 < 3)$ and $\Pi_2(U(1)) = \Pi_2(S^1) = 0$, according to (E.4.11), it follows from (E.4.12) that $\Pi_2(U(2)) = 0$. Hence, from (E.4.13):

$$\Pi_2(U(N)) = 0 \quad \text{for any } N. \tag{E.4.15}$$

E.4.3 $\Pi_D(V(n,\, N))$

As the Stiefel manifold $V(n,\, N) = U(N)/U(N-n)$, the corresponding exact homotopy sequence is

$$\ldots \to \Pi_D(U(N)) \to \Pi_D(V(n, N)) \to \Pi_{D-1}(U(N-n)) \to \Pi_{D-1}(U(N))$$
$$\to \Pi_{D-1}(V(n, N)) \to \ldots \qquad \text{(E.4.16)}$$

which is, unfortunately, inconclusive for us to compute $\Pi_D(V(n,\, N))$.

To get around with this we shall use the fibre bundle picture $(V(n,\, N),\, V(n-1,\, N),\, S^{2(N-n)+1}, p)$ whose setting is explained as follows [55]. Let $Y = (\mathbf{w}^1, \ldots, \mathbf{w}^{n-1})$ be a fixed orthonormal $(n-1)$-frame and $S^{2(N-n)+1}$ be the unit sphere which is tangent to Y, *i.e.* $\mathbf{v}^\dagger Y = 0$ for $\mathbf{v} \in S^{2(N-n)+1}$. Thus, locally $V(n,\, N) = V(n-1,\, N) \times S^{2(N-n)+1} = \Big\{Z = (Y,\, \mathbf{v})\Big\}$, and $p:\, V(n,\, N) \to V(n-1,\, N)$ is the projector that assigns to each n-frame Z the $(n-1)$- frame Y obtained by omitting one vector (here, the last vector $\mathbf{v}$) with $p^{-1}(V(n-1,\, N)) = S^{2(N-n)+1}$ is the fibre. For this bundle, the corresponding exact homotopy sequence is

$$\ldots \to \Pi_D(S^{2(N-n)+1}) \to \Pi_D(V(n,\, N)) \to \Pi_D(V(n-1,\, N))$$
$$\to \Pi_{(D-1)}(S^{2(N-n)+1}) \to \ldots \qquad \text{(E.4.17)}$$

Since for $D < 2(N-n)+1$, $\Pi_D(S^{2(N-n)+1}) = 0$, we have

$$\Pi_D(V(n,\, N)) = \Pi_D(V(n-1,\, N)). \qquad \text{(E.4.18)}$$

For $n = 2$, $V(1,\, N) = S^{2N-1}$, so $\Pi_D(V(1,\, N)) = 0$, and by induction we obtain from (E.4.18) that [50]

$$\Pi_D(V(n,\, N)) = 0, \qquad \text{for} \quad D < 2(N-n)+1. \qquad \text{(E.4.19)}$$

For $D = 2(N-n)+1$, $\Pi_D(S^{2(N-n)+1}) = \mathbb{Z}$, and so the exact sequence (E.4.17) gives us [50]

$$\Pi_D(V(n,\, N)) = \Pi_D(S^{2(N-n)+1}) = \mathbb{Z}. \qquad \text{(E.4.20)}$$

E.4.4 $\Pi_2(Gr(n,\, N))$

Now we come to our main aim. As $Gr(n,\, N) = V(n,\, N)/U(n)$, the corresponding exact homotopy sequence is

$$\ldots \to \Pi_D(V(n,N)) \to \Pi_D(Gr(n,N)) \to \Pi_{D-1}(U(n)) \to \Pi_{D-1}(V(n,N)) \to \ldots \tag{E.4.21}$$

If $D < 2(N-n)+1$, then using (E.4.19) in (E.4.21) gives us

$$\Pi_D(Gr(n,N) = \Pi_{D-1}(U(n)). \tag{E.4.22}$$

Thus, for $(N-n) \geq 1$ we ultimately obtain from (E.4.22) that

$$\Pi_2(Gr(n,\, N)) = \Pi_1(U(n)) = \mathbb{Z}, \tag{E.4.23}$$

where the last equality (isomorphism) follows from (E.4.14).

E.5 Computation of $\Pi_3(SU(N))$

To compute $\Pi_3(SU(N))$, we use exact homotopy sequence for the coset relation $S^{2N-1} = SU(N)/SU(N-1)$. In analogy with the computation of $\Pi_D(U(N))$ that we have carried out in section 4.2.3, we get

$$\Pi_D(SU(N-1)) = \Pi_D(SU(N)), \quad \text{for} \quad D < 2(N-1). \tag{E.5.24}$$

As $SU(2)$ is topologically a 3-sphere S^3, $\Pi_1(SU(2)) = \Pi_2(SU(2)) = 0$ whereas $\Pi_3(SU(2)) = \mathbb{Z}$. Then by induction we obtain

$$\begin{aligned} \Pi_1(SU(N)) &= \Pi_2(SU(N)) = 0 \quad \text{for all } N, \\ \Pi_3(SU(N)) &= \mathbb{Z}, \quad \text{for } N \geq 2. \end{aligned} \tag{E.5.25}$$

Appendix F

Reduced Formula for Evaluating $\mathrm{Tr}\left(\left[|M_{k+1}|^2|M_k|^{-2}\right]^2\right)$

In this appendix, we derive a formula for simplifying the calculation of the trace in $\mathcal{I}$ for the case $n = 2$. Using the formula,

$$\mathrm{Tr}\left(H^2\right) = \left(\mathrm{Tr}\, H\right)^2 - 2\, Det\, H, \tag{F.0.1}$$

which is true for any (2×2) matrix H we note that for $D_k = Det\,|M_k|^2 \neq 0$, we have

$$\mathrm{Tr}\left(\left[|M_{k+1}|^2|M_k|^{-2}\right]^2\right) = \left(\mathrm{Tr}\left[|M_{k+1}|^2|M_k|^{-2}\right]\right)^2 - 2\,\frac{D_{k+1}}{D_k}. \tag{F.0.2}$$

Thus, if $D_{k+1} = 0$, then from (F.0.2) we have

$$\mathrm{Tr}\left(\left[|M_{k+1}|^2|M_k|^{-2}\right]^2\right) = \left(\mathrm{Tr}\left[|M_{k+1}|^2|M_k|^{-2}\right]\right)^2. \tag{F.0.3}$$

Appendix G

Condition (4.96) from the $SU(N)$ Chiral Models

To simplify our search for finding a condition for exact solution of the $SU(N)$ Skyrme model equations (4.89), in this appendix, we look at the corresponding $SU(N)$ chiral model equations

$$\partial_r \left(r^2 L_r\right) + \frac{(1+|\xi|^2)^2}{2} \left(\partial_{\bar{\xi}} L_\xi + \partial_\xi L_{\bar{\xi}}\right) = 0, \tag{G.0.1}$$

i.e. (4.89) without the Skyrme terms. In terms of the rank-n projectors P_k, equations (G.0.1), after we have put in (4.37) and (4.90), become

$$\sum_{k=0}^{\lambda-1} \Bigg[\left(P_k - \frac{nI}{N}\right) \partial_r \left(r^2 \dot{g}_k\right) + (1+|\xi|^2)^2 \sin F_k$$

$$\times \left(M_{l+1}|M_l|^{-2}M_{l+1}^\dagger - M_l|M_l|^{-2}|M_{l+1}|^2|M_l|^{-2}M_l^\dagger\right) \Bigg] = 0, \tag{G.0.2}$$

where $F_k = g_k - g_{k+1}$, $g_\lambda = 0$.

Next we multiply equations (G.0.2) from the right by M_s. Using $P_k M_s = (M_k|M_k|^{-2}M_k^\dagger)M_s = M_s\delta_{sk}$ and noting that M_s are independent matrix fields, we see that the requirement of the vanishing of the corresponding coefficients leaves us with

$$\partial_r \left[r^2 \left(\dot{g}_s - \frac{n}{N} \sum_{k=0}^{\lambda-1} \dot{g}_k \right) \right] I_n$$

$$+ (1+|\xi|^2)^2 \left(|M_{s-1}|^{-2}|M_s|^2 \sin F_{s-1} - |M_s|^{-2}|M_{s+1}|^2 \sin F_s\right) = 0. \tag{G.0.3}$$

Finally, summing over s from 0 to l, and noting that $M_{-1} = 0$ (by definition), we find that (G.0.3) gives us

$$\partial_r \left[r^2 \left(\sum_{p=0}^{l} \sum_{q=p}^{\lambda-1} \dot{F}_q - \frac{n(l+1)}{N} \sum_{p=0}^{\lambda-1} (p+1) \dot{F}_p \right) \right] I_n$$

$$- (1 + |\xi|^2)^2 |M_l|^{-2} |M_{l+1}|^2 \sin F_l = 0. \tag{G.0.4}$$

Thus, in order to have a compatible and a consistent set of equations for the functions F_l in (G.0.4), the matrices M_l must satisfy

$$|M_l|^{-2} |M_{l+1}|^2 = \mathcal{K}_l (1 + |\xi|^2)^{-2} I_n, \tag{G.0.5}$$

i.e. the condition (4.96).

Appendix H

Derivation of Energy from Energy-Momentum Density Tensor

In this appendix we shall derive static energies of the field theories we are considering in this thesis. Here, instead of using the canonical method, we use the familiar method in General Relativity theory for deriving the corresponding energy-momentum density tensor $T^{\mu\nu}$ from which the corresponding energy density $\mathcal{E}$ is given by the component T^{00}.

Let $g_{\mu\nu}$ be the metric tensor of the curved spacetime $\mathcal{M}_0$, then the corresponding action is

$$S = \int d^4x\sqrt{-g}\mathcal{L}, \tag{H.0.1}$$

where $\mathcal{L}$ is the curved spacetime Lagrangian density and where $g = det(g_{\mu\nu}) < 0$.

By definition, the variation of the action (H.0.1) with respect to the variation of the metric tensor $g_{\mu\nu}$ *alone* is [67]

$$\delta_g S = -\frac{1}{2}\int d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}. \tag{H.0.2}$$

Using

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\rho\sigma}\delta g_{\rho\sigma}, \qquad \delta g^{\mu\nu} = -g^{\mu\rho}\left(\delta g_{\rho\sigma}\right)g^{\sigma\nu}, \tag{H.0.3}$$

then

$$\delta_g S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}g^{\rho\sigma}\left(\delta g_{\rho\sigma}\right)\mathcal{L} + \delta_g\mathcal{L}\Big], \tag{H.0.4}$$

where $\delta_g \mathcal{L}$ is the corresponding variation of $\mathcal{L}$.

By comparing (H.0.4) and (H.0.2), we can read-off the corresponding energy-momentum density tensor $T^{\mu\nu}$.

In the following two subsections we shall apply this formalism to the usual $SU(N)$ Skyrme models and pure massive $SU(N)$ Yang-Mills fields.

H.1 $SU(N)$ Skyrme Models

Here, the flat spacetime Lagrangian density is given by (4.2), so its curved spacetime version is

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{F^2}{16}g^{\mu\nu}L_\mu L_\nu + \frac{1}{32a^2}g^{\mu\alpha}g^{\nu\beta}[L_\mu, L_\nu][L_\alpha, L_\beta] + \frac{F^2}{16}M_\pi^2\left(U^{-1}+U-2I\right)\Big]. \tag{H.1.5}$$

Under the variation of the metric tensor alone, $\delta g_{\mu\nu}$, it transforms as

$$\delta_g \mathcal{L} = \frac{F^2}{16}g^{\mu\rho}(\delta g_{\rho\sigma})g^{\sigma\nu}\mathrm{Tr}\left(L_\mu L_\nu\right)$$

$$-\frac{1}{32a^2}\left(g^{\mu\rho}g^{\sigma\alpha}g^{\nu\beta} + g^{\mu\alpha}g^{\nu\rho}g^{\sigma\beta}\right)g_{\rho\sigma}\mathrm{Tr}[L_\mu, L_\nu][L_\alpha, L_\beta]. \tag{H.1.6}$$

Hence, the energy-momentum density tensor that we read-off from (H.0.4) is

$$T^{\rho\sigma} = -g^{\rho\sigma}\mathcal{L} - \frac{F^2}{8}g^{\mu\rho}g^{\sigma\nu}\mathrm{Tr}\left(L_\mu L_\nu\right)$$

$$+\frac{1}{16a^2}\left(g^{\mu\rho}g^{\sigma\alpha}g^{\nu\beta} + g^{\mu\alpha}g^{\nu\rho}g^{\sigma\beta}\right)\mathrm{Tr}[L_\mu, L_\nu][L_\alpha, L_\beta]. \tag{H.1.7}$$

We now let $g_{\mu\nu}$ to be Minkowskian metric. Then the energy of the $SU(N)$ Skyrme models is

$$E = \int d^3x T^{00} = E_{stat} + E_{rot}, \tag{H.1.8}$$

where

$$E_{stat} = -\int d^3x \mathrm{Tr}\Big[\frac{F^2}{16}L_a^2 + \frac{1}{32a^2}[L_a, L_b]^2 + \frac{F^2}{16}M_\pi^2\left(U^{-1}+U-2I\right)\Big], \tag{H.1.9}$$

which determines the mass of the configuration, and

$$E_{rot} = -\int d^3x \mathrm{Tr}\Big[\frac{F^2}{16}L_0^2 + \frac{1}{16a^2}[L_0, L_a]^2\Big], \tag{H.1.10}$$

for rotating finite mass configuration and its quantisation [9].

H.2 Pure Massive $SU(N)$ Yang-Mills Fields

Here, the curved spacetime Lagrangian density of the pure massive $SU(N)$ Yang-Mills fields as we read-off from the action (5.12) is

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} + M^2\left(g^{\mu\alpha}A_\alpha A_\mu\right)\right]. \tag{H.2.11}$$

Under the metric tensor fields variation $\delta g_{\mu\nu}$, it varies as

$$\delta_g\mathcal{L} = \mathrm{Tr}\left[F^{\rho\nu}F^{\sigma}{}_{\nu} - M^2A^\rho A^\sigma\right]\delta g_{\rho\sigma}. \tag{H.2.12}$$

Hence, the energy-momentum density tensor that we read-off from (H.0.4) is

$$T^{\mu\nu} = -\mathrm{Tr}\Big[2\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) - 2M^2\left(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}A^\mu A_\mu\right)\Big]. \tag{H.2.13}$$

Thus the static energy density of the pure massive $SU(N)$ Yang-Mills fields, for which $A_0 = 0$ and $F_{0a} = 0$, in flat spacetime, is

$$\mathcal{E} = T^{00} = \mathrm{Tr}\Big[\frac{1}{2}F_{ab}F_{ab} + M^2A_aA_a\Big]. \tag{H.2.14}$$

Appendix I

Numerical Methods

In this appendix we outline the numerical methods that we have used to solve the boundary value problems of the profile functions of the models in this thesis. These equations, which are of the type of ordinary differential equations (ODE), are highly nonlinear, thus require iterative methods to approach the corresponding exact solutions $F_k(r)$, $k = 0, 1, \dots, (\lambda - 1)$. In this thesis we apply the Newton-Raphson iterative method [68] which proceeds as follows:

(a) Choose an initial guess $F_k^{(0)}(r)$ and prescribe a tolerance value $TOL \simeq 0$.

(b) Set the iteration sequence:

$$F_k^{(i)}(r) = F_k^{(i-1)}(r) + f_k^{(i)}(r), \quad i = 1, 2, \dots, I, \tag{I.0.1}$$

with $|f_k^{(i)}|^2 \sim 0$.

(c) At each iteration step i, substitute (I.0.1) to the profile equations and solve the resulting system of linear ordinary differential equations:

$$a_{2k}\ddot{f}_k^{(i)} + a_{1k}\dot{f}_k^{(i)} + a_{0k}f_k^{(i)} + d_k = 0, \tag{I.0.2}$$

where the coeficients $a_{0k}, \dots, d_k$ are functions of $F_k^{(i-1)}$. Eq. (I.0.2) are subjected to the boundary conditions: $f_k^{(i)}(0) = 0$, $f_k^{(i)}(\infty) = 0$.

(d) Stop the iteration when

$$|F_k^{(I)} - F_k^{(I-1)}| \leq TOL. \tag{I.0.3}$$

(e) The I-th iterated functions $F_k^{(I)}(x)$ are the required (numerical) solutions.

In this thesis, we solve the linear ordinary differential equations (I.0.2) by using the *finite difference scheme* [68]. For the first derivative df/dr we use the central difference approximation:

$$\dot{f}\Big|_{r=r_j} \equiv \frac{df}{dr}\Big|_{r=r_j} \simeq \frac{f_{j+1} - f_{j-1}}{2h} + O(h^2), \tag{I.0.4}$$

whereas for the second derivative with an equally sensible order of accuracy

$$\ddot{f}\Big|_{r=r_j} \equiv \frac{d^2 f}{dr^2}\Big|_{r=r_j} \simeq \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2} + O(h^2). \tag{I.0.5}$$

where $f_j = f(r_j)$ and $h = r_{j+1} - r_j$.

Having the results of the computed profile functions $F_k^{(I)}$ available, we then proceed to compute the corresponding energy integral E using the simple *trapesoidal* rule [68].

In executing these numerical computations, we have used the computer C-language program to implement the above algorithm [68].