Aspects of the affine superalgebra sl(2/1) at fractional level

Johnstone, Gavin Balfour

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Gavin Balfour Johnstone

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Department of Mathematical Sciences
University of Durham

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Abstract

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In this thesis we study the affine superalgebra $\hat{sl}(2|1;\mathbb{C})$ at fractional levels of the form $k = 1/u - 1$, $u \in \mathbb{N}\setminus\{1\}$. It is for these levels that admissible representations exist, which transform into each other under modular transformations.

In the second chapter we review background material on conformal field theory, particularly the Wess-Zumino-Witten model and the connection with modular transformations. The superalgebra $sl(2|1;\mathbb{C})$ is introduced, as is its affine version.

The next chapter studies the modular transformation properties of $\hat{sl}(2|1;\mathbb{C})$ characters. We derive formulae for these transformations for all levels of the form $k = 1/u - 1$, $u \in \mathbb{N}\setminus\{1\}$. We also investigate some modular invariant combinations of characters and find two series of modular invariants, analogous to the A- and D-series of the classification of $\hat{sl}(2)$ modular invariants.

In chapter 4 we turn to the study of fusion rules. We concentrate on the case $k = -1/2$. By considering the decoupling of singular vectors, we are able to find consistent fusion rules for this particular level. These fusion rules correspond to a modular invariant found in chapter 3.

This study suggests that one may consistently define a conformal field theory based on $\hat{sl}(2|1;\mathbb{C})$ at fractional level.

Declaration

This thesis is the result of research carried out by the author between October 1997 and March 2001 in the Department of Mathematical Sciences at the University of Durham. No part of this thesis has been submitted for a degree at this or any other university.

Chapter 2 is a review of necessary background material. The work of chapters 3 and 4 is entirely that of the author unless otherwise acknowledged. The results of chapter 3 have been published as [1] and those of chapter 4 are available as the preprint [2].

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Chapter 1

Introduction

One of the most prevalent features of 20th century physics was the widespread application made of the concept of symmetry. While this has long been a guiding principle in many aspects of scientific endeavour, perhaps its most direct successes have come in the field of theoretical particle physics. The Standard Model of elementary particles, governed by the $SU(3) \times SU(2) \times U(1)$ symmetry group classification, is a triumph of agreement between theoretical prediction and experimental observation. Symmetry has also transformed our understanding of phase transitions and critical phenomena. Recent times have seen a move towards ever more theoretical concepts awaiting the next generation of experimental corroboration: the introduction of supersymmetry has deep appeal and offers an elegant solution to a number of unresolved problems within theoretical physics. However, the most intractable issue within the subject remains the problem of unifying electromagnetism and the nuclear forces of weak and strong interactions with gravity. The most promising candidate for such a unified framework is provided by the theory of superstrings. An essential tool in its description is that of two-dimensional conformal symmetry. The methods of conformal field theory underpin string theory, as well as the aforementioned study of critical phenomena. These and other areas of application maintain theoretical physicists' interest in the subject of conformal field theory. That much of the activity in this field is essentially of an abstract nature cannot be denied: (two-dimensional) conformal field theory is one of the few arenas in modern physics where the complete solution of models is obtainable. As such, intense study is devoted to the analysis of this paradigm. A wide variety of sophisticated mathematical structures are encompassed in conformal field theory, resulting in a rich and fruitful interplay on the boundary of mathematical and physical research.

This thesis is particularly concerned with expanding the boundaries of conformal field theory. While generally extremely well understood, there remain certain challenges to the researcher in this field. One of these is the status of conformal field theories with fractional level affine algebra symmetry. The simplest case to consider is fractional level $\hat{sl}(2)$ and even for this situation the picture is not yet completely clear. From an algebraic point of view, one may consistently define a Wess-Zumino-Witten (WZW) model with a fractional level spectrum-generating algebra. However, when formulated in terms of an action, such a model is not well-defined. A longstanding problem is how to reconcile these two points of view. It may be the case that one may not define a consistent conformal field theory using a fractional level affine algebra, although the evidence would seem to indicate otherwise. In any case, one may use such models as building blocks of a coset theory which is consistent.

One way in which this study should be generalised is to consider algebras of higher rank than simply $\hat{sl}(2)$; another to consider superalgebras, which is the path adopted here. The extension to $\hat{sl}(2|1;\mathbb{C})$ is significant in that this superalgebra involves many features of more complicated superalgebras, in contrast to the simplest affine superalgebra $\widehat{osp}(1|2)$. In particular, in $\hat{sl}(2|1;\mathbb{C})$ we see the appearance of zero length roots, bringing additional complexity to the problem. This superalgebra is also important in the study of a particular string theory, that of the $N=2$ non-critical superstring.

In this thesis we will attempt to unravel some aspects of a WZW model based on fractional level $\hat{sl}(2|1;\mathbb{C})$. We concentrate on two fundamental areas: modular transformations of characters and fusion rules. Combining characters into combinations invariant under modular transformations typically gives partition functions corresponding to some physical conformal field theory. In the situation of fractional level, it is not clear that this is necessarily the case. One should also examine fusion rules, which are related to modular transformations by the Verlinde formula. Again, in the situation of fractional level, it is not clear how this relation should be interpreted; hence the necessity of considering this as a separate problem. Although finding consistent fusion rules and modular invariants does not entirely answer the question of whether a conformal field theory may be well defined in this context, it does at least give a strong indication that this may be the case.

To begin with, we give a general overview of some features of conformal field theory, particularly WZW models. We will also give an introduction to superalgebras and in particular to $sl(2|1)$. The question of establishing the general modular transformations of $\hat{sl}(2|1;\mathbb{C})$ characters will then be addressed. We then consider fusion rules for a specific value of the level, $k = -1/2$. Finally, conclusions about the results of this work will be drawn and further possibilities for study arising from this research will be discussed.

Chapter 2

Conformal Field Theory and $\hat{sl}(2|1;\mathbb{C})$

This chapter examines background material that will be used throughout the thesis. The main tools of this work are those of conformal field theory and an introduction to this vast subject is provided. In particular, the Wess-Zumino-Witten model is considered in its algebraic formulation as a conformal field theory with affine algebra symmetry. We then introduce the main object of study—the affine superalgebra $\hat{sl}(2|1;\mathbb{C})$.

2.1 Conformal Transformations in Two Dimensions

Although conformal symmetry had been incorporated into quantum field theory in the early 1970s, it was the seminal work of Belavin, Polyakov and Zamolodchikov [3] in 1984 that ushered in a new era for the subject. One of the main attributes of their work was to realise that conformal symmetry, while certainly a powerful feature, took on another level of potency for the case of a two-dimensional euclidean quantum field theory[1]. Considering the general case of a d-dimensional spacetime, with metric $g_{\mu\nu}$, then a conformal transformation $x \to x'$ is one under which this metric tensor is left invariant up to a change of scale

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{2.1}$$

These transformations are such that the angle between two vectors is preserved, where this angle is given by $v \cdot w/|v||w| = g_{\alpha\beta} v^\alpha w^\beta / (g_{\gamma\delta} v^\gamma v^\delta g_{\rho\sigma} w^\rho w^\sigma)^{1/2}$.

The infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$ gives rise to a change in the metric

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.2}$$

In order for this to be a conformal transformation, the change in the metric is required to satisfy

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\rho \epsilon^\rho \eta_{\mu\nu}, \tag{2.3}$$

now in a flat spacetime of signature (p, q). This equation reveals the fact that the case of $d = 2$ may indeed be markedly different from the general situation. For the case $d > 2$, the finite transformations corresponding to solutions of the above restriction on ϵ are translations, Lorentz transformations, dilatations and special conformal transformations (a combination of translation and inversion):

$$\begin{aligned}
x &\to x' = x + a, \\
x &\to x' = \Lambda x \quad (\Lambda \in SO(p,q)), \\
x &\to x' = \lambda x, \\
x &\to x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2}.
\end{aligned} \tag{2.4}$$

For $d \geqslant 2$ spacetime dimensions, the conformal group is described by $\frac{1}{2}(d + 2)(d+1)$ parameters. In the case of $d = 2$, in a spacetime with euclidean signature,

[1] As well as [3] we draw heavily on the lectures by Ginsparg [4] and the comprehensive textbook by di Francesco, Mathieu and Sénéchal [5] in this and following sections. The works by Fuchs [6] have also been much consulted.

the equation (2.3) takes the form of the Cauchy-Riemann equations

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{2.5}$$

It is now clear why the case of two dimensions is somewhat distinct from higher dimensions. We may use the language of complex analysis, the natural variables to consider being the complex coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. Within this context, the (holomorphic) Cauchy-Riemann equations are $\partial_{\bar{z}} f(z, \bar{z}) = 0$, with solution any analytic mapping $z \to f(z)$: the set of local conformal transformations is thus coincident with the infinite-dimensional set of analytic coordinate transformations. The number of parameters (six) obtained from the above formula is indeed the correct number specifying the global conformal group. However, in two dimensions this is distinct from the set of local symmetries, which are not well-defined at all points on the Riemann sphere. It is this distinction which furnishes two-dimensional conformal field theory with an infinite-dimensional symmetry algebra, giving rise to a vast number of constraints that permits, at least in principle, the complete solution of the theory.

2.2 The Virasoro Algebra

The analytic coordinate transformations $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ may be represented, on an infinitesimal level, by the transformations $z \to z' = z + \epsilon(z)$ and similarly for $\bar{z}$, where it is assumed that the Laurent expansion $\epsilon(z) = -\sum a_n z^{n+1}$ holds. Acting on a function $\phi(z, \bar{z})$ these mappings yield

$$\begin{aligned} \phi(z', \bar{z}') &= \phi(z, \bar{z}) + \epsilon(z)\partial\phi(z, \bar{z}) + \bar{\epsilon}(\bar{z})\bar{\partial}\phi(z, \bar{z}) \\ &= \phi(z, \bar{z}) + \sum_n \{a_n l_n \phi(z, \bar{z}) + \bar{a}_n \bar{l}_n \phi(z, \bar{z})\} \end{aligned} \tag{2.6}$$

where the generators

$$l_n = -z^{n+1}\partial, \qquad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \tag{2.7}$$

have been introduced ($\partial \equiv \partial_z, \bar{\partial} \equiv \partial_{\bar{z}}$). These generators satisfy (two copies of) the Witt algebra

$$
\begin{aligned}
[l_m, l_n] &= (m-n)l_{m+n}, \\
[\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n}, \\
[l_m, \bar{l}_n] &= 0.
\end{aligned}
\tag{2.8}
$$

The appearance of these two isomorphic, independent Lie algebras invites us to consider the coordinates z and $\bar{z}$ as independent: we shall generally do so and often will only consider the dependence on the holomorphic coordinate z, with properties for $\bar{z}$ assumed to hold in similar fashion. Imposing the condition $\bar{z} = z^*$ recovers the original physical coordinates.

Consideration of the limiting behaviour of $\epsilon(z)$ as $z \to 0$ and $z \to \infty$ tells us that it is only those transformations involving the generators l_{-1}, l_0 and l_1 which are defined on the whole Riemann sphere. Similarly, only those anti-holomorphic transformations involving $\bar{l}_{-1}$, $\bar{l}_0$ and $\bar{l}_1$ are globally well-defined. These six elements are the generators of the global conformal group mentioned earlier, giving rise to the finite transformations

$$
z \to \frac{az+b}{cz+d}, \qquad \bar{z} \to \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}
\tag{2.9}
$$

where $ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1$. This forms (twice) the group of projective conformal transformations $SL(2, \mathbb{C})/\mathbb{Z}_2$.

The discussion so far has been essentially classical. Consider now a two-dimensional euclidean spacetime with space coordinate σ and time coordinate τ, such that for any field $X(\sigma, \tau)$ the identification $X(\sigma + 2\pi, \tau) = X(\sigma, \tau)$ is made. This is a typical scenario in string theory, where such a spacetime may be taken to be the worldsheet of a closed string. The cylinder thus defined may be conformally mapped to the complex plane via $z = e^{\tau + i\sigma}$ and $\bar{z} = e^{\tau - i\sigma}$, z and $\bar{z}$ taken to be independent. The infinite past $\tau = -\infty$ is mapped to the origin of the z-plane and the infinite future $\tau = \infty$ is mapped to the point at infinity on the Riemann sphere. Curves of equal time are mapped to concentric circles (with centre $z = 0$) on the complex plane. This is of course similar to the picture described previously, but now we have in mind that we are dealing with a quantum field theory in terms of continuous operator-valued fields obeying canonical commutation relations. The above choice of space and time directions (somewhat arbitrary in the euclidean situation) leads to what is known as the radial quantisation of two-dimensional conformal field theories.

One object which plays a central role in a two-dimensional conformal field theory is the energy-momentum tensor $T_{\mu\nu}$. This may be obtained by varying the action of the theory with respect to the metric, $T_{\mu\nu} \propto \delta S/\delta g^{\mu\nu}$. In a two-dimensional conformally invariant theory, the energy-momentum tensor may be taken to be symmetric, conserved ($\partial^{\mu} T_{\mu\nu} = 0$) and traceless. In terms of the coordinates z and $\bar{z}$, these features mean that the energy-momentum tensor has two independent components, $T_{zz}(z) = T(z)$ and $T_{\bar{z}\bar{z}}(\bar{z}) = \bar{T}(\bar{z})$. They may be expanded in Laurent series

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}. \tag{2.10}$$

Through the Noether prescription, it may be shown that the field $T(z)$ generates conformal transformations of local fields $A(z, \bar{z})$. More precisely, under the infinitesimal transformation $z \to z + \epsilon(z)$ the variation of the field $A(z, \bar{z})$ is given by

$$\delta_{\epsilon} A(z, \bar{z}) = \frac{1}{2\pi i} \oint_{C_z} dw\, \epsilon(w) T(w) A(z), \tag{2.11}$$

with a similar relation holding for $\bar{T}(\bar{z})$. In the above, the contour of integration is understood as being around z in an anticlockwise direction. A little more than this should be specified: in the scheme of radial quantisation, products of operators $A(w)B(z)$ are only defined for $|w| > |z|$. To this end, radial ordering is understood:

$$R(A(w)B(z)) = \begin{cases} A(w)B(z) & \text{if} \quad |w| > |z|, \\ B(z)A(w) & \text{if} \quad |w| < |z| \end{cases} \tag{2.12}$$

with the further requirement that if the operators are fermionic, then a minus sign must be introduced on the change of order. The contour integral should be split into two pieces accordingly. This amounts to time ordering within correlation functions and we will generally assume that when products of operators are written down, they are implicitly radially ordered.

The operator product expansion tells us what singularities appear when two local fields become coincident. In conformal field theories, it is assumed that the set of local fields $A_i(z,\bar{z})$ is complete, *i.e.*

$$A_i(z,\bar{z})A_j(w,\bar{w}) = \sum_k C_{ijk}(z-w,\bar{z}-\bar{w})A_k(w,\bar{w}), \tag{2.13}$$

where the C_{ijk} are numerical coefficients. In the case of the product of the energy-momentum tensor with itself, the operator product expansion is given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots, \tag{2.14}$$

where the unwritten terms are non-singular as $z \to w$. Together with the inversion of the Laurent expansion (2.10)

$$L_n = \frac{1}{2\pi i}\oint_{C_0} dz\, T(z) z^{n+1} \tag{2.15}$$

this operator product expansion implies the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \qquad [L_n, c] = 0. \tag{2.16}$$

There is also a corresponding algebra in terms of the modes $\bar{L}_n$. It is this algebra which characterises a two-dimensional conformal field theory (at least in part). It is in fact possible to define the Virasoro algebra as the central extension of the Witt algebra previously discussed, the central term being the conformal anomaly telling us how the classical symmetry of the Witt algebra is modified at the quantum level.

The subalgebra $\{L_{-1}, L_0, L_1\}$ of the Virasoro algebra is isomorphic to the subalgebra $\{l_{-1}, l_0, l_1\}$ of the Witt algebra, since for commutators of these elements the central piece vanishes. Hence, this subalgebra generates the same $SL(2,\mathbb{C})/\mathbb{Z}_2$ global conformal group as before, which remains an exact symmetry.

The Hilbert space of states of a conformal field theory is characterised by the local conformal algebra. This need not be simply the Virasoro algebra, but for the moment we shall discuss states assuming that it is. The vacuum of the theory $|0\rangle$ must be invariant under the global $SL(2,\mathbb{C})$ transformations, which is to say that it must be annihilated by the elements L_{-1}, L_0 and L_1 (and their anti-holomorphic counterparts). In fact, given the mode expansion of the energy-momentum tensor (2.10), this is a requirement of having $T(z)|0\rangle$ well-defined as $z \to 0$, which gives rise to the condition

$$L_n|0\rangle = 0, \qquad n \geqslant -1. \tag{2.17}$$

A similar requirement holds for the corresponding anti-holomorphic quantities. Further, the requirement that $T(z)$ be self-adjoint under $z \to 1/\bar{z}$ means that $L_n^\dagger = L_{-n}$. The relationship between states and fields of the theory is given by

$$|A\rangle = \lim_{z,\bar{z}\to 0} A(z,\bar{z})|0\rangle. \tag{2.18}$$

In order to define the state $\langle A|$, a similar construction for $z \to \infty$ is required. Using the map $z \to 1/w$, this suggests that we take

$$\langle A| = \lim_{w,\bar{w}\to 0} \langle 0|\tilde{A}(w,\bar{w}) \tag{2.19}$$

where the relation between A and $\tilde{A}$ is given by

$$\tilde{A}(w,\bar{w}) = A\left(\frac{1}{z},\frac{1}{\bar{z}}\right) z^{-2h}\bar{z}^{-2\bar{h}}. \tag{2.20}$$

Hence

$$\langle A| = \lim_{z,\bar{z}\to\infty} \langle 0|A(z,\bar{z}) z^{2h}\bar{z}^{2\bar{h}}. \tag{2.21}$$

The conformal transformation relating A and $\tilde{A}$ is a specific example of a general result for the transformation of so-called primary fields, defined by

$$\phi_{h,\bar{h}}(z,\bar{z}) \to (\partial f)^h (\bar{\partial}\bar{f})^{\bar{h}} \phi_{h,\bar{h}}(f(z),\bar{f}(\bar{z})) \tag{2.22}$$

under the analytic coordinate transformations $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$.

The quantities h and $\bar{h}$ may be shown to be the eigenvalues of L_0 and $\bar{L}_0$:

$$\begin{aligned} L_n|h,\bar{h}\rangle &= \lim_{z,\bar{z}\to 0}[L_n, \phi_{h,\bar{h}}(z,\bar{z})]|0\rangle \\ &= \lim_{z,\bar{z}\to 0}\{h(n+1)z^n\phi_{h,\bar{h}}(z,\bar{z}) + z^{n+1}\partial\phi_{h,\bar{h}}(z,\bar{z})\}|0\rangle \end{aligned} \tag{2.23}$$

with the result that

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle, \qquad L_n|h,\bar{h}\rangle = 0, \quad n > 0 \tag{2.24}$$

for the highest weight state $|h,\bar{h}\rangle$ created by a primary field $\phi_{h,\bar{h}}(z,\bar{z})$. Generally, we will only consider the holomorphic part $|h\rangle$, created by $\phi_h(z)$. Now the Virasoro algebra tells us that

$$[L_0, L_m] = -mL_m. \tag{2.25}$$

Together with the condition $L_n|h\rangle = 0$ for $n > 0$, this shows that the operators L_{-m}, $m > 0$ act as raising operators on the state $|h\rangle$. The state $L_{-n}|h\rangle$ is an eigenvector of L_0, with eigenvalue $h + n$. Applying these raising operators in all possible combinations to $|h\rangle$ yields the descendant states corresponding to the descendant fields of the primary field $\phi_h(z)$. The descendant fields $(L_{-n}\phi)(z)$ are those appearing in the operator product expansion of $T(z)$ with a primary field:

$$\begin{aligned} T(z)\phi_h(w) = &\frac{1}{(z-w)^2}(L_0\phi)(w) + \frac{1}{z-w}(L_{-1}\phi)(w) \\ &+ (L_{-2}\phi)(w) + (z-w)(L_{-3}\phi)(w) + \dots . \end{aligned} \tag{2.26}$$

The set of primary field and its descendants comprise a conformal family $[\phi_n]$ $(\phi_n \equiv \phi_{h_n})$ and form a representation of the conformal (here Virasoro) algebra, transforming amongst themselves under a conformal transformation. The set of states $|h\rangle$ and descendants form a (reducible) Verma module. We will discuss this further in a more general setting in the next two sections. The point of organising the fields of a conformal field theory into conformal families is that properties of descendant fields follow from those of primary fields. In particular, correlation functions of descendants are determined by those of the primary fields: knowing these quantities one may justifiably claim to have completely specified the theory in question.

We conclude this section by discussing how the Virasoro generators may be used to calculate correlation functions of primary fields. It is a postulate of conformal field theory that these be invariant under global conformal transformations. Together with the assumption of an $SL(2,\mathbb{C})$ invariant vacuum state, we may write

$$\begin{aligned}\langle 0|\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)|0\rangle &= \prod_j (\partial w)^{h_j}(\bar{\partial}\bar{w})^{\bar{h}_j}\langle 0|\phi_1(w_1,\bar{w}_1)\cdots\phi_n(w_n,\bar{w}_n)|0\rangle \\ &= \langle 0|U^{-1}\phi_1(z_1,\bar{z}_1)U\cdots U^{-1}\phi_n(z_n,\bar{z}_n)U|0\rangle \qquad (2.27)\end{aligned}$$

where $w = f(z)$ is of the form (2.9) and $U \in SL(2,\mathbb{C})$. Since such transformations are generated by L_{-1}, L_0 and L_1, we may rewrite this in the infinitesimal version

$$\langle 0|[L_k,\phi_1(z_1)]\cdots\phi_n(z_n)|0\rangle + \cdots + \langle 0|\phi_1(z_1)\cdots[L_k,\phi_n(z_n)]|0\rangle = 0 \qquad (2.28)$$

with one such equation for each of $k = 0, \pm 1$. Then using the expression for the commutator of a primary field and Virasoro generator

$$[L_n, \phi_m(z)] = h_m(n+1)z^n\phi_m(z) + z^{n+1}\partial\phi_m(z), \qquad (2.29)$$

these may be written as

$$\begin{aligned}&\sum_{i=1}^{n}\partial_i\langle 0|\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = 0,\\ &\sum_{i=1}^{n}(z_i\partial_i + h_i)\langle 0|\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = 0,\\ &\sum_{i=1}^{n}(z_i^2\partial_i + 2z_ih_i)\langle 0|\phi_1(z_1)\cdots\phi_n(z_n)|0\rangle = 0. \qquad (2.30)\end{aligned}$$

The above expressions embody the invariance of correlation functions under translations, dilatations and special conformal transformations, respectively, and may be solved for n-point functions. From these equations it is possible to determine 2- and 3-point functions exactly. For the 4-point function additional input is required, but knowledge of the 4-point functions of primary fields is enough to give the coefficients C_{ijk} of (2.13), enabling the complete solution of the theory.

2.3 The Wess-Zumino-Witten Model

As already mentioned, conformally invariant field theories may have symmetries associated to them beyond that specified by conformal invariance, encapsulated in the Virasoro algebra. One particular class of such theories are Wess-Zumino-Witten (WZW) models [7, 8], reviewed in (for example) [5, 6, 9]. The WZW model arose from the search for a modification of the nonlinear sigma model that would be a conformally invariant theory. Witten added the Wess-Zumino term to the sigma model action, giving the action

$$\begin{aligned} S_{WZW} = \frac{k}{16\pi} \int d\sigma d\tau \, \mathrm{Tr}((\partial_m g)(\partial^m g^{-1})) \\ + \frac{k}{24\pi} \int d^3 y \epsilon^{mnl} \, \mathrm{Tr}((g^{-1}\partial_m g)(g^{-1}\partial_n g)(g^{-1}\partial_l g)). \end{aligned} \quad (2.31)$$

The field g takes its values in a representation of the Lie group G associated with the sigma model. The Wess-Zumino term is defined on a three-dimensional space which has as its boundary the two-dimensional worldsheet of the sigma model action. In order for the WZW model to be well-defined (that is, for the path integral to be single-valued) the parameter k must be quantised and in fact should be an integer.

The point about this action is that, as well as exhibiting conformal invariance, it is also invariant under

$$g(\tau, \sigma) \to \Omega(\tau + \sigma) g(\tau, \sigma) \bar{\Omega}^{-1}(\tau - \sigma), \quad (2.32)$$

where Ω and $\bar{\Omega}$ take their values in the Lie group G in the same way as g. As previously discussed, we may take $g(\tau, \sigma + 2\pi) = g(\tau, \sigma)$: the WZW action then describes a closed string moving on a Lie group manifold. Mapping this closed string worldsheet (the infinite cylinder) to the complex plane as before, via $z = e^{\tau + i\sigma}$ and $\bar{z} = e^{\tau - i\sigma}$, this symmetry becomes

$$g(z, \bar{z}) \to \Omega(z) g(z, \bar{z}) \bar{\Omega}^{-1}(\bar{z}). \quad (2.33)$$

The theory is therefore seen to have a local $G(z) \times G(\bar{z})$ invariance.

In terms of the variables z and $\bar{z}$, the equations of motion of the WZW model are

$$\partial_z(g^{-1}\partial_{\bar{z}}g) = 0, \qquad \partial_{\bar{z}}((\partial_z g)g^{-1}) = 0. \tag{2.34}$$

Defining the currents $J = -k(\partial_z g)g^{-1}$ and $\bar{J} = kg^{-1}\partial_{\bar{z}}g$ we see that

$$\partial_{\bar{z}}J = 0, \qquad \partial_z\bar{J} = 0 \tag{2.35}$$

and hence that $J = J(z)$ and $\bar{J} = \bar{J}(\bar{z})$. The currents J and $\bar{J}$ may be expanded in the Lie algebra elements T^a, which form a basis for the algebra $\mathfrak{g}$ associated to the group G:

$$J(z) = \sum_{a=1}^{\dim\mathfrak{g}} J^a(z)T^a, \qquad \bar{J}(\bar{z}) = \sum_{a=1}^{\dim\mathfrak{g}} \bar{J}^a(\bar{z})T^a. \tag{2.36}$$

These J^a and $\bar{J}^a$, being also functions of z and $\bar{z}$ alone, respectively, can therefore be expanded in the Laurent series

$$J^a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n, \qquad \bar{J}^a(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-1}\bar{J}^a_n. \tag{2.37}$$

The modes of these expansions may be shown to satisfy two commuting (untwisted) affine Lie algebras

$$\begin{aligned}
[J^a_m, J^b_n] &= \sum_{c=1}^{\dim\mathfrak{g}} if^{ab}{}_c J^c_{m+n} + \tilde{k}m\delta^{ab}\delta_{m+n,0}, \\
[\bar{J}^a_m, \bar{J}^b_n] &= \sum_{c=1}^{\dim\mathfrak{g}} if^{ab}{}_c \bar{J}^c_{m+n} + \tilde{k}m\delta^{ab}\delta_{m+n,0}, \\
[J^a_m, \bar{J}^b_n] &= 0,
\end{aligned} \tag{2.38}$$

where the $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$, $[T^a, T^b] = \sum_c if^{ab}{}_c T^c$.

At this point we make a short digression on the subject of affine Lie algebras. This subject is covered, for example, in [10, 5, 6]. For every Lie algebra $\mathfrak{g}$, one may consider the set of mappings $\tilde{\mathfrak{g}}$ from the circle into $\mathfrak{g}$. The unit circle has a description in terms of the coordinate $z = e^{2\pi it}$, which gives rise to a basis of $\tilde{\mathfrak{g}}$ as $\{J^a_n | a = 1, 2, \ldots, \dim\mathfrak{g}; n \in \mathbb{Z}\}$, where $J^a_n \equiv J^a \otimes z^n$. The J^a are elements of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ has the multiplication rule

$$[J^a_m, J^b_n] = \sum_{c=1}^{\dim\mathfrak{g}} if^{ab}{}_c J^c_{m+n}. \tag{2.39}$$

This algebra may be supplemented by a central extension in a unique non-trivial way to give

$$[J^a_m, J^b_n] = \sum_{c=1}^{\dim \mathfrak{g}} i f^{ab}{}_c J^c_{m+n} + \tilde{k} m \delta^{ab} \delta_{m+n,0},$$
$$[J^a_m, \tilde{k}] = 0. \tag{2.40}$$

To completely specify the affine Lie algebra $\hat{\mathfrak{g}}$, we must also add the derivation d, such that

$$[d, J^a_m] = m J^a_m, \qquad [d, \tilde{k}] = 0. \tag{2.41}$$

This additional generator is required since upon choosing the Cartan-Weyl basis for $\mathfrak{g}$ we see that the algebra without d has Cartan subalgebra $\{H^1_0, \ldots, H^r_0, \tilde{k}\}$ (r the rank of $\mathfrak{g}$). Then choosing the remaining generators E^α_n in the usual way to be combinations of the J^a_n such that

$$[H^i_0, E^\alpha_n] = \alpha^i E^\alpha_n, \tag{2.42}$$

we see that the root vector $(\alpha^1, \ldots, \alpha^r, 0)$ (0 from the relation $[\tilde{k}, E^\alpha_n] = 0$) is the same for each E^α_n. The addition of d (in the Cartan subalgebra) removes this infinite degeneracy in the roots as they are then given by $(\alpha^1, \ldots, \alpha^r, 0, n)$.

The vector space structure of $\hat{\mathfrak{g}}$ is thus

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}\tilde{k} \oplus \mathbb{C}d. \tag{2.43}$$

The affine Lie algebra $\hat{\mathfrak{g}}$ is infinite-dimensional, being made up of the generators $\{J^a_n, \tilde{k}, d | n \in \mathbb{Z}\}$. It contains as a subalgebra the finite Lie algebra $\mathfrak{g}$ whose generators are the zero modes of $\hat{\mathfrak{g}}$, $\{J^a_0\}$.

With these generalities established, we move on to consider states of the WZW model as defined by the affine algebra structure. In the same way as the Virasoro algebra, we can consider a vacuum state $|0\rangle$ which we can act on with $J^a(z)$. Then requiring regularity of this expression at $z = 0$ gives rise to the condition

$$J^a_n |0\rangle = 0, \qquad n \geqslant 0. \tag{2.44}$$

As Virasoro primary fields may be defined by their operator product expansion with the energy-momentum tensor (2.26), so the affine algebra primary fields are defined (following the conventions of [5]) by

$$\begin{aligned} J^a(z)\phi_{\lambda,\mu}(w,\bar{w}) &= \frac{-T^a_\lambda \phi_{\lambda,\mu}(w,\bar{w})}{z-w} + \dots, \\ \bar{J}^a(\bar{z})\phi_{\lambda,\mu}(w,\bar{w}) &= \frac{\phi_{\lambda,\mu}(w,\bar{w})T^a_\mu}{z-w} + \dots . \end{aligned} \tag{2.45}$$

In the above, T^a_λ denotes the matrix T^a in the representation which has λ as its highest weight. We remind the reader that the equation (2.42) describing root vectors may be generalised from the adjoint representation to an arbitrary representation with basis $\{|\lambda\rangle\}$ such that

$$H^i_0|\lambda\rangle = \lambda(H^i_0)|\lambda\rangle \quad (i=1,\dots,r), \quad \tilde{k}|\lambda\rangle = \lambda(\tilde{k})|\lambda\rangle, \quad d|\lambda\rangle = \lambda(d)|\lambda\rangle, \tag{2.46}$$

the latter two relations being specific to the affine case. For a particular highest weight representation, there will be a unique highest weight state $|\lambda\rangle$ which is annihilated by all the raising operators of the algebra,

$$E^\alpha_0|\lambda\rangle = E^{\pm\alpha}_n|\lambda\rangle = H^i_n|\lambda\rangle = 0, \quad n>0 \tag{2.47}$$

for α a so-called positive root. Then

$$H^i_0|\lambda\rangle = \lambda^i|\lambda\rangle \quad (i=1,\dots,r), \quad \tilde{k}|\lambda\rangle = \tilde{k}|\lambda\rangle, \quad d|\lambda\rangle = -h_\lambda|\lambda\rangle. \tag{2.48}$$

States in the representation space are generated by acting with lowering operators on $|\lambda\rangle$. As the generator $\tilde{k}$ commutes with all other generators, these states will all have the same eigenvalue and we use the notation $\tilde{k}$ for this also. The highest weight vector is given by[2] $\lambda = (\lambda^1,\dots,\lambda^r,\tilde{k},-h_\lambda)$, with the eigenvalue of d suggestively chosen.

Concentrating on the holomorphic sector, we associate the primary field $\phi_\lambda(z)$ to the highest weight state $|\lambda\rangle$ via

$$\lim_{z\to 0}\phi_\lambda(z)|0\rangle = \phi_\lambda(0)|0\rangle = |\lambda\rangle. \tag{2.49}$$

[2]To make a clear distinction between affine weights and finite weights, one might label the former $\hat{\lambda}$. As we will not make use of finite weights, we simply denote affine weights by λ.

Then the operator product expansion (2.45) gives the following conditions defining a WZW primary field in terms of modes and states:

$$\begin{aligned} J_0^a|\lambda\rangle &= -T_\lambda^a|\lambda\rangle \\ J_n^a|\lambda\rangle &= 0, \quad n>0, \end{aligned} \tag{2.50}$$

which is to say that $|\lambda\rangle$ is annihilated by all the raising operators (now denoted J_n^a with $n > 0$) and provides a representation of the finite algebra.

We have mentioned that the WZW model has conformal symmetry, so expect to see the Virasoro algebra making an appearance. It turns out that one may associate a Virasoro algebra to every affine Lie algebra using the Sugawara construction. The natural definition of the energy-momentum tensor $T(z)$ in this setting is

$$T(z) = \gamma \sum_{a=1}^{\dim \mathfrak{g}} :J^a(z)J^a(z): \ = \gamma \lim_{z\to w} \left[\sum_{a=1}^{\dim \mathfrak{g}} J^a(z)J^a(w) - \frac{\tilde{k}\dim \mathfrak{g}}{(z-w)^2} \right]. \tag{2.51}$$

The colons :: denote the normal-ordered product, the term of order $(z-w)^0$ in the operator product expansion of $J^a(z)J^a(w)$;

$$J^a(z)J^b(w) = \frac{\tilde{k}\delta_{ab}}{(z-w)^2} + \sum_c if_{abc}\frac{J^c(w)}{(z-w)} + :J^a(z)J^a(z): + \mathcal{O}(z-w). \tag{2.52}$$

The antisymmetry of f_{abc} ensures the disappearance of the $(z-w)^{-1}$ term in (2.51).

The value of the constant γ may be fixed by, for example, requiring $T(z)$ to have an operator product expansion with itself of the form (2.14). Then

$$\gamma = \frac{1}{2\tilde{k} + h^\vee \psi^2} \tag{2.53}$$

where ψ is the highest root of $\mathfrak{g}$ (typically, but not necessarily, normalised to 2) and $h^\vee$ is its dual Coxeter number. The central charge of the Virasoro algebra is given by

$$c = \frac{\tilde{k}\dim \mathfrak{g}}{\tilde{k} + h^\vee \psi^2/2}. \tag{2.54}$$

In terms of modes, the expression (2.51) becomes

$$L_n = \frac{1}{2\tilde{k} + h^\vee \psi^2} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m=-\infty}^{\infty} :J_m^a J_{n-m}^a:, \tag{2.55}$$

the normal ordering now meaning that when $n = 0$ the term with larger subindex must be placed to the right, *i.e.*

$$\sum_m :J^a_m J^a_{n-m}: \; = \sum_{m\leqslant -1} J^a_m J^a_{n-m} + \sum_{m\geqslant 0} J^a_{n-m} J^a_m . \tag{2.56}$$

When $n \neq 0$, the modes simply commute. The link between the Virasoro (2.16) and affine Lie algebras (2.38) is completed with the relation

$$[L_m, J^a_n] = -nJ_{m+n}, \tag{2.57}$$

from which we see that L_0 is to be identified with the generator $-d$. Examining the conditions for a WZW primary field (2.50), it is clear that a WZW highest weight state is also a Virasoro highest weight state:

$$L_n|\lambda\rangle = \frac{1}{2\tilde{k} + h^\vee \psi^2} \sum_{a=1}^{\dim \mathfrak{g}} \sum_{m=-\infty}^{\infty} :J^a_m J^a_{n-m}: \, |\lambda\rangle = 0, \quad n > 0, \tag{2.58}$$

since each term involves $J^a_m|\lambda\rangle = 0$ with $m > 0$. As the generators of the Virasoro algebra are given in terms of the affine algebra generators, it suffices to consider states of the theory as

$$J_{-n_1} J_{-n_2} \cdots J_{-n_i}|\lambda\rangle. \tag{2.59}$$

With the n_i positive integers, the J_{-n_i} then are raising[3] operators and the states correspond to descendant fields.

Lastly, we define the level of the affine Lie algebra $\hat{\mathfrak{g}}$. This is simply given by $k = 2\tilde{k}/\psi^2$: in the situation where $\psi^2 = 2$ we have $k = \tilde{k}$. It is the same parameter as appears in the WZW action (2.31). We mentioned that this should be an integer for a well-defined theory and in this situation particular highest weight representations, the integrable highest weight representations, appear. However, the WZW model may also be considered in a totally algebraic context, without reference to an action, as has been essentially done above. In that case we may

[3]There is the usual confusion between the terms "raising" and "lowering" operators: while J^a_{-n} and L_{-n} raise the eigenvalue of L_0 by $+n$, one can really only go down from a highest weight state, hence the mathematical description of $|\lambda\rangle$ annihilated by raising operators.

consider non-integer values for the level k. Specific choices of values of the level organise the representations of the affine Lie algebra, and hence the states of the theory, in a remarkable way: we will discuss this in the next section.

2.4 Characters, Modular Transformations and Fusion Rules

We have seen that a conformal field theory splits naturally into two chiral halves, with the chiral algebra of one half commuting with that of the other. In statistical mechanics, we imagine a physical system as having some continuous parameters associated with it. At some specific values of these parameters, the system reaches a critical point at which the system becomes conformally invariant: it is at this point that the decoupling between chiral halves occurs. One might imagine that the physical spectrum of the theory should deform continuously in moving away from this point, leading to restrictions on how the two chiral sectors can be joined together. It turns out that considering the theory still at the conformally invariant point but defined on higher genus Riemann surfaces[4], principally the torus, gives these additional constraints. Other motivations are that the torus is equivalent to the plane with opposite sides identified, relevant in many physical applications; and the perturbative treatment of string theory relies on summing over higher genus worldsheets in the calculation of scattering amplitudes.

The complex plane may be transformed to the torus by considering two linearly independent vectors, ω_1 and ω_2 which define a lattice: on identifying points that differ by integer combinations of these vectors, we obtain a torus. There will be several choices of vectors that yield the same lattice and hence the same torus; our conformal field theory should be independent of such choices. For two new vectors

[4]We have thus far defined a conformal field theory on the complex plane plus the point at infinity, *i.e.* the Riemann sphere of genus zero.

(periods) ω_1' and ω_2' describing the same lattice we must have

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{2.60}$$

where a, b, c and d are all integers. Since the area of the basic cell of the lattice should remain the same, and since ω_1 and ω_2 can also be written as an integer combination of the primed periods, the determinant of the transformation matrix must be equal to 1. In terms of the modular parameter τ, defined as the ratio of the two periods $\tau = \omega_2/\omega_1$, this transformation becomes

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1. \tag{2.61}$$

The transformation matrices form the group $SL(2, \mathbb{Z})$: since changing the sign of the parameters simultaneously does not affect the transformation of τ, the relevant symmetry group here is $SL(2, \mathbb{Z})/\mathbb{Z}_2$. The whole group is generated by two transformations, the modular $\mathcal{S}$ and modular $\mathcal{T}$ transformations:

$$\mathcal{S} : \tau \to -\frac{1}{\tau}, \qquad \mathcal{T} : \tau \to \tau + 1 \tag{2.62}$$

corresponding to the matrices

$$\mathsf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.63}$$

Before making contact with conformal field theory, we must consider those quantities of particular relevance in terms of modular transformations: the characters of the affine Lie algebra $\hat{\mathfrak{g}}$. These are defined as

$$\chi_\lambda^{\hat{\mathfrak{g}}_k} = q^{-c/24} \mathrm{Tr}_{\lambda,k} q^{L_0} e^{2i\pi \sum_i \rho_i H_0^i}. \tag{2.64}$$

In this expression, we sum over elements H_0^i of the Cartan subalgebra of the finite Lie algebra $\mathfrak{g}$. The parameter q is defined as $q = e^{2i\pi\tau}$, where τ will turn out to be the modular parameter. If we are only dealing with Virasoro symmetry, the expression remains valid with this sum set to zero. In either case, the parameter

c is the central charge of the (associated) Virasoro algebra, given by (2.54) in the WZW situation. The character associated with a particular representation space generated by the action of J^a_{-m} on $|\lambda\rangle$ is a generating functional for the number of states at a given grade (the number of J^a_{-m} that have acted on $|\lambda\rangle$). Specifically, evaluating (2.64) at $\rho_i = 0$ for all i gives the Virasoro specialised character

$$\chi_\lambda^{\hat{\mathfrak{g}}_k}(\tau, \{0\}) = q^{-c/24}\mathrm{Tr}_{\lambda,k} q^{L_0} = q^{h_\lambda - c/24} \sum_{n=0}^{\infty} d(n) q^n, \tag{2.65}$$

where $d(n)$ is the number of states at grade n. The other parameters in (2.64) allow us to keep track of the other quantum numbers of the states.

It may happen that in a given module, one produces a state which is also a highest weight state through the action of the J^a_{-m}, that is, the state $J^a_{-n_1} J^a_{-n_2} \cdots J^a_{-n_k} |\lambda\rangle$ is also annihilated by all the J^a_n, $n > 0$. Such a state is called a singular vector and has zero inner product with all other states of the module, as do its descendants. It generates its own submodule which forms a representation of the algebra, so the original module is said to be reducible. An irreducible module may be constructed by factoring out all such submodules generated by singular vectors, which amounts to identifying states that differ by states of zero norm. Information about which states of a module are singular vectors is encoded in the Kac-Khazdan determinant formula, knowledge of which then permits the construction of irreducible modules and the characters corresponding to them. It is these irreducible representations which are of interest here.

In the WZW model, the quantisation of the level k of the affine algebra $\hat{\mathfrak{g}}$ as an integer arose out of the need for a well-defined action. In purely algebraic terms, we have mentioned that this corresponds to the integrable highest weight representations. These are irreducible representations which are also unitary, by which we mean that the generators J^a_n satisfy the condition $(J^a_{-n})^\dagger = J^a_n$. Requiring representations to be integrable results in k being an integer. It so happens that there is a finite number of such highest weights for some specific value of k, which correspondingly means that there is a finite number of primary fields in the model. What is particularly remarkable about this situation is that the characters corresponding to these primary fields transform into each other under the modular transformations $\mathcal{S}$ and $\mathcal{T}$.

It would seem from this discussion that the only representations of interest in the study of WZW models are integrable ones, at integer level k. However, Kac and Wakimoto [11] asked if it was possible to relax this condition yet retain a finite set of highest weight representations whose characters closed under the action of the modular group. They discovered the admissible representations, which retain these important properties, but where now the level k need not be an integer. These representations are now generically non-unitary. The requirement for admissible representations is that $k = t/u$, where $t \in \mathbb{Z}$, $u \in \mathbb{N}$ and $\gcd(t, u) = 1$. Additionally, $k + h^\vee \geqslant h^\vee/u$, where $h^\vee$ is the dual Coxeter number of $\hat{\mathfrak{g}}_k$. Restricting to positive values of t and $u = 1$ recovers the integrable representations. Clearly there is now a problem of interpretation of the WZW model: although the algebraic formulation is sound, the WZW action is not well-defined. However, many systems are described by fractional level conformal field theories. It is also possible to relate fractional level models to integer level models via the coset construction. Additionally, probing the fractional level situation may allow for greater understanding of the relation between the two formulations of WZW models.

Why should this area of modular transformations and their relation to characters be of any interest? The answer lies in the fact that the partition function of a WZW model may be written as a modular invariant combination of characters. Considering the complex plane, the Hamiltonian operator H of the theory will generate translations in time, along the imaginary axis; the momentum operator P will generate translations in space, along the real axis. The evolution operator corresponding to a translation along the period ω_2 (in distance and direction) is $e^{-H\,\mathrm{Im}\,\omega_2 + iP\,\mathrm{Re}\,\omega_2}$. The partition function of the theory is given by the trace of this operator

$$Z(\omega_1, \omega_2) = \mathrm{Tr}\; e^{-H\,\mathrm{Im}\,\omega_2 + iP\,\mathrm{Re}\,\omega_2}, \tag{2.66}$$

the dependence on ω_1 being through H and P. The Hamiltonian on a cylinder is given by $H = (2\pi/\omega_1)(L_0 + \bar{L}_0 - c/12)$ and the momentum operator by $P = (2\pi i/\omega_1)(L_0 - \bar{L}_0)$, where the cylinder has circumference ω_1 (chosen to be real). We have taken $c = \bar{c}$, the factor $c/24$ coming about from mapping the plane to the cylinder: in physical terms, this means that the vacuum energy density vanishes in the limit $\omega_1 \to \infty$. Inserting the expressions for H and P in (2.66) gives

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \tag{2.67}$$

where $\bar{q} = e^{-2i\pi\bar{\tau}}$, with the identification $\bar{\tau} = \tau^*$. We recognise this expression as involving the Virasoro specialised characters (2.65). With more care [12] one may derive the complete partition function as

$$Z(\tau, \{\rho_i\}) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - c/24} e^{2i\pi \sum_i \rho_i H_0^i} \bar{q}^{\bar{L}_0 - c/24} e^{2i\pi \sum_i \bar{\rho}_i \bar{H}_0^i}. \tag{2.68}$$

This trace over the full Hilbert space of states of the theory may be decomposed into a trace over irreducible modules and so we find

$$Z(\tau, \{\rho_i\}) = \sum \chi_\lambda^{\hat{\mathfrak{g}}_k}(\tau, \{\rho_i\}) N_{\lambda,\mu} \bar{\chi}_\mu^{\hat{\mathfrak{g}}_k}(\bar{\tau}, \{\bar{\rho}_i\}), \tag{2.69}$$

the sum being over all the weight labels of the set of characters closed under the action of modular transformations.

Clearly, this partition function should be invariant under reparametrisations of the torus, *i.e.* modular transformations. We must therefore find matrices N such that under the action of $\mathcal{S}$ and $\mathcal{T}$, the partition function is unchanged. Dropping the ρ_i dependence for clarity, with

$$\begin{aligned} \chi_\lambda^{\hat{\mathfrak{g}}_k}(-1/\tau) &= \sum S_{\lambda\mu} \chi_\mu^{\hat{\mathfrak{g}}_k}(\tau), \\ \chi_\lambda^{\hat{\mathfrak{g}}_k}(\tau + 1) &= \sum T_{\lambda\mu} \chi_\mu^{\hat{\mathfrak{g}}_k}(\tau) \end{aligned} \tag{2.70}$$

this means that

$$T^\dagger N T = S^\dagger N S = N \tag{2.71}$$

or $[N, S] = [N, T] = 0$. For the partition function to describe some physical theory, the entries of N must be positive-integer valued. Additionally, on combining the holomorphic and anti-holomorphic sectors of the theory, there should be a unique vacuum state, requiring $N_{00} = 1$ (χ_0 denoting the vacuum character). With these requirements, one may search for modular invariant combinations of characters of some given affine Lie algebra, which then stand a chance of describing the space of states of some physical conformal field theory.

In general, it is an extremely difficult problem to classify all modular invariants of some particular algebra. The canonical example of a complete classification is that for $\hat{sl}(2)$ [13], implying the classification of the minimal models. It involves a totally unexpected correspondence with the simply-laced simple Lie algebras, the discovery of which has prompted many years of intensive research by both physicists and mathematicians: the mystery of the A-D-E classification is as yet unsolved. The case $\hat{sl}(3)$ has also been completed [14], revealing a similar pattern. The $N = 1$ [15] and $N = 2$ [16] superconformal minimal models have also been classified, with the former again showing an A-D-E pattern. Beyond these examples, only special cases are known, with a complete investigation of the simple affine Lie algebras a distant goal. These classifications have all been carried out at integer level. The case of fractional level $\hat{sl}(2)$ has also been investigated, with a full list of invariants for admissible representations given by [17] and [18]. The work in [1] and this thesis is the first attempt at discussing the problem for fractional level superalgebras.

A sensible physical theory is not guaranteed to arise solely from a modular invariant combination of characters. Conversely, not all conformal field theories exhibit modular invariance, although we will not be concerned with such here. One question one might ask is whether the theory admits a consistent fusion algebra. The operator product algebra (2.13) is generally quite a complicated structure. However, it is in principle possible to derive it from knowledge of the fusion rules and the symmetry algebra of the theory. Fusion rules tell us which primary fields appear in the operator product of two primary fields,

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k. \tag{2.72}$$

The fusion rule coefficients ${N_{ij}}^k$ are non-negative integers: it may be that the family $[\phi_k]$ can couple in several distinct ways to ϕ_i and ϕ_j, resulting in coefficients that are greater than 1. If ${N_{ij}}^k = 0$, then $[\phi_k]$ does not appear in the operator product $\phi_i(z)\phi_j(w)$. Alternatively, this determines whether or not the 3-point function $\langle \phi_k^* \phi_j \phi_i \rangle$ vanishes, where ϕ_k^* is the field conjugate to ϕ_k. This will be discussed more fully in chapter 4.

Remarkably, there exists a formula for the fusion rule coefficients in terms of the modular transformation matrix S of (2.70), the Verlinde formula [19] (modified to include fermionic theories in [20]). This states that

$$ {N_{ij}}^k = \sum_m \frac{S_{im} S_{jm} (S^{-1})_{mk}}{S_{0m}}, \tag{2.73} $$

the entries S_{ij} being the same as in (2.70), 0 again labelling the vacuum. This formula means that, in general, once the modular transformations of characters are known, fusion rules may be determined; consistency of the operator algebra follows. We are thus no further along the road to a consistent conformal field theory. However, in the case of fractional level, the Verlinde formula does not always yield meaningful results: the coefficients may be negative. Some modification of the Verlinde formula would seem to be required, although when fractional level algebras are used as building blocks in the coset construction, these problems disappear. This perhaps indicates that it is not possible to define a consistent non-unitary conformal field theory based on a fractional level affine algebra. Other results do not support this conclusion and the issue remains unresolved. One may calculate fusion rules from the decoupling of singular vectors, which will be discussed later, allowing a comparison between these results and the Verlinde formula. As yet, results on both these approaches are only available for $\hat{sl}(2)$ as discussed in [21] and partially $\hat{sl}(3)$ [22]. The modular transformation matrices for the affine superalgebra $\widehat{osp}(1|2)$ were calculated in [23], where the authors made a preliminary attempt at calculating fusion rules from the Verlinde formula. Fusion rules for $\widehat{osp}(1|2)$ have been calculated through singular vector decoupling in [24], but no comparison was made there with the results of [23] and indeed the fusion rules given in these two works do not agree. The work in this thesis is a first step in presenting both the modular transformation results and the fusion rules for the affine superalgebra $\hat{sl}(2|1;\mathbb{C})$ at fractional level.

2.5 Overview of $\hat{sl}(2|1;\mathbb{C})$

This section introduces the principal object of study in this thesis: the affine superalgebra $\hat{sl}(2|1;\mathbb{C})_k$. The finite superalgebra $sl(2|1;\mathbb{C})$ is not quite the simplest superalgebra one may consider, that being $osp(1|2;\mathbb{C})$, but it differs in important ways from this case which means it exhibits many features more common to superalgebras as a whole.

First, we review some general material on Lie superalgebras. These were first investigated by mathematicians in the early 1960s, before being rediscovered by physicists in the context of supersymmetry. A full classification by Kac appeared in 1977 [25] and a comprehensive modern review may be found in [26] as well as the book by Cornwell [27].

In the theory of Lie algebras, one considers a vector space together with a multiplication which is antisymmetric and satisfies the Jacobi identity. For Lie superalgebras, the vector space concerned may be split into an even part $\mathfrak{g}_0$ and an odd part $\mathfrak{g}_1$. Elements of the even part are assigned a degree of zero, while elements of the odd part have degree one. The vector space is then $\mathbb{Z}_2$-graded, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The multiplication is such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \mod 2}\,. \tag{2.74}$$

The bracket $[\cdot,\cdot]$ satisfies the properties of graded antisymmetry

$$[a, b] = -(-1)^{\deg a \cdot \deg b}[b, a] \tag{2.75}$$

and the generalised Jacobi identity

$$(-1)^{\deg a \cdot \deg c}[a, [b, c]] + (-1)^{\deg b \cdot \deg a}[b, [c, a]] + (-1)^{\deg c \cdot \deg b}[c, [a, b]] = 0. \tag{2.76}$$

The elements a, b and c are chosen from a homogeneous basis for $\mathfrak{g}$, that is, one in which they are wholly even or wholly odd. The property (2.75) defines an anti-commutator when a and b are both odd and the usual commutator otherwise: we retain the notation $[\cdot,\cdot]$ for the "superbracket", bearing in mind its dependence on the elements within it and will often use $\{\cdot,\cdot\}$ when it is clear that the anticommutator is required. The even part of $\mathfrak{g}$, $\mathfrak{g}_0$, forms a Lie algebra structure on its own. While $\mathfrak{g}_1$ does not share this property, it does carry a representation of the even subalgebra.

The most immediately useful example of objects which obey the properties necessary to be a superalgebra are a set of matrices, divided into those which are block diagonal and those which are block anti-diagonal. That is, matrices of the form

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \tag{2.77}$$

should be taken as the even elements comprising $\mathfrak{g}_0$ and

$$\left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \tag{2.78}$$

behave as the odd elements of $\mathfrak{g}_1$, with the partitioning being of the same form throughout. The supertrace of such a matrix is defined as

$$\mathrm{STr}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \mathrm{Tr}\, A - \mathrm{Tr}\, D, \tag{2.79}$$

from which we see that the supertrace of an odd matrix is always zero.

While definitions of quantities from the theory of Lie algebras generally carry over to superalgebras, there is a significant area which requires serious modification. It may happen that the metric on the root space of a superalgebra is not euclidean, as is the case for Lie algebras. The superalgebra of interest here, $sl(2|1;\mathbb{C})$, is one such, whereas the simplest Lie superalgebra $osp(1|2;\mathbb{C})$ actually has a euclidean metric on its space of roots. This is why we mentioned earlier that $sl(2|1;\mathbb{C})$ is more typical of superalgebras as a whole, having this additional complexity over $osp(1|2;\mathbb{C})$. The reason that a non-euclidean metric is of concern is that roots may then have zero length, so any definitions involving division by the length of roots (such as coroots and the dual Coxeter number) must be modified to allow for this situation. Additionally, it is not entirely straightforward to imagine a Weyl reflection about a root of zero length. With these provisos in mind, we proceed by looking at the relevant example: $sl(2|1;\mathbb{C})$.

The Lie superalgebra $sl(2|1;\mathbb{C})$[5] (also denoted $A(1,0)$) has a representation as the set of 3×3 matrices with supertrace zero, that is, $m_{11}+m_{22}-m_{33}=0$ with the following partition:

$$\left(\begin{array}{cc|c} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ \hline m_{31} & m_{32} & m_{33} \end{array}\right). \tag{2.80}$$

Matrices with $m_{13}=m_{23}=m_{31}=m_{32}=0$ are even, while those with $m_{11}=m_{12}=m_{21}=m_{22}=m_{33}=0$ are odd. The basis chosen in [28] is

$$\begin{aligned} \mathbf{h}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathbf{h}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathbf{e}_{\alpha_1+\alpha_2} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{e}_{-(\alpha_1+\alpha_2)} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{e}_{\alpha_1} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & \mathbf{e}_{-\alpha_1} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{e}_{\alpha_2} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbf{e}_{-\alpha_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.81}$$

[5]Most of the information in the remainder may be found in the references [28], [29] and [30].

Writing $\mathbf{h}_\pm = \mathbf{h}_1 \pm \mathbf{h}_2$, one may calculate the (anti)commutation relations of the four bosonic generators $H^\pm, E^{\pm(\alpha_1+\alpha_2)}$ and four fermionic generators $E^{\pm\alpha_1}, E^{\pm\alpha_2}$ (with the obvious correspondence) of $sl(2|1;\mathbb{C})$ as

$$
\begin{aligned}
[E^{\alpha_1+\alpha_2}, E^{-(\alpha_1+\alpha_2)}] &= H^-, & [H^-, E^{\pm(\alpha_1+\alpha_2)}] &= \pm 2E^{\pm(\alpha_1+\alpha_2)}, \\
[E^{\pm(\alpha_1+\alpha_2)}, E^{\mp\alpha_1}] &= \pm E^{\pm\alpha_2}, & [E^{\pm(\alpha_1+\alpha_2)}, E^{\mp\alpha_2}] &= \mp E^{\pm\alpha_1}, \\
[H^-, E^{\pm\alpha_1}] &= \pm E^{\pm\alpha_1}, & [H^-, E^{\pm\alpha_2}] &= \pm E^{\pm\alpha_2}, \\
[H^+, E^{\pm\alpha_1}] &= \pm E^{\pm\alpha_1}, & [H^+, E^{\pm\alpha_2}] &= \mp E^{\pm\alpha_2}, \\
\{E^{\alpha_1}, E^{-\alpha_1}\} &= (H^+ - H^-)/2, & \{E^{\alpha_2}, E^{-\alpha_2}\} &= (H^+ + H^-)/2, \\
\{E^{\pm\alpha_1}, E^{\pm\alpha_2}\} &= E^{\pm(\alpha_1+\alpha_2)}. & & \qquad (2.82)
\end{aligned}
$$

The even subalgebra of $sl(2|1;\mathbb{C})$ is a direct sum of a $u(1)$ algebra generated by H^+ and an $sl(2)$ algebra generated by H^- and $E^{\pm(\alpha_1+\alpha_2)}$. The Cartan subalgebra is made up of H^- and H^+. The non-zero roots are made up of the even roots $\Delta_0 = \{\pm(\alpha_1+\alpha_2)\}$ and odd roots $\Delta_1 = \{\pm\alpha_1, \pm\alpha_2\}$, where $\alpha_1 = (1,1)$ and $\alpha_2 = (1,-1)$ (with order (H^-, H^+)). The root diagram is shown in figure 2.1.

Here we meet a new superalgebra feature: the Killing form must be generalised to incorporate the $\mathbb{Z}_2$ grading. The definition is

$$K(X^a, X^b) = g^{ab} = f^{ac}{}_p f^{bp}{}_c (-1)^{d(p)} = \mathrm{STr}\,(\mathrm{ad}(X^a)\mathrm{ad}(X^b)), \qquad (2.83)$$

where $d(p) = 0$ for p a bosonic index and 1 for p a fermionic index and the f are the structure constants of the superalgebra. These as usual define the adjoint representation, hence the second expression above. The supertrace is defined as in (2.79). Considering only the Cartan subalgebra elements H^+ and H^-, the Cartan-Killing metric is Minkowski, with

$$g_{\mathrm{root}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.84)$$

As in the case of ungraded Lie algebras, the Killing form relates the Cartan subalgebra and weight space of a superalgebra. For every element γ of the weight

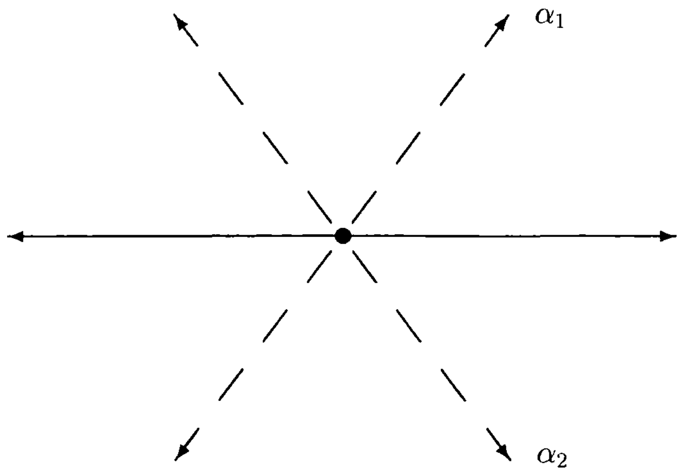

Figure 2.1: The root diagram of $sl(2|1;\mathbb{C})$.

space there corresponds a unique element of the Cartan subalgebra H^γ through

$$K(H, H^\gamma) = \gamma(H). \tag{2.85}$$

Then for a root α this element is given by $H^\alpha = \sum_i \alpha^i H^i$, with the scalar product of roots defined by

$$\langle \alpha_i, \alpha_j \rangle = K(H^{\alpha_i}, H^{\alpha_j}). \tag{2.86}$$

From this we see that the fermionic roots have zero length, $\alpha_1^2 = \alpha_2^2 = 0$, while the bosonic roots have $(\pm(\alpha_1 + \alpha_2))^2 = 4$ (in this normalisation). The root diagram is in a two-dimensional Minkowski space, with the fermionic roots along lightlike directions indicating their zero length.

As previously mentioned, it is not obvious how to consider Weyl reflections about a zero length root. Although the definition of Weyl group has been extended to deal with this fact, we consider only the standard situation. The Weyl group then is $W = \mathbb{Z}_2$ generated by the reflection in the plane perpendicular to $\alpha_1 + \alpha_2$. This leads to different inequivalent choices for the set of positive roots. One may choose $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ which yields the simple roots as $\{\alpha_1, \alpha_2\}$. The Cartan matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.87}$$

where the entries are calculated using $A_{ij} = \langle\alpha_i,\alpha_j\rangle/\langle\alpha_i,\alpha_{i'}\rangle$, $i \neq i'$ to avoid dividing by zero [27].

Another choice of positive roots might have been $\{\alpha_1+\alpha_2, \alpha_1, -\alpha_2\}$ with simple roots $\{\alpha_1+\alpha_2, -\alpha_2\}$. This variety of simple root systems available leads to the non-uniqueness of the Dynkin diagram [25, 26] although one may always choose a distinguished basis of simple roots, containing the smallest possible number of odd roots (the second alternative given here). The definition of highest weight state also depends on which simple roots we choose: we will use the (not distinguished) set $\{\alpha_1,\alpha_2\}$ and define highest weight states shortly in the context of the affine version of $sl(2|1;\mathbb{C})$. A quantity also of importance is the dual Coxeter number $h^\vee$: however, it is independent of the choice of simple roots and is equal to 1 for $sl(2|1;\mathbb{C})$.

A superalgebra may be extended to an affine superalgebra in exactly the same way as for an ungraded Lie algebra, as previously described. Although we may extend the relations (2.82) to the affine case, as we do wish to make contact with conformal field theory, we first introduce the currents of $\hat{sl}(2|1;\mathbb{C})$ (the $J^a(z)$ of (2.37)):

$$\begin{aligned}
J(\mathbf{e}_{\pm(\alpha_1+\alpha_2)})(z) &= \sum_n J_n^\pm z^{-n-1}, \\
J(\mathbf{h}_-)(z) &= 2\sum_n J_n^3 z^{-n-1}, \qquad & J(\mathbf{h}_+)(z) &= 2\sum_n U_n z^{-n-1}, \\
J(\mathbf{e}_{\pm\alpha_1})(z) &= \sum_n j_n'^{\pm} z^{-n-1}, \qquad & J(\mathbf{e}_{\pm\alpha_2})(z) &= \sum_n j_n^\pm z^{-n-1}.
\end{aligned} \tag{2.88}$$

The modes introduced here satisfy the (anti)commutation relations

$$\begin{aligned}
[J^+_m, J^-_n] &= 2J^3_{m+n} + 2\tilde{k}m\delta_{m+n,0}, & [U_m, U_n] &= -\tilde{k}m\delta_{m+n,0},\\
[J^3_m, J^\pm_n] &= \pm J^\pm_{m+n}, & [J^3_m, J^3_n] &= \tilde{k}m\delta_{m+n,0},\\
[J^\pm_m, j'^\mp_n] &= \pm j^\pm_{m+n}, & [J^\pm_m, j^\mp_n] &= \mp j'^\pm_{m+n},\\
[2J^3_m, j'^\pm_n] &= \pm j'^\pm_{m+n}, & [2J^3_m, j^\pm_n] &= \pm j^\pm_{m+n},\\
[2U_m, j'^\pm_n] &= \pm j'^\pm_{m+n}, & [2U_m, j^\pm_n] &= \mp j^\pm_{m+n},\\
\{j'^+_m, j'^-_n\} &= (U_{m+n} - J^3_{m+n}) - 2\tilde{k}m\delta_{m+n,0}, & &\\
\{j^+_m, j^-_n\} &= (U_{m+n} + J^3_{m+n}) + 2\tilde{k}m\delta_{m+n,0}, & \{j'^\pm_m, j^\pm_n\} &= J^\pm_{m+n}.
\end{aligned} \tag{2.89}$$

Together with the relation $[L_0, X_n] = -nX_n$ for all generators X_n and $\tilde{k}$ commuting with all generators, this defines the affine Lie superalgebra $\hat{sl}(2|1;\mathbb{C})$ at arbitrary level k. Taking the zero modes only gives the finite algebra $sl(2|1;\mathbb{C})$ (2.82), though now with a different normalisation of the Cartan subalgebra generators. The form of L_0 is given through the Sugawara construction. Rewriting (2.55) for a generic basis (*i.e.* one for which the generators are not necessarily orthonormal with respect to the Killing form) we have

$$L_n = \frac{1}{2\tilde{k} + h^\vee\psi^2} \sum_{a,b=1}^{\dim sl(2|1;\mathbb{C})} \sum_{m=-\infty}^{\infty} :g_{ab}X^a_m X^b_{n-m}:, \tag{2.90}$$

where $g_{ab} = (g^{ab})^{-1}$ and the X^a_n are the superalgebra generators. Then

$$\begin{aligned} L_0 = \frac{1}{2(k+1)} \sum_{m\in\mathbb{Z}} :&2J^3_m J^3_{-m} - 2U_m U_{-m} + J^+_m J^-_{-m} - J^-_m J^+_{-m}\\
&+ j'^+_m j'^-_{-m} - j'^-_m j'^+_{-m} - j^+_m j^-_{-m} + j^-_m j^+_{-m}:, \end{aligned} \tag{2.91}$$

bearing in mind that when two fermionic generators are interchanged upon normal ordering a minus sign appears. In the present normalisation, $\psi^2 = (\alpha_1 + \alpha_2)^2 = 1$ and so $k = 2\tilde{k}$. The central charge formula (2.54) is modified to

$$c = \frac{\tilde{k}\,\mathrm{sdim}\, sl(2|1;\mathbb{C})}{\tilde{k} + h^\vee\psi^2/2} \tag{2.92}$$

with sdim denoting the superdimension, equal to the number of bosonic minus the number of fermionic generators. For $sl(2|1;\mathbb{C})$, this is zero and hence $c = 0$.

With affine superalgebras, the fermionic generators have mode indices which may be integers or half-integers. This gives rise to two different versions (sectors) of the same algebra. In the Ramond sector, the odd generators have $n \in \mathbb{Z}$ whereas in the Neveu-Schwarz sector, they have $n \in \mathbb{Z}+\frac{1}{2}$. The even generators always have mode index $n \in \mathbb{Z}$. This structure naturally incorporates the periodic (Ramond) and anti-periodic (Neveu-Schwarz) boundary conditions for fermions on the torus. Indeed, it was in the construction of fermionic string theories (in the guise of dual models) by Ramond [31] and Neveu and Schwarz [32] that superalgebras (supersymmetric extensions of the Virasoro algebra) came to prominence amongst physicists.

The weights of $\hat{sl}(2|1;\mathbb{C})$ are specified by their eigenvalues with respect to the Cartan subalgebra $\{J_0^3, U_0, \tilde{k}, L_0\}$, as discussed earlier ((2.48)). The set of simple roots is augmented from the finite situation by $\alpha_0 = (-(\alpha_1 + \alpha_2), 0, 1)$, where the other simple roots are $\alpha_i = (\alpha_i, 0, 0)$, $i = 1, 2$ (using the same notation for the affine roots as the finite roots). Highest weights are parametrised by the quantum numbers h_- and h_+ as

$$\Lambda = (\tfrac{1}{2}h_-(\alpha_1 + \alpha_2) + \tfrac{1}{2}h_+(\alpha_1 - \alpha_2), \tilde{k}, -h) \tag{2.93}$$

which, with the present choice of $\alpha_1 = (1/2, 1/2)$ and $\alpha_2 = (1/2, -1/2)$ means that $\Lambda = (\frac{1}{2}h_-, \frac{1}{2}h_+, \tilde{k}, -h)$.

Highest weight states $|\Lambda\rangle = |\frac{1}{2}h_-, \frac{1}{2}h_+, h\rangle$ have isospin, charge and conformal weight, respectively, defined by

$$J_0^3|\Lambda\rangle = \frac{1}{2}h_-|\Lambda\rangle, \quad U_0|\Lambda\rangle = \frac{1}{2}h_+|\Lambda\rangle, \quad L_0|\Lambda\rangle = h|\Lambda\rangle. \tag{2.94}$$

The highest weight states are annihilated by all the raising operators of the algebra:

$$
\begin{aligned}
J_n^+|\Lambda\rangle &= 0, & J_{n+1}^-|\Lambda\rangle &= 0, \\
J_{n+1}^3|\Lambda\rangle &= 0, & U_{n+1}|\Lambda\rangle &= 0, \\
j_n^+|\Lambda\rangle &= 0, & j_n'^+|\Lambda\rangle &= 0, \\
j_{n+1}^-|\Lambda\rangle &= 0, & j_{n+1}'^-|\Lambda\rangle &= 0,
\end{aligned}
\tag{2.95}
$$

where $n \in \mathbb{Z}_+$. Additionally, $L_{n+1}|\Lambda\rangle = 0$. Note that the mode indices are all integers, so this properly defines a Ramond highest weight state. The conditions (2.95) may all be obtained from

$$
J_1^-|\Lambda\rangle = j_0^+|\Lambda\rangle = j_0'^+|\Lambda\rangle = 0. \tag{2.96}
$$

To conclude this section, we relate the Ramond and Neveu-Schwarz sectors of $\hat{sl}(2|1;\mathbb{C})$. There are several possible "spectral flows" which achieve this, but one in particular which means that a Ramond highest weight state is mapped to a Neveu-Schwarz highest weight state [29, 30]. This is of some importance in computing $\hat{sl}(2|1;\mathbb{C})$ character formulae, since then the transformation of a Ramond character built on a highest weight yields a Neveu-Schwarz highest weight character (as used in [29]). The relevant spectral flow corresponds to an order 2 automorphism γ (with $\gamma^2 = 1$) of the finite $sl(2|1;\mathbb{C})$ algebra, yielding

$$
\begin{aligned}
\gamma(J_n^3) &= -J_n^3 + \tilde{k}\delta_{n,0}, & \gamma(U_n) &= U_n, \\
\gamma(J_n^\pm) &= J_{n\pm 1}^\mp, & \gamma(j_n^\pm) &= j'^\mp_{n\pm\frac{1}{2}}, \\
\gamma(j_n'^\pm) &= j^\mp_{n\pm\frac{1}{2}}, & \gamma(\tilde{k}) &= \tilde{k}.
\end{aligned}
\tag{2.97}
$$

There is also the relation $\gamma(L_n) = L_n - J_n^3 + \frac{1}{2}\tilde{k}\delta_{n,0}$. Then γ maps between the Ramond and Neveu-Schwarz sectors, with highest weights mapping to highest weights. The fact that γ is order 2 means that the fermionic Neveu-Schwarz generators are 1/2 integer indexed. On flowing between the two sectors, the quantum numbers of states are related in the following way:

$$
\begin{aligned}
\tfrac{1}{2}h_-^{NS} &= -\tfrac{1}{2}h_-^R + \tfrac{1}{2}k, \\
\tfrac{1}{2}h_+^{NS} &= \tfrac{1}{2}h_+^R, \\
h^{NS} &= h^R - \tfrac{1}{2}h_-^R + \tfrac{1}{4}k. \qquad (2.98)
\end{aligned}
$$

This tool of spectral flow will be used in later calculations: for now, it ends our overview of $\hat{sl}(2|1;\mathbb{C})$.

Chapter 3

Modular Transformations and Invariants of $\hat{sl}(2|1;\mathbb{C})$ Characters

This chapter examines the particular problem of calculating modular transformations of $\hat{sl}(2|1;\mathbb{C})_k$ characters, for fractional levels $k = 1/u - 1$, $u \in \mathbb{N} \setminus \{1\}$. With the solution of this problem, we are able to examine some modular invariant combinations of characters, which may be taken as a first step in attempting to determine partition functions for conformal field theories. The work is entirely contained in [1].

3.1 Introduction

The properties of the affine Lie superalgebra $\hat{sl}(2|1;\mathbb{C})_k$ at fractional level were much investigated in a series of papers, variously by Bowcock, Hayes and Taormina [28, 29, 33]. The motivation for this work was not solely abstract, although this is certainly not an insignificant one, given the limited study of affine superalgebras generally. In fact, the representation theory of $\hat{sl}(2|1;\mathbb{C})_k$ seems to play a role in the study of non-critical string theory, specifically the $N = 2$ non-critical superstring [34]. It has long been held that a non-critical (super)string may be described in terms of a topological G/G Wess-Zumino-Witten model, with G a Lie (super)group

[23, 35, 36]. The physical states of the G/G model are believed to be in a one-to-one correspondence with those of the relevant string theory. Establishing this correspondence between the states of the $SL(2|1)/SL(2|1)$ model and those of the $N = 2$ non-critical superstring was one of the primary motivations of the works [28, 29, 33]. Then with the relationship in place, one might hope to say something about the string theory from purely algebraic investigations. Non-critical string theory may be described in terms of a matter sector, a Liouville sector describing the gravitational degrees of freedom and a ghost system. When related to the relevant G/G WZW model, in the case of the $N = 2$ string it is found that the level of the $\hat{sl}(2|1;\mathbb{C})_k$ algebra describing the matter sector is given by

$$k = \frac{p}{u} - 1, \qquad p, u \in \mathbb{N}, \qquad \gcd(p, u) = 1. \tag{3.1}$$

In the introduction we stated that Kac and Wakimoto [11] showed that for levels of the form $k = t/u$, $\gcd(t, u) = 1$, admissible representations of an affine algebra exist. We see that this is precisely the case here, motivating the study of $\hat{sl}(2|1;\mathbb{C})_k$ at fractional level. Additionally, fractional level $\hat{sl}(2|1;\mathbb{C})$ has also arisen in the study of Gaussian disordered systems [37], though not at levels of the form which we will consider.

While the promise of this particular approach to the $N = 2$ non-critical string has yet to run its course, the work of this chapter continues a more general study based on the understanding developed in [28, 29, 33]. In particular, in [33] the branching functions of some $\hat{sl}(2|1;\mathbb{C})_k$ characters were derived (for $k = 1/u-1$), re-expressing these complicated objects in terms of simpler ones with known modular transformation properties. This was done with a view to calculating the behaviour of $\hat{sl}(2|1;\mathbb{C})_k$ characters under modular transformations. While the behaviour under $\mathcal{T} : \tau \to \tau+1$ was found to be standard, under $\mathcal{S} : \tau \to -1/\tau$ only a few specific examples were worked out, namely for $k = -1/2$ [29] and the Neveu-Schwarz sector for $k = -2/3$ [30], being the first two cases of $u = 2$ and $u = 3$ respectively (the case $u = 1$ corresponds to integrable representations). The importance of finding modular transformations lies in the fact that one may then address the question of finding modular invariants, extending the understanding of the physical space of states of the $SL(2|1)/SL(2|1)$ model. More generally, one may also attempt to define a conformal field theory based on fractional level $\hat{sl}(2|1;\mathbb{C})_k$. Attempts to develop fractional level conformal field theories have thus far mainly concentrated on $\hat{sl}(2)$ and $\widehat{osp}(1|2)$, more of which later.

We begin by reviewing previous work leading to the branching formulae for $\hat{sl}(2|1;\mathbb{C})_k$ characters. These must be manipulated somewhat in order to facilitate the calculation of their modular transformations. Having achieved this, we describe how the behaviour of $\hat{sl}(2|1;\mathbb{C})_k$ characters under the $\mathcal{S}$ transformation may be found and give the form of this transformation for all levels $k = 1/u-1$, $u \in \mathbb{N}\backslash\{1\}$. In the final section we identify some modular invariant combinations of characters, which are analogous to the A- and D-series in the famous classification of $\hat{sl}(2)$ modular invariants [13].

3.2 Branching $\hat{sl}(2|1;\mathbb{C})$ Characters

In [33], branching formulae for the Neveu-Schwarz class IV and class V characters of the affine superalgebra $\hat{sl}(2|1;\mathbb{C})_k$ ($k = 1/u - 1$) were conjectured, based on patterns appearing for low values of u. These branching formulae have now been firmly established in [38]. The characters were branched into products of $\hat{sl}(2;\mathbb{C})_k$ characters, generalised theta functions and string functions, the modular transformation properties of which are known. This decomposition was found where the level is of the form $k = 1/u - 1$, rather than the more general $k = p/u - 1$ and it is for this reason that we are restricted to $p = 1$. Class IV and class V refer to specific sets of representations identified in [28, 29], satisfying certain restrictions on their quantum numbers. The reason for considering these classes only is that class I characters are modular forms rather than functions, transforming with additional factors under modular transformations, so it is not clear that they may be used to construct modular invariants. Classes II and III are believed to contain subsingular vectors, making the computation of their characters extremely difficult. Class IV and class V characters are thought not to be afflicted with these difficulties, being modular functions and not containing subsingular vectors, so stand a chance of forming a modular invariant set. The work of [38] further reinforces the fact that the relevant representations are class IV and class V, since in considering certain decompositions of $\hat{sl}(2|1;\mathbb{C})$ characters at level $k = 1/u - 1$ it is only these which appear.

has now shown that for levels of the form considered here ($k = 1/u - 1$), the $\hat{sl}(2|1;\mathbb{C})_k$ characters which appear are all equivalent to class IV and class V characters.

From the definition (2.64), $\hat{sl}(2|1;\mathbb{C})_k$ characters are given by

$$\chi^{\hat{sl}(2|1;\mathbb{C})_k}_{h_-,h_+}(\sigma,\nu,\tau) = \operatorname{Tr}\exp\{2\pi i(\sigma J^3_0 + \nu U_0 + \tau L_0)\}, \tag{3.2}$$

where J^3_0 and U_0 are the zero mode Cartan generators of $\hat{sl}(2|1;\mathbb{C})_k$. Note that there is no factor $q^{-c/24}$ in this definition, since for $\hat{sl}(2|1;\mathbb{C})$, $c = 0$. We label the characters by the isospin and charge quantum numbers which characterise the $\hat{sl}(2|1;\mathbb{C})_k$ highest weight states $|\Lambda\rangle$ (defined in (2.95)) of the associated representations:

$$J^3_0|\Lambda\rangle = \frac{1}{2}h_-|\Lambda\rangle, \quad U_0|\Lambda\rangle = \frac{1}{2}h_+|\Lambda\rangle. \tag{3.3}$$

We now examine some properties of $\hat{sl}(2;\mathbb{C})_k$ characters, generalised theta functions and string functions which are combined together in the expressions which we will use for $\hat{sl}(2|1;\mathbb{C})$ characters. From [39] we have the following expression for $\hat{sl}(2;\mathbb{C})_k$ characters:

$$\chi^{\hat{sl}(2;\mathbb{C})_k}_{n,n'}(\sigma,\tau) = \frac{\vartheta_{v_+,w}\left(\frac{\sigma}{u},\tau\right) - \vartheta_{v_-,w}\left(\frac{\sigma}{u},\tau\right)}{\vartheta_{1,2}(\sigma,\tau) - \vartheta_{-1,2}(\sigma,\tau)}, \tag{3.4}$$

where the level is parametrised as

$$k = \frac{t}{u}, \qquad \gcd(t,u) = 1, \quad u \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{3.5}$$

with $0 \leqslant n \leqslant 2u+t-2$ and $0 \leqslant n' \leqslant u-1$ and

$$v_{\pm} = u(\pm(n+1) - n'(k+2)), \qquad w = u^2(k+2). \tag{3.6}$$

In the above, the generalised theta functions $\vartheta_{m,m'}$ [40] are defined as

$$\vartheta_{m,m'}(\sigma,\tau) = \sum_{n\in\mathbb{Z}} q^{m'(n+\frac{m}{2m'})^2} z^{m'(n+\frac{m}{2m'})}. \tag{3.7}$$

The variables q and z are defined by

$$\begin{aligned} q &= \exp(2\pi i\tau), \quad \tau\in\mathbb{C},\ \mathrm{Im}(\tau)>0 \Rightarrow |q|<1, \\ z &= \exp(2\pi i\sigma). \end{aligned} \tag{3.8}$$

The cases of interest here are those for which $k = 1/u - 1$, that is, where $t = 1-u$: the $\hat{sl}(2|1;\mathbb{C})_k$ characters at level k involve $\hat{sl}(2;\mathbb{C})_k$ characters at the same level.

Under the modular $\mathcal{S}$ transformation $\mathcal{S}: (\sigma,\nu,\tau) \to (\frac{\sigma}{\tau}, \frac{\nu}{\tau}, -\frac{1}{\tau})$, the $\hat{sl}(2;\mathbb{C})_k$ characters (3.4) transform via [39]

$$\chi_{m,m'}^{\hat{sl}(2;\mathbb{C})_k}\left(\frac{\sigma}{\tau}, -\frac{1}{\tau}\right) = e^{-i\pi k\sigma^2/\tau} \sum_{n=0}^{2u+t-2}\sum_{n'=0}^{u-1} S_{mm',nn'} \chi_{n,n'}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau), \tag{3.9}$$

where

$$S_{mm',nn'} = \sqrt{\frac{2}{u^2(k+2)}}(-1)^{m'(n+1)+(m+1)n'} e^{-i\pi(k+2)m'n'} \sin\left(\frac{\pi(m+1)(n+1)}{k+2}\right). \tag{3.10}$$

For the generalised theta functions (3.7) we have [40]

$$\vartheta_{m,m'}\left(\frac{\sigma}{\tau}, -\frac{1}{\tau}\right) = e^{-i\pi m'\sigma^2/\tau}\sqrt{\frac{-i\tau}{2m'}} \sum_{r=0}^{2m'-1} e^{-i\pi rm/m'} \vartheta_{r,m'}(\sigma,\tau) \tag{3.11}$$

and for the string functions [41]

$$c_{a,b}^{(u-1)}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{(-i\tau)(u-1)(u+1)}} \sum_{a'=0}^{u-1} \sum_{\substack{b'=-u+2 \\ a'\equiv b' \bmod 2}}^{u-1} s(a,b,a',b')\, c_{a',b'}^{(u-1)}(\tau), \tag{3.12}$$

where

$$s(a,b,a',b') = e^{i\pi bb'/(u-1)} \sin\left(\frac{\pi(a+1)(a'+1)}{u+1}\right). \tag{3.13}$$

The string functions have the following useful symmetry properties [42]:

$$c_{a,b}^{(u-1)}(\tau) = c_{a,-b}^{(u-1)}(\tau) = c_{a,b+2(u-1)\mathbb{Z}}^{(u-1)}(\tau) = c_{u-1-a,u-1-b}^{(u-1)}(\tau),$$
$$c_{a,b}^{(u-1)}(\tau) = 0 \quad \text{for} \quad a-b \neq 0 \mod 2. \tag{3.14}$$

As already mentioned, in [33] branching formulae for $\hat{sl}(2|1;\mathbb{C})$ Neveu-Schwarz characters were derived which expressed them in terms of the quantities discussed above. The result for class IV characters reads

$$\chi_{h_-^{NS},h_+^{NS}}^{NS,IV,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} \chi_{a,u-m-1}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau)$$
$$\times \sum_{b=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}^{(u-1)}(\tau) \tag{3.15}$$

and for those in class V it is

$$\chi_{h_-^{NS},h_+^{NS}}^{NS,V,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} \chi_{a,M+M'+1}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau)$$
$$\times \sum_{b=0}^{u-2} \vartheta_{(u-1)(M'-M)+u(u-1)a+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}^{(u-1)}(\tau), \tag{3.16}$$

where $[\frac{u}{2}]$ denotes the integer part of $\frac{u}{2}$.

For the Neveu-Schwarz sector, the isospin $(\frac{1}{2}h_-^{NS})$ and charge $(\frac{1}{2}h_+^{NS})$ quantum numbers of highest weight states are given by

$$h_-^{NS} = -\frac{1}{u}(u-m-1), \quad h_+^{NS} = \frac{1}{u}(2m'-m), \quad 0 \leqslant m' \leqslant m \leqslant u-1 \tag{3.17}$$

in class IV and by

$$h_-^{NS} = -\frac{1}{u}(M+M'+1+u), \quad h_+^{NS} = \frac{1}{u}(M-M'), \quad 0 \leqslant M+M' \leqslant u-2 \tag{3.18}$$

in class V. We use the quantum numbers of the highest weight state upon which a character is built to label that character.

Under the modular transformations $\mathcal{S}$ and $\mathcal{T}$, characters of the Neveu-Schwarz and Ramond sectors mix according to table 3.1. This has been discussed in [43] in the context of superconformal field theory. In string theory, these transformations relate boundary conditions for fermions living on the string worldsheet, as discussed by Seiberg and Witten [44] (amongst others). Here we see the appearance of the supercharacters $S\chi^R$ and $S\chi^{NS}$. Though these do not have the physical relevance of χ^R and χ^{NS}, they are useful here as can be seen in table 3.1, in that they are necessary for writing down a set of characters closed under modular transformations. In fact, the Ramond supercharacters form a closed set on their own; we discuss them here for completeness. The difference between a supercharacter and a character is an insertion of the factor $(-1)^F$ in the definition (3.2). This fermion number operator highlights states that are fermionic, giving rise to a minus sign in their presence.

	$\mathcal{S}$	$\mathcal{T}$
χ^{NS}	χ^{NS}	$S\chi^{NS}$
χ^R	$S\chi^{NS}$	χ^R
$S\chi^{NS}$	χ^R	χ^{NS}
$S\chi^R$	$S\chi^R$	$S\chi^R$

(3.19)

Table 3.1: Effect of modular transformations on characters.

Given then that we will come to consider the characters and supercharacters of the Ramond sector and the supercharacters of the Neveu-Schwarz sector, we mention that the Neveu-Schwarz supercharacters have the same quantum numbers as the characters while the Ramond characters and supercharacters have

$$h^R_- = -\frac{m}{u}, \quad h^R_+ = h^{NS}_+, \quad 0 \leqslant m' \leqslant m \leqslant u-1 \tag{3.20}$$

in class IV and

$$h^R_- = \frac{1}{u}(M+M'+2), \quad h^R_+ = h^{NS}_+, \quad 0 \leqslant M+M' \leqslant u-2 \tag{3.21}$$

in class V. The conformal weight in both classes for the Ramond sector is given by

$$h^R = \frac{u}{4}\left((h_-^R)^2 - (h_+^R)^2\right) \tag{3.22}$$

and in the Neveu-Schwarz sector by

$$h^{NS} = h^R - \frac{1}{2}h_-^R + \frac{1-u}{4u}. \tag{3.23}$$

At this point, we have given only the branching formulae for Neveu-Schwarz characters. We can obtain those for Ramond characters from (3.15) and (3.16) by the spectral flow (2.97). Neveu-Schwarz characters are defined as in (3.2):

$$\chi_{h_-^{NS},h_+^{NS}}^{\hat{sl}(2|1;\mathbb{C})_k,NS}(\sigma,\nu,\tau) = \mathrm{Tr}\exp\{2\pi i(\sigma J_0^{3,NS} + \nu U_0^{NS} + \tau L_0^{NS})\}, \tag{3.24}$$

to which the application of (2.97) gives

$$\begin{aligned}
\chi_{h_-,h_+}^{\hat{sl}(2|1;\mathbb{C})_k,NS}(\sigma,\nu,\tau) &= \mathrm{Tr}\exp\{2\pi i(\sigma(-J_0^{3,R} + \tfrac{1}{2}k) + \nu U_0^R + \tau(L_0^R - J_0^{3,R} + \tfrac{1}{4}k))\} \\
&= q^{\frac{1}{4}k} z^{\frac{1}{2}k}\mathrm{Tr}\exp\{2\pi i(-(\sigma+\tau)J_0^{3,R} + \nu U_0^R + \tau L_0^R)\} \\
&= q^{k/4} z^{k/2}\chi_{h_-^R,h_+^R}^{R,\hat{sl}(2|1;\mathbb{C})_k}(-\sigma-\tau,\nu,\tau),
\end{aligned} \tag{3.25}$$

with q and z defined in (3.8). Then we may write

$$\begin{aligned}
\chi_{h_-^R,h_+^R}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) &= q^{-k/4} z'^{-k/2}\chi_{h_-^{NS},h_+^{NS}}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(-\sigma-\tau,\nu,\tau) \\
&= q^{(1-u)/4u} z^{(1-u)/2u}\chi_{h_-^{NS},h_+^{NS}}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(-\sigma-\tau,\nu,\tau)
\end{aligned} \tag{3.26}$$

where $z' = z^{-1}q^{-1}$ and recalling that we are considering only $k = 1/u - 1$.

This means that we must shift the variable σ to $-\sigma-\tau$ in the $\hat{sl}(2;\mathbb{C})_k$ characters, the only place where σ appears. The effect of this may be calculated by using the definition of the generalised theta functions (3.7) and the definition of the $\hat{sl}(2;\mathbb{C})_k$ characters in terms of these given by (3.4).

The theta functions which appear in (3.4) are of the form $\vartheta_{m,m'}(\frac{\sigma}{r},\tau)$ with $r=1$ or u, $m = \pm 1$ or $v_\pm$ and $m' = 2$ or w. From the definition (3.7) we find that

$$\vartheta_{m,m'}\left(\frac{-\sigma-\tau}{r},\tau\right) = \vartheta_{\frac{m'}{r}-m,m'}\left(\frac{\sigma}{r},\tau\right) q^{-m'/4r^2} z^{-m'/2r^2}. \tag{3.27}$$

In the case of $\vartheta_{\pm1,2}(\sigma,\tau)$ appearing in the denominator of (3.4), the index $m'/r-m$ is simply equal to 1 for $\vartheta_{1,2}$ and 3 for $\vartheta_{-1,2}$—but from (3.7) we see that $\vartheta_{3,2}=\vartheta_{-1,2}$. So spectral flow on the denominator of (3.4) introduces an overall factor $q^{1/2}z$ in $\chi_{n,n'}^{\hat{sl}(2;\mathbb{C})_k}(-\sigma-\tau,\tau)$.

For the case of $\vartheta_{v_\pm,w}(\frac{-\sigma-\tau}{u},\tau)=\vartheta_{\frac{w}{u}-v_\pm,w}(\frac{\sigma}{u},\tau)q^{-w/4u^2}z^{-w/2u^2}$, it can easily be verified from the definitions (3.6) and (3.7) that $\vartheta_{\frac{w}{u}-v_\pm,w}(\frac{\sigma}{u},\tau)=\vartheta_{v'_\pm,w}(\frac{\sigma}{u},\tau)$ where $v'_\pm=u(\pm(u-1-n)+1)-(u-1-n')(u+1)$, again recalling that we take $k=1/u-1$. Hence

$$\chi_{n,n'}^{\hat{sl}(2;\mathbb{C})_k}(-\sigma-\tau,\tau)=q^{-(1-u)/4u}z^{-(1-u)/2u}\chi_{u-1-n,u-1-n'}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau) \tag{3.28}$$

and thus

$$\begin{aligned}\chi_{h_-^R,h_+^R}^{R,IV,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau)=&\sum_{a=0}^{u-1}\chi_{u-1-a,m}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau)\\&\times\sum_{b=0}^{u-2}\vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)\\&\qquad\times c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau)\end{aligned} \tag{3.29}$$

and in class V

$$\begin{aligned}\chi_{h_-^R,h_+^R}^{R,V,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau)=&\sum_{a=0}^{u-1}\chi_{u-1-a,u-M-M'-2}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau)\\&\times\sum_{b=0}^{u-2}\vartheta_{(u-1)(M'-M)+u(u-1)a+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)\\&\qquad\times c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau),\end{aligned} \tag{3.30}$$

with definitions as given previously.

To obtain the Neveu-Schwarz supercharacters, we must shift the variable $\sigma\to\sigma+1$ in (3.15) and (3.16), then divide the results by $e^{i\pi h_-^{NS}}$ (this procedure corresponding to an insertion of the operator $(-1)^F$ in the definition of the character

[29]). For class IV we find

$$S\chi^{NS,IV,\hat{sl}(2|1;\mathbb{C})_k}_{h_-^{NS},h_+^{NS}}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} e^{i\pi(a-(u-m-1))}\chi^{\hat{sl}(2;\mathbb{C})_k}_{a,u-m-1}(\sigma,\tau)$$
$$\times \sum_{b=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau) \quad (3.31)$$

and for class V we obtain

$$S\chi^{NS,V,\hat{sl}(2|1;\mathbb{C})_k}_{h_-^{NS},h_+^{NS}}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} e^{i\pi(a-(M+M'))}\chi^{\hat{sl}(2;\mathbb{C})_k}_{a,M+M'+1}(\sigma,\tau)$$
$$\times \sum_{b=0}^{u-2} \vartheta_{(u-1)(M'-M)+u(u-1)a+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau). \quad (3.32)$$

The Ramond sector supercharacters are obtained in similar fashion from (3.29) and (3.30), shifting $\sigma \to \sigma+1$ and dividing by $e^{i\pi h_-^R}$:

$$S\chi^{R,IV,\hat{sl}(2|1;\mathbb{C})_k}_{h_-^{R},h_+^{R}}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} e^{i\pi(u-1-a-m)}\chi^{\hat{sl}(2;\mathbb{C})_k}_{u-1-a,m}(\sigma,\tau)$$
$$\times \sum_{b=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau) \quad (3.33)$$

and

$$S\chi^{R,V,\hat{sl}(2|1;\mathbb{C})_k}_{h_-^{R},h_+^{R}}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} e^{i\pi(M+M'-a)}\chi^{\hat{sl}(2;\mathbb{C})_k}_{u-1-a,u-M-M'-2}(\sigma,\tau)$$
$$\times \sum_{b=0}^{u-2} \vartheta_{(u-1)(M'-M)+u(u-1)a+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau). \quad (3.34)$$

On applying the modular $\mathcal{S}$ transformation to a particular character, a linear combination of class IV and class V characters is obtained. The calculation of the effect of $\mathcal{S}$ in the general case is thus simplified by combining the branched

$\hat{sl}(2|1;\mathbb{C})_k$ formulae. Examining the Neveu-Schwarz branching formulae (3.15) and (3.16), it is clear that the substitution $M = m' - m - 1$, $M' = u - 1 - m'$ in the class V formula (3.16) gives us precisely the class IV formula (3.15). Since $0 \leqslant M + M' \leqslant u - 2$, these new values of $m = u - 2 - M - M'$ and $m' = u - 1 - M'$ are those for which $0 \leqslant m < m' \leqslant u - 1$, whereas for class IV we have $0 \leqslant m' \leqslant m \leqslant u - 1$. We can use the same branching formula (3.15) for both class IV and class V, with $0 \leqslant m \leqslant u - 1$ and now the range of m' extended to $0 \leqslant m' \leqslant u - 1$, but the relevant M and M' and expressions for the class V quantum numbers must be used to obtain the correct values of h_-^{NS} and h_+^{NS}.

The story for the Ramond characters (3.29) and (3.30) is exactly the same; for the Neveu-Schwarz and Ramond supercharacters we must modify (3.31) and (3.33) slightly. The final versions of the branching formulae read (now labelling the $\hat{sl}(2|1;\mathbb{C})_k$ characters by m and m'):

$$\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} \chi_{a,u-m-1}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau) \times \sum_{b=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right) \times c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau); \quad (3.35)$$

$$\chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} \chi_{u-1-a,m}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau) \times \sum_{b=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right) \times c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau); \quad (3.36)$$

$$S\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) = \sum_{a=0}^{u-1} (-1)^{G+a-(u-m-1)} \chi_{a,u-m-1}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau) \times \sum_{b=0}^{u-2} \vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right) \times c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau); \quad (3.37)$$

and

$$S\chi^{R,\hat{sl}(2|1;\mathbb{C})_k}_{m,m'}(\sigma,\nu,\tau)=\sum_{a=0}^{u-1}(-1)^{G+u-1-a-m}\chi^{\hat{sl}(2;\mathbb{C})_k}_{u-1-a,m}(\sigma,\tau)$$
$$\times\sum_{b=0}^{u-2}\vartheta_{(u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\, c^{(u-1)}_{a,a(u-1)+2a(\frac{u}{2}-[\frac{u}{2}])-2b}(\tau),\quad (3.38)$$

where

$$G=\begin{cases}0 & \text{if} \quad m\geqslant m',\\ 1 & \text{if} \quad m<m'\end{cases}$$

and $0\leqslant m,m'\leqslant u-1$ in both sectors.

3.3 Modular $\mathcal{S}$ Transformation of $\hat{sl}(2|1;\mathbb{C})_k$ Characters

With the branched form of the $\hat{sl}(2|1;\mathbb{C})$ characters now established, the action of $\mathcal{S}:(\sigma,\nu,\tau)\to\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right)$ on the branched $\hat{sl}(2|1;\mathbb{C})_k$ characters (3.35), (3.36), (3.37) and (3.38) may be obtained by use of (3.9), (3.11) and (3.12). For example, in the case of the Neveu-Schwarz characters (3.35) we find

$$\chi^{NS,\hat{sl}(2|1;\mathbb{C})_k}_{m,m'}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right)=\frac{e^{i\pi(u-1)\left(\sigma^2-\nu^2\right)/u\tau}}{(u-1)\sqrt{2u(u+1)}}\sum_{a,n,n'=0}^{u-1}S_{a(u-m-1),nn'}\chi^{\hat{sl}(2;\mathbb{C})_k}_{n,n'}(\sigma,\tau)$$
$$\times\sum_{b=0}^{u-2}\sum_{r=0}^{2u(u-1)-1}e^{-i\pi r((u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub)/(u(u-1))}\vartheta_{r,u(u-1)}\left(\frac{\nu}{u},\tau\right)$$
$$\times\sum_{a'=0}^{u-1}\sum_{\substack{b'=-u+2\\ a'\equiv b' \bmod 2}}^{u-1}s\left(a,l,a',b'\right)c^{(u-1)}_{a',b'}(\tau),\quad (3.39)$$

where $l=a(u-1)+2a\left(\frac{u}{2}-[\frac{u}{2}]\right)-2b$.

At first sight the problem of extracting from this expression a linear combination of Neveu-Schwarz characters (3.35) would appear a fairly challenging task. However, having calculated the transformation matrices for the cases $u=2,3$ (the results of which are listed in the appendices) an alternative way of proceeding presents itself. Looking at (3.35), we see that taking only the $a = b = 0$ term (say) provides us with a unique linear "signature" term for each Neveu-Schwarz character, which appears only in that particular character. Hence the coefficient of this term is necessarily the coefficient of that particular character. It should be stated that this procedure makes the assumption that on the right hand side of (3.39) it is terms which may be rearranged exactly into class IV and class V characters which appear, *i.e.* that these characters are closed under modular transformations. This was certainly found to be the case for $u = 2$ and $u = 3$ which were considered fully by "brute force" (as described in the appendices) and this conclusion is also supported by [38], which shows that for certain decompositions of $\hat{sl}(2|1;\mathbb{C})$ characters only class IV and class V characters appear at levels $k = 1/u - 1$.

The signature term for a Neveu-Schwarz character takes the following form:

$$\chi_{n,n'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) \sim \chi_{0,u-n-1}^{\hat{sl}(2;\mathbb{C})_k}(\sigma,\tau)\vartheta_{(u-1)(n-2n'+u),u(u-1)}\left(\frac{\nu}{u},\tau\right)c_{0,0}^{(u-1)}(\tau). \quad (3.40)$$

The problem of computing the coefficients of $\hat{sl}(2|1;\mathbb{C})_k$ Neveu-Schwarz characters in (3.39) is reduced to a simple matter of extracting the correct coefficients for the signatures (3.40). We therefore have

$$\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{i\pi(u-1)(\sigma^2-\nu^2)/u\tau}\sum_{n=0}^{u-1}\sum_{n'=0}^{u-1} S_{mm',nn'}^{NS}\chi_{n,n'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) \quad (3.41)$$

where

$$S_{mm',nn'}^{NS} = \frac{1}{u(u-1)}\sqrt{\frac{u}{2(u+1)}}\sum_{a=0}^{u-1}\sum_{b=0}^{u-2} S_{a(u-m-1),0(u-n-1)}$$
$$\times e^{-i\pi(n-2n'+u)((u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub)/u}$$
$$\times \{s(a,l,0,0)+s(a,l,u-1,u-1)\}, \quad (3.42)$$

making use of the fact that, by (3.14), $c_{u-1,u-1}^{(u-1)}(\tau) = c_{0,0}^{(u-1)}(\tau)$. Expanding the various factors gives

$$S^{NS}_{mm',nn'} = \frac{1}{u(u-1)(u+1)}\sum_{a=0}^{u-1}\sum_{b=0}^{u-2}(-1)^{(u-m-1)+(a+1)(u-n-1)}$$
$$\times\, e^{-i\pi(u+1)(u-m-1)(u-n-1)/u}\sin\left(\frac{\pi(a+1)u}{u+1}\right)$$
$$\times\, e^{-i\pi(n-2n'+u)((u-1)(m-2m')+u(u-1)(a+1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub)/u}$$
$$\times\left\{\sin\left(\frac{\pi(a+1)}{u+1}\right)+e^{i\pi l}\sin\left(\frac{\pi(a+1)u}{u+1}\right)\right\} \qquad (3.43)$$

with as before $l = a(u-1)+2a\left(\frac{u}{2}-[\frac{u}{2}]\right)-2b$.

However, this expression simplifies considerably. The sum over b can trivially be performed to give a factor $u-1$. As for the sum over a, we have

$$\sum_{a=0}^{u-1}(-1)^{a(u-n-1)}e^{-i\pi(n-2n'+u)(au(u-1)+2au(\frac{u}{2}-[\frac{u}{2}])-2ub)/u}\sin\left(\frac{\pi(a+1)u}{u+1}\right)$$
$$\times\left(\sin\left(\frac{\pi(a+1)}{u+1}\right)+e^{i\pi\left(a(u-1)+2a\left(\frac{u}{2}-[\frac{u}{2}]\right)\right)}\sin\left(\frac{\pi(a+1)u}{u+1}\right)\right). \qquad (3.44)$$

For the case of u odd, this is equal to

$$\sum_{a=0}^{u-1}(-1)^{an}e^{-i\pi(n+u)au}\sin\left(\frac{\pi(a+1)u}{u+1}\right)$$
$$\times\left(\sin\left(\frac{\pi(a+1)}{u+1}\right)+e^{i\pi au}\sin\left(\frac{\pi(a+1)u}{u+1}\right)\right) \qquad (3.45)$$

and for u even it is

$$\sum_{a=0}^{u-1}(-1)^{a(n+1)}e^{-i\pi an(u-1)}\sin\left(\frac{\pi(a+1)u}{u+1}\right)$$
$$\times\left(\sin\left(\frac{\pi(a+1)}{u+1}\right)+e^{i\pi a(u-1)}\sin\left(\frac{\pi(a+1)u}{u+1}\right)\right). \qquad (3.46)$$

In both cases the summation is equal to

$$\begin{aligned}
\sum_{a=0}^{u-1}(-1)^a \sin\left(\frac{\pi(a+1)u}{u+1}\right)&\left(\sin\left(\frac{\pi(a+1)}{u+1}\right)+(-1)^a\sin\left(\frac{\pi(a+1)u}{u+1}\right)\right)\\
&=\sum_{a=0}^{u-1} 2\sin^2\left(\frac{\pi(a+1)}{u+1}\right)\\
&=\sum_{a=0}^{u-1}\left(1-\cos\left(\frac{2\pi(a+1)}{u+1}\right)\right)\\
&=u-\sum_{a=1}^{u}\cos\left(\frac{2\pi a}{u+1}\right)\\
&=u-\sum_{a=0}^{u}\operatorname{Re} e^{2i\pi a/(u+1)}+1\\
&=u+1.
\end{aligned}$$

Hence the final expression for the matrix entries $S^{NS}_{mm',nn'}$ in (3.41) is

$$S^{NS}_{mm',nn'}=\frac{1}{u}(-1)^{m+n}e^{-i\pi(u+1)(u-m-1)(u-n-1)/u}e^{-i\pi(u-1)(m-2m'+u)(n-2n'+u)/u}, \tag{3.47}$$

a perhaps unexpectedly elegant result.

For the Ramond characters and Neveu-Schwarz and Ramond supercharacters we proceed in an essentially similar way: the signature for the Ramond characters is

$$\chi^{R,\hat{sl}(2|1;\mathbb{C})_k}_{n,n'}(\sigma,\nu,\tau)\sim\chi^{\hat{sl}(2;\mathbb{C})_k}_{u-1,n}(\sigma,\tau)\vartheta_{(u-1)(n-2n'+u),u(u-1)}\left(\frac{\nu}{u},\tau\right)c^{(u-1)}_{0,0}(\tau); \tag{3.48}$$

for the Neveu-Schwarz supercharacters it is

$$\begin{aligned}S\chi^{NS,\hat{sl}(2|1;\mathbb{C})_k}_{n,n'}(\sigma,\nu,\tau)\sim(-1)^{G-(u-n-1)}&\chi^{\hat{sl}(2;\mathbb{C})_k}_{0,u-n-1}(\sigma,\tau)\\
&\times\vartheta_{(u-1)(n-2n'+u),u(u-1)}\left(\frac{\nu}{u},\tau\right)c^{(u-1)}_{0,0}(\tau);\end{aligned} \tag{3.49}$$

and for the Ramond supercharacters it is

$$\begin{aligned}\chi^{R,\hat{sl}(2|1;\mathbb{C})_k}_{n,n'}(\sigma,\nu,\tau)\sim(-1)^{G+u-1-n}&\chi^{\hat{sl}(2;\mathbb{C})_k}_{u-1,n}(\sigma,\tau)\\
&\times\vartheta_{(u-1)(n-2n'+u),u(u-1)}\left(\frac{\nu}{u},\tau\right)c^{(u-1)}_{0,0}(\tau),\end{aligned} \tag{3.50}$$

with G defined as before. As $\mathcal{S}$ transforms Ramond characters into Neveu-Schwarz supercharacters and vice versa we must extract the coefficient of (3.48) in the $\mathcal{S}$-transformed Neveu-Schwarz supercharacter (3.37). Similarly, we extract the coefficient of (3.49) in the $\mathcal{S}$-transformed Ramond character (3.36). Again, the expressions simplify along similar lines to the Neveu-Schwarz case. We find

$$\chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right)=e^{i\pi(u-1)(\sigma^2-\nu^2)/u\tau}\sum_{n=0}^{u-1}\sum_{n'=0}^{u-1}S^{R}_{mm',nn'}S\chi_{n,n'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) \tag{3.51}$$

where

$$S^{R}_{mm',nn'}=\frac{1}{u}(-1)^{G'+m+n+u(u-n-1)}e^{-i\pi(u+1)m(u-n-1)/u}e^{-i\pi(u-1)(m-2m'+u)(n-2n'+u)/u}; \tag{3.52}$$

$$S\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right)=e^{i\pi(u-1)(\sigma^2-\nu^2)/u\tau}\sum_{n=0}^{u-1}\sum_{n'=0}^{u-1}S^{SNS}_{mm',nn'}\chi_{n,n'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) \tag{3.53}$$

where

$$S^{SNS}_{mm',nn'}=\frac{1}{u}(-1)^{G+m+n+u(u-m-1)}e^{-i\pi(u+1)(u-m-1)n/u}e^{-i\pi(u-1)(m-2m'+u)(n-2n'+u)/u}; \tag{3.54}$$

and

$$S\chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right)=e^{i\pi(u-1)(\sigma^2-\nu^2)/u\tau}\sum_{n=0}^{u-1}\sum_{n'=0}^{u-1}S^{SR}_{mm',nn'}S\chi_{n,n'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau) \tag{3.55}$$

where

$$S^{SR}_{mm',nn'}=\frac{1}{u}(-1)^{G+G'+(u-1)(m+n)}e^{-i\pi(u+1)mn/u}e^{-i\pi(u-1)(m-2m'+u)(n-2n'+u)/u}, \tag{3.56}$$

with

$$G=\begin{cases}0 & \text{if } m\geqslant m',\\ 1 & \text{if } m<m'\end{cases}$$

and

$$G'=\begin{cases}0 & \text{if } n\geqslant n',\\ 1 & \text{if } n<n'.\end{cases}$$

This completes the derivation of the $\mathcal{S}$ transformation of $\hat{sl}(2|1;\mathbb{C})_k$ characters. Some expected properties of these matrices are that S^{NS} and S^{SR} are symmetric; that $S^R = (S^{SNS})^\mathsf{T}$; that all of these matrices are unitary; and that the matrices as calculated by brute force for $u = 2$ (as found in [29]) and $u = 3$ (also calculated in [30] for the Neveu-Schwarz characters) given in the appendices agree with the above results.

In order to consider modular invariant combinations of $\hat{sl}(2|1;\mathbb{C})_k$ characters, we must also know how they transform under the modular $\mathcal{T}$ transformation $\mathcal{T}$: $(\sigma,\nu,\tau) \to (\sigma,\nu,\tau+1)$. It can be shown that the action of $\mathcal{T}$ is as follows [29, 30]:

$$\begin{aligned}
\chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau+1) &= e^{2\pi i h^R} \chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau), \\
\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau+1) &= e^{2\pi i h^{NS}} S\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau), \\
S\chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau+1) &= e^{2\pi i h^{NS}} \chi_{m,m'}^{NS,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau), \\
S\chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau+1) &= e^{2\pi i h^R} S\chi_{m,m'}^{R,\hat{sl}(2|1;\mathbb{C})_k}(\sigma,\nu,\tau),
\end{aligned} \tag{3.57}$$

recalling that for class V characters $(m < m')$ we must use the appropriate M and M' and the class V formulae to calculate the conformal weights.

3.4 Modular Invariants

With the behaviour of $\hat{sl}(2|1;\mathbb{C})_k$ characters under the modular $\mathcal{S}$ and $\mathcal{T}$ transformations now established, we can now proceed by looking for modular invariant combinations of characters. Recalling the discussion of section 2.4, one can usually say that modular invariant combinations of characters correspond to partition functions of conformal field theories, which is why such objects are of considerable interest. For Wess-Zumino-Witten models based on affine algebras, this is generally (though not always) true when the level of the algebra is an integer. Work has been done to define a rigorous model based on the affine superalgebra $\widehat{osp}(1|2)$ at integer level in [45] (though not through looking for modular invariants). However, even for integer levels, full classifications of modular invariants (and indeed the matrices of the $\mathcal{S}$ transformation) are very rare. The proof of the classification of $\hat{sl}(2)$ invariants [13] has not been found to adapt easily even to other $\hat{sl}(n)$ algebras (but was extended successfully to classify $N = 1$ superconformal minimal models [15]), though Gannon [14] achieved the classification for the $\hat{sl}(3)$ case and also classified the $N = 2$ superconformal minimal models [16]. The mysterious emergence of an A-D-E pattern (relating each modular invariant to a simply-laced Lie algebra), which these classifications appear to fall into (to greater or lesser degrees of definiteness) continues to defy explanation.

When the symmetry algebra is fractional, it is still unclear whether sound conformal field theories can be defined, though there has been much work on the subject for the case of $\hat{sl}(2)$ which we will mention in the next chapter. As far as modular transformations are concerned, admissible $\hat{sl}(2)$ representations were dealt with by Koh and Sorba [18] and Lu [17], who obtained a complete classification of modular invariants. These also fell into an A-D-E pattern, though now with a whole series of invariants associated to each Lie algebra.

In general, the problem of classifying all modular invariants given the matrices of the $\mathcal{S}$ and $\mathcal{T}$ transformations is not at all straightforward and one which we have not attempted to solve. However, we have found all invariants for the cases $u = 2$ and $u = 3$, special cases of which are analogous to the A- and D-series obtained in the $\hat{sl}(2)$ case. We will also show that such modular invariants exist for all $u \geqslant 2$.

Modular invariant combinations of characters take the form

$$Z = \sum_{m,m',n,n'=0}^{u-1} N^{R}_{mm',nn'} \chi^{R}_{m,m'} \overline{\chi^{R}}_{n,n'} + N^{NS}_{mm,nn'} \chi^{NS}_{m,m'} \overline{\chi^{NS}}_{n,n'} + N^{SNS}_{mm',nn'} S\chi^{NS}_{m,m'} \overline{S\chi^{NS}}_{n,n'} + \sum_{a,a',b,b'=0}^{u-1} N^{SR}_{aa',bb'} S\chi^{R}_{a,a'} \overline{S\chi^{R}}_{b,b'}, \quad (3.58)$$

written in this way to emphasise the fact that the Ramond supercharacters form a closed set under modular transformations, whereas the remaining sectors mix as detailed previously. For "physical" modular invariants, the $N_{mm',nn'}$ must be non-negative integers. In addition, the identity should be unique so $N^{R}_{00,00}$ must be equal to 1 (the identity character in this context is $\chi^R_{0,0}$).

As discussed in section 2.4, on applying the $\mathcal{S}$ transformation, we see that to find these combinations we must solve the matrix equation $S^\dagger NS = N$, using the appropriate matrix S for each sector. We can make life easier for ourselves by noting that for the character combinations to be invariant under $\mathcal{T}$, non-zero entries in N may only occur where the factors $e^{2\pi ih}$ arising in this transformation conspire to give non-negative integers. The cases $u = 2$ and $u = 3$ have been investigated exhaustively (see appendices) for which this amounts to the requirement that the characters have the same weight h associated with them, as is clear from the tables of values of h; in addition, as $\mathcal{T}$ interpolates between Neveu-Schwarz characters and supercharacters, $N^{NS} = N^{SNS}$.

For the case of $u = 2$, we find two possibilities:

$$(i) \quad N^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & a-1 & 0 \\ 0 & a-1 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad N^{NS} = \begin{pmatrix} a & 0 & 0 & a-1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a-1 & 0 & 0 & a \end{pmatrix} \tag{3.59}$$

or

$$(ii) \quad N^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a-1 & a & 0 \\ 0 & a & a-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad N^{NS} = \begin{pmatrix} a & 0 & 0 & a-1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ a-1 & 0 & 0 & a \end{pmatrix}. \tag{3.60}$$

Clearly for non-negative integer entries, $a \in \mathbb{N}$. For $u = 3$ we find a similar situation, with an additional parameter:

$$
(i) \quad N^R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & b & b & 0 & a-1 & 0 & 0 & 0 \\
0 & b & a & a-1 & 0 & b & 0 & 0 & 0 \\
0 & b & a-1 & a & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a+b & 0 & a+b-1 & 0 & 0 \\
0 & a-1 & b & b & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a+b-1 & 0 & a+b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{3.61}
$$

$$
N^{NS} = \begin{pmatrix}
a+b & 0 & 0 & 0 & 0 & 0 & a+b-1 & 0 & 0 \\
0 & a & b & 0 & 0 & 0 & 0 & a-1 & b \\
0 & b & a & 0 & 0 & 0 & 0 & b & a-1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
a+b-1 & 0 & 0 & 0 & 0 & 0 & a+b & 0 & 0 \\
0 & a-1 & b & 0 & 0 & 0 & 0 & a & b \\
0 & b & a-1 & 0 & 0 & 0 & 0 & b & a
\end{pmatrix} \tag{3.62}
$$

or

$$
(ii) \quad N^R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & a & a-1 & 0 & b & 0 & 0 & 0 \\
0 & a & b & b & 0 & a-1 & 0 & 0 & 0 \\
0 & a-1 & b & b & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a+b & 0 & a+b-1 & 0 & 0 \\
0 & b & a-1 & a & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a+b-1 & 0 & a+b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \tag{3.63}
$$

$$
N^{NS} = \begin{pmatrix}
a+b & 0 & 0 & 0 & 0 & 0 & a+b-1 & 0 & 0 \\
0 & b & a & 0 & 0 & 0 & 0 & b & a-1 \\
0 & a & b & 0 & 0 & 0 & 0 & a-1 & b \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
a+b-1 & 0 & 0 & 0 & 0 & 0 & a+b & 0 & 0 \\
0 & b & a-1 & 0 & 0 & 0 & 0 & b & a \\
0 & a-1 & b & 0 & 0 & 0 & 0 & a & b
\end{pmatrix}. \tag{3.64}
$$

For non-negative integer entries we require $a \in \mathbb{N}$, $b \in \mathbb{Z}_+$. The ordering of terms in these matrices is as given in the tables 3.2, 3.3, 3.4 and 3.5. We recall that the supercharacters in each sector have the same quantum numbers as the corresponding characters. The relation between M and M' values in class V and the m and m' values which allowed us to combine classes IV and V in the branching formulae (3.35), (3.36), (3.37) and (3.38) is $m = u - 2 - M - M'$, $m' = u - 1 - M'$.

In the cases (3.59) and (3.61), when $a = 1$ and $b = 0$, we find that $N^R = N^{NS} = I$. This diagonal invariant we find at all levels (analogous to the $\hat{sl}(2)$ A-series),

	m	m'	h^R_-	h^R_+	h^R	h^{NS}_-	h^{NS}_+	h^{NS}
$\chi_{0,0}$	0	0	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{8}$
$\chi_{1,0}$	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{8}$
$\chi_{1,1}$	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{8}$

Table 3.2: Class IV $\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}$ characters

	$M(m)$	$M'(m')$	h^R_-	h^R_+	h^R	h^{NS}_-	h^{NS}_+	h^{NS}
$\chi_{0,1}$	0(0)	0(1)	1	0	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{8}$

Table 3.3: Class V $\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}$ characters

since the matrices S and T are unitary. With $a = 1$ and $b = 0$ in (3.60) and (3.63), the resulting expressions are permutation invariants of the form

$$\sum \chi_{m,m'}\overline{\chi}_{\Pi(m,m')}, \tag{3.65}$$

where

$$\Pi(m,m') = (m, (m - m') \mod u). \tag{3.66}$$

Alternatively, labelling the characters by (h_-, h_+) this is equivalent (see the definitions (3.17), (3.18), (3.20) and (3.21)) to $\Pi(h_-, h_+) = (h_-, -h_+)$. This pattern is also apparent in the Ramond supercharacters (analysis of which we leave to the appendices). Does this permutation give rise to a series of modular invariants? The easiest way to investigate this is by calculating NS and SN and seeing if these are equal, for then $S^\dagger NS = N$. We take as an ansatz

$$(N)_{mm',nn'} = \begin{cases} 1 & \text{if} \quad n = m,\ n' = (m - m') \mod u \\ 0 & \text{otherwise.} \end{cases} \tag{3.67}$$

	m	m'	h^R_-	h^R_+	h^R	h^{NS}_-	h^{NS}_+	h^{NS}
$\chi_{0,0}$	0	0	0	0	0	$-\frac{2}{3}$	0	$-\frac{1}{6}$
$\chi_{1,0}$	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0
$\chi_{1,1}$	1	1	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0
$\chi_{2,0}$	2	0	$-\frac{2}{3}$	$-\frac{2}{3}$	0	0	$-\frac{2}{3}$	$\frac{1}{6}$
$\chi_{2,1}$	2	1	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{2}$
$\chi_{2,2}$	2	2	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{1}{6}$

Table 3.4: Class IV $\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}$ characters

	$M(m)$	$M'(m')$	h^R_-	h^R_+	h^R	h^{NS}_-	h^{NS}_+	h^{NS}
$\chi_{1,2}$	0(1)	0(2)	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{4}{3}$	0	$-\frac{1}{6}$
$\chi_{0,1}$	0(0)	1(1)	1	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	0
$\chi_{0,2}$	1(0)	0(2)	1	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	0

Table 3.5: Class V $\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}$ characters

Then in the Neveu-Schwarz case we have

$$\begin{aligned}(NS^{NS})_{mm',nn'} &= \sum_{a,a'} N_{mm',aa'} S^{NS}_{aa',nn'} \\ &= S^{NS}_{m((m-m') \mod u),nn'} \\ &= (-1)^{m+n} e^{-i\pi(u+1)(u-m-1)(u-n-1)/u} \\ &\qquad \times e^{-i\pi(u-1)(m-2((m-m') \mod u)+u)(n-2n'+u)/u}\end{aligned} \tag{3.68}$$

and

$$\begin{aligned}(S^{NS}N)_{mm',nn'} &= S^{NS}_{mm',n((n-n') \mod u)} \\ &= (-1)^{m+n} e^{-i\pi(u+1)(u-m-1)(u-n-1)/u} \\ &\qquad \times e^{-i\pi(u-1)(m-2m'+u)(n-2((n-n') \mod u)+u)/u}.\end{aligned} \tag{3.69}$$

Considering each possible combination of $m \geqslant m'$ or $m < m'$ and $n \geqslant n'$ or $n < n'$ it is a simple matter to verify that the phases in (3.68) and (3.69) are equal; similar analysis applies to NS^R and NS^{SNS} for the same matrix N (3.67). We can also check that N commutes with T. Hence we have found another series of modular invariants, analogous to the D-series of the $\hat{sl}(2)$ case.

We might expect the D-type invariants to be related to an automorphism of the fusion rules, in the manner of Schellekens and Yankielowicz [46]. However, establishing fusion rules for fractional level algebras is a difficult problem and is the subject of the next chapter. These invariants might additionally have some description in terms of automorphisms of the $\hat{sl}(2|1;\mathbb{C})$ Dynkin diagram [5]: however, as previously discussed, there is a choice of simple roots available for a superalgebra and so the Dynkin diagram is not unique, making such an interpretation not entirely straightforward. Other means of discovering modular invariants include looking for conformal embeddings of the algebra into larger algebras of the same central charge. For $\hat{sl}(2|1;\mathbb{C})$, with central charge zero, there are no obvious candidate algebras since the only possible embeddings are in algebras which are essentially similar, in fact $\hat{sl}(m+1|m)$ [47]. The problem of finding all modular invariants, while interesting, remains uninvestigated at present.

3.5 Conclusion

In this chapter, we have found expressions for the modular S transformation of $\hat{sl}(2|1;\mathbb{C})_k$ characters at fractional level $k = 1/u - 1$. This has allowed us to calculate all modular invariants for the cases $u = 2$ and $u = 3$, leading to the discovery of an A-series and D-series of modular invariants. The derivation of the general S transformation of $\hat{sl}(2|1;\mathbb{C})_k$ characters solves the problem left open in [33] and has enabled us to begin looking at modular invariants in this framework of affine superalgebras at fractional level, a subject not previously studied. It remains to fully investigate possible modular invariant combinations of characters and indeed establish beyond doubt that the set of class IV and class V characters at level $k = 1/u - 1$ is closed under the action of modular transformations, for example by solving expressions of the type (3.39). However, the recent work of [38] also corroborates the results found here. As to whether these modular invariants may be regarded as partition functions—this is a question which we will begin to address in considering fusion rules, investigated in the next chapter.

Chapter 4

Fusion Rules at $k = -1/2$

In this chapter, we investigate the structure of fusion rules for $\hat{sl}(2|1;\mathbb{C})_k$ at level $k = -1/2$. The method discussed here is particularly practicable for this case and results are obtained in the Ramond sector, though as this is isomorphic to the Neveu-Schwarz sector we are able to remark on the form of fusion rules there as well. In the course of this work use is made of the $N = 2$ superconformal 3-point function, allowing us to relate work on this subject contained in [48], [49], [50] and [51]. The results of this chapter are contained in [2].

4.1 Introduction

In the previous chapter, we were able to establish general expressions for the modular transformations of $\hat{sl}(2|1;\mathbb{C})_k$ characters for fractional levels of the form $k = 1/u - 1$, $u \in \mathbb{N} \setminus \{1\}$. The obvious course of action, given matrices for the $\mathcal{S}$ transformation of characters, is to apply the Verlinde formula and derive fusion rules, which encode information on which fields may appear in the operator product expansion of two primary fields (2.13), through

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k. \tag{4.1}$$

As was discussed in section 2.4, the Verlinde formula (2.73), when applied in the situation of fractional level, does not seem to yield meaningful results. In particular, the entries ${N_{ij}}^k$ in (4.1), which should be non-negative integers, may turn out to be negative. It is therefore important to study fusion rules by other means, which, in cases thus far investigated, yield non-negative integer fusion coefficients. This furthers the attempt to define conformal field theories at fractional level, specifically, those permitting admissible representations for which the set of characters closes under the action of modular transformations. It is as yet still unclear whether fractional level affine algebras may define conformal field theories in their own right, though the consensus seems to be that this is possible. That this issue remains unresolved and that other models may be derived through the coset construction from fractional level theories means that interest in this subject is maintained.

The case of $\hat{sl}(2)$ is the one where most effort has been concentrated. In particular, fusion rules for fractional level have been investigated in [52, 53, 54, 55], with more abstract analysis carried out in [56] and [57]. Recently, this question has also been addressed in the case of $\hat{sl}(3)$ in [22], where an overview of the $\hat{sl}(2)$ situation is also given. Conformal blocks for fractional level $\hat{sl}(2)$ have also been studied extensively, for example in [55, 58, 59]. As for superalgebras, the case of $\widehat{osp}(1|2)$ has been considered in [23] and [24]. This chapter sees the techniques of [24] extended to discuss the superalgebra $\hat{sl}(2|1;\mathbb{C})$. The situation for $\hat{sl}(2|1;\mathbb{C})$ is far more complex, essentially arising from the fact that $\hat{sl}(2|1;\mathbb{C})$ is the simplest superalgebra where zero length roots appear, in contrast to $\widehat{osp}(1|2)$. As mentioned earlier, it is not only worthwhile studying this from an abstract point of view, but is also of importance given the intimate link between non-unitary $\hat{sl}(2|1;\mathbb{C})$ and the $N=2$ non-critical superstring [34]. An understanding of fusion rules on the $\hat{sl}(2|1;\mathbb{C})_k$ side and the string theory side would deepen the supposed correspondence between the $SL(2|1)/SL(2|1)$ WZW model and the $N=2$ non-critical string.

Common to most of the works mentioned above is the understanding that for fractional level representations, one must work with fields which are not only a function of the usual coordinate z, but also of an isotopic coordinate x, representing an internal $sl(2)$ symmetry. This technique was first applied to the unitary $\hat{sl}(2)$

case in [60] and was also developed in [61]. For fractional level, this overcomes the problem of needing to consider general infinite-dimensional representations, which are neither highest nor lowest weight. In [24], the authors extended this approach by including not only the coordinate x but also dependence on a Grassmann coordinate θ to represent the supersymmetry present in $\widehat{osp}(1|2)$ (being the $N = 1$ extension of $\hat{sl}(2)$): for the case of $\hat{sl}(2|1;\mathbb{C})$ we must additionally augment this by another Grassmann coordinate $\bar{\theta}$, now with two supersymmetries present. Our basic approach will be to determine 3-point correlation functions involving fields with this dependence (amounting to a determination of the 3-point function for $N = 2$ superconformal field theory) and then determine this correlator involving a singular vector. We may then rewrite this as an expression involving differential operators acting on the 3-point function, which must equal zero since it involves a singular vector (which is, by definition, orthogonal to all other states). This will provide relations between the quantum numbers of the three fields present, determining which fusion rules are permitted.

We begin by recalling some essential features of $\hat{sl}(2|1;\mathbb{C})$ from section 2.5. We then discuss the relation between highest weight states and primary fields, before going on to determine a realisation of $\hat{sl}(2|1;\mathbb{C})$ in terms of differential operators. This will allow us to determine the $\hat{sl}(2|1;\mathbb{C})$ invariant 3-point function, an issue discussed in the context of $N = 2$ superconformal field theory; in examining this issue we will relate the work of [48], [49], [50] and [51] on this subject. A brief summary of singular vectors required (as determined in [28]) will then be given, before putting these pieces together in the calculation of fusion rules. The results of this calculation will be compared with those of the Verlinde formula, as applied to the results of the previous chapter.

4.2 Review of $\hat{sl}(2|1;\mathbb{C})$

We begin with a brief reminder of some of the important features of $\hat{sl}(2|1;\mathbb{C})$, already introduced in section 2.5.

The affine superalgebra $\hat{sl}(2|1;\mathbb{C})$ is made up of the even generators $\{J_n^\pm, J_n^3, U_n\}$ and odd generators $\{j_n^\pm, j_n'^\pm\}$, supplemented by the usual affine generators $\tilde{k}$, the central generator, and d, the derivative operator. This is identified with the generator $-L_0$ of the Virasoro algebra associated to $\hat{sl}(2|1;\mathbb{C})_k$ via the Sugawara construction. The non-zero (anti)commutation relations of $\hat{sl}(2|1;\mathbb{C})_k$ are

$$
\begin{aligned}
&[J_m^+, J_n^-] = 2J_{m+n}^3 + 2\tilde{k}m\delta_{m+n,0}, && [U_m, U_n] = -\tilde{k}m\delta_{m+n,0},\\
&[J_m^3, J_n^\pm] = \pm J_{m+n}^\pm, && [J_m^3, J_n^3] = \tilde{k}m\delta_{m+n,0},\\
&[J_m^\pm, j_n'^\mp] = \pm j_{m+n}^\pm, && [J_m^\pm, j_n^\mp] = \mp j_{m+n}'^\pm,\\
&[2J_m^3, j_n'^\pm] = \pm j_{m+n}'^\pm, && [2J_m^3, j_n^\pm] = \pm j_{m+n}^\pm,\\
&[2U_m, j_n'^\pm] = \pm j_{m+n}'^\pm, && [2U_m, j_n^\pm] = \mp j_{m+n}^\pm,\\
&\{j_m'^+, j_n'^-\} = (U_{m+n} - J_{m+n}^3) - 2\tilde{k}m\delta_{m+n,0}, && \\
&\{j_m^+, j_n^-\} = (U_{m+n} + J_{m+n}^3) + 2\tilde{k}m\delta_{m+n,0}, && \{j_m'^\pm, j_n^\pm\} = J_{m+n}^\pm.
\end{aligned}
\tag{4.2}
$$

In addition, we have that

$$[d, X_n] = nX_n \tag{4.3}$$

and the central generator $\tilde{k}$ commutes with all other generators.

The even generators have mode index $n \in \mathbb{Z}$, whereas the odd generators have $n \in \mathbb{Z}$ in the Ramond sector and $n \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector. Setting this index to zero recovers the finite $sl(2|1;\mathbb{C})$ algebra. For the work in this chapter, we consider only the Ramond sector unless stated otherwise.

One natural way of splitting up an affine Lie (super)algebra is through a triangular decomposition. Utilising the generators d and J_0^3, we may define the principal gradation $\tilde{d} = ad(J_0^3) + 2ad(d)$ which has the following action on $\hat{sl}(2|1;\mathbb{C})$ generators:

$$\tilde{d}(J_n^{\pm}) = 2n \pm 1, \quad \tilde{d}(J_n^3) = 2n, \quad \tilde{d}(j_n^{\pm}) = 2n \pm \frac{1}{2}, \tag{4.4}$$

$$\tilde{d}(j_n'^{\pm}) = 2n \pm \frac{1}{2}, \quad \tilde{d}(U_n) = 2n, \quad \tilde{d}(\tilde{k}) = \tilde{d}(d) = 0. \tag{4.5}$$

Denoting the algebra by $\hat{\mathfrak{g}}$, we obtain the decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}^- \oplus \hat{\mathfrak{g}}^0 \oplus \hat{\mathfrak{g}}^+, \tag{4.6}$$

$\hat{\mathfrak{g}}^-$ consisting of those elements having $\tilde{d} < 0$, $\hat{\mathfrak{g}}^0$ of elements with $\tilde{d} = 0$ and $\hat{\mathfrak{g}}^+$ of elements with $\tilde{d} > 0$.

Highest weight states of $\hat{sl}(2|1;\mathbb{C})_k$ are characterised by their isospin $\frac{1}{2}h_-$, charge $\frac{1}{2}h_+$ and conformal weight h:

$$|\Lambda\rangle = |\tfrac{1}{2}h_-, \tfrac{1}{2}h_+, h\rangle \tag{4.7}$$

where

$$J_0^3|\Lambda\rangle = \frac{1}{2}h_-|\Lambda\rangle, \quad U_0|\Lambda\rangle = \frac{1}{2}h_+|\Lambda\rangle, \quad L_0|\Lambda\rangle = h|\Lambda\rangle. \tag{4.8}$$

$|\Lambda\rangle$ is annihilated by all the raising operators, which we can see to be simply the elements of $\hat{\mathfrak{g}}^+$. This condition is equivalent to the following three:

$$j_0^+|\Lambda\rangle = j_0'^+|\Lambda\rangle = J_1^-|\Lambda\rangle = 0 \tag{4.9}$$

and corresponds to a particular choice of simple roots of $sl(2|1;\mathbb{C})$ (see section 2.5).

The relationship between quantum numbers h, h_- and h_+ is given by

$$h = \frac{1}{4(k+1)}(h_-^2 - h_+^2) \tag{4.10}$$

where k is the level of the representation concerned (taken to be $-1/2$ for the bulk of this chapter). The Kac-Khazdan determinant formula dictates that the Verma module built on a highest weight state with certain specific h_- and h_+ values will contain singular vectors, a full analysis of which was carried out in [28] and [29]. We shall make extensive use of the results of these references for the $k = -1/2$ case.

4.3 Fields and States

In section 2.3 we saw that, in conformal field theory, the link between highest weight states and primary fields is given by the state-field correspondence (2.49), namely

$$\lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle = |\Lambda\rangle, \tag{4.11}$$

where $|0\rangle$ denotes the vacuum state of the theory. It has been noted by several authors—in, for example, [53, 54, 55, 56, 57, 58, 59]—that for the case of admissible $\hat{sl}(2)$ (and $\hat{sl}(3)$ [22]) representations dependence on a second $sl(2)$ coordinate should be added. This overcomes the difficulty in describing these infinite-dimensional representations, which are in general neither highest nor lowest weight. Various ways of seeing how this difficulty arises exist. When considering a free field approach, it is related to the fact that screening currents are required to involve rational powers of fields [55]. Most straightforwardly, as stated in [59], in the $\hat{sl}(2)$ case the conformal weight of a state $|j\rangle$ (created by the primary field $\phi_j(z)$) is given by $h_j = j(j+1)/(k+2)$. This is equal to h_{-j-1} and so operator product expansions are defined up to an identification of j and $-j-1$, which can be overcome by the introduction of an additional parameter. For the case of integrable representations, the operator algebra closes with $0 \leqslant j \leqslant k/2$ where the level $k \in \mathbb{N}$ and this is not an issue.

The case of $\widehat{osp}(1|2)$ was studied in [24] where additionally a Grassmann coordinate θ was introduced to represent the supersymmetry present. For the case of $\hat{sl}(2|1;\mathbb{C})_k$, with two supersymmetries, it is necessary to consider not only the coordinate z, but also an additional $sl(2)$ coordinate x as well as two Grassmann coordinates θ and $\bar{\theta}$. This approach has been formalised for superalgebras in [62], which we essentially follow (the definitions of Ennes and Ramallo [24] for fields and states are the same, arrived at in a different framework). Anti-holomorphic counterparts of all these coordinates should also be included, although here we suppress any dependence on these coordinates for clarity. Thus in our case, the state-field correspondence is given by

$$\lim_{z,x,\theta,\bar{\theta}\to 0} \phi_a(z,x,\theta,\bar{\theta})\,|0\rangle = |\Lambda_a\rangle. \tag{4.12}$$

We will see shortly that it is possible to represent the action of the $\hat{sl}(2|1;\mathbb{C})$ modes on primary fields through the action of certain differential operators, using the commutation relations

$$\begin{aligned}
[J_n^\alpha, \phi_a(z,x,\theta,\bar{\theta})] &= z^n D_a^\alpha \phi_a(z,x,\theta,\bar{\theta}),\\
[j_n^\alpha, \phi_a(z,x,\theta,\bar{\theta})] &= z^n d_a^\alpha \phi_a(z,x,\theta,\bar{\theta}),\\
[j_n^{'\alpha}, \phi_a(z,x,\theta,\bar{\theta})] &= z^n d_a^{'\alpha} \phi_a(z,x,\theta,\bar{\theta}),\\
[U_n, \phi_a(z,x,\theta,\bar{\theta})] &= z^n D_a^U \phi_a(z,x,\theta,\bar{\theta}).
\end{aligned} \tag{4.13}$$

It should be noted that for the case of the fermionic modes j_n^α, $j_n^{'\alpha}$ the commutator should be replaced by an anticommutator as appropriate: we always take highest weight states to be bosonic. In addition, we have the standard relation with Virasoro modes (2.29) given by

$$[L_n, \phi_a(z,x,\theta,\bar{\theta})] = \{z^{n+1}\partial_z + (n+1)h_a z^n\}\phi_a(z,x,\theta,\bar{\theta}). \tag{4.14}$$

We will also find the definition of the action of ϕ_a on the vacuum:

$$\phi_a(z,x,\theta,\bar{\theta})\,|0\rangle = e^{zL_{-1}+xJ_0^- +\theta j_0^- +\bar{\theta} j_0^{'-}}|\Lambda_a\rangle. \tag{4.15}$$

4.4 Differential Operators for $sl(2|1)$

In [62], Rasmussen applied longstanding techniques (see for example [63] and references therein) to discuss free field representations of affine superalgebras in the context of general (*i.e.* not just integrable) representations. The key to this approach, as mentioned in the previous section, is in extending the dependence of primary fields to additional coordinates. These arise in the context of the Wakimoto free field representation of affine algebras where the finite algebra is realised as differential operators on a polynomial ring $\mathbb{C}[x^{\alpha_i}]$ (as detailed, for example, in [64]), with a coordinate x^{α_i} for each positive root α_i of the finite algebra. In the case of superalgebras, which involve fermionic roots, one must introduce fermionic (Grassmann) coordinates. The finite superalgebra is then realised on the ring $\mathbb{C}[x^{\alpha_i}, \theta^{\dot{\alpha}_j}]$, with $\dot{\alpha}_j$ denoting the positive fermionic roots. The full affine currents are defined by expressions involving free fields and these differential operators representing the finite (super)algebra.

Above we mentioned the triangular decomposition of affine superalgebras: this arises from the corresponding decomposition of finite superalgebras, where

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+ = (\mathfrak{g}_0^- \oplus \mathfrak{g}_1^-) \oplus \mathfrak{g}^0 \oplus (\mathfrak{g}_0^+ \oplus \mathfrak{g}_1^+). \tag{4.16}$$

Here, the subspaces of lowering and raising operators are split into even and odd parts (recall $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ for superalgebras, where even elements have degree 0 and odd elements have degree 1). Elements in the superalgebra may be parametrised by coordinates x^{α_i} and $\theta^{\dot{\alpha}_j}$ in the following way [62]:

$$g^+(x^{\alpha_i}, \theta^{\dot{\alpha}_j}) = \sum_{i,j} x^{\alpha_i} E^{\alpha_i} + \theta^{\dot{\alpha}_j} e^{\dot{\alpha}_j}, \qquad g^-(x^{\alpha_i}, \theta^{\dot{\alpha}_j}) = \sum_{i,j} x^{\alpha_i} F^{\alpha_i} + \theta^{\dot{\alpha}_j} f^{\dot{\alpha}_j} \tag{4.17}$$

where

$$E^{\alpha_i} \in \mathfrak{g}_0^+, \quad e^{\dot{\alpha}_j} \in \mathfrak{g}_1^+, \quad F^{\alpha_i} \in \mathfrak{g}_0^-, \quad f^{\dot{\alpha}_j} \in \mathfrak{g}_1^-. \tag{4.18}$$

Now states of the Verma module are defined as

$$e^{g^-(x^{\alpha_i}, \theta^{\dot{\alpha}_j})}|\Lambda\rangle = e^{\sum_{i,j} x^{\alpha_i} F^{\alpha_i} + \theta^{\dot{\alpha}_j} f^{\dot{\alpha}_j}}|\Lambda\rangle, \tag{4.19}$$

with the dual states given correspondingly in terms of g^+. The highest weight states should as usual be created by the action of a primary field on the vacuum: we should additionally have a dependence on the original variable z, which may be included by incorporating the translation (in z) operator L_{-1} into the exponential [61], giving rise to the relation

$$\phi_a(z, x^{\alpha_i}, \theta^{\dot{\alpha}_j})\,|0\rangle = e^{zL_{-1} + \sum_{i,j} x^{\alpha_i} F^{\alpha_i} + \theta^{\dot{\alpha}_j} f^{\dot{\alpha}_j}}|\Lambda_a\rangle. \tag{4.20}$$

In the case of $sl(2|1;\mathbb{C})$, the lowering operators (elements of $\mathfrak{g}^-$) corresponding to the negative roots $-(\alpha_1 + \alpha_2)$, $-\alpha_1$ and $-\alpha_2$ (as described in section 2.5) are

J_0^-, j_0^- and $j_0^{\prime -}$, where we have included the zero mode index to make contact with the affine generators. Hence we arrive at the advertised relation (4.15):

$$\phi_a(z,x,\theta,\bar\theta)\,|0\rangle = e^{zL_{-1}+xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle. \tag{4.21}$$

On the states $e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle$ the $sl(2|1;\mathbb{C})$ generators act as differential operators:

$$X_0 e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle = D^X e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle. \tag{4.22}$$

We may write

$$e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} X_0 e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle = e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} D^X e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle, \tag{4.23}$$

where D^X is the differential operator corresponding to the $sl(2|1;\mathbb{C})$ generator X_0. Differentiation of the left hand side with respect to x and application of the relations (4.2) gives rise to the following conjugation formulae:

$$\begin{aligned}
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} J_0^3 e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (J_0^3 - xJ_0^- - \frac{1}{2}\theta j_0^- - \frac{1}{2}\bar\theta j_0^{\prime -})|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} J_0^+ e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (2xJ_0^3 - x^2 J_0^- - x\theta j_0^- - x\bar\theta j_0^{\prime -} + \theta\bar\theta U_0)|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} J_0^- e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= J_0^-|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} j_0^+ e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (-xj_0^{\prime -} + x\theta J_0^- - \theta(J_0^3+U_0) + \frac{1}{2}\theta\bar\theta j_0^{\prime -})|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} j_0^- e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (j_0^- - \bar\theta J_0^-)|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} j_0^{\prime +} e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (xj_0^- - x\bar\theta J_0^- + \bar\theta(J_0^3-U_0) + \frac{1}{2}\theta\bar\theta j_0^-)|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} j_0^{\prime -} e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (j_0^{\prime -} - \theta J_0^-)|\Lambda_a\rangle,\\
e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}} U_0 e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle &= (U_0 + \frac{1}{2}\theta j_0^- - \frac{1}{2}\bar\theta j_0^{\prime -} - \frac{1}{2}\theta\bar\theta J_0^-)|\Lambda_a\rangle,
\end{aligned} \tag{4.24}$$

where we have eliminated those terms which annihilate $|\Lambda_a\rangle$. For example, in the case of J_0^3, differentiating once with respect to x yields

$$e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}}[J_0^3, J_0^-]e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle = e^{-xJ_0^--\theta j_0^--\bar\theta j_0^{\prime -}}(-J_0^-)e^{xJ_0^-+\theta j_0^-+\bar\theta j_0^{\prime -}}|\Lambda_a\rangle \tag{4.25}$$

which differentiated again gives 0. The overall expression is therefore linear in x. Setting $x=0$ in $e^{-xJ_0^- - \theta j_0^- - \bar{\theta} j_0^{'-}} J_0^3 e^{xJ_0^- + \theta j_0^- + \bar{\theta} j_0^{'-}}$ and expanding gives

$$e^{-\theta j_0^- - \bar{\theta} j_0^{'-}} J_0^3 e^{\theta j_0^- + \bar{\theta} j_0^{'-}} =$$
$$(1 - \theta j_0^- - \bar{\theta} j_0^{'-} - \theta\bar{\theta} j_0^- j_0^{'-} + \frac{1}{2}\theta\bar{\theta} J_0^-) J_0^3 (1 + \theta j_0^- + \bar{\theta} j_0^{'-} - \theta\bar{\theta} j_0^- j_0^{'-} + \frac{1}{2}\theta\bar{\theta} J_0^-)$$
$$= J_0^3 - \frac{1}{2}\theta j_0^- - \frac{1}{2}\bar{\theta} j_0^{'-}, \quad (4.26)$$

which is therefore the constant term. Setting $x=0$ in (4.25) yields

$$e^{-\theta j_0^- - \bar{\theta} j_0^{'-}} (-J_0^-) e^{\theta j_0^- + \bar{\theta} j_0^{'-}} = -J_0^- \quad (4.27)$$

since J_0^- commutes with j_0^- and $j_0^{'-}$, so this is the x term and we arrive at the stated result.

Comparing the conjugation formulae (4.24) with the result of calculating

$$e^{-xJ_0^- - \theta j_0^- - \bar{\theta} j_0^{'-}} (a\partial_x + b\theta\partial_x + c\bar{\theta}\partial_x + d\partial_\theta + e\theta\partial_\theta + f\bar{\theta}\partial_\theta$$
$$+ g\partial_{\bar{\theta}} + h\theta\partial_{\bar{\theta}} + l\bar{\theta}\partial_{\bar{\theta}} + m\theta\bar{\theta}\partial_\theta + n\theta\bar{\theta}\partial_{\bar{\theta}}) e^{xJ_0^- + \theta j_0^- + \bar{\theta} j_0^{'-}} |\Lambda_a\rangle =$$
$$(aJ_0^- + b\theta J_0^- + c\bar{\theta} J_0^- + d j_0^- - \frac{1}{2} d\bar{\theta} J_0^- + e\theta j_0^- - \frac{1}{2} e\theta\bar{\theta} J_0^-$$
$$+ f\bar{\theta} j_0^- + g j_0^{'-} - \frac{1}{2} g\theta J_0^- + h\theta j_0^{'-} + l\bar{\theta} j_0^{'-} + \frac{1}{2} l\theta\bar{\theta} J_0^- + m\theta\bar{\theta} j_0^- + n\theta\bar{\theta} j_0^{'-})|\Lambda_a\rangle$$
$$(4.28)$$

we obtain the following differential operator realisation of $sl(2|1;\mathbb{C})$:

$$\begin{aligned}
J_0^3 &\to D^3 = -x\partial_x - \frac{1}{2}\theta\partial_\theta - \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}} + \frac{1}{2}h_-,\\
J_0^+ &\to D^+ = -x^2\partial_x - x\theta\partial_\theta - x\bar{\theta}\partial_{\bar{\theta}} + xh_- + \frac{1}{2}\theta\bar{\theta}h_+,\\
J_0^- &\to D^- = \partial_x,\\
j_0^+ &\to d^+ = -x\partial_{\bar{\theta}} + \frac{1}{2}x\theta\partial_x - \frac{1}{2}\theta(h_- + h_+) + \frac{1}{2}\theta\bar{\theta}\partial_{\bar{\theta}},\\
j_0^- &\to d^- = \partial_\theta - \frac{1}{2}\bar{\theta}\partial_x,\\
j_0^{'+} &\to d^{'+} = x\partial_\theta - \frac{1}{2}x\bar{\theta}\partial_x + \frac{1}{2}\bar{\theta}(h_- - h_+) + \frac{1}{2}\theta\bar{\theta}\partial_\theta,\\
j_0^{'-} &\to d^{'-} = \partial_{\bar{\theta}} - \frac{1}{2}\theta\partial_x,\\
U_0 &\to D^U = \frac{1}{2}\theta\partial_\theta - \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}} + \frac{1}{2}h_+.
\end{aligned} \quad (4.29)$$

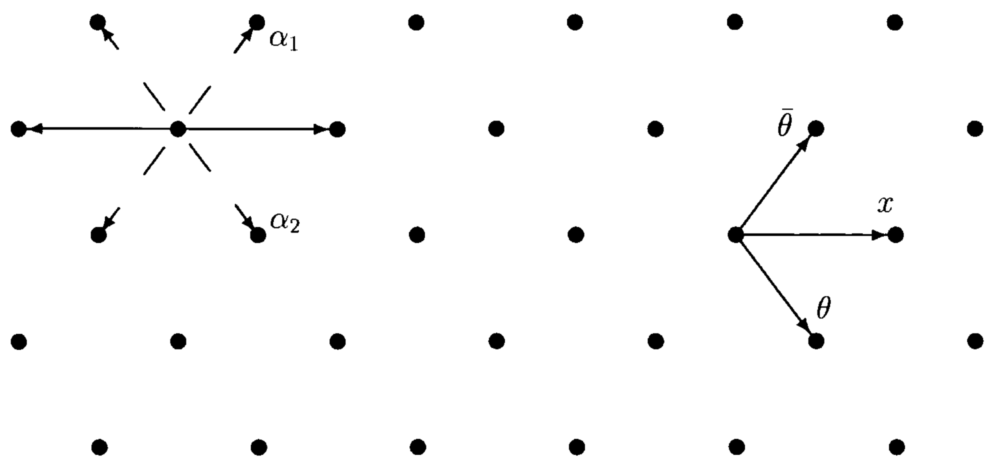

Figure 4.1: The action of x, θ and $\bar{\theta}$ in root space.

The operator product expansion between $\hat{sl}(2|1;\mathbb{C})$ currents and primary fields is given by the expression (compare (2.45))

$$X(z)\phi_a(w,x,\theta,\bar{\theta}) = \frac{D_a^X}{z-w}\phi_a(w,x,\theta,\bar{\theta}) + \ldots . \tag{4.30}$$

With the expansion of the currents $X(z) = \sum_n X_n z^{-n-1}$ (as in (2.88), bearing in mind that we are restricting to the Ramond sector) we find that

$$\begin{aligned}[X_n, \phi_a(w,x,\theta,\bar{\theta})] &= \frac{1}{2\pi i}\oint dz \frac{z^n D_a^X}{z-w}\phi_a(w,x,\theta,\bar{\theta}) \\ &= w^n D_a^X \phi_a(w,x,\theta,\bar{\theta}),\end{aligned} \tag{4.31}$$

which is the result already described in (4.13).

4.5 $sl(2|1)$ Invariant 3-point Function

In order to discover $\hat{sl}(2|1;\mathbb{C})_k$ fusion rules, we wish to consider quantities such as

$$\langle \Lambda_1^* | \phi_2(z,x,\theta,\bar{\theta}) | \omega\Lambda_3\rangle, \tag{4.32}$$

where $\omega\Lambda_3$ is a singular vector. Then (4.32) will be equal to zero and we may rewrite this expression as

$$(\text{differential operators})\langle \Lambda_1^* | \phi_2(z,x,\theta,\bar{\theta}) | \Lambda_3\rangle = 0 \tag{4.33}$$

using the relations (4.13). Evaluating (4.33) we then obtain conditions on possible quantum numbers of ϕ_2 and ϕ_1^* given those of ϕ_3, amounting to a specification of allowed fusings (the appearance of the conjugate field ϕ_1^* will be addressed shortly). This procedure relies on knowledge of $\hat{sl}(2|1;\mathbb{C})_k$ singular vectors, which we have from [28], and knowledge of the $sl(2|1;\mathbb{C})$ invariant 3-point function, which we now move on to discuss.

The differential operator realisation (4.29) allows us to determine the $sl(2|1;\mathbb{C})$ invariant 3-point function (for the variables x, θ and $\bar{\theta}$—the z dependence is standard), which is in fact the 3-point function for $N = 2$ superconformal field theory, since $sl(2|1;\mathbb{C})$ is isomorphic to the set of generators of the $N = 2$ super-Möbius group. This has been considered by several authors [48, 49, 50, 51], the interrelation of whose work will be clarified here. We note that our discussion of the Ramond sector directly parallels the discussion of the Neveu-Schwarz sector in these (and other) works on superconformal field theory: the algebra (4.2) has vanishing central piece for the Ramond sector zero mode subalgebra, whereas the equivalent subalgebra in the usual superconformal discussion is in the Neveu-Schwarz sector. Indeed, there it is Neveu-Schwarz generators which give rise to the super-Möbius group, as discussed in [49]. Although our terminology derives from the fact that we are considering the situation where the fermionic modes have integer labels, when we discuss Ramond fields they are in fact more akin to Neveu-Schwarz fields, in the sense that they do not introduce branch cuts in the operator product expansion with fermionic currents: our fermionic currents are expanded as, for example, $J(\mathrm{e}_{\pm\alpha_1})(z) = \sum_n j_n'^{\pm} z^{-n-1}$ whereas a typical fermionic current in superconformal field theory is $G(z) = \sum_n G_n z^{-n-3/2}$. This relative difference between the mode numbers and powers of z in the mode expansions means that the behaviour of operator products in the Ramond (Neveu-Schwarz) sector of superconformal field theory is like that of the Neveu-Schwarz (Ramond) sector in the present situation. As we will find that the fusion of two Ramond fields gives rise to another Ramond field, this interpretation means that our results are not in conflict with those of (for example) [65] for the $N = 1$ superconformal case and [48] for $N = 2$, where it is discussed how the fusion of two Ramond fields produces a Neveu-Schwarz field, whereas the fusion of two Neveu-Schwarz fields gives another Neveu-Schwarz field.

For a 3-point function to be $sl(2|1;\mathbb{C})$ invariant, we require that

$$\langle 0|[X_0,\phi_1]\phi_2\phi_3|0\rangle + \langle 0|\phi_1[X_0,\phi_2]\phi_3|0\rangle + \langle 0|\phi_1\phi_2[X_0,\phi_3]|0\rangle = 0 \tag{4.34}$$

for each of the $sl(2|1;\mathbb{C})$ generators X_0. Using the relations (4.13) yields

$$\sum_{i=1}^{3} D_i^X \langle 0|\phi_1(z_1,x_1,\theta_1,\bar{\theta}_1)\phi_2(z_2,x_2,\theta_2,\bar{\theta}_2)\phi_3(z_3,x_3,\theta_3,\bar{\theta}_3)|0\rangle = 0, \tag{4.35}$$

where D_a^X is the differential operator corresponding to X_0, taking its parameters h_+ and h_- from the primary field ϕ_a. This assumes that the vacuum $|0\rangle$ is annihilated by elements of $sl(2|1;\mathbb{C})$. As previously related in section 1.2, in conformal field theory solving the resulting differential equations determines the 3-point function exactly. In the $N = 2$ superconformal case, the 3-point function can depend on the nine variables x_i, θ_i and $\bar{\theta}_i$. Finding a 3-point function which satisfies the differential equations arising from the generators J_0^+, j_0^- and $j_0^{'-}$ will result in the automatic solution of the remaining equations, from the commutation relations (4.2). With nine parameters but only three independent equations available (although we make use of five equations for simplicity), there will naturally be some ambiguity in the final answer. With this in mind, we proceed with our analysis, closely following the above references and particularly [49]. We concentrate on the variables x, θ and $\bar{\theta}$, the z dependence being straightforward to consider.

From the equations for J_0^-, j_0^- and $j_0^{'-}$, it is clear that the 3-point function depends on the following variables:

$$s_{ij} = x_i - x_j - \frac{1}{2}\theta_i\bar{\theta}_j - \frac{1}{2}\bar{\theta}_i\theta_j, \quad \theta_{ij} = \theta_i - \theta_j, \quad \bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j, \quad i,j = 1,2,3. \tag{4.36}$$

Then from the U_0 equation (suppressing the z dependence of the fields for ease of notation)

$$\sum_{i=1}^{3}(\frac{1}{2}\theta_i\partial_{\theta_i} - \frac{1}{2}\bar{\theta}_i\partial_{\bar{\theta}_i} + \frac{1}{2}h_{+,i})\langle 0|\phi_1(x_1,\theta_1,\bar{\theta}_1)\phi_2(x_2,\theta_2,\bar{\theta}_2)\phi_3(x_3,\theta_3,\bar{\theta}_3)|0\rangle = 0 \tag{4.37}$$

we find that there are several distinct cases to be examined, that is, for which

$$H_+ = \sum_{i=1}^{3} h_{+,i} = 0, \pm 1, \pm 2. \tag{4.38}$$

To give an indication of how this comes about, note that we may only have terms involving up to $\theta_{12}\theta_{23}\bar{\theta}_{12}\bar{\theta}_{23}$, since $\theta_{13} = \theta_{12} + \theta_{23}$ and similarly for $\bar{\theta}_{13}$. When the differential piece $\sum \theta_i \partial_{\theta_i} - \bar{\theta}_i \partial_{\bar{\theta}_i}$ in the U_0 equation acts on $(s_{ij})^n$, it gives zero so it is only the θ_{ij} and $\bar{\theta}_{kl}$ pieces which are of concern. Terms involving an odd number of θ_{ij}s and $\bar{\theta}_{jk}$s require $H_+ = \pm 1$, while terms such as $\theta_{ij}\bar{\theta}_{kl}$, $\theta_{ij}\theta_{kl}\bar{\theta}_{mn}\bar{\theta}_{pq}$ and the constant term require $H_+ = 0$. The only remaining options are $\theta_{12}\theta_{23}$ and the barred counterpart, which require $H_+ = 2$ and $H_+ = -2$, respectively. However, consideration of the J_0^+ condition eliminates the cases where $H_+ = \pm 2$, so there are in fact three distinct 3-point functions to be obtained. It remains to solve the J_0^+ equation subject to the conditions $H_+ = 0, \pm 1$.

Through the use of an ansatz based on the results of [48] and [51], for the case $H_+ = 0$, we find (again suppressing the z dependence)

$$\begin{aligned} \langle 0|\phi_1\phi_2\phi_3|0\rangle = C_{123} s_{12}^{a_3} s_{23}^{a_1} s_{13}^{a_2} \Bigg[1 &+ \frac{h_{+,1}\theta_{12}\bar{\theta}_{12}}{2s_{12}} \\ &+ \frac{(h_{+,1}+h_{+,2})\theta_{23}\bar{\theta}_{23}}{2s_{23}} + \frac{h_{+,1}(h_{+,1}+h_{+,2})\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{4s_{12}s_{23}} \\ &+ \alpha\frac{\theta_{12}\bar{\theta}_{12}s_{23}}{s_{12}s_{13}} - \alpha\frac{(\theta_{12}\bar{\theta}_{23}+\theta_{23}\bar{\theta}_{12})}{s_{13}} + \alpha\frac{\theta_{23}\bar{\theta}_{23}s_{12}}{s_{23}s_{13}} \\ &+ \alpha\frac{h_{+,1}\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{2s_{23}s_{13}} + \alpha\frac{(h_{+,1}+h_{+,2})\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{2s_{12}s_{13}} \Bigg], \end{aligned} \tag{4.39}$$

where α is an undetermined parameter and $a_1 = \frac{1}{2}(h_{-,2}+h_{-,3}-h_{-,1})$, $a_2 = \frac{1}{2}(h_{-,1}+h_{-,3}-h_{-,2})$ and $a_3 = \frac{1}{2}(h_{-,1}+h_{-,2}-h_{-,3})$. This is essentially the answer of [48] and [51], though the authors in both cases have additional (different) restrictions on α involving the parameters h_-. It is straightforward to see that this is not the case. Writing the J_0^+ equation for the above answer schematically as

$$\sum_i (D - h_{-,i}x_i - \frac{1}{2}h_{+,i}\theta_i\bar{\theta}_i)(sss)[\ldots] = 0 \tag{4.40}$$

we require

$$D(sss)[\dots] + (sss)D[\dots] = \sum_i (h_{-,i}x_i + \frac{1}{2}h_{+,i}\theta_i\bar{\theta}_i)(sss)[\dots]. \tag{4.41}$$

Now $D(s_{ij})^a = a(x_i + x_j)(s_{ij})^a$ and so the term $D(s_{12}^{a_3}s_{23}^{a_1}s_{13}^{a_2})[\dots]$ is precisely equal to the term $\sum h_{-,i}x_i(sss)[\dots]$ on the right hand side, from the definition of a_i. We may cancel (sss) in the remaining terms and see that there is no more involvement of the parameters h_-, which therefore cannot enter into the undetermined parameter α.

The system of equations given by (4.35) is invariant under permutations of 1,2 and 3: this is not reflected in the solution (4.39) and indeed any permutation of these labels will also give a solution. If we interchange 1 and 3 (keeping the same value of α) and add the resulting answer to the one above, we obtain a solution

$$\begin{aligned}\langle 0|\phi_1\phi_2\phi_3|0\rangle = 2C_{123}s_{12}^{a_3}s_{23}^{a_1}s_{13}^{a_2}\Bigg[1 + \frac{h_{+,1}\theta_{12}\bar{\theta}_{12}}{2s_{12}} + \frac{(h_{+,1}+h_{+,2})\theta_{23}\bar{\theta}_{23}}{2s_{23}} \\ + \frac{h_{+,1}(h_{+,1}+h_{+,2})\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{4s_{12}s_{23}}\Bigg],\end{aligned} \tag{4.42}$$

where we have used the fact that $H_+ = 0$ and taken $C_{123} = C_{321}$. Strictly speaking, this is not the exact answer obtained by this procedure, since $s_{ij} = -s_{ji}$ means that the permuted answer differs from (4.39) by a factor $(-1)^{-a_1-a_3-a_2}$. However, when the full dependence on anti-holomorphic variables is included, the overall multiplicative factor in (4.39) is modified to $C_{123}|s_{12}|^{-2a_3}|s_{23}|^{-2a_1}|s_{13}|^{-2a_2}$ (with $h_{-,i} = \bar{h}_{-,i}$) and this discrepancy disappears. The expression (4.42) is in fact a particular case of the general solution obtained by Kiritsis [50]. Understanding (4.39) as written for the labelling $\{123\}$ we find that the solution obtained by adding (4.39) written with $\{213\}$ to that with $\{312\}$ is also a particular Kiritsis solution, as is the expression resulting from the addition of $\{132\}$ to $\{231\}$. When all six versions of (4.39) are added together, the solution obtained is precisely that given by Howe and West [49], which is again a specific instance of the solution described by Kiritsis, distinguished by the fact that it is a permutation invariant solution of the equations (4.35). Before going on to clarify this situation, we consider the other 3-point functions for the cases $H_+ = \pm 1$.

When $H_+ = -1$, we find that

$$\langle 0|\phi_1\phi_2\phi_3|0\rangle = C'_{123}s_{12}^{a_3}s_{23}^{a_1}s_{13}^{a_2-1/2}\Bigg[\frac{\theta_{12}}{s_{12}^{1/2}s_{23}^{-1/2}} - \frac{\theta_{23}}{s_{12}^{-1/2}s_{23}^{1/2}} - \frac{(h_{+,1}\theta_{23}\theta_{12}\bar{\theta}_{12} + (1 - h_{+,1} - h_{+,2})\theta_{12}\theta_{23}\bar{\theta}_{23})}{2s_{12}^{1/2}s_{23}^{1/2}}\Bigg]; \quad (4.43)$$

when $H_+ = 1$ we have

$$\langle 0|\phi_1\phi_2\phi_3|0\rangle = C''_{123}s_{12}^{a_3}s_{23}^{a_1}s_{13}^{a_2-1/2}\Bigg[\frac{\bar{\theta}_{12}}{s_{12}^{1/2}s_{23}^{-1/2}} - \frac{\bar{\theta}_{23}}{s_{12}^{-1/2}s_{23}^{1/2}} - \frac{(h_{+,1}\bar{\theta}_{23}\theta_{12}\bar{\theta}_{12} + (-1 - h_{+,1} - h_{+,2})\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23})}{2s_{12}^{1/2}s_{23}^{1/2}}\Bigg]. \quad (4.44)$$

These are identical to the expressions given in [51] and corrected from [48]. Again, they are not invariant under permutations of the field labels. However, we find that (with the proviso discussed above that anti-holomorphic coordinates should be included) the form of these expressions for $\{123\}$ is the same as that for $\{321\}$, *etc.* Indeed, if we sum the resulting three variants in each case, we obtain the solutions found by Howe and West [49].

To proceed with the calculation of fusion rules, we note that the expression (4.33) is in terms of highest weight states rather than fields acting on the vacuum. We have the definition (4.12) to give us the highest weight in-state. The global superconformal transformations may be found by exponentiating the generators to be [50]

$$\begin{aligned}
x' &= \frac{ax+b}{cx+d} + \frac{e^q\theta((1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\bar{\epsilon}_1 x + (1+\frac{1}{2}\epsilon_2\bar{\epsilon}_1)\bar{\epsilon}_2)}{(cx+d)^2} \\
&\quad + \frac{e^{-q}\bar{\theta}((1+\frac{1}{2}\epsilon_2\bar{\epsilon}_1)\epsilon_1 x + (1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_2)}{(cx+d)^2} \\
&\quad + \frac{\theta\bar{\theta}((2d\epsilon_1\bar{\epsilon}_1 - c(\epsilon_1\bar{\epsilon}_2+\epsilon_2\bar{\epsilon}_1))x + d(\epsilon_1\bar{\epsilon}_2+\epsilon_2\bar{\epsilon}_1) - 2c\epsilon_2\bar{\epsilon}_2)}{(cx+d)^3}, \\
\theta' &= \frac{\epsilon_1 x+\epsilon_2}{cx+d} + \frac{e^q\theta(1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1-\epsilon_1\bar{\epsilon}_2) - \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2)}{cx+d} + \frac{\theta\bar{\theta}(d\epsilon_1 - c\epsilon_2)}{(cx+d)^2}, \\
\bar{\theta}' &= \frac{\bar{\epsilon}_1 x+\bar{\epsilon}_2}{cx+d} + \frac{e^{-q}\bar{\theta}(1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1-\epsilon_1\bar{\epsilon}_2) - \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2)}{cx+d} - \frac{\theta\bar{\theta}(d\bar{\epsilon}_1 - c\bar{\epsilon}_2)}{(cx+d)^2},
\end{aligned} \quad (4.45)$$

here corrected from [50] by checking that these transformations are superanalytic. In these expressions, a, b, c and d are the $SL(2)$ parameters $|ad - bc| = 1$, ϵ and $\bar{\epsilon}$ are anticommuting parameters associated with the supersymmetry transformations and q with the transformation arising from U_0.

For a suitable definition of out-state $\langle\Lambda|$, we wish to take $x \to \infty$ via the transformation

$$x' = \frac{1}{x}. \tag{4.46}$$

For the global transformation to be of this form, we require that $\epsilon_1 = \epsilon_2 = \bar{\epsilon}_1 = \bar{\epsilon}_2 = 0$. Consequently, the transformations for θ and $\bar{\theta}$ are given by

$$\begin{aligned} \theta' &= \frac{e^q\theta}{x}, \\ \bar{\theta}' &= \frac{e^{-q}\bar{\theta}}{x}. \end{aligned} \tag{4.47}$$

The point $(0,0,0)$ is mapped to $(\infty,0,0)$ as its natural inverse, which is then the limit for the out-state

$$\langle\Lambda| = \lim_{\substack{x\to 0\\ \theta=\bar{\theta}=0}} x^{h_-}\langle 0|\phi\left(\frac{1}{x},\frac{\theta}{x},\frac{\bar{\theta}}{x}\right). \tag{4.48}$$

The factor x^{h_-} arises from the transformation law for superprimary fields [50]

$$\tilde{\phi}(x,\theta,\bar{\theta}) = \phi(x',\theta',\bar{\theta}')[(\partial_\theta + \frac{1}{2}\bar{\theta}\partial_x)\theta']^{(-h_- - h_+)/2}[(\partial_{\bar{\theta}} + \frac{1}{2}\theta\partial_x)\bar{\theta}']^{(-h_- + h_+)/2} \tag{4.49}$$

the factor in which reduces to x^{h_-} for the transformation given, evaluated at $\theta = \bar{\theta} = 0$ and with the choice $q = 0$.

Alternatively, we may consider the expansion of a superprimary field as given by

$$\phi(x,\theta,\bar{\theta}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}(x) + \theta\bar{\theta}g(x). \tag{4.50}$$

We see that with our previous definition of in-state (4.12) we have

$$\varphi(0)|0\rangle = |\Lambda\rangle, \tag{4.51}$$

so then

$$\langle\Lambda| = \lim_{x\to\infty}\langle 0|\varphi(x)x^{-h_-} \tag{4.52}$$

with again $\theta = \bar{\theta} = 0$.

With the definition of out-state established, we arrive at the form of 3-point function which we will use for our calculations of fusion rules. For the even case (4.39) for which $h_{+,1} + h_{+,2} + h_{+,3} = 0$, the result is

$$\langle \Lambda_1 | \phi(z_2, x_2, \theta_2, \bar{\theta}_2) | \Lambda_3 \rangle = C_{123} z_2^{h_1 - h_2 - h_3} x_2^{(h_{-,2} + h_{-,3} - h_{-,1})/2} \left[1 - \frac{(h_{+,3} - 2\alpha)}{2x_2} \theta_2 \bar{\theta}_2 \right]. \tag{4.53}$$

For the odd case (4.43) for which $h_{+,1} + h_{+,2} + h_{+,3} = -1$ we find

$$\langle \Lambda_1 | \phi(z_2, x_2, \theta_2, \bar{\theta}_2) | \Lambda_3 \rangle = \tilde{C}'_{123} z_2^{h_1 - h_2 - h_3} x_2^{(h_{-,2} + h_{-,3} - h_{-,1} - 1)/2} \theta_2 \tag{4.54}$$

and the other odd case (4.44), where $h_{+,1} + h_{+,2} + h_{+,3} = 1$, becomes

$$\langle \Lambda_1 | \phi(z_2, x_2, \theta_2, \bar{\theta}_2) | \Lambda_3 \rangle = \tilde{C}''_{123} z_2^{h_1 - h_2 - h_3} x_2^{(h_{-,2} + h_{-,3} - h_{-,1} - 1)/2} \bar{\theta}_2. \tag{4.55}$$

In the above, the z dependence is determined, as usual, by taking the commutator with L_0:

$$\langle 0 | [L_0, \phi_1] \phi_2 \phi_3 | 0 \rangle + \langle 0 | \phi_1 [L_0, \phi_2] \phi_3 | 0 \rangle + \langle 0 | \phi_1] \phi_2 [L_0] \phi_3 | 0 \rangle = 0 \tag{4.56}$$

which means that (from (4.14))

$$\sum_{i=1}^{3} (z_i \partial_{z_i} + h_i) \langle 0 | \phi_1 \phi_2 \phi_3 | 0 \rangle = 0. \tag{4.57}$$

As the z dependence will not influence our discussion of fusion rules, we shall generally omit it in what follows.

We should mention at this point that in the limit discussed above, where $x_1 \to \infty$, $x_3 = 0$ and $\theta_1 = \bar{\theta}_1 = \theta_3 = \bar{\theta}_3 = 0$, the odd 3-point functions as given by Howe and West [49] reduce to the expressions (4.54) and (4.55). However, the expression obtained by this procedure from their even 3-point function differs from (4.53). We will show that (4.53) leads to sensible fusion rules, whereas use of the corresponding Howe and West expression only gives these in part. Beyond this, we can give no formal justification of why one might start from a non-permutation invariant expression for the 3-point function (which is thus intrinsically non-local).

We might also note that each of the possible ways of writing (4.39) leads to the same expression (4.53), that is

$$\lim_{\substack{z_i,x_i\to\infty \\ z_k=x_k=0 \\ \theta_{i,k}=\bar\theta_{i,k}=0}} z_i^{2h_i} x_i^{-h_{-,i}} \langle 0|\phi_i(z_i,x_i,\theta_i,\bar\theta_i)\phi_j(z_j,x_j,\theta_j,\bar\theta_j)\phi_k(z_k,x_k,\theta_k,\bar\theta_k)|0\rangle =$$

$$C_{ijk} z_j^{h_1-h_2-h_3} x_j^{(h_{-,j}+h_{-,k}-h_{-,i})/2}\left[1-\frac{(h_{+,k}-2\alpha)}{2x_j}\theta_j\bar\theta_j\right], \quad (4.58)$$

where $i,j,k = 1,2,3,\ i \neq j \neq k$.

4.6 Singular Vectors for $k = -1/2$

In the Ramond sector of $\hat{sl}(2|1;\mathbb{C})_k$ at level $k = -1/2$, there are four primary fields $\phi_{m,m'}$, $0 \leqslant m,m' \leqslant 1$ in the class IV and class V representations, corresponding to the $\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}$ characters described in the tables 3.2 and 3.3. As discussed in the previous chapter, these are the characters which close under modular transformations, motivating their relevance. In [28], the authors calculated general expressions for singular vectors using a Malikov-Feigin-Fuks type construction; we will make use of these expressions here. The embedding diagram describing singular vectors of these representations takes the form [29] of figure 4.2.

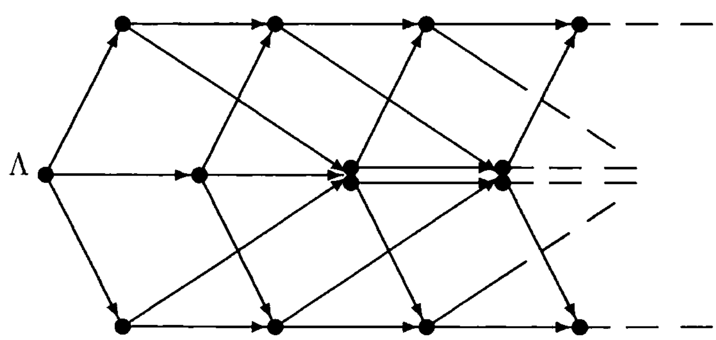

Figure 4.2: The embedding diagram for classes IV and V.

For each of the four highest weight representations specified by $|\Lambda_{m,m'}\rangle$ there are three singular vectors at the first level of the embedding diagram, from which conditions obtained through the calculation of (4.33) must be simultaneously satisfied. We shall list the appropriate form of singular vectors as given in [28] and go on to make use of them to calculate fusion rules in the next section.

For $|\Lambda_{0,0}\rangle$, with quantum numbers $h_- = h_+ = h = 0$, the three singular vectors are

$$
\begin{array}{ll}
(i) & j_0^- |\Lambda_{0,0}\rangle, \\
(ii) & j_0^{\prime -} |\Lambda_{0,0}\rangle, \\
(iii) & (J_{-1}^+)^{3/2}(j_0^- j_0^{\prime -} - j_0^{\prime -} j_0^-)(J_{-1}^+)^{1/2}|\Lambda_{0,0}\rangle.
\end{array}
\tag{4.59}
$$

For $|\Lambda_{1,0}\rangle$, $h_- = h_+ = -\frac{1}{2}$, $h = 0$ the singular vectors are

$$
\begin{array}{ll}
(i) & j_0^{\prime -} |\Lambda_{1,0}\rangle, \\
(ii) & J_{-1}^+ |\Lambda_{1,0}\rangle, \\
(iii) & (J_0^-)^{1/2}(j_0^- j_0^{\prime -} - 2 j_0^{\prime -} j_0^-) J_{-1}^+ j_0^- J_{-1}^+ (J_0^-)^{-1/2}(-j_0^{\prime -} j_0^-)|\Lambda_{1,0}\rangle.
\end{array}
\tag{4.60}
$$

The state $|\Lambda_{1,1}\rangle$ has $h_- = -\frac{1}{2}$, $h_+ = \frac{1}{2}$, $h = 0$ and singular vectors

$$
\begin{array}{ll}
(i) & j_0^- |\Lambda_{1,1}\rangle, \\
(ii) & J_{-1}^+ |\Lambda_{1,1}\rangle, \\
(iii) & (J_0^-)^{1/2}(3 j_0^- j_0^{\prime -} - J_0^-) J_{-1}^+ j_0^{\prime -} J_{-1}^+ (J_0^-)^{-3/2}(-j_0^- j_0^{\prime -} + J_0^-)|\Lambda_{1,1}\rangle
\end{array}
\tag{4.61}
$$

and for $|\Lambda_{0,1}\rangle$ with $h_- = 1$, $h_+ = 0$, $h = \frac{1}{2}$ the singular vectors are

$$
\begin{array}{ll}
(i) & (\frac{3}{2} j_{-1}^+ + j_0^{\prime -} J_{-1}^+)|\Lambda_{0,1}\rangle, \\
(ii) & (j_0^- j_0^{\prime -} - j_0^{\prime -} j_0^-)|\Lambda_{0,1}\rangle, \\
(iii) & (-\frac{3}{2} j_{-1}^{\prime +} + j_0^- J_{-1}^+)|\Lambda_{0,1}\rangle.
\end{array}
\tag{4.62}
$$

These expressions for singular vectors may be used in (4.32) to give expressions of the form (4.33), utilising the equations (4.13). The singular vectors generally involve fractional powers of generators, which may be rearranged using

$$
AB^a = \sum_{i=0}^{\infty} \binom{a}{i} B^{a-i} \overbrace{[\cdots[[A, B], B], \cdots, B]}^{i}
\tag{4.63}
$$

to give expressions with integer powers. For the purposes of calculation, it is more convenient to keep the expressions as they stand, using the techniques of fractional calculus [66] which allow the manipulation of these quantities involving fractional powers. We should modify (4.13) accordingly, using

$$\phi_j(x,\theta,\bar{\theta})(X_0)^a = \sum_{i=0}^{\infty} \binom{a}{i} (X_0)^{a-i}(-D_j^X)^i \phi_j(x,\theta,\bar{\theta}) = (X_0 - D_j^X)^a \phi_j(x,\theta,\bar{\theta}), \tag{4.64}$$

with an overall minus sign as required for the case of fermionic generators and a fermionic field ϕ_j. This results in the differential operators in (4.33) also involving fractional powers, in our case, of the differential operators corresponding to the generators J_0^- and J_0^+. To deal with this situation, we will make use of the following expressions in our calculations:

$$\begin{aligned}(D^-)^a\theta^\gamma\bar{\theta}^{\bar{\gamma}}x^b &= (\partial_x)^a\theta^\gamma\bar{\theta}^{\bar{\gamma}}x^b \\ &= \frac{\Gamma(b+1)}{\Gamma(b-a+1)}\theta^\gamma\bar{\theta}^{\bar{\gamma}}x^{b-a};\end{aligned} \tag{4.65}$$

$$\begin{aligned}(D^+)^a\theta^\gamma\bar{\theta}^{\bar{\gamma}}x^b &= (-x^2\partial_x - x\theta\partial_\theta - x\bar{\theta}\partial_{\bar{\theta}} + xh_- + \frac{1}{2}\theta\bar{\theta}h_+)^a\theta^\gamma\bar{\theta}^{\bar{\gamma}}x^b \\ &= \frac{\Gamma(h_- - b - \gamma - \bar{\gamma} + 1)}{\Gamma(h_- - b - a - \gamma - \bar{\gamma} + 1)}\theta^\gamma\bar{\theta}^{\bar{\gamma}}x^{b+a} \\ &\quad + \frac{ah_+\Gamma(h_- - b)}{2\Gamma(h_- - b - a + 1)}\theta^{\gamma+1}\bar{\theta}^{\bar{\gamma}+1}x^{b+a-1}.\end{aligned} \tag{4.66}$$

These expressions may be verified as holding for integer values of a, with the validity for fractional values of a following by analytic continuation.

4.7 Calculation of Fusions

The information presented in the above sections allows us now to calculate fusion rules. As we wish to calculate expressions of the form (4.33)

$$\langle \Lambda_1^* | \phi_2(z,x,\theta,\bar{\theta}) | \omega\Lambda_3 \rangle = 0, \tag{4.67}$$

we note that since these are equal to zero, the procedure for deriving (4.33) from (4.32) essentially amounts to replacing the generators by their corresponding differential operators, with any quantum numbers involved in those expressions being the ones associated to the field ϕ_2, through which we are commuting. We may ignore factors of z arising from those generators with mode numbers not equal to zero and the possibility of having to use anticommutators between fields and fermionic generators, since this only gives rise to an overall minus sign. Consider the singular vector (iii) of (4.59). Using (4.64) we have

$$\begin{aligned}\langle\Lambda_1^*|\phi_2(z,x,\theta,\bar\theta)(J_{-1}^+)^{3/2}(j_0^- j_0^{'-} - j_0^{'-} j_0^-)(J_{-1}^+)^{1/2}|\Lambda_3\rangle &= \\ \langle\Lambda_1^*|(J_{-1}^+ - z^{-1}D_2^+)^{3/2}(j_0^- - d_2^-)(j_0^{'-} - d_2^{'-}) - \\ (j_0^{'-} - d_2^{'-})(j_0^- - d_2^-)(J_{-1}^+ - z^{-1}D_2^+)^{1/2}\phi_2(z,x,\theta,\bar\theta)|\Lambda_3\rangle &= 0. \end{aligned} \quad (4.68)$$

This expression may be rearranged using (4.63), with the generators arising from this procedure all such that they annihilate the out-state, as can be seen from the mode numbers involved. The only remaining part is the piece made up of the derivative terms, with an overall factor involving powers of (-1) and z, which can be eliminated. The derivative terms may then be "unrearranged" to give the expression with fractional powers and we have

$$\begin{aligned}\langle\Lambda_1^*|\phi_2(x,\theta,\bar\theta)(J_{-1}^+)^{3/2}(j_0^- j_0^{'-} - j_0^{'-} j_0^-)(J_{-1}^+)^{1/2}|\Lambda_3\rangle = 0 \rightarrow \\ (D_2^+)^{3/2}(d_2^- d_2^{'-} - d_2^{'-} d_2^-)(D_2^+)^{1/2}\langle\Lambda_1^*|\phi_2(x,\theta,\bar\theta)|\Lambda_3\rangle = 0. \end{aligned} \quad (4.69)$$

The only instance where some care needs to be taken is for the case of $|\Lambda_{0,1}\rangle$, where there are singular vectors made up of a term involving one generator and a term involving two generators, which will lead to a minus sign difference on commuting with ϕ_2. It remains to apply each of the three singular vectors for each field to the three 3-point functions (4.53), (4.54) and (4.55).

The results obtained tell us which 3-point functions are permitted, which in turn gives the fusion rules. One slight subtlety is the appearance of a conjugate field in this calculation. The fusion rules will take the form

$$\phi_i \times \phi_j = \sum N_{ij}{}^k \phi_k = \sum N_{ijk}\phi_k^* \quad (4.70)$$

corresponding to the non-vanishing of the 3-point function $\langle\phi_k^*\phi_j\phi_i\rangle$. To obtain the correct fusion rules from the calculations ahead, we need to bear in mind that the conjugate quantum numbers for ϕ_1^* will appear. The representation conjugate to that labelled by the highest weight vector Λ is given by $-w_0\Lambda$, where w_0 is the longest element of the affine Weyl group. In the present situation, the Weyl group is just generated by the reflection in the plane perpendicular to the (affine) root $\alpha_1+\alpha_2$, as described in section 2.5. As an example, consider the case of $\Lambda_{1,0} = (-\frac{1}{4}(\alpha_1+\alpha_2) - \frac{1}{4}(\alpha_1-\alpha_2), -\frac{1}{2}, 0)$ (recalling the parametrisation of highest weight vectors (2.93)). For Weyl reflections with respect to roots associated to the zero mode subalgebra, only the non-affine part of Λ is affected. Then

$$\begin{aligned}
-(s_{(\alpha_1+\alpha_2)}\Lambda_{1,0}) &= \left(-\Lambda_{1,0} + 2\frac{\langle\Lambda_{1,0},\alpha_1+\alpha_2\rangle}{\langle\alpha_1+\alpha_2,\alpha_1+\alpha_2\rangle}(\alpha_1+\alpha_2), -\frac{1}{2}, 0\right) \\
&= \left(\frac{1}{2}\alpha_1 - \frac{\langle\alpha_1,\alpha_1+\alpha_2\rangle}{\langle\alpha_1+\alpha_2,\alpha_1+\alpha_2\rangle}(\alpha_1+\alpha_2), -\frac{1}{2}, 0\right) \\
&= \left(\frac{1}{2}\alpha_1 - \frac{1}{2}(\alpha_1+\alpha_2), -\frac{1}{2}, 0\right) = \left(-\frac{1}{2}\alpha_2, -\frac{1}{2}, 0\right) \\
&= \left(-\frac{1}{4}(\alpha_1+\alpha_2) + \frac{1}{4}(\alpha_1-\alpha_2), -\frac{1}{2}, 0\right) \\
&= \Lambda_{1,1}.
\end{aligned} \tag{4.71}$$

This result may easily be deduced from the root diagram as shown in figure 4.3. Clearly $\Lambda_{1,1}^* = \Lambda_{1,0}$, while $\Lambda_{0,0}$ and $\Lambda_{0,1}$ are self-conjugate.

$\phi_{0,0} : h_- = h_+ = 0$

We begin by examining $\phi_{0,0}$, the identity field in this context, where we hope the behaviour to be fairly transparent. For the even 3-point function (4.53), where now $H_+ = h_{+,1} + h_{+,2} = 0$ we calculate

$$\begin{aligned}
&(i) \qquad d_2^-\langle\Lambda_1^*|\phi_2(x,\theta,\bar\theta)|\Lambda_{0,0}\rangle = 0 \\
&(ii) \qquad d_2'^-\langle\Lambda_1^*|\phi_2(x,\theta,\bar\theta)|\Lambda_{0,0}\rangle = 0 \\
&(iii) \qquad (D_2^+)^{3/2}(d_2^-d_2'^- - d_2'^-d_2^-)(D_2^+)^{1/2}\langle\Lambda_1^*|\phi_2(x,\theta,\bar\theta)|\Lambda_{0,0}\rangle = 0,
\end{aligned} \tag{4.72}$$

with

$$\langle\Lambda_1^*|\phi_2(x,\theta,\bar\theta)|\Lambda_{0,0}\rangle = C[x^{a_1} + \frac{1}{2}(h_{+,1}+h_{+,2}+2\alpha)\theta\bar\theta x^{a_1-1}]. \tag{4.73}$$

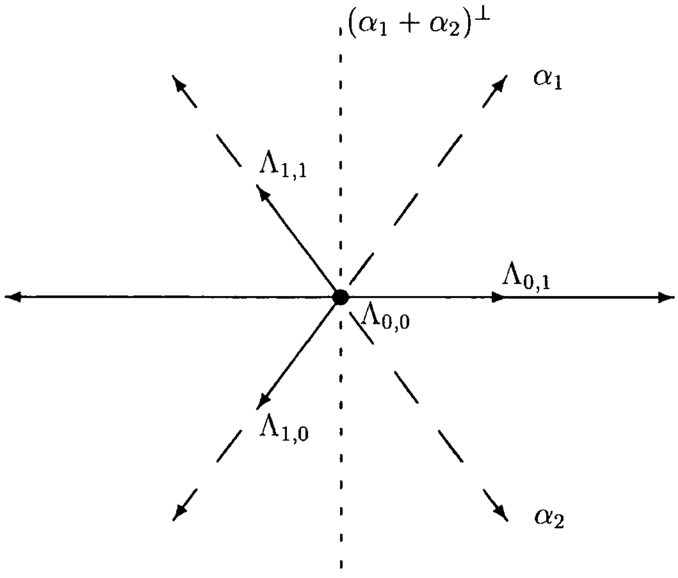

Figure 4.3: The finite parts of the $\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}$ Ramond weights.

From the first two equations, we find that $a_1 = \frac{1}{2}(h_{-,2} - h_{-,1}) = 0$ and $\alpha = 0$. Then from the third equation we have two conditions:

$$\begin{aligned} &h_{-,2}(h_{-,2} - 1) - 3h_{+,2}^2 = 0, \\ &h_{+,2}(h_{-,2} + \frac{1}{2}) = 0. \end{aligned} \tag{4.74}$$

When $h_{+,2} = 0$ we have $h_{-,2} = 0$ or $h_{-,2} = 1$. When $h_{-,2} = -\frac{1}{2}$ we have $h_{+,2} = -\frac{1}{2}$ or $h_{+,2} = \frac{1}{2}$. This unambiguously identifies the following possibilities:

$$\begin{aligned} \text{when } \phi_3 = \phi_{0,0}, \text{ then } \quad & \phi_2 = \phi_{0,0} \text{ and } \phi_1^* = \phi_{0,0} \\ \text{or} \quad & \phi_2 = \phi_{1,0} \text{ and } \phi_1^* = \phi_{1,1} \\ \text{or} \quad & \phi_2 = \phi_{1,1} \text{ and } \phi_1^* = \phi_{1,0} \\ \text{or} \quad & \phi_2 = \phi_{0,1} \text{ and } \phi_1^* = \phi_{0,1}. \end{aligned} \tag{4.75}$$

This is the sort of behaviour one would wish for, given that $\phi_{0,0}$ is the identity field. However, when this calculation is performed with the even 3-point function of Howe and West (with limits taken as described), we only find the first and last of these results arising, with no coupling between the identity and $\phi_{1,0}$ or $\phi_{1,1}$.

Repeating the exercise with the odd 3-point functions (4.54) and (4.55) requires that these are identically zero. For example, considering case (i) of (4.72) using (4.54) gives

$$(\partial_{\theta_2} - \frac{1}{2}\bar{\theta}_2\partial_{x_2})\tilde{C}'_{123}x_2^{a_1-1/2}\theta_2 = \tilde{C}'_{123}\left(x_2^{a_1-1/2} + \frac{1}{2}\left(a_1 - \frac{1}{2}\right)\theta\bar{\theta}x_2^{a_1-3/2}\right) = 0 \quad (4.76)$$

which implies that $\tilde{C}'_{123} = 0$. Hence the even case exhausts all possibilities.

$\underline{\phi_{0,1} : h_- = 1, h_+ = 0}$

The next case we examine is that of $\phi_{0,1}$. The even 3-point function (with $h_{+,1} + h_{+,2} = 0$) yields:

$$\begin{aligned} \text{when } \phi_3 = \phi_{0,1}, \quad & \phi_2 = \phi_{0,0} \text{ and } \phi_1^* = \phi_{0,1} \\ \text{or} \quad & \phi_2 = \phi_{0,1} \text{ and } \phi_1^* = \phi_{0,0}. \end{aligned} \quad (4.77)$$

The odd 3-point function (4.54), for which $h_{+,1} + h_{+,2} = -1$ gives:

$$\text{when } \phi_3 = \phi_{0,1}, \quad \phi_2 = \phi_{1,0} \text{ and } \phi_1^* = \phi_{1,0} \quad (4.78)$$

while the other odd 3-point function $(h_{+,1} + h_{+,2} = 1)$ reveals:

$$\text{when } \phi_3 = \phi_{0,1}, \quad \phi_2 = \phi_{1,1} \text{ and } \phi_1^* = \phi_{1,1}. \quad (4.79)$$

$\underline{\phi_{1,0} : h_- = -\frac{1}{2}, h_+ = -\frac{1}{2}}$

Turning now to $\phi_{1,0}$ we notice that for this particular case of quantum numbers, the singular vector in case (iii) of (4.60) gives no additional information over case (i). Once the fact that $j_0'^-|\Lambda_{1,0}\rangle = 0$ has been imposed, case (iii) vanishes after the first step and this singular vector need not be considered.

For the even 3-point function, where $h_{+,1} + h_{+,2} - \frac{1}{2} = 0$, we find:

$$h_{-,2} = -h_{+,2} = a_1 = \frac{1}{2}\left(h_{-,2} + \left(-\frac{1}{2}\right) - h_{-,1}\right). \quad (4.80)$$

While the quantum numbers of ϕ_1^* and ϕ_2 are not given explicitly, we can allow ϕ_2 to take the quantum numbers of all the $\phi_{m,m'}$ in turn and see what results this gives for ϕ_1^*. In fact, since $h_{-,2} = -h_{+,3}$ we are immediately restricted to taking $\phi_2 = \phi_{0,0}$ or $\phi_2 = \phi_{1,1}$. Then

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,0}, \text{ and } \phi_2 = \phi_{0,0} \text{ then } \phi_1^* = \phi_{1,1} \\ \text{and when } \phi_2 = \phi_{1,1} \text{ then } \phi_1^* = \phi_{0,0} \end{aligned} \quad (4.81)$$

which is in agreement with results from the $\phi_{0,0}$ calculation (although with different values of the parameter α).

In the case of the odd 3-point function (4.55) we find that this is identically zero. However, the 3-point function (4.54) (for which $h_{+,1} + h_{+,2} - \frac{1}{2} = -1$) gives

$$h_{-,2} = a_1 + \frac{1}{2} = \frac{1}{2} - h_{-,1}. \tag{4.82}$$

Again, letting ϕ_2 take the quantum numbers of $\phi_{m,m'}$ yields

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,0}, \text{ and } \phi_2 = \phi_{0,0} \text{ then } h^*_{-,1} = \frac{1}{2} \text{ and } h^*_{+,1} = -\frac{1}{2} \\ \text{when } \phi_2 = \phi_{1,0} \text{ then } \phi^*_1 = \phi_{0,1} \\ \text{when } \phi_2 = \phi_{1,1} \text{ then } h^*_{-,1} = 1 \text{ and } h^*_{+,1} = -1 \\ \text{when } \phi_2 = \phi_{0,1} \text{ then } \phi^*_1 = \phi_{1,0}. \end{aligned} \tag{4.83}$$

There are two cases here for which the quantum numbers do not correspond to any of the fields available. However, these results do not appear when considering fusions for $\phi_{0,0}$ above and $\phi_{1,1}$ below: not being in the intersection of these rules, they may be discarded. The last result is as already obtained in the consideration of $\phi_{0,1}$.

$\phi_{1,1} : h_- = -\frac{1}{2}, h_+ = \frac{1}{2}$

The situation for $\phi_{1,1}$ is very similar to that for $\phi_{1,0}$. The singular vector (iii) of (4.61) is of the form $(\dots)(-j_0^- j_0'^- + J_0^-)|\Lambda_{1,1}\rangle$ which may be rearranged as $(\dots)(j_0'^- j_0^-)|\Lambda_{1,1}\rangle$. This will again give no additional information over the result of using the singular vector (i) in (4.61), which is $j_0^-|\Lambda_{1,1}\rangle$.

The even 3-point function, with $h_{+,1} + h_{+,2} + \frac{1}{2} = 0$ shows that

$$h_{-,2} = h_{+,2} = a_1 = \frac{1}{2}\left(h_{-,2} + \left(-\frac{1}{2}\right) - h_{-,1}\right). \tag{4.84}$$

We see that the only options for ϕ_2 are $\phi_{0,0}$ and $\phi_{1,0}$. Hence

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,1}, \text{ and } \phi_2 = \phi_{0,0} \text{ then } \phi^*_1 = \phi_{1,0} \\ \text{and when } \phi_2 = \phi_{1,0} \text{ then } \phi^*_1 = \phi_{0,0}, \end{aligned} \tag{4.85}$$

again with different values of α. As for the odd 3-point functions, it is now (4.54) which is identically zero and (4.55) (where $h_{+,1} + h_{+,2} = \frac{1}{2} = 1$) that gives

$$h_{-,2} = a_1 + \frac{1}{2} = \frac{1}{2} - h_{-,1}. \tag{4.86}$$

Considering the remaining options for ϕ_2, we find

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,1}, \text{ and } \phi_2 = \phi_{1,1} \text{ then } \phi_1^* = \phi_{0,1} \\ \text{and when } \phi_2 = \phi_{0,1} \text{ then } \phi_1^* = \phi_{1,1}. \end{aligned} \tag{4.87}$$

The first of these results has already been seen in considering $\phi_{1,0}$ while the second echoes the result of the $\phi_{0,1}$ calculation.

To summarise the above results, replacing the fields ϕ_1^* by their relevant conjugates, we have found that the following fusion rules hold for the Ramond fields of $\hat{sl}(2|1;\mathbb{C})_k$ with $k = -1/2$:

$$\begin{aligned} &\phi_{0,0} \times \phi_{0,0} = \phi_{0,0}, && \phi_{1,0} \times \phi_{1,1} = \phi_{0,0}, \\ &\phi_{0,0} \times \phi_{1,0} = \phi_{1,0}, && \phi_{1,0} \times \phi_{0,1} = \phi_{1,1}, \\ &\phi_{0,0} \times \phi_{1,1} = \phi_{1,1}, && \phi_{1,1} \times \phi_{1,1} = \phi_{0,1}, \\ &\phi_{0,0} \times \phi_{0,1} = \phi_{0,1}, && \phi_{1,1} \times \phi_{0,1} = \phi_{1,0}, \\ &\phi_{1,0} \times \phi_{1,0} = \phi_{0,1}, && \phi_{0,1} \times \phi_{0,1} = \phi_{0,0}. \end{aligned} \tag{4.88}$$

These fusion rules form an associative algebra, as they should. One immediately obvious statement about these results is that on interchanging $\phi_{1,0}$ and $\phi_{1,1}$ the form of the fusion rules is unchanged. This precisely reflects what was discovered in the investigation of modular invariants in the previous chapter. There we found the permutation invariants (3.65), involving $\Pi(m, m') = (m, (m - m') \bmod u)$. This permutation leaves $\phi_{0,0}$ and $\phi_{0,1}$ unchanged, but interchanges $\phi_{1,0}$ and $\phi_{1,1}$ and so the modular invariant (3.65) would seem to be a consequence of the fusion rule automorphism (though we have not explicitly established the form of fusion rules for the remaining sectors involved in the modular invariant).

4.8 The Neveu-Schwarz Sector and the Verlinde Formula

We close with a few remarks on the fusion rules for Neveu-Schwarz fields and the results of the Verlinde formula applied to the S matrices derived in the previous chapter. The Ramond sector and Neveu-Schwarz sector of $\hat{sl}(2|1;\mathbb{C})$ are isomorphic and as such we expect the fusion rules for Neveu-Schwarz fields to bear a strong resemblance to those derived above. However, as previously detailed, Neveu-Schwarz fields have operator product expansions that introduce branch cuts with fermionic currents. The moding of generators is changed on commuting with Neveu-Schwarz fields, so one has to be very careful about defining commutation relations. Approaches to this problem (in the context of superconformal field theory) are described in (for example) [48], [65] and [67]. In superconformal field theory, where it is Ramond fields that introduce branch cuts (recall that our definitions of Ramond and Neveu-Schwarz are interchanged on this point) one considers a (degenerate) Ramond vacuum state, from which Ramond highest weight states are obtained by the action of Neveu-Schwarz primary fields. The Ramond vacuum is created from the true Neveu-Schwarz vacuum by the action of a so-called spin field σ. Then correlators are (loosely) given by

$$\langle 0^{NS}|\sigma\phi_1^{NS}\phi_2^{NS}\cdots\phi_n^{NS}\sigma|0^{NS}\rangle = \langle 0^R|\phi_1^{NS}\phi_2^{NS}\cdots\phi_n^{NS}|0^R\rangle. \tag{4.89}$$

The correlators may also be seen as $\langle 0^{NS}|\phi_1^R\phi_2^{NS}\cdots\phi_n^R|0^{NS}\rangle$ which in the appropriate limit reduces to $\langle\Lambda_1^R|\phi_2^{NS}\cdots\phi_{n-1}^{NS}|\Lambda_n^R\rangle$. The outcome of work on superconformal field theory, as discussed particularly in [67], is that the fusion rules respect the relevant algebra automorphism. To obtain the appropriate fusion rules in the present case involving Neveu-Schwarz fields, we note that these should be of the form

$$\phi_i^{NS}\times\phi_j^R = \sum N_{ij}{}^k\phi_k^{NS} \tag{4.90}$$

and we should simply replace the fields ϕ_1 and ϕ_3 in (4.88) by their Neveu-Schwarz counterparts.

As to the Verlinde formula, [20] established the appropriate extension of (2.73) to fermionic theories. There use was made of the fact that the form of fusion rules in superconformal field theories is $\phi^{NS} \times \phi^{NS} = \phi^{NS}$ and $\phi^{R} \times \phi^{NS} = \phi^{R}$. This was held in [20] to reflect the behaviour of characters under modular transformations, where Neveu-Schwarz characters are closed under the $\mathcal{S}$ transformation and Ramond characters and Neveu-Schwarz supercharacters are transformed into each other. The behaviour of characters remains the same in the present context, yet the behaviour of fusion rules is different and so we are not sure what to make of this point. However, naïve application of the Verlinde formula introduced in [20]

$$N_{ij}{}^{k} = \sum_m \frac{S^{NS}_{im} S^{NS}_{jm} (S^{NS})^{-1}_{mk}}{S^{NS}_{0m}} \tag{4.91}$$

which is the expression used in [20] to describe the $\phi^{NS} \times \phi^{NS} = \phi^{NS}$ fusion gives (using (A.2))

$$\begin{aligned}
\phi_{0,0} \times \phi_{0,0} &= \phi_{0,0}, & \phi_{1,0} \times \phi_{1,1} &= -\phi_{0,1}, \\
\phi_{0,0} \times \phi_{1,0} &= \phi_{1,0}, & \phi_{1,0} \times \phi_{0,1} &= \phi_{1,1}, \\
\phi_{0,0} \times \phi_{1,1} &= \phi_{1,1}, & \phi_{1,1} \times \phi_{1,1} &= -\phi_{0,0}, \\
\phi_{0,0} \times \phi_{0,1} &= \phi_{0,1}, & \phi_{1,1} \times \phi_{0,1} &= \phi_{1,0}, \\
\phi_{1,0} \times \phi_{1,0} &= -\phi_{0,0}, & \phi_{0,1} \times \phi_{0,1} &= \phi_{0,0}.
\end{aligned} \tag{4.92}$$

This is typical of results for fractional level, as seen in [18] and [53] for $\hat{sl}(2)$ and [23] for $\widehat{osp}(1|2)$. These last two references argued for a prescription to change the negative signs into positive ones: in (4.92) we see that replacing $-\phi_{0,0}$ by $\phi_{0,1}$ and $-\phi_{0,1}$ by $\phi_{0,0}$ reproduces the fusion rules (4.88). When the formula in [20] appropriate to the Ramond sector is used (with (A.2) and (A.4)),

$$N_{ij}{}^{k} = \sum_m \frac{S^{R}_{im} S^{R}_{jm} (S^{NS})^{-1}_{mk}}{S^{NS}_{0m}} \tag{4.93}$$

describing the fusion $\phi^{R} \times \phi^{R} = \phi^{NS}$, a very similar set of results is obtained, with negative signs appearing in different relations. Again, the replacement of $-\phi_{0,0}$ by $\phi_{0,1}$ and $-\phi_{0,1}$ by $\phi_{0,0}$ reproduces the fusion rules (4.88). While this indicates that a replacement prescription along the lines of [23] and [53] could be established, it is in the first instance more important to justify an appropriate extension of the Verlinde formula. In studying $\widehat{osp}(1|2)$ the authors of [23] simply applied the Verlinde formula to the $\mathcal{S}$ transformation matrix for Ramond supercharacters, which do close under modular transformations. If we do the same, we obtain fusion rules which do not agree with (4.88), such as $\phi_{1,0} \times \phi_{1,0} = \phi_{0,0}$. It remains to suitably adapt the expressions of [20].

4.9 Conclusion

In this chapter we have considered fusion rules for the Ramond sector of $\hat{sl}(2|1;\mathbb{C})_k$, at level $k = -1/2$. The approach used to examine this problem is that of studying the decoupling of singular vectors, giving rise to conditions that determine non-vanishing 3-point functions and hence fusion rules. The expressions for singular vectors are vastly more complicated than for the case of $\widehat{osp}(1|2)$, which was studied in [24]. There the authors were able to determine fusion rules for all admissible levels using this approach. We have been unable to discuss the general case of $k = 1/u - 1$, simply due to the difficulty in dealing with $\hat{sl}(2|1;\mathbb{C})_k$ singular vectors. It is not impossible that the general solution may be obtained through these methods. However, other approaches, notably the Coulomb gas formalism as used in [48] to study $N = 2$ superconformal field theory, would seem to be much more promising. Such an approach would allow a direct consideration of the Neveu-Schwarz sector (not mentioned in [24]) and also the $sl(2|1)$ invariant 4-point function. We have been able to determine consistent fusion rules for $k = -1/2$ and found these to be related to a modular invariant found in the previous chapter. These results indicate that one may consistently define a conformal field theory based on fractional level $\hat{sl}(2|1;\mathbb{C})$.

Chapter 5

Conclusion

In this thesis we have considered the affine superalgebra $\hat{sl}(2|1;\mathbb{C})$ at fractional level $k = 1/u - 1$, corresponding to admissible representations. These representations are more general than integrable ones but keep the important property that at a particular value of the level, the characters transform into each other under modular transformations. As such, one may construct modular invariant combinations of characters which one might hope could correspond to some physical conformal field theory.

With this end in mind, we first began by establishing the general form of the modular transformations of $\hat{sl}(2|1;\mathbb{C})$ characters. Although this was based on the assumption that the characters would indeed transform into each other, we did find this to be the case for explicit calculations at $k = -1/2$ and $k = -2/3$. Additionally, the work of [38], showing that all $\hat{sl}(2|1;\mathbb{C})$ characters at admissible levels of the form $k = 1/u - 1$ are equivalent to class IV and class V characters supports the assumption that it is these classes of characters which form a closed set under modular transformations. The expressions for modular transformations allowed the beginning of the study of modular invariant combinations of characters. Finding all possible modular invariants remains an unsolved problem, which is still a challenge in conformal field theory generally.

The Verlinde formula gives the relation between matrices of the modular S

transformation and fusion rules. In fractional level theories ($\hat{sl}(2)$ studied in [18], [21], [52] and [53], and $\widehat{osp}(1|2)$ in [23]) it is observed that the Verlinde formula gives fusion rules with negative coefficients. Although prescriptions have been given to deal with these situations, the form of the Verlinde formula appropriate to fractional level (there does still seem to be a relation between S-matrices and fusion rules) has yet to be established. In the present work, we have found fusion rules which are consistent at fractional level, as also discovered in [24] for $\widehat{osp}(1|2)$. The relation with the Verlinde formula is even more confused here due to the behaviour of the Ramond and Neveu-Schwarz sectors, which seems to be different to superconformal field theory.

The study of conformal field theories based on affine algebras at fractional level is one that has been tackled somewhat sporadically over the last decade. As yet, no absolute consensus has been reached even for $\hat{sl}(2)$ as to whether these can actually define *bona fide* conformal field theories in their own right. However, the evidence does seem to suggest that this is possible; in any case, other models may be obtained through hamiltonian reduction or the coset construction. The work of [24] is a first indication that fusion rules are well-defined for fractional level superalgebras, a conclusion which is also borne out by this work. The authors of [24] were able to determine consistent fusion rules for all levels at which admissible representations of $\widehat{osp}(1|2)$ exist (in the Ramond sector). Due to the far more complex nature of singular vectors of $\hat{sl}(2|1;\mathbb{C})_k$, we have only examined a particular case, that of $k = -1/2$. Even here we have restricted to Ramond fields, although fusion rules for Neveu-Schwarz fields may be at least strongly conjectured from these. It should be possible to calculate Neveu-Schwarz fusion rules explicitly through an adaptation of the techniques used in the previous chapter. Additionally, one may use these techniques to calculate fusion rules for higher values of u, although this would be hugely laborious. It may be possible to consider the general case $k = 1/u - 1$, although we have had little success in this regard. More promising (as the authors of [48] found for $N = 2$ superconformal field theory) would be to use an approach based on the Coulomb gas formalism. Then Neveu-Schwarz fusion rules could be considered in a more straightforward manner and one could also consider the 4-point function. As yet, the technology to implement this approach for fractional level superalgebras (other than $\widehat{osp}(1|2)$) has not been developed [62].

Conformal field theory has been an area of study popular amongst mathematical physicists for the last 20 years. Its successes have been many and varied, from statistical mechanics to string theory. Yet there remains much that is not well understood in this field, which will provide challenges for the researcher for some time to come.

Appendix A

$u = 2$

Here we list the explicit forms of the matrices S for each sector at $u = 2$. We then list the possible modular invariants satisfying the condition that the matrices N have non-negative integer entries. In what follows, we understand a sum over the repeated index β.

$$\chi_\alpha^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{\pi i\left(\sigma^2-\nu^2\right)/2\tau}\, S^{NS}_{\alpha\beta}\,\chi_\beta^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,3,4, \quad \text{(A.1)}$$

where

$$S^{NS} = \frac{1}{2}\begin{pmatrix} i & 1 & 1 & i \\ 1 & -i & i & -1 \\ 1 & i & -i & -1 \\ i & -1 & -1 & i \end{pmatrix}; \qquad \text{(A.2)}$$

$$\chi_\alpha^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{\pi i\left(\sigma^2-\nu^2\right)/2\tau}\, S^{R}_{\alpha\beta} S\chi_\beta^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,3,4, \quad \text{(A.3)}$$

where

$$S^R = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ i & -i & i & i \\ i & i & -i & i \\ 1 & -1 & -1 & -1 \end{pmatrix}; \tag{A.4}$$

$$S\chi_\alpha^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{\pi i\left(\sigma^2-\nu^2\right)/2\tau}\, S^{SNS}_{\alpha\beta}\,\chi_\beta^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,3,4, \tag{A.5}$$

where

$$S^{SNS} = \frac{1}{2}\begin{pmatrix} 1 & i & i & 1 \\ 1 & -i & i & -1 \\ 1 & i & -i & -1 \\ -1 & i & i & -1 \end{pmatrix}; \tag{A.6}$$

$$S\chi_\alpha^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{\pi i\left(\sigma^2-\nu^2\right)/2\tau}\, S^{SR}_{\alpha\beta}\,S\chi_\beta^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,3,4, \tag{A.7}$$

where

$$S^{SR} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{A.8}$$

We use the definitions as laid out in the tables A.1 and A.2. The supercharacters in each sector have the same quantum numbers as the corresponding characters. The relation between M and M' values in class V and the m and m' values which allow us to combine classes IV and V in the branching formulae (3.35), (3.36), (3.37) and (3.38) is $m = u - 2 - M - M'$, $m' = u - 1 - M'$.

With the above information, we have calculated modular invariant matrices N

	m	m'	h^R_-	h^R_+	h^R	h^{NS}_-	h^{NS}_+	h^{NS}
χ_1	0	0	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{8}$
χ_2	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{8}$
χ_3	1	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{8}$

Table A.1: Class IV $\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}$ characters

	$M(m)$	$M'(m')$	h^R_-	h^R_+	h^R	h^{NS}_-	h^{NS}_+	h^{NS}
χ_4	0(0)	0(1)	1	0	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{8}$

Table A.2: Class V $\hat{sl}(2|1;\mathbb{C})_{-\frac{1}{2}}$ characters

in

$$Z = \sum_{m,m',n,n'=0}^{u-1} N^R_{mm',nn'}\chi^R_{m,m'}\overline{\chi^R}_{n,n'} + N^{NS}_{mm,nn'}\chi^{NS}_{m,m'}\overline{\chi^{NS}}_{n,n'}$$

$$+ N^{SNS}_{mm',nn'}S\chi^{NS}_{m,m'}\overline{S\chi^{NS}}_{n,n'} + \sum_{a,a',b,b'=0}^{u-1} N^{SR}_{aa',bb'}S\chi^R_{a,a'}\overline{S\chi^R}_{b,b'}, \quad \text{(A.9)}$$

that is to say, N such that $[S,N] = [T,N] = 0$, using the appropriate matrices S and T. We find that the general form of these N is

$$N^R = \begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & c+b & a-c & 0 \\ 0 & a-c & c+b & 0 \\ 0 & 0 & 0 & a-b \end{pmatrix},$$

$$N^{NS} = N^{SNS} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & a-b-c & 0 \\ 0 & a-b-c & c & 0 \\ b & 0 & 0 & a \end{pmatrix}. \quad \text{(A.10)}$$

With the requirements that all the $N_{mm',nn'}$ are non-negative integers and

$N^R_{00,00} = 1$, we find two possible cases:

$$(i) \quad N^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & a-1 & 0 \\ 0 & a-1 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad N^{NS} = \begin{pmatrix} a & 0 & 0 & a-1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a-1 & 0 & 0 & a \end{pmatrix} \tag{A.11}$$

or

$$(ii) \quad N^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a-1 & a & 0 \\ 0 & a & a-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad N^{NS} = \begin{pmatrix} a & 0 & 0 & a-1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ a-1 & 0 & 0 & a \end{pmatrix}, \tag{A.12}$$

with $a \in \mathbb{N}$. For the Ramond supercharacters we find

$$N^{SR} = \begin{pmatrix} d+e+f+g-h & d+f-h & e+g-h & 0 \\ d+e-h & d & e & 0 \\ f+g-h & f & g & 0 \\ 0 & 0 & 0 & h \end{pmatrix}. \tag{A.13}$$

Setting $d = g = h = 1$, $e = f = 0$ gives the identity matrix and $e = f = h = 1$, $d = g = 0$ gives us the permutation.

Appendix B

$u = 3$

$$\chi_\alpha^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{2\pi i\left(\sigma^2-\nu^2\right)/3\tau}\, S^{NS}_{\alpha\beta}\,\chi_\beta^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,\ldots,9, \quad \text{(B.1)}$$

where

$$S^{NS} = \frac{1}{3}\begin{pmatrix}
e^{2\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & 1 & 1 & 1 & e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} \\
e^{\pi i/3} & 1 & e^{-2\pi i/3} & e^{-\pi i/3} & -1 & e^{\pi i/3} & e^{2\pi i/3} & e^{-\pi i/3} & -1 \\
e^{\pi i/3} & e^{-2\pi i/3} & 1 & e^{\pi i/3} & -1 & e^{-\pi i/3} & e^{2\pi i/3} & -1 & e^{-\pi i/3} \\
1 & e^{-\pi i/3} & e^{\pi i/3} & e^{-2\pi i/3} & 1 & e^{2\pi i/3} & -1 & e^{2\pi i/3} & e^{-2\pi i/3} \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & e^{\pi i/3} & e^{-\pi i/3} & e^{2\pi i/3} & 1 & e^{-2\pi i/3} & -1 & e^{-2\pi i/3} & e^{2\pi i/3} \\
e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & -1 & -1 & -1 & e^{2\pi i/3} & e^{\pi i/3} & e^{\pi i/3} \\
e^{2\pi i/3} & e^{-\pi i/3} & -1 & e^{2\pi i/3} & 1 & e^{-2\pi i/3} & e^{\pi i/3} & 1 & e^{-2\pi i/3} \\
e^{2\pi i/3} & -1 & e^{-\pi i/3} & e^{-2\pi i/3} & 1 & e^{2\pi i/3} & e^{\pi i/3} & e^{-2\pi i/3} & 1
\end{pmatrix};$$

$$\text{(B.2)}$$

$$\chi_\alpha^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{2\pi i\left(\sigma^2-\nu^2\right)/3\tau}\, S^{R}_{\alpha\beta} S\chi_\beta^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,\ldots,9, \quad \text{(B.3)}$$

where

$$S^R = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ e^{\pi i/3} & -1 & e^{\pi i/3} & e^{-\pi i/3} & -1 & e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & 1 \\ e^{\pi i/3} & e^{\pi i/3} & -1 & e^{\pi i/3} & -1 & e^{-\pi i/3} & e^{2\pi i/3} & 1 & e^{2\pi i/3} \\ e^{2\pi i/3} & 1 & e^{2\pi i/3} & e^{-2\pi i/3} & 1 & e^{2\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & -1 \\ e^{2\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} & 1 & 1 & 1 & e^{\pi i/3} & e^{-\pi i/3} & e^{-\pi i/3} \\ e^{2\pi i/3} & e^{2\pi i/3} & 1 & e^{2\pi i/3} & 1 & e^{-2\pi i/3} & e^{\pi i/3} & -1 & e^{\pi i/3} \\ e^{\pi i/3} & e^{-\pi i/3} & e^{-\pi i/3} & -1 & -1 & -1 & e^{2\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & 1 & e^{-2\pi i/3} & -1 & e^{\pi i/3} & e^{-\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} & 1 & e^{2\pi i/3} & -1 & e^{-\pi i/3} & e^{\pi i/3} \end{pmatrix}; \tag{B.4}$$

$$S\chi_\alpha^{NS,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{2\pi i\left(\sigma^2-\nu^2\right)/3\tau}\, S^{SNS}_{\alpha\beta}\, \chi_\beta^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,\ldots,9, \tag{B.5}$$

where

$$S^{SNS} = \frac{1}{3}\begin{pmatrix} 1 & e^{\pi i/3} & e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & e^{\pi i/3} & 1 & 1 \\ 1 & -1 & e^{\pi i/3} & 1 & e^{-2\pi i/3} & e^{2\pi i/3} & e^{-\pi i/3} & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{\pi i/3} & -1 & e^{2\pi i/3} & e^{-2\pi i/3} & 1 & e^{-\pi i/3} & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-\pi i/3} & e^{\pi i/3} & e^{-2\pi i/3} & 1 & e^{2\pi i/3} & -1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & e^{\pi i/3} & e^{-\pi i/3} & e^{2\pi i/3} & 1 & e^{-2\pi i/3} & -1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ -1 & e^{2\pi i/3} & e^{2\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & e^{2\pi i/3} & -1 & -1 \\ -1 & e^{2\pi i/3} & 1 & e^{\pi i/3} & e^{-\pi i/3} & -1 & e^{-2\pi i/3} & e^{\pi i/3} & e^{-\pi i/3} \\ -1 & 1 & e^{2\pi i/3} & -1 & e^{-\pi i/3} & e^{\pi i/3} & e^{-2\pi i/3} & e^{-\pi i/3} & e^{\pi i/3} \end{pmatrix}; \tag{B.6}$$

$$S\chi_\alpha^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}\left(\frac{\sigma}{\tau},\frac{\nu}{\tau},-\frac{1}{\tau}\right) = e^{2\pi i\left(\sigma^2-\nu^2\right)/3\tau}\, S^{SR}_{\alpha\beta} S\chi_\beta^{R,\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}}(\sigma,\nu,\tau),$$

$$\alpha,\beta = 1,2,\ldots,9, \tag{B.7}$$

	m	m'	h_-^R	h_+^R	h^R	h_-^{NS}	h_+^{NS}	h^{NS}
χ_1	0	0	0	0	0	$-\frac{2}{3}$	0	$-\frac{1}{6}$
χ_2	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0
χ_3	1	1	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0
χ_4	2	0	$-\frac{2}{3}$	$-\frac{2}{3}$	0	0	$-\frac{2}{3}$	$\frac{1}{6}$
χ_5	2	1	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	0	$\frac{1}{2}$
χ_6	2	2	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{1}{6}$

Table B.1: Class IV $\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}$ characters

	$M(m)$	$M'(m')$	h_-^R	h_+^R	h^R	h_-^{NS}	h_+^{NS}	h^{NS}
χ_7	0(1)	0(2)	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{4}{3}$	0	$-\frac{1}{6}$
χ_8	0(0)	1(1)	1	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	0
χ_9	1(0)	0(2)	1	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	0

Table B.2: Class V $\hat{sl}(2|1;\mathbb{C})_{-\frac{2}{3}}$ characters

where

$$S^{SR} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & e^{-2\pi i/3} & 1 & e^{-2\pi i/3} & e^{2\pi i/3} & e^{-\pi i/3} & e^{\pi i/3} & e^{-\pi i/3} \\ 1 & e^{-2\pi i/3} & 1 & e^{2\pi i/3} & e^{-2\pi i/3} & 1 & e^{-\pi i/3} & e^{-\pi i/3} & e^{\pi i/3} \\ 1 & 1 & e^{2\pi i/3} & 1 & e^{2\pi i/3} & e^{-2\pi i/3} & e^{\pi i/3} & e^{-\pi i/3} & e^{\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{-2\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} & e^{\pi i/3} & -1 & -1 \\ 1 & e^{2\pi i/3} & 1 & e^{-2\pi i/3} & e^{2\pi i/3} & 1 & e^{\pi i/3} & e^{\pi i/3} & e^{-\pi i/3} \\ -1 & e^{-\pi i/3} & e^{-\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & e^{2\pi i/3} & 1 & 1 \\ -1 & e^{\pi i/3} & e^{-\pi i/3} & e^{-\pi i/3} & -1 & e^{\pi i/3} & 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ -1 & e^{-\pi i/3} & e^{\pi i/3} & e^{\pi i/3} & -1 & e^{-\pi i/3} & 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{pmatrix}. \tag{B.8}$$

In the above we use the definitions of tables B.1 and B.2.

For the $u = 3$ modular invariants we find:

$$N^R = \begin{pmatrix} a-b+c-d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & c & b & 0 & d & 0 & 0 & 0 \\ 0 & c & a & d & 0 & b & 0 & 0 & 0 \\ 0 & b & d & a & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a+c & 0 & b+d & 0 & 0 \\ 0 & d & b & c & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b+d & 0 & a+c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a-d & c-b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c-b & a-d \end{pmatrix},$$

$$N^{NS} = N^{SNS} = \begin{pmatrix} a+c & 0 & 0 & 0 & 0 & 0 & b+d & 0 & 0 \\ 0 & a & c & 0 & 0 & 0 & 0 & d & b \\ 0 & c & a & 0 & 0 & 0 & 0 & b & d \\ 0 & 0 & 0 & a-d & 0 & c-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a-b+c-d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c-b & 0 & a-d & 0 & 0 & 0 \\ b+d & 0 & 0 & 0 & 0 & 0 & a+c & 0 & 0 \\ 0 & d & b & 0 & 0 & 0 & 0 & a & c \\ 0 & b & d & 0 & 0 & 0 & 0 & c & a \end{pmatrix}. \tag{B.9}$$

Requiring that entries be non-negative integers and that $N^R_{00,00} = 1$ leads to the following invariants:

$$(i)\quad N^R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & b & 0 & a-1 & 0 & 0 & 0 \\ 0 & b & a & a-1 & 0 & b & 0 & 0 & 0 \\ 0 & b & a-1 & a & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a+b & 0 & a+b-1 & 0 & 0 \\ 0 & a-1 & b & b & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a+b-1 & 0 & a+b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$N^{NS} = \begin{pmatrix} a+b & 0 & 0 & 0 & 0 & 0 & a+b-1 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & a-1 & b \\ 0 & b & a & 0 & 0 & 0 & 0 & b & a-1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ a+b-1 & 0 & 0 & 0 & 0 & 0 & a+b & 0 & 0 \\ 0 & a-1 & b & 0 & 0 & 0 & 0 & a & b \\ 0 & b & a-1 & 0 & 0 & 0 & 0 & b & a \end{pmatrix} \tag{B.10}$$

or

$$(ii)\quad N^R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & c & c-1 & 0 & a & 0 & 0 & 0 \\ 0 & c & a & a & 0 & c-1 & 0 & 0 & 0 \\ 0 & c-1 & a & a & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a+c & 0 & a+c-1 & 0 & 0 \\ 0 & a & c-1 & c & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a+c-1 & 0 & a+c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$N^{NS} = \begin{pmatrix} a+c & 0 & 0 & 0 & 0 & 0 & a+c-1 & 0 & 0 \\ 0 & a & c & 0 & 0 & 0 & 0 & a & c-1 \\ 0 & c & a & 0 & 0 & 0 & 0 & c-1 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a+c-1 & 0 & 0 & 0 & 0 & 0 & a+c & 0 & 0 \\ 0 & a & c-1 & 0 & 0 & 0 & 0 & a & c \\ 0 & c-1 & a & 0 & 0 & 0 & 0 & c & a \end{pmatrix} . \qquad \text{(B.11)}$$

For the Ramond supercharacters we find

$$N^{SR} = \begin{pmatrix} e_1 & e_2 & e_3 & e_2 & 0 & e_3 & 0 & 0 & 0 \\ g & j & f_3 & f_4 & 0 & h & 0 & 0 & 0 \\ e & g_2 & g_3 & f & 0 & l & 0 & 0 & 0 \\ g & f_4 & h & j & 0 & f_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 & j_7 & 0 & 0 \\ e & f & l & g_2 & 0 & g_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j_7 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_8 & m_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_9 & m_8 \end{pmatrix} \qquad \text{(B.12)}$$

where $e_1 = f + g + k + l$, $e_2 = f + g - h$, $e_3 = e - f + h$, $f_3 = g - j + k$, $f_4 = -e + f + g - h + l$, $g_2 = f + g - h - j + k$, $g_3 = e - f - g + h + j$, $j_7 = e - f - l$, $m_8 = -g + h + j$, $m_9 = -e + f + g - h - j + k + l$. Setting $j = k = 1$, $e = f = g = h = l = 0$ gives us the identity and $k = 1$, $e = f = g = h = j = l = 0$ the permutation invariant.

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