Fitting ideals and module structure

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Abstract

Let R be a commutative ring with a 1. Original work by H. Fitting showed how we can associate to each finitely generated R-module a unique sequence of R-ideals, which are known as Fitting Ideals. The aim of this thesis is to undertake an investigation of Fitting Ideals and their relation with module structure and to construct a notion of Fitting Invariant for certain non-commutative rings.

We first of all consider the commutative case and see how Fitting Ideals arise by considering determinantal ideals of presentation matrices of the underlying module and we describe some applications. We then study the behaviour of Fitting Ideals for certain module structures and investigate how useful Fitting Ideals are in determining the underlying module.

The main part of this work considers the non-commutative case and constructs Fitting Invariants for modules over hereditary orders and shows how, by considering maximal orders and projectives in the hereditary order, we can obtain some very useful invariants which ultimately determine the structure of torsion modules. We then consider what we can do in the non-hereditary case, in particular for twisted group rings. Here we construct invariants by adjusting presentation matrices which generalises the previous work done in the hereditary case.

Declaration

The work in this thesis is based on research carried out in the Geometry and Arithmetic Group, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction - Fitting Ideals

Throughout this chapter let R be a Noetherian commutative ring with a 1, unless stated otherwise. Given any finitely generated R-module M, we can associate with M a sequence of ideals of R known as the Fitting Invariants or Fitting Ideals of M. The Fitting Invariants are named after H. Fitting who investigated their properties in [6] in 1936.

In this chapter we recall the definition of these Fitting Ideals which is in terms of the determinantal ideals of a presentation matrix for the R-module, M. It can be shown that this sequence of ideals is independent of the particular presentation for M. We will also look at relations within the Fitting Ideals and define the Fitting Length of a module.

In section 1.4 we discuss some applications for Fitting Ideals. They can tell us information about the underlying R-module and in particular they can provide estimates for the annihilator of the module. We also come across Fitting Ideals in Knot Theory where they are known as Alexander Ideals.

1.1 Presentations of finitely generated modules

In this section suppose R is any Noetherian ring with a 1, not necessarily commutative. Since R is Noetherian, for a finitely generated left R-module ${}_RM$, there exists a finite presentation for ${}_RM$:

$$R^n \xrightarrow{\alpha} R^g \xrightarrow{\pi} M \longrightarrow 0$$

with respect to g generators and n relations. We can represent the R-module homomorphism $\alpha : R^n \longrightarrow R^g$ by an $n \times g$ matrix $A = (a_{ij})$, where $a_{ij} \in R$ for $i = 1, ..., n$ and $j = 1, ..., g$. As we are thinking of left R-modules the matrix A acts on the right and we say A is a *presentation matrix* for ${}_RM$.

We can find a set $\underline{e} = \{e_1, ..., e_g\}$ which generates ${}_RM$ over R. A relation for ${}_RM$ is an equation of the form $r_1e_1 + ... + r_ge_g = 0$ for some $r_j \in R$. The coefficient vector of this relation is the row vector $(r_1, ..., r_g)$. As ${}_RM$ is finitely generated and R is Noetherian we can find a matrix B with g columns such that B has the following properties. Firstly, each row is a coefficient vector of some relation for ${}_RM$, with respect to the generating set $\underline{e}$. Secondly, coefficient vectors from all relations for the generating set $\underline{e}$ can be generated as R-linear combinations of the rows of B. Such a matrix B is known as a *relations matrix* for ${}_RM$ over R.

In fact it is clear that a relations matrix is a presentation matrix and vice-versa. Thus, from now on, we will simply use the term presentation matrix.

We next prove a lemma about what the presentation matrix looks like when we extend the generating set of the finitely generated module. This is useful in showing Fitting Invariants are not dependent on the particular generating set chosen and we will make further use of this lemma in section 6.1

Lemma 1.1 Let $\underline{e} = \{e_1, ..., e_g\}$ be a generating set for ${}_RM$ and $\underline{f} = \{f_1, \ldots, f_k\}$ be a set of elements of ${}_RM$, not necessarily a generating set. Let $Q = (q_{l,j})$ be the matrix in $M_{k\times g}(R)$, for $l = 1, \ldots, k$ and for $j = 1, \ldots, g$, such that:

$$f_l + \sum_{j=1}^{g} q_{l,j}e_j = 0$$

Then:

$$\left(\begin{array}{c|c} Q & I_k \\ \hline A & 0 \end{array}\right)$$

is a presentation matrix for ${}_RM$, with respect to the extended generating set $\underline{ef} = \{e_1, \ldots, e_g, f_1, \ldots, f_k\}$.

Proof:

We have a presentation for ${}_RM$ with respect to $\underline{e}$:

$$R^n \xrightarrow{\alpha} R^g \xrightarrow{\pi} M$$

where α is represented by the presentation matrix $A \in M_{n\times g}(R)$. Also there is a homomorphism π' such that:

$$R^k \xrightarrow{\pi'} M$$

where:

$$\pi'(\underline{w}) = \sum_{l=1}^{k} w_l f_l \ \forall\ \underline{w} \in R^k$$

We now find a presentation for ${}_RM$ with respect to $\underline{ef}$:

$$R^{n'} \longrightarrow R^{g+k} \xrightarrow{\epsilon} M$$

for some positive integer n', where the map ϵ is:

$$\epsilon(\underline{v}, \underline{w}) = \pi(\underline{v}) + \pi'(\underline{w}) = \pi(\underline{v}) + \sum_{l=1}^{k} w_l f_l = \pi(\underline{v}) - \sum_{j=1}^{g}\sum_{l=1}^{k} w_l q_{l,j} e_j = \pi(\underline{v}) - \pi(\underline{w}Q)$$

for all $\underline{v} \in R^g, \underline{w} \in R^k$. Hence:

$$\ker\ \epsilon = \{(\underline{v}|\underline{w}) \mid \pi(\underline{v} - \underline{w}Q) = 0\}$$

Now,

$$\ker\ \pi = \mathrm{Im}\ \alpha = \left\{\underline{y}A \mid \underline{y} \in R^n\right\}$$

and therefore:

$$\ker\ \epsilon = \left\{(\underline{v}|\underline{w}) \mid \underline{v} = \underline{w}Q + \underline{y}A\right\} = \left\{(\underline{w}Q + \underline{y}A|\underline{w}) \mid \underline{w} \in R^k, \underline{y} \in R^n\right\}$$

Hence:

$$\ker\ \epsilon = \left\{ (\underline{w}|\underline{y}) \left(\begin{array}{c|c} Q & I_k \\ \hline A & 0 \end{array}\right) \mid \underline{w} + \underline{y} \in R^{k+n} \right\}$$

and therefore the matrix

$$\left(\begin{array}{c|c} Q & I_k \\ \hline A & 0 \end{array}\right)$$

is indeed a presentation matrix for ${}_RM$ with respect to the generating set $\underline{ef}$.

$\square$

1.2 Construction of Fitting Invariants for finitely generated modules

For the remainder of this chapter R denotes a Noetherian commutative ring with a 1. As above, let M be a finitely generated left R-module with a generating set, $\underline{e} = \{e_1, ..., e_g\}$. Given any $n \times g$ matrix A, with entries in R, for $s \in \mathbb{Z}$ we define the determinantal ideals of A:

Definition 1.2 For $0 < s \leq \min(n, g)$ we let $I_s(A)$ be the R-ideal generated by all the $s \times s$ minors of A, or the determinants of the $s \times s$ submatrices of A. $I_s(A)$ is known as the s-th determinantal ideal of A, and we let

$$I_s(A) = \begin{cases} 0 & \text{for } s > \min(n, g) \\ R & \text{for } s \leq 0 \end{cases}$$

Note that $I_s(A)$ is invariant under elementary row and column operations. Let B be a matrix with g columns whose rows are coefficient vectors of relations with respect to the generating set $\underline{e}$, for the R-module M. That is, B is a matrix of relations for M. Define

Definition 1.3

$$I_s(M|\underline{e}) = \sum_B I_s(B)$$

where the summation is over all such matrices B.

As M is finitely generated and R Noetherian there exists a presentation matrix A for M whose rows generate the coefficient vectors of all the relations for M with respect to the generating set $\underline{e}$. Hence, there exists a matrix T such that $B = TA$. Then:

$$I_s(B) = I_s(TA) \subseteq I_s(T) \cap I_s(A) \subseteq I_s(A)$$

(see [13], page 7). Hence, $\sum_B I_s(B) \subseteq I_s(A)$and since clearly $I_s(A) \subseteq \sum_B I_s(B)$ we can write

$$I_s(M|\underline{e}) = I_s(A)$$

Suppose now that $\underline{f} = \{f_1, ..., f_k\}$ is another generating set for M. Then it is easily shown that $I_s(M|\underline{f})$ and $I_s(M|\underline{e})$ are not always equal, if the size of $\underline{e}$ and $\underline{f}$

are not the same. So just looking at the s-th determinantal ideals of a presentation matrix will not give us invariants of M, as this is dependent on the generating set used. However, if we now consider the extended generating set $\underline{ef}$ it can be shown that (for details see [13], pages 57-58):

$$I_{g-s}(M|\underline{e}) = I_{g+k-s}(M|\underline{ef}) = I_{k-s}(M|\underline{f}) \text{ for } 0 \leq s \leq \min(g,k) \tag{1.1}$$

and that, w.l.o.g., if $g < k$, then

$$I_{k-t}(M|\underline{f}) = I_{g+k-t}(M|\underline{ef}) = R \text{ for } g < t \leq k \tag{1.2}$$

This now leads us to define:

Definition 1.4

$$\mathcal{F}_s(M) = \begin{cases} I_{g-s}(M|\underline{e}) & \text{for } 0 \leq s \leq g \\ R & \text{for } s > g \end{cases}$$

for $s = 0, 1, 2, 3, \ldots$.

Equations (1.1) and (1.2) show that $\mathcal{F}_s(M)$ is independent of the particular set of generators given for M. In other words $\mathcal{F}_s(M)$ is an invariant of M. We now define this Invariant.

Definition 1.5 Let M be a finitely generated R-module, then the R-ideals $\mathcal{F}_s(M)$ are defined to be the s-th Fitting Invariants or s-th Fitting Ideals of M for the integers $s = 0, 1, 2, 3, \ldots$.

As we remarked earlier, for a finitely generated module, M, we can replace $I_{g-s}(\underline{M}|e)$ by $I_{g-s}(A)$, where A is a presentation matrix for M, with respect to $\underline{e}$ and in this case we obtain:

$$\mathcal{F}_s(M) = \begin{cases} I_{g-s}(A) & \text{for } 0 \leq s \leq g \\ R & \text{for } s > g \end{cases} \tag{1.3}$$

Example 1.6 Let us work in the polynomial ring, $R = \mathbb{Z}[t]$ and consider the R-module, M with generators e_1, e_2 given by:

$$M = R/I \oplus R/J = Re_1 \oplus Re_2$$

where I, J are R-ideals given by $I = (2, t), J = (4, t^2)$. Now, a presentation matrix for M is:

$$A = \begin{pmatrix} 2 & 0 \\ t & 0 \\ 0 & 4 \\ 0 & t^2 \end{pmatrix}$$

since $2e_1 = te_1 = 0$ and $4e_2 = t^2e_2 = 0$. Calculating the Fitting Ideals of M we have:

$$\begin{aligned} \mathcal{F}_0(M) &= I_2(A) = (8, 4t, 2t^2, t^3) \\ \mathcal{F}_1(M) &= I_1(A) = (2, t, 4, t^2) = (2, t) \\ \mathcal{F}_2(M) &= R = \mathbb{Z}[t] \end{aligned}$$

□

Remark 1.7 Since we can choose the same presentation matrix for any two R-isomorphic modules the Fitting Ideals of R-isomorphic modules are equal. We will see later that the converse of this statement is false; namely there exist R-modules with the same Fitting Invariants which are not isomorphic as R-modules.

1.3 Fitting Length of a module

We now look at the relationship between the sequence of Fitting Ideals we have derived. From the Laplace expansion formula for determinants (see [9], 107D), we can show that the sequences of Fitting Ideals associated to each finitely generated R-module, M, is increasing. Let us consider $I_r(A)$, for $r \geq 1$. Let B_r be any $r \times r$ submatrix of A. Then the Laplace Expansion formula for determinants tells us det B_r can be written as a sum of products of $s \times s$ minors of B_r times $[r-s] \times [r-s]$ minors of B_r, for $0 \leq s \leq r$. Hence:

$$\det B_r \in I_s(A)I_{r-s}(A) \Rightarrow I_r(A) \subseteq I_s(A)I_{r-s}(A)$$

If A is a presentation matrix for M with g generators, setting $s = 1$ we see that $I_r(A) \subseteq I_1(A)I_{r-1}(A) \subseteq I_{r-1}(A)$. Hence, we get a sequence of increasing Fitting Ideals:

$$\mathcal{F}_0(M) \subseteq \mathcal{F}_1(M) \subseteq \mathcal{F}_2(M) \subseteq \cdots \mathcal{F}_t(M) \subseteq \mathcal{F}_{t+1}(M) \subseteq \cdots \subseteq \mathcal{F}_{g-1}(M) \subseteq \mathcal{F}_g(M) = R$$

We can obtain a more refined sequence of increasing Fitting Ideals to the one above by noting that, for $0 \leq t \leq g-1$ we have:

$$\mathcal{F}_{g-1}(M)\mathcal{F}_t(M) \subseteq \mathcal{F}_t(M) \subseteq \mathcal{F}_{g-1}(M)\mathcal{F}_{t+1}(M) \subseteq \mathcal{F}_{t+1}(M) \tag{1.4}$$

We see that, if M can be generated by g generators, then the sequence of Fitting Ideals always becomes trivial at the g-th Fitting Invariant, i.e. $\mathcal{F}_g(M) = R$. We ask ourselves what is the least t such that $\mathcal{F}_t(M) = R$? This leads us to define the Fitting Length of a module.

Definition 1.8 For any finitely generated R-module, M, the Fitting Length of M, which we denote by $\mathcal{L}_R(M)$, is the smallest integer $t \geq 0$ such that $\mathcal{F}_t(M) = R$.

We know that

$$\mathcal{L}_R(M) \leq \min_g\{g \text{ is the cardinality of a generating set for } M\}$$

In fact equality does not always hold as the following example shows:

Example 1.9 Let $R = \mathbb{Z}[\sqrt{-6}]$ and let J be the R-ideal $J = (2, \sqrt{-6})$. Then viewed as a left R-module $_RJ$ has a minimal generating set of cardinality 2 and has relations:

$$\begin{aligned} \sqrt{-6} \times 2 - 2 \times \sqrt{-6} &= 0 \\ 3 \times 2 + \sqrt{-6} \times \sqrt{-6} &= 0 \end{aligned}$$

Hence, a presentation matrix for J as an R-module is simply:

$$A = \begin{pmatrix} \sqrt{-6} & -2 \\ 3 & \sqrt{-6} \end{pmatrix}$$

Then, the Fitting Ideals are:

$$\begin{aligned} \mathcal{F}_0(J) &= (0) \\ \mathcal{F}_1(J) &= R \end{aligned}$$

So we have $\mathcal{L}_R(J) = 1 < 2$ in this case.

$\square$

We will discuss Fitting Lengths in more detail in chapters 2 and 3.

1.4 Applications

In this section we give some uses of Fitting Invariants. Fitting Invariants can provide us with useful information about the structure of a module. We will see later that in some cases if we know all the Fitting information about a module then we can determine the stucture of the R-module completely (this is in fact the case when R is a Principal Ideal Domain). Even when this is not the case, the Fitting information can still help us to understand the structure of a module and we may be able to say something about the relationship between non-isomorphic modules when we compare the information from their Fitting Invariants. One frequent use of Fitting Invariants is to say something about the annihilator of a module.

1.4.1 Annihilators

First, we define the initial Fitting Ideal.

Definition 1.10 The initial Fitting Ideal of a finitely generated R-module M is $\mathcal{F}_0(M)$.

We then have the following theorem which gives us a relationship between the initial Fitting Ideal and annihilator of a module:

Theorem 1.11 If M is a finitely generated R-module which can be generated by g generators then:

$$[\mathrm{Ann}_R(M)]^g \subseteq \mathcal{F}_0(M) \subseteq \mathrm{Ann}_R(M)$$

$$\text{where } \mathrm{Ann}_R(M) = \{r \in R | rm = 0 \ \forall \ m \in M\}$$

Proof:

See [13], Theorem 5, pages 60-61.

$\square$

Recent work done by Cornacchia and Greither in [2], shows that the initial Fitting Ideal of the Galois module of the ideal class group of a real abelian number field with prime power conductor is in fact equal to the initial Fitting Ideal of the Galois module of the units modulo the cyclotomic units of the number field. Here the Fitting Ideals are calculated in the group ring $\mathbb{Z}G$, where G is an abelian group, so the group ring $\mathbb{Z}G$ is commutative. Thus, an estimate for the annihilator of the ideal class group as a $\mathbb{Z}G$-module is obtained by calculating the initial Fitting Ideal of the Galois module of the units modulo cyclotomic units. In fact the initial Fitting Ideals give us a relation between these two $\mathbb{Z}G$-modules which are non-isomorphic Galois modules; in fact they are not even isomorphic as abelian groups in general. We shall see later in Chapter 6 how we can generalise these $\mathbb{Z}G$-Fitting Ideals to the case where G is a metacyclic group, so the group ring we are working with is non-commutative. The following example shows that we can sometimes find a better estimate for the annihilator than the initial Fitting Ideal.

Example 1.12 Let us work in the polynomial ring $R = \mathbb{Z}[t]$ and consider the R-module M given by:

$$M = R/(2) \oplus R/(6t)$$

M has generators e_1, e_2 with relations $2e_1 = 0$ and $6te_2 = 0$, so a presentation matrix for M is:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 6t \end{pmatrix}$$

Calculating the Fitting Ideals we have that:

$$\mathcal{F}_0(M) = I_2(A) = (12t) \text{ and } \mathcal{F}_1(M) = I_1(A) = (2, 6t) = (2)$$

Certainly, $12te_1 = 0$ and $12te_2 = 0$ which tells us that

$$\mathcal{F}_0(M) \subsetneqq \mathrm{Ann}_R(M)$$

But, now consider the quotient ideal $(\mathcal{F}_0(M) : \mathcal{F}_1(M))$, where for R-ideals I and J we have $(I : J) = \{r \in R \text{ s.t. } rJ \subseteq I\}$. Here,

$$\mathcal{F}_0(M) \subsetneqq (\mathcal{F}_0(M) : \mathcal{F}_1(M)) = (6t) = \mathrm{Ann}_R(M)$$

So in this example, the quotient ideal lies in the annihilator and it is a better estimate of the annihilator than the initial Fitting Ideal. However, the following example shows us that it is not always the case that $(\mathcal{F}_0(M) : \mathcal{F}_1(M)) \subseteq \mathrm{Ann}_R(M)$.

$\square$

Example 1.13 Consider the non-Dedekind ring, $R = \mathbb{Z}[\sqrt{-3}]$ and let J denote the R-ideal, $J = (2, 1+\sqrt{-3})$. Consider the R-module:

$$M = R/(2) \oplus R/J$$

which has a presentation matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1+\sqrt{-3} \\ 0 & 2 \end{pmatrix}$$

Calculating the Fitting ideals we obtain:

$$\begin{aligned} \mathcal{F}_0(M) &= (2(1+\sqrt{-3}), 4) &&= (2)J = J^2 \\ \mathcal{F}_1(M) &= (2, 1+\sqrt{-3}) &&= J \end{aligned}$$

Then:

$$(\mathcal{F}_0(M) : \mathcal{F}_1(M)) = J \not\subseteq \mathrm{Ann}_R(M)$$

since $1+\sqrt{-3} \in J$ but $1+\sqrt{-3} \notin \mathrm{Ann}_R(M)$. However, note here that

$$(\mathcal{F}_0(M) : \mathcal{F}_1(M))^2 (= J^2) \subseteq \mathrm{Ann}_R(M)$$

and it may well be true in general that a power of the quotient ideal lies in the annihilator.

$\square$

1.4.2 Alexander Ideals

In Knot Theory, to each knot, or oriented link, we can associate a unique family of Fitting Ideals which are known as the Alexander Ideals of the knot. These arise as follows.

Working over the Laurent polynomial ring, $R = \mathbb{Z}[t, t^{-1}]$, the first homology group of the infinite cyclic cover of the complement of a knot, $H_1(X_\infty, \mathbb{Z})$, is an R-module, which is an invariant for the knot. The Alexander Ideals of the knot are then the determinantal ideals of a square presentation matrix for the $R = \mathbb{Z}[t, t^{-1}]$-module, $H_1(X_\infty, \mathbb{Z})$. The Alexander polynomial of the knot is then defined to be the determinant of this square presentation matrix; hence, the initial Alexander Ideal is the principal ideal generated by the Alexander polynomial. There exist examples of different knots with the same Alexander polynomial which are also indistinguishable when considering the knot invariants of Jones, Kauffmann and HOMFLY. However, the higher Alexander Ideals may be different for these particular knots which tells us that the Alexander Ideals can be useful in distinguishing between these knots (for details see [14]), where the other invariants cannot.

Given any R-ideal we ask does this ideal belong to a sequence of Alexander Ideals for some knot? Necessary and sufficient conditions for this to be true are given in [11]. However, given two or more R-ideals it is not yet known whether these ideals belong to the same family of Alexander Ideals for some knot (subject to the conditions for each ideal to belong to a family of Alexander Ideals for some knot).

This leads us to a question in general, namely; for a general ring R, if we are given a sequence of one or more Fitting Ideals from an R-module, then what can we say about the remaining Fitting Ideals in the sequence? For example, suppose we know that two of the Fitting Invariants of a given module are equal then, can we say anything about the remaining Fitting Invariants? We will study this question further in section 2.4.

Chapter 2

Behaviour of Fitting Ideals with Module Theoretic Constructions

In this chapter we consider how Fitting Invariants behave for certain module theoretic constructions. We will consider the Fitting Invariants of exact sequences and direct sums of modules and also consider Fitting Ideals when we work over an inflated ring. Section 2.4 is quite an important section as it considers localisation and shows that when the ring is Noetherian Fitting Ideals are a local phenomena - that is we can calculate the global Invariants from the local Invariants with respect to maximal ideals of the ring we are working in. This will become quite an important tool to use in our later work. We will prove that the sequence of Fitting Ideals is in fact strictly increasing for modules over commutative Noetherian rings (except where the ideals are zero or the whole ring). We end this chapter by considering how we can obtain another family of invariants for a module over a Dedekind ring, in terms of the initial Fitting Invariants of the alternating product of the module.

As in chapter 1, we take R to be a Noetherian commutative ring with a 1, unless stated otherwise.

2.1 Free Modules

Suppose the non-zero module M is a free R-module of rank g with generating set $\underline{e} = \{e_1, ..., e_g\}$. As M is free the only relation for the generating set $\underline{e}$ is:

$$0e_1 + \cdots + 0e_g = 0$$

so a presentation matrix for M is the $1 \times g$ zero matrix, $A = (0 \cdots 0)$. Therefore, since $I_{g-s}(M|\underline{e}) = I_{g-s}(A)$ we have:

$$\mathcal{F}_s(M) = \begin{cases} 0 & \text{if } 0 \leq s < g \\ R & \text{if } s \geq g \end{cases}$$

Hence, for free modules the Fitting Ideals are trivial.

2.2 Exact sequences

Here, we consider the relationship between the Fitting Ideals of modules in exact sequences.

Theorem 2.1 For an exact sequence of finitely generated R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

we have for integers $r, s \geq 0$:

$$\mathcal{F}_r(L)\mathcal{F}_s(N) \subseteq \mathcal{F}_{r+s}(M)$$

Futhermore, if the above sequence splits, i.e. $M = L \oplus N$, then for each integer $t \geq 0$ we can write:

$$\mathcal{F}_t(L \oplus N) = \sum_{r+s=t} \mathcal{F}_r(L)\mathcal{F}_s(N) \tag{2.1}$$

Proof:

See [13], pages 90-93.

□

We can obtain a stronger relationship between initial Fitting Ideals when N is an *elementary* module.

Definition 2.2 We call N an elementary R-module if there exists a square presentation for N, that is there is a positive integer k such that:

$$0 \longrightarrow R^k \longrightarrow R^k \longrightarrow N \longrightarrow 0$$

is a presentation for N.

In this case we get a multiplicative version of Theorem 2.1.

Theorem 2.3 If N is an elementary R-module in the exact sequence of Theorem 2.1 then:

$$\mathcal{F}_0(L)\mathcal{F}_0(N) = \mathcal{F}_0(M)$$

and here $\mathcal{F}_0(N)$ is a principal ideal generated by the determinant of the square presentation matrix.

Proof:

See [13], pages 80-81.

□

2.3 Ring Extension

Let S be another Noetherian commutative ring with a 1 and let $\theta : R \to S$ be a ring homomorphism. We can make S into a right R-module by defining $sr = s\theta(r)$ for all $s \in S, r \in R$. If $_RN$ is a finitely generated left R-module we can tensor over R to obtain a left S-module $S \otimes_R N$ which we call the extension of N over S. Then we have a relationship between the Fitting Ideals of N as an R-module and the Fitting Ideals of the extended module $S \otimes_R N$ as an S-module.

Theorem 2.4 For any integer $t \geq 0$

$$\theta(\mathcal{F}_t^R(N))S = \mathcal{F}_t^S(S \otimes_R N)$$

Proof:

Suppose

$$R^n \xrightarrow{\alpha} R^g \xrightarrow{\pi} N \longrightarrow 0$$

is a presentation for N. Then, since $(S \otimes_R -)$ is a right exact functor the sequence:

$$S \otimes_R R^n \xrightarrow{\alpha'} S \otimes_R R^g \xrightarrow{\pi'} S \otimes_R N \longrightarrow 0$$

is exact. If the map α is represented by a presentation matrix $A = (a_{ij})$ then $\theta(A)(= \theta(a_{ij}))$ is the presentation matrix for the extended module $S \otimes_R N$, since

$\theta(A)$ is a presentation matrix representing the map α'. But, then it is clear that, for each positive integer $v \geq 0$:

$$\theta(I_v(A))S = I_v(\theta(A))$$

and we are done.

□

2.4 Localisation

We now want to consider what happens when we just consider local Fitting Invariants. Can we retrieve the global Fitting information when the local Fitting information is known? If we let $\theta_P : R \longrightarrow R_P$, where P is a maximal ideal of R, be the localisation map then Theorem 2.4 tells us that for any finitely generated R-module, M:

$$\theta_P(\mathcal{F}_t^R(M))R_P = \mathcal{F}_t^{R_P}(M_P) \tag{2.2}$$

where we denote by M_P the extended module, $R_P \otimes_R M$. If we know the local Fitting Ideals (i.e. we know the RHS of equation (2.2)) for all maximal ideals, P, does this determine the global Fitting Ideals, $\mathcal{F}_t^R(M)$? This is an important question because it is sometimes easier to calculate the local Fitting Ideals, for example where R_P is a PID, than the global Fitting Ideals. In fact we can answer yes to the above question if we take R to be a Noetherian ring as the following Theorem shows:

Theorem 2.5 If R is a Noetherian ring then the local Fitting Ideals completely determine the global Fitting Ideals, Indeed, if we denote by θ_P the localisation map, $\theta_P : R \longrightarrow R_P$, for P a maximal ideal of R and M is any finitely generated R-module, then:

$$\mathcal{F}_t^R(M) = \bigcap_P \theta_P^{-1}(\mathcal{F}_t^{R_P}(M_P))$$

Furthermore, for any two finitely generated R-modules M and N then:

$$\mathcal{F}_t^{R_P}(M_P) = \mathcal{F}_t^{R_P}(N_P) \ \forall \ P \Rightarrow \mathcal{F}_t^R(M) = \mathcal{F}_t^R(N)$$

Proof:

We need to prove that if I and J are R-ideals with $I_P = J_P$ (where $I_P = \theta_P(I)R_p$) for all P then this tells us that $I = J$. Since R is Noetherian every R-ideal, I, can be represented as an irredundant primary decomposition:

$$I = \bigcap_{i=1}^{n} Q_i$$

where the Q_i are primary ideals, for $i = 1, ..., n$ (see [21], Theorem 4, page 209). We fix P and choose the Q_i such that $Q_i \bigcap R\backslash P = \varnothing$ for $1 \leq i \leq r$ and $Q_i \cap R\backslash P \neq \varnothing$ for $r+1 \leq i \leq n$. Then

$$I_P = \bigcap_{i=1}^{r} (Q_i)_P$$

since $(Q_i)_P = R_P$ for $r+1 \leq i \leq n$. Let

$$I(P) = \bigcap_{i=1}^{r} Q_i \text{ which tells us that } I_P = (I(P))_P$$

Now, $R\backslash P$ is prime to $I(P)$ since $R\backslash P$ is prime to each Q_i for $i = 1, ..., r$. Thus, under the localisation map $\theta_P : R \longrightarrow R_P$, $I(P)$ is a contracted ideal (see [21], Theorem 15b, page 223). Hence,

$$I(P) = \theta_P^{-1}\theta_P(I(P)) = \theta_P^{-1}(I_P)$$

and so I_P must determine $I(P)$. But, since

$$I = \bigcap_{P} I(P) = \bigcap_{P} \theta_P^{-1}(I_P)$$

we see that I_P must determine I as P runs through the maximal ideals of R. Thus,

$$I_P = J_P \ \forall\ P \Rightarrow I(P) = J(P) \ \forall\ P \Rightarrow I = J$$

Now substitute $I = \mathcal{F}_t^R(M)$. □

One aspect of localisation is that we can say more precisely what the Fitting Length of a localised module is. Before we do this we need a lemma.

Lemma 2.6 Let R be any commutative ring with a 1 and P some maximal ideal of R. If A is an $m \times h$ matrix with entries in the localised ring R_P satisfying $I_s(A) = R_P$, for some positive integer s such that $1 \leq s \leq h$, then A can be brought to the form:

$$A' = \left(\begin{array}{c|c} I_s & 0 \\ \hline 0 & A_s \end{array} \right)$$

where $A_s \in M_{[m-s]\times[h-s]}(R_P)$, by means of elementary row and column operations.

Proof:

See [13], Theorem 12, page 18.

□

We are now in a position to prove a result about Fitting Lengths.

Theorem 2.7 Let R be any commutative ring with a 1 and M some finitely generated R-module. Then if P is some maximal ideal of R we have:

$$\mathcal{L}_{R_P}(M_P) = \text{ minimum number of generators of } M_P \text{ over } R_P$$

Proof:

Let h equal the minimum number of generators of M_P. Then we know that $\mathcal{L}_{R_P}(M_P) \leq h$. Suppose for a contradiction that $\mathcal{L}_{R_P}(M_P) = h - s$, for some positive integer s such that $1 \leq s \leq h$. Then if A is a presentation matrix for M_P with respect to a minimum generating set of size h, we know that:

$$\mathcal{F}_{h-s}^{R_P}(M_P) = I_s(A) = R_P$$

Then Lemma 2.6 tells us that A can be brought to the form:

$$A' = \left(\begin{array}{c|c} I_s & 0 \\ \hline 0 & A_s \end{array}\right)$$

Hence, the first s generators in this minimum generating set are redundant, so we have found a generating set of size $h - s < h$ for M_P, a contradiction.

□

One consequence of Theorem 2.7 is that we can refine the increasing sequence of Fitting Ideals we obtained in section 1.3 to obtain a strictly increasing sequence of Fitting Ideals. We obtain:

Theorem 2.8 Suppose R is a Noetherian ring, M a finitely generated R-module and P runs through the maximal ideals of R. Then there is a maximal ideal Q of R such that:

$$\mathcal{L}_R(M) = \mathcal{L}_{R_Q}(M_Q) = h$$

for some positive integer h. Furthermore, if $\mathcal{F}_t^R(M) \neq (0)$ then we obtain a strictly increasing sequence of Fitting Ideals:

$$\mathcal{F}_{t-1}^R(M) \subsetneq \mathcal{F}_t^R(M) \subsetneq \mathcal{F}_{t+1}^R(M) \subsetneq \cdots \subsetneq \mathcal{F}_{h-1}^R(M) \subsetneq \mathcal{F}_h^R(M) = R$$

Proof:

We know from Theorem 2.5 that

$$\mathcal{F}_t^R(M) = \bigcap_P \theta_P^{-1}(\mathcal{F}_t^{R_P}(M_P))$$

so we obtain:

$$\mathcal{L}_R(M) = \max_P\{\mathcal{L}_{R_P}(M_P)\}$$

Thus, there exists a maximal ideal Q of R such that:

$$\mathcal{L}_R(M) = \mathcal{L}_{R_Q}(M_Q)$$

We know from Theorem 2.7 that we can find some minimum generating set for M_Q of size h such that $\mathcal{L}_{R_Q}(M_Q) = h$. We also know from equation (1.4) that:

$$\mathcal{F}_s^{R_Q}(M_Q) \subseteq \mathcal{F}_{h-1}^{R_Q}(M_Q)\mathcal{F}_{s+1}^{R_Q}(M_Q) \subseteq \mathcal{F}_{s+1}^{R_Q}(M_Q)$$

for $0 \leq s \leq h-1$. But since $\mathcal{L}_{R_Q}(M_Q) = h$ we must have:

$$\mathcal{F}_{h-1}^{R_Q}(M_Q) \subsetneq \mathcal{F}_h^{R_Q}(M_Q) = R_Q$$

and therefore:

$$\mathcal{F}_{h-1}^{R_Q}(M_Q) \subseteq QR_Q$$

where QR_Q is the maximal ideal. Now suppose that we have:

$$\mathcal{F}_{h-1}^{R_Q}(M_Q)\mathcal{F}_{s+1}^{R_Q}(M_Q) = \mathcal{F}_{s+1}^{R_Q}(M_Q)$$

Then Nakayama's Lemma tells us that:

$$\mathcal{F}_{s+1}^{R_Q}(M_Q) = (0)$$

which is a contradiction for $s \geq t-1$. Thus, we must have:

$$\mathcal{F}_s^{R_Q}(M_Q) \subsetneq \mathcal{F}_{s+1}^{R_Q}(M_Q)$$

for $t-1 \leq s \leq h-1$. Hence, we obtain a strictly increasing sequence of ideals in the local ring R_Q:

$$\mathcal{F}_{t-1}^{R_Q}(M_Q) \subsetneq \mathcal{F}_t^{R_Q}(M_Q) \subsetneq \cdots \subsetneq \mathcal{F}_{h-1}^{R_Q}(M_Q) \subsetneq \mathcal{F}_h^{R_Q}(M) = R_Q$$

Now, we know that the local Fitting Ideals determine the global Fitting Ideals so we must have a strictly increasing sequence of ideals in the global ring R:

$$\mathcal{F}_{t-1}^{R}(M) \subsetneq \mathcal{F}_t^{R}(M) \subsetneq \cdots \subsetneq \mathcal{F}_{h-1}^{R}(M) \subsetneq \mathcal{F}_h^{R}(M) = R$$

as required.

□

We now return to the question we posed at the end of section 1.4.2 - that is, if any two Fitting Ideals in a sequence are equal then what can we say about the remaining Fitting Ideals in a sequence? Well, for Noetherian rings the equal Fitting Ideals must be trivial as the following corollary shows:

Corollary 2.9 Suppose R is a Noetherian ring and M a finitely generated R-module with g generators. Then if we have:

$$(0) \neq \mathcal{F}_s^R(M) = \mathcal{F}_{s+1}^R(M)$$

for some s such that $0 \leq s \leq g-1$, then we must have:

$$\mathcal{F}_s^R(M) = R$$

Proof:

Theorem 2.8 tells us that $\mathcal{L}_R(M) \leq s$ and hence $\mathcal{F}_s^R(M) = R$.

□

An important theme throughout this work will be that of localising problems in order to simplify them. For example, by localisation we can express the annihilator of a projective R-module quite succinctly, as the following theorem shows:

Theorem 2.10 Suppose R is a Noetherian ring and N is a projective R-module. Then:

$$\mathrm{Ann}_R(N) = \mathcal{F}_0^R(N)$$

Proof:

If we localise at any maximal ideal P of R then we know that, since N is projective over R, N_P is a free R_P-module. Thus, either $\mathrm{Ann}_{R_P}(N_P) = (0)$ or $\mathrm{Ann}_{R_P}(N_P) = R_P$. But, then Theorem 1.11 tells us that:

$$\begin{aligned} \mathrm{Ann}_{R_P}(N_P) = (0) &\Rightarrow \mathcal{F}_0^{R_P}(N_P) = (0) = \mathrm{Ann}_{R_P}(N_P) \\ \mathrm{Ann}_{R_P}(N_P) = R_P &\Rightarrow \mathcal{F}_0^{R_P}(N_P) = R_P = \mathrm{Ann}_{R_P}(N_P) \end{aligned}$$

Thus,

$$\mathrm{Ann}_{R_P}(N_P) = \mathcal{F}_0^{R_P}(N_P) \ \forall\ P \Rightarrow \mathrm{Ann}_R(N) = \mathcal{F}_0^R(N)$$

□

2.5 Ring Inflation

Suppose J is an ideal of R and we know the Fitting Ideals of a finitely generated R/J-module, N. Is it possible to determine the Fitting ideals of the R-module, N? In other words, given Fitting Invariants in the ring R/J can we find Fitting Ideals in the inflated ring, R? Before we answer these questions we first need a lemma.

Lemma 2.11 If M is a finitely generated R-module, $t \geq 0$ a positive integer and π_J is the natural map $\pi_J : R \to R/J$ then:

$$\pi_J(\mathcal{F}_t^R(M))R/J = \mathcal{F}_t^{R/J}(M/JM)$$

Proof:

Since

$$R/J \otimes_R M \cong M/JM$$

Theorem 2.4 tells us that:

$$\pi_J(\mathcal{F}_t^R(M))R/J = \mathcal{F}_t^{R/J}(R/J \otimes_R M) = \mathcal{F}_t^{R/J}(M/JM)$$

as required.

□

Theorem 2.12 If N is an R/J-module and $t \geq 0$ a positive integer then, considering N as an R-module in the obvious way, we have:

$$\pi_J(\mathcal{F}_t^R(N))R/J = \mathcal{F}_t^{R/J}(N) \tag{2.3}$$

Consequently,

$$\mathcal{F}_t^R(N) \subseteq \pi_J^{-1}(\mathcal{F}_t^{R/J}(N)) \tag{2.4}$$

But equality will not necessarily hold in (2.4).

Proof:

Lemma 2.11 tells us that $\pi_J(\mathcal{F}_t^R(N))R/J = \mathcal{F}_t^{R/J}(N/JN)$ and since $N = N/JN$ equation (2.3) follows.

We know that $\mathcal{F}_t^R(N) \subseteq \pi_J^{-1}(\mathcal{F}_t^{R/J}(N))$ and that there exists a one-one correspondence between ideals of R/J and ideals of R containing J, so equality holds in (2.4) if and only if $\mathcal{F}_t^R(N) \supseteq J$. This is not always the case as we can see in the following counter-example.

Example 2.13 Suppose $R = \mathbb{Z}[t]$, $J = (t)$ and $\overline{B}$ is a presentation matrix for N as an R/J-module where:

$$\overline{B} = \begin{pmatrix} \overline{2} & 0 \\ 0 & \overline{2} \end{pmatrix}$$

Then B is a presentation matrix for N as an R-module where:

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ t & 0 \\ 0 & t \end{pmatrix}$$

Then

$$\mathcal{F}_0^R(N) = (4, 2t, t^2) \not\supseteq (t) = J$$

and therefore,

$$\mathcal{F}_0^R(N) \neq \pi_J^{-1}(\mathcal{F}_0^{R/J}(N))$$

and so equality does not hold for equation (2.4) in this case.

□

Suppose now that we are given the Fitting Ideals of M as an R-module. Consider the R/J-module, M/JM. We know from Lemma 2.11 how to calculate the Fitting Ideals of M/JM as an R/J-module. The following theorem tells us how we can go a step further and calculate the Fitting Ideals of M/JM in the inflated ring, R.

Theorem 2.14 Let ${}_RM$ be a finitely generated left R-module with g generators. Then, for any integer r, such that $0 \leq r \leq g$, we have that:

$$\mathcal{F}^R_{g-r}(M/JM) = \sum_{t=0}^{r} J^{r-t}\mathcal{F}^R_{g-t}(M)$$

Proof:

Let $B = (b_{ij})$ be an $m \times g$ presentation matrix for the left R-module, M with respect to the generating set, $\underline{e} = \{e_j\}$, for $j = 1, \ldots, g$ and for some $b_{i,j} \in R$. If C is the presentation matrix for M/JM as an R-module then we need to prove that:

$$I_r(C) = \sum_{t=0}^{r} J^{r-t}I_t(B)$$

From section 2.4 we know that $\overline{B} = (\overline{b}_{ij})$ is a presentation matrix for M/JM as an R/J-module. Note that $\overline{\underline{e}} = \{\overline{e_j}\}$ is a generating set for M/JM as an R-module. Suppose $J = (x_1, ..., x_s)_R$ for some $x_k \in R$, for $k = 1, ..., s$. If we let D denote the $sg \times g$ matrix

$$D = \begin{pmatrix} I_g x_1 \\ \vdots \\ I_g x_s \end{pmatrix}$$

and C denote the $[m + sg] \times g$ matrix

$$C = \begin{pmatrix} B \\ D \end{pmatrix}$$

then we claim that C is indeed the presentation matrix for M/JM as an R-module. Note that every row of C is a row relation for M/JM with respect to the generating set $\overline{\underline{e}}$, since

$$\sum_{j=1}^{g} \overline{b}_{ij}\overline{e_j} = \overline{0}$$

which tells us that:

$$\sum_{j=1}^{g} b_{ij}\overline{e_j} = \overline{0}$$

and we also have $x_k\overline{e_j} = 0$ for $j = 1, ..., g$. We now need to show every row relation is an R-linear combination of the rows of C. Suppose we have a relation

$$\sum_{j=1}^{g} r_j\overline{e_j} = 0 \text{ for some } r_j \in R$$

then we have:

$$\sum_{j=1}^{g} \overline{r}_j\overline{e_j} = 0$$

Hence, the row vector $(\overline{r}_1, ..., \overline{r}_g)$ is an R/J-linear combination of the rows of $\overline{B}$, say

$$(\overline{r}_1, ..., \overline{r}_g) = \sum_{i=1}^{m} \overline{\lambda}_i(\overline{b}_{i1}, ..., \overline{b}_{ig})$$

for some $\overline{\lambda}_i \in R/J$. We then have

$$\overline{r}_j = \sum_{i=1}^{m} \overline{\lambda}_i\overline{b}_{ij} \Rightarrow r_j = \sum_{i=1}^{m} \lambda_i b_{ij} + y_j$$

for some $y_j \in J$. Thus, we can write

$$r_j = \sum_{i=1}^{m} \lambda_i b_{ij} + \sum_{k=1}^{s} \mu_{kj} x_k$$

for some $\mu_{kj} \in R$. Hence,

$$(r_1, ..., r_g) = \sum_{i=1}^{m} \lambda_i(b_{i1}, ..., b_{ig}) + \sum_{j=1}^{g}\sum_{k=1}^{s} \mu_{kj}(0, ..., 0, x_k, 0, ..., 0)_j$$

where $(0, ..., 0, x_k, 0, ..., 0)_j$ is a row vector consisting of x_k in the j^{th} column and zeroes elsewhere, and we have written $(r_1, ..., r_g)$ as an R-linear combination of the rows of C. Thus, C is indeed a presentation matrix for M/JM as an R-module. Now, we set out to prove the formula.

Consider the case $\underline{r = 0}$: then $I_0(C) = R = I_0(B)$

Consider the case $\underline{r = 1}$: then $I_1(C) = (b_{11}, \ldots, b_{mg}, x_1, \ldots, x_s) = I_1(B) + J$, so we can see the formula holds for $r = 0, 1$.

Consider the case $\underline{r = 2}$: $I_2(C)$ is generated by all the 2×2 minors of C. Such a minor must be either:

1. A 2×2 minor of B, i.e. it belongs to $I_2(B)$; or

2. A 2×2 minor of D, i.e. it belongs to J^2; or

3. The determinant of a 2×2 matrix, E_2 say, which is made up as a 1×2 submatrix of B and a 1×2 submatrix of D.

For det $E_2 \neq 0$, E_2 must be of the form

$$E_2 = \begin{pmatrix} b_{iu} & b_{iu'} \\ 0 & x_j \end{pmatrix}$$

for some i, j, u, u' with $u \neq u'$. So, det $E_2 = x_j b_{iu} \in JI_1(B)$ and we have

$$I_2(C) \subseteq J^2 + JI_1(B) + I_2(B)$$

For the reverse inclusion it is clear $J^2 \subseteq I_2(C)$ and $I_2(B) \subseteq I_2(C)$, so it remains to check whether $JI_1(B) \subseteq I_2(C)$. Well, each generator of $JI_1(B)$ must be of the form $x_l b_{iw}$ for some l, i, w. But, we can always find a 2×2 submatrix of C of the form:

$$E_2' = \begin{pmatrix} b_{iw} & b_{iw'} \\ 0 & x_l \end{pmatrix}$$

for some w' such that $w \neq w'$. Hence, taking the determinant of this submatrix we see that det $E_2' = x_l b_{iw} \in I_2(C)$ which tells us that $JI_1(B) \subseteq I_2(C)$ and thus equality holds and the formula is true for $r = 2$.

Consider the case $\underline{2 \leq r \leq g}$: $I_r(C)$ is generated by all the $r \times r$ minors of C. Such a minor must be either:

1. An $r \times r$ minor of B, i.e. it belongs to $I_r(B)$; or

2. An $r \times r$ minor of D, i.e. it belongs to J^r; or

3. The determinant of a $r \times r$ matrix, E_r say, which is made up as a $t \times r$ submatrix of B, which we can call B_t, and a $[r-t] \times r$ submatrix of D, which we can call D_{r-t}, with $1 \leq t \leq r-1$.

Thus, we write:

$$E_r = \begin{pmatrix} B_t \\ D_{r-t} \end{pmatrix}$$

and using the Laplace Expansion formula for determinants (see section 1.3), we see that det E_r can be written as a sum of $t \times t$ minors of B_t times $[r-t] \times [r-t]$ minors of D_{r-t}. Hence, det $E_r \in I_t(B)J^{r-t}$ and thus $I_r(C) \subseteq \sum_{t=0}^{r} J^{r-t} I_t(B)$.

For the reverse inclusion it is clear $J^r \subseteq I_r(C)$ and $I_r(B) \subseteq I_r(C)$, so it remains to check whether $J^{r-t}I_t(B) \subseteq I_r(C)$, for $1 \leq t \leq r-1$. Well, each generator of $J^{r-t}I_t(B)$ must be the determinant of an $[r-t] \times [r-t]$ submatrix of D, which we call D'_{r-t}, times the determinant of a $t \times t$ submatrix of B, which we call B'_t. So. such a generator must be of the form det $E'_r =$ det B'_tdet D'_{r-t} where:

$$E'_r = \begin{pmatrix} B'_t & B''_t \\ 0 & D'_{r-t} \end{pmatrix}$$

and where B''_t is some $t \times [r-t]$ submatrix of B. But, E'_r is an $r \times r$ submatrix of C (up to permutation of columns), so we have det $E'_r \in I_r(C)$ and hence $J^{r-t}I_t(B) \subseteq I_r(C)$. Thus equality holds and our formula holds for $2 \leq r \leq g$.

$\square$

2.6 Alternating Products

For each finitely generated R-module M, we have associated to it a unique family of Fitting Invariants, $\mathcal{F}_t(M)$, for $t \geq 0$. In this section we ask whether we can obtain a different family of Invariants which is better or easier to use. For example, we know that the initial Fitting Ideal $\mathcal{F}_0(M)$, is quite useful - it annihilates the underlying module and for elementary modules it is a principal ideal generated by the determinant of the square presentation matrix. Can we obtain Invariants just in terms of initial Fitting Ideals? One avenue of investigation is to consider the alternating product of a module. We will show that for R a Dedekind domain we can express the annihilator and higher Fitting Ideals of a torsion R-module in terms of the initial Fitting Ideal of the r-th alternating product of the module. Suppose M is a torsion R-module and denote by $\bigwedge^r M$ the r-th alternating product of M, for some $r \in \mathbb{N} \cup \{0\}$. Suppose M is generated by n generators with m relations so we have a map:

$$\gamma : R^m \longrightarrow R^n$$

which is represented by some presentation matrix $A \in M_{m \times n}(R)$. Then the map between r-th exterior powers is given by:

$$\bigwedge^r \gamma : \bigwedge^r R^m \longrightarrow \bigwedge^r R^n$$

which is represented by the r-th exterior power of the matrix A, which we denote by $A^{(r)}$. In fact $A^{(r)}$ is the matrix whose entries are the $r \times r$ minors of A (see [12], page 739). Thus, the entries in $A^{(r)}$ generate the $(n-r)$-th Fitting Ideal, $\mathcal{F}_{n-r}(M)$. So we can see there is some relationship between determinantal ideals of a presentation matrix and determinantal ideals of the exterior powers of the matrix.

We know from section 2.4 that we only need to calculate local Fitting Invariants so we will assume that R is a local Dedekind ring, hence PID. Thus, we can represent each finitely generated R-torsion module as a sum of cyclic modules, where the representation is arranged so that each torsion coefficient divides the next. Let P be the unique maximal ideal of R which is generated by some prime element π. Then we can write each torsion R-module, M, as a direct sum of cyclic modules:

$$M = \bigoplus_{i=1}^{n} R/P^{r_i} \text{ where } r_i \text{ are positive integers s.t. } r_1 \leq r_2 \leq \cdots \leq r_n$$

2.6.1 Initial Fitting Invariants

We firstly consider what we can say about the initial Fitting Ideals of the exterior powers of the module. As the following theorem shows we can obtain an increasing sequence of Invariants in terms of the initial Fitting Ideals of the alternating product:

Theorem 2.15 Let R be a local Dedekind domain and M a finitely generated torsion R-module with n generators then:

$$\mathcal{F}_0(\textstyle\bigwedge^{\frac{n}{2}} M) \subseteq \mathcal{F}_0(\bigwedge^{\frac{n}{2}+1} M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{\frac{n}{2}+l} M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{n-1} M) \subseteq \mathcal{F}_0(\bigwedge^{n} M)$$

for n even and l s.t. $0 \leq l \leq \frac{n}{2}$, and

$$\mathcal{F}_0(\textstyle\bigwedge^{\frac{n-1}{2}} M) \subseteq \mathcal{F}_0(\bigwedge^{\frac{n+1}{2}} M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{\frac{n-1}{2}+l} M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{n-1} M) \subseteq \mathcal{F}_0(\bigwedge^{n} M)$$

for n odd and l s.t. $0 \leq l \leq \frac{n+1}{2}$.

Proof:

Since

$$\bigwedge^2 (R/P^{r_i} \oplus R/P^{r_j}) = (R/P^{r_i} \otimes_R R/P^{r_j}) = R/P^{\min(r_i,r_j)}$$

for $1 \leq i, j \leq n$, it can be shown that:

$$\bigwedge^l M = \bigoplus_{i=1}^{n-l+1} \binom{n-i}{l-1} R/P^{r_i} \text{ for } 1 \leq l \leq n$$

where

$$\binom{m}{s} = \frac{m!}{(m-s)s!}$$

is the binomial coefficient for $m, s \in \mathbb{N} \cup \{0\}$. Thus, calculating initial Fitting Ideals we obtain:

$$\mathcal{F}_0(\bigwedge^l M) = P^{\sum_{i=1}^{n-l+1} \binom{n-i}{l-1} r_i} \text{ and } \mathcal{F}_0(\bigwedge^n M) = P^{r_1}$$

Note here that if we know $\mathcal{F}_0(\bigwedge^l M)$, for $1 \leq l \leq n$, this sequence of Fitting Invariants enables us to determine the r_i, i.e. we can determine the torsion coefficients of M (see section 3.2 for more details).

Suppose now that n is even so we can write $n = 2m$ for some $m \in \mathbb{N}$. We will show that $\mathcal{F}_0(\bigwedge^{m+l} M) \subseteq \mathcal{F}_0(\bigwedge^{m+l+1} M)$ for $0 \leq l \leq m-1$. Well,

$$\begin{aligned} \mathcal{F}_0(\bigwedge^{m+l} M) &= P^{\sum_{i=1}^{m-l+1} \binom{2m-i}{m+l-1} r_i} \\ \mathcal{F}_0(\bigwedge^{m+l+1} M) &= P^{\sum_{i=1}^{m-l} \binom{2m-i}{m+l} r_i} \end{aligned}$$

and since

$$\binom{2m-i}{m+l-1} \geq \binom{2m-i}{m+l}$$

for $i \geq 1 - 2l$, which holds since $i \geq 1$, we obtain $\mathcal{F}_0(\bigwedge^{m+l} M) \subseteq \mathcal{F}_0(\bigwedge^{m+l+1} M)$ for $0 \leq l \leq m-1$, as required. Thus, we obtain the sequence:

$$\mathcal{F}_0(\bigwedge^{\frac{n}{2}} M) \subseteq \mathcal{F}_0(\bigwedge^{\frac{n}{2}+1} M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{\frac{n}{2}+l} M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{n-1} M) \subseteq \mathcal{F}_0(\bigwedge^{n} M)$$

for n even.

Suppose now that n is odd so we can write $n = 2m+1$ for some $m \in \mathbb{N}$. We will show that $\mathcal{F}_0(\bigwedge^{m+l} M) \subseteq \mathcal{F}_0(\bigwedge^{m+l+1} M)$ for $0 \leq l \leq m$. Well,

$$\begin{aligned} \mathcal{F}_0(\bigwedge^{m+l} M) &= P^{\sum_{i=1}^{m-l+2} \binom{2m+1-i}{m+l-1} r_i} \\ \mathcal{F}_0(\bigwedge^{m+l+1} M) &= P^{\sum_{i=1}^{m-l+1} \binom{2m+1-i}{m+l} r_i} \end{aligned}$$

and since

$$\binom{2m+1-i}{m+l-1} \geq \binom{2m+1-i}{m+l}$$

for $i \geq 2-2l$ we obtain $\mathcal{F}_0(\bigwedge^{m+l}M) \subseteq \mathcal{F}_0(\bigwedge^{m+l+1}M)$ for $1 \leq l \leq m$. So it remains to consider the case $l=0$. Now, for $l=0$, we have:

$$\begin{aligned} \mathcal{F}_0(\bigwedge^{m}M) = \; & P^{\sum_{i=1}^{m+2}\binom{2m+1-i}{m-1}r_i} \\ \mathcal{F}_0(\bigwedge^{m+1}M) = \; & P^{\sum_{i=1}^{m+1}\binom{2m+1-i}{m}r_i} \end{aligned}$$

We claim that

$$\sum_{i=1}^{m+2}\binom{2m+1-i}{m-1} - \sum_{i=1}^{m+1}\binom{2m+1-i}{m} = 0$$

Suppose our claim is true. Then, since $r_1 \leq \cdots \leq r_{m+2}$ and $\binom{2m+1-i}{m-1} \geq \binom{2m+1-i}{m}$ for $i \geq 2$, we must have:

$$\min\left\{\sum_{i=1}^{m+2}\binom{2m+1-i}{m-1}r_i - \sum_{i=1}^{m+1}\binom{2m+1-i}{m}r_i\right\}$$

occurs when $r_1 = r_2 = \cdots = r_{m+1} = r_{m+2} = r$, for some r. Thus, the minimum value is:

$$\left\{\sum_{i=1}^{m+2}\binom{2m+1-i}{m-1} - \sum_{i=1}^{m+1}\binom{2m+1-i}{m}\right\}r = 0$$

since we are assuming the claim is true. Then this last result tells us that:

$$\mathcal{F}_0(\bigwedge^{m}M) \subseteq \mathcal{F}_0(\bigwedge^{m+1}M)$$

To prove the claim let $r = m-i+1$ so we want to calculate:

$$1 + \sum_{r=0}^{m}\binom{m+r}{r+1} - \sum_{r=0}^{m}\binom{m+r}{r}$$

Now,

$$\sum_{r=0}^{m}\binom{m+r+1}{r+1} - \sum_{r=0}^{m}\binom{m+r}{r+1} = \sum_{r=0}^{m}\binom{m+r}{r} = \binom{2m+1}{m} = \binom{2m+1}{m+1}$$

and thus

$$\sum_{r=0}^{m}\binom{m+r}{r+1} = \sum_{r=0}^{m}\binom{m+r+1}{r+1} - \binom{2m+1}{m+1} = \sum_{r=1}^{m}\binom{m+r}{r}$$

which tells us that

$$\sum_{r=0}^{m}\binom{m+r}{r+1} - \sum_{r=0}^{m}\binom{m+r}{r} = \sum_{r=1}^{m}\binom{m+r}{r} - \sum_{r=0}^{m}\binom{m+r}{r} = -\binom{m}{0} = -1$$

as required. Thus we obtain the sequence:

$$\mathcal{F}_0(\bigwedge^{\frac{n-1}{2}}M) \subseteq \mathcal{F}_0(\bigwedge^{\frac{n+1}{2}}M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{\frac{n-1}{2}+l}M) \subseteq \cdots \subseteq \mathcal{F}_0(\bigwedge^{n-1}M) \subseteq \mathcal{F}_0(\bigwedge^{n}M)$$

for n odd.

□

2.6.2 Annihilators

We now show how we can express the annihilator of a module in terms of the initial Fitting Ideals of the alternating product of the module.

Theorem 2.16 Let R be a local Dedekind domain and M a finitely generated torsion R-module with n generators then:

$$\mathrm{Ann}_R(M) = \prod_{l=1}^{n}[\mathcal{F}_0(\bigwedge^l M)]^{(-1)^{l+1}}$$

Proof:

From the proof of Theorem 2.15 we know that $\mathrm{Ann}_R(M) = P^{r_n}$, since we have $r_1 \leq r_2 \leq \cdots \leq r_n$, and that we can determine the r_i from $\mathcal{F}_0(\bigwedge^l M)$ for $1 \leq l \leq n$. In fact since we know that:

$$\mathcal{F}_0(\bigwedge^l M) = P^{\sum_{i=1}^{n-l+1}\binom{n-i}{l-1}r_i} \text{ for } 1 \leq l \leq n$$

we obtain:

$$\prod_{l=1}^{n}[\mathcal{F}_0(\bigwedge^l M)]^{(-1)^{l+1}} = P^{\gamma}$$

where

$$\gamma = \sum_{l=1}^{n-1}\sum_{i=1}^{n-l+1}(-1)^{l+1}\binom{n-i}{l-1}r_i + (-1)^{n+1}r_1$$

Now, the coefficient of r_1 in γ is obtained from all exterior l-th powers where l satisfies $1 \leq l \leq n$. So, the required coefficient is:

$$\sum_{l=1}^{n-1}(-1)^{l+1}\binom{n-1}{l-1} + (-1)^{n+1}$$

If we now consider

$$\sum_{l=1}^{n-1}(-1)^{l+1}\binom{n-1}{l-1}$$

and let $r = l-1$ we get:

$$\sum_{r=0}^{n-2}(-1)^r\binom{n-1}{r} = \sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r} - (-1)^{n-1} = 0 + (-1)^n = (-1)^n$$

Hence, the coefficient of r_1 in γ is $(-1)^n + (-1)^{n+1} = 0$. Next, the coefficient of r_i in γ, for $1 < i < n$ is obtained from those exterior l-th powers where $1 \leq l \leq n-i+1$.

So, the required coefficient is:

$$\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}=0$$

Lastly, the coefficient of r_n in γ only comes from the 1-st exterior power and the coefficient is simply 1. Thus we have shown that $\gamma = r_n$ as required.

□

2.6.3 Higher Fitting Invariants

In this subsection we deduce the higher Fitting Invariants from the initial Fitting Ideals of the alternating product of a module. Before we state and prove a theorem about the higher Fitting Invariants we need a lemma concerning the sums of binomial coefficients.

Lemma 2.17 For l, k, s non-negative integers:

$$\sum_{s=0}^{k}\binom{l+k}{s+l}\binom{s+l-1}{l-1}(-1)^s=1$$

Proof:

We have that for positive integers, t, w:

$$(1+y)^{-t} = \sum_{w=0}^{\infty}\binom{t-1+w}{t-1}y^w(-1)^w \tag{2.5}$$

$$(1+y)^{t} = \sum_{w=0}^{t}\binom{t}{w}y^w \tag{2.6}$$

In equation (2.5) put $t = l, s = w$ and $y = x$ to obtain:

$$(1+x)^{-l}=\sum_{s=0}^{k}\binom{s+l-1}{l-1}x^s(-1)^s+\sum_{s=k+1}^{\infty}\binom{s+l-1}{l-1}x^s(-1)^s$$

and in equation (2.6) put $t = l + k$ and $y = \frac{1}{x}$ to obtain:

$$\left(1+\frac{1}{x}\right)^{l+k}=\sum_{w=0}^{l-1}\binom{l+k}{w}x^{-w}+\sum_{w=l}^{l+k}\binom{l+k}{w}x^{-w}=\sum_{w=0}^{l-1}\binom{l+k}{w}x^{-w}+\sum_{s=0}^{k}\binom{l+k}{s+l}x^{-(s+l)}$$

when we substitute $w = s + l$ in the second summation. Now,

$$\begin{aligned}\left(1+\frac{1}{x}\right)^{l+k}(1+x)^{-l} &= \left[\sum_{s=0}^{k}\binom{l+k}{s+l}x^{-(s+l)}+\sum_{w=0}^{l-1}\binom{l+k}{w}x^{-w}\right]\\ &\times\left[\sum_{s=0}^{k}\binom{s+l-1}{l-1}x^s(-1)^s+\sum_{s=k+1}^{\infty}\binom{s+l-1}{l-1}x^s(-1)^s\right]\end{aligned}$$

Hence the coefficient of x^{-l} in the RHS of the above equation is:

$$\sum_{s=0}^{k} \binom{l+k}{s+l} \binom{s+l-1}{l-1} (-1)^s$$

However, the coefficient of x^{-l} in

$$\left(1+\frac{1}{x}\right)^{l+k} (1+x)^{-l} = x^{-(l+k)}(1+x)^k$$

is equal to 1, and we thus obtain:

$$\sum_{s=0}^{k} \binom{l+k}{s+l} \binom{s+l-1}{l-1} (-1)^s = 1$$

□

We now have the following theorem which shows the relationship between higher Fitting Ideals and the initial Fitting Ideals of the alternating product of a module.

Theorem 2.18 Let R be a local Dedekind domain and M a finitely generated torsion R-module with n generators then, for $0 \leq l \leq n$:

$$\mathcal{F}_l(M) = \prod_{r=l+1}^{n} [\mathcal{F}_0(\bigwedge^r M)]^{\beta_r}$$

where

$$\beta_r = \binom{r-2}{l-1}(-1)^{r-l+1}$$

.

Proof:

We know that

$$\mathcal{F}_0(\bigwedge^r M) = P^{\sum_{i=1}^{n-r+1}\binom{n-i}{r-1} r_i}$$

Hence,

$$\prod_{r=l+1}^{n} [\mathcal{F}_0(\bigwedge^r M)]^{\beta_r} = P^{\gamma}$$

where

$$\gamma = \sum_{r=l+1}^{n} \left[\sum_{i=1}^{n-r+1} \binom{n-i}{r-1} r_i \right] \binom{r-2}{l-1} (-1)^{r-l+1}$$

Now, the coefficient of r_i, for $i = 1, ..., n-l$, in γ is:

$$\sum_{r=l+1}^{n-i+1} \binom{n-i}{r-1} \binom{r-2}{l-1} (-1)^{r-l+1} = 1$$

(by substituting $s = r - l - 1$ and $k = n - i - l$ in Lemma 2.17). Thus,

$$\gamma = \sum_{i=1}^{n-l} r_i \Rightarrow \prod_{r=l+1}^{n} [\mathcal{F}_0(\bigwedge^r M)]^{\beta_r} = P^{\sum_{i=1}^{n-l} r_i} = \mathcal{F}_l(M)$$

as required.

□

2.6.4 Elementary Modules

To end this section we briefly study elementary modules and see whether we can say anything further about these particular modules and their relationship with the alternating product. Suppose that E is an elementary R-module, so there exists $n \in \mathbb{N}$ such that the sequence:

$$0 \longrightarrow R^n \xrightarrow{\alpha} R^n \xrightarrow{\delta} E \longrightarrow 0$$

is a presentation for E. We now ask whether the sequence:

$$0 \longrightarrow \bigwedge^r R^n \xrightarrow{\bigwedge^r \alpha} \bigwedge^r R^n \xrightarrow{\bigwedge^r \delta} \bigwedge^r E \longrightarrow 0$$

is exact? The following counter-example shows that this is not always the case.

Example 2.19 Suppose E is the R-module: $E = R/P \oplus R/P^2$. Then a presentation for E is:

$$0 \longrightarrow R^2 \xrightarrow{\alpha} R^2 \xrightarrow{\delta} R/P \oplus R/P^2 \longrightarrow 0$$

where α is given by the presentation matrix

$$A = \begin{pmatrix} \pi & 0 \\ 0 & \pi^2 \end{pmatrix}$$

and where $\pi R = P$. Now $\bigwedge^2 E = R/P$ so we get a sequence:

$$0 \longrightarrow R \xrightarrow{\bigwedge^2 \alpha} R \xrightarrow{\bigwedge^2 \delta} R/P \longrightarrow 0$$

Now if S is any commutative ring with a 1 and the map $\gamma : S^n \to S^n$ is represented by the matrix B, then we know the map $\bigwedge^r \gamma : \bigwedge^r S^n \to \bigwedge^r S^n$ is given by the

r-th exterior power of B, namely $B^{(r)}$. Thus, returning to our example, $A^{(2)} = (\pi^3)$ represents $\bigwedge^2 \alpha$. Hence:

$$\mathrm{Im} \bigwedge^2 \alpha = P^3 \neq P = \ker \bigwedge^2 \delta$$

and the sequence is not exact.

□

However, we can still obtain an exact sequence after taking the alternating product by defining a new module. Suppose N is any finitely generated S-module (not necessarily elementary) with presentation:

$$0 \longrightarrow S^n \xrightarrow{\alpha} S^g \xrightarrow{\delta} N \longrightarrow 0$$

where α is given by the presentation matrix A. If we now let $A^{(r)}(N) = \operatorname{coker} \bigwedge^r \alpha$, then, for $r \leq g$:

$$0 \longrightarrow \bigwedge^r S^n \xrightarrow{\bigwedge^r \alpha} \bigwedge^r S^g \xrightarrow{\bigwedge^r \delta} A^{(r)}(N) \longrightarrow 0$$

is an exact sequence of S-modules. However, $A^{(r)}(N)$ is not an invariant of N as it depends on the number of generators used in the presentation. Consider the following example:

Example 2.20 Let N be an S-module with presentation:

$$0 \longrightarrow S^2 \xrightarrow{\alpha} S^2 \xrightarrow{\delta} N \longrightarrow 0$$

So, we obtain:

$$0 \longrightarrow S \xrightarrow{\bigwedge^2 \alpha} S \xrightarrow{\bigwedge^2 \delta} A^{(2)}(N) \longrightarrow 0$$

and $\bigwedge^2 \alpha$ is given by the presentation matrix $A^{(2)} = (\det A)$. But of course we can obtain another presentation for N by just adding a redundant generator to get:

$$0 \longrightarrow S^3 \xrightarrow{\beta} S^3 \xrightarrow{\delta} N \longrightarrow 0$$

where β is given by the presentation matrix:

$$B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Then we obtain:

$$B^{(2)} = \begin{pmatrix} \det A & 0 \\ 0 & A \end{pmatrix}$$

and calculating Fitting Invariants we see that:

$$\mathcal{F}_0(B^{(2)}(N)) = (\det A)^2 \neq (\det A) = \mathcal{F}_0(A^{(2)}(N)) \Rightarrow B^{(2)}(N) \not\cong A^{(2)}(N) \text{ as } S\text{-modules}$$

□

These results work for local Dedekind rings since, after localising, we can rearrange the cyclic summands of a torsion module such that the torsion coefficients divide each other. For rings where this arrangement is not possible we can maybe obtain some similar results to the ones above. However, we leave this study as an area for future possible research. Instead we investigate in chapter 3 how useful our Fitting Invariants really are and we consider what information they can tell us about the underlying module structure.

Chapter 3

Determination of the Module

In this chapter we consider how the Fitting information can be used to determine the structure of a module. We will see that for certain rings, such as Principal Ideal Domains and Dedekind rings, if we know what the Fitting Ideals of a module are, then we can completely determine the underlying module. For other rings the Fitting information may tell us something useful about the module and its relation with other modules over that particular ring, but it may not tell us the whole picture. We firstly look at cyclic modules.

3.1 Cyclic modules

A cyclic R-module, M is of the form $M = R/J$ for some R-ideal J. As this module has only one generator, Theorem 1.11 tells us that $\mathcal{F}_0(M) = \mathrm{Ann}_R(M) = J$ and $\mathcal{F}_t(M) = R$ for $t > 0$. So, if we have two cyclic modules M and N with the same Fitting Ideals we have $M = R/\mathcal{F}_0(M) = R/\mathcal{F}_0(N) = N$. Hence, the Fitting Ideals completely determine cyclic modules.

3.2 Principal Ideal Domains (PIDs)

Suppose now that we are working in a PID, R. For any finitely generated torsion module over R the Fitting information completely determines the module structure.

Theorem 3.1 Let M be a finitely generated torsion module over a PID, R. Then the Fitting Ideals over R uniquely determine the module structure of M.

Proof:

M can be written as a sum of cyclic modules over R, namely:

$$M = \bigoplus_{i=1}^{g} R/(d_i) \text{ where } d_1 \mid d_2 \mid \cdots \mid d_{g-1} \mid d_g$$

The $\{d_i\}_{i=1}^{g} \in R$ are the torsion coefficients of M. So a presentation matrix for M is the diagonal matrix, A, where:

$$A = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_g \end{pmatrix}$$

Calculating the Fitting Ideals we have:

$$\begin{aligned} \mathcal{F}_0(M) &= (d_1 \cdots d_g) \\ \mathcal{F}_1(M) &= (d_1 \cdots d_{g-1}) \\ &\vdots \\ \mathcal{F}_i(M) &= (d_1 \cdots d_{g-i}) \\ &\vdots \\ \mathcal{F}_{g-1}(M) &= (d_1) \\ \mathcal{F}_g(M) &= R \end{aligned}$$

We ask how, if we know this Fitting information, we can then find the torsion coefficients? Well, if we simply consider quotients of these Fitting Ideals we can indeed retrieve the torsion coefficients of the underlying module. Calculating the quotient ideals we obtain:

$$\begin{aligned}
(\mathcal{F}_0(M) : \mathcal{F}_1(M)) &= (d_g) \\
(\mathcal{F}_1(M) : \mathcal{F}_2(M)) &= (d_{g-1}) \\
&\vdots \\
(\mathcal{F}_{g-i}(M) : \mathcal{F}_{g-i+1}(M)) &= (d_i) \\
&\vdots \\
(\mathcal{F}_{g-1}(M) : \mathcal{F}_g(M)) &= (d_1)
\end{aligned}$$

Hence, the Fitting Ideals completely determine the torsion coefficients of the module and we can write:

$$M = \bigoplus_{i=0}^{g-1} R/(\mathcal{F}_i(M) : \mathcal{F}_{i+1}(M))$$

□

Remark 3.2 From the above analysis we can see that, working over a PID, the quotients of successive Fitting Ideals generalise the invariant factors of a finite abelian group, or finitely generated $\mathbb{Z}$-module. We also note that the initial Fitting Ideal $\mathcal{F}_0$, generalises the order of a finite abelian group.

3.3 Dedekind Rings

We can do a similar thing for Dedekind rings and use the Fitting Ideals to tell us the underlying structure of torsion modules.

Theorem 3.3 Let R be a Dedekind Ring and M a torsion R-module. Then the R-ideals $\mathcal{F}_t(M)$, for $t \geq 0$, uniquely determine M.

Proof:

As R is Dedekind we can write the initial Fitting Ideal $\mathcal{F}_0(M)$ as a product of prime ideals in R, say,

$$\mathcal{F}_0(M) = \prod_{i=1}^{r} P_i^{a_i} \text{ for some distinct prime ideals } P_i \text{ and } a_i \in \mathbb{N}$$

Since $\mathcal{F}_0(M) \subseteq \mathrm{Ann}_R(M)$ we have $\mathcal{F}_0(M)M = \{0\}$. Then the Chinese Remainder theorem tells us that:

$$M = M/\{0\} = M/[\mathcal{F}_0(M)M] = \prod_{i=1}^{r} M/[P_i^{a_i}M]$$

Suppose now that N is another torsion R-module with the same Fitting Ideals as M. We must show that M and N are isomorphic R-modules. Let $J_i = P_i^{a_i}$ for $i = 1, \ldots, r$, then we firstly show that

$$M/J_iM \cong N/J_iN$$

as R-modules. Now, Theorem 2.14 tells us that the Fitting Ideals of M/J_iM as an R-module can be calculated from the Fitting Ideals of M as an R-module. Since M and N have the same Fitting Ideals this tells us that M/J_iM and N/J_iN have the same Fitting Ideals. If we now localise at a prime ideal P_i then equation (2.2) tells us that:

$$\mathcal{F}_t^{R_{P_i}}((M/J_iM)_{R_{P_i}}) = \mathcal{F}_t^{R_{P_i}}((N/J_iN)_{R_{P_i}})$$

Since R is Dedekind the ring R_{P_i} is a PID. Therefore, we know from Theorem 3.1 that, since the Fitting Ideals in a PID determine the module uniquely,

$$(M/J_iM)_{R_{P_i}} \cong (N/J_iN)_{R_{P_i}}$$

as R_{P_i}-modules. But then (see [16], Theorem 3.13, page 36), there is an R_{P_i}-isomorphism:

$$(M/J_iM)_{R_{P_i}} \cong M/J_iM$$

and therefore

$$M/J_iM \cong N/J_iN$$

as R_{P_i}-modules. Hence

$$M/J_iM \cong N/J_iN$$

as R-modules. Therefore, we obtain:

$$M = \prod_{i=1}^{r} M/[J_iM] \cong \prod_{i=1}^{r} N/[J_iN] = N$$

Hence, we have shown that the Fitting Ideals in a Dedekind ring uniquely determine the underlying torsion module.

□

However, for torsion-free modules over Dedekind rings which are not PIDs we will show the Fitting Ideals do not determine the module structure. We first consider a theorem about the Fitting Invariants of R-ideals.

Theorem 3.4 Let R be a Dedekind ring and I be an R-ideal. Then:

$$\begin{aligned} \mathcal{F}_0(I) &= (0) \\ \mathcal{F}_t(I) &= R \quad \text{for } t \geq 1 \end{aligned}$$

Proof:

I is a torsion-free R-module so we must have $\mathcal{F}_0(I) = (0)$. Now, we know from Theorem 2.5 that if P runs through the prime ideals of R then:

$$\mathcal{L}_R(I) = \max_P\{\mathcal{L}_{R_P}(I_P)\}$$

But, since R is a Dedekind ring we know R_P is a PID so each I_P can be generated by one generator. Thus, $\mathcal{L}_{R_P}(I_P) = 1$ for all P, which tells us $\mathcal{L}_R(I) = 1$. Hence, $\mathcal{F}_t(I) = R$ for $t \geq 1$.

□

Corollary 3.5 Let R be a Dedekind ring which is not a PID. Then the Fitting Ideals of torsion-free modules over R do not determine the module.

Proof:

We can easily choose an R-ideal I which is not isomorphic to R when viewed as R-modules. Then we know from Theorem 3.4 that:

$$\begin{aligned} \mathcal{F}_0(I) &= (0) = \mathcal{F}_0(R) \\ \mathcal{F}_t(I) &= R = \mathcal{F}_t(R) \quad \text{for } t \geq 1 \end{aligned}$$

But, I is not isormorphic to R as R-modules so the Fitting Ideals do not determine the module.

□

3.4 Non-Dedekind Rings

When we work over non-Dedekind rings the Fitting Invariants do not determine the module structure. Let us return to example 1.13 and work in the ring $R = \mathbb{Z}[\sqrt{-3}]$, which is not integrally closed in $\mathbb{Q}[\sqrt{-3}]$, so is non-Dedekind. As before, let $J = (2, 1 + \sqrt{-3})$ and let M be the R-module:

$$M = R/(2) \oplus R/J$$

Now if we let N be the R-module,

$$N = R/J \oplus R/J$$

then, on calculating the Fitting Ideals of M and N using the direct sum formula, equation (2.1), we obtain:

$$(0) \neq \mathcal{F}_0(M) = (2)J = J^2 = \mathcal{F}_0(N) \text{ and } \mathcal{F}_1(M) = (2) + J = J = \mathcal{F}_1(N)$$

Thus, M and N are torsion R-modules with the same R-Fitting Invariants, however, they are not isomorphic as R-modules. In fact they are not even isomorphic as $\mathbb{Z}$-modules. If we let $\mathbb{J}_n$ denote the ring of integers modulo n, then we find that as $\mathbb{Z}$-modules

$$(R/J)_{\mathbb{Z}} \cong \mathbb{J}_2$$

under the $\mathbb{Z}$-module homomorphism:

$$a + b\sqrt{-3} \mapsto a - b(\text{mod } 2)$$

and

$$(R/(2))_{\mathbb{Z}} \cong \mathbb{J}_2 \oplus \mathbb{J}_2$$

under the $\mathbb{Z}$-module homomorphism:

$$a + b\sqrt{-3} \mapsto (a(\text{mod } 2), b(\text{mod } 2))$$

Hence as $\mathbb{Z}$-modules,

$$(M)_{\mathbb{Z}} \cong \mathbb{J}_2 \oplus \mathbb{J}_2 \oplus \mathbb{J}_2 \text{ and } (N)_{\mathbb{Z}} \cong \mathbb{J}_2 \oplus \mathbb{J}_2$$

which are non-isomorphic $\mathbb{Z}$-modules. So, M and N cannot be isomorphic as R-modules and the Fitting Ideals of torsion modules in non-Dedekind rings do not determine the module structure. However, note that in this example, the Fitting machinery is still useful. Firstly, we have a relationship between M and N as R-modules, in terms of equal Fitting Invariants. Secondly, we have a relationship between them as $\mathbb{Z}$-modules or Abelian Groups, which again can be expressed in terms of Fitting Invariants; here the Fitting Invariants are not equal so they tell us M and N are different as Abelian Groups. In general, we may have two non-isomorphic R-modules with the same R-Fitting Invariants and try to find some subring S of R, such that the S- Fitting Ideals are not equal.

3.5 Unique Factorisation Domains (UFDs)

We saw in section 3.4 that for torsion modules over the ring $\mathbb{Z}[\sqrt{-3}]$, which is not a UFD, the Fitting Ideals do not determine the module structure. Indeed even for torsion modules over UFDs we still cannot use the Fitting Invariants to tell us about the module structure as the following counter-example shows.

Example 3.6 Let R be the polynomial ring $R = \mathbb{Z}[t]$ over the indeterminate t. Let $I = (2, t)$ and $J = (4, t^2)$ be R-ideals. Further, let

$$\begin{aligned} M &= R/I \oplus R/J \\ N &= R/I \oplus R/I^2 \end{aligned}$$

be torsion R-modules. Then, calculating Fitting Invariants we obtain:

$$(0) \neq \mathcal{F}_0(M) = IJ = I^3 = \mathcal{F}_0(N) \text{ and } \mathcal{F}_1(M) = I + J = I = I + I^2 = \mathcal{F}_1(N)$$

Thus, M and N are R-torsion modules with the same R-Fitting Invariants. However, we will show that they are not isomorphic as R-modules. A presentation matrix for M with respect to generators $\{e_1, e_2\}$ is:

$$A = \begin{pmatrix} 2 & 0 \\ t & 0 \\ 0 & 4 \\ 0 & t^2 \end{pmatrix}$$

and a presentation matrix for N with respect to generators $\{f_1, f_2\}$ is:

$$B = \begin{pmatrix} 2 & 0 \\ t & 0 \\ 0 & 4 \\ 0 & 2t \\ 0 & t^2 \end{pmatrix}$$

Suppose for a contradiction that in fact M is isomorphic to N as R-modules. Then there exists an R-isomorphism $\phi : N \longrightarrow M$, such that:

$$\phi(rf_1 + sf_2) = e_2 \text{ for some } r, s \in R$$

Thus:

$$0 = \phi(2t[rf_1 + sf_2]) = 2te_2$$

and the hence row $(0, 2t)$ is a row relation for M with respect to generators $\{e_1, e_2\}$ and therefore must be an R-linear combination of the rows of A. But, $2t \notin (4, t^2)$ so we have our contradiction and no such isomorphism can exist. So we see that there exist torsion modules over UFDs where the Fitting information does not determine what the underlying module is.

$\square$

Chapter 4

Modules over Group Rings

In this chapter we turn our attention to the group ring $\Lambda = \mathbb{Z}G$, where G is a cyclic group of order p^k. We wish to calculate the Fitting Invariants of all Λ-lattices, M. Each such M is a direct sum of indecomposable Λ-lattices so we do this by calculating the Fitting Invariants of each such indecomposable. Once we have calculated the Fitting Ideals of the indecomposables we will investigate how useful they are in telling us about the underlying structure of the Λ-lattices. In fact in the case $k = 1$, we prove in Theorem 4.2 that the Fitting Ideals uniquely determine the structure of Λ-lattices. We first define what we mean by a lattice.

Definition 4.1 Let R be a commutative ring with a 1. Then an R-lattice is a finitely generated projective R-module. In the commutative group ring RG an RG-lattice is an RG-module whose underlying R-module is an R-lattice.

Note that if R is a Dedekind ring then a module is an R-lattice if and only if it is a finitely generated torsion-free module. In fact if we allow $\mathbb{Z}$-torsion modules then there are infinitely many non-isomorphic indecomposable $\mathbb{Z}G$-modules. Of course in the case $G = \{1\}$ non-isomorphic modules are determined by the Fitting Ideals (see Theorem 3.1) but this seems unlikely to be true for more general G. So we will study Λ-modules which are torsion-free as $\mathbb{Z}$-modules, in cases where there are only finitely many indecomposables. We know from Theorem 2.5 that the local Fitting Invariants will determine the global Fitting Invariants in this case. So we will work locally over the ring $\Lambda_p = \mathbb{Z}_pC_{p^k}$. Now, the Krull-Schmidt-Azumaya (KSA) Theorem (see [3], page 128), tells us that every Λ_p-lattice can be written as a sum of indecomposable Λ_p-lattices, so we will study the indecomposable Λ_p-lattices. In fact M is an indecomposable Λ-lattice if and only if M_p is an indecomposable Λ_p-lattice, so global indecomposables are equivalent to local indecomposables. Indeed, once we have determined M_p this determines M_q, for q an integer prime not equal to p. So we only need consider localising at the integer prime p.

Throughout let ω_k be a primitive p^k-th root of unity and $\Phi_{p^k}(X)$ denote the cyclotomic polynomial of order p^k over the indeterminate X. We first consider the indecomposables when $k = 1$ and we are working in the group ring $\mathbb{Z}_pC_p$.

4.1 Fitting Invariants of $\mathbb{Z}_pC_p$ - lattices

In this section let $\Lambda = \mathbb{Z}C_p$ and let $C_p =< z >$. We know from [4] and [15] that every indecomposable Λ_p-lattice is isomorphic to one of the following:

1. $\mathbb{Z}_p[\omega_1]$;
2. $\mathbb{Z}_p$;
3. the non-split extensions of $\mathbb{Z}_p[\omega_1]$ over $\mathbb{Z}_p$. In fact it can be shown that these are all isomorphic to $\mathbb{Z}_pC_p$

We can easily calculate Fitting Invariants of these indecomposables.

1. $\mathbb{Z}_p[\omega_1] \cong \Lambda_p/\Phi_p(z)\Lambda_p$, so $\mathcal{F}_0^{\Lambda_p}(\mathbb{Z}_p[\omega_1]) = (\Phi_p(z))\Lambda_p$ and $\mathcal{F}_t^{\Lambda_p}(\mathbb{Z}_p[\omega_1]) = \Lambda_p$, for $t > 0$;
2. $\mathbb{Z}_p \cong \Lambda_p/(z-1)\Lambda_p$, so $\mathcal{F}_0^{\Lambda_p}(\mathbb{Z}_p) = (z-1)\Lambda_p$ and $\mathcal{F}_t^{\Lambda_p}(\mathbb{Z}_p) = \Lambda_p$, for $t > 0$;
3. $\Lambda_p = \mathbb{Z}_pC_p$, so $\mathcal{F}_0^{\Lambda_p}(\Lambda_p) = (0)$ and $\mathcal{F}_t^{\Lambda_p}(\Lambda_p) = \Lambda_p$, for $t > 0$.

We see that each indecomposable Λ_p-lattice is either an indecomposable cyclic module or the whole ring. Thus, each finitely generated Λ_p-lattice, M, can be written as a sum of cyclic modules:

$$M = (\Lambda_p/(\Phi_p(z)))^a \oplus (\Lambda_p/(z-1))^b \oplus (\Lambda_p/(0))^c$$

for some $a, b, c \in \mathbb{N} \cup \{0\}$. We now prove the Fitting Ideals uniquely determine the lattice.

Theorem 4.2 The Fitting Ideals of a finitely generated $\Lambda_p = \mathbb{Z}_pC_p$-lattice uniquely determine the lattice.

Proof:

First note that we can generalise the direct sum formula (see Theorem 2.1) to:

$$\mathcal{F}_t(M_1 \oplus \cdots \oplus M_r) = \sum_{n_1+\cdots+n_r=t} \mathcal{F}_{n_1}(M_1)\ldots\mathcal{F}_{n_r}(M_r) \tag{4.1}$$

for modules M_i and where $t \geq 0$ and $n_i \geq 0$, for $i = 1, \ldots, r$. Let $l = \mathcal{L}_{\Lambda_p}(M)$. Then, for $t = a+b+c$ we see from equation (4.1) that:

$$\mathcal{F}_t^{\Lambda_p}(M) \supseteq \mathcal{F}_a^{\Lambda_p}((\Lambda_p/(\Phi_p(z)))^a)\mathcal{F}_b^{\Lambda_p}((\Lambda_p/(z-1))^b)\mathcal{F}_c^{\Lambda_p}((\Lambda_p/(0))^c) = \Lambda_p$$

If $t < a+b+c$ then $\mathcal{F}_t^{\Lambda_p}(M)$ contains the zero ideal (since $(z-1)\Phi_p(z) = 0$), or non-zero ideals strictly contained in Λ_p. Thus,

$$l = a+b+c$$

Next, let the s-th Fitting Ideal be the first non-zero Fitting Ideal, i.e. $(0) \neq \mathcal{F}_s^{\Lambda_p}(M)$. Then using (4.1) again, we see that if $t = c + \min(a,b)$ then

$$\mathcal{F}_t^{\Lambda_p}(M) \supseteq \mathcal{F}_{\min(a,b)}^{\Lambda_p}((\Lambda_p/(\Phi_p(z)))^a \oplus (\Lambda_p/(z-1))^b)\mathcal{F}_c^{\Lambda_p}((\Lambda_p/(0))^c)$$

which is a non-zero ideal. For $t < c + \min(a,b)$, $\mathcal{F}_t^{\Lambda_p}(M)$ contains only zero ideals. Hence:

$$s = c + \min(a,b)$$

Then:

$$\mathcal{F}_s^{\Lambda_p}(M) = \begin{cases} (\Phi_p(z))^a & \text{for } a > b \\ (z-1)^b & \text{for } a < b \\ ((\Phi_p(z))^a, (z-1)^a) & \text{for } a = b \end{cases}$$

If $a = b$ then $\mathcal{F}_s^{\Lambda_p}(M)$ determines a and we know $c = s - a$. Now suppose $a > b$ so $\mathcal{F}_s^{\Lambda_p}(M)$ determines a. Then using (4.1) we find that:

$$\mathcal{F}_{c+a-1}^{\Lambda_p}(M) = (\Phi_p(z))^{b+1}$$

and

$$\mathcal{F}_{c+a}^{\Lambda_p}(M) = ((\Phi_p(z))^b, (z-1)^b)$$

so we can determine c and b. We can do a similar thing for $a < b$.

□

In the next section we will see what the Fitting Invariants tell us when G is a cyclic group of order p^2.

4.2 $\mathbb{Z}_pC_{p^2}$ - lattices

Now let $\Lambda = \mathbb{Z}C_{p^2}$. From the work of Heller and Reiner [7] we know we can classify all indecompoable $\mathbb{Z}C_{p^2}$ - lattices. Let us denote by (A, B) the non-split extensions of A over B as Λ_p-modules. Then, working locally there are exacly $4p+1$ non-isomorphic indecomposable Λ_p-lattices as follows (see [3], page 736):

1. $\mathbb{Z}_p, \mathbb{Z}_p[\omega_1], \mathbb{Z}_pC_p, \mathbb{Z}_p[\omega_2]$;
2. The indecomposable lattices in:

   $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p)$ 1 indecomposable here;

   $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$ p indecomposables here;

   $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p[\omega_1])$ $p-1$ indecomposables here.
3. The indecomposable lattices in:

   $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$ $p-2$ indecomposables here;

   $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_p[\omega_1])$ $p-1$ indecomposables here.

The Fitting Ideals of the lattices in 1. are straightforward to calculate as they are just Fitting Ideals of cyclic modules. In order to illustrate the type of techniques used, we will only consider how to calculate Fitting Ideals for the indecomposable lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p), (\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$ and $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$. The Fitting Ideals of the remaining indecomposable lattices in 2. and 3. can then be calculated using similar methods. Now, let $C_{p^2} =< x >$ and $C_p =< z >$. We first consider:

Indecomposable Λ_p-lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p)$

The following sequence, E_1, is a non-split exact sequence:

$$\mathbb{Z}_p \xrightarrow{\theta_1} \mathbb{Z}_pC_{p^2}/N \xrightarrow{\theta_2} \mathbb{Z}_p[\omega_2]$$

where N is the Λ_p-ideal, $N = ((x-1)\Phi_{p^2}(x))$ and

$$\theta_1(m) = m\Phi_{p^2}(x) + N \ \forall \ m \in \mathbb{Z}_p \text{ and } \theta_2(f(x) + N) = f(\omega_2) \ \forall \ f(x) \in \mathbb{Z}_pC_{p^2}$$

In fact every non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_pC_{p^2}/N$, so we have found the unique indecomposable up to isomorphism.

We next consider:

Indecomposable Λ_p-lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$

Consider the following non-split short exact sequence, E_2:

$$\mathbb{Z}_pC_p \xrightarrow{\psi_1} \mathbb{Z}_pC_{p^2} \xrightarrow{\psi_2} \mathbb{Z}_p[\omega_2]$$

where

$$\psi_1(f(z)) = f(x)\Phi_{p^2}(x) \ \forall \ f(z) \in \mathbb{Z}_pC_p \text{ and } \psi_2(g(x)) = g(\omega_2) \ \forall \ g(x) \in \mathbb{Z}_pC_{p^2}$$

Before we classify the non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$ we first prove a lemma.

Lemma 4.3

$$\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p) \cong \mathbb{Z}_p[\omega_2]/(1-{\omega_2}^p)\mathbb{Z}_p[\omega_2]$$

Proof:

From our non-split exact sequence, E_2, we obtain a long exact sequence:

$$\mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p) \longrightarrow \mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_{p^2}) \xrightarrow{(\psi_2)_*} \mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p[\omega_2])$$
$$\xrightarrow{\delta} \mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p) \longrightarrow \mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_{p^2}) \longrightarrow \cdots$$

where $(\psi_2)_*$ is the composite map induced by ψ_2 such that:

$$(\psi_2)_*(\alpha) = \psi_2 \circ \alpha \ \forall \ \alpha \in \mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_{p^2})$$

Now, since $\Lambda_p = \mathbb{Z}_pC_{p^2}$ is a Gorenstein order (see [3], pages 778-779) Λ_p is weakly injective, so $\mathrm{Ext}^1_{\Lambda_p}(N, \Lambda_p) = 0$ for every Λ_p-lattice N. Then take $N = \mathbb{Z}_p[\omega_2]$ to obtain $\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_{p^2}) = 0$. Hence,

$$\mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p[\omega_2])/\ker\delta \cong \mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$$

If we denote by ϕ the isomorphism which identifies $\mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p[\omega_2])$ with $\mathbb{Z}_p[\omega_2]$, then we claim that $\ker\delta = \phi^{-1}((1-\omega_2{}^p)\mathbb{Z}_p[\omega_2])$.

Well, suppose that $\mu = (1-\omega_2{}^p)g(\omega_2) \in (1-\omega_2{}^p)\mathbb{Z}_p[\omega_2]$ for some $g(x) \in \mathbb{Z}_pC_{p^2}$. Define $\alpha \in \mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_{p^2})$ such that

$$\alpha(f(\omega_2)) = (1-x^p)g(x)f(x) \text{ for some } f(x) \in \mathbb{Z}_pC_{p^2}$$

Now α is well defined since if we have $f(\omega_2) = h(\omega_2)$ for some $h(x) \in \mathbb{Z}_pC_{p^2}$, then we must have $\Phi_{p^2}(x) \mid f(x) - h(x)$. This tells us that:

$$\alpha(f(\omega_2)) = (1-x^p)g(x)[h(x) + \Phi_{p^2}(x)k(x)] \text{ for some } k(x) \in \mathbb{Z}_pC_{p^2}$$

so we obtain:

$$\alpha(f(\omega_2)) = (1-x^p)g(x)h(x) = \alpha(h(\omega_2))$$

If we denote by $\hat{\mu}$ the map left multiplication by μ then:

$$\hat{\mu}(f(\omega_2)) = (1-\omega_2{}^p)g(\omega_2)f(\omega_2) = \psi_2 \circ \alpha(f(\omega_2)) = ((\psi_2)_*(\alpha))(f(\omega_2))$$

or in other words,

$$\hat{\mu} \in \mathrm{Im}\ (\psi_2)_* = \ker\delta \Rightarrow \phi^{-1}((1-\omega_2{}^p)\mathbb{Z}_p[\omega_2]) \subseteq \ker\delta$$

For the reverse inclusion suppose $\gamma \in \ker\delta = \mathrm{Im}\,(\psi_2)_*$, so we can write $\gamma = \psi_2 \circ \beta$ for some $\beta \in \mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_{p^2})$. Now

$$0 = \beta(\Phi_{p^2}(\omega_2)) = \beta(\Phi_{p^2}(x)1) = \Phi_{p^2}(x)\beta(1)$$

Suppose we write

$$\beta(1) = \sum_{i=0}^{p^2-1} b_i x^i \text{ for some } b_i \in \mathbb{Z}$$

Then:

$$\left(\sum_{i=0}^{p-1} x^{pi}\right)\left(\sum_{i=0}^{p^2-1} b_i x^i\right) = 0 \Rightarrow \sum_{i=0}^{p-1} b_{pi+r} = 0 \text{ for } r = 0, 1, ..., p-1$$

So we can write

$$\beta(1) = \sum_{r=0}^{p-1}\sum_{i=0}^{p-2} b_{pi+r}(1 - x^{p(p-i-1)})x^{pi+r} \Rightarrow 1 - x^p \mid \beta(1) \Rightarrow \beta(1) = h(x)(1 - x^p)$$

for some $h(x) \in \mathbb{Z}_pC_{p^2}$. Then

$$\phi(\gamma) = \gamma(1) = (\psi_2\circ\beta)(1) = h(\omega_2)(1-{\omega_2}^p) \in (1-{\omega_2}^p)\mathbb{Z}_p[\omega_2] \Rightarrow \phi(\ker\delta) \subseteq (1-{\omega_2}^p)\mathbb{Z}_p[\omega_2]$$

and thus we have $\ker\delta \subseteq \phi^{-1}((1 - {\omega_2}^p)\mathbb{Z}_p[\omega_2])$ and our claim is true.

Hence, we have shown there exists a one-one correspondence between $\ker\delta$ and $(1 - {\omega_2}^p)\mathbb{Z}_p[\omega_2]$ so we must have

$$\mathbb{Z}_p[\omega_2]/(1 - {\omega_2}^p)\mathbb{Z}_p[\omega_2] \cong \mathrm{Hom}_\Lambda(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p[\omega_2])/\ker\delta \cong \mathrm{Ext}^1_\Lambda(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$$

as required.

□

We will now classify all non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$.

Theorem 4.4 The indecomposable non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$ are isomorphic to the $\mathbb{Z}_pC_{p^2}$-ideals:

$$I_r = (\Phi_{p^2}(x), (x-1)^r)\mathbb{Z}_pC_{p^2} \text{ for } r = 0, 1, ..., p-1.$$

Proof:

Let $K = (1-\omega_2{}^p)\mathbb{Z}_p[\omega_2]$ be a $\mathbb{Z}_p[\omega_2]$-ideal and $J = (1-\omega_2)\mathbb{Z}_p[\omega_2]$ be the unique maximal ideal in $\mathbb{Z}_p[\omega_2]$. From Lemma 4.3 we know that

$$\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p) \cong \mathbb{Z}_p[\omega_2]/K$$

so we can identify the non-split extension class $[\mathbb{Z}_pC_{p^2}]$ with $1+K \in \mathbb{Z}_p[\omega_2]/K$.

Since $(1-\omega_2)^r \notin K$ for $0 \leq r \leq p-1$, any non-zero element of $\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$ can be identified with the element $(1-\omega_2)^r u + K$ for some $u \in \mathbb{Z}_p[\omega_2] \setminus J$. Suppose P is a non-split extension of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$ such that we can identify $[P]$ with the element $(1-\omega_2)^r + K$. Suppose Q is another non-split extension of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$ such that we identify $[Q]$ with the element $(1-\omega_2)^r u + K$. Then we claim that $[P] = [Q]$. We start with our short exact sequence E_2:

$$\mathbb{Z}_pC_p \longrightarrow \mathbb{Z}_pC_{p^2} \xrightarrow{\psi_2} \mathbb{Z}_p[\omega_2]$$

Then we can form succesive pullbacks P_r' and Q_r' as in the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{Z}_p & \longrightarrow & Q_r' & \longrightarrow & \mathbb{Z}_p[\omega_2] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \hat{u}} \\
\mathbb{Z}_p & \longrightarrow & P_r' & \longrightarrow & \mathbb{Z}_p[\omega_2] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \hat{\beta}_r} \\
\mathbb{Z}_p & \longrightarrow & \mathbb{Z}_pC_{p^2} & \xrightarrow{\psi_2} & \mathbb{Z}_p[\omega_2]
\end{array}$$

where $\beta_r = (1-\omega_2)^r$ and $\hat{u}$ and $\hat{\beta}_r$ denote left multiplication by u and β_r respectively. Now, $u \in \mathbb{Z}_p[\omega_2] \setminus J$ which tells us that $u \in U(\mathbb{Z}_p[\omega_2])$. Thus, $\hat{u}$ is an isomorphism and Q_r' and P_r' must be equivalent extensions so we have $[Q_r'] = [P_r']$. But, since

$$[Q_r'] = [Q] \text{ and } [P_r'] = [P] \Rightarrow [P] = [Q]$$

as required. Hence, $P \cong Q$, and the non-isomorphic non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$ can be identified with the following elements of $\mathbb{Z}_p[\omega_2]/K$:

$$a_r = (1-\omega_2)^r + K \text{ for } r = 0, 1, ..., p-1$$

We wish to find the corresponding $\mathbb{Z}_pC_{p^2}$-modules. Now, each a_r comes from a pullback

$$P_r' = \{(f(\omega_2), g(x)) \mid f(\omega_2)(1-\omega_2)^r = g(\omega_2) \text{ where } f(\omega_2) \in \mathbb{Z}_p[\omega_2], g(x) \in \mathbb{Z}_pC_{p^2}\}$$

Let us define a map $\tau : P'_r \longrightarrow \mathbb{Z}_pC_{p^2}$ by

$$\tau(f(\omega_2), g(x)) = g(x)$$

Then τ is a $\mathbb{Z}_pC_{p^2}$-module homomorphism. Note that

$$g(x) = 0 \Rightarrow g(\omega_2) = 0 = f(\omega_2)(1-\omega_2)^r \Rightarrow f(\omega_2) = 0$$

so τ is injective. Suppose now that $g(x) \in \text{Im } \tau$ then we must have

$$f(\omega_2)(1-\omega_2)^r = g(\omega_2) \Rightarrow \Phi_{p^2}(x) \mid g(x)-(1-x)^r f(x) \Rightarrow \Phi_{p^2}(x)l(x) = g(x)-(1-x)^r f(x)$$

for some $l(x) \in \mathbb{Z}_pC_{p^2}$. Hence, we see that

$$g(x) \in I_r = (\Phi_{p^2}(x), (1-x)^r)\mathbb{Z}_pC_{p^2} \Rightarrow \text{Im } \tau \subseteq I_r$$

It is easy to show that $I_r \subseteq \text{Im } \tau$ and so we see that:

$$P'_r \cong I_r = (\Phi_{p^2}(x), (1-x)^r)\mathbb{Z}_pC_{p^2}$$

Note that $I_r \ncong I_s$ for $r \neq s$ so we have calculated a set of p non-isomorphic indecomposable non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_pC_p$.

□

We lastly consider in this section:

Indecomposable Λ_p-lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$

Consider the following non-split exact sequence, E_3:

$$\mathbb{Z}_p \oplus \mathbb{Z}_pC_p \xrightarrow{\eta_1} \mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2} \xrightarrow{\eta_2} \mathbb{Z}_p[\omega_2]$$

where

$$\eta_1(m, f(z)) = (m, f(x)\Phi_{p^2}(x)) \text{ and } \eta_2(n, g(x)) = g(\omega_2)$$

$\forall\, m, n \in \mathbb{Z}_p, f(z) \in \mathbb{Z}_pC_p, g(x) \in \mathbb{Z}_pC_{p^2}$. From E_3 we obtain a long exact sequence

$$\begin{aligned} &\text{Hom}_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p) \longrightarrow \text{Hom}_{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2}, \mathbb{Z}_p \oplus \mathbb{Z}_pC_p) \\ &\xrightarrow{(\eta_1)^*} \text{Hom}_{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_p, \mathbb{Z}_p \oplus \mathbb{Z}_pC_p) \xrightarrow{\delta} \text{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p) \\ &\longrightarrow \text{Ext}^1_{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2}, \mathbb{Z}_p \oplus \mathbb{Z}_pC_p) \longrightarrow \cdots \end{aligned}$$

where $(\eta_1)^*$ is the composite map induced by η_1 such that:

$$(\eta_1)^*(\alpha) = \alpha \circ \eta_1 \ \forall \ \alpha \in \mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2}, \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$$

We wish to consider the elements of $\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$. Well if α is some element in $\mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2}, \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$ then α must be represented by the matrix:

$$\begin{pmatrix} a & b\epsilon_{p^2} \\ f(z)\Phi_p(z) & g(z)\tau \end{pmatrix}$$

for some $a, b \in \mathbb{Z}_p$, $f(z), g(z) \in \mathbb{Z}_pC_p$. Here, ϵ_{p^2} is the augmentation map given by:

$$\epsilon_{p^2} : h(x) \mapsto h(1) \ \forall \ h(x) \in \mathbb{Z}_pC_{p^2}$$

and τ is the map:

$$\tau : h(x) \mapsto h(z)$$

Now we know that η_1 is represented by the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & \Phi_{p^2}(x) \end{pmatrix}$$

and therefore $(\eta_1)^*(\alpha) = \alpha \circ \eta_1$ is represented by the matrix:

$$\begin{pmatrix} a & b\epsilon_{p^2} \\ f(z)\Phi_p(z) & g(z)\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Phi_{p^2}(x) \end{pmatrix} = \begin{pmatrix} a & b\epsilon_{p^2}\Phi_{p^2}(x) \\ f(z)\Phi_p(z) & g(z)\tau\Phi_{p^2}(x) \end{pmatrix} = \begin{pmatrix} a & bp \\ f(z)\Phi_p(z) & g(z)p \end{pmatrix}$$

since $\Phi_{p^2}(1) = p$ and $\Phi_{p^2}(z) = \Phi_p(z^p) = p$. Hence,

$$\mathrm{Hom}_{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_p, \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)/\mathrm{Im}\ \eta_1^* = \left\{ \begin{pmatrix} 0 & d + (p) \\ 0 & h(z) + (p) \end{pmatrix} \mid d \in \mathbb{Z}_p, h(z) \in \mathbb{Z}_pC_p \right\}$$

The non-split extensions correspond to the non-zero elements of $\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$ which can be represented by the matrix:

$$C_r = \begin{pmatrix} 0 & \epsilon_p \\ 0 & (1-z)^r \end{pmatrix}$$

for $0 \leq r \leq p-1$, where ϵ_p is the augmentation map given by:

$$\epsilon_p : f(z) \mapsto f(1) \ \forall \ f(z) \in \mathbb{Z}_pC_p$$

We can write the matrix in this way since if $d \notin (p)$ then $d \in U(\mathbb{Z}_p)$ and if $g(z) \notin (p)$ then we can write:

$$g(z) + (p) = (1-z)^r u + (p) \text{ for some } u \in U(\mathbb{Z}_pC_p) \text{ and } 0 \leq r \leq p-1.$$

Now, if we denote by $\gamma_r : \mathbb{Z}_p \oplus \mathbb{Z}_pC_p \longrightarrow \mathbb{Z}_p \oplus \mathbb{Z}_pC_p$ the map which is multiplication on the left by C_r, then each non-zero element of $\mathrm{Ext}^1_{\Lambda_p}(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$ can be represented as a pushout or fibre sum of the exact sequence E_3 as follows:

$$\begin{array}{ccccc}
\mathbb{Z}_p \oplus \mathbb{Z}_pC_p & \xrightarrow{\eta_1} & \mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2} & \longrightarrow & \mathbb{Z}_p[\omega_2] \\
{\scriptstyle\gamma_r}\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_p \oplus \mathbb{Z}_pC_p & \longrightarrow & P_r & \longrightarrow & \mathbb{Z}_p[\omega_2]
\end{array}$$

where P_r denotes the pushout of η_1 and γ_r. If $r=0$ then P_0 gives the decomposable non-split extension, $\mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2}$. If $r = p-1$ then P_{p-1} gives the decomposable non-split extension, $\mathbb{Z}_pC_p \oplus \mathbb{Z}_pC_{p^2}/N$. In fact for $1 \leq r \leq p-2$ we obtain indecomposable non-split extensions as required.

From our work above we obtain a theorem:

Theorem 4.5 There are precisely $p-2$ non-split indecomposable Λ_p-lattices which arise from the non-split extensions of $\mathbb{Z}_p[\omega_2]$ over $\mathbb{Z}_p \oplus \mathbb{Z}_pC_p$. These are given by the pushouts P_r, as above, for $1 \leq r \leq p-2$.

4.3 Fitting Invariants of $\mathbb{Z}_pC_{p^2}$ - lattices

We now come to calculate the Fitting Invariants of the indecomposable non-split extensions discussed in section 4.2.

Indecomposable Λ_p-lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p)$

We know that any such non-split extension is isomorphic to the cyclic module, $\mathbb{Z}_pC_{p^2}/N$. Hence, calculating the Fitting Ideals we simply obtain:

$$\begin{aligned}
\mathcal{F}_0^{\Lambda_p}(\mathbb{Z}_pC_{p^2}/N) &= ((x-1)\Phi_{p^2}(x))\Lambda_p \\
\mathcal{F}_t^{\Lambda_p}(\mathbb{Z}_pC_{p^2}/N) &= \Lambda_p \qquad \text{for } t \geq 1
\end{aligned}$$

Indecomposable Λ_p-lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_pC_p)$

We know from Theorem 4.4 that any such extensions are isomorphic to the $\mathbb{Z}_pC_{p^2}$-ideals:

$$I_r = (\Phi_{p^2}(x), (x-1)^r)\Lambda_p$$

for $r = 0, 1, ..., p-1$. Well, we claim that a presentation matrix for I_r as a $\mathbb{Z}C_{p^2}$-module is:

$$A_r = \begin{pmatrix} (x-1)^r & -\Phi_{p^2}(x) \\ (x^p-1) & 0 \end{pmatrix}$$

with respect to the generating set $\{\Phi_{p^2}(x), (x-1)^r\}$, for $1 \leq r \leq p-1$. Indeed, the rows of A_r are row relations for this generating set. Suppose now that we have a relation

$$f(x)\Phi_{p^2}(x) + g(x)(x-1)^r = 0$$

with repect to this generating set, for some $f(x), g(x) \in \mathbb{Z}_pC_{p^2}$. Then for A_r to be a presentation matrix we need to show that the row vector $(f(x), g(x))$ is a $\mathbb{Z}_pC_{p^2}$-linear combination of the rows of A_r. Now, working in the ring $\mathbb{Z}_p[X]$, where X is an indeterminate, we must have:

$$X^{p^2} - 1 \mid f(X)\Phi_{p^2}(X) + g(X)(X-1)^r$$

and thus

$$(X-1)\Phi_p(X)\Phi_{p^2}(X)h(X) = f(X)\Phi_{p^2}(X) + g(X)(X-1)^r$$

for some $h(X) \in \mathbb{Z}_p[X]$. Therefore

$$X - 1 \mid f(X) \text{ and } \Phi_{p^2}(X) \mid g(X)$$

thus

$$f(X) = f_1(X)(X-1) \text{ and } g(X) = g_1(X)\Phi_{p^2}(X)$$

for some $f_1(X), g_1(X) \in \mathbb{Z}_p[X]$. Hence

$$\Phi_p(X)h(X) = f_1(X) + g_1(X)(X-1)^{r-1}$$

This tells us that:

$$\begin{aligned}
& -g_1(X)((X-1)^r, -\Phi_{p^2}(X)) + h(X)((X^p-1),0) \\
= \;& (-g_1(X)(X-1)^r + h(X)(X^p-1), g_1(X)\Phi_{p^2}(X)) \\
= \;& (f_1(X)(X-1), g_1(X)\Phi_{p^2}(X)) \\
= \;& (f(X), g(X))
\end{aligned}$$

Hence, we have shown any relation for this generating set is a $\mathbb{Z}_p[X]$-linear combination and thus a $\mathbb{Z}_pC_{p^2}$-linear combination of the rows of A_r, and we conclude A_r is indeed a presentation matrix for I_r, for $1 \le r \le p-1$. When $r=0$, we have $I_0 = \Lambda_p$. Then, calculating the Fitting Invariants we obtain:

$$\begin{aligned}
\mathcal{F}_0^{\Lambda_p}(I_r) &= (0)\Lambda_p \\
\mathcal{F}_1^{\Lambda_p}(I_r) &= (\Phi_{p^2}(x), (x-1)^r, x^p-1)\Lambda_p \\
\mathcal{F}_t^{\Lambda_p}(I_r) &= \Lambda_p \text{ for } t \ge 2.
\end{aligned}$$

for $0 \le r \le p-1$.

Indecomposable Λ_p-lattices in $(\mathbb{Z}_p[\omega_2], \mathbb{Z}_p \oplus \mathbb{Z}_pC_p)$

We know from Theorem 4.5 that any such extension is in fact a pushout P_r for $r = 1, \dots, p-2$. Now P_r can be written in the form $P_r = M/L$ where

$$\begin{aligned}
M &= \mathbb{Z}_p \oplus \mathbb{Z}_pC_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2} \\
L &= \{(\gamma_r(m, f(z)), -\eta_1(m, f(z))) \;\forall\; (m, f(z)) \in \mathbb{Z}_p \oplus \mathbb{Z}_pC_p\}
\end{aligned}$$

Now a presentation matrix for M as a $\mathbb{Z}_pC_{p^2}$-module is:

$$\begin{pmatrix}
x-1 & 0 & 0 & 0 \\
0 & x^p-1 & 0 & 0 \\
0 & 0 & x-1 & 0
\end{pmatrix}$$

Also we have:

$$\begin{aligned}
L &= \{(f(1), f(z)(1-z)^r, 0, 0) - (0, 0, m, f(x)\Phi_{p^2}(x))\} \\
&= \{f(x)(1, (1-z)^r, 0, -\Phi_{p^2}(x)) - m(0,0,1,0)\}
\end{aligned}$$

Hence a presentation matrix for $P_r = M/L$ is:

$$\begin{pmatrix} x-1 & 0 & 0 & 0 \\ 0 & x^p-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 1 & (1-x)^r & 0 & -\Phi_{p^2}(x) \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} x-1 & 0 & 0 \\ 0 & x^p-1 & 0 \\ 1 & (1-x)^r & -\Phi_{p^2}(x) \end{pmatrix}$$

We then calculate the Fitting Ideals to be:

$$\begin{aligned} \mathcal{F}_0^{\Lambda_p}(P_r) &= (0)\Lambda_p \\ \mathcal{F}_1^{\Lambda_p}(P_r) &= (x-1)(\Phi_p(x), \Phi_{p^2}(x), (x-1)^r)\Lambda_p \\ \mathcal{F}_t^{\Lambda_p}(P_r) &= \Lambda_p \text{ for } t \geq 2. \end{aligned}$$

Note that:

$$\mathcal{F}_s^{\Lambda_p}(P_0) = \mathcal{F}_s^{\Lambda_p}(\mathbb{Z}_p \oplus \mathbb{Z}_pC_{p^2})$$

and since we can write:

$$(X-1)^{p-1} = \Phi_p(X) + \Phi_{p^2}(X)h(X) + (X^p-1)g(X)$$

for some $h(X), g(X) \in \mathbb{Z}_p[X]$ then:

$$\mathcal{F}_s^{\Lambda_p}(P_{p-1}) = \mathcal{F}_s^{\Lambda_p}(\mathbb{Z}_pC_p \oplus \mathbb{Z}_pC_{p^2}/N)$$

for all integers $s \geq 0$, as we would expect. We want to investigate whether the Fitting Ideals do in fact determine the $\mathbb{Z}G$-lattices in the case $k = 2$. Well, we can consider calculating Fitting Ideals of certain combinations of indecomposables. For example, let us consider the decomposable module $\mathbb{Z}_p \oplus I_{p-1}$. On calculating its Fitting Ideals we obtain:

$$\begin{aligned} \mathcal{F}_0^{\Lambda_p}(\mathbb{Z}_p \oplus I_{p-1}) &= (0)\Lambda_p \\ \mathcal{F}_1^{\Lambda_p}(\mathbb{Z}_p \oplus I_{p-1}) &= (x-1)(\Phi_{p^2}(x), (x-1)^{p-1}, x^p-1)\Lambda_p = (x-1)(\Phi_p(x), \Phi_{p^2}(x))\Lambda_p \\ \mathcal{F}_2^{\Lambda_p}(\mathbb{Z}_p \oplus I_{p-1}) &= (x-1, \Phi_{p^2}(x))\Lambda_p = (p, (x-1))\Lambda_p \end{aligned}$$

If we now compare these Fitting Ideals with those of the decomposable module $\mathbb{Z}_pC_p \oplus \mathbb{Z}_pC_{p^2}/N$ (which has the same $\mathbb{Z}_p$-rank as $\mathbb{Z}_p \oplus I_{p-1}$, namely p^2+1) we see that:

$$\mathcal{F}_s^{\Lambda_p}(\mathbb{Z}_p \oplus I_{p-1}) = \mathcal{F}_s^{\Lambda_p}(\mathbb{Z}_pC_p \oplus \mathbb{Z}_pC_{p^2}/N)$$

for $s = 0, 1$, but equality breaks down for $s = 2$. Further research has failed to find two non-isomorphic Λ_p-lattices M_1 and M_2 such that:

$$\mathcal{F}_s^{\Lambda_p}(M_1) = \mathcal{F}_s^{\Lambda_p}(M_2)$$

for all values of $s \geq 0$. Therefore, it is possible that the Fitting Invariants do in fact determine the indecomposable $\mathbb{Z}G$-lattices when G is a cyclic group of order p^2. However, it is difficult to prove that this is indeed the case.

When G is a cyclic group of order p^k, for $k > 2$, a complete classification of all indecomposable $\Lambda = \mathbb{Z}G$-lattices is not possible, in fact there are infinitely many non-isomorphic indecomposable $\mathbb{Z}G$-lattices (see [8]). Similarly, Heller and Reiner in [7] show that for G an arbitrary finite group containing a non-cyclic p-Sylow subgroup, for some integer prime p, then the indecomposable $\mathbb{Z}G$-lattices are also of infinite representation type. In these cases we may be able to obtain some partial results and obtain Fitting Invariants which are useful in telling us some information about the underlying $\mathbb{Z}G$-lattices, but not the whole picture. However, we leave this area of research for future consideration.

Chapter 5

Fitting Invariants over Non-Commutative Rings

In the first four chapters we have defined Fitting Ideals for modules over commutative rings and considered various properties of Fitting Ideals in the commutative case. We will now generalise this to construct a useful definition of Fitting Invariants for modules over certain non-commutative rings, and to extend some of our earlier results. In this chapter we first review what has been done to date in the non-commutative case and construct Fitting Invariants for modules over matrix rings, where the underlying ring is commutative. The main part of this chapter will deal with a construction of Fitting Invariants for modules over hereditary orders by considering the effect of maximal orders and projectives in the hereditary order. This will enable us to prove the main result of this chapter, Corollary 5.24, that this set of invariants will enable us to completely determine the isomorphism class of torsion modules over hereditary orders.

5.1 Review of work to date

The literature to date shows that little has been written about Fitting Ideals for modules over non-commutative rings. Some work has been done by Susperregui in [19] who constructs Fitting Invariants for modules over skewcommutative graded rings and rings of differential operators satisfying the left Ore condition. In these two cases a concept of determinant can be defined and hence a definition of determinantal ideals given. In the work in [20] Susperregui constructs Fitting Invariants for modules over anticommutative graded rings, where a reasonable definition of determinant is given in terms of the exterior algebra of the anticommutative graded ring. Fitting Invariants are then derived with respect to a particular set of homogenous generators. Other than this little has been done in the non-commutative case.

We first turn our attention to the case of modules over the matrix ring $M_n(R)$, where the underlying ring, R, is Noetherian and commutative.

5.2 Matrix Rings over Commutative Rings

Let R be a Noetherian commutative ring with a 1 and let $\Lambda = M_n(R)$ denote the ring of $n \times n$ matrices over R. Suppose that ${}_\Lambda N$ is a finitely generated left Λ-module. We will construct Fitting Invariants for ${}_\Lambda N$. Our problem is that here Λ is a non-commutative ring so we need a notion of determinant to obtain determinantal ideals. To do this we first notice that there is an equivalence of categories between the category of all finitely generated left R-modules, ${}_R\mathrm{Mod}$ and the category of all finitely generated left Λ-modules ${}_\Lambda\mathrm{Mod}$, via the Morita map, $\mathcal{M}$. We have:

$$\mathcal{M}: \ {}_R\mathrm{Mod} \longrightarrow_\Lambda \mathrm{Mod} \qquad \text{via} \qquad \mathcal{M}(M) = R^n \otimes_R M$$

where M is a finitely generated left R-module. As this Morita map defines an equivalence of categories there is an inverse map, $\mathcal{N}$, where:

$$\mathcal{N}: \ {}_\Lambda\mathrm{Mod} \longrightarrow_R \mathrm{Mod} \qquad \text{via} \qquad \mathcal{N}(N) = \mathrm{Hom}_\Lambda(R^n, N)$$

and in fact $\mathcal{N}(N)$ is a finitely generated left R-module. We now ask given a presentation matrix for N as a Λ-module, what is the presentation matrix for $\mathcal{N}(N)$ as an R-module? We have the following lemma:

Lemma 5.1 Suppose that ${}_\Lambda N$ has a presentation matrix $B = (B_{i,j})$ for some $B_{i,j} \in M_n(R)$, for $i = 1, \ldots, m$ and $j = 1, \ldots, g$. Then $\tilde{B} = (b_{k,l})$ is a presentation matrix for $\mathcal{N}(N)$, where we write $B_{i,j} = (b_{k,l})$ for $k = n(i-1) + u, l = n(j-1) + v$ and $u, v = 1, \ldots, n$ and $b_{k,l} \in R$. We refer to $\tilde{B} \in M_{mn \times gn}(R)$ as the *unboxed* matrix of $B \in M_{m \times g}(\Lambda)$.

Before we prove this lemma we illustrate what it is saying in the case $m = g = 2$.

Example 5.2 Suppose $m = g = 2$ and $B = (B_{i,j})$ is a presentation matrix for N. Now we can write B as:

$$B = \begin{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} \\ \\ \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} & \begin{pmatrix} b_{23} & b_{24} \\ b_{43} & b_{44} \end{pmatrix} \end{pmatrix}$$

and hence the unboxed matrix is simply:

$$\tilde{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

which is a presentation matrix for $\mathcal{N}(N)$.

Proof: (of Lemma 5.1)

Suppose B is a presentation matrix for N with respect to a generating set $\{\hat{e}_j\}$. Let $\gamma \in \mathcal{N}(N) = \mathrm{Hom}_\Lambda(R^n, N)$ and let y_u be the n-column vector with a 1 in the u-th row and zeroes elsewhere. Then, γ is determined by

$$\gamma y_1 = \sum_{j=1}^{g} A_j \hat{e}_j$$

for some $A_j \in \Lambda$. Let $E_{u,v}$ denote the $n \times n$ matrix with a 1 in the (u, v)-th position and zeroes elsewhere. Then, we can write

$$\gamma y_2 = \gamma E_{2,1} y_1 = E_{2,1} \gamma y_1 = \sum_{j=1}^{g} (\underline{0}^T, \underline{a_j}^T, \underline{0}^T, \ldots, \underline{0}^T)^T \hat{e}_j$$

where $\underline{a_j}$ denotes the first row of A_j. But,

$$\gamma y_1 = E_{1,2} \gamma y_2 = E_{1,2} E_{2,1} \gamma y_1 = \sum_{j=1}^{g} (\underline{a_j}^T, \underline{0}^T, \ldots, \underline{0}^T)^T \hat{e}_j$$

and so we see that γ is determined by the first rows of each of the A_j.

Now, $\mathcal{N}(N)$ has ng generators as an R-module, namely $\{\hat{f}_{j,w}\}$ for $w = 1, \ldots, n$, where $\hat{f}_{j,w}y_1 = E_{1,w}\hat{e}_j$. Let $\underline{b^{i,j}_w}$ denote the w-th row of $B_{i,j}$. Then

$$\sum_{j=1}^{g} \underline{b^{i,j}_w}(\hat{f}_{j,1}, \ldots, \hat{f}_{j,n})^T y_1 = \sum_{j=1}^{g} (\underline{b^{i,j}_w}^T, \underline{0}^T, \ldots, \underline{0}^T)^T \hat{e}_j = E_{1,w} \sum_{j=1}^{g} B_{i,j}\hat{e}_j = 0$$

since the i-th row of B is a coefficient vector of a relation for N with respect to the generating set $\{\hat{e}_j\}$. Hence, the row vector $\underline{b^{i,j}_w}$ is a coefficient vector of a relation for $\mathcal{N}(N)$ with respect to the generating set $\{\hat{f}_{j,w}\}$. That is, every row of the unboxed matrix $\tilde{B} = (b_{k,l})$ is the coefficient vector of a relation for $\mathcal{N}(N)$. It remains to show that any relation for $\mathcal{N}(N)$ with respect to the generating set $\{\hat{f}_{j,w}\}$ is an R-linear combination of the rows of $\tilde{B}$.

Well, suppose that we have a relation

$$\sum_{j=1}^{g} \sum_{w=1}^{n} c_{j,w}\hat{f}_{j,w} = 0$$

for $\mathcal{N}(N)$ as an R-module, for some $c_{j,w} \in R$. Then

$$\sum_{j=1}^{g} \sum_{w=1}^{n} c_{j,w}\hat{f}_{j,w}y_1 = \sum_{j=1}^{g} \sum_{w=1}^{n} c_{j,w}E_{1,w}\hat{e}_j = \sum_{j=1}^{g} (\underline{c_j}^T, \underline{0}^T \ldots, \underline{0}^T)^T \hat{e}_j = 0$$

where $\underline{c_j} = (c_{j,1}, \ldots, c_{j,n})$. If we let $C_j = (\underline{c_j}^T, \underline{0}^T, \ldots, \underline{0}^T)^T \in M_n(R)$ then we have the relation:

$$\sum_{j=1}^{g} C_j\hat{e}_j = 0$$

Now, since B is a presentation matrix the row vector $(C_1, \ldots, C_g)$ must be a Λ-linear combination of the rows of B. So,

$$(C_1, \ldots, C_g) = \sum_{i=1}^{m} \lambda_i(B_{i,1}, \ldots, B_{i,g})$$

for some $\lambda_i \in \Lambda$. So we can write

$$C_j = \sum_{i=1}^{m} \lambda_i B_{i,j}$$

which tells us that:

$$(\underline{c_1}, \ldots, \underline{c_g}) = \sum_{k=1}^{mg} x_k \underline{b_k}$$

for some $x_k \in R$ and where $\underline{b_k}$ is the k-th row of the unboxed matrix, $\tilde{B}$. Hence, we have shown every row relation for $\mathcal{N}(N)$ is an R-linear combination of the rows of $\tilde{B}$ and thus the unboxed matrix $\tilde{B}$ is indeed a presentation matrix for $\mathcal{N}(N)$ as an R-module.

□

Now let Γ be a Noetherian ring such that we have a representation, $\rho : \Gamma \longrightarrow \Lambda = M_n(R)$. Let ${}_\Gamma M$ be a finitely generated left Γ-module with presentation matrix $A \in M_{m\times g}(\Gamma)$. Then by extending the scalars we can make M into the left Λ-module $\Lambda \otimes_{\rho(\Gamma)} M$. Then, since $\rho(A)$ is a presentation matrix for $\Lambda \otimes_{\rho(\Gamma)} M$, Lemma 5.1 tells us that the unboxed matrix $\widetilde{\rho(A)}$ is a presentation matrix for the left R-module $\mathcal{N}(\Lambda \otimes_{\rho(\Gamma)} M)$. This leads us to define a set of Fitting Invariants as follows:

Definition 5.3 The s-th Fitting Invariant of ${}_\Gamma M$ with respect to ρ is defined to be:

$$\mathcal{F}_s^\rho(M) = I_{gn-s}(\widetilde{\rho(A)})$$

From this definition we deduce that $\mathcal{F}_s^\rho(M)$ does not depend on A, i.e. it is independent of the particular generating set used for ${}_\Gamma M$. For, if $C \in M_{l\times h}(\Gamma)$ is another presentation matrix for ${}_\Gamma M$ then Definition 1.4 tells us that:

$$I_{gn-s}(\widetilde{\rho(A)}) = \mathcal{F}_s^R(\mathcal{N}(\Lambda \otimes_{\rho(\Gamma)} M)) = I_{hn-s}(\widetilde{\rho(C)})$$

Note that in the special case $\Gamma = \Lambda$ we just take ρ to be the identity map, Id_Λ and the s-th Fitting Invariant of ${}_\Lambda M$ is simply $\mathcal{F}_s^{\mathrm{Id}_\Lambda}(M) = I_{gn-s}(\tilde{A})$.

However, as it stands the invariants derived from Definition 5.3 are in general quite crude. The invariants $\mathcal{F}_s^\rho(M)$ will in fact determine the structure of a torsion module, M in the special case $\Gamma = \Lambda$ and R a Dedekind ring. But for more general cases, for example where Γ is a hereditary order spanning Λ (see section 5.3), these invariants will only provide us with partial information about the structure of torsion modules. We will show how we can construct a finer set of invariants in the next section for modules over Hereditary Orders.

5.3 Hereditary Orders

We start this section with a definition and short discussion about hereditary rings.

Definition 5.4 A ring S with a 1 is (left) hereditary if every (left) ideal of S is a projective S-module.

We use the term left hereditary here as there exist examples of rings which are left hereditary but not right hereditary. For example, the ring

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a \in \mathbb{Z}, b, c, \in \mathbb{Q} \right\}$$

is left but not right hereditary (see [18]). However, it can be shown that if S is a left and right Noetherian ring then S is left hereditary if and only if S is right hereditary. From now on we will only be concerned with rings which are both left and right hereditary which we will just refer to as hereditary rings.

Example 5.5 Every Dedekind domain is a hereditary ring. Indeed Dedekind domains are precisely hereditary integral domains.

□

In fact hereditary rings are precisely those rings where submodules of free modules are projective. This last fact enables us to find a structure theorem for submodules of free S-modules, where S is hereditary. This structure theorem extends the classical structure theorem for modules over PIDs (since PIDs are hereditary).

We now say what we mean by an order. Let R be a Noetherian integral domain and K its field of fractions. Let A be a finite dimensional K-algebra. Then we first define:

Definition 5.6 For any finite dimensional K-space, V, a full R-lattice in V is a finitely generated R-submodule, M in V, such that $K.M = V$.

By an order we mean:

Definition 5.7 The ring S is an R-order in the K-algebra A if it is a subring of A such that S is a full R-lattice in A, i.e. $K.S = A$.

Examples of R-orders are:

Example 5.8 Let $A = M_n(K)$ and $S = M_n(R)$ then S is an R-order in A.

□

Example 5.9 Let R be a Dedekind domain and let L be a finite separable extension of K. If S denotes the integral closure of R in L, then S is an R-order in L.

□

Example 5.10 Let G be a finite group and let $A = KG$ be its group algebra. Then the group ring $S = RG$ is an R-order in A.

□

We now present a structure theorem for finitely generated modules over hereditary orders due to Drozd [5].

Theorem 5.11 Let S be a hereditary R-order in A and ${}_SM$ a finitely generated left S-module. Then we have:

$$ {}_SM = \bigoplus_{i=1}^{k} \text{coker } f_i $$

where for each i, f_i is an S-homomorphism of indecomposable projective S-modules.

Proof:

See [5] Theorem 1.

□

We use this theorem in section 5.5 in order to construct our Fitting Invariants for modules over hereditary orders.

5.4 Hereditary Orders in central simple division algebras

The problem of determining the structure of hereditary orders can be reduced to the study of hereditary orders in division algebras over local fields. By a local field we mean a complete discrete valuation field with finite residue field. Let R be a Dedekind ring with quotient field K. Suppose S is a hereditary R-order in a separable finite dimensional K-algebra, A. Then we can write A as the sum of its Wedderburn components. That is:

$$A = \bigoplus_{i=1}^{t} A_i$$

where each Wedderburn component, A_i, is a central simple algebra over its centre K_i. So we can write $A_i = M_{m_i}(D_i)$ for some division algebra, D_i, over K_i and where the integer m_i is such that $(A_i : K_i) = m_i^2(D_i : K_i)$.

If we let $\{e_i\}$ denote the corresponding idempotents of $\{A_i\}$ then we can write the hereditary order S as:

$$S = \bigoplus_{i=1}^{t} Se_i$$

where each Se_i is a hereditary R-order in A_i. Thus, we have reduced to the case of hereditary R-orders in central simple algebras over K_i. Furthermore, if R is a complete discrete valuation ring (dvr) then the above remains true whenever A is a semisimple K-algebra, whether or not separable over K.

We now reduce further to the local case. Let P range over all the maximal ideals of R and let R_P denote localisation at P. Then the local ring, S_P, is a hereditary R_P-order in A and we can write:

$$S = \bigcap_{P} S_P$$

Thus, we need only consider hereditary orders in central simple algebras over local fields. So, from now on we will work locally and we will let R be a complete dvr with quotient field, K. We can now obtain a structure theorem for hereditary orders in this reduced case. In fact, it can be shown that there exists a unique maximal R-order in D which we denote by Δ_D. It turns out that Δ_D is the integral closure of R in D and behaves very much like a dvr, except it is not necessarily commutative. This fact allows us to describe hereditary orders in matrix rings as the following theorem shows:

Theorem 5.12 Let R be a complete dvr with quotient field K and D be a finite dimensional division algebra over its centre, K. Denote by Δ_D the integral closure of R in D and let $P = \text{rad}\ \Delta_D$, which is generated by some prime element π. Suppose

$A = M_t(D)$ and that S is a hereditary R-order in A, then there is an isomorphism such that

$$S \cong S_{(n_1,\dots,n_r)}(\Delta_D) = \begin{pmatrix} (\Delta_D) & (P) & \cdots & (P) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & (P) \\ (\Delta_D) & \cdots & \cdots & (\Delta_D) \end{pmatrix}^{(n_1,\dots,n_r)}$$

where the (i, j)-th entry in $S_{(n_1,\dots,n_r)}(\Delta_D)$ is the $n_i \times n_j$ matrix with entries in Δ_D for $i \geq j$ or P for $i < j$. Furthermore, the cycle $(n_1, \dots, n_r)$ of type r are isomorphism invariants of $S_{(n_1,\dots,n_r)}(\Delta_D)$ up to cyclic permutation such that $\sum_{i=0}^{r-1} n_{i+1} = t$.

Proof:

See [16], Theorem 39.14, page 358.

□

From now on we identify S with $S_{(n_1,\dots,n_r)}(\Delta_D)$. We will call S *principal* if each $n_{i+1} = w$ for some positive integer w. We wish to calculate Fitting Invariants for finitely generated S-modules. The problem is that Δ_D may not be a commutative ring. If it is commutative then we can use our construction for Fitting Invariants for modules over matrix rings, as in section 5.2. However, if it is not commutative we can overcome this problem by considering splitting fields for the division algebra, D.

Suppose now that S is simply a hereditary R-order in a finite dimensional division algebra D over K. We wish to embed D in a matrix ring over a commutative ring by considering splitting fields of D. In fact $(D : K) = n^2$ for some $n \in \mathbb{N}$, and as we are working locally, we can find a unique unramified extension, L over K of degree n such that L is a splitting field for D (see [17], page 183). In other words, we can find L such that:

$$L \otimes_K D = M_n(L)$$

If we denote by $\mathcal{O}_L$ the valuation ring of the field L then $\mathcal{O}_L \otimes_{\mathcal{O}_K} S$ is a hereditary $R = \mathcal{O}_K$-order in $M_n(L)$. This is true since hereditary orders remain hereditary under unramified extensions of the ground ring (for details see Janusz [10], Theorem 6). So in fact, when D is non-commutative, our strategy will be to work over the ring $\mathcal{O}_L \otimes_{\mathcal{O}_K} S$ which we know is a hereditary order in the matrix ring $M_n(L)$.

We are now in a much stronger position to construct our Fitting Invariants.

5.5 Construction of Fitting Invariants

In this section we use the results of the previous section, in particular the structure theorem of Drozd, Theorem 5.11, to construct Fitting Ideals. Let S denote the hereditary R-order $S_{(n_1,\ldots,n_r)}(\Delta_D)$ of type r. Then each indecomposable S-lattice, indeed each indecomposable projective of S, is isomorphic to:

$$P_i = (\text{rad } S)^i[\Delta_D]^t$$

for some $i = 0, 1, \ldots, r-1$, where $[\Delta_D]^k$ denotes the set of k-column vectors with each entry in Δ_D. But,

$$\text{rad } S = \begin{pmatrix} (P) & \cdots & \cdots & (P) \\ (\Delta_D) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ (\Delta_D) & \cdots & (\Delta_D) & (P) \end{pmatrix}^{(n_1,\ldots,n_r)}$$

and hence,

$$P_i = \begin{pmatrix} [P]^{n_1} \\ \vdots \\ [P]^{n_i} \\ \hline [\Delta_D]^{n_{i+1}} \\ \vdots \\ [\Delta_D]^{n_r} \end{pmatrix}$$

i.e., each P_i consists of the column vector which has the first $\sum_{v=0}^{i} n_v$ entries lying in P and the next $\sum_{v=i+1}^{r} n_v$ entries lying in Δ_D (where we set $n_0 = 0$). Note that in a very natural way:

$$S \cong \bigoplus_{i=0}^{r-1} n_{i+1} P_i \tag{5.1}$$

Now the structure theorem due to Drozd, Theorem 5.11, tells us that every finitely generated S-module consists of direct summands of the form coker f where f is a S-homomorphism of indecomposable S-lattices, i.e. $f : P_j \longrightarrow P_i$ for some $i, j =$

$0, 1, \ldots, r-1$. But the map f must be just the map which is right multiplication by some element of Δ_D, that is right multiplication by $u\pi^l$ for some $u \in U(\Delta_D)$ and for some integer $l \geq 0$. So we write each f as:

$$f^l_{i,j} : P_j \xrightarrow{\times u\pi^l} P_i$$

Then coker $f^l_{i,j} = P_i/P_j\pi^l$, is a quotient of projective modules, where if $l = 0$ we must have $i < j$. Now let:

$$A^k_{i,j} = \begin{cases} P_i/P_j\pi^k & \text{for } i \geq j \\ P_i/P_j\pi^{k-1} & \text{for } i < j \end{cases}$$

for integers $k \geq 1$. We obtain the following structure theorem for finitely generated S-modules.

Theorem 5.13 If S is a hereditary R-order in $M_t(D)$ of type r, then every finitely generated left S-module, ${}_SM$, can be written as a sum of quotients of projectives in the form:

$${}_SM = \bigoplus_{i,j,k} a^k_{i,j} A^k_{i,j}$$

where the sum is over i, j and k, for $0 \leq i, j \leq r-1$, for some $a^k_{i,j} \in \mathbb{N} \cup \{0\}$.

Therefore, we will calculate the Fitting Ideals of the quotient modules $A^k_{i,j}$ in order to determine the Fitting Ideals of ${}_SM$. We will first of all consider what we can say in the simplified case when $D = E$ for some field E. In this situation S is a hereditary R-order in $M_t(E)$. Then the integral closure of R in E, which we denote by Δ_E, is of course a commutative ring. Thus, we are working in the hereditary R-order $S = S_{(n_1,\ldots,n_r)}(\Delta_E)$ which lies inside the matrix ring $M_t(\Delta_E)$. If we denote by ρ the inclusion map $\rho : S \longrightarrow M_t(\Delta_E)$, then Definition 5.3 tells us we can calculate the s-th Fitting Ideal of $A^k_{i,j}$ to be:

$$\mathcal{F}^\rho_s(A^k_{i,j})$$

We next calculate these Fitting Ideals explicitly. In order to do this we will find a presentation matrix for the $A^k_{i,j}$ in S. It is instructive to initially consider the case $n_i = 1$, for all i, i.e. when S is a principal hereditary R-order in $M_r(E)$. Then a presentation matrix for the module $A^k_{i,j}$ (for $i \geq j$) is given by $B = (B_s) \in M_{r\times 1}(S)$, where:

$$B_s = \begin{cases} E_{j+1,i+1}(\pi^k) & \text{for } s = i+1 \\ E_{s,s}(1) & \text{for } 1 \leq s \leq r \text{ and } s \neq i+1 \end{cases}$$

for $1 \leq s \leq r$. Here $E_{u,v}(\kappa)$ denotes the $r \times r$ matrix with κ in the (u,v)-th position and zeroes elsewhere, for $1 \leq u, v \leq r$. We can see this is the case since the map $\beta : S^r \longrightarrow S$, represented by the matrix B, has

$$\text{Im } \beta = S^r B = \bigoplus_{s=1}^{r} SB_s$$

Now, we have:

$$\begin{aligned} SB_{i+1} &= (\overbrace{0 \mid \cdots \mid 0}^{i} \mid P_j\pi^k \mid 0 \mid \cdots \mid 0) \\ SB_s &= (\overbrace{0 \mid \cdots \mid 0}^{s-1} \mid P_{s-1} \mid 0 \mid \cdots \mid 0) \quad \text{for } s \neq i+1 \end{aligned}$$

which tells us that:

$$\text{Im } \beta = (P_0 | \cdots | P_{i-1} | P_j\pi^k | P_{i+1} | \cdots | P_{r-1}) \Rightarrow \text{coker } \beta = P_i / P_j\pi^k$$

as required. For $i < j$ we just replace π^k by π^{k-1} throughout.

The above analysis carries through to the case when S is a hereditary R-order in $M_t(E)$, for some positive integer t. Again we initially assume that $i \geq j$. Then, the presentation matrix $B = (B_s) \in M_{t\times 1}(S)$, where $S = S_{(n_1,\ldots,n_r)}(\Delta_E)$ and where $s = 1, \ldots, t$. We have that:

$$B_s = \begin{cases} E_{j+1,i+1}(C_k) & \text{for } s = \sum_{v=0}^{i} n_v + 1 \\ E_{l+1,l+1}(E_{w,w}(1)) & \text{for } s \neq \sum_{v=0}^{i} n_v + 1 \end{cases}$$

Here $C_k = \text{diag}(\pi^k, 0, \ldots, 0)$ and we write s in the form $s = \sum_{v=0}^{l} n_v + w$ for $1 \leq w \leq n_{l+1}$. Then the map $\beta : S^t \longrightarrow S$ is represented by the matrix B and we obtain:

$$\text{Im } \beta = (n_1 P_0 | \cdots | n_i P_{i-1} \underbrace{| P_j\pi^k | P_i | \cdots | P_i}_{n_{i+1}} | n_{i+2} P_{i+1} | \cdots | n_r P_{r-1})$$

which tells us that:

$$\text{coker } \beta = P_i / P_j\pi^k$$

as required. Again, for $i < j$ we just replace π^k by π^{k-1}.

We can now calculate the Fitting Invariants of $A^k_{i,j}$ in terms of the presentation matrix given above. We obtain:

$$\mathcal{F}^{\rho}_s(A^k_{i,j}) = \mathcal{F}^{\Delta_E}_s(\mathcal{N}(M_t(\Delta_E) \otimes_S A^k_{i,j})) = I_{t-s}(\tilde{B})$$

Hence, dropping the symbol ρ since it is just the inclusion map, calculating the Fitting Ideals we obtain:

$$\mathcal{F}_0(A^k_{i,j}) = \begin{cases} \pi^k & \text{for } i \geq j \\ \pi^{k-1} & \text{for } i < j \end{cases}$$

and

$$\mathcal{F}_s(A^k_{i,j}) = \Delta_E \text{ for } s \geq 1$$

We know from Definition 5.3 that these Fitting Ideals are independent of the presentation matrix used for each $A^k_{i,j}$. The $A^k_{i,j}$ are cyclic modules and we just use the simple presentation matrix B with one generator to calculate the Fitting Ideals. Thus, the Fitting Ideals of some finitely generated left S-module ${}_SM$ can be calculated using the direct sum formula, equation (2.1), as a sum of the Fitting Ideals of each of the cyclic modules. However, these Fitting Ideals do not in fact reveal too much information about the underlying module structure, as the following example shows:

Example 5.14 Let $r = 2$, then calculating Fitting Ideals of certain cyclic modules we obtain:

$$\mathcal{F}_0(A^2_{0,0}) = \mathcal{F}_0(A^2_{1,0}) = \mathcal{F}_0(A^2_{1,1}) = \mathcal{F}_0(A^3_{0,1}) = \pi^2$$

so we can see that the definition so far, even though it tells us some information, does not allow us to distinguish between these four different cyclic modules.

□

In the next section we go one step further and refine our definition in order to obtain more information about the underlying module structure.

5.6 Fitting Invariants with respect to Maximal Orders

If S is a hereditary R-order of the form $S = S_{(n_1,\ldots,n_r)}(\Delta_D)$ then S is contained in precisely r maximal orders. If we denote these maximal orders by Γ_i, for $i = 0, 1, \ldots, r-1$, then in fact we can write S as the intersection of all maximal orders containing it, i.e.

$$S = \bigcap_{i=0}^{r-1} \Gamma_i$$

If we take S to be a hereditary R-order in $A = M_t(D)$, where $t = \sum_{i=0}^{r-1} n_{i+1}$, then each maximal order is given by:

$$\Gamma_i = \{y \in A \mid yP_i \subseteq P_i\}$$

where P_i are the indecomposable S-lattices. We can then calculate the Γ_i to be:

$$\Gamma_i = \begin{array}{c} \\ i\left\{\vphantom{\begin{array}{c}(\Delta_D)\\ \vdots\\ (\Delta_D)\end{array}}\right. \\ \vphantom{\begin{array}{c}(P^{-1})\\ \vdots\\ (P^{-1})\end{array}} \end{array} \left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{(\Delta_D) \quad \cdots \quad (\Delta_D)}}^{i}} & & & \\ (\Delta_D) & \cdots & (\Delta_D) & (P) & \cdots & (P) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (\Delta_D) & \cdots & (\Delta_D) & (P) & \cdots & (P) \\ \hline (P^{-1}) & \cdots & (P^{-1}) & (\Delta_D) & \cdots & (\Delta_D) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (P^{-1}) & \cdots & (P^{-1}) & (\Delta_D) & \cdots & (\Delta_D) \end{array}\right)^{(n_1,\ldots,n_r)}$$

for $1 \leq i \leq r-1$ and where $\Gamma_0 = M_t(\Delta_D)$. The point of considering maximal orders which contain the hereditary order is that we can think of the presentation matrix for some finitely generated S-module with respect to each of the maximal orders. In the case we are dealing with, when S is a hereditary R-order with K a local field, even stronger results are available and all maximal order are conjugate to each other (see [16], Theorem 17.3, page 171 for details). In this case we have an element $x_k \in U(A)$, for some integer k where:

$$x_k = \begin{array}{r} \\ k\left\{\vphantom{\begin{array}{c}0\\ \vdots\\ \vdots\\ 0\end{array}}\right. \\ \vphantom{\begin{array}{c}0\\ \vdots\\ 0\\ 1\end{array}} \end{array} \begin{array}{l} \quad\overbrace{\hphantom{0 \;\; \cdots \;\; \cdots \;\; 0}}^{t-k} \\ \left(\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & \pi \\ \vdots & \ddots & & \vdots & \vdots & & & 0 \\ \vdots & & \ddots & \vdots & 0 & & & \vdots \\ 0 & \cdots & \cdots & 0 & \pi & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & 0 & \vdots & \ddots & & \vdots \\ 0 & & & \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array}\right) \end{array}$$

and such that each maximal order Γ_i can be expressed as:

$$\Gamma_i = x_{t_i}\Gamma_0 x_{t_i}^{-1} \qquad \text{where } t_i = \sum_{s=0}^{i} n_s$$

We now need a lemma which shows how we can find presentation matrices for certain Γ_i-modules when we know the presentation matrix as a Γ_0-module.

Lemma 5.15 Suppose ${}_SN$ is a finitely generated left S-module with presentation matrix $B = (B_{k,l})$, for some $B_{k,l} \in S$. Then B is also a presentation matrix for $\Gamma_i \otimes_S N$ as a left Γ_i-module. Furthermore, the matrix $x_{t_i}^{-1}Bx_{t_i} = (x_{t_i}^{-1}B_{k,l}x_{t_i})$ is a presentation matrix for $\Gamma_i \otimes_S N$ as a left Γ_0-module (where the action of Γ_0 on $\Gamma_i \otimes_S N$ is given by $\gamma_0.n = x_{t_i}\gamma_0 x_{t_i}^{-1}n$ for all $\gamma_0 \in \Gamma_0, n \in \Gamma_i \otimes_S N$).

Proof:

Let θ_i be the inclusion mapping $\theta_i : S \longrightarrow \Gamma_i$. Then we know from the proof of Theorem 2.4 that $\theta_i(B) = B$ is a presentation matrix for ${}_{\Gamma_i}N' = \Gamma_i \otimes_S N$ as a left Γ_i-module. Now, we know $\Gamma_0 = x_{t_i}^{-1}\Gamma_i x_{t_i}$ hence $x_{t_i}^{-1}Bx_{t_i}$ must be a presentation matrix for ${}_{\Gamma_i}N'$ as a left Γ_0-module.

□

Thus, we can see that from Definition 5.3, the calculation of the Fitting Ideals of ${}_SN$ is equivalent to calculating Fitting Ideals for ${}_{\Gamma_i}N' = \Gamma_i \otimes_S N$ as a Γ_i-module. If instead we calculate Fitting Ideals for ${}_{\Gamma_i}N'$ as a Γ_0-module we get a new set of Fitting Invariants for each maximal order Γ_i. We do this by considering determinantal ideals of the unboxed matrix, $\widetilde{x_{t_i}^{-1}Bx_{t_i}}$. Let us now return again to the simplified case where $D = E$ for some field E. This leads us to a new definition.

Definition 5.16 Let $S = S_{(n_1,\dots,n_r)}(\Delta_E)$ be a hereditary R-order in $M_t(E)$, for some field E. If ${}_SN$ is a finitely generated S-module then we define the s-th Fitting Invariant of ${}_SN$ with respect to the i-th maximal order, Γ_i to be:

$$\mathcal{F}_s^i({}_SN) = \mathcal{F}_s^{\rho_i}(\Gamma_i \otimes_S N)$$

where ρ_i is the map $\rho_i : \Gamma_i \longrightarrow \Gamma_0$, for $i = 0, 1, \dots, r-1$.

Note that when $i = 0$ we obtain:

$$\mathcal{F}_s^0({}_SN) = \mathcal{F}_s^{\rho_0}(\Gamma_0 \otimes_S N)$$

which agrees with Definition 5.3.

If B is a presentation matrix for ${}_SN$ with g generators then, since $\rho_i(B) = x_{t_i}^{-1}Bx_{t_i}$ we obtain:

$$\mathcal{F}_s^i({}_SN) = I_{gt-s}(\widetilde{x_{t_i}^{-1}Bx_{t_i}})$$

Thus, we have a whole new set of Fitting Invariants for each maximal order which gives us finer information about the underlying module structure. However, as the following example shows, it is still not quite enough to tell us completely what the module structure is.

Example 5.17 Let $r = 2$ and consider the Fitting Invariants of the cyclic modules we considered in Example 5.14. On calculating initial Fitting Ideals we obtain:

${}_SN$	$\mathcal{F}_0^0$	$\mathcal{F}_0^1$
$A_{0,0}^2$	π^2	π^2
$A_{0,1}^3$	π^2	π^3
$A_{1,0}^2$	π^2	π^1
$A_{1,1}^2$	π^2	π^2

So here we can see we have improved on the situation in Example 5.14 in that we can now distinguish between most of the cyclic modules if we consider the Fitting Ideals with respect to each maximal order. Whereas previously, when we calculated the Fitting Ideals with respect to only one maximal order we could not distinguish between any of these cyclic modules. However, in this example we still require more information as these Fitting Ideals do not allow us to distinguish between $A^2_{0,0}$ and $A^2_{1,1}$.

□

We will remedy this in the next section where we construct an even finer set of invariants dependent on the projectives inside the hereditary order. We will then see how this finer set of invariants will enable us to completely determine the isomorphism class of torsion modules.

5.7 Fitting Invariants with respect to Projectives

Recall that in the decomposition of a hereditary R-order, S in $A = M_t(D)$ of type r (see (5.1)), there are n_{i+1} copies of each P_i, for $i = 0, 1, \ldots, r-1$. Let P_i' denote one of these copies which is the left ideal of S which consists of all matrices in S in which all columns are zero except the $(\sum_{v=0}^{i} n_v + 1)$-st column which is an element of P_i. That is:

$$P_i' = \{(\underbrace{0|\cdots|0}_{\sum_{v=0}^{i} n_v}|\underline{x}|0|\cdots|0) \mid \underline{x} \in P_i\}$$

We can obtain further invariants by 'hitting' each S-module by one of the P_i', or more precisely by calculating $P_i'N$, where N is a finitely generated left S-module. Hitting with projectives in this way we obtain:

$$P_i'P_j = \begin{cases} P_i & \text{for } i \geq j \\ \pi P_i & \text{for } i < j \end{cases}$$

Note that here we are considering $P_i'P_j$ as a submodule of P_j since P_i and πP_i are isomorphic as abstract modules. This gives us a new module and a new presentation matrix for each projective, P_i and thus a new set of invariants. Thus, for each N, we have a total set of r^2 Fitting Invariants, with respect to the projectives, P_i and the maximal orders Γ_j, for $i, j = 0, 1, \ldots, r-1$. So we can now construct a notion of Fitting Invariant with respect to the two variables P_i and Γ_j. We first do this in the case $D = E$.

Definition 5.18 Keeping the notation of Definition 5.16 we define the s-th Fitting Invariant of ${}_SN$ with respect to P_i and Γ_j to be:

$$\mathcal{F}_s^{i,j}({}_SN) = \mathcal{F}_s^{j}(P_i'{}_SN)$$

for $i, j = 0, 1, \ldots, r-1$.

We can easily extend this definition to the case of D a division algebra (not necessarily commutative) over K.

Definition 5.19 Let S be a hereditary $\mathcal{O}_K$-order in $M_t(D)$, where D is a division algebra over a local field, K. Further let L denote the unique unramified extension of K of degree $n = \sqrt{D : K}$. Then the s-th Fitting Invariant of the finitely generated left S-module, ${}_SM$ with respect to P_i and Γ_j is defined to be:

$$\mathcal{F}_s^{i,j}({}_SM) = \mathcal{F}_s^{i,j}(\mathcal{O}_L \otimes_S M)$$

for $i, j = 0, 1, \ldots, n-1$.

Since $\mathcal{O}_L \otimes_{\mathcal{O}_K} S$ is a hereditary $\mathcal{O}_K$-order in $M_{nt}(L)$ Definition 5.19 tells us that the Fitting Ideals we obtain are in fact ideals in the ring $\mathcal{O}_L$.

Before we go further we illustrate the effectiveness of the above definition by continuing with Example 5.17.

Example 5.20 In this example $r = 2$ so the S-projectives are just P_0 and P_1. Calculating $P_i'N$ for $i = 0, 1$ we obtain:

${}_SN$	$P_0'N$	$P_1'N$
$A_{0,0}^2 = P_0/\pi^2 P_0$	$P_0'P_0/\pi^2P_0 = P_0/\pi^2P_0 = A_{0,0}^2$	$P_1'P_0/\pi^2P_0 = P_1/\pi^2P_0 = A_{1,0}^2$
$A_{0,1}^3 = P_0/\pi^2 P_1$	$P_0'P_0/\pi^2P_1 = P_0/\pi^2P_1 = A_{0,1}^3$	$P_1'P_0/\pi^2P_1 = P_1/\pi^2P_1 = A_{1,1}^2$
$A_{1,0}^2 = P_1/\pi^2 P_0$	$P_0'P_1/\pi^2P_0 = \pi P_0/\pi^2P_0 = A_{0,0}^1$	$P_1'P_1/\pi^2P_0 = P_1/\pi^2P_0 = A_{1,0}^2$
$A_{1,1}^2 = P_1/\pi^2 P_1$	$P_0'P_1/\pi^2P_1 = \pi P_0/\pi^2P_1 = A_{0,1}^2$	$P_1'P_1/\pi^2P_1 = P_1/\pi^2P_1 = A_{1,1}^2$

and on calculating the initial Fitting Invariants $\mathcal{F}_0^{i,j}(N)$ we obtain:

${}_SN$	$\mathcal{F}_0^{0,0}$	$\mathcal{F}_0^{0,1}$	$\mathcal{F}_0^{1,0}$	$\mathcal{F}_0^{1,1}$
$A_{0,0}^2$	π^2	π^2	π^2	π^1
$A_{0,1}^3$	π^2	π^3	π^2	π^2
$A_{1,0}^2$	π^1	π^1	π^2	π^1
$A_{1,1}^2$	π^1	π^2	π^2	π^2

In this example we have $r^2 = 4$ sets of Fitting Invariants for each cyclic module and we see that the sequence of invariants, $\mathcal{F}_0^{i,j}$, is different for each cyclic module. So in this case we can tell which of the cyclic modules we are dealing with just by looking at the Fitting information.

$\square$

In the next section we extend this idea to the general case and we will prove the main result of this chapter, Corollary 5.24, that this finer set of Fitting Invariants uniquely determines the structure of torsion modules over hereditary orders.

5.8 Determination of the Module

We will initially continue to study the simplified case $D = E$ and $r = 2$ and will show that if ${}_SN$ is any finitely generated S-module, where $S = S_{(n_1,n_2)}(\Delta_E)$, such that:

$$ {}_SN = \bigoplus_{i,j,k} a_{i,j}^k A_{i,j}^k $$

then our Fitting Invariants will completely determine the structure of ${}_SN$. We extend Example 5.20 to the case for general cyclic modules $A_{i,j}^k$ and we find that the initial Fitting Invariants are:

${}_SN$	$\mathcal{F}_0^{0,0}$	$\mathcal{F}_0^{0,1}$	$\mathcal{F}_0^{1,0}$	$\mathcal{F}_0^{1,1}$
$A_{0,0}^k$	π^k	π^k	π^k	π^{k-1}
$A_{0,1}^k$	π^{k-1}	π^k	π^{k-1}	π^{k-1}
$A_{1,0}^k$	π^{k-1}	π^{k-1}	π^k	π^{k-1}
$A_{1,1}^k$	π^{k-1}	π^k	π^k	π^k

We can actually write ${}_SN$ as:

$$ {}_SN = \bigoplus_{k=1}^{m} N_k \text{ where } N_k = \bigoplus_{i,j} a_{i,j}^k A_{i,j}^k $$

for some integers $a_{i,j}^k \geq 0$ and for $i,j = 0,1$. We will call N_k the k-th stratum occurring in ${}_SN$. Thus, N_m is the maximum stratum occurring. The point of decomposing ${}_SN$ this way is that we can now calculate succesive quotients of the $\mathcal{F}_s^{i,j}(N)$ (as we did in section 3.2 for PIDs) to determine the coefficients in the m-th stratum, $a_{i,j}^m$.

Considering quotients of:

$$\begin{array}{lll} \mathcal{F}_s^{0,0}(N) & \text{determines} & a_{0,0}^m \\ \mathcal{F}_s^{1,1}(N) & \text{determines} & a_{1,1}^m \\ \mathcal{F}_s^{0,1}(N) & \text{determines} & a_{0,0}^m + a_{0,1}^m + a_{1,1}^m \Rightarrow a_{0,1}^m \\ \mathcal{F}_s^{1,0}(N) & \text{determines} & a_{0,0}^m + a_{1,0}^m + a_{1,1}^m \Rightarrow a_{1,0}^m \end{array}$$

Thus, we have determined N_m and we repeat this procedure for the $(m-1)$-st stratum, N_{m-1}. By considering quotients of the Fitting Ideals again and, since we know the $a_{i,j}^m$, we can determine the $a_{i,j}^{m-1}$ and therefore determine N_{m-1}. We repeat this procedure until all the strata, N_k, are known and thus we have determined $_SN$, which is the direct sum of all the strata. We illustrate this procedure with an example:

Example 5.21 Suppose we are given the Fitting Invariants of some finitely generated S-module, $_SN$, as powers of π as follows:

$_SN$	$\text{ord}_\pi(\mathcal{F}_s^{0,0})$	$\text{ord}_\pi(\mathcal{F}_s^{0,1})$	$\text{ord}_\pi(\mathcal{F}_s^{1,0})$	$\text{ord}_\pi(\mathcal{F}_s^{1,1})$
$s=0$	22	27	25	22
$s=1$	18	23	21	18
$s=2$	14	19	17	14
$s=3$	11	15	13	11
$s=4$	8	11	9	8
$s=5$	5	8	6	6
$s=6$	3	5	4	4
$s=7$	1	2	2	2
$s=8$	0	0	0	0

We immediately see that as the Fitting Ideals become trivial at $s=8$ then $_SN$ must consist of exactly 8 cyclic summands. Let us now consider the quotient ideals of $\mathcal{F}_s^{i,j}$ for $i,j=0,1$. Since, $(\mathcal{F}_0^{i,j} : \mathcal{F}_1^{i,j}) = \pi^4$ for $i,j=0,1$ we see that the maximum stratum is N_4. Next consider the quotient ideals of $\mathcal{F}_s^{0,0}$:

$$\begin{aligned} (\mathcal{F}_0^{0,0} : \mathcal{F}_1^{0,0}) &= \pi^4 \\ (\mathcal{F}_1^{0,0} : \mathcal{F}_2^{0,0}) &= \pi^4 \\ (\mathcal{F}_2^{0,0} : \mathcal{F}_3^{0,0}) &= \pi^3 \end{aligned}$$

which tells us that $a^4_{0,0} = 2$. If we now look at quotients of $\mathcal{F}^{1,1}_s$ we see that $a^4_{1,1} = 2$. Now consider quotients of $\mathcal{F}^{0,1}_s$:

$$\begin{aligned}
(\mathcal{F}^{0,1}_0 : \mathcal{F}^{0,1}_1) &= \pi^4 \\
(\mathcal{F}^{0,1}_1 : \mathcal{F}^{0,1}_2) &= \pi^4 \\
(\mathcal{F}^{0,1}_2 : \mathcal{F}^{0,1}_3) &= \pi^4 \\
(\mathcal{F}^{0,1}_3 : \mathcal{F}^{0,1}_4) &= \pi^4 \\
(\mathcal{F}^{0,1}_4 : \mathcal{F}^{0,1}_5) &= \pi^3
\end{aligned}$$

which tells us that:

$$a^4_{0,0} + a^4_{0,1} + a^4_{1,1} = 4 \Rightarrow a^4_{0,1} = 0$$

and repeating for quotients of $\mathcal{F}^{1,0}_s$ we obtain:

$$a^4_{0,0} + a^4_{1,0} + a^4_{1,1} = 4 \Rightarrow a^4_{1,0} = 0$$

Hence, we have determined the maximum stratum N_4 to be:

$$N_4 = 2A^4_{0,0} \oplus 2A^4_{1,1}$$

We then repeat the procedure for the next stratum down, N_3. Again, considering quotients of $\mathcal{F}^{0,0}_s$ we obtain:

$$\begin{aligned}
(\mathcal{F}^{0,0}_2 : \mathcal{F}^{0,0}_3) &= \pi^3 \\
(\mathcal{F}^{0,0}_3 : \mathcal{F}^{0,0}_4) &= \pi^3 \\
(\mathcal{F}^{0,0}_4 : \mathcal{F}^{0,0}_5) &= \pi^3 \\
(\mathcal{F}^{0,0}_5 : \mathcal{F}^{0,0}_6) &= \pi^2
\end{aligned}$$

which tells us that:

$$(a^4_{0,1} + a^4_{1,0} + a^4_{1,1}) + a^3_{0,0} = 3 \Rightarrow a^3_{0,0} = 1$$

Again, considering quotients of $\mathcal{F}^{1,1}_s$ we obtain:

$$(a^4_{0,0} + a^4_{0,1} + a^4_{1,0}) + a^3_{1,1} = 2 \Rightarrow a^3_{1,1} = 0$$

and repeating for quotients of $\mathcal{F}^{0,1}_s$ we obtain:

$$a^4_{1,0} + (a^3_{0,0} + a^3_{0,1} + a^3_{1,1}) = 3 \Rightarrow a^3_{0,1} = 2$$

and repeating for quotients of $\mathcal{F}_s^{1,0}$ we obtain:

$$a_{0,1}^4 + (a_{0,0}^3 + a_{1,0}^3 + a_{1,1}^3) = 1 \Rightarrow a_{1,0}^3 = 0$$

and hence:

$$N_3 = A_{0,0}^3 \oplus 2A_{0,1}^3$$

We now see that $N_4 \oplus N_3$ has 7 cyclic summands so we just need to find the remaining cyclic summand. If this time we consider the remaining quotients of $\mathcal{F}_s^{1,1}$ we obtain:

$$(a_{0,0}^3 + a_{0,1}^3 + a_{1,0}^3) + a_{1,1}^2 = 4 \Rightarrow a_{1,1}^2 = 1$$

and thus:

$$N_2 = A_{1,1}^2$$

We therefore uniquely determine ${}_SN$ to be:

$${}_SN = N_4 \oplus N_3 \oplus N_2 = 2A_{0,0}^4 \oplus 2A_{1,1}^4 \oplus A_{0,0}^3 \oplus 2A_{0,1}^3 \oplus A_{1,1}^2$$

□

We will now extend this idea in the case $r = 2$ to the case of general r. As before, given a finitely generated S-module, ${}_SN$, we write it as a direct sum of its strata:

$${}_SN = \bigoplus_{k=1}^{m} N_k \text{ where } N_k = \bigoplus_{i,j} a_{i,j}^k A_{i,j}^k$$

for some integers $a_{i,j}^k \geq 0$ and $i, j = 0.1, \ldots, r-1$. We can calculate the initial Fitting Ideals, $\mathcal{F}_0^{i,j}$ of the cyclic modules $A_{i,j}^k$ for $i, j = 0, 1, \ldots, r-1$ and put this information into the form of an $r^2 \times r^2$ matrix. Denote this matrix by C_k, where the (u, v)-th entry of C_k is $\mathrm{ord}_\pi(\mathcal{F}_0^{m_1,m_2-1}(A_{l_1,l_2-1}^k))$, where we write $u = rl_1 + l_2$ for $0 \leq l_1 \leq r-1$ and $1 \leq l_2 \leq r$, and $v = rm_1 + m_2$ for $0 \leq m_1 \leq r-1$ and $1 \leq m_2 \leq r$. Thus, C_k will only consist of entries of the form $k, k-1$ or $k-2$ (if $k \geq 2$), since $k - 2 \leq \mathrm{ord}_\pi(\mathcal{F}_0^{m_1,m_2-1}(A_{l_1,l_2-1}^k)) \leq k$. For example, if $r \geq 4$ then $\mathcal{F}_0^{3,2}(A_{0,1}^k) = \pi^{k-2}$.

If we now consider the maximum stratum N_m, then the contributions to N_m will come from those entries of C_m which are equal to m; the entries equal to $m-1$ or $m-2$ do not contribute to N_m. Thus, we can concentrate on those entries m, and ignore the rest. So we replace C_k by B_k where B_k is the $r^2 \times r^2$ matrix with a 1 replacing k and zeroes replacing $k-1$ and $k-2$ in C_k. So from the m-th stratum we obtain a matrix B_m. We wish to determine the $a^m_{i,j}$ and by considering the contributions to N_m we get a series of simultaneous equations of the form:

$$B^T_m \underline{a}_m = \underline{b}_m$$

where $\underline{x}_m$ is the r^2-column vector given by:

$$\underline{x}^T_m = (x^m_{0,0}, \ldots, x^m_{0,r-1}, \ldots, x^m_{r-1,0}, \ldots, x^m_{r-1,r-1})$$

Thus, $\underline{a}_m$ denotes the r^2-column vector of the coefficients of the maximum stratum, N_m and $\underline{b}_m$ denotes the r^2-column vector of positive integers which are derived from the quotients of the Fitting Ideals (as in Example 5.21). We calculate the positive integers $b^m_{i,j}$ as follows:

1. Calculate $\mathrm{ord}_\pi(\mathcal{F}^{i,j}_0) - \mathrm{ord}_\pi(\mathcal{F}^{i,j}_1) = c^{i,j}_1$ for some positive integer $c^{i,j}_1$ which may not always be equal to m;

2. Keep repeating this calculation for higher Fitting Invariants until we find the positive integer $b^m_{i,j}$ such that:

$$\begin{array}{rcl} \mathrm{ord}_\pi(\mathcal{F}^{i,j}_{[b^m_{i,j}-1]}) - \mathrm{ord}_\pi(\mathcal{F}^{i,j}_{b^m_{i,j}}) & = & c^{i,j}_1 \\ \mathrm{ord}_\pi(\mathcal{F}^{i,j}_{b^m_{i,j}}) - \mathrm{ord}_\pi(\mathcal{F}^{i,j}_{[b^m_{i,j}+1]}) & = & c^{i,j}_2 \quad < c^{i,j}_1 \end{array}$$

   for some positive integer $c^{i,j}_2$.

We repeat this procedure for each i,j so we can determine $\underline{b}_m$. Then we know we can uniquely determine the $a^m_{i,j}$ if and only if the equation $B^T_m \underline{a}_m = \underline{b}_m$ has a unique solution, or in other words, if and only if $\det\ B_m \neq 0$. Furthermore, if we show that $\det\ B_m = \pm 1$ then we can uniquely determine the $a^m_{i,j}$ as integers. We will determine $\det\ B_m$ in the following lemma:

Lemma 5.22 With the notation as above, the $r^2 \times r^2$ matrix B_m, derived from the Fitting information of the cyclic modules $A^m_{i,j}$, has a determinant equal to $+1$.

Proof:

Let us write $C_m = (C_{q,p})$ for $p, q = 0, 1, \ldots, r-1$, where $C_{q,p}$ is the $r \times r$ matrix with the following entries:

${}_SN$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,0})$	$\cdots\cdots$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,r-1})$
$A^m_{q,0}$	$\square$	$\cdots\cdots$	$\square$
$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$
$A^m_{q,r-1}$	$\square$	$\cdots\cdots$	$\square$

for some entries $\square$, which are equal to $m, m-1$ or $m-2$. Let us now denote the $r \times r$ matrix $B_{p,q}$ to be the matrix derived from $C_{q,p}$ by replacing each entry m with a 1 and each entry $m-1$ or $m-2$ with a zero. Then we see that $B_m^T = (B_{p,q}^T)$. We will first consider what $B_{p,p}^T$ looks like. Let w and z be integers such that $0 \leq w, z \leq r-1$. Then:

$$P'_p A^m_{p,z} = A^m_{p,z}$$

Thus,

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{p,z})) = \begin{cases} m & \text{for } p \geq z \\ m-1 & \text{for } p < z \end{cases}$$

Then for $w \leq p$ we have:

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{p,z})) = \begin{cases} \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{p,z})) - 1 & \text{for } z < w \\ \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{p,z})) & \text{for } z \geq w \end{cases}$$

and for $w > p$ we have:

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{p,z})) = \begin{cases} \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{p,z})) & \text{for } z < w \\ \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{p,z})) + 1 & \text{for } z \geq w \end{cases}$$

Hence:

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{p,z})) = \begin{cases} m & \text{for } p \geq z, w \leq p \text{ and } z \geq w \\ m & \text{for } p \geq z, w > p \text{ and } z < w \\ m & \text{for } p < z, w > p \text{ and } z \geq w \\ m-1 & \text{otherwise} \end{cases}$$

Note that $\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{p,z})) = m-2$ is not a possibility since this requires the condition $p < z, w \leq p$ and $z < w$ which gives $w > p$, a contradiction. Similarly, $\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{p,z})) = m+1$ is not a possibility since this requires the condition $p \geq z$, $w > p$ and $z \geq w$ which gives $p \geq w$, a contradiction. We can put these values into tabular form as follows:

${}_SN$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,0})$	$\cdots$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w})$	$\cdots$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,p})$	$\cdots$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w'})$	$\cdots$	$\mathrm{ord}_\pi(\mathcal{F}_0^{p,r-1})$
$A^m_{p,0}$	m	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	m
$A^m_{p,1}$	m	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	m
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
$A^m_{p,w-1}$	m	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	m
$A^m_{p,w}$	m	$\cdots$	m	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	m
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
$A^m_{p,p-1}$	m	$\cdots$	m	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	m
$A^m_{p,p}$	m	$\cdots$	m	$\cdots$	m	$\cdots$	m	$\cdots$	m
$A^m_{p,p+1}$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
$A^m_{p,w'-1}$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$
$A^m_{p,w'}$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	$m-1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
$A^m_{p,r-2}$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	$m-1$
$A^m_{p,r-1}$	$m-1$	$\cdots$	$m-1$	$\cdots$	$m-1$	$\cdots$	m	$\cdots$	m

for $1 \le w < p$ and $p < w' \le r-1$. We replace the entries which are m's by 1's and put zeroes elsewhere in order to determine the matrix:

$$B^T_{p,p} = \begin{array}{r} \\ p+1\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \end{array} \left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\overbrace{\qquad\qquad\qquad}^{p+1}} & & & & \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & 1 & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 & 1 \end{array}\right)$$

Now we will consider $B^T_{p,q}$ for $p < q$. Well,

$$P'_p A^m_{q,z} = \begin{cases} A^{m-1}_{p,z} & \text{for } z \le p \\ A^m_{p,z} & \text{for } p < z \le q \\ A^{m-1}_{p,z} & \text{for } z > q \end{cases}$$

Thus,

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) = \begin{cases} m-1 & \text{for } z \le q \\ m-2 & \text{for } z > q \end{cases}$$

Then for $w \leq p$ we have:

$$\operatorname{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = \begin{cases} \operatorname{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) - 1 & \text{for } z < w \\ \operatorname{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) & \text{for } z \geq w \end{cases}$$

and for $w > p$ we have:

$$\operatorname{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = \begin{cases} \operatorname{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) & \text{for } z < w \\ \operatorname{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) + 1 & \text{for } z \geq w \end{cases}$$

Hence:

$$\operatorname{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = \begin{cases} m & \text{for } z \leq q, w > p \text{ and } z \geq w \\ m-1 & \text{for } z \leq q, w \leq p \text{ and } z \geq w \\ m-1 & \text{for } z \leq q, w > p \text{ and } z < w \\ m-1 & \text{for } z > q, w > p \text{ and } z \geq w \\ m-2 & \text{for } z \leq q, w \leq p \text{ and } z < w \\ m-2 & \text{for } z > q, w \leq p \text{ and } z \geq w \\ m-2 & \text{for } z > q, w > p \text{ and } z < w \end{cases}$$

Note that $\operatorname{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = m - 3$ is not a possibility since this requires the condition $z > q, w \leq p$ and $z < w$ which gives $p > q$, a contradiction. We can go straight to the matrix:

$$B^T_{p,q} = \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad\qquad\qquad}^{p+1} & \overbrace{\qquad\qquad\qquad}^{q-p} & \qquad\qquad\qquad \end{array} \\ \left(\begin{array}{cccc|cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & 0 & \ddots & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array}\right) \end{array}$$

Now we will consider $B^T_{p,q}$ for $p > q$. Well,

$$P'_p A^m_{q,z} = \begin{cases} A^m_{p,z} & \text{for } z \leq q \\ A^{m-1}_{p,z} & \text{for } q < z \leq p \\ A^m_{p,z} & \text{for } z > p \end{cases}$$

Thus,

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) = \begin{cases} m & \text{for } z \leq q \\ m-1 & \text{for } z > q \end{cases}$$

Then for $w \leq p$ we have:

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = \begin{cases} \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) - 1 & \text{for } z < w \\ \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) & \text{for } z \geq w \end{cases}$$

and for $w > p$ we have:

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = \begin{cases} \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) & \text{for } z < w \\ \mathrm{ord}_\pi(\mathcal{F}_0^{p,0}(A^m_{q,z})) + 1 & \text{for } z \geq w \end{cases}$$

Hence:

$$\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = \begin{cases} m & \text{for } z \leq q, w \leq p \text{ and } z \geq w \\ m & \text{for } z \leq q, w > p \text{ and } z < w \\ m & \text{for } z > q, w > p \text{ and } z \geq w \\ m-1 & \text{for } z \leq q, w \leq p \text{ and } z < w \\ m-1 & \text{for } z > q, w \leq p \text{ and } z \geq w \\ m-1 & \text{for } z > q, w > p \text{ and } z < w \\ m-2 & \text{for } z > q, w \leq p \text{ and } z < w \end{cases}$$

Note that $\mathrm{ord}_\pi(\mathcal{F}_0^{p,w}(A^m_{q,z})) = m+1$ is not a possibility since this requires the condition $z \leq q, w > p$ and $z \geq w$ which gives $q > p$, a contradiction. We thus obtain the matrix:

$$B_{p,q}^T = \begin{array}{c} \begin{array}{ccc} \overbrace{\qquad\qquad\qquad}^{q+1} & \overbrace{\qquad\qquad\qquad}^{p-q} & \qquad\qquad\qquad \end{array} \\ \left(\begin{array}{cccc|cccc|cccc}
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\hline
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & & & \vdots & \vdots & & & \vdots & 0 & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1
\end{array}\right) \end{array}$$

We wish to calculate det B_m and to do this we will rearrange B_m^T into a simpler form by permuting rows and columns of this matrix. Consider the $r \times r$ matrix $B_{p,q}^T$ and suppose we perform the following operations:

Column	Permutes to	Row	Permutes to
1	$r-q$	1	$r-p$
2	$r-q+1$	2	$r-p+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
q	$r-1$	p	$r-1$
$q+1$	r	$p+1$	r
$q+2$	1	$p+2$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$r-q-1$	r	$r-p-1$

for every p, q. Denote by $\widehat{B_m^T} = (\widehat{B_{p,q}^T})$ the resulting matrix after these column and row operations have been made on B_m^T. In fact these column permutations are given by the $(r-q-1)$-st power of the r-cycle $(1\ 2\ \cdots\ r)$ and so the effect on $\det B_{p,q}^T$ is a sign change of $(-1)^{(r-1)(r-q-1)}$. Similarly, the effect on $\det B_{p,q}^T$ of performing the row operations is a sign change of $(-1)^{(r-1)(r-p-1)}$. Thus, the overall effect of these operations gives $\det \widehat{B^T_{p,q}} = (-1)^{(r-1)(p+q)} \det B^T_{p,q}$. Hence, when $p = q$ we obtain $\det \widehat{B^T_{p,p}} = \det B^T_{p,p}$. Now

$$\widehat{B^T_{p,p}} = \begin{array}{r@{}l} & \quad \overbrace{\hphantom{1 \quad \cdots \quad \cdots \quad 1}}^{p+1} \\ p+1\left\{\vphantom{\begin{array}{c}1\\0\\\vdots\\0\end{array}}\right. & \left(\begin{array}{cccc|cccc} 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & 1 & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \\ \hline 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \end{array}\right) \end{array}$$

and for $p < q$ we have:

$$\widehat{B^T_{p,q}} = \begin{array}{r} \overbrace{\hphantom{1 \quad \cdots \quad \cdots \quad 1}}^{q-p} \\ \left(\begin{array}{cccc|cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots & \vdots & & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array}\right) \end{array}$$

and for $p > q$ we obtain:

$$
\widehat{B^T_{p,q}} =
\begin{array}{c}
\begin{array}{ccc}
\overbrace{\hphantom{0 \;\cdots\; \cdots \;0}}^{p-q} & \hphantom{1 \;\cdots\; \cdots \;1} & \overbrace{\hphantom{1 \;\cdots\; \cdots \;1}}^{q+1}
\end{array} \\
\left(\begin{array}{cccc|cccc|cccc}
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & & & \vdots & 0 & \ddots & & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & & & \vdots & \vdots & & & \vdots & 0 & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0
\end{array}\right)
\end{array}
$$

If we denote by E_k the $r \times r$ matrix:

$$
E_k =
\begin{array}{c}
\begin{array}{cc}
\hphantom{0 \;\cdots\; \cdots \;0} & \overbrace{\hphantom{1 \;\cdots\; \cdots \;1}}^{k}
\end{array} \\
\left(\begin{array}{cccc|cccc}
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & & & \vdots & 0 & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0
\end{array}\right)
\end{array}
$$

for $k = 1, \ldots, r$, then we have that:

$$
\widehat{B^T_{p,q}} = \begin{cases} E_r & \text{for } p = q \\ E_{q-p} & \text{for } p < q \\ E_{r-p+q} & \text{for } p > q \end{cases}
$$

We can then write:

$$\widehat{B_m^T} = \begin{pmatrix} E_r & E_1 & \cdots & E_{r-1} \\ E_{r-1} & E_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & E_1 \\ E_1 & \cdots & E_{r-1} & E_r \end{pmatrix}$$

Now, let T be the $r \times r$ matrix:

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

then we can see that:

$$T^u = \begin{array}{r} \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0}}^{r-u} \\ \left(\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array}\right) \end{array}$$

for $u = 1, \ldots, r-1$ and $T^r = 0$. Thus, we can write $E_{r-u} = T^u E_r$ and hence:

$$\widehat{B_m^T} = \begin{pmatrix} E_r & T^{r-1}E_r & \cdots & TE_r \\ TE_r & E_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & T^{r-1}E_r \\ T^{r-1}E_r & \cdots & TE_r & E_r \end{pmatrix}$$

Now think of $\widehat{B_m^T}$ as a block $r \times r$ matrix, with each entry equal to $T^u E_r$, for some u. Now we can perform block row operations:

$$\text{Block row } u - T^u[\text{Block row } r] \text{ for } u = 1, \ldots, r-1$$

to obtain:

$$\widehat{B^T_m} \sim \begin{pmatrix} E_r & 0 & \cdots & 0 \\ TE_r & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ T^{r-1}E_r & \cdots & TE_r & E_r \end{pmatrix}$$

Note that these block operations are achieved by a sequence of ordinary elementary row operations and thus the determinant is unchanged. Then, since each $r \times r$ block entry $T^u E_r$ is a square matrix we have:

$$\det\ \widehat{B^T_m} = (\det\ E_r)^r = 1$$

Then:

$$\det\ B_m = (\det\ B^T_{p,p})^r = (\det\ \widehat{B^T_{p,p}})^r = \det\ \widehat{B^T_m} = 1$$

□

We are now in a position to prove the main results of this chapter.

Theorem 5.23 Let S be a hereditary R-order in $M_t(E)$, for some field E, and $_SN$ be a finitely generated left torsion S-module. Then the set of Fitting Ideals $\mathcal{F}^{i,j}_s({}_SN)$ uniquely determine $_SN$ up to isomorphism.

Proof:

From Lemma 5.22 we know det $B_m = 1$ so we can therefore uniquely determine $\underline{a}_m$ and thus uniquely determine the maximum stratum, N_m of $_SN$. We next consider the $(m-1)$-st stratum. We know C_k is a matrix consisting of the entries k, $k-1$ or $k-2$ in exactly the same places as the entries m, $m-1$ or $m-2$ respectively in C_m. Thus, $B_k = B_m$ and det $B_k =$ det B_m. Since the $\underline{a}_m$ are known and det $B_{m-1} = 1$ we can uniquely determine the $\underline{a}_{m-1}$. Thus, we can determine N_{m-1} and we repeat this procedure in order to determine N_k for $k = m-2, \ldots, 1$. Since $_SN = \bigoplus N_k$ we have determined $_SN$ uniquely up to isomorphism.

□

We can easily extend this result for hereditary orders in matrix rings over division algebras.

Corollary 5.24 Let S' be a hereditary R-order in $M_t(D)$, for some division algebra D over K, and ${}_{S'}M$ a finitely generated left torsion S'-module. Then the set of Fitting Ideals $\mathcal{F}_s^{i,j}({}_{S'}M)$ uniquely determine ${}_{S'}M$ up to isomorphism.

Proof:

Now we know that $\mathcal{O}_L \otimes_{\mathcal{O}_K} S' \cong S$, where S is a hereditary R-order in $M_w(L)$, for some w and where L is the unique unramified extension of K of degree $n = \sqrt{D : K}$. Suppose we let ${}_SN ={}_S (\mathcal{O}_L \otimes_{S'} M)$, then Theorem 5.23 tells us that the $\mathcal{F}_s^{i,j}({}_{S'}M)$ uniquely determine ${}_SN$ as an S-module. However, the functor:

$$(\mathcal{O}_L \otimes_{\mathcal{O}_K} -) : \ {}_{S'}\text{Mod} \longrightarrow_S \text{Mod}$$

is one-one on isomorphism classes. To see this suppose that:

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} M_1 \cong \mathcal{O}_L \otimes_{\mathcal{O}_K} M_2$$

for some ${}_{S'}M_1$ and ${}_{S'}M_2$. Then, since $(\mathcal{O}_L)_{\mathcal{O}_K} \cong (\mathcal{O}_K)^n$, we have:

$$\mathcal{O}_L \otimes_{\mathcal{O}_K} M_1 \cong (\mathcal{O}_K)^n \otimes_{\mathcal{O}_K} M_1 \cong M_1^n \cong M_2^n$$

Then by the KSA Theorem we must have $M_1 \cong M_2$. Therefore, ${}_{S'}M$ must be uniquely determined by the Fitting Invariants up to isomorphism.

□

Thus, we have constructed a notion of Fitting Invariant for modules over hereditary orders with respect to maximal orders and the projectives in the hereditary order. We have shown that the Fitting Ideals derived give us enough information to completely determine the structure of torsion modules over hereditary orders. In the next chapter we consider modules over non-hereditary orders. In this case the option of considering projectives is not a viable one, so we need an alternative viewpoint. We do this by considering how we can adjust the presentation matrix of a finitely generated module over a non-hereditary order in order to derive some invariants. It will be shown that if we then use this 'matrix' procedure back in the hereditary case we again obtain enough information to tell us the structure of torsion modules.

Chapter 6

Non-Herditary Orders

In this chapter we will construct a notion of Fitting Invariant for modules over certain orders, which may not be hereditary. We will prove in Theorem 6.5 that we can derive a series of invariants by considering adjustments or operations on the presentation matrix of a finitely generated module over such an order. We will call these $\mathcal{G}$-invariants. This construction works well for modules over principal hereditary orders where we prove in Corollary 6.15 that there exists a sequence of $\mathcal{G}$-invariants which enable us to completely determine the isomorphism class of torsion modules. As an application we see what these $\mathcal{G}$-invariants can tell us in the case of modules over twisted group rings and we then obtain significant invariants for modules over group rings of metacyclic groups.

6.1 Construction of $\mathcal{G}$-invariants

In sections 5.6 and 5.7 we derived finer Fitting Invariants for modules over hereditary orders when we considered the action of the maximal orders and projectives in the hereditary order on the modules. We saw that this procedure gave us some new information which enabled us to completely determine the structure of torsion modules. The problem with this approach is that it is difficult to extend to more general orders and also it involves operations on modules, which can be quite cumbersome. So, instead we consider how we can derive extra information by considering adjustments to the presentation matrix of a finitely generated module.

Let Γ be a Noetherian ring such that Γ is a subring of some Noetherian ring Ω. Let R be an integral domain with field of fractions L. Suppose that we have a representation $\rho : \Omega \longrightarrow M_n(L)$, such that $\rho(\Gamma) \subseteq \Lambda = M_n(R)$. Define a fractional R-ideal to be a finitely generated R-submodule of L. We now define the Fitting Ideals or Fitting Invariants of a matrix, which we will need for our subsequent discussion.

Definition 6.1 Let $C \in M_{m\times g}(\Omega)$ be a matrix over Ω. Then the s-th Fitting Invariant or Fitting Ideal of C with respect to ρ is defined to be:

$$\mathcal{F}_s^\rho(C) = I_{gn-s}(\widetilde{\rho(C)})$$

Note that the ideals we obtain are fractional R-ideals. We next consider a simple example which will give us some idea as to how we should adjust the presentation matrix in order to provide us with more Fitting information.

Example 6.2 Let us work over the 2-adic ring, $\mathbb{Z}_2$ and let ${}_{\mathbb{Z}_2}M$ be a finitely generated $\mathbb{Z}_2$-module, with presentation matrix:

$$A = \begin{pmatrix} 2^2 & 0 \\ 0 & 2^3 \end{pmatrix}$$

Of course we know from Theorem 3.1 that the Fitting Ideals $\mathcal{F}_0({}_{\mathbb{Z}_2}M)$, $\mathcal{F}_1({}_{\mathbb{Z}_2}M)$ and $\mathcal{F}_2({}_{\mathbb{Z}_2}M)$ will tell us the structure of ${}_{\mathbb{Z}_2}M$ completely. Indeed,

$${}_{\mathbb{Z}_2}M = \mathbb{Z}_2/2^2\mathbb{Z}_2 \oplus \mathbb{Z}_2/2^3\mathbb{Z}_2$$

Suppose however we consider the matrix:

$$A2^{-1} = \begin{pmatrix} 2^1 & 0 \\ 0 & 2^2 \end{pmatrix}$$

then the Fitting Ideals of this matrix tell us $A2^{-1}$ is a presentation matrix for:

$$\mathbb{Z}_2/2^1\mathbb{Z}_2 \oplus \mathbb{Z}_2/2^2\mathbb{Z}_2 = 2[{}_{\mathbb{Z}_2}M]$$

and similarly the Fitting information derived from the matrix $A2^{-2}$ tells us this is a presentation matrix for $2^2[{}_{\mathbb{Z}_2}M]$.

□

This example shows us that in general maybe we can consider deriving invariants from $A\alpha^{-r}$, for some presentation matrix A, and some appropriately chosen element $\alpha \in \Gamma$. We want to construct invariants which are independent of the particular generating sets or presentation matrices chosen. Suppose A is a presentation matrix for some finitely generated left Γ-module, ${}_{\Gamma}M$. Then we see that:

$$B = \left(\begin{array}{c|c} 0 & 1 \\ \hline A & 0 \end{array}\right)$$

is also a presentation matrix for ${}_{\Gamma}M$, but that the matrices $A\alpha^{-r}$ and $B\alpha^{-r}$ will not in general have the same Fitting Ideals. Indeed, in example 6.2, where $\alpha = 2$, we see that the matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 2^3 \end{pmatrix}$$

is also a presentation matrix for ${}_{\mathbb{Z}_2}M$ but the matrix:

$$B2^{-1} = \begin{pmatrix} 2^{-1} & 0 & 0 \\ 0 & 2^1 & 0 \\ 0 & 0 & 2^2 \end{pmatrix}$$

has different Fitting Ideals to the Fitting Ideals of $A2^{-1}$ (e.g. $\mathcal{F}_0(A2^{-1}) = (2)^3 \neq (2)^2 = \mathcal{F}_0(B2^{-1})$).

If $A = (a_{i,j})$ for some $a_{i,j} \in \Gamma$, then we write $\omega_1 A \omega_2 = (\omega_1 a_{i,j} \omega_2)$ for all $\omega_1, \omega_2 \in \Omega$. We will choose $\alpha \in \Gamma$ such that α is a unit in Ω and consider determinantal ideals of the unboxed matrix of $\rho(\alpha^{-b} A \alpha^{-r} \alpha^b)$ for integers b and integers $r \geq 0$. We first of all need to choose an appropriate α.

Definition 6.3 We say that a subring Σ of Γ is α-stable with respect to Γ if $\alpha^{-b}\Sigma\alpha^b \subseteq \Gamma$ for all $b \in \mathbb{Z}$. Simply say that Γ is α-stable whenever Γ is α-stable with respect to Γ.

We suppose now that Γ is α-stable. We will construct invariants for finitely generated Γ-modules which are independent of the particular generating set chosen.

To do this let $A \in M_{m\times g}(\Gamma)$ be a presentation matrix for ${}_\Gamma M$ with respect to some generating set $\underline{e}$. Let us define the matrix

$$A^t_{r,b} = \left(\begin{array}{c|c} 0 & \alpha^{-b} I_t \alpha^{-r+b} \\ \hline \alpha^{-b} A \alpha^{-r+b} & 0 \end{array}\right)$$

for $b \in \mathbb{Z}$, $r, t \geq 0$ and where I_t is the $t \times t$ identity matrix. We will consider Fitting Ideals of $A^t_{r,b}$. We first need a lemma:

Lemma 6.4 If $t \geq s \geq 0$, then:

$$\mathcal{F}^\rho_s(A^{t+1}_{r,b}) = \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_s(A^t_{r,b})$$

Proof:

First note that we can obtain an analogous result to the direct sum formula (see equation (2.1)) for non-integral matrices, namely:

$$\mathcal{F}^\rho_s(A^{t+1}_{r,b}) = \sum_{k=0}^{s} \mathcal{F}^\rho_k(\alpha^{-r})\mathcal{F}^\rho_{s-k}(A^t_{r,b}) \tag{6.1}$$

We next proceed by induction on t. Consider $t = 0$. Then, using equation (6.1) we obtain:

$$\mathcal{F}^\rho_0(A^1_{r,b}) = \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_0(A^0_{r,b})$$

and so the lemma is true for $t = 0$. Next assume the lemma is true for $t = k-1$, for an integer $k \geq 1$, so that:

$$\mathcal{F}^\rho_s(A^k_{r,b}) = \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_s(A^{k-1}_{r,b})$$

for $0 \leq s \leq k-1$. Consider $t = k$. Then, using equation (6.1) we obtain:

$$\begin{aligned}
\mathcal{F}^\rho_s(A^{k+1}_{r,b}) &= \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_s(A^k_{r,b}) + \mathcal{F}^\rho_1(\alpha^{-r})\mathcal{F}^\rho_{s-1}(A^k_{r,b}) + \cdots + \mathcal{F}^\rho_s(\alpha^{-r})\mathcal{F}^\rho_0(A^k_{r,b}) \\
&= \mathcal{F}^\rho_0(\alpha^{-r})\left[\mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_s(A^{k-1}_{r,b}) + \mathcal{F}^\rho_1(\alpha^{-r})\mathcal{F}^\rho_{s-1}(A^{k-1}_{r,b}) + \cdots + \mathcal{F}^\rho_s(\alpha^{-r})\mathcal{F}^\rho_0(A^{k-1}_{r,b})\right] \\
&= \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_s(A^k_{r,b})
\end{aligned}$$

for $0 \leq s \leq k-1$. So it remains to prove the assertion for $s = k$. Well,

$$\begin{aligned}
\mathcal{F}^\rho_k(A^{k+1}_{r,b}) &= \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_k(A^k_{r,b}) + \mathcal{F}^\rho_1(\alpha^{-r})\mathcal{F}^\rho_{k-1}(A^k_{r,b}) + \cdots + \mathcal{F}^\rho_k(\alpha^{-r})\mathcal{F}^\rho_0(A^k_{r,b}) \\
&= \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_k(A^k_{r,b}) + \mathcal{F}^\rho_0(\alpha^{-r})\left[\mathcal{F}^\rho_1(\alpha^{-r})\mathcal{F}^\rho_{k-1}(A^{k-1}_{r,b}) + \cdots + \mathcal{F}^\rho_k(\alpha^{-r})\mathcal{F}^\rho_0(A^{k-1}_{r,b})\right] \\
&= \mathcal{F}^\rho_0(\alpha^{-r})\mathcal{F}^\rho_k(A^k_{r,b})
\end{aligned}$$

since using equation (6.1) again we see that:

$$\left[\mathcal{F}_1^\rho(\alpha^{-r})\mathcal{F}_{k-1}^\rho(A_{r,b}^{k-1}) + \cdots + \mathcal{F}_k^\rho(\alpha^{-r})\mathcal{F}_0^\rho(A_{r,b}^{k-1})\right] \subseteq \mathcal{F}_k^\rho(A_{r,b}^k)$$

□

In view of Lemma 6.4, and since $\mathcal{F}_0^\rho(\alpha^{-r})$ is a principal fractional R-ideal and hence invertible, we define:

$$\mathcal{G}_s^\rho(A)_{r,b} = [\mathcal{F}_0^\rho(\alpha^{-r})]^{-[g+t]}\mathcal{F}_s^\rho(A_{r,b}^t)$$

for $0 \leq s \leq t$.

The main theorem of this section shows that the $\mathcal{G}_s^\rho(A)_{r,b}$ are in fact invariants of ${}_\Gamma M$.

Theorem 6.5 Suppose Γ is α-stable. Then, for ${}_\Gamma M$ a finitely generated left Γ-module the R-ideals $\mathcal{G}_s^\rho(A)_{r,b}$, as defined above, are independent of the particular presentation matrix A chosen to compute them.

Proof:

Let $C \in M_{m' \times k}(\Gamma)$ be another presentation matrix for ${}_\Gamma M$ with respect to some generating set $\underline{f}$. Now,

$$C_{r,b}^{g+2t} = \left(\begin{array}{c|c} 0 & \alpha^{-b}I_{g+2t}\alpha^{-r+b} \\ \hline \alpha^{-b}C\alpha^{-r+b} & 0 \end{array}\right)$$

We will show that $A_{r,b}^{k+2t}$ and $C_{r,b}^{g+2t}$ both have the same Fitting Ideals with respect to ρ. Well, we know from Lemma 1.1 that we can find a matrix $Q \in M_{[k+2t]\times g}(\Gamma)$ such that the matrix:

$$B_A = \left(\begin{array}{c|c} Q & I_{k+2t} \\ \hline A & 0 \end{array}\right)$$

is a presentation matrix for ${}_\Gamma M$ with respect to the extended generating set $\underline{e} \cup \underline{f} \cup \underline{0}^{(2t)}$, where $\underline{0}^{(2t)}$ is the set consisting of $2t$ generators $\{0, \ldots, 0\}$. Similarly, we can find a matrix $P \in M_{[g+2t]\times k}(\Gamma)$ such that:

$$B_C = \left(\begin{array}{c|c} P & I_{g+2t} \\ \hline C & 0 \end{array}\right)$$

is a presentation matrix for ${}_\Gamma M$ with respect to the extended generating set $\underline{f} \cup \underline{e} \cup \underline{0}^{(2t)}$. We can perform elementary column operations on B_C to bring it to the form B'_C, where B'_C is a presentation matrix with respect to $\underline{e} \cup \underline{f} \cup \underline{0}^{(2t)}$. Hence, we can write:

$$\begin{aligned} B_A &= T_1 B'_C \\ B'_C &= T_2 B_A \end{aligned}$$

for some matrices T_1 and T_2 over Γ. Then:

$$\alpha^{-b} B_A \alpha^{-r+b} = \alpha^{-b} T_1 B'_C \alpha^{-r+b} = \alpha^{-b} T_1 \alpha^b \alpha^{-b} B'_C \alpha^{-r+b}$$

Hence:

$$\mathcal{F}^\rho_s(\alpha^{-b} B_A \alpha^{-r+b}) \subseteq \mathcal{F}^\rho_s(\alpha^{-b} T_1 \alpha^b) \mathcal{F}^\rho_s(\alpha^{-b} B'_C \alpha^{-r+b})$$

(see [13], page 7). Furthermore, since $\alpha^{-b} T_1 \alpha^b$ is a matrix over Γ (as Γ is α-stable) we obtain:

$$\mathcal{F}^\rho_s(\alpha^{-b} B_A \alpha^{-r+b}) \subseteq \mathcal{F}^\rho_s(\alpha^{-b} B'_C \alpha^{-r+b})$$

Similarly, we can show that:

$$\mathcal{F}^\rho_s(\alpha^{-b} B'_C \alpha^{-r+b}) \subseteq \mathcal{F}^\rho_s(\alpha^{-b} B_A \alpha^{-r+b})$$

and therefore:

$$\mathcal{F}^\rho_s(\alpha^{-b} B_A \alpha^{-r+b}) = \mathcal{F}^\rho_s(\alpha^{-b} B'_C \alpha^{-r+b}) = \mathcal{F}^\rho_s(\alpha^{-b} B_C \alpha^{-r+b}) \tag{6.2}$$

Now, we can perform column operations on $\alpha^{-b} B_A \alpha^{-r+b}$ to obtain:

$$A^{k+2t}_{r,b} = \left(\begin{array}{c|c} \alpha^{-b} Q \alpha^{-r+b} & \alpha^{-b} I_{k+2t} \alpha^{-r+b} \\ \hline \alpha^{-b} A \alpha^{-r+b} & 0 \end{array}\right) \left(\begin{array}{c|c} I_g & 0 \\ \hline -\alpha^{-b+r} Q \alpha^{-r+b} & I_{k+2t} \end{array}\right)$$

since $\alpha^{-b+r} Q \alpha^{-r+b} \in M_{[k+2t]\times g}(\Gamma)$. Hence, using the same argument as in the proof of equation (6.2), we have $\mathcal{F}^\rho_s(A^{k+2t}_{r,b}) = \mathcal{F}^\rho_s(\alpha^{-b} B_A \alpha^{-r+b})$. Similarly, $\mathcal{F}^\rho_s(C^{g+2t}_{r,b}) = \mathcal{F}^\rho_s(\alpha^{-b} B_C \alpha^{-r+b})$ and hence:

$$\mathcal{F}^\rho_s(A^{k+2t}_{r,b}) = \mathcal{F}^\rho_s(C^{g+2t}_{r,b})$$

Now, using Lemma 6.4 we obtain:

$$[\mathcal{F}^\rho_0(\alpha^{-r})]^{[k+t]} \mathcal{F}^\rho_s(A^t_{r,b}) = [\mathcal{F}^\rho_0(\alpha^{-r})]^{[g+t]} \mathcal{F}^\rho_s(C^t_{r,b})$$

for $0 \leq s \leq t$, which tells us that:

$$[\mathcal{F}_0^\rho(\alpha^{-r})]^{-[g+t]}\mathcal{F}_s^\rho(A_{r,b}^t) = [\mathcal{F}_0^\rho(\alpha^{-r})]^{-[k+t]}\mathcal{F}_s^\rho(C_{r,b}^t)$$

for $0 \leq s \leq t$ and hence $\mathcal{G}_s^\rho(A)_{r,b}$ is independent of the particular presentation matrix chosen. □

We now define our invariants.

Definition 6.6 For a given α, if Γ is α-stable and ${}_\Gamma M$ has a presentation matrix A, we define the s-th $\mathcal{G}$-invariant of ${}_\Gamma M$ with respect to ρ, r and b to be:

$$\mathcal{G}_s^\rho({}_\Gamma M)_{r,b} = \mathcal{G}_s^\rho(A)_{r,b}$$

Remark 6.7 We can calculate $\mathcal{G}_s^\rho({}_\Gamma M)_{r,b}$ for all values of s by just making the positive integer t sufficiently large. We also note that when $r = b = 0$, we obtain determinantal ideals of the unboxed matrix of $\rho(A)$ and we thus obtain the Fitting Ideals $\mathcal{F}_s^\rho({}_\Gamma M)$, which coincide with our definition of Fitting Invariants for rings which can be represented inside matrix rings, Definition 5.3.

Remark 6.8 Now suppose $\alpha^{-b}\Gamma\alpha^b \not\subseteq \Gamma$, but we can find a subring Σ of Γ such that $\alpha^{-b}\Sigma\alpha^b \subseteq \Gamma$ for all $b \in \mathbb{Z}$ (i.e. Σ is α-stable with respect to Γ). Suppose we choose a generating set $\underline{e}$ which generates ${}_\Gamma M$ over Σ. In this case, the argument given in the proof of Theorem 6.5 breaks down for $b \neq 0$, since the T_i may not be matrices over Σ and therefore it may not be true that $\alpha^{-b}T_i\alpha^b \subseteq \Gamma$, for $i = 1, 2$. However, we can still obtain a partial series of $\mathcal{G}$-invariants by taking $b = 0$. When $b = 0$ equation (6.2) holds and the matrix Q is such that $Q \in M_{[k+2t]\times g}(\Sigma)$, since $\underline{e}$ generates ${}_\Gamma M$ over Σ. Thus, $\alpha^r Q\alpha^{-r} \in M_{[k+2t]\times g}(\Gamma)$. Therefore, the proof of Theorem 6.5 holds for $b = 0$ and we obtain the partial set of $\mathcal{G}$-invariants, $\mathcal{G}_s^\rho({}_\Gamma M)_{r,0}$. We will construct partial $\mathcal{G}$-invariants for the case $b = 0$ in section 6.4.

We now obtain generalisations of some of the results we considered in the commutative case in chapters 1 and 2. We first consider the relationship between $\mathcal{G}_s^\rho({}_\Gamma M)_{r,b}$ and $\mathcal{G}_{s+1}^\rho({}_\Gamma M)_{r,b}$, where for $b \neq 0$ we have Γ is α-stable and for $b = 0$ we have Σ is α-stable with respect to Γ.

Theorem 6.9 For any finitely generated left Γ-module, ${}_\Gamma M$:

$$\mathcal{G}_s^\rho({}_\Gamma M)_{r,b} \subseteq \mathcal{F}_{n-1}^\rho(\alpha^{-r})\mathcal{G}_{s+1}^\rho({}_\Gamma M)_{r,b}$$

Proof:

Consider $\mathcal{F}_s^\rho(A_{r,b}^t)$. Every $[(g+t)n-s] \times [(g+t)n-s]$ minor of $\widetilde{\rho(A_{r,b}^t)}$ can be written, by the Laplace expansion formula for determinants, as a sum:

$$\sum_v J_{v,1} J_{v,[(g+t)n-s-1]}$$

where $J_{v,w}$ is some $w \times w$ minor of $\widetilde{\rho(A_{r,b}^t)}$ (where the minor includes the sign). Now, each $J_{v,[(g+t)n-s-1]} \in \mathcal{F}_{s+1}^\rho(A_{r,b}^t)$. Let us consider $J_{v,1}$. Such a minor is either a 1×1 minor of $\rho(\widetilde{\alpha^{-b} I_t \alpha^{-r+b}})$, and so lies in $\mathcal{F}_{n-1}^\rho(\alpha^{-r})$, or, it is a 1×1 minor of $\rho(\widetilde{\alpha^{-b} A \alpha^{-r+b}})$. But,

$$I_1(\rho(\widetilde{\alpha^{-b} A \alpha^{-r+b}})) \subseteq I_1(\rho(\widetilde{\alpha^{-b} A \alpha^{b}}))I_1(\widetilde{\rho(\alpha^{-r})}) \subseteq I_1(\widetilde{\rho(\alpha^{-r})})$$

since $\alpha^{-b} A \alpha^b$ is a matrix over Γ. Therefore, $I_1(\rho(\widetilde{\alpha^{-b} A \alpha^{-r+b}})) \subseteq \mathcal{F}_{n-1}^\rho(\alpha^{-r})$ and we have:

$$\mathcal{F}_s^\rho(A_{r,b}^t) \subseteq \mathcal{F}_{n-1}^\rho(\alpha^{-r})\mathcal{F}_{s+1}^\rho(A_{r,b}^t)$$

as required.

□

We next obtain some results concerning exact sequences of finitely generated Γ-modules.

Theorem 6.10 For a short exact sequence of finitely generated left Γ-modules:

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and for integers $s, s' \geq 0$ we have:

$$\mathcal{G}_s^\rho({}_\Gamma L)_{r,b}\mathcal{G}_{s'}^\rho({}_\Gamma N)_{r,b} \subseteq \mathcal{G}_{s+s'}^\rho({}_\Gamma M)_{r,b}$$

Proof:

Suppose $A \in M_{m\times g}(\Gamma)$ is a presentation matrix for ${}_\Gamma L$ and $C \in M_{l\times k}(\Gamma)$ is a presentation matrix for ${}_\Gamma N$. The first part of the proof of Theorem 2.1 generalises to the non-commutative case giving us a matrix $B \in M_{l\times g}(\Gamma)$ such that the matrix:

$$E = \left(\begin{array}{c|c} C & B \\ \hline 0 & A \end{array}\right)$$

is a presentation matrix for ${}_\Gamma M$. Now let us consider the matrix:

$$E^{2t}_{r,b} = \left(\begin{array}{c|c} 0 & \alpha^{-b} I_{2t} \alpha^{-r+b} \\ \hline \alpha^{-b} E \alpha^{-r+b} & 0 \end{array}\right)$$

Then:

$$E^{2t}_{r,b} \sim \left(\begin{array}{cc|cc} 0 & 0 & 0 & \alpha^{-b} I_t \alpha^{-r+b} \\ \alpha^{-b} B \alpha^{-r+b} & 0 & \alpha^{-b} C \alpha^{-r+b} & \\ \hline 0 & \alpha^{-b} I_t \alpha^{-r+b} & 0 & 0 \\ \alpha^{-b} A \alpha^{-r+b} & 0 & 0 & 0 \end{array}\right)$$

Now let

$$\begin{aligned} &\Delta_1 \text{ be a } [(g+t)n - s] \times [(g+t)n - s] \text{ minor of } \widetilde{\rho(A^t_{r,b})} \\ \text{and} \quad &\Delta_2 \text{ be a } [(k+t)n - s'] \times [(k+t)n - s'] \text{ minor of } \widetilde{\rho(C^t_{r,b})} \end{aligned}$$

then $\Delta_1 \Delta_2 \in \mathcal{F}^\rho_{s+s'}(E^{2t}_{r,b})$, since $\Delta_1\Delta_2$ must be a

$$[(g+k+2t)n - (s+s')] \times [(g+k+2t)n - (s+s')]$$

minor of $\widetilde{\rho(E^{2t}_{r,b})}$ and such a minor lies in $\mathcal{F}^\rho_{s+s'}(E^{2t}_{r,b})$. But,

$$\begin{aligned} &\mathcal{F}^\rho_s(A^t_{r,b}) \text{ is generated by all such minors of the form } \Delta_1 \\ \text{and} \quad &\mathcal{F}^\rho_{s'}(C^t_{r,b}) \text{ is generated by all such minors of the form } \Delta_2 \end{aligned}$$

which tells us that:

$$\mathcal{F}^\rho_s(A^t_{r,b})\mathcal{F}^\rho_{s'}(C^t_{r,b}) \subseteq \mathcal{F}^\rho_{s+s'}(E^{2t}_{r,b})$$

and hence:

$$\mathcal{G}^\rho_s({}_\Gamma L)_{r,b}\mathcal{G}^\rho_{s'}({}_\Gamma N)_{r,b} \subseteq \mathcal{G}^\rho_{s+s'}({}_\Gamma M)_{r,b}$$

as required.

□

We next consider how we can calculate $\mathcal{G}$-invariants of direct sums of Γ-modules.

Theorem 6.11 If $L \oplus N$ is a finitely generated left Γ-module then for each integer $s \geq 0$ we have:

$$\mathcal{G}_s^\rho(L \oplus N)_{r,b} = \sum_{u+v=s} \mathcal{G}_u^\rho(L)_{r,b}\mathcal{G}_v^\rho(N)_{r,b}$$

Proof:

We have a short exact sequence:

$$0 \longrightarrow L \longrightarrow L \oplus N \longrightarrow N \longrightarrow 0$$

and so by Theorem 6.10 we see immediately that:

$$\mathcal{G}_u^\rho(L)_{r.b}\mathcal{G}_v^\rho(N)_{r,b} \subseteq \mathcal{G}_{u+v}^\rho(L \oplus N)_{r,b}$$

and hence:

$$\sum_{u+v=s} \mathcal{G}_u^\rho(L)_{r,b}\mathcal{G}_v^\rho(N)_{r,b} \subseteq \mathcal{G}_s^\rho(L \oplus N)_{r,b}$$

We now consider the reverse inclusion. Keeping the notation used in the proof of Theorem 6.10, it can be shown that the matrix:

$$D = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right)$$

is a presentation matrix for $L \oplus N$ and hence:

$$D_{0,0}^{2t} = \left(\begin{array}{c|c} 0 & I_{2t} \\ \hline D & 0 \end{array}\right)$$

is also a presentation matrix for $L \oplus N$. Now,

$$D_{r,b}^{2t} \sim \left(\begin{array}{cc|cc} 0 & 0 & 0 & \alpha^{-b}I_t\alpha^{-r+b} \\ 0 & 0 & \alpha^{-b}C\alpha^{-r+b} & 0 \\ \hline 0 & \alpha^{-b}I_t\alpha^{-r+b} & 0 & 0 \\ \alpha^{-b}A\alpha^{-r+b} & 0 & 0 & 0 \end{array}\right)$$

so every $[(g+k+2t)-s] \times [(g+k+2t)-s]$ minor of $\rho(\widetilde{D_{r,b}^{2t}})$ must be of the form det J where:

$$J = \left(\begin{array}{c|c} 0 & J_2 \\ \hline J_1 & 0 \end{array}\right)$$

where

$$J_1 \text{ is a } w_1 \times [(g+t)n-u] \text{ submatrix of } \widetilde{\rho(A^t_{r,b})}$$
$$\text{and} \quad J_2 \text{ is a } w_2 \times [(k+t)n-v] \text{ submatrix of } \widetilde{\rho(C^t_{r,b})}$$

and where $w_1 + w_2 = (g+k+2t)n - s$ and $u + v = s$. So, we must have det $J = 0$ unless $w_1 = (g+t)n - u$ which tells us that $w_2 = (k+t)n - v$ and hence:

$$\det\ J = \det\ J_1 \det\ J_2 \in \mathcal{F}^\rho_u(A^t_{r,b})\mathcal{F}^\rho_v(C^t_{r,b})$$

which tells us:

$$[\mathcal{F}^\rho_0(\alpha^{-r})]^{-[g+k+2t]} \det\ J \in [\mathcal{F}^\rho_0(\alpha^{-r})]^{-[g+t]}\mathcal{F}^\rho_u(A^t_{r,b})[\mathcal{F}^\rho_0(\alpha^{-r})]^{-[k+t]}\mathcal{F}^\rho_v(C^t_{r,b})$$

But $\mathcal{G}^\rho_{u+v}(L \oplus N)_{r,b}$ is generated by all elements of the form $[\mathcal{F}^\rho_0(\alpha^{-r})]^{-[g+k+2t]} \det\ J$ so we must have:

$$\mathcal{G}^\rho_{u+v}(L \oplus N)_{r,b} \subseteq \sum_{u+v=s} \mathcal{G}^\rho_u(L)_{r,b}\mathcal{G}^\rho_v(N)_{r,b}$$

as required.

□

6.2 Determination of torsion modules over hereditary orders

Having now defined some different invariants we will check how useful they are. We want these $\mathcal{G}$-invariants to provide us with as much information as possible about the underlying module structure. We return to the case of modules over hereditary orders to show us that in this case, if we choose α appropriately then we can find a series of $\mathcal{G}$-invariants which completely determine the structure of torsion modules. We return to the notation of chapter 5 and let R be a complete dvr with $S = S_{(1,\ldots,1)}(\Delta_D)$ a principal hereditary R-order in $M_n(D)$ of type n, for some division algebra, D and where K, the field of fractions of R, is a local field. First of all we consider the simplified case $D = E$. Suppose that ${}_SN$ is a finitely generated torsion S-module. We will use the $\mathcal{G}$-invariants to determine ${}_SN$. Let α be the element:

$$\alpha = \left(\begin{array}{c|c} 0 & \pi \\ \hline I_{n-1} & 0 \end{array}\right) \in S$$

Note that $\alpha^{-1} \in M_n(E)$ and that $\alpha^{-1}S\alpha = S$. Thus, S is α-stable with respect to S and we can choose any generating set for $_SN$ to calculate a full set of $\mathcal{G}$-invariants. Note also that:

$$\alpha^n = \pi I_n$$

Now, if $m = kn + l$ for $0 \leq l \leq n-1$, we let $P_m = \pi^k P_l$. Then:

$$\alpha^r P_i = P_{i+r} \text{ for } i = 0, 1, \ldots, n-1$$

so we can write each quotient of projectives as:

$$A^k_{i,j} = P_i/\pi^k P_j = \alpha^i P_0/\alpha^{kn+j} P_0$$

for $i \geq j$ and $i, j = 0, 1, \ldots, n-1$. We will now determine what the $\mathcal{G}$-invariants of $_SN$ can tell us.

Lemma 6.12 Let $_SN$ be a finitely generated torsion S-module. Then the $\mathcal{G}$-invariants, $\mathcal{G}^\rho_s({}_SN)_{r,b}$ determine the torsion coefficients of $\Gamma_b \otimes_S \alpha^r N$ when viewed as a left Γ_0-module, where Γ_b is one of the maximal R-orders containing S, for $b = 0, 1, \ldots, n-1$ (as defined in section 5.6).

Proof:

Here, ρ is just the inclusion map, $\rho : S \longrightarrow M_n(\Delta_E)$, where Δ_E is the integral closure of R in E. Suppose $A \in M_{m\times g}(S)$ is a presentation matrix for $_SN$. Then:

$$_SN \cong S^g/S^m A$$

and thus

$$\alpha^r N \cong \alpha^r S^g/[S^m A \cap \alpha^r S^g]$$

Now, we have an isomorphism, $S \cong S\alpha^{-r}$ under the map: $\theta : s \mapsto s\alpha^{-r}$ for all $s \in S$. Thus, due to the way we have chosen α, we have $\theta(\alpha^r S) = \alpha^r S \alpha^{-r} = S$, which tells us that:

$$\alpha^r N \cong S^g/[S^m A\alpha^{-r} \cap S^g]$$

Now, since S is hereditary, any S-lattice U is S-projective and thus we can write $\Gamma_b \otimes_S U \cong \Gamma_b U$. Also, since S is hereditary, Γ_b is S-projective and thus it is flat over S. Hence, from the exact sequence

$$[S^m A\alpha^{-r} \cap S^g] \longrightarrow S^g \longrightarrow \alpha^r N$$

we obtain the exact sequence:

$$\Gamma_b \otimes_S [S^m A\alpha^{-r} \cap S^g] \longrightarrow \Gamma_b \otimes_S S^g \longrightarrow \Gamma_b \otimes_S \alpha^r N$$

Therefore, we can write:

$$\Gamma_b \otimes_S \alpha^r N \cong \Gamma_b(\alpha^r N) \cong \Gamma_b S^g / \Gamma_b[S^m A\alpha^{-r} \cap S^g]$$

Suppose that V is another S-lattice. Then, from the short exact sequence:

$$0 \longrightarrow U \cap V \longrightarrow U \oplus V \xrightarrow{\theta} U + V \longrightarrow 0$$

(where $\theta(u, v) = u - v$ for all $(u, v) \in U \oplus V$) we obtain the short exact sequence:

$$0 \longrightarrow \Gamma_b[U \cap V] \longrightarrow \Gamma_b U \oplus \Gamma_b V \xrightarrow{\theta'} \Gamma_b U + \Gamma_b V \longrightarrow 0$$

Then $\Gamma_b U \cap \Gamma_b V = \ker\ \theta' = \Gamma_b[U \cap V]$ and therefore:

$$\Gamma_b \otimes_S \alpha^r N \cong (\Gamma_b)^g / [(\Gamma_b)^m A\alpha^{-r} \cap (\Gamma_b)^g]$$

Note that we can write:

$$\Gamma_0 = \alpha^{-b} \Gamma_b \alpha^b$$

and thus:

$$\Gamma_b \otimes_S \alpha^r N \cong (\Gamma_0)^g / [(\Gamma_0)^m (\alpha^{-b} A \alpha^{-r} \alpha^b) \cap (\Gamma_0)^g]$$

Now, on calculating the $\mathcal{G}_s^\rho(N)_{r,b}$ we are effectively calculating determinantal ideals of the unboxed matrix of $\rho(A^t_{r,b})$. Thus, it can be seen that the torsion coefficients derived from quotients of the $\mathcal{G}_s^\rho(N)_{r,b}$ are precisely the torsion coefficients of the Δ_E-module $\mathcal{N}(\Gamma_0 \otimes_{\rho_b(\Gamma_b)} \Gamma_b \otimes_S \alpha^r N)$, where ρ_b is the representation $\rho_b : \Gamma_b \longrightarrow \Gamma_0$ (as in Definition 5.16). Here, $\Gamma_0 \otimes_{\rho_b(\Gamma_b)} \Gamma_b \otimes_S \alpha^r N$ is simply $\Gamma_b \otimes_S \alpha^r N$ when viewed as a left Γ_0-module. Since Δ_E is a complete dvr hence PID, we know from Theorem 3.1 that the torsion coefficients will uniquely determine $\mathcal{N}(\Gamma_0 \otimes_{\rho_b(\Gamma_b)} \Gamma_b \otimes_S \alpha^r N)$ as a Δ_E-module. But, since $\mathcal{N}$ is a bijective functor this means that $(\Gamma_0 \otimes_{\rho_b(\Gamma_b)} \Gamma_b \otimes_S \alpha^r N)$ must be uniquely determined as a Γ_0-module. Therefore, $\Gamma_b \otimes_S \alpha^r N$ is uniquely determined as a Γ_0-module by $\mathcal{G}_s^\rho(N)_{r,b}$.

$\square$

Now, let r_0 be the positive integer such that:

$$\Gamma_b \otimes_S \alpha^{r_0} N = 0 \; \forall \; b$$
$$\text{and} \quad \Gamma_b \otimes_S \alpha^{r_0 - 1} N \neq 0 \text{ for some } b$$

By Lemma 6.12 we see that if we calculate $\mathcal{G}^{\rho}_{s}(N)_{r,b}$ we will be able to determine r_0. Let $\mathrm{Rk}(N)$ denote the rank or number of cyclic summands of N. Then the $\mathcal{G}$-invariants will tell us that $\mathrm{Rk}(\alpha_0^r N) = 0$ and that $\mathrm{Rk}(\alpha^{r_0-1}N) = R_0$, for some $R_0 \in \mathbb{N}$. We can think of α^{r_0} as the smallest annihilator of ${}_S N$. Hence, we can write:

$${}_S N = N^{(0)} \oplus N_{r_0}$$

where $\alpha^{r_0-1} N^{(0)} = 0$ and

$$N_{r_0} = \bigoplus_{j=0}^{n-1} n_j^{(r_0)} \alpha^j P_0 / \alpha^{j+r_0} P_0$$

for some $n_j^{(r_0)} \in \mathbb{N} \cup \{0\}$. We can think of N_{r_0} as the direct summand of ${}_S N$ which is annihilated by α^{r_0} but not annihilated by α^{r_0-1}. We will now determine N_{r_0} by considering $\Gamma_b \otimes_S \alpha^{r_0-1} N$. Let us first consider:

$$\Gamma_b \otimes_S P_m \cong \Gamma_b P_m$$

Well, if $m = kn + l$ for some l such that $0 \leq l \leq n-1$, then:

$$\Gamma_b P_m = \Gamma_b \pi^k P_l = \begin{cases} \pi^k P_b & \text{for } b \leq l \\ \pi^{k-1} P_b & \text{for } b > l \end{cases}$$

which tells us that:

$$\Gamma_b P_m = \begin{cases} \alpha^{nk+b} P_0 & \text{for } b \leq l \\ \alpha^{nk+b-n} P_0 & \text{for } b > l \end{cases}$$

But we can write:

$$\left[\frac{m-b}{n}\right] n = \begin{cases} nk & \text{for } b \leq l \\ n(k-1) & \text{for } b > l \end{cases}$$

Thus we obtain:

$$\Gamma_b P_m = \pi^{\left[\frac{m-b}{n}\right]} \alpha^b P_0$$

We now use this to obtain:

$$\Gamma_b \otimes_S \alpha^{r_0-1} N = \bigoplus_{j=0}^{n-1} n_j^{(r_0)} \alpha^b P_0 / \pi^{\delta_j} \alpha^b P_0$$

where

$$\delta_j = \begin{cases} 1 & \text{if } j + r_0 \equiv b(n) \\ 0 & \text{if } j + r_0 \not\equiv b(n) \end{cases}$$

So we can see that each b will determine j uniquely where $j + r_0 \equiv b(n)$ and thus:

$$\Gamma_b \otimes_S \alpha^{r_0-1} N = n_j^{(r_0)} \alpha^b P_0 / \pi \alpha^b P_0$$

Then the set of $\mathcal{G}$-invariants $\mathcal{G}_s^{\rho}(N)_{r_0-1,b}$ will determine the torsion coefficients of $\Gamma_b \otimes_S \alpha^{r_0-1} N$ which tells us $\mathrm{Rk}(\Gamma_b \otimes_S \alpha^{r_0-1} N) = n_j^{(r_0)}$. Thus, we have determined the $n_j^{(r_0)}$ and we then know what N_{r_0} is. Furthermore, we can then calculate

$$R_0 = \mathrm{Rk}(\alpha^{r_0-1} N) = \sum_{j=0}^{n-1} n_j^{(r_0)}$$

Next, from the $\mathcal{G}$-invariants $\mathcal{G}_s^{\rho}(N)_{r,b}$ we continue to calculate $\mathrm{Rk}(\Gamma_b \otimes_S \alpha^{r_0-t} N)$, for $t \in \mathbb{N}$, until we find the greatest positive integer $r_1 < r_0$ such that, for some b we have $\mathrm{Rk}(\Gamma_b \otimes_S \alpha^{r_1-1} N) > R_0$ which tells us that there exists $R_1 \in \mathbb{N}$ such that:

$$R_1 = \mathrm{Rk}(\alpha^{r_1-1} N) > R_0 = \mathrm{Rk}(\alpha^{r_1} N)$$

We can then write:

$${}_S N = N^{(1)} \oplus N_{r_1} \oplus N_{r_0}$$

where $\alpha^{r_1-1} N^{(1)} = 0$ and

$$N_{r_1} = \bigoplus_{j=0}^{n-1} n_j^{(r_1)} \alpha^j P_0 / \alpha^{j+r_1} P_0$$

for some $n_j^{(r_1)} \in \mathbb{N} \cup \{0\}$. Again, we can think of N_{r_1} as the direct summand of ${}_S N$ which is annihilated by α^{r_1} but not annihilated by α^{r_1-1}. We will now determine N_{r_1} by considering $\Gamma_b \otimes_S \alpha^{r_1-1} N$. Well,

$$\Gamma_b \otimes_S \alpha^{r_1-1} N = \left[\bigoplus_{j=0}^{n-1} n_j^{(r_1)} \alpha^b P_0 / \pi^{\delta_j} \alpha^b P_0 \right] \oplus \left[\bigoplus_{j=0}^{n-1} n_j^{(r_0)} \alpha^b P_0 / \pi^{\gamma_j} \alpha^b P_0 \right]$$

where $\gamma_j \geq 1$ are known from our previous work in determining N_{r_0} and:

$$\delta_j = \begin{cases} 1 & \text{if } j + r_1 \equiv b(n) \\ 0 & \text{if } j + r_1 \not\equiv b(n) \end{cases}$$

So we can again see that each b will determine j uniquely where $j+r_1 \equiv b(n)$. Then the $\mathcal{G}$-invariants $\mathcal{G}_S^{\rho}(N)_{r_1-1,b}$ will tell us the torsion coefficients of $\Gamma_b \otimes_S \alpha^{r_1-1}N$, which give us $\mathrm{Rk}(\Gamma_b \otimes_S \alpha^{r_1-1}N)$ and we have the equation:

$$\mathrm{Rk}(\Gamma_b \otimes_S \alpha^{r_1-1}N) = n_j^{(r_1)} + \sum_{j=0}^{n-1} n_j^{(r_0)}$$

so we can determine the $n_j^{(r_1)}$, and hence determine N_{r_1}.

We then repeat the above procedure to find the greatest positive integer r_2 such that

$$R_2 = \mathrm{Rk}(\alpha^{r_2-1}N) > R_1 = \mathrm{Rk}(\alpha^{r_1}N)$$

for some $R_2 \in \mathbb{N}$. This will enable us to write:

$${}_SN = N^{(2)} \oplus N_{r_2} \oplus N_{r_1} \oplus N_{r_0}$$

and we then determine N_{r_2}. We continue in this way until the rank of N is determined and we obtain a sequence:

Module	Rank
$\alpha^{r_0}N = 0$	0
$\alpha^{r_0-1}N \neq 0$	R_0
$\vdots$	$\vdots$
$\alpha^{r_1}N \neq 0$	R_0
$\alpha^{r_1-1}N \neq 0$	R_1
$\vdots$	$\vdots$
$\vdots$	$\vdots$
N	R_k

where the ranks satisfy: $R_k > R_{k-1} > \cdots > R_1 > R_0$. We will consequently obtain:

$${}_SN = \bigoplus_{i=0}^{k} N_{r_i}$$

and hence we have determined ${}_SN$. We have thus given a procedure for using the $\mathcal{G}$-invariants for determining the structure of ${}_SN$. The case where S is the principal order $S_{(t,\dots,t)}(\Delta_E)$ generalises easily. Hence, we have proved the main theorem of this section:

Theorem 6.13 Let R be a complete dvr with $S = S_{(t,\dots,t)}(\Delta_E)$ a principal hereditary R-order in $M_n(E)$, for some field E. Suppose that ${}_SN$ is a finitely generated torsion S-module. Then we can find an appropriate $\alpha \in S$ such that the set of $\mathcal{G}$-invariants $\mathcal{G}_s^\rho(N)_{r,b}$ completely determine the structure of ${}_SN$.

We now extend Theorem 6.13 to the case where D is a division algebra (not necessarily commutative). In order to do this we first need a lemma.

Lemma 6.14 Let $S' = S_{(w,\dots,w)}(\Delta_D)$ be a principal hereditary R-order of type k in $M_t(D)$, for some division algebra D over K and where $kw = t$. Let L be the unique unramified extension of K of degree $m = \sqrt{D:K}$. Then there is an isomorphism:

$$\gamma : \mathcal{O}_L \otimes_{\mathcal{O}_K} S' \cong S_{(w,\dots,w)}(\mathcal{O}_L)$$

where $S = S_{(w,\dots,w)}(\mathcal{O}_L)$ is of type mk.

Proof:

We sketch the proof. Consider the case $t = 1$. Suppose the residue class field $\overline{\mathcal{O}_K}$ has cardinality q. Then $L = K(\omega)$, where ω is a primitive $(q^m - 1)$-st root of unity (see [16], Thereorm 5.10, pages 72-73). The galois group $\mathrm{Gal}(L/K)$ is cyclic of order m and has as a generator the Frobenius automorphism σ, where $\sigma(\omega) = \omega^q$. Let $\pi\mathcal{O}_L = \mathrm{rad}\ \mathcal{O}_L$. Then there is a prime element $z \in D$ such that:

$$D = \sum_{i=0}^{m-1} z^i L$$

where $z^m = \pi$ and $z^{-1}\beta z = \sigma^k(\beta)$ for all $\beta \in L$. D uniquely determines the integer k modulo m, where $\gcd(k,m) = 1$. We can then write S' as:

$$S' = \sum_{i=0}^{m-1} z^i \mathcal{O}_L$$

Now, $\mathcal{O}_L = \mathcal{O}_K[\omega]$, so we can view S' as an $\mathcal{O}_K$-module with $\mathcal{O}_K$-basis $\{z^u\omega^v\}$, for $u, v = 0, 1, \dots, m-1$. Define the map:

$$\gamma : \mathcal{O}_L \otimes_{\mathcal{O}_K} S' \longrightarrow S_{(1,\dots,1)}(\Delta_L)$$

(where $\Delta_L = \mathcal{O}_L$ and $S_{(1,\dots,1)}(\Delta_L)$ is of type m) by:

$$\gamma(1 \otimes_{\mathcal{O}_K} z^u\omega^v) = (\widehat{z^u\omega^v})^*$$

where $\hat{\mu}$ denotes left multiplication by $\mu \in L$ and $(\hat{\mu})^*$ is the matrix representation of $\hat{\mu}$. Then, it can be shown that the map γ is one-one and, since L is an unramified extension of K, γ is onto. The case of general t extends easily.

□

Corollary 6.15 Let $S' = S_{(w,\dots,w)}(\Delta_D)$ be a principal hereditary R-order of type k in $M_t(D)$, for some division algebra D over K and where $kw = t$. Let L be the unique unramified extension of K of degree $m = \sqrt{D : K}$. Let ${}_{S'}M$ be a finitely generated torsion S'-module. Then there exists a representation τ and an appropriate $\alpha \in S'$ such that the set of $\mathcal{G}$-invariants $\mathcal{G}_s^\tau(M)_{r,b}$ completely determine the structure of ${}_{S'}M$.

Proof:

Δ_D may not be commutative so we use the splitting field L of D and calculate the $\mathcal{G}$-invariants of $\mathcal{O}_L \otimes_{\mathcal{O}_K} M$. We have a composite map:

$$\delta : S' \longrightarrow \mathcal{O}_L \otimes_{\mathcal{O}_K} S' \xrightarrow{\gamma} S_{(w,\dots,w)}(\Delta_L)$$

where γ is as given in Lemma 6.14. This tells us that we have a composite map:

$$\tau : S' \xrightarrow{\delta} S_{(w,\dots,w)}(\mathcal{O}_L) \xrightarrow{\rho} M_{mt}(\mathcal{O}_L)$$

where ρ is just the inclusion map. Then we know that:

$$\mathcal{G}_s^\tau(M)_{r,b} = \mathcal{G}_s^\rho(\mathcal{O}_L \otimes_{\mathcal{O}_K} M)_{r,b}$$

Now, Theoem 6.13 tells us the $\mathcal{G}$-invariants $\mathcal{G}_s^\rho(\mathcal{O}_L \otimes_{\mathcal{O}_K} M)_{r,b}$ (and thus the $\mathcal{G}$-invariants $\mathcal{G}_s^\tau(M)_{r,b}$) will completely determine the isomorphism class of $\mathcal{O}_L \otimes_{\mathcal{O}_K} M$. But, we know from the proof of Corollary 5.24 that the functor $(\mathcal{O}_L \otimes_{\mathcal{O}_K} -)$ is one-one on isomorphism classes.

□

We next illustrate the above procedure and Theorem 6.13 with an example.

Example 6.16 Let $S = S_{(1,1)}(\Delta_E)$ be a hereditary R-order in $M_2(E)$ and let ${}_SN$ be a finitely generated left torsion S-module. We wish to use the $\mathcal{G}$-invariants to determine ${}_SN$ precisely. Well, suppose we are given a complete set of $\mathcal{G}$-invariants as follows:

$\mathrm{ord}_\pi \mathcal{G}_s^\rho(N)_{r,0}$	$r=6$	$r=5$	$r=4$	$r=3$	$r=2$	$r=1$	$r=0$
$s=0$	5	11	16	22	27	36	43
$s=1$	4	10	14	20	24	33	39
$s=2$	3	9	12	18	21	30	35
$s=3$	2	8	10	16	18	27	31
$s=4$	1	7	8	14	15	24	27
$s=5$	0	6	6	12	12	21	23
$s=6$	0	5	5	10	10	18	20
$s=7$	0	4	4	8	8	15	17
$s=8$	0	3	3	6	6	12	14
$s=9$	0	2	2	4	4	9	11
$s=10$	0	1	1	2	2	6	8
$s=11$	0	0	0	0	0	3	5
$s=12$	0	0	0	0	0	2	4
$s=13$	0	0	0	0	0	1	3
$s=14$	0	0	0	0	0	0	2
$s=15$	0	0	0	0	0	0	1
$s=16$	0	0	0	0	0	0	0

where $\mathcal{G}_s^\rho(N)_{r,0} = (0)$ for $r \geq 7$.

$\mathrm{ord}_\pi \mathcal{G}_s^\rho(N)_{r,1}$	$r=6$	$r=5$	$r=4$	$r=3$	$r=2$	$r=1$	$r=0$
$s=0$	2	7	13	18	24	31	40
$s=1$	1	6	11	16	21	28	36
$s=2$	0	5	9	14	18	25	32
$s=3$	0	4	8	12	16	22	29
$s=4$	0	3	7	10	14	19	26
$s=5$	0	2	6	8	12	16	23
$s=6$	0	1	5	6	10	13	20
$s=7$	0	0	4	4	8	10	17
$s=8$	0	0	3	3	6	8	14
$s=9$	0	0	2	2	4	6	11
$s=10$	0	0	1	1	2	4	8
$s=11$	0	0	0	0	0	2	5
$s=12$	0	0	0	0	0	1	4
$s=13$	0	0	0	0	0	0	3
$s=14$	0	0	0	0	0	0	2
$s=15$	0	0	0	0	0	0	1
$s=16$	0	0	0	0	0	0	0

where $\mathcal{G}_s^\rho(N)_{r,1} = (0)$ for $r \geq 7$. We see immediately that $\alpha^7 N = 0$, but $\alpha^6 N \neq 0$ so we can write

$$_S N = N^{(0)} \oplus N_7$$

where $\alpha^6 N^{(0)} = 0$ and

$$N_7 = n_0^{(7)} P_0/\alpha^7 P_0 \oplus n_1^{(7)} \alpha P_0/\alpha^8 P_0$$

Now, the columns $\mathrm{ord}_\pi \mathcal{G}_s^\rho(N)_{6,b}$, for $b = 0, 1$, tell us that:

$$\begin{aligned} \mathrm{Rk}(\Gamma_0 \otimes_S \alpha^6 N) &= 5 = n_1^{(7)} \quad \text{since } 1+7 \equiv 0(2) \\ \mathrm{Rk}(\Gamma_1 \otimes_S \alpha^6 N) &= 2 = n_0^{(7)} \quad \text{since } 0+7 \equiv 1(2) \end{aligned}$$

and hence:

$$\begin{aligned} N_7 &= 2P_0/\alpha^7 P_0 \oplus 5\alpha P_0/\alpha^8 P_0 \\ R_0 &= \mathrm{Rk}(\alpha^6 N) = 2+5 = 7 \end{aligned}$$

Next we consider the columns $\mathrm{ord}_\pi \mathcal{G}_s^\rho(N)_{5,b}$, which tell us that:

$$\begin{aligned} \mathrm{Rk}(\Gamma_0 \otimes_S \alpha^5 N) &= 11 = R_0 + n_0^{(6)} \Rightarrow n_0^{(6)} = 4 \\ \mathrm{Rk}(\Gamma_1 \otimes_S \alpha^5 N) &= 7 = R_0 + n_1^{(6)} \Rightarrow n_1^{(6)} = 0 \end{aligned}$$

and hence we can write:

$${}_S N = N^{(1)} \oplus N_6 \oplus N_7$$

where

$$\begin{aligned} N_6 &= 4P_0/\alpha^6 P_0 \\ R_1 &= \mathrm{Rk}(\alpha^5 N) = 11 \end{aligned}$$

Looking at the next set of $\mathcal{G}$-invariants we find that:

$$R_1 = \mathrm{Rk}(\alpha^4 N) = \mathrm{Rk}(\alpha^3 N) = \mathrm{Rk}(\alpha^2 N) = 11$$

so we next consider the columns $\mathrm{ord}_\pi \mathcal{G}_s^\rho(N)_{1,b}$, which tell us that:

$$\begin{aligned} \mathrm{Rk}(\Gamma_0 \otimes_S \alpha N) &= 14 = R_1 + n_0^{(2)} \Rightarrow n_0^{(2)} = 3 \\ \mathrm{Rk}(\Gamma_1 \otimes_S \alpha N) &= 13 = R_1 + n_1^{(2)} \Rightarrow n_1^{(2)} = 2 \end{aligned}$$

and hence we can write:

$${}_S N = N^{(2)} \oplus N_2 \oplus N_6 \oplus N_7$$

where

$$N_2 = 3P_0/\alpha^2 P_0 \oplus 2\alpha P_0/\alpha^3 P_0$$

But then

$$\mathrm{Rk}(N_2 \oplus N_6 \oplus N_7) = 5 + 4 + 7 = 16 = \mathrm{Rk}(N)$$

and so we must have determined N. Thus, we can write:

$$\begin{aligned} {}_S N &= N_2 \oplus N_6 \oplus N_7 \\ &= \left[3P_0/\alpha^2 P_0 \oplus 2\alpha P_0/\alpha^3 P_0\right] \oplus \left[4P_0/\alpha^6 P_0\right] \oplus \left[2P_0/\alpha^7 P_0 \oplus 5\alpha P_0/\alpha^8 P_0\right] \\ &= 3A_{0,0}^1 \oplus 2A_{1,1}^1 \oplus 4A_{0,0}^3 \oplus 2A_{0,1}^4 \oplus 5A_{1,0}^4 \end{aligned}$$

□

6.3 Twisted Group Rings

In this section we consider how we can apply the $\mathcal{G}$-invariants we constructed in section 6.1 to the case of modules over a twisted group ring. Let T be a commutative ring with a 1 and G be a finite group which acts as a group of automorphisms on T. Then, by the twisted group ring $T \circ G$ of G over T, we mean:

$$T \circ G = \bigoplus_{g \in G} Tg$$

where $\{g | g \in G\}$ form a T-basis for $T \circ G$ and multiplication in $T \circ G$ is defined as:

$$(t_1 g_1)(t_2 g_2) = t_1 g_1(t_2) g_1 g_2 \ \forall \ t_1, t_2 \in T, g_1, g_2 \in G$$

Let R be an integral domain with quotient field K and let L be a finite Galois extension of K with Galois group G, of order n. Let T be an R-order in L such that T is stable under G. Now if $g \in G$ then g determines a K-automorphism of L which induces an R-automorphism of T. Then we may form the twisted group rings:

$$L \circ G = \bigoplus_{g \in G} Lg \text{ and } T \circ G = \bigoplus_{g \in G} Tg$$

In order to construct $\mathcal{G}$-invariants we will embed $L \circ G$ into $M_n(L)$, via a particular representation which we denote by ρ_0. Let $G = \{g_1, \ldots, g_n\}$ and write:

$$A = L \circ G = \left\{ \sum_{i=1}^{n} g_i a_i \mid a_i \in L, g_i \in G \text{ where } g.a = g(a)g \ \forall \ g \in G, a \in L \right\}$$

We view A as a right L-module A_L, and consider the L-endomorphisms of A_L which will act on the left. Now, let $a \in L$ and let $\hat{a}$ denote left multiplication by a. Then, since $\{g_i\}$ form an L-basis for A_L we see that:

$$\hat{a}(g_i) = a g_i = g_i.[g_i^{-1}(a)]$$

and hence the matrix representation of $\hat{a}$ (where the matrix acts on the left) is:

$$(\hat{a})^* = \text{diag}(g_1^{-1}(a), \ldots, g_n^{-1}(a))$$

Similarly, since $g g_s = g_r$ for some r, s such that $1 \leq r, s \leq n$, the matrix representation of $\hat{g}$ which we call $(\hat{g})^*$, is the $n \times n$ matrix:

$$(\hat{g})^* = (\delta_{g_i, g g_j}) \text{ for } 1 \leq i, j \leq n$$

where:

$$\delta_{g,h} = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}$$

for all $g, h \in G$. Then the map ρ_0 defined by:

$$\rho_0 : a \mapsto (\hat{a})^*, g \mapsto (\hat{g})^*$$

yields an n-th degree representation of $L \circ G$ over L:

$$\rho_0 : L \circ G \longrightarrow \mathrm{End}_L(L \circ G) \cong \mathrm{End}_L(L^n) \cong M_n(L)$$

which when restricted to T gives us:

$$\rho_0 : T \circ G \longrightarrow M_n(T)$$

Let $\Lambda = T \circ G$, then for ${}_\Lambda M$ a finitely generated left Λ-module, we can construct the $\mathcal{G}$-invariants with respect to ρ_0 to be $\mathcal{G}_s^{\rho_0}({}_\Lambda M)_{r,b}$, which give us invariants as ideals in the ring T. We ask ourselves can we do any better than this in order to obtain finer invariants? For example, given a $\mathcal{G}$-invariant $\mathcal{G}_s^{\rho_0}$, which is an ideal of the ring T, can we find an ideal J of the fixed ring T^G, such that when J is extended to the larger ring T we obtain $JT = \mathcal{G}_s^{\rho_0}$? The following example shows we cannot always do this.

Example 6.17 Let $T = \mathbb{Z}[i\sqrt{3}]$ and let $G = C_2 =< g >$ be such that G acts on T by complex conjugation. Suppose we have a finitely generated left $\Lambda = T \circ G$-module, ${}_\Lambda M$, with presentation matrix:

$$C = \begin{pmatrix} 2.1 + 0.g \\ 0.1 + (1 + i\sqrt{3})g \end{pmatrix}$$

with respect to one generator e, say. Then

$$\rho_0(C) = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 + i\sqrt{3} \\ 1 - i\sqrt{3} & 0 \end{pmatrix} \end{pmatrix}$$

and thus

$$\widetilde{\rho_0(C)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 1+i\sqrt{3} \\ 1-i\sqrt{3} & 0 \end{pmatrix}$$

We will calculate $\mathcal{G}_s^{\rho_0}({}_\Lambda M)_{0,0}$ in this case which is the same as calculating $\mathcal{F}_s^{\rho_0}({}_\Lambda M)$. We obtain:

$$\begin{aligned} \mathcal{F}_0^{\rho_0}({}_\Lambda M) &= (4, 2(1+i\sqrt{3}))T \\ \mathcal{F}_1^{\rho_0}({}_\Lambda M) &= (2, 1+i\sqrt{3})T \end{aligned}$$

Let us consider $\mathcal{F}_0^{\rho_0}$. A short calculation shows that $(\mathcal{F}_0^{\rho_0} \cap T^G)T = (4)T$. But, $2(1+i\sqrt{3}) \in \mathcal{F}_0^{\rho_0} \setminus (4)T$ so we see that $(\mathcal{F}_0^{\rho_0} \cap T^G)T \subsetneq \mathcal{F}_0^{\rho_0}$ and thus we cannot find a T^G-ideal J such that the extended ideal $JT = \mathcal{F}_0^{\rho_0}$. However, note that:

$$\begin{aligned} g(\mathcal{F}_0^{\rho_0}) &= (4, 2-2i\sqrt{3}) &= \mathcal{F}_0^{\rho_0} \\ g(\mathcal{F}_1^{\rho_0}) &= (2, 1-i\sqrt{3}) &= \mathcal{F}_1^{\rho_0} \end{aligned}$$

So, in this case, we see that each $\mathcal{F}_s^{\rho_0}$ is fixed under the action of $G = C_2$.

□

We can extend this idea to the general case as the following theorem shows:

Theorem 6.18 Let $\Lambda = T \circ G$. Then for any finitely generated Λ-module, ${}_\Lambda M$, we have:

$$g(\mathcal{G}_s^{\rho_0}({}_\Lambda M)_{r,b}) = \mathcal{G}_s^{\rho_0}({}_\Lambda M)_{r,b}$$

for all $g \in G$ and for all integers $s \geq 0$.

If G has order n then we know that ${}_T\Lambda \cong T^n$ as left T-modules. Thus, we can view ${}_\Lambda M$ as a left T-module which we denote by ${}_T M$. In order to prove Theorem 6.18 we first need a lemma which shows the relationship between certain presentation matrices for ${}_\Lambda M$ and certain presentation matrices for ${}_T M$.

Lemma 6.19 Suppose $C = (\lambda_{k,l})$ is a presentation matrix for ${}_\Lambda M$, for some $\lambda_{k,l} \in \Lambda$, for $k = 1, \ldots, m$ and $l = 1, \ldots, h$. Then, if $g \in G$:

1. $gC = (g(\lambda_{k,l}))$ is also a presentation matrix for ${}_\Lambda M$;

2. $gCg^{-1} = (g\lambda_{k,l}g^{-1})$ is also a presentation matrix for ${}_\Lambda M$; and

3. $\widetilde{\rho_0(C)}, \widetilde{\rho_0(gC)}$ and $\widetilde{\rho_0(gCg^{-1})}$ are all presentation matrices for ${}_TM$.

Proof:

Suppose C is a presentation matrix with respect to the generating set $\underline{e} = \{e_l\}$. Then we have a set of m relations:

$$\sum_{l=1}^{h} \lambda_{k,l}e_l = 0 \Leftrightarrow g\sum_{l=1}^{h} \lambda_{k,l}e_l = 0 \Leftrightarrow \sum_{l=1}^{h} g\lambda_{k,l}e_l = 0$$

for $k = 1, \ldots, m$. Hence gC is indeed a presentation matrix with respect to $\underline{e}$. Furthermore,

$$\sum_{l=1}^{h} g\lambda_{k,l}e_l = 0 \Leftrightarrow \sum_{l=1}^{h} g\lambda_{k,l}g^{-1}[ge_l] = 0$$

and hence gCg^{-1} is a presentation matrix with respect to the generating set $\{ge_l\}$. Thus, it remains to prove 3. We have a presentation for ${}_\Lambda M$:

$$\Lambda^m \xrightarrow{C} \Lambda^h \longrightarrow_\Lambda M$$

Now, since ${}_T\Lambda \cong T^n$ as left T-modules, we see that there is a presentation for ${}_TM$:

$$T^{nm} \xrightarrow{A} T^{nh} \longrightarrow_T M$$

We claim that A and $\widetilde{\rho_0(C)}$ are equivalent matrices by means of elementary row and column operations; if this is the case we write $A \sim \widetilde{\rho_0(C)}$. We first consider the simplified case $m = h = 1$. Then we can write $C = (\lambda)$ for some $\lambda \in \Lambda$. We also write:

$$\lambda = \sum_{j=1}^{n} a_jg_j \text{ for some } a_j \in T$$

so we have the relation

$$\lambda e_1 = 0 \text{ i.e. } \sum_{j=1}^{n} a_jg_je_1 = 0$$

If we multiply the relation $\lambda e_1 = 0$ by g_i^{-1} on the left we obtain:

$$g_i^{-1}\lambda e_1 = [\ldots + g_i^{-1}(a_j)g_i^{-1}g_j + \ldots]e_1 = 0$$

If we let $g_i^{-1}g_j = g_r$ for some r, such that $1 \leq r \leq n$, then $g_j = g_ig_r$. Hence, $\rho_0(g_i) = (\hat{g}_i)^*$ has a 1 in the (j,r)-th entry and consequently $\rho_0(C)$ has $g_j^{-1}(a_i)$ in the (j,r)-th entry. Next, multiplying the relation $\lambda e_1 = 0$ by g_j^{-1} on the left we obtain:

$$g_j^{-1}\lambda e_1 = [\ldots + g_j^{-1}(a_i)g_j^{-1}g_i + \ldots]e_1 = 0$$

If we let $g_j^{-1}g_i = g_s$ for some s, such that $1 \le s \le n$, then as a T-module the above is a relation for ${}_TM$ with respect to the generating set $\{g_je_1\}$. So this equation gives us the entry $g_j^{-1}(a_i)$ in the (j,s)-th position of the matrix A. Since $g_s = g_r^{-1}$ each s is uniquely determined by r. Thus, if we take the matrix A and interchange columns s and r for each s, which is equivalent to swapping the generator g_se_1 with g_re_1, we obtain $\widetilde{\rho_0(C)}$. Thus, we have proved that in the case $m = h = 1$, $A \sim \widetilde{\rho_0(C)}$ as required. The general case follows easily. For if $C = (\lambda_{i,j})$ then $\rho_0(C) = (\rho_0(\lambda_{i,j}))$ is an $m \times h$ matrix with entries in $M_n(T)$. If we take the m relations:

$$\sum_{l=1}^{h} \lambda_{k,l}e_l = 0$$

and, as before, multiply on the left by g_i^{-1}, then we see that each $\rho_0(\lambda_{i,j})$ is the matrix with $g_j^{-1}(a_i)$ in the (j,r)-th position, as we had in the $m = g = 1$ case. Similarly, if we now multiply the above relations on the left by g_j^{-1} then we obtain relations with respect to the nh generators $\{g_je_l\}$. Thus, A is an $nm \times nh$ matrix. If we think of it as an $m \times k$ block matrix, with each block an $n \times n$ matrix, then each of these $n \times n$ blocks has the entry $g_j^{-1}(a_i)$ in the (j,s)-th position, as before. Thus, swapping columns s and r for each s, in each of the $n \times n$ blocks, we obtain $A \sim \widetilde{\rho_0(C)}$.

Hence, as the matrices C, gC and gCg^{-1} are presentation matrices for ${}_\Lambda M$ we see that $\widetilde{\rho_0(C)}, \widetilde{\rho_0(gC)}$ and $\widetilde{\rho_0(gCg^{-1})}$ must be presentation matrices for ${}_TM$.

□

We are now in a position to prove Theorem 6.18.

Proof: (of Theorem 6.18)

Assume that C is a presentation matrix for ${}_\Lambda M$ and that Λ is α-stable with respect to Λ. We know from Lemma 6.19 that $\widetilde{\rho_0(C)}$ and $\widetilde{\rho_0(gCg^{-1})}$ are both presentation matrices for ${}_TM$ with respect to the same size generating sets so we must have $\mathcal{F}_s^{\rho_0}(C) = \mathcal{F}_s^{\rho_0}(gCg^{-1})$. Since Λ is α-stable with respect to Λ we obtain $\mathcal{F}_s^{\rho_0}(C_{r,b}^t) = \mathcal{F}_s^{\rho_0}(gC_{r,b}^tg^{-1})$. Suppose we also know that $\widetilde{\rho_0(gCg^{-1})} \sim g(\widetilde{\rho_0(C)})$ and therefore $\mathcal{F}_s^{\rho_0}(gC_{r,b}^t g^{-1}) = \mathcal{F}_s^{\rho_0}(gC_{r,b}^t)$. Then we have that:

$$\mathcal{F}_s^{\rho_0}(C_{r,b}^t) = \mathcal{F}_s^{\rho_0}(gC_{r,b}^t g^{-1}) = \mathcal{F}_s^{\rho_0}(gC_{r,b}^t) = g\mathcal{F}_s^{\rho_0}(C_{r,b}^t)$$

which tells us that:

$$g(\mathcal{G}_s^{\rho_0}({}_\Lambda M)_{r,b}) = \mathcal{G}_s^{\rho_0}({}_\Lambda M)_{r,b}$$

So it remains to prove that $\widetilde{\rho_0(gCg^{-1})} \sim \widetilde{g(\rho_0(C))}$. Let $g = g_r$. Again, we first consider the case $m = h = 1$ and write $C = (\lambda)$ where $\lambda = \sum_{j=1}^n a_j g_j$. Let us consider:

$$g_r \lambda g_r^{-1} = g_r(a_1) g_r g_1 g_r^{-1} + \cdots + g_r(a_i) g_r g_i g_r^{-1} + \cdots + g_r(a_n) g_r g_n g_r^{-1}$$

Now, using similar methods to those used in the proof of Lemma 6.19, we can show that $\rho_0(g_r \lambda g_r^{-1})$ is the $n \times n$ matrix with the entry $g_k^{-1} g_r(a_i)$ in the (k,s)-th position where $g_k = g_r g_i g_r^{-1} g_s$. Now, we can find a p such that $g_p^{-1} g_r = g_r g_k^{-1}$. Further, if we let:

$$g_{\sigma_r(p)} = g_r^{-1} g_p g_r$$

then we see that $g_{\sigma_r(p)} = g_k$. Define t such that:

$$g_p = g_r g_i g_r^{-1} g_t$$

then the (p,t)-th entry of $\rho_0(g_r \lambda g_r^{-1})$ is equal to $g_p^{-1} g_r(a_i) = g_r g_k^{-1}(a_i)$. Let us now consider $g_r \rho_0(\lambda)$. Well, the (k,u)-th entry of $\rho_0(\lambda)$ is $g_k^{-1}(a_i)$ where $g_k = g_i g_u$. Hence, the (k,u)-th entry of $g_r \rho_0(\lambda)$ is $g_r g_k^{-1}(a_i)$. Let us revert back to the matrix $\rho_0(g_r \lambda g_r^{-1})$ and perform the following permutations:

$$\text{Col } t \mapsto \text{Col } u$$

$$\text{Row } p \mapsto \text{Row } k$$

and call the resulting matrix $\rho_0'(g_r \lambda g_r^{-1})$. Then the (k,u)-th entry of $\rho_0'(g_r \lambda g_r^{-1})$ is equal to $g_r g_k^{-1}(a_i)$ which is precisely the (k,u)-th entry of $g_r \rho_0(\lambda)$. It remains to verify that $g_{\sigma_r(t)} = g_u$. Well,

$$g_{\sigma_r(t)} = g_r^{-1} g_t g_r = g_i^{-1} g_r^{-1} g_p g_r = g_i^{-1} g_k = g_u$$

Hence,

$$\rho_0(g_r C g_r^{-1}) \sim g_r(\rho_0(C))$$

For the general case, we can easily repeat the above argument, performing the permutations on each of the $n \times n$ blocks of $\rho_0(g_r C g_r^{-1})$ to obtain:

$$\rho_0(\widetilde{g_r C g_r^{-1}}) \sim g_r(\widetilde{\rho_0(C)})$$

as required.

□

We finish this section by proving a result about the relationship between the initial Fitting Ideal and the annihilator of a finitely generated Λ-module.

Theorem 6.20 For any finitely generated left Λ-module, ${}_\Lambda M$, we have:

$$\mathcal{F}_0^{\rho_0}({}_\Lambda M) \cap T^G \subseteq \mathrm{Ann}_\Lambda M$$

Proof:

Suppose we have the following presentation for ${}_\Lambda M$:

$$\Lambda^m \xrightarrow{\tau} \Lambda^h \xrightarrow{\pi} {}_\Lambda M$$

where the map τ is represented by the matrix $C \in M_{m \times h}(\Lambda)$ acting on the right, with respect to the generating set $\underline{e} = \{e_l\}$, for $l = 1, \ldots, h$. Now, since ${}_T\Lambda \cong T^n$ as left T-modules, we see that there is a presentation for ${}_T M$:

$$T^{nm} \xrightarrow{\tau'} T^{nh} \xrightarrow{\pi'} {}_T M$$

where we can represent τ' by the unboxed matrix $\widetilde{\rho_0(C)}$ acting on the right. Since T is a commutative ring $\mathcal{F}_0^{\rho_0}({}_\Lambda M) \subseteq \mathrm{Ann}_T M$ (see Theorem 1.11) and thus $T^{nh}\mathcal{F}_0^{\rho_0} \subseteq T^{nm}\widetilde{\rho_0(C)}$. Now choose $x \in \mathcal{F}_0^{\rho_0} \cap T^G$, so $T^{nh}x \subseteq T^{nm}\widetilde{\rho_0(C)}$ and hence, $\Lambda^h x \subseteq \Lambda^m C$. But, $x \in T^G$ so $\Lambda^h x = x\Lambda^h$ and we obtain:

$$x\Lambda^h \subseteq \Lambda^m C$$

Now, suppose $m \in_\Lambda M$, so we can write:

$$m = \sum_{l=1}^{h} \lambda_l e_l \text{ for some } \lambda_l \in \Lambda$$

Then:

$$(x\lambda_1, \ldots, x\lambda_h) \in \mathrm{Im}\ \tau = \ker\ \pi$$

and thus

$$0 = \pi(x\lambda_1, \ldots, x\lambda_h) = \sum_{l=1}^{h} x\lambda_l e_l = xm$$

Therefore

$$x \in \mathrm{Ann}_\Lambda M$$

□

6.4 Metacyclic Groups

We now consider how we can apply the work in section 6.3 to obtain $\mathcal{G}$-invariants for modules over group rings, where the underlying group is metacyclic. Let G_s be the metacyclic group given by:

$$G_s = \langle x, y \mid x^{p^s} = 1, y^q = 1, yx = x^t y \rangle$$

where p and q are distinct primes with p odd and t is a primitive q-th root of 1 modulo p^s. Let Γ_s denote the group ring, $\Gamma_s = \mathbb{Z}_p G_s$. Then we can represent Γ_s as a twisted group ring:

$$\Gamma_s = \mathbb{Z}_p G_s = \mathbb{Z}_p[x] \circ C_q$$

where $C_q =< \gamma >$ and γ acts as y on x; the action given by $\gamma(x) = x^t$.

We first consider the case $s = 1$. If we denote by A the semisimple $\mathbb{Q}$-algebra $A = \mathbb{Q}_p G_1$, then we have a representation for A:

$$A = \mathbb{Q}_p G_1 = \mathbb{Q}_p G_1 e_1 \oplus \mathbb{Q}_p G_1 e_2 \cong \mathbb{Q}_p C_q \oplus \mathbb{Q}_p[\zeta] \circ C_q$$

where $e_1 = \frac{\Phi_p(x)}{p}$ and $e_2 = 1 - e_1$ are primitive idempotents, $\Phi_p(x)$ is the cyclotomic polynomial of order p and ζ is a primitive p-th root of 1. However, for Γ_1 the analogous result is only an inclusion:

$$\Gamma_1 = \mathbb{Z}_p G_1 \subseteq \mathbb{Z}_p G_1 e_1 \oplus \mathbb{Z}_p G_1 e_2 \cong \mathbb{Z}_p C_q \oplus \mathbb{Z}_p[\zeta] \circ C_q$$

since e_1 and e_2 do not lie in Γ_1. If we let $S = \mathbb{Z}_p[\zeta] \circ C_q$ then, since $\mathbb{Z}_p[\zeta]$ is a tamely ramified extension of its fixed ring $\mathbb{Z}_p[\zeta]^{C_q}$, S is a hereditary $\mathbb{Z}_p[\zeta]^{C_q}$-order in $\mathbb{Q}S = \mathbb{Q}_p[\zeta] \circ C_q$ (see [1], Corollary 3.6). Thus, Γ_1 can be projected onto a hereditary order, namely S.

We will calculate a set of $\mathcal{G}$-invariants for finitely generated Γ_1-modules. We have a representation for Γ_1:

$$\rho_0 : \mathbb{Z}_p[x] \circ C_q \longrightarrow M_q(\mathbb{Z}_p[x])$$

where

$$\rho_0(a_i\gamma^i) = (\hat{a}_i)^*(\hat{\gamma^i})^* \ \forall \ a_i \in \mathbb{Z}_p[x], \gamma^i \in C_q \text{ for } i = 0, 1, \ldots, q-1$$

where ρ_0 is the representation we defined in section 6.3. Thus, calculating our $\mathcal{G}$-invariants with respect to ρ_0 we will obtain a set of ideals in $\mathbb{Z}_p[x]$. Let ${}_{\Gamma_1}M$ be a finitely generated left torsion Γ_1-module. We want to choose an appropriate α to provide us with a useful set of $\mathcal{G}$-invariants. We also choose α such that the projected element $\alpha e_2 \in S$ can be used to calculate $\mathcal{G}$-invariants for the underlying module over the hereditary order, S, such that they completely determine the S-module structure. For a full set of $\mathcal{G}$-invariants (i.e. $b \neq 0$) we want an α such that Γ_1 is α-stable with respect to Γ_1 and which provides a useful set of $\mathcal{G}$-invariants. However, research has failed to find such an α (of course we can always take $\alpha = 1$ but the subsequent $\mathcal{G}$-invariants we obtain are not very useful). However, we can take $\Sigma = \mathbb{Z}_p[x]$ and choose:

$$\alpha = pe_1 + (1-x)e_2 = \Phi_p(x) + (1-x) \in \Gamma_1 = \mathbb{Z}_p[x] \circ C_q$$

Then $\alpha^{-1}\mathbb{Z}_p[x]\alpha = \mathbb{Z}_p[x] \subsetneq \Gamma_1$ and thus $\mathbb{Z}_p[x]$ is α-stable wih respect to Γ_1. We then need to choose a set which generates ${}_{\Gamma_1}M$ over Σ to give us a partial set of $\mathcal{G}$-invariants. Note that α is not a zero-divisor in Γ_1 so the element $\alpha^{-1} \in A = \mathbb{Q}\Gamma_1$. The projection of the element α onto $\Gamma_1 e_2$ is $\alpha e_2 = 1 - \zeta \in S$. Let $R = \mathbb{Z}_p[\zeta]$ and $R_0 = R^{c_q}$. Then we have a representation:

$$\tau_2 : S \longrightarrow S_{(1,\ldots,1)}(R_0) = \begin{pmatrix} R_0 & P_0 & \cdots & P_0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & P_0 \\ R_0 & \cdots & \cdots & R_0 \end{pmatrix}^{(1,\ldots,1)}$$

where $S_{(1,\ldots,1)}(R_0)$ is of type q. Here P_0 is the unique prime ideal in R_0 such that $P_0 = \pi_0 R_0$, for some $\pi_0 \in R_0$, and such that $P_0 R = \pi^q R$, where $\pi = 1 - \zeta$. In fact τ_2

is an isomorphism. To see this note that since $R = R_0[\pi]$, we can write $S = R_0[\pi, \gamma]$. Thus, $\{\pi^i\gamma^j\}$ constitutes an R_0-basis for S, for $i, j = 0, 1, \ldots, q-1$. Then, we can make $S = R \circ C_q$ act on the R_0-module, R, via left multiplication and we define:

$$\tau_2(\lambda) = (\hat{\lambda})^* \text{ for all } \lambda \in S$$

So, we have a representation:

$$\tau_2 : S = R \circ C_q \longrightarrow \mathrm{End}_{R_0}(R) \cong \mathrm{End}_{R_0}(R_0^q) \cong M_q(R_0)$$

and

$$\tau_2(\alpha e_2) = \left(\begin{array}{cccc|c} 0 & \cdots & \cdots & 0 & s_0\pi_0 \\ \hline 1 & 0 & \cdots & 0 & s_1\pi_0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & s_{q-1}\pi_0 \end{array}\right)$$

for some $s_i \in R_0$, for $i = 1, \ldots, q-1$ and $s_0 \in U(R_0)$. Then, using similar techniques to those used in the proof of Lemma 6.14, it can be shown that τ_2 is both one-one and onto. Now, the element $\tau_2(\alpha e_2)$ has the same properties as the α we chose in Theorem 6.13, so it can be used to calculate $\mathcal{G}$-invariants with respect to τ_2 which will completely determine the underlying S-module structure of ${}_{\Gamma_1}M$.

Remark 6.21 Since $\Sigma = \mathbb{Z}_p[x] \subsetneq \Gamma_1$ we only obtain a partial set of $\mathcal{G}$-invariants, $\mathcal{G}_s^{\rho_0}({}_{\Gamma_1}M)_{r,0}$. Indeed, if we could find an α such that Γ_1 is α-stable (with respect to Γ_1), it is debatable whether the fuller set of $\mathcal{G}$-invariants, $\mathcal{G}_s^{\rho_0}({}_{\Gamma_1}M)_{r,b}$ would in fact give us any more useful information.

Remark 6.22 The partial set of $\mathcal{G}$-invariants we obtain for Γ_1-modules via ρ_0 are ideals in the ring $\mathbb{Z}_p[x]$, whereas the $\mathcal{G}$-invariants we obtain for S-modules via τ_2 are ideals in the ring R_0. However, under the projection $\theta_2 : \Gamma_1 \longrightarrow \Gamma_1 e_2 \cong S$, the $\mathcal{G}$-invariants with respect to ρ_0 do not necessarily correspond to the $\mathcal{G}$-invariants with respect to τ_2. That is, in general:

$$\theta_2(\mathcal{G}_s^{\rho_0}({}_{\Gamma_1}M)_{r,0})Z(S) \neq \mathcal{G}_s^{\tau_2}(\Gamma_1 e_2 \otimes_{\Gamma_1} M)_{r,0}$$

since $\mathcal{G}_s^{\rho_0}({}_{\Gamma_1}M)_{r,0}$ is an ideal of $\mathbb{Z}_p[x]$ which is projected under θ_2 into an ideal of $\mathbb{Z}_p[\zeta]$ which may not lie in $Z(S)$, the centre of S.

However, we will remedy these above problems for the case $s > 1$, which we now study showing a slightly different approach. Consider the element $\beta = 1 - x \in \Gamma_s$. Now β is not an invertible element of $\mathbb{Q}\Gamma_s$ since $1-x$ is a zero divisor in Γ_s. However, if we quotient out by the ideal $J = (\sum_{j=0}^{p^s-1} x^j)\mathbb{Z}_p[x]$ to obtain the quotient ring:

$$\overline{\Gamma_s} = (\mathbb{Z}_p[x]/J) \circ C_q$$

then $\overline{\beta} = \beta + J$ is not a zero divisor in $\overline{\Gamma_s}$. In fact since we can write:

$$(\mathbb{Q}_p C_{p^s}/J) \circ C_q = \bigoplus_{j=1}^{s} \mathbb{Q}_p[\zeta_{p^j}] \circ C_q$$

where ζ_{p^j} is a primitive p^j-th root of unity, then we can express $\overline{\beta}$ as $\overline{\beta} = (\beta_1, \ldots, \beta_s)$, for some $\beta_j \in \mathbb{Q}_p[\zeta_{p^j}]$. Thus, $\overline{\beta}$ is invertible in $(\mathbb{Q}_p C_{p^s}/J) \circ C_q$ if and only if each β_j is invertible in $\mathbb{Q}_p[\zeta_{p^j}] \circ C_q$. But, $\beta_j = 1 - \zeta_{p^j}$ is a non-zero element in the field $\mathbb{Q}_p[\zeta_{p^j}]$ so must be invertible. We next prove a lemma which shows that the β we have chosen satisfies $\overline{\beta}^{-1}\overline{\Gamma_s}\,\overline{\beta} = \overline{\Gamma_s}$.

Lemma 6.23

$$\overline{\beta}^{-1}\overline{\Gamma_s}\,\overline{\beta} = \overline{\Gamma_s}$$

Proof:

Let

$$\lambda = \sum_{i=0}^{q-1} a_i\gamma^i \in \Gamma_s \text{ for some } a_i \in \mathbb{Z}_p[x]$$

Then

$$[\lambda + J]\overline{\beta} = \sum_{i=0}^{q-1} a_i\gamma^i.[1-x] + J = \sum_{i=0}^{q-1} a_i\gamma^i(1-x)\gamma^i + J$$

But, $\gamma^i(1-x) = 1 - x^{t^i}$ so we obtain:

$$[\lambda + J]\overline{\beta} = \sum_{i=0}^{q-1} a_i(1-x^{t^i})\gamma^i + J = \sum_{i=0}^{q-1} a_i(1-x)h_i(x)\gamma^i + J$$

where $h_i(x) = \sum_{w=0}^{t^i-1} x^w$. Hence:

$$[\lambda + J]\overline{\beta} = (1-x)\left[\sum_{i=0}^{q-1} a_i h_i(x)\gamma^i + J\right] \in \overline{\beta}\,\overline{\Gamma_s}$$

and thus $\overline{\Gamma_s}\,\overline{\beta} \subseteq \overline{\beta}\,\overline{\Gamma_s}$.

The reverse inclusion can be proved by a similar argument.

□

Lemma 6.23 tells us that $\overline{\Gamma_s}$ is $\overline{\beta}$-stable with respect to $\overline{\Gamma_s}$. Therefore, using the element $\overline{\beta}$, if ${}_{\overline{\Gamma_s}}M$ is a finitely generated left torsion $\overline{\Gamma_s}$-module we can choose any generating set for ${}_{\overline{\Gamma_s}}M$ to give us a full set of $\mathcal{G}$-invariants. Now, let $T = \mathbb{Z}_p[x]$ and $\overline{T} = T/J$, so we can write:

$$\overline{\Gamma_s} = \overline{T} \circ C_q$$

We can view the twisted group ring $\overline{T} \circ C_q$ as a right $\overline{T}$-module and let $\overline{T} \circ C_q$ act on $\overline{T} \circ C_q$ as a $\overline{T}$-module to obtain the representation:

$$\overline{\rho_0} : \overline{T} \circ C_q \longrightarrow \mathrm{End}_{\overline{T}}(\overline{T} \circ C_q) \cong \mathrm{End}_{\overline{T}}(\overline{T}^q) \cong M_q(\overline{T})$$

This is one way of unboxing presentation matrices over $\overline{T} \circ C_q$ with respect to $\overline{T}$ to give $\mathcal{G}$-invariants $\mathcal{G}_s^{\overline{\rho_0}}(M)_{r,b}$, which are ideals in the ring $\overline{T}$. However, if we unbox the presentation matrices in a different way, via an alternative representation of $\overline{T} \circ C_q$ into a matrix ring over a commutative ring, we can obtain ideals in the fixed ring $\overline{T}^{C_q}$. To do this, suppose we view $\overline{T}$ as a right $\overline{T}^{C_q}$-module. Now let $\overline{T} \circ C_q$ act on $\overline{T}$ as a right $\overline{T}^{C_q}$-module via the action:

$$(\overline{t_1}\gamma^i).\overline{t_2} = \overline{t_1}\gamma^i(\overline{t_2}) \ \forall\ \overline{t_1}, \overline{t_2} \in \overline{T}$$

to obtain a representation:

$$\delta : \overline{T} \circ C_q \longrightarrow \mathrm{End}_{\overline{T}^{C_q}}(\overline{T})$$

Before we see the significance of this representation we need a lemma.

Lemma 6.24 We have an isomorphism of $\overline{T}^{C_q}$-modules:

$$\overline{T} \cong (\overline{T}^{C_q})^q$$

Proof:

Let $\overline{L} = \mathbb{Q}\overline{T}$ and let η be the natural map, $\eta : T \longrightarrow \overline{T}$. Let $\eta(x) = \overline{x}$ so then the minimum polynomial of $\overline{x}$ over $\overline{L}^{C_q}$ is of degree q and we denote it by $f(X) \in \overline{L}^{C_q}[X]$. Thus, we can write:

$$\overline{L} = \langle 1, \overline{x}, \ldots, \overline{x}^{q-1} \rangle_{\overline{L}^{C_q}}$$

In fact we know that if $\overline{x}$ is a root of $f(X)$ then $\gamma^i(\overline{x})$ must also be a root of $f(X)$ and therefore:

$$f(X) = \prod_{i=0}^{q-1}(X - \gamma^i(\overline{x}))$$

and thus the coefficients of $f(X)$ lie in $\overline{T}^{C_q}$. Thus, we can write:

$$\overline{T} = \left\langle 1, \overline{x}, \ldots, \overline{x}^{q-1} \right\rangle_{\overline{T}^{C_q}}$$

and since the $\{\overline{x}^i\}_{i=0}^{q-1}$ are linearly independent over $\overline{T}^{C_q}$, we obtain $\overline{T} \cong (\overline{T}^{C_q})^q$ as required.

□

Lemma 6.24 tells us we have a representation:

$$\delta : \overline{T} \circ C_q \longrightarrow \mathrm{End}_{\overline{T}^{C_q}}(\overline{T}) \cong \mathrm{End}_{\overline{T}^{C_q}}((\overline{T}^{C_q})^q) \cong M_q(\overline{T}^{C_q})$$

Therefore, we can construct the $\mathcal{G}$-invariants in an alternative manner as $\mathcal{G}_s^\delta(M)_{r,b}$, in order to obtain ideals in the fixed ring $\overline{T}^{C_q}$.

We ask ourselves what is the significance of the alternative construction via δ? Well, we can project the $\mathcal{G}$-invariants lying in the ring $\overline{T}^{C_q}$, derived via δ, onto a hereditary order in order to determine the original torsion module extended by that particular hereditary order. For let $T_j = \mathbb{Z}_p[\zeta_{p^j}]$ for $j = 1, \ldots, s$. Then we have an inclusion:

$$\overline{T} \circ C_q \subseteq \bigoplus_{j=1}^{s} T_j \circ C_q$$

If we let $S_j = T_j \circ C_q$ then S_j is a hereditary $T_j^{C_q}$-order in $\mathbb{Q}S_j$, since T_j is tamely ramified over $T_j^{C_q}$, for $j = 1, \ldots, s$. Thus, we can project $\overline{\Gamma_s}$ onto each hereditary order S_j via the natural map:

$$\theta_j : \overline{T} \circ C_q \longrightarrow T_j \circ C_q = S_j$$

Let us denote by $\rho_j' : S_j \longrightarrow M_q(T_j)$ the inclusion map of the hereditary order S_j into a matrix ring over a commutative ring. Then, on extending the $\mathcal{G}$-invariants we obtain:

$$\theta_j(\mathcal{G}_s^\delta({}_{\overline{\Gamma_s}}M)_{r,b})Z(S_j) = \mathcal{G}_s^{\rho_j'}(S_j \otimes_{\overline{\Gamma_s}} M)_{r,b}$$

But, since S_j is a principal hereditary $T_j^{C_q}$-order, Theorem 6.13 tells us that $\mathcal{G}_s^{\rho'_j}(S_j \otimes_{\overline{\Gamma_s}} M)_{r,b}$ will uniquely determine the extended modules $S_j \otimes_{\overline{\Gamma_s}} M$ and we conclude that the invaraints $\mathcal{G}_s^{\delta}({}_{\overline{\Gamma_s}}M)_{r,b}$ uniquely determine the $S_j \otimes_{\overline{\Gamma_s}} M$, up to isomorphism.

Remark 6.25 If we only consider Γ_s-modules then $T \ncong (T^{c_q})^q$, so we are unable to find a representation which gives $\mathcal{G}$-invariants in the fixed ring T^{C_q} in the same manner as δ does for $\overline{\Gamma_s}$-modules as above. One question for future investigation is to ask ourselves whether the alternative $\mathcal{G}$-invariants which lie in the smaller ring $\overline{T}^{C_q}$ give us finer invariants than the $\mathcal{G}$-invariants which lie in $\overline{T}$. It may be possible to find examples of modules where the $\mathcal{G}$-invariants with respect to δ can provide us with more useful information about the module than the $\mathcal{G}$-invariants with respect to $\overline{\rho_0}$.

Chapter 7

Conclusions

To summarise, in the first part of this thesis we have reviewed and extended some results about Fitting Ideals and their relation with module structure, where the underlying ring is commutative. In chapter 2 we showed that for modules over Noetherian rings we can improve upon the sequence of increasing Fitting Ideals to obtain a strictly increasing sequence of ideals (except where the ideals are zero or the whole ring). We also showed that for modules over Dedekind rings there are some intimate relationships between initial Fitting Ideals and annihilators and higher Fitting Ideals. For non-Dedekind rings we could try to obtain similar relationships, in particular try to find a better estimate for annihilators, or attempt to show the relationships between modules in terms of their higher Fitting Invariants.

In chapter 3 we considered to what extent the Fitting Ideals could be used to tell us about the underlying module structure. Our investigation concentrated on certain rings - for example, we showed that the Fitting Ideals completely determine the module structure for torsion modules over Dedekind rings. We could consider other rings and try to say more precisely what the Fitting information tells us, especially when it does not provide us with the whole picture.

In chapter 4 we showed that the Fitting Invariants could determinine the underlying lattices over integral group rings, where the underlying group is cyclic of order p. For cyclic groups of order p^2 we saw that the situation is more complex and it is not clear whether the Fitting Ideals determine the lattices over the integral group ring in this case. For other finite groups the group ring may not have finite representation type. However, we may still be able to make use of Fitting Invariants to tell us something about the underlying group ring modules in these cases.

In chapter 5 we extended the notion of Fitting Invariants to modules over certain non-commutative rings. As mentioned previously, not much work has been done before in this area. We constructed finer Fitting Invariants for modules over hereditary orders from a module-theoretic view. In chapter 6 we constructed a set of invariants for modules over more general orders, which we called $\mathcal{G}$-invariants, by considering adjustments to presentation matrices. In both these cases the construction works well for modules over hereditary orders where the invariants completely determine the structure of torsion modules. We could try to generalise this construction for other non-commutative rings - for example, rings which can be projected onto hereditary orders. In the twisted group ring case further work needs to be done in order to say more precisely what the invariants tell us. Indeed, for modules over the group ring of metacyclic groups we could only obtain a full set of $\mathcal{G}$-invariants when we worked over a quotient ring. Further work needs to be done to see if a full set of $\mathcal{G}$-invariants can be obtained in the whole ring, and if this can be done, then we need to investigate what this extra information tells us. One application is to generalise the work on Galois modules in [2] to the case where the Galois group is non-Abelian, for example where the Galois group is a metacyclic group.

Overall, we have demonstrated that Fitting Invariants can be a powerful tool in telling us information about certain modules. In some cases it may be difficult to compare the structure of different modules but we can use Fitting Invariants to tell us something about the relationships between these modules.

Bibliography

[1] M. Auslander and D. S. Rim, *Ramification index and multiplicity*, Illinois J. Math **7** (1963), 566-581.

[2] P. Cornacchia and C. Greither, *Fitting Ideals of Class Groups of Real Fields with Prime Power Conductor*, Journal of Number Theory **73** (1998), 459-471.

[3] C. W. Curtis and I. Reiner, *Methods of Representation Theory, Volume 1*, John Wiley and Sons (1981).

[4] F. E. Diederichsen, *Über die Ausreduktion ganzzahliger Gruppendarstellungen bei arithmetischer Aquivalenz*, Abh. Math. Sem. Univ. Hamburg **14** (1940), 357-412.

[5] Y. A. Drozd, *Modules over hereditary orders*, Mat. Zametki **29** (1981), no. 6, 813-816.

[6] H. Fitting, *Die Determinantenideale eines Moduls*, Jahresbericht, Deutsch Math.-Verein, **46** (1936), 195-228.

[7] A. Heller and I. Reiner, *Representations of cyclic groups in rings of integers I*, Ann. of Math.(2) **76** (1962), 73-92.

[8] A. Heller and I. Reiner, *Representations of cyclic groups in rings of integers II*, Ann. of Math.(2) **77** (1963), 318-328.

[9] S. Iyanga and Y. Kawada, *Encyclopedic Dictionary of Mathematics Volume I*, MIT Press (1980).

[10] G. J. Janusz, *Tensor products of orders*, J. London Math. Soc. (2) **20** (1979), 186-192.

[11] C. Kearton and S. M. J. Wilson, *Alexander Ideals of Classical Knots*, Publicacions Matemàtiques, Vol **41** (1997), 489-494.

[12] S. Lang, *Algebra*, 3rd Ed., Addison-Wesley (1993).

[13] D. G. Northcott, *Finite Free Resolutions*, Cambridge University Press (1976).

[14] K. M. O'Brien, *Seifert's Algorithm, Châtelet Bases and the Alexander Ideals of Classical Knots*, PhD Thesis, University of Durham, 2002.

[15] I. Reiner, *Integral representations of cyclic groups of prime order*, Proc. Amer. Math. Soc. **8** (1957), 142-146.

[16] I. Reiner, *Maximal Orders*, Academic Press (1975).

[17] J. P. Serre, *Local Fields*, Springer-Verlag (1979).

[18] L. Small, *Hereditary Rings*, Proc. Nat. Acad. Sci. USA **55** (1966), 25-27.

[19] J. Susperregui, *On determinantal ideals over certain non commutative rings*, Ring theory (Granada, 1986), Lecture Notes in Math., 1328, Springer (1988), 269-282.

[20] J. Susperregui, *Fitting Invariants for modules over anticommutative graded rings*, Comm. Algebra **17** (1989), no. 8, 2035-2054.

[21] O. Zariski and P. Samuel, *Commutative Algebra Volume I*, University Series in Higher Mathematics (1958).