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ABSTRACT

This thesis aims at presenting results and remarks concerning the study of subvarieties of the projective space $|2\Theta|$ associated to a smooth projective curve C of genus at least 3 and its connections to the moduli space $\mathcal{SU}_C(2)$ of rank 2 semi-stable vector bundles with trivial determinant.

In the first part of the thesis, I present a review of Narasimhan and Ramanan's embedding of $\mathcal{SU}_C(2)$ in $|2\Theta|$ for non-hyperelliptic curves of genus 3 ([N-R2]). In particular, I clarify some of the points of their construction (2.3.6) and give complete proofs of lemma 5.1 and lemma 5.2 (see 2.3.4 and 2.3.17). Moreover in section 2.3 I show that lemma 5.4 of [N-R2] is false, providing an extensive counterexample (2.4.3).

In the second part, I discuss the Abel-Jacobi stratification of $|2\Theta|$ for non-hyperelliptic curves of genus at least 3 as introduced in [O-P], which generalises classical subvarieties of $|2\Theta|$ such as the Kummer variety. I show that the top element of these stratifications is always a hypersurface and compute its degree (3.2.5), then I provide insight into the characterisation of the general element of the stratification (§3.3).

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences, University of Durham, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

The aim of this thesis is to present some results and remarks concerning the study of subvarieties of the projective space $|2\Theta|$ and the moduli space $\mathcal{SU}_C(2)$ associated to a non-hyperelliptic curve C of genus at least 3.

This chapter presents all those notions which are essential to the development of this thesis, such as Jacobians, the moduli space $\mathcal{SU}_C(2)$ and extension spaces. The description is by no means exhaustive and has, as its main objective, that of providing definitions and results, while references are given for proofs and further reading. Sections 1.4 and 1.5 are the only ones where more proofs and details are provided, since they will often be referred to in the successive chapters. In particular, section 1.5 is the key to understanding the counterexample of §2.3.

Chapter 2 is the central part of the thesis. It presents a very detailed review of [N-R2] of Narasimhan and Ramanan, in particular the proof that the natural map δ from $\mathcal{SU}_C(2)$ to $|2\Theta|$ is an embedding for non-hyperelliptic curves of genus 3. The motivation for this is twofold. On the one hand, the original paper is in many parts unclear and sketchy, many steps of the proof are either only hinted or not given at all and the central statement, i.e., *lemma* 5.2 goes unproved. Hence I have striven to produce a coherent interpretation of section 5 of [N-R2]. On the other hand, I show that *lemma* 5.4 of [N-R2] is false and produce a counterexample (see proposition 2.4.3). In more detail, I show that there exist 64 cones over some Veronese surfaces in $\mathcal{SU}_C(2)$ of vector bundles that fail to satisfy the above mentioned lemma.

Chapter 3 presents the construction of the Abel-Jacobi stratification of $|2\Theta|$ for curves of genus at least 3, as first done by Oxbury and Pauly in [O-P]. This construction partially mirrors and formalises the one used by Narasimhan and Ramanan in [N-R2]. In this section I prove that the top element of every Abel-Jacobi stratification for non-hyperelliptic curves of genus at least 4 is a hypersurface and compute its degree (see proposition 3.2.5). Moreover, recent results by Pareschi and Popa, see [Pa-Po], allow me to make some remarks on how to study the general element of the stratification.

Finally, the Appendix presents some results concerning the intersection rings of the d-*th* symmetric product of a given curve, C_d, and the product variety $C_d \times J^d$. In particular, I consider a generalisation of the *gamma* class, $\gamma \in H^2(C \times J, \mathbb{Z})$ (see [ACGH], p.335) and show how to compute integrals that contain powers of it. Moreover, I construct a Poincaré line bundle parametrising line bundles of C_d induced by line bundles on the curve C and compute its first Chern class. These computations are interesting in their own right and are also used in the course of chapter 3 to give some explicit results and examples.

1.1 Basic notions and notation

In the whole thesis, the underlying field will always be $\mathbb{C}$, though in many cases any algebraically closed field of characteristic zero would do. In general, for any given $\mathbb{C}$-vector space V, V^* denotes the dual and $\mathbb{P}V$ the space of one-dimensional subspaces of V.

C will always be a smooth projective curve of genus $g \geq 2$, although the main definitions hold for curves of any genus. Knowledge of basic properties of sheaves and vector bundles on algebraic curves is assumed throughout the thesis, in particular we shall make the usual identification of invertible sheaves with line bundles and of locally free sheaves with vector bundles. We will denote by $\mathcal{O}_C$ the trivial line bundle, i.e., the sheaf of regular functions on C, while K_C will be the canonical line bundle, i.e., the dual of the tangent bundle of C. When no confusion can arise, we will write $\mathcal{O}$ and K for $\mathcal{O}_C$ and K_C. A theta characteristic will be a line bundle κ over C which is a square root of the canonical line bundle, i.e., $\kappa^2 = K$. If C is a curve of genus g the set of all theta characteristics, $\vartheta(C)$, consists of 2^{2g} elements (see [L-B], p.331), as they are in one to one correspondence with the sets of two-torsion points of $Pic^0(C)$.

For any given vector bundle E on a variety X we denote by $H^0(X, E)$ the space of global sections and by $h^0(X, E)$ its complex dimension. Similar notions hold for sheaves and higher cohomology spaces. Moreover, for a given line bundle L on X we shall write $|L|$ to denote the projective space $\mathbb{P}H^0(X, L)$. If s is a non zero global section of L and D_s is the corresponding divisor, then we identify $|L|$ with the linear series $|D_s|$.

Finally, when tensoring line bundles we shall often drop the tensor product symbol, in particular for any line bundle ξ on C and divisor D of C, $\xi(D)$ will indicate the tensor product $\xi \otimes \mathcal{O}(D)$.

1.1.1 The Jacobian

A principally polarised abelian variety is a pair consisting of a complex torus A of dimension g, the abelian variety, together with the (first) Chern class η of an ample line bundle Θ on A such that $h^0(A, \Theta) = \frac{1}{g!}\int_A \eta^g = 1$, the principal polarisation. Thus a principal polarisation η on A is the fundamental class of a divisor, which we still denote Θ and call theta divisor. This divisor is unique up to translation and can be explicitly constructed (see [ACGH], chapter 1, §4 for this and the other relevant notions).

A smooth projective curve C gives rise to a canonical principally polarised abelian variety, the Jacobian, where the abelian variety is defined as $J(C) = H^0(C,K)^*/H_1(C,\mathbb{Z})$, while the theta divisor $\Theta(C)$ is defined as the zero set of Riemann's theta function θ (see [ACGH], p.23). From the definition it is also clear that $J(C)$ is a g-dimensional variety, since $H^0(C,K)$ has dimension g.

For all relevant notions on Jacobians the main references are Arbarello et al. ([ACGH]) and Mumford ([M1], lecture III).

For every non negative integer d one denotes by J^d the Picard variety $Pic^d(C)$ of degree d line bundles over C identified with the Jacobian $J(C)$ (see [ACGH], p.19 and recall that the isomorphism depends on the choice of a point of C). For each positive integer d, W_d is the image in J^d of the d-fold symmetric product of C, C_d, via the Abel-Jacobi map $u_d : C_d \longrightarrow J^d$ given by $D \longmapsto \mathcal{O}(D)$, i.e., W_d is the set of degree d line bundles over C with non zero global sections (see [M1], lecture III, pp.261-164 for low genus examples). When $d = g-1$, the divisor Θ is identified with W_{g-1}, the divisor of degree $g-1$ line bundles with non zero global sections. This is the identification which we will take as natural, J^{g-1} and $\Theta = W_{g-1}$. Another isomorphism which will often be used is that of $Pic^0(C)$ with $J(C)$, in which case the theta divisor is denoted by Θ^0. It is important to notice that Θ and Θ^0 are non-canonically isomorphic, an isomorphism being given by any choice of a theta characteristic $\kappa \in \vartheta(C)$, $\Theta^0 = \Theta_\kappa = \{\eta \in J^0 : h^0(\eta \otimes \kappa) > 0\}$, where for given $\xi \in J^d$ the convention is to denote by Θ_ξ the divisor in J^{g-1-d} with support $\{\eta : h^0(C, \eta \otimes \xi) > 0\}$.

The divisor Θ is ample, yet since $h^0(J, \Theta) = 1$ not much information on C can be gained by studying the associated linear map $\phi_{|\Theta|}$. The divisor $n\Theta$ is very ample for $n \geq 3$ and, hence, the associated linear map $\phi_{|n\Theta|}$ is an embedding of J in the projective space $|n\Theta|^*$ (see [K], p.16). We will focus our attention on the base-point-free linear series $|2\Theta|$ (see [K], p.15). It is worth remembering that this system has been particularly studied in connection to the Schottky problem, i.e., the problem of characterising the g-dimensional principally polarised abelian varieties that arise as Jacobians of curves of genus g. For further notions on the Schottky problem we refer to [M1] (lecture IV and the Survey) with its very exhaustive list of publications on this problem by Beauville, Debarre, Donagi, van Geemen, Welters and many others (pp.299-304).

Note that $|2\Theta|$ has dimension $2^g - 1$ (see [K], p.27) so that $\phi_{|2\Theta|}$ is a regular map to $\mathbb{P}^{2^g-1}$. Finally, to keep in line with conventions we will use the symbol $\mathcal{L}$ to denote the line bundle $2\Theta^0$ and $|\mathcal{L}|$ for the corresponding linear series.

Remark 1.1.1. It is possible to describe some of the divisors in the linear series $|2\Theta|$ and $|\mathcal{L}|$ explicitly. For each $\zeta \in J^{g-1}$ one has $\Theta_\zeta + \Theta_{K\zeta^{-1}} \in |\mathcal{L}|$ and similarly for each $\eta \in J^0$, $\Theta_\eta + \Theta_{\eta^{-1}} \in |2\Theta|$.

The first part of the statement is just a consequence of the Theorem of the Square (see [K], p.14), that is, $\Theta^0 + \Theta^0$ is linearly equivalent to $\Theta^0_\xi + \Theta^0_{\xi^{-1}}$ for any $\xi \in J^0$. In particular, if $\Theta^0 = \Theta_\kappa$ and $\xi = \zeta \otimes \kappa^{-1}$ one obtains the required result. As for the second part, this is proved by translating by the given theta characteristic.

From here onward, we will call *split* 2Θ divisors the divisors in $|2\Theta|$ of the above form. As remarked above, Θ and Θ^0 are non-canonically isomorphic, yet the following holds.

Lemma 1.1.2. *Wirtinger Duality (see [M2], pp.335-336) There is a canonical isomorphism between $|2\Theta|$ and $|\mathcal{L}|^*$.*

Proof. Consider the map

$$\begin{array}{rccc} \beta: & J^{g-1}\times J^0 & \longrightarrow & J^{g-1}\times J^{g-1} \\ & (\xi,\eta) & \longmapsto & (\xi\eta,\xi\eta^{-1}) \end{array}$$

and use the symbol $\boxtimes$ for the tensor product of pull-backs of line bundles living on the "factor" varieties J^{g-1} and J^0. One can see that $\beta^*(\Theta\boxtimes\Theta)=2\Theta\boxtimes\mathcal{L}$, as if one denotes $\mathcal{D}=\Theta\boxtimes\Theta$ then, for $\eta\in J^0$, $\beta^*(\mathcal{D})|_{J^{g-1}\times\{\eta\}}=\Theta_\eta+\Theta_{\eta^{-1}}\sim 2\Theta$ and similarly, for $\xi\in J^{g-1}$, $\beta^*(\mathcal{D})|_{\{\xi\}\times J^0}=\Theta_\xi+\Theta_{K\xi^{-1}}\sim\mathcal{L}$ by the previous remark.

Let θ be a generator of $H^0(J^{g-1},\Theta)$ and denote by $\{s_i\}$ and $\{t_j\}$ two bases of $H^0(J^{g-1},2\Theta)$ and $H^0(J^0,\mathcal{L})$ respectively. Then the above equality can be written as

$$\beta^*(\theta\otimes\theta)=\textstyle\sum_{i,j}c_{i,j}\,s_i\otimes t_j$$

for a matrix $\{c_{i,j}\}$ of coefficients. In turn, this says that for any pair of line bundles $(\xi,\eta)\in J^{g-1}\times J^0$ one has

$$\theta(\xi\eta)\theta(\xi\eta^{-1})=\sum_{i,j}c_{i,j}\,s_i(\xi)\otimes t_j(\eta). \tag{1.1.1}$$

Recall (see [M3], §1) that one can associate a group, denoted $H(\mathcal{L})$, to the line bundle $\mathcal{L}$: $H(\mathcal{L})$ is the set of points $x\in J^0$ such that $T_x^*\mathcal{L}\cong\mathcal{L}$, where T_x denotes the translation by x. The group $H(\mathcal{L})$ has cardinality 2^{2g} (see [M3], p.289) and can be identified with the group J_2^0 of 2-torsion points of J^0 (see [M3], proposition 4, p.310). One can then define a set the elements of which are pairs (x,ϕ) where $x\in H(\mathcal{L})$ and ϕ is an isomorphism of $\mathcal{L}$ with $T_x^*\mathcal{L}$. This is denoted $\mathcal{G}(\mathcal{L})$ and it can be shown to be a group. The map that "forgets" the isomorphism induces a short exact sequence

$$0\longrightarrow\mathbb{C}^*\longrightarrow\mathcal{G}(\mathcal{L})\longrightarrow H(\mathcal{L})\longrightarrow 0$$

where one can verify that the kernel is $\mathbb{C}^*$ since it is the group of automorphisms of $\mathcal{L}$ (see [M3], p.290). The group $\mathcal{G}(\mathcal{L})$ acts on the vector space $H^0(J^0,\mathcal{L})$ in the following way: let $z = (x, \phi) \in \mathcal{G}(\mathcal{L})$, then one defines $U_z : H^0(J^0, \mathcal{L}) \longrightarrow H^0(J^0, \mathcal{L})$ by $U_z(s) = T_x^*(\phi(s))$ for all sections $s \in H^0(J^0, \mathcal{L})$ (see [M3], p.295). Note that in this action $\mathbb{C}^*$ acts by its natural character: if $\alpha \in \mathbb{C}^*$ then the action is multiplication by α. There is a unique irreducible representation of $\mathcal{G}(\mathcal{L})$ in which $\mathbb{C}^*$ acts in this way ([M3], theorem 1, p.295) and it is precisely the one described above (see [M3], theorem 2, p.297).

A similar construction and equivalent results can also be applied to 2Θ and the vector space $H^0(J^{g-1}, 2\Theta)$, where again $H(2\Theta)$ can be identified with J_2^0 since the translation maps on J^{g-1} are given by tensoring with points of J^0.

The construction can be repeated once more for the product abelian variety $J^{g-1} \times J^0$ and the line bundle $2\Theta \boxtimes \mathcal{L}$, and one obtains a short exact sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(2\Theta \boxtimes \mathcal{L}) \longrightarrow H(2\Theta \boxtimes \mathcal{L}) \longrightarrow 0$$

as well as an action of $\mathcal{G}(2\Theta \boxtimes \mathcal{L})$ on $H^0(J^{g-1} \times J^0, 2\Theta \boxtimes \mathcal{L})$, which by the Künneth decomposition can be thought of as $H^0(J^{g-1}, 2\Theta) \otimes H^0(J^0, \mathcal{L})$. Considering again the map β defined at the outset of this proof, this is a map of degree 2^{2g} ([M3], p.322) and its kernel K can be naturally identified with the set of 2-torsion points of J^0.

By the general theory described by Mumford ([M3], p.290), K can be thought of as a proper subgroup of $H(2\Theta \boxtimes \mathcal{L})$ and can hence be lifted to a subgroup $\mathbf{K}$ of $\mathcal{G}(2\Theta \boxtimes \mathcal{L})$. Then $\mathbf{K}$ acts on $H^0(J^{g-1}, 2\Theta) \otimes H^0(J^0, \mathcal{L})$ with the action induced by $\mathcal{G}(2\Theta \boxtimes \mathcal{L})$ and one can note that $\beta^*(\theta \otimes \theta)$ is invariant under the action of $\mathbf{K}$. Moreover, since the action of $\mathbf{K}$ restricts to that of $\mathcal{G}(2\Theta)$ on $H^0(J^{g-1}, 2\Theta)$ and is hence irreducible (and similarly on $H^0(J^0, \mathcal{L})$), $\beta^*(\theta \otimes \theta)$ cannot lie in any proper subspace $\Lambda_1 \otimes \Lambda_2$ of $H^0(J^{g-1}, 2\Theta) \otimes H^0(J^0, \mathcal{L})$. This implies that $det\,\{c_{i,j}\} \neq 0$ and, hence, that $\{c_{i,j}\}$ defines a non-degenerate bilinear form $H^0(J^{g-1}, 2\Theta) \otimes H^0(J^0, \mathcal{L}) \to \mathbb{C}$, i.e., $H^0(J^{g-1}, 2\Theta)$ is canonically isomorphic to $H^0(J^0, \mathcal{L})^*$. □

Remark 1.1.3. Note that equation (1.1.1) implies that $\xi \in J^{g-1}$ belongs to the support of $\Theta_\eta + \Theta_{\eta^{-1}}$ if and only if it lies in the zero set of $\sum_{i,j} c_{i,j}\, t_j(\eta) s_i(\cdot)$.

1.1.2 The Kummer map

The *Kummer* map is defined as the map $J^0 \to |2\Theta|$ such that $\eta \longmapsto \Theta_\eta + \Theta_{\eta^{-1}}$. It is a regular map whose quotient by the natural involution $\iota : \eta \longmapsto \eta^{-1}$ is an embedding (see [L-B], p.101, theorem 8.2).

It is also customary to consider a map from J^0 to $|\mathcal{L}|^*$ induced by the base-point-free linear system $|\mathcal{L}|$, $\psi_{|\mathcal{L}|} : J^0 \longrightarrow |\mathcal{L}|^*$. This, again, can be composed with the natural involution ι on J^0 to give an embedding of J^0/ι in $|\mathcal{L}|^*$. Wirtinger duality allows one to identify these maps.

Corollary 1.1.4. *(see [M2], pp.335-336) The following diagram is commutative*

$$\begin{tikzcd} {|2\Theta|} \arrow[rr, "\sim"] & & {|\mathcal{L}|^*} \\ & J^0 \arrow[ul, "Kum"] \arrow[ur, "\psi_{|\mathcal{L}|}"'] & \end{tikzcd}$$

Proof. By using notation and construction from the proof of Wirtinger duality (see lemma 1.1.2), it is known that

$$Kum(\eta) = \Theta_\eta + \Theta_{\eta^{-1}} = \beta^*(\mathcal{D})|_{J^{g-1}\times\{\eta\}}$$

hence, what one needs to show is that the set of global sections of $\mathcal{L}$ which are zero at η, i.e., $|\mathcal{L}_{-\eta}|$ is dual to $\Theta_\eta + \Theta_{\eta^{-1}}$. Yet note that, by remark 1.1.3, $\xi \in J^{g-1}$ lies in the support of $\Theta_\eta + \Theta_{\eta^{-1}}$ if and only if $\sum_{i,j} c_{i,j} t_j(\eta) s_i(\xi) = 0$, i.e., if and only if $\sum_{i,j} c_{i,j}\, s_i(\xi) t_j(\cdot)$ is a section of $\mathcal{L}$ which vanishes at η. Hence the duality is proved. $\square$

In particular this lemma explicitly describes Wirtinger duality for *split* 2Θ divisors, i.e., divisors of the form $\Theta_\eta + \Theta_{\eta^{-1}}$ with $\eta \in J^0$.

Note that in the following we may use the expression *Kummer* map to indicate either of the above maps on J^0. Moreover similar results hold for J^{g-1}, its natural map to $|2\Theta|^*$ and an analogous *Kummer* map to $|\mathcal{L}|$.

It is a well known result (see [L-B], chapter 10, §3) that for curves of genus 2, this map embeds J/ι as a quartic in $\mathbb{P}^3$ which has 16 nodes and is *Heisenberg* invariant, the *Kummer surface.*

1.2 $\mathcal{SU}_C(2)$ and $\mathcal{SU}_C(2, K)$

Recall that a vector bundle E of $rk\ n$ on a curve C of genus $g \geq 2$ is defined to be *(semi)stable* if $\frac{\deg F}{rk\, F} < \frac{\deg E}{rk\, E}$ (*resp.* $\leq$) for every proper vector subbundle F of E. The quotient $\frac{\deg}{rk}$ is called the slope of a vector bundle, hence a vector bundle E of slope μ is stable if and only if every vector subbundle of E has slope strictly less than μ (see [LP], p.73). In particular every line bundle is stable, while a $rk\, 2$ vector bundle E is stable if and only if $\deg\, \xi < \frac{1}{2} \deg\, E$ for every line subbundle ξ of E.

Following the work of Seshadri (see [S]), the set of semi-stable vector bundles can be given an equivalence relation, *S-equivalence*, which we will now briefly sketch. Let E be a semi-stable vector bundle of $rk\, n$ and slope μ, it admits an increasing sequence (S_E) of vector subbundles

$$(S_E) \qquad E_1 \subset E_2 \subset \cdots \subset E_m = E$$

such that E_1 and all the quotients E_{i+1}/E_i are stable of slope μ, called a Jordan-Hölder series for E (see [LP], chapter 5, p.76). The vector bundle $gr(E) \stackrel{def}{=} E_1 \oplus E_2/E_1 \oplus \cdots \oplus E_m/E_{m-1}$ is unique up to isomorphism and is called the graded quotient of E (see [S], theorem 2.1). Two semi-stable vector bundles E and E' are said to be S-equivalent if $gr(E) \cong gr(E')$, in which case we write $[E] = [E']$. In particular two stable vector bundles E and E' are S-equivalent if and only if they are isomorphic as vector bundles, since in this case $E \cong gr(E)$ and similarly for E'.

Remark 1.2.1. A semi-stable, non stable rank 2 vector bundle E admits a Jordan-Hölder series $L_1 \subset E$ where L_1 is a line bundle of degree equal to the slope μ_E of E, i.e., $\deg L_1 = \frac{1}{2}\deg E$ and hence $gr(E) = L_1 \oplus L_2$, where $L_2 = E/L_1$ and $\deg L_2 = \mu_E$. Thus E is S-equivalent to a direct sum $L_1 \oplus L_2$ if and only if there exists an extension $0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$ (see §1.3) with $\deg L_i = \mu_E$, yet E is not isomorphic to $L_1 \oplus L_2$ unless the above sequence splits. Moreover all extensions of L_2 by L_1 give rise to S-equivalent semi-stable vector bundles of slope μ_E.

Let $\mathcal{SU}_C(2, L)$ be the moduli space of (S-equivalence classes of) semi-stable rank 2 vector bundles with determinant $L \in Pic(C)$. It is well known that $\mathcal{SU}_C(2, L)$ has dimension $3g - 3$. For example, a stable vector bundle $E \in \mathcal{SU}_C(2) = \mathcal{SU}_C(2, \mathcal{O})$ admits line subbundles of degree at most -1, while a semi-stable, non stable vector bundle admits also line subbundles of degree 0.

We will only consider vector bundles with determinant either trivial, $\mathcal{SU}_C(2)$, or canonical, $\mathcal{SU}_C(2, K)$. Note that these moduli spaces are isomorphic though not canonically, hence for the time being we will make some remarks only on $\mathcal{SU}_C(2, K)$. Let Δ be the Cartier divisor on $\mathcal{SU}_C(2, K)$ whose support is the subvariety of vector bundles with non zero sections and $\tilde{\Theta} = \mathcal{O}(\Delta)$ the associated ample line bundle, then $\mathcal{SU}_C(2, K)$ has Picard group isomorphic to $\mathbb{Z}$ and $\tilde{\Theta}$ is a generator, i.e., $Pic\,(\mathcal{SU}_C(2, K)) \cong \mathbb{Z}(\tilde{\Theta})$ (see [B1], proposition 3.1, p.442 or [D-N], theorem B, p.55). We will often identify the line bundle $\tilde{\Theta}$ with the divisor Δ and call either a generalised theta divisor. Now consider the map

$$\begin{aligned} \psi : \; J^{g-1} &\longrightarrow \mathcal{SU}_C(2, K) \\ \eta &\longmapsto [\eta \oplus K\eta^{-1}]. \end{aligned}$$

It is easy to verify that $\eta \oplus K\eta^{-1}$ is a semi-stable vector bundle. It was shown by Beauville (see [B1], proposition2.5) that $\psi^*(\tilde{\Theta}) = 2\Theta$, where ψ^* denotes the pull-back map associated to ψ, and that there exists a canonical isomorphism

$$H^0(\mathcal{SU}_C(2,K),\tilde{\Theta}) \cong H^0(J^{g-1},2\Theta).$$

Similar results hold for $\mathcal{SU}_C(2)$ and $|\mathcal{L}|$, in which case one denotes by $\mathcal{L}'$ the ample generator of $Pic\,(\mathcal{SU}_C(2))$.

1.3 Extensions

Given vector bundles G and F over a curve C, an extension of F by G is a short exact sequence $0 \to G \to E \to F \to 0$. Every extension gives rise to an element $\delta(E) \in H^1(C, Hom(F,G))$ which is the image of the identity homomorphism in $H^0(C, Hom(F,F))$ by the connecting homomorphism of cohomology δ: $H^0(C, Hom(F,F)) \longrightarrow H^1(C, Hom(F,G))$. Two extensions of F by G are said to be equivalent if the corresponding exact sequences are isomorphic, i.e., if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d, "I_G"] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "I_F"] & 0 \\ 0 \arrow[r] & G \arrow[r] & E' \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

where I_G and I_F are the identity map of G and F, respectively. Atiyah proved (see [At], proposition 2, p.184) that there is a 1 to 1 correspondence between the set of equivalence classes of extensions of F by G and $H^1(C, Hom(F,G))$, with the trivial (split) extension corresponding to the zero element. The following remark was made by Narasimhan and Ramanan (see [N-R1], lemma 3.3).

Remark 1.3.1. If E and E' are two extensions of F by G, then E and E' are isomorphic as bundles if $\delta(E) = \lambda\delta(E')$ for some $\lambda \in \mathbb{C}^*$.

It is important to notice, however, that the converse does not hold in general. The following lemma will be used in remark 1.5.2.

Lemma 1.3.2. *(see [N-R1], lemma 3.2) Let* $0 \to G \to E \to F \to 0$ *be an extension of* F *by* G*,* W *a vector bundle and* f *a homomorphism* $W \longrightarrow F$*. Then* f *can be lifted to a homomorphism* $\tilde{f} : W \longrightarrow E$ *if and only if*

$$\delta(E) \in Ker\,[H^1(C, Hom(F,G)) \longrightarrow H^1(C, Hom(W,G))].$$

We conclude this section with some remarks on maximal line subbundles of given bundles, which we will need in section 2.2 (see the proof of theorem 2.2.4).

Lemma 1.3.3. *(see [N-R1], lemma 5.3) Let* E *be a vector bundle and* ζ *a line bundle on* C*. A morphism* $\phi : \zeta \to E$ *fails to be injective at* $p \in C$ *if and only if it factorises through a map* $\zeta(p) \to E$*.*

The following result on maximal line subbundles is a consequence of lemma 1.3.3, however since no explicit reference is available we give a complete proof.

Lemma 1.3.4. *If* E *is a rank* 2 *(semi)stable vector bundle and* η *is a line bundle of maximal degree with respect to* E*, i.e.,* $\deg \eta = max \deg \xi$ *where* ξ *varies among all line subbundles of* E*, then* η *is a maximal line subbundle of* E *if and only if* $h^0(C, E \otimes \eta^{-1}) \geq 1$*.*

Proof. If η is a maximal line subbundle of E then $h^0(C, E \otimes \eta^{-1}) \geq 1$. On the other hand, if $h^0(C, E \otimes \eta^{-1}) \geq 1$, there is a non-zero morphism $\eta \to E$. Lemma 1.3.3 assures that this morphism fails to be injective only if it factorises through an injective map $\eta(D) \to E$, where D is a suitable effective divisor of positive degree on the curve C. In this case the vector bundle E would have a maximal line subbundle, $\eta(D)$, of degree strictly higher that $\deg \eta$. This contradicts the hypothesis that $\deg \eta$ is the maximal degree of a line subbundle of E, so the original morphism $\eta \to E$ has to be an injection of vector bundles. □

Recall that a vector subbundle of a given vector bundle E can be thought of as a pair (F, ϕ) consisting of a vector bundle, F, and an embedding ϕ of F in

E. In particular if E is a rank 2 vector bundle over a curve C, then there is a natural map from the set of all its line subbundles to $Pic(C)$, the map that forgets the embedding. By considering its restriction to the set of maximal line subbundles one has the following.

Proposition 1.3.5. *(see [L-N], lemma 2.1) If E is a stable rank 2 vector bundle, the natural map from the set of maximal line subbundles of E to $Pic(C)$ is injective.*

Remark 1.3.6. This means that the condition in lemma 1.3.4 becomes simply $h^0(C, E \otimes \eta^{-1}) = 1$. In fact, a line bundle η that has maximal degree with respect to a given stable bundle E is a maximal line subbundle of E if and only if there is a unique homomorphism, up to a multiplicative factor, of η in E. Note that such a map is necessarily an embedding by lemma 1.3.3.

1.4 A result of Lange and Narasimhan

In this section we will review some results on maximal line subbundles of given rank 2 stable vector bundles, presented by Lange and Narasimhan in [L-N], §§1 and 2, with particular emphasis on the aspects that will be needed in the following chapters.

If E is a rank 2 vector bundle, one defines the *Segre invariant* of E to be

$$s(E) = \deg\, E - 2\, max\, \deg(\eta)$$

where the maximum is taken among all line subbundles η of E. By comparing with the notion of (semi)stability, E is (semi)stable if and only if $s(E) > 0$ (resp. ≥ 0). Note, moreover, that according to the definition $s(E)$ is congruent to $d = \deg(E)$ $(mod\, 2)$.

Remark 1.4.1. The Segre invariant is an invariant with respect to tensoring with line bundles, so to study $s(E)$ one may always assume that the given vector bundle admits $\mathcal{O}_C$ as a maximal line subbundle.

To see this consider an extension (f) of a rank 2 vector bundle F:

$$0 \to \eta \to F \to L' \to 0$$

tensoring it with η^{-1} gives an extension (e):

$$0 \to \mathcal{O}_C \to E = F \otimes \eta^{-1} \to L = L'\eta^{-1} \to 0$$

and so $s(E) = s(F)$. Moreover (f) and (e) can be identified as points in the extension space $\mathbb{P}H^1(C, Hom(L'\eta^{-1}, \mathcal{O})) = \mathbb{P}H^1(C, L^{-1}) = \mathbb{P}H^1(C, Hom(L', \eta))$ which we denote simply by $\mathbb{P}_L \cong \mathbb{P}H^0(C, KL)^*$.

From now onward, in order to study the Segre invariant of a vector bundle E we will always make the above assumption. In the following we will describe some results of Lange and Narasimhan (see [L-N]) which allow us to study $s(E)$ for a given rank two vector bundle E admitting an extension (e) of the form $0 \to \mathcal{O}_C \to E \to L \to 0$ in terms of the corresponding point e in the projective extension space $\mathbb{P}_L$.

Remark 1.4.2. We will denote the linear map induced by the linear series $|KL|$ by $\phi_L : C \to \mathbb{P}_L$. Recall moreover, that if C is a smooth curve and ϕ: $C \to \mathbb{P}^r$ is a regular mapping, for every effective divisor D on C one denotes by $\overline{\phi(D)}$, or just by $\overline{D}$, the intersection of all the hyperplanes H such that either $\phi(C) \subset H$ or the pull-back of H satisfies $\phi^*(H) \geq D$ (see [ACGH], p.12). In particular if the mapping is non-degenerate the first condition is empty, while if ϕ is an embedding and $D = \sum n_i x_i$ then $\overline{D}$ is the linear subspace spanned by the n_i^{th}-osculating spaces at $\phi(x_i)$ to $\phi(C)$.

Moreover, if the mapping ϕ_L is non-degenerate, a point $f \in \mathbb{P}_L$ belongs to $\overline{D}$ if and only if the corresponding line in $H^0(C, KL)^*$ lies in the kernel of the map $H^0(C, KL)^* \longrightarrow H^0(C, KL(-D))^*$, i.e., if and only if every $H \in \mathbb{P}H^0(C, KL(-D))$, a hyperplane containing $\overline{D}$, vanishes at f.

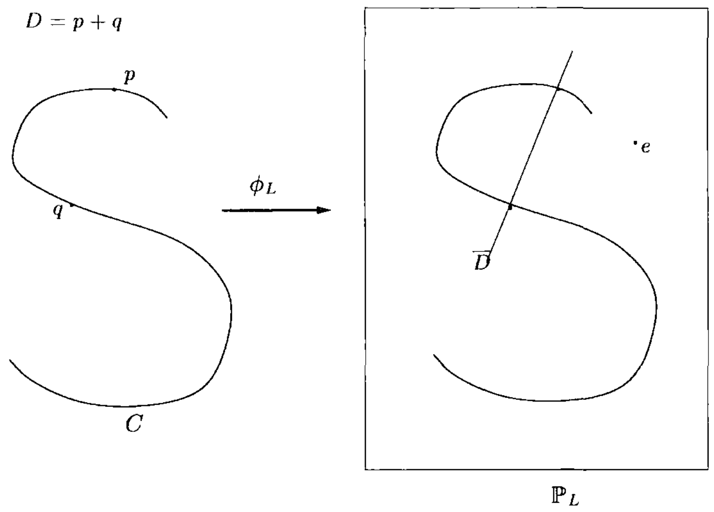

Figure 1.4.1: The span of a degree two divisor D.

If deg $L = d$, for any j such that $0 \le j \le d-1$ one denotes by $\mathcal{S}ec_j(C)$ the union of all the linear spaces $\overline{\phi_L(D)} = \overline{D}$ of $\mathbb{P}_L$ as D varies among all effective divisors of degree j on C. The variety $\mathcal{S}ec_j(C)$ in called the j^{th}-secant variety of C with respect to ϕ_L.

Recall that all secant varieties form a flag in $\mathbb{P}_L$

$$C = \mathcal{S}ec_1(C) \subset \mathcal{S}ec_2(C) \subset \cdots \subset \mathcal{S}ec_{d-1}(C) \subset \mathbb{P}_L.$$

In [L-N], Lange and Narasimhan prove a basic result which links the Segre invariant of a vector bundle E isomorphic to an extension (e) (see remark 1.4.1) to the index j such that the corresponding extension point $e \in \mathbb{P}_L$ lies on $Sec_j(C)$ but on no smaller secant variety.

Proposition 1.4.3. *(see [L-N], proposition 1.1, p.57) Let L be a line bundle of degree $d > 0$ and E a vector bundle given as an extension*

$$(e) : 0 \to \mathcal{O} \to E \to L \to 0,$$

i.e., E is parametrised by a point e in the extension space $\mathbb{P}_L$. For any integer $s \equiv d \ (mod\, 2)$ with $s \le d$ and $s \ge 4-d$ the following are equivalent

1. $s(E) \geq s$;
2. $e \notin Sec_{(d+s-2)/2}(C)$.

Remark 1.4.4. In particular, if $d = 2$, the Segre invariant $s(E)$ has to be even, so it is positive if and only if it is at least 2. Therefore, if $\deg(L) = 2$, E is stable if and only if $e \notin Sec_1(C)$.

Moreover, Lange and Narasimhan show how to find all maximal line bundles of a stable vector bundle E with an extension of type (e). If $\deg L = d$, then $e \in \mathbb{P}_L$ corresponding to E must lie in the span of some degree d divisor D of C.

Theorem 1.4.5. *(see [L-N], proposition 2.4, p.59) Let E be a rank 2 stable vector bundle corresponding to a point e in the extension space $\mathbb{P}_L$ where $\deg L = d \geq 1$. There is a bijection between*

1. *maximal line subbundles ξ of E different from $\mathcal{O}_C$;*
2. *line bundles $\mathcal{O}(D)$ of C such that D is a degree d divisor on C and $e \in \overline{D}$;*

given by $\mathcal{O}(D) = L \otimes \xi^{-1}$, i.e., $\xi = L(-D)$.

This theorem will be extremely useful in the rest of this thesis.

1.5 An example: curves of genus 3

The aim of this paragraph is to give a concrete example of how the previous results can be applied to curves of genus 3. This will turn out to be very useful throughout the following chapter. Moreover this section is a key to understanding the counterexample in §2.3.

In this final section, C will be a non-hyperelliptic curve of genus 3. In this context, every non trivial equivalence class of extensions (e) of $\xi \in J^1$ by ξ^{-1} can be identified with a point e in the projective space $\mathbb{P}(\xi) = \mathbb{P}H^1(C, \xi^{-2})$.

Lemma 1.5.1. *(see [N-R2], lemma 4.2) Let ξ be a degree 1 line bundle on C. Let $0 \longrightarrow \xi^{-1} \longrightarrow E \longrightarrow \xi \longrightarrow 0$ be a non trivial extension of ξ by ξ^{-1}, then the vector bundle E is semi-stable.*

Proof. First note that by construction the determinant of E is $\mathcal{O}_C$, so all we have to show is semi-stability. Assume that the vector bundle E is not semi-stable, i.e., there is a line subbundle $\mu \subset E$ of positive degree, then defining $\eta = E/\mu$ one has the extension $0 \to \mu \to E \to \eta \to 0$. So there exists a non zero homomorphism $\alpha : \mu \to \xi$ and this implies $\deg \mu \leq \deg \xi = 1$, i.e., $\deg \mu = 1$. As the only non trivial homomorphisms between line bundles of same degree are isomorphisms $\mu \cong \xi$ and the original extension $0 \to \xi^{-1} \to E \to \xi \to 0$ splits, against the hypothesis. $\square$

We will often denote by e a point in $\mathbb{P}(\xi)$ and by E the corresponding semi-stable rank two vector bundle with trivial determinant given by the extension (e), or rather its S-equivalence class. Moreover we will call ε_ξ the regular extension map $\mathbb{P}(\xi) \longrightarrow \mathcal{SU}_C(2)$ given by the correspondence $e \longmapsto E$. It is well known that ε_ξ is a regular injective map (see [Be], p.430 and p.461, corollary 4.4) and that $\mathcal{O}(\tilde{\Theta})$ pulls-back to $\mathcal{O}_{\mathbb{P}(\xi)}(1)$, that is, it is linear. The extension map ε_ξ embeds $\mathbb{P}(\xi)$ as a 3 dimensional projective subspace of $\mathcal{SU}_C(2)$.

Note that there always exists a regular map from C to $\mathbb{P}(\xi) \cong \mathbb{P}H^0(C, K\xi^2)^*$ given by the linear series $|K\xi^2|$ and denoted by ϕ_ξ.

Remark 1.5.2. One can describe exactly which points of $\mathbb{P}(\xi)$ give extensions of semi-stable, non stable vector bundles and characterise the corresponding vector bundles.

The image of C in $\mathbb{P}(\xi)$ composed with the extension map ε_ξ gives semi-stable, non stable S-equivalence classes of vector bundles of the form $[\xi(-p) \oplus \xi^{-1}(p)]$ as p varies in C.

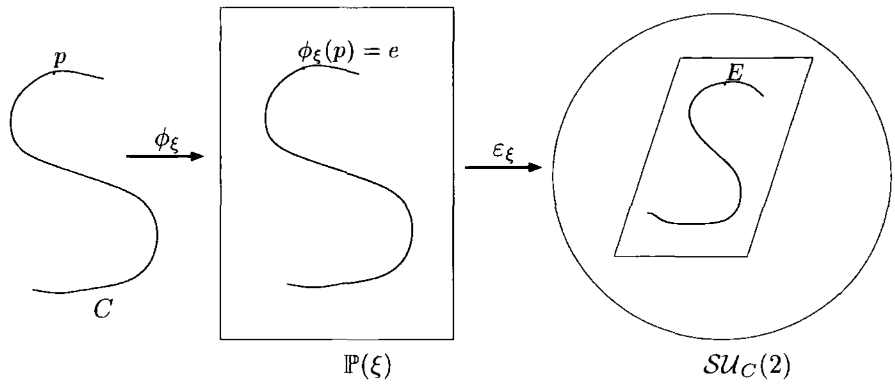

Figure 1.5.1: The composition of ε_ξ with ϕ_ξ.

In fact, if $e = \phi_\xi(p)$, i.e., e represents the hyperplane of divisors in the linear series $|K\xi^2|$ which contain p in their support, it implies that the line corresponding to e lies in the kernel of $H^0(C, K\xi^2)^* \to H^0(C, K\xi^2(-p))^*$ or, by Serre duality, in that of $H^1(C, \xi^{-2}) \to H^1(C, \xi^{-2}(p))$. This, in turn, shows that a map $\xi(-p) \to \xi$ lifts to a map $\xi(-p) \to E$, by lemma 1.3.2. Finally this implies that $\xi(-p) \in J^0$ is a maximal line subbundle of E, so E is a semi-stable, non stable vector bundle. Moreover, since $det\, E = \mathcal{O}$, it must necessarily be isomorphic to the vector bundle $\xi^{-1}(p) \oplus \xi(-p)$.

Conversely if E is any semi-stable, non stable vector bundle with trivial determinant, then it must be S-equivalent to $L \oplus L^{-1}$ for some $L \in J^0$. So, if it is isomorphic to an extension of ξ by ξ^{-1}, $h^0(C, E\xi) > 0$ implies $h^0(C, (L \oplus L^{-1}) \otimes \xi) > 0$, which is equivalent to having either $h^0(C, L\xi) > 0$ or $h^0(C, L^{-1}\xi) > 0$, i.e., either $L = \xi^{-1}(p)$ or $L^{-1} = \xi^{-1}(p)$. Both cases imply that E is S-equivalent to $\xi(-p) \oplus \xi^{-1}(p)$.

Note that as $\deg \xi^2 = d = 2$, by remark 1.4.4 we know that points outside the image of C in $\mathbb{P}(\xi)$ map to stable vector bundles. Remark 1.5.2 shows that the points on the image of C map to semi-stable, non stable vector bundles, and gives an explicit description of these bundles which will be used in chapter 2.

The following lemma is implied in lemmas 4.2 and 4.4 of [N-R2] and the comments that follow them.

Lemma 1.5.3. *The S-equivalence class of a semi-stable vector bundle is the S-equivalence class of an extension of ξ by ξ^{-1}, where $\xi \in J^1(C)$, if and only if $h^0(C, E\xi) > 0$ for some E in the same S-equivalence class.*

Proof. One implication is just a consequence of the fact that if E is an extension of ξ by ξ^{-1} there is an injection $\xi^{-1} \to E$ and hence $h^0(C, E\xi) > 0$.
Conversely if there is a non-zero map ϕ: $\xi^{-1} \to E$, it can fail to be injective only if there exists a point $p \in C$ such that ϕ factorises as $\xi^{-1} \to \xi^{-1}\mathcal{O}(p) \to E$, i.e., p is a point of C where the rank of the map ϕ drops. This cannot happen if E is stable since any degree -1 line bundle gives an injective map in each fibre (see lemma 1.3.2 and [N-R1], lemma 5.3). If E is semi-stable but not stable, then the assumption implies that E is S-equivalent to $\xi(-p) \oplus \xi^{-1}(p)$ for some $p \in C$. Now let F be any vector bundle obtained from a non trivial extension corresponding to the kernel of the map

$$H^1(C, \xi^{-2}) \longrightarrow H^1(C, \xi^{-2}(p)).$$

Then F is S-equivalent to the vector bundle $\xi(-p) \oplus \xi^{-1}(p)$ and hence to E. $\square$

The theorem of Lange and Narasimhan quoted in the previous section as theorem 1.4.5 can now be rephrased in this more specific context.

Theorem 1.5.4. *For every point $e \in \mathbb{P}(\xi) \setminus \phi_\xi(C)$ there is a bijection given by $\mathcal{O}(D) = \eta\xi$ between*

1. *maximal line subbundles η^{-1} of E such that $\eta \neq \xi$;*

2. *degree 2 divisors D of C such that $e \in \overline{D}$.*

It is particularly interesting for future reference to observe how the theorem of Lange and Narasimhan can actually be used to find how many maximal line subbundles there are for any stable vector bundle of a smooth non-hyperelliptic curve of genus $g = 3$. This example will be used in chapter 2, proposition 2.4.3.

Example 1.5.5. Let C be a non-hyperelliptic, non-bielliptic curve of genus 3, then we know that for any line bundle $\xi \in J^1$, C can be mapped to the 3-dimensional projective space $\mathbb{P}(\xi)$ by the linear series $|K\xi^2|$. If ξ is generic, i.e., $h^0(C, \xi^2) = 0$ the map is an embedding of C as a smooth sextic in $\mathbb{P}^3$, which we still denote by C. Then projection from a general point $e \notin C$ to a

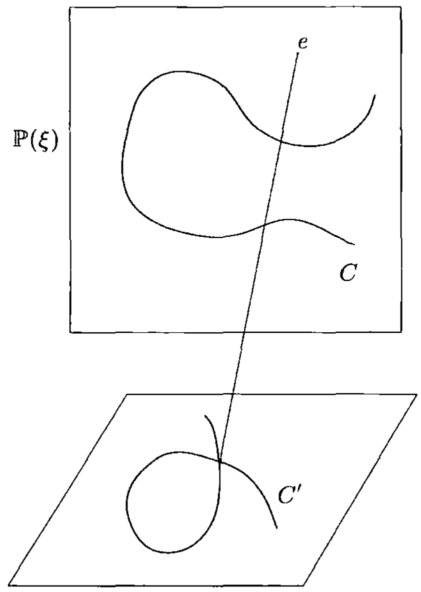

Figure 1.5.2: Bisecants of C through e give nodes of the projection C'.

plane not containing it will be 1 to 1 onto a plane sextic C' of genus 3 having r nodes. Clearly each simple node of C' corresponds to a bisecant of C passing through e and, by theorem 1.5.4, each of these gives a maximal line subbundle of $E = \varepsilon_\xi(e)$ distinct from ξ^{-1}. The number of nodes r can be computed explicitly using the genus formula for plane curves

$$g = \frac{(d-1)(d-2)}{2} - r.$$

As $d = 6$ and $g = 3$, it is clear that the general stable vector bundle will have 7 other distinct maximal line subbundles apart from ξ^{-1}.

Another possibility is that $h^0(C,\xi^2) = 1$ in which case $\xi^2 = \mathcal{O}(p+q)$ with $p,q \in C$ uniquely determined since $h^0(C,\xi^2) = 1$. In general p and q are distinct and C maps via ϕ_ξ to a sextic in $\mathbb{P}^3$, since the degree of $K\xi^2$ is 6, with a node given by $\phi_\xi(p) = \phi_\xi(q)$ (see [Gr], p.59). In this case the projection from a general point $e \notin C$ will still give 7 nodes one of which comes from the singularity of the image of C in $\mathbb{P}(\xi)$. However, by applying theorem 1.5.4 one can verify that the node coming from the singularity of C does not give a maximal line subbundle since e does not lie in the span $\overline{\phi_\xi(p)+\phi_\xi(q)}$: there exist planes H in $\mathbb{P}(\xi)$ that belong to $\mathbb{P}H^0(C,K)$ but don't pass through e. Hence in this case we obtain a total of 7 maximal line subbundles.

A similar thing happens when $p = q$, in which case $\xi = \mathcal{O}(p)$ and C maps to $\mathbb{P}(\xi)$ as a sextic with a cuspidal singularity, the projection from a general point e to a plane gives 6 nodes and a cusp and the corresponding stable vector bundle admits 7 maximal line bundles (see [L-N], lemma 5.2).

However, it is also of great interest to see if there are points of $\mathbb{P}(\xi)$ from which the projection is *not* general. A priori there are two other possibilities when projecting C

- C projects $2:1$ to a plane cubic;
- C projects $3:1$ to a plane conic.

The first case has, in fact, two subcases, either C projects to a smooth cubic of genus 1 or to a singular cubic of genus 0, however both can be ruled out by requiring C to be general; in particular we require C to be neither hyperelliptic nor bielliptic. It is easy to verify that the general smooth curve of genus 3 is not bielliptic by a parameter count since the space of bielliptic curves of genus 3 has dimension at most 4 while the moduli space of curves of genus 3, $\mathcal{M}_3$, has dimension 6.

As for the second case, C projects $3:1$ to a plane conic whenever it lies on a quadric cone whose generators are trisecants to the curve and it is projected from the vertex of the cone to a plane not containing it. In particular this means that C has a degree 3 pencil. Since C is a genus 3 curve all its degree 3 pencils are of the form $K(-p)$ for some choice of a point $p \in C$ (this can be verified by applying the Riemann-Roch formula to any degree 3 line bundle L on C satisfying the requirement $h^0(C, L) = 2$). This implies that the required family of trisecants is given by the linear system $|K(-p)|$ (see [Gr], lemma 4.1.2 and remark 4.4.8). Hence, if C, as a curve in $\mathbb{P}(\xi)$, lies on such a quadric cone, the projection π_e from the vertex e of the cone satisfies:

$$\pi_e^*\mathcal{O}(1) = L \qquad \pi_e^*\mathcal{O}(2) = K\xi^2$$

and this implies that $\xi^2 = K(-2p)$, i.e., $\xi = \kappa(-p)$ with $\kappa \in \vartheta(C)$ a theta characteristic. Conversely, if $\xi = \kappa(-p)$ then C always lies on a quadric cone whose vertex e is not on C and whose generators are trisecants (see [Gr], proposition 4.4.7, p.59). Note moreover that $\xi = \kappa(-p)$ implies $h^0(C, \xi^2) > 0$, hence C maps in $\mathbb{P}(\xi)$ as a singular sextic.

Summarising, if $\xi = \kappa(-p)$ with $\kappa \in \vartheta(C)$ and $p \in C$, then there exists exactly one point e, namely the vertex of the quadric cone on which C lies, such that the projection π_e is $3:1$, i.e., such that the corresponding stable bundle E has infinitely many maximal line subbundles. Given a non-hyperelliptic, non-bielliptic curve C of genus 3, this is the only case when a stable vector bundle admits infinitely many maximal line subbundles.

Chapter 2

The Coble quartic

The aim of this chapter is to review the construction of the natural embedding of $\mathcal{SU}_C(2)$ in $|2\Theta|$, as originally described by Narasimhan and Ramanan in [N-R2] for non-hyperelliptic curves of genus 3. The motivation for reconsidering Narasimhan and Ramanan's results, in particular the embedding of $\mathcal{SU}_C(2)$ as the Coble quartic shown in [N-R2], is strong. [N-R2], together with [N-R1], is a seminal paper in the study of the moduli space $\mathcal{SU}_C(2)$ and many results and generalisations for higher genus curves rest on some of the ideas presented in it (see [Br-V] and [vG-I]). However, the original proof presents a number of gaps and inaccuracies, which successive works have not clarified. Hence, the work presented in this chapter provides a complete and detailed analysis of §§4 and 5 of [N-R2], highlighting the relation between the ideas of Narasimhan and Ramanan and the geometry of the genus 3 case (see for example lemma 2.3.10, which holds only for curves of genus 3). Moreover, it gives a proof of the central *lemma* 5.2 of [N-R2]. Since the hint provided in [N-R2] does not appear to lead to any clear understanding of how to obtain the required result I propose an independent proof, which still maintains the point of view of the work of Narasimhan and Ramanan. Throughout the chapter I have introduced lemmas and remarks which should help the reader in understanding the ideas

behind [N-R2]. In addition, the paper presents a few mistakes, in particular the authors claim that the result relies on *lemma* 5.4. However I prove that that lemma is false by producing a 3-dimensional subvariety of $\mathcal{SU}_C(2)$ that does not satisfy the claim, see proposition 2.4.3. This affects some of the original statements, where this is the case I prove more general results than those of [N-R2] (compare *lemma* 5.1 with lemma 2.3.4 of this thesis).
Finally note that even the computation of the degree of $\mathcal{SU}_C(2)$ as a subvariety of $|2\Theta|$ as presented in [N-R2] is inaccurate, however corrections to this have been pointed out by Oxbury and Pauly in [O-P], §7.

Throughout the chapter we maintain notations and conventions introduced at the outset of chapter 1.

2.1 The map δ from $\mathcal{SU}_C(2)$ to $|2\Theta|$

Given a smooth curve C of genus $g \geq 2$, the morphism $\delta : \mathcal{SU}_C(2) \longrightarrow |2\Theta|$ is defined in the following way. Consider the product variety $C \times J^{g-1}$, denote by p_1 and p_2 the projections to the first and second factor, respectively, and let $\mathcal{P}$ be a Poincaré line bundle on $C \times J^{g-1}$ (see [ACGH], p.166).

$$\begin{tikzcd} \mathcal{P} \arrow[r] & C \times J^{g-1} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C & & J^{g-1} \end{tikzcd}$$

Then, for any semi-stable vector bundle $E \in \mathcal{SU}_C(2)$, $\delta(E) \stackrel{def}{=} \delta_E$ is the determinantal divisor associated to $\mathcal{P} \otimes p_1^* E$.

Here we briefly review the construction of determinantal divisors in this context, as done by Raynaud (see [Ra], p.109). Fix $E \in \mathcal{SU}_C(2)$ and consider the total direct image $Rp_{2*}(\mathcal{P} \otimes p_1^* E)$. All direct images of order greater than 1 are zero and we can find a perfect complex $\mathcal{M}$ of length 2 such that ([Ra], p.109)

$$0 \longrightarrow p_{2*}(\mathcal{P} \otimes p_1^* E) \longrightarrow M_0 \xrightarrow{u} M_1 \longrightarrow R^1 p_{2*}(\mathcal{P} \otimes p_1^* E) \longrightarrow 0$$

where M_0 and M_1 are vector bundles over J^{g-1}. Note that by the Riemann-Roch formula for vector bundles, the Euler characteristic of $E \otimes \zeta$, $\chi(E \otimes \zeta)$, is zero for any line bundle $\zeta \in J^{g-1}$. Hence, M_0 and M_1 must have the same rank. Moreover, if ζ is generic $h^0(C, E \otimes \zeta) = h^1(C, E \otimes \zeta) = 0$ as a consequence of [Ra], proposition 1.6.2 - Raynaud actually proves the result for any $F \in \mathcal{SU}_C(2, K)$ and general $\xi \in J^0$ but it is enough to select a theta characteristic κ and translate by it so that $F = E \otimes \kappa$ and $\xi = \zeta \kappa^{-1}$. Hence the map u is generically a bijection and one can consider its determinant, $det(u)$. The determinant $det(u)$ defines an effective divisor on J^{g-1} called the determinantal divisor and usually denoted $det(Rp_{2*}(\mathcal{P} \otimes p_1^* E))$ for any $E \in \mathcal{SU}_C(2)$. The support of this divisor is given by those line bundles $\zeta \in J^{g-1}$ for which $h^0(C, \zeta \otimes E) \geq 1$. This definition is independent of the representative in the S-equivalence class of E and of the choice of complex $\mathcal{M}$. Thus for any $E \in \mathcal{SU}_C(2)$, $\delta(E) \stackrel{def}{=} \delta_E$ is the determinantal divisor $det(Rp_{2*}(\mathcal{P} \otimes p_1^* E))$. Moreover Raynaud proves (see [Ra], proposition 1.8.1) that δ_E is linearly equivalent to 2Θ for all $E \in \mathcal{SU}_C(2)$ and so δ is a well defined map. One can easily verify this when E is a semi-stable, non stable vector bundle; in fact in this case E is S-equivalent to $\eta^{-1} \oplus \eta$ for some $\eta \in J^0$ by remark 1.2.1, and $\delta_E = \delta_{\eta^{-1} \oplus \eta} = \Theta_{\eta^{-1}} + \Theta_\eta$ since $h^0(C, (\eta \oplus \eta^{-1}) \otimes \zeta) > 0$ if and only if $h^0(C, \eta \otimes \zeta) > 0$ or $h^0(C, \eta^{-1} \otimes \zeta) > 0$. In particular this shows that (S-equivalence classes of) semi-stable bundles map to split 2θ divisors.

For curves of genus 2, Narasimhan and Ramanan proved in [N-R1] that δ is an isomorphism and that $\mathcal{SU}_C(2)$ is smooth.

Remark 2.1.1. The map δ has been defined for curves of all positive genera and is well known to be an embedding for every non-hyperelliptic curve of genus at least 3 as a consequence of works by Laszlo, Brivio and Verra, van Geemen and Izadi, and many others (see [Br-V] and [vG-I]). Yet, these approaches all tackle the problem from a different prospective and use more general techniques, while the approach of Narasimhan and Ramanan is based on the specific geometry of genus 3 curves and precisely for this reason it is interesting to understand it in more detail.

Just like in [N-R2], we start the study of the map δ by showing it is an injective morphism. Before one can prove this result it is necessary to consider some lemmas and construct a vector bundle over J^1. First we consider a few results that hold for curves of any genus $g \geq 2$.

Fix a smooth curve C of genus $g \geq 2$, for any line bundle $\xi \in J^{g-2}$ one can consider an embedding $\alpha_\xi : C \longrightarrow J^{g-1}$ given by $p \longmapsto \mathcal{O}(p) \otimes \xi = \xi(p)$, the image of which is denoted C_ξ. Similarly, for every degree 1 line bundle η one can consider an embedding $\beta_\eta : C \longrightarrow J^0$ defined by $p \longmapsto \mathcal{O}(p) \otimes \eta^{-1} = \eta^{-1}(p)$, the image of which is denoted $C_{\eta^{-1}}$. One can then consider the restriction of 2Θ along C_ξ, or equivalently, its pull-back to C via the above morphism α_ξ. Note that the following result is well known, but a proof is given for the convenience of the reader since no complete reference is available.

Lemma 2.1.2. *For any line bundle $\xi \in J^{g-2}$, the restriction of 2Θ to C_ξ can be identified with the line bundle $K^2\xi^{-2}$ of C.*

Proof. Fix a theta characteristic $\kappa \in \vartheta(C)$ and take $\eta = \kappa\xi^{-1}$ a line bundle of degree 1, then consider the commutative diagram

$$\begin{tikzcd} & J^1 \arrow[dr, "\otimes\xi"] & \\ C \arrow[ur, "u_1"] \arrow[dr, "\beta_\eta"'] & & J^{g-1} \\ & J^0 \arrow[ur, "\otimes\kappa"'] & \end{tikzcd}$$

where the map u_1 is the Abel-Jacobi map $p \longmapsto \mathcal{O}(p)$ and the composition of the two upper maps is just α_ξ. Note that it is enough to show that the pull-back of Θ

via α_ξ is isomorphic to $K\xi^{-1}$. So we start by showing that the two line bundles have the same degree, i.e., that deg $\Theta|_{C_\xi}$ is equal to deg $K\xi^{-1} = g$. The degree of $\Theta|_{C_\xi}$ can easily be computed using Poincaré's formula (see [ACGH], chapter 1, p.25), once it has been noticed that the degree is given by the intersection number of $\Theta \cong W_{g-1}$ with the translate of $W_1 \subset J^1$ in J^{g-1}. Hence

$$\deg \Theta|_{C_\xi} = [\Theta]\cdot[W_1] = [\Theta]\cdot\frac{[\Theta]^{g-1}}{(g-1)!} = \frac{[\Theta]^g}{(g-1)!} = g$$

where we use [ ] to denote the fundamental class of a subvariety and we recall that since Θ is a principal polarisation the integral of its first Chern class satisfies $\int \theta^g = g!$ (see §1.1.1).

If ξ is general, i.e., restricting to an open set in $Pic^1(C)$, one can assume $h^0(C,\xi) = 0$, in which case $h^0(C, K\xi^{-1}) = 1$ by the Riemann-Roch formula and there exists a unique degree g effective divisor D on C such that $K\xi^{-1} \cong \mathcal{O}(D)$. Moreover by requiring ξ to be general we can also assume that every point in the support of the divisor D has multiplicity one. Hence $\Theta|_{C_\xi}$ is isomorphic to $K\xi^{-1}$ if and only if one can show that they have the same support

$$\xi(p) \in \operatorname{supp}\Theta \quad \overset{\alpha_\xi}{\Longleftrightarrow} \quad p \in \operatorname{supp} D.$$

On the one hand, if $\xi(p)$ belongs to the support of Θ, then by definition $h^0(C,\xi(p)) > 0$ and there exists an effective divisor D' of degree $g-1$ such that $\xi(p) \cong \mathcal{O}(D')$; in particular $p \notin \operatorname{supp} D'$ since $h^0(C,\xi) = 0$. Hence, $\mathcal{O}(D) = K(p - D')$, equivalently, $K(p) = \mathcal{O}(D + D')$, where deg $D + D' = 2g - 1$. By the Riemann-Roch formula, $h^0(C, \mathcal{O}(D + D')) = g$ and $h^0(C, \mathcal{O}(D + D' - p)) = h^0(C, K) = g$, so p is in the support of D. On the other hand if p is a point in the support of D, it is immediate to verify that $\xi(p)$ has a non zero global section and hence lies in the support of Θ.

If ξ is not general, consider the map:

$$\begin{array}{ccc} C \times J^{g-2} & \xrightarrow{m} & J^{g-1} \\ (q,\zeta) & \longmapsto & \zeta(q). \end{array}$$

Then $m^*(\Theta)$ can be thought of as a family of degree g line bundles over C. So we have a morphism $J^{g-2} \longrightarrow J^g$ and it is known, from the above part of the proof, that on a Zariski open set $S \subset J^{g-2}$ this map coincides with the map: $\zeta \longmapsto K\zeta^{-1}$. Hence the maps coincide everywhere on J^{g-2} and the result is proved. $\square$

Remark 2.1.3. It is easy to verify that for every line bundle $\eta \in J^1$, the restriction of $\mathcal{L}$ (see page 5 for the definition) to $C_{\eta^{-1}}$, i.e., the pull-back of $\mathcal{L}$ via β_η (as defined on page 26), is isomorphic to $K\eta^2$, by fixing a theta characteristic κ and applying the above result to $\xi = \kappa\eta^{-1}$.

The following lemma is just *lemma* 4.1 of [N-R2], but we present a more extensive proof.

Lemma 2.1.4. *For any line bundle $\eta \in J^1$ the embedding $\beta_\eta : C \longrightarrow J^0$ given by $p \longmapsto \eta^{-1}(p)$ induces a surjective map*

$$H^0(J^0, \mathcal{L}) \longrightarrow H^0(C, K\eta^2).$$

Proof. We already know, by lemma 2.1.2 and the following remark, that the map β_η induces a cohomology map $H^0(J^0, \mathcal{L}) \longrightarrow H^0(C, K\eta^2)$. The aim is to prove that the induced map $|\mathcal{L}| \to |K\eta^2| \cong \mathbb{P}^g$ is surjective. Consider the *Kummer* map

$$\begin{array}{ccc} J^{g-1} & \longrightarrow & |\mathcal{L}| \\ \zeta & \longmapsto & \Theta_\zeta + \Theta_{K\zeta^{-1}}. \end{array}$$

One can actually prove a stronger statement, i.e., the composed map $J^{g-1} \longrightarrow |K\eta^2| = \mathbb{P}^g$, given by $\zeta \longmapsto \Theta_\zeta + \Theta_{K\zeta^{-1}}|_{C_{\eta^{-1}}}$, is surjective.

Any divisor $D \in |K\eta^2|$ may be written as a sum of two divisors of degree g each, D' and D'' (notice that this can be done in only a finite number of ways). These divisors belong, respectively, to the classes of $K\eta\zeta^{-1}$ and $\eta\zeta$ for some line bundle $\zeta \in J^{g-1}$ (e.g. choose $\zeta = \eta^{-1}(D')$). Note that by requiring D to be general, i.e., by restricting attention to an open set of $|K\eta^2|$, one can assume that $h^0(C, K\eta\zeta^{-1}) = h^0(C, \eta\zeta) = 1$ so that one can identify the line bundle $K\eta\zeta^{-1}$ with the corresponding divisor D' and similarly for $\eta\zeta$ and D''. To verify that this is the case, note that the general degree g divisors satisfies D' satisfies the condition $h^0(C, \mathcal{O}(D')) = h^0(C, K\eta^2(-D')) = 1$, hence there is a Zariski open set in the space of degree g divisors on the curve C on which one can define a map to $|K\eta^2|$ by $D' \longmapsto D' + D''$, where D'' is the only element of $|K\eta^2(-D')|$; as this map is finite to one, the image is Zariski dense. Moreover, one can also assume that each point in the support of D', and D'', appears with multiplicity one. Then, to prove the statement, it is enough to show that the support of $\Theta_\zeta|_{C_{\eta^{-1}}}$ coincides with that of D' and similarly for $\Theta_{K\zeta^{-1}}|_{C_{\eta^{-1}}}$ and D''. However the support of $\Theta_\zeta|_{C_{\eta^{-1}}}$ can be identified with the set of points $p \in C$ such that the line bundles $\eta^{-1}(p)$ satisfy the condition that their translates by ζ have non-zero global sections, i.e., $h^0(C, \zeta\eta^{-1}(p)) > 0$ or, by the Serre duality and the Riemann-Roch formula, such that $K\zeta^{-1}\eta(-p)$ has non-zero global sections. This simply means that p is a point of the support of D'. Analogous reasoning works for $\Theta_{K\zeta^{-1}}|_{C_{\eta^{-1}}}$ and D''. Hence the map $|\mathcal{L}| \to |\,K\eta^2\,| \cong \mathbb{P}^g$ is surjective on its image set, a Zariski open, thus everywhere. □

Remark 2.1.5. Similar results hold also for 2Θ, i.e., for any line bundle $\xi \in J^{g-2}$ the embedding $\alpha_\xi : C \longrightarrow J^{g-1}$ given by $p \longmapsto \xi(p)$, has the property that the induced cohomology map $H^0(J^{g-1}, 2\Theta) \longrightarrow H^0(C, K^2\xi^{-2})$ is surjective.

2.2 The genus 3 case

From now on we will restrict our attention exclusively to non-hyperelliptic, non-bielliptic curves of **genus 3**. However, throughout the chapter we will use results that are valid also for curves of higher genus, these results are all presented in the following chapter 3 and here are only quoted in their form for curves of genus 3.

The above lemmas imply that for all line bundles ξ, η in J^1 there exist short exact sequences

$$0 \longrightarrow \mathcal{K}_1(\eta) \longrightarrow H^0(J^0, \mathcal{L}) \xrightarrow{\beta_\eta^*} H^0(C, K\eta^2) \longrightarrow 0 \tag{2.2.1}$$

$$0 \longrightarrow \mathcal{K}_2(\xi) \longrightarrow H^0(J^2, 2\Theta) \xrightarrow{\alpha_\eta^*} H^0(C, K^2\xi^{-2}) \longrightarrow 0. \tag{2.2.2}$$

In particular, if $\eta = \kappa\xi^{-1}$, where κ is a fixed theta characteristic, the two sequences are isomorphic. Moreover, from what was said above, it is natural to identify $\mathcal{K}_2(\xi)$ with the space $H^0(J^2, \mathcal{I}_{C_\xi}(2\Theta))$, where $\mathcal{I}_{C_\xi}$ is the ideal sheaf of C_ξ, i.e., with the space of 2Θ sections which vanish along C_ξ. A similar identification holds also for $\mathcal{K}_1(\eta)$. We now consider the dual of sequence (2.2.1)

$$0 \longrightarrow H^0(C, K\eta^2)^* \longrightarrow H^0(J^0, \mathcal{L})^* \longrightarrow \mathcal{K}_1(\eta)^* \longrightarrow 0 \tag{2.2.3}$$

and compare it to (2.2.2). When $\eta = \xi$, Oxbury and Pauly have shown in [O-P], §7, that by identifying $H^0(J^0, \mathcal{L})^*$ with $H^0(J^2, 2\Theta)$ via Wirtinger duality, the two sequences are canonically isomorphic. In particular $H^0(C, K\xi^2)^*$ is isomorphic to $H^0(J^2, \mathcal{I}_{C_\xi}(2\Theta))$ (see proposition 3.1.6). By allowing ξ to vary in J^1, Narasimhan and Ramanan obtain a bundle Q with fibre $H^0(C, K\xi^2)^*$ at ξ and, hence, of rank 4. This vector bundle fits into a short exact sequence

$$0 \longrightarrow Q \longrightarrow H^0(J^2, 2\Theta) \otimes \mathcal{O}_{J^1} \longrightarrow N \longrightarrow 0 \tag{2.2.4}$$

where N is the cokernel and has fibre $H^0(C, K^2\xi^{-2})$ (see a review of Oxbury and Pauly's formal construction in §3.1). Finally, note that the projectivised vector bundle $\mathbb{P}Q$ associated to Q comes with a map $f\colon \mathbb{P}Q \longrightarrow |2\Theta|$, which is the projectivisation of the embedding $Q \longrightarrow H^0(J^2, 2\Theta) \otimes \mathcal{O}_{J^1}$ composed with the natural projection to $|2\Theta|$. By denoting f_ξ the restriction of f to the fibre $\mathbb{P}Q_\xi$ we have an injective map

$$f_\xi : \mathbb{P}Q_\xi \longrightarrow |2\Theta| \tag{2.2.5}$$

which is linear since it is just the projectivisation of a linear map of vector spaces. Hence, $\mathbb{P}Q_\xi$ maps isomorphically to a 3-dimensional subspace of $|2\Theta| \cong \mathbb{P}^7$.

By the Serre duality $\mathbb{P}Q_\xi = \mathbb{P}H^0(C, K\xi^2)^*$ is isomorphic to the extension space $\mathbb{P}(\xi) = \mathbb{P}H^1(C, \xi^{-2})$ and in this sense it has a natural linear map to $\mathcal{SU}_C(2)$, the extension map described in §1.5 and denoted by ε_ξ. The following lemma considers the composition of ε_ξ with δ and compares it to f_ξ. It was originally stated in [N-R2], but not proved (see *lemma* 4.3).

Lemma 2.2.1. *Given any line bundle $\xi \in J^1$, the diagram*

$$\begin{tikzcd} & \mathcal{SU}_C(2) \arrow[d, "\delta"] \\ \mathbb{P}(\xi) \arrow[ur, "\varepsilon_\xi"] \arrow[r, "f_\xi"'] & {|2\Theta|} \end{tikzcd} \tag{2.2.6}$$

commutes.

Proof. The proof of this lemma relies on the fact that all the maps in the diagram are linear. This has already been shown for ε_ξ in §1.5 and for f_ξ above. As for the map δ linearity means proving that the pullback of $\mathcal{O}_{|2\Theta|}(1)$ via δ is $\tilde{\Theta}^0$, the generalised theta divisor, which is the generator of the Picard group of $\mathcal{SU}_C(2)$ (see §1.2). This has been shown to be true by Beauville ([B1], lemma 2.3). Now consider the diagram

$$\begin{tikzcd} & \mathbb{P}(\xi) \arrow[r, "\varepsilon_\xi"] \arrow[d, equal, "\text{Serre duality}"] & \mathcal{SU}_C(2) \arrow[r, "\delta"] & {|2\Theta|} \arrow[d, equal, "\text{Wirtinger duality}"] \\ C \arrow[ur] \arrow[r] & \mathbb{P}H^0(C, K\xi^2)^* \arrow[rr, "g_\xi^*"] & & {|\mathcal{L}|^*} \end{tikzcd}$$

where g_ξ^* denotes the map obtained by composing f_ξ of diagram (2.2.5) with the Wirtinger duality. Since the image of C in $\mathbb{P}Q_\xi$ via the regular map $\phi_{|K\xi^2|}$ is non degenerate and all the maps involved are linear, it is enough to prove that $\delta \circ \varepsilon_\xi$ and g_ξ^* agree on the spanning set given by the image of C. We know that the upper line is given by $p \longmapsto \Theta_{\xi(-p)} + \Theta_{\xi^{-1}(p)}$, see remark 1.5.2. As for the lower part of the diagram, a point $p \in C$ is mapped to $\mathbb{P}H^0(C, K\xi^2)^*$ by the linear series $|\, K\xi^2\,|$; its image is the hyperplane of divisors linearly equivalent to $K\xi^2$ which contain p in their support. Now consider the map

$$g_\xi : |\mathcal{L}| \longrightarrow \mathbb{P}H^0(C, K\xi^2)$$

which is the dual of g_ξ^* and is surjective by lemma 2.1.4. The map g_ξ is such that the hyperplane $|\, K\xi^2(-p)\,|$ pulls back to the hyperplane in $|\mathcal{L}|$ of divisors containing $\xi(-p)$ in their support, which we will denote by $|\, \mathcal{L}_{-\xi(-p)}\,|$. It has already been shown in the proof of corollary 1.1.4 that $|\, \mathcal{L}_{-\eta}\,|$ can be identified with $\Theta_\eta + \Theta_{\eta^{-1}}$, for any $\eta \in J^0$, via Wirtinger duality (1.1.2). Hence the two maps can be identified. $\square$

Remark 2.2.2. It is worth noticing that Beauville's proof of the linearity of δ is consistent with the point of view presented in this thesis. In fact, though he mentions the results of Narasimhan and Ramanan in [N-R2], he makes no use of them in his detailed proof of the result. We refer the reader to §§1 and 2 of [B1] for the construction and proof.

The above lemma implies that if E is a vector bundle in $\mathcal{SU}_C(2)$ coming, as an extension, from an element $e \in \mathbb{P}(\xi)$, then $f_\xi(e) = \delta_E$.

Lemma 2.2.3. *Let ξ be a degree one line bundle. A 2θ divisor D contains a translate C_ξ in its support if and only if $D = \delta_E$ for some $E \in \mathcal{SU}_C(2)$ and $h^0(C, E\xi) > 0$.*

Proof. By the characterisation of sequence (2.2.2) we know that

$$f_\xi(\mathbb{P}(\xi)) = \{D \in |2\Theta| \,|\, C_\xi \subset \mathit{supp}\, D\}$$

while by commutativity of diagram (2.2.6) one also has

$$f_\xi(\mathbb{P}(\xi)) = \{\delta_E \,|\, E \in \mathcal{SU}_C(2) \text{ and } h^0(C, \xi \otimes E) > 0\}$$

from which we can deduce that δ_E contains C_ξ in its support if and only if $h^0(C, \xi \otimes E)$ is not zero. $\square$

It is then possible to prove injectivity of δ.

Theorem 2.2.4. *The morphism δ is injective.*

Proof. Let E and F be two vector bundles in $\mathcal{SU}_C(2)$ such that $\delta_E = \delta_F$. One wants to show that they must lie in the same S-equivalence class.

First of all we prove that there exists $\xi \in J^1$ such that $h^0(C, \xi \otimes E) \neq 0$. In fact, there is a bound for the degree of a maximal line subbundle η of a vector bundle $E \in \mathcal{SU}_C(2)$ over a curve C of genus g, $[(g-1)/2] - g + 1 \leq \deg \eta \leq 0$ where the upper bound is reached only when E is a semi-stable, non stable vector bundle (see [O], p.10). Since we are working with curves of genus 3, if $E \in \mathcal{SU}_C(2)$ is stable, it must have a maximal line subbundle η of degree -1, hence $h^0(C, \eta^{-1} \otimes E) > 0$ by lemma 1.3.4 and $\xi = \eta^{-1}$ is the required line bundle in J^1. While if E is semi-stable, non stable it admits a maximal line subbundle $\zeta \in J^0$ and hence $\xi = \zeta^{-1}(q)$ will do for any $q \in C$.

The fact that there exists a line bundle ξ such that $h^0(C, \xi \otimes E) \neq 0$ implies that $h^0(C, E \otimes \xi(p)) \neq 0$ for every point $p \in C$, i.e., $C_\xi \subset supp\,\delta_E = supp\,\delta_F$. It also implies that E is an extension of ξ by ξ^{-1}, i.e., there exists $e \in \mathbb{P}(\xi)$ such that $\varepsilon_\xi(e) = [E]$ by lemma 1.5.3 and $f_\xi(e) = \delta_E$ by the commutativity of diagram (2.2.6) in lemma 2.2.1. Note, moreover, that F satisfies the assumption of the previous lemma and hence $H^0(C, \xi \otimes F) \neq 0$, so by lemma 1.5.3 there exists an e' such that $[F] = \varepsilon_\xi(e')$. In turn this shows that $f_\xi(e) = f_\xi(e')$, but it is known that f_ξ is injective (see its definition on page 30), hence $e = e'$. $\square$

Remark 2.2.5. The fact that for every semi-stable vector bundle $E \in \mathcal{SU}_C(2)$ there exists a line bundle $\xi \in J^1$ such that $h^0(C, \xi \otimes E) \neq 0$ also implies that the map $\varepsilon\colon \mathbb{P}Q \longrightarrow \mathcal{SU}_C(2)$ previously defined is surjective.

Remark 2.2.6. The proof of theorem 2.2.4 has essentially the same argument as the one given by Narasimhan and Ramanan for curves of genus 2 in [N-R1]. Note, moreover that it fails to be true for curves of higher genus since, in general, not every stable vector bundle admits a maximal line subbundle of degree -1.

2.3 δ is an embedding

In the following I will give a detailed proof of the fact that δ is an embedding. As previously stated, I will prove all the results of [N-R2], §5, with the exception of *lemma* 5.4 which is not true and for which I will provide a counterexample in the next section. However, it turns out that *lemma* 5.4 is not necessary to show that δ is an embedding. The layout of this section broadly follows that of [N-R2], §5, yet parts from it in order to explain in detail the construction, introduce additional lemmas and prove *lemma* 5.2, for which an original approach is given.

As Narasimhan and Ramanan noticed, the proof follows from remarking that the statements in the previous section imply the commutativity of the diagram

$$\begin{tikzcd} & \mathcal{SU}_C(2) \arrow[d, "\delta"] \\ \mathbb{P}Q \arrow[ur, "\varepsilon"] \arrow[r, "f"'] & {|2\Theta|} \end{tikzcd} \tag{2.3.1}$$

and that injectivity of the differential of δ follows from proving

The image G of $\mathcal{SU}_C(2)$ in $|2\Theta|$ is a normal variety.

In fact, once it has been proved that G is a normal variety, as a consequence we have

Theorem 2.3.1. *The map $\delta : \mathcal{SU}_C(2) \longrightarrow |2\Theta|$ is an embedding for any non-hyperelliptic curve of genus 3.*

Proof. δ is an injective morphism which is a bijection onto its image G, a normal variety, hence by Zariski's *Main Theorem* (see [M1], p.209) it must be an isomorphism. $\square$

To prove normality of G, it would be enough to know that

> *The differential of f is injective at every point in $\mathbb{P}Q$ lying over a stable bundle $E \in \mathcal{SU}_C(2)$.*

Assuming this statement, normality of G is easily shown.

Theorem 2.3.2. *$G = \delta(\mathcal{SU}_C(2))$ is a normal variety.*

Proof. First of all note that since $\mathcal{SU}_C(2)$ is a variety of dimension 6 mapping injectively to $|2\Theta| \cong \mathbb{P}^7$, G is a hypersurface, hence a complete intersection. Recall that a complete intersection is normal if and only if its singular locus has codimension at least 2. Assume G is singular along a divisor D and let $D' = \delta^{-1}(D) \subset \mathcal{SU}_C(2)$. As $\mathcal{SU}_C(2)$ is singular exactly along the 3 dimensional Kummer subvariety of semi-stable, non stable vector bundles, D' must be smooth at its generic point. In order to have a *singular image*, $d\delta$ must fail to have maximal rank and, by the commutativity of diagram (2.3.1), this would imply that df fails to have maximal rank at the generic point of $D'' = \varepsilon^{-1}(D') \subset \mathbb{P}Q$ (note that $D'' \neq \varnothing$ since by remark 2.2.5 ε is surjective). However, if the differential of f is injective at every point in $\mathbb{P}Q$ that lies over a stable vector bundle, it means that df is injective at the generic point of D'', since the subvariety of $\mathbb{P}Q$ mapping to non stable bundles has codimension at least 2 (see example 1.5.5 in the previous chapter). Hence G can't be singular along a divisor. $\square$

Before we can proceed we need to recall some notions given in section §1.5 of the previous chapter. When C is a non-hyperelliptic, non-bielliptic curve of genus 3, we have seen that if $\xi \in J^1$ is a line bundle of the form $\kappa(-p)$ where κ is a theta characteristic and $p \in C$, then the extension space $\mathbb{P}(\xi) \cong \mathbb{P}Q_\xi$ has a unique point e that maps to a stable vector bundle with infinitely many maximal line subbundles (see page 22). Moreover we saw that this is the only case in which the extension map ε maps to a stable vector bundle with infinitely many subbundles. We will characterise these vector bundles in more detail at the end of this chapter, in §2.4. For the time being it is enough to know that there are only finitely many of them, one for every choice of a theta characteristic κ, and that if E_κ is such a vector bundle its maximal line subbundles are $\{\kappa^{-1}(p),\, p \in C\}$.

Definition 2.3.3. *Throughout the chapter we will say that a point $e \in \mathbb{P}Q$ satisfies **property** $(\star)$ if it does not map to one of the stable vector bundles E_κ via the extension map ε. Note that the locus of points of $\mathbb{P}Q$ not satisfying $(\star)$ has codimension* 5.

We now return to the diagram (2.3.1) and the study of the differential of the map f. The result one can prove is the following.

Lemma 2.3.4. *The differential of $f : \mathbb{P}Q \longrightarrow \mathcal{SU}_C(2)$ is injective at every point $e \in \mathbb{P}Q$ lying over a stable bundle E and satisfying property $(\star)$.*

Note that the statement of lemma 2.3.4 differs from the analogous one present in [N-R2], in fact as already noticed the locus of points where we fail to show injectivity for *df* is smaller than the locus where Narasimhan and Ramanan fail to prove injectivity (see *lemma* 5.1 of [N-R2]). In particular, while in both cases one does not expect the differential to be injective at points that map to semi-stable, non stable vector bundles, i.e., to the 3-dimensional Kummer variety of $\mathcal{SU}_C(2)$, outside this locus I show that *df* is not injective at points that don't satisfy property $(\star)$, a locus of codimension 5 compared to the locus of codimension 2 of fibres $\mathbb{P}Q_\xi$ with $\xi^2 \in \Theta$ that Narasimhan and Ramanan exclude. Moreover, the proof that G is a normal variety is not affected, since it is still true that the set of points where df is not injective has codimension at least 2 in $\mathbb{P}Q$.

The proof of lemma 2.3.4 is very long and will occupy the rest of this section. One starts by noting that to the vector bundle Q is associated a map ϕ from the Jacobian J^1 into the Grassmannian of 4-dimensional sub-spaces of $H^0(J^2, 2\Theta)$, denoted $G_4(2\Theta)$, and that one has the following canonical diagram

$$\begin{tikzcd} \mathbb{P}Q \arrow[r, "\iota"] \arrow[d, "\pi"] & \mathbb{P}U \arrow[r, "f'"] \arrow[d, "\pi'"] & {|2\Theta|} \\ J^1 \arrow[r, "\phi"'] & G_4(2\Theta) \end{tikzcd} \tag{2.3.2}$$

where the bottom map sends ξ to Q_ξ and $\mathbb{P}U$ is the projectivised universal subbundle of the Grassmannian, while our map f is just the composition $f' \circ \iota$ and π, π' are the structural projections. In fact the left hand box is just a fibred square.

This is a general construction. Let V be an n-dimensional vector space, $k \leq n$ a positive integer and $G_k(V)$ the Grassmannian of k-dimensional subspaces of V. Let $\mathbb{P}U \subset \mathbb{P}V \times G_k(V)$ be the projectivised universal bundle over $G_k(V)$, then one can consider the following diagram where π and f' are the natural projections

$$\begin{tikzcd} \mathbb{P}U \arrow[r, "f'"] \arrow[d, "\pi'"] & \mathbb{P}V \\ G_k(V) \end{tikzcd} \tag{2.3.3}$$

By construction f' is linear when restricted to a fibre $\mathbb{P}U_x$ of $\mathbb{P}U$.

We start by studying df' at a general point $e \in \mathbb{P}U$. Let $e \in \mathbb{P}U$, $x = \pi'(e)$ and $p = f'(e)$, where p is also used to indicate the corresponding line in V. Note that at a point $e \in \mathbb{P}U$ the tangent space $T_e\mathbb{P}U$ naturally splits in a direct sum, $T_e\mathbb{P}U = T_xG_k(V) \oplus T_e^{vert}\mathbb{P}U$, where $T_e^{vert}\mathbb{P}U$ is the tangent space to the fibre ${\pi'}^{-1}(x)$ and can be identified with the fibre of U over x. Then, clearly, if $v \in T_e\mathbb{P}U$ lies in $T_e^{vert}\mathbb{P}U$ one has $df'_e(v) \neq 0$ (since f' is linear on each fibre $\mathbb{P}U_x$), while $d\pi'_e(v) = 0$. Note, moreover, that this is the only case when $d\pi'_e(v)$ can be zero.

Before proceeding we need to recall the characterisation of the tangent space to a Grassmannian at a point x. If $x \in X$ is any closed point of a scheme X over $\mathbb{C}$ the tangent space T_xX can be described as the set of morphisms $\mathcal{M}_x(Spec\, D, X)$ where $D = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ is called the algebra of dual numbers and one allows in $\mathcal{M}_x$ only the morphisms that map the closed point of $Spec\, D$ to the point $x \in X$ (see [Sh], chapter V, §3.4). However in the context of this thesis it will be more useful to identify the tangent space to the Grassmannian $G_k(V)$ at a point x with the open set $Hom(U_x, V/U_x)$ of $G_k(V)$, where U_x is the k-dimensional subspace of V parametrised by x. The identification of $\mathcal{M}_x(Spec\, D, G_k(V))$ with $Hom(U_x, V/U_x)$ can be found in [Sh], example 3, page 100. In the following a descriptive interpretation of this identification is presented (see [H], pp.200-201). Recall that the sets of the form $Hom(U_x, V/U_x)$ give an affine covering of $G_k(V)$. The identification can be viewed in the following way. Fix any vector $v \in T_xG_k(V)$, let Δ be a complex disc centred at the origin and let γ: $\Delta \longrightarrow G_k(V)$ be a holomorphic arc such that $\gamma(0) = x$ and $\frac{d}{dt}\gamma(t)|_{t=0} = v$. We can denote $\gamma(t)$ as x_t for every $t \in \Delta$, in particular $x_0 = x$. Select any point e in the space U_x. Then, for any choice of a holomorphic arc $\{e_t\} \subset U$ such that $e_t \in U_{x_t}$ and $e_0 = e$, we can compute $e'(0) = \frac{d}{dt}(e_t)|_{t=0}$. This vector is not unique, but it is easy to check that it is unique modulo U_x and can be thought of as a vector in V/U_x. Hence every tangent vector v to the Grassmannian at a point x, can be thought of as a morphism $U_x \longrightarrow V/U_x$ such that $e \longmapsto e'(0)$. In particular $d\pi'_e(v)$ is such a morphism.

Lemma 2.3.5. *With the above hypotheses, let $v \in T_e\mathbb{P}U$ be such that $d\pi'_e(v) \neq$*

0. *Then*

$$v \in \mathcal{K}er\, df'_e \quad \text{if and only if} \quad p \subset \mathcal{K}er\, d\pi'_e(v).$$

Proof. First of all note that $v \in \mathcal{K}er\, df'_e$ if and only if it lies in the tangent space to the fibre ${f'}^{-1}(p)$ at e. Since $d\pi'_e(v) \neq 0$, this is the same as saying that $d\pi'_e(v) \in T_x(X_p)$, where $X_p = \{k\text{-dimensional subspaces of V containing p}\}$ is a fixed subspace of $G_k(V)$. As the above is a condition on the tangent space of X_p, we can restrict X_p to its intersection with any affine open set in $G_k(V)$ containing x, in particular we can consider $Hom(U_x, V/U_x)$, where U_x is the fibre of U containing e since $x = \pi'(e)$. We denote this restriction also by X_p. In this way, $X_p = \{\alpha \in Hom(U_x, V/U_x) \,: p \subset \mathcal{K}er\, \alpha\}$ and can be identified with its tangent space at x. In particular, this happens if and only if $d\pi'_e(v) \in X_p$, i.e., $p \subset \mathcal{K}er\, d\pi'_e(v)$. $\square$

Now, let X be a complex manifold with a morphism ϕ: $X \to G_k(V)$, this defines a rank k bundle over X as $W = \phi^*(U)$. Then one has the following commutative diagram

$$\begin{tikzcd} \mathbb{P}W \arrow[r, "\iota"] \arrow[d, "\pi"] & \mathbb{P}U \arrow[r, "f'"] \arrow[d, "\pi'"] & \mathbb{P}V \\ X \arrow[r, "\phi"'] & G_k(V) & \end{tikzcd} \qquad (2.3.4)$$

Consider a point $e \in \mathbb{P}W$ and the composition $f = f' \circ \iota$, we are interested in studying the differential df_e of f at e. Let $x = \pi(e) \in X$, $e' = \iota(e) \in \mathbb{P}U$ and $p = f(e) = f'(e') \in \mathbb{P}V$ is identified with the corresponding line in V. Consider a vector $v \in T_e\mathbb{P}W$, again note that $v \in T_e^{vert}\mathbb{P}W$ if and only if $d\pi_e(v) = 0$, i.e, it is tangent to the fibre $\mathbb{P}W_x$, yet in this case $df_e(v)$ is certainly not zero. Moreover, given $v \in T_e\mathbb{P}W$ such that $d\pi_e(v) \neq 0$, one can consider $d\phi_x \circ d\pi_e(v) \in T_{\phi(x)}G_k(V) \cong Hom(W_x, V/W_x)$. Under these assumptions the following holds.

Lemma 2.3.6. *With the above notation, for every $e \in \mathbb{P}W_x$ and $v \in T_e\mathbb{P}W$ such that $d\pi_e(v) \neq 0$*

$$v \in \mathcal{K}er\, df_e \quad \textit{if and only if} \quad p \subset \mathcal{K}er\, d\phi_x \circ d\pi_e(v).$$

Proof. Notice that since diagram (2.3.4) is commutative, i.e, $\pi' \circ \iota = \phi \circ \pi$, the following holds, $d\phi_x \circ d\pi_e(v) = d\pi'_{e'} \circ d\iota_e(v)$.

Now, let $w = d\iota_e(v) \in T_{e'}\mathbb{P}U$. Two cases can occur

- $w \neq 0$

  notice that $d\pi_e(v) \neq 0$ implies $d\pi'_{e'}(w) \neq 0$ and that, since $w = d\iota_e(v)$, $v \in \mathcal{K}er\, df_e$ if and only if $w \in \mathcal{K}er\, df'_e$. Hence one can apply the previous lemma to w, since it is a non-zero vector in $T_{e'}\mathbb{P}U$: $w \in \mathcal{K}er\, df'_e$ if and only if $p \subset \mathcal{K}er\, d\pi'_{e'}(w) = \mathcal{K}er\, d\pi'_{e'} \circ d\iota_e(v) = \mathcal{K}er\, d\phi_x \circ d\pi_e(v)$;

- $w = 0$

  in this case $df_e(v) = df'_{e'}(w) = 0$ and, from the commutativity noticed above, $d\phi_x \circ d\pi_e(v) = d\pi'_{e'} \circ d\iota_e(v) = d\pi'_{e'}(w)$ is the zero map, hence its kernel is the whole $W_x \subset V$.

Hence the statement is proved. □

Diagram (2.3.2) is just a special case of the situation presented in the above lemma. In order to study the differential of f: $\mathbb{P}Q \longrightarrow |2\Theta|$ at a point $e \in \mathbb{P}Q$ lying over a stable vector bundle, we study instead the differential of ϕ: $J^1 \longrightarrow G_4(2\Theta)$ at $\xi = \pi(e)$. One aims at proving that given a line bundle $\xi \in J^1$ the differential of f is injective at all points $e \in \mathbb{P}Q_\xi$ which satisfy property $(\star)$ (see definition 2.3.3).

It is then necessary to study the following differential map

$$T_\xi J^1 \cong H^0(C, K)^* \xrightarrow{d\phi_\xi} T_{\phi(\xi)}G_4(2\Theta) \cong Hom(Q_\xi, H^0(J^2, 2\Theta)/Q_\xi) \qquad (2.3.5)$$

where one knows that $Q_\xi = H^0(C, K\xi^2)^*$ (see page 30). Moreover, one can identify Q_ξ with $H^0(J^2, \mathcal{I}_{C_\xi}(2\Theta))$, where $\mathcal{I}_{C_\xi}$ represents the ideal sheaf of the embedded curve C_ξ in J^2 (see proposition 3.1.6). Equivalently Q_ξ can be thought of as the kernel of the surjective restriction map $H^0(J^2, 2\Theta) \longrightarrow H^0(C, K^2\xi^{-2})$ of remark 2.1.5, and the fibre $N_\xi = H^0(J^2, 2\Theta)/Q_\xi$ of sequence (2.2.4) is isomorphic to $H^0(C, K^2\xi^{-2})$, i.e., we have the following

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(J^2, \mathcal{I}_{C_\xi}(2\Theta)) & \longrightarrow & H^0(J^2, 2\Theta) & \longrightarrow & H^0(C, K^2\xi^{-2}) & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & Q_\xi & & & & N_\xi & & \end{array} \tag{2.3.6}$$

To study $d\phi_\xi$ we need to understand, for every tangent vector $v \in T_\xi J^1$, the associated map $d\phi_\xi(v) : Q_\xi \longrightarrow N_\xi$.

As suggested by Narasimhan and Ramanan, one can consider the natural short exact sequence

$$0 \to \mathcal{I}^2_{C_\xi} \longrightarrow \mathcal{I}_{C_\xi} \longrightarrow \mathcal{N}^* \to 0$$

where $\mathcal{N} \overset{def}{=} \mathcal{N}_{C_\xi/J^2}$ is the normal bundle of the image of C in J^2 via the map α_ξ and is just the cokernel of the map $\mathcal{I}^2_{C_\xi} \longrightarrow \mathcal{I}_{C_\xi}$. By tensoring with 2Θ we then obtain

$$0 \to \mathcal{I}^2_{C_\xi}(2\Theta) \longrightarrow \mathcal{I}_{C_\xi}(2\Theta) \longrightarrow \mathcal{N}^* \otimes K^2\xi^{-2} \to 0. \tag{2.3.7}$$

The study of the associated long exact sequence of cohomology provides results that are essential in proving that the differential of ϕ is injective. In particular the approach suggested in [N-R2] relies vitally on *lemma* 5.3 (see lemma 2.3.10), which holds only for curves of genus 3. However, before it is possible to prove it, it is necessary to introduce some lemmas. Though both lemma 2.3.7 and lemma 2.3.9 use well known ideas, no explicit reference is available, hence complete proofs are given.

Lemma 2.3.7. *Let E be a stable vector bundle with trivial determinant which is isomorphic to an extension of $\xi \in J^1$ by ξ^{-1}, then $h^0(C, E \otimes \xi(p)) = 1$ for general $p \in C$.*

Proof. By assumption, $\xi^{-1} \in J^{-1}$ is a maximal line subbundle of E, hence $h^0(C, E \otimes \xi) = 1$ (see remark 1.3.6) while the Riemann-Roch formula gives $h^1(C, E \otimes \xi) = 3$. By considering the short exact sequence

$$0 \to K\xi^{-1}(-p) \otimes E^\vee \longrightarrow K\xi^{-1} \otimes E^\vee \longrightarrow \mathbb{C}_p^2 \to 0 \tag{2.3.8}$$

where $E^\vee$ denotes the dual of E and the associated long exact sequence

$$0 \to H^0(C, K\xi^{-1}(-p) \otimes E^\vee) \longrightarrow H^0(C, K\xi^{-1} \otimes E^\vee) \xrightarrow{f_p} H^0(C, \mathbb{C}_p^2) \to \cdots$$

it is clear that the result would follow from f_p being surjective as $h^0(C, \mathbb{C}_p^2) = 2$ and $h^0(C, K\xi^{-1} \otimes E^\vee) = 3$ as by the Riemann-Roch formula and Serre duality one has $h^0(C, K\xi^{-1}(-p) \otimes E^\vee) = h^1(C, \xi(p) \otimes E) = h^0(C, \xi(p) \otimes E)$.

Assume that, for all $p \in C$, f_p is not surjective, then its rank can be at most 1. This means that the evaluation map $H^0(C, K\xi^{-1} \otimes E^\vee) \otimes \mathcal{O} \longrightarrow K\xi^{-1} \otimes E^\vee$ has as image a line subbundle $L \subset K\xi^{-1} \otimes E^\vee$, where $h^0(C, L) = 3$. By semi-stability $\deg L \leq \frac{1}{2}\deg(K\xi^{-1} \otimes E^\vee) = 3$. This contradicts Clifford's Theorem (see [ACGH], p.107) as $h^0(C, L) \neq 0$ implies $h^0(C, L) - 1 \leq \frac{1}{2}\deg L$, but it has just been shown that $h^0(C, L) = 3$ and $\deg L \leq 3$. $\square$

Remark 2.3.8. Note that every rank two vector bundle F is isomorphic to $F^\vee \otimes \det F$ (see [Ha], exercise 5.16 b, page 127). Hence, from now onward, we will not distinguish between a semi-stable rank two vector bundle with trivial determinant E and its dual $E^\vee$.

Lemma 2.3.9. *Let $E \in \mathcal{SU}_C(2)$ be a stable vector bundle isomorphic to an extension of $\xi \in J^1$ by ξ^{-1}. The associated divisor δ_E is smooth at $\xi(p)$ for general $p \in C$.*

Proof. The claim follows from a result of Laszlo (see [La], p.343)

$$mult_x \delta_E \geq h^0(C, E \otimes x) \qquad x \in supp\, \delta_E \subset J^{g-1}$$

with equality holding if and only if the *Brill-Noether* map

$$\mathcal{P}_x : \quad \begin{array}{ccc} H^0(C, E \otimes x) \otimes H^0(C, E \otimes Kx^{-1}) & \longrightarrow & H^0(C, K) \\ s \otimes t & \longmapsto & s \wedge t \end{array} \qquad (2.3.9)$$

is not identically zero.

By lemma 2.3.7, the Riemann-Roch formula and Serre duality it is known that, for general $p \in C$, $h^0(C, E \otimes K\xi^{-1}(-p)) = 1$, hence one can select a non-zero global section of the bundle $E \otimes K\xi^{-1}(-p)$, t_p. Note that this section is unique up to scalar. Consider the zero set of t_p. This is either the empty set or a divisor D_p in C, in this case, however, stability of E and hence of $E \otimes K\xi^{-1}(-p)$ imply that $deg\, D_p = 1$, i.e., D_p is a point of C. One can treat these two cases separately.

1. $(t_p) = \varnothing$

The section t_p induces a short exact sequence

$$0 \to \mathcal{O} \longrightarrow E \otimes K\xi^{-1}(-p) \longrightarrow K^2\xi^{-2}(-2p) \to 0$$

and by dualising one obtains

$$0 \to \xi^2(2p)K^{-1} \longrightarrow E \otimes \xi(p) \longrightarrow K \to 0.$$

Finally one can consider the long exact sequence associated to it

$$0 \to H^0(C, \xi^2(2p)K^{-1}) \longrightarrow H^0(C, E \otimes \xi(p)) \xrightarrow{\cup t_p} H^0(C, K) \to \dots$$

Hence, the map (2.3.9) is not identically zero if and only if $h^0(C, \xi^2(2p)K^{-1}) = 0$ as, by lemma 2.3.7, $h^0(C, E \otimes \xi(p)) = 1$ for general p. This is exactly the case since, for given $\xi \in J^1$ and general p, $\xi^2(2p)K^{-1}$ has degree 0 and is not the trivial line bundle.

2. $(t_p) = D_p$

The idea is to show that for given ξ and general p this case cannot occur. Again there is a short exact sequence induced by the section t_p

$$0 \to \mathcal{O}(D_p) \longrightarrow E \otimes K\xi^{-1}(-p) \longrightarrow K^2\xi^{-2}(-2p - D_p) \to 0$$

and by dualising one obtains

$$0 \to \xi^2(2p + D_p)K^{-1} \longrightarrow E \otimes \xi(p) \longrightarrow K(-D_p) \to 0.$$

By tensoring this sequence with $\xi^{-1}(-p)$, one obtains a maximal extension of E by $\xi(p + D_p)K^{-1} \in J^{-1}$,

$$0 \to \xi(p + D_p)K^{-1} \longrightarrow E \longrightarrow K\xi^{-1}(-p - D_p) \to 0.$$

By using theorem 1.5.4 and the fact that $\xi(p + D_p)K^{-1}$ is a maximal line subbundle of E, one obtains one of the following

1. $\xi(p + D_p)K^{-1} = \xi^{-1}$, which implies $\xi^2 = K(-p - D_p)$ and hence cannot occur for general p;

2. $\xi^{-1}(-p - D_p)K \otimes \xi = \mathcal{O}(D)$, i.e., $K(-p - D_p) = \mathcal{O}(D)$ where D is a suitable divisor of degree 2 given by theorem 1.5.4. Whenever the number of such allowable divisors is finite one finds that, again, the identity cannot hold for general p. As we will show in proposition 2.4.3, there are only 64 stable vector bundles in $\mathcal{SU}_C(2)$ which admit infinitely many maximal line subbundles and these are exactly those of the form $E_\kappa = L \otimes \kappa^{-1}$, where L is the Laszlo bundle, the unique stable vector bundle in $\mathcal{SU}_C(2, K)$ such that $h^0(C, L) \geq 3$, and κ is any theta characteristic. On the other hand, if $E_\kappa = L \otimes \kappa^{-1}$ then it follows from proposition 2.4.3 that E is isomorphic only to an extension of $\kappa(-p)$ by $\kappa^{-1}(p)$ with $p \in C$ and, hence, that the curve $C_{\kappa(-p)}$ lies in the support of δ_E for every $p \in C$. One can show directly that δ_E is smooth at the generic point of each of these curves. Consider the divisor δ_L' in J^0 with support $\{\, \eta \in J^0 |\, h^0(C, L \otimes \eta) > 0 \,\}$; δ_L' is just the translate of $\delta_E \subset J^2$ via the theta characteristic κ. By

proposition IV.9 of [La] we know that δ'_L can be identified with the surface $C - C = \{\mathcal{O}(q - s) \mid q, s \in C\}$, which is swept by the curves $C_{-p} = \{\mathcal{O}(q-p) \mid q \in C\}$ and has a unique singularity at the origin $\mathcal{O}$ of J^0 (see also [ACGH], p.223). The required result is then obtained by translating back to δ_E, which is then smooth at every point except at κ.

Hence the proof is completed. □

Note that these two lemmas give as a result a proof of *lemma* 5.3 of [N-R2], but it must be remarked that the hypotheses have been weakened as it is not required for the line bundle ξ to satisfy $\xi^2 \notin \Theta$.

Lemma 2.3.10. *([N-R2], lemma 5.3) For any* $\xi \in J^1$, $H^0(J^2, \mathcal{I}^2_{C_\xi}(2\Theta)) = 0$, *that is, no divisor linearly equivalent to* 2Θ *can vanish along* C_ξ *with multiplicity* 2.

Proof. Suppose there exists $D \in |2\Theta|$ which is singular along C_ξ, in particular C_ξ must lie in the support of D, so there is a vector bundle $E \in \mathcal{SU}_C(2)$ such that $D = \delta_E$ for some extension $0 \to \xi^{-1} \to E \to \xi \to 0$ by lemmas 1.5.3 and 2.2.3. This clearly contradicts lemma 2.3.9, which assures that δ_E is smooth at the generic point of C_ξ. □

Remark 2.3.11. It is worth noticing that this result holds **only** for curves of genus 3. For a curve C of higher genus g it is possible to produce 2Θ divisors of the form δ_E, where δ is defined as for curves of genus 3, which vanish doubly along translates of the curve, C_ξ, where now ξ must be in J^{g-2} (recall the definition of C_ξ given on page 26).

At this point we are in a position to understand the differential of ϕ at $\xi \in J^1$ (see map (2.3.5)). The following result is original, in the sense that it is neither stated nor proved in [N-R2], however it is the natural result one needs to have in order to prove *lemma* 5.2. Recall that for every vector $v \in T_\xi J^1$, $d\phi_\xi(v)$ is a map $Q_\xi \longrightarrow N_\xi$, hence for every global section $s \in Q_\xi$ it gives a section in $N_\xi = H^0(C, K^2\xi^{-2})$ (see page 38).

Lemma 2.3.12. *Let $\xi \in J^1$ and $v \in T_\xi J^1$, then the map $d\phi_\xi(v)$ has the following characterisation: for any section $s \in Q_\xi$ corresponding to the 2theta divisor D_s, $d\phi_\xi(v)(s)$ is a section that is zero on the set of points $p \in C$ such that $v \in T_{\xi(p)}D_s$.*

Proof. First of all note that it is possible to think of v as a tangent vector to the Jacobian J^2 at $\xi(p)$ since all Jacobians are (non canonically) isomorphic and at each point the tangent space is isomorphic to $T_0J^0 = H^0(C,K)^*$. So the claim is that v is tangent to D_s at $\xi(p)$ if and only if the section $d\phi_\xi(v)(s)$ of $K^2\xi^{-2}$ is zero at $p \in C$.

Let Δ be a complex disc and let γ be a holomorphic map from Δ to J^1 such that $\gamma(0) = \xi$ and $\frac{d}{dt}\gamma(t)|_{t=0} = v$, where t is the variable in Δ. Denote by ξ_t the image of $t \in \Delta$, so that $\xi_0 = \xi$. Consider the composition of ϕ with γ, this gives a holomorphic map from Δ to $G_4(2\Theta)$. Let $\{Q_{\xi_t}\}_{t\in\Delta}$ denote the image of this map, in particular $Q_{\xi_0} = Q_\xi$. Now let $\{s_t\}$ be any holomorphic arc parametrised by Δ, with the only requirement that $s_t \in Q_{\xi_t}$ and $s_0 = s$. From the characterisation of the tangent space to a Grassmannian at a point, we know that $d\phi_\xi(v)(s) = \frac{d}{dt}s_t|_{t=0}$. To compute this derivative we need to fix a point $p \in C$ and choose local coordinates u_i in some open neighbourhood U_p of $\xi(p) \in J^2$. Then, up to having to restrict s to a smaller disc, $s_t = s_t(u_i)$, i.e., each holomorphic section s_t is a holomorphic function of the coordinates u_i on U_p. Notice moreover that, for each $t \in \Delta$, $s_t(\xi_t(p)) \equiv 0$ since s_t is a section of Q_{ξ_t} and hence vanishes along the curve C_{ξ_t}. Using the chain rule, one can easily verify that

$$0 = \frac{d}{dt}s_t(\xi_t(p))|_{t=0} = \sum \frac{\partial s_0((\xi_t(p))_i)}{\partial u_i}|_{u_i=0} \cdot v_i + \frac{ds_t}{dt}(\xi(p))|_{t=0} \qquad (2.3.10)$$

where v_i is the i-th coordinate of v and is given by $\frac{d(\xi_t(p))_i}{dt}$ evaluated at $t = 0$.

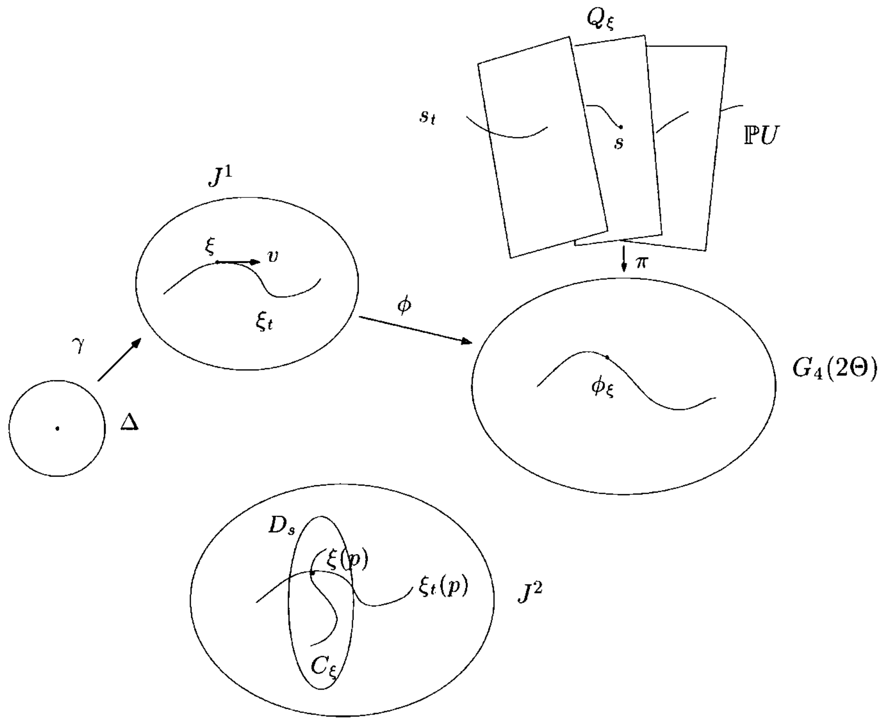

Figure 2.3.1: Interpretation of the differential of ϕ at ξ.

Hence $\frac{ds_t}{dt}|_{t=0}(\xi(p)) = 0$ if and only if $\sum \frac{\partial s_0(u_i)}{\partial u_i}|_{u_i=0} \cdot v_i = 0$. Note that for general p, the divisor D_s is smooth at $\xi(p)$, by lemmas 2.3.9 and 2.3.10 (if D_s is split then we will see on p.49 that it is smooth at the generic point of C_ξ). Hence $\sum \frac{\partial s_0(u_i)}{\partial u_i}|_{u_i=0} \cdot u_i = 0$ is the equation of the tangent space to D_s at $\xi(p)$ and the statement is proved. If $\xi(p)$ is not a smooth point of D_s then the tangent space at $\xi(p)$ is degenerate and coincides with that of J^2, so any vector v is tangent to D_s at $\xi(p)$. On the other hand, because of equation (2.3.10), $\frac{ds_t}{dt}|_{t=0}(\xi(p))$ is also zero and the claim is proved. □

The remaining part of the section is devoted to explaining how one obtains the results stated in *lemma* 5.2 of [N-R2]. None of the following can be found in the paper of Narasimhan and Ramanan, though it is the natural explanation for the results they state.

First note that the differential map $d\phi_\xi : T_\xi J^1 \longrightarrow T_{\phi(\xi)} G_4(2\Theta)$ can be thought of as a pairing

$$\begin{array}{ccccc} H^0(C,K)^* & \otimes & H^0(J^2,\mathcal{I}_{C_\xi}(2\Theta)) & \longrightarrow & H^0(C,K^2\xi^{-2}). \\ \| & & \| & & \| \\ T_\xi J^1 & & Q_\xi & & N_\xi \end{array}$$

Recall that the Gauss map of a smooth n-dimensional variety X embedded in projective space $\mathbb{P}^m$ is a map $\gamma : X \longrightarrow \mathbb{G}(n,m)$ such that at each point $p \in X$ the image $\gamma(p)$ is just the projectivised tangent space $\mathbb{P}T_pX$. The Gauss map is still defined if X has isolated singularities, in which case it is a rational map.

From now onward we will denote by e a point in $\mathbb{P}Q$ (see the definition on page 30) and by s_e any corresponding section in Q_ξ (since s_e is identified only up to a multiplicative constant), by E the semi-stable vector bundle $\varepsilon(e)$ and by D_e or δ_E the associated 2Θ divisor, while $\xi = \pi(e) \in J^1$ is the line bundle such that $e \in \mathbb{P}Q_\xi$. Moreover, we will denote by $\mathbf{v}$ the *projectivisation* of a non-zero vector $v \in H^0(C,K)^*$.

Remark 2.3.13. Let γ_D be the Gauss map of the 2Θ divisor $D = D_e$ corresponding to $e \in \mathbb{P}Q_\xi$, restricted to the image of the curve $C_\xi \subset D$ or, rather, to C. Then $\gamma_D : C \longrightarrow (\mathbb{P}^{g-1})^* = |K|$ is such that $p \longmapsto \mathbb{P}T_{\xi(p)}D$. Under these assumptions the zero set of a section $d\phi_\xi(v)(s_e)$ consists of those points $p \in C$ such that $v \in H^0(C,K)^*$ lies in the tangent space $T_{\xi(p)}D \subset H^0(C,K)^*$.
Clearly the point $e \in \mathbb{P}Q_\xi = \mathbb{P}H^0(J^2,\mathcal{I}_{C_\xi}(2\Theta))$ belongs to the kernel of $d\phi_\xi(v) : Q_\xi \to H^0(J^2,2\Theta)/Q_\xi$ if and only if $\mathbf{v} \in \bigcap_{p\in C} \gamma_{D_e}(p)$, where we think of $\gamma_{D_e}(p)$ as a hyperplane in $|K|^*$ (for those points p for which it does not degenerate to the whole space).

At this point it is clear why lemma 2.3.10, i.e., *lemma* 5.3 of [N-R2] is so important. It assures that the Gauss map of any 2Θ divisor is well defined along the restriction to the translates C_ξ. Without it the whole construction fails. With this basic remark in mind it is easy to prove the next corollary.

Corollary 2.3.14. *The kernel of df_e can be identified with the intersection of the tangent planes to D_e along C_ξ and so, up to projectivising, with $\bigcap_{p\in C}\gamma_{D_e}(p) \subset (\mathbb{P}^{g-1})^*$.*

Proof. This is just lemma 2.3.6 together with the above remark. In fact by lemma 2.3.6, v lies in $\mathcal{K}er\, df_e$ if and only if e belongs to $\mathcal{K}er\, d\phi_\xi \circ d\pi(v) = \mathcal{K}er\, d\phi_\xi(\tilde{v})$, if we denote $d\pi(v)$ by $\tilde{v}$. We have already seen (see page 38) that it is enough to consider vectors v that give a non-zero projection to J^1. Consequently, $v \in \mathcal{K}er\, df_e$ if and only if $d\phi_\xi(\tilde{v})(s_e)$ is the zero section of N_ξ. This, in turn, is the same as stating that $\tilde{\mathbf{v}}$ lies in $\bigcap_{p\in C}\gamma_{D_e}(p) = \bigcap_{p\in C}\mathbb{P}T_{\xi(p)}D_e$. $\square$

The next step consists in understanding the Gauss map γ_D, for any given 2Θ divisor D. Clearly $\bigcap_{p\in C}\gamma_D(p) = \varnothing$ if and only if the Gauss map γ_D is non-degenerate.

We shall divide the study of the Gauss map in two cases, when the divisor D_e is a split divisor and when it is not.

Split Divisors

Let D_e be a split 2Θ divisor, then $D_e = D_q = \Theta_{\xi(-q)} + \Theta_{\xi^{-1}(q)}$ for some $q \in C$. In this case we also write s_q instead of s_e for a corresponding section in Q_ξ. One can easily verify that $C_\xi \subset \Theta_{\xi(-q)}$ as

$$C_\xi = \{\zeta \in J^{g-1} \,:\, \zeta = \xi(p) \;\; and \;\; p \in C\}$$
$$\Theta_{\xi(-q)} = \{\zeta \in J^{g-1} \,:\, h^0(C, \zeta \otimes \xi^{-1}(q)) > 0\}$$

and clearly if $\zeta = \xi(p)$ then $\xi(p) \otimes \xi^{-1}(q) = \mathcal{O}(p+q)$ always has sections. Moreover one can show that $C_\xi \cap \Theta_{\xi^{-1}(q)} \neq \varnothing$ and consists of 3 points as the fundamental class of C in J^{g-1} is given, again, by Poincaré formula, $\frac{[\Theta]^{g-1}}{(g-1)!}$, and hence $|C \cap \Theta| = \frac{\theta^g}{(g-1)!} = g = 3$.

It is easy to describe the map γ_{D_q} when $D_q = \Theta_{\xi(-q)} + \Theta_{\xi^{-1}(q)}$. As noticed above $\xi(p) \in \Theta_{\xi(-q)}$ if and only if $\mathcal{O}(p+q) \in \Theta$ and by Riemann-Kempf *Singularity Theorem* (see [M1], p.264) this is always a smooth point of Θ, as C is non-hyperelliptic, and it is possible to explicitly describe the tangent space $T_{\xi(p)}\Theta_{\xi(-q)} \cong T_{\mathcal{O}(p+q)}\Theta = \bigcup_{D\in|\mathcal{O}(p+q)|}\overline{D} = \overline{pq}$ where $\overline{D}$ is the linear span of the canonical image of D. Hence, the Gauss map of $\Theta_{\xi(p)}$ is

$$\begin{aligned} \gamma_{\Theta_{\xi(-q)}} : \quad & C \longrightarrow |K| \\ & p \longmapsto \overline{pq} \end{aligned}$$

Clearly $\gamma_{\Theta_{\xi(-q)}}$ is a *star* in $|K|^*$ with centre q. The Gauss map γ_{D_q} will coincide with $\gamma_{\Theta_{\xi(-q)}}$ except at the 3 intersection points $C_\xi \cap \Theta_{\xi^{-1}(q)}$ where it degenerates, as each of these is a singular point of D_q the projectivised tangent space is the whole $|K|^*$. Hence we have that $\bigcap_{p\in C}\gamma_{D_q}(p) = \{q\}$. In particular if v is a tangent vector to $\mathbb{P}Q$ at e and $\tilde{v}$ is its non zero projection to J^1, then $df_e(v) = 0$ if and only if $\tilde{\mathbf{v}} = q$ as a point of $|K|^*$.

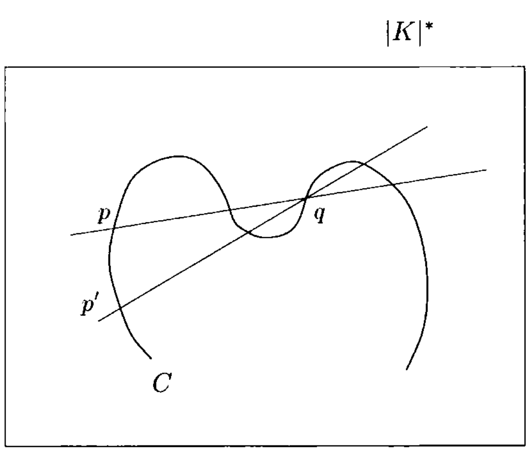

Figure 2.3.2: Tangent spaces to $\Theta_{\xi(-q)}$ at $\xi(p)$ and $\xi(p')$ viewed in $|K|^*$.

Remark 2.3.15. Recalling 2.3.13 one can give a description of the zero set of each section $d\phi_\xi(v)(s_q)$ in $H^0(C, K^2\xi^{-2})$. If $v \in H^0(C,K)^*$ is a fixed vector then one of the following occurs:

- if $\mathbf{v} \in |K|^*/C$ then the zero set of the global section $d\phi_\xi(v)(s_q)$ is given by $\{C \cap \overline{qv}\}/\{q\}$ plus the 3 singular points;

- if $\mathbf{v} \in C \subset |K|^*$ then the zero set of $d\phi_\xi(v)(s_q)$ can be one of the following: either $\{C \cap \overline{qv}\}/\{q\}$ plus the 3 singular points if $\mathbf{v} \neq q$, while, if $\mathbf{v} = q$, then $\mathbf{v}$ lies in every tangent space, i.e., the corresponding global section in $H^0(C, K^2\xi^{-2})$ is the zero section.

Summarising, $\gamma_{D_e} : C \longrightarrow |K|$ is degenerate whenever D_e is a split divisor; in particular if $D_e = \Theta_{\xi(-q)} + \Theta_{\xi^{-1}(q)}$ then we have seen that $\bigcap_{p \in C} \gamma_{D_e}(p) = \{q\}$. Note that this is not a problem since one only needs to prove the injectivity of the differential df at points of $\mathbb{P}Q$ that map to stable bundles with finitely many maximal line bundles (see lemma 2.3.4), while by the commutativity of diagram (2.2.6) in lemma 2.2.1 it is known that split 2Θ divisors come from semi-stable, non stable vector bundles. However it is interesting to have the complete description of df at every point.

Non-Split Divisors

Now one has to describe the Gauss map associated to non-split divisors D corresponding to global sections of $\mathcal{I}_{C_\xi}(2\Theta)$. Fix a point $e \in \mathbb{P}Q_\xi$ corresponding to a stable vector bundle E and to a non-split divisor $D_e = \delta_E$.

Recall the *Brill-Noether* map introduced in lemma 2.3.9 and defined as

$$\begin{array}{rrcl} \mathcal{P}_{\xi(p)} : & H^0(C, \xi(p) \otimes E) \otimes H^0(C, K\xi^{-1}(-p) \otimes E) & \longrightarrow & H^0(C, K) \\ & (s \otimes t) & \longmapsto & s \wedge t \end{array}$$

where again E is identified with its dual (see remark 2.3.8). As in classical Brill-Noether theory for line bundles (see [ACGH], chapter IV), the above map governs the local structure of the determinantal divisor D_e, in the sense that the Gauss map of D_e is given by

$$\begin{array}{rrcl} \gamma_{D_e} : & C & \longrightarrow & |K| \\ & p & \longmapsto & \mathbb{P}(Im\, \mathcal{P}_{\xi(p)}). \end{array} \tag{2.3.11}$$

With this in mind one can prove

Proposition 2.3.16. *The Gauss map γ_{D_e} is a non degenerate rational map for any $e \in \mathbb{P}Q_\xi$ mapping to a stable vector bundle and satisfying property $(\star)$.*

Proof. As seen in lemma 2.3.9 the *Brill-Noether* map $\mathcal{P}_{\xi(p)}$ is injective at the general point $p \in C$ as long as E is stable.

Consider now the rational map induced by the global sections of the vector bundle $K\xi^{-1} \otimes E$, which satisfies $h^0(C, K\xi^{-1} \otimes E) = 3$ and $h^0(C, K\xi^{-1}(-p) \otimes E) = 1$ at the general point p of C (see lemma 2.3.7). This is, by definition, a map to the Grassmannian $G_2(H^0(C, K\xi^{-1} \otimes E)^*)$, yet it can be thought of as a map to $\mathbb{P}H^0(C, K\xi^{-1} \otimes E)$,

$$\begin{tikzcd} C \arrow[r] \arrow[dr] \arrow[ddr, "\psi_\xi"'] & G_2(H^0(C, K\xi^{-1} \otimes E)^*) \arrow[d, equal] \\ & G_1(H^0(C, K\xi^{-1} \otimes E)) \arrow[d, equal] \\ & \mathbb{P}H^0(C, K\xi^{-1} \otimes E) \end{tikzcd}$$

where ψ_ξ maps the general point $p \in C$ to the space of global sections of $K\xi^{-1} \otimes E$ vanishing at p. By construction ψ_ξ is a non-degenerate map. One can consider another *Brill-Noether type* map

$$\mathcal{P}_\xi \,:\, H^0(C, \xi \otimes E) \otimes H^0(C, K\xi^{-1} \otimes E) \longrightarrow H^0(C, K)$$

introduced in [O], p.12. This map is injective if and only if the set of maximal line bundles of E is smooth in J^{-1} and has codimension 3 (see [O], proposition 2.4). If e satisfies property $(\star)$, $\mathcal{P}_\xi$ is injective and, by a dimension count, an isomorphism; however if it doesn't then $\mathcal{P}_\xi$ fails to be injective.

We restrict to points e that satisfy $(\star)$ and hence can consider the diagram

$$\begin{tikzcd} & \mathbb{P}H^0(C, K\xi^{-1} \otimes E) \arrow[d, "\mathcal{P}"] \\ C \arrow[ur, "\psi_\xi"] \arrow[r, "\gamma_{D_e}"'] & {|K|} \end{tikzcd} \tag{2.3.12}$$

where $\mathcal{P}$ is obtained by composing with a Segre embedding the projectivisation of $\mathcal{P}_\xi$. Hence even $\mathcal{P}$ is an isomorphism and the composition with ψ_ξ is non-degenerate. So it is enough to show that the diagram (2.3.12) commutes to prove that γ_{D_e} is non-degenerate. Take a generic point $p \in C$, then

$$\begin{aligned}
\mathcal{P}(\psi_\xi(p)) &= \mathcal{P}(\mathbb{P}H^0(C, K\xi^{-1}(-p) \otimes E)) \\
&= \mathbb{P}(\mathcal{P}_\xi(H^0(C, \xi \otimes E) \otimes H^0(C, K\xi^{-1}(-p) \otimes E) \to H^0(C, K))) \\
&\cong \mathbb{P}(\mathcal{P}_{\xi(p)}(H^0(C, \xi(p) \otimes E) \otimes H^0(C, K\xi^{-1}(-p) \otimes E) \to H^0(C, K))) \\
&= \mathbb{P}(Im\, \mathcal{P}_{\xi(p)}) \\
&= \gamma_{D_e}(p)
\end{aligned}$$

where in the third step we have used the fact that the inclusion of the space $H^0(C, \xi \otimes E)$ in $H^0(C, \xi(p) \otimes E)$ is an isomorphism for general p, since the two spaces have the same dimension. □

Note that it is precisely during the proof of proposition 2.3.16 that we see why it is necessary to introduce property $(\star)$ (see page 36). If $e \in \mathbb{P}Q$ fails to satisfy this requirement then the corresponding Gauss map is degenerate and one cannot show injectivity of the differential df_e. On the other hand, if $e \in \mathbb{P}Q$ satisfies $(\star)$ and maps to a stable vector bundle then $\bigcap_{p \in C} \gamma_{D_e}(p) = \varnothing$ and df_e is injective. Now we can gather all this information in a result which is the analogue of *lemma* 5.2 in [N-R2].

Finally, this proves injectivity of the differential of f at $e \in \mathbb{P}Q$ as long as e maps to a stable vector bundle and satisfies property $(\star)$, i.e., it maps to a stable vector bundles with only finitely many maximal line subbundles.

We can at last conclude the proof of lemma 2.3.4. By applying corollary 2.3.14, $v \in Ker\, df_e$ if and only if any section s_e corresponding to e in $\mathcal{K}er\, d\phi_\xi(\tilde{v})$, where $\tilde{v} = d\pi(v) \neq 0$. equivalently, corollary 2.3.14 sates that $v \in \mathcal{K}er\, df_e$ if and only if $\tilde{\mathbf{v}} \in \bigcap_{p \in C} \gamma_{D_e}(p)$. Then the previous proposition says that if e maps to a stable vector bundle and has property $(\star)$, the corresponding Gauss map is non degenerate and hence no vector v can lie in the kernel of df_e. On the other hand, if e maps to a semi-stable vector bundle or does not satisfy $(\star)$ then df_e is not injective.

It is also worth noticing that if e maps via ε to a stable vector bundle with infinitely many maximal line subbundles then it is easy to see directly that df_e cannot have maximal rank, since $d\varepsilon_e$ maps to zero any vector v tangent to the 1-dimensional fibre $\varepsilon^{-1}(\varepsilon(e))$ at e and by commutativity of diagram (2.2.6) this implies that $df_e(v) = 0$ too.

Moreover one can re-obtain Narasimhan and Ramanan's result concerning the description of the map $d\phi_\xi(v)$ in terms of the vector v. This allows us to say exactly which vectors lie in the kernel of df_e, if any, in terms of their projection to $|K|^*$.

Proposition 2.3.17. *Let $\xi \in J^1$ and $\tilde{v} \in T_\xi J^1$ be a non-zero projection of a tangent vector v to $\mathbb{P}Q$. If all $e \in \mathbb{P}Q_\xi$ satisfy property $(\star)$ or map to non-stable vector bundles, then one of the following occurs:*

- *if $\tilde{\mathbf{v}} \in |K|^*/C$ then $d\phi_\xi(\tilde{v}) \in Hom(H^0(J^2, \mathcal{I}_{C_\xi}(2\Theta)), H^0(C, K^2\xi^{-2}))$ is injective;*
- *if $\tilde{\mathbf{v}} \in C \subset |K|^*$ then $d\phi_\xi(\tilde{v}) \in Hom(H^0(J^2, \mathcal{I}_{C_\xi}(2\Theta)), H^0(C, K^2\xi^{-2}))$ has a 1-dimensional kernel and this, as a point of $\mathbb{P}Q_\xi$, corresponds to the point mapping to the semi-stable, non stable vector bundle $\Theta_{\xi(-p)} + \Theta_{\xi^{-1}(p)}$.*

Proof. It is clear from the above constructions that if $e \in \mathbb{P}Q_\xi$ represents a stable bundle then $\bigcap_{p\in C} \gamma_{D_e}(p) = \varnothing$, while if e represents a semi-stable, non stable bundle then $D_e = \Theta_{\xi(-q)} + \Theta_{\xi^{-1}(q)}$ for some $q \in C$ and $\bigcap_{p\in C} \gamma_{D_e}(p) = \{q\}$. Hence, by remark 2.3.13 and corollary 2.3.14, the map $d\phi_\xi(\tilde{v})$ is injective unless $\tilde{\mathbf{v}} = q \in C \subset |K|^*$, in which case the kernel consists of the line in Q_ξ corresponding to the semi-stable, non stable vector bundle $\Theta_{\xi(-q)} + \Theta_{\xi^{-1}(q)}$. □

Note that to require every point $e \in \mathbb{P}Q_\xi$ to satisfy property $(\star)$ or map to a non stable vector bundle, implies that $\xi \in J^1$ must satisfy the condition $\xi^2 \notin \Theta$. In fact, there are points of $\mathbb{P}Q_\xi$ mapping to stable bundles with infinitely many maximal line bundles only if $\xi = \kappa(-p)$ for some choice of a theta characteristic κ and point $p \in C$ as will be shown in proposition 2.4.3 and in this case it is immediate to verify that ξ^2 has non zero global sections. However the above description is not needed in the proof of proposition 2.3.4.

This concludes the proof of the fact that δ is an embedding of $\mathcal{SU}_C(2)$ in $|2\Theta|$. Note, that contrary to [N-R2], we have given a complete account of the behaviour of the differential df at every point e in the vector bundle $\mathbb{P}Q$.

2.3.1 The Coble quartic

Before ending this section I will briefly talk about the degree of $\mathcal{SU}_C(2)$ as a subvariety of $|2\Theta|$. It is a well known result (see [N-R2] and [O-P]) that the image G of $\mathcal{SU}_C(2)$ in $|2\Theta|$ is a degree 4 hypersurface singular along $Kum(J^0)$. In fact it is the only quartic in $\mathbb{P}^7$ with this property.

Moreover it is also the only Heisenberg invariant quartic of $\mathbb{P}^7$ singular along $Kum(J^0)$ and it is classically known as the Coble quartic.

The degree of $G = \delta(\mathcal{SU}_C(2))$ is computed using, again, the commutativity of diagram (2.2.6) and the fact that δ is an embedding. In fact, $G = f(\mathbb{P}Q)$ and, as f is a finite map, its degree is given be the well known formula (see [F], example 8.3.14, p.144)

$$\deg\, f \cdot \deg\,(Im\, f) = \int_{\mathbb{P}Q} c_1(f^*\mathcal{O}_{|2\Theta|}(1))^{\dim \mathbb{P}Q}.$$

From what has been said in the previous section it is clear that $\deg\, f = \deg\, \varepsilon$ since δ is an embedding. Moreover $\deg\, \varepsilon = 8$ since the general stable vector bundle $E \in \mathcal{SU}_C(2)$ has 8 maximal line subbundles (see example 1.5.5).
Oxbury and Pauly showed in [O-P], p.310, that $\int c_1(f^*\mathcal{O}_{|2\Theta|}(1))^{\dim \mathbb{P}Q} = 32$, from which the fact that G is a quartic immediately follows (see remark 3.2.4).

2.4 A counterexample to a claim of Narasimhan and Ramanan

Let C continue to be a smooth non-hyperelliptic, non-bielliptic curve of genus 3. In order to prove the statement about the differential of f it has been necessary to characterise the maximal line subbundles of stable vector bundles $E \in \mathcal{SU}_C(2)$. In doing so, I have found a counterexample to *lemma* 5.4 of [N-R2], more precisely there are 64 Veronese cones in $\mathcal{SU}_C(2)$ of vector bundles that fail to satisfy the claim. To show that this is the case, it is first necessary to recall some facts about $\mathcal{SU}_C(2,K)$, the moduli space of rank two semi-stable vector bundles with canonical determinant.

Remark 2.4.1. Note that in analogy to what has been done in the previous paragraphs, it is possible to define a map δ' from $\mathcal{SU}_C(2,K)$ to the linear system $|\mathcal{L}|$ such that for any $F \in \mathcal{SU}_C(2,K)$, $\delta'(F)$ has as support the set $\{\eta \in J^0 \mid h^0(C, F \otimes \eta) > 0\}$. Clearly this map has similar properties to δ.

An important feature of $\mathcal{SU}_C(2,K)$ is its *Brill-Noether* stratification, we will refer to the work of Oxbury et al. in [O-P-Pr] for the results concerning it.

Proposition 2.4.2. *(see [O-P-Pr], §5) If $\mathcal{W}^i$ is the closure of the set of stable vector bundles $E \in \mathcal{SU}_C(2,K)$ such that $h^0(C,E) > i$, then for non-hyperelliptic curves of genus 3 the stratification consists of 3 varieties*

1. *$\mathcal{W}^0$ is just the generalised theta divisor of the moduli space $\mathcal{SU}_C(2,K)$ and consists of those vector bundles that have non zero global sections;*

2. *$\mathcal{W}^1$ is a cone over a Veronese surface, its generators are exactly the trisecants to $Kum(J^2)$ in $\mathcal{SU}_C(2,K)$;*

3. *$\mathcal{W}^2$ is the vertex of the cone, i.e., there is a unique stable vector bundle in $\mathcal{SU}_C(2,K)$, called the Laszlo bundle and denoted L, such that $h^0(C,L) = 3$ (see [La], p.342).*

Moreover it is immediate to verify that if a stable bundle $E \in \mathcal{SU}_C(2,K)$ satisfies $h^0(C,E) \geq 2$ then all its maximal line subbundles, which by stability must have degree 1, are of the form $\mathcal{O}(p)$ for some $p \in C$. Conversely if $E \in \mathcal{SU}_C(2,K)$ has no non-zero global section then none of its maximal line subbundles can have sections either, so they cannot be of the form $\mathcal{O}(p)$ with $p \in C$. To verify the above statements, simply consider the long exact sequence associated to the extension $0 \to \zeta \longrightarrow E \longrightarrow K\zeta^{-1} \to 0$ where $\zeta \in J^1$ is a maximal line subbundle of E,

$$0 \longrightarrow H^0(C,\xi) \longrightarrow H^0(C,E) \longrightarrow H^0(C,K\xi^{-1}) \longrightarrow \dots$$

Note that by the Riemann-Roch formula and Serre duality, $h^0(C,K\xi^{-1}) = h^0(C,\xi)+1$, while the exactness of the above sequence then implies that $h^0(C,E) \leq 2h^0(C,\xi)+1$. Hence $h^0(C,E) \geq 2$ forces $h^0(C,\xi) \geq 1$, i.e., $\xi = \mathcal{O}(p)$ for some choice of a point $p \in C$. On the other hand, if $h^0(C,E) = 0$ then necessarily $h^0(C,\xi) = 0$.

Any choice of a theta characteristic $\kappa \in \vartheta(C)$ allows to pull-back the cone over the Veronese surface $\mathcal{W}^1$ from $\mathcal{SU}_C(2,K)$ to $\mathcal{SU}_C(2)$, so that in $\mathcal{SU}_C(2)$ it is possible to identify $64 = 2^{2g}$ distinct such cones, one for each choice of κ. Moreover, the above remark allows one to conclude that every vector bundle on these "translated" cones admits only maximal line subbundles of the form $\mathcal{O}(p) \otimes \kappa^{-1} = \kappa^{-1}(p) \in J^{-1}$, where κ is fixed by the translation.

Proposition 2.4.3. *If C is a general non-hyperelliptic, non-bielliptic curve of genus 3 then $\mathcal{SU}_C(2)$ contains exactly 64 Veronese cones whose general point is a stable vector bundle with all maximal line subbundles $\xi^{-1} \in J^{-1}$ satisfying $\xi^2 \in \Theta$.*

In particular the cone obtained by translating $\mathcal{W}^1$ by $\kappa \in \vartheta(C)$ has, as vertex, a stable vector bundle $E_\kappa = L \otimes \kappa^{-1}$ which admits an infinite family of maximal line subbundles $\{\kappa^{-1}(p) : p \in C\}$; while the general point is a stable vector *bundle with 7 maximal line subbundles* $\{\kappa^{-1}(p_i) : p_i \in C\}$.

Proof. The first statement is just a consequence of the above remarks, as it is enough to notice that if E is a vector bundle in the cone translated by κ then all its maximal line subbundles are of the form $\xi^{-1} = \mathcal{O}(p)\kappa^{-1}$ and hence $\xi^2 = K(-2p)$ is a degree 2 line bundle with non zero global sections, i.e., an element of Θ.

As for the second part of the proposition it is necessary to recall that if a stable vector bundle $E \in \mathcal{SU}_C(2)$ admits a maximal line subbundle $\xi^{-1} = \mathcal{O}(p)\kappa^{-1}$ then it can be thought of as a point in the 3 dimensional space of extensions $\mathbb{P}(\xi)$. Moreover, as seen in example 1.5.5, the general stable bundle with this property has 7 maximal line subbundles. These are all of the form $\kappa^{-1}(q_i)$ since they are all translates of the maximal line subbundles of $F = E \otimes \kappa \in \mathcal{SU}_C(2, K)$ with $h^0(C, F) = 2$, whose maximal line subbundles are $\mathcal{O}(q_i)$ for some finite number of $q_i \in C$.

As for the vertex, the only thing left to prove is that it admits infinitely many maximal line subbundles. Note that the Laszlo bundle has this property since its maximal line subbundles are $\{\mathcal{O}(p) : p \in C\}$, in fact, $h^0(C, L) = 3$ being strictly bigger than $rk(L)$, for every $p \in C$ there exists a non zero section $s \in H^0(C, L)$ which vanishes at p and hence $0 \to \mathcal{O}(p) \to L \to K(-p) \to 0$ is exact (see [La], proposition IV.7). Moreover we also know that all its maximal line subbundles are of this form. Hence the vertex of the Veronese cone $\mathcal{W}^1$ translated by κ^{-1} has infinitely many maximal line subbundles $\{\kappa^{-1}(p) : p \in C\}$.

Finally one has to verify that the 64 Veronese cones are distinct. Assume there are two cones intersecting at a stable point $E \in \mathcal{SU}_C(2)$, then there exist two distinct vector bundles F, G in the Veronese cone of $\mathcal{SU}_C(2, K)$ and two distinct theta characteristics κ, λ such that $E = F \otimes \kappa^{-1} = G \otimes \lambda^{-1}$. In particular this means that $F = G \otimes \eta$ where $\eta = \lambda^{-1}\kappa \in J^0$ is a two-torsion point, i.e., $\eta^2 = \mathcal{O}_C$. Moreover both F and G lie in the Veronese cone, hence $h^0(C, F) \geq 2$

and $h^0(C,G) \geq 2$. In particular this implies that there exists $p \in C$ such that $\mathcal{O}(p)$ is a maximal line subbundle of G, giving an exact sequence

$$0 \to \mathcal{O}(p) \longrightarrow G \longrightarrow K(-p) \to 0.$$

This gives the following short exact sequence for F

$$0 \to \eta(p) \longrightarrow F \longrightarrow K\eta(-p) \to 0.$$

Hence $h^0(C,\eta(p)) \geq 1$, i.e., there exists $q \in C$ such that $\eta(p) = \mathcal{O}(q)$. Then $\eta^2(2p) = \mathcal{O}(2p)$ has to be equal to $\mathcal{O}(2q)$ and since C is non-hyperelliptic this implies $p = q$ and, thus, $\eta = \mathcal{O}$, i.e., $\kappa = \lambda$ which contradicts the hypotheses. $\square$

Remark 2.4.4. Note that by example 1.5.5, we already know that each extension space $\mathbb{P}(\xi)$ with $\xi = \kappa(-p)$ has exactly one point which parametrises a stable vector bundle with infinitely many maximal line subbundles. The above proof shows that this vector bundle is $L \otimes \kappa^{-1}$.

Concluding, as shown above, *lemma* 5.4 of [N-R2] is clearly false. However, we have also seen that this does not affect the proof of the fact that the map δ is an embedding, since there is no need to restrict attention to line bundles with any particular property.

Chapter 3

Abel-Jacobi stratification

This chapter recalls the construction of the Abel-Jacobi stratification of $|2\Theta|$ as defined by Oxbury and Pauly in [O-P] and presents some new results on the characterisation of some of the varieties that compose it. The idea behind the Abel-Jacobi stratification is to generalise the construction of the 3-ruling $\mathbb{P}Q$ over the Jacobian J^1 outlined in [N-R2] for non-hyperelliptic curves of genus 3 (see §2.2) to curves of higher genus g. These varieties are interesting since they are related to classical subvarieties of $|2\Theta|$, for example the first element of the stratification, G_1, is the Kummer variety $Kum(J^0)$, while the second, denoted G_2, contains the trisecants to the Kummer (see [O-P-Pr], §2). However, little is known about the general element, in fact apart from the two cases mentioned above, even the dimension of the other varieties is unknown. In §3.2, I will show that the top variety of the stratification, G_{g-1}, is always a hypersurface and compute its degree for all curves of genus at least 4. Finally, recent results by Pareschi and Popa in [Pa-Po], allow me to make some additional remarks and computations on the general element of the stratification.

From now on C will be a nonsingular, complete, non-hyperelliptic curve of genus $g \geq 3$ and we will continue to use the notations and conventions of chapter 1.

3.1 The stratification

Recall that W_d is the image of the d-th symmetric product of C, C_d, in the Jacobian J^d via the Abel-Jacobi map u_d; in particular $W_1 \cong C$ and $W_{g-1} \cong \Theta$ (see [ACGH], p.25). The idea is to construct and characterise subvarieties described, set-theoretically, in the following way

$$G_{d+1} = \{D \in |2\Theta| \mid W_{g-d-1,\eta^{-1}} \subset supp\, D, \text{ for some } \eta \in J^d\} \qquad 0 \leq d \leq g-2$$
$$G^*_{g-d} = \{D \in |\mathcal{L}| \mid W_{d,\eta} \subset supp\, D, \text{ for some } \eta \in J^d\} \qquad 1 \leq d \leq g-1$$

where, for any line bundle $\lambda \in J^i$, $W_{d,\lambda}$ is the translate of W_d to J^{d-i} obtained by tensoring with λ^{-1}.

Remark 3.1.1. Wirtinger duality (see lemma 1.1.2), assures that $|2\Theta|$ and $|\mathcal{L}|^*$ are canonically isomorphic, yet an isomorphism between $|2\Theta|$ and $|\mathcal{L}|$ depends on the choice of a theta characteristic $\kappa \in \vartheta(C)$. Once such a choice has been made, it is easy to verify that $G_d \simeq G^*_d$.

Because of the above remark it is enough to consider the study of the varieties G_d contained in $|2\Theta|$ and then translate by κ to obtain analogous results for the varieties G^*_d.

Remark 3.1.2. It is immediate to verify that $G_d \subset G_{d+1}$, since if $D \in |2\Theta|$ belongs to G_d, there exists a line bundle $\eta \in J^{d-1}$ such that $W_{g-d,\eta^{-1}} \subset supp\, D$. This implies that $W_{g-d-1,\eta^{-1}(-p)} \subset supp\, D$ for all line bundles $\eta(p) \in J^d$ with $p \in C$.

The defined objects constitute a stratification, the Abel-Jacobi stratification of $|2\Theta|$ and $|\mathcal{L}|$, respectively. Some of the varieties of these stratifications are easily identified

- $G_1 = \{D \in |2\Theta| \mid \Theta_{\eta^{-1}} \subset supp\, D, \eta \in J^0\}$ and hence is just $Kum(J^0)$. Clearly $Kum(J^0) \subset G_1$ as every split 2Θ divisor $\Theta_\mu + \Theta_{\mu^{-1}}$ with $\mu \in J^0$ lies in G_1. Conversely, if $D \in |2\Theta|$ belongs to G_1 it means that D is linearly equivalent to 2Θ and there exists a line bundle $\eta \in J^0$ such that $\Theta_{\eta^{-1}}$ is contained in the support of D. Hence, if we denote by D' the residue divisor $D - \Theta_{\eta^{-1}}$, $D' + \Theta_{\eta^{-1}}$ is linearly equivalent to 2Θ, as is $\Theta_\eta + \Theta_{\eta^{-1}}$ by remark 1.1.1. This implies that D' and Θ_η are linearly equivalent. Since Θ is a principal polarisation $h^0(J^{g-1}, \Theta_\eta) = 1$, i.e., $D' = \Theta_\eta$ and $D = \Theta_\eta + \Theta_{\eta^{-1}} \in Kum(J^0)$.

Similarly $G_1^* = Kum(J^{g-1})$ in $|\mathcal{L}|$.

- If $g = 3$ there are only two varieties, $G_1 = Kum(J^0)$, and $G_2 = \{D \in |2\Theta| \mid C_{\eta^{-1}} \subset supp\, D, \eta \in J^1\}$, which is just the embedded image of $\mathcal{SU}_C(2)$ in $|2\Theta|$ as seen in chapter 2 (see the proof of lemma 2.2.3 and remark 2.2.5).

Remark 3.1.3. One could define an Abel-Jacobi stratification for non-singular complete curves of genus 2, in which case the stratification would consist of just one element $G_1 = Kum(J^0)$, the Kummer surface, as can be seen by repeating the argument for G_1 presented above.

One has the following inclusions

$$G_1 = Kum(J^0) \subset G_2 \subset \cdots \subset G_{g-1} \subset |2\Theta|.$$

In particular, in the genus 3 case this reduces to

$$G_1 = Kum(J^0) \subset G_2 = \mathcal{SU}_C(2) \subset |2\Theta|.$$

Moreover, it is well known that G_1 is the singular locus of G_2.

In general it is not known how these varieties are related to one another, or even what their dimensions are. To try to tackle some of these problems, we now review the formal construction of these varieties as given in [O-P], §§7 and 10.

Consider the diagram

$$\begin{tikzcd} J^d \times C_d \arrow[r, "\beta_d"] \arrow[d, "\pi_d"] & J^0 \\ J^d & \end{tikzcd} \tag{3.1.1}$$

where β_d is defined as a "difference" map $\beta_d(\eta, D) = \eta^{-1}(D)$, while π_d is just the projection to the first factor. Note that when $d = 1$ and $\eta \in J^1$ is fixed, $\beta_1(\eta, \cdot)$ is just the map β_η of chapter 2, page 26, its image being $C_{\eta^{-1}}$, the η translate of the curve $C = W_1$ in J^0. In general, for any line bundle $\eta \in J^d$ the image of $\beta_d(\eta, \cdot)$, i.e., the restriction of β_d to $\{\eta\} \times C_d$, is the translate $W_{d,\eta}$ of W_d in the Jacobian J^0. Then one can define a coherent sheaf on J^d, $Q_d \overset{def}{=} \pi_{d*}\beta_d^*\mathcal{L}$.

At this point we recall a result of Oxbury and Pauly. Note that from here on we use the notation of §A.1 of the Appendix. However, we briefly recall that x_d and θ_d are classes on the symmetric product C_d. Moreover, if L is a line bundle on the curve C, $(L)_d$ is the induced line bundle on C_d (see §A.3), while $\overline{\Delta}$ is the diagonal divisor on C_d.

Proposition 3.1.4. *(see [O-P], proposition 10.1) Let $\mathcal{L}_\eta$ be the pull-back $(\beta_d^*\mathcal{L})|_{\{\eta\}\times C_d}$ for any line bundle $\eta \in J^d$, then the following equalities hold:*

1. $\mathcal{L}_\eta \cong (K\eta^2)_d \otimes \mathcal{O}(-\overline{\Delta})$;
2. $c_1(\mathcal{L}_\eta) = 2\theta_d$, *where c_1 is the first Chern class;*
3. $\chi(C_d, \mathcal{L}_\eta) = \sum_{i=0}^{d} \binom{g}{i}$, *where χ denotes the Euler characteristic.*

The first equality implies that for every line bundle $\eta \in J^d$

$$H^0(C_d, \mathcal{L}_\eta) \cong H^0(C_d, (K\eta^2)_d \otimes \mathcal{O}(-\overline{\Delta})).$$

If $d = 1$, the right-hand side becomes $H^0(C, K\eta^2)$, so that this generalises the result of lemma 2.1.2 or, rather, the following remark 2.1.3 (Oxbury and Pauly's proof is in fact based on induction on d starting from this older result). In [O-P] it is conjectured that the coherent sheaves Q_d are locally free, i.e., vector bundles over J^d, for all integers d for which they are defined. This result was eventually proved by Pareschi and Popa in [Pa-Po] by showing that all cohomology groups $H^i(C_d, \mathcal{L}_\eta)$ are zero for $i \geq 1$, since this result together with proposition 3.1.4 assures that the fibres of Q_d are equidimensional. In more detail, they prove the following proposition.

Proposition 3.1.5. *Oxbury-Pauly conjecture (see [Pa-Po], corollary 4.3) With the same notation as above*

1. $h^0(C_d, \mathcal{L}_\eta) = \sum_{i=0}^{d} \binom{g}{i}$;
2. *the restriction map* $\beta_d(\eta, \cdot)^* : H^0(J^0, \mathcal{L}) \longrightarrow H^0(C_d, \mathcal{L}_\eta)$ *is surjective.*

As previously said, this proposition assures that Q_d is a vector bundle for every integer d and that for every line bundle $\eta \in J^d$ the corresponding fibre of Q_d is:

$$Q_{d,\eta} = H^0(C_d, \mathcal{L}_\eta) \cong H^0(C_d, (K\eta^2)_d \otimes \mathcal{O}(-\overline{\Delta})).$$

Note that for curves of genus 3 and $d = 1$ these results are well known as seen in lemma 2.1.2 and lemma 2.1.4, i.e., *lemma 4.1* of [N-R2]. The cases $d = 1$, 2, $g-1$ and g are mentioned in the appendix of [O-P], where a complete proof is given only for curves of genera 3 and 4. However, in oder to prove part *1.*when $d = 2$, the authors refer to [Br-V], proposition 4.9, where the statement is proved for curves of any genus.

Finally, part *2.* of proposition 3.1.5 is equivalent to stating that the restriction map $\mathcal{O}_{J^d} \otimes H^0(J^0, \mathcal{L}) \to Q_d$ is surjective. Hence one obtains a short exact sequence of vector bundles on J^d

$$0 \to N_d \longrightarrow \mathcal{O}_{J^d} \otimes H^0(J^0, \mathcal{L}) \xrightarrow{\beta_d^*} Q_d \to 0 \tag{3.1.2}$$

where N_d is by definition the kernel of β_d^* and, as such, has fibres $N_{d,\eta} = H^0(J^0, \mathcal{I}_{W_{d,\eta^{-1}}}(\mathcal{L}))$, if we denote by $\mathcal{I}_{W_{d,\eta}}$ the ideal sheaf of $W_{d,\eta}$. By considering the projectivised bundle associated to N_d for every integer $1 \leq d \leq g-1$, $\mathbb{P}N_d \subset J^d \times |\mathcal{L}|$, one can define the following ruled varieties

$$G^*_{g-d} \stackrel{def}{=} im\,\{\mathbb{P}N_d \subset J^d \times |\mathcal{L}| \longrightarrow |\mathcal{L}|\}.$$

It is immediate to verify that these are the same as the varieties described set-theoretically at the outset of the chapter since we have just noticed that $\mathbb{P}N_{d,\eta}$ consists of divisors in $|\mathcal{L}|$ which contain $W_{d,\eta}$ in their support, that is, the image of $\{\eta\} \times C_d$ in J^0 via the map β_d. Oxbury and Pauly also consider the dual of sequence (3.1.2) and, by identifying the space $H^0(J^0, \mathcal{L})^*$ with $H^0(J^{g-1}, 2\Theta)$ via Wirtinger duality (see proposition 1.1.2), obtain another short exact sequence on each Jacobian J^d

$$0 \to Q^*_d \longrightarrow \mathcal{O}_{J^d} \otimes H^0(J^{g-1}, 2\Theta) \longrightarrow N^*_d \to 0. \tag{3.1.3}$$

Again, they consider the projective bundle $\mathbb{P}Q^*_d$ associated to the vector bundle Q^*_d and define ruled varieties in $|2\Theta|$ for every integer $0 \leq d \leq g-2$

$$G_{d+1} \stackrel{def}{=} im\,\{\mathbb{P}Q^*_d \subset J^d \times |2\Theta| \longrightarrow |2\Theta|\}.$$

However in this case it is not so obvious that these are the same as the varieties considered at the beginning of this chapter, but this follows from the next proposition.

Proposition 3.1.6. *(see [O-P], proposition 7.2) For any line bundle $\eta \in J^d$ with $0 \leq d \leq g-1$, the following characterisation holds: $\mathbb{P}Q^*_{d,\eta}$ can be identified with $\{D \in |2\Theta| \mid W_{g-1-d,\eta^{-1}} \subset supp\, D\}$.*

Proof. The first step consists in evaluating the ranks of the vector bundles Q_d and N_d. The rank of Q_d is given by proposition 3.1.5 (*1.*), since the rank is the dimension of the general fibre and $\dim Q_{d,\eta} = \sum_{i=0}^{d} \binom{g}{i}$. Moreover it is well known that $h^0(J^0, \mathcal{L}) = 2^g$, and hence using sequence (3.1.2) one can evaluate the rank of the bundles N_d:

$$rk\, N_d = 2^g - rk\, Q_d = \sum_{i=d+1}^{g} \binom{g}{i} = \sum_{i=0}^{g-d-1} \binom{g}{i} = rk\, Q_{g-d-1}.$$

Secondly Oxbury and Pauly observe (see [O-P], proposition 7.2), that the isomorphism of $H^0(J^{g-1}, 2\Theta)$ with $H^0(J^0, \mathcal{L})$ given by a choice of a theta characteristic κ restricts to

$$Q^*_{d,\eta} \xrightarrow{\sim} N_{g-1-d,\kappa\eta^{-1}}.$$

Hence, if for any divisor $D \in |2\Theta|$ one denotes by D_κ the corresponding divisor in $|\mathcal{L}|$ obtained by translating via κ, one obtains the required identity

$$\begin{aligned} \mathbb{P}Q^*_{d,\eta} &= \{D \in |2\Theta| \mid W_{g-d-1,\eta\kappa^{-1}} \subset supp\, D_\kappa\} \\ &= \{D \in |2\Theta| \mid W_{g-d-1,\eta^{-1}} \subset supp\, D\}. \end{aligned}$$

□

Remark 3.1.7. Though Oxbury and Pauly define the varieties G_d and G^*_d for every integer d in the correct range, it is only because of the result of Pareschi and Popa (see proposition 3.1.5) that the construction is proved to be meaningful in all cases.

At this point it is natural to enquire about the dimension and degree of all G_d's as subvarieties of $|2\Theta|$. Apart from the trivial case of $G_1 = Kum(J^0)$, this has been explicitly done in just one case, that of G_2 or, rather, G^*_2. In his Ph.D. thesis, Gronow observes that proposition 3.3 of [L-N] implies that the projection from $J^{g-2} \times |\mathcal{L}|$ to $|\mathcal{L}|$ is injective on an open set of $\mathbb{P}N_{g-2}$ for any curve of genus at least 4 (see [Gr], remark 2.0.4). As an immediate consequence one has the dimension of G^*_2 since it is that of $\mathbb{P}N_{g-2}$, dim $G^*_2 = rk\, N_{g-2} - 1 + \dim\, J^{g-2} = 2g$. Moreover, he explicitly computes the degree of G_2 (see [Gr], proposition 3.2.9, p.39)

$$\deg\, G_2 = \deg\, G^*_2 = (-4)^g \sum_{k=0}^{g} (-1)^k \frac{k!}{2^k} \binom{g}{k} \binom{2g}{k}. \tag{3.1.4}$$

In order to consider the other G_d's, recall that we have the following diagram

$$\begin{tikzcd} \mathbb{P}Q_d^* \arrow[r, hook] \arrow[d, "f_d"] & J^d \times |2\Theta| \arrow[d, "\pi"] \\ G_{d+1} \arrow[r, hook] & |2\Theta| \end{tikzcd}$$

If f_d were a map of finite degree, which is a "reasonable" expectation by a dimension count, then the dimension of each G_{d+1} would be known - being the same as that of the corresponding $\mathbb{P}Q_d^*$ - and the degree could be computed using the following classical result (see [F], example 8.3.14, p.144).

Let X^m be a variety of dimension m obtained as projectivisation of a vector bundle over a give variety Y and f a map of finite degree $X^m \longrightarrow \mathbb{P}^n$, the degree of the image $f(X^m)$ is given by

$$\deg\, f \cdot \deg\, f(X^m) = \int_{X^m} c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))^m.$$

For the moment we leave aside the problem of determining whether f_d has finite degree and end this section with some remarks.

Remark 3.1.8. The following remarks are implied in [O-P], §7:

1. $f^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}Q_d^*}(1)$ hence it is enough to compute the first Chern class of the twisted structure sheaf of $\mathbb{P}Q_d^*$;

2. the self intersection of the hyperplane class in $\mathbb{P}Q_d^*$ is the top Segre class of Q_d^* and, in turn, this is the top Chern class of N_d^* because of sequence (3.1.3);

3. since $Q_d \simeq N_{g-d-1}^*$ via a choice of a theta characteristic κ (see the proof of proposition 3.1.6), the top Chern class of N_d^* is the same as $c_g(Q_{g-d-1})$;

4. by making the substitution $d \leftrightarrow (g-d-1)$ the degree of G_{g-d} should be given by

$$\deg f_{g-d-1} \cdot \deg G_{g-d} = \int_{\mathbb{P}Q^*_{g-d-1}} c_g(Q_d) \tag{3.1.5}$$

where $\deg f_{g-d-1}$ is the number of translates of W_d contained in the support of the general $D \in G_{g-d}$, i.e., the number of line bundles $\eta \in J^{g-d}$ such that $W_{d,\eta^{-1}} \subset supp\, D$.

3.2 The hypersurface case: G_{g-1}

In this section the attention will be focused on the top element of the Abel-Jacobi stratification, G_{g-1}. In this case $d = 1$, hence we are restricting attention to 2Θ divisors which contain translates of $W_1 \cong C$ in their support. If the map $f_{g-d-1} = f_{g-2}$ can be shown to be of finite degree, then the dimension of G_{g-1} is $2^g - 2$, i.e., it should be a hypersurface of $|2\Theta|$ (see [O-P], remark 7.1). This is known only for non-hyperelliptic curves of genus 3, in which case $G_{g-1} = G_2$ is the embedded image of $\mathcal{SU}_C(2)$ in $|2\Theta|$ and the map f_1 has degree 8 (see page 55). In [O-P], §7, Oxbury and Pauly show how to compute the top Chern class of the vector bundle Q_1. In proposition 3.2.2 I provide a closed formula for $c_g(Q_1)$. However, [O-P] fails to prove that f_{g-2} is a finite degree map and find its degree. In proposition 3.2.5 I show that f_{g-2} is a map of finite degree map of degree 1 for all non-hyperelliptic curves of genus $g \geq 4$. Moreover this fact and the evaluation of $c_g(Q_1)$ allow me to use formula (3.1.5) and compute the degree of G_{g-1} for all $g \geq 4$.

First we give Oxbury and Pauly's computation of the top Chern class of the vector bundle Q_1 over the Jacobian J^1, as presented in [O-P], §7. Consider the product variety $J^1 \times C$ and denote by π and p the projections to J^1 and C, respectively. Let $\mathcal{P}$ be the Poincaré line bundle on $J^1 \times C$ such that $\mathcal{P}|_{J^1 \times \{p_0\}}$

is trivial for some choice of a point $p_0 \in C$. Define a coherent sheaf on J^1 as the push-forward $Q' = \pi_*(\mathcal{P}^2 \otimes p^* K_C)$. Then consider the translation map t_{p_0}, $J^1 \longrightarrow J^0$ such that $\eta \longmapsto \eta(-p_0)$ and define a line bundle on J^1 as $\mathcal{N} = t_{p_0}^* \mathcal{L}$.

$$\begin{tikzcd}
 & \mathcal{P}^2 \otimes p^* K \arrow[d] & \\
Q' \arrow[d] & J^1 \times C \arrow[r, "\beta_1"] \arrow[dl, "\pi"'] \arrow[dr, "p"] & J^0 \\
J^1 & & C
\end{tikzcd}$$

One can show that Q_1 and $Q' \otimes \mathcal{N}$ are isomorphic. To see this, recall that $Q_1 = \pi_*(\beta_1^* \mathcal{L})$ by definition and that, hence, it is enough to show that the line bundle $\beta_1^* \mathcal{L}$ is isomorphic to $(\mathcal{P}^2 \otimes p^* K_C) \otimes \pi^* \mathcal{N}$, since the isomorphism is then preserved by the projection to J^1. Consider the line bundles $\beta_1^* \mathcal{L} \otimes \pi^* \mathcal{N}^{-1}$ and $\mathcal{P}^2 \otimes p^* K_C$ and note that whenever restricted to $\{\eta\} \times C$, for any choice of a line bundle $\eta \in J^1$, they are are both isomorphic to $\eta^2 \otimes K$. Note that for $\mathcal{P}^2 \otimes p^* K_C$ this follows from the properties of the Poincaré line bundle, while for $\beta_1^* \mathcal{L} \otimes \pi^* \mathcal{N}^{-1}$ it is a consequence of proposition 3.1.4 and the fact that $\pi^* \mathcal{N}^{-1}$ is trivial on $\{\eta\} \times C$. Moreover the line bundle $\mathcal{P}^2 \otimes p^* K_C$ is trivial over $J^1 \times \{p_0\}$, while the line bundle $\mathcal{N}$ was chosen in such a way that $\beta_1^* \mathcal{L} \otimes \pi^* \mathcal{N}^{-1}$ has the same property. This is enough to assure that the two line bundles are isomorphic on $J^1 \times C$ and hence also their push-forwards to J^1 (see [Ha], exercise 12.4, p.291). Note, moreover that this also proves that Q' is a vector bundle.

To compute the Chern classes of Q_1, we will first compute those of Q'. The results presented in what follows can be found in [O-P], §7, they are presented here for clarity and because they will be needed again in §3.3 of this thesis. One starts by finding the Chern character of Q' via the Grothendieck-Riemann-Roch formula ([ACGH], p.333)

$$ch(\pi_!(\mathcal{P}^2 \otimes p^* K_C)) \cdot td(J^1) = \pi_*(ch(\mathcal{P}^2 \otimes p^* K_C) \cdot td(J^1 \times C))$$

where td denotes the Todd class (see [F], p.56), while for any coherent sheaf $\mathcal{F}$ on $J^1 \times C$, $\pi_!(\mathcal{F}) = \sum_i (-1)^i R^i \pi_* \mathcal{F}$ (see [ACGH], p.331). Note that $td(J^1)$ is trivial since the Jacobian has trivial tangent bundle and $td(J^1 \times C) = td(C)$.

Remark 3.2.1. One can verify that $\pi_!(\mathcal{P}^2 \otimes p^* K_C) = \pi_*(\mathcal{P}^2 \otimes p^* K_C) = Q'$. One starts by noting that all higher direct images of $\beta_1^* \mathcal{L}$ are zero as by proposition 3.1.5 in this thesis all higher cohomology groups $H^i(C, \mathcal{L}_\eta)$ are zero as η varies in J^1 and hence one can use corollary 12.9 of [Ha], p.288, to show that $R^i \pi(\beta_1 \mathcal{L})$ are zero for i positive. Finally, by the projection formula (exercise 8.3 in [Ha], p.253) the same applies to $\mathcal{P}^2 \otimes p^* K_C$, since $\mathcal{P}^2 \otimes p^* K_C = \beta_1^* \mathcal{L} \otimes \pi^* \mathcal{N}^{-1}$.

The above implies

$$ch(Q') = \pi_*(ch(\mathcal{P}^2 \otimes p^* K_C) \cdot (td(C))). \tag{3.2.1}$$

In the following we will compute all the terms on the right hand side of the equation (3.2.1):

- $td(C) = 1 - (g-1)x$

  This can be seen by recalling that $td(C)$ is the Todd class of the tangent bundle of C, which is a line bundle with first Chern class $c_1(\mathcal{T}_C) = -c_1(K_C) = (2g-2)x$ where x is the class of a point on C (see §A.1), hence (see [F], p.56)

  $$td(C) = 1 + \frac{1}{2} c_1(\mathcal{T}_C) = 1 - (g-1)x.$$

- $ch(\mathcal{P}) = 1 + x + \gamma - x\theta$

  The class $\gamma \in H^1(J^1, \mathbb{Z}) \otimes H^1(C, \mathbb{Z})$ is constructed in [ACGH], p.335, θ is the class of Θ in the Jacobian (see §A.1 of the Appendix) and the proof of the result can be found in [ACGH], p.336.

- $ch(p^* K_C) = 1 + (2g-2)x$

  This is a direct consequence of the fact that the first Chern class of K_C is $(2g-2)x$.

It is then possible to compute the Chern character of Q':

$$\begin{aligned} ch(Q') &= \pi_*((1+x-\gamma-x\theta)^2(1+(2g-2)x)(1-(g-1)x)) \\ &= \pi_*(1+(g-1)x-4\theta x+2x+2\gamma) \\ &= g+1-4\theta \end{aligned} \tag{3.2.2}$$

where one uses well known identities for x, γ and θ (see [ACGH], pp. 335-336) and the fact that $\pi_*(1)=\pi_*(\gamma)=0$ while $\pi_*(x)=1$, i.e., π_* *selects* only the coefficients of x. Hence, by applying Newton's formula (see [F], p.56) one can compute the Chern classes of Q'

$$p_n - c_1 p_{n-1} + c_2 p_{n-2} + \cdots (-1)^{n-1} c_{n-1} p_1 + (-1)^n n c_n = 0$$

where $p_j/j!$ is the j-th term of the expansion of the Chern character of Q' and the only non-zero one is $p_1 = -4\theta$. The i-th Chern class of Q' is then given by

$$c_i(Q') = \frac{1}{i!}(-4\theta)^i.$$

On the other hand the first Chern class of $\mathcal{N}$ is clearly

$$c_1(\mathcal{N}) = 2\theta$$

since $\mathcal{N}$ is a translate of $\mathcal{L}$. It is then possible to use classical results to compute the top Chern class of Q_1. In particular, the following result gives an expression for the top Chern class of Q_1, which cannot be found in [O-P].

Proposition 3.2.2. *The top Chern class of Q_1 is given by*

$$c_g(Q_1) = 2^g\theta^g \sum_{i=0}^{g} \frac{(-2)^i}{i!}(g+1-i).$$

Proof. One can use example 3.2.2 of [F] to obtain

$$\begin{aligned} c_g(Q_1) &= c_g(Q'\otimes\mathcal{N}) \\ &= \textstyle\sum_{i=0}^g \binom{g+1-i}{g-i} c_i(Q') c_1(\mathcal{N})^{g-i} \\ &= \textstyle\sum_{i=0}^g (g+1-i)\frac{1}{i!}(-4\theta)^i(2\theta)^{g-i} \\ &= (2\theta)^g \textstyle\sum_{i=0}^g \frac{(-2)^i}{i!}(g+1-i). \end{aligned}$$

□

Example 3.2.3. The top Chern class of Q_1 for low genus curves is then easily computed

$$
\begin{array}{ll}
g = 3 & \int_{\mathbb{P}Q_1^*} c_3(Q) = 32 \\
g = 4 & \int_{\mathbb{P}Q_1^*} c_4(Q) = 384 \\
g = 5 & \int_{\mathbb{P}Q_1^*} c_5(Q) = 4096
\end{array}
$$

where we recall that since θ is the class of the Θ divisor, $\int \theta^g = g!$. Note that these values are consistent with those given in [O-P], p.310.

However, to evaluate the dimension of G_{g-1} and compute its degree via formula (3.1.5) it is necessary to study the map f_{g-2}.

Remark 3.2.4. If $g = 3$ then we already know that $f_{g-d-1} = f_1$ is a map of finite degree and deg $f_1 = 8$, so together with the above computation this gives deg $G_2 = 4$. This is the result referred to in §2.3.1 and it completes the proof that $\mathcal{SU}_C(2)$ embeds as a quartic in $|2\Theta|$.

The following results extend the study of f_{g-2} to non-hyperelliptic curves of genus higher than 3 in an "unexpected" way. Oxbury and Pauly had conjectured that at least for curves of genus 4 the degree of G_{g-1} should be 4, i.e., deg $f_{g-2} =$ 96, however this turns out not to be the case.

Recall that $G_{g-1} = \{D \in |2\Theta| \,|\, C_\xi \subset supp\, D,\, \xi \in J^{g-2}\}$ is the image of the vector bundle $\mathbb{P}Q^*_{g-2} \subset |2\Theta| \times J^{g-2}$ in $|2\Theta|$ via the natural projection f_{g-2} (see proposition 3.1.6). Hence the degree of f_{g-2}, if finite, corresponds to the number of translates of the curve C that lie in the support of the general 2Θ divisor $D \in G_{g-1}$.

Proposition 3.2.5. *Given a non-hyperelliptic curve C of genus $g \geq 4$, the map f_{g-2} has degree* 1, *hence $G_{g-1} \subset |2\Theta|$ is a hypersurface and*

$$\deg\, G_{g-1} = \int_{\mathbb{P}Q_1^*} c_g(Q_1).$$

Proof. Recall that, as seen in the course of the proof of proposition 3.1.6, the isomorphism between $H^0(J^{g-1}, 2\Theta)$ and $H^0(J^0, \mathcal{L})$ given by tensoring with a fixed theta characteristic $\kappa \in \vartheta(C)$, induces an isomorphism between the fibres $Q^*_{g-2,\eta}$ and $N_{1,\kappa\eta^{-1}}$ for every line bundle $\eta \in J^{g-2}$. Moreover, as a consequence of propositions 3.1.4 and 3.1.5 as well as the characterisations on page 65, $N^*_{1,\kappa\eta^{-1}} = H^0(C, K(\kappa\eta^{-1})^2) = H^1(C, K^{-1}\eta^2)^*$, i.e., $N_{1,\kappa\eta^{-1}} = H^1(C, K^{-1}\eta^2)$. Hence one has the following commutative diagram

$$\begin{tikzcd}
H^0(J^{g-1}, 2\Theta) \arrow[r, leftrightarrow] & H^0(J^0, \mathcal{L}) = H^0(J^{g-1}, 2\Theta)^* \\
Q^*_{g-2,\eta} \arrow[u, "f_{g-2}"] \arrow[r, leftrightarrow, "\sim"] & H^1(C, K^{-1}\eta^2) \arrow[u]
\end{tikzcd}$$

where we recall that $H^0(J^0, \mathcal{L})$ is canonically isomorphic to $H^0(J^{g-1}, 2\Theta)^*$ via the Wirtinger duality (see lemma 1.1.2). Finally, by projectivising the embedding on the right hand side of the above diagram one can obtain another commutative diagram

$$\begin{tikzcd}
\mathbb{P}H^1(C, K^{-1}\eta^2) = \mathbb{P}(\eta) \arrow[r] \arrow[dr, "\varepsilon_\eta"] & {|2\Theta|^* \cong \mathbb{P}^{2^g-1}} \\
& \mathcal{SU}_C(2, K) \arrow[u, "\delta'"]
\end{tikzcd}$$

where ε_η is the extension map parametrising S-equivalence classes of semi-stable vector bundles E isomorphic to an extension of the form $0 \to \eta \to E \to K\eta^{-1} \to 0$ (see §1.4). As seen in previous sections ε_η is a linear moduli map, while δ' is an embedding (see remark 2.1.1). In this way, the problem of finding the degree of f_{g-2} becomes that of studying the intersection of the spaces $\mathbb{P}(\eta)$ in $\mathcal{SU}_C(2, K)$ as η varies in J^{g-2}. This latter problem turns out to be easier to tackle because of a result of Oxbury et al. in [O-P-Pr], proposition 1.2:

> for any non-hyperelliptic curve C and any pair of distinct line bundles $\xi, \eta \in J^{g-2}$ the intersection $\mathbb{P}(\xi) \cap \mathbb{P}(\eta)$ is either empty or

1. the secant line $\overline{pq}$ of the curve (in either $\mathbb{P}(\xi)$ or $\mathbb{P}(\eta)$) if $\xi \otimes \eta = K(-p-q)$;

2. the point $\xi(p) \oplus K\eta^{-1}(-p) \in Kum\,(J^{g-1})$ if $h^0(C, K\xi^{-1}\eta^{-1})$ is zero and $\xi \otimes \eta^{-1} = \mathcal{O}(q-p)$.

Hence for any choice of a line bundle $\eta \in J^{g-2}$, in $\mathbb{P}(\eta)$ there will be a family of lines, parametrised by C_2 (by condition 1. above) and a family of points parametrised by $C \times C$ (by condition 2.) where $\mathbb{P}(\eta)$ intersects other extension spaces. As η varies in J^{g-2}, this means that there are two subvarieties in $\mathbb{P}Q^*_{g-2}$ where the map f_{g-2} fails to be injective: $\mathcal{S}_1$, a family of lines parametrised by $J^{g-2} \times C_2$, and $\mathcal{S}_2$, a family of points parametrised by $J^2 \times C \times C$. Since $\dim \mathcal{S}_1 = g+3$ and $\dim \mathcal{S}_2 = g+2$, while $\dim \mathbb{P}Q^*_{g-2} = 2^g - 2$ the general point of $\mathbb{P}Q^*_{g-2}$ fails to lie on either of them whenever $g \geq 4$. Equivalently, the general 2Θ divisor $D \in G_{g-1}$ contains only one translate of the curve C in its support and the map f_{g-2} has degree one. In turn, this implies that G_{g-1} is a hypersurface whose degree can be computed using formula (3.1.5):

$$\deg\, G_{g-1} = \int_{\mathbb{P}Q_1^*} c_g(Q_1).$$

□

Remark 3.2.6. If C has genus 3, then $\dim \mathbb{P}Q_1^* = 2^3 - 2 = 6 = g+3 = \dim \mathcal{S}_1$ and hence the general 2Θ divisor $D \in G_2$ contains more than one translate of C in its support. This is consistent with what already known, i.e., that in this case the degree of f_1 is 8 (see §2.3.1).

3.3 Remarks on the general case

This section tackles some of the issues involved in studying the general element of the Abel-Jacobi stratification. In analogy with the hypersurface case there are two aspects to this problem. The first consists in determining the top Chern class of the vector bundle Q_d over J^d where there is no immediate analogous construction to the one used for $d = 1$ (see page 68 and proposition 3.2.2). The second consists in showing that the map f_{g-d-1} is has finite degree and computing it, yet this time it is not possible to use a result like that of [O-P-Pr], proposition 1.2 or [Gr], remark 2.0.4.

In the following I will show how to proceed in order to deal with the first problem. In detail, a method is given that allows to compute the top Chern class of all the vector bundles Q_d. Note that in the case of curves of genus 4, where there is only one variety between the Kummer and $G_{g-1} = G_3$, that is G_2, and the map f_2 is already known to be finite, this gives the same result as [Gr] (see formula (3.1.4)) as will be verified explicitly at the end of the next section.

The last section presents some remarks on other interesting questions concerning the Abel-Jacobi stratification and its relation to classical configurations present in the projective space $|2\Theta|$.

3.3.1 The top Chern class of Q_d

The idea is to repeat the construction used by Oxbury and Pauly, that is, to compute the top Chern class of Q_d by introducing a new vector bundle Q'_d over J^d and showing that Q_d differs from it only by tensoring with a line bundle (see page 68). One can then use Grothendieck-Riemann-Roch computations to compute the Chern character of Q'_d and then use classical results to obtain the top Chern class of Q_d. However, in order to define Q'_d it is necessary to introduce a "symmetric" Poincaré line bundle, i.e., a Poincaré line bundle, $\mathcal{P}_d$, over the product space $J^d \times C_d$ parametrising line bundles of degree d on C_d that are induced by line bundles on C. For the details of this construction we refer the reader to the Appendix, §A.3.

Throughout the section we will use notation and results provided in the Appendix. In particular, if L is a line bundle on the curve C, we denote by $(L)_d$ or L_d the induced line bundle on the symmetric product C_d.

$$\begin{tikzcd}
 & \mathcal{P}_d^2 \otimes p^*(K_d(-\overline{\Delta})) \arrow[d] & \\
Q'_d \arrow[d] & J^d \times C_d \arrow[r, "\beta_d"] \arrow[dl, "\pi"'] \arrow[dr, "p"] & J^0 \\
J^d & & C_d
\end{tikzcd}$$

Consider the product variety $J^d \times C_d$ and let π and p be the natural projections to J^d and C_d, respectively. Take the "symmetric" Poincaré bundle $\mathcal{P}_d$ such that $\mathcal{P}_d|_{J^d \times \{D_0\}}$ is trivial for some choice of a point $D_0 \in C_d$, where we think of C_d as parametrising effective divisors of degree d on C. Define a coherent sheaf over J^d as $Q'_d = \pi_*(\mathcal{P}_d^2 \otimes p^*(K_d(-\overline{\Delta})))$, where $\overline{\Delta}$ is the diagonal divisor in C_d (see proposition 3.1.4). Finally, consider the translation $t_{D_0} : J^d \longrightarrow J^0$ given by $\eta \longmapsto \eta(-D_0)$ and the line bundle over J^d defined as $\mathcal{N}_d = t_{D_0}^*\mathcal{L}$. Then, in analogy to the hypersurface case, one can prove the following result.

Proposition 3.3.1. *Q'_d is a vector bundle. In fact, it is isomorphic to $Q_d \otimes \mathcal{N}^{-1}$.*

Proof. Recall that $Q_d = \pi_*\beta_d^*\mathcal{L}$ and $Q'_d = \pi_*(\mathcal{P}_d^2 \otimes p^*(K_d(-\overline{\Delta})))$. Consider the line bundles $\beta_d^*\mathcal{L}$ and $(\mathcal{P}_d^2 \otimes p^*(K_d(-\overline{\Delta}))) \otimes \pi^*\mathcal{N}$ defined over $J^d \times C_d$, we start by showing that they are isomorphic. This is equivalent to showing

$$\mathcal{P}_d^2 \otimes p^*(K_d(-\overline{\Delta})) \cong \beta_d^*\mathcal{L} \otimes \pi^*\mathcal{N}^{-1}. \tag{3.3.1}$$

Note that by construction and propositions 3.1.4 and 3.1.5 these two line bundles are isomorphic when restricted to $\{\eta\} \times C_d$ for any choice of a line bundle $\eta \in J^d$. Moreover they are both trivial when restricted to $J^d \times \{D_0\}$ and this is enough to assure that they are isomorphic as bundles on $J^d \times C_d$ (see also page 69). Pushing-forward with the projection π preserves the isomorphism, hence Q_d is isomorphic to $Q'_d \otimes \mathcal{N}$. Finally, since Q_d and $\mathcal{N}$ are vector bundles, it follows that Q'_d is a vector bundle. $\square$

The result presented in proposition 3.3.1 allows us to compute the Chern classes of Q_d by finding the Chern polynomial of Q'_d or, rather, its Chern character, using the Grothendieck-Riemann-Roch formula (see [ACGH], p.333 and page 69 of this thesis). Let $\mathcal{E} = \mathcal{P}_d^2 \otimes p^*(K_d(-\overline{\Delta}))$, then

$$ch(\pi_!\mathcal{E}) \cdot td(J^d) = \pi_*(ch(\mathcal{E}) \cdot td(J^d \times C_d)) \tag{3.3.2}$$

where $td(J^d) = 1$ and that $td(J^d \times C_d) = td(C_d)$ is given in closed formula by Oxbury and Pauly in [O-P], p.316

$$td(C_d) = \Big(\frac{x_d}{1 - e^{-x_d}}\Big)^{d-2g+1} \prod_{i=1}^{g} \Big(\frac{x_d - \sigma_{d,i}}{1 - e^{-x_d + \sigma_{d,i}}}\Big).$$

As in the hypersurface case discussed on page 69, $ch(\pi_!\mathcal{E}) = ch(Q'_d)$ since there are no higher direct images of $\mathcal{E}$, i.e., $R^i\pi_*\mathcal{E} = 0$ if $i \geq 1$ as can be verified by applying once more the reasoning of remark 3.2.1. One can compute the first Chern class of $\mathcal{E}$ as

$$\begin{aligned} c_1(\mathcal{P}_d) &= \gamma_d + dx_d, & (3.3.3)\\ c_1(K_d) &= (\deg K)x_d = (2g-2)x_d, & (3.3.4)\\ c_1(\mathcal{O}(-\overline{\Delta})) &= -2(d+g-1)x_d + 2\theta_d & (3.3.5) \end{aligned}$$

where (3.3.4) and (3.3.5) are obtained in §A.1, while (3.3.3) is just A.3.2 of §A.3, and hence

$$c_1(\mathcal{E}) = 2\gamma_d + 2\theta_d.$$

This in turn gives the Chern character of $\mathcal{E}$

$$ch(\mathcal{E}) = e^{c_1(\mathcal{E})} = e^{2\gamma_d + 2\theta_d}.$$

The Grothendieck-Riemann-Roch formula then allows the computation of the Chern character of $Q'_d = \pi_!\mathcal{E} = \pi_*\mathcal{E}$ as

$$ch(Q'_d) = \pi_*\Big(e^{2\gamma_d + 2\theta_d}\Big(\frac{x_d}{1 - e^{-x_d}}\Big)^{d-2g+1} \prod_{i=1}^{g} \Big(\frac{x_d - \sigma_{d,i}}{1 - e^{-x_d + \sigma_{d,i}}}\Big)\Big). \tag{3.3.6}$$

Note that, though it is not too difficult to see how the push-forward π_* acts and a detailed analysis of how to compute integrals is given in the Appendix (see §A.2), there are a number of computational difficulties when dealing with high genus curves and large values of d . In the following I demonstrate the procedure explicitly in the case $d = 2$.

Example 3.3.2. The case when $d = 2$, which correspond to the varieties G_{g-2} because of formula (3.1.5), can be dealt with easily using notation and results of the Appendix. First of all note that since we are working on $J^2 \times C_2$ all monomials in θ_2, x_2 and γ_2^2 of degree higher than 2 are zero, hence one can explicitly express the Chern character of $\mathcal{E}$. Note, moreover, that since the class γ_2 only appears in the expression of the Chern character of $\mathcal{E}$ and one knows that all monomials containing an odd power of γ_2 map to zero under π_* (as they don't lie in $H^*(J^2,\mathbb{Z}) \otimes H^4(C,\mathbb{Z})$, see §A.2) one can "forget" all terms containing an odd power of γ_2:

$$\begin{aligned} e^{2\gamma_d+2\theta_d} &= \sum \frac{1}{k!}(2\gamma_2 + 2\theta_2)^k \\ &= 1 + 2\gamma_2 + 2\theta_2 + 2\gamma_2^2 + 4\gamma_2\theta_2 + 2\theta_2^2 + \frac{4}{3}\gamma_2^3 + 4\gamma_2^2\theta_2 + \frac{2}{3}\gamma_2^4 \\ &\sim 1 + 2\theta_2 + 2\gamma_2^2 + 2\theta_2^2 + 4\gamma_2^2\theta_2 + \frac{2}{3}\gamma_2^4 \end{aligned}$$

where the symbol $\sim$ denotes that one can use the expression on the last line instead of $e^{2\gamma_d+2\theta_d}$ in the calculation of $ch(Q_2')$ in (3.3.6). The other two factors in the expression of the Todd class of C_2 can also be simplified

$$\left(\frac{x_2}{1-e^{-x_2}}\right)^{3-2g} = \left(1 - \frac{1}{2}x_2 + \frac{1}{6}x_2^2\right)^{2g-3} = 1 - \frac{2g-3}{2}x_2 + \frac{6g^2-17g+12}{12}x_2^2$$

$$\begin{aligned} \prod_{i=1}^{g}\left(\frac{x_2-\sigma_{2,i}}{1-e^{-x_2+\sigma_{2,i}}}\right) &= \prod_{i=1}^{g}\left(1 + \frac{1}{2}(x_2-\sigma_{2,i}) + \frac{1}{12}(x_2^2 - 2x_2\sigma_{2,i})\right) \\ &= 1 + \frac{g}{2}x_2 - \frac{1}{2}\theta_2 + \frac{g(3g-1)}{24}x_2^2 - \frac{3g-1}{12}x_2\theta_2 + \frac{1}{8}\theta_2^2. \end{aligned}$$

Hence the Todd class of C_2 is given by

$$td(C_2) = 1 + \frac{3-g}{2}x_2 - \frac{1}{2}\theta_2 + \frac{3g^2-17g+24}{24}x_2^2 + \frac{3g-8}{12}x_2\theta_2 + \frac{1}{8}\theta_2^2;$$

in this case one could have alternatively used the expression for the Todd class of a surface which only requires the knowledge of the two Chern classes of C_2 (see [F], p.56). Altogether this gives

$$\begin{aligned} ch(Q_2') &= \pi_*\Big(\Big(1+2\theta_2+2\gamma_2^2+2\theta_2^2+4\gamma_2^2\theta_2+\frac{2}{3}\gamma_2^4\Big) \\ &\quad \cdot\Big(1-\frac{g-3}{2}x_2-\frac{1}{2}\theta_2+\frac{3g^2-17g+24}{24}x_2^2+\frac{3g-8}{12}x_2\theta_2+\frac{1}{8}\theta_2^2\Big)\Big) \\ &= \pi_*\Big(1+\frac{3-g}{2}x_2+\frac{3}{2}\theta_2+\frac{3g^2-17g+24}{24}x_2^2+\frac{28-9g}{12}x_2\theta_2+\frac{9}{8}\theta_2^2 \\ &\quad +2\gamma_2^2+(3-g)\gamma_2^2x_2+3\gamma_2^2\theta_2+\frac{2}{3}\gamma_2^4\Big). \end{aligned}$$

In order to obtain the Chern character of Q_2' it is necessary to push forward the above expression via π_*. The only classes that do not map to zero are those of degree 2 in x_2, θ_2 and γ_2^2. Moreover if a is any class of J^2, identified with its pull-back to $J^2 \times C_2$, then

$$\pi_*(ax_2^2)=a \qquad \pi_*(ax_2\theta_2)=ga \qquad \pi_*(a\theta_2^2)=g(g-1)a$$

and note that these push-forwards are, in fact, known for all integers $d<g$, that is, $\pi_*(ax_d^i\theta_d^{d-i})=g\cdots(g-d+i+1)a$ for all integers $0\geq i\geq d$ and classes a of J^d (see [ACGH], p.26). As for the push-forwards of classes that contain powers of γ_2^2, these have been computed in §A.2, in particular example A.2.5 of the Appendix deals explicitly with the case $d=2$. All together this gives

$$ch(Q_2') = \frac{g^2+g+2}{2} - 4g\theta + 8\theta^2$$

where θ is the class of the theta divisor in the Jacobian. First of all note that the constant term of the Chern character gives the rank of the bundle Q_2' and it is easy to verify that this value coincides with the one previously computed for Q_2, i.e., $rk\, Q_2' = rk\, Q_2 = \sum_{i=0}^{2}\binom{g}{i}$ (see the proof of proposition 3.1.6). Next, one can use Newton's formula ([F], p.56 or page 71) to compute the Chern classes of Q_2'. Let

$$p_1=-4g\theta \qquad p_2=16\theta^2$$

then, denoting by c_i the i-*th* Chern class of Q'_2, one has

$$\begin{aligned} c_1 &= p_1 = -4g\theta \\ c_2 &= \frac{1}{2}(c_1p_1 - p_2) = 8(g^2-1)\theta^2 \\ c_k &= \frac{1}{k}(c_{k-1}p_1 - c_{k-2}p_2) \qquad k \geq 3. \end{aligned}$$

It is possible to give a closed formula for the k-*th* Chern class of Q'_2.

Lemma 3.3.3. *The k-th Chern class of the vector bundle Q'_2 is given by*

$$c_k = \frac{4^k}{k!}\theta^k \sum_{i=0}^{\left[\frac{k}{2}\right]} (-1)^{k-i} g^{k-2i} \binom{k}{2i} \frac{(2i)!}{2^i i!}$$

where [ ] *denotes the integer part of a number.*

Proof. The statement is easily checked to be true for $k=1$ and $k=2$. As for $k>2$, this result can be proved by induction, though one has to pay attention to distinguish between two cases, k even and k odd. The main tool is the recursive relation

$$c_k = \frac{1}{k}(c_{k-1}p_1 - c_{k-2}p_2) \qquad k \geq 3$$

which, assuming the statement to be true for integers smaller than k, gives

$$\begin{aligned} c_k &= \frac{1}{k}(c_{k-1}(-4g\theta) - c_{k-2}(16\theta^2)) \\ &= \frac{1}{k}\Bigg[\Bigg(\frac{4^{k-1}}{(k-1)!} \sum_{i=0}^{\left[\frac{k-1}{2}\right]} (-1)^{k-1-i} g^{k-1-2i} \binom{k-1}{2i} \frac{(2i)!}{2^i i!}\Bigg)(-4g) + \\ &\quad - \Bigg(\frac{4^{k-2}}{(k-2)!} \sum_{i=0}^{\left[\frac{k-2}{2}\right]} (-1)^{k-2-i} g^{k-2-2i} \binom{k-2}{2i} \frac{(2i)!}{2^i i!}\Bigg)(16)\Bigg]\theta^k. \end{aligned} \tag{3.3.7}$$

It is easy to see that $\left[\frac{k-2}{2}\right] = \left[\frac{k}{2}\right] - 1$, while $\left[\frac{k-1}{2}\right]$ depends on the parity of k.

Assume k is even, in which case $\left[\frac{k-1}{2}\right] = \left[\frac{k-2}{2}\right]$ and

$$\begin{aligned} c_k &= \frac{4^k}{k}\frac{1}{(k-2)!}\Big[\sum_{i=0}^{\left[\frac{k-2}{2}\right]} \frac{1}{k-1}(-1)^{k-i}g^{k-2i}\binom{k-1}{2i}\frac{(2i)!}{2^i i!} \\ &\quad - \sum_{i=0}^{\left[\frac{k-2}{2}\right]}(-1)^{k-i}g^{k-2-2i}\binom{k-2}{2i}\frac{(2i)!}{2^i i!}\Big]\theta^k \qquad (3.3.8) \\ &= \frac{4^k}{k}\frac{1}{(k-2)!}\Big[\frac{1}{k-1}(-1)^k g^k + \sum_{i=1}^{\left[\frac{k-2}{2}\right]}\Big(\frac{1}{k-1}(-1)^{k-i}g^{k-2i}\binom{k-1}{2i}\frac{(2i)!}{2^i i!} \\ &\quad +(-1)^{k-i}g^{k-2i}\binom{k-2}{2i-2}\frac{(2i-2)!}{2^{i-1}(i-1)!}\Big) + (-1)^{\frac{k}{2}}\frac{(k-2)!}{2^{(k-2)/2}((k-2)/2)!}\Big]\theta^k \end{aligned}$$

where the first term is just the value corresponding to $i = 0$ in the first sum while the last one comes from $i = \frac{k-2}{2}$ in the second sum, moreover in the second sum there has been a change of variable $i \to i + 1$. Rearranging these terms gives

$$\begin{aligned} c_k &= \frac{4^k}{k}\frac{1}{(k-2)!}\Big[\frac{1}{k-1}(-1)^k g^k + \frac{1}{k-1}\sum_{i=1}^{\left[\frac{k-2}{2}\right]}(-1)^{k-i}g^{k-2i}\Big(\binom{k-1}{2i}\frac{(2i)!}{2^i i!} \\ &\quad +(k-1)\binom{k-2}{2i-2}\frac{(2i-2)!}{2^{i-1}(i-1)!}\Big) + (-1)^{\frac{k}{2}}\frac{(k-2)!}{2^{\frac{k-2}{2}}(\frac{k-2}{2})!}\Big]\theta^k \\ &= \frac{4^k}{k!}\Big[(-1)^k g^k\binom{k}{0}\frac{(2\cdot 0)!}{2^0 0!} + \sum_{i=1}^{\left[\frac{k}{2}\right]-1}(-1)^{k-i}g^{k-2i}\binom{k}{2i}\frac{(2i)!}{2^i i!} \\ &\quad +(-1)^{k-\frac{k}{2}}g^0\binom{k}{k}\frac{k!}{2^{k/2}(\frac{k}{2})!}\Big]\theta^k \\ &= \frac{4^k}{k!}\theta^k\sum_{i=0}^{\left[\frac{k}{2}\right]}(-1)^{k-i}g^{k-2i}\binom{k}{2i}\frac{(2i)!}{2^i i!} \qquad (3.3.9) \end{aligned}$$

and hence one has the required result when k is an even integer. The case when k is an odd integer presents no conceptual difference. The main point is that at step (3.3.8) one only needs to take out the term corresponding to $i = 0$ in the first sum of (3.3.7) and rearrange everything else accordingly. □

If C is a curve of genus 4 then the vector bundle Q_2' has only 4 Chern classes

$$\begin{aligned} c_1 &= -4g\theta = -16\theta \\ c_2 &= 8(g^2-1)\theta^2 = 120\theta^2 \\ c_3 &= -\frac{32}{3}g^3\theta^3 + 32g\theta^3 = -\frac{1664}{3}\theta^3 \\ c_4 &= \frac{32}{3}g^4\theta^4 - 64g^2\theta^4 + 32\theta^4 = \frac{5216}{3}\theta^4. \end{aligned}$$

Recall that $Q_2 = Q_2' \otimes \mathcal{N}$ by proposition 3.3.1 and hence the top Chern class of Q_2, that is, $c_g(Q_2)$, can be computed using a classic formula (see [F], p.56), the above lemma 3.3.3 and the fact that $c_1(\mathcal{N}) = 2\theta$ since $\mathcal{N}$ is just a translate of the line bundle $\mathcal{L}$ (see page 69):

$$\begin{aligned} c_g(Q_2) &= \sum_{k=0}^{g} \binom{rk-k}{g-k} c_k \cdot c_1(\mathcal{N})^{g-k} \\ &= \sum_{k=0}^{g} \binom{rk-k}{g-k} c_k \cdot (2\theta)^{g-k} \\ &= (2\theta)^g \sum_{k=0}^{g} \frac{2^k}{k!} \binom{rk-k}{g-k} \left(\sum_{i=0}^{\left[\frac{k}{2}\right]} (-1)^{k-i} g^{k-2i} \binom{k}{2i} \frac{(2i)!}{2^i i!} \right) \quad (3.3.10) \end{aligned}$$

where $rk = \frac{g^2+g+2}{2}$ is the rank of Q_2. So, if f_{g-3} were a finite map, one could use formula (3.1.5) to compute the degree of G_{g-2},

$$\deg\, G_{g-2} = \frac{1}{\deg\, f_{g-3}} \int_{\mathbb{P}Q_2} c_g(Q_2).$$

In particular, if the curve C has genus 4, then $rk = 11$ and the top Chern class of Q_2 is given by

$$c_4(Q_2) = 64\theta^4.$$

Moreover in this case $G_{g-2} = G_2$ and Gronow has already shown that f_1 is a finite morphism of degree one ([Gr], remark 2.0.4), hence recalling that $\int \theta^g = g!$ since it is the class of a principal polarisation one obtains

$$\deg\, G_2 = \int_{\mathbb{P}Q_2} c_g(Q_2) = \int_{J^2} 64\theta^4 = 1536$$

which we already know to be true (see [Gr], proposition 3.2.9 or (3.1.4)).

3.3.2 Open problems

The problem of studying the Abel-Jacobi stratification is still full of unanswered questions. As previously said, there is the problem of verifying whether the maps f_d are maps of finite degree for all integers d for which they are defined and, should this be the case, finding their degrees. This, together with the results presented in the above section would answer the question concerning the dimension and degree of each variety G_d.

It would also be of great interest to see what are the relations between the different elements of the stratification. It is well known that for curves of genus 3, where there are only two elements, $G_1 = Kum(J^0)$ and $G_2 = \mathcal{SU}_C(2)$, the first is precisely the singular locus of the second. It is reasonable to expect, in general, that G_{d-1} will be contained in the singular locus of G_d because of the characterisation given at the outset of the chapter (see remark 3.1.2). However this problem has not been tackled, not even for curves of genus 4.

Moreover there are questions concerning the relations between these objects and classical structures of $|2\Theta|$. For example, the variety G_2 is known to contain all trisecants to the Kummer variety $Kum(J^0)$ (see [O-P-Pr], §2) and a detailed study of all trisecant loci has been carried out by Gronow ([Gr], chapter 4). Similarly, Oxbury and Pauly have shown that for all curves of genus at least 4 the ruling of G_3 cuts out all Beauville-Debarre quadrisecants (see [B-D]) to each Prym Kummer in $|2\Theta|$ ([O-P-Pr], §9). Again, however, nothing is known of the other objects of this stratification.

Appendix A

Symmetric products of a curve

This appendix deals with classes and structures related to symmetric products of a given smooth algebraic curve C. The first section recalls the description of the integral cohomology ring of the curve C and the rational cohomology ring of its d-*th* symmetric product C_d, as done by Macdonald (see [Mc]), with particular attention to the classes of some specific cycles. Section A.2 deals with product varieties of the form $C_d \times J^d$ and in particular with a special class, denoted $\gamma_d \in H^2(C_d \times J^d, \mathbb{Z})$, which generalises a well known class on the products $C \times J$ (see [ACGH], p.335). Most of this section is devoted to understanding how to compute integrals of top degree classes of $C_d \times J^d$ that contain powers of γ_d. At the end of this section we also show how one can generalise this class to a class of the product variety $C_d \times C_e$. Finally, section A.3 provides the construction of a "symmetric" Poincaré line bundle, i.e., a line bundle $\mathcal{P}_d$ over $C_d \times J^d$ parametrising line bundles of C_d that are induced by line bundles on C. Most of the notions given in this appendix are used throughout chapter three. In particular, §A.2 allows the computation of the integral of certain top Chern classes needed in formula (3.1.5), while §A.3 is essential in finding the corresponding Chern classes (see proposition 3.3.1). However both these sections carry out original computations and constructions which might be of interest outside the context of this thesis.

A.1 Symmetric products of C

Let C be a complex algebraic curve of genus g. Following Macdonald, [Mc], we consider the integral cohomology of C

$$H^0(C,\mathbb{Z}) = \mathbb{Z} \qquad H^1(C,\mathbb{Z}) = \mathbb{Z}^{2g} \qquad H^2(C,\mathbb{Z}) = \mathbb{Z}$$

where $H^2(C,\mathbb{Z})$ has a generator x. The ring structure of $H^*(C,\mathbb{Z})$ is then determined by a selection of generators $\alpha_1, \ldots \alpha_{2g}$ of $H^1(C,\mathbb{Z})$ such that

$$\alpha_i\alpha_{i+g} = -\alpha_{i+g}\alpha_i = x \quad \text{class of a point}$$
$$\alpha_i\alpha_j = 0 \qquad i \neq j \pm g$$

where juxtaposition means "cup product". This is a complete set of relations for $H^*(C,\mathbb{Z})$ and it implies

$$\alpha_i x = x\alpha_i = 0 \qquad\qquad x^2 = 0$$

for all indices i. From now on we will work over the field of rational numbers $\mathbb{Q}$, recalling that $H^*(C,\mathbb{Q}) = H^*(C,\mathbb{Z}) \otimes \mathbb{Q}$ (see [Mc], p.321). Now, let C^d denote the d-*th* direct product of C. It is possible to give an explicit description of the rational cohomology of C^d. The ring $H^*(C^d,\mathbb{Q})$ is the d-*th* tensor power of $H^*(C,\mathbb{Q})$. In particular one can set

$$\alpha_l^k = 1 \otimes \cdots \otimes \alpha_l \otimes \cdots \otimes 1 \in H^1(C^d,\mathbb{Q}) \qquad l = 1,\ldots,2g \qquad k = 1,\ldots,d$$
$$\beta^k = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1 \in H^2(C^d,\mathbb{Q}) \qquad\qquad k = 1,\ldots,d.$$

These elements satisfy the following relations

$$\begin{array}{ll}
\alpha_i^k\alpha_{i+g}^k = -\alpha_{i+g}^k\alpha_i^k = \beta^k & \\
\alpha_i^k\alpha_j^k = 0 & i \neq j \pm g \\
\alpha_i^k\alpha_j^h = -\alpha_j^h\alpha_i^k & k \neq h \\
\alpha_i^k\beta^k = \beta^k\alpha_i^k = \beta^k\beta^k = 0 & \\
\beta^k\beta^h = \beta^h\beta^k & k \neq h.
\end{array}$$

Moreover every β^k commutes with every element in $H^*(C^d, \mathbb{Q})$.

Let S_d denote the symmetric group on d elements and consider its natural action on C^d; denote by C_d the quotient space C^d/S_d, this is the d-*th* symmetric product of the curve C. A result of Macdonald, ([Mc], §4, p.322), assures that the cohomology ring of C_d with rational coefficients is the invariant ring of the cohomology ring of C^d with respect to the action of the symmetric group S_d, i.e., $H^*(C_d, \mathbb{Q})$ is isomorphic to $H^*(C^d, \mathbb{Q})^{S_d}$ via the natural map $p^* : H^*(C_d, \mathbb{Q}) \longrightarrow H^*(C^d, \mathbb{Q})$ induced by the projection $p : C^d \longrightarrow C_d$. This implies ([Mc], §5, p.322 and (6.3), p.325) that $H^*(C_d, \mathbb{Q})$ is generated by $2g+1$ elements

$$\begin{array}{ll}
\xi_{d,i} \stackrel{def}{=} \sum_{k=1}^{d} \alpha_i^k \in H^1(C_d, \mathbb{Q}) & i = 1, \dots, 2g \\
x_d \stackrel{def}{=} \sum_{k=1}^{d} \beta^k \in H^2(C_d, \mathbb{Q}) &
\end{array}$$

satisfying the following relations

$$\xi_{d,i}\xi_{d,j} = -\xi_{d,j}\xi_{d,i} \qquad \xi_{d,i}x_d = x_d\xi_{d,i}.$$

It is then natural to define the following classes

$$\begin{array}{lll}
\sigma_{d,i} & \stackrel{def}{=} & \xi_{d,i}\xi_{d,i+g} = -\xi_{d,i+g}\xi_{d,i} \qquad i = 1, \dots g \\
\theta_d & \stackrel{def}{=} & \sum_{i=1}^{g} \sigma_{d,i}
\end{array}$$

where the σ_i have some immediate properties:

1. $\sigma_{d,i}^2 = 0$ for any $i = 1, \dots, g$;

2. $\sigma_{d,i}\sigma_{d,j} = \sigma_{d,j}\sigma_{d,i}$ for all i, j.

Remark A.1.1. Note that the integer cohomology ring of C_d is a subring of $H^*(C_d, \mathbb{Q})$ and that since all the classes defined above are integer classes we can think of them as being in $H^*(C_d, \mathbb{Z})$.

Remark A.1.2. One may identify the points of C_d with the effective divisors of degree d on C, hence there is a mapping $C_{d-1} \longrightarrow C_d$ given by $D \longmapsto D + p$ for any choice of a point $p \in C$ (see [Mc], p.329). Note that this map is an embedding and that x_d is the cohomology class of C_{d-1} embedded in C_d (see [Mc], proof of (14.2), p.332).

Recall that if Δ denotes the union of the diagonals in C^d, i.e., the ramification divisor of p: $C^d \longrightarrow C_d$ and $\overline{\Delta}$ the diagonal divisor in C_d, then $2\Delta = p^*(\overline{\Delta})$. Moreover the first Chern class of $\overline{\Delta}$ is given by

$$c_1(\overline{\Delta}) = 2(d + g - 1)x_d - 2\theta_d$$

as proved by Macdonald (see [Mc], (15.4) with $s = 2$).
Finally recall that the isomorphism u^* of $H^1(J, \mathbb{Z})$ with $H^1(C, \mathbb{Z})$, induced by the Abel-Jacobi map $u : C \longrightarrow J$, allows us to obtain a basis for the first integer cohomology group of the Jacobian, J, via pull-back along u^*. This basis is denoted by $\alpha'_1, \dots, \alpha'_{2g}$ (see [ACGH], p.334). Moreover one considers a class $\theta \in H^2(J, \mathbb{Z})$ defined as $\theta = \sum_i \alpha'_i \alpha'_{i+g}$. This is the class of the theta divisor in the Jacobian (see [ACGH], p.26). In the following we will find it convenient to use the symbol σ'_i to denote the cup products $\alpha'_i \alpha'_{i+g}$.

The notation introduced above will be used throughout the rest of the chapter.

A.2 Classes on the product $C_d \times J^d$

In the following we intend to describe some classes in the integer cohomology rings of the product variety of a symmetric product of a curve C of genus g, C_d, with the Jacobian J^d. There are "obvious" classes obtained by pulling back the classes of C_d and J^d and for which we maintain the same notation as given in the previous section, i.e., we identify $\alpha_{d,i}$ with its pull-back to $C_d \times J^d$ etc. We introduce a class in $H^2(C_d \times J^d, \mathbb{Z})$ which, via the Künneth decomposition, can be thought of as being in $H^1(C_d, \mathbb{Z}) \otimes H^1(J^d, \mathbb{Z})$

$$\gamma_d \stackrel{def}{=} -\sum_{i=1}^{g}(\xi_{d,i}\,\alpha'_{i+g} - \xi_{d,i+g}\,\alpha'_i). \tag{A.2.1}$$

This class is defined by analogy with the γ class of $C \times J$, see [ACGH], p.335, which is just γ_1 since $\xi_{1,i} = \alpha_i$ for all i. The aim is to be able to compute the integrals

$$\int_{C_d \times J^d} x_d^i\,\theta_d^j\,\gamma_d^{2k}\,A_{g-k} \tag{A.2.2}$$

where $i + j + k = d$ and $k \leq \mathcal{M} = min\{d, g\}$, while A_{g-k} is the pull-back of any class of degree $2g - 2k$ on the Jacobian. Clearly, if $k = 0$ this integral is easily computed as it "splits" in two integrals over C_d and J^d. Note, moreover, that γ_d must be raised to an even power for the integral to be non zero. More in detail, in the course of this section we will show that the following holds:

$$\int_{C_d \times J^d} x_d^i\,\theta_d^j\,\gamma_d^{2k}\,A_{g-k} = (-1)^k(2k)!\frac{j!}{k!}\Big(\sum_{h=0}^{min\{j,k\}}(-1)^h\binom{k}{h}\binom{g-h}{g-j}\Big)\int_{J^d}\theta^k\,A_{g-k}.$$

Remark A.2.1. In order to simplify notation, the subscript d will be dropped from the classes α, β, ξ and σ from here until the end of the section, since no confusion can arise.

Remark A.2.2. By unraveling the definitions of x_d and θ_d and recalling the rules of multiplication for the α_j^i's, one can see that what one really needs to compute are the integrals

$$\int_{C_d \times J^d} \beta^{i_1} \dots \beta^{i_l} \alpha_{j_1}^{i_{l+1}} \dots \alpha_{j_s}^{i_{l+s}} \alpha_{j_1+g}^{i_{l+s+1}} \dots \alpha_{j_s+g}^{i_{l+2s}} \gamma_d^{2k} A_{g-k} \tag{A.2.3}$$

over $C_d \times J^d$, where the indices satisfy the conditions

$$l + s = d - k$$
$$i_1, \dots, i_{l+2s} \in \{1, \dots, d\} \text{ are all different}$$
$$j_i, \dots, j_s \in \{1, \dots, g\} \text{ are all different.}$$

Note that the condition $s \leq k$ follows from the assumption that $i_1, \dots i_{l+2s}$ are all different, since this is possible only if $l + 2s \leq d$, which, together with $l + s = d - k$, gives the bound. In fact, it is easy to verify that

$$x_d^l = \sum_{|I|=l} \beta^{i_1} \dots \beta^{i_l} \tag{A.2.4}$$

where the I's are multi-indices without repetitions of length $|I| = l$ such that each $i_j \in \{1, \dots, d\}$. Let $\mathcal{N} = min\{d - l, l\}$ and consider the whole expression of θ_d to verify that

$$\begin{aligned} \theta_d^l &= \Big(\sum_{i=1}^{g} \sigma_i\Big) = \sum_{|I|=l} \sigma_{i_1} \dots \sigma_{i_l} \quad \text{since } \sigma_i^2 = 0 \\ &= \sum_{|I|=l} \xi_{i_1} \xi_{i_1+g} \dots \xi_{i_l} \xi_{i_l+g} \quad \text{since } \sigma_i = \xi_i \xi_{i+g} \end{aligned}$$

where I is a multi-index without repetitions of length l such that each index $i_j \in \{1, \dots, g\}$. Recall that $\xi_i = \sum_{k=1}^{d} \alpha_i^k$ and that $\alpha_i^k \alpha_j^k = 0$ unless $j = i \pm g$, hence

$$\begin{aligned} \theta_d^l &= \sum_{|I|=l} \Big(\sum_{k=1}^{d} \alpha_{i_1}^k \sum_{k=1}^{d} \alpha_{i_1+g}^k \cdots \sum_{k=1}^{d} \alpha_{i_l}^k \sum_{k=1}^{d} \alpha_{i_l+g}^k\Big) \\ &= \sum_{|I|=l} \Big(\sum_{k_1, \dots, k_l'=1}^{d} \alpha_{i_1}^{k_1} \alpha_{i_1+g}^{k_1'} \cdots \alpha_{i_l}^{k_l} \alpha_{i_l+g}^{k_l'} \Big). \end{aligned}$$

Note that all the terms in the above sum such that $k_i = k_j$ are zero since $\alpha_i^k \alpha_j^k = 0$ when i, $j \in \{1, \ldots, g\}$ and similarly for the multi-index K', so one can write

$$\begin{aligned}\theta_d^l &= \sum_{|I|=l} \Big(\sum_{|K|,|K'|=l} \alpha_{i_1}^{k_1} \alpha_{i_1+g}^{k_1'} \cdots \alpha_{i_l}^{k_l} \alpha_{i_l+g}^{k_l'} \Big) \\ &= \sum_{|K|,|K'|=l} \Big(\sum_{|I|=l} \alpha_{i_1}^{k_1} \alpha_{i_1+g}^{k_1'} \cdots \alpha_{i_l}^{k_l} \alpha_{i_l+g}^{k_l'} \Big)\end{aligned}$$

where K and K' are multi-indices without repetitions of length l such that each k_i, $k_i' \in \{1, \ldots, d\}$, and in the second step the order of summation has been inverted. Observe that if one has $k_j = k_j'$ then $\alpha_{i_j}^{k_j} \alpha_{i_j+g}^{k_j} = \beta^{k_j}$. So if the multi-indices K and K' have the first h indices in common, for example, one would obtain:

$$\sum_{|I|=l} \alpha_{i_1}^{k_1} \alpha_{i_1+g}^{k_1'} \cdots \alpha_{i_l}^{k_l} \alpha_{i_l+g}^{k_l'} = \frac{(g-l+h)!}{(g-l)!} \beta^{k_1} \cdots \beta^{k_h} \sum_{|I|=l-h} \alpha_{i_1}^{k_{h+1}} \alpha_{i_1+g}^{k_{h+1}'} \cdots \alpha_{i_{l-h}}^{k_l} \alpha_{i_{l-h}+g}^{k_l'}$$

while if the h indices in common are not the first h, the result still applies but one has to consider some permutation of the superscripts k_j. Note that two multi-indices K, K' as described above can be identical or have as few as $l - \mathcal{N}$ indices $k_j = k_j'$, where $\mathcal{N} = min\{d-l, d\}$. Moreover, in order for the corresponding term to be non-zero, it also must happen that the remaining set of indices k_j, k_j' are disjoint, i.e., if K and K' have exactly h indices in common and these are $k_1 = k_1', \ldots, k_h = k_h'$ then the remaining $k_{h+1}, \ldots k_l, k_{h+1}', \ldots, k_l'$ have to be all different (recall that $\alpha_i^k \alpha_j^k = 0$ unless $j = i \pm g$). Finally, as there are $\binom{l}{h}$ ways of fixing h pairs (k_j, k_j') from l given pairs of indices, one has

$$\begin{aligned}\theta_d^l &= \sum_{h=l-\mathcal{N}}^{l} \binom{l}{h} \frac{(g-l+h)!}{(g-l)!} \sum_{|K|=2l+h} \beta^{k_1} \cdots \beta^{k_h} \sum_{|I|=l-h} \alpha_{i_1}^{k_{h+1}} \alpha_{i_1+g}^{k_{h+2}} \cdots \alpha_{i_{l-h}}^{k_{2l-2h+1}} \alpha_{i_{l-h}+g}^{k_{2l-2h}} \\ &= \sum_{h=0}^{\mathcal{N}} \binom{l}{h} \frac{(g-h)!}{(g-l)!} \sum_{|J|=h} \sum_{|I|=l+h} \beta^{i_1} \ldots \beta^{i_{l-h}} \alpha_{j_1}^{i_{l-h+1}} \alpha_{j_1+g}^{i_{l+1}} \cdots \alpha_{j_1+g}^{i_{l+1}} \cdots \alpha_{j_h+g}^{i_{l+h}} \\ &= \sum_{h=0}^{\mathcal{N}} (-1)^{\frac{h^2-h}{2}} \binom{l}{h} \frac{(g-h)!}{(g-l)!} \sum_{|J|=h} \sum_{|I|=l+h} \beta^{i_1} \ldots \beta^{i_{l-h}} \alpha_{j_1}^{i_{l-h+1}} \cdots \alpha_{j_h}^{i_l} \alpha_{j_1+g}^{i_{l+1}} \cdots \alpha_{j_h+g}^{i_{l+h}}\end{aligned} \tag{A.2.5}$$

where in the second step one exchanges h with $l-h$, while in the last step the α's have been reordered and so there is a (possible) sign change. Hence it is clear that once one knows how to compute integrals of the form (A.2.3) then it is possible to compute (A.2.2).

One starts by expressing γ_d^{2k} for any $k \in \{1, \ldots, \mathcal{M}\}$, where $\mathcal{M} = min\{d, g\}$. This can be done explicitly for $k=1$,

$$\begin{aligned}
\gamma_d^2 &= \Big(-\sum_{i=1}^{g}(\xi_i\,\alpha'_{i+g} - \xi_{i+g}\,\alpha'_i)\Big)^2 \\
&= \sum_{i,j=1}^{g}\Big[(\xi_i\alpha'_{i+g})(\xi_j\alpha'_{j+g}) + (\xi_i\alpha'_{i+g})(-\xi_{j+g}\alpha'_j) + (-\xi_{i+g}\alpha'_i)(\xi_j\alpha'_{j+g}) \\
&\quad +(-\xi_{i+g}\alpha'_i)(-\xi_{j+g}\alpha'_j)\Big]
\end{aligned}$$

where one observes that the first and last term are non-zero only if $i \neq j$, while up to reordering the second and third term are the same. Hence one can write

$$\begin{aligned}
\gamma_d^2 &= \sum_{i\neq j}(\xi_i\alpha'_{i+g}\xi_j\alpha'_{j+g}) - 2\sum_{i,j}(\xi_i\alpha'_{i+g}\xi_{j+g}\alpha'_j) + \sum_{i\neq j}(\xi_{i+g}\alpha'_i\xi_{j+g}\alpha'_j) \\
&= -\sum_{i\neq j}(\xi_i\xi_j\alpha'_{i+g}\alpha'_{j+g}) - 2\sum_{i,j}(\xi_i\xi_{j+g}\alpha'_j\alpha'_{i+g}) - \sum_{i\neq j}(\xi_{i+g}\xi_{j+g}\alpha'_i\alpha'_j)
\end{aligned}$$

If one denotes by $(\gamma_d^{2k})_h$ the sum of the terms in the expansion of γ_d^{2k} that have exactly $h-1$ factors of the form ξ_{j+g}, then one can see directly that:

$$\begin{aligned}
(\gamma_d^2)_1 &= \textstyle\sum_{i\neq j}(-\xi_i\,\xi_j\,\alpha'_{i+g}\,\alpha'_{j+g}) \\
(\gamma_d^2)_2 &= \textstyle\sum_{i,j}(-2\xi_i\,\xi_{j+g}\,\alpha'_j\,\alpha'_{i+g}) \\
(\gamma_d^2)_3 &= \textstyle\sum_{i\neq j}(-\xi_{i+g}\,\xi_{j+g}\,\alpha'_i\,\alpha'_j)
\end{aligned}$$

where $i,\, j \in \{1, \ldots, g\}$. One then is interested in expressing $(\gamma_d^{2k})_h$ for every $h \in \{1, \ldots, 2k+1\}$ and this can be done following the $k=1$ case

$$\begin{aligned}
(\gamma_d^{2k})_h &= \binom{2k}{h-1}\sum_{\substack{|I|=2k+1-h\\|J|=h-1}}(\xi_{i_1}\alpha'_{i_1+g})\cdots(\xi_{i_{2k+1-h}}\alpha'_{i_{2k+1-h}+g})(\xi_{j_1+g}\alpha'_{j_1})\cdots(\xi_{j_{h-1}+g}\alpha'_{j_{h-1}}) \\
&= (-1)^k\binom{2k}{h-1}\sum_{\substack{|I|=2k-h+1\\|J|=h-1}}\xi_{i_1}\cdots\xi_{i_{2k-h+1}}\xi_{j_1+g}\cdots\xi_{j_{h-1}+g}\alpha'_{j_1}\cdots\alpha'_{j_{h-1}} \\
&\quad \cdot\alpha'_{i_1+g}\cdots\alpha'_{i_{2k-h+1}+g}
\end{aligned} \tag{A.2.6}$$

where $\binom{2k}{h-1}$ counts the number of terms in the expression of γ_d^{2k} that have exactly $h-1$ factors of type ξ_{j+g} out of the $2k$ factors of type ξ, while $(-1)^k$ accounts for the sign change caused by the terms being rearranged in the last step.

Remark A.2.3. Note that every integral (A.2.2), or rather (A.2.3), is the sum of integrals

$$\int_{C_d \times J^d} \beta^{i_1} \cdots \beta^{i_l} \alpha_{j_1}^{i_{l+1}} \cdots \alpha_{j_s}^{i_{l+s}} \alpha_{j_1+g}^{i_{l+s+1}} \cdots \alpha_{j_s+g}^{i_{l+2s}} (\gamma_d^{2k})_h A_{g-k}$$

where $(\gamma_d^{2k})_h$ has been defined above and the indices satisfy the conditions

$$l + s = d - k$$
$$i_1, \ldots, i_{l+2s} \in \{1, \ldots, d\} \text{ all different}$$
$$j_1, \ldots, j_s \in \{1, \ldots, g\} \text{ all different.}$$

Recall that $\xi_i = \sum_{k=1}^{d} \alpha_i^k$ and note that, if $h < k+1$, the above integral will consist of monomials each of which will have at least $s+k+1$ terms of the form α_i^k with $i \in \{1, \ldots, g\}$, this means that one of the following must happen: either there are two α_i^k's with the same superscript k or there exist an α_i^k and a β^k with the same superscript, since $l+s+k+1 > d$ and all the k's lie in the set $\{1, \ldots, d\}$. In turn this implies that the corresponding integral is zero since one knows that $\alpha_i^k \alpha_j^k = 0$ and $\alpha_i^k \beta^k = 0$ for all $i,\ j \in \{1, \ldots, g\}$. Similarly, if $h > k+1$ the same reasoning applied to the α_{i+g}^k's says that the corresponding integral must be zero. Hence, as far as integration is concerned, we can substitute γ_d^{2k} with the $(k+1)^{th}$ monomial

$$(\gamma_d^{2k})_{k+1} = \sum_{|J|,|J'|=k} (-1)^k \binom{2k}{k} \xi_{j_1} \cdots \xi_{j_k} \xi_{j'_1+g} \cdots \xi_{j'_k+g}$$
$$\cdot \alpha'_{j'_1} \cdots \alpha'_{j'_k} \alpha'_{j_1+g} \cdots \alpha'_{j_k+g}$$

in the integral (A.2.3) since all the other terms will not give any contribution.

Special case:

In order to understand how to compute the integrals (A.2.3) we study first of all a special case, $s = 0$, i.e., $l = d - k$

$$\int_{C_d \times J^d} \beta^{i_1} \dots \beta^{i_{d-k}} \gamma_d^{2k} A_{g-k} \tag{A.2.7}$$

which, from the previous remark, is the same as

$$\int_{C_d \times J^d} \beta^{i_1} \cdots \beta^{i_{d-k}} \cdot \Big(\sum_{|J|,|J'|=k} (-1)^k \binom{2k}{k} \xi_{j_1} \cdots \xi_{j_k} \xi_{j'_1+g} \cdots \xi_{j'_k+g}$$
$$\cdot \alpha'_{j'_1} \cdots \alpha'_{j'_k} \alpha'_{j_1+g} \cdots \alpha'_{j_k+g} \Big) A_{g-k}$$

where $j_i, j'_i \in \{i, \dots, g\}$, $\xi_{i_h} = \sum_{n=1}^d \alpha_{i_h}^n$ and $\xi_{i_h+g} = \sum_{m=1}^d \alpha_{i_h+g}^m$. Note that the indices $i_1, \dots, i_{d-k}$ are fixed, hence the only terms in the sum which might give a non-zero contribution are those for which the n's belong to $\{1, \dots, d\} \setminus \{i_1, \dots i_{d-k}\}$, which is a set of cardinality k and similarly for the m's. Then the integral can be rewritten as

$$\int_{C_d \times J^d} \Big(\sum_{|J|,|J'|=k} \sum_{|N|,|M|=k} (-1)^k \binom{2k}{k} \beta^{i_1} \cdots \beta^{i_{d-k}} \alpha_{j_1}^{n_1} \cdots \alpha_{j_k}^{n_k}$$
$$\cdot \alpha_{j'_1+g}^{m_1} \cdots \alpha_{j'_k+g}^{m_k} \alpha'_{j'_1} \cdots \alpha'_{j'_k} \alpha'_{j_1+g} \cdots \alpha'_{j_k+g} \Big) A_{g-k}.$$

Now observe that for every choice of two multi-indices N and M of length k consisting of indices without repetitions picked from a fixed set of cardinality k (in this case $\{1, \dots, d\} \setminus \{i_1, \dots i_{d-k}\}$), there must be a permutation π such that $n_i = m_{\pi(i)}$ for all $i \in \{1, \dots, k\}$ and indices $n_i \in N$, $m_j \in M$. Next, recall that $\alpha_{j_i}^{n_i} \alpha_{j'_{\pi(i)}+g}^{n_i} = 0$ unless $j_i = j'_{\pi(i)}$, hence the above integral becomes

$$\int_{C_d \times J^d} \Big(k! \sum_{|J|=k} \sum_{|N|=k} (-1)^k \binom{2k}{k} \beta^{i_1} \dots \beta^{i_{d-k}} \alpha_{j_1}^{i_1} \cdots \alpha_{j_k}^{n_k} \alpha_{j_1+g}^{n_1} \cdots \alpha_{j_k+g}^{n_k}$$
$$\cdot \alpha'_{j_1} \alpha'_{j_1+g} \cdots \alpha'_{j_k} \alpha'_{j_k+g} \Big) A_{g-k}$$

where $k!$ accounts for all the possible permutations π. This can be rewritten as

$$\int_{C_d \times J^d} \Big(k!k! \sum_{|J|=k} (-1)^k \binom{2k}{k} \beta^{i_1} \dots \beta^{i_d} \sigma'_{\jmath} \cdots \sigma'_{j_k} \Big) A_{g-k}$$

where one uses the fact that $\alpha_{j_i}^{n_i}\alpha_{j_i+g}^{n_i} = \beta^{n_i}$ and that $\alpha'_{j_i}\alpha'_{j_i+g} = \sigma'_{j_i}$, while the second factor $k!$ accounts for all possible multi-indices N satisfying the above conditions. Finally, one obtains

$$\int_{C_d\times J^d} (-1)^k(2k)!\,\beta^1\cdots\beta^d\,\theta^k\,A_{g-k} \tag{A.2.8}$$

using the expression for θ given at the end of section A.1. Remembering that $x_d^d = d!\,\beta^1\cdots\beta^d$ (see (A.2.4)), this allows to compute the integrals in which θ_d does not appear

$$\begin{aligned}\int_{C_d\times J^d} x_d^{d-k}\,\gamma_d^{2k}\,A_{g-k} &= \int_{C_d\times J^d}\sum_{|I|=d-k}\beta^1\cdots\beta^{d-k}\,\gamma_d^{2k}\,A_{g-k}\\ &= \int_{C_d\times J^d}\sum_{|I|=d-k}\left((-1)^k\,(2k)!\,\beta^1\cdots\beta^d\right)\theta^k\,A_{g-k}\\ &= \int_{C_d\times J^d}(-1)^k(2k)!\,\frac{d!}{k!}\,\beta^1\cdots\beta^d\,\theta^k\,A_{g-k}.\end{aligned}$$

Since $d!\,\beta^1\cdots\beta^d$ generates the top integer cohomology of C^d and C^d is a d-fold cover of C_d, it also generates the top integer cohomology of C_d, hence

$$\begin{aligned}\int_{C_d\times J^d} x_d^{d-k}\,\gamma_d^{2k}\,A_{g-k} &= \int_{C_d\times J^d}(-1)^k\,\frac{(2k)!}{k!}\,x_d^d\,\theta^k\,A_{g-k}\\ &= \int_{J^d}(-1)^k\,\frac{(2k)!}{k!}\,\theta^k\,A_{g-k}\end{aligned} \tag{A.2.9}$$

where the final equality is given by projection formula.

Remark A.2.4. In particular, when $d = 1$ this says that

$$\int_{C\times J^1}\gamma_1^2\,A_{g-1} = -\int_{C\times J^1} 2x\,\theta\,A_{g-1} = -2\int_{J^1}\theta\,A_{g-1}$$

where x is the class of a point in $H^2(C,\mathbb{Z})$. This is the same as the usual result that $\gamma^2 = -2x\theta$ of [ACGH], p.335, since $\gamma_1 = \gamma$.

<u>*General case:*</u>

We now consider the case when $s > 0$, i.e., we want to compute integrals like

(A.2.3) and as has been shown before this is the same as computing integrals of the form

$$\int_{C_d\times J^d} \beta^{i_1}\cdots\beta^{i_l}\,\alpha_{h_1}^{i_{l+1}}\cdots\alpha_{h_s}^{i_{l+s}}\,\alpha_{h_1+g}^{i_{l+s+1}}\cdots\alpha_{h_s+g}^{i_{l+2s}}\cdot \Big(\sum_{|J|,|J'|=k}(-1)^k\binom{2k}{k}\xi_{j_1}\cdots\xi_{j_k}\cdot\xi_{j'_1+g}\cdots\xi_{j'_k+g} \cdot\alpha'_{j'_1}\ldots\alpha'_{j'_k}\,\alpha'_{j_1+g}\cdots\alpha'_{j_k+g}\Big)A_{g-k}.$$

As before, having fixed indexes $i_1,\ldots,i_l$ for the β's means that in each of the sums $\xi_{j_h}=\sum_{n=1}^{d}\alpha_{j_h}^{n}$ the n's must belong to $\mathcal{B}=\{1,\ldots,d\}\setminus\{i_1,\ldots i_l\}$,

$$\int_{C_d\times J^d} \beta^{i_1}\cdots\beta^{i_l}\,\alpha_{h_1}^{i_{l+1}}\cdots\alpha_{h_s}^{i_{l+s}}\,\alpha_{h_1+g}^{i_{l+s+1}}\ldots\alpha_{h_s+g}^{i_{l+2s}}\cdot \Big(\sum_{|J|,|J'|=k}\sum_{|N|,|M|=k}(-1)^k\binom{2k}{k}\alpha_{j_1}^{n_1}\cdots\alpha_{j_k}^{n_k}\,\alpha_{j'_1+g}^{m_1}\cdots\alpha_{j'_k+g}^{m_k} \cdot\alpha'_{j'_1}\ldots\alpha'_{j'_k}\,\alpha'_{j_1+g}\cdots\alpha'_{j_k+g}\Big)A_{g-k}$$

where N and M only take values in the set $\mathcal{B}$. Note moreover that the sets $\{i_{l+1},\ldots,i_{l+s}\}$ and $\{i_{l+s+1},\ldots,i_{l+2s}\}$ are disjoint by hypothesis and $l+s+k=d$, hence each of the $\alpha_{h_t}^{i_{l+t}}$ must "pair" with an $\alpha_{j'_{t'}+g}^{m_{t'}}$, i.e., $m_{t'}=i_{l+t}$ and similarly $\alpha_{h_t+g}^{i_{l+s+t}}$ with an $\alpha_{j_t}^{n_t}$ in order to have a non-zero term for all $t=1,\ldots s$. Note that the condition $s\le k$ assures that such pairings are always possible and "exhaust" the $\alpha_{h(+g)}^{i}$'s. The two remaining groups of $(k-s)$ α^n's and α^m's must pair-up too, since the remaining n's belong to $\{1,\ldots d\}\setminus\{i_{l+1},\ldots,i_{l+s}\}$ and the m's to $\{1,\ldots d\}\setminus\{i_{l+s+1},\ldots,i_{l+2s}\}$ which are sets of cardinality k that intersect in a set of cardinality $k-s$, $\{n_1,\ldots n_{k-s}\}$. Moreover if $n_t=m_{t'}$ this forces $j_t=j'_{t'}$ otherwise $\alpha_{j_t}^{n_t}\alpha_{j'_{t'}}^{n_t}$ is zero. The difficulty consists in evaluating in how many different ways this can happen and computing the associated permutation of the indices. This eventually gives

$$\int_{C_d\times J^d} \sum_{|J|=k-s} (-1)^k(-1)^{(s^2+s)/2}\binom{2k}{k} k!\,k!\,\beta^{i_1}\cdots\beta^{i_d}\cdot$$
$$\alpha'_{h_1}\,\alpha'_{h_1+g}\cdots\alpha'_{h_s}\,\alpha'_{h_s+g}\cdot\alpha'_{j_1}\,\alpha'_{j_1+g}\cdots\alpha'_{j_{k-s}}\,\alpha'_{j_{k-s}+g}\,A_{g-k}$$
$$= \int_{C_d\times J^d}(-1)^k(-1)^{(s^2+s)/2}\,(2k)!\,\beta^1\cdots\beta^d\,\sigma'_{h_1}\cdots\sigma'_{h_s}\,\theta^{k-s}\,A_{g-k}.$$

This allows the computation of all integrals like (A.2.2). By recalling the expressions for x_d^l and θ_d^l, plus the comments on γ_d^{2k}, we will be able to prove the following

$$\int_{C_d\times J^d} x_d^i\,\theta_d^j\,\gamma_d^{2k}\,A_{g-k} = \int_{J^d}(-1)^k\,(2k)!\,\frac{j!}{k!}\Big(\sum_{h=0}^{min\{j,k\}}(-1)^h\binom{k}{h}\binom{g-h}{g-j}\Big)\,\theta^k\,A_{g-k} \tag{A.2.10}$$

To do so, first of all notice that $\int_{C_d\times J^d} x_d^i\,\theta_d^j\,\gamma_d^{2k}\,A_{g-k}$ equals

$$\int_{C_d\times J^d}\Big(\sum_{|I|=d-k-j}\beta^{i_1}\ldots\beta^{i_{d-k-j}}\Big)\cdot$$
$$\Big(\sum_{h=0}^{min\{j,d-j\}}(-1)^{\frac{h^2-h}{2}}\binom{j}{h}\frac{(g-h)!}{(g-j)!}\sum_{\substack{|R|=h\\|L|=j+h}}\beta^{j_1}\ldots\beta^{l_{j-h}}\,\alpha_{r_1}^{l_{j-h+1}}\cdots\alpha_{r_h}^{l_j}\,\alpha_{r_1+g}^{l_{j+1}}\cdots\alpha_{r_h+g}^{l_{j+h}}\Big)\cdot$$
$$\Big(\sum_{|J|,|J'|=k}(-1)^k\binom{2k}{k}\xi_{j_1}\ldots\xi_{j_k}\xi_{j'_1+g}\ldots\xi_{j'_k+g}\alpha'_{j_1}\ldots\alpha'_{j_k}\alpha'_{j'_1+g}\ldots\alpha'_{j'_k+g}\Big)\cdot A_{g-k}.$$

Note that the sets of indices $\{i_1,\ldots,i_{d-k-j}\}$ and $\{l_1,\ldots l_{j+h}\}$ must be disjoint, hence one can rewrite the above as

$$\int_{C_d\times J^d}\sum_{h=0}^{min\{j,d-j\}}(-1)^{(h^2-h)/2}\binom{j}{h}\frac{(g-h)!}{(g-j)!}\cdot$$
$$\Big(\sum_{|R|=h}\sum_{|I|=d-k-h}\beta^{i_1}\cdots\beta^{i_{d-k+h}}\alpha_{r_1}^{i_{d-k-h+1}}\cdots\alpha_{r_h}^{i_{d-k}}\alpha_{r_1+g}^{i_{d-k+1}}\cdots\alpha_{r_h+g}^{i_{d-k+h}}\Big)\cdot$$
$$\Big(\sum_{|J|,|J'|=k}(-1)^k\binom{2k}{k}\xi_{j_1}\cdots\xi_{j_k}\,\xi_{j'_1+g}\cdots\xi_{j'_k+g}\alpha'_{j_1}\cdots\alpha'_{j_k}\,\alpha'_{j'_1+g}\cdots\alpha'_{j'_k+g}\Big)\cdot A_{g-k}$$

$$= \int_{C_d \times J^d} \sum_{h=0}^{min\{j,k\}} (-1)^{(h^2-h)/2} \binom{j}{h} \frac{(g-h)!}{(g-j)!} \sum_{|R|=h} \Big((-1)^k (-1)^{(h^2+h)/2} \frac{d!}{(k-h)!}$$

$$(2k)!\, \beta^1 \cdots \beta^d \cdot \sigma_{r_1} \cdots \sigma_{r_h} \Big)\, \theta^{k-h}\, A_{g-k}$$

$$= \int_{C_d \times J^d} \sum_{h=0}^{min\{j,k\}} (-1)^{k+h} \frac{(2k)!}{(k-h)!} \binom{j}{h} \frac{(g-h)!}{(g-j)!}\, x_d^d\, \theta^k\, A_{g-k}$$

$$= \int_{C_d \times J^d} (2k)!\, \frac{j!}{k!} \Big(\sum_{h=0}^{min\{j,k\}} (-1)^{k+h} \binom{k}{h} \binom{g-h}{g-j} \Big)\, x_d^d\, \theta^k\, A_{g-k}$$

$$= \int_{J^d} (-1)^k\, (2k)!\, \frac{j!}{k!} \Big(\sum_{h=0}^{min\{j,k\}} (-1)^h \binom{k}{h} \binom{g-h}{g-j} \Big)\, \theta^k\, A_{g-k}$$

which concludes the proof.

Example A.2.5. If $d = 2$ this gives the following integrals

$$\begin{aligned}
\int_{C_2 \times J^2} x_2\, \gamma_2^2\, A &= -2 \int_{C_2 \times J^2} x_2^2 \theta\, A = -2 \int_{J^2} \theta\, A \\
\int_{C_2 \times J^2} \theta_2\, \gamma_2^2\, A &= -2(g-1) \int_{C_2 \times J^2} x_2^2 \theta\, A = -2(g-1) \int_{J^2} \theta\, A \\
\int_{C_2 \times J^2} \gamma_2^4\, A &= 12 \int_{C_2 \times J^2} x_2^2 \theta^2\, A = 12 \int_{J^2} \theta^2\, A
\end{aligned}$$

where A is the pull-back of a class over the Jacobian J^2, of the appropriate degree.

Remark A.2.6. Note moreover that even if the class $A \in H^*(J^d, \mathbb{Z})$ does not have degree $2g - 2k$, the previous result still allows us to compute the push-forward of the class $x_d^i\, \theta_d^j\, \gamma_d^{2k} A$ from $C_d \times J^d$ to J^d by using Poincaré duality on J^d.

Remark A.2.7. One might also consider the product variety $C_d \times C_e$ and define a similar degree 2 class $\gamma_{d,e}$ in $H^1(C_d, \mathbb{Z}) \otimes H^1(C_e, \mathbb{Z})$ as

$$\gamma_{d,e} \stackrel{def}{=} -\sum_{i=1}^{g} (\xi_{d,i}\, \xi_{e,i+g} - \xi_{d,i+g}\, \xi_{e,i}). \tag{A.2.11}$$

It is not difficult to verify that all the above computations can be repeated in this new case, since one only needs to use the properties of classes on C_d and write $\xi_{e,j}$ wherever there was α'_j. Note only that in the final expressions of the integrals one has to write $\theta_e = \sum_{k=1}^{g} \sigma_{e,i}$ instead of $\theta \in H^2(J, \mathbb{Z})$. These new classes $\gamma_{d,e}$ are also useful, since the pull-back of the class θ_{d+e} on the variety C_{d+e} via the addition map r: $C_d \times C_e \longrightarrow C_{d+e}$ is given by

$$r^*(\theta_{d+e}) = \theta_d + \gamma_{d,e} + \theta_e \tag{A.2.12}$$

as can be seen by generalising the computations of [ACGH], p.368.

A.3 "Symmetric" Poincaré line bundle

This section deals with the construction of a "symmetric" Poincaré line bundle, that is, a line bundle that parametrises certain line bundles over a symmetric product C_d.

Let $p : C^d \longrightarrow C_d$ denote the quotient map described at the outset of the Appendix and denote by π_i the projection from C^d to its i-*th* factor C. For any line bundle $\eta \in Pic(C)$ one can define a line bundle on C_d, denoted $(\eta)_d$ or η_d, as the line bundle on C_d such that $p^*(\eta_d) = \otimes \pi_i^*(\eta)$. Moreover the Chern class of η_d is easily computed, $c_1(\eta_d) = (\deg\, \eta) x_d$. Then, a "symmetric" Poincaré line bundle $\mathcal{P}_d$ over $C_d \times J^d$ should parametrise precisely these line bundles. Our aim is to construct such a bundle and compute its Chern class.

Let $\mathcal{P}$ be a Poincaré line bundle on $C \times J^d$. Then one can consider the following diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r] & C \times J^d & & \\
\mathcal{P} \arrow[r] & C \times J^d & \mathcal{P}^1 \otimes \cdots \otimes \mathcal{P}^d \arrow[d] & \\
\vdots & \vdots & C^d \times J^d \arrow[uul] \arrow[ul, "\pi_1 \times 1"'] \arrow[ul, "\pi_2 \times 1"] \arrow[dl, "\pi_d \times 1"] \arrow[r, "p \times 1"] & C_d \times J^d \\
\mathcal{P} \arrow[r] & C \times J^d & &
\end{tikzcd}$$

where $\mathcal{P}^i \stackrel{def}{=} (\pi_i \times 1)^*(\mathcal{P})$ for every choice of a factor and p is the quotient map. Since the action of the symmetric group S_d lifts to $\mathcal{P}^1 \otimes \cdots \otimes \mathcal{P}^d$ and is trivial on the fibres of fixed points, this diagram defines a line bundle on $C_d \times J^d$, $\mathcal{P}_d$. Hence the "symmetric" Poincaré line bundle is a line bundle that has the property

$$(p \times 1)^*\mathcal{P}_d = \otimes_{i=1}^d (\pi_i \times 1)^*\mathcal{P}^i.$$

Recall that the Poincaré line bundle over $C \times J^d$ is uniquely defined up to translation, i.e., it is unique once one asks it to be trivial on $\{p_0\} \times J^d$ for some choice of a point $p_0 \in C$ (where one uses the identification used in remark A.1.2). The same property holds for $\mathcal{P}_d$, it is uniquely defined once it is required to be trivial over $\{D_0\} \times J^d$ for some divisor $D_0 \in C_d$, where we identify effective divisors of degree d on C with points of C_d.

Our aim is, now, to calculate the first Chern class of $\mathcal{P}_d$. Recall that the first Chern class of $\mathcal{P}$ is given by $c_1(\mathcal{P}) = \gamma + dx$ where γ satisfies $\gamma^2 = -2x\theta$ and $x \in H^1(C, \mathbb{Z})$ is the class of a point. Then, clearly, $c_1(\mathcal{P}^i) = \gamma^i + d\beta^i$, which, in turn, gives

$$c_1(\mathcal{P}^1 \otimes \cdots \otimes \mathcal{P}^d) = \sum_{i=1}^d c_1(\mathcal{P}^i) = \sum_{i=1}^d \gamma^i + d\sum_{i=1}^d \beta^i. \tag{A.3.1}$$

Then, since both sums in (A.3.1) give classes of $C_d \times J^d$, this is precisely the Chern class of $\mathcal{P}_d$

$$c_1(\mathcal{P}_d) = \gamma_d + dx_d \tag{A.3.2}$$

where the definition of x_d as $\sum_{i=1}^d \beta^i$ was given in §A.1 while the class γ_d was defined in the previous section as

$$\gamma_d = \sum_{i=1}^d \gamma^i = -\sum_{i=1}^g [\xi_{d,i}\, \alpha'_{i+g} - \xi_{d,i+g}\, \alpha'_i].$$

These bundles are used in §3.3 in order to compute the top Chern class of the vector bundles Q_d, for all integers d.

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