Hypercomplex hyperbolic geometry

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A thesis presented for the degree of
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Dedicated to

my parents

CONTENTS.

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Abstract

The rank one symmetric spaces of non-compact type are the real, complex, quaternionic and octonionic hyperbolic spaces. Real hyperbolic geometry is widely studied, complex hyperbolic geometry less so, whilst quaternionic hyperbolic geometry is still in its infancy. The purpose of this thesis is to investigate the conditions for discrete group action in quaternionic and octonionic hyperbolic 2-spaces and their geometric consequences, in the octonionic case, in terms of lower bounds on the volumes of non-compact manifolds. We will also explore the eigenvalue problem for the 3×3 octonionic matrices germane to the Jordan algebra model of the octonionic hyperbolic plane.

In Chapters One and Two we concentrate on discreteness conditions in quaternionic hyperbolic 2-space. In Chapter One we develop a quaternionic Jørgensen's inequality for non-elementary groups of isometries of quaternionic hyperbolic 2-space generated by two elements, one of which is either loxodromic or boundary elliptic. In Chapter Two we give a generalisation of Shimizu's Lemma to groups of isometries of quaternionic hyperbolic 2-space containing a screw-parabolic element. In Chapter Three we present the Jordan algebra model of the octonionic hyperbolic plane and develop a generalisation of Shimizu's Lemma to groups of isometries of octonionic hyperbolic 2-space containing a parabolic map. We use this result to determine estimates of lower bounds on the volumes of non-compact closed octonionic 2-manifolds. In Chapter Four we construct an octonionic Jørgensen's inequality for non-elementary groups of isometries of octonionic hyperbolic 2-space generated by two elements, one of which is loxodromic. In Chapter Five we solve the real eigenvalue problem $X\nu = \lambda\nu$, for the 3×3 Φ-Hermitian matrices, X, of the Jordan algebra model of the octonionic hyperbolic plane. Finally, in Chapter Six we consider the embedding of collars about real geodesics in complex hyperbolic 2-space, quaternionic hyperbolic 2-space and octonionic hyperbolic 2-space.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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CHAPTER ONE.

A JØRGENSEN'S INEQUALITY FOR QUATERNIONIC HYPERBOLIC 2-SPACE.

In this chapter we will construct a quaternionic Jørgensen's inequality for certain types of subgroups of isometries of quaternionic hyperbolic 2-space $\mathbf{H}_{\mathbb{H}}^2$. The classical Jørgensen's inequality gives a necessary algebraic condition for a non-elementary 2-generator group of isometries of real hyperbolic space to be discrete. This inequality uses the traces of one of the generators and the commutator of both generators to express this condition. To be more specific the classical Jørgensen's inequality is as follows.

Let U, V be two elements of $SL(2,\mathbb{C})$. Then if

$$|tr^2(V) - 4| + |tr(VUV^{-1}U^{-1}) - 2| < 1,$$

then the group $\langle V, U\rangle$ is either elementary or not discrete.

Remark. A **non-elementary group** G is one which acts in such a way that there are no finite G-orbits in the hyperbolic space.

Wielenberg generalised Jørgensen's inequality to isometries of higher dimensional Möbius transformations using the group $SO(n,1)$ (see [39]). In [3], Basmajian and Miner gave a version of Jørgensen's inequality for 2-generator groups of complex hyperbolic isometries by considering the action of the group on the boundary of complex hyperbolic space. In [23], [30] and [31], Kamiya and Parker considered generalisations of Jørgensen's inequality to groups of complex hyperbolic isometries where one generator was a Heisenberg translation. In these papers they used a method similar to that of Jørgensen which involves considering the generators as matrices in $SU(n,1)$ and constructing a particular sequence of distinct elements of the group and determining conditions on the entries of the matrices that force the sequence to tend to the identity, thus violating discreteness. In [21], Jiang, Kamiya and Parker extended this method to subgroups of $SU(2,1)$, reformulating Jørgensen's inequality in terms of cross-ratios of fixed points of a loxodromic or elliptic generator and one of its conjugate maps. In this chapter the same method is extended to subgroups of $Sp(2,1)$ where one generator is loxodromic or boundary elliptic and fixing a quaternionic line.

The principal results of Jiang, Kamiya and Parker are as follows.

Suppose that V is a loxodromic or boundary elliptic element of $SU(2,1)$ with complex dilation factor $\lambda(V)$. We define $M = |\lambda(V) - 1| + |\lambda(V)^{-1} - 1|$ and let $[.,.,.,.]$ denote a complex cross-ratio (see page 8 of [21]) which is the complex analogue of the cross-ratio defined in section 1 of this chapter. Then we have the following theorem.

Theorem (pages 2-3 [21]). *Suppose that either:*

(i) V is a loxodromic element of $SU(2,1)$ with fixed points μ and ν and that U is any element of $SU(2,1)$, or

(ii) V is a boundary elliptic element of $SU(2,1)$ with fixed complex line L_V and U is any other element of $SU(2,1)$. Also let μ, ν be distinct points in $L_V \cap \partial\mathbf{H}_{\mathbb{C}}^2$. Suppose that L_V and $U(L_V)$ do not intersect orthogonally.

Suppose that one of the following conditions holds:

(1) $M\left(|[U(\mu),\nu,\mu,U(\nu)]|^{1/2} + 1\right) < 1$,

(2) $M\left(|[U(\mu),\mu,\nu,U(\nu)]|^{1/2} + 1\right) < 1$,

(3) $M < \sqrt{2} - 1$ *and*

$$|[U(\mu),\nu,\mu,U(\nu)]| + |[U(\mu),\mu,\nu,U(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

(4) $M + \left(|[\mu, \nu, U(\mu), U(\nu)]|^{1/2} + 1\right) < 1$,
then the group $\langle V, U\rangle$ *is elementary or not discrete.*

The purpose of this chapter is to develop a quaternionic Jørgensen's inequality for non-elementary groups of isometries of quaternionic hyperbolic 2-space generated by 2 elements, one of which is either loxodromic or boundary elliptic. In order to achieve this aim we have followed closely the methods used in [21] to develop a Jørgensen's inequality for complex hyperbolic 2-space.

Suppose that V is an element of $Sp(2,1)$ (this group is defined in section 1 of this chapter) with dilation factor λ and rotation factors p and q where $\lambda \in \mathbb{R}_+$ and p and q are unit quaternions. We define

$$M = |\lambda p - 1| + 2|pq - 1| + |\lambda^{-1} p - 1|$$

and let $[.,.,.,.]$ denote a quaternionic cross-ratio (see section 1 of this chapter). Then we have the following theorem.

Theorem. *Suppose that either:*

(i) V *is a loxodromic element of* $Sp(2,1)$ *(with fixed points* μ *and* ν*) and that* U *is any element of* $Sp(2,1)$*, or*

(ii) V *is a boundary elliptic element of* $Sp(2,1)$ *which rotates through the order* m *unit quaternion* p *about a fixed quaternionic line* L_V *with polar vector* $\mathbf{v}_V$ *(normalised so that* $\langle \mathbf{v}_V | \mathbf{v}_V\rangle = 1$ *and* U *is any other element of* $Sp(2,1)$*. Also let* μ*,* ν *be distinct points in* $L_V \cap \partial\mathbf{H}^2_{\mathbb{H}}$*. Suppose that* L_V *and* $U(L_V)$ *do not intersect orthogonally.*

Suppose that one of the following conditions holds:

(1) $M\left(|[U(\mu), \nu, \mu, U(\nu)]|^{1/2} + 1\right) < 1$,

(2) $M\left(|[U(\mu), \mu, \nu, U(\nu)]|^{1/2} + 1\right) < 1$,

(3) $M < \sqrt{2} - 1$ *and*

$$|[U(\mu), \nu, \mu, U(\nu)]| + |[U(\mu), \mu, \nu, U(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group $\langle V, U\rangle$ *is elementary or not discrete.*

The first two conditions are considered in Theorems 1.1 and 1.4. The third condition is considered in Theorems 1.2 and 1.5. The three conditions are independent of each other. We discuss the relationship between them in section 3.

In section 1 we give the necessary background material for quaternionic hyperbolic space. In sections 2 and 3 we discuss groups with loxodromic generators and in section 4 we discuss groups with boundary elliptic generators.

1. *Quaternionic Hyperbolic Space.*

The following section includes material found in section 1 of [25]. The upper half space model of real hyperbolic space can be generalised to the quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ as the Siegel domain which is equivalent to the direct product of the generalised Heisenberg group

$$\mathfrak{N}_n = \{(\zeta, v) | \zeta \in \mathbb{H}^{n-1}, v \in Im(\mathbb{H})\}$$

and the positive real numbers $\mathbb{R}^+$. Note that $\mathfrak{N}_n$ is a simply connected nilpotent Lie group of order 2 and dimension $4n - 1$ and is associated with a Iwasawa decomposition

$$KAN = PSp(n,1).$$

The Heisenberg group has the multiplication rule

$$(\zeta, v)(w, s) = (\zeta + w, v + s + 2Im(\langle\zeta, w\rangle))$$

where $\langle\langle \zeta, w\rangle\rangle = \overline{w}\zeta$. The centre of $\mathfrak{N}_n$ is clearly $\{0\}\times\mathbb{R}^3$. The coordinate change between the Siegel domain and the horospherical coordinate system is

$$(w, w_n) \mapsto (w/\sqrt{2}, Im(w_n), -Re(w_n) - 1/2\langle\langle w, w\rangle\rangle).$$

We consider an embedding from the Siegel domain to $P\mathbb{H}^{n,1}$

$$(w, w_n) \mapsto [w_n, w, 1].$$

The corresponding embedding from the horospherical coordinate system to $P\mathbb{H}^{n,1}$ is

$$(\zeta, v, u) \mapsto \left[-|\zeta|^2 - u + v, \sqrt{2}\zeta, 1\right]$$

where $\zeta = w/\sqrt{2}$.

The point $[0, 0', 1]$ in $P\mathbb{H}^{n,1}$ (which has horospherical coordinates $(0,0,0)$) corresponds to the origin o in $\partial\mathbf{H}^n_{\mathbb{H}}$ (see below) and $[1, 0', 0]$ is a distinguished point at ∞ where $0'$ denotes the zero vector in $\mathbb{H}^{n-1}$. For the sake of convenience, we choose the standard Hermitian form on $\mathbb{H}^{n,1}$ to be represented by the matrix

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(where I is the $(n-1)\times(n-1)$ identity matrix), that is $\langle \mathbf{w}, \mathbf{z}\rangle = \mathbf{z}^* J\mathbf{w}$. The induced Hermitian product on horospherical coordinates is

$$\langle(\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)\rangle = 2\overline{\zeta_2}\zeta_1 - |\zeta_1|^2 - |\zeta_2|^2 - u_1 - u_2 + v_1 - v_2.$$

(Alternatively we could have used the (*first*) Hermitian form given by the diagonal matrix with entries $(1, ..., 1, -1)$.)

Definition. *The distance ρ (according to the Bergman metric) between two points with homogeneous coordinates z and $w \in \mathbb{H}^{n+1}$ is given by*

$$\cosh(\rho(z,w)/2) = \frac{|\langle z, w\rangle|}{|\langle z, z\rangle|^{1/2}|\langle w, w\rangle|^{1/2}}.$$

Definition. *The ideal boundary $\partial\mathbf{H}^n_{\mathbb{H}}$ of quaternionic hyperbolic space can be identified with the one point compactification of the generalised Heisenberg group $\mathfrak{N}_n$. The* **Cygan metric** *on $\mathfrak{N}_n$ corresponds to the norm*

$$|(\zeta, v)|_0 = \left||\zeta|^2 + v\right|^{1/2} = \left(|\zeta|^4 + |v|^2\right)^{1/4}.$$

The associated metric is

$$\rho_0\big((\zeta, v), (w, s)\big) = |(\zeta, v)^{-1}(w, s)|_0 = \big|(\zeta - w, -v + s - 2Im\langle\langle \zeta, w\rangle\rangle\big|_0.$$

The Cygan metric can also be expressed explicitly in terms of the Hermitian product on $P\mathbb{H}^{n,1}$ as follows. Let $\mathbf{x}$, $\mathbf{y} \in P\mathbb{H}^{n,1}$ correspond to points in $\partial\mathbf{H}^n_{\mathbb{H}}$, then

$$\rho_0(\mathbf{x}, \mathbf{y}) = |\langle \mathbf{x}, \mathbf{y}\rangle|^{1/2}.$$

From now on we will identify $PSp(n,1)$ with the set of matrices preserving the above Hermitian form J, also known as the *second* Hermitian form. We adopt the convention that matrices in $Sp(n,1)$ always act on the left and we projectivise ($\mathbb{H}^{n,1} \mapsto P\mathbb{H}^{n,1}$) on the right. We observe that this is the opposite convention to that which we will use later when working with octonionic hyperbolic 2-space. Such a matrix U has by definition the property that $U^{-1} = JU^*J$. Restricting to the case $n = 2$ as we will be working in $\mathbf{H}^2_{\mathbb{H}}$ such a matrix U and its inverse $U^{-1} \in Sp(2,1)$ have the general forms

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \tag{1}$$

and

$$U^{-1} = \begin{bmatrix} \bar{j} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{bmatrix} \tag{2}$$

where $\{a, b, c, d, e, f, g, h, j\} \subset \mathbb{H}$. As the composition of the inverse of an element of $Sp(2,1)$ with itself is the identity we obtain a list of equations that the matrix entries in (1) must satisfy. Four of these are

$$1 = \bar{j}a + \bar{f}d + \bar{c}g,$$

$$|d|^2 = -\bar{g}a - \bar{a}g,$$

$$1 = |e|^2 + \bar{h}b + \bar{b}h,$$

$$|f|^2 = -\bar{j}c - \bar{c}j.$$

In order to represent the isometries of $\mathbf{H}^2_{\mathbb{H}}$ uniquely we work with $Sp(2,1)$ throughout this chapter. Elements of $Sp(2,1)$ are classified as in the familiar case of Möbius transformations.

Elements of $Sp(2,1)$ fall into three distinct classes:

1) A *parabolic* element has exactly one fixed point which is on the boundary of $\mathbf{H}^2_{\mathbb{H}}$. Conjugating if necessary, we can always assume that this is the distinguished point at infinity q_∞. A parabolic element can be imagined as corresponding to a rotation around a boundary point.
2) A *loxodromic* element has exactly 2 fixed points both on the boundary of $\mathbf{H}^2_{\mathbb{H}}$ and can be imagined as corresponding to a rotation about a point which lies outside the closure of $\mathbf{H}^2_{\mathbb{H}}$.
3) An *elliptic* element has at least one fixed point within $\mathbf{H}^2_{\mathbb{H}}$ and may have others on the boundary.

We observe that an elliptic element is *regular elliptic* if its eigenvalues are all distinct. We also observe that there are two types of parabolic elements:

(i) If a parabolic element may be written as an element of $PSp(2,1)$ with 1 as its only eigenvalue, then it is said to be *unipotent* or *pure parabolic.* The group of pure parabolic maps fixing a given point is isomorphic to the 7-dimensional Heisenberg group and once the fixed point is specified are Heisenberg translations.
(ii) If a parabolic element is not unipotent it is said to be *ellipto-parabolic* or *screw-parabolic.* An ellipto-parabolic element preserves a unique quaternionic geodesic on which it acts as a parabolic element of $PSp(1,1)$, i.e. as a translation.

For later use we give some inequalities on the entries of a matrix in $Sp(2,1)$.

Proposition 1.1. *Let U be an element of $Sp(2,1)$ of the form (1). Then*

$$|df| \leq 2|aj|^{1/2}|cg|^{1/2},$$

$$|bh| \leq 2|aj|^{1/2}|cg|^{1/2},$$

$$|e| \leq |aj|^{1/2} + |cg|^{1/2},$$

$$|aj|^{1/2} \leq |cg|^{1/2} + 1,$$

$$|cg|^{1/2} \leq |aj|^{1/2} + 1,$$

$$1 \leq |aj|^{1/2} + |cg|^{1/2}.$$

Proof. From

$$|d|^2 = -\overline{g}a - \overline{a}g = -2Re(\overline{a}g)$$

and

$$|f|^2 = -\overline{c}j - \overline{j}c = -2Re(\overline{c}j),$$

we have

$$|df|^2 = 4Re(\overline{a}g)Re(\overline{c}j) \leq 4|ag||cj| = 4|aj||cg|.$$

Similarly for $|bh|^2 \leq 4|aj||cg|$. Taking square roots gives the first two parts of the result.

Now

$$\begin{aligned} |e|^2 =& 1 - d\overline{f} - f\overline{d} = \overline{j}a + \overline{c}g - d\overline{f} \\ &\leq |aj| + |df| + |cg| \leq \left(|aj|^{1/2} + |cg|^{1/2}\right)^2 \end{aligned}$$

using $|df| \leq 2|aj|^{1/2}|cg|^{1/2}$. This completes the result.

Finally we will adopt a more geometric method of proof and use the Cygan metric ρ_0 to prove the last three results. Now by direct calculation we have

$$\rho_0\left(U(\infty), o\right) = \left|\frac{a}{g}\right|^{1/2},$$

$$\rho_0\left(U(o), o\right) = \left|\frac{c}{j}\right|^{1/2}$$

and

$$\rho_0\left(U(\infty), U(o)\right) = \left|\frac{1}{gj}\right|^{1/2}.$$

Therefore

$$|aj|^{1/2} = \frac{\rho_0\left(U(\infty), o\right)}{\rho_0\left(U(\infty), U(o)\right)}$$

and

$$|cg|^{1/2} = \frac{\rho_0\left(U(o), o\right)}{\rho_0\left(U(\infty), U(o)\right)}.$$

And so by the triangle inequality

$$\begin{aligned} |aj|^{1/2} &= \frac{\rho_0\left(U(\infty), o\right)}{\rho_0\left(U(\infty), U(o)\right)} \\ &\leq \frac{\rho_0\left(U(\infty), U(o)\right) + \rho_0\left(U(o), o\right)}{\rho_0\left(U(\infty), U(o)\right)} \\ &= 1 + |cg|^{1/2}. \end{aligned}$$

The last two results follow similarly.

Remark. We observe that the rest of Proposition 1.1 could also be proved by applying the triangle inequality to the respective distances between the origin and a selected point at infinity and their images under U. Similarly all the convergence criteria proved in this chapter possess a geometrical interpretation in terms of the fixed points, invariant axes and dilation factors of the generators.

Definition. *Suppose that z_1, z_2, w_1, w_2 are four distinct points of $\mathbf{H}^2_{\mathbb{H}}$, we define the magnitude of their quaternionic cross-ratio to be*

$$|[z_1, z_2, w_1, w_2]| = \frac{|\langle w_1, z_1\rangle||\langle w_2, z_2\rangle|}{|\langle w_2, z_1\rangle||\langle w_1, z_2\rangle|}.$$

We will only use the absolute value $|[z_1, z_2, w_1, w_2]|$ which we call the *cross-ratio*. Observe that if two of the entries are the same then the cross-ratio is still defined and equals one of $0, 1$ or ∞. The cross-ratio possesses the following properties:

(i) The cross-ratio is invariant under the action of elements of $Sp(2,1)$.

(ii) The cross-ratio is invariant under projectivisation on the right.

(iii) The cross-ratio possesses the same properties as the corresponding complex cross-ratio under permutations of S_4 (see [16] page 225). To be specific these are:

$$\begin{aligned}
|[z_1, z_2, w_1, w_2]| &= |[z_2, z_1, w_2, w_1]| \\
&= |[w_1, w_2, z_1, z_2]| \\
&= |[w_2, w_1, z_2, z_1]| \\
&= |[z_2, z_1, w_1, w_2]| \\
&= |[z_1, z_2, w_2, w_1]| \\
&= |[w_2, w_1, z_1, z_2]| \\
&= |[w_1, w_2, z_2, z_1]|.
\end{aligned}$$

(iv) The cross-ratio possesses the same properties as the corresponding complex cross-ratio on taking cyclic products (see [16] page 225). To be specific these are:

$$\begin{aligned}
|[z_1, z_2, w_1, w_2][z_1, w_2, z_2, w_1][z_1, w_1, w_2, w_1]| &= |[z_1, z_2, w_1, w_2][w_2, z_2, z_1, w_1][w_1, z_2, w_2, z_1]| \\
&= |[z_1, z_2, w_1, w_2][z_2, w_2, w_1, z_1][w_2, z_1, z_2, w_2]| \\
&= |[z_1, z_2, w_1, w_2][z_2, w_1, z_1, w_2][w_1, z_1, w_1, w_2]| \\
&= 1.
\end{aligned}$$

2. *The iterative step with loxodromic maps.*

We consider U and V in $PSp(2,1)$ with V loxodromic fixing μ and ν in $\partial \mathbf{H}^2_{\mathbb{H}}$. Suppose that V has a dilation factor specified by $\lambda \in \mathbb{R}_+$ and rotation factors specified by the unit quaternions p and q. (See below for justification of the following.) In fact p is a measure of the "rotation angle" around the invariant axis of V in $\mathbf{H}^2_{\mathbb{H}}$. We now define a further conjugation invariant factor M by

$$M = |\lambda p - 1| + 2|pq - 1| + |\lambda^{-1} p - 1|.$$

Because $Sp(2,1)$ acts transitively on pairs of points in $\partial \mathbf{H}^2_{\mathbb{H}}$ we may assume, without loss of generality, that V fixes $\nu = o$ and $\mu = \infty$. The matrix group $Sp(1)$ acts by *Heisenberg rotations*. In horospherical coordinates this action is given by

$$(\zeta, v, u) \mapsto (q\zeta, v, u)$$

where q is a unit quaternion. The corresponding matrix in $PSp(2,1)$ acting on $P\mathbb{H}^{2,1}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In $\mathbf{H}^2_{\mathbb{H}}$ there is another type of elliptic element fixing o and ∞. This is given as follows. Let p be another unit quaternion, i.e. $p \in Sp(1)$. Consider the matrix pI. This acts on $\mathbb{H}$ on the left. When we projectivise on the right, we obtain the following action on horospherical coordinates

$$(\zeta, v, u) \mapsto (p\zeta\overline{p}, pv\overline{p}, pu\overline{p}) = (p\zeta\overline{p}, pv\overline{p}, u).$$

This action by conjugation is the identity on the real part and gives the action of $SO(3)$ on the imaginary part. In this way the composition of all elliptic elements of $PSp(2,1)$ fixing o and ∞ is $SO(3) \times Sp(1)$ (which is isomorphic to $SO(4)$) and corresponds to matrices of the form

$$\begin{bmatrix} p & 0 & 0 \\ 0 & pq & 0 \\ 0 & 0 & p \end{bmatrix}. \tag{3}$$

The positive real numbers $r \in \mathbb{R}^+$ act by Heisenberg dilation. In horospherical coordinates this is given by

$$(\zeta, v, u) \mapsto (r\zeta, r^2 v, r^2 u).$$

Then in $PSp(2,1)$ the corresponding matrix is

$$\begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{bmatrix}. \tag{4}$$

(Note that r corresponds to the λ mentioned above in the case when $r \geq 1$.) Therefore since V has quaternionic dilation factor determined by λ, p and q we assume U and V have the form

$$V = \begin{bmatrix} \lambda p & 0 & 0 \\ 0 & pq & 0 \\ 0 & 0 & p/\lambda \end{bmatrix}, \tag{5}$$

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

Thus refering back to definition of the quaternionic cross-ratio we have

$$|[U(\mu), \nu, \mu, U(\nu)]| = \frac{|\langle \mu, U(\mu)\rangle||\langle U(\nu), \nu\rangle|}{|\langle U(\nu), U(\mu)\rangle||\langle \mu, \nu\rangle|} = |\overline{g}c|, \tag{6}$$

$$|[U(\mu), \mu, \nu, U(\nu)]| = \frac{|\langle \nu, U(\mu)\rangle||\langle U(\nu), \mu\rangle|}{|\langle U(\nu), U(\mu)\rangle||\langle \nu, \mu\rangle|} = |\overline{a}j|. \tag{7}$$

Remark. Here $\mu = \infty$ and $\nu = o$.

Let U and V be elements of $Sp(2,1)$ with the above respective forms. Following Jørgensen, we form the sequence (U_n) by defining $U_0 = U$ and $U_{n+1} = U_n V {U_n}^{-1}$. By multiplication of matrices we have

$$U_{n+1} = \begin{bmatrix} a_{n+1} & b_{n+1} & c_{n+1} \\ d_{n+1} & e_{n+1} & f_{n+1} \\ g_{n+1} & h_{n+1} & j_{n+1} \end{bmatrix} = U_n \begin{bmatrix} \lambda p\overline{j_n} & \lambda p\overline{f_n} & \lambda p\overline{c_n} \\ pq\overline{h_n} & pq\overline{e_n} & pq\overline{b_n} \\ \lambda^{-1}p\overline{g_n} & \lambda^{-1}p\overline{d_n} & \lambda^{-1}p\overline{a_n} \end{bmatrix}. \tag{8}$$

Performing further matrix multiplication using $U_n {U_n}^{-1} = I$, the 3×3 identity matrix, we find that

$$a_{n+1} = \lambda a_n p\overline{j_n} + b_n pq\overline{h_n} + \lambda^{-1} c_n p\overline{g_n}, \tag{9}$$

$$c_{n+1} = \lambda a_n p\overline{c_n} + b_n pq\overline{b_n} + \lambda^{-1} c_n p\overline{a_n}, \tag{10}$$

$$g_{n+1} = \lambda g_n p\overline{j_n} + h_n pq\overline{h_n} + \lambda^{-1} j_n p\overline{g_n}, \tag{11}$$

$$j_{n+1} = \lambda g_n p\overline{c_n} + h_n pq\overline{b_n} + \lambda^{-1} j_n p\overline{a_n}. \tag{12}$$

From here on we shall assume that V has the form

$$V = \begin{bmatrix} \lambda p & 0 & 0 \\ 0 & pq & 0 \\ 0 & 0 & p/\lambda \end{bmatrix} \tag{13}$$

where p and q are arbitrary unit quaternions. Thus using the triangle inequality together with the identities derived from $U_n U_n{}^{-1} = I$ on equations (9) to (12) we have

$$|a_{n+1}| \leq |\lambda p - 1||a_n j_n| + 2|pq - 1|(|a_n j_n||c_n g_n|)^{1/2} + |\lambda^{-1}p - 1||c_n g_n| + 1, \tag{14}$$

$$|c_{n+1}| \leq (|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)|a_n c_n|, \tag{15}$$

$$|g_{n+1}| \leq (|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)|g_n j_n|, \tag{16}$$

$$|j_{n+1}| \leq |\lambda p - 1||c_n g_n| + 2|pq - 1|(|c_n g_n||a_n j_n|)^{1/2} + |\lambda^{-1}p - 1||a_n j_n| + 1. \tag{17}$$

Lemma 1.1. *Suppose that V is loxodromic, that is $\lambda > 1$. If $c_n = 0$ or $g_n = 0$ for some $n \geq 0$, then the group $\langle V, U \rangle$ is either elementary or not discrete.*

Proof. First suppose that there exists some n such that $c_n = 0$, $g_n \neq 0$ or $c_n \neq 0$, $g_n = 0$. This means that V and U_n share exactly one common fixed point and so $\langle V, U_n \rangle$ is not discrete. Hence $\langle V, U \rangle$ is not discrete.

We shall suppose that $c_{n+1} = g_{n+1} = 0$ for some $n \geq 0$. Now

$$\begin{aligned} c_{n+1} = 0 &\iff U_{n+1}(o) = o \\ &\iff U_n V U_n{}^{-1}(o) = o \\ &\iff V U_n{}^{-1}(o) = U_n{}^{-1}(o) \\ &\iff U_n{}^{-1}(o) = o, \infty \\ &\iff c_n = 0 \text{ or } a_n = 0. \end{aligned} \tag{18}$$

(Here we have used the fact that V fixes o and ∞.) Thus $c_{n+1} = 0$ implies $c_n = 0$ or $a_n = 0$. Likewise as $g_{n+1} = 0$ if and only if $U_{n+1}(\infty) = \infty$, $g_{n+1} = 0$ implies $g_n = 0$ or $j_n = 0$. We must now deal with 4 cases:
(1) $a_n = j_n = 0$,
(2) $c_n = j_n = 0$,
(3) $g_n = a_n = 0$,
(4) $c_n = g_n = 0$.

We claim that cases (2) and (3) cannot occur. Using $U_n U_n{}^{-1} = I$ we see that if $c_n = j_n = 0$, then $f_n = 0$ and if $a_n = g_n = 0$, then $d_n = 0$. In either case this means that one of the columns of U_n is zero which contradicts $U_n \in Sp(2,1)$.

Next we claim that if $n \geq 1$ then case (1) can not occur. Otherwise we have

$$U_n = \begin{bmatrix} 0 & 0 & c_n \\ 0 & e_n & 0 \\ g_n & 0 & 0 \end{bmatrix} \tag{19}$$

where $|e_n| = 1$. Thus by direct calculation we see that U_n^2 fixes the origin and infinity, whereas U_n does not, hence U_n cannot be loxodromic. However we assumed that V (and therefore U_n) is loxodromic and so we have a contradiction.

Therefore we have shown that if $n \geq 1$, then $c_{n+1} = g_{n+1} = 0$ implies $c_n = g_n = 0$. By induction this means that $c_1 = g_1 = 0$. This in turn means that $c_0 = g_0 = 0$ or $a_0 = j_0 = 0$. Thus V either fixes or interchanges the fixed points of U and so in either case $\langle V, U \rangle$ is elementary.

3. *Subgroups with loxodromic generators.*

In this section we give our results about the subgroups with loxodromic elements. Here we, like Jiang, Kamiya and Parker in [21], are inspired by the methods in [3] where a 'stable basin theorem' (see section 2.1 of Chapter Four) is used to determine a discreteness criterion in terms of the complex multiplier of the loxodromic generator V and the cross-ratio of the fixed points of V and UVU^{-1}. We are also motivated by the work on higher dimensional Möbius groups in [38] where Clifford algebras are used.

Theorem 1.1. *Let V be a loxodromic element of $Sp(2,1)$ fixing μ and ν and let U be any element of $Sp(2,1)$. If either*

$$M\left(|[U(\mu),\nu,\mu,U(\nu)]|^{1/2}+1\right)<1 \tag{20}$$

or

$$M\left(|[U(\mu),\mu,\nu,U(\nu)]|^{1/2}+1\right)<1, \tag{21}$$

then the group $\langle V,U\rangle$ is elementary or not discrete.

Proof. Since the quaternionic dilation factor (determined by λ, p and q) and the cross-ratio are invariant under conjugation, without loss of generality, we may assume that V and U have the forms given by equations (13) and (1) respectively. Furthermore by consideration of the dependence of M on λ we may (for the purposes of this proof) assume that $\lambda>1$. Using Lemma 1.1 we only need to consider the case where $c_n\neq 0$ and $g_n\neq 0$ for all n. Suppose that

$$M\left(|[B(\mu),\nu,\mu,U(\nu)]|^{1/2}+1\right)<1,$$

in other words $M(|cg|^{1/2}+1)<1$. By recursion and using Proposition 1.1, we have

$$0<|c_1g_1|^{1/2}\leq M|cg|^{1/2}|aj|^{1/2}\leq M(|cg|^{1/2}+1|)|cg|^{1/2}<|cg|^{1/2}.$$

Now suppose that

$$M\left(|[U(\mu),\mu,\nu,U(\nu)]|^{1/2}+1\right)<1,$$

in other words $M(|aj|^{1/2}+1)<1$. Similarly we have

$$0<|c_1g_1|^{1/2}\leq M|aj|^{1/2}|cg|^{1/2}\leq M(|aj|^{1/2}+1|)|aj|^{1/2}<|aj|^{1/2}.$$

In both cases we have

$$M(|c_1g_1|^{1/2}+1|<1.$$

Using induction a similar argument shows that

$$\begin{aligned}|c_{n+1}g_{n+1}|^{1/2}&\leq M(|c_ng_n|+1)^{1/2}|c_ng_n|^{1/2}\\&<M(|c_1g_1|+1)^{1/2}|c_ng_n|^{1/2}\\&<\left(M(|c_1g_1|+1)^{1/2}\right)^n|c_1g_1|^{1/2}.\end{aligned}$$

Thus we know that $|c_ng_n|$ is not equal to zero and tends to 0 as $n\to\infty$. Using Proposition 1.1 again, we see that $|a_nj_n|$ is bounded for all n. Thus

$$|f_nd_n|^2\leq 4|a_nj_n||c_ng_n|\to 0$$

as $n\to\infty$. Observing that $\overline{a_n}j_n+\overline{d_n}f_n+\overline{g_n}c_n=1$, we find

$$\overline{a_n}j_n=1-\overline{d_n}f_n-\overline{g_n}c_n\to 1$$

as $n\to\infty$. Then from equation (9) and the fact that $|b_nh_n|\leq 2|a_nj_n|^{1/2}|c_ng_n|^{1/2}$, we have

$$a_{n+1}=\lambda a_np\overline{j_n}+b_npq\overline{h_n}+\lambda^{-1}c_np\overline{g_n}\to\lambda a^*$$

as $n \to \infty$ (where a^* is a unit quaternion). Similarly

$$j_{n+1} \to \lambda^{-1} j^*$$

as $n \to \infty$ (where j^* is a unit quaternion). Furthermore $e_{n+1} \to e^*$, a unit quaternion as $n \to \infty$ as from equation (8) we have

$$e_{n+1} = d_n \lambda p \overline{f_n} + e_n pq \overline{e_n} + f_n \lambda^{-1} p \overline{d_n}$$

and from $U_n {U_n}^{-1} = I$ we have $|e_n|^2 = 1 - d_n \overline{f_n} - f_n \overline{d_n} \to 1$ as $n \to \infty$. Since

$$|c_{n+1}| \leq M|a_n||c_n| \leq M(|a_n - \lambda a^*| + |\lambda a^*|)|c_n|,$$

$$|g_{n+1}| \leq M|j_n||g_n| \leq M(|j_n - \lambda^{-1} j^*| + |\lambda^{-1} j^*|)|g_n|$$

and $M < 1$, we can find $N > 0$ so that

$$M(|a_n - \lambda a^*| + |\lambda a^*|) < \lambda$$

and

$$M(|j_n - \lambda^{-1} j^*| + |\lambda^{-1} j^*|) < \lambda^{-1}$$

for all $n \geq N$. This means that $|c_n|\lambda^{-n} \to 0$ and $|g_n|\lambda^n \to 0$ as $n \to \infty$.

Following Jørgensen, we now define the sequence $W_n = V^{-n} U_{2n} V^n$ where $U_0 = U$. As a matrix in $Sp(2,1)$ this is given by

$$W_n = \begin{bmatrix} \overline{p}^n a_{2n} p^n & \lambda^{-n} \overline{p}^n b_{2n} (pq)^n & \lambda^{-2n} \overline{p}^n c_{2n} p^n \\ \lambda^n (\overline{q}\,\overline{p})^n d_{2n} p^n & (\overline{q}\,\overline{p})^n e_{2n} (pq)^n & \lambda^{-n} (\overline{q}\,\overline{p})^n f_{2n} p^n \\ \lambda^{2n} \overline{p}^n g_{2n} p^n & \lambda^n \overline{p}^n h_{2n} (pq)^n & \overline{p}^n j_{2n} p^n \end{bmatrix}. \tag{22}$$

We already have $a_{2n} - \lambda a^*$, $\lambda^{-2n} c_{2n}$, $\lambda^{2n} g_{2n}$, $e_{2n} - e^*$ and $j_{2n} - \lambda^{-1} j^*$ tending to zero as n tends to infinity. Moreover $\lambda^{-2n} \overline{p}^n c_{2n} p^n$ and $\lambda^{2n} \overline{p}^n g_{2n} p^n$ are distinct and non-zero. We need to show that $\lambda^{-n} \overline{p}^n b_{2n} (pq)^n$, $\lambda^n (\overline{q}\,\overline{p})^n d_{2n} p^n$, $\lambda^{-n} (\overline{q}\,\overline{p})^n f_{2n} p^n$ and $(\lambda \overline{p})^n h_{2n} (pq)^n$ all tend to 0 as $n \to \infty$. From $W_n {W_n}^{-1} = I$ it follows that

$$|\lambda^{-n} \overline{p}^n b_{2n} (pq)^n|^2 = -\lambda^{-2n} \overline{p}^n a_{2n} \overline{c_{2n}} p^n - \lambda^{-2n} \overline{p}^n c_{2n} \overline{a_{2n}} p^n,$$
$$|\lambda^n (\overline{q}\,\overline{p})^n d_{2n} p^n|^2 = -\lambda^{-2n} \overline{p}^n \overline{g}_{2n} a_{2n} p^n - \lambda^{2n} \overline{a_{2n}} g_{2n} p^n,$$
$$|\lambda^{-n} (\overline{q}\,\overline{p})^n f_{2n} p^n|^2 = -\lambda^{-2n} \overline{p}^n \overline{j_{2n}} c_{2n} p^n - \lambda^{-2n} \overline{p}^n \overline{c_{2n}} j_{2n} p^n,$$
$$|\lambda^n \overline{p}^n h_{2n} (pq)^n|^2 = -\lambda^{2n} \overline{p}^n g_{2n} \overline{j_{2n}} p^n - \lambda^{2n} \overline{p}^n j_{2n} \overline{g_{2n}} p^n.$$

Therefore we have $\lambda^{-n} \overline{p}^n b_{2n} (pq)^n$, $\lambda^n (\overline{q}\,\overline{p})^n d_{2n} p^n$, $\lambda^{-n} (\overline{q}\,\overline{p})^n f_{2n} p^n$ and $\lambda^n \overline{p}^n h_{2n} (pq)^n$ all tending to 0 as $n \to \infty$.

From all of this we know that the W_n are all distinct and tend to a finite map W^* as $n \to \infty$. Therefore the group $\langle V, U \rangle$ is not discrete.

Our second version of Jørgensen's inequality for groups with loxodromic elements is the following.

Theorem 1.2. *Let V be a loxodromic element of $Sp(2,1)$ fixing μ and ν and let U be any element of $Sp(2,1)$. If $M < \sqrt{2} - 1$ and*

$$|[U(\mu), \mu, \nu, U(\nu)]| + |[U(\mu), \nu, \mu, U(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group $\langle V, U \rangle$ is elementary or not discrete.

As before we assume that V and U have the forms given by equations (13) and (1) respectively and form the sequence $U_0 = U$ and $U_{n+1} = U_n V {U_n}^{-1}$. Also if $c_n = 0$ or $g_n = 0$, then by Lemma 1.1 the group is elementary or not discrete. Thus we assume that c_n and g_n are nonzero.

Lemma 1.2. *We have*

$$|a_{n+1}j_{n+1}| + |c_{n+1}g_{n+1}| \leq \frac{1}{2}M^2(|a_nj_n| + |c_ng_n|)^2 + M(|a_nj_n| + |c_ng_n|) + 1,$$

where $M = |\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|$.

Proof. From equation (9) and $U_nU_n{}^{-1} = I$ we have

$$\begin{aligned}
|a_{n+1}| &= |\lambda a_n p\overline{j_n} + b_n pq\overline{h_n} + \lambda^{-1}c_n p\overline{g_n}| \\
&= |1 + a_n(\lambda p - 1)\overline{j_n} + b_n(pq - 1)\overline{h_n} + c_n(\lambda^{-1}p - 1)\overline{g_n}| \\
&\leq 1 + |a_nj_n||\lambda p - 1| + |b_nh_n||pq - 1| + |c_ng_n||\lambda^{-1}p - 1| \\
&\leq 1 + |a_nj_n|(|\lambda p - 1| + |pq - 1|) + |c_ng_n|(|\lambda^{-1}p - 1| + |pq - 1|).
\end{aligned}$$

Here we have used $|b_nh_n| \leq 2|a_nj_n|^{1/2}|c_ng_n|^{1/2} \leq |a_nj_n| + |c_ng_n|$.
Similarly we have

$$|j_{n+1}| \leq 1 + |c_ng_n|(|\lambda p - 1| + |pq - 1|) + |a_nj_n|(|\lambda^{-1}p - 1| + |pq - 1|),$$

$$|c_{n+1}| \leq |a_nc_n|(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)$$

and

$$|g_{n+1}| \leq |g_nj_n|(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|).$$

Thus

$$\begin{aligned}
&|a_{n+1}j_{n+1}| + |c_{n+1}g_{n+1}| \\
&\leq 1 + (|a_nj_n| + |c_ng_n|)(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|) \\
&\quad + |a_nj_n|^2(|\lambda p - 1| + |pq - 1|)(|pq - 1| + |\lambda^{-1}p - 1|) \\
&\quad + |a_nj_n||c_ng_n|\left((|\lambda p - 1| + |pq - 1|)^2 + (|pq - 1| + |\lambda^{-1}p - 1|)^2 + (|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2\right) \\
&\quad + |c_ng_n|^2(|\lambda p - 1| + |pq - 1|)(|pq - 1| + |\lambda^{-1}p - 1|) \\
&\leq 1 + (|a_nj_n| + |c_ng_n|)(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|) + |a_nj_n||c_ng_n|(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2 \\
&\quad + |a_nj_n|^2(|\lambda p - 1| + |pq - 1|)(|pq - 1| + |\lambda^{-1}p - 1|) \\
&\quad + \frac{1}{2}(|a_nj_n|^2 + |c_ng_n|^2)\left((|\lambda p - 1| + |pq - 1|)^2 + (|pq - 1| + |\lambda^{-1}p - 1|)^2\right) \\
&\quad + |c_ng_n|^2(|\lambda p - 1| + |pq - 1|)(|pq - 1| + |\lambda^{-1}p - 1|) \\
&= 1 + (|a_nj_n| + |c_ng_n|)(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|) + |a_nj_n||c_ng_n|(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2 \\
&+ |a_nj_n|^2\left(\frac{1}{2}(|\lambda p - 1| + |pq - 1|)^2 + (|\lambda p - 1| + |pq - 1|)(|pq - 1| + |\lambda^{-1}p - 1|) + \frac{1}{2}(|pq - 1| + |\lambda^{-1}p - 1|)^2\right) \\
&+ |c_ng_n|^2\left(\frac{1}{2}(|\lambda p - 1| + |pq - 1|)^2 + (|\lambda p - 1| + |pq - 1|)(|pq - 1| + |\lambda^{-1}p - 1|) + \frac{1}{2}(|pq - 1| + |\lambda^{-1}p - 1|)^2\right) \\
&= 1 + (|a_nj_n| + |c_ng_n|)(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|) + |a_nj_n||c_ng_n|(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2 \\
&\quad + \frac{1}{2}|a_nj_n|^2(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2 + \frac{1}{2}|c_ng_n|^2(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2 \\
&= 1 + (|a_nj_n| + |c_ng_n|)(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|) \\
&\quad + \frac{1}{2}(|a_nj_n| + |c_ng_n|)^2(|\lambda p - 1| + 2|pq - 1| + |\lambda^{-1}p - 1|)^2 \\
&= 1 + (|a_nj_n| + |c_ng_n|)M + \frac{1}{2}(|a_nj_n| + |c_ng_n|)^2M^2.
\end{aligned}$$

This has proved the lemma. If we define $x_n = |a_n j_n| + |c_n g_n|$, then Lemma 1.2 says

$$x_{n+1} \leq \frac{M^2}{2} {x_n}^2 + M x_n + 1.$$

The next two lemmas will be stated without proof. Their proofs can be obtained directly from the proofs of Lemmas 4.4 and 4.5 in [21].

Lemma 1.3. *Assume that $0 < M \leq \sqrt{2} - 1$. If*

$$0 \leq x_0 < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then there exists a number N such that $M^2 x_n < 1 - M$ for all $n > N$.

Lemma 1.4. *If*

$$0 \leq x_0 < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then $|a_n j_n|$ is bounded and $|c_n g_n|$ tends to 0 as $n \to \infty$.

Lemma 1.5. *Assume $M < \sqrt{2} - 1$. If $|a_n j_n|$ is bounded and $|c_n g_n|$ tends to 0 as $n \to \infty$, then a subsequence of (U_n) converges as n tends to infinity.*

Proof. As in the proof of Theorem 1.1 we find that as n tends to infinity

$$|d_n f_n| \to 0,$$

$$|a_n j_n| \to 1,$$

$$a_n \to \lambda a^*$$

where a^* is a unit quaternion,

$$e_n \to e^*$$

where e^* is a unit quaternion and

$$j_n \to \lambda^{-1} j^*$$

where j^* is a unit quaternion. Since $M = |\lambda p - 1| + 2|pq - 1| + |\lambda^{-1} p - 1| < \sqrt{2} - 1$ we obtain

$$\lambda \leq |\lambda p - 1| + 1 < (\sqrt{2} - 1) + 1 = \sqrt{2}.$$

Since we have $a_n \to \lambda a^*$ and $\lambda < 2$ we know that there is an N so that $|a_n| < 2$ for all $n > N$. Hence for all $n > N$ we have

$$M|a_n| < 2(\sqrt{2} - 1) < 1$$

which means

$$|c_{n+m}| < \big(2(\sqrt{2} - 1)^m\big)|c_n|.$$

Hence $c_{n+1} \to 0$ as $n \to \infty$. Similarly $g_{n+1} \to 0$ as $n \to \infty$. Also from $|f_n|^2 < 2|c_n j_n|$ we have $f_{n+1} \to 0$ as $n \to \infty$. Similarly as $n \to \infty$ we have $d_{n+1} \to 0$, $h_{n+1} \to 0$ and $b_{n+1} \to 0$. Therefore we see that U_{n+1} tends to a finite map U^* as $n \to \infty$.

Finally since we have assumed that c_n and g_n are both non-zero for all n we can extract a sequence of distinct c_n and g_n tending to zero. Thus (at least on this subsequence) the U_n are distinct and so the group $\langle V, U \rangle$ is not discrete. This completes our proof of Theorem 1.2.

Proposition 1.2. *If* $|aj|^{1/2} = |cg|^{1/2} + 1$ *or* $|cg|^{1/2} = |aj|^{1/2} + 1$, *then Theorem 1.2 follows from Theorem 1.1. If* $|aj| = |cg|$ *then for* $0 < M < M_0$ *where*

$$M_0 = 1 + (54 + 6\sqrt{87})^{-1/3} - (54 + 6\sqrt{87})^{1/3}/6 < \sqrt{2} - 1,$$

Theorem 1.1 follows from Theorem 1.2.

Proof. The proof of this theorem follows almost identically the arguments used by the authors of [21] to prove their Proposition 4.7.

4. *Subgroups with a boundary elliptic generator.*

In this section we give an analogy of Jørgensen's inequality for groups with a boundary elliptic generator. Suppose that $V \in Sp(2,1)$ is a boundary elliptic element. We define the order of V as

$$ord(V) = inf\{m \in \mathbb{Z}_+ : V^m = I\}.$$

If V has order n, then taking powers of V if necessary, we may assume that V rotates through a unit quaternion p (where $p^n = 1$). This is due to the fact that as in the case of real hyperbolic geometry a discrete subgroup of $Sp(2,1)$ can not contain elliptic elements of infinite order. We may assume that V fixes o and ∞. Let V be a boundary elliptic element of $Sp(2,1)$, then V fixes a subspace in $\mathbf{H}^2_{\mathbb{H}}$. We denote this quaternionic subspace by L_V. The precise form of L_V depends on V.

Proposition 1.3. *Let* L_V *be the quaternionic subspace fixed by* V. *Then, if* $p \neq 1$

1) L_V *is a quaternionic line if* $V = diag(1,p,1)$,
2) L_V *is a complex hyperbolic 2-space if* $V = diag(p,p,p)$,
3) L_V *is a complex line if* $V = diag(p,pq,p)$ *and* $q \neq 1$.

Remark. We observe that a complex line is the intersection of a quaternionic line with a copy of complex hyperbolic 2-space embedded in $\mathbf{H}^2_{\mathbb{H}}$.

Proof. In horospherical coordinates (ζ, v, u) a point in $\mathbf{H}^2_{\mathbb{H}}$ fixed by V can be represented by the vector

$$\mathbf{v}_V = \begin{bmatrix} -|\zeta|^2 - u + v \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}.$$

In case 1) $V(\mathbf{v}_V)$ has the form

$$\begin{bmatrix} -|\zeta|^2 - u + v \\ \sqrt{2}p\zeta \\ 1 \end{bmatrix}.$$

Therefore $diag(1,p,1)$ fixes a quaternionic line given by $\zeta = 0$.

In case 2) $V(\mathbf{v}_V)$, after normalising, has the form

$$\begin{bmatrix} -|\zeta|^2 - u + pv\overline{p} \\ \sqrt{2}p\zeta\overline{p} \\ 1 \end{bmatrix}.$$

There exists a non-real quaternion z_0 such that $pz_0\overline{p} = z_0$. Therefore $\mathbb{R} + z_0\mathbb{R} \subset \mathbb{H}$ which is isomorphic to $\mathbb{C}$ is fixed by conjugation by p. If v, $\zeta \in \mathbb{R} + z_0\mathbb{R}$ we see that (ζ, v, u) is fixed by $diag(p,p,p)$. These points must form a copy of complex hyperbolic 2-space.

In case 3) $V(\mathbf{v}_V)$ after normalising has the form

$$\begin{bmatrix} -|\zeta|^2 - u + pv\overline{p} \\ \sqrt{2}pq\zeta\overline{p} \\ 1 \end{bmatrix}.$$

Therefore $diag(p, pq, p)$ fixes a complex line with $\zeta = 0$ and $v \in \mathbb{R} + z_0\mathbb{R}$ where z_0 is as above.

We will consider only case 1) here.

Let $\mathbf{v}_V$ be the **polar vector** to L_V, by which we mean that the orthogonal complement of L_V is spanned by $\mathbf{v}_V$. Thus the fixed quaternionic line of UVU^{-1} is $U(L_V)$, which is polar to $U(\mathbf{v}_V)$. Normalising $\mathbf{v}_V$ and $U(\mathbf{v}_V)$ so that $\langle \mathbf{v}_V, \mathbf{v}_V \rangle = \langle U(\mathbf{v}_V), U(\mathbf{v}_V) \rangle = 1$ there are three cases (see page 100 of [16]):

(1) $|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| < 1$, in this case L_V and $U(L_V)$ intersect at a point in $\mathbf{H}^2_{\mathbb{H}}$. Moreover

$$|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = \cos(\phi),$$

where ϕ corresponds to the angle between the polar vectors to L_V and $U(L_V)$. In particular if

$$|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = 0,$$

then L_V and $U(L_V)$ intersect orthogonally.

(2) $|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = 1$, in this case either $U(L_V) = L_V$ or else L_V and $U(L_V)$ are asymptotic at a point in $\partial\mathbf{H}^2_{\mathbb{H}}$.

(3) $|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| > 1$, in this case L_V and $U(L_V)$ are ultraparallel, that is they are disjoint and have a common quaternionic geodesic. Moreover $|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = \cosh(\rho/2)$ where ρ is the distance between L_V and $U(L_V)$.

We have assumed that V fixes o and ∞. This means that L_V is the line spanned by o and ∞. In other words

$$\mathbf{v}_V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We are taking V and U to have the forms

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

and

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}. \tag{1}$$

This gives

$$|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = |e|.$$

First we show that if U preserves L_V, then the group is elementary.

Lemma 1.6. *Let V be a boundary elliptic element of $Sp(2,1)$ which fixes pointwise a quaternionic line L_V. Let U be any element of $Sp(2,1)$. Suppose that $L_V = U(L_V)$. Then the group $\langle V, U \rangle$ is elementary.*

Proof. Because U maps $\overline{L_V}$ to itself we know by Brouwer's Fixed Point Theorem that U has a fixed point in $\overline{L_V}$. As V fixes all points of $\overline{L_V}$ pointwise we see that $\langle V, U \rangle$ is elementary.

The main purpose of this section is to prove the following theorem.

Theorem 1.3. *Let V be a boundary elliptic element of $Sp(2,1)$ which rotates through the order m unit quaternion p (i.e. $p^m = 1$) about a quaternionic line L_V with polar vector $\mathbf{v}_V$ normalised so that $\langle \mathbf{v}_V, \mathbf{v}_V \rangle = 1$. Let U be any element of $Sp(2,1)$ so that $U(L_V) \neq L_V$.*

(1) Suppose that $|\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = \cos(\phi) \neq 0$. If $|1 - p| \leq 1$, then the group $\langle V, U \rangle$ is not discrete.

(2) *Suppose that* L_V *and* $U(L_V)$ *are asymptotic. If* $|1-p|<1$, *then the group* $\langle V,U\rangle$ *is not discrete.*

(3) *Suppose that* L_V *and* $U(L_V)$ *are ultraparallel and denote the distance between them by* ρ. *If*

$$\cosh(\rho/2)|1-p|<1, \tag{24}$$

then the group $\langle V,U\rangle$ *is not discrete.*

(4) *Suppose that* L_V *and* $U(L_V)$ *intersect orthogonally, i.e.* $\phi=\pi/2$ *and* $|\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle|=0$. *If*

$$\cosh(\rho'/2)|1-p|<1 \tag{25}$$

where ρ' *is the distance between* L_V *and* $U^2(L_V)$, *then the group* $\langle V,U\rangle$ *is elementary or not discrete.*

Proof. Note that $\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle$ and $|p|$ are invariant under conjugation in $Sp(2,1)$. We recall that V has the form

$$V=\begin{bmatrix}1&0&0\\0&p&0\\0&0&1\end{bmatrix}.$$

We will also assume that U has the form given in equation (1). If $U_0=U$ and $U_{n+1}=U_nVU_n^{-1}$ we have

$$\begin{aligned}U_{n+1}&=\begin{bmatrix}a_n&b_n&c_n\\d_n&e_n&f_n\\g_n&h_n&j_n\end{bmatrix}\begin{bmatrix}1&0&0\\0&p&0\\0&0&1\end{bmatrix}\begin{bmatrix}\overline{j_n}&\overline{f_n}&\overline{c_n}\\\overline{h_n}&\overline{e_n}&\overline{b_n}\\\overline{g_n}&\overline{d_n}&\overline{a_n}\end{bmatrix}\\&=\begin{bmatrix}a_n\overline{j_n}+b_np\overline{h_n}+c_n\overline{g_n}&a_n\overline{f_n}+b_np\overline{e_n}+c_n\overline{d_n}&a_n\overline{c_n}+b_np\overline{b_n}+c_n\overline{a_n}\\d_n\overline{j_n}+e_np\overline{h_n}+f_n\overline{g_n}&d_n\overline{f_n}+e_np\overline{e_n}+f_n\overline{d_n}&d_n\overline{c_n}+e_np\overline{b_n}+f_n\overline{a_n}\\g_n\overline{j_n}+h_np\overline{h_n}+j_n\overline{g_n}&g_n\overline{f_n}+h_np\overline{e_n}+j_n\overline{d_n}&g_n\overline{c_n}+h_np\overline{b_n}+j_n\overline{a_n}\end{bmatrix}.\end{aligned}$$

Using the identities from $U_nU_n^{-1}=I$ we see

$$U_{n+1}=\begin{bmatrix}1+b_n(p-1)\overline{h_n}&b_n(p-1)\overline{e_n}&b_n(p-1)\overline{b_n}\\e_n(p-1)\overline{h_n}&1+e_n(p-1)\overline{e_n}&e_n(p-1)\overline{b_n}\\h_n(p-1)\overline{h_n}&h_n(p-1)\overline{e_n}&1+h_n(p-1)\overline{b_n}\end{bmatrix}.$$

Computation shows that

$$|e_{n+1}|^2-1=|1-p|^2|e_n|^2(|e_n|^2-1),$$
$$|b_{n+1}|=|1-p||b_n||e_n|$$

and

$$|h_{n+1}|=|1-p||h_n||e_n|.$$

We now consider the four cases separately.

(1) First assume that L_V and $U(L_V)$ are such that

$$0<\cos(\phi)=|\langle \mathbf{v}_V,U(\mathbf{v}_V)\rangle|=|e|<1.$$

Using induction we find

$$0<|e_n|<1$$

and

$$1-|e_{n+1}|^2\le(1-|e_n|^2)|e_n|^2.$$

We claim that $1-|e_n|^2<\frac{1}{(n+1)}$ for all $n\ge 0$. Clearly this is true for $n=0$ and $n=1$. Now assume that $1-|e_n|^2<\frac{1}{(n+1)}$. Since $x(1-x)$ is an increasing function for all $0<x<1/2$ we have

$$\begin{aligned}1-|e_{n+1}|^2&\le(1-|e_n|^2)|e_n|^2\\&<\frac{1}{n+1}\left(1-\frac{1}{n+1}\right)\\&=\frac{n}{(n+1)^2}<\frac{1}{n+2}.\end{aligned}$$

Thus $|e_n|$ tends to 1 as n tends to infinity.

When $|1-p|<1$ we have

$$|b_{n+1}| = |1-p||b_n||e_n| \leq |1-p||b_n| < |b_0||1-p|^{n+1},$$

$$|h_{n+1}| = |1-p||h_n||e_n| \leq |1-p||h_n| < |h_0||1-p|^{n+1}.$$

When $|1-p|=1$ using induction as above we can show that $1-|e_n|^2 > (|e_1|^2)/(n+1)$. Therefore

$$|b_{n+1}|^2 = |b_n|^2|e_n|^2 < |b_n|^2\left(\frac{n+|e_1|^2}{n+1}\right) < |b_n|^2\left(1-\frac{1}{2(n+1)}\right).$$

As n tends to infinity the product

$$\left(1-\frac{1}{2(n+1)}\right)\left(1-\frac{1}{2n}\right)\cdots\left(1-\frac{1}{2}\right)$$

diverges to zero. Thus $|b_n|$ and similarly $|h_n|$ tend to zero as n tends to infinity.

Thus in both cases we have

$$|e_n| \to 1,$$

$$|b_n| \to 0,$$

$$|h_n| \to 0$$

as $n\to\infty$. Now it is easy to see that U_n tends to V. Moreover as $0<|e_n|\neq 1$ and $|e_n|\to 1$ we see that the $|e_n|$ are distinct and so are the U_n. So the group $\langle V,U\rangle$ is not discrete.

(2) Secondly assume that L_V and $U(L_V)$ are asymptotic. That is $|\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle| = |e| = 1$ and $\mathbf{v}_V$ is not projectively equal to $U(\mathbf{v}_V)$. Hence either $b\neq 0$ or $h\neq 0$ (or both).

Assume that L_V and $U(L_V)$ are asymptotic at a point in $\partial\mathbf{H}^2_{\mathbb{H}}$. In this case we have $|e|=1$. Moreover both

$$|b_{n+1}| = |1-p||b_n| = |1-p|^{n+1}|b| \to 0$$

and

$$|h_{n+1}| = |1-p||h_n| = |1-p|^{n+1}|h| \to 0$$

as $n\to\infty$. Therefore U_n tends to V as n tends to infinity. If $b\neq 0$, then the b_n are distinct and if $h\neq 0$, then the h_n are distinct. Since one of these possibilities occurs the U_n are also distinct. This means that the group $\langle V,U\rangle$ is not discrete.

(3) We now suppose that L_V and $U(L_V)$ are ultraparallel. This means that

$$1 < |e| = |\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle| = \cosh(\rho/2) < \frac{1}{|1-p|}.$$

Therefore

$$0 < |e_{n+1}|^2 - 1 = (|e_n|^2-1)|e_n|^2|1-p|^2 < |e_n|^2 - 1$$

so $|e_n|$ is a decreasing sequence of numbers between 1 and $1/|1-p|$. Hence using induction and the fact that $|e||1-p|<1$ we find that as n tends to infinity

$$|e_{n+1}|^2 - 1 < \left(|e||1-p|\right)^{2(n+1)}\left(|e|^2-1\right) \to 0,$$

$$|b_{n+1}| < \left(|e||1-p|\right)^{n+1}|b| \to 0$$

and

$$|h_{n+1}| < \left(|e||1-p|\right)^{n+1}|h| \to 0.$$

Again we find that $U_n \to V$. Since $|e_n| > 1$ and $|e_n| \to 1$ we see that the e_n are distinct. Hence the U_n are distinct and the group $\langle V, U\rangle$ is not discrete.

(4) In this case we have

$$|e| = |\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle| = 0.$$

Using the relations between entries of U arising from $UU^{-1} = I$ we find that none of the other entries of U is equal to zero. Consider the sequence $U_0 = U^2$ and $U_{n+1} = U_n V {U_n}^{-1}$. We claim that L_V and $U(L_V)$ are either ultraparallel or coincide. In the former case the result follows from case (3). In the latter case we will show that $\langle V, U\rangle$ is elementary. By squaring U we see that

$$\begin{aligned} |\langle \mathbf{v}_V, U^2(\mathbf{v}_V)\rangle|^2 &= |db + fh|^2 \\ &= |b|^2|d|^2 + db\overline{h}\,\overline{f} + fh\overline{b}\,\overline{d} + |f|^2|h|^2. \end{aligned} \tag{26}$$

From $U^{-1}U = I$ we obtain

$$\begin{aligned} 1 &= \overline{h}b + \overline{b}h \\ &= \left(\frac{h}{|h|^2}\right)(\overline{h}b + \overline{b}h)\overline{h} \\ &= b\overline{h} + h\overline{b}. \end{aligned}$$

Substituting this into equation (26), then using $d\overline{f} + f\overline{d} = 1$ (obtained from $UU^{-1} = I$) we find

$$\begin{aligned} |\langle \mathbf{v}_V, U^2(\mathbf{v}_V)\rangle|^2 &= |b|^2|d|^2 + d\overline{f} + f\overline{d} - dh\overline{b}\,\overline{f} - fb\overline{h}\,\overline{d} + |f|^2|h|^2 \\ &= 1 + |b|^2|d|^2 - dh\overline{b}\,\overline{f} - fb\overline{h}\,\overline{d} + |f|^2|h|^2 \\ &= 1 + \left| dh\frac{|b|}{|h|} - fb\frac{|h|}{|b|}\right|^2. \end{aligned}$$

This shows that $|\langle \mathbf{v}_V, U^2(\mathbf{v}_V)\rangle| \geq 1$ and hence L_V and $U^2(L_V)$ are ultraparallel, asymptotic or coincide. First consider the case where L_V and $U^2(L_V)$ are ultraparallel. We have

$$|\langle \mathbf{v}_V, U^2(\mathbf{v}_V)\rangle| = \cosh(\rho'/2)$$

where ρ' is the distance between L_V and $U^2(L_V)$. If

$$\cosh(\rho'/2)|1-p| < 1,$$

then (3) implies that $\langle V, U^2\rangle$ ($\subset \langle V, U\rangle$) is not discrete.

Secondly consider the case where $|\langle \mathbf{v}_V, U^2(\mathbf{v}_V)\rangle| = 1$, that is L_V and $U^2(L_V)$ are either asymptotic or coincide. By the computations above this means that $|b|^2dh = |h|^2fb$. Setting

$$\mathbf{u} = \begin{bmatrix} -h|b|^2 \\ 0 \\ b|h|^2 \end{bmatrix}$$

and using the $U^{-1}U = I$ derived identity $b\overline{h} + h\overline{b} = \overline{h}b + \overline{b}h = 1$, we observe that

$$\langle \mathbf{u}, \mathbf{u}\rangle = -|b|^2|h|^2(h\overline{b} + b\overline{h}) = -|b|^2|h|^2 < 0.$$

This proves that $\mathbf{u} \in \mathbf{H}^2_{\mathbb{H}}$.

Using $|b|^2 dh = |h|^2 fb$ we see that

$$\begin{bmatrix} a & b & c \\ d & 0 & f \\ g & h & j \end{bmatrix} \begin{bmatrix} -h|b|^2 \\ 0 \\ b|h|^2 \end{bmatrix} = \begin{bmatrix} -ah|b|^2 + cb|h|^2 \\ -dh|b|^2 + fb|h|^2 \\ -gh|b|^2 + jb|h|^2 \end{bmatrix}$$
$$= \begin{bmatrix} -h|b|^2\left(\frac{\bar{h}ah}{|h|^2}\right) + h(\bar{h}c)b \\ 0 \\ -b\bar{b}gh + b|h|^2\left(\frac{\bar{b}jb}{|b|^2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} -h|b|^2\left(\frac{\bar{h}ah}{|h|^2} + \frac{\bar{b}jb}{|b|^2}\right) \\ 0 \\ b|h|^2\left(\frac{\bar{h}ah}{|h|^2} + \frac{\bar{b}jb}{|b|^2}\right) \end{bmatrix}.$$

We observe that the ($U^{-1}U = I$ derived) identities $\bar{h}c + \bar{b}j = 0$ and $\bar{h}a + \bar{b}g = 0$ were used to prove the last equation. Thus we have proved that $U(\mathbf{u}) = \mathbf{u}$. Furthermore direct calculation shows that $V(\mathbf{u}) = \mathbf{u}$. Therefore $\mathbf{u}$ is fixed by both U and V.

Thus $\langle V, U\rangle$ is elementary. This completes our proof of Theorem 1.3.

Let V be a boundary elliptic element which rotates through a unit quaternion p about the fixed quaternionic line L_V. Then we get

$$M = 2|p-1|.$$

Next we show two theorems similar to Theorems 1.1 and 1.2 for groups containing such a boundary elliptic element.

Theorem 1.4. *Let V be a boundary elliptic element of $Sp(2,1)$ fixing the line L_V spanned by μ and ν in $\partial\mathbf{H}^2_{\mathbb{H}}$. Suppose that U is any element of $Sp(2,1)$ for which L_V and $U(L_V)$ do not intersect orthogonally. If either*

$$M\left(|[U(\mu),\nu,\mu,U(\nu)]|^{1/2} + 1\right) < 1$$

or

$$M\left(|[U(\mu),\mu,\nu,U(\nu)]|^{1/2} + 1\right) < 1,$$

then the group $\langle V, U\rangle$ is elementary or not discrete.

Proof. If L_V and $U(L_V)$ coincide then $\langle V, U\rangle$ is elementary. From now on assume that $U(L_V) \neq L_V$. The inequality in Theorem 1.4 implies $M < 1$ (where $M = 2|p-1|$). If L_V and $U(L_V)$ are asymptotic or intersect at an angle $\phi \neq \pi/2$ then using Theorem 1.3 we see that $\langle V, U\rangle$ is not discrete.

Now suppose that L_V and $U(L_V)$ are ultraparallel. Conjugating by an element of $Sp(2,1)$, if necessary, we may assume that V and U have the forms given by equations (23) and (1) respectively. Thus we have

$$|\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle| = |e|,$$

$$|[U(\mu),\mu,\nu,U(\nu)]| = |aj|$$

and

$$|[U(\mu),\nu,\mu,U(\nu)]| = |cg|.$$

By Proposition 1.1 we have

$$|e| \le |aj|^{1/2} + |cg|^{1/2} \le 2|aj|^{1/2} + 1 < 2(|aj|^{1/2} + 1).$$

It follows that

$$\begin{aligned} 1 &> M(|aj|^{1/2} + 1) \\ &= 2|1-p|(|aj|^{1/2} + 1) \\ &> |1-p||e| \\ &= |1-p||\langle \mathbf{v}_V, U(\mathbf{v}_V)\rangle| \\ &= |1-p|\cosh(\rho/2). \end{aligned}$$

The result now follows from Theorem 1.3 part (3).

Similarly we have

$$|e| \leq 2|cg|^{1/2} + 1 < 2(|cg|^{1/2} + 1)$$

and the second part of the theorem follows.

Theorem 1.5. *Let V be a boundary elliptic element of $Sp(2,1)$ fixing the line L_V spanned by μ and ν in $\partial\mathbf{H}^2_{\mathbb{H}}$. Suppose that U is any element of $Sp(2,1)$ for which L_V and $U(L_V)$ do not intersect orthogonally. If $M \leq \sqrt{2} - 1$ and*

$$|[U(\mu), \mu, \nu, U(\nu)]| + |[U(\mu), \nu, \mu, U(\nu)]| < \frac{1 - M + \sqrt{1 - 2M - M^2}}{M^2},$$

then the group $\langle V, U \rangle$ is elementary or not discrete.

Proof. If L_V and $U(L_V)$ coincide then $\langle V, U \rangle$ is elementary. From now on we assume that $U(L_V) \neq L_V$. Furthermore as in the proof of Theorem 1.4 if L_V and $U(L_V)$ are asymptotic or intersect at an angle $\phi \neq \pi/2$ from $M \leq \sqrt{2} - 1$ we have $M < 1$. Thus the group $\langle V, U \rangle$ is not discrete.

If L_V and $U(L_V)$ are ultraparallel, then we have

$$|e| = |\langle \mathbf{v}_V, U(\mathbf{v}_V) \rangle| = \cosh(\rho/2)$$

where ρ is the distance between L_V and $U(L_V)$. Using Proposition 1.1 we see that

$$|e|^2 \leq |aj| + |cg| + 2|aj|^{1/2}|cg|^{1/2} \leq 2(|aj| + |cg|).$$

It then follows from $|[U(\mu), \mu, \nu, U(\nu)]| + |[U(\mu), \nu, \mu, U(\nu)]| = |aj| + |cg|$ and the hypothesis of Theorem 1.5 that

$$\begin{aligned} \cosh^2(\rho/2)|1 - p|^2 &= |e|^2 M^2/4 \\ &\leq (|aj| + |cg|)M^2/2 \\ &\leq \left(1 - M + \sqrt{1 - 2M - M^2}\right)/2 \\ &< 1. \end{aligned}$$

Finally by using Theorem 1.3 the group $\langle V, U \rangle$ is not discrete.

5. *Discreteness Regions.*

Theorem 1.6. *Suppose that either*

(1) V is a loxodromic element of $Sp(2,1)$ with fixed points μ and ν and that U is any element of $Sp(2,1)$ or

(2) V is a boundary elliptic element of $Sp(2,1)$ which rotates through an order m unit quaternion p about a fixed quaternionic line L_V and U is any element of $Sp(2,1)$. Also let μ, ν be distinct points in $L_V \cap \partial\mathbf{H}^2_{\mathbb{H}}$. Let $r \in (0,1)$. If both

$$|[\mu, \nu, U(\mu), U(\nu)]|^{1/2} < r^2$$

and

$$2|\lambda p - 1| + r^2 < 1,$$

then the group $\langle V, U \rangle$ is elementary or not discrete.

Remark. Observe that if V is a boundary elliptic element fixing the quaternionic line L_V and L_V intersects $U(L_V)$ orthogonally, then $|[\mu, \nu, U(\mu), U(\nu)]| = 1$. This case is automatically excluded from Theorem 1.6.

Proof. The proof of this theorem is in effect identical to the proof of Theorem 6.1 in [21] and will not be included in this chapter.

CHAPTER TWO.

A SHIMIZU'S LEMMA FOR HEISENBERG SCREW MOTIONS IN QUATERNIONIC HYPERBOLIC 2-SPACE.

Shimizu's Lemma [34] gives a necessary condition for a subgroup of $PSL(2,\mathbb{R})$ containing a parabolic element fixing ∞ to be discrete. It was generalised for discrete groups of higher dimensional real hyperbolic isometries containing a parabolic element by Leutbecher [26], Wielenberg [39], Ohtake [29] and Waterman [38]. The hyperbolic plane is not only real hyperbolic 2-space, but also complex hyperbolic 1-space, therefore it is natural to generalise Shimizu's Lemma to discrete groups of isometries of higher dimensional complex hyperbolic space containing a parabolic element. This comes within the scope of generalisations of Jørgensen's inequality for complex hyperbolic space (see [3] and [21]).

In [30] Parker showed that for groups containing a non-vertical Heisenberg translation (Proposition 7.3) or a screw-parabolic map with infinite order rotational part (Proposition 6.4) there is no precisely invariant horoball. Other results of Parker's in [30] are extensions of earlier work by Kamiya. In [31] Parker showed that for non-vertical Heisenberg translations there is a version of Shimizu's Lemma where the radius of an isometric sphere is bounded in terms of the translation length of the Heisenberg translation at its centre. The analogous result for real hyperbolic space is given in [38]. In the appendix of [18] a different result for Heisenberg translations is given.

In [22] Jiang and Parker give a version of Shimizu's Lemma for groups of complex hyperbolic isometries one of whose generators is a Heisenberg screw motion and interpreted this result in terms of the relation between radii of isometric spheres and their distance from the axis of the Heisenberg screw motion. Recently a Shimizu's Lemma for discrete groups of isometries of quaternionic hyperbolic space containing a Heisenberg translation has been developed by Kim and Parker (see Proposition 4.4 and Proposition 4.5 of [25]).

We will now quote Shimizu's Lemma in the hyperbolic plane (Lemma 4 of [34]) followed by Parker's extensions to vertical and non-vertical Heisenberg translations in complex hyperbolic space (see page 300 of [30] and page 492 of [31] respectively, where the Hermitian form corresponding to the $(n+1)\times(n+1)$ diagonal matrix with diagonal entries $(1,...,1,-1)$ is used). We observe that we quote Parker's results in terms of the second Hermitian form (which is our choice of Hermitian form in Chapter Six where we deal with the embedding of collars in complex hyperbolic manifolds).

Proposition (Lemma 4 [34]). *Let G be a discrete subgroup of $PSL(2,\mathbb{R})$ containing the map $g: z\mapsto z+t$ for $t>0$. Let $h: z\mapsto \frac{az+b}{cz+d}$ be any element of G not fixing infinity, that is $ad-bc=1$ and $c\neq 0$. Then*

i) $|tc|\geq 1$.

ii) the radius of the isometric sphere of h is at most t.

iii) $tr[g,h]\geq 3$.

iv) every horocycle of height greater than t does not intersect its image under h.

Proposition (page 300 [30]). *Let G be a discrete subgroup of $PU(n,1)$ that contains g; vertical translation by $t\in\mathbb{R}_+$ written in the form*

$$\begin{bmatrix} 1 & 0 & it \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let h be any element of G not fixing infinity q_∞ written in the form

$$\begin{bmatrix} a & \gamma^* & b \\ \alpha & A & \beta \\ c & \delta^* & d \end{bmatrix}$$

with $A \in U(n-1)$, α, β, γ, δ column vectors in $\mathbb{C}^{n-1}$ and a, b, c, $d \in \mathbb{R}$ such that $c \neq 0$. Then

i) $|tc| \geq 1$.

ii) the radius of the Ford isometric sphere of h is at most $\sqrt{t}$.

iii) $tr[g,h] \geq n+2$.

iv) every horocycle of height greater than t does not intersect its image under h.

Theorem (page 492 [31]). *Let G be a discrete subgroup of $PU(n,1)$ that contains the Heisenberg translation $g = T_{(\tau,t)}$. Let h be any element of G not fixing infinity q_∞ and with isometric sphere of radius r_h. Then*

$$r_h{}^2 \leq t_g\big(h^{-1}(q_\infty)\big)t_g\big(h(q_\infty)\big) + 4\|\tau\|^2.$$

The purpose of this chapter is to give a generalisation of Shimizu's Lemma for groups of isometries of $\mathbf{H}^2_{\mathbb{H}}$ containing a screw-parabolic element. Screw motions are the most interesting of all parabolic maps, for by allowing the holonomy to tend to the identity, a screw motion tends to a vertical translation and similarly letting the translation length tend to zero, a screw motion tends to a boundary elliptic map. We will demonstrate that our theorem interpolates between similar results for these types of isometry.

Our principal result depends on normalising our screw-parabolic map as a specific Heisenberg screw motion. In the final section of this paper we will restate the result to give a bound on the radii of isometric spheres in terms of the distance of their centres from the axis of the screw-parabolic map. Our result will demonstrate that it is necessary for the isometric spheres of very large radius to have centres far away from the axis of the screw-parabolic map (which has a large translation length at its centre).

To be specific these results are as follows.

Theorem 2.1. *Suppose that U, $V \in PSp(2,1)$ have the form*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \tag{1}$$

and

$$V = \begin{bmatrix} 1 & 0 & t \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $|t| \geq 0$, $p \in Sp(1)$ with $|p-1| < 1$ and $g \neq 0$. For any real $k > \frac{2|t||p-1|}{1-|p-1|^2}$ let $M(k)$ denote $max\{|e|^2 - 1, |d|^2k, |h|^2k, |g|^2k^2\}$. If

$$(M(k)+1)\left(|p-1| + \frac{|t|}{k}\right)^2 < 1 + \frac{|t|^2}{k^2}, \tag{27}$$

then the group $\langle U, V\rangle$ is not discrete.

Corollary 2.1.1 (see also Proposition 4.4 [25]). *Suppose U and V in $PSp(2,1)$ are as in Theorem 2.1 with $p = 1$, i.e. $V = T_{(0,t)}$ (see section 1). If*

$$0 < |g||t| < 1,$$

then the group $\langle U, V\rangle$ is not discrete.

Corollary 2.1.2 (see also Theorem 1.3 (3) of Chapter One). *Suppose U and V in $PSp(2,1)$ are as in Theorem 2.1 with $t = 0$ and $|e| > 1$, i.e. $V = R_{p,0}$ (see section 1). If*

$$0 < |e||p-1| < 1,$$

then the group $\langle U, V\rangle$ is not discrete.

Theorem 2.2. *Suppose that $\langle U, V\rangle$ is a non-elementary discrete subgroup of $PSp(2,1)$ where U and V have the form*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \tag{1}$$

and

$$V = \begin{bmatrix} 1 & 0 & t \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $|t| \geq 0$, $p \in Sp(1)$ with $|p-1| < 1$ and $g \neq 0$. Let $L_V = \{(0, v) \in \mathfrak{N}_2\}$ be the axis of V and let

$$R = max\{\rho_0(L_V, U(\infty)), \rho_0(L_V, U^{-1}(\infty))\}.$$

Then

$$r_U{}^2 \leq \frac{2R^2|p-1|}{(1-|p-1|)} + \frac{|t|}{(1-|p-1|^{1/2})^2}$$

where r_U denotes the radius of I_U.

This work was inspired by Jiang's and Parker's 2002 preprint 'Uniform discreteness and Heisenberg screw motions' (see [22]) which tackles similar questions in complex hyperbolic 2-space and proves similar results. Therefore our method of proof follows that introduced by Shimizu (see [34]) and used in Chapter One and in most of the references mentioned above. The principal idea is that if two members of a Lie group are close to the identity then their commutator is even closer to the identity. Therefore given a screw-parabolic map V we consider a sequence $(U_n) \subset \langle U, V\rangle$ such that $U_0 = U$ and $U_{n+1} = U_n V U_n{}^{-1}$. We then determine iterative relations between the elements of U_n and U_{n+1}. This iterative system has a fixed point corresponding to the solution $U_n = V$ for all n. We aim to formulate criteria for all U_n to lie in a basin of attraction of this fixed point. Finally we demonstrate that under these conditions $U_n \to V$ as $n \to \infty$.

1. *Action of $Sp(2,1)$ on Quaternionic Hyperbolic 2-Space.*

The boundary of quaternionic hyperbolic 2-space $\partial\mathbf{H}^2_{\mathbb{H}}$ corresponds to the boundary of the unit ball in $\mathbb{H}^2$ and to the sets $\{(w, w_n) | w_n + \overline{w_n} + \langle\langle w, w\rangle\rangle = 0\}$ in the Siegel domain model. Refering to the standard Hermitian form mentioned in section 1 of Chapter One, let V_0 denote the set $\{\mathbf{z} \in \mathbb{H}^{2,1} - [0,0,0] : \langle \mathbf{z}, \mathbf{z}\rangle = 0\}$. There exists an identification between $\mathfrak{N}_2 \cup \{\infty\}$ and $\mathbb{P}(V_0)$ given by

$$(\zeta, v) \mapsto \begin{bmatrix} -|\zeta|^2 + v \\ \sqrt{2}\zeta \\ 1 \end{bmatrix},$$

$$\infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{P}(V_0)$$

where $(\zeta, v) \in \mathfrak{N}_2$ and $\zeta = w/\sqrt{2}$.

We now recall the following definition from section 1 of Chapter One.

Definition. *The Cygan metric ρ_0 corresponds to the norm $|.|_0$ on $\partial\mathbf{H}^n_{\mathbb{H}}$ which is such that if a point has horospherical coordinates (ζ, v) then*

$$|(\zeta, v)|_0 = ||\zeta|^2 + v|^{1/2} = ||\zeta|^4 + |v|^2|^{1/4}.$$

The distance ρ_0 (according to the Cygan metric) between two points with horospherical coordinates (ζ, v) and $(\omega, s) \in \partial\mathbf{H}^n_{\mathbb{H}}$ is given by

$$\rho_0((\zeta, v), (\omega, s)) = |(\zeta, v)^{-1}(\omega, s)|_0 = |(\zeta - \omega, -v + s - 2Im\langle\langle\zeta, \omega\rangle\rangle)|_0 = |\langle(\zeta, v), (\omega, s)\rangle|^{1/2}.$$

One can extend the Cygan metric to $\mathbf{H}^n_{\mathbb{H}}$ via the following extension of $|.|_0$

$$|(\zeta, v, u)|_0 = \left||\zeta|^2 + u - v\right|^{1/2}.$$

Thus we have

$$\rho_0\big((\zeta, v, u), (\omega, s, t)\big) = \left|(\zeta - \omega, -v + s - 2Im\langle\langle\zeta, \omega\rangle\rangle, |u - t|)\right|_0.$$

The Cygan metric can also be expressed explicitly in terms of the Hermitian product on $P\mathbb{H}^{n,1}$ as follows. Let $\mathbf{x}$, $\mathbf{y} \in P\mathbb{H}^{n,1}$ correspond to points in $\partial\mathbf{H}^n_{\mathbb{H}}$, then

$$\rho_0(\mathbf{x}, \mathbf{y}) = |\langle\mathbf{x}, \mathbf{y}\rangle|^{1/2}.$$

The Heisenberg group $\mathfrak{N}_2$ acts on itself by (left) *Heisenberg translation*: translation $T_{(\tau,t)} \in \mathfrak{N}_2$ is given by

$$T_{(\tau,t)} : (w, v) \mapsto (\tau, t)(w, v) = (\tau + w, v + t + 2Im\langle\langle\tau, w\rangle\rangle).$$

Heisenberg translation by $(0, t)$ for any given $t \in Im(\mathbb{H})$ is called *vertical translation* by t.

We recall from Chapter One that the group of unit quaternions $Sp(1)$ acts on the Heisenberg group by *Heisenberg rotation*: rotation $R_{p,0}$ with holonomy $p \in Sp(1)$ and *axis* the chain $(0, v) \subset \mathfrak{N}_2$ is given by

$$R_{p,0} : (w, v) \mapsto (pw, v).$$

For any $\tau \in \mathbb{H}$ the Heisenberg rotation $R_{p,\tau}$ by $p \in Sp(1)$ with axis $(\tau, v) \subset \mathfrak{N}_2$ is given by conjugating $R_{p,0}$ by $T_{(\tau,0)}$.

The product of a Heisenberg translation and a Heisenberg rotation is a *Heisenberg screw motion*. The simplest example is the product of a vertical translation $T_{(0,t)}$ with $R_{p,0}$. This is $S_{p,0,t}$ given by

$$S_{p,0,t} : (w, v) \mapsto (pw, v + t).$$

It has axis the chain $(0, v) \subset \mathfrak{N}_2$, rotates about the axis with *holonomy* $p \in Sp(1)$ and translates along the axis by a Cygan distance $\sqrt{|t|} \in \mathbb{R}$. Other screw motions can be obtained by conjugating this one by a Heisenberg translation or by composing other Heisenberg translations with other Heisenberg rotations. Heisenberg translations, rotations and screw motions are all isometries of the Cygan metric, in fact the group of Heisenberg isometries is generated by $T_{(\tau,t)}$, $R_{p,0}$ where (τ, t) and p vary over $\mathfrak{N}_2$ and $Sp(1)$ respectively.

The action of Heisenberg isometries can be extended to quaternionic hyperbolic space. The Heisenberg translation $T_{(\tau,t)}$, Heisenberg rotation $R_{p,0}$ and Heisenberg screw motion $S_{p,0,t}$ correspond to the following matrices in $PSp(2,1)$.

$$T_{(\tau,t)} = \begin{bmatrix} 1 & -\sqrt{2}\overline{\tau} & -|\tau|^2 + t \\ 0 & 1 & \sqrt{2}\tau \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_{p,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$S_{p,0,t} = \begin{bmatrix} 1 & 0 & t \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We recall from section 1 of Chapter One that elements of $Sp(2,1)$ are classified as parabolic, loxodromic or elliptic as in the familiar case of Möbius transformations. An element is called parabolic if and only if it has a unique fixed point which is in $\partial\mathbf{H}^2_{\mathbb{H}}$.

i) A parabolic element is called *pure parabolic* if it is conjugate to a Heisenberg translation.

ii) A parabolic element is called *screw parabolic* if it is conjugate to a Heisenberg screw motion.

2. *Main Theorem.*

Theorem 2.1. *Suppose that U, $V \in PSp(2,1)$ have the form*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \tag{1}$$

and

$$V = \begin{bmatrix} 1 & 0 & t \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $|t| \geq 0$, $p \in Sp(1)$ with $|p-1| < 1$ and $g \neq 0$. For any real $k > \frac{2|t||p-1|}{1-|p-1|^2}$ let $M(k)$ denote $max\{|e|^2 - 1, |d|^2 k, |h|^2 k, |g|^2 k^2\}$. If

$$(M(k)+1)\left(|p-1| + \frac{|t|}{k}\right)^2 < 1 + \frac{|t|^2}{k^2}, \tag{27}$$

then the group $\langle U, V\rangle$ is not discrete.

Remark. If we set $V = S_{p,w,t}$ with axis the chain $(w,v) \subset \mathfrak{N}_2$ then we need to define $M(k)$ to be

$$max\left\{\left|e - d\overline{w}\sqrt{2} - wh\sqrt{2} + 2wg\overline{w}\right|^2 - 1, |d + wg\sqrt{2}|^2 k, |h - g\overline{w}\sqrt{2}|^2 k, |g|^2 k^2\right\}.$$

We observe that the inequality (27) compels U to be sufficiently close to I.

We will now give two straightforward corollaries from our theorem. If $p = 1$ then V is a vertical translation. In this case the following corollary shows that Theorem 2.1 reduces to Shimizu's Lemma for vertical Heisenberg translations. Here the discreteness criterion only involves the term $|g| = 1/r_U{}^2$ where r_U is the radius of the isometric sphere of U. The geometrical consequence of this result is that the radii of isometric spheres are bounded by the Cygan translation length of V. (A very large horosphere requires a very long translation.)

Corollary 2.1.1 (see also Proposition 4.4 [25]). *Suppose U and V in $PSp(2,1)$ are as in Theorem 2.1 with $p = 1$, i.e. $V = T_{(0,t)}$. If*

$$0 < |g||t| < 1,$$

then the group $\langle U, V\rangle$ is not discrete.

Proof. Choose k sufficiently large so that $M(k) = |g|^2 k^2$. Since $|g||t| < 1$ we have

$$(M(k)+1)\frac{|t|^2}{k^2} < 1 + \frac{|t|^2}{k^2}.$$

This is the inequality (27) with $|p-1|=0$. So Theorem 2.1 implies that the group $\langle U,V\rangle$ is not discrete.

If $t=0$ then V is a boundary elliptic element. In this case under the condition $|e|>1$ which is a criterion for the axes of V and UVU^{-1} to be disjoint (see page 20 in Chapter One), the theorem reduces to the generalisation of Jørgensen's inequality for boundary elliptic elements of $Sp(2,1)$. Here the condition for discreteness only involves $|e|=\cosh\big(\rho(L_V,U(L_V))/2\big)$ (see page 20 in Chapter One) where ρ is the Bergman metric on quaternionic hyperbolic space and L_V is the axis of V. The geometrical consequence of this result is that if V rotates through a small angle then the axis of V must be displaced very far by U.

Corollary 2.1.2 (see also Theorem 1.3 (3) of Chapter One). *Suppose U and V in $PSp(2,1)$ are as in Theorem 2.1 with $t=0$ and $|e|>1$, i.e. $V=R_{p,0}$. If*

$$0<|e||p-1|<1,$$

then the group $\langle U,V\rangle$ is not discrete.

Proof. Since $t=0$ and $|e|>1$, we can choose a sufficiently small k so that $M(k)=|e|^2-1$. Now from $|e||p-1|<1$ we have

$$(M(k)+1)|p-1|^2<1.$$

This is the inequality (27) with $t=0$. So Theorem 2.1 implies that the group $\langle U,V\rangle$ is not discrete.

It is interesting to compare our Theorem 2.1 for screw parabolic elements with Kim and Parker's Proposition 4.4 and Proposition 4.5 of [25] for pure parabolic elements conjugate to vertical and non-vertical Heisenberg translations respectively.

Proposition 4.4 [25] (see Corollary 2.1.1 of this chapter). *Let G be a discrete subgroup of $PSp(n,1)$ which contains a vertical translation g by $t\in Im(\mathbb{H})$. Let h be any element of G not fixing ∞ and written in a standard matrix form (see section 1 of [25]). Then $|t||c|\geq 1$.*

Proposition 4.5 [25]. *Let G be a discrete subgroup of $PSp(n,1)$ whose stabiliser G_∞ of ∞ is a Heisenberg lattice. Let $T_{(\tau,t)}$ be a non-vertical translation (so $\tau\neq 0$) and let s be the radius of the Dirichlet domain centred at the origin of the vertical lattice $ker\Pi\cap G_\infty$ where $\Pi:(\zeta,v)\mapsto\zeta$. Let h be any element of G not fixing ∞ and written in a standard matrix form (see section 1 of [25]). Then*

$$\big||\tau|^2+is\big||c|\geq 1.$$

Remark. According to section 1 of [25] any element of $PSp(n,1)$ not fixing ∞ can be represented by a matrix of the form

$$\begin{bmatrix} a & \gamma^* & b\\ \alpha & A & \beta\\ c & \delta^* & d\end{bmatrix}$$

where a, b, c, $d\in\mathbb{H}$, α, β, γ and δ are column vectors in $\mathbb{H}^{n-1}$ and A is a $(n-1)\times(n-1)$ matrix over $\mathbb{H}$. (Here $*$ denotes the conjugate transpose.)

2.1. *Proof of the Main Theorem.*

In this section we will prove the main theorem using methods similar to those established by Shimizu [34]. First consider the sequence (U_n) defined by setting $U_0=U$ and $U_{n+1}=U_nVU_n{}^{-1}$. We write

$$U_n=\begin{bmatrix} a_n & b_n & c_n\\ d_n & e_n & f_n\\ g_n & h_n & j_n\end{bmatrix}.$$

Then U_{n+1} is given by

$$U_{n+1} = \begin{bmatrix} a_{n+1} & b_{n+1} & c_{n+1} \\ d_{n+1} & e_{n+1} & f_{n+1} \\ g_{n+1} & h_{n+1} & j_{n+1} \end{bmatrix} = \begin{bmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & j_n \end{bmatrix} \begin{bmatrix} 1 & 0 & t \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{j_n} & \overline{f_n} & \overline{c_n} \\ \overline{h_n} & \overline{e_n} & \overline{b_n} \\ \overline{g_n} & \overline{d_n} & \overline{a_n} \end{bmatrix}.$$

Performing the matrix multiplication and using the identities given by $U_n {U_n}^{-1} = I$ we have

$$a_{n+1} = 1 + b_n(p-1)\overline{h_n} + a_n t\overline{g_n},$$

$$\begin{aligned} b_{n+1} &= b_n(p-1)\overline{e_n} + a_n t\overline{d_n}, \\ e_{n+1} &= 1 + e_n(p-1)\overline{e_n} + d_n t\overline{d_n}, \\ d_{n+1} &= e_n(p-1)\overline{h_n} + d_n t\overline{g_n}, \\ h_{n+1} &= h_n(p-1)\overline{e_n} + g_n t\overline{d_n}, \\ g_{n+1} &= h_n(p-1)\overline{h_n} + g_n t\overline{g_n}. \end{aligned}$$

We observe that

$$\begin{aligned} |e_{n+1}|^2 - 1 &= |e_n|^2(|e_n|^2-1)|p-1|^2 + 2Re(d_n t\overline{d_n} e_n(\overline{p}-1)\overline{e_n}) + |d_n|^4|t|^2 \\ &\leq \big((|e_n|^2-1)+1\big)(|e_n|^2-1)|p-1|^2 + 2\big((|e_n|^2-1)+1\big)|d_n|^2|p-1||t| + |d_n|^4|t|^2. \end{aligned}$$

The remainder of the proof is the same as Jiang and Parker's proof of the corresponding result in complex hyperbolic 2-space (see [22]). They construct a dynamical system relating the entries of U_{n+1} to those of U_n. Inspired by the idea in the appendix of [18], for any positive real number k they define the quantity

$$M(k) = max\{|e|^2 - 1, |d|^2 k, |h|^2 k, |g|^2 k^2\}$$

and use it to prove a criterion for distinct U_n which implies that $\langle U, V\rangle$ is not discrete.

3. *A Bound on the Radii of Isometric Spheres.*

In Theorem 4.8 of [25], Kim and Parker proved the following bound on the radii of isometric spheres of U and U^{-1}.

Theorem 4.8 [25]. *Let G be a discrete subgroup of $PSp(n,1)$ which contains a Heisenberg translation $T = T_{(\tau,t)}$. Let U be any element of G not fixing ∞ with isometric sphere of radius r_U. Then*

$$r_U{}^2 \leq \rho_0(U^{-1}(\infty), TU^{-1}(\infty))\rho_0(U(\infty), TU(\infty)) + 4|\tau|^2.$$

In this section we will use Theorem 2.1 to give a bound on the radii of the isometric spheres of U and U^{-1} in terms of the Cygan distance between their centres and the axis of V. First we will give some definitions.

3.1. *Isometric Spheres.*

The *isometric sphere* I_h with respect to a distinguished point at infinity $q_\infty \in \partial\mathbf{H}^n_{\mathbb{H}}$ of a map h in $PSp(n,1)$ which does not projectively fix q_∞ (i.e. the *Ford isometric sphere* of h) is the spinal hypersurface given by

$$I_h = \{z \in \mathbf{H}^n_{\mathbb{H}} : |\langle Z, Q_\infty\rangle| = |\langle Z, h^{-1}(Q_\infty)\rangle|\}$$

where Z and Q_∞ are vectors in $\mathbb{H}^{n,1}$ that project to z and q_∞ respectively. The automorphism h maps I_h to $I_{h^{-1}}$ and maps the component of $\overline{\mathbf{H}^n_{\mathbb{H}}} - I_h$ containing q_∞ to the component of $\overline{\mathbf{H}^n_{\mathbb{H}}} - I_{h^{-1}}$ not containing q_∞.

First we will quote Kim and Parker's Proposition 4.3 (see [25]).

Proposition 4.3 [25]. *Let U be an element of $PSp(2,1)$ which does not projectively fix q_∞ (i.e. $g \neq 0$). The Ford isometric sphere of U is a round sphere in the Cygan metric with centre $U^{-1}(q_\infty)$ and radius $r_U = \sqrt{1/|g|}$ where*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

We will now interpret the Ford isometric sphere in terms of ρ_0, the Cygan metric on $\partial\mathbf{H}^n_{\mathbb{H}}$.

Theorem 2.2. *Suppose that $\langle U, V\rangle$ is a non-elementary discrete subgroup of $PSp(2,1)$, where U and V have the form*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \tag{1}$$

and

$$V = \begin{bmatrix} 1 & 0 & t \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $|t| \geq 0$, $p \in Sp(1)$ with $|p-1| < 1$ and $g \neq 0$. Let $L_V = \{(0,v) \in \mathfrak{N}_2\}$ be the axis of V and let

$$R = max\{\rho_0(L_V, U(\infty)), \rho_0(L_V, U^{-1}(\infty))\}.$$

Then

$${r_U}^2 \leq \frac{2R^2|p-1|}{(1-|p-1|)} + \frac{|t|}{(1-|p-1|^{1/2})^2}$$

where r_U denotes the radius of I_U.

Proof. The proof of this theorem is identical to that given for the complex hyperbolic case by Jiang and Parker in [22] provided we substitute $|t|$ for their $t \geq 0$.

We observe that all the quantities in this theorem are defined in terms of the Cygan metric on the complement of a fixed point of V. Therefore the theorem can be restated in an invariant form.

CHAPTER THREE.

OCTONIONIC HYPERBOLIC GEOMETRY.

The purpose of this chapter is to build on the inchoate foundations of octonionic hyperbolic geometry (see [27]). We will find that the essential formalism and many results from complex hyperbolic geometry hold in the octonionic case. The exceptions arise where we employ commutativity, associativity and the isomorphism between $\mathbb{R}$ and the imaginary complex numbers.

In this chapter we will use the Jordan algebra (J) model of octonionic hyperbolic 2-space $\mathbf{H}^2_{\mathbb{O}}$ to develop the Siegel domain model with horospherical coordinates. In this model $\mathbf{H}^2_{\mathbb{O}}$ becomes a direct product of $\mathbb{R}_+$ with horospheres. Each horosphere can be identified with the generalised Heisenberg group where the centre is seven dimensional, corresponding to the purely imaginary octonions. We describe a Cygan-like metric (on the generalised Heisenberg group) and the automorphism group $\mathrm{Aut}(J)$ of $\mathbf{H}^2_{\mathbb{O}}$. We then proceed to prove a generalised octonionic Shimizu's Lemma for discrete subgroups of $\mathrm{Aut}(J)$ containing a parabolic map. We use this result to determine criteria for horoballs and sub-horospherical regions to be precisely invariant under the action of discrete subgroups of $\mathrm{Aut}(J)$ containing parabolic maps. Furthermore we also explore the action of discrete subgroups of $\mathrm{Aut}(J)$ containing ellipto-parabolic maps on the octonionic hyperbolic plane. Our ultimate goal is to use the octonionic Shimizu's Lemma to determine estimates of lower bounds on the volumes of non-compact octonionic hyperbolic 2-manifolds. First we determine a disjointness criterion for horoballs based at distinct parabolic fixed points on $\partial\mathbf{H}^2_{\mathbb{O}}$. Then we use this condition together with a conjecture regarding a certain property of 7-dimensional Euclidean geometry to derive estimates of lower bounds on the volumes of disjoint embedded cusp neighbourhoods. Finally we use these lower bounds to determine estimates of lower bounds on the volumes of non-compact octonionic hyperbolic 2-manifolds.

To be specific our generalised Shimizu's Lemma is as follows.

Theorem. *Let G be a discrete subgroup of* $\mathrm{Aut}(J)$ *that contains the parabolic map $g = T_{\tau,t'}$ where $\tau \in \mathbb{O}$ and $t' \in Im(\mathbb{O})$. Suppose G also contains a lattice of vertical translations. Let h be any element of G not fixing a distinguished point at infinity B and with isometric sphere of radius r_h. Then*

$$r_h \leq \min\{k, (|\tau|^4/4 + r_{\mathrm{D}}{}^2)^{1/4}\}$$

where $k = \sqrt{|t|}$ is the translation length of the shortest vertical translation $T_{0,t}$ in G and r_{D} is the radius of the Dirichlet domain of the vertical lattice in the stabiliser of B, G_B centred at the origin.

Furthermore our conjectured lower bound on the volume of a non-compact octonionic hyperbolic 2-manifold is given by the following estimate.

Volume Estimate. Let M be a 2-dimensional non-compact octonionic hyperbolic manifold with k' cusps and let G be a discrete, cofinite volume group of isometries of $\mathbf{H}^2_{\mathbb{O}}$. Let G_i be a parabolic subgroup of G fixing a parabolic fixed point $q_i \in \partial\mathbf{H}^2_{\mathbb{O}}$ of G. Let H_i be a canonical horoball left invariant by G_i. Then

$$Vol(M) \geq k' \min\{Vol(H_i)/G_i\} \geq k' \frac{2}{99m}$$

where m is the maximal index of a lattice L_i in G_i.

This chapter was inspired by the work of Allcock (see [1]), Kamiya (see [23]) and Parker (see [30], [31] and [32]). Many of our results are generalisations of Parker's results for complex hyperbolic space, however due to the inflexibility of the non-associative division algebra of the octonions as compared to the complex numbers, there are often significant differences in the means of proof.

The material in the first two sections is taken from [2], [5], [7] and the papers [10] and [11] of Dray and Manogue as listed in the Bibliography. Furthermore throughout this paper we will use Dray and Manogue's convention regarding multiplication of the octonions as opposed to the convention of Allcock (see [1]) which appears less frequently in existing literature on the octonions.

1. *Preliminaries.*

1.1. *The Octonions.*

The algebra of octonions $\mathbb{O}$ is a non-commutative, non-associative, normed division algebra over the reals. Therefore the set of non-zero octonions is closed upon taking the multiplicative inverse. The automorphism group of the octonions is G_2, the smallest of the exceptional Lie groups. In terms of a natural basis an octonion x can be written

$$x = \sum_{i=0}^{7} x_i e_i$$

where the coefficients x_i are real and the basis vectors e_i for $i = 0, 1, ...7$ are such that

$$e_0 = 1$$

and

$$e_i e_i = -1$$

for $i = 1, ...7$.

We call $\{e_i | i = 1, ...7\}$ the *imaginary basis units*; they anti-commute, i.e. for $i \neq j : i, j = 1, ..., 7$ we have

$$e_i e_j = -e_j e_i$$

and products of two different imaginary units yield a third, i.e. $e_i e_j = \pm e_k$ for some $k \in \{1, ..., 7\} \subset \mathbb{Z}$. In fact the following index cycling identity holds. If $e_i e_j = e_k$ then the two equations generated by cyclic permutation of the indices also hold. The above identities together with the products

$$e_1 = e_2 e_3 = e_4 e_5 = e_7 e_6,$$

$$e_2 = e_5 e_7$$

and

$$e_3 = e_4 e_7 = e_6 e_5$$

are sufficient to define the octonionic multiplication table. We observe that the full multiplication table for the octonions is encoded in the 7-point projective plane (i.e. the projective plane over the 2-element field $\mathbb{Z}_2$ otherwise known as the *Fano plane*) together with the rules given by $e_0 = 1$ and $e_i e_i = -1$.

The *commutator* of two octonions is

$$[x, y] = xy - yx.$$

The *associator* of three octonions is

$$[x, y, z] = (xy)z - x(yz).$$

Both the commutator and the associator are totally antisymmetric in their arguments, have no real part and change sign if any one of their arguments is replaced by the conjugate (defined below). Although the associator does not vanish in general, the octonions do satisfy the following weak form of associativity known as *alternativity*

$$[x, y, x] = (xy)x - x(yx) = 0.$$

The underlying justification for this result is that two octonions determine a quaternionic subalgebra of $\mathbb{O}$ so that any product containing only two linearly independent octonions is associative. For the same reason we have

$$[x,x,y] = (xx)y - x(xy) = (x^2)y - x(xy) = 0$$

and

$$x^n x^m = x^m x^n,$$

for all integers n and m.

Octonionic conjugation is given by reversing the sign of the imaginary basis units so if we denote $Re(x) = x_0$ and $Im(x) = \Sigma_{i=1}^{7} x_i e_i$ then we have

$$\overline{x} = -x + 2Re(x).$$

Conjugation is an antiautomorphism since it satisfies

$$\overline{xy} = (\overline{y})(\overline{x}).$$

The symmetric *inner product* $\langle .,. \rangle$ on $\mathbb{O}$ is

$$\langle x,y \rangle = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(\overline{y}x + \overline{x}y).$$

The square of the norm $|x|$ of x is equal to $\langle x,x \rangle = x\overline{x} = \sum_{i=0}^{7} x_i x_i$ which satisfies the defining property of a normed division algebra, i.e.

$$|xy| = |x||y| = |y||x|.$$

We also observe that x is a unit if and only if $|x| = 1$.

The octonions can also be represented as the direct sum of two copies of the quaternions. Letting i, j, k denote the three standard imaginary quaternionic units, then

$$\mathbb{O} = \mathbb{H} + \mathbb{H}l = (\mathbb{C} + k\mathbb{C}) + (\mathbb{C} + k\mathbb{C})l$$

where l is another (distinct) square root of -1. The octonions are thus spanned by the identity element 1 and the 7 imaginary units i, j, k, kl, jl, il and l. These units can be divided into (the imaginary parts of) quaternionic subspaces in 7 different ways. Any three of these imaginary units which do not lie in such a triple anti-associate.

We will find the following identities (which hold for all x, y, $z \in \mathbb{O}$ and all imaginary units $\mu \in \mathbb{O}$) useful in describing the octonionic hyperbolic plane and its geometry. (They can also be found on page 471 of [1].)

i) $z(xy)z = (zx)(yz)$,

ii) $Re\big((xy)z\big) = Re\big(x(yz)\big) = Re\big((yz)x\big)$,

iii) $(\mu x\overline{\mu})(\mu y) = \mu(xy)$,

iv) $(x\mu)(\overline{\mu}y\mu) = (xy)\mu$

and

v) $xy + yx = (x\overline{\mu})(\mu y) + (y\overline{\mu})(\mu x)$.

We write $Re(xyz)$ for any of the 3 expressions in ii).

These identities are readily proven using the automorphism group of the octonions (namely the compact form of the exceptional Lie group G_2) and the fact that the stabiliser of a subalgebra isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ acts transitively on the units orthogonal to the algebra.

1.2. $\mathbf{G_2}$.

G_2 the smallest of the exceptional Lie groups is the automorphism group of the octonions. Any octonion, say i whose square is -1 generates a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{C}$. If we then pick any octonion, say j with square equal to -1 that anticommutes with i, the elements i, j generate a subalgebra isomorphic to $\mathbb{H}$. Finally if we choose any octonion, say l with square equal to -1 that anti-commutes with i, j and ij, the elements i, j and l generate all of $\mathbb{O}$. We call such a triple a *basic triple*.

It follows that given any two basic triples there exists a unique automorphism of $\mathbb{O}$ mapping the first to the second. Conversely it is clear that any automorphism maps basic triples to basic triples. This fact can be used to describe G_2.

Fix a basic triple e_1, e_2 and e_3. There is a unique automorphism of the octonions mapping this to any other basic triple e_1', e_2' and e_3'. A basic triple is any triple of unit imaginary octonions such that each is orthogonal to the algebra generated by the other two. This means that our automorphism can map e_1 to any point e_1' on the 6-sphere of imaginary unit octonions, then map e_2 to e_2' on the 5-sphere of imaginary unit octonions that are orthogonal to e_1' and then map e_3 to any point e_3' on the 3-sphere of imaginary unit octonions that are orthogonal to e_1', e_2' and $e_1'e_2'$. It follows that

$$dim G_2 = dim S^6 + dim S^5 + dim S^3 = 14.$$

Using the vector representation of $Spin(8)$ on $\mathbb{O}$ we get homomorphisms

$$G_2 \mapsto Spin(8) \mapsto SO(\mathbb{O})$$

where $SO(\mathbb{O})$ is the rotation group of the octonions viewed as a real vector space with inner product $\langle x, y\rangle = Re(\overline{x}y)$ (which is the same as the inner product on $\mathbb{O}$ mentioned in the previous section) and is isomorphic to $SO(8)$. The map from $Spin(8)$ to $SO(\mathbb{O})$ is two-to-one, but when we restrict it to G_2 we get a one-to-one inclusion

$$G_2 \mapsto SO(\mathbb{O}).$$

G_2 has a 7-dimensional representation $Im(\mathbb{O})$. This is the smallest faithful representation of G_2. Both $Im(\mathbb{H})$ and $Im(\mathbb{O})$ are closed under the commutator. In the case of $Im(\mathbb{H})$ the commutator divided by two is the familiar cross product in three dimensions

$$a \times b = \frac{1}{2}[a, b].$$

We can make the same definition for $Im(\mathbb{O})$ obtaining a 7-dimensional analogue of the cross product. For both $Im(\mathbb{H})$ and $Im(\mathbb{O})$ the cross product is bilinear and anti-commutative. The cross product makes $Im(\mathbb{H})$ into a Lie algebra, but not $Im(\mathbb{O})$. For both $Im(\mathbb{H})$ and $Im(\mathbb{O})$ the norm of the cross product is given by

$$||a \times b||^2 + \langle a, b\rangle^2 = ||a||^2||b||^2$$

or equivalently

$$||a \times b|| = |\sin\theta|\,||a||\,||b||$$

where θ is the angle between a and b. We observe that the norm of this cross product on $Im(\mathbb{O})$ is the same as the norm on $\mathbb{O}$ defined in the previous section on the octonions. Direct computation shows that $a \times b$ is orthogonal to a and b. We further observe that the group of real-linear transformations of $Im(\mathbb{O})$ preserving the cross product is exactly G_2.

Finally we note that the following formula defines a derivation on the octonions

$$D_{x,y}a = [[x, y], a] - 3[x, y, a]$$

where x, y and z are octonions and $[a, x, y]$ denotes the associator $(ax)y - a(xy)$. As $\mathbb{O}$ is a normed division algebra every derivation is a linear combination of derivations of this form.

1.3. *The Octonionic Hyperbolic Plane.*

The octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$ is a rank one symmetric space of non-compact type. We can also define the *octonionic hyperbolic line* $\mathbf{H}^1_{\mathbb{O}}$ which is isometric with the real hyperbolic space $\mathbf{H}^8_{\mathbb{R}}$. In this setting a hyperplane is just a point and the octonionic reflection therein is just central inversion.

To model $\mathbf{H}^2_{\mathbb{O}}$ we can't treat triples of octonions as elements of an octonionic vector space due to the non-associativity, therefore many of the approaches to real, complex and quaternionic hyperbolic geometry are invalid in the case of the octonionic hyperbolic plane. Therefore we have to work harder and use Jordan algebras J of matrices X instead. These are associative which means that PJ is well defined. In formulating and then applying matrix representations of the automorphism group $\mathrm{Aut}(J)$ of $\mathbf{H}^2_{\mathbb{O}}$ we use the fact that every 2-generator subalgebra of $\mathbb{O}$ is associative together with the octonionic identities listed at the end of section 1.1 which hold for all x, y, $z \in \mathbb{O}$ and all imaginary units μ.

1.4. *An Octonionic Jordan Algebra.*

This section uses material from [1], [10], [11] and [33].

Let $M_3(\mathbb{O})$ denote the real vector space of 3×3 matrices X with octonionic entries and X^* denote the conjugate transpose of X. Furthermore let Φ denote a real symmetric matrix (therefore $\Phi = \Phi^*$) in $M_3(\mathbb{O})$ which gives an Hermitian form on a space of octonion triples $\mathbb{O}^3$ which we will use to model the elements of J. The Jordan algebra associated to Φ is then defined as

$$J = \{X \in M_3(\mathbb{O}) | X\Phi = \Phi X^*\}.$$

J is closed under *Jordan multiplication* $X * Y = (XY + YX)/2$ which is commutative but not associative. X, Y Jordan commute if and only if $X * Y = 0$. We have in particular that

$$X^2 = X * X$$

and we define (see page 3 of [11])

$$X^3 := X^2 * X = X * X^2 = \frac{1}{2}\left((X^2)X + X(X^2)\right)$$

which differs from the cube of X using ordinary matrix multiplication. Other useful operations on Jordan matrices are the *trace* denoted by $tr(X)$ and the *Freudenthal product*

$$X \circ Y = X * Y - \frac{1}{2}\big(Xtr(Y) + Ytr(X)\big) + \frac{1}{2}\big(tr(X)tr(Y) - tr(X * Y)\big)I$$

where I denotes the 3 by 3 identity matrix. This has the important special case

$$X \circ X = X^2 - \big(tr(X)\big)X + \sigma(X)I \tag{28}$$

where

$$\sigma(X) = \frac{1}{2}\big((tr(X))^2 - tr(X^2)\big) = tr(X \circ X).$$

There is also *trace reversal*

$$X' := X - tr(X)I = -2I \circ X$$

and the *determinant*

$$det(X) = \frac{1}{3}tr\big((X \circ X) * X\big)$$

which can be equivalently defined by

$$\big((X \circ X) * X\big) = (det(X))\, I. \tag{29}$$

Expanding (29) using (28) we observe that Jordan matrices X satisfy the usual characteristic equation

$$X^3 - \big(tr(X)\big)X^2 + Re(\sigma(X))X - \big(det(X)\big)I = 0.$$

We choose to work with the following inner product matrix

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which may also be regarded as an indefinite Hermitian form.

Remark. For any invertible symmetric matrix Ψ in $M_3(\mathbb{R})$ corresponding to a nondegenerate form there exists $g \in GL_3(\mathbb{R})$ such that $g\Psi g^* \in \{\pm I, \pm\Phi\}$ so J_Ψ is isomorphic to J_I or J_Φ.

If we choose Φ as above, then $X\Phi = \Phi X^*$ means that an element X of J has the form

$$X = (a, b, c, u, v, w) = \begin{bmatrix} a & w & \overline{v} \\ v & \overline{u} & b \\ \overline{w} & c & u \end{bmatrix}$$

with a, b, $c \in \mathbb{R}$ and u, v, $w \in \mathbb{O}$. We observe that $tr(X) \in \mathbb{R}$.

Remark. The automorphism group of J is a real form of F_4 and as Φ is indefinite the group is

$$F_{4(-20)} = Aut(\mathbf{H}^2_{\mathbb{O}}).$$

We define X to be

0) a λ-potent if $X * X = \lambda X$,
1) a negpotent if $X * X = \lambda X$ where $\lambda < 0$,
2) a nilpotent if $X * X = 0$,
3) an idempotent if $X * X = \lambda X$ where $\lambda > 0$,
4) good if a) X is nonzero, b) X is a member of the image space of the map q defined below.

The map q is such that

$$q : {\mathbb{O}^3}_0 \mapsto J$$

$$q : v \mapsto \Phi v^* v$$

where ${\mathbb{O}^3}_0$ denotes the set of octonionic triples $\{v = (x, y, z)\}$ such that x, y and z lie in an asssociative subalgebra. Thus if $v = (v_1, v_2, v_3)$ we have

$$q(v_1, v_2, v_3) = \begin{bmatrix} |v_1|^2 & \overline{v_1}v_2 & \overline{v_1}v_3 \\ \overline{v_3}v_1 & \overline{v_3}v_2 & |v_3|^2 \\ \overline{v_2}v_1 & |v_2|^2 & \overline{v_2}v_3 \end{bmatrix}.$$

Furthermore if $X = q(v_1, v_2, v_3)$ then we have

$$tr(X) = |v_1|^2 + 2Re(\overline{v_2}v_3) \in \mathbb{R}$$

and

$$\begin{aligned} &\sigma(X) \\ &= |v_1|^2\overline{v_3}v_2 + |v_1|^2\overline{v_2}v_3 + 1/2(\overline{v_3}v_2)(\overline{v_2}v_3) + 1/2(\overline{v_2}v_3)(\overline{v_3}v_2) \\ &- |\overline{v_2}v_3|^2 - 1/2(\overline{v_1}v_2)(\overline{v_3}v_1) - 1/2(\overline{v_3}v_1)(\overline{v_1}v_2) - 1/2(\overline{v_1}v_3)(\overline{v_2}v_1) - 1/2(\overline{v_2}v_1)(\overline{v_1}v_3). \end{aligned}$$

Definition. *We define the indefinite norm* $|v|^2{}_\Phi$ *of* $v \in \mathbb{O}^3{}_0$ *to be the single entry of* $v\Phi v^*$ *which is automatically real. (Here* v^* *denotes the conjugate transpose of* v.*)*

Then we have
$$q(v)q(v) = \Phi v^* v \Phi v^* v = \Phi v^* |v|^2{}_\Phi v = |v|^2{}_\Phi q(v)$$
so we see that $q(v)$ is $|v|^2{}_\Phi$-potent. We observe that the norm $|v|^2{}_\Phi$ of $v \in \mathbb{O}^3{}_0$ is not the same as the norm in J of $q(v)$ which is defined as the inner product (see below) of $q(v)$ with itself. In fact the latter norm is the square of the former. Thus we also have

$$tr(q(v)) = Re\, tr(q(v)) = Re \sum_{i,j} \Phi_{i,j} \overline{v_j} v_i = Re \sum_{i,j} v_i \Phi_{i,j} \overline{v_j} = |v|^2{}_\Phi.$$

Therefore a necessary condition for $X \in J$ to be good is that there exists $\lambda \in \mathbb{R}$ such that X is λ-potent and has trace λ. This is also a sufficient condition (see Theorem 4.3. of [1]).

We define an equivalence relation $\sim$ on J by $X \sim Y$ if and only if $X = \epsilon Y$ for some $\epsilon \in \mathbb{R}_+$.

Definition. *The octonionic hyperbolic plane consists of those equivalence classes of Jordan matrices* X *which satisfy the restrictions*
$$X * X = \lambda X$$
where $\lambda < 0$ *and*
$$tr(X) = \lambda.$$

These conditions force the entries of X to lie in a quaternionic subalgebra of $\mathbb{O}$ determined by X. Elementary (associative) linear algebra then shows that each element of $\mathbf{H}^2_{\mathbb{O}}$ is a primitive negpotent (a negpotent that is not the sum of other negpotents) and can be written as

$$X = q(v)$$

as defined above. Similarly the good nilpotents correspond to points in $\partial\mathbf{H}^2_{\mathbb{O}}$ and the good idempotents correspond to points in the octonionic projective space PJ outside $\overline{\mathbf{H}^2_{\mathbb{O}}}$.

We recall that $\mathbb{O}^3{}_0$ denotes the set of octonion triples (x, y, z) such that x, y, z lie in an associative subalgebra. For the purposes of this thesis $\mathbb{O}^3{}_0$ can be further restricted (to a set $\mathbb{O}^3{}_{00}$) by requiring that y be real. It is proved in [1] that in the Siegel upper half space model $\mathfrak{S}_2$ of the hyperbolic octonionic plane $\mathbf{H}^2_{\mathbb{O}}$ is described as the image in PJ of the good negpotents, i.e. $\mathbf{H}^2_{\mathbb{O}} = \{(x, 1, z) \in \mathbb{O}^3_{00} | Re(z) < -x\overline{x}/2\}$. The space at ∞ $\partial\mathbf{H}^2_{\mathbb{O}}$ is the image of the good nilpotents, namely (0,0,1) (whose image in $\overline{\mathbf{H}^2_{\mathbb{O}}}$ we denote by $B = q(0, 0, 1)$) and $\{(x, 1, z) | Re(z) = -x\overline{x}/2\}$ which form the 15 dimensional Heisenberg space $\mathcal{H}^{15}$. (We will henceforth refer to $\{(x, 1, z) \in \mathbb{O}^3{}_{00} | Re(z) \leq -|x|^2/2\} \cup \{(0, 0, 1)\}$ as the coordinates of the closure of the Siegel domain of points X in $\mathbf{H}^2_{\mathbb{O}}$.) This realises the boundary of the octonionic hyperbolic plane topologically as the 15-sphere S^{15} and the octonionic hyperbolic plane as the open 16-ball bounded by S^{15}. A line in $\mathbf{H}^2_{\mathbb{O}}$ is the set of good negpotents that Jordan commute with some fixed good idempotent X. The line associated to a given good idempotent X is said to be *polar* to X.

Alternately we may say that in the upper half space model of $\mathbf{H}^2_{\mathbb{O}}$ the points of PJ (except those on the line at infinity) are described by pairs $(x, -z - |x|^2/2)$.

Definition. *If* $X = q(x, 1, z)$, *then the horospherical coordinates* (ζ, v, u) *on* $\overline{\mathbf{H}^2_{\mathbb{O}}} - \{B\}$ *are defined as follows.*

$$\zeta = x, \tag{30}$$

$$v = Im(z), \tag{31}$$

$$u = -Re(z) - |x|^2/2. \tag{32}$$

We observe that in horospherical coordinates C the image of $(0,1,0)$ under the map q is denoted by $(0,0,0)$.

Definition. *In $\mathbf{H}^2_{\mathbb{O}}$ the subspaces defined by*

$$H_t = \{(\zeta, v, u) | u > t\}$$

are defined as horoballs of height t and those defined by

$$\partial H_t = \{(\zeta, v, u) | u = t\}$$

are called horospheres of height t. We observe that $\partial \mathbf{H}^2_{\mathbb{O}} = \partial H_0 \cup \{\infty\}$.

Definition. *$\langle X|Y \rangle$ the symmetric inner product on J is defined as*

$$\langle X|Y \rangle = Re\, tr(X * Y) = tr(X * Y).$$

The rationale behind this definition is as follows. If $X = \Phi v^* v$, $Y = \Phi w^* w$ so that $XY = (\Phi v^* v)(\Phi w^* w)$, then if we also had commutativity and associativity $XY = (v\Phi w^*)(\Phi v^* w)$. This would give the following equality

$$tr\big(1/2(XY + YX)\big) = 1/2(v\Phi w^*)(w\Phi v^*) + 1/2(w\Phi v^*)(v\Phi w^*)$$

which is a natural generalisation of the square of the 'standard' Hermitian form $v\Phi w^*$. We observe that this sets up an isomorphism between J and its dual J^*. We also observe that if $X = q(v)$, $Y = q(w)$ then $tr(X * Y) \in \mathbb{R}$.

1.5. *Norm and Metric on the Boundary of the Octonionic Hyperbolic Plane.*

We will now define a Heisenberg norm $|.|_{\mathbb{O}}$ and metric $\rho_{\mathbb{O}}$ on the boundary of the octonionic hyperbolic plane. Treating the boundary of the octonionic hyperbolic plane as the one point compactification of a 15 real dimensional Heisenberg group $\mathfrak{N}_2$ we can define a group product $\circ$ (not to be confused with the Freudenthal matrix product defined in section 1.4)

$$(x, 1, z) \circ (y, 1, w) = (x + y, 1, z + w - x\overline{y}).$$

Using $z + \overline{z} + |x|^2 = 0$, the inverse of $(x, 1, z)$ is $(-x, 1, \overline{z})$.

The norm of $(x, 1, z)$ is $|(x, 1, z)|_{\mathbb{O}} = \sqrt{|z|}$.

The metric relation between points on $\partial \mathbf{H}^2_{\mathbb{O}}$ is

$$\rho_{\mathbb{O}}((x, 1, z), (y, 1, w)) = \rho_{\mathbb{O}}(s, t) = |(t \circ s^{-1})|_{\mathbb{O}}$$

where $s = (x, 1, z)$ and $t = (y, 1, w)$.

Switching to Heisenberg coordinates and performing a direct calculation using equations (30), (31) and (32) we find that

$$\rho_{\mathbb{O}}\big((\zeta, v, 0), (\zeta_0, v_0, 0)\big) = \big||\zeta - \zeta_0|^2/2 + v - v_0 - Im(\zeta_0\overline{\zeta})\big|^{1/2}.$$

It is straightforward to check that $|.|_{\mathbb{O}}$ and $\rho_{\mathbb{O}}$ satisfy the appropriate conditions for a norm and a metric.

Remark. This metric should be thought of as the counterpart of the Euclidean metric on the upper half space model of real hyperbolic space. This metric is a natural metric intrinsic to $\mathfrak{N}_2$ (and so to each horosphere) but is not intrinsic to $\mathbf{H}^2_{\mathbb{O}}$.

The metric ρ_0 on $\mathfrak{N}_2$ can be extended to a metric also denoted by ρ_0 on the octonionic hyperbolic plane as follows. Letting (ζ, v, u) and (ζ_0, v_0, u_0) denote the horospherical coordinates of two points in $\mathbf{H}^2_{\mathbb{O}}$ we have

$$\begin{aligned}\rho_0((\zeta, v, u), (\zeta_0, v_0, u_0)) &= \left| -v_0 + v - |\zeta|^2/2 - |\zeta_0|^2/2 + \zeta\bar{\zeta}_0 - |u_0 - u| \right|^{1/2} \\ &= \left| |\zeta - \zeta_0|^2/2 + |u - u_0| + v - v_0 - Im(\zeta_0\bar{\zeta}) \right|^{1/2}.\end{aligned}$$

The **Bergman metric** is the intrinsic metric on $\mathbf{H}^2_{\mathbb{O}}$, it corresponds to the hyperbolic metric on real hyperbolic space. According to the Bergman metric, the distance ρ between two points X, Y in $\mathbf{H}^2_{\mathbb{O}}$ is given by

$$\cosh^4\left(\frac{\rho(X,Y)}{2}\right) = \frac{\langle X|Y\rangle\langle Y|X\rangle}{\langle X|X\rangle\langle Y|Y\rangle}.$$

1.6. *The Automorphism Group* $\mathrm{Aut}(J)$.

The automorphism group of J, $\mathrm{Aut}(J)$ is isomorphic to the group $F_{4(-20)}$, a connected simply connected simple exceptional Lie group of 52 dimensions which acts simply transitively with compact stabiliser on $\mathbf{H}^2_{\mathbb{O}}$. This is proved in Theorem 4.5 of [1]. (In fact $\mathbf{H}^2_{\mathbb{O}}$ is isomorphic to $F_{4(-20)}/Spin(9)$ as $Spin(9)$ is the stabiliser of a point in $\mathbf{H}^2_{\mathbb{O}}$.) Furthermore Theorem 4.4 of [1] shows that $\mathrm{Aut}(J)$ acts transitively on the points and lines of $\mathbf{H}^2_{\mathbb{O}}$ and 2-transitively on the points of $\partial\mathbf{H}^2_{\mathbb{O}}$. Elements of $\mathrm{Aut}(J)$ are classified as in the familiar case of Möbius transformations (and elements of $Sp(2,1)$) as described in section 1 of Chapter One. We also observe that the 78-dimensional exceptional Lie group E_6 has a real noncompact form $E_{6(-26)}$ which acts as the group of collineations on PJ. Equivalently $E_{6(-26)}$ acts as the determinant preserving linear transformations of J. F_4 is a maximal compact subgroup of $E_{6(-26)}$.

We will now show that any element h of $AutJ$ not fixing $B = q(0,0,1)$ has a **canonical decomposition** of the following form

$$h = T_1 D\{S\} R T_2^{-1}$$

where

* T_1 is a translation such that $T_1(C) = h(B)$ where $C = q(0,1,0)$ and B correspond respectively to the origin and infinity on the boundary of the octonionic hyperbolic plane. Similarly $T_2(C) = h^{-1}(B)$. Furthermore such translations fix B.

* D is a dilation, dilation factor λ, i.e. $|D(X)|_0 = \lambda|X|_0$, fixing B and C.

* R is the inversion interchanging B and C.

* $\{S\}$ is a sequence of "rotations" S_μ where μ is an imaginary unit of the octonions, fixing B and C. The set of maps S_μ generate a subgroup of $\mathrm{Aut}(J)$ which is isomorphic to $Spin_7\mathbb{R}$.

Proof. By inspection we find that $T_1^{-1}h : B \mapsto C$, but $T_1^{-1}hT_2R$ fixes B and C (see below for an explicit definition of R) and any map which fixes infinity and the origin must be of the form $D\{S\}$. Therefore $h = T_1 D\{S\} R T_2^{-1}$.

1.6.1. *Transformation Formulae.*

Since $\mathbb{O}$ is nonassociative, computations with it are sometimes complicated. In the development and use of the following transformation formulae this problem is dealt with by using five useful identities i) to v) listed at the end of section 1.1 of this chapter (see also page 471 of [1]). The transformation formulae are as follows.

(i) Inversion.

$$R(x, 1, z) = (-z^{-1}x, 1, z^{-1}).$$

We observe that the form of this definition of R differs slightly from that given in [1] where the action of R is defined as $R(x,1,z) = (x, -z, -1)$. This is to ensure that R preserves ${\mathbb{O}_0}^3$. A further consequence of our definition of R is that it acts as inversion in the boundary of the unit ρ_0-ball centred at C.

(ii) Rotation.

$$S_{\overline{\mu}}(x,1,z) = (\overline{\mu}x, 1, \overline{\mu}z\mu),$$

where μ is a unit imaginary octonion.

(iii) Dilation.

$$D_\lambda(x,1,z) = (\lambda x, 1, \lambda^2 z),$$

where $\lambda \in \mathbb{R}_+$.

(iv) Translation.

$$T_{a,b}(x,1,z) = (x+a, 1, z - x\overline{a} - a\overline{a}/2 + b)$$

where a is an octonion and b is an imaginary octonion. (Here $a = 0$ corresponds to a *central* or *vertical* translation.)

By consideration of the Heisenberg coordinates and direct calculation we see that the translations are parabolic transformations which form a 15 dimensional Lie group which is nilpotent of class 2 and acts transitively on $\partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$. Furthermore the translations are normalised by the S_μ.

Remark. For proof that R, S_μ and $T_{a,b}$ are automorphisms of J see Lemma 4.1 of [1]. That D_λ is an automorphism of J follows by inspection.

The corresponding matrix representations are as follows. (We observe here that in performing the calculations we first perform matrix multiplication on the right of a triple $(x,1,z)$, then project on the left when necessary in order to preserve the form of the $(.,1,.)$ triples. This is the opposite convention to that used earlier with transformations on quaternionic hyperbolic 2-space.)

$S_{\overline{\mu}}$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

D_λ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

$T_{a,b}$ is given by

$$\begin{bmatrix} 1 & 0 & -\overline{a} \\ a & 1 & -|a|^2/2 + b \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 1. The $T_{t,0}$ ($t \in \mathbb{R}$) together with their conjugates under $Spin_7\mathbb{R}$ generate a 15 dimensional nilpotent Lie group.

Remark 2. The above representation of $T_{a,b}$ is justified by the following two facts.

1) the quaternionic subalgebra generated by any pair of octonions is associative.

2) the associator of any three octonions is purely imaginary.

We observe that each of these matrices M satisfies the following identity.

$$M\Phi M^* = \Phi$$

where $*$ denotes the conjugate transpose.

Proposition 3.1. *$T_{a,b}$ and S_μ preserve ρ_0, i.e. they are Heisenberg isometries.*

Proof. Elementary calculations show that

$$T_{a,b}(x,1,z) = (x,1,z) \circ (a, 1, b - a\overline{a}/2)$$

and

$$\rho_0((x,1,z),(y,1,w)) = |(y-x,1,w+\overline{z}+y\overline{x})|_0.$$

Now

$$\begin{aligned}\rho_0(T_{a,b}(x,1,z),T_{a,b}(y,1,w)) &= \rho_0((y+a,1,w-y\overline{a}-a\overline{a}/2+b)\circ(-x-a,1,\overline{z}-a\overline{x}-a\overline{a}/2-b))\\ &= |(y-x,1,w+\overline{z}+y\overline{x})|_0.\end{aligned}$$

(In this last calculation we use $\overline{b}=-b$ as b is imaginary.) Thus $T_{a,b}$ preserves ρ_0.

The proof for S_μ is very similar although the following identities must be used

$$\overline{\mu}zu = -\mu z\mu$$

and

$$(\mu y)(\overline{\mu x}) = -(\mu y)(\overline{x}\mu) = \mu(y\overline{x})\overline{\mu}.$$

Remark. Any group of Heisenberg similarities (dilations and isometries) containing both a translation $T_{a,b}$ and a dilation D_λ with $\lambda > 1$ is not discrete. This can be seen by applying the proof of Proposition 3.3 [30] to our octonionic Heisenberg similarities $T_{a,b}$ and D_λ ($\lambda > 1$).

2. *An Octonionic Shimizu's Lemma.*

In this section we will prove an octonionic analogue of Shimizu's Lemma for vertical parabolic maps on real hyperbolic spaces. We recall here that in Chapter Two we proved a Shimizu's Lemma for Heisenberg screw motions in quaternionic hyperbolic 2-space. Given a discrete subgroup of $PSL(2,\mathbb{R})$ containing a parabolic map, a classical result of Shimizu (see [34]) gives a uniform bound on the radii of isometric circles of elements of the group not fixing infinity. The uniform bound is the (Euclidean) translation length of the parabolic map. Equivalently every horocycle whose height is greater than the translation length of the parabolic map is precisely invariant under such a group. Both of these statements involve the use of an intrinsic metric (the Poincaré metric) and a non-intrinsic metric (the Euclidean metric) on the upper half plane model of the hyperbolic plane. For further details of Shimizu's Lemma and its extensions see Chapter Two.

Before we prove the main theorem, Theorem 3.1 we will prove some preliminary propositions and a lemma.

Proposition 3.2. *For all X, $Y \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$, the symmetric inner product $\langle .|. \rangle$ on J and the metric on the boundary ρ_0 satisfy the following relation*

$$\langle X|Y\rangle = \rho_0(X,Y)^4.$$

Proof. By straightforward calculations using the matrix representations $q(x,1,z)$ and $q(0,1,0)$ of X and C respectively we find that

$$\begin{aligned}\langle X|C\rangle &= Re\,tr(X*C)\\ &= z\overline{z}\\ &= {|X|_0}^4\\ &= \rho_0(X,C)^4.\end{aligned}$$

Applying a suitable translation T which is an automorphism preserving $\langle .|. \rangle$ and also an isometry of ρ_0 we get the general result

$$\langle X|Y\rangle = \rho_0(X,Y)^4.$$

This can be seen from the following explicit argument. Let X, $Y \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$ and let the translation T be such that $T(Y) = C$, then we have

$$\begin{aligned}\langle X|Y\rangle &= \langle T(X)|T(Y)\rangle \\ &= \langle T(X)|C\rangle \\ &= \rho_0\left(T(X), C\right)^4 \\ &= \rho_0\left(T(X), T(Y)\right)^4 \\ &= \rho_0\left(X, Y\right)^4.\end{aligned}$$

Definition. *Let $h \in \mathrm{Aut}(J)$ not fixing B, then the isometric sphere I_h of h is the sphere center $h^{-1}(B)$ which is mapped by h to $I_{h^{-1}}$, the isometric sphere (of equal radius with respect to the Cygan metric ρ_0) of h^{-1}.*

Proposition 3.3. *If h is a member of $\mathrm{Aut}(J)$ not fixing B, then the radius of the corresponding isometric sphere is $r = r(h) = \sqrt{\lambda}$ where λ is the dilation factor of h.*

Proof. Let h be decomposed as

$$h = T_1 D\{S\} R T_2^{-1}.$$

Then we have for all $X \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$

$$\rho_0(h^{-1}(B), X) = \rho_0(T_2 R\{S\}^{-1} D^{-1} T_1^{-1}(B), X) = \rho_0(T_2(C), X) \tag{33a}$$

as $T_1(B) = D(B) = S(B) = B$ and $R(B) = C$. Now

$$\rho_0(h(B), h(X)) = \rho_0(T_1 D\{S\} R T_2^{-1}(B), T_1 D\{S\} R T_2^{-1}(X)) = \lambda\rho_0(R(B), R T_2^{-1}(X)). \tag{33b}$$

Multiplying the above two expressions gives

$$\begin{aligned}\rho_0\left(h^{-1}(B), X\right)\rho_0\left(h(B), h(X)\right) &= \lambda\rho_0\left(C, T_2^{-1}(X)\right)\rho_0\left(C, R T_2^{-1}(X)\right) \\ &= \lambda|T_2^{-1}(X)|_0 \frac{1}{|T_2^{-1}(X)|_0} \\ &= \lambda\end{aligned} \tag{34}$$

as

$$\rho_0\left(C, R(Z)\right) = \frac{1}{|Z|_0}$$

for all $Z \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$. If $D := \rho_0(h^{-1}(B), X) = \sqrt{\lambda}$ then $E := \rho_0(h(B), h(X)) = \sqrt{\lambda}$ by equation (34), but $D = r$ and $E = r$ as h maps its isometric sphere, centre $h^{-1}(B)$ to the isometric sphere of h^{-1}, centre $h(B)$. Therefore $r = \sqrt{\lambda}$.

The following lemma was inspired by Lemma 2 on page 811 of [29].

Lemma 3.1. *If h is a member of $\mathrm{Aut}(J)$ such that $h(B) \neq B$, then for all X, $Y \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B, h^{-1}(B)\}$*

$$\rho_0(h(X), h(Y)) = \lambda\rho_0(X, Y)/(\rho_0(X, h^{-1}(B))\rho_0(Y, h^{-1}(B))).$$

Proof.

$$\begin{aligned}\rho_0(h(X), h(Y)) &= \rho_0(T_1 D(S) R T_2^{-1}(X), T_1 D(S) R T_2^{-1}(Y)) \\ &= \lambda\rho_0(R T_2^{-1}(X), R T_2^{-1}(Y)) \\ &= \lambda\rho_0(T_2^{-1}(X), T_2^{-1}(Y))/(|T_2^{-1}(X)|_0 |T_2^{-1}(Y)|_0) \\ &= \lambda\rho_0(X, Y)/(\rho_0(X, h^{-1}(B))\rho_0(Y, h^{-1}(B))).\end{aligned} \tag{35}$$

To justify the last but one equality, i.e.

$$\rho_0(R(W), R(Z)) = \rho_0(W, Z)/(|W|_0|Z|_0)$$

we first prove

$$\langle R(W)|R(Z)\rangle = \langle W|Z\rangle/(st\bar{s}\bar{t}) \tag{36}$$

by elementary matrix calculation where $W = (w, 1, s)$, $Z = (z, 1, t)$. Then we use the relationship between the inner product and the metric proved in Proposition 3.2.

We will now prove the following geometric consequence of Lemma 3.1.

Proposition 3.4. *Let h be an element of $\mathrm{Aut}(J)$ not fixing B. Then the ρ_0-sphere of radius r centred at $h^{-1}(B)$ is mapped by h to the ρ_0-sphere of radius ${r_h}^2/r$ centred at $h(B)$.*

Proof. Let h be a map in $\mathrm{Aut}(J)$ not fixing B.

Recalling equation (34)

$$\rho_0(h(X), h(B))\rho_0(X, h^{-1}(B)) = \lambda \tag{34}$$

and considering the case $X \in I_h$ we have $\lambda = {r_h}^2$. Suppose now $\rho_0(X, h^{-1}(B)) = r \neq r_h$ and $\rho_0(X, h(B)) = r'$ then by the above we have

$$rr' = {r_h}^2.$$

Therefore

$$r' = {r_h}^2/r.$$

That is h sends the ρ_0-sphere centred at $h^{-1}(B)$ of radius r to the ρ_0-sphere centred at $h(B)$ of radius ${r_h}^2/r$.

We will now prove an octonionic analogue of Shimizu's Lemma for vertical parabolic maps. This proof was influenced by page 812 of [29].

Theorem 3.1. *Suppose a subgroup G of $\mathrm{Aut}(J)$ acts discretely on the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$ and contains a vertical translation T of translation length k. Then for all elements h of G such that $h(B) \neq B$ the radius of the corresponding isometric sphere does not exceed k.*

Proof. Let h be a member of G a discrete subgroup of $\mathrm{Aut}(J)$ such that $h(B) \neq B$, $h = T_1 D\{S\} R T_2^{-1}$ and $T = T_{0,m}$. Assume $\lambda \in \mathbb{R}$ exceeds k^2 where $k = \sqrt{|m|}$ is the translation length of T. Then define $Y_0 = h$, $Y_{j+1} = Y_j T Y_j^{-1}$. Let λ_j denote the real dilation factor of Y_j. Setting $h = Y_{j+1}$, $X = Y_j(B)$ in (34) we see that

$$\begin{aligned}\lambda_{j+1} &= \rho_0(Y_{j+1}(Y_j(B)), Y_{j+1}(B))\rho_0(Y_j(B), {Y_{j+1}}^{-1}(B)) \\ &= \rho_0(Y_j T(B), Y_j T {Y_j}^{-1}(B))\rho_0(Y_j(B), Y_j T^{-1} {Y_j}^{-1}(B)).\end{aligned} \tag{37}$$

Consider equation (37). (A review of the proof of Ohtake's Lemma 2 (see page 812 of [29]) would be helpful at this point.) If we substitute $h = Y_j$, $X = T{Y_j}^{-1}(B)$ into equation (34), then substitute this result into the first factor of equation (37) (remembering that $T(B) = B$) and then repeat these two actions with $h = Y_j$, $X = T^{-1}{Y_j}^{-1}(B)$ and the second factor of equation (37), we see that

$$\lambda_{j+1} = (\lambda_j)\left(\frac{1}{\rho_0(TY_j^{-1}(B), Y_j^{-1}(B))}\right)(\lambda_j)\left(\frac{1}{\rho_0(T^{-1}Y_j^{-1}(B), Y_j^{-1}(B))}\right) = \frac{{\lambda_j}^2}{k^2}. \tag{38}$$

Therefore

$$\lambda_j/k^2 = (\lambda/k^2)^{2^j} \tag{39}$$

which implies that the λ_j form a monotonic increasing sequence of dilation factors.

The next proposition will complete the proof of our octonionic analogue of Shimizu's Lemma for vertical parabolic maps, Theorem 3.1.

Proposition 3.5. *For $Z \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$, $\rho_0(Y_{j+1}(B), Z)^2/\lambda_{j+1}$ tends to 1 as j tends to infinity.*

Proof. Using equation (34) and the triangle inequality we have

$$\begin{aligned}
\rho_0(Y_{j+1}(B), Z) - \lambda_j/k &= \rho_0(Y_{j+1}(B), Z) - \rho_0(Y_{j+1}(B), Y_j(B)) \\
&\le \rho_0(Y_j(B), Z) \\
&\le \sum_{s=1}^{j} \rho_0(Y_s(B), Y_{s-1}(B)) + \rho_0(Y_0(B), Z) \\
&= \sum_{s=1}^{j} \lambda_{s-1}/k + \rho_0(Y_0(B), Z) \\
&= k\sum_{s=1}^{j} \lambda_{s-1}/k^2 + \rho_0(Y_0(B), Z) \\
&= k\sum_{s=1}^{j} (\lambda/k^2)^{2^{s-1}} + \rho_0(Y_0(B), Z),
\end{aligned}$$

which is asymptotically equal to $k\ (\lambda/k^2)^{2^{j-1}}$ by equation (39). Therefore

$$(\rho_0(Y_{j+1}(B), Z)/k)(k^2/\lambda)^{2^j} - 1 \le (\lambda/k^2)^{2^{j-1}-2^j}.$$

The right hand side of the above inequality tends to 0 as j tends to ∞. Therefore $\frac{\rho_0(Y_{j+1}(B),Z)}{k(\lambda_j/k^2)}$ tends to 1 as j tends to infinity. Taking the square of both sides and using the relation between λ_{j+1} and λ_j it is now obvious that we have proved the proposition.

Following Shimizu [34] we will now investigate the orbits of the good negpotent $W = (0, 1, -1)$ under distinct (as the λ_j form a monotonic increasing sequence of dilation factors) Y_j. We observe that in order to check that W is a good negpotent we calculate $W * W$ and compare it to W.

Let $T_{j,1}$ denote T_{a_j,b_j} (given $Y_j(B) = q(a_j, 1, b_j - a_j\bar{a}_j/2)$ and $T_{j,2}(C) = Y_j^{-1}(B)$) and let f_j, g_j, n_j denote $\mathbb{R}$-linear modulus preserving maps (acting on the set $\mathbb{O}$). The maps f_j, g_j, n_j correspond to the action of $\{S\}_j{}^{-1}$ (see page 479 of [1]) and therefore preserve the norm of elements of $\mathbb{O}$. This is because the $\{S_\mu\}$ generate a subgroup of $\mathrm{Aut}(J)$ isomorphic to $Spin_7\mathbb{R}$. (See Theorem 4.4 of [1].)

By direct calculation consistent with the unprojected transformation formulae given in [1] we have

$$\begin{aligned}
Y_{j+1}(W) &= Y_j T Y_j^{-1}(W) \\
&= T_{j,1} D_j \{S\}_j R T_{j,2}^{-1} T T_{j,2} R \{S\}_j^{-1} D_j^{-1} T_{j,1}^{-1}(0, 1, -1) \\
&= T_{j,1} D_j \{S\}_j R T R \{S\}_j^{-1} D_j^{-1}\left(-a_j, 1, -1 - a_j\overline{a_j}/2 - b_j\right) \\
&= T_{j,1} D_j \{S\}_j R T R\left(-f_j(a_j), \lambda_j g_j(1), n_j\left(-(1/\lambda_j)(1 + a_j\bar{a}_j/2 + b_j)\right)\right) \\
&= \left(-g_j{}^{-1}(m/\lambda_j)a_j, 1 - g_j{}^{-1}(m/\lambda_j), -1 + a_j\bar{a}_j g_j{}^{-1}(m/\lambda_j)/2 - g_j{}^{-1}(m/\lambda_j)b_j\right).
\end{aligned}$$

As j tends to infinity, λ_j tends to infinity, therefore using the previous proposition it can be seen that

1) $g_j{}^{-1}(m/\lambda_j)a_j$ tends to 0 as $g_j{}^{-1}$ preserves modulus and $a_j\overline{a_j}$ is of order λ_j.

2) $1 - g_j{}^{-1}(m/\lambda_j)$ tends to 1.

3) $-1 + a_j\bar{a}_j g_j{}^{-1}(m/\lambda_j)/2 - g_j{}^{-1}(m/\lambda_j)b_j$ tends to a point on a 6-sphere of radius k^2 as by direct calculation

$$|(a_j\bar{a}_j/2 - b_j)m|/\lambda_j = (k\rho_0(Y_j(B), C))^2/\lambda_j$$

which tends to k^2 by the previous proposition.

Therefore the orbits of the good negpotent W tend to points on a 6-sphere of finite radius. The sphere is compact thus there exists a convergent subsequence which lies entirely in the octonionic hyperbolic plane. Therefore the orbits have an accumulation point which contradicts the discreteness of G.

Therefore $r(h)$ does not exceed k, the translation length of T. This proves Theorem 3.1.

3. *An Octonionic Shimizu's Lemma for non-vertical parabolic maps.*

We will now extend the octonionic Shimizu's Lemma to non-vertical parabolic maps.

3.1. *A skew-symmetric form on $J - \{B\}$.*

We can define a skew-symmetric form $\omega : (J - \{B\})^2 \mapsto \mathbb{R}^7$ on $(J - \{B\})^2$ by requiring

$$[T_{a,b}, T_{c,d}] = T_{0,2\omega_{(X,Y)}}$$

where X, Y are points in $\partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$ possessing Heisenberg coordinates (a, b) and (c, d) respectively. Direct calculation shows that this implies

$$\omega(X, Y) = -Im(a\bar{c}) = -Im\langle\langle a, c\rangle\rangle$$

where $\langle\langle ., .\rangle\rangle$ denotes the usual positive-definite Hermitian form on $\mathbb{O}^1$.

The proof of the following lemma is modeled on that of the lemma on page 490 of [31].

Lemma 3.2. *Let g be a non-vertical Heisenberg translation $T_{\tau,t}$ by (τ, t) where $\tau \in \mathbb{O}$ and $t \in Im(\mathbb{O})$. Let X and Y be two points in $\partial\mathbf{H}^2_{\mathbb{O}}$ distinct from B with horospherical coordinates $(\zeta, v, 0)$ and $(\zeta', v', 0)$ respectively, then the translation lengths of g at X and Y are related by*

$$t_g^2(X) \leq t_g^2(Y) + 2|\tau||\zeta - \zeta'|.$$

Proof. In our original coordinates $X = (x, 1, z)$, $Y = (x', 1, z')$ and by our original definition of the metric ρ_0 on $\partial\mathbf{H}^2_{\mathbb{O}}$ we find

$$\begin{aligned} t_g^2(X) &= {\rho_0}^2(X, g(X)) \\ &= {\rho_0}^2\big((x, 1, z), (x + \tau, 1, z - x\bar{\tau} - |\tau|^2/2 + t)\big) \\ &= |(\tau, 1, -x\bar{\tau} - |\tau|^2/2 + t + \tau\bar{x})|_0{}^2 \\ &= \big|-|\tau|^2/2 - t + x\bar{\tau} - \tau\bar{x}\big|. \end{aligned}$$

Using equation (30) we find

$$\begin{aligned} t_g^2(X) &= \big|-|\tau|^2/2 - t + \zeta\bar{\tau} - \tau\bar{\zeta}\big| \\ &= \big||\tau|^2/2 + t + 2Im(\tau\bar{\zeta})\big|. \end{aligned} \tag{40}$$

Similarly

$$t_g^2(Y) = \big||\tau|^2/2 + t + 2Im(\tau\overline{\zeta'})\big|.$$

Therefore by comparison of the previous two equations, then applying the triangle inequality we see

$$\begin{aligned} t_g^2(X) &= \big|t_g^2(Y) - 2Im(\tau\overline{\zeta'}) + 2Im(\tau\bar{\zeta})\big| \\ &\leq t_g^2(Y) + 2|\tau||\zeta - \zeta'|. \end{aligned}$$

Remark. We observe that we can simplify this argument slightly by restricting to the case $t = 0$ (by conjugating within the group of Heisenberg translations). Then by direct calculation we find

$$\begin{aligned} t_g(X) &= \rho_0(T_{\tau,0}(X), X) \\ &= \Big||\tau|^2/2 + 2Im(\tau\bar{\zeta})\Big|^{1/2} \end{aligned} \tag{41}$$

and

$$t_g(Y) = \left||\tau|^2/2 + 2Im(\tau\overline{\zeta'})\right|^{1/2}.$$

Therefore

$$t_g^2(X) = \left|t_g^2(Y) + 2Im(\tau\overline{\zeta} - \tau\overline{\zeta'})\right|$$

which using the triangle inequality does not exceed $t_g^2(Y) + 2|\tau||\zeta - \zeta'|$.

3.2. *Improving the Upper Bound on the radius of an Isometric Sphere.*

Definition. *A lattice is a discrete subgroup such that the quotient space has finite invariant volume.*

Let G be a discrete subgroup of $\mathrm{Aut}(J)$ containing a lattice of Heisenberg translations L. Let $T_{0,t}$ and $T_{\tau,t'}$ denote vertical and non-vertical translations in this lattice. By Theorem 3.1 if $h \in G$ is such that $h(B) \neq B$ then r_h, the radius of the isometric sphere of h does not exceed $\sqrt{|t|}$. Consider now non-vertical translations $T_{\tau,t'}$. The translation length of $T_{\tau,t'}$ is given by (40), substituting t' for t, i.e.

$$t_g^2(X) = \left||\tau|^2/2 + t' + 2Im(\tau\overline{\zeta})\right|.$$

We aim to minimise the component of the translation length of $T_{\tau,t'}$ dependent on its vertical parameter t'. For each $\zeta \in \mathbb{O}$ we consider the translation lengths at (ζ, v) of elements of G_B. We want to find a number M so that there exists $T_{\tau,t''}$ (where t'' is dependent on (ζ, v)) with translation length at (ζ, v) at most M. Typically t'' will be such that $T_{\tau,t''}$ is the translation of shortest translation length at (ζ, v). To be mathematically explicit we aim to find the following limit:

$$\max_{\{(\zeta,v)\in\mathcal{H}^{15}\}}\left\{\min_{\{T_{\tau,t'}\in L_B\}} t_{T_{\tau,t'}(\zeta,v)}\right\}$$

where L_B is a lattice in $\mathfrak{N}_2 \subset G_B \subset G$. The lattice L^{V} of vertical translations in G, a discrete subgroup of $\mathrm{Aut}(J)$ has maximal rank (i.e. rank 7), therefore we can tessellate $\mathbb{R}^7$, the vertical part of the Heisenberg space $\mathcal{H}^{15}$ with fundamental domains for the action of these vertical translations. We consider a closed ball centred at the origin of $\mathbb{R}^7$ containing such a fundamental domain. To make the ball as small as possible we need to find a fundamental domain for L^{V} centred at the origin of $\mathbb{R}^7$ and of minimal diameter. This is the Dirichlet domain for the action of L^{V} centred at the origin in $\mathbb{R}^7$. (For details of the lattice theory used in this chapter see the Appendix A and [6].)

Definition. *The Dirichlet domain* D *centred at the origin of a lattice is the set of points closer to the origin than to any other lattice point.*

No matter where our point corresponding to $v \in Im(\mathbb{O})$ is originally located in the lattice L^{V} we can always translate it into the Dirichlet domain centred at the origin. To say this in another way we can always pull back the vertical translation length to be less than the radius of any compact fundamental domain of the lattice and therefore of the Dirichlet domain centred at the origin of the lattice. Let r_D denote the radius of the Dirichlet domain. For instance the lattice $\mathbb{Z}^7$ in Euclidean 7-space with minimal lattice translation of 1 has a Dirichlet domain centred at the origin of radius $\sqrt{7}/2$. We observe that $\mathbb{R}^7 = Im(\mathbb{O})$. Therefore for each point (ζ, v) there exists a 7 member sequence (t_i) of elements of $Im(\mathbb{O})$ and a 7 member sequence of integers (k_i) such that the product $T = T_{\tau,t'+\sum_{i=1}^{7} k_i t_i} = T_{\tau,t'}T_{0,t_1}{}^{k_1}...T_{0,t_7}{}^{k_7}$ of a non-vertical translation $T_{\tau,t'}$ and a finite sequence of vertical translations $T_{0,k_i t_i}$ (where the order is unimportant as the vertical translations commute) has a translation length denoted by t_T (given by equation (40) by substituting $t' + \sum_{i=1}^{7} k_i t_i$ for t) which is less than or equal to $\left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/4}$ and is exactly attained when at an extreme point of the Dirichlet domain.

We observe that in the case of the quaternions we have

$$r_h \leq \left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/4}.$$

We will prove that the same inequality is satisfied in the case of the octonions.

3.2.1. *Proof of an Octonionic Shimizu's Lemma for non-vertical parabolic maps.*

Theorem 3.2. *Let G be a discrete subgroup of* $\mathrm{Aut}(J)$ *that contains the Heisenberg translation $g = T_{\tau,t'}$ together with a lattice of vertical translations which includes a vertical translation of translation length t. Let h be any element of G not fixing B and with isometric sphere of radius r_h. Then*

$$r_h \leq \left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/4}$$

where r_{D} is the radius of the Dirichlet domain of the vertical lattice in G_B centred at the origin.

Remark. This has been done in the vertical lattice E_7 where

$$r_{\mathrm{D}} = \frac{\sqrt{3}}{2}t^2$$

(see pages 124 and 125 of [6]).

Proof.

We will argue as follows. Suppose that for each $(\zeta, v) \in \mathcal{H}^{15}$ there exists $T \in L_B$ (as defined above) such that $t_T(\zeta, v) \leq t = t_0$. If $\left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/4} \leq t_0$ and $r_h > \left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/4}$, then using equation (38) we see that for a sequence of maps $h_{j+1} = h_j T h_j^{-1}$ not fixing B

$$\lambda_{j+1} = \frac{\lambda_j^2}{t_T(\zeta, v)^2} > \frac{\lambda_j^2}{t_0^2}.$$

Thus λ_{j+1} tends to ∞ as j tends to ∞ if $\frac{\lambda_0}{t_0^2} > 1$. This contradicts the fact that r_{j+1} and therefore λ_{j+1} are bounded above (by Theorem 3.1). Hence $\lambda_0 \leq t_0^2$.

Using equation (34) and the same arguments as those used to prove equation (39) we find

$$\begin{aligned}
\lambda_{j+1} &= \rho_0(Y_jT(B), Y_jTY_j{}^{-1}(B))\rho_0(Y_j(B), Y_jT^{-1}Y_j{}^{-1}(B)) \\
&= \rho_0(Y_j(B), Y_jTY_j{}^{-1}(B))\rho_0(Y_j(B), Y_jT^{-1}Y_j{}^{-1}(B)) \\
&= \left(\frac{\lambda_j}{\rho_0(TY_j{}^{-1}(B), Y_j{}^{-1}(B))}\right)\left(\frac{\lambda_j}{\rho_0(T^{-1}Y_j{}^{-1}(B), Y_j{}^{-1}(B))}\right) \\
&= \left(\frac{\lambda_j}{\rho_0(T^{-1}Y_j{}^{-1}(B), Y_j{}^{-1}(B))}\right)^2.
\end{aligned} \tag{42}$$

Recalling that $T = T_{\tau,t'+\sum_{i=1}^{7} k_i t_i} = T_{\tau,t'}\Pi_{i=1}{}^{7}T_{0,t_i}{}^{k_i}$ and that we can always find a sequence (k_i) so that

$$\rho_0\left(T\left((\zeta, v)\right), (\zeta, v)\right) < \left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/4}$$

where r_{D} is the radius of the Dirichlet domain centred at the origin, then according to equation (42) we have

$$\frac{\lambda_{j+1}}{\left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/2}} \geq \frac{\lambda_j{}^2}{\left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)}.$$

Therefore if

$$\frac{\lambda_0}{\left(|\tau|^4/4 + r_{\mathrm{D}}{}^2\right)^{1/2}} > 1,$$

then

$$\frac{\lambda_{j+1}}{\left(|\tau|^4/4 + r_D{}^2\right)^{1/2}} > \left(\frac{\lambda_0}{\left(|\tau|^4/4 + r_D{}^2\right)^{1/2}}\right)^{2^j}$$

which tends to infinity as j tends to infinity. This is a contradiction, therefore

$$\lambda_0 \leq \left(|\tau|^4/4 + r_D{}^2\right)^{1/2}.$$

4. *A precisely invariant sub-horospherical region.*

We will now explore the behaviour of sub-horospherical regions under the action of discrete subgroups of $\mathrm{Aut}(J)$ containing parabolic maps. The material in this section is a generalisation to octonionic hyperbolic geometry of the corresponding results for complex hyperbolic geometry given on pages 498 to 500 of [31].

Definitions. *Let G be a subgroup of $\mathrm{Aut}(J)$, then U, a subset of $\mathbf{H}^2_{\mathbb{O}}$ is said to be precisely invariant under K, a subgroup of G if for all $g \in K$, $gU = U$ and for all g which are elements of G but not of K, U and gU are disjoint.*

Given a point of $\partial\mathbf{H}^2_{\mathbb{O}}$ which without loss of generality we take to be B, a subset U of $\mathbf{H}^2_{\mathbb{O}}$ is called a sub-horospherical region if

i) there exists $u_0 > 0$ so that U is contained in the horoball of height u_0,

ii) for every $(\zeta, v) \in \mathfrak{N}_2$ there is a $u_1 > 0$ so that $(\zeta, v, u) \in U$ for all $u > u_1$

but iii) U does not contain any horoball.

The point B is fixed by all parabolic elements of G, a discrete subgroup of $\mathrm{Aut}(J)$. (Denote the subgroup of G stabilising B by G_B.) Observe that if U, a horoball or sub-horospherical region based at B is precisely invariant under a subgroup of G then that subgroup must be G_B. This is because the intersection of the boundaries of U and $\mathbf{H}^2_{\mathbb{O}}$ is B and so if $g(U) = U$ then $g(B) = B$ and so $g \in G_B$. Conversely if $g(B) = B$ then for each $(\zeta, v) \in \partial\mathbf{H}^2_{\mathbb{O}}$ there is a $u > 0$ so that U contains both (ζ, v, u) and $g(\zeta, v, u)$. Thus the intersection of U and $g(U)$ is non-empty and so $g(U) = U$. Thus we may speak without ambiguity of a horoball or a sub-horospherical region being precisely invariant under elements of G.

We know (see section 7.3 of [30]) that in general there is no horoball centred at B that is precisely invariant under G_B in G. We will now show that if G_B is a group of Heisenberg translations, then provided a certain condition is met there is either a precisely invariant horoball or a precisely invariant sub-horospherical region. We know that if G is discrete then G_B cannot contain loxodromic elements (see the proof of Corollary 4.1 of [31] as the proof of the octonionic hyperbolic case follows that of the complex hyperbolic case), but G_B may contain screw parabolic or elliptic maps provided it also contains a Heisenberg translation. We will not consider those cases.

Let g be a pure parabolic element of $\mathrm{Aut}(J)$ fixing B. Say g is a Heisenberg translation $T_{\tau,t}$ by $(\tau, t) \in \mathfrak{N}_2$. We define an open subset U_g of $\mathfrak{S}_2$ by

$$U_g := \{(\zeta, v, u) \in \mathfrak{S}_2 : u > t_g^2\left((\zeta, v, u)\right) + 4|\tau|^2\}.$$

If g is a vertical translation by $(0, t) \in \mathfrak{N}_2$ with $t \neq 0$, then U_g is just the horoball of height $t_g^2 = |t|$ centred at B. If g is not a vertical translation then U_g is clearly a sub-horospherical region. We will prove the following result.

Theorem 3.3. *Let G be a discrete subgroup of $\mathrm{Aut}(J)$. Denote the stabilizer of B in G by G_B and suppose that G_B is a group of Heisenberg translations. Then*

1) if G contains a vertical Heisenberg translation g, then the horoball U_g is precisely invariant under G.

2) if G_B contains no vertical Heisenberg translations and G contains an element h that does not fix B, then for any $g = T_{\tau,t} \in G_B - \{I\}$ (where $\tau \neq 0$) the sub-horospherical region U_g is precisely invariant under G provided that

$$r_h{}^2 \leq t_g\left(h^{-1}(B)\right)t_g\left(h(B)\right) + 2|\tau|^2.$$

Remark. We prove 1) in Proposition 3.6 and Corollary 3.8.2.1. We will now prove 2).

Lemma 3.4. *Every point (ζ, v, u) of the ρ_0-sphere of radius r centred at $(\zeta_0, v_0, 0)$ where*

$$r^2 = t_g^2\big((\zeta_0, v_0, 0)\big) + 2|\tau|^2$$

is contained in the region

$$u \leq t_g^2\big((\zeta_0, v_0, 0)\big) + 4|\tau|^2.$$

Proof. As $X = (\zeta, v, u)$ is on the sphere centred at $X_0 = (\zeta_0, v_0, 0)$ of radius r we know that

$$\begin{aligned} r^2 &= \big||\zeta - \zeta_0|^2/2 + u + v - v_0 - Im(\zeta_0\bar{\zeta})\big| \\ &\geq |\zeta - \zeta_0|^2/2 + u. \end{aligned}$$

Thus

$$\begin{aligned} u &\leq r^2 - |\zeta - \zeta_0|^2/2 \\ &= t_g^2(X_0) + 2|\tau|^2 - |\zeta - \zeta_0|^2/2 \\ &\leq t_g^2(X) + 2|\tau||\zeta - \zeta_0| + 2|\tau|^2 - |\zeta - \zeta_0|^2/2 \\ &= t_g^2(X) + 4|\tau|^2 - 2||\tau| - |\zeta - \zeta_0|/2|^2 \\ &\leq t_g^2(X) + 4|\tau|^2 \end{aligned}$$

where Lemma 3.2 is used to compare $t_g(X_0)$ and $t_g(X)$. So $X = (\zeta, v, u)$ is contained in the required region.

We now use Lemma 3.4 to show that U_g is disjoint from all its images under elements of $G - G_\infty$.

Lemma 3.5. *Let G be a discrete subgroup of* $\mathrm{Aut}(J)$ *containing the Heisenberg translation $g = T_{\tau,t}$ (where $\tau \neq 0$). Suppose that G contains an element h that does not fix B. Then U_g does not intersect its image under h provided that*

$${r_h}^2 \leq t_g\big(h^{-1}(B)\big)t_g\big(h(B)\big) + 2|\tau|^2.$$

Proof. Let X be a point in U_g with horospherical coordinates (ζ, u, v). Therefore

$$u > t_g^2(X) + 4|\tau|^2.$$

Then X lies outside the sphere centre $h^{-1}(B)$ and radius r where $r^2 = t_g^2\big(h^{-1}(B)\big) + 2|\tau|^2$ by Lemma 3.4. Therefore $h(X)$ lies inside the ρ_0-sphere with centre $h(B)$ and radius r' where $r' = {r_h}^2/r$ by Proposition 3.4. By hypothesis we know that

$${r_h}^2 \leq t_g\big(h^{-1}(B)\big)t_g\big(h(B)\big) + 2|\tau|^2.$$

Therefore

$$\begin{aligned} r'^2 &= {r_h}^4/r^2 \\ &\leq \frac{\big(t_g\big(h^{-1}(B)\big)t_g\big(h(B)\big) + 2|\tau|^2\big)^2}{t_g^2\big(h^{-1}(B)\big) + 2|\tau|^2} \\ &\leq t_g^2\big(h(B)\big) + 2|\tau|^2. \end{aligned}$$

Therefore by Lemma 3.4, $h(X)$ lies in the region where

$$u \leq t_g^2(X) + 4|\tau|^2,$$

that is the complement of U_g, i.e. if X is in U_g, then $h(X)$ is not in U_g as required.

Lemma 3.6. *Let G be as in Theorem 3.3. Suppose that G contains no vertical Heisenberg translations. Then $f(U_g) = U_g$ for any elements f and g of $G_B - \{I\}$.*

Proof. Suppose that f is a Heisenberg translation by $(\sigma, s) \in \mathfrak{N}_2$. Suppose that $f(U_g) \neq U_g$ and so without loss of generality there is a point X with horospherical coordinates $(\zeta, u, v) \in U_g$ with $f(X)$ not in U_g. As f preserves horospheres this means that

$$t_g^2\big(f(X)\big) + 4|\tau|^2 \geq u > t_g^2(X) + 4|\tau|^2.$$

By definition of $t_g(X)$ this is true if and only if

$$||\tau|^2/2 + t + 2Im(\tau\overline{(\zeta + \sigma)})| > ||\tau|^2/2 + t + 2Im(\tau\overline{\zeta})|.$$

This implies that $Im(\tau\overline{\sigma}) \neq 0$. Thus f and g do not commute and their commutator is a vertical translation by $2Im(\tau\overline{\sigma})$. This contradicts our assumption that G contains no vertical translations.

5. *Ellipto-Parabolic Maps.*

We will now explore the action on the octonionic hyperbolic plane of discrete subgroups of Aut(J) containing ellipto-parabolic maps. The material in this section is a generalisation to octonionic hyperbolic geometry of the corresponding results for complex hyperbolic geometry given on pages 300 to 302 of [30].

Proposition 3.6. *Let G be a discrete subgroup of* Aut(J) *that contains $T_{0,t}$, a vertical translation by $t \in Im(\mathbb{O})$ with uniform translation length $|t|^{1/2}$. Let h be any element of* Aut(J) *that doesn't fix B. Then*
1) r_h, the radius of the Ford isometric sphere of h is at most $|t|^{1/2}$.
2) Every horosphere of height greater than $|t|$ does not intersect its image under h.

Proof.
1) See the proof of Theorem 3.1.
2) We claim that the exterior of the Ford isometric sphere of h which contains H_{u_0}, the horosphere of height u_0 for $u_0 > |t|$ is mapped to the interior of the Ford isometric sphere of h^{-1} which has radius at most $\sqrt{|t|}$ by part 1). The proof of this claim is as follows.

Let (ζ, v, u) denote a point on I_h and $(\zeta', v', 0)$ denote the centre of I_h. Then

$$\rho_0((\zeta, v, u), (\zeta', v', 0)) = \sqrt{|v' - v - |\zeta|^2/2 - |\zeta'|^2/2 + \zeta'\overline{\zeta} - u|}.$$

Without loss of generality we can set $\zeta' = 0$ and $v' = 0$ by applying the appropriate Heisenberg translation (an isometry). Therefore

$$\begin{aligned} r_h &= \rho_0((\zeta, v, u), (0,0,0)) \\ &= \sqrt{| - v - |\zeta|^2/2 - u|}. \end{aligned}$$

We see that u attains its upper bound ${r_h}^2$ when $\zeta = 0$ and $v = 0$, but $r_h \leq \sqrt{|t|}$, therefore as $u_0 > |t|$ we have $u < u_0$ and so H_{u_0} lies outside I_h. Hence by Proposition 3.4, $h(H_{u_0})$ lies inside $I_{h^{-1}}$. Part 2) follows immediately from this claim.

Let f be a sequence of octonionic rotations $\{S_{\mu_i}\}$, $i = 1, ...n$ and $g_0 = T_{0,t}$ $(t \neq 0)$, a vertical translation. It is immediate that $g = fg_0$ has a single fixed point B. Maps of this type are called *ellipto-parabolic* or *screw-parabolic.* The cyclic group generated by g is automatically discrete. Suppose that f has order $m < \infty$. We claim that the translation length of g^m does not exceed $\sqrt{m|t|}$. The proof of this claim is as follows.

Let $(x, 1, z)$ denote $X \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$. Then

$$g_0(x, 1, z) = (x, 1, z + t)$$

and

$$f(x, 1, z) = ((\mu_n(...(\mu_1 x)...), 1, ((\mu_n(...\mu_2(\mu_1 z\overline{\mu}_1)\overline{\mu}_2)...)\overline{\mu}_n)).$$

Therefore as the order of f is m, by direct calculation we see

$$(fg_0)^m(x, 1, z) = (x, 1, z + \sum_{i=0}^{m-1} t_i)$$

where t_i denotes the third entry in the 3-tuple given by $f^i(0,1,t)$. Let $s = \sum_{i=0}^{m-1} t_i$. Therefore

$$g^m = (fg_0)^m = T_{0,s}$$

where by the triangle inequality we find that $|s| \leq m|t|$ as the μ_i are unit octonions.

Therefore Proposition 3.6 gives the following result.

Proposition 3.7. *Let G be a discrete subgroup of* Aut(J) *that contains the ellipto-parabolic map $g = fg_0$ as above, f having order $m < \infty$. Let h be any element of G not fixing B. Then*

1) the radius of the Ford isometric sphere of h is at most $\sqrt{m|t|}$.

2) every horosphere of height greater than $m|t|$ does not intersect its image under h.

Corollary 3.7.1. *Let G be as above. Every horosphere of height greater than $m|t|$ is precisely invariant under g.*

The case of infinite m has yet to be considered.

6. *Disjointness Criteria for Canonical Horoballs.*

In this section we will construct a disjointness criterion for horoballs based at distinct parabolic fixed points on the boundary of the octonionic hyperbolic plane. We recall that *horoballs* of height t are subspaces H_t of $\mathbf{H}^2_{\mathbb{O}}$ defined by

$$H_t = \{q(x,1,z) | - Re(z) - x\overline{x}/2 > t\}.$$

We observe that $\partial H_0 = \partial \mathbf{H}^2_{\mathbb{O}} - \{B\}$.

The form of the next section follows that given for the complex hyperbolic case on pages 440 to 444 of [32]. The corresponding results for the quaternionic case can be found in section 4.2 of [25].

A basic octonionic analogue of Shimizu's Lemma is given by Theorem 3.1 at the beginning of this chapter. We recall it now renaming it Proposition 3.8.1 for the purposes of stating the following corollary.

Proposition 3.8.1. *Suppose a subgroup G of* Aut(J) *acts discretely on the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$ and contains a vertical translation T of translation length k. Then for all elements h of G such that $h(B) \neq B$ the radius of the corresponding isometric circle does not exceed k.*

Proof. See Theorem 3.1.

We will now recall our improved version of Shimizu's Lemma, Theorem 3.2 and rename it Proposition 3.8.2 for the purposes of stating the following corollary.

Proposition 3.8.2. *Let G be a discrete subgroup of* Aut(J) *whose stabilizer G_B of B is a Heisenberg lattice L. Let $T_{\tau,t'}$ be a non-vertical translation (so $\tau \neq 0$) and let r_{D} be the radius of the Dirichlet domain centred at the origin of the vertical lattice $L^{\mathrm{V}} = ker\Pi \cap L$ where Π denotes vertical projection. Let h be any element of G not fixing B. Then*

$$\left||\tau|^4/4 + r_{\mathrm{D}}^2\right|^{1/4} \geq r_h.$$

Proof. See Theorem 3.2.

Corollary 3.8.2.1. *Let G be as in Proposition 3.8.2 and let $T_{\tau,t'}$ and $T_{0,t} \in G$. Then every horosphere of height greater than*

$$u_* = \min\left\{|t|, \left||\tau|^4/4 + r_{\mathrm{D}}^2\right|^{1/2}\right\}$$

is precisely invariant under G.

Proof. Suppose that h doesn't fix B. The radius of the Ford isometric sphere of h is at most $\sqrt{u_*}$ by Proposition 3.8.1 and Proposition 3.8.2. Then the distance of a point in a horosphere of height $|t|$ greater than u_* from $h^{-1}(B)$ is greater than $\sqrt{u_*}$. So any horoball based at B with height greater than u_* is outside the Ford isometric sphere of h with centre $h^{-1}(B)$. But the exterior of the Ford isometric sphere of h is mapped to the interior of the Ford isometric sphere of h^{-1} with centre $h(B)$ and radius at most $\sqrt{u_*}$. So the image under h of the horoball is disjoint from itself.

If $h(B) = B$, then h has no dilation part, i.e. $\lambda_h=1$. The reason is that if G_B contains both a Heisenberg translation $b = T_{\tau,t}$ and a dilation $a = D_\lambda$, then by direct calculation using the formulae given in section 1.6.1 $a^{-j}ba^j$ is a Heisenberg translation by $(\lambda^{-j}\tau, \lambda^{-2j}t)$ which converges to the identity as j tends to ∞ (without loss of generality we may assume $\lambda > 1$). Then G_B is not discrete. This shows that $h(H_{|t|}) = H_{|t|}$ for $|t| > u_*$.

Let G be a discrete subgroup of $\mathrm{Aut}(J)$ of cofinite volume. We want to be able to determine at each parabolic fixed point of G a horoball whose quotient by the stabiliser of the parabolic fixed point will embed in $\mathbf{H}^2_{\mathbb{O}}/G$ disjointly from other chosen quotients of horoballs at the inequivalent parabolic fixed points of G. This will enable us to demonstrate that the lower fixed bound on the volume of the manifold is linear with respect to the number of inequivalent parabolic fixed points, i.e. the number of cusps of the manifold.

Let G be a discrete subgroup of $\mathrm{Aut}(J)$ with $T_{0,t}$ and $T_{\tau,t'}$ both elements of G_B, the stabiliser in G of B. Let r_{D} be the radius of the Dirichlet domain of the vertical lattice corresponding to $T_{0,t}$. The horoball based at B of height $u_H = \min_{t,\tau}\left\{|t|, \left||\tau|^4/4 + r_{\mathrm{D}}^2\right|^{1/2}\right\}$ is the *canonical horoball* at B for G. We observe that without loss of generality we can take $T_{0,t}$ to correspond to the shortest vertical translation in G_B and $T_{\tau,t'}$ to be the non-vertical translation $T = T_{\tau,t'+\sum_{i=1}^{7}k_it_i}$ (defined in section 3.2), which by construction has translation length $t_T \leq \left||\tau|^4/4 + r_{\mathrm{D}}^2\right|^{1/4}$ for all $T_{\tau,t'}$.

Proposition 3.9. *Canonical horoballs based at distinct parabolic fixed points are disjoint.*

Proof. See next three lemmas.

Horoballs based at C are the images under R of those based at B. In general if H is a horoball based at some point (ζ, v) in the Heisenberg group, i.e. in $\partial\mathbf{H}^2_{\mathbb{O}} - B$, then the image $RT_{-\zeta,-v}(H)$ is a horoball based at B. The height of H is defined to be the height of $RT_{-\zeta,-v}(H)$. Using the conjugation by R we may also talk about the Heisenberg translation on the horosphere based at any point. We recall (see section 1.6.1) that

$$R(x, 1, z) = (-z^{-1}x, 1, z^{-1}).$$

Therefore in horospherical coordinates

$$\begin{aligned} R(\zeta, v, u) &= \left(-z^{-1}x, Im(z^{-1}), -Re(\overline{z}/|z|^2) - |x|^2/(2|z|^2)\right) \\ &= \left((u + |\zeta|^2/2 + v)\zeta/F, -v/F, u/F\right) \end{aligned}$$

where $z^{-1} = \frac{\overline{z}}{|z|^2}$ and

$$F := u^2 + |\zeta|^2u + |\zeta|^4/4 + v^2.$$

Elements of $\mathrm{Aut}(J)$ fixing C may be obtained from those fixing B by conjugation by R.

We will now characterise disjoint horoballs H with respect to their height u.

Lemma 3.7. *Let $H^B_{u_B}$ denote a horoball height u_B based at B and $H^C_{u_C}$ a horoball height u_C based at C. Then these two horoballs are disjoint if and only if $u_Bu_C \geq 1$.*

Proof. The point $(\zeta, v, u) \in H^C_{u_C}$ implies $R(\zeta, v, u) \in H^B_{u_C}$. Therefore using the above notation $u/F > u_C$. If (ζ, v, u) lies in the intersection of $H^B_{u_B}$ with $H^C_{u_C}$ then $u > u_B$, therefore

$$u_C < u/F \leq 1/u < 1/u_B.$$

Therefore $u_C u_B \geq 1$ implies $H^B_{u_B}$ and $H^C_{u_C}$ are disjoint.

Conversely if $u_C u_B < 1$ consider the point $(0, 0, u)$ where

$$u = \frac{(1 + u_B u_C)}{2u_C}.$$

Then

$$u_B < u < \frac{1}{u_C}$$

which implies $(0, 0, u) \in H^B_{u_B}$. Now $R(0, 0, u) = (0, 0, 1/u)$, but

$$\frac{1}{u} = \frac{2u_C}{(1 + u_B u_C)} > u_C.$$

Therefore $R(0, 0, u)$ is a member of the complement of $H^C_{u_C}$ and so $(0, 0, u)$ is an element of $H^C_{u_C}$. Thus if $H^C_{u_C}$ and $H^B_{u_B}$ are disjoint then $u_C u_B \geq 1$.

It remains to prove that canonical horoballs are disjoint. For general groups this is done as follows.

Lemma 3.8. *Let G be a discrete subgroup of* $\mathrm{Aut}(J)$. *Suppose that the stabilizer G_B of B contains the Heisenberg translations by (τ, t) and $(0, t')$ where τ is nonzero. Suppose also that the stabilizer G_C of C in G contains Heisenberg translations by (σ, s) and $(0, s')$ where σ is nonzero. Then*

$$\min\{|t'|, \left||\tau|^4/4 + r_{\mathrm{D}_B}{}^2\right|^{1/2}\}.\min\{|s'|, \left||\sigma|^4/4 + r_{\mathrm{D}_C}{}^2\right|^{1/2}\} \geq 1$$

where r_{D_B} and r_{D_C} are the radii of the smallest Dirichlet domains in G_B and G_C respectively.

Proof. Let G be a discrete subgroup of $\mathrm{Aut}(J)$. Suppose that G contains Heisenberg translations $T = T_{\tau,t}$ and $U = T_{\sigma,s}$ (which fix B). Suppose the parabolic element $h = RUR$ (which fixes C but not B) is also in G, we observe that it is an isometry with respect to a "C"-analogue of the ("B") boundary metric ρ_0.

For all 7 element sequences of independent Heisenberg translations (T_i) for $i = 1, ..., 7$ there exists a bound b_T such that for all $X \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$ there exists a 7 element sequence $(j_i) \subset \mathbb{Z}$ such that

$$\rho_0\big(TT_1^{j_1}...T_7^{j_7}(X), X\big) \leq b_T \leq \sqrt{u_B}.$$

Similarly for all 7 element sequences of independent Heisenberg translations (U_i) for $i = 1, ..., 7$ there exists a bound b_U such that for all $X \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$ there exists a 7 element sequence $(k_i) \subset \mathbb{Z}$ such that

$$\rho_0\big(UU_1^{k_1}...U_7^{k_7}(X), X\big) \leq b_U \leq \sqrt{u_C}.$$

We observe that without loss of generality we can restrict $X \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$ as translations preserve horospheres and translation lengths are horosphere invariant. By Theorem 3.2 we have that

$$\lambda \leq \sqrt{(|\tau|^4/4 + r^2_{\mathrm{D}_B})}. \tag{43}$$

Recalling equation (34) and using the fact that $h(C) = C$ we have

$$\lambda = \rho_0\big(C, h^{-1}(B)\big)\rho_0\big(C, h(B)\big). \tag{44}$$

Using equation (4.3) on page 476 of [1] and letting $U = T_{\sigma,s}$ we find

$$h(B) = RUR(0,0,1) = (-\sigma, -|\sigma|^2/2 - s, 1)$$

and

$$h^{-1}(B) = RU^{-1}R(0,0,1) = (\sigma, -|\sigma|^2/2 + s, 1).$$

Therefore

$$\rho_0(C, h(B)) = \frac{1}{\left|-|\sigma|^2/2 - s\right|^{1/2}}$$

and

$$\rho_0(C, h^{-1}(B)) = \frac{1}{\left|-|\sigma|^2/2 + s\right|^{1/2}}.$$

Substituting these two results into equation (44) and using equation (43) we have

$$\begin{aligned} 1 &= \lambda\left|-|\sigma|^2/2 + s\right|^{1/2}\left|-|\sigma|^2/2 - s\right|^{1/2} \\ &= \lambda\left||\sigma|^2/2 + s\right| \\ &\leq \sqrt{\left||\tau|^4/4 + r_{\mathrm{D}_B}^2\right|}\left||\sigma|^2/2 + s\right| \\ &= \sqrt{\left||\tau|^4/4 + r_{\mathrm{D}_B}^2\right|}\sqrt{\left||\sigma|^4/4 + |s|^2\right|}. \end{aligned}$$

This inequality holds for all Heisenberg translations by (σ, s) and so it holds for the Heisenberg translation with the shortest corresponding translation length. This will be at most $\sqrt{|\sigma|^4/4 + r_{\mathrm{D}_C}{}^2}$. Repeating the above arguments with $T' = TT_1^{j_1}...T_7^{j_7}$ and $U' = UU_1^{k_1}...U_7^{k_7}$ replacing T and U we have proved Lemma 3.8.

Lemma 3.9. *Let G be a discrete subgroup of* $\mathrm{Aut}(J)$. *Let G_X be a Heisenberg lattice in G fixing X and let r_{D} be the radius of the smallest Dirichlet domain of the vertical lattice of G_X. If $T_{\tau,t}$ and $T_{0,t'}$ are Heisenberg translations in G_X then the horoballs of height* $\min\{|t'|, \left||\tau|^4/4 + |r_{\mathrm{D}}|^2\right|^{1/2}\}$ *at distinct parabolic fixed points are disjoint.*

Proof. By Lemma 3.7 we see that if $H^B_{u_B}$ and $H^C_{u_C}$ are horoballs of height u_B and u_C based at B and C respectively, they are disjoint if and only if $u_B u_C \geq 1$. Also if G_B contains $T_{\tau,t}$, $T_{0,t'}$ and G_C contains $RT_{\sigma,s}R$, $RT_{0,s'}R$, then

$$\min\{|t'|, \left||\tau|^4/4 + r_{\mathrm{D}_B}{}^2\right|^{1/2}\}.\min\{|s'|, \left||\sigma|^4/4 + r_{\mathrm{D}_C}{}^2\right|^{1/2}\} \geq 1$$

where r_{D_B} and r_{D_C} are the radii of the smallest Dirichlet domains of vertical lattices in G_B and G_C respectively. This is proved in Lemma 3.8. So if H_{u_B} and H_{u_C} are horoballs of height u_B and u_C respectively, then by conjugation we can assume that they are horoballs based at B and C. Then by the above argument the claim follows.

Remark. We observe that if we make the additional assumption that a cusped finite-volume octonionic 2-manifold has only one end, then we only need to find one precisely invariant horoball based at the parabolic fixed point.

7. *On the Volumes of Non-compact Octonionic Hyperbolic 2-Manifolds.*

The following treatment of the non-compact octonionic hyperbolic 2-manifold volumes was influenced by a similar treatment for the quaternionic hyperbolic manifold volumes given in section 5 of [25]. The corresponding treatment for complex hyperbolic manifold volumes can be found in [32].

We will now use the disjointness criterion for canonical horoballs derived in the previous section to derive lower bounds for the volumes of disjoint embedded cusp neighbourhoods of finite volume octonionic hyperbolic 2-manifolds with a finite number of ends. We will then use these lower bounds to derive lower bounds for the volumes of closed, cusped octonionic hyperbolic 2-manifolds.

Let M be a complete locally symmetric Riemannian manifold of negative curvature over the Cayley octonionic algebra. The universal cover of M is isometric to $\mathbf{H}^2_{\mathbb{O}}$. Furthermore by a theorem of H.C. Wang (see [37] Theorem 8.1) there do exist manifolds covered by $\mathbf{H}^2_{\mathbb{O}}$ of smallest volume. Moreover the minimum is obtained by only finitely many manifolds (up to isometry). The manifold M is the quotient of $\mathbf{H}^2_{\mathbb{O}}$ by G, a discrete cofinite subgroup of $\mathrm{Aut}(J)$. The ends of M correspond one-to-one with conjugacy classes of maximal parabolic subgroups of G. If X is the fixed point of a parabolic element then the stabiliser of X in G, G_X preserves every horosphere ∂H_t and $\partial H_t/G_X$ is compact. Furthermore there exists $t_0 > 0$ such that the end of M corresponding to the maximal parabolic subgroup G_X has a neighbourhood isometric to a cusp H_{t_0}/G_X. Therefore $Vol(M)$ is bounded below by $Vol(H_{t_0}/G_X)$.

7.1. *The Octonionic Volume Form.*

Any rank 1 symmetric space has a metric invariant under the automorphism group of the form $g_q * dq^2$ where $*$ denotes the direct sum, q is a parameter along the real geodesic l contained in an $\mathbb{O}$-line $\mathbf{H}^1_{\mathbb{O}}$ and g_q is the invariant induced metric on the horosphere ∂H_q parametrised by q. We will compute all volumes in the Siegel domain using the Bergman metric with constant sectional curvature -1. We claim that the Riemannian metric given by

$$\mathbf{g_1} = (dv - 2Im\langle\langle \zeta, d\zeta\rangle\rangle)^2 + 4\langle\langle d\zeta, d\zeta\rangle\rangle, \tag{45}$$

where

$$\langle\langle \zeta, \zeta'\rangle\rangle = \zeta\overline{\zeta'}$$

is invariant under the Heisenberg isometry group (which is generated by the Heisenberg rotations S_μ and the Heisenberg translations $T_{\tau,t}$). This invariance follows by direct application of any S_μ and $T_{\tau,t}$ to equation (45). The calculation for $T_{\tau,t}$ is straightforward but tedious and will not be presented here. Regarding the Heisenberg rotations S_μ a slightly different, but relatively uncomplicated argument must be used involving the invariance of the metric under conjugation by purely imaginary octonionic units. (We recall here that the S_μ generate $Spin_7\mathbb{R}$.)

The metric $\mathbf{g_1}$ is the pullback of the Bergman metric under geodesic perspective from the Siegel domain $\mathfrak{S}_2$ to $\mathfrak{N}_2$ identified with the horosphere ∂H_1 of height 1. Therefore the volume form $dVol_{\mathfrak{N}_2}$ is

$$dVol_{\mathfrak{N}_2} = 2^8 dVol_{Im(\mathbb{O})}\, dVol_{\mathbb{O}}.$$

Let $\mathbf{v}$ be a vector tangent to l at $x = \partial H_q \cap l$. We observe that $\mathbf{v}$ lies in a tangent space of $\mathbf{H}^2_{\mathbb{O}}$ given by a copy of $\mathbb{O}^2 = \mathcal{H}^{15} \times \mathbb{R}$. If we apply the standard basis vectors e_i, $i = 1, \ldots, 7$ of $Im(\mathbb{O})$ to $\mathbf{v}$ these vectors become tangent to $\partial H_q \cap \mathbf{H}^1_{\mathbb{O}}$. The subspace spanned by these vectors is **vertical** and the orthogonal complement in $T_x\partial H_q$ is **horizontal**. We now make the further observation that $\mathcal{H}^{15} = \mathbb{O} \times Im(\mathbb{O})$. Under the canonical projection along the geodesics which are parallel to l at ∞ the length of the image at ∂H_{q+s} of a vector $\mathbf{w}$ in the vertical direction is $e^s|\mathbf{w}|$ if one normalises the metric so that the sectional curvature of $\mathbf{H}^1_{\mathbb{O}}$ is -1. However the length of the image at ∂H_{q+s} of a vector $\mathbf{w}$ in the horizontal direction is $e^{s/2}|\mathbf{w}|$. Therefore the invariant metric on $\{(\zeta, v, u) | (\zeta, v) \in \mathfrak{N}_2, u \in \mathbb{R}_+\} = \mathfrak{N}_2 \times \mathbb{R}_+$, i.e. the Bergman metric $\mathbf{g}$ can be written as

$$\mathbf{g} = \frac{du^2 + (dv - 2Im\langle\langle \zeta, d\zeta\rangle\rangle)^2 + 4\langle\langle d\zeta, d\zeta\rangle\rangle}{u^2}.$$

This means that the volume form $dVol_{\mathbf{H}^2_{\mathbb{O}}}$ is

$$dVol_{\mathbf{H}^2_{\mathbb{O}}} = \frac{2^8}{u^{12}} du\, dVol_{Im(\mathbb{O})}\, dVol_{\mathbb{O}} = \frac{1}{u^{12}} du\, dVol_{\mathfrak{N}_2}.$$

Let $\Pi : (\zeta, v) \mapsto \zeta$ be vertical projection. If L is a Heisenberg lattice, i.e a lattice of Heisenberg translations, we call $L^V = ker\Pi \cap L$ a vertical lattice and $\Pi(L)$ a horizontal lattice of L. We observe that $L^V \subset L_B$.

Lemma 3.10. *Let G_B be a discrete cocompact group of Heisenberg isometries of* $\mathrm{Aut}(J)$ *fixing B. Then for any horoball $H_q = \{(\zeta, v, u) \in \mathfrak{N}_2 \times \mathbb{R}^+ | u \geq q\}$ we have*

$$Vol_{\mathbf{H}^2_{\mathbb{O}}}(H_q/G_B) = \frac{1}{11q^{11}} Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/G_B).$$

Proof. Using geodesic perspective it is clear that the intersection of H_q/G_B with each horosphere has the same area. Therefore we may split the integral into one integral over $\mathfrak{N}_2$ and another one over (q, ∞).

$$\begin{aligned} Vol_{\mathbf{H}^2_{\mathbb{O}}}(H_q/G_B) &= \int_{\mathfrak{N}_2/G_B} \int_q^\infty \frac{1}{x^{12}} dx\, dVol_{\mathfrak{N}_2} \\ &= Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/G_B) \frac{1}{11q^{11}}. \end{aligned}$$

We now want to compute $Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/G_B)$. As any discrete cocompact group of Heisenberg isometries contains a lattice of finite index, it is sufficient to find $Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/L)$ where L is a Heisenberg lattice.

Lemma 3.11. *Let L be a Heisenberg lattice. Then*

$$Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/L) = 2^8 Vol_{Im(\mathbb{O})}(Im(\mathbb{O})/L \cap ker\Pi) Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L)).$$

Proof.

$$Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/L) = \int_{\mathbb{O}/\Pi(L)} \int_{Im(\mathbb{O})/L \cap ker\Pi} 2^8 dvol_{Im(\mathbb{O})}\, dvol_{\mathbb{O}}.$$

7.2. *Volume Estimates of Cusped Octonionic 2-manifolds.*

Theorem 3.4. *Let G be a discrete, cofinite volume group of isometries of $\mathbf{H}^2_{\mathbb{O}}$. Let G_B be a parabolic subgroup of G fixing a parabolic fixed point B of G. Suppose G_B contains a Heisenberg lattice L as a subgroup of index m and let t denote the square of the shortest (Cygan) translation length of an element of $ker\Pi \cap L$. (From here on in order to be brief we will refer to such t as the shortest translation length.) Let H_u be a horoball left invariant by G_B. Then*

$$Vol(H_u/G_B) \geq 2^5 \frac{1}{11mu^{11}} Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L))\, t^7.$$

Proof. From Lemma 3.10 we have

$$Vol(H_u/G_B) = \frac{1}{11u^{11}} Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/G_B).$$

Now with respect to our scaling the volume of the unit cell in E_7 is $t^7/8$, but E_7 is the densest lattice in 7 real dimensions, therefore

$$Vol_{\mathbb{R}^7}(\mathbb{R}^7/L \cap ker\Pi) \geq \frac{t^7}{8}.$$

Thus by Lemma 3.11 we have

$$\begin{aligned} Vol_{\mathfrak{N}_2}(\mathfrak{N}_2/L) &= 2^8 Vol_{Im(\mathbb{O})}(Im(\mathbb{O})/L \cap ker\Pi) Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L)) \\ &= 2^8 Vol_{\mathbb{R}^7}(\mathbb{R}^7/L \cap ker\Pi) Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L)) \\ &\geq 2^8 Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L))\,(t^7/8) \\ &= 2^5 Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L))\, t^7 \end{aligned}$$

and as L has index m in G_B we have

$$Vol(\mathfrak{N}_2/G_B) \geq \frac{1}{m} 2^5 Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L))\, t^7.$$

Therefore

$$Vol(H_u/G_B) \geq \frac{2^5}{11mu^{11}} Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L))\, t^7.$$

We will now construct the following conjectured volume estimate.

Conjecture 3.1. *Let G be a discrete, cofinite volume group of isometries of $\mathbf{H}^2_{\mathbb{O}}$. Let G_B be a parabolic subgroup of G fixing a parabolic fixed point B of G. Suppose G_B contains a Heisenberg lattice L as a subgroup of index m and let t denote the shortest translation length of an element of $ker\Pi \cap L$. Let H_u be a horoball left invariant by G_B. Then*

$$Vol(H_u/G_B) \geq \frac{2}{99mu^{11}}t^{11}.$$

Evidence for Conjecture 3.1. Let $\zeta_1, \zeta_2 \in \mathbb{O}$. Set $\omega : \mathbb{O}^2 \mapsto Im(\mathbb{O})$;

$$\omega(\zeta_1, \zeta_2) = \big(\omega_1(\zeta_1, \zeta_2), ..., \omega_7(\zeta_1, \zeta_2)\big) = -Im\langle\langle \zeta_1, \zeta_2 \rangle\rangle$$

(see section 3.1), where if $\zeta = x_0 + \sum^7{}_{i=1} x_i e_i$, then according to the octonion multiplication table, i.e.

$$e_1 = e_0 e_1 = e_2 e_3 = e_4 e_5 = e_7 e_6,$$

etc., we have the following standard symplectic forms on $\mathbb{R}^8$.

$$\omega_1 = dx_0 \times dx_1 + dx_2 \times dx_3 + dx_4 \times dx_5 + dx_7 \times dx_6,$$

$$\omega_2 = dx_0 \times dx_2 + dx_3 \times dx_1 + dx_4 \times dx_6 + dx_5 \times dx_7,$$

$$\omega_3 = dx_0 \times dx_3 + dx_1 \times dx_2 + dx_4 \times dx_7 + dx_6 \times dx_5,$$

$$\omega_4 = dx_0 \times dx_4 + dx_5 \times dx_1 + dx_6 \times dx_2 + dx_7 \times dx_3,$$

$$\omega_5 = dx_0 \times dx_5 + dx_1 \times dx_4 + dx_7 \times dx_2 + dx_3 \times dx_6,$$

$$\omega_6 = dx_0 \times dx_6 + dx_2 \times dx_4 + dx_1 \times dx_7 + dx_5 \times dx_3,$$

$$\omega_7 = dx_0 \times dx_7 + dx_6 \times dx_1 + dx_2 \times dx_5 + dx_3 \times dx_4.$$

In the matrix representation the ω_i are block diagonal matrices.

$$\omega_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\omega_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\omega_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\omega_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\omega_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\omega_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\omega_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{v_1}, ..., \mathbf{v_7}$ be the shortest linearly independent vectors in the vertical lattice $ker\Pi \cap L$ (that is $\{\mathbf{v_1}, ..., \mathbf{v_7}\}$ is a Minkowski-reduced basis for $ker\Pi \cap L$ and $t \leq |\mathbf{v_1}| \leq |\mathbf{v_2}| \leq ... \leq |\mathbf{v_7}|$). Let A be a matrix in $GL(7, \mathbb{R})$ whose ith row is a unit vector in the direction orthogonal to all $\mathbf{v_j}$ where j takes integer values from 1 to 7 and is distinct from i. Then A maps the $\mathbb{R}\mathbf{v_j}$ (the lines spanned by the $\mathbf{v_j}$) to the mutually orthogonal axes in Euclidean 7-space $\mathbb{R}^7$. Define ω' by

$$A \begin{bmatrix} \omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \omega_7 \end{bmatrix} = \begin{bmatrix} \omega'_1 \\ \cdot \\ \cdot \\ \cdot \\ \omega'_7 \end{bmatrix}.$$

Writing $A = (a_{ij})$ we claim that $\omega'_i = \sum^7{}_{k=1} a_{ik}\omega_k$ is symplectic. It is clear that it is antisymmetric and closed. Now

$$det(\omega'_i) = det(\Sigma_{j=1}^7 a_{ij}\omega_j) = (\Sigma_{j=1}^7 a_{ij}^2)^4 = 1$$

(this was verified using Maple 6) therefore ω'_i is non-degenerate.

Let $B\mathbb{Z}^8$ be the lattice $\Pi(L) \subset \mathbb{O}$ isomorphic to $\mathbb{R}^8$ and set $\theta_i(\zeta, \eta) = \omega'{}_i(B\zeta, B\eta)$. We observe that since $det(\omega'_i) = 1$ we have

$$det(\theta_i) = \big(det(B)\big)^2 = Vol(\mathbb{O}/\Pi(L))^2.$$

If $B\zeta$ and $B\eta$ are in $\Pi(L)$, then $(B\zeta, x)$, $(B\eta, y) \in L$ for some x, $y \in \mathbb{R}^7$. The commutator of these two elements is

$$-2Im\langle\langle B\zeta, B\eta\rangle\rangle = 2\omega(B\zeta, B\eta) \in ker\Pi \cap L,$$

where $\langle\langle ., .\rangle\rangle$ denotes the usual positive-definite Hermitian form on $\mathbb{O}^1$. Therefore we can write

$$2\omega(B\zeta, B\eta) = \sum_{i=1}^{7} n_i \mathbf{v_i}$$

for some $\{n_i\} \subset \mathbb{Z}$. Applying A we find

$$\begin{aligned} 2\omega'(B\zeta, B\eta) &= 2A\omega(B\zeta, B\eta) \\ &= \sum_{i=1}^{7} n_i A\mathbf{v_i}. \end{aligned}$$

Since $A\mathbf{v_i}$ is on the x_i axis in $\mathbb{R}^7$ for $i = 1$ to 7, i.e.

$$A\mathbf{v_i} = \lambda_i \mathbf{e_i}$$

where $\mathbf{e_i}$ is the unit vector along the x_i-axis in $\mathbb{R}^7$ and $\lambda_i \in \mathbb{R}$, we find that

$$2\omega_i'(B\zeta, B\eta) = n_i A\mathbf{v_i}$$

and so

$$\theta_i(\zeta, \eta) = n_i A\mathbf{v_i}/2.$$

Therefore for all ζ, $\eta \in \mathbb{Z}^8$ we have either $\theta_i(\zeta, \eta) = 0$ or $|\theta_i(\zeta, \eta)| \geq |A\mathbf{v_i}|/2$. Note that $\{|\theta_i(\zeta, \eta)|\} \neq \{0\}$ since θ is nondegenerate. By the argument in Lemma 5.6 of [18] we obtain $|det(\theta_i)| \geq (|A\mathbf{v_i}|/2)^8$. Thus

$$Vol\big(\mathbb{O}/\Pi(L)\big) = \sqrt{det(\theta_i)} \geq (|A\mathbf{v_i}|/2)^4.$$

Thus given any lattice $L' \subset \mathbb{R}^7$ there exists a matrix $A \in GL(7, \mathbb{R})$ such that

$$A : \mathbb{R}^7 \mapsto \mathbb{R}^7$$

and

$$A(L') = \mathbb{Z}^7.$$

We want to find a lower bound on $|A\mathbf{v_i}|$ in terms of $|\mathbf{v_i}|$. One approach is to apply the Gram-Schmidt orthogonalisation process to the set of $\mathbf{v_i}$ to obtain 7 orthonormal bases of vectors $\{\mathbf{u_i}^{(j)} | i = 1, ..., 7\}$ for $j = 1, ..., 7$. This gives us (starting with a specific $\mathbf{v_j}$)

$$\mathbf{u_j^{(j)}} = \mathbf{v_j}/|\mathbf{v_j}|,$$

$$\mathbf{u_{j+_71}^{(j)}} = \frac{\mathbf{v_{j+_71}} - \langle \mathbf{v_{j+_71}}, \mathbf{u_j^{(j)}}\rangle \mathbf{u_j^{(j)}}}{|\mathbf{v_{j+_71}} - \langle \mathbf{v_{j+_71}}, \mathbf{u_j^{(j)}}\rangle \mathbf{u_j^{(j)}}|},$$

and so on. This means that $\mathbf{v_j}$ lies in the span of $\mathbf{u_j^{(j)}}$, $\mathbf{v_{j+_71}}$ lies in the span of $\mathbf{u_j^{(j)}}$ and $\mathbf{u_{j+_71}^{(j)}}$, ... and $\mathbf{v_{j+_76}}$ lies in the span of $\mathbf{u_j^{(j)}}, ..., \mathbf{u_{j+_76}^{(j)}}$. (Here $+_7$ denotes addition modulo 7.) By inspection we see that by the orthonormality of the bases $\{\mathbf{u_i}^{(j)}\}$ for $j = 1, ..., 7$, $(\mathbf{v_{j+_76}})_7$ is the component of $\mathbf{v_{j+_76}}$ orthogonal to the $\mathbf{u_{j+_7k}^{(j)}}$ for $k = 0, ..., 5$, i.e. $(\mathbf{v_{j+_76}})_7 = A\mathbf{v_{j+_76}}$. By the Gram-Schmidt orthogonalisation process we have

$$\mathbf{u_{j+_76}^{(j)}} = \frac{\mathbf{v_{j+_76}} - \Sigma^5{}_{k=0}\langle \mathbf{v_{j+_76}}, \mathbf{u_{j+_7k}^{(j)}}\rangle \mathbf{u_{j+_7k}^{(j)}}}{|\mathbf{v_{j+_76}} - \Sigma^5{}_{k=0}\langle \mathbf{v_{j+_76}}, \mathbf{u_{j+_7k}^{(j)}}\rangle \mathbf{u_{j+_7k}^{(j)}}|},$$

which implies

$$\mathbf{v_{j+_76}} = |\mathbf{v_{j+_76}} - \Sigma^5{}_{k=0}\langle \mathbf{v_{j+_76}}, \mathbf{u^{(j)}_{j+_7k}}\rangle \mathbf{u^{(j)}_{j+_7k}}|\mathbf{u^{(j)}_{j+_76}} + \Sigma^5{}_{k=0}\langle \mathbf{v_{j+_76}}, \mathbf{u^{(j)}_{j+_7k}}\rangle \mathbf{u^{(j)}_{j+_7k}}.$$

Therefore

$$|A\mathbf{v_{j+_76}}|^2 = |\mathbf{v_{j+_76}} - \Sigma^5{}_{k=0}\langle \mathbf{v_{j+_76}}, \mathbf{u^{(j)}_{j+_7k}}\rangle \mathbf{u^{(j)}_{j+_7k}}|^2.$$

Without loss of generality a lower bound on the set of $|A\mathbf{v_{j+_76}}|$ for $j = 1, ..., 7$ is a lower bound on the set of $|A\mathbf{v_i}|$ for $i = 1, ..., 7$.

Attempts have been made using Maple 6 and Maple 7 to perform the calculations necessary to implement this theoretical approach to a determination of a greatest lower bound on the set of $|A\mathbf{v_i}|$ for $i = 1, ..., 7$. However Maple seems to lack the computational capacity required to produce certain of the above expressions and we have yet to find a greatest lower bound.

We will now demonstrate an alternative theoretical route to the determination of the greatest lower bound on the set of $|A\mathbf{v_i}|$ for $i = 1, ..., 7$. We seek a greatest lower bound of

$$\frac{Vol(L')}{Vol(L'')}$$

as L' varies through all lattices in $\mathbb{R}^7$ and L'' is the sublattice of L' generated by the 6 reduced basis vectors $\mathbf{v_i}$ for $i = 1, ..., 6$ excluding the seventh reduced basis vector of the reduced vector basis of L'. This ratio is equal to the projection of $\mathbf{v_7}$ in the direction orthogonal to the span of the sublattice L''. Let $vol(L')$ denote the volume of the unit cell spanned by the above mentioned $\mathbf{v_1}, ..., \mathbf{v_7}$ and let l_i denote $|\mathbf{v_i}|$ for $i = 1, ..., 7$. Then we have

$$vol(L')^2 = (\Pi_{i=1}{}^7 l_i{}^2) det(\mathfrak{D})$$

where $\mathfrak{D}$ denotes the matrix

$$\mathfrak{D} = \begin{bmatrix} 1 & \cos(\theta_{12}) & \cos(\theta_{13}) & \cos(\theta_{14}) & \cos(\theta_{15}) & \cos(\theta_{16}) & \cos(\theta_{17}) \\ \cos(\theta_{12}) & 1 & \cos(\theta_{23}) & \cos(\theta_{24}) & \cos(\theta_{25}) & \cos(\theta_{26}) & \cos(\theta_{27}) \\ \cos(\theta_{13}) & \cos(\theta_{23}) & 1 & \cos(\theta_{34}) & \cos(\theta_{35}) & \cos(\theta_{36}) & \cos(\theta_{37}) \\ \cos(\theta_{14}) & \cos(\theta_{24}) & \cos(\theta_{34}) & 1 & \cos(\theta_{45}) & \cos(\theta_{46}) & \cos(\theta_{47}) \\ \cos(\theta_{15}) & \cos(\theta_{25}) & \cos(\theta_{35}) & \cos(\theta_{45}) & 1 & \cos(\theta_{56}) & \cos(\theta_{57}) \\ \cos(\theta_{16}) & \cos(\theta_{26}) & \cos(\theta_{36}) & \cos(\theta_{46}) & \cos(\theta_{56}) & 1 & \cos(\theta_{67}) \\ \cos(\theta_{17}) & \cos(\theta_{27}) & \cos(\theta_{37}) & \cos(\theta_{47}) & \cos(\theta_{57}) & \cos(\theta_{67}) & 1 \end{bmatrix}$$

and θ_{jk} denotes the angle between $\mathbf{v_j}$ and $\mathbf{v_k}$ for $1 \leq j, k \leq 7$. We require a greatest lower bound on $\sqrt{\left(\frac{vol(L')^2}{(\Pi_{i=1}{}^7 l_i{}^2)det(\mathfrak{D}')}\right)}$ where $det(\mathfrak{D}')$ denotes the leading minor of $\mathfrak{D}$. Once again computational difficulties appear to be, for the moment insurmountable.

In lieu of computational instrumentation sufficiently sophisticated to implement the above orthogonalisation process we will now attempt to produce a conjecture of the greatest lower bound on the set of $|A\mathbf{v_i}|$ for $i = 1, ..., 7$.

Conjecture. The E_6 lattice and E_7 lattice are the densest lattices in 6 and 7 real dimensions respectively. If t is the minimal vector length in E_7 (where $E_6 \subset E_7$), then the ratio of their volumes is $t/\sqrt{3}$ (as calculated by taking the square root of the ratio of the determinants of their respective Gram-matrices; calculated in this instant using the even coordinate (reduced) lattice bases on pages 124 and 126 of [6]). This volume ratio is equal to the length of the projection of the E_7 lattice vector $(1, 0, 0, 0, 0, 0, 0)$ orthogonal to the E_6 lattice. We conjecture that this ratio is a greatest lower bound on $|A\mathbf{v_i}|$. Therefore $|A\mathbf{v_i}| \geq \frac{1}{\sqrt{3}}|\mathbf{v_i}| \geq \frac{1}{\sqrt{3}}t$.

Attempts to deform the aforementioned basis of reduced lattice vectors (of the E_7 lattice) to reduced bases of other 7 dimensional lattices; for instance replacing $(1, 0, 0, 0, 0, 0, 0)$ with $(1, a, 0, 0, 0, 0, 0)$ (where $a \in \mathbb{R}$) or $(1, 0, 0, 0, 0, 0, 0)$ with $(1, 0, a, 0, 0, 0, 0)$ have failed to produce a smaller ratio. This corroborates, but of course does not prove the above conjecture.

By this conjecture we have

$$Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L)) \geq \frac{t^4}{2^4.9}.$$

Applying this estimate to Theorem 3.4 we have

$$\begin{aligned} Vol(H_u/G_B) &\geq 2^5 \frac{1}{11mu^{11}} Vol_{\mathbb{O}}(\mathbb{O}/\Pi(L))\, t^7 \\ &\geq 2\frac{t^{11}}{99mu^{11}}. \end{aligned}$$

Furthermore if H_u is a canonical horoball, then $u \leq t$ and so

$$Vol(H_u/G_B) \geq \frac{2}{99m}.$$

As a further consequence of our conjecture we have the following sub-conjecture. If k' is the number of ends (cusps) of the octonionic hyperbolic 2-manifold $M = \mathbf{H}^2_{\mathbb{O}}/G$ where G is a discrete, cofinite group of isometries of $\mathbf{H}^2_{\mathbb{O}}$ and $p_1, \ldots, p_{k'}$ are pairwise inequivalent representatives of all parabolic fixed points, then with H_i, the canonical horoball at p_i and G_i, its stabiliser in G, we have (as the canonical horoballs at distinct cusps are disjoint) the following conjectured result.

Conjecture 3.1.1. *Let M be a 2-dimensional non-compact octonionic hyperbolic manifold with k' cusps and let G be a discrete, cofinite volume group of isometries of $\mathbf{H}^2_{\mathbb{O}}$. Let G_i be a parabolic subgroup of G fixing a parabolic fixed point $q_i \in \partial\mathbf{H}^2_{\mathbb{O}}$ of G. Let H_i be a canonical horoball left invariant by G_i. Then*

$$Vol(M) \geq k' \min\{Vol(H_i)/G_i\} \geq k' \frac{2}{99m},$$

where m is the maximal index of a lattice L_i in G_i.

Remark. The parabolic subgroups G_i of G fixing a parabolic fixed point $q_i \in \partial\mathbf{H}^2_{\mathbb{O}}$ of G correspond to the subgroups of $\pi_1(M)$ stabilising the cusps of M.

CHAPTER FOUR.

A JØRGENSEN'S INEQUALITY FOR OCTONIONIC HYPERBOLIC 2-SPACE.

We recall from Chapter One that the classical Jørgensen's inequality gives a necessary algebraic criterion for a non-elementary 2-generator group of isometries of real hyperbolic space to be discrete. In this chapter we will formulate an octonionic analogue of Jørgensen's inequality for non-elementary 2-generator subgroups of the automorphism group $\mathrm{Aut}(J)$ of the octonionic hyperbolic plane. Our octonionic Jørgensen's inequality replaces algebraic hypotheses on traces with geometric hypotheses on fixed points and dilation factors. This is in part due to the non-associativity of the octonions which means that we will not be able to consider the generators as matrices in the appropriate matrix group. Instead we will follow the methods of Chapter Three and Allcock (see [1]) and use a canonical decomposition of elements of $\mathrm{Aut}(J)$, the isometry group of the octonionic hyperbolic plane. We will then, as in Chapter One use a method similar to that of Jørgensen and construct a particular sequence of distinct elements of the group using an octonionic version of Basmajian and Miner's stable basin argument (see [3]) (which in our case is a compactness theorem for elements close to the identity in $\mathrm{Aut}(J)$) to determine conditions on the parameters of the automorphism decompositions that force the sequence to tend to the identity, thus violating discreteness.

One of the generators, U of the non-elementary 2-generator subgroups in question, $\langle U, V\rangle$ will be a general element of $\mathrm{Aut}(J)$ not fixing C or B and the other, V will correspond to the composition of a rotation and a dilation, i.e. a loxodromic map (which is a type of **octonionic dilation**). The map V has the form

$$V = DS$$

with respect to the canonical decomposition of elements of $\mathrm{Aut}(J)$. The map D is a dilation with dilation factor $\lambda > 1$ and S is a rotation which is parametrised by a nested finite sequence of imaginary units μ_i for $i = 1, ..., n$ which for the purposes of our results can, on repeated application of the second of the five octonionic identities stated at the end of section 1.1 of Chapter Three, be considered to act as a single imaginary unit $\mu = (...(((\mu_n)\mu_{n-1})\mu_{n-2})...)\mu_1$. We observe here that if $\lambda = 1$, then V is a boundary elliptic map. In this chapter we prove the following theorem (where $[.,.,.,.]$ is a cross-ratio defined in section 2).

Theorem. *Let U and V be elements of* $\mathrm{Aut}(J)$ *such that U doesn't fix C or B and $V = D_\lambda S_\mu$ where $\lambda > 1$ and μ is an imaginary unit octonion. Let*

$$M = |\lambda\mu - 1| + |\lambda^{-1}\overline{\mu} - 1|.$$

If $M({\mathbb{X}_0}^{1/4} + 1) < 1$ with $\mathbb{X}_0 = [C, U^{-1}(B), U^{-1}(C), B]$, then either U and V commute or the group $\langle U, V\rangle$ is not discrete.

Remark. The hypotheses of this theorem involve distances between fixed points of U measured invariantly with respect to $\mathrm{Aut}(J)$ by cross-ratios.

1. *Preliminary Lemmas.*

First we will give two alternate versions of Lemma 3.1 of Chapter Three.

Lemma 4.1.1. *If h is a member of* $\mathrm{Aut}(J)$ *with dilation factor λ_h such that $h(B) \neq B$, then for all X, $Y \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B, h^{-1}(B)\}$*

$$\rho_0(h(X), h(Y)) = \frac{\lambda_h \rho_0(X, Y)}{\rho_0(X, h^{-1}(B))\rho_0(Y, h^{-1}(B))}.$$

Proof. See Lemma 3.1.

Similarly we have the following lemma.

Lemma 4.1.2. *If h is a member of* $\mathrm{Aut}(J)$ *with dilation factor λ_h such that $h(B) \neq B$, then for all $X \in \partial\mathbf{H}^2_{\mathbb{O}} - \{B\}$*

$$\rho_0(h(X), h(B)) = \frac{\lambda_h}{\rho_0(X, h^{-1}(B))}.$$

Proof.

$$\begin{aligned}
\rho_0(h(X), h(B)) &= \rho_0(T_1 D(S) R T_2^{-1}(X), T_1 D(S) R T_2^{-1}(B)) \\
&= \lambda_h \rho_0(R T_2^{-1}(X), R T_2^{-1}(B)) \\
&= \lambda_h \rho_0(R T_2^{-1}(X), C) \\
&= \frac{\lambda_h}{\rho_0(T_2^{-1}(X), C)} \\
&= \frac{\lambda_h}{\rho_0(X, h^{-1}(B))}.
\end{aligned}$$

Here we have used the fact that as R acts as inversion in the unit ρ_0-ball centred at C, then for all $Z \in \overline{\mathbf{H}^2_{\mathbb{O}}} - \{B\}$

$$\rho_0(R(Z), C) = \frac{1}{\rho_0(Z, C)}.$$

2. *Proof of the Principal Results.*

To determine discreteness conditions for the group $\langle U, V\rangle$ we will investigate the convergence of a sequence of cross-ratios. Consider the following cross-ratio (here we use Proposition 3.2)

$$\begin{aligned}
[X_1, X_2, Y_1, Y_2] &= \frac{\langle Y_1|X_1\rangle\langle Y_2|X_2\rangle}{\langle Y_2|X_1\rangle\langle Y_1|X_2\rangle} \\
&= \left(\frac{\rho_0(Y_1, X_1)\rho_0(Y_2, X_2)}{\rho_0(Y_2, X_1)\rho_0(Y_1, X_2)}\right)^4
\end{aligned}$$

where X_1, X_2, Y_1 and Y_2 all lie in $\partial\mathbf{H}^2_{\mathbb{O}}$. Throughout this chapter we will use the normalisation $\langle X|B\rangle = 1$ for all $X \in \partial\mathbf{H}^2_{\mathbb{O}}$. Thus we observe here that if say $X_1 = B$ which corresponds to 'infinity', then the boundary metric ρ_0 tends to infinity, but the cross-ratio becomes

$$[B, X_2, Y_1, Y_2] = \left(\frac{\rho_0(Y_2, X_2)}{\rho_0(Y_1, X_2)}\right)^4.$$

Remark. We observe that the cross-ratio is preserved by $\mathrm{Aut}(J)$. This can be seen using the canonical decomposition $h = T_1 D\{S\} R {T_2}^{-1}$ of the elements h of $\mathrm{Aut}(J)$ not fixing B given in section 1.6 of Chapter Three. The Heisenberg translations T_1 and T_2 together with the nested sequence of rotations $\{S\}$ preserve the inner product and therefore the cross-ratio by Proposition 3.1 and Proposition 3.2 of Chapter Three. That the dilation D with dilation factor λ preserves the cross-ratio can be seen from the definition $|D(X)|_0 = \lambda |X|_0$. Similarly using equation (36) of Chapter Three it follows by direct calculation and cancelling equal factors that the reflection R preserves the cross-ratio.

We will now define $M = |\lambda\mu - 1| + |\lambda^{-1}\overline{\mu} - 1|$. Recall that $|x|^2 + z + \overline{z} = 0$ for points $X = (x, 1, z)$ on the boundary of octonionic hyperbolic 2-space. We will use Allcock's sextuple representation for points $X = (x, 1, z) \in \overline{\mathbf{H}_{\mathbb{O}}^2} - \{B\}$; namely $X = (|x|^2, |z|^2, 1, z, \overline{z}x, \overline{x})$ (see equation (2.3) of [1]). Using the formula for $\langle X_1 | X_2 \rangle$ (where X_1, $X_2 \in J$) given in section 1 of Chapter Five we find that the following two lemmas hold.

Lemma 4.2. *If V is a loxodromic isometry of the octonionic hyperbolic plane fixing B and C, parametrised by the dilation factor λ and the imaginary unit octonion μ, then for all $X \in \partial\mathbf{H}_{\mathbb{O}}^2 - \{B\}$ we have*

$$\langle V(X) | X \rangle \leq \lambda^2 M^2 \langle X | C \rangle. \tag{46}$$

Proof. Using the second of the five octonionic identities listed at the end of section 1.1 of Chapter Three together with the fact that $\overline{\mu} = -\mu$ for all imaginary unit octonions we have

$$\begin{aligned}
\langle V(X) | X \rangle &= \langle (\lambda^2|x|^2, \lambda^4|z|^2, 1, \lambda^2\mu z\overline{\mu}, \lambda^3\mu(\overline{z}x), \lambda\overline{x}\,\overline{\mu}) | (|x|^2, |z|^2, 1, z, \overline{z}x, \overline{x}) \rangle \\
&= \lambda^2|x|^4 + (\lambda^4 + 1)|z|^2 + 2Re(\lambda^2\mu z\overline{\mu}z) + 2Re(\lambda^3\mu\overline{z}|x|^2 + \lambda\overline{z}|x|^2\overline{\mu}) \\
&= \lambda(\lambda z - \mu z)(z - \lambda\overline{\mu}z) + \lambda(\overline{z} - \lambda\overline{z}\mu)(\lambda\overline{z} - \overline{z}\,\overline{\mu}) + (\lambda z - \mu z)(\lambda\overline{z} - \overline{z}\,\overline{\mu}) + \lambda^2(\lambda z - z\mu)(\lambda\overline{z} - \overline{\mu}\,\overline{z}) \\
&\leq (\lambda + 1)^2|\lambda\mu - 1|^2|z|^2 = \lambda^2 M^2 |z|^2 = \lambda^2 M^2 \langle X | C \rangle.
\end{aligned}$$

We note here that $|z|^2 = \langle X | C \rangle$ where $C = (0, 0, 1, 0, 0, 0)$.

Lemma 4.3. *If V is a loxodromic isometry of the octonionic hyperbolic plane parametrised by the dilation factor λ and the imaginary unit octonion μ and $X_1 = q(x_1, 1, z_1)$, $X_2 = q(x_2, 1, z_2) \in \partial\mathbf{H}_{\mathbb{O}}^2 - \{B\}$, then*

$$\langle VX_1 | VX_2 \rangle = \lambda^4 \langle X_1 | X_2 \rangle. \tag{47}$$

Remark. Recalling Proposition 3.2 of Chapter Three one of the consequences of this lemma is that due to our normalisation (i.e. $\langle X | B \rangle = 1$ for all $X \in \partial\mathbf{H}_{\mathbb{O}}^2$), dilations, despite being automorphisms fail to preserve the inner product but scale the boundary metric linearly.

Proof. Using the first two of the five octonionic identities listed at the end of section 1.1 of Chapter Three we have

$$\begin{aligned}
&\langle VX_1 | VX_2 \rangle \\
&= \lambda^4|x_1|^2|x_2|^2 + \lambda^4|z_1|^2 + \lambda^4|z_2|^2 + 2Re\big((\lambda^2\mu z_1\overline{\mu})(\lambda^2\mu z_2\overline{\mu})\big) \\
&\quad + 2Re\big(\lambda^3(\mu(\overline{z_1}x_1))\lambda(\overline{x_2}\,\overline{\mu}) + \lambda^3(\mu(\overline{z_2}x_2))\lambda(\overline{x_1}\,\overline{\mu})\big) \\
&= \lambda^4\big(|x_1|^2|x_2|^2 + |z_1|^2 + |z_2|^2 + 2Re(\mu(z_1z_2)\overline{\mu}) + 2Re\big(\mu((\overline{z_1}x_1)\overline{x_2} + (\overline{z_2}x_2)\overline{x_1})\overline{\mu}\big) \\
&= \lambda^4\big(|x_1|^2|x_2|^2 + |z_1|^2 + |z_2|^2 + 2Re(z_1z_2) + 2Re\big((\overline{z_1}x_1)\overline{x_2} + (\overline{z_2}x_2)\overline{x_1}\big) \\
&= \lambda^4\langle X_1 | X_2 \rangle.
\end{aligned}$$

We will now define the sequence (U_n) such that $U_0 = U$ and $U_{n+1} = U_n V {U_n}^{-1}$ for $n > 0$. The map $U_{n+1} = U_n V {U_n}^{-1}$ is loxodromic with dilation factor λ and a pair of fixed points belonging to compact sets; one belonging to a compact set containing C, the other to a compact set containing B.

If $\langle U, V \rangle$ is non-elementary, then $V \neq U$. If V and U do not commute which we will assume from here on, then $V \neq U_1$. Therefore by induction we see that (U_n) is a sequence of distinct elements. We want to relate the terms

$$\frac{\langle {U_{n+1}}^{-1}(C) | C \rangle}{\langle {U_{n+1}}^{-1}(C) | {U_{n+1}}^{-1}(B) \rangle}$$

and

$$\frac{\langle {U_n}^{-1}(C) | C \rangle}{\langle {U_n}^{-1}(C) | {U_n}^{-1}(B) \rangle}$$

in order to determine convergence criteria for the corresponding sequence. Here, as we are using the normalisation $\langle X|B\rangle = 1$, $\frac{\langle U_n{}^{-1}(C)|C\rangle}{\langle U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle}$ is equal to the cross-ratio $[C, U_n{}^{-1}(B), U_n{}^{-1}(C), B]$.

Lemma 4.4.

$$\frac{\langle U_{n+1}^{-1}(C)|C\rangle}{\langle U_{n+1}{}^{-1}(C)|U_{n+1}{}^{-1}(B)\rangle} = \frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(C)\rangle}{\langle V^{-1}U_n{}^{-1}(C)|V^{-1}U_n{}^{-1}(B)\rangle}\frac{\langle V^{-1}U_n{}^{-1}(B)|U_n{}^{-1}(B)\rangle}{\langle U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle}. \tag{48}$$

Proof. Using the definition of U_{n+1}, then applying Lemma 4.1.1 twice we see that

$$\begin{aligned}\frac{\langle U_{n+1}^{-1}(C)|C\rangle}{\langle U_{n+1}{}^{-1}(C)|U_{n+1}{}^{-1}(B)\rangle} &= \frac{\langle U_nV^{-1}U_n{}^{-1}(C)|U_nU_n{}^{-1}(C)\rangle}{\langle U_nV^{-1}U_n{}^{-1}(C)|U_nV^{-1}U_n{}^{-1}(B)\rangle}\\ &= \frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(C)\rangle}{\langle V^{-1}U_n{}^{-1}(C)|V^{-1}U_n{}^{-1}(B)\rangle}\frac{\langle V^{-1}U_n{}^{-1}(B)|U_n{}^{-1}(B)\rangle}{\langle U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle}.\end{aligned}$$

Lemma 4.5.

$$\langle U_{n+1}{}^{-1}(C)|C\rangle\langle U_{n+1}(B)|C\rangle = \frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(C)\rangle}{\langle V^{-1}U_n{}^{-1}(B)|U_n{}^{-1}(B)\rangle}\langle C|U_n(B)\rangle^2. \tag{49}$$

Proof. Using Lemma 4.1.1, then Lemma 4.1.2 and Lemma 4.3 we see that

$$\begin{aligned}\frac{\langle U_{n+1}{}^{-1}(C)|C\rangle\langle U_{n+1}(B)|C\rangle}{\langle U_n(B)|C\rangle^2} &= \frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(C)\rangle\langle VU_n{}^{-1}(B)|U_n{}^{-1}(C)\rangle}{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle\langle VU_n{}^{-1}(B)|U_n{}^{-1}(B)\rangle}\\ &= \frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(C)\rangle}{\langle U_n{}^{-1}(B)|V^{-1}U_n{}^{-1}(B)\rangle}.\end{aligned}$$

Lemma 4.6.

$$\begin{aligned}\lambda\left(1-\lambda^{-1/2}M^{1/2}\frac{\rho_0(U_n{}^{-1}(C),C)}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))}\right)^2 &\le \frac{\rho_0(U_{n+1}(B),C)}{\rho_0(U_{n+1}{}^{-1}(C),U_{n+1}{}^{-1}(B))}\\ &\le \lambda\left(1+\lambda^{-1/2}M^{1/2}\frac{\rho_0(U_n{}^{-1}(C),C)}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))}\right)^2.\end{aligned} \tag{50}$$

Proof. Using Lemma 4.1.1 twice, followed by Lemma 4.3 and Proposition 3.2, in that order we see that

$$\begin{aligned}\frac{\langle U_{n+1}(B)|C\rangle}{\langle U_{n+1}{}^{-1}(B)|U_{n+1}^{-1}(C)\rangle} &= \frac{\langle VU_n{}^{-1}(B)|U_n{}^{-1}(C)\rangle}{\langle VU_n{}^{-1}(B)|U_n{}^{-1}(B)\rangle}\frac{\langle V^{-1}U_n{}^{-1}(B)|U_n{}^{-1}(B)\rangle}{\langle U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle}\frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle}{\langle V^{-1}U_n{}^{-1}(B)|V^{-1}U_n{}^{-1}(C)\rangle}\\ &= \lambda^4\frac{\langle V^{-1}U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle^2}{\langle U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle^2}\\ &= \left(\lambda\frac{\rho_0(V^{-1}U_n{}^{-1}(C),U_n{}^{-1}(B))^2}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))^2}\right)^4.\end{aligned}$$

Now using the triangle inequality we have

$$\left(1-\frac{\rho_0(V^{-1}U_n{}^{-1}(C),U_n{}^{-1}(C))}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))}\right) \le \frac{\rho_0(V^{-1}U_n{}^{-1}(C),U_n{}^{-1}(B))}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))} \le \left(1+\frac{\rho_0(V^{-1}U_n{}^{-1}(C),U_n{}^{-1}(C))}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))}\right).$$

Finally using Lemma 4.2 we obtain equation (50) and the lemma is proved.

2.1. *The Stable Basin Theorem.*

Regarding octonionic dilations we will now state a weaker version of the stable basin theorem (as proved by Basmajian and Miner in [3]). It will tell us how elements of $\mathrm{Aut}(J)$, conjugate to octonionic dilations, move points in $\partial\mathbf{H}^2_{\mathbb{O}}$ in terms of characteristic geometric parameters.

Given r and s with $r < s$, the pair of open sets $(B_r, R(B_{1/s}))$ (where $B_r \subset \partial\mathbf{H}^2_{\mathbb{O}}$ denotes the ρ_0-ball of radius r centred at C) is said to be stable with respect to a set of elements $\mathcal{S}$ in $\mathrm{Aut}(J)$ if for any $V \in \mathcal{S}$

$$V(C) \in B_r$$

and

$$V(B) \in R(B_{1/s}).$$

The stable basin theorem shows that neighbourhoods of C and B are stable with respect to the family of elements of $\mathrm{Aut}(J)$ conjugate to octonionic dilations which have fixed points near the origin C and infinity B and dilation factors close to 1.

Let $\mathcal{S}(r,\epsilon)$ denote the family of elements conjugate to octonionic dilations with fixed points in B_r and $R(B_r)$ and dilation factors λ satisfying $M = |\lambda\mu - 1| + |\lambda^{-1}\overline{\mu} - 1| < \epsilon$. We observe that $\mathcal{S}(r,\epsilon)$ is closed under conjugation by the inversion R. The stable basin theorem imposes conditions both on fixed points and dilation factors. The condition on the fixed points is determined by the quantity r and the condition on the dilation factors is determined by $\epsilon(r,r')$.

Stable Basin Theorem. *There exist positive real numbers r and r' such that the pair of open sets $(B_{r'}, R(B_{r'}))$ is stable with respect to the family $\mathcal{S}(r,\epsilon(r,r'))$.*

Proof. See Appendix B.

Remark. A version of the proof of Basmajian and Miner (see [3], pages 113-117) works for the case of octonionic dilations and their conjugate maps as both fixed points of members of $\mathcal{S}(r,\epsilon)$ and the corresponding dilation factors lie in compact sets. In particular it can be shown that the neighbourhoods of C and B are stable with respect to octonionic dilations having fixed points near C and B and dilation factor close to 1.

Remark. In [23] Kamiya proves a modified version of the Basmajian and Miner stable basin theorem which avoids the complication of their uniform bound Lipschitz theorem (used to estimate how much a pure parabolic in the stabiliser of C distorts distance in a neighbourhood of C, by putting a bound on the dilation factor of the pure parabolic). Once again it is clear from compactness arguments that there exists an octonionic analogue of Kamiya's proof (see Appendix B for an exact proof).

2.2. *Application of a Stable Basin Argument.*

By means of a stable basin argument we derive necessary conditions for discreteness as follows. Suppose V and W are loxodromic maps whose fixed points are close together (measured invariantly in terms of the size of a specific cross-ratio) and whose dilation factors are close to 1. By conjugation we assume that the fixed points of W lie in B_r and $R(B_r)$ for some sufficiently small r. Our stable basin argument then implies that the fixed points of WVW^{-1} also lie in B_r and $R(B_r)$. Furthermore the dilation factor of WVW^{-1} is the same as the dilation factor of V. Thus provided V and W don't commute, by repeated conjugation it is possible to produce an infinite sequence of elements whose fixed points are restricted to B_r and $R(B_r)$ and whose dilation factors are constant. By compactness one can select a convergent subsequence and thereby conclude that $\langle W, V\rangle$ is not discrete.

2.3. *Proof of the Theorem.*

Let $\mathbb{X}_n$ denote the cross-ratio $[C, {U_n}^{-1}(B), {U_n}^{-1}(C), B]$ where C and B are the fixed points of V and ${U_n}^{-1}(C)$ and ${U_n}^{-1}(B)$ are the fixed points of ${U_n}^{-1}VU_n$. Then by the triangle inequality we have the following proposition.

Proposition 4.1. *If* $M(\mathbb{X}_0{}^{1/4}+1)<1$, *then*
$$\mathbb{X}_n{}^{1/4} \leq \big(M(\mathbb{X}_0{}^{1/4}+1)\big)^n \mathbb{X}_0{}^{1/4}$$
so the sequence $\mathbb{X}_n{}^{1/4}$ *tends exponentially to 0.*

Proof. By applying Lemma 4.2, Lemma 4.3 and the triangle inequality to the right hand side of equation (48) in Lemma 4.4 we see that
$$\mathbb{X}_{n+1}{}^{1/4} \leq M(\mathbb{X}_n{}^{1/2}+\mathbb{X}_n{}^{1/4}) = \mathbb{X}_n{}^{1/4}\big(M(\mathbb{X}_n{}^{1/4}+1)\big).$$
Thus by induction
$$\mathbb{X}_n{}^{1/4} \leq \big(M(\mathbb{X}_0{}^{1/4}+1)\big)^n \mathbb{X}_0{}^{1/4}.$$
So if $M(\mathbb{X}_0{}^{1/4}+1)<1$ we see that the cross-ratios $\mathbb{X}_n$ form a monotonically decreasing sequence tending exponentially to zero. The sequence is strictly monotonic decreasing provided $\mathbb{X}_n>0$ for all $n\in\mathbb{Z}_+$. This is so if $\mathbb{X}_n\neq 0$ for all $n\in\mathbb{Z}_+$. We recall that
$$\mathbb{X}_n = \frac{\langle U_n{}^{-1}(C)|C\rangle}{\langle U_n{}^{-1}(C)|U_n{}^{-1}(B)\rangle}.$$
This is non-zero provided $U_n{}^{-1}(C)\neq C$ (numerator non-zero) for all $n\in\mathbb{Z}_+$.

Suppose there exists n such that $U_n{}^{-1}(C)=C$. This implies $V^{-1}U_{n-1}{}^{-1}(C)=U_{n-1}{}^{-1}(C)$ and therefore as U_{n-1} isn't an inversion, considering the fixed points of V we have $U_{n-1}{}^{-1}(C)=C$. By induction we see that $U^{-1}(C)=C$ which implies $U(C)=C$. This is a contradiction. Therefore $U_n{}^{-1}(C)\neq C$.

Proposition 4.2.
$$\rho_0(U_{n+1}{}^{-1}(C),C) \leq \lambda^{1/2}M^{1/2}\frac{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^2}{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})}\rho_0(U_n{}^{-1}(C),C). \tag{51}$$

Proof. Using first Lemma 4.4, Lemma 4.5 and Lemma 4.6, then Lemma 4.3, followed by Lemma 4.2 and Lemma 4.6 again, together with the abbreviation
$$\delta_n = \lambda^{-1/2}M^{1/2}\frac{\rho_0(U_n{}^{-1}(C),C)}{\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))} = \lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4},$$
we see that
$$\begin{aligned}
&\rho_0(U_{n+1}{}^{-1}(C),C)^2\\
&= \frac{\rho_0(U_{n+1}{}^{-1}(C),C)}{\rho_0(U_{n+1}{}^{-1}(C),U_{n+1}{}^{-1}(B))}\rho_0(U_{n+1}{}^{-1}(C),C)\rho_0(U_{n+1}(B),C)\frac{\rho_0(U_{n+1}{}^{-1}(C),U_{n+1}{}^{-1}(B))}{\rho_0(U_{n+1}(B),C)}\\
&\leq \frac{\rho_0(V^{-1}U_n{}^{-1}(C),U_n{}^{-1}(C))^2\rho_0(U_n(B),C)^2}{\lambda^{-1}\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))^2}\frac{1}{\lambda\left(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4}\right)^2}\\
&\leq M\rho_0(U_n{}^{-1}(C),C)^2\lambda^2\left(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4}\right)^4\frac{1}{\lambda\left(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4}\right)^2}\\
&= \lambda M\frac{(1+\delta_{n-1})^4}{(1-\delta_n)^2}\rho_0(U_n{}^{-1}(C),C)^2.
\end{aligned}$$

Proposition 4.3.
$$\rho_0(U_{n+1}{}^{-1}(B),U_{n+1}{}^{-1}(C)) \geq \lambda^{1/2}M^{-1/2}\frac{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^2}{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})}\frac{1}{(1+\mathbb{X}_n{}^{1/4})}\rho_0(U_n{}^{-1}(B),U_n{}^{-1}(C)). \tag{52}$$

Proof. Using Lemmas 4.4, 4.5, 4.6 and 4.2 and the triangle inequality in that order, together with Lemma 4.3 we see that

$$\begin{aligned}
&\frac{1}{\rho_0(U_{n+1}{}^{-1}(B),U_{n+1}{}^{-1}(C))^2}\\
&= \frac{\rho_0(U_{n+1}{}^{-1}(C),C)}{\rho_0(U_{n+1}{}^{-1}(B),U_{n+1}{}^{-1}(C))}\frac{1}{\rho_0(U_{n+1}{}^{-1}(C),C)\rho_0(U_{n+1}(B),C)}\frac{\rho_0(U_{n+1}(B),C)}{\rho_0(U_{n+1}{}^{-1}(B),U_{n+1}{}^{-1}(C))}\\
&\leq \frac{\lambda}{\rho_0(U_n{}^{-1}(B),U_n{}^{-1}(C))^2}\frac{\rho_0(V^{-1}U_n{}^{-1}(B),U_n{}^{-1}(B))^2}{\rho_0(U_n(B),C)^2}\lambda\left(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4}\right)^2\\
&\leq M\lambda^{-1}\frac{1}{\rho_0(U_n{}^{-1}(B),U_n{}^{-1}(C))^2}\frac{\rho_0(U_n{}^{-1}(B),C)^2}{\rho_0(U_n{}^{-1}(B),U_n{}^{-1}(C))^2}\frac{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})^2}{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^4}\\
&\leq M\lambda^{-1}\frac{1}{\rho_0(U_n{}^{-1}(B),U_n{}^{-1}(C))^2}(1+\mathbb{X}_n{}^{1/4})^2\frac{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})^2}{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^4}.
\end{aligned}$$

Thus from Propositions 4.2 and 4.3 we have

$$\rho_0(U_{n+1}{}^{-1}(C),C) \leq \lambda^{1/2}M^{1/2}\frac{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^2}{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})}\rho_0(U_n{}^{-1}(C),C) \tag{53.1}$$

and

$$\rho_0(U_{n+1}{}^{-1}(C),U_{n+1}{}^{-1}(B)) \geq \lambda^{1/2}M^{-1/2}\frac{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^2}{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})}\frac{1}{(1+\mathbb{X}_n{}^{1/4})}\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B)). \tag{53.2}$$

As $M(\mathbb{X}_0{}^{1/4}+1)<1$, i.e. λ is close to 1 and $\mathbb{X}_0>0$, we see that $M<1$. Therefore as $\mathbb{X}_n \to 0$ as $n\to\infty$, there exists a positive integer N such that for all $n>N$

$$M^{1/2}\frac{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^2}{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})}<1$$

and

$$M^{1/2}\frac{(1+\lambda^{-1/2}M^{1/2}\mathbb{X}_n{}^{1/4})}{(1-\lambda^{-1/2}M^{1/2}\mathbb{X}_{n-1}{}^{1/4})^2}(1+\mathbb{X}_n{}^{1/4})<1.$$

Therefore by induction

$$\rho_0(U_n{}^{-1}(C),C)\lambda^{-n/2}\to 0$$

and

$$\rho_0(U_n{}^{-1}(C),U_n{}^{-1}(B))\lambda^{-n/2}\to\infty$$

as $n\to\infty$. Consideration of these two limits and application of the first, together with the triangle inequality, to the second implies that $(\rho_0(U_n{}^{-1}(C),C)\lambda^{-n/2})$ and $\left(\frac{1}{\rho_0(U_n{}^{-1}(B),C)\lambda^{-n/2}}\right)$ are bounded sequences. Therefore the images of the fixed points C and B of V under the sequence of maps (U_n) are bounded about C and B respectively. This implies that (U_n) is a bounded sequence of distinct maps. Following Jørgensen we define a sequence of distinct maps (W_n) such that $W_0=U$ and for $n>0$

$$W_n = V^{-n}U_{2n}V^n.$$

We see that for $n>0$ the W_n are all conjugate to V and therefore have the same dilation and rotation factors. The map W_n is also conjugate to U_{2n}, thus as (U_n) is a sequence of distinct elements of $\mathrm{Aut}(J)$ so is (W_n).

We will now seek to apply our octonionic version of the stable basin argument to $\mathcal{S}(r,\epsilon) = \{W_n | n > 0\}$ (instead of $\mathcal{S}(r,\epsilon) = \{U_n | n > 0\}$), i.e. we will prove that (W_n) has a convergent subsequence under certain conditions and therefore generates a non-discrete group. To do this we need the following proposition.

Proposition 4.4. *For all n (say $n > N$) such that*

$$M^{1/2}\frac{(1+\lambda^{-1/2}M^{1/2}{\mathbb{X}_{n-1}}^{1/4})^2}{1-\lambda^{-1/2}M^{1/2}{\mathbb{X}_n}^{1/4}} < \epsilon \tag{54.1}$$

and

$$M^{1/2}\frac{(1+\lambda^{-1/2}M^{1/2}{\mathbb{X}_n}^{1/4})}{(1-\lambda^{-1/2}M^{1/2}{\mathbb{X}_{n-1}}^{1/4})^2}(1+{\mathbb{X}_n}^{1/4}) < \epsilon \tag{54.2}$$

where $0 < \epsilon < 1$, the following inequalities hold.

(1)

$$\langle {W_n}^{-1}(C)|C\rangle < \epsilon^{8(n-N)}\langle {W_N}^{-1}(C)|C\rangle,$$

(2)

$$\langle {W_n}^{-1}(B)|{W_n}^{-1}(C)\rangle > \epsilon^{-8(n-N)}\langle {W_N}^{-1}(B)|{W_N}^{-1}(C)\rangle.$$

Proof.

(1) Using the fact that $V(C) = C$, then Lemma 4.3, we have for $n > 0$

$$\begin{aligned}\langle {W_n}^{-1}(C)|C\rangle &= \langle V^{-n}{U_{2n}}^{-1}V^n(C)|C\rangle \\ &= \langle V^{-n}{U_{2n}}^{-1}(C)|V^{-n}(C)\rangle \\ &= \lambda^{-4n}\langle {U_{2n}}^{-1}(C)|C\rangle.\end{aligned}$$

Now applying both inequality (53.1) and inequality (54.1) twice we have

$$\begin{aligned}\langle {W_n}^{-1}(C)|C\rangle &< \lambda^{-4n+4}\epsilon^8\langle {U_{2n-2}}^{-1}(C)|C\rangle \\ &= \epsilon^8\langle {W_{n-1}}^{-1}(C)|C\rangle.\end{aligned}$$

(2) Similarly using the fact that V fixes both C and B, then Lemma 4.3, we have for $n > 0$

$$\begin{aligned}\langle {W_n}^{-1}(B)|{W_n}^{-1}(C)\rangle &= \langle V^{-n}{U_{2n}}^{-1}V^n(B)|V^{-n}{U_{2n}}^{-1}V^n(C)\rangle \\ &= \langle V^{-n}{U_{2n}}^{-1}(B)|V^{-n}{U_{2n}}^{-1}(C)\rangle \\ &= \lambda^{-4n}\langle {U_{2n}}^{-1}(B)|{U_{2n}}^{-1}(C)\rangle.\end{aligned}$$

Now applying both inequality (53.2) and inequality (54.2) twice we have

$$\begin{aligned}\langle {W_n}^{-1}(B)|{W_n}^{-1}(C)\rangle &> \lambda^{-4n+4}\epsilon^{-8}\langle {U_{2n-2}}^{-1}(B)|{U_{2n-2}}^{-1}(C)\rangle \\ &= \epsilon^{-8}\langle {W_{n-1}}^{-1}(B)|{W_{n-1}}^{-1}(C)\rangle.\end{aligned}$$

3. *Conclusion.*

By Proposition 4.4 we see that $\langle {W_n}^{-1}(C)|C\rangle \to 0$ and $\langle {W_n}^{-1}(B)|{W_n}^{-1}(C)\rangle \to \infty$ as $n \to \infty$. Now by the triangle inequality $\langle {W_n}^{-1}(B)|C\rangle \to \infty$ as $n \to \infty$. Therefore the two sequences $\left(\langle {W_n}^{-1}(C)|C\rangle\right)$ and $\left(\frac{1}{\langle {W_n}^{-1}(B)|C\rangle}\right)$ are bounded. By construction the maps W_n are distinct with dilation factor λ and equal rotation factors. Furthermore their fixed points lie in compact sets B_r and $R(B_r)$ (r finite) containing C and B respectively. Therefore applying our octonionic version of the stable basin argument to $\mathcal{S}(r,\epsilon) = \{W_n | n > 0\}$ (where $\epsilon(r,r) = \frac{1}{({\mathbb{X}_0}^{1/4}+1)^2}$) we see that if $M({\mathbb{X}_0}^{1/4}+1) < 1$, the sequence (W_n) lies in a compact region of $\mathrm{Aut}(J)$. By compactness we argue that there exists a convergent subsequence of (W_n) and so the group spanned by the W_n is not discrete. Therefore the larger group $\langle U, V\rangle$ is not discrete when $M({\mathbb{X}_0}^{1/4}+1) < 1$ and we have proved our theorem.

CHAPTER FIVE.

THE EIGENVALUE PROBLEM FOR Φ-HERMITIAN OCTONIONIC 3 BY 3 MATRICES.

The eigenvalue problem is usually considered over the real and complex number fields. In this chapter we study an aspect of the generalisation of this problem to the other normed division algebras, i.e. the quaternions and the octonions. We discover that many of the standard properties hold provided they are reinterpreted in terms of the lack of commutativity of $\mathbb{H}$ and $\mathbb{O}$ and the lack of associativity of $\mathbb{O}$.

To be specific we solve the real eigenvalue problem $X\nu = \lambda\nu$ for the 3×3 Φ-Hermitian matrices X of the Jordan algebra model of the octonionic hyperbolic plane. We show that the real eigenvalues λ of these 3×3 Φ-Hermitian matrices form two independent families each consisting of three real eigenvalues which satisfy a modified characteristic equation.

This chapter was inspired by the work of Dray, Janesky, Manogue and Okubo (see [8]-[13]) on the eigenvalue problem for I-Hermitian octonionic three by three matrices (where I denotes the 3×3 identity matrix). Other good references are [14] and [15].

Our principal results are as follows.

Lemma 5.1. *The real eigenvalues λ of the 3×3 octonionic Φ-Hermitian matrix X satisfy the modified characteristic equation*

$$det(\lambda I - X) = \lambda^3 - (tr(X))\lambda^2 + \sigma(X)\lambda - det(X) = r$$

where r is either of the roots of

$$r^2 + 4\Psi(v, -u, w)r - |[v, -u, w]|^2 = 0.$$

The octonions u, v and w are components of X and Ψ is defined by

$$\Psi(v, -u, w) = \frac{1}{2}Re([v, -\overline{u}]w)$$

where $[v, u] = vu - uv$ denotes the standard commutator of two octonions u and v and $[v, u, w]$ denotes the associator of three octonions u, v and w (see section 1.1 of Chapter Three).

Remark. The octonionic determinant $det(X)$ is defined on page 493 of [1].

These results and the associated proofs are a generalisation of the corresponding results and proofs given in [10] for the more symmetric and therefore simpler to treat, standard Hermitian matrices (which are invariant upon taking the conjugate transpose).

Remark. The geometrical significance of these results with respect to the octonionic hyperbolic plane is not yet understood although they may be of relevance to Lie algebra theory.

1. *An Octonionic Jordan Algebra.*

As in Chapter Three we choose to work with the following inner product matrix (an indefinite non-degenerate Hermitian form)

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We recall that the Jordan algebra associated to Φ is then defined as $J = \{ X \in M_3(\mathbb{O}) \mid X\Phi = \Phi X^* \}$ where X^* denotes the conjugate transpose of the matrix X. Such elements X of J are called **Φ-Hermitian matrices** and have the form

$$X = (a, b, c, u, v, w) = \begin{bmatrix} a & w & \overline{v} \\ v & \overline{u} & b \\ \overline{w} & c & u \end{bmatrix}$$

with a, b, $c \in \mathbb{R}$ and u, v, $w \in \mathbb{O}$. It is obvious that X has real trace. Computation reveals

$$\langle X|X \rangle = a^2 + 2bc + 2Re(u^2) + 4Re(vw),$$

and so by polarisation we obtain

$$\langle X_1|X_2 \rangle = a_1 a_2 + b_1 c_2 + b_2 c_1 + 2Re(u_1 u_2) + 2Re(v_1 w_2 + v_2 w_1).$$

Further computation shows that

$$\begin{aligned} \sigma(X) &:= 1/2\big((tr(X))^2 - tr(X^2)\big) \\ &= 2aRe(u) + 2\big(Re(u)\big)^2 - bc - Re(vw) - Re(wv) - Re(u^2) \end{aligned}$$

and it is proved in Theorem 6.1 of [1] that

$$det(X) = a|u|^2 + b|w|^2 + c|v|^2 - abc - 2Re(uvw). \tag{55}$$

2. *The Φ-Hermitian eigenvalue problem.*

The *left* and *right* eigenvalue problems for 3×3 octonionic Φ-Hermitian matrices X are given explicitly by

$$X\nu = \lambda\nu \tag{56}$$

and

$$X\nu = \nu\lambda \tag{57}$$

respectively, where $\lambda \in \mathbb{O}$ and ν denotes a 3×1 octonionic vector

$$\nu = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{58}$$

Remark. We observe that the phrase 'octonionic vector' is a slight abuse of terminology as $\mathbb{O}^3$ is not a vector space. However as our meaning is clear we will continue to use this nomenclature.

3. *Orthogonality of Eigenvectors.*

We will now introduce an asymmetric notion of orthogonal eigenvectors.

Definition. *Let ν and ω be two octonionic eigenvectors. We will say that ω is orthogonal to ν if*

$$(\nu\nu^*\Phi)\omega = 0. \tag{59}$$

Remark. This is an asymmetric relation.

The vectors $\{\nu, \omega\}$ are *orthonormal* if in addition

$$\nu^* \Phi \nu = 1 = \omega^* \Phi \omega. \tag{60}$$

We observe here that if

$$\nu \nu^* \Phi = \begin{bmatrix} |v_1|^2 & v_1 \overline{v_3} & v_1 \overline{v_2} \\ v_2 \overline{v_1} & v_2 \overline{v_3} & |v_2|^2 \\ v_3 \overline{v_1} & |v_3|^2 & v_3 \overline{v_2} \end{bmatrix},$$

then

$$\nu^* \Phi \nu = |v_1|^2 + \overline{v_2} v_3 + \overline{v_3} v_2. \tag{61}$$

Therefore if ν and ω lie in the same associative subalgebra, then equation (59) is equivalent to

$$\nu^* \Phi \omega = \overline{v_1} \omega_1 + \overline{v_2} \omega_3 + \overline{v_3} \omega_2 = 0.$$

If we assume that the eigenvalues of X are real then the left eigenvalue problem is equivalent to the right eigenvalue problem.

4. *The real eigenvalue problem.*

We will now consider the case of $\lambda \in \mathbb{R}$ and show that such matrices, X do admit real eigenvalues. A (real) eigenvalue λ of a Φ-Hermitian matrix X does not in general satisfy the characteristic equation

$$X^3 - \big(tr(X)\big) X^2 + \sigma(X) X - \big(det(X)\big) I = 0. \tag{62}$$

To see this consider the eigenvalue equation (56) with $\lambda \in \mathbb{R}$ and ν as in expression (58). Explicit computation yields

$$(\lambda - a) x = wy + \overline{v} z, \tag{63}$$

$$(\lambda - \overline{u}) y = vx + bz, \tag{64}$$

$$(\lambda - u) z = \overline{w} x + cy. \tag{65}$$

From equations (63) and (64) and the fact that $(\lambda - a) \in \mathbb{R}$ we have

$$\begin{aligned} (\lambda - a)(\lambda - \overline{u}) y &= (\lambda - a)(vx + bz) \\ &= (\lambda - a) bz + v(wy + \overline{v} z) \end{aligned}$$

which implies

$$\big((\lambda - a) b + |v|^2\big) z = \big((\lambda - a)(\lambda - \overline{u}) y - v(wy)\big). \tag{66}$$

From equations (63) and (65) we also find in a similar manner that

$$\begin{aligned} (\lambda - a)(\lambda - u) z &= (\lambda - a)(\overline{w} x + cy) \\ &= \overline{w}(wy + \overline{v} z) + (\lambda - a) cy \end{aligned}$$

which implies

$$\big((\lambda - a) c + |w|^2\big) y = (\lambda - a)(\lambda - u) z - \overline{w}(\overline{v} z). \tag{67}$$

Using equations (67) then (66) we find

$$\big((\lambda - a) b + |v|^2\big)\big((\lambda - a) c + |w|^2\big) y = \big((\lambda - a) b + |v|^2\big)\big((\lambda - a)(\lambda - u) z - \overline{w}(\overline{v} z)\big)$$

$$= (\lambda - a)(\lambda - u)\big((\lambda - a)(\lambda - \overline{u})y - v(wy)\big) - \overline{w}\big(\overline{v}\big((\lambda - a)(\lambda - \overline{u})y - v(wy)\big)\big).$$

Assuming first that $\lambda \neq a$ this implies that

$$0 = (\lambda - a)(\lambda - u)(\lambda - \overline{u})y - (\lambda - u)\big(v(wy)\big) - \overline{w}\big(\overline{v}\big((\lambda - \overline{u})y\big)\big) - (\lambda - a)bcy - b|w|^2y - c|v|^2y. \tag{68}$$

Thus expanding $\big(det(\lambda I - X)\big)y$ we have

$$\big(det(\lambda I - X)\big)y = \big((\lambda - a)(\lambda - u)(\lambda - \overline{u}) - b|w|^2 - c|v|^2 - (\lambda - a)bc - 2Re((\lambda - u)vw)\big)y.$$

Finally, comparing the expansion with equation (68) we find that

$$\begin{aligned} \big(det(\lambda I - X)\big)y &= \big(\lambda^3 - tr(X)\lambda^2 + \sigma(X)\lambda - det(X)\big)y \\ &= -2Re\big((\lambda - u)vw)\big)y + (\lambda - u)\big(v(wy)\big) + \overline{w}\big(\overline{v}\big((\lambda - \overline{u})y\big)\big). \end{aligned} \tag{69}$$

Remark. In obtaining equation (68) we have used the relation

$$|v|^2|w|^2y = \overline{w}(\overline{v}(v(wy))).$$

Proof. Considering the associative subalgebra generated by v and wy we have

$$\overline{v}(v(wy)) = \overline{v}v(wy) = |v|^2(wy).$$

Similarly considering the associative subalgebra generated by w and y we have

$$\overline{w}(|v|^2wy) = |v|^2|w|^2y.$$

We now consider the case $\lambda = a$. In this case it is clear from equation (63) that we have

$$|v|^2z = -v(wy), \tag{70}$$

$$|w|^2y = -\overline{w}(\overline{v}z), \tag{71}$$

$$wy = -\overline{v}z. \tag{72}$$

Thus using equations (71), (64), (65) and (70) in that order we find

$$\begin{aligned} det(\lambda I - X)y &= -b|w|^2y - c|v|^2y - 2Re\big((\lambda - u)vw\big)y \\ &= b\overline{w}(\overline{v}z) - c|v|^2y - 2Re\big((\lambda - u)vw\big)y \\ &= \overline{w}\big(\overline{v}\big((\lambda - \overline{u})y\big)\big) - \overline{w}|v|^2x - c|v|^2y - 2Re\big((\lambda - u)vw\big)y \\ &= \overline{w}\big(\overline{v}\big((\lambda - \overline{u})y\big)\big) - |v|^2(\lambda - u)z - 2Re\big((\lambda - u)vw\big)y \\ &= \overline{w}\big(\overline{v}\big((\lambda - \overline{u})y\big)\big) + (\lambda - u)\big(v(wy)\big) - 2Re\big((\lambda - u)vw\big)y. \end{aligned} \tag{73}$$

Thus we see that equation (69) is satisfied in the case $\lambda = a$.

Consider equation (69), if v, $\lambda - u$ and w associate the right hand side vanishes and λ does indeed satisfy the characteristic equation; this will not happen in general. However since the left hand side of (69) is a real multiple of y, this must also be true of the right hand side, so that

$$-2Re\big((\lambda - u)vw)\big)y + (\lambda - u)\big(v(wy)\big) + \overline{w}\big(\overline{v}\big((\lambda - \overline{u})y\big)\big) = ry \tag{74}$$

(where $r \in \mathbb{R}$) which can be solved to yield a quadratic equation for r as well as constraints on y.

Lemma 5.1. *The real eigenvalues of the 3×3 octonionic Φ-Hermitian matrix X satisfy the modified characteristic equation*

$$det(\lambda I - X) = \lambda^3 - (tr(X))\lambda^2 + \sigma(X)\lambda - det(X) = r, \tag{75}$$

where r is either of two roots of

$$r^2 + 4\Psi(v, -u, w)r - |[v, -u, w]|^2 = 0. \tag{76}$$

The octonions u, v and w are components of X and Ψ is defined by

$$\Psi(v, -u, w) = \frac{1}{2}Re([v, -\overline{u}]w) \tag{77}$$

where $[v, u] = vu - uv$ denotes the standard commutator of two elements u and v.

Proof. Equation (74) is identical in form to the equation (60) in [10] upon substituting v, $\lambda - u$, w and y for a, b, c and z respectively. In [10] Dray and Manogue solved this equation for real r and octonionic y given generic octonions u, v and w using *Mathematica* to obtain the equation

$$r^2 + 4\Psi(v, \lambda - u, w)r - |[v, \lambda - u, w]|^2 = 0. \tag{78}$$

But

$$\begin{aligned} \Psi(v, \lambda - u, w) &= \frac{1}{2}Re\big((v(\lambda - \overline{u}))w\big) - \big((\lambda - \overline{u})v)w\big) \\ &= \frac{1}{2}Re\big((v(-\overline{u}))w - ((-\overline{u})v)w\big) \\ &= \Psi(v, -u, w) \end{aligned}$$

and

$$\begin{aligned} [v, \lambda - u, w] &= \big(v(\lambda - u)\big)w - v\big((\lambda - u)w\big) \\ &= \big(v(-u)\big)w - v(-uw) \\ &= [v, -u, w]. \end{aligned}$$

Therefore we obtain equation (76).

We note several interesting properties of these results. If X is in fact complex, then due to commutativity and associativity the only solution of (76) is $r = 0$ and we recover the usual characteristic equation with a unique set of 3 (real) eigenvalues. If X is quaternionic, then one solution of equation (76) is $r = 0$ leading to the standard set of 3 real eigenvalues and their corresponding quaternionic eigenvectors. However unless u, v, w involve only two independent imaginary quaternionic directions (in which case $\Psi(v, -u, w) = 0 = [v, u, w]$), there will also be a non-zero solution for r, leading to a second set of 3 real eigenvalues. Finally if X is octonionic (so that in particular $[v, u, w] \neq 0$), then there are two distinct non-zero solutions for r and hence two different sets of real eigenvalues, with the corresponding eigenvectors.

5. *The Φ-Hermitian Matrix Eigenvalue Problem.*

We will now show how to construct left eigenmatrices $\mathfrak{V}$ of X, i.e. matrices $\mathfrak{V}$ which satisfy the following equation

$$X * \mathfrak{V} = \lambda \mathfrak{V} \tag{79}$$

where λ is a real eigenvalue satisfying the standard characteristic equation

$$det(\lambda I - X) = \lambda^3 - \big(tr(X)\big)\lambda^2 + \sigma(X)\lambda - \big(det(X)\big)I = 0. \tag{80}$$

From the definitions of the determinant, the Jordan multiplication $*$ and the Freudenthal product $\circ$ given in section 1.4 of Chapter Three we have for real λ satisfying equation (80)

$$(X - \lambda I) * \big((X - \lambda I) \circ (X - \lambda I)\big) = (det(X - \lambda I))I = 0. \tag{81}$$

Setting

$$\mathfrak{L}_\lambda = (X - \lambda I) \circ (X - \lambda I), \tag{82}$$

equation (81) can now be rewritten as

$$(X - \lambda I) * \mathfrak{L}_\lambda = 0, \tag{83}$$

i.e. $\mathfrak{L}_\lambda$ is a solution of the left eigenmatrix problem.

6 . *Conclusion.*

We will now summarise the main results of this chapter.

If X is a 3 by 3 Φ-Hermitian octonionic matrix the solutions of the real eigenvalue equation

$$X\nu = \lambda\nu, \tag{56}$$

where $\nu \in \mathbb{O}^3$ and $\lambda \in \mathbb{R}$, satisfy a modified characteristic equation of the form

$$det(\lambda I - X) = \lambda^3 - tr(X)\lambda^2 + \sigma(X)\lambda - det(X) = r. \tag{75}$$

The real number r is a root of the quadratic equation

$$r^2 + 4\Psi(v, -u, w)r - |[v, -u, w]|^2 = 0 \tag{76}$$

with u, v and w components of X and

$$\Psi(v, -u, w) = \frac{1}{2} Re([v, -\overline{u}]w), \tag{77}$$

where $[v, u] = vu - uv$ denotes the standard commutator of two octonions u and v and $[v, u, w]$ denotes the associator of three octonions u, v and w. There are three real solutions for λ corresponding to each solution for r (which labels the families). If the roots r_1 and r_2 are the same, then over the real octonions $\Psi = 0 = [v, -u, w]$, which forces X to be quaternionic (and $r_1 = 0 = r_2$).

CHAPTER SIX.

COLLARS IN PU(2,1), PSp(2,1) and Aut(J).

In this chapter we construct a collar theorem for complex hyperbolic 2-space, quaternionic hyperbolic 2-space and octonionic hyperbolic 2-space (each denoted below by $\mathbf{H}$). We then show for the complex and quaternionic cases that in the corresponding quotient manifolds $M = \mathbf{H}/\Gamma$ where Γ is an orientation preserving discrete torsionfree non-elementary subgroup of the corresponding isometry group, two disjoint geodesics of bounded length have non-intersecting collars and that the collars are disjoint from the canonical cusps associated to parabolic elements in Γ. Finally we determine a bound for the injectivity radius $i(M)$ in the complex case. Our results extend the collar theorems for Riemannian surfaces of genus > 1, hyperbolic 3-manifolds and 4-dimensional oriented hyperbolic manifolds of finite volume to orientation preserving discrete subgroups of the isometry groups of $\mathbf{H}^2_{\mathbb{C}}$, $\mathbf{H}^2_{\mathbb{H}}$, $\mathbf{H}^2_{\mathbb{O}}$ and their quotient manifolds.

This chapter was inspired by Ruth Kellerhals paper, "Collars in $PSL(2,\mathbb{H})$" (see [24]), in which she constructs embedded tubular neighbourhoods around short simple closed geodesics in the 4-dimensional oriented hyperbolic manifolds of finite volume $M = \mathbf{H}^4_{\mathbb{R}}/\Gamma$ (where Γ is an orientation preserving discrete torsionfree non-elementary subgroup of $Isom(\mathbf{H}^4_{\mathbb{R}})$ which can be identified with the group $PSL(2,\mathbb{H})$ of Clifford matrices with quaternion coefficients) whose collar width depends on the length of the simple closed geodesic only. She also demonstrates that two non-intersecting short geodesics have disjoint collars and that the constructed collars fail to intersect the canonical cusps associated to parabolic elements in Γ. Finally she proves bounds for the injectivity radius and the number of simple closed geodesics in M.

To be specific Kellerhals principal result is the following theorem in which she presents a non-trivial lower bound for the radius r of a collar depending only on the length l of the corresponding simple closed geodesic γ.

Theorem. *Let $l_0 = (\sqrt{3}/4\pi)\log^2 2$. Then each simple closed geodesic γ in M of length $l < l_0$ has a collar $T_\gamma(r)$ of radius r satisfying*

$$\cosh(2r) = \frac{1-3k}{k}$$

where

$$k = \cosh\sqrt{\frac{4\pi l}{\sqrt{3}}} - 1.$$

Here a **collar** $T_\gamma(r)$ around γ embedded in M of collar width $r \geq 0$ is defined to be the set of points

$$T_\gamma(r) = \{p \in M | dist(p,\gamma) < r\}$$

that is homeomorphic to $\gamma \times B^3$ where B^3 denotes a 3-dimensional ball of radius r. A collar is **precisely invariant** if its translates under the group G form a disjoint collection. She goes on to prove that the volume $vol_4(T_\gamma(r))$ of the collar $T_\gamma(r)$ is a strictly decreasing function of the length l of γ.

These results can be contrasted with our principal result for the complex hyperbolic case (stated below) which is a generalisation of the above theorem to 2-dimensional oriented complex hyperbolic manifolds $M = \mathbf{H}^2_{\mathbb{C}}/\Gamma$ where Γ is an orientation preserving discrete torsionfree non-elementary subgroup of $Isom(\mathbf{H}^2_{\mathbb{C}})$ which can be identified with an appropriate subgroup of $PU(2,1)$. The quaternionic and octonionic hyperbolic collar theorems are very similar.

Theorem 6.1. *Let the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{C}}$ be the axis of a loxodromic map V (with complex dilation factor $\lambda e^{i\theta}$) represented by an element of $PU(2,1)$. Let $N = |\lambda e^{i\theta} - 1| + |\lambda^{-1}e^{-i\theta} - 1|$ and suppose $N \leq 1/2$. Let U be an element of $PU(2,1)$ not fixing o or ∞ represented by*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

and satisfying

$$N(1+|aj|^{1/2}) \geq 1.$$

Then the r-neighbourhood of γ is disjoint from its images under elements U of $PU(2,1)$ when

$$\cosh(r) = \frac{1-N}{N}.$$

Here $|aj| = |[U(\omega_1), \omega_1, \omega_2, U(\omega_2)]|$ (see page 8 of [21]) where ω_1 and ω_2 are the fixed points of V.

Corollary 6.1. *Let V be a complex dilation in $PU(2,1)$ with axis γ and complex dilation factor $\lambda e^{i\theta}$. Let $N = |\lambda e^{i\theta} - 1| + |\lambda^{-1}e^{-i\theta} - 1|$ and suppose $N \leq 1/2$. Let $U \in PU(2,1)$ so that $\langle U, V\rangle$ is discrete. Then in the manifold $M = \mathbf{H}^2_{\mathbb{C}}/\langle U, V\rangle$ the simple closed geodesic $\gamma' = \gamma/\langle V\rangle$ corresponding to $\gamma \subset \mathbf{H}^2_{\mathbb{C}}$ has a precisely invariant collar $T(r)$ of width r where r is such that*

$$\cosh(r) = \frac{1-N}{N}.$$

First we will present some background material on complex hyperbolic geometry.

1. *Complex Hyperbolic Geometry.*

The majority of the material in this section is taken from [16], [21], [30], [31] and [32].

1.1 Models of Complex Hyperbolic Space.

Let $\mathbb{C}^{n,1}$ denote a complex vector space of dimension $n+1$ equipped with an indefinite Hermitian form $\langle .,. \rangle$ of signature $(n,1)$. This means that $\langle .,. \rangle$ is given by a non-singular $(n+1) \times (n+1)$ Hermitian matrix with n positive eigenvalues and one negative eigenvalue. A vector $\mathbf{z}$ in $\mathbb{C}^{n,1}$ is defined as negative, null or positive according as $\langle \mathbf{z}, \mathbf{z}\rangle$ is negative, null or positive. Complex hyperbolic n-space $\mathbf{H}^n_{\mathbb{C}}$ is defined as the complex projectivisation of the negative vectors in $\mathbb{C}^{n,1}$.

Restricting to the case $n = 2$, $\mathbb{C}^{2,1}$ is a copy of the complex vector space $\mathbb{C}^3$ equipped with say the first Hermitian form, a specific case of the Hermitian form of signature $(2,1)$. It is defined to be

$$\langle \mathbf{z}, \mathbf{w}\rangle_1 = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3$$

and is given by the Hermitian matrix J_1 where

$$J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Using $\langle .,. \rangle_1$, $\mathbf{H}^2_{\mathbb{C}}$ and its boundary are respectively the projective images in the projective complex plane $P\mathbb{C}^2$ of the sets

$$V = \{\mathbf{z} \in \mathbb{C}^{2,1} \,|\, |z_1|^2 + |z_2|^2 - |z_3|^2 < 0\},$$

$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} - \{[0,0,0]\} \,|\, |z_1|^2 + |z_2|^2 - |z_3|^2 = 0\}.$$

Since a negative vector $\mathbf{z}$ must have $z_3 \neq 0$, V may be identified with the unit ball in $\mathbb{C}^2$ by normalising the last coordinate and then ignoring it. We observe that the first Hermitian form as described above is best adapted to the unit ball model as setting $z_3 = 1$ in the condition $|z_1|^2 + |z_2|^2 - |z_3|^2 \leq 0$ describes the unit ball in $\mathbb{C}^2$. To be precise we define the map $\pi : \mathbb{C}^{2,1} - \{[0,0,0]\} \mapsto P\mathbb{C}^2$ to be the canonical projection assigning to $\mathbf{z} \in \mathbb{C}^{2,1} - \{[0,0,0]\}$ the complex line through the origin and $\mathbf{z}$. When $z_3 \neq 0$ (in particular on $V \cup V_0$), π is given by

$$\pi : [z_1, z_2, z_3] = [z_1/z_3, z_2/z_3].$$

It maps V and V_0 (excluding the zero element) respectively to the open unit ball B^2 and its boundary the unit sphere S^3 (3 real dimensions) in $\mathbb{C}^2$. Given a point z of $\overline{B^2}$ we call a $\pi-preimage$ a lift of z. The lift of form $(z_1, z_2, 1)$ is called the *standard lift.*

The second Hermitian form for $\mathbb{C}^{2,1}$ is defined to be

$$\langle \mathbf{z}, \mathbf{w} \rangle_2 = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1$$

and is given by the Hermitian matrix J_2 where

$$J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Using $\langle .,. \rangle_2$, $\mathbf{H}^2_{\mathbb{C}}$ and its boundary are respectively the projective images in the projective complex plane $P\mathbb{C}^2$ of the sets

$$V = \{\mathbf{z} \in \mathbb{C}^{2,1} | z_1\overline{z_3} + |z_2|^2 + z_3\overline{z_1} < 0\},$$

$$V_0 = \{\mathbf{z} \in \mathbb{C}^{2,1} - \{[0,0,0]\} | z_1\overline{z_3} + |z_2|^2 + z_3\overline{z_1} = 0\}.$$

This forms the *Siegel domain model* $\mathfrak{S}_2$ of $\mathbf{H}^2_{\mathbb{C}}$. The boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ of $\mathbf{H}^2_{\mathbb{C}}$ is a paraboloid defined by

$$z_1\overline{z_3} + |z_2|^2 + z_3\overline{z_1} = 0$$

together with a distinguished point $[1,0,0] \in \mathbb{C}^3$. This point is called the *point at infinity* and is denoted ∞.

The upper half plane model of 1 dimensional complex hyperbolic space (the hyperbolic plane) is generalised by the Siegel model $\mathfrak{S}_n$ in higher dimensions. In horospherical coordinates (ζ, v, u) the Siegel domain of complex dimension n is $\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$ a subset of $\mathbb{C}^n$.

Definition. *The Bergman distance ρ between points z,w in the Siegel domain is given by*

$$\cosh^2(\rho(z,w)/2) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.$$

For the ball model and the Siegel domain model one can calculate the distance between points z and w by substituting their standard lifts z' and w' into the above formula.

We observe that the boundary of the Siegel domain is the one point compactification of the Heisenberg space $\mathfrak{N}_n$, i.e. $\partial\mathfrak{S}_n = \mathfrak{N}_n \cup \{\infty\}$. The Siegel domain embeds in complex projective n-space in such a way that restricting for example to the case $n = 2$, a homeomorphism of $\mathfrak{S}_2$ onto the projection to $P\mathbb{C}^2$ of the subset $V \cup V_0$ of $\mathbb{C}^{2,1}$ maps (ζ, v, u) to

$$\begin{bmatrix} -(|\zeta|^2 + u - iv) \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}$$

and ∞ to

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Here

$$\sqrt{2}\zeta = z_2 \in \mathbb{C},$$

$$v = Im(z_1) \in \mathbb{R},$$

$$u = -Re(z_1) - |z_2|^2/2 \in \mathbb{R}_+$$

(given $z_3 = 1$) . We observe that $\partial\mathbf{H}^2_{\mathbb{C}}$ can be represented by the set of coordinates $\{(\zeta, v)\} \cup \{\infty\}$.

1.2 The Heisenberg Group.

Definition. *The Heisenberg group* $\mathfrak{N}_2$ *corresponding to the Siegel domain* $\mathfrak{S}_2$ *of complex dimension 2 is defined as the set of pairs* $(\zeta, v) \in \mathbb{C} \times \mathbb{R}$ *with the group law*

$$(\zeta_1, v_1) \circ (\zeta_2, v_2) = \big(\zeta_1 + \zeta_2, v_1 + v_2 + 2Im(\overline{\zeta_2}\zeta_1)\big)$$

where $\circ$ *denotes the group product. The inverse of* (ζ, v) *is*

$$(\zeta, v)^{-1} = (-\zeta, -v).$$

We observe the inverse of a product is the product of inverses with the order reversed. The Heisenberg group is 2-step nilpotent (and therefore non-commutative). In order to see this we note that

$$(\zeta_1, v_1) \circ (\zeta_2, v_2) \circ (-\zeta_1, -v_1) \circ (-\zeta_2, -v_2) = \big(0, 4Im(\overline{\zeta_2}\zeta_1)\big),$$

i.e. translating around a horizontal square gives a vertical translation. Thus using stereographic projection we can identify $\partial \mathbf{H}^2_{\mathbb{C}}$ with the one point compactification $\overline{\mathfrak{N}_2}$ of $\mathfrak{N}_2$. Projection onto the first factor ζ of (ζ, ν) is called vertical projection. One can envisage ζ as corresponding to the "horizontal" xy-plane and ν to the "vertical" z-axis of Euclidean 3-space.

The *Heisenberg norm* assigns to (ζ, v) the non-negative real number

$$|\zeta, v|_0 = \left(||\zeta||^4 + v^2\right)^{1/4} = \left|||\zeta||^2 - iv\right|^{1/2},$$

where $||\zeta||^2 = \langle\langle \zeta, \zeta \rangle\rangle = |\zeta|^2$. This enables us to define the Cygan metric ρ_0 on the Heisenberg group.

$$\begin{aligned}\rho_0\big((\zeta_1, v_1)(\zeta_2, v_2)\big) &= |\zeta_1 - \zeta_2, v_1 - v_2 + 2Im\langle\langle \zeta_1, \zeta_2 \rangle\rangle|_0 \\ &= \left|||\zeta_1 - \zeta_2||^2 + |u_1 - u_2| + iv_1 - iv_2 + 2iIm\langle\langle \zeta_1, \zeta_2 \rangle\rangle\right|^{1/2} \\ &= |(\zeta_1, v_1)^{-1} \circ (\zeta_2, v_2)|_0.\end{aligned}$$

The Cygan metric can also be expressed explicitly in terms of the Hermitian product on $P\mathbb{C}^n$ as follows. Let $\mathbf{x}$, $\mathbf{y} \in P\mathbb{C}^n$ correspond to points in $\partial \mathbf{H}^n_{\mathbb{C}}$, then

$$\rho_0(\mathbf{x}, \mathbf{y}) = |\langle \mathbf{x}, \mathbf{y} \rangle|^{1/2}.$$

The Cygan metric is a natural metric intrinsic to $\mathfrak{N}_2$ but is not intrinsic to $\mathfrak{S}_2$ as it depends on the choice of $q_\infty \in \partial \mathbf{H}^2_{\mathbb{C}}$.

We will now define the group of *Heisenberg similarities* to be the semi-direct product of $\big(\mathbb{R}_+ \times U(1)\big)$ and $\mathfrak{N}_2$ acting on the Heisenberg group (now viewed as a Heisenberg space).

1) Heisenberg Translations

The Heisenberg group acts on itself by (left) *Heisenberg translations* $T_{\zeta', v'}$. This action is described by

$$T_{\zeta', v'}(\zeta, v) = (\zeta + \zeta', v + v' + 2Im\langle\langle \zeta, \zeta' \rangle\rangle),$$

where $\langle\langle ., . \rangle\rangle$ denotes the standard positive-definite Hermitian form on $\mathbb{C}$. Heisenberg translations by $(0, v')$ for $v' \in \mathbb{R}$ where 0 is the origin in $\mathbb{C}$ are called *vertical translations*. We observe that Heisenberg translations are ordinary translations in the horizontal direction and shears in the vertical direction.

2) Heisenberg Rotations

The unitary group $U(1)$, i.e. $\{A \in \mathbb{C} | A\overline{A} = 1\}$ acts on Heisenberg space by *Heisenberg rotations*. Let A be an element of $U(1)$, then A determines a Heisenberg rotation R_A

$$R_A(\zeta, v) = (A\zeta, t).$$

All other Heisenberg rotations may be obtained from these by conjugating by a Heisenberg translation. Heisenberg translations and rotations are isometries with respect to the Cygan metric and form the Heisenberg isometry group. The central elements in the Heisenberg isometry group are precisely the vertical translations.

3) Heisenberg Inversions

Heisenberg inversions are the restriction to Heisenberg space of the antiholomorphic isometries of complex space. The simplest example is

$$\iota(\zeta, v) = (\overline{\zeta}, -v).$$

4) Heisenberg Dilations

Heisenberg dilations d_k act by positive scalars k:

$$d_k(\zeta, v) = (k\zeta, k^2 v).$$

Heisenberg translations, rotations, inversions and dilations generate the group of Heisenberg similarities. They extend trivially to the compactification $\overline{\mathfrak{N}_2}$ of $\mathfrak{N}_2$ and represent transformations in $Isom(\mathbf{H}^2_{\mathbb{C}})$ acting on $\partial\mathbf{H}^2_{\mathbb{C}}$.

1.3 $PU(2,1)$.

The isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is generated by $PU(2,1)$, the projective unitary group of Hermitian forms of signature (2,1) and the anti-holomorphic transformations (complex conjugation is sufficient for this purpose). $SU(2,1)$ is the group of determinant 1, complex linear maps which preserve the Hermitian form of signature (2,1). The projectivisation map $SU(2,1) \to PU(2,1)$ is 3 to 1. Elements of $PU(2,1)$ the automorphism group of $\mathbb{C}^{2,1}$ act transitively on $\mathbf{H}^2_{\mathbb{C}}$ by complex projective transformations and the stabilizer of a point in $\mathbf{H}^2_{\mathbb{C}}$ under $PU(2,1)$ is conjugate to $U(2)$ on $\mathbf{H}^2_{\mathbb{C}}$ by complex projection. We observe that the Heisenberg similarities correspond to a subgroup of $PU(2,1)$ stabilising the point at ∞.

Elements of $PU(2,1)$ fall into three distinct classes:

1) A *parabolic* element has exactly one fixed point which is on the boundary. Conjugating if necessary we can always assume that this is the distinguished point at infinity q_∞. A parabolic element can be imagined as corresponding to a rotation around a boundary point.
2) A *loxodromic* element has exactly 2 fixed points both on the boundary and can be imagined as corresponding to a rotation about a point which lies outside the closure of the unit ball.
3) An *elliptic* element has at least one fixed point within the unit ball and may have others on the boundary.

We observe that an elliptic element is *regular elliptic* if its eigenvalues are all distinct. We also observe that there are two types of parabolic elements:

i) If a parabolic element may be written as an element of $U(2,1)$ with 1 as its only eigenvalue, then it is said to be *unipotent* or *pure parabolic.* The group of pure parabolic maps fixing a given point is isomorphic to $\mathfrak{N}_2$ and once the fixed point is specified are Heisenberg translations.

ii) If a parabolic element is not unipotent it is said to be *ellipto-parabolic* or *screw-parabolic.* An ellipto-parabolic element preserves a unique complex geodesic on which it acts as a parabolic element of $PU(2,1)$.

1.4 Special structures on Complex Hyperbolic Space.

Real hyperbolic space $\mathbf{H}^2_{\mathbb{R}}$ is contained as a totally geodesic submanifold in $\mathbf{H}^2_{\mathbb{C}}$ in 2 distinct ways. *Complex lines* in $\mathbf{H}^2_{\mathbb{C}}$ are complex geodesics represented by $\mathbf{H}^1_{\mathbb{C}} \subset \mathbf{H}^2_{\mathbb{C}}$ and *Lagrangian planes* in $\mathbf{H}^2_{\mathbb{C}}$ are totally geodesic 2-planes represented by $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$. Each of these totally geodesic submanifolds is a model of the real hyperbolic plane. All totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$ are either complex linear or Lagrangian (see below). We observe that there are no totally geodesic real hypersurfaces. This makes it harder to construct polyhedra (for example fundamental polyhedra for discrete groups of complex hyperbolic isometries). To get round this problem we use *bisectors* which are minimal hypersurfaces of cohomogeneity one.

A discrete subgroup of $PU(2,1)$ preserving a complex line is called $\mathbb{C}$-*Fuchsian* and is isomorphic to a subgroup of $P\big(U(1)\times U(1,1)\big) \subset PU(2,1)$. A discrete subgroup of $PU(2,1)$ preserving a Lagrangian plane is called $\mathbb{R}$-*Fuchsian* and is isomorphic to a subgroup of $SO(2,1)$ included in $PU(2,1)$ by the projectivisation of the obvious inclusion $SO(2,1) \subset SU(2,1)$.

In the unit ball model a complex geodesic is the intersection of a complex line with $\mathbf{H}^2_{\mathbb{C}}$. Given two points z' and w' in the closure of $\mathbf{H}^2_{\mathbb{C}}$, the complex geodesic L containing these points is generated by lifting z' and w' to z and w respectively and then taking L to be the complex span of these two lifted points. L' is therefore the projectivisation of L, a projective subspace of complex dimension 1. L' is unique by construction and as it has complex dimension 1 can be referred to as a complex line. We observe that consideration of dimensions shows that two complex geodesics are either identical, disjoint or meet in a single point. It is impossible for two complex geodesics to meet in a real line.

Definition. *The polar vector associated to the complex geodesic L' is the unique (up to non-zero scalar multiplication), positive vector Hermitian orthogonal to both z and w. It is given by their Hermitian Cross Product. If z and w are two vectors in $\mathbb{C}^{2,1}$ then their Hermitian Cross Product with respect to the second Hermitian form is given by*

$$\begin{bmatrix} \overline{z}_1\overline{w}_2 - \overline{z}_2\overline{w}_1 \\ \overline{z}_3\overline{w}_1 - \overline{z}_1\overline{w}_3 \\ \overline{z}_2\overline{w}_3 - \overline{z}_3\overline{w}_2 \end{bmatrix}.$$

Therefore any positive vector $\mathbf{p}$ *defines a unique complex geodesic* $\mathbf{p}^\perp = \{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{p}\rangle = 0\}$.

Two complex geodesics that do not intersect inside complex hyperbolic space either intersect in the boundary in which case they are called parallel or asymptotic or are disjoint in which case they are called ultraparallel. A chain (or $\mathbb{C}$-circle) is the corresponding intersection of the complex geodesic with the boundary. In the unit ball model a complex geodesic is homeomorphic to a disc (complex lines in $\mathbb{C}^2$ intersect the 2-dimensional Siegel domain $\mathfrak{S}_2$ corresponding to $\mathbf{H}^2_{\mathbb{C}}$ in discs where the Bergman metric is the Poincaré metric). Therefore its intersection with the boundary of complex hyperbolic space is a geometrically round circle. As any two distinct points on the chain determine the corresponding complex geodesic there exists a bijection between chains and complex geodesics.

Consider a complex line passing through the point at infinity. By applying a suitable Heisenberg translation we may suppose that it also passes through the origin $o = (0,0) \in \mathfrak{N}_2$. Thus we have the complex geodesic spanned by

$$\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This intersects the finite part of the boundary in the vertical line $\{(0,v) | v \in \mathbb{R}\}$. By applying a Heisenberg translation we observe that any other complex line passing through infinity intersects the finite part of the boundary in the vertical line $\{(\zeta_0, v) | v \in \mathbb{R}\}$ for some fixed $\zeta_0 \in \mathbb{C}$. This is called an *infinite chain* or *infinite* $\mathbb{C}$-*circle*. The corresponding *vertical* complex geodesic is characterised by the set of points $\{(\zeta_0, v, u) \in \mathfrak{S}_2\}$ for the same fixed $\zeta_0 \in \mathbb{C}$ and is a copy of the hyperbolic upper half plane with coordinates $\{v + iu | v \in \mathbb{R}, u \in \mathbb{R}_+\}$.

Consider a complex line not passing through the point at infinity. The simplest example of such a line which intersects $\mathbf{H}^2_{\mathbb{C}}$ is the line spanned by

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This intersects the boundary of the Siegel domain in the circle $\{(e^{i\theta},0)|\theta \in [0,2\pi)\}$. By applying a Heisenberg dilation we observe that the circle $\{(r_0e^{i\theta},0)|\theta \in [0,2\pi)\}$ for any fixed $r_0 \in \mathbb{R}_+$ is also the boundary of a complex line. Furthermore by applying a Heisenberg translation by (x_0+iy_0,v_0) we observe that the most general complex line not passing through infinity intersects the boundary of the Siegel domain in the following ellipse whose vertical projection is a circle

$$\{(r_0e^{i\theta}+x_0+iy_0, v_0+2r_0y_0\cos(\theta)-2r_0x_0\sin(\theta))|\theta \in [0,2\pi)\}$$

for fixed $r_0 \in \mathbb{R}_+$ and $(x_0+iy_0,v_0) \in \mathfrak{N}_2$. It can be seen that the eccentricity of the ellipse increases with $|x_0+iy_0|$. This is called a *finite chain* or *finite* $\mathbb{C}$*-circle*.

If L is a complex line, then there exists a unique involution in $PU(2,1)$ whose fixed point set is L. This involution is called an inversion in L. This inversion acts on the boundary fixing the chain corresponding to L. The inversion is given by the standard inversion in a vector space. To be exact if $\mathbf{p}$ denotes the polar vector for L, then inversion in L is the unitary transformation of $\mathbb{C}^{2,1}$ defined by

$$\iota_L : \mathbf{z} \mapsto \mathbf{z} - 2\frac{\langle \mathbf{z},\mathbf{p}\rangle}{\langle \mathbf{p},\mathbf{p}\rangle}\mathbf{p}$$

as well as the corresponding element of $PU(2,1)$. Either $q_\infty \in \partial L$ (in which case ∂L corresponds to a linear subspace of $\partial\mathbf{H}^2_{\mathbb{C}}$) or ι_L maps q_∞ to a point in $\partial\mathbf{H}^2_{\mathbb{C}}$. In the latter case the *centre* of the chain ∂L is defined to be the image, $\iota_L(q_\infty) \in \partial\mathbf{H}^2_{\mathbb{C}}$. By applying a Heisenberg translation we may assume that ∂L is centred at the origin in $\partial\mathbf{H}^2_{\mathbb{C}}$.

A Lagrangian totally geodesic subspace in $\mathbf{H}^2_{\mathbb{C}}$ is $PU(2,1)$ equivalent to $\mathbf{H}^2_{\mathbb{R}}$ embedded as a real linear subspace. To be more precise let U be a subspace of the underlying real vector space $\mathbb{C}^{2,1}$. The subspace U is said to be Lagrangian if $\mathbb{J}(U)$ is orthogonal to U where orthogonality is determined under the standard nondegenerate real-valued symmetric bilinear form $Re\langle .,. \rangle$ and $\mathbb{J}$, (called a complex structure) acts on $\mathbb{C}^{2,1}$ as an anti-involution. The projectivisation of U is then a Lagrangian geodesic submanifold of complex hyperbolic 2-space to which the metric restricts yielding the Klein model of $\mathbf{H}^2_{\mathbb{R}}$.

An $\mathbb{R}$*-circle* is the intersection of a Lagrangian totally geodesic submanifold with $\partial\mathbf{H}^2_{\mathbb{C}}$. For example consider the Lagrangian subspace passing through $(0,0) \in \mathfrak{N}_2$ and ∞ which is fixed by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \end{bmatrix}.$$

Finite points in the boundary of the Siegel domain fixed by this involution have the form

$$\begin{bmatrix} -x^2 \\ \sqrt{2}x \\ 1 \end{bmatrix}$$

where $x \in \mathbb{R}$. Therefore the subset of $\mathfrak{N}_2$ given by $\{(x,0)|x \in \mathbb{R}\}$ is fixed by this inversion. By applying Heisenberg rotations we see that for any $\theta_0 \in [0,2\pi)$ the line $\{(xe^{i\theta_0},0)|x \in \mathbb{R}\}$ is also the finite part of the boundary of a Lagrangian plane containing $(0,0) \in \mathfrak{N}_2$ and ∞. By applying a Heisenberg translation by (x_0+iy_0,v_0) we find the general form for the boundary of a Lagrangian plane passing through ∞. It is

$$\{(xe^{i\theta_0}+x_0+iy_0, v_0+2xy_0\cos(\theta_0)-2xx_0\sin(\theta_0))|x \in \mathbb{R}\}$$

for fixed $\theta_0 \in [0,2\pi)$ and $(x_0+iy_0,v_0) \in \mathfrak{N}_2$. Observe that the gradient of the line increases with $|x_0+iy_0|$. This is called an *infinite* $\mathbb{R}$-circle.

Finite $\mathbb{R}$-circles do not pass through ∞. For example consider the *purely imaginary* $\mathbb{R}$-circle which forms the boundary of the Lagrangian subspace fixed by the following involution

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \overline{z_3} \\ \overline{z_2} \\ \overline{z_1} \end{bmatrix}.$$

Points in the boundary of the Siegel domain fixed by this involution have the form

$$\begin{bmatrix} -e^{2i\theta} \\ i\sqrt{2\cos(2\theta)}e^{i\theta} \\ 1 \end{bmatrix}$$

where $\theta \in [-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4]$. The values of θ are chosen to make $\cos(2\theta)$ non-negative. Thus this non-planar space curve is given by the following subset of $\mathfrak{N}_2$

$$\{p(\theta) = (i\sqrt{\cos(2\theta)}e^{i\theta}, -\sin(2\theta)) | \theta \in [-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4]\}.$$

Note that this $\mathbb{R}$-circle is connected in spite of the fact that the values of the parameter θ are contained in two disjoint intervals. In order to see this we observe that $p(-\pi/4) = p(3\pi/4) = (0,1)$ and $p(\pi/4) = p(5\pi/4) = (0,-1)$.

As in the case of a chain, an $\mathbb{R}$-circle R defines a unique inversion ι_{R} under which it is pointwise invariant. For instance the transformation $\iota_0[\zeta, v] = [\overline{\zeta}, -v]$ on the Heisenberg group is the inversion that fixes pointwise the $\mathbb{R}$-circle $Im(\zeta) = 0$.

2. *Collars in PU(2,1).*

Theorem 6.1. *Let the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{C}}$ be the axis of a loxodromic map V (with complex dilation factor $\lambda e^{i\theta}$) represented by an element of $PU(2,1)$. Let $N = |\lambda e^{i\theta} - 1| + |\lambda^{-1}e^{-i\theta} - 1|$ and suppose $N \leq 1/2$. Let U be an element of $PU(2,1)$ not fixing o or ∞ represented by*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

and satisfying

$$N(1 + |aj|^{1/2}) \geq 1.$$

Then the r-neighbourhood of γ is disjoint from its images under elements U of $PU(2,1)$ when

$$\cosh(r) = \frac{1-N}{N}.$$

Here $|aj| = |[U(\omega_1), \omega_1, \omega_2, U(\omega_2)]|$ (see page 8 of [21]) where ω_1 and ω_2 are the fixed points of V.

Corollary 6.1. *Let V be a complex dilation in $PU(2,1)$ with axis γ and complex dilation factor $\lambda e^{i\theta}$. Let $N = |\lambda e^{i\theta} - 1| + |\lambda^{-1}e^{-i\theta} - 1|$ and suppose $N \leq 1/2$. Let $U \in PU(2,1)$ so that $\langle U, V\rangle$ is discrete. Then in the manifold $M = \mathbf{H}^2_{\mathbb{C}}/\langle U, V\rangle$ the simple closed geodesic $\gamma' = \gamma/\langle V\rangle$ corresponding to $\gamma \subset \mathbf{H}^2_{\mathbb{C}}$ has a precisely invariant collar $T(r)$ of width r where r is such that*

$$\cosh(r) = \frac{1-N}{N}.$$

Proof. Let U, $V \in PU(2,1)$ and let $\lambda e^{i\theta}$ be the complex dilation factor of the loxodromic map V. Without loss of generality V is a dilation with fixed points

$$o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore let

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

Let the real geodesics γ_1 and γ_2 be the respective axes of the loxodromic maps V and UVU^{-1} with respective pairs of endpoints o, ∞ and $U(o)$, $U(\infty)$ all lying on the boundary of $\mathbf{H}^2_{\mathbb{C}}$. We note that using the second Hermitian form we have

$$\begin{aligned}[U(\infty), o, \infty, U(o)] &= \frac{\langle \infty, U(\infty)\rangle \langle U(o), o\rangle}{\langle U(o), U(\infty)\rangle \langle \infty, o\rangle} \\ &= \overline{g}c\end{aligned}$$

and

$$\begin{aligned}[U(\infty), \infty, o, U(o)] &= \frac{\langle o, U(\infty)\rangle \langle U(o), \infty\rangle}{\langle U(o), U(\infty)\rangle \langle o, \infty\rangle} \\ &= \overline{a}j.\end{aligned}$$

Parametrising points p_t and q_s on γ_1 and γ_2 respectively, by t, $s \in \mathbb{R}_+$ we have

$$p_t = \begin{bmatrix} -t \\ 0 \\ 1 \end{bmatrix} \in \gamma_1$$

and

$$q_s = \begin{bmatrix} -as + c \\ -ds + f \\ -gs + j \end{bmatrix} \in \gamma_2.$$

Let $d = d(s,t)$ denote the Bergman distance between the two points p_t and q_s on γ_1 and γ_2 respectively, then by direct calculation we find

$$\cosh^2(d/2) = \frac{|gst - tj - as + c|^2}{4st}.$$

On further calculation using the matrix identity $UU^{-1} = I$ and the inequality $A^2 + B^2 \geq 2AB$ where A, $B \in \mathbb{R}_+$, we see that

$$\begin{aligned}&\frac{|gst - tj - as + c|^2}{4st} \\ &= \frac{1}{4}(|g|^2 st + |c|^2/st + |a|^2 s/t + |j|^2 t/s + g\overline{c} + c\overline{g} + a\overline{j} + j\overline{a} + |h|^2 t + |b|^2/t + |d|^2 s + |f|^2/s) \\ &\geq \frac{1}{4}(2|cg| + 2|aj| + 2|hb| + 2|df| + g\overline{c} + c\overline{g} + a\overline{j} + j\overline{a}) \\ &\geq \frac{1}{4}(2|cg| + 2|aj| + 2|hb| + d\overline{f} + f\overline{d} + g\overline{c} + c\overline{g} + a\overline{j} + j\overline{a}) \\ &= \frac{1}{4}(2|cg| + 2|aj| + 2|hb| + 2) \\ &\geq |aj|.\end{aligned}$$

Here the $UU^{-1} = I$ derived identity $1 = a\bar{j} + b\bar{h} + c\bar{g}$ is used at the end of the above calculation. We observe that the penultimate line of the above set of relations is also greater than or equal to $|cg|$. Now a necessary condition for $\langle U, V\rangle$ to be a discrete and non-elementary group is $N(1 + |aj|^{1/2}) \geq 1$ (see [21], pages 2 and 3). Therefore if we let $\cosh(r) = \frac{1-N}{N}$ where $N = |\lambda e^{i\theta} - 1| + |\lambda^{-1}e^{-i\theta} - 1|$, then we have proved that the r-neighbourhood of γ_1 doesn't intersect any of its images under elements of $\langle U, V\rangle - Stab\{\gamma_1\}$. Thus the simple closed geodesic in the manifold $M = \mathbf{H}^2_{\mathbb{C}}/\langle U, V\rangle$ corresponding to $\gamma_1 \subset \mathbf{H}^2_{\mathbb{C}}$ has a precisely invariant collar $T(r)$ of width r where r is such that

$$\cosh(r) = \frac{1-N}{N}$$

and

$$N(1 + |aj|^{1/2}) \geq 1.$$

Remark. We recall that $|aj| = |[U(\omega_1), \omega_1, \omega_2, U(\omega_2)]|$ where ω_1 and ω_2 are the fixed points of V.

We can extend these results to collars embedded in hypercomplex hyperbolic manifolds.

3. *Collars in* $PSp(2,1)$.

The above proof together with the results of the main theorem of Chapter One (see page 8) applied to quaternionic hyperbolic 2-space yield the following theorem.

Theorem 6.2. *Let the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{H}}$ be the axis of a loxodromic map V with quaternionic dilation factor determined by λ, μ and ν (where $\lambda \in \mathbb{R}_+$ and μ and ν are unit quaternions). Let V be represented by an element of $PSp(2,1)$. Let $N = |\lambda\mu - 1| + 2|\mu\nu - 1| + |\lambda^{-1}\mu - 1|$ and suppose $N \leq 1/2$. Let U be an element of $PSp(2,1)$ not fixing o or ∞ represented by*

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

and satisfying

$$N(1 + |aj|^{1/2}) \geq 1.$$

Then the r-neighbourhood of γ is disjoint from its images under elements U of $PSp(2,1)$ when

$$\cosh(r) = \frac{1-N}{N}.$$

Here $|aj| = |[U(\omega_1), \omega_1, \omega_2, U(\omega_2)]|$ (see equation (7) of Chapter One) where ω_1 and ω_2 are the fixed points of V.

Corollary 6.2. *Let V be a quaternionic dilation in $PSp(2,1)$ with axis γ and quaternionic dilation factor determined by λ, μ and ν (where $\lambda \in \mathbb{R}_+$ and μ and ν are unit quaternions). Let*

$$N = |\lambda\mu - 1| + 2|\mu\nu - 1| + |\lambda^{-1}\mu - 1|$$

and suppose $N \leq 1/2$. Let $U \in PSp(2,1)$ so that $\langle U, V\rangle$ is discrete. Then in the manifold $M = \mathbf{H}^2_{\mathbb{H}}/\langle U, V\rangle$ the simple closed geodesic $\gamma' = \gamma/\langle V\rangle$ corresponding to $\gamma \subset \mathbf{H}^2_{\mathbb{H}}$ has a precisely invariant collar $T(r)$ of width r where r is such that

$$\cosh(r) = \frac{1-N}{N}.$$

4. *Collars in* $\mathrm{Aut}(J)$.

First we recall the principal result of Chapter Four.

Theorem. *Let U and V be elements of* $\mathrm{Aut}(J)$ *such that U doesn't fix C or B and $V = D_\lambda S_\mu$ where $\lambda > 1$ and μ is an imaginary unit octonion. Let*

$$N^{1/2} = |\lambda\mu - 1| + |\lambda^{-1}\overline{\mu} - 1|.$$

If $N^{1/2}({\mathbb{X}_0}^{1/4} + 1) < 1$ with $\mathbb{X}_0 = [C, U^{-1}(B), U^{-1}(C), B]$, then either U and V commute or the group $\langle U, V\rangle$ is not discrete.

Using this theorem the proof of Theorem 6.1 can be modified to apply to octonionic hyperbolic 2-space. This yields the following theorem.

Theorem 6.3. *Let the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{O}}$ be the axis of a loxodromic element $V = D_\lambda S_\mu$ of* $\mathrm{Aut}(J)$ *(where $\lambda > 1$ and μ is an imaginary unit octonion). Let $N^{1/2} = |\lambda\mu - 1| + |\lambda^{-1}\overline{\mu} - 1|$. Let U be an element of* $\mathrm{Aut}(J)$ *not fixing B and C and not commuting with V, represented according to the standard decomposition of such elements of* $\mathrm{Aut}(J)$ *by*

$$U = T_1 D\{S\} R T_2^{-1}$$

and satisfying

$$N^{1/2}({\mathbb{X}_0}^{1/4} + 1) \geq 1.$$

Then the r-neighbourhood of γ is disjoint from its images under elements U of $\mathrm{Aut}(J)$ *when*

$$\cosh^2(r) = {\mathbb{X}_0}^{1/2}.$$

Here $\mathbb{X}_0 = [C, U^{-1}(B), U^{-1}(C), B]$ where C and B are the fixed points of V.

Proof.

Let U, $V \in \mathrm{Aut}(J)$ where $V = D_\lambda S_\mu$ is a dilation with fixed points C and B and

$$U = T_1 D\{S\} R T_2^{-1}$$

(see section 1.6 of Chapter Three). Let the real geodesics γ_1 and γ_2 be the respective axes of the loxodromic maps V and $U^{-1}VU$ with respective pairs of endpoints C, B and $U^{-1}(C)$, $U^{-1}(B)$ all lying on the boundary of $\mathbf{H}^2_{\mathbb{O}}$. We note that

$$[C, U^{-1}(B), U^{-1}(C), B] = \frac{\langle U^{-1}(C)|C\rangle\langle B|U^{-1}(B)\rangle}{\langle B|C\rangle\langle U^{-1}(C)|U^{-1}(B)\rangle}. \tag{84a}$$

Parametrising points p_s and q_t on γ_1 and $\gamma_2 = U^{-1}(\gamma_1)$ respectively, by s and $t \in \mathbb{R}_+$ we have, using Allcock's associative triple notation

$$p_s = (0, 1, -s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s & s^2 \\ 0 & 1 & -s \end{bmatrix} \in \gamma_1$$

and

$$q_t = U^{-1}(0, 1, -t) \in \gamma_2.$$

Let $d = d(s,t)$ denote the Bergman distance between the two points p_s and q_t on γ_1 and γ_2 respectively, we recall from section 1.5 of Chapter Three that

$$\cosh^2(d/2) = \frac{|\langle X|Y\rangle|}{\langle X|X\rangle^{1/2}\langle Y|Y\rangle^{1/2}}. \tag{84b}$$

Let $U^{-1}(C) = p \in \partial\mathbf{H}^2_{\mathbb{O}}$ and $U^{-1}(B) = q \in \partial\mathbf{H}^2_{\mathbb{O}}$. Without loss of generality there exist translations $T_{\alpha,\beta}$ and $T_{\delta,\epsilon}$ such that

$$T_{\alpha,\beta} R T_{\delta,\epsilon} R(C) = p$$

and

$$T_{\alpha,\beta} R T_{\delta,\epsilon} R(B) = q.$$

Now $(T_{\alpha,\beta}RT_{\delta,\epsilon}R)^{-1}U^{-1}$ fixes C and B, therefore $(T_{\alpha,\beta}RT_{\delta,\epsilon}R)^{-1}U^{-1}$ fixes γ_1 and so

$$U^{-1}(\gamma_1) = T_{\alpha,\beta}RT_{\delta,\epsilon}R(\gamma_1).$$

Let $U' = T_{\alpha,\beta}RT_{\delta,\epsilon}R$. Direct calculation of $T_{\alpha,\beta}RT_{\delta,\epsilon}R(x,y,z)$ using the transformation formulae of section 1.6.1 of Chapter Three (and section 4 of [1]) together with Allcock's associative triples gives the following

$$\begin{aligned}
R(x,y,z) &= (x,-z,-y),\\
T_{\delta,\epsilon}R(x,y,z) &= (x - z\delta, -z, -y - x\overline{\delta} + z|\delta|^2/2 - z\epsilon),\\
RT_{\delta,\epsilon}R(x,y,z) &= (x - z\delta, y + x\overline{\delta} - z|\delta|^2/2 + z\epsilon, z) = (x', y', z'),\\
T_{\alpha,\beta}RT_{\delta,\epsilon}R(x,y,z) &= (x' + y'\alpha, y', z' - x'\overline{\alpha} - y'|\alpha|^2/2 + y'\beta).
\end{aligned}$$

Thus $T_{\alpha,\beta}RT_{\delta,\epsilon}R(x,y,z)$ has the general form

$$\begin{aligned}
&T_{\alpha,\beta}RT_{\delta,\epsilon}R(x,y,z) =\\
&\big((xa_1)a_2 + (yb_1)b_2 + (zc_1)c_2, (xd_1)d_2 + (ye_1)e_2 + (zf_1)f_2, (xg_1)g_2 + (yh_1)h_2 + (zj_1)j_2 + (zl_1)l_2\big)
\end{aligned}$$

where a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, j_i and $l_i \in \mathbb{O}$ for $i = 1,2$. Further calculation using the transformation formulae of section 1.6.1 of Chapter Three shows that there exist b, c, e, f, h and $j \in \mathbb{O}$ such that

$$U'(0,1,-t) = (b - ct, e - ft, h - jt)$$

where, as $t \in \mathbb{R}$ we have

$$b = b_1b_2, \quad c = c_1c_2,$$
$$e = e_1e_2, \quad f = f_1f_2,$$

and

$$h = h_1h_2, \quad j = j_1j_2 + l_1l_2.$$

Similarly there exist $d = d_1d_2$ and $g = g_1g_2 \in \mathbb{O}$ such that the following $U'U'^{-1} = I$ derived identities hold

$$|d|^2 + \overline{e}f + \overline{f}e = 0, \tag{84c}$$

$$|g|^2 + \overline{h}j + \overline{j}h = 0, \tag{84d}$$

$$g\overline{d} + d\overline{g} + j\overline{e} + e\overline{j} + h\overline{f} + f\overline{h} = 2, \tag{84e}$$

$$\overline{b}c + \overline{e}j + \overline{h}f + \overline{c}b + \overline{j}e + \overline{f}h = 2. \tag{84f}$$

Now $U'(C) = U'(0,1,0) = (b,e,h) \in \partial\mathbf{H}^2_{\mathbb{O}}$, therefore

$$|b|^2 + \overline{e}h + \overline{h}e = 0. \tag{84}$$

Similarly $U'(B) = U'(0,0,1) = (c,f,j) \in \partial\mathbf{H}^2_{\mathbb{O}}$, therefore

$$|c|^2 + \overline{f}j + \overline{j}f = 0. \tag{85}$$

Using equations (84c), (84d), (84), (85) followed by the inequality $A^2 + B^2 \geq 2AB$ (where A, $B \in \mathbb{R}_+$) and equation (84f) we have

$$\begin{aligned}
&\langle p_s | q_t \rangle\\
&= \langle (0,1,-s) | U^{-1}(0,1,-t) \rangle\\
&= \langle (0,1,-s) | T_{\alpha,\beta}RT_{\delta,\epsilon}R(0,1,-t) \rangle\\
&= s^2|e - tf|^2 + |h - tj|^2 - s(\overline{h} - t\overline{j})(e - tf) - s(\overline{e} - t\overline{f})(h - tj)\\
&= s^2t^2|f|^2 + |h|^2 - s(\overline{h}e + \overline{e}h) - st^2(\overline{j}f + \overline{f}j) - s^2t(\overline{e}f + \overline{f}e) - t(\overline{h}j + \overline{j}h)\\
&\quad + st(\overline{h}f + \overline{j}e + \overline{f}h + \overline{e}j) + s^2|e|^2 + t^2|j|^2\\
&\geq st(2|f||h| + 2|b||c| + 2|d||g| + 2|e||j| + \overline{h}f + \overline{f}h + \overline{j}e + \overline{e}j)\\
&\geq st(2|f||h| + 2|b||c| + 2|e||j| + 2)\\
&\geq 4st|f||h|.
\end{aligned}$$

Similarly we find

$$\begin{aligned}
&\langle U^{-1}(0,1,-t)|U^{-1}(0,1,-t)\rangle^{1/2} \\
&= |b|^2 - \bar{c}bt - \bar{b}ct + t^2|c|^2 + \bar{h}e - \bar{h}ft - \bar{j}et + t^2\bar{j}f + \bar{e}h - \bar{f}ht - \bar{e}jt + t^2\bar{f}j \\
&= -2Re(\bar{b}c + \bar{e}j + \bar{h}f)t.
\end{aligned} \tag{86}$$

Using equation (84a) by direct calculation we have

$$\begin{aligned}
[C, U^{-1}(B), U^{-1}(C), B] &= \frac{|f|^2|h|^2}{|b|^2|c|^2 + |e|^2|j|^2 + |f|^2|h|^2 + \bar{c}f\bar{h}b + \bar{c}j\bar{e}b + \bar{j}c\bar{b}e + \bar{j}f\bar{h}e + \bar{f}c\bar{b}h + \bar{f}j\bar{e}h} \\
&= \frac{|f|^2|h|^2}{|e\bar{j} + b\bar{c} + h\bar{f}|^2}.
\end{aligned} \tag{87}$$

Similarly using equations (84b) and (86) together with the above inequality for $\langle p_s|q_t\rangle$ followed by equation (87) we have

$$\begin{aligned}
\cosh^2(d(s,t)/2) &= \frac{|\langle p_s|q_t\rangle|}{\langle p_s|p_s\rangle^{1/2}\langle q_t|q_t\rangle^{1/2}} \\
&\geq \frac{4st|f||h|}{(-2s)\big(-2Re(\bar{b}c + \bar{e}j + \bar{h}f)t\big)} \\
&\geq \frac{4st|f||h|}{4st|\bar{b}c + \bar{e}j + \bar{h}f|} \\
&= [C, U^{-1}(B), U^{-1}(C), B]^{1/2}.
\end{aligned} \tag{88}$$

Now a necessary condition for $\langle U, V\rangle$ to be a discrete and non-elementary group is

$$N^{1/2}(\mathbb{X}_0{}^{1/4} + 1) \geq 1$$

where $\mathbb{X}_0 = [C, U^{-1}(B), U^{-1}(C), B]$ (see page 65, Chapter Four). Therefore if we let $\cosh^2(r) = \mathbb{X}_0{}^{1/2}$, then we have proved that the r-neighbourhood of γ_1 doesn't intersect any of its images under elements of $\langle U, V\rangle - Stab\{\gamma_1\}$. Thus the simple closed geodesic in the manifold $M = \mathbf{H}^2_{\mathbb{O}}/\langle U, V\rangle$ corresponding to $\gamma_1 \subset \mathbf{H}^2_{\mathbb{O}}$ has a precisely invariant collar $T(r)$ of width r where r is such that

$$\cosh^2(r) = \mathbb{X}_0{}^{1/2}$$

and

$$N^{1/2}(\mathbb{X}_0{}^{1/4} + 1) \geq 1.$$

(Here $N^{1/2} = |\lambda\mu - 1| + |\lambda^{-1}\bar{\mu} - 1|$.) Thus we have the following corollary.

Corollary 6.3. *Let V be an octonionic dilation in* Aut(J) *such that $V = D_\lambda S_\mu$ (where $\lambda > 1$ and μ is an imaginary unit octonion). Let $N^{1/2} = |\lambda\mu - 1| + |\lambda^{-1}\bar{\mu} - 1|$. Let $U \in$* Aut(J) *so that $\langle U, V\rangle$ is discrete. Then in the manifold $M = \mathbf{H}^2_{\mathbb{O}}/\langle U, V\rangle$ the simple closed geodesic $\gamma' = \gamma/\langle V\rangle$ corresponding to $\gamma \subset \mathbf{H}^2_{\mathbb{O}}$ has a precisely invariant collar $T(r)$ of width r where r is such that*

$$\cosh^2(r) = \mathbb{X}_0{}^{1/2}.$$

We will now consider some of the applications of the above collar theorems.

5. *Lengths of Simple Closed Geodesics.*

(i) **Complex hyperbolic 2-space.**

Let V and $U \in PU(2,1)$. Let V be a loxodromic map and U a general map such that $\langle U, V\rangle$ is discrete. Let $\lambda e^{i\theta}$ be the complex dilation factor of V where $\lambda \in \mathbb{R}_+$. Without loss of generality V is a dilation with fixed points

$$o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and can be represented as the following matrix

$$V = \begin{bmatrix} \lambda e^{-i\theta/3} & 0 & 0 \\ 0 & e^{2i\theta/3} & 0 \\ 0 & 0 & \lambda^{-1}e^{-i\theta/3} \end{bmatrix}.$$

Let the real geodesic γ be the axis of V. Let the points p_s on γ be parametrised by $s \in \mathbb{R}_+$, then we have

$$p_s = \begin{bmatrix} -s \\ 0 \\ 1 \end{bmatrix} \in \gamma.$$

The (Bergman) length l of the simple closed geodesic γ' in the complex hyperbolic manifold $\mathbf{H}^2_{\mathbb{C}}/\langle V, U\rangle$ corresponding to the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{C}}$ is equal to the distance between (without loss of generality) p_1 and $V(p_1)$, i.e.

$$\begin{aligned} \cosh^2(l/2) &= \cosh^2(\rho(p_1, V(p_1))/2) \\ &= \frac{(-\lambda^2-1)^2}{(-2)(-2\lambda^2)} \\ &= \frac{(-\lambda^2-1)^2}{4\lambda^2}. \end{aligned}$$

Therefore

$$\lambda = \exp(l/2). \tag{89}$$

(ii) **Quaternionic hyperbolic 2-space.**

Let V and $U \in PSp(2,1)$. Let V be a loxodromic map and U a general map such that $\langle U, V\rangle$ is discrete. Let λ be the magnitude of the quaternionic dilation factor of V. Without loss of generality V is a dilation with fixed points

$$o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and can be represented as the following matrix

$$V = \begin{bmatrix} \lambda\mu & 0 & 0 \\ 0 & \mu\nu & 0 \\ 0 & 0 & \lambda^{-1}\mu \end{bmatrix}$$

where μ and ν are unit quaternions. Let the real geodesic γ be the axis of V. Let the points p_s on γ be parametrised by $s \in \mathbb{R}_+$, then we have

$$p_s = \begin{bmatrix} -s \\ 0 \\ 1 \end{bmatrix} \in \gamma.$$

The (Bergman) length l of the simple closed geodesic γ' in the quaternionic hyperbolic manifold $\mathbf{H}^2_{\mathbb{H}}/\langle V, U\rangle$ corresponding to the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{H}}$ is equal to the distance between (without loss of generality) p_1 and $V(p_1)$, i.e.

$$\begin{aligned}\cosh^2(l/2) &= \cosh^2(\rho(p_1, V(p_1))/2) \\ &= \frac{(-\lambda^2 - 1)^2}{(-2)(-2\lambda^2)} \\ &= \frac{(-\lambda^2 - 1)^2}{4\lambda^2}.\end{aligned}$$

Therefore

$$\lambda = \exp(l/2). \tag{90}$$

(iii) **Octonionic hyperbolic 2-space.**

Let V and $U \in \mathrm{Aut}(J)$. Let $V = DS$ be a loxodromic map and U a general map such that $\langle U, V\rangle$ is discrete (where $\lambda \in \mathbb{R}_+$ is the dilation factor of D and S represents a nested sequence of octonionic rotations which for our purposes can be parametrised by the imaginary octonionic unit μ). Let the real geodesic γ with endpoints C and B be the axis of the loxodromic map V. Parametrising points on γ by $s \in \mathbb{R}_+$ we have using Allcock's associative triple notation

$$p_s = (0, 1, -s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -s & s^2 \\ 0 & 1 & -s \end{bmatrix} \in \gamma.$$

The (Bergman) length l of the simple closed geodesic γ' in the octonionic hyperbolic manifold $\mathbf{H}^2_{\mathbb{O}}/\langle V, U\rangle$ corresponding to the real geodesic $\gamma \subset \mathbf{H}^2_{\mathbb{O}}$ is equal to the distance between (without loss of generality) p_1 and $V(p_1)$, i.e.

$$\begin{aligned}\cosh^2(l/2) &= \cosh^2(\rho(p_1, V(p_1))/2) \\ &= \cosh^2(\rho((0,1,-1),(0,1,-\lambda^2))/2) \\ &= \frac{(\lambda^4 + 2\lambda^2 + 1)}{(2)(2\lambda^2)} \\ &= \frac{(\lambda^2 + 1)^2}{4\lambda^2}.\end{aligned}$$

Therefore

$$\lambda = \exp(l/2). \tag{91}$$

Remark. We observe that using equations (89), (90) and (91) we can express the above collar theorems in terms of the length l of the simple closed geodesic (in the quotient manifold) instead of the magnitude λ of the dilation factor of the loxodromic group generator. We will now consider the relationship between the length l of the simple closed geodesic and the radius $r(l)$ of the corresponding collar in complex and quaternionic hyperbolic manifolds.

Proposition 6.1. *Let γ' denote a simple closed geodesic in a (quotient) complex hyperbolic 2-manifold M of length l. Then the radius $r = r(l)$ of the collar $T_{\gamma'}(r)$ is a strictly decreasing function of l.*

Proof. We must investigate the growth of r where

$$\cosh(r) = \frac{1 - N}{N}$$

and

$$N = |e^{l/2+i\theta} - 1| + |e^{-l/2-i\theta} - 1|$$
$$= |e^{l/2+i\theta} - 1|(1 + |e^{l/2+i\theta}|^{-1}).$$

Now $\cosh(r)$ is a strictly decreasing function of N. Differentiating the square of the above expression for N we find that N is a strictly increasing function of l for $\lambda = e^{l/2} > 1$. This is always true as $l > 0$. Thus $\cosh(r)$ (and therefore r) is a strictly decreasing function of l.

Similarly we can prove the following proposition for the quaternionic case.

Proposition 6.2. *Let γ' denote a simple closed geodesic in a (quotient) quaternionic hyperbolic 2-manifold M of length l. Then the radius $r(l)$ of the collar $T_{\gamma'}(r)$ is a strictly decreasing function of l.*

Remark. We observe that in both the above cases and in the octonionic case $N = N(\lambda)$ is a strictly increasing function of $l = l(\lambda)$. Furthermore N is bounded below by a (discreteness) relation of the form $N \geq \frac{1}{P}$ where $P \in \mathbb{R}_+$. Therefore l is also bounded below.

6. *The Relative Size of a Collar in Complex and Quaternionic Hyperbolic 2-Manifolds.*

In this section we explore some of the properties of the collars constructed above around sufficiently short (see Proposition 6.3 and Proposition 6.5) closed geodesics in complex and quaternionic hyperbolic 2-manifolds. We calculate the volumes of collars and consider the relative position of the collars around disjoint loops.

The boundary of a collar $T(r(l))$ is a hyperbolic cylinder $Cyl(r,l)$. We will now calculate the volume $vol\big(Cyl(r,l)\big)$ of such a cylinder. The volume $vol\big(Cyl(r,l)\big)$ gives a lower bound on the volume of the corresponding manifold.

(i) Complex Hyperbolic 2-Manifolds.

We will first consider the complex hyperbolic case. Let γ' be a simple closed geodesic in a complex hyperbolic 2-manifold (which is also a quotient manifold) of length l, then Theorem 6.1 yields a precisely invariant collar $T(r)$ around γ' of radius r given by

$$\cosh(r) = \frac{1-N}{N}$$

where

$$N = |e^{l/2+i\theta} - 1| + |e^{-l/2-i\theta} - 1|. \tag{92}$$

Lemma 6.1. *Let $Cyl(r,l) \subset \mathbf{H}^2_{\mathbb{C}}$ denote a complex hyperbolic 2-cylinder of radius r with axis of length l. Then the volume $vol\big(Cyl(r,l)\big)$ of $Cyl(r,l)$ is given by*

$$vol\big(Cyl(r,l)\big) = \frac{32\pi l}{3}\sinh^3(r/2)\cosh(r/2). \tag{93}$$

Proof. The volume form in $\mathbf{H}^2_{\mathbb{C}}$ is (see page 436 of [32])

$$dvol_2 = \frac{4}{x^3}dx\,dy\,dx'\,dy'. \tag{94}$$

First we observe that the distance between the points with $y = x' = y' = 0$ and $x = x_1$, $x = x_2$ for $x_2 > x_1$ is $\log(x_2/x_1)$.

The fundamental domain D of the loxodromic element in $PU(2,1)$ whose axis projects to the simple closed geodesic γ' in M lies in the real plane described by the coordinates x and y. This fundamental domain D is a segment of an annulus of inner and outer radii 1 and e^l and angular separation $2\psi_0$. The geodesic γ corresponding to the axis of the loxodromic element lies on the line of reflection symmetry of the segment (i.e. γ is defined by the equations $y = x' = y' = 0$). Consider discs of radii $\sqrt{x'^2 + y'^2}$ (in the complex line orthogonal to the (x, y) plane) centred on an interior arc of this fundamental domain. The volume $vol\big(Cyl(r,l)\big)$ is calculated by integrating the "banana"-shaped region formed by these discs over the fundamental domain D.

To determine the relation between $\cosh^2(r/2)$ and the aforementioned coordinates we must find the distance between a general point X in the fundamental domain D and the geodesic γ. To do this we must minimise the distance $r(s)$ between X and a general point p_s on the geodesic γ (parametrised by $s \in \mathbb{R}_+$). Let

$$X = \begin{bmatrix} -x'^2 - y'^2 - x + iy \\ \sqrt{2}(x' + iy') \\ 1 \end{bmatrix}$$

and

$$p_s = \begin{bmatrix} -s \\ 0 \\ 1 \end{bmatrix}.$$

Now according to the Bergman metric

$$\cosh^2(r(s)/2) = \frac{(x'^2 + y'^2 + x + s)^2 + y^2}{4sx}.$$

Elementary calculus shows that this expression is minimised by

$$s_0{}^2 = x^2 + y^2 + (x'^2 + y'^2)^2 + 2x(x'^2 + y'^2).$$

Thus the locus of points $(x,\, y,\, x',\, y')$, a distance

$$r = r(s_0) = \min_{s\in\mathbb{R}_+} r(s)$$

from the geodesic γ, i.e. $y = x' = y' = 0$ is given by

$$\begin{aligned}\cosh^2(r/2) &= \frac{\left(\sqrt{(x'^2 + y'^2 + x)^2 + y^2} + x'^2 + y'^2 + x\right)^2 + y^2}{4x\sqrt{(x'^2 + y'^2 + x)^2 + y^2}} \\ &= \frac{x'^2 + y'^2 + x + \sqrt{(x'^2 + y'^2 + x)^2 + y^2}}{2x}.\end{aligned}$$

In particular if $x' = y' = 0$ and $x = q\cos(\psi_0)$, $y = q\sin(\psi_0)$ (using polar coordinates), then

$$\cosh^2(r/2) = \frac{1 + \cos(\psi_0)}{2\cos(\psi_0)}$$

which is equivalent to $\sinh(r) = \tan(\psi_0)$. We observe that this also follows from consideration of the area of the fundamental domain D. Rearranging the above formula gives

$$t_0{}^2 := x'^2 + y'^2 = \frac{x^2\sinh^2(r) - y^2}{4x\cosh^2(r/2)}.$$

Choosing to work in polar coordinates we have

$$q := \sqrt{x^2 + y^2},$$

$$t := \sqrt{x'^2 + y'^2}$$

with the angular coordinate ψ measured from γ and the angular coordinate ϕ describing the discs in the (x', y') plane. Therefore we can write

$$\begin{aligned}
vol_2(Cyl(r,l)) &= \int_1^{e^l} \int_{-\psi_0}^{\psi_0} \int_0^{t_0} \int_0^{2\pi} \frac{4t}{q^2 \cos^3(\psi)} dq\, d\psi\, dt\, d\phi \\
&= \int_1^{e^l} \int_{-\psi_0}^{\psi_0} \frac{4\pi {t_0}^2}{q^2 \cos^3(\psi)} dq\, d\psi \\
&= \int_1^{e^l} \int_{-\psi_0}^{\psi_0} \frac{4\pi}{q^2 \cos^3(\psi)} \frac{q^2 \cos^2(\psi) \sinh^2(r) - q^2 \sin^2(\psi)}{4q \cos(\psi) \cosh^2(r/2)} dq\, d\psi \\
&= \pi \int_1^{e^l} \frac{1}{q} dq \int_{-\psi_0}^{\psi_0} \left(\frac{\sinh^2(r)}{\cos^2(\psi) \cosh^2(r/2)} - \frac{\sin^2(\psi)}{\cos^4(\psi) \cosh^2(r/2)} \right) d\psi \\
&= \pi l \left[\frac{\sinh^2(r) \tan(\psi)}{\cosh^2(r/2)} - \frac{\tan^3(\psi)}{3 \cosh^2(r/2)} \right]_{-\psi_0}^{\psi_0} \\
&= 2\pi l \left(\frac{\sinh^3(r)}{\cosh^2(r/2)} - \frac{\sinh^3(r)}{3 \cosh^2(r/2)} \right) \\
&= \frac{4\pi l}{3} \frac{\sinh^3(r)}{\cosh^2(r/2)} \\
&= \frac{32\pi l}{3} \sinh^3(r/2) \cosh(r/2).
\end{aligned}$$

Proposition 6.3. *Let γ' denote a simple closed geodesic in a (quotient) complex hyperbolic 2-manifold M of length l. Then there exists an upper bound l_0 on l.*

Proof. By Theorem 6.1

$$\frac{1-N}{N} = \cosh(r) \geq 1.$$

This implies the condition $N = N(l) \leq 1/2$, i.e. there exists an upper bound on N. By the proof of Proposition 6.1, N is a strictly increasing function of l, therefore there exists an upper bound, say l_0 on l.

Proposition 6.4. *Let ${\gamma_1}'$, ${\gamma_2}'$ be two disjoint simple short closed geodesics in a (quotient) complex hyperbolic 2-manifold M of length l_1, $l_2 \leq l_0$. Then the collars $T_{\gamma_1'}$, $T_{\gamma_2'}$ of radius $r(l_1)$, $r(l_2)$ are disjoint.*

Proof. Let γ_1, γ_2 be lifts to $\mathbf{H}^2_{\mathbb{C}}$ of ${\gamma_1}'$, ${\gamma_2}'$ in $M = \mathbf{H}^2_{\mathbb{C}}/\Gamma$ (where Γ is a discrete non-elementary subgroup of $PU(2,1)$) and denote by U_1, $U_2 \in \Gamma$ loxodromic elements with axes γ_1, γ_2. Without loss of generality U_1 has fixed points o and ∞ and let U_2 have fixed points p and q. Now recalling the Cygan metric (see section 1.2 of Chapter Six) and the *complex cross-ratio* defined on page 8 of [21], we have by the triangle inequality and the complex analogues of Lemmas 4.2 and 4.3 (verified by direct calculation)

$$\begin{aligned}
|[U_1(p), q, p, U_1(q)]|^{1/2} &= \frac{\rho_0\left(U_1(p), p\right) \rho_0\left(U_1(q), q\right)}{\rho_0(p,q) \rho_0\left(U_1(p), U_1(q)\right)} \\
&\leq N_1 \frac{\rho_0\left(p, o\right) \rho_0\left(q, o\right)}{\rho_0(p,q)^2} \\
&\leq N_1 \left(\frac{\rho_0\left(p, o\right)}{\rho_0(p,q)} \right) \left(\frac{\rho_0\left(p, o\right)}{\rho_0(p,q)} + 1 \right) \\
&= N_1 \left(|[o,q,p,\infty]|^{1/2} \right) \left(|[o,q,p,\infty]|^{1/2} + 1 \right).
\end{aligned}$$

Therefore by Theorem 6.1 we have

$$N_1 |[o,q,p,\infty]|^{1/2} \left(|[o,q,p,\infty]|^{1/2} + 1 \right) \geq |[U_1(p), q, p, U_1(q)]|^{1/2} \geq \left(\frac{1 - N_2}{N_2} \right)$$

which implies

$$|[o,q,p,\infty]|^{1/2}\left(|[o,q,p,\infty]|^{1/2}+1\right) \geq \left(\frac{1-N_2}{N_2}\right)\left(\left(\frac{1-N_1}{N_1}\right)+1\right).$$

By symmetry

$$|[o,q,p,\infty]|^{1/2}\left(|[o,q,p,\infty]|^{1/2}+1\right) \geq \left(\frac{1-N_1}{N_1}\right)\left(\left(\frac{1-N_2}{N_2}\right)+1\right).$$

Therefore

$$\begin{aligned}|[o,q,p,\infty]|^{1/2}\left(|[o,q,p,\infty]|^{1/2}+1\right) &\geq \frac{1-N_1^{1/2}N_2^{1/2}}{N_1N_2}\\ &= \left(\frac{1-N_1^{1/2}N_2^{1/2}}{N_1^{1/2}N_2^{1/2}}\right)\left(\frac{1-N_1^{1/2}N_2^{1/2}}{N_1^{1/2}N_2^{1/2}}+1\right).\end{aligned}$$

Thus

$$|[o,q,p,\infty]|^{1/2} \geq \frac{1-N_1^{1/2}N_2^{1/2}}{N_1^{1/2}N_2^{1/2}}$$

as $x(x+1)$ is an increasing function of x for all $x>0$. This implies

$$\begin{aligned}|[o,q,p,\infty]| &\geq \frac{1-2N_1^{1/2}N_2^{1/2}+N_1N_2}{N_1N_2}\\ &\geq \frac{1-N_1-N_2+N_1N_2}{N_1N_2}\\ &= \left(\frac{1-N_1}{N_1}\right)\left(\frac{1-N_2}{N_2}\right).\end{aligned}$$

Therefore

$$\begin{aligned}|[o,q,p,\infty]| &\geq \left(\frac{1-N_1}{N_1}\right)\left(\frac{1-N_2}{N_2}\right)\\ &= \cosh(r_1)\cosh(r_2).\end{aligned}$$

Now let $\delta = dist(\gamma_1, \gamma_2)$. Let U be the loxodromic map such that $\gamma_2 = U^{-1}(\gamma_1)$, then arguing as in the proof of Theorem 6.3 we find

$$\cosh^2(\delta/2) \geq |[o,q,p,\infty]|.$$

Therefore by the above result

$$\begin{aligned}\cosh^2(\delta/2) &\geq \cosh(r_1)\cosh(r_2)\\ &= \left(\cosh(r_1+r_2)+\cosh(r_1-r_2)\right)/2\\ &\geq \frac{\cosh(r_1+r_2)+1}{2}\\ &= \cosh^2\left(\frac{r_1+r_2}{2}\right).\end{aligned}$$

As $\cosh(k)$ is a strictly monotonic increasing function of k for all $k \in \mathbb{R}_+$, $\delta \geq r_1+r_2$ if and only if $\cosh(\delta/2) \geq \cosh\left((r_1+r_2)/2\right)$ and we are done.

(ii) Quaternionic Hyperbolic 2-Manifolds.

We will now consider the quaternionic hyperbolic case which is very similar to the complex hyperbolic case. Let γ' be a simple closed geodesic in a quaternionic hyperbolic 2-manifold (which is also a quotient manifold) of length l, then Theorem 6.2 yields a precisely invariant collar $T(r)$ around γ' of radius r given by

$$\cosh(r) = \frac{1-N}{N}$$

where

$$N = |e^{l/2}\mu - 1| + 2|\mu\nu - 1| + |e^{-l/2}\mu - 1|. \tag{95}$$

(Here μ and ν are unit quaternions.)

Lemma 6.2. *Let $Cyl(r,l) \subset \mathbf{H}^2_{\mathbb{H}}$ denote a quaternionic hyperbolic 2-cylinder of radius r with axis of length l. Then the volume $vol\big(Cyl(r,l)\big)$ of $Cyl(r,l)$ is given by*

$$vol\big(Cyl(r,l)\big) = \frac{2^{11}\pi^3 l}{105}\sinh^7(r/2)\cosh^3(r/2). \tag{96}$$

Proof. The volume form in $\mathbf{H}^2_{\mathbb{H}}$ is (see section 3 of [25])

$$\begin{aligned} dvol_2 &= \frac{2^4}{u^6} du\, dVol_{\mathbb{R}^3}\, dVol_{\mathbb{H}} \\ &= \frac{2^4}{u^6} du\, da\, db\, dc\, dx\, dy\, dz\, dw. \end{aligned}$$

The fundamental domain D of the loxodromic element in $PSp(2,1)$ whose axis projects to the simple closed geodesic γ' in M lies in the real 4-space described by the coordinates u, a, b and c. This fundamental domain D is a segment of an annulus of inner and outer radii 1 and e^l and angular separation $2\alpha_0$. The geodesic γ corresponding to the axis of the loxodromic element lies in the 3-space of reflection symmetry of the segment. Consider 3-spheres of radii $\sqrt{x^2+y^2+z^2+w^2}$ (in the quaternionic line orthogonal to the (u,a,b,c) real 4-space) centred on an interior arc of this fundamental domain. The volume $vol\big(Cyl(r,l)\big)$ is calculated by integrating the region formed by these 3-spheres over the fundamental domain D.

To determine the relation between $\cosh^2(r/2)$ and the aforementioned coordinates we must find the distance between a general point X in the fundamental domain D and the geodesic γ. To do this we must minimise the distance $r(s)$ between X and a general point p_s on the geodesic γ (parametrised by $s \in \mathbb{R}_+$). Let

$$X = \begin{bmatrix} -x^2-y^2-z^2-w^2-u+ia+jb+kc \\ \sqrt{2}(x+iy+jz+kw) \\ 1 \end{bmatrix}$$

and

$$p_s = \begin{bmatrix} -s \\ 0 \\ 1 \end{bmatrix}.$$

Now according to the Bergman metric

$$\cosh^2(r(s)/2) = \frac{(x^2+y^2+z^2+w^2+u+s)^2+a^2+b^2+c^2}{4su}.$$

Elementary calculus shows that this expression is minimised by

$${s_0}^2 = u^2+a^2+b^2+c^2+(x^2+y^2+z^2+w^2)^2+2u(x^2+y^2+z^2+w^2).$$

Thus the locus of points (u,a,b,c,x,y,z,w), a distance $r = r(s_0) = \min_{s\in\mathbb{R}_+} r(s)$ from the geodesic γ, i.e.

$$a=b=c=x=y=z=w=0$$

is given by

$$\cosh^2(r/2) = \frac{\left(\sqrt{(x^2+y^2+z^2+w^2+u)^2+a^2+b^2+c^2}+x^2+y^2+z^2+w^2+u\right)^2+a^2+b^2+c^2}{4u\sqrt{(x^2+y^2+z^2+w^2+u)^2+a^2+b^2+c^2}}$$
$$= \frac{x^2+y^2+z^2+w^2+u+\sqrt{(x^2+y^2+z^2+w^2+u)^2+a^2+b^2+c^2}}{2u}.$$

As in the complex case the following relation is true

$$\sinh(r) = \tan(\alpha_0).$$

Rearranging the above formula gives

$$t_0{}^2 := x^2+y^2+z^2+w^2 = \frac{u^2\sinh^2(r)-a^2-b^2-c^2}{4u\cosh^2(r/2)}.$$

Therefore choosing to work in the following polar coordinates

$$u = q\cos(\alpha),$$

$$a = q\sin(\alpha)\cos(\beta),$$

$$b = q\sin(\alpha)\sin(\beta)\cos(\omega),$$

$$c = q\sin(\alpha)\sin(\beta)\sin(\omega),$$

$$x = t\cos(\alpha'),$$

$$y = t\sin(\alpha')\cos(\beta'),$$

$$z = t\sin(\alpha')\sin(\beta')\cos(\omega'),$$

$$w = t\sin(\alpha')\sin(\beta')\sin(\omega'),$$

we have

$$q = \sqrt{u^2+a^2+b^2+c^2},$$

$$t = \sqrt{x^2+y^2+z^2+w^2}$$

with the angular coordinates α (measured from the geodesic γ), β and ω describing the 3-spheres in the (u, a, b, c) real 4-space. Similarly the angular coordinates α', β' and ω' describe the 3-spheres in the (x, y, z, w) real 4-space. Direct calculation of the appropriate Jacobian shows

$$du\,da\,db\,dc = q^3\sin^2(\alpha)\sin(\beta)dq\,d\alpha\,d\beta\,d\omega$$

and

$$dx\,dy\,dz\,dw = t^3\sin^2(\alpha')\sin(\beta')dt\,d\alpha'\,d\beta'\,d\omega'.$$

Thus we can write

$$
\begin{aligned}
&vol_2(Cyl(r,l)) \\
&= \int_1^{e^l}\int_0^{\alpha_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{t_0}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} \frac{16t^3}{q^3\cos^6(\alpha)}\sin^2(\alpha)\sin(\beta)\sin^2(\alpha')\sin(\beta')dq\,d\alpha\,d\beta\,d\omega\,dt\,d\alpha'\,d\beta'\,d\omega' \\
&= 32\pi^3\int_1^{e^l}\int_0^{\alpha_0}\frac{{t_0}^4}{q^3\cos^6(\alpha)}\sin^2(\alpha)dq\,d\alpha \\
&= 32\pi^3\int_1^{e^l}\int_0^{\alpha_0}\frac{1}{q^3\cos^6(\alpha)}\left(\frac{q^2\cos^2(\alpha)\sinh^2(r)-q^2\sin^2(\alpha)}{4q\cos(\alpha)\cosh^2(r/2)}\right)^2\sin^2(\alpha)dq\,d\alpha \\
&= 2\pi^3\int_1^{e^l}\int_0^{\alpha_0}\frac{1}{q\cos^6(\alpha)}\left(\frac{\cos^2(\alpha)\sinh^2(r)-\sin^2(\alpha)}{\cosh^2(r/2)}\right)^2\tan^2(\alpha)dq\,d\alpha \\
&= 2\pi^3 l\int_0^{\alpha_0}\frac{1}{\cos^6(\alpha)}\left(\frac{\cos^2(\alpha)\sinh^2(r)-\sin^2(\alpha)}{\cosh^2(r/2)}\right)^2\tan^2(\alpha)d\alpha \\
&= \frac{2\pi^3 l}{\cosh^4(r/2)}\left[\frac{1}{3}\sinh^4(r)\tan^3(\alpha)-\frac{2}{5}\sinh^2(r)\tan^5(\alpha)+\frac{1}{7}\tan^7(\alpha)\right]_0^{\alpha_0} \\
&= \frac{16\pi^3 l}{105}\frac{\sinh^7(r)}{\cosh^4(r/2)} \\
&= \frac{2^{11}\pi^3 l}{105}\sinh^7(r/2)\cosh^3(r/2).
\end{aligned}
$$

We observe that in the above calculation we used the relation $\sinh(r) = \tan(\alpha_0)$.

Proposition 6.5. *Let γ' denote a simple closed geodesic in a (quotient) quaternionic hyperbolic 2-manifold M of length l. Then there exists an upper bound ${l_0}'$ on l.*

Proof. By Theorem 6.2

$$\frac{1-N}{N} = \cosh(r) \geq 1.$$

This implies the condition $N = N(l) \leq 1/2$, i.e. there exists an upper bound on N. By the proof of Proposition 6.2, N is a strictly increasing function of l, therefore there exists an upper bound, say ${l_0}'$ on l.

Proposition 6.6. *Let ${\gamma_1}'$, ${\gamma_2}'$ be two disjoint simple short closed geodesics in a (quotient) quaternionic hyperbolic 2-manifold M of lengths l_1, $l_2 \leq {l_0}'$. Then the collars $T_{{\gamma_1}'}$, $T_{{\gamma_2}'}$, of radius $r(l_1)$, $r(l_2)$ are disjoint.*

Proof. Proposition 6.6 can be proved following the same arguments used to prove Proposition 6.4.

7. *Cusps and Collars.*

In this section assume that $M = H/\Gamma$ (where $H = \mathbf{H}^2_{\mathbb{C}}$ or $\mathbf{H}^2_{\mathbb{H}}$ respectively) is a non-compact oriented manifold of finite volume. That is, Γ is a discrete non-elementary subgroup without torsion of $PU(2,1)$ or $PSp(2,1)$ respectively, containing parabolic elements which give rise to cusps in M.

In general a cusp $C \subset H/\Gamma$ can be written as $C = C_q = \mathrm{H}_q/\Gamma_q$ for some point $q \in \partial H$ where Γ_q is a subgroup of Γ of the parabolic type with fixed point q and where $\mathrm{H}_q \subset H$ is some precisely invariant horoball based at q. Actually one can associate to Γ_q a specific horoball H_q based at q such that H_q/Γ_q embeds in H/Γ. Assume for simplicity that $q = \infty$. The subgroup $\Phi = \Phi(\infty) \subset \Gamma_\infty$ consisting of all translations is a lattice of finite index and rank 1 or 3 respectively. Let T denote the translation of shortest translation length $t^{1/2}$, then

$$\mathrm{H}(t) = \mathrm{H}_\infty(t) = \{(\zeta, v, u) \in H | u > t\}$$

is the **canonical horoball** of Γ_∞. In Proposition 2.4 of [32], Parker showed that for the complex hyperbolic case $H_\infty(t)/\Gamma$ embeds in M and that the canonical horoballs associated to inequivalent parabolic elements in Γ are disjoint. In Proposition 4.7 of [25], Kim and Parker proved the corresponding result for the quaternionic hyperbolic case.

We will now, using the appropriate normalisations, quote the complex and quaternionic analogues of Shimizu's Lemma taken from the aforementioned papers.

Theorem 2.2 [32]. *Let G be a discrete subgroup of $PU(n,1)$ so that the stabiliser G_∞ of q_∞ in G contains a Heisenberg translation by (τ, t') where $\tau \neq 0$ and also vertical translation by $t \in \mathbb{R}$. Let h be any element of G not fixing q_∞ written in a standard matrix form (see equation (1.3) of [32] relabeling the matrix element c as g). Then*

$$\min\{|t|, \big||\tau^2| + it\big|\}|g| \geq 1.$$

Propositions 4.4, 4.5 [25]. *Let G be a discrete subgroup of $PSp(n,1)$ whose stabiliser G_∞ of ∞ is a Heisenberg lattice. Let $T_{(\tau,t')}$ be a non-vertical translation (so $\tau \neq 0$) and $T_{(0,t)}$ a vertical translation. Denote by s the radius of the Dirichlet domain centred at the origin of the vertical lattice $\ker\Pi \cap G_\infty$. Let h be any element of G not fixing ∞ and written in a standard matrix form (see equation (2) of [25] and section 2 of Chapter 2 and relabel the matrix element c as g and vice versa). Then*

$$\min\{|t|, \big||\tau|^2 + is\big|\}|g| \geq 1.$$

Consider an oriented complex or quaternionic hyperbolic 2-manifold M with cusps. We will now prove that canonical cusps, i.e. those covered by canonical horoballs and the collars around closed geodesics, i.e. simple closed geodesics γ' of length $l(\gamma')$ do not intersect. We will first consider the complex hyperbolic case.

(i) **Complex Hyperbolic 2-Manifolds.**

Theorem 6.4. *Let M denote a non-compact oriented complex hyperbolic 2-manifold of finite volume. Then the canonical cusps and the collars around simple short closed geodesics in M are disjoint.*

Proof. Let $M = \mathbf{H}^2_{\mathbb{C}}/\Gamma$ where Γ is a discrete non-elementary subgroup of $PU(2,1)$ and assume without loss of generality that Γ contains a parabolic element fixing ∞. Denote by

$$T = \begin{bmatrix} 1 & * & c_T \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \in \Gamma_\infty$$

the translation of minimal translation length in $\Phi(\Gamma_\infty)$. Hence $H(|c_T|)/\Gamma_\infty$ is a cusp in M.

Let γ' be a closed geodesic in M. Denote by U a loxodromic element whose axis γ projects to γ' and which is represented by a matrix which has the form

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \in PU(2,1).$$

Here T and U generate a discrete, non-elementary subgroup of Γ. Consider an embedded collar $T_{\gamma'}(r)$ around γ' satisfying

$$\cosh(r) = \frac{1-N}{N},$$

where

$$N = |e^{l/2+i\theta} - 1| + |e^{-l/2-i\theta} - 1| \tag{92}$$

and

$$\lambda = e^{l/2}.$$

We must prove that $T_{\gamma'}(r)$ is disjoint from $\mathrm{H}(|c_T|)/\Gamma_\infty$. We will first calculate the maximum horospherical height of a point on the axis of U. Let $U = WVW^{-1}$ where

$$V = \begin{bmatrix} \lambda e^{-i\theta/3} & 0 & 0 \\ 0 & e^{2i\theta/3} & 0 \\ 0 & 0 & \lambda^{-1}e^{-i\theta/3} \end{bmatrix}$$

and

$$W = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & j' \end{bmatrix}.$$

Points p_s on the axis γ of U can be represented by

$$p_s = W \begin{bmatrix} -s \\ 0 \\ 1 \end{bmatrix},$$

where $s \in \mathbb{R}_+$. Let $p_s = (\zeta_s, v_s, u_s)$ (in horospherical coordinates), then by direct calculation we find

$$-2u_s = \frac{-2s}{|g'|^2 s^2 + |h'|^2 s + |j'|^2}.$$

Elementary calculus shows that u_s is maximised by $s = s^* = |j'|/|g'|$. Thus

$$u_{s^*} = \frac{1}{2|j'||g'| + |h'|^2}.$$

We observe that direct calculation shows

$$|g| = \left|g'\overline{j'}\lambda e^{-i\theta} + |h'|^2 + j'\overline{g'}\lambda^{-1}e^{-i\theta}\right|.$$

Now the horospherical height of the point on the boundary of the r-neighbourhood of γ which is nearest to the horoball $\mathrm{H}(|c_T|)$ is

$$e^r u_{s^*} = \frac{e^r}{2|j'||g'| + |h'|^2}.$$

Therefore in order to prove that the boundary of the r-neighbourhood of γ does not intersect the horoball $\mathrm{H}(|c_T|)$ we must show that

$$\frac{e^r}{2|j'||g'| + |h'|^2} \leq |c_T|.$$

Now by Theorem 2.2 of [32] we have

$$|c_T||g| = |c_T|\left|g'\overline{j'}\lambda e^{-i\theta} + |h'|^2 + j'\overline{g'}\lambda^{-1}e^{-i\theta}\right| \geq 1.$$

Therefore if we can prove that

$$\frac{e^r}{2|j'||g'| + |h'|^2} \leq \frac{1}{|g|} = \frac{1}{\left|g'\overline{j'}\lambda e^{-i\theta} + |h'|^2 + j'\overline{g'}\lambda^{-1}e^{-i\theta}\right|},$$

then we are done. This can be proved as follows.

$$\begin{aligned} e^r\left|g'\overline{j'}\lambda e^{-i\theta} + |h'|^2 + j'\overline{g'}\lambda^{-1}e^{-i\theta}\right| &< 2\cosh(r)\left|g'\overline{j'}\lambda e^{-i\theta} + |h'|^2 + j'\overline{g'}\lambda^{-1}e^{-i\theta}\right| \\ &\leq 2\frac{(1-N)}{N}|g'\overline{j'}|N \\ &= 2(1-N)|g'\overline{j'}| \\ &< 2|g'\overline{j'}| \\ &\leq 2|j'||g'| + |h'|^2. \end{aligned}$$

We will now consider the quaternionic hyperbolic case and find that the results and proof are very similar to the complex hyperbolic case.

(ii) **Quaternionic Hyperbolic 2-Manifolds.**

Theorem 6.5. *Let M denote a non-compact oriented quaternionic hyperbolic 2-manifold of finite volume. Then the canonical cusps and the collars around simple short closed geodesics in M are disjoint.*

Proof. Let $M = \mathbf{H}^2_{\mathbb{H}}/\Gamma$ where Γ is a discrete non-elementary subgroup of $PSp(2,1)$ and assume without loss of generality that Γ contains a parabolic element fixing ∞. Denote by

$$T = \begin{bmatrix} 1 & * & c_T \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \in \Gamma_\infty$$

the translation of minimal translation length in $\Phi(\Gamma_\infty)$. Hence $\mathrm{H}(|c_T|)/\Gamma_\infty$ is a cusp in M.

Let γ' be a closed geodesic in M. Denote by U a loxodromic element whose axis γ projects to γ' and which can be represented by a matrix which has the form

$$U = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \in PSp(2,1).$$

Here T and U generate a discrete, non-elementary subgroup of Γ. Consider an embedded collar $T_{\gamma'}(r)$ around γ' satisfying

$$\cosh(r) = \frac{1-N}{N}$$

where

$$N = |e^{l/2}\mu - 1| + 2|\mu\nu - 1| + |e^{-l/2}\mu - 1| \tag{95}$$

and

$$\lambda = e^{l/2}.$$

We must prove that $T_{\gamma'}(r)$ is disjoint from $\mathrm{H}(|c_T|)/\Gamma_\infty$. We will first calculate the maximum horospherical height of a point on the axis of U. Let $U = WVW^{-1}$ where

$$V = \begin{bmatrix} \lambda\mu & 0 & 0 \\ 0 & \mu\nu & 0 \\ 0 & 0 & \lambda^{-1}\mu \end{bmatrix}$$

and

$$W = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & j' \end{bmatrix}.$$

Points p_s on the axis γ of U can be represented by

$$p_s = W \begin{bmatrix} -s \\ 0 \\ 1 \end{bmatrix}$$

where $s \in \mathbb{R}_+$. Let $p_s = (\zeta_s, v_s, u_s)$ (in horospherical coordinates), then by direct calculation we find

$$-2u_s = \frac{-2s}{|g'|^2 s^2 + |h'|^2 s + |j'|^2}.$$

Elementary calculus shows that u_s is maximised by $s = s^* = |j'|/|g'|$. Thus

$$u_{s^*} = \frac{1}{2|j'||g'| + |h'|^2}.$$

We observe that direct calculation shows

$$|g| = \left|g'\mu\overline{j'}\lambda + h'\mu\nu\overline{h'} + j'\mu\overline{g'}\lambda^{-1}\right|.$$

Now the horospherical height of the point on the boundary of the r-neighbourhood of γ which is nearest to the horoball $\mathrm{H}(|c_T|)$ is

$$e^r u_{s^*} = \frac{e^r}{2|j'||g'| + |h'|^2}.$$

Therefore in order to prove that the boundary of the r-neighbourhood of γ does not intersect the horoball $\mathrm{H}(|c_T|)$ we must show that

$$\frac{e^r}{2|j'||g'| + |h'|^2} \le |c_T|.$$

Now by Theorem 4.5 of [25] we have

$$|c_T||g| = |c_T|\left|g'\mu\overline{j'}\lambda + h'\mu\nu\overline{h'} + j'\mu\overline{g'}\lambda^{-1}\right| \ge 1.$$

Therefore if we can prove that

$$\frac{e^r}{2|j'||g'| + |h'|^2} \le \frac{1}{|g|} = \frac{1}{\left|g'\mu\overline{j'}\lambda + h'\mu\nu\overline{h'} + j'\mu\overline{g'}\lambda^{-1}\right|},$$

then we are done. As $U \in PSp(2,1)$ we have $|h'|^2 = -g'\overline{j'} - j'\overline{g'}$, therefore letting $g'\overline{j'} = m$ we need to prove

$$e^r\left|g'(\lambda\mu - 1)\overline{j'} + h'(\mu\nu - 1)\overline{h'} + j'(\lambda^{-1}\mu - 1)\overline{g'}\right| \le 2|m| - m - \overline{m}.$$

Now

$$e^r < 2\cosh(r) = 2\left(\frac{1-N}{N}\right)$$

and

$$\left|g'(\lambda\mu - 1)\overline{j'} + h'(\mu\nu - 1)\overline{h'} + j'(\lambda^{-1}\mu - 1)\overline{g'}\right| \le |m|N.$$

Thus

$$e^r\left|g'(\lambda\mu - 1)\overline{j'} + h'(\mu\nu - 1)\overline{h'} + j'(\lambda^{-1}\mu - 1)\overline{g'}\right| < 2|m|(1-N).$$

Therefore if we can prove

$$-2|m|N \le -m - \overline{m},$$

then we are done. However this inequality is true for all $m \in \mathbb{H}$ as $-m - \overline{m} = -g'\overline{j'} - j'\overline{g'} = |h'|^2 \ge 0$ and $-2|m|N \le 0$. Therefore we are done.

8. *Injectivity Radii.*

Let M be a compact oriented complex hyperbolic 2-manifold. Closed geodesics in M possess embedded tubular neighbourhoods whose volumes we can calculate (see Lemma 6.1). This allows us to derive a result concerning the geometry of M.

Before we prove any results we will state Zagier's Lemma as quoted on page 62 of [24].

Lemma. *For $0 < \tau_1 < \pi\sqrt{3}$ and $\tau_2 \in [0, 2\pi)$ there exists a number $n \in \mathbb{N}$ such that*

$$\cosh(n\tau_1) - \cos(n\tau_2) \le \cosh\sqrt{\frac{4\pi\tau_1}{\sqrt{3}}} - 1.$$

We observe that in $\mathbf{H}^2_{\mathbb{C}}$, $2\tau_1 = l$, the translation length of the loxodromic generator and $\tau_2 = \theta$, the angle of rotation of the loxodromic generator.

Consider the **injectivity radius** $i(M)$ of M which is related to the length of a shortest (and necessarily simple) closed geodesic γ_0' in M by $i(M) = \frac{1}{2}l(\gamma_0')$.

Assume that there is a simple closed geodesic γ' of length l in M. By Theorem 6.1 there is a tubular neighbourhood (embedded collar) $T_{\gamma'}(r) \subset M$ around γ' of radius r satisfying

$$\cosh(r) = \frac{1-N}{N}$$

where

$$N = |e^{l/2+i\theta} - 1| + |e^{-l/2-i\theta} - 1|. \tag{92}$$

This collar corresponds to the loxdromic map $V = V^1$ in the proof of Theorem 6.1. Similarly there exists a collar of radius $r_n > 0$ corresponding to the loxodromic map V^n for all n such that $N_n = N(\lambda^n, n\theta) < \frac{1}{2}$. We will show below that by Zagier's Lemma there exists one value n' of n such that ${N_{n'}}^2 \leq f(l)$ where

$$f(l)^{1/2} = 2\sinh\left(\sqrt{2\pi l/\sqrt{3}}\right).$$

Therefore there exists a collar of radius $r_{n'}$ where

$$\cosh(r_{n'}) = \frac{1 - f(l)^{1/2}}{f(l)^{1/2}}.$$

Furthermore the condition that $f(l)^{1/2} < 1/2$ requires that $l < 0.01688$ (approx.).

Now in accordance with Zagier's Lemma we have

$$\begin{aligned}
{N_{n'}}^2 &= \left(|e^{l_{n'}/2}e^{i\theta_{n'}} - 1| + |e^{-l_{n'}/2}e^{-i\theta_{n'}} - 1|\right)^2 \\
&= (e^{l_{n'}/4} + e^{-l_{n'}/4})^2 |e^{l_{n'}/4}e^{i\theta_{n'}/2} - e^{-l_{n'}/4}e^{-i\theta_{n'}/2}|^2 \\
&= (2\cosh(l_{n'}/4))^2 \left(4\cosh^2(l_{n'}/4) - 4\cos^2(\theta_{n'}/2)\right) \\
&= (2\cosh(l_{n'}/4))^2 \left(2\cosh(l_{n'}/2) - 2\cos(\theta_{n'})\right) \\
&= (2\cosh(l_{n'}/2) + 2)\left(2\cosh(l_{n'}/2) - 2\cos(\theta_{n'})\right) \\
&\leq 4\left(\cosh\left(\sqrt{2\pi l/\sqrt{3}}\right) + 1\right)\left(\cosh\left(\sqrt{2\pi l/\sqrt{3}}\right) - 1\right) \\
&= \left(2\sinh\left(\sqrt{2\pi l/\sqrt{3}}\right)\right)^2
\end{aligned}$$

where $l_{n'} = n'l$ and $\theta_{n'} = n'\theta$.

Remark. We observe that in the third line of the above calculation we have used the following relation. If $z = x + iy \in \mathbb{C}$, then

$$\begin{aligned}
\sinh^2(x+iy) &= \sinh^2(x)\cos^2(y) + \cosh^2(x)\sin^2(y) \\
&= \cosh^2(x)\cos^2(y) - \cos^2(y) + \cosh^2(x) - \cosh^2(x)\cos^2(y) \\
&= \cosh^2(x) - \cos^2(y).
\end{aligned}$$

We will now prove our theorem concerning the injectivity radius $i(M)$ of M.

Proposition 6.7. *Let M denote a compact oriented complex hyperbolic 2-manifold and $T_{\gamma'}(r') \subset M$ an embedded collar of radius r' around a short simple closed geodesic γ' in M. Then the injectivity radius of M satisfies the relation*

$$i(M) \geq \frac{\sqrt{3}}{64\pi}\left(vol\,(T_{\gamma'}(r'))^2 - \frac{1}{3}\right)^2.$$

Proof. Assume that there is a simple closed geodesic γ' of length $l < 0.01688$ (approx.) in M. As proved above there is a tubular neighbourhood $T_{\gamma'}(r') \subset M$ around γ' of radius r' satisfying

$$\cosh(r') = \frac{1 - N'}{N'}$$

where

$$N' \leq 2\sinh\left(\sqrt{2\pi l/\sqrt{3}}\right) < 1/2.$$

This implies

$$\begin{aligned} N' &\leq 2\sinh\left(\sqrt{2\pi l/\sqrt{3}}\right) \\ &= 2\sqrt{\frac{2\pi l}{\sqrt{3}}}\left(1 + \frac{2\pi l}{3!\sqrt{3}} + \frac{1}{5!}\left(\frac{2\pi l}{\sqrt{3}}\right)^2 + \ldots\right) \end{aligned}$$

for $l < 0.01688$ (approx.). By Lemma 6.1 we have

$$vol(T_{\gamma'}(r')) = \frac{32\pi l}{3}\sinh^3(r'/2)\cosh(r'/2).$$

Therefore

$$\begin{aligned} vol(T_{\gamma'}(r')) &= \frac{32\pi l}{3}\sinh^3(r'/2)\cosh(r'/2) \\ &= \frac{32\pi l}{3}\frac{1}{4}\sinh(r')(\cosh(r') - 1) \\ &= \frac{32\pi l}{12}\sqrt{(\cosh^2(r') - 1)}(\cosh(r') - 1) \\ &= \frac{8\pi l}{3}\frac{(1 - 2N')^{3/2}}{N'^2}. \end{aligned}$$

Thus using the above asymptotic expansion of our upper bound on N' we find

$$\begin{aligned} vol\,(T_{\gamma'}(r'))^2 &= \left(\frac{8\pi}{3}\right)^2 l^2 \frac{(1 - 2N')^3}{N'^4} \\ &\approx \frac{1}{3}\left(1 - 12\sqrt{\frac{2\pi l}{\sqrt{3}}}\right) \end{aligned}$$

for $l < 0.01688$ (approx.). Therefore

$$\left(vol\,(T_{\gamma'}(r'))^2 - \frac{1}{3}\right)^2 \approx \frac{32\pi l}{\sqrt{3}}$$

which implies that

$$\frac{l}{2} \approx \frac{\sqrt{3}}{64\pi}\left(vol\,(T_{\gamma'}(r'))^2 - \frac{1}{3}\right)^2$$

for $l < 0.01688$ (approx.).

APPENDIX A.

Special Notes on Lattices (see section 8 of Chapter 4 of [6]).

Consider the lattice $\mathbb{Z}^7$ in $\mathbb{R}^7$ generated by $(1,0,0,0,0,0,0)$, $(0,1,0,0,0,0,0)$, etc. The Dirichlet domain D centred at the origin $(0,0,0,0,0,0,0)$ has vertices at the 2^7 points

$$(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2).$$

The radius r_{D} of D, i.e. the length of a lattice vector furthest from the origin equals $\sqrt{7}/2$. Applying $r_{\mathrm{D}} = \sqrt{7}/2 > 1$ to Theorem 3.2 we see that

$$\left(|\tau|^4/4 + \left(\sqrt{7}t/2\right)^2\right)^{1/2} > t$$

and so Theorem 3.2 doesn't give an improvement on Theorem 3.1 when $\mathbb{Z}^7$ is the vertical lattice. However we will show below that when the vertical lattice is E_7 we do get an improvement.

E_7 is the densest lattice in $\mathbb{R}^7$ (see section 8.2 of Chapter 4 of [6]). If we normalise so that the minimal root vector length is 1, then the covering radius of the lattice (which is equal to r_{D}) is $\sqrt{3}/2 < 1$. Therefore there exists at least one lattice in $\partial \mathbf{H}^2_{\mathbb{O}}$ to which we can apply the length arguments of section 3.2 of Chapter Three (in order to obtain an improved upper bound on r_h).

We will now give a more detailed description of E_7 (see section 8.2 of chapter 4 of [6]). The lattice E_7 is contained in a hyperplane of $\mathbb{R}^8$ whose Euclidean coordinates satisfy the following equation

$$X_1 + X_2 + \ldots + X_8 = 0.$$

E_7 has the following basis

$\{(1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0, 0, 0, 0),$
$(0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0, 0, 0),$
$(0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0, 0),$
$(0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0),$
$(0, 0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0),$
$(0, 0, 0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0),$
$(\frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})\}.$

Thus E_7 has

i) minimal vector length (norm) 1.
ii) volume $\frac{1}{8}$.
iii) covering radius $\sqrt{3}/2$.
iv) extreme point (point furthest from any vertex) $(\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{-3}{4\sqrt{2}}, \frac{-3}{4\sqrt{2}})$.
v) automorphism group of order $2^{10}.3^4.5.7 = 2903040$.
vi) 126 units, i.e lattice points at unit distance from the origin.

Alternatively, in $\mathbb{R}^7$ the lattice E_7 has the following basis

$\{(1,0,0,0,0,0,0),$
$(0,1,0,0,0,0,0),$
$(0,0,1,0,0,0,0),$
$(0,0,0,1,0,0,0),$

$(1/2, 1/2, 1/2, 0, 1/2, 0, 0)$,
$(0, 1/2, 1/2, 1/2, 0, 1/2, 0)$,
$(0, 0, 1/2, 1/2, 1/2, 0, 1/2)\}$.

The E_8 lattice (see section 8.1 of chapter 4 of [6]) is the densest lattice in $\mathbb{R}^8$; it has the following basis

$\{(\sqrt{2}, 0, 0, 0, 0, 0, 0, 0)$,
$(-1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0, 0, 0, 0)$,
$(0, -1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0, 0, 0)$,
$(0, 0, -1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0, 0)$,
$(0, 0, 0, -1/\sqrt{2}, 1/\sqrt{2}, 0, 0, 0)$,
$(0, 0, 0, 0, -1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$,
$(0, 0, 0, 0, 0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$,
$(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})\}$.

Therefore E_8 has

i) minimal vector length (norm) 1.
ii) volume $\frac{1}{16}$.
iii) covering radius $\frac{1}{\sqrt{2}}$.
iv) automorphism group of order $2^{14}.3^5.5^2.7 = 696729600$.
v) 240 units, i.e lattice points at unit distance from the origin.

If octonionic coordinates are chosen so that a given minimal vector in E_8 is $+1$, the vectors in E_8 that are perpendicular to $+1$ make up a spacelike E_7 lattice. To build the E_8 lattice first start with an 8-dimensional spacetime $\mathbb{R}^8 = \mathbb{O}$ using the usual basis for $\mathbb{O}$. The vertices of the E_8 lattice are of the form

$$\frac{1}{2}\sum_{i=0}^{7} a_i e_i$$

where the a_i may be either all even integers, all odd integers, or four of each (even and odd) with residues mod 2 being $(1, 0, 0, 0, 1, 1, 0, 1)$ or $(0, 1, 1, 1, 0, 0, 1, 0)$ or the same with the last seven integers cyclically permuted.

APPENDIX B.

An Octonionic Stable Basin Theorem.

Let

$$B_r = \{z \in \partial\mathbf{H}^2_{\mathbb{C}} | \rho_0(z, o) < r\},$$

where ρ_0 denotes the Cygan metric on the boundary of complex hyperbolic space. Let $\overline{B_s}^c = \partial\mathbf{H}^2_{\mathbb{C}} - \overline{B_s}$. Given r and s with $r < s$, the pair of open sets $(B_r, \overline{B_s}^c)$ is said to be *stable* with respect to a set $\mathcal{S}$ of elements in $PU(2,1)$, if for any element $g \in \mathcal{S}$

$$g(o) \in B_r$$

and

$$g(\infty) \in \overline{B_s}^c.$$

Let $\mathcal{S}(r, \epsilon)$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B_{1/r}}^c$ and satisfying $|\lambda_f - 1| < \epsilon$ where λ_f denotes the magnitude of the dilation factor of f. For positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$ we define $\epsilon(r, r')$ by

$$\epsilon(r, r') = \sup_{f \in \mathcal{S}(r,\epsilon)} \{|\lambda_f - 1|\} \qquad (i)$$

where $|\lambda_f - 1|$ satisfies the inequality

$$|\lambda_f - 1| < \sqrt{2 + \left(\frac{1 - (3 + |\lambda_f - 1|)r^2}{1 - 2r^2}\right)^2 \left(\frac{1 - 3r^2}{1 - r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - \sqrt{2}. \qquad (ii)$$

A triple of non-negative numbers (r, r', ϵ) is said to be a *basin point* provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\epsilon < \epsilon(r, r')$. In particular if $r' \leq r$ we call (r, r', ϵ) a *stable basin point.* We call the set of all such points the *stable basin region.* For simplicity we shall abbreviate (r, r, ϵ) to (r, ϵ).

In [23] Kamiya proves the following modified (and improved) version of the stable basin theorem of Basmajian and Miner (see [3]) for complex hyperbolic 2-space.

Theorem (page 831 of [23]). *Given positive numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, the pair of open sets $(B_{r'}, \overline{B_{1/r'}}^c)$ is stable with respect to the family $\mathcal{S}(r, \epsilon(r, r'))$ where $\epsilon(r, r')$ is given by (i) and (ii).*

In this appendix we will prove the following octonionic stable basin theorem.

Theorem B. *Given positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, the pair of open sets $(B_{r'}(C), R(B_{r'}(C)))$ is stable with respect to the family $\mathcal{S}(r, \epsilon(r, r'))$ where ϵ is given by (i) and (x).*

Remark. The following proof of Theorem B treats only the case of loxodromic elements of $\mathcal{S}(r, \epsilon(r, r'))$ parametrised by a single imaginary unit octonion μ.

To prove this theorem we will construct a generalisation to $\mathbf{H}^2_{\mathbb{O}}$ of Kamiya's proof in $\mathbf{H}^2_{\mathbb{C}}$. First we need three lemmas which are generalisations of Proposition 3.3 of [3], Lemma 3.13 of [3] and Lemma 3.5 of [23] respectively.

Let the loxodromic element V (an octonionic dilation) be such that

$$V = DS \qquad (iii)$$

where D is a dilation with real dilation factor λ and S is an octonionic rotation parametrised by an imaginary unit octonion μ (see section 1.6.1 of Chapter Three).

Lemma B1. *Let b, c be given. If V is a loxodromic map whose dilation factor λ satisfies*

$$|\lambda - 1| < \sqrt{1 + (b/c)^2} - 1,$$

then $V(X) \in B_b(X)$ for $X \in \overline{B_c(C)}$.

Remark. The closed ball $\overline{B_c(C)}$ lies in the boundary of octonionic hyperbolic 2-space.

Proof. Suppose $X = (x, 1, z) \in \overline{B_c(C)}$, that is

$$\sqrt{|z|} \leq c.$$

Now $V(X) = (\lambda\mu x, 1, \lambda^2 \mu z \overline{\mu})$ and $|x|^2 = -z - \overline{z}$ as $X \in \partial\mathbf{H}_{\mathbb{O}}^2 - \{B\}$, therefore if $|\lambda - 1| < \sqrt{1 + (b/c)^2} - 1$ we have

$$\begin{aligned}
\rho_0(X, V(X)) &= |(\lambda\mu x - x, 1, \lambda^2 \mu z \overline{\mu} + \overline{z} + \lambda\mu x \overline{x})|_0 \\
&= |\lambda^2 \mu z \overline{\mu} + \overline{z} + \lambda\mu x \overline{x}|^{1/2} \\
&= |\lambda^2 \mu z \overline{\mu} + \overline{z} - \lambda\mu z - \lambda\mu\overline{z}|^{1/2} \\
&= |\lambda\mu z(\lambda\overline{\mu} - 1) - (\lambda\mu - 1)\overline{z}|^{1/2} \\
&= |(\lambda\mu - 1)z(\lambda\overline{\mu} - 1) + z(\lambda\overline{\mu} - 1) - (\lambda\mu - 1)\overline{z}|^{1/2} \\
&\leq (|\lambda\mu - 1|^2 |z| + 2|\lambda\mu - 1||z|)^{1/2} \\
&\leq (|\lambda\mu - 1|^2 c^2 + 2|\lambda\mu - 1|c^2)^{1/2} < b.
\end{aligned}$$

Lemma B2. *Let T_X be an octonionic translation with $T_X(C) = X$ and let $s > r$. For $X \in B_r(C)$ and $Y \in R(B_{1/s}(C))$,*

$$R(T_X{}^{-1}(Y)) \in B_{1/(s-r)}(C).$$

Proof. By the triangle inequality

$$\rho_0(X, Y) > s - r.$$

Since translation is an isometry for the metric

$$\rho_0(X, Y) = \rho_0(C, T_X{}^{-1}(Y)).$$

Therefore

$$T_X{}^{-1}(Y) \in R(B_{1/(s-r)}(C)).$$

Hence

$$R(T_X{}^{-1}(Y)) \in R^2(B_{1/(s-r)}(C)) = B_{1/(s-r)}(C).$$

Lemma B3. *Let $G = RTR$, a pure parabolic element with fixed point C (where $T = T_{a,b}$ is an octonionic translation fixing B). Let $X \in B_r(C)$ and $Y \in B_s(C)$. If $\rho_0(G^{-1}(B), C) > c > \max\{r, s\}$, then*

$$\rho_0(G(X), G(Y)) \leq \left(\frac{c}{c-r}\right)\left(\frac{c}{c-s}\right)\rho_0(X, Y).$$

Proof. As $G = RT_{a,b}R$ we have

$$\begin{aligned}
G^{-1}(B) &= RT_{-a,-b}R(B) \\
&= RT_{-a,-b}(C) \\
&= (-(|a|^2/2 + b)^{-1}a, 1, -(|a|^2/2 + b)^{-1}).
\end{aligned}$$

This implies

$$\begin{aligned}
\rho_0(G^{-1}(B), C) &= 1/\sqrt{|(|a|^2/2 + b)^{-1}|} \\
&= 1/\sqrt{|(|a|^2/2 - b)^{-1}|} \\
&= \rho_0(G(B), C).
\end{aligned}$$

Therefore $\sqrt{\lambda_G}$ the common radius of the isometric spheres of G and G^{-1} (see Proposition 3.3 of Chapter Three) is equal to $\rho_0(G^{-1}(B), C)$. And so by Lemma 3.1 of Chapter Three we have

$$\begin{aligned}\rho_0(G(X), G(Y)) &= \frac{\lambda_G}{\rho_0(G^{-1}(B), X)\rho_0(G^{-1}(B), Y)}\rho_0(X, Y) \\ &= \frac{\rho_0(G^{-1}(B), C)^2}{\rho_0(G^{-1}(B), X)\rho_0(G^{-1}(B), Y)}\rho_0(X, Y).\end{aligned}$$

We observe that by the triangle inequality

$$\begin{aligned}\frac{\rho_0(G^{-1}(B), C)}{\rho_0(G^{-1}(B), X)} &\leq \frac{\rho_0(G^{-1}(B), X) + \rho_0(X, C)}{\rho_0(G^{-1}(B), X)} \\ &= 1 + \frac{\rho_0(X, C)}{\rho_0(G^{-1}(B), X)} \\ &\leq 1 + \frac{\rho_0(X, C)}{\rho_0(G^{-1}(B), C) - \rho_0(X, C)} \\ &\leq 1 + \frac{r}{c - r} = \frac{c}{c - r}.\end{aligned}$$

Similarly

$$\frac{\rho_0(G^{-1}(B), C)}{\rho_0(G^{-1}(B), Y)} \leq \frac{c}{c - s}.$$

Thus

$$\rho_0(G(X), G(Y)) \leq \left(\frac{c}{c - r}\right)\left(\frac{c}{c - s}\right)\rho_0(X, Y).$$

We now come to the proof of Theorem B.

Proof of Theorem B.

Since $\mathcal{S}(r, \epsilon(r, r'))$ is closed under conjugation by the inversion R, it is sufficient to determine conditions which guarantee that $f(C) \in B_r(C)$ for all $f \in \mathcal{S}(r, \epsilon(r, r'))$ in order to establish the pair

$$(B_{r'}(C), R(B_{r'}(C)))$$

is stable under $\mathcal{S}(r, \epsilon(r, r'))$. Let p and q denote the fixed points of f in $B_{r'}(C)$ and $R(B_{r'}(C))$ respectively. We may assume $p \neq C$. There exists an element h such that $h(p) = C$ and $h(q) = B$. This element h can be written as the product of two elements $G_{(a,b)}$ and T_p, i.e.

$$h = G_{(a,b)}{}^{-1}T_p{}^{-1}$$

where T_p is an octonionic translation with $T_p(C) = p$ and $G_{(a,b)}$ is a strictly parabolic element with its fixed point C and $G_{(a,b)}(B) = T_p{}^{-1}(q)$. Thus by Lemma B2

$$R(G_{(a,b)}(B)) = R(T_p{}^{-1}(q)) \in B_{r/(1-r^2)}(C)$$

since $p \in B_r(C)$ and $q \in R(B_{1/r}(C))$. Set $f' = hfh^{-1}$. As $f'(C) = C$ and $f'(B) = B$, f' is of the form $f' = DS$ as in expression (iii).

To simplify notation let $T = T_p$ and $G = G_{(a,b)}$. Since the metric ρ_0 is invariant under octonionic translations

$$\begin{aligned}\rho_0(C, f(C)) &= \rho_0(C, h^{-1}f'h(C)) \\ &= \rho_0(C, TGf'G^{-1}T^{-1}(C)) \\ &= \rho_0(T^{-1}(C), Gf'G^{-1}T^{-1}(C)) \\ &= \rho_0(G(G^{-1}T^{-1}(C)), G(f'G^{-1}T^{-1}(C))) \\ &= \rho_0(G(h(C)), G(f'h(C))).\end{aligned} \qquad (iv)$$

We will now use Lemma B3 to estimate how much G distorts the distance $\rho_0(h(C), f'h(C))$, i.e. we will estimate $\rho_0(G(h(C)), G(f'h(C)))$ in terms of $\rho_0(h(C), f'h(C))$. As $T(C) = p$ and $p \in B_r(C)$ we see that $T^{-1}(C) \in B_r(C)$. We observe that from

$$R(G_{(a,b)}(B)) = R(T_p{}^{-1}(q)) \in B_{r/(1-r^2)}(C)$$

we have

$$\rho_0(C, G(B)) > \frac{1-r^2}{r}. \tag{v}$$

From (v) we see that $\rho_0(C, G(B)) > \rho_0(C, T^{-1}(C))$ for $r < \frac{1}{\sqrt{3}}$. By Lemma B3 we have

$$\begin{aligned}
\rho_0(h(C), C) &= \rho_0(G^{-1}T^{-1}(C), C) \\
&= \rho_0(G^{-1}(T^{-1}(C)), G^{-1}(C)) \\
&\leq \left(\frac{1-r^2}{1-2r^2}\right) \rho_0(T^{-1}(C), C) \\
&\leq \frac{(1-r^2)r}{1-2r^2}
\end{aligned}$$

which means

$$h(C) \in B_{\frac{(1-r^2)r}{1-2r^2}}. \tag{vi}$$

Since f' has the same dilation factor λ_f as f

$$f'h(C) \in B_{\lambda_f\left(\frac{(1-r^2)r}{1-2r^2}\right)}. \tag{vii}$$

Direct calculation shows that $\rho_0(G^{-1}(B), C) = \rho_0(G(B), C)$. By (v), $\rho_0(G^{-1}(B), C) > (1 - r^2)/r$. We assume that $\rho_0(G^{-1}(B), C) > \rho_0(C, f'h(C))$ which is implied by

$$\lambda_f < \frac{1-2r^2}{r^2}. \tag{$viii$}$$

We see that $(viii)$ implies $1 - (2 + \lambda_f)r^2$ is positive. Consequently application of Lemma B1 with

$$b = \frac{(1-(2+\lambda_f)r^2)(1-3r^2)r'}{(1-2r^2)^2}$$

and

$$c = \frac{(1-r^2)r}{1-2r^2}$$

implies that

$$\rho_0(h(C), f'h(C)) < b$$

provided that

$$\begin{aligned}
|\lambda_f - 1| &< \sqrt{1 + (b/c)^2} - 1 \\
&= \sqrt{1 + \left(\frac{1-(2+\lambda_f)r^2)}{1-2r^2}\right)^2 \left(\frac{1-3r^2}{1-r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - 1.
\end{aligned} \tag{ix}$$

Lemma B3 together with (iv), (vi) and (vii) gives

$$\begin{aligned}
\rho_0(C, f(C)) &= \rho_0(G(h(C)), G(f'h(C))) \\
&\leq \left(\frac{1-2r^2}{1-(2+\lambda_f)r^2}\right) \left(\frac{1-2r^2}{1-3r^2}\right) \rho_0(h(C), f'h(C)) \\
&< r'.
\end{aligned}$$

To determine the family $\mathcal{S}(r, \epsilon(r,r'))$ for which $(B_{r'}(C), R(B_{r'}(C)))$ is stable we need to consider the assumptions we have made on r, r' and λ_f. There are two assumptions $(viii)$ and (ix). We shall show that it is sufficient to assume

$$|\lambda_f - 1| < \sqrt{1 + \left(\frac{1-(3+|\lambda_f - 1|)r^2}{1-2r^2}\right)^2 \left(\frac{1-3r^2}{1-r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - 1. \qquad (x)$$

We have $1 - 2r^2 > r^2$ for $r < 1/\sqrt{3}$ so that

$$\begin{aligned} |\lambda_f - 1| &< \sqrt{1 + \left(\frac{1-3r^2}{r^2}\right)^2 r'^2} - 1 \\ &< \frac{1-3r^2}{r^2} \end{aligned} \qquad (xi)$$

which is equivalent to

$$1 - (3 + |\lambda_f - 1|)r^2 > 0. \qquad (xii)$$

It follows from (xii) that

$$\lambda_f < |\lambda_f - 1| + 1 < \frac{1-2r^2}{r^2}.$$

Hence we have $(viii)$. It is obvious that (ix) follows from (x) and (xii). Thus our proof is complete.

Special Note: Improved Results.

In his recent preprint "On the stable basin theorem" J.R.Parker proves the following modified (and improved) version of the stable basin theorems of Basmajian and Miner (see [3]) and Kamiya (see [23]) for complex hyperbolic 2-space.

Theorem. *Given positive numbers r and r' with $0 < r, r' < 1$, the pair of open sets $(B_{r'}, \overline{B_{1/r'}}^{c})$ is stable with respect to the family $\mathcal{S}(r, \epsilon(r,r'))$ where $\epsilon(r,r')$ is given by*

$$\epsilon = \frac{\sqrt{1+(1-r^4)s^2} - 1 - r^2(1-r^2)s^2}{1 - r^4 s^2}$$

and $s = r'/r$.

We observe that using our lemmas 4.1.1, 4.1.2, 4.2 and 4.3 (see Chapter Four) in place of his lemmas 2.2(i), 2.2(ii), 2.1(ii) and 2.1(i) as used in his proof of the above theorem, we can prove the corresponding result in octonionic hyperbolic 2-space.

APPENDIX C.

Strengthening the Quaternionic Stable Basin Theorem.

Kim and Parker proved the following quaternionic stable basin theorem for quaternionic hyperbolic 2-space (see Theorem 4.1 of [25]).

Theorem. *Given positive real numbers r and r' with $r < 1/2$ and $r' < 1$, the pair of open sets $(B_{r'}, \iota(B_{r'}))$ is stable with respect to the family of loxodromic elements g with fixed points in B_r and $\iota(B_r)$ satisfying $X(g) < \epsilon(r, r')$ where $\epsilon(r, r')$ is a function of r, r' such that for a fixed r', $\epsilon(r, r') = O(1/r)$ as $r \to \infty$.*

Remark. In the above theorem ι refers to an inversion which transposes o and ∞ and $X(g)$ represents an upper bound on the deviation from 1 of the quaternionic dilation factors of the loxodromic maps g.

In his recent preprint "On the stable basin theorem" J.R.Parker proves the following modified (and improved) version of the stable basin theorems of Basmajian and Miner (see [3]) and Kamiya (see [23]) for complex hyperbolic 2-space.

Theorem. *Given positive numbers r and r' with $0 < r, r' < 1$, the pair of open sets $(B_{r'}, \overline{B_{1/r'}}^{c})$ is stable with respect to the family $\mathcal{S}(r, \epsilon(r, r'))$ where $\epsilon(r, r')$ is given by*

$$\epsilon = \frac{\sqrt{1 + (1 - r^4)s^2} - 1 - r^2(1 - r^2)s^2}{1 - r^4 s^2}$$

and $s = r'/r$.

Consider the following lemma.

Lemma C. *If g is in a Heisenberg similarity group fixing the origin and λ is the magnitude of the quaternionic dilation factor of g, then for all $X \in \partial\mathbf{H}^2_{\mathbb{H}}$ the following inequality holds*

$$\rho_0(g(X), X) \leq \left(|\lambda\mu - 1|(\lambda + 1) + 2|\mu\nu - 1|\lambda\right)^{1/2} \rho_0(X, o)$$

where μ, ν are unit quaternions.

Proof. Let the loxodromic element g be represented by the matrix

$$\begin{bmatrix} \lambda\mu & 0 & 0 \\ 0 & \mu\nu & 0 \\ 0 & 0 & \lambda^{-1}\mu \end{bmatrix} \in PSp(2, 1).$$

Let $X \in \partial\mathbf{H}^2_{\mathbb{H}}$ be represented in projective space by the vector

$$\begin{bmatrix} -|\zeta|^2 + v \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}.$$

Projecting on the right hand side we have

$$g(X) = \begin{bmatrix} \lambda^2\mu(-|\zeta|^2 + v)\overline{\mu} \\ \lambda\sqrt{2}\mu\nu\zeta\overline{\mu} \\ 1 \end{bmatrix}.$$

This implies the following inequality

$$\begin{aligned}
\rho_0(g(X), X) &= \left|\lambda^2\mu(-|\zeta|^2 + v)\overline{\mu} + 2\lambda\overline{\zeta}\mu\nu\zeta\overline{\mu} - |\zeta|^2 - v\right|^{1/2} \\
&= \left|(\lambda\mu - 1)(-|\zeta|^2 + v)\lambda\overline{\mu} + 2\lambda\overline{\zeta}(\mu\nu - 1)\zeta\overline{\mu} - (-|\zeta|^2 - v)(\lambda\overline{\mu} - 1)\right|^{1/2} \\
&\leq \left(|\lambda\mu - 1||\lambda| - |\zeta|^2 + v| + 2|\mu\nu - 1|\lambda|\zeta|^2 + |\lambda\mu - 1|| - |\zeta|^2 + v|\right)^{1/2} \\
&\leq (|\lambda\mu - 1|(\lambda + 1) + 2|\mu\nu - 1|\lambda)^{1/2} \,| - |\zeta|^2 + v|^{1/2} \\
&= (|\lambda\mu - 1|(\lambda + 1) + 2|\mu\nu - 1|\lambda)^{1/2} \,\rho_0(X, o).
\end{aligned}$$

We observe that by imitating Parker's proof of the above theorem (for complex hyperbolic 2-space) step by step, we can (using our Lemma C in place of his Lemma 2.1(ii)) show that the corresponding result holds in quaternionic hyperbolic 2-space.

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