Representations of the space of n-theta functions

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Abstract

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Let X be a smooth complex projective curve with group of automorphisms G. In this thesis we apply the *Holomorphic Lefschetz Theorem* in certain cases to compute the decomposition of the space $H^0(J_X, \mathcal{O}(n\Theta))$ into a sum of irreducible representations of G, where J_X is the Jacobian variety of X and $\mathcal{O}(\Theta)$ is the theta line bundle of J_X. Namely we compute this decomposition in the cases when X is the Klein quartic curve, the Bring curve of genus 4 and the Macbeath curve of genus 7.

Declaration

The work in this thesis is based on research carried out betwen October 1999 and March 2003 under the supervision of Dr. W.M. Oxbury and Dr. W. Klingenberg at Department of Mathematical Sciences of the University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Contents

Chapter 0

Introduction

Throughout this work all curves will be assumed to be smooth projective curves over $\mathbb{C}$ and we sometimes will make no distinction between a curve and its corresponding compact Riemann surface. For any smooth projective curve X there is an abelian variety, the Jacobian J_X and a divisor Θ of J_X. We briefly mention some well known facts about Jacobians of curves and we refer to [13] or [2] for some background reading. We can say that the Jacobian is the moduli space of line bundles of degree 0 on X, that is, J_X parametrizes isomorphism classes of line bundles of degree 0 on the curve X. Let S^bX denote the symmetric product of the curve X. For each positive integer b, there is a morphism

$$\alpha : S^bX \to J_X$$

called the Abel-Jacobi map. Let g denote the genus of X. If $b \leq g$ then α is generically injective and if $b = g$ then α is a birational map. By tensoring with a fixed line bundle of degree $g-1$ we get an isomorphism of J_X with the moduli space J_X^{g-1} of line bundles of degree $g-1$ on X. The theta divisor can be defined as the image of the map

$$\alpha : S^{g-1} \to J_X^{g-1}.$$

That is, the theta divisor is the reduced divisor supported on the set

$$\{L \in J_X^{g-1} \mid h^0(L) > 0\}.$$

Riemann proved that the singular locus of Θ is the set of line bundles for which $h^0(L) > 1$. The space $H^0(J_X, \mathcal{O}(n\Theta))$ of global sections $\mathcal{O}(n\Theta)$ can be identified with certain quasi-periodic functions on the universal cover of J_X, the theta functions of order n, the dimension of $H^0(J_X, \mathcal{O}(n\Theta))$ is n^g, the linear system $\mid n\Theta \mid$ is base point free for $n \geq 2$ and very ample for $n \geq 3$.

Although we do not talk about vector bundles, the motivation for this work initially was to study moduli spaces of vector bundles. Let $SU_X(n)$ be the moduli space of semi stable rank n vector bundles with trivial determinant. In [4] Beauville surveys some problems about these moduli spaces and one can find there the following situation. Given $L \in J_X^{g-1}$ the set

$$\Theta_L = \{E \in SU_X(n) \mid h^0(X, E \otimes L) \neq 0\}$$

is a Cartier divisor on $SU_X(n)$ and the associated line bundle $\mathcal{L} = \mathcal{O}(\Theta_L)$ is the generator of the Picard group of $SU_X(n)$. There is a rational map

$$\Psi : SU_X(n) \rightarrow \mathbb{P}H^0(J_X, \mathcal{O}(n\Theta)) = \mid n\Theta \mid$$

given by $\Psi(E) = \{L \in J_X^{g-1} \mid h^0(X, E \otimes L) \neq 0\}$. It is known that $\Psi(E)$ is either a divisor in the linear system $\mid n\Theta \mid$ or equal to J_X^{g-1}. There is a canonical isomorphism

$$H^0(SU_X(n), \mathcal{L}) \cong H^0(J_X^{g-1}, \mathcal{O}(n\Theta))^*$$

making the following diagram commutative:

$$\begin{tikzcd}
& \mid \mathcal{L} \mid^* \arrow[dd, "\wr"] \\
SU_X(n) \arrow[ur, "\phi_{\mathcal{L}}"] \arrow[dr, "\Psi"'] & \\
& \mid n\Theta \mid
\end{tikzcd}$$

So the base points of the linear system $\mid \mathcal{L} \mid^*$ are the vector bundles $E \in SU_X(n)$ such that $\Psi(E) = J_X^{g-1}$.

Suppose that the curve X has a non trivial group of automorphisms G. By functoriality G acts on $SU_X(n)$. Our initial intention was to study the behaviour of the fixed points of G in $SU_X(n)$ with respect to the map Ψ, in particular we wanted to know whether these fixed points are base points for the linear system $\mid \mathcal{L} \mid^*$ when $n \geq 3$. Thus one first has to find a way to compute these fixed points.

The group G acts on $H^0(J_X, \mathcal{O}(n\Theta))$. Dolgachev computed the decomposition of $H^0(J_X, \mathcal{O}(2\Theta))$ as a sum of irreducible representations of G when X is the Klein quartic and used this decomposition to prove the existence of fixed points of G in $SU_X(2)$, see proof of corollary 6.3 in [7]. So we thought that it might be useful to have the decomposition of $H^0(J_X, \mathcal{O}(n\Theta))$ as a sum of irreducible representations of G. The problem of finding this decomposition for a given curve X is interesting in its own right and this is the central problem of this work. In the first chapter we describe the strategy and tools to find this decomposition. As one may suspect the problem is reduced to computing the traces of automorphisms $h \in G$ on $H^0(J_X, \mathcal{O}(n\Theta))$ for which we make use of the Holomorphic Lefschetz Theorem 1.4 . One notices that it is enough to compute the trace of h on $H^0(S^{g-1}X, K^{\otimes n}_{S^{g-1}X})$, where $S^{g-1}X$ is the $g-1$ symmetric product of the curve X and g is the genus of X. In the second chapter we give some information about curves with non trivial automorphisms and then we present examples of curves on which we have computed the decomposition, namely the Klein quartic, the Macbeath curve of genus 7, and Bring's curve of genus 4. In the appendix there is a Maple worksheet which contains a program code that computes the trace of $h \in G$ on $H^0(S^{g-1}X, K^{\otimes n}_{S^{g-1}X})$ if h has prime order and if $\langle h \rangle \setminus \{1\}$ is contained in a conjugacy class of G. We used this program to compute some of those traces and the values obtained have been verified by hand.

Chapter 1

1.1 Decomposition of $H^0(J, n\Theta)$.

Let X be a complex smooth curve with group of automorphisms G. Let J be the Jacobian of X and let Θ be the theta divisor of J. Then G acts on J and Θ is invariant under the action of G.

Our goal will be to find the decomposition of $H^0(J, \mathcal{O}(n\Theta))$ as sum of irreducible representations of G.

We will do it by induction. Consider the exact sequence

$$0 \to \mathcal{O}(n\Theta) \to \mathcal{O}((n+1)\Theta) \to \mathcal{O}_\Theta((n+1)\Theta) \to 0. \tag{1.1}$$

By the Kodaira Vanishing Theorem we have

$$H^0\left(J, \mathcal{O}\left((n+1)\Theta\right)\right) = H^0\left(J, \mathcal{O}(n\Theta)\right) \oplus H^0\left(\Theta, \mathcal{O}((n+1)\Theta)\right)$$

for $n \geq 1$. Then all we have to do is to compute the decomposition for $H^0(\Theta, \mathcal{O}(n\Theta))$. Let $\alpha : S^{g-1}X \to \Theta$ be the Abel-Jacobi map and let $\mathcal{L} = \alpha^* \mathcal{O}_\Theta(\Theta)$.

Lemma 1.1. *We have $\mathcal{L}^n = K_{S^{g-1}X}^{\otimes n}$.*

Proof. (See [2] pg. 258) Let W be the singular locus of Θ and let $D = \alpha^{-1}(W)$. Then W has codimension at least 3 in Θ (see [2] pg. 250) and therefore D has codimension at least 2 in $S^{g-1}X$. Consider the inclusion of Θ in J^{g-1}. The tangent bundle of J^{g-1} is trivial and by adjunction we see that $K_{S^{g-1}}^{\otimes n} \cong \mathcal{L}^n$ away from D. Now by Hartogs' theorem, two line bundles that are isomorphic on an open set whose complement has codimension at least 2 are isomorphic on the whole variety. ■

Lemma 1.2. *a)* $\chi(K^{\otimes n}_{S^{g-1}X}) = n^g - (n-1)^g$.

b) For $n \geq 2$ *we have* $H^i(\Theta, \mathcal{O}(n\Theta)) \cong H^i(S^{g-1}X, \mathcal{L}^n)$.

We need the theorem below to prove Lemma 1.2.

A line bundle λ on a variety X is called semi-ample, if for some $\mu > 0$ the sheaf λ^μ is generated by global sections.

Let X be a projective variety and λ be an invertible sheaf on X. If $H^0(X, \lambda^\mu) \neq 0$, the sections of λ^μ define a rational map

$$\phi_\mu = \phi_{\lambda^\mu} : X \to \mathbb{P}(H^0(X, \lambda^\mu)^*).$$

The *Iitaka dimension* $\kappa(\lambda)$ of λ is given by

$$\kappa(\lambda) = \begin{cases} -\infty & \text{if } H^0(X, \lambda^\mu) = 0 \ \forall \ \mu \\ \max\{\dim \phi_\mu(X) \mid H^0(X, \lambda^\mu) \neq 0\} & \text{otherwise.} \end{cases}$$

Theorem 1.3. *Let X be a projective manifold defined over a field K of characteristic zero and let λ be an invertible sheaf on X. If λ is semi-ample and $\kappa(\lambda) = n = \dim X$, then*

$$H^b(X, \lambda^{-1}) = 0 \quad for \quad b < n.$$

Proof. See corollary 5.6 b) in [10]. ■

Proof of Lemma 1.2. We postpone the proof of a) to section 1.3. For b), notice that since α is surjective the natural map

$$\alpha^* : H^0(\Theta, \mathcal{O}(n\Theta)) \to H^0(S^{g-1}X, \alpha^*\mathcal{O}(n\Theta))$$

is injective. From the exact sequence (1.1) we see that $h^0(\Theta, \mathcal{O}(n\Theta)) = n^g - (n-1)^g$, then by a), $\chi(\mathcal{L}^n) = h^0(\Theta, \mathcal{O}(n\Theta))$. On the other hand, since $\mathcal{O}(n\Theta)$ is ample, $\alpha^*\mathcal{O}_\Theta(n\Theta)$ is semi-ample. Notice that the Iitaka dimension of $\mathcal{L}^n$ is $g - 1 =$

$\dim S^{g-1}X$ because α is a birational map between $S^{g-1}X$ and Θ. Then by Theorem 1.3 and Serre Duality Theorem we have

$$H^i(S^{g-1}X, \alpha^*\mathcal{O}(n\Theta) \otimes K_{S^{g-1}X}) = 0 \quad \text{for} \quad i > 0 \quad \text{and} \quad n \geq 1.$$

So α^* is an isomorphism for $n \geq 2$. ■

To compute the decomposition of $H^0(S^{g-1}X, \mathcal{L})$ we first need to compute the trace on $H^0(S^{g-1}X, \mathcal{L})$ of one element in each conjugacy class of G. Once we have these traces we only have to solve a system of linear equations (if we know the character table of G). The traces will be computed using Lefschetz theorem 1.4.

1.2 The fixed point Theorem

Let E be a vector bundle on a smooth variety X. Let G be a finite group acting on X. We say that G acts on E if for each $g \in G$ there is an isomorphism of vector bundles $\phi_g : g^*E \to E$ such that given $g, h \in G$ we have $\phi_{g \cdot h} = \phi_h \circ h^*(\phi_g)$ (see definition of G-linearized vector bundle in [7]).

Suppose that X is a variety with a trivial action of a finite group G, i.e. every element of G acts as the identity. Let $V_1, \cdots, V_m$ be the irreducible representations of G. Then any vector bundle E on X with action of G is isomorphic to a vector bundle of the form

$$\bigoplus_i V_i \otimes E_i,$$

where E_i is a unique vector bundle (which of course depends on E) with trivial action of G. Moreover if we have an exact sequence of vector bundles on X

$$0 \to A \to B \to C \to 0$$

with action of G, then for each i we have an exact sequence

$$0 \to V_i \otimes A_i \to V_i \otimes B_i \to V_i \otimes C_i \to 0.$$

This how the exact sequence (1.19) below is obtained.

For $h \in G$ and E as before define

$$ch_h(E) = \sum_i \chi_i(h) \cdot ch(E_i), \tag{1.2}$$

where $ch(E_i)$ is the Chern character of E_i and $\chi_i(h)$ represents the trace of $h|V_i$ (see definition of $ch\ u(g)$ in [3] just before 3.1).

If G acts on E and $h \in G$ acts trivially on X then E has a decomposition $E = \oplus_{j=0}^{n} E(\nu^j)$, where $E(\nu^j)$ is the subvector bundle of E on which h acts as ν^j, $\nu = e^{2i\pi/n}$ and n is the order of h. For each vector bundle $E(\nu^i)$ define the characteristic class

$$\mathcal{U}(E(\nu^i)) = \prod_j \left(\frac{1 - \frac{e^{-x_j}}{\nu^i}}{1 - \frac{1}{\nu^i}} \right)^{-1}, \tag{1.3}$$

where $\{x_j\}_{j=1}^{\dim X}$ are the Chern roots of $E(\nu^i)$, see (4.5) in [3].
For instance, if $1 + \sum_{i=1}^{3} c_i$ is the total Chern class of $E(\nu)$ and the variety X has dimension at most three we have

$$\begin{aligned}\mathcal{U}(E(\nu)) = \; & \left(-\frac{({c_1}^3 - 3\,c_1\,c_2 + 3\,c_3)\,(\nu^2 + 4\,\nu + 1) + 6c_3}{6(\nu-1)^3} \right. \\ & \left. - \frac{3(\nu+1)\,(({c_1}^2 - 2\,c_2)\,c_1 - {c_1}^3 + 3\,c_1\,c_2 - 3\,c_3)}{6(\nu-1)^3} \right) t^3 \\ & + \left(\frac{({c_1}^2 - 2\,c_2)\,(\nu+1) + 2c_2}{2(\nu-1)^2} \right) t^2 - \frac{c_1\,t}{\nu - 1} + 1.\end{aligned}$$

If ν is a cube root of unity then

$$\begin{aligned}\mathcal{U}(E(\nu)) = \; & \left(-\frac{1}{18}\,c_3 + \frac{5}{18}\,c_1\,c_2 + \left(-\frac{1}{9}\,c_3 + \frac{2}{9}\,c_1\,c_2 - \frac{1}{18}\,{c_1}^3 \right) \nu - \frac{1}{9}\,{c_1}^3 \right) t^3 \\ & + \left(\frac{1}{6}\,{c_1}^2\,\nu + \frac{1}{3}\,c_2 \right) t^2 + \left(\frac{1}{3}\,c_1\,\nu + \frac{2}{3}\,c_1 \right) t + 1.\end{aligned}$$

If $\nu = -1$ then

$$\mathcal{U}(E(-1)) = 1 + \frac{c_1}{2} + \frac{c_1 c_2}{8} - \frac{c_1^3}{24} + \frac{c_2}{4}. \tag{1.4}$$

Theorem 1.4. (Holomorphic Lefschetz Theorem) (see [3] Theorem 4.6). *Let X be a compact complex manifold, V a holomorphic vector bundle over X, and g be a finite order automorphism of the pair (X, V). Let X^g denote the fixed point set of g. Then we have*

$$\sum(-1)^i \text{trace}\,(g \mid H^i(X, V)) = \int_{X^g} \frac{ch_g(V\mid_{X^g}) \cdot \Pi_j \mathcal{U}(N(\nu^j)) \cdot \text{td}\,(X^g)}{\det(1 - g_|(N^g)^\vee)}.$$

Assume that the set of fixed points of g on X is finite. Then for each fixed point x we have $\text{td}(x) = 1$, $N^g = T_{X_x}$. So $(N^g)^\vee = \Omega_{X_x}$, $\mathcal{U}(N(\nu^j))$=1. We have $ch(E\mid_x)(g) = trace(g) \mid E_x$, where E_x is the fibre of E at x. Then we have

Corollary 1.5. (Atiyah-Bott fixed point Theorem). *Let g be an automorphism of a compact complex manifold X with a finite set fix(g) of fixed points. Suppose that g acts on a holomorphic vector bundle $E \to X$. Then*

$$\sum(-1)^i \text{trace}\, g \mid H^i(X, E) = \sum_{x \in fixg} \frac{trace(g) \mid E_x}{\det(I - dg_x)},$$

where dg_x is the automorphism induced by g on the fibre of the cotangent bundle Ω_X of X at x.

1.3 Preliminaries on symmetric products

In this section we summarize some facts about symmetric products of curves. We refer to [20] for more details. Let X be a compact connected Riemann surface of genus g. We have

$$H^0(X, \mathbb{Z}) = \mathbb{Z}, \quad H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}, \quad H^2(X, \mathbb{Z}) = \mathbb{Z}\beta,$$

where β represents the fundamental class of a point $p \in X$. The ring structure of $H^*(X, \mathbb{Z})$ can be described as follows: One can choose a basis

$$\alpha_1, \cdots, \alpha_{2g}$$

of $H^1(X, \mathbb{Z})$ such that,

$$\alpha_i \alpha_j = 0 \text{ unless } \quad i - j = \pm g; \quad \alpha_i \alpha_{i+g} = -\alpha_{i+g}\alpha_i = \beta \quad (1 \le i \le g). \tag{1.5}$$

Then (1.5) is a complete set of relations for $H^*(X, \mathbb{Z})$. These relations imply

$$\alpha_i\beta = \beta\alpha_i = 0, \quad \beta^2 = 0.$$

If K is any field then $H^*(X, K) = H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} K$, and if X^n is the product of n copies of X, then $H^*(X^n, K) = H^*(X, K)^{\otimes n}$. We shall assume that K is a field of characteristic 0. Let

$$\alpha_{ik} = \underbrace{1 \otimes \cdots \otimes 1 \otimes \alpha_i}_{k^{th}\ place} \otimes 1 \otimes \cdots \otimes 1 \in H^1(X^n, K),$$

$$\beta_k = \overbrace{1 \otimes \cdots \otimes 1 \otimes \beta} \otimes 1 \otimes \cdots \otimes 1 \in H^2(X^n, K).$$

Then $H^*(X^n, K)$ is generated as a ring by the α_{ik}'s and the $\beta_k's$ $(1 \leq i \leq 2g$, $1 \leq k \leq n)$ subject to the relations

$$\alpha_{ik}\alpha_{jk} = 0 \ \text{ unless } \quad i - j = \pm g;$$

$$\alpha_{ik}\alpha_{i+g,k} = -\alpha_{i+g,k}\alpha_{ik} = \beta_k (1 \leq i \leq g);$$

$$\alpha_{ik}\alpha_{jl} = -\alpha_{jl}\alpha_{ik}.$$

We also have that

$$\alpha_{ik}\beta_k = \beta_k\alpha_{ik} = 0; \ \beta_k^2 = 0;$$

and that each β_k commutes with every element of $H^*(X^n, K)$. The symmetric group S_n acts on $H^*(X^n, K)$ by the rule

$$g(\alpha_{ik}) = \alpha_{i,g^{-1}(k)}, \quad g(\beta_k) = \beta_{g^{-1}(k)} \ \text{ for } \ g \in S_n.$$

The ring $H^*(S^nX, K)$ can be identified with the ring $H^*(X^n, K)^{S_n}$ of S_n-invariants of $H^*(X^n, K)$. The ring $H^*(X^n, K)^{S_n}$ is generated as a K-algebra by η and $\xi_1, \cdots, \xi_{2g}$ where

$$\xi_i = \sum_j^n \alpha_{ij} \quad \text{for} \quad 1 \leq i \leq 2g, \tag{1.6}$$

$$\eta = \sum_j^n \beta_j.$$

The ξ_i's anticommute with each other and commute with η. Moreover, let $\sigma_i = \xi_i\xi_{i+g} \in H^2(S^dX, \mathbb{Z})$, for $i = 1, \cdots, g$; then from (5.4) in [20] we have

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \ \sigma_i^2 = 0, \ \text{and} \ \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_a}\eta^b = \eta^d, \tag{1.7}$$

for $d = a + b$, $a, b \geq 0$, and distinct $i_1, \cdots, i_a$.

Then if

$$\vartheta = \sum_{i=1}^{g} \sigma_i \tag{1.8}$$

and $a + b = d$, we have

$$\vartheta^a \eta^b = a! \binom{g}{a} \eta^d. \tag{1.9}$$

In general the total Chern class of $S^d X$ is given by

$$ch(S^d X) = (1 + \eta\, t)^{(d-g+1)}\, e^{-\frac{\vartheta\, t}{1+\eta\, t}}, \tag{1.10}$$

where g is the genus of X (see [2] pg 339). Now can we prove Lemma 1.2 a):

Proof of Lemma 1.2 a) . (See Proposition 10.1 (3) in [23]) We have

$$\begin{aligned} td(S^d X) = & \left(\tfrac{\eta}{1-e^{-\eta}}\right)^{d-g+1} \textstyle\prod_{i=1}^{g} (1 + \sigma_i \tau), \\ = & \left(\tfrac{\eta}{1-e^{-\eta}}\right)^{d-g+1} \textstyle\sum_{i=0}^{g} \frac{\tau^i \vartheta^i}{i!} \end{aligned} \tag{1.11}$$

where

$$\tau = \frac{\eta e^{-\eta} + e^{-\eta} - 1}{\eta(1 - e^{-\eta})}.$$

By formula (1.10) the Chern class of $K^{\otimes n}_{S^{g-1}X}$ is $1 + n\vartheta$, therefore the Chern character of $K^{\otimes n}_{S^{g-1}X}$ is

$$ch(K^{\otimes n}_{S^{g-1}X}) = e^{n\vartheta} = \prod_{i=1}^{g} (1 + n\sigma_i).$$

So by Hirzebruch-Riemann-Roch

$$\chi(K^{\otimes n}_{S^{g-1}X}) = \deg \left\{ \prod_{i=1}^{g} (1 + \sigma_i(\tau + n)) \right\}_{g-1}.$$

Notice that none of the terms of the expression

$$\prod_{i=1}^{g} (1 + \sigma_i(\tau + n))$$

is divisible by a square of a σ_i, so by (1.7) we can assume $\sigma_1 = \cdots = \sigma_g = \eta$. Then what we want to compute is the coefficient of η^{g-1} in the following expression

$$(1 + \eta(\tau + n))^g = \left(\frac{\eta}{1 - e^{-\eta}} \right)^g (n - (n-1)e^{-\eta})^g,$$

that is

$$\chi(K^{\otimes n}_{S^{g-1}X}) = \mathrm{Res}_{\eta=0} \left(\frac{n - (n-1)e^{-\eta}}{1 - e^{-\eta}} \right)^g,$$

setting $z = 1 - e^{-\eta}$ the last is equal to

$$\mathrm{Res}_{z=0}\frac{((n-1)z+1)^g}{z^g(1-z)} = \sum_{i=0}^{g-1}\binom{g}{i}(n-1)^i = n^g - (n-1)^g. \qquad \blacksquare$$

The universal effective divisor of degree d on X is the divisor

$$\Delta \subset X \times S^d X$$

that, for any $D \in S^d X$, cuts on $X \cong X \times \{D\}$ exactly the divisor D. Let

$$f : S \to S^d X$$

be a morphism and consider the divisor $\Delta' = (\mathrm{Id}_X \times f)^*\Delta$ on $X \times S$. Let $\phi : X \times S \to S$ and $\pi : X \times S^d X \to S^d X$ be the natural projections.

$$\begin{tikzcd} X \times S \arrow[r, "\mathrm{Id}_X \times f"] \arrow[d, "\phi"'] & X \times S^d X \arrow[d, "\pi"] \\ S \arrow[r, "f"] & S^d X \end{tikzcd}$$

We will need the following Lemma to prove Lemma 1.16 below.

Lemma 1.6. *We have*

a) $\phi_*\mathcal{O}_{\Delta'}(\Delta')$ *is locally free and* $\pi_*\mathcal{O}_\Delta(\Delta) \cong T_{S^d X}$.

b) $f^*\pi_*\mathcal{O}_\Delta(\Delta) \cong \phi_*\mathcal{O}_{\Delta'}(\Delta')$ *and* $f^*R^1\pi_*\mathcal{O}(\Delta) \cong R^1\phi_*\mathcal{O}(\Delta')$.

Proof. See [2], **IV**, §2. ■

1.4 Fixed points in the symmetric products

Suppose that X is a curve with an automorphism h. In order to apply Theorem 1.4 when h is acting on the symmetric product $S^b X$ we will need some information about the fixed point set of h at $S^b X$ and its normal bundle at $S^b X$. For instance, we need to know the Todd class of the fixed point set and the product $\Pi_j \mathcal{U}(N(\nu^j))$.

Suppose that h has order p. If we consider the map

$$f_{k,0} : S^k X \to S^{pk} X$$

$$D \mapsto \sum_{i=0}^{p-1} h^i D,$$

then $f_{k,0}(S^k X)$ is a subset of fixed points of h in $S^{pk}X$. Let D be an effective divisor of degree d invariant under the action of h. Consider the embedding

$$\mathcal{A}_D : S^{pk}X \hookrightarrow S^{pk+d}X$$

$$u \mapsto u + D.$$

The image of $S^k X$ under the map $f_{k,D} = \mathcal{A}_D \circ f_{k,0}$ is a subset of fixed points of h in $S^{pk+d}X$. Notice that when $k = 0$ the image of $f_{0,D}$ is the divisor D.

Now we will describe the fixed point set $\mathrm{fix}\,(h)$ of h in $S^b X$. If $b = 1$ then see Theorem 2.1 below. Take m, l such that

$$b = pm + l$$

and $m \geq 0$ and $p > l \geq 0$.

For each integer k such that $m \geq k \geq 0$, let $d_k = b - kp$. Let $(S^{d_k}X)^h$ denote the fixed point set of h in $S^{d_k}X$. Define A_k as the set of divisors $D \in (S^{d_k}X)^h$ satisfying the following property: if x is a point in the support of D then $D - \sum_{i=0}^{p-1} h^i x$ is not an effective divisor.

Now consider the set

$$F_k = \bigcup_{D \in A_k} f_{k,D}(S^k X).$$

Notice that $F_i \cap F_j = \emptyset$ and $f_{D_1}(S^i X) \cap f_{D_2}(S^i X) = \emptyset$ for $D_1, D_2 \in A_i$. It is easy to verify the following:

Lemma 1.7.

$$\bigcup_{k=0}^{m} F_k = \mathrm{fix}\,(h).$$

Notice that if p is a prime number, then

$$A_k = \left\{ D = a_1 x_1 + \cdots + a_s x_s \mid 0 \leq a_j \leq p-1 \quad \text{and} \quad \sum_{j=1}^{s} a_j = d_k \right\},$$

where $x_1, ..., x_s$ are the fixed points of h in X and there are

$$\sum_{j=0}^{m-k} (-1)^j \binom{s}{j} \binom{s-1+d_k-jp}{d_k-jp} \tag{1.12}$$

divisors in A_k.

If p is not a prime number then the divisors in A_k are not necessarily supported on the fixed points of h in X. For instance there are situations in which h has no fixed points in X but h^2 has finitely many fixed points.

Let $f : X \to Y$ be a morphism of degree p of smooth curves. Then there is an embedding $i : S^kY \to S^{pk}X$ which sends $D \in S^kY$ to the divisor $f^*D \in S^{pk}X$.

If we take f to be the quotient map

$$f : X \to X/\langle h \rangle = Y,$$

then the map $f_{k,0}$ splits as

$$f_{k,0} : S^kX \overset{a}{\twoheadrightarrow} S^kY \overset{i}{\hookrightarrow} S^{pk}X,$$

where a is the natural map induced by f on the symmetric product. From this we see that the fixed point set of h in S^bX is a disjoint union of varieties which are isomorphic to symmetric products of the quotient curve Y.

Remember from section 1.3 the definition of the cohomology classes η, ϑ on the symmetric product of a curve. The proof of the following Lemma involves at least two different symmetric products and we will use the same notation to represent these cohomology classes regardless of the symmetric product on which they are defined as this will be clear from the context.

Lemma 1.8. *Consider the induced map $i^* : H^*(S^{pk}X, \mathbb{Z}) \to H^*(S^kY, \mathbb{Z})$. Then we have $i^*\eta = \eta$ and $i^*\vartheta = p\vartheta$.*

Proof. Consider the maps

$$f_{k,0}^* : H^*(S^{pk}X, \mathbb{Z}) \xrightarrow{i^*} H^*(S^kY, \mathbb{Z}) \xrightarrow{a^*} H^*(S^kX, \mathbb{Z}).$$

We first will show that a^* is injective and that $a^*\eta = p\eta$, then we will see that $f_{k,0}^*\vartheta = pa^*\vartheta$ and $f_{k,0}^*\eta = p\eta$. From this we deduce that $i^*\eta = \eta$ and $i^*\vartheta = p\vartheta$.

Notice that the natural map

$$f^* : H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C})$$

is injective by (1.2) in [20]. The commutative diagram

$$\begin{array}{ccc} X^k & \longrightarrow & Y^k \\ \big\downarrow & & \big\downarrow \\ S^kX & \rightarrow & S^kY \end{array}$$

induces another commutative diagram

$$\begin{array}{ccc} H^*(S^kY,\mathbb{C}) & \xrightarrow{a^*} & H^*(S^kX,\mathbb{C}) \\ \big\downarrow & & \big\downarrow \\ H^*(Y^k,\mathbb{C}) & \longrightarrow & H^*(X^k,\mathbb{C}) \end{array}$$

where the vertical maps and the lower map are injective, therefore a^* is injective. Now fix a symplectic basis

$$\alpha_1, \cdots, \alpha_\gamma, \alpha_{\gamma+1}, \cdots, \alpha_{2\gamma}$$

for $H_1(Y,\mathbb{Z})$. Above each cycle α_i there are p cycles $r_i, hr_i, \cdots, h^{p-1}r_i$ on X, and they satisfy

$$h^m r_i h^l r_j = \begin{cases} r_i r_j = \alpha_i\alpha_j & \text{if } m = l \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

The set $\mathcal{A} = \{h^m r_i \mid m = 0, \cdots, p-1 \text{ and } i = 1, \cdots, 2\gamma\}$ forms part of a symplectic basis

$$\mathcal{B} = \{\alpha'_1, \cdots, \alpha'_g, \alpha'_{g+1}, \cdots, \alpha'_{2g}\}$$

of $H_1(X,\mathbb{Z})$ in which

$$\alpha'_m = h^j r_{q+1}$$
$$\alpha'_{m+g} = h^j r_{q+1+\gamma}$$

for $m = qp + j$, where $1 \leq j \leq p$ and $0 \leq q \leq \gamma - 1$. Abusing our notation we will write α_m instead of α'_m. Consider the map

$$f^* : H^*(Y,\mathbb{Z}) \to H^*(X,\mathbb{Z}).$$

Under this map we have

$$f^*\alpha_i = \sum_{j=1}^{p-1} h^j r_i = \sum_{j=0}^{p-1} \alpha_{j+p(i-1)} \tag{1.14}$$

and

$$f^*\alpha_{i+\gamma} = \sum_{j=0}^{p-1} h^j r_{i+\gamma} = \sum_{j=1}^{p-1} \alpha_{j+p(i-1)+g}.$$

Claim: If $\alpha_i \in \mathcal{B} \setminus \mathcal{A}$ then

$$\sum_{j=0}^{p-1} h^j \alpha_i = 0.$$

We first prove the claim. If $\alpha_i \in \mathcal{B} \setminus \mathcal{A}$ then we have

$$\alpha \cdot \alpha_i = 0 \ \ \forall \alpha \in \mathcal{A}.$$

The automorphism h is compatible with the product $\cdot$, so

$$\alpha \cdot h^j \alpha_i = 0 \ \ \forall \alpha \in \mathcal{A} \ \text{ and } j = 0, \cdots, p-1.$$

Now $\sum_{j=0}^{p-1} h^j \alpha_i \in H^1(X, \mathbb{C})^{\langle h \rangle}$. By (1.2) in [20], $H^1(X, \mathbb{C})^{\langle h \rangle}$ is the image of the map

$$f^* : H^1(Y, \mathbb{C}) \to H^1(X, \mathbb{C})$$

which is obtained by tensoring the map

$$f^* : H^1(Y, \mathbb{Z}) \to H^1(X, \mathbb{Z})$$

with $\mathbb{C}$. Thus $H^1(X, \mathbb{C})^{\langle h \rangle}$ is generated by

$$\left\{ \sum_{j=1}^{p-1} h^j r_1, \cdots, \sum_{j=1}^{p-1} h^j r_\gamma, \sum_{j=1}^{p-1} h^j r_{\gamma+1}, \cdots, \sum_{j=1}^{p-1} h^j r_{2\gamma} \right\}.$$

Notice that $H^1(X, \mathbb{Z})^{\langle h \rangle}$ is not necessarily the image of the map

$$f^* : H^1(Y, \mathbb{Z}) \to H^1(X, \mathbb{Z}).$$

Since

$$\alpha \cdot \sum_{j=0}^{p-1} h^j \alpha_i = 0 \ \forall \alpha \in \mathcal{A},$$

we see that

$$\sum_{j=0}^{p-1} h^j \alpha_i = 0.$$

And this proves the claim.

We refer to section 1.3 for the definition of the cohomology classes η, $\alpha_{i,l}$, β_l, and ξ_i. Using the relations (1.13) we have

$$f^*(\beta) = f^*(\alpha_i \alpha_{i+\gamma}) = \left(\sum_{j=0}^{p-1} h^j r_i\right)\left(\sum_{j=0}^{p-1} h^j r_{i+\gamma}\right) = p\beta. \tag{1.15}$$

Now under the map

$$a^* : H^*(Y^k, \mathbb{Z}) \to H^*(X^k, \mathbb{Z})$$

we have from (1.14)

$$a^*(\alpha_{i,l}) = \sum_{j=0}^{p-1} \alpha_{j+p(i-1),l}$$

and

$$a^*(\beta_l) = p\beta_l.$$

Suppose $i \leq \gamma$. Then using the definition of ξ_i, we get

$$a^*(\xi_i) = \sum_{l=1}^{k} \sum_{j=0}^{p-1} \alpha_{j+p(i-1),l}.$$

Notice from the definition of $\mathcal{B}$ that

$$\sum_{r=0}^{p-1} h^r \alpha_{pi,l} = \sum_{j=1}^{p} \alpha_{j+p(i-1),l}.$$

Then

$$a^*(\xi_i) = \sum_{j=0}^{p-1} h^j \xi_{pi}.$$

and

$$a^*(\xi_{i+\gamma}) = \sum_{l=1}^{k} \sum_{j=0}^{p-1} \alpha_{j+p(i-1)+g,l} = \sum_{j=0}^{p-1} h^j \xi_{pi+g}.$$

From the definition of η and from (1.15) we have

$$a^*(\eta) = \sum_{l=1}^{k} p\beta_l = p\eta.$$

Consider the map

$$f_{k,0}^* : H^*(X^{pk}, \mathbb{Z}) \to H^*(X^k, \mathbb{Z}).$$

In this case $f_{k,0} : X^k \to (X^k)^p$ is defined by the rule $D \mapsto (D, hD, \cdots, h^{p-1}D)$. Now we will compute $f_{k,0}^*(\xi_m)$. We first will compute $f_{k,0}^*(\alpha_{il})$. Notice

$$H^*(X^{pk}, \mathbb{Z}) = H^*(X, \mathbb{Z})^{\otimes pk} = H^*(X^k, \mathbb{Z})^{\otimes p}.$$

In particular

$$H^1(X^{pk}, \mathbb{Z}) =$$

$$\bigoplus_{i=1}^{p} \underbrace{H^0(X^k, \mathbb{Z}) \otimes \cdots \otimes H^0(X^k, \mathbb{Z}) \otimes H^1(X^k, \mathbb{Z})}_{i^{th}\ place} \otimes H^0(X^k, \mathbb{Z}) \otimes \cdots \otimes H^0(X^k, \mathbb{Z})$$

$$\left(\cong \bigoplus_{i=1}^{p} H^1(X^k, \mathbb{Z}) \right).$$

Suppose that $l = sk + j$, where s, j are non-negative integers and $1 \leq j \leq k$. We now can see that

$$f_{k,0}^*(\alpha_{il}) = h^s \alpha_{i,j}$$

and

$$f_{k,0}^*(\beta_l) = h^s(\alpha_{i,j}\alpha_{i+g,j}) = h^s \beta_j = \beta_j.$$

Thus

$$f_{k,0}^*(\eta) = \sum_{l=1}^{pk} f_{k,0}^*(\beta_l) = p \sum_{j=1}^{k} \beta_j = p\eta$$

and

$$f_{k,0}^*(\xi_i) = f_{k,0}^* \left(\sum_{l=1}^{pk} \alpha_{i,l} \right) = \sum_{s=0}^{p-1} h^s \sum_{j=1}^{k} \alpha_{i,j} = \sum_{j=0}^{p-1} h^j \xi_i = \sum_{j=1}^{k} \sum_{s=0}^{p-1} h^s \alpha_{i,j}.$$

From this we see that if $\alpha_m \in \mathcal{B} \setminus \mathcal{A}$ then

$$f_{k,0}^*(\xi_m) = 0.$$

Let $m \leq g$ such that $\alpha_m \in \mathcal{A}$. Write $m = qp + j$ with $1 \leq j \leq p$. So

$$f_{k,0}^*(\xi_m) = \sum_{j=0}^{p-1} h^j \xi_{p(q+1)} = a^*(\xi_{q+1}).$$

and

$$f_{k,0}^*(\xi_{m+g}) = \sum_{j=0}^{p-1} h^j \xi_{p(q+1)+\gamma} = a^*(\xi_{q+1+\gamma}).$$

Then we have

$$f_{k,0}^*(\vartheta) = \sum_{m=1}^{g} f_{k,0}^*(\xi_m \xi_{m+g})) = p \sum_{q=0}^{\gamma-1} a^*(\xi_{(q+1)}\xi_{(q+1)+\gamma}) = pa^*(\vartheta). \blacksquare$$

1.5 Normal bundles of the fixed point sets

In this section we will consider the normal bundles of the components of the fixed point set of h in S^bX. The aim will be to find a way to compute the characteristic classes of their eigenvector bundles as defined in (1.3). Consider the quotient map

$$f : X \to X/\langle h \rangle = Y.$$

Let g and γ be the genus of X and Y respectively and let R be the ramification divisor of f at X.

From section 1.4 we know that these components are symmetric products of the quotient curve Y. A component of dimension k is the image of S^kX under the map $f_{k,D}$ for some $D \in A_k$ and we identify it with S^kY, the embedding of S^kY into S^bX is given by the composition map

$$S^kY \overset{i}{\hookrightarrow} S^{pk}X \overset{\mathcal{A}_D}{\hookrightarrow} S^{pk+d_k}X. \tag{1.16}$$

We will use the following notation:

- N_i for the normal bundle of S^kY in $S^{pk}X$,
- $N_{\mathcal{A}_D \circ i}$ for the normal bundle of S^kY in $S^{pk+d_k}X$ and
- $N_{\mathcal{A}_D}$ for the normal bundle of $S^{pk}X$ in $S^{pk+d_k}X$.

The total Chern class of N_i is given by

$$c(N_i) = \frac{i^*c(S^{pk}X)}{c(S^kY)}.$$

Using formula (1.10) and Lemma 1.8 we obtain

$$c(N_i) = \left((1+\eta\, t)^A e^{-\frac{\vartheta\, t}{1+\eta\, t}}\right)^{p-1}, \tag{1.17}$$

where

$$A = k + \frac{\gamma - g}{p-1} = k + 1 - \gamma - \frac{\deg(R)}{2(p-1)}.$$

In particular when $k = 1$ the normal bundle has degree

$$(p-1)\left(2 - 2\gamma - \frac{\deg(R)}{2(p-1)}\right).$$

Let $D \in S^dX$. Using (1.10) and that $\mathcal{A}_D^*\eta = \eta$ and $\mathcal{A}_D^*\vartheta = \vartheta$ we see that the normal bundle $N_{\mathcal{A}_D}$ has total Chern class

$$c(N_{\mathcal{A}_D}) = (1+\eta)^d. \tag{1.18}$$

Lemma 1.9. *Let $x \in X$ be a fixed point of an automorphism h of X of order p. Let d be a positive integer and let $Q = dx \in S^dX$. Suppose that h acts as ν^a on the tangent space T_x of x at X. Then $h_|(T_{S^dX})_Q$ has eigenvalues $\nu^a, \nu^{2a}, \cdots, \nu^{da}$, where $\nu = e^{2i\pi/p}$.*

Proof. In a neighbourhood of $(x, x, \cdots, x) \in X^d$ choose coordinates $(x_1, \cdots, x_d)$ so that $(x, x, \cdots, x)$ is the origin. Then in a neighbourhood of $Q = dx \in S^dX$ there are coordinates $(\sigma_1, \cdots, \sigma_d)$ (see [2] chap. **IV**, §2) defined by the property that the natural morphism

$$X^d \to S^dX$$

is given in a neighbourhood of $(x, x, \cdots, x)$ by

$$\sigma_i(x_1, \cdots, x_d) = i^{th} \text{ symmetric function of } (x_1, \cdots, x_d).$$

Now, if x is a fixed point of an automorphism h of X, we can assume that, in a neighbourhood of x, the action of h is the multiplication by a scalar λ. Then in our system of coordinates

$$hx_i = \lambda x_i$$

implies

$$h\sigma_i = \lambda^i \sigma_i. \qquad \blacksquare$$

Lemma 1.10. *Consider the map $f_{k,0}$ of section 1.4. Let*

$$Q = x + hx + \cdots + h^{p-1}x$$

be a point in the image of this map (that means $x \in S^kX$). Then $h \mid (T_{S^{pk}X})_Q$ has eigenvalues $1, \nu, \cdots \nu^{p-1}$, $\nu = e^{2i\pi/p}$, and the eigenspace of ν^i has dimension k.

Proof. The automorphism h acts on the vector bundle $i^*T_{S^{pk}X}$, and since h acts trivially on S^kY we have

$$i^*T_{S^{pk}X} = \bigoplus_{i=0}^{p-1}(i^*T_{S^{pk}X})(\nu^i),$$

so it is enough to consider a fibre of $i^*T_{S^{pk}X}$ at a general point x of S^kX. Let $x = p_1 + \ldots + p_k \in S^kX$ such that all the p_i's belong to different h-orbits and the orbit of p_i has exactly p elements for all $i = 1, \cdots, k$.

For each i, j choose disjoint open neighbourhoods $V_{i,j}$ of h^ip_j in X. So

$$\prod_{\substack{i=0,1\cdots,p-1 \\ j=1,\cdots,k}} V_{i,j}$$

is isomorphic to an open neighbourhood of

$$Q = \sum_{i=0}^{p-1} h^i x$$

in $S^{pk}X$. Then

$$(T_{S^{pk}X})_Q = \bigoplus_{\substack{i=0,1\cdots,p-1 \\ j=1,\cdots,k}} T_{h^ip_j}$$

From this we see that $h \mid (T_{S^{pk}X})_Q$ is a matrix conjugate to

$$\begin{pmatrix} B & 0 & \cdots & & & 0 \\ 0 & B & 0 & & \cdots & \cdot \\ \vdots & 0 & \ddots & & & \vdots \\ & & & \ddots & 0 & \\ & \vdots & & 0 & B & 0 \\ 0 & \cdot & \cdots & & 0 & B \end{pmatrix}$$

where B is a cyclic matrix of dimension $p \times p$ i.e.

$$\begin{pmatrix} 0 & \cdots & & & \cdot & 0 & 1 \\ 1 & 0 & \cdots & & & \vdots & 0 \\ 0 & 1 & 0 & & & & \\ \vdots & 0 & \ddots & \ddots & \vdots & & \\ & \vdots & & \ddots & 0 & & \\ & & & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & & & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of B is $\lambda^p - 1$, so the characteristic polynomial of $h \mid (T_{S^{pk}X})_Q$ is $(\lambda^p - 1)^k$. ■

Lemma 1.11. *Consider the divisor $D = d_1x_1 + \cdots + d_sx_s$, where x_i is a fixed point of h in X ($x_i \neq x_j$) and d_i is a positive integer. Suppose that h acts as ν^{a_i} (notice $(a_i, p) = 1$) on the tangent space T_{X,x_i} of x_i. Consider the composition map (1.16). Let $Q = x + hx + \cdots + h^{p-1}x + D$ be a point in the image of this map. Then the dimension of the eigenspace for ν^i of $h \mid (T_{S^{pk+d}X})_Q$ is $k + r_i$, where r_i is the number of times that ν^i appears in the following list*

$$\begin{gathered}\nu^{a_1}, \nu^{2a_1}, \cdots, \nu^{d_1a_1},\\ \nu^{a_2}, \nu^{2a_2}, \cdots, \nu^{d_2a_2},\\ \vdots\\ \nu^{a_s}, \nu^{2a_s}, \cdots, \nu^{d_sa_s}.\end{gathered}$$

Proof. Let x be a general point of S^kX (as in proof of Lemma 1.10) and put $P_0 = x + hx + \cdots + h^{p-1}x$. Then one can choose neighbourhoods V_0 of $P_0 \in S^{pk}X$ and V_i of $d_ix_i \in S^{d_i}X$ such that if D_1, D_2 belong to different neighbourhoods then D_1, D_2 have no common points in their supports. In this way

$$\prod_{i=0}^{s} V_i$$

is isomorphic to a neighbourhood of $Q = P_0 + D \in S^{pk+d}X$. So

$$(T_{S^{pk+d}X})_Q = (T_{S^{pk}X})_{P_0} \oplus \bigoplus_{i=1}^{l} (T_{S^{d_i}X})_{d_ip_i}$$

and we can apply Lemmas 1.9 and 1.10. ■

The normal bundle $N_{\mathcal{A}_D oi}$ has a decomposition

$$N_{\mathcal{A}_D oi} = \bigoplus_{i=0}^{p-1} N_{\mathcal{A}_D oi}(\nu^i).$$

We will need to know the Chern classes of the vector bundles $N_{\mathcal{A}_D oi}(\nu^i)$ in order to compute their characteristic classes. We have an exact sequence

$$0 \to N_i \to N_{\mathcal{A}_D oi} \to i^* N_{\mathcal{A}_D} \to 0$$

from which we obtain exact sequences

$$0 \to N_i(\nu^j) \to N_{\mathcal{A}_D \circ i}(\nu^j) \to i^* N_{\mathcal{A}_D}(\nu^j) \to 0. \tag{1.19}$$

Remark 1.12. Consider the exact sequence

$$0 \to T_{S^k Y} \to (\mathcal{A}_D \circ i)^* T_{S^{nk+d} X} \to N_{\mathcal{A}_D \circ i} \to 0.$$

Since h acts trivially on $T_{S^k Y}$ we have

$$(\mathcal{A}_D \circ i)^* T_{S^{nk+d} X}(\nu^j) \cong N_{\mathcal{A}_D \circ i}(\nu^j)$$

for $\nu^j \neq 1$. Then from the exact sequence (1.19) and from Lemma 1.11 we see that the rank of $(i^* N_{\mathcal{A}_D})(\nu^j)$ is r_j, where r_j is the number of times that ν^j appears in the list

$$\begin{gathered} \nu^{a_1}, \nu^{2a_1}, \dots, \nu^{d_1 a_1}, \\ \nu^{a_2}, \nu^{2a_2}, \dots, \nu^{d_2 a_2}, \\ \vdots \\ \nu^{a_s}, \nu^{2a_s}, \dots, \nu^{d_s a_s}. \end{gathered}$$

Lemma 1.13. *Let r be the rank of $(i^* N_{\mathcal{A}_D})(\nu^j)$. Then*

$$c\left((i^* N_{\mathcal{A}_D})(\nu^j)\right) = (1+\eta)^r.$$

Proof. It is enough to notice that the Chern class of $i^* N_{\mathcal{A}_D}$ is $(1+\eta)^d$ by (1.18) and Lemma 1.8. ■

Lemma 1.14. *Suppose that h is an automorphism of X of order p such that h is conjugate to h^j in* $\mathrm{Aut}(X)$. *Then $N_i(\nu^s) \cong N_i(\nu^{sj})$.*

Proof. Let h be an automorphism of a variety W and suppose that $h^j = u^{-1} h u$ where $u \in \mathrm{Aut}(W)$. If Z is a subvariety contained in the fixed point set of h at W such that u acts on Z, then the action of u on the tangent bundles of W and Z extends to an action on the normal bundle $N_{Z/W}$. From this one can see that under the isomorphism $u : N_{Z/W} \to N_{Z/W}$ the eigenvector bundle $N_{Z/W}(\nu^s)$ is mapped to $N_{Z/W}(\nu^{sj})$.

In our case the embedding of $Z = S^k Y$ into $W = S^{pk} X$ is equivariant with respect to u because $f_{k,0}(ux) = \sum_{i=0}^{p-1} u h^{ij} x$ and since $(p,j) = 1$, this is equal to

$uf_{k,0}(x)$. Notice that the composition map (1.16) is not necessarily equivariant with respect to u. ■

Lemma 1.15. *Let h be an automorphism of X of order prime p. Assume that $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of* $\mathrm{Aut}(X)$. *Then*

$$\mathcal{U}(N_i(\nu)) = \left(1 - \frac{1}{\nu}\right)^A \left(1 - \frac{e^{-t\eta}}{\nu}\right)^{-A} e^{t\vartheta\left(\frac{e^{-t\eta}}{\nu - e^{-t\eta}}\right)},$$

where $\nu \neq 1$ is a power of $e^{2i\pi/p}$.

Proof. Using Lemma 1.14 and (1.17), the Chern class of $N_i(\nu^s)$ is given by

$$(1 + t\eta)^A e^{\frac{-\vartheta t}{1+\eta t}}.$$

The last can be written as

$$(1 + t\eta)^{A-\gamma} \prod_{i=1}^{\gamma} (1 + t\eta - t\sigma_i).$$

So using (1.3), the characteristic class of $N_i(\nu)$ is given by

$$\mathcal{U}(N_i(\nu)) = \left(\frac{1 - \frac{e^{-t\eta}}{\nu}}{1 - \frac{1}{\nu}}\right)^{\gamma - A} \prod_{i=1}^{\gamma} \left(\frac{1 - \frac{e^{t\sigma_i - t\eta}}{\nu}}{1 - \frac{1}{\nu}}\right)^{-1}.$$

Using (1.7) we have

$$1 - \frac{e^{t\sigma_i - t\eta}}{\nu} = \frac{\nu - e^{-t\eta} - e^{-t\eta}t\sigma_i}{\nu} = \left(\frac{\nu - e^{-t\eta}}{\nu}\right)\left(1 - \frac{e^{-t\eta}t\sigma_i}{\nu - e^{-t\eta}}\right).$$

Then

$$\mathcal{U}(N_i(\nu)) = \left(\frac{1 - \frac{e^{-t\eta}}{\nu}}{1 - \frac{1}{\nu}}\right)^{\gamma - A} \left(\frac{\nu - e^{-t\eta}}{\nu(1 - \frac{1}{\nu})}\right)^{-\gamma} \prod_{i=1}^{\gamma} \left(1 - \frac{e^{-t\eta}t\sigma_i}{\nu - e^{-t\eta}}\right)^{-1}$$

$$= \left(1 - \tfrac{1}{\nu}\right)^A \left(1 - \tfrac{e^{-t\eta}}{\nu}\right)^{-A} e^{t\vartheta\left(\frac{e^{-t\eta}}{\nu - e^{-t\eta}}\right)}.$$ ■

Let

$$m(z) = \sum_{i=0}^{p-1} z^i,$$

$$q(z) = -\frac{zm'(z)}{m(z)},$$

then

$$\begin{aligned}\prod_{j=1}^{p-1}\mathcal{U}(N_i(\nu^j)) &= p^A m(e^{-\eta t})^{-A} e^{t\vartheta q(e^{-\eta t})} \\ &= p^A m(e^{-\eta t})^{-A} \sum_{i=0}^{\gamma} \frac{\left(t\vartheta q(e^{-\eta t})\right)^i}{i!}.\end{aligned} \tag{1.20}$$

We have no method to compute $\mathcal{U}(N_i(\nu^j))$ for any $h \in \mathrm{Aut}(X)$, not even in the case when h has order prime (unless it satisfies the condition that $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of $\mathrm{Aut}(X)$). In what follows we will explain a way to compute it if enough information about the quotient map $f : X \to Y$ is known and for the case when N_i is the normal bundle of the curve Y in S^pX under the embedding

$$i : Y \hookrightarrow S^pX.$$

.

Lemma 1.16. *Let $f : X \to Y$ be a degree p morphism of smooth curves. We have*

$$i^*T_{S^pX} \cong f_*f^*(K_Y^{-1}) = K_Y^{-1} \otimes f_*\mathcal{O}_X.$$

Proof. Consider the graph map

$$\Gamma : X \to X \times Y$$

$$x \mapsto (x, f(x)).$$

Let Δ be the universal divisor of degree p on X and let Δ' denote $(\mathrm{Id}_X \times i)^*(\Delta)$. By Lemma 2.1 in [2], **IV**,§2 we have $\Delta' = \Gamma(X) \cong X$. Thus by the adjunction formula we have

$$\mathcal{O}_{\Delta'}(\Delta') \cong f^*K_Y^{-1}.$$

The result follows from Lemma 1.6. ■

Using the following Lemma we can compute the degrees of the eigen line bundles of $i^*T_{S^nX}$ and since $i^*T_{S^nX}(\nu^j) \cong N_i(\nu^j)$ for $\nu^j \neq 1$, that is all we need to compute $\mathcal{U}(N_i(\nu^j))$. Let Z be a smooth projective variety defined over $\mathbb{C}$ and let $\mathcal{L}$ be a line bundle on Z such that a positive power $\mathcal{L}^p$ admits a global section s and its corresponding divisor D has normal crossings. Write D as $C + \sum a_jE_j$ where C denotes the components of multiplicity 1 and E_j is a component of multiplicity a_j.

For every real number x, $[x]$ represents the integral part of x, defined as the only integer such that

$$[x] \leq x < [x] + 1.$$

Consider the line bundles

$$\mathcal{L}^{(i)} = \mathcal{L}^i \otimes \mathcal{O}_Z \left(-\sum_j \left[\frac{ia_j}{p} \right] E_j \right). \tag{1.21}$$

The sheaf of $\mathcal{O}_Z$-modules

$$\bigoplus_{i=0}^{p-1} \mathcal{L}^{-i}$$

admits a structure of $\mathcal{O}_Z$-algebra, given by the inclusion

$$s^\vee : \mathcal{L}^{-p} \hookrightarrow \mathcal{O}_Z.$$

Let

$$Z' = \mathrm{Spec}_Z \left(\bigoplus_{i=0}^{p-1} \mathcal{L}^{-i} \right),$$

let $\tau' : Z' \to Z$ be the associated morphism and $n : \bar{Z} \to Z'$ the normalization of Z' and τ the composition of n and τ'.

Lemma 1.17. *With the previous notation we have*

$$\tau_* \mathcal{O}_{\bar{Z}} = \bigoplus_{i=0}^{p-1} \mathcal{L}^{(i)^{-1}}.$$

Moreover τ is a galois cyclic cover of degree p, then we have an automorphism h of $\bar{Z}$ which acts on $\tau_*\mathcal{O}_{\bar{Z}}$ and h acts as multiplication by ν^i on $\mathcal{L}^{(i)^{-1}}$, where $\nu = e^{2\pi I/p}$. If $\bar{Z}$ is irreducible then $\bar{Z}$ is nothing but the normalization of Z in $K(Z)(\sqrt[p]{f})$, where $K(Z)$ is the function field of Z and f is a rational function giving the section s.

Proof. See Lemma 2 in [11]. ■

Example. Let X be the Klein quartic curve. If h is an automorphism of order 7 then we have that h, h^2, h^4 are in the same conjugacy class whereas h^3, h^5, h^6 belong to another conjugacy class of $\mathbf{PSL}_2(\mathbb{F}_7)$. In this case we have

$$X/\langle h \rangle \cong \mathbb{P}^1$$

and if we consider the normal bundle N of

$$i : \mathbb{P}^1 \hookrightarrow S^7X$$

then we see that the eigenvector bundles of N don't have the same degree because the number A in formula (1.17) is not an integer.

We know that X can be constructed by adding a seventh root of a polynomial $q(z)$ (see formula (2.6)) to $\mathbb{C}(z)$. The divisor defined by $q(z)$ at $\mathbb{P}^1$ has the form $4p_0 + 2p_1 + p_2$, then by Lemma 1.16 and by Lemma 1.17 we have that

$$N \cong K_{\mathbb{P}^1}^{-1} \otimes \bigoplus_{i=1}^{6} \mathcal{L}^{(i)^{-1}}.$$

Using (1.21) we see that

$$\mathcal{L}^{(i)^{-1}} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) & \text{for } i = 1, 2, 4 \\ \mathcal{O}_{\mathbb{P}^1}(-2) & \quad i = 3, 5, 6. \end{cases}$$

Notice this agrees with Lemma 1.14. Using these values the degree of N is 3 which also agrees with the value obtained using formula (1.17).

1.6 The computation

Now we shall assume that h is an automorphism of prime order p. Let K denote the canonical line bundle K_{S^bX} of S^bX. In the appendix there is a Maple worksheet which computes

$$\sum (-1)^i \text{trace}\,(h \mid H^i(S^bX, K^n)).$$

Notice that if $n \geq 2$ then by Lemma 1.2 this is trace $(h \mid H^0(S^bX, K^n))$. The only assumption made there is that $\langle h \rangle \setminus \{1\}$ is contained in a conjugacy class of $\mathrm{Aut}(X)$. The data required is

- The dimension b of the symmetric product S^bX,
- the order p of the automorphism h,
- the genus g of the curve X,
- s the number of fixed points of h on the curve X and
- a vector $u = (a_1, \cdots, a_s)$ in which a_i is a positive integer such that the automorphism h acts as ν^{a_i} on the tangent space T_{x_i} of the fixed point $x_i \in X$.

To apply the Theorem 1.4 we first need to know the fixed point set fix(h) of the automorphism h in S^bX. So let $b = mp + l$ such that $m \geq 0$ and $0 \leq l < p$. From section 1.4 we have

$$\mathrm{fix}(h) = \bigcup_{k=0}^{m} F_k$$

where

$$F_k = \bigcup_{D \in A_k} f_{k,D}(S^kX)$$

and

$$A_k = \{D = a_1x_1 + \cdots + a_sx_s \mid 0 \leq a_j \leq p-1 \quad \text{and} \quad \sum_{j=1}^{s} a_j = b - kp\}.$$

Since $F_i \cap F_j = \emptyset$ and $f_{i,D_1}(S^iX) \cap f_{i,D_2}(S^iX) = \emptyset$ for $D_1, D_2 \in A_i$, we have

$$\sum (-1)^i \mathrm{trace}\,(h \mid H^i(S^bX, \mathcal{L}^n)) = \sum_{k=0}^{m} \sum_{D \in A_k} \lambda(k, D),$$

where

$$\lambda(k,D) = \int_{f_{k,D}(S^kX)} \frac{ch_h(K^n \mid_{f_{k,D}(S^kX)}) \cdot \prod_i \mathcal{U}(N(\nu^i)) \cdot td(f_{k,D}(S^kX))}{\det(1 - h_{|N^\vee})}.$$

So we need to compute all the divisors $D \in A_k$. This is done by **monadd**() and **exponents**() in the appendix. The idea is the following: a divisor $\sum_{i=1}^{s} a_ix_i$ can be seen as a monomial $\prod_{i=1}^{s} x_i^{a_i}$. The terms of the expanded polynomial $q(x_1, \cdots, x_s) = (\sum_{i=1}^{s} x_i)^{d_k}$ are (ignoring their coefficients) all the monomials of degree d_k in the variables $x_1, \cdots, x_s$. Then taking residues modulo x_i^p for $i = 1, \cdots, s$ we obtain a polynomial **monadd**(s, p, d_k) whose terms represent all the elements of A_k (taking $d_k = b - kp$). So **exponents**(**monadd**(s, p, d_k)$, s$) returns (matrix, number of divisors), a matrix whose rows are the coefficients of the divisors $D \in A_k$.

Now, given $D \in A_k$ how to compute $\lambda(k, D)$? The tangent space $T_{(S^{d_k}X)_D}$ of $D \in T_{(S^{d_k}X)}$ has a decomposition

$$T_{(S^{d_k}X)_D} = \bigoplus_{i=1}^{p-1} T_{(S^{d_k}X)_D}(\nu^i).$$

Let $r_i = \dim T_{(S^{d_k}X)_D}(\nu^i)$ we call $(r_1, \cdots, r_{p-1})$ the class of D, notice $r_0 = 0$. The class of $D = d_1x_1 + \cdots + d_sx_s$ is computed using Lemma 1.11, that is r_i is the number of times that ν^i appears in the list

$$\begin{gathered}
\nu^{a_1}, \nu^{2a_1}, \cdots, \nu^{d_1 a_1}, \\
\nu^{a_2}, \nu^{2a_2}, \cdots, \nu^{d_2 a_2}, \\
\vdots \\
\nu^{a_s}, \nu^{2a_s}, \cdots, \nu^{d_s a_s}.
\end{gathered}$$

From the following remark it should be clear how to compute $\lambda(k, D)$ if we have the class of the divisor D.

Remark 1.18. Assume that the class of D is $(r_1, \cdots, r_{p-1})$.

1) $f_{k,D}(S^k X)$ can be identified with $S^k Y$, where Y is the quotient curve of X by the automorphism h, so $td(f_D(S^k X))$ does not depend on the divisor D. In the appendix the function **tdsdx**(d, g) computes the Todd class of the symmetric product $S^d C$, where C is a genus g curve, it is defined using formula (1.11).

2) N is the normal bundle of $f_D(S^k X)$ in $S^b X$. Now, $b = pk + d_k$ and d_k is the degree of D. If we consider the composition map (1.16) then N is the normal bundle $N_{\mathcal{A}_D \circ i}$. Notice from remark 1.12 and Lemma 1.11 we have

$$\det(1 - h_{|_{N^\vee}}) = p^k \prod_{j=1}^{p-1} (1 - \nu^{p-j})^{r_j}.$$

3) From the exact sequence (1.19) we have

$$\prod_j \mathcal{U}(N(\nu^j)) = \left(\prod_j \mathcal{U}(N_i(\nu^j)) \right) \left(\prod_j \mathcal{U}(i^*(N_{\mathcal{A}_D})(\nu^j)) \right).$$

The first factor is independent of D and if h is conjugate to h^i for $i = 1, \cdots, p-1$ then it can be computed using formula (1.20), in the appendix the function **charclass1**(k, p, γ, g) computes it.

As for the second factor, from Lemma 1.13 we have

$$\mathcal{U}\left(i^*(N_{\mathcal{A}_D})(\nu^j)\right) = \left(\frac{1 - \frac{e^{-\eta t}}{\nu^j}}{1 - \frac{1}{\nu^j}} \right)^{-r_j}.$$

In the appendix this is equal to **precharclass**$(k, p, j)^{r_j}$.

4) Now, h acts on $K_{|f_{k,D}(S^k X)} = (\mathcal{A}_D \circ i)^* K$ as $\det(h_|(\mathcal{A}_D \circ i)^* T_{S^b X})^{-1}$. From Lemma 1.11 this is equal to

$$\nu^{\mu - \sum_{j=1}^{p-1} j r_j}, \tag{1.22}$$

where

$$\mu = \begin{cases} k & \text{if } p = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Using formula (1.2) we have

$$ch_h(K^n_{|_{f_D(S^k X)}}) = (\nu^{\mu - \sum_{j=1}^{p-1} j r_j})^n ch((\mathcal{A}_D \circ i)^* K^{\otimes n}). \tag{1.23}$$

The Chern class of $K_{S^b X}$ is $1 + [(g-1-b)\eta + \vartheta]t$ so using Lemma 1.8 we see that $(\mathcal{A}_D \circ i)^* K$ has Chern class $1 + [(g-1-b)\eta + p\vartheta]t$. Thus

$$ch((\mathcal{A}_D \circ i)^* K^{\otimes n}) = e^{[(g-1-b)\eta + p\vartheta]nt}.$$

Example. Let X be a hyperelliptic curve of genus g. In this case h has order 2, the quotient curve is $\mathbb{P}^1$ and h has $2g+2$ fixed points on X. We assume $b = g-1$. Each set A_k has

$$\binom{2g+2}{g-1-2k}$$

elements and all the divisors have the same class because there is only one eigenvalue, in fact, $r_1 = g - 1 - 2k (= \dim T_{(S^{d_k}X)_D}(\nu^i))$. For each component of dimension k of the fixed point set of h in $S^{g-1}X$ we have, following the points of the previous remark:

1) Because $\mathbb{P}^1$ has genus 0, the ϑ class is 0 in the cohomology ring of $S^k\mathbb{P}^1 \cong \mathbb{P}^k$ and η is the class of a hyperplane. The Todd class of $S^k\mathbb{P}^1$ is given by

$$\left(\frac{\eta}{1-e^{-\eta}}\right)^{k+1}.$$

2)

$$\det(1 - h_{|_{N^\vee}}) = 2^{k+r_1} = 2^{g-1-k}.$$

3) For the first factor in remark 1.18 3) we have $A = k - g$, so it is equal to

$$\left(\frac{2}{1+e^{-\eta}}\right)^{k-g}.$$

The second factor is equal to

$$\left(\frac{1+e^{-\eta}}{2}\right)^{2k+1-g}.$$

4)

$$ch_h(K^n_{S^{g-1}\mathbb{P}^1}|_{S^k\mathbb{P}^1}) = (-1)^{n(k-r_1)} ch(K^n_{S^{g-1}\mathbb{P}^1}|_{S^k\mathbb{P}^1}),$$
$$ch(K^n_{S^{g-1}\mathbb{P}^1}|_{S^k\mathbb{P}^1}) = 1.$$

So

$$ch_h(K^n_{S^{g-1}\mathbb{P}^1}|_{S^k\mathbb{P}^1}) = (-1)^{-n(g-1-k)}.$$

Then we have

$$\lambda(k, D) = \int_{S^k\mathbb{P}^1} 2^{-g}(-1)^{-n(g-1-k)} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1}.$$

Now

$$\int_{S^k\mathbb{P}^1} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1}$$

is the coefficient of η^k in $\left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1}$. So

$$\int_{S^k\mathbb{P}^1} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1} = Res_{\eta=0} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} = Res_{z=0} \left(\frac{2-z}{z}\right)^{k+1} \frac{dz}{1-z}$$
$$= 1 - (-1)^{k+1}.$$

Then

$$\sum (-1)^i \operatorname{trace}(h \mid H^i(S^{g-1}X, K^n)) =$$

$$2^{-g} \sum_{k=0}^{[\frac{g-1}{2}]} (-1)^{-n(g-1-k)} (1 - (-1)^{k+1}) \binom{2g+2}{g-1-2k}.$$

An induction shows that for $n \geq 2$,

$$\operatorname{trace}(h \mid H^0(J, \mathcal{O}(n\Theta))) =$$

$$1 + 2^{-g} \sum_{k=0}^{[\frac{g-1}{2}]} \left[n \frac{(-1)^{g-k-1}+1}{2} + \frac{(-1)^{n(g-k-1)}-1}{2} + (-1)^{g-k} \right] (1-(-1)^{k+1}) \binom{2g+2}{g-1-2k}.$$

With respect to this involution, we have

$$H^0(J, \mathcal{O}(n\Theta)) = \mathbb{C}^{\alpha(n)} \oplus V^{\beta(n)},$$

where

$$\alpha(n) = \frac{1}{2} \left[n^g + \operatorname{trace}(h \mid H^0(J, \mathcal{O}(n\Theta))) \right]$$

and

$$\beta(n) = \frac{1}{2}\left[n^g - \operatorname{trace}\left(h \mid H^0(J, \mathcal{O}(n\Theta))\right)\right],$$

and $\mathbb{C}$ and V are the 1 dimensional representations on which h acts as 1 and -1 respectively.

Chapter 2

2.1 Curves with automorphisms.

Let X be a smooth complex curve of genus $g > 1$. Let $\mathrm{Aut}(X)$ be the group of automorphisms of X. For every $g \geq 2$, there is a maximum order $\mu(g)$ for an automorphism group of a curve of genus g. Hurwitz proved that

$$\mid \mathrm{Aut}(X) \mid \leq 84(g-1). \tag{2.1}$$

He also proved that a finite group can be realized as a group of $84(g-1)$ automorphisms of a curve of genus g if and only if the group is generated by elements t, u such that $t^2 = u^3 = (tu)^7 = 1$. In fact we have

$$8g + 8 \leq \mu(g) \leq 84(g-1).$$

Macbeath (see [19] §6) has proved that there is an infinite number of g for which $\mu(g) = 84(g-1)$, on the other hand Accola [1] and Maclachan [21] have found infinite families of g with $\mu(g) = 8g + 8$.

The following information about Fuchsian groups and curves with automorphisms comes from [17] and as an application of it we will see how to prove (2.1). For a different proof of (2.1) and other details about curves with automorphisms see [2] pg 45.

Let H be the upper half plane. Let Γ be a discrete subgroup of $\mathrm{Aut}(H) = \mathbf{PSL}_2(\mathbb{R})$ such that acts freely on H, that is, the only element of Γ with fixed points on H is the identity. Then the orbit space H/Γ can be given an analytic structure such that the quotient map $H \to H/\Gamma$ is holomorphic. We will say that Γ is a

Fuchsian group if H/Γ is a compact Riemann surface. This is a particular case of the so-called Fuchsian groups of the first kind (see [25] pg 19).

Curves with automorphisms can be characterized in terms of their uniformization by the hyperbolic plane: any Riemann surface of genus > 1 can be identified with $H/\pi_1(X)$; conversely any Fuchsian subgroup N of $\mathbf{PSL}_2(\mathbb{R})$ that acts freely on H produces a Riemann surface H/N of genus > 1 whose fundamental group is N. The automorphism group of H/N is Γ/N, where Γ is the normalizer of N in $\mathbf{PSL}_2(\mathbb{R})$.

An element of a Fuchsian group Γ has a fixed point in H if and only if it has finite order, and the stabilizer of a point of H in Γ is always a finite cyclic subgroup of Γ. An element of Γ cannot fix more than one point of H, so every element of finite order in Γ belongs to a maximal finite cyclic subgroup. There are infinitely many of these maximal finite cyclic groups if there is one, and they fall into a finite number of conjugacy classes. The orders of these maximal finite cyclic subgroups are called the periods of Γ. The multiplicity of a period is the number of distinct conjugacy classes of maximal finite subgroups with that period for their order.
The algebraic structure of Γ is completely determined when the periods and the genus of the orbit space H/Γ are known. In fact, if $m_1, \cdots, m_r$ are the periods of Γ in some order, each one repeated according to its multiplicity (we call $(m_1, \cdots, m_r)$ the period partition of H/Γ), and if γ is the genus of the surface H/Γ, then Γ is defined by generators

$$x_1, \cdots, x_r \quad a_1, b_1, \cdots, a_\gamma, b_\gamma$$

and relations

$$x_1^{m_1} = x_2^{m_2} = \cdots = x_r^{m_r} = x_1 x_2 \cdots x_r a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1. \tag{2.2}$$

The following Theorem tells us how to count the number of fixed points of an automorphism of a Riemann surface. Assume that our Riemann surface is H/N has automorphism group $G = \Gamma/N$, where N is a Fuchsian group acting freely on H and Γ is the normalizer of N in $\mathbf{PSL}_2(\mathbb{R})$.

Theorem 2.1. *Let $x_1, \ldots, x_r \in \Gamma$ of orders $m_1, \ldots, m_r$ be generators of maximal finite cyclic subgroups of Γ, including exactly one for each conjugacy class. Let*

$q : \Gamma \to \Gamma/N = G$ be the quotient map. For $1 \neq h \in G$ let $\xi_i(h)$ be 1 or 0 according as h is or not conjugate to a power of $q(x_i)$. Then the number $|\operatorname{fix}(h)|$ of points of H/N fixed by h is given by the formula

$$|\operatorname{fix}(h)| = |N_G(\langle h \rangle)| \sum_{i=1}^{r} \xi_i(h)/m_i,$$

where $N_G(\langle h \rangle)$ is the normalizer of $\langle h \rangle$ in G.

Proof. See Theorem 1 in [18]. ■

Assume that Γ is a Fuchsian group and let r be the sum of the multiplicities of the periods of Γ. Then there are r points of H/Γ at which the covering map

$$H \to H/\Gamma \tag{2.3}$$

is branched. If N is a normal subgroup of Γ acting freely on H, then it gives rise to a covering map

$$H/N \to H/\Gamma. \tag{2.4}$$

Since

$$H \to H/N$$

is unbranched, the r points at which the covering (2.3) branches are the same points at which the branching of (2.4) occurs and the orders of the branching will be the periods $m_1, m_2 \cdots, m_r$ of Γ.

A fundamental domain for a discrete subgroup $\Gamma \subset \mathbf{PSL}_2(\mathbb{R})$ is an open subset F of H, such that no two points are Γ-equivalent and every point in H is Γ-equivalent to a point in the closure of F.

If N is a subgroup of Γ of index k, a fundamental region for N can be obtained by taking the union of k copies tF of a fundamental region for Γ, where the elements t form a complete system of representatives of cosets Nt of N in Γ. Hence

$$\Delta(N) = k\Delta(\Gamma).$$

The area of a fundamental region for Γ is given by the formula (see Theorem 2.20 in [25]):

$$\Delta(\Gamma) = 2\pi \left(2\gamma - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) \right). \tag{2.5}$$

In particular if N acts freely on H and g is the genus of its orbit space, we have

$$\Delta(N) = 4\pi(g-1).$$

So

$$|G| = \frac{\Delta(N)}{\Delta(\Gamma)} = \frac{4\pi(g-1)}{2\pi\left(2\gamma - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})\right)}.$$

This formula can also be obtained using the Riemann-Hurwitz theorem. A theorem of Siegel (Theorem 5 in [26]) states that $\Delta(\Gamma) \geq \frac{\pi}{21}$ and that the equality occurs only if Γ is the *triangle group* with period partition (2,3,7) and orbit space of genus 0. Notice in this case that from (2.2) Γ is defined by generators t, u and relations

$$t^2 = u^3 = (tu)^7 = 1.$$

From this it follows that $|G| \leq 84(g-1)$.

2.1.1 Equations of curves with automorphisms.

In general there is no algorithm to find equations of curves with known group of automorphisms. In [17] Macbeath proposes the following idea in order to find equations of curves with automorphisms. The group of automorphisms G acts on the function field $K(X)$ of X: suppose that the field of invariant functions is the field of a rational curve, that is the quotient map

$$X \longrightarrow X/G \cong \mathbb{P}^1$$

is induced by a field extension $\mathbb{C}(z) \subset K(X)$. If G is a soluble group, then by the theory of Galois $K(X)$ is an extension by radicals of $\mathbb{C}(z)$. If enough is known about the branch points of the quotient map this will enable us to compute equations defining the curve. We illustrate this idea by computing the equation for the Klein quartic.

The Klein quartic.

This curve was discovered by Felix Klein in the 1870s and is better known for being the only curve of genus 3 with automorphism group G of size 168, namely $G = \mathbf{PSL}_2(\mathbb{F}_7)$, the maximum for its genus. Klein computed the ring of invariants of G for a 3-dimensional irreducible representation of G: the equation of this curve is the only quartic invariant for G in this representation. In [16] one can find several survey articles about this curve.

We reproduce from [5] some information about G and the character table of G. This group has 6 conjugacy classes, say 1A, 2A, 3A, 4A, 7A, 7B. The number of elements in each conjugacy class is 1, 21, 56, 42, 24 and 24 respectively, an element in each conjugacy class has order 1, 2, 3, 4, 7, 7 respectively. Therefore there are 6 irreducible representations of $\mathbf{PSL}_2(\mathbb{F}_7)$. If $h \in 7A$ then $h^2, h^4 \in 7A$ and $h^3, h^5, h^6 \in 7B$.

CHARACTER TABLE OF $\mathbf{PSL}_2(\mathbb{F}_7)$.

	1A	2A	3A	4A	7A	7B
χ_1	1	1	1	1	1	1
χ_3	3	-1	0	1	α	$\bar{\alpha}$
$\bar{\chi}_3$	3	-1	0	1	$\bar{\alpha}$	α
χ_6	6	2	0	0	-1	-1
χ_7	7	-1	1	-1	0	0
χ_8	8	0	-1	0	1	1

$\alpha = \frac{-1+i\sqrt{7}}{2}$.

Now to follow Macbeath's idea, a way to start could be this: from [6] one knows that G is generated by elements t, u such that

$$t^2 = u^3 = (tu)^7 = 1.$$

Thus from the previous section there is a curve X admitting G as its automorphism group. Let T be the triangle group with period partition $(2,3,7)$ and orbit space of genus 0. There is a normal subgroup N of T acting freely on the upper half plane H such that $X \cong H/N$ and $G \cong T/N$.

Let G_2 be a subgroup of G of order 7. Let G_1 be the normalizer of G_2 in G. Then G_1 has order 21.

Consider the quotient map $\phi : T \to T/N$. Let

$$\Gamma_1 = \phi^{-1}(G_1), \Gamma_2 = \phi^{-1}(G_2).$$

Since the order of G_1 is 21, it has periods of order 3 and 7. So

$$8 = \frac{\Delta(\Gamma_1)}{\Delta(T)} = \frac{2\pi(2\gamma - 2 + \frac{2m}{3} + \frac{6n}{7})}{\frac{\pi}{21}}.$$

Then one sees that $\gamma = 0$, $m = 1$ and $n = 2$. Similarly Γ_2 has periods (7,7,7) and $\gamma = 0$.

Consider the following maps

$$X \longrightarrow X/G_2 \longrightarrow X/G_1.$$

Notice G_1/G_2 is a group of order three acting on X/G_2 and the map

$$X/G_2 \longrightarrow X/G_1$$

is the quotient map of X/G_2 by G_1/G_2. Let h be a generator of G_1/G_2. Since $X/G_2 \cong \mathbb{P}^1$, we can assume that the two fixed points of h in X/G_2 are 0 and ∞ and that h acts on X/G_2 by the rule $z \mapsto \omega z$ where ω is a cube root of unity. Notice if a is a branch point in $\frac{X}{G_2}$ for $X \longrightarrow \frac{X}{G_2}$, then $a, \omega a, \omega^2 a$ are the three branch points for $X \longrightarrow \frac{X}{G_2}$. Therefore $a \neq 0, \infty$.

Let K(X) be the function field of X. The map $X \longrightarrow X/G_2$ corresponds to a field extension $\mathbb{C}(z) \subset K(X)$ with Galois group G_2.

Then $K(X) = \mathbb{C}(z)(\sqrt[7]{q(z)})$, where $q(z) \in \mathbb{C}[z]$. Notice $q(\omega^i a) = 0$ since these are the only points where the branching occurs. Since the action of h on $\mathbb{C}(z)$ extends to $K(X)$, we have that if $u^7 \in \mathbb{C}(z)$ with $u \in K(X)$ then $h \cdot u^7 = v^7$ for some $v \in K(X)$.

A natural candidate for $q(z)$ should be $z^3 - a^3$, however this makes ∞ a branch point. So one can choose

$$q(z) = A_0^4 A_1^2 A_2, \tag{2.6}$$

where $A_i = \omega^i z - a$.

We have $y^7 = q(z)$. Notice

$$q(\omega z) = \left(\frac{y^2}{A_0}\right)^7$$

$$q(\omega^2 z) = \left(\frac{y^4}{A_0^2 A_1}\right)^7$$

One can define a curve with field of functions $K(X)$ in $\mathbb{A}^4$ as the set of points

$$(x_0, x_1, x_2, z)$$

satisfying the following equations

$$x_0^7 = \omega^2 q(z)$$

and for i=1,2

$$x_i = \frac{\omega^{2i} x_{i-1}^2}{A_{i-1}}.$$

The last equation is valid for $i = 0 \,(\text{modulo}\, 3)$. Then we have

$$\begin{vmatrix} \omega^2 x_0 & x_0 & {x_2}^2 \\ x_1 & x_1 & \omega^2 {x_0}^2 \\ \omega\, x_2 & x_2 & \omega\, {x_1}^2 \end{vmatrix} = -({x_0}^3 x_2 + x_1 {x_2}^3 + x_0 {x_1}^3)(\omega - 1) = 0.$$

The expressions $x_i dz$ are abelian differentials on the curve and notice that the set $\{x_0, x_1, x_2, x_0^3, x_0^5, x_0^6, 1\}$ forms a basis of $K(X)$ as a $\mathbb{C}(z)$-vector space. Therefore $x_0 dz, x_1 dz, x_2 dz$ are three linearly independent abelian differentials over $\mathbb{C}$, and since 3 is the genus of the curve, the ratios $(x_0 : x_1 : x_2)$ determine the canonical embedding of the curve in $\mathbb{P}^2$.

2.2 Decomposition for the Klein quartic.

Let X be the Klein quartic curve. Now we present the information about the fixed points of an automorphism of X that we need in order to compute the decomposition. The number of fixed points of $h \in G = \mathbf{PSL}_2(\mathbb{F}_7)$ in X acording to its conjugacy class can be worked out using Theorem 2.1 knowing that the Fuchsian group that yields G and X has period partition (2,3,7):

- if $h \in 1A$. The whole curve is the fixed point set of the identity.
- if $h \in 2A$, then h has 4 fixed points on X.
- if $h \in 3A$, then h has 2 fixed points.
- if $h \in 4A$, then h has no fixed points on X.
- if $h \in 7A$ then $h^2, h^4 \in 7A$ too, and $h^3, h^5, h^6 \in 7B$. In this case h has 3 fixed points and these are the fixed points of any power of h.

Now let x be a fixed point of $h \in G$, then h acts on the fibre K_{Xx} at x of the cotangent bundle K_X of X as

- 1 if $h \in 1A$.
- -1 if $h \in 2A$.
- If $h \in 3A$, let ω be the primitive cubic root of unit. There are two fixed points and h acts as ω on the fibre of one of them and as ω^2 on the other.
- If $h \in 4A$ there are no fixed points so this case does not apply.
- If $h \in 7A$ or $7B$, let ζ be a primitive seventh root of unity. There are 3 fixed points, the action of h on the fibre of the cotangent bundle at these points is multiplication by ζ^6, ζ^5 and ζ^3.

For an element $h \in 7B$ the action of h on the cotangent space at its three fixed points are multiplication by ζ, ζ^2 and ζ^4.

As an example we can use these values in 1.5, for the trivial line bundle $E = \mathcal{O}$. We have

If $g \in 2A$

$$1 - \text{trace}\left(g \mid H^1(X, \mathcal{O})\right) = \sum_{x \in \text{fix}(g)} \frac{1}{2} = 4(\frac{1}{2}) = 2.$$

If $g \in 3A$

$$1 - \text{trace}\left(g \mid H^1(X, \mathcal{O})\right) = \frac{1}{1-\omega} + \frac{1}{1-\omega^2} = 1$$

If $g \in 7A$

$$1 - \text{trace}\left(g \mid H^1(X,\mathcal{O})\right) = \frac{1}{1-\zeta^3} + \frac{1}{1-\zeta^5} + \frac{1}{1-\zeta^6} = 1 + \frac{1}{2} - \frac{i\sqrt{7}}{2}$$

If $g \in 7B$

$$1 - \text{trace}\left(g \mid H^1(X,\mathcal{O})\right) = \frac{1}{1-\zeta} + \frac{1}{1-\zeta^2} + \frac{1}{1-\zeta^4} = 1 + \frac{1}{2} + \frac{i\sqrt{7}}{2}$$

Let $H^1(X,\mathcal{O}) = \chi_1^a \oplus \chi_3^b \oplus \bar{\chi}_3^c \oplus \chi_6^d \oplus \chi_7^e \oplus \chi_8^f$. So from the charcter table we have:

$$\begin{array}{cccccccrl}
a & +3b & +3c & +6d & +7e & +8f & = & \text{trace}\,(g \mid H^1(X,\mathcal{O})) = 3, & g \in 1A \\
a & -b & -c & +2d & -e & & = & -1, & g \in 2A \\
a & & & & +e & -f & = & 0, & g \in 3A \\
a & +\alpha b & +\bar{\alpha} c & -d & & +f & = & -\frac{1}{2} + \frac{i\sqrt{7}}{2}, & g \in 7A \\
a & +\bar{\alpha} b & +\alpha c & -d & & +f & = & -\frac{1}{2} - \frac{i\sqrt{7}}{2}, & g \in 7B.
\end{array}$$

Since $\dim H^1(X,\mathcal{O}) = 3$ we have $d = e = f = 0$. Therefore $b = 1$, $a = c = 0$, i.e $H^0(X, K_X) = \bar{\chi}_3$.

Now to compute the decomposition of $H^0(J_X, \mathcal{O}(n\Theta))$ into irreducible representations of G we first have to compute

$$\sum (-1)^i \text{trace}\,(h \mid H^i(S^{g-1}X, K^n)).$$

For $h \in 2A$ and $3A$ this can be done using our program:

$$\sum (-1)^i \text{trace}\,(h \mid H^i(S^{g-1}X, K^n)) = \tfrac{3}{2} + \tfrac{(-1)^n(2n-1)}{2} \quad \text{for } h \in 2A$$

$$\sum (-1)^i \text{trace}\,(h \mid H^i(S^{g-1}X, K^n)) = 1 \quad \text{for } h \in 3A.$$

Although not all the powers of an automorphism of order seven belong to the same conjugacy class, we can use our program because its fixed point set in S^2X has dimension 0.

So for h in 7A we obtain:

$$\begin{aligned}\sum(-1)^i\text{trace}\,(h \mid H^i(S^{g-1}X, K^n)) &= \tfrac{2\zeta^{4n}}{(1-\zeta^6)(1-\zeta^5)} + \tfrac{2\zeta^{n}}{(1-\zeta^5)(1-\zeta^3)} + \tfrac{2\zeta^{2n}}{(1-\zeta^3)(1-\zeta^6)} \\ &= 2\zeta^{4n}(\tfrac{1}{7} + \tfrac{2}{7}\zeta^5 + \tfrac{1}{7}\zeta^3 - \tfrac{2}{7}\zeta - \tfrac{2}{7}\zeta^2) \\ &\quad + 2\zeta^{n}(\tfrac{1}{7}\zeta^3 - \tfrac{3}{7}\zeta^2 - \tfrac{3}{7}\zeta^4 - \tfrac{1}{7}\zeta - \tfrac{1}{7}\zeta^5) \\ &\quad + 2\zeta^{2n}(-\tfrac{1}{7} - \tfrac{1}{7}\zeta^5 - \tfrac{2}{7}\zeta^2 - \tfrac{2}{7}\zeta^3 - \tfrac{4}{7}\zeta - \tfrac{4}{7}\zeta^4).\end{aligned}$$

For h in 7B we have:

$$\begin{aligned}\sum(-1)^i\text{trace}\,(h \mid H^i(S^{g-1}X, K^n)) &= \tfrac{2\zeta^{3n}}{(1-\zeta^1)(1-\zeta^2)} + \tfrac{2\zeta^{6n}}{(1-\zeta^2)(1-\zeta^4)} + \tfrac{2\zeta^{5n}}{(1-\zeta^4)(1-\zeta^1)} \\ &= 2\zeta^{3n}(\tfrac{3}{7} + \tfrac{2}{7}\zeta + \tfrac{4}{7}\zeta^2 + \tfrac{3}{7}\zeta^4 + \tfrac{2}{7}\zeta^3) \\ &\quad + 2\zeta^{6n}(\tfrac{1}{7} + \tfrac{2}{7}\zeta^4 - \tfrac{2}{7}\zeta^3 - \tfrac{2}{7}\zeta^5 + \tfrac{1}{7}\zeta) \\ &\quad + 2\zeta^{5n}(\tfrac{3}{7} + \tfrac{4}{7}\zeta + \tfrac{3}{7}\zeta^2 + \tfrac{2}{7}\zeta^4 + \tfrac{2}{7}\zeta^5).\end{aligned}$$

For h in 4A, let $p_1, \cdots, p_4$ be the four fixed points of h^2 in X. We can assume that $p_3 = hp_1$ and $p_4 = hp_2$. Then the fixed points of h in S^2X are $p_1 + hp_1$ and $p_2 + hp_2$. We have

$$(T_{S^2X})_{p_1+hp_1} = T_{X\,p_1} \oplus T_{X\,hp_1}$$

and h induces two linear maps $\alpha : T_{X\,p_1} \to T_{X\,hp_1}$, $\beta : T_{X\,hp_1} \to T_{X\,p_1}$.

Then the automorphism induced on $(T_{S^2X})_{p_1+hp_1}$ has a matrix conjugate to

$$A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

Since $p_1 + hp_1$ is a fixed point of h^2 $\in$2A, we see that $A^2 = -\mathrm{Id}_{(T_{S^2X})_x}$. Since trace $(A) = 0$ we see that A is conjugate to

$$\begin{pmatrix} i & * \\ 0 & -i \end{pmatrix}.$$

Thus we see that

$$h \mid K_{S^2X\,p_j+hp_j} = 1.$$

Using Atiyah-Bott fixed point Theorem (Corollary 1.5) we have

$$\sum(-1)^i\text{trace}\,(h \mid H^i(S^{g-1}X, K^n)) = 1.$$

Now using an induction we see that on the Jacobian of X we have:

$$\begin{aligned}
\sum(-1)^i \operatorname{trace}(h \mid H^i(J, \mathcal{O}(n\Theta))) &= n^3 && \text{for } h \in 1A \\
&= \tfrac{n(3+(-1)^n)}{2} && \text{for } h \in 2A, \\
&= n && \text{for } h \in 3A, \\
&= n && \text{for } h \in 4A.
\end{aligned}$$

Let $\mu = \zeta^{n+1}$ and $\zeta = e^{2i\pi/7}$, then for $h \in 7A$

$$\begin{aligned}
\sum(-1)^i \operatorname{trace}(h \mid H^i(J, \mathcal{O}(n\Theta))) &= (-\tfrac{2}{7}\zeta^3 - \tfrac{4}{7}\zeta^4 - \tfrac{4}{7}\zeta^5 - \tfrac{4}{7})\mu^4 \\
&+ (\tfrac{2}{7}\zeta^5 + \tfrac{4}{7}\zeta^4 + \tfrac{4}{7}\zeta + \tfrac{4}{7}\zeta^3)\mu^2 \\
&+ (-\tfrac{2}{7}\zeta + \tfrac{2}{7}\zeta^5 - \tfrac{2}{7}\zeta^3 + \tfrac{2}{7}\zeta^4 + \tfrac{2}{7}\zeta^2 - \tfrac{2}{7})\mu \\
&- \tfrac{2\zeta}{7} - \tfrac{1}{7} - \tfrac{2\zeta^4}{7} - \tfrac{2\zeta^2}{7}.
\end{aligned}$$

For $h \in 7B$

$$\begin{aligned}
\sum(-1)^i \operatorname{trace}(h \mid H^i(J, \mathcal{O}(n\Theta))) &= (\tfrac{4}{7}\zeta^3 + \tfrac{2}{7}\zeta + \tfrac{4}{7}\zeta^2 + \tfrac{4}{7}\zeta^5)\,\mu^6 \\
&+ (-\tfrac{4}{7}\zeta^5 - \tfrac{2}{7}\zeta^2 - \tfrac{4}{7}\nu - \tfrac{4}{7})\,\mu^5 \\
&+ (-\tfrac{4}{7} - \tfrac{2}{7}\zeta^4 - \tfrac{4}{7}\zeta^3 - \tfrac{4}{7}\zeta^2)\,\mu^3 \\
&+ \tfrac{1}{7} + \tfrac{2\zeta^4}{7} + \tfrac{2\zeta^2}{7} + \tfrac{2\zeta}{7}.
\end{aligned}$$

Now the general solution of

$$\begin{array}{ccccccccc}
a & +3b & +3c & +6d & +7e & +8f & = & tr1a & 1A \\
a & -b & -c & +2d & -e & & = & tr2a & 2A \\
a & & & & +e & -f & = & tr3a & 3A \\
a & +b & +c & & -e & & = & tr4a & 4A \\
a & +\alpha b & +\bar{\alpha} c & -d & & +f & = & tr7a & 7A \\
a & +\bar{\alpha} b & +\alpha c & -d & & +f & = & tr7b & 7B
\end{array}$$

(where $\alpha = \frac{-1+i\sqrt{7}}{2}$) is given by

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{bmatrix} \frac{tr1a}{168} + \frac{tr2a}{8} + \frac{tr7b}{7} + \frac{tr7a}{7} + \frac{tr4a}{4} + \frac{tr3a}{3} \\ -\frac{tr2a}{8} + \frac{tr1a}{56} - \frac{i\sqrt{7}tr7a}{14} + \frac{i\sqrt{7}tr7b}{14} - \frac{tr7b}{14} - \frac{tr7a}{14} + \frac{tr4a}{4} \\ \frac{tr4a}{4} - \frac{tr7a}{14} + \frac{i\sqrt{7}tr7a}{14} - \frac{tr7b}{14} - \frac{i\sqrt{7}tr7b}{14} - \frac{tr2a}{8} + \frac{tr1a}{56} \\ \frac{tr1a}{28} + \frac{tr2a}{4} - \frac{tr7b}{7} - \frac{tr7a}{7} \\ \frac{tr3a}{3} + \frac{tr1a}{24} - \frac{tr2a}{8} - \frac{tr4a}{4} \\ \frac{tr7b}{7} + \frac{tr1a}{21} + \frac{tr7a}{7} - \frac{tr3a}{3} \end{bmatrix}.$$

Then if $H^0(J_X^{g-1}, \mathcal{O}(n\Theta)) = \chi_1^{a(n)} \oplus \chi_3^{b(n)} \oplus \bar{\chi}_3^{c(n)} \oplus \chi_6^{d(n)} \oplus \chi_7^{e(n)} \oplus \chi_8^{f(n)}$, we have for $n = 1, \cdots, 10$:

$$\begin{pmatrix} a(n) \\ b(n) \\ c(n) \\ d(n) \\ e(n) \\ f(n) \end{pmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 & 4 & 6 & 7 & 10 & 11 & 14 \\ 0 & 0 & 1 & 1 & 3 & 4 & 6 & 9 & 14 & 18 \\ 0 & 0 & 1 & 1 & 3 & 4 & 8 & 9 & 14 & 18 \\ 0 & 1 & 2 & 4 & 6 & 11 & 14 & 22 & 28 & 41 \\ 0 & 0 & 1 & 2 & 5 & 8 & 14 & 20 & 30 & 40 \\ 0 & 0 & 0 & 2 & 4 & 8 & 14 & 22 & 32 & 44 \end{bmatrix}.$$

2.3 Decomposition on the Macbeath curve.

There exists a Hurwitz curve of genus 7 with group of automorphisms $G = \mathbb{P}SL_2(\mathbb{F}_8)$. Equations for this curve were first computed in [17] by Macbeath and we refer to his paper for more details. The group G is simple and has 504 elements. There are 9 conjugacy classes 1A, 2A, 3A, 7A, 7B*2, 7C*4, 9A, 9B*2, 9C*4. An element in each class has order 1, 2, 3, 7, 7, 7, 9, 9, 9 respectively. We reproduce from [5] the character table of G.

Character table of $\mathbf{P}SL_2(F_8)$.

	1A	2A	3A	7A	B*2	C*4	9A	B*2	C*4
χ_1	1	1	1	1	1	1	1	1	1
χ_2	7	-1	-2	0	0	0	1	1	1
χ_3	7	-1	1	0	0	0	$-\alpha_1$	$-\alpha_2$	$-\alpha_4$
χ_4	7	-1	1	0	0	0	$-\alpha_4$	$-\alpha_1$	$-\alpha_2$
χ_5	7	-1	1	0	0	0	$-\alpha_2$	$-\alpha_4$	$-\alpha_1$
χ_6	8	0	-1	1	1	1	-1	-1	-1
χ_7	9	1	0	β_1	β_2	β_4	0	0	0
χ_8	9	1	0	β_4	β_1	β_2	0	0	0
χ_9	9	1	0	β_2	β_4	β_1	0	0	0

Here $\alpha_i = \mu^i + \mu^{-i}$, $\mu = e^{2i\pi/9}$ and $\beta_i = \zeta^i + \zeta^{-i}$, $\zeta = e^{2i\pi/7}$.

Now we will compute trace $(h|\ H^0(S^6X, K^n))$ for an element h in each conjugacy class of G.

2.3.1 Case 1A.

By Lemma 1.2 we have

$$\sum(-1)^i \text{trace}\,(h \mid H^i(S^6X, K^n)) = n^7 - (n-1)^7.$$

2.3.2 Case 2A.

An element h of order 2 has 4 fixed points $\{p_1, \cdots, p_4\}$ on X. Then using our program we have

$$\sum(-1)^i \mathrm{trace}\,(h \mid H^i(S^6X, K^n)) =$$
$$\frac{(2\,n-1)\,(n^2-n+1)\,(-1)^n}{2} + \frac{3}{2} - \frac{9\,n}{2} + \frac{9\,n^2}{2}$$

2.3.3 Case 3A.

An automorphism h of order 3 has 6 fixed points at X. Since there is only one conjugacy class of order 3, all we need to now to apply our program is the following:

Lemma 2.2. *Let $p_1, \cdots, p_6$ be the fixed points of h. Then h acts as $\omega = e^{2i\pi/3}$ on the tangent spaces of three points and as ω^2 on the tangent spaces of the other three points.*

Proof. Let $p \in X$ be a fixed point of h. We first will see what are the other fixed points of h. Let $H = \langle\, h \,\rangle$. The normalizer $N(H)$ of H has order 18. Choose $t \in N(H)$ such that $t^2 = 1$, then $tht = h^2$ because G has elements of order $2, 3, 7$ and 9 only. We can assume that $h = z^3$, with z an element of order 9. $N(H) = \langle\ z, t\ \rangle$ and $N(H)/H = \{\bar{t}, \bar{tz}, \bar{tz^2}, \bar{z^2}, \bar{z}, 1\}$ $(\cong S_3)$. Then the 6 fixed points of h are tp, tzp, tz^2p, zp, z^2p and p.

Now one can verify the lemma, for instance, if h acts as w on T_p, consider the composition

$$T_p \xrightarrow{t} T_{tp} \xrightarrow{h} T_{tp} \xrightarrow{t^{-1}} T_p,$$

induced by $X \xrightarrow{t} X \xrightarrow{h} X \xrightarrow{t^{-1}} X$. Then one sees that h acts as w^2 on T_{tp}. ∎

So

$$\sum(-1)^i \mathrm{trace}\,(h \mid H^i(S^6X, K^n)) =$$
$$\left(\left(\frac{8}{3} - \frac{8}{3}\,n\right)\,\omega + \frac{8}{3}\,n\right)\,(\omega^n)^2 + \left(\left(-\frac{8}{3} + \frac{8}{3}\,n\right)\,\omega - \frac{8}{3} + \frac{16}{3}\,n\right)\,\omega^n + \frac{11}{3}.$$

2.3.4 Case 7A,B,C

Let $h \in G$ be of order 7. Let $H = \langle h \rangle$. Since h has 2 fixed points on X, the normalizer $N(H)$ of H has order 14. Let $t \in N(H)$ be of order 2. We have $N(H) = \langle t, h \rangle$. So if p_1 is a fixed point of h, then the other fixed point is tp_1. Now $tht = h^k$. Notice $k \neq 1$, otherwise $N(H)$ would be cyclic and there is no element of order 14 in G. So $k \equiv -1 \mod 7$. From this we see that if h acts as ζ^α on T_{p_1} then h acts as $\zeta^{-\alpha}$ on T_{tp_1}. The value of α depends on the conjugacy class of h, we do not know these values, however in this case we obtain the same result for any value of $\alpha (= 1, 2, 3, 4, 5, 6)$.

Although not all the powers of h belong to the same conjugacy class, we can apply our program because the fixed point set of h on S^6X has dimension 0 i.e. the normal bundles of the components of the fixed point set have total Chern class equal to 1. In this case we obtain

$$\sum (-1)^i \text{trace}\,(h \mid H^i(S^6X, K^n)) = 1.$$

2.3.5 Case 9A,B,C

Let $z \in G$ has order 9. The 6 fixed points of z^3 at X have the form p, zp, z^2p, p_2, zp_2, z^2p_2, where z^3 acts on T_p as ω and as ω^2 on T_{p_2}. The fixed point set of z at S^6X consists of the 3 points $2p + 2zp + 2z^2p$, $2p_2 + 2zp_2 + 2z^2p_2$ and $p + zp + z^2p + p_2 + zp_2 + z^2p_2$. So we need to know the action on the tangent spaces of these points at S^6X. If we identify the tangent spaces T_p, T_{zp} and T_{z^2p} with $\mathbb{C}$ then we can assume that the induced maps $T_p \to T_{zp}$, $T_{zp} \to T_{z^2p}$ and $T_{z^2p} \to T_{zp}$ are mulptiplication by scalars a, b and c respectively. Since z^3 acts as ω on T_p we have $abc = \omega$. Similarly we choose scalars d, e, f such that $edf = \omega^2$ for the case p_2, zp_2, z^2p_2. From this we see that the action on the tangent space of $p + zp + z^2p + p_2 + zp_2 + z^2p_2$ is given by the matrix

$$\begin{bmatrix} 0 & 0 & c & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 \end{bmatrix}$$

whose determinant is 1 and its characteristic polynomial is $q(\lambda) = (\lambda^3 - \omega)(\lambda^3 - \omega^2) = \lambda^6 + \lambda^3 + 1$. Now, for the other 2 points, let $V_1, V_2, V_3 \subset S^2X$ be disjoint open neighbourhoods of $2p, 2zp, 2z^2p$ respectively. Then $V_1 \times V_2 \times V_3$ is isomorphic to a neighbourhood of $2p + 2zp + 2z^2p$. Choosing coordinates σ_1, σ_2 for S^2X as in the proof of Lemma 1.9, one can see that the action on the tangent space of this point is given by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b^2 & 0 & 0 \end{bmatrix}$$

which also has determinant 1 and characteristic polynomial $q(\lambda)$. Now we can use 1.5 to obtain

$$\sum (-1)^i \mathrm{trace}\,(z \mid H^i(S^6X, K^n)) = 3\binom{1}{3} = 1.$$

2.3.6 Final result.

Let $H^0(J, \mathcal{O}(n\Theta)) = \mathbb{C}^{a_1(n)} \oplus V_2^{a_2(n)} \oplus V_3^{a_3(n)} \oplus \cdots V_9^{a_9(n)}$, then

$$a_1 = \frac{16}{21} n + \frac{1}{9} trj3 + \frac{1}{504} trj1 + \frac{1}{8} trj2,$$

$$a_2 = \frac{1}{72} trj1 - \frac{1}{8} trj2 + \frac{1}{3} n - \frac{2}{9} trj3,$$

$$a_3 = a_4 = a_5 = \frac{1}{9} trj3 + \frac{1}{72} trj1 - \frac{1}{8} trj2,$$

$$a_6 = \frac{2}{21}n + \frac{1}{63}trj1 - \frac{1}{9}trj3,$$

$$a_7 = a_8 = a_9 = -\frac{1}{7}n + \frac{1}{56}trj1 + \frac{1}{8}trj2,$$

where

$$trj1 = n^7,$$

$$trj2 = \frac{1}{2}n^3\left((-1)^n + 3\right),$$

$$trj3 = \frac{8}{3}w^n\,n + \frac{8}{3}(w^n)^2\,n + \frac{11}{3}n.$$

For the first 10 values of n we have

$$\begin{bmatrix} a_1(n) \\ a_2(n) \\ a_3(n) \\ a_4(n) \\ a_5(n) \\ a_6(n) \\ a_7(n) \\ a_8(n) \\ a_9(n) \end{bmatrix} = \begin{bmatrix} 1 & 4 & 13 & 52 & 175 & 620 & 1683 & 4296 & 9597 & 20100 \\ 0 & 0 & 22 & 212 & 1070 & 3824 & 11396 & 29000 & 66324 & 138640 \\ 0 & 0 & 30 & 212 & 1070 & 3840 & 11396 & 29000 & 66348 & 138640 \\ 0 & 0 & 30 & 212 & 1070 & 3840 & 11396 & 29000 & 66348 & 138640 \\ 0 & 0 & 30 & 212 & 1070 & 3840 & 11396 & 29000 & 66348 & 138640 \\ 0 & 2 & 32 & 260 & 1240 & 4438 & 13072 & 33288 & 75912 & 158730 \\ 0 & 4 & 42 & 308 & 1410 & 5052 & 14748 & 37576 & 85500 & 178820 \\ 0 & 4 & 42 & 308 & 1410 & 5052 & 14748 & 37576 & 85500 & 178820 \\ 0 & 4 & 42 & 308 & 1410 & 5052 & 14748 & 37576 & 85500 & 178820 \end{bmatrix}.$$

2.4 The Bring curve of genus 4.

The Bring curve is the only genus 4 curve admiting the symmetric group $G := S_5$ as its group of automorphisms. It is another example of a curve with maximal group of automorphisms. Some information about this curve can be found in [24], [8] or [9]. This curve can be defined in $\mathbb{P}^4$ using the equations

$$\sum_{i=1}^{5} x_i = 0, \quad \sum_{i=1}^{5} x_i^2 = 0, \quad \sum_{i=1}^{5} x_i^3 = 0.$$

The group acts permuting coordinates. We use [12] to produce the character table of S_5 and some information about its subgroups. There are 9 conjugacy classes for G, say $\overline{1}$, $\overline{(1,2)}$, $\overline{(1,2)(3,4)}$, $\overline{(1,2,3)}$, $\overline{(1,2,3)(4,5)}$, $\overline{(1,2,3,4)}$, $\overline{(1,2,3,4,5)}$ of orders 1,2,2,3,6,4,5 and sizes 1,10,15,20,20,30,24 respectively. Denote by $1a, 2a, 2b, 3a, 6a, 4a, 5a$ the conjugacy classes of G. The character table of S_5

	$1a$	$2a$	$2b$	$3a$	$6a$	$4a$	$5a$
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	-2	0	1	1	0	-1
χ_4	4	2	0	1	-1	0	-1
χ_5	5	1	1	-1	1	-1	0
χ_6	5	-1	1	-1	-1	1	0
χ_7	6	0	-2	0	0	0	1

The Fuchsian group which yields X and G has period partition (2,4,5). Next we will compute trace$(h \mid H^0(S^3X, K^n))$ for an element h in each conjugacy class of G.

2.4.1 Case 1a.

By Lemma 1.2 we have

$$\sum (-1)^i \text{trace}\,(h \mid H^i(S^3X, K^n)) = n^4 - (n-1)^4.$$

On the Jacobian we have

$$tr1a := \text{trace}\,(h \mid H^0(J_X, \mathcal{O}(n\Theta))) = n^4.$$

2.4.2 Case 2a.

The normalizer of $\langle(1,2)\rangle$ in G is $H = \langle(4,5),(3,4),(1,2)\rangle$, H has order 12. An element in this class is the image of a maximal cyclic subgroup of order 2 of the Fuchsian group that yields S_5 as the group of automorphisms of X. So an element in this class has 6 fixed points in X.

$$\sum(-1)^i \mathrm{trace}\,(h \mid H^i(S^3X, K^n)) = \frac{5(-1)^n - 3}{2} + 3n.$$

Then on the Jacobian we have

$$tr2a := \mathrm{trace}\,(h \mid H^0(J_X, \mathcal{O}(n\Theta))) = \frac{3}{4} + \frac{5\,(-1)^n}{4} + \frac{3\,n\,(n+1)}{2} - \frac{3\,n}{2}.$$

2.4.3 Case 2b.

The normalizer of $\langle(1,2)(3,4)\rangle$ is $H = \langle(3,4),(1,2),(1,3)(2,4)\rangle$, H has 8 elements. An element in this class is the saquare of an element in 4a, so it is the image of an element in a maximal cyclic subgroup of order 4 of the Fuchsian group that yields S_5 as the group of automorphisms of X. Therefore there are 2 fixed points in X for an automorphism in this class.

$$\sum(-1)^i \mathrm{trace}\,(h \mid H^i(S^3X, K^n)) = 2n - 1.$$

Then on the Jacobian we have

$$tr2b := \mathrm{trace}\,(h \mid H^0(J_X, \mathcal{O}(n\Theta))) = n\,(n+1) - n.$$

2.4.4 Case 3a.

The automorphisms in this class have no fixed points in X. So we have

$$\sum(-1)^i \mathrm{trace}\,(h \mid H^i(S^3X, K^n)) = 2n - 1.$$

Then on the Jacobian we have

$$tr3a := \mathrm{trace}\,(h \mid H^0(J_X, \mathcal{O}(n\Theta))) = n\,(n+1) - n.$$

2.4.5 Case 6a.

An element in this class has no fixed points at X. If $h \in 6a$ then $h^3 \in 2a$. So the fixed points of h^3 at X are of the form $p_1, hp_1, h^2p_1, p_2, hp_2, h^2p_2$ and the fixed points of h at S^3X are $p_1 + hp_1 + h^2p_1,\ p_2 + hp_2 + h^2p_2$. The action on the tangent spaces at these points is given by matrixes of the form

$$\begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix},$$

where $abc = -1$. This matrix has characteristic polynomial $\lambda^3 + 1$. So using Corollary 1.5 we get

$$\sum(-1)^i \text{trace}\,(h \,|\, H^i(S^3X, K^n)) = (-1)^n.$$

Then on the Jacobian we have

$$tr6a := \text{trace}\,(h \,|\, H^0(J_X, \mathcal{O}(n\Theta))) = \frac{3}{2} + \frac{(-1)^n}{2}.$$

2.4.6 Case 4a.

The normalizer of $\langle(1,2,3,4)\rangle$ is $\langle(1,2,3,4),(2,4)\rangle$ and has 8 elements. So if $h \in 4a$ then the 2 fixed points $p_1,\ p_2$ of h^2 are the fixed points of h in X. Since h and h^3 are conjugate to each other we see that h acts as i and $-i$ on the tangent spaces of the two fixed points. The fixed points of h in S^3X are $3p_1, 2p_1 + p_2, p_1 + 2p_2, 3p_2$. From Lemma 1.11 we see that the eigenvalues of h on the tangent spaces of these points are $-1, -i, i$. So h acts as -1 on the fibres of the canonical line bundle at these points. Using 1.5 we get

$$\sum(-1)^i \text{trace}\,(h \,|\, H^i(S^3X, K^n)) = (-1)^n.$$

Then on the Jacobian we have

$$tr4a := \text{trace}\,(h \,|\, H^0(J_X, \mathcal{O}(n\Theta))) = \frac{3}{2} + \frac{(-1)^n}{2}.$$

2.4.7 Case 5a.

The normalizer of $\langle(1,2,3,4,5)\rangle$ is $\langle(1,2,3,4,5),(2,5)(3,4),(2,4,5,3)\rangle$ and has 20 elements. So there are 4 fixed points in X for an automorphism h in this class and since the 4 powers of h belong to this same class we see that h acts as $\nu^1,\cdots,\nu^4$ ($\nu = e^{2i\pi/5}$) on the tangent spaces of these points.

$$\begin{aligned}\sum(-1)^i \mathrm{trace}\,(h\,|\,H^i(S^3X,K^n)) &= -\frac{4\,(\nu^n)^4\,(\nu-1)}{5} - \frac{4\,(\nu^n)^3\,(\nu-1)\,(\nu+1)}{5} \\ &\quad - \frac{4\,(\nu^n)^2\,(\nu-1)\,(\nu^2+\nu+1)}{5} \\ &\quad + \frac{4\,\nu^n\,(2+\nu^3+\nu^2+\nu)}{5}.\end{aligned}$$

Then on the Jacobian we have

$$\begin{aligned} tr5a := \mathrm{trace}\,(h\,|\,H^0(J_X,\mathcal{O}(n\Theta))) &= \frac{9}{5} + \frac{4\,\nu\,\mu^4}{5} + \frac{4\,\nu^2\,\mu^3}{5} + \frac{4\,\nu^3\,\mu^2}{5} \\ &\quad + (-\frac{4}{5}\,\nu^3 - \frac{4}{5} - \frac{4}{5}\,\nu - \frac{4}{5}\,\nu^2)\,\mu,\end{aligned}$$

where $\mu = \nu^{n+1}$.

2.4.8 Final result.

Let $H^0(J,\mathcal{O}(n\Theta)) = \mathbb{C}^{a_1(n)} \oplus V_2^{a_2(n)} \oplus V_3^{a_3(n)} \oplus \cdots V_9^{a_7(n)}$. Then

$$\begin{bmatrix} a_1(n) \\ a_2(n) \\ a_3(n) \\ a_4(n) \\ a_5(n) \\ a_6(n) \\ a_7(n) \end{bmatrix} = \begin{bmatrix} \frac{tr1a}{120} + \frac{tr6a}{6} + \frac{tr5a}{5} + \frac{tr4a}{4} + \frac{tr2b}{8} + \frac{tr2a}{12} + \frac{tr3a}{6} \\ -\frac{tr2a}{12} + \frac{tr1a}{120} - \frac{tr6a}{6} + \frac{tr5a}{5} - \frac{tr4a}{4} + \frac{tr2b}{8} + \frac{tr3a}{6} \\ \frac{tr1a}{30} + \frac{tr6a}{6} - \frac{tr2a}{6} + \frac{tr3a}{6} - \frac{tr5a}{5} \\ -\frac{tr6a}{6} + \frac{tr1a}{30} + \frac{tr2a}{6} + \frac{tr3a}{6} - \frac{tr5a}{5} \\ -\frac{tr3a}{6} + \frac{tr1a}{24} + \frac{tr2b}{8} - \frac{tr4a}{4} + \frac{tr2a}{12} + \frac{tr6a}{6} \\ \frac{tr4a}{4} + \frac{tr1a}{24} - \frac{tr2a}{12} + \frac{tr2b}{8} - \frac{tr6a}{6} - \frac{tr3a}{6} \\ \frac{tr5a}{5} + \frac{tr1a}{20} - \frac{tr2b}{4} \end{bmatrix}.$$

For $n = 1, \cdots, 10$ we have

$$\begin{bmatrix} a_1(n) \\ a_2(n) \\ a_3(n) \\ a_4(n) \\ a_5(n) \\ a_6(n) \\ a_7(n) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 10 & 17 & 27 & 41 & 62 & 89 & 127 \\ 0 & 0 & 2 & 4 & 10 & 16 & 28 & 44 & 68 & 100 \\ 0 & 0 & 2 & 7 & 18 & 40 & 76 & 131 & 212 & 324 \\ 0 & 2 & 6 & 15 & 30 & 58 & 100 & 163 & 252 & 374 \\ 0 & 1 & 4 & 12 & 28 & 57 & 104 & 176 & 280 & 425 \\ 0 & 0 & 2 & 8 & 22 & 48 & 92 & 160 & 260 & 400 \\ 0 & 0 & 2 & 9 & 26 & 56 & 108 & 189 & 308 & 476 \end{bmatrix}.$$

Appendix A

The program.

Although from what is said in section 1.6 it would not be difficult to write a program to compute

$$\sum(-1)^i \text{trace}\,(h \mid H^i(S^b X, K^n))$$

we present one here. The program is for Maple and we will explain what some of its subprograms do.

We start with some notation and we refer to section 1.6 for this. What we want to compute is

$$\sum_{k=0}^{m} \sum_{D \in A_k} \lambda(k, D).$$

Let $sum1(k)$ represent

$$\sum_{D \in A_k} \lambda(k, D).$$

This can be written as

$$\int_{S^k Y} sum0(k).$$

Now $sum0(k)$ can be written as

$$sum0(k) = tdsdx(k,\gamma) charclass1(k, p, \gamma, g)(\sum_{D \in A_k} prod(D)),$$

where

$$prod(D) = \frac{ch_h(K^n \mid_{f_{k,D}(S^k X)}) \cdot \prod_j \mathcal{U}(i^*(N_{A_D})(\nu^j))}{det(1 - h_{|_{N^\vee}})}. \tag{A.1}$$

There can be several divisors with the same class, and $prod(D)$ depends only on the class of D. If we compute the set of classes $A_k/class$ of divisors in A_k and the number of elements $\#class(D)$ in each class, then we have

$$\sum_{D\in A_k} prod(D) = \sum_{class(D)\in A_k/class} \#class(D)prod(class(D)).$$

The function **classpts**() computes the classes of the divisors $D \in A_k$ and the function **classes**() counts number of classes and the number of divisors in each class. We use several subprograms, namely **trg**, **sum1**, **lefpts**, **newnops**, **newop**, **monadd**, **exponents**, **classpts**, **classes**, **tdsdx**, **charclass1**, **invers1mnu**, **precharclass**, **ptstheta**. Now we are going to explain three of these maple programs, namely, **trg**, **sum1** and **lefpts**.

•**trg**. This is the main program, that is

$$trg(g,s,u,p,b) = \sum(-1)^i \text{trace}\,(h \mid H^i(S^bX, K^n)).$$

What trg does is to compute

$$\sum_{j=0}^{m} ptstheta(sum1(j), \gamma).$$

We use **ptstheta** because **sum1**(j) is a homogeneous polynomial of degree j in the variables η, θ and to make sense of it we use formula (1.9), that is, *ptstheta* converts $\sum a_{rs}\eta^s\theta^r$ to $\sum a_{rs}\binom{\gamma}{r}r!$, where γ is the genus of the quotient curve $Y = X/<h>$ and a_{rs} is free of η or θ.

•**sum1**. Roughly what this program does is to compute the classes of divisors $A_j/class$ and then use them to compute $\int_{S^jY} sum0(j)$. Let $nfix$ denote the number of fixed points of the automorphism h on the curve X.

If

$$a = monadd(nfix, p, b - p * j)$$

then

$$exponents(a, nfix)$$

is a matrix of dimension $\hbar \times nfix$ whose rows represent all the divisors in A_j, i.e. if $(\alpha_1, ..., \alpha_{nfix})$ is a row of this matrix , then it represents the divisor $\sum_{i=1}^{nfix} \alpha_i x_i$, where x_i are the fixed points of the automorphism h at the curve X . Notice $\hbar$ is the number of divisors in A_j and it can be computed using (1.12), notice also that $b - p * j$ is the degree d_j of the divisors in A_j.

Now B:=classpts(u,nfix,exponents(a,nfix),p) computes the class of each divisor in A_j. Here a matrix of dimension $\hbar \times (p-1)$ is produced; each row of this matrix represents the class of a divisor in A_j.

With classes(B,p) we obtain a matrix cc of dimension $t \times p$. The first $p-1$ coordinates of each row of this matrix represent the class of a divisor in A_j and the p^{th} coordinate is the number of divisors with that class. There are t different classes of divisors in A_j.

At the end $sum0$ is computed and the coefficient of t^j of $sum0$ is what we called $\int_{S^jY} sum0(j)$.

Notice *classpts* is a function of five arguments but since *exponents* returns two values, then classpts(u,nfix,exponents(a,nfix),p) is well used. Notice also that *classpts* and *classes* return two values.

•**lefpts**. This program is to compute

$$\sum_{class(D)\in A_k/class} \#class(D)prod(class(D)).$$

Here there is a cycle in which i runs over the number of of classes of divisors in the A_j in question. Inside this cycle there is another cycle in which j runs from 1 to $p-1$. The final value of $l[i]$ in the cycle for j will be the factor

$$\prod_j \mathcal{U}(i^*(N_{A_D})(\nu^j))$$

in formula (A.1).

The final value of $invdtr[i]$ will be $\frac{1}{\det(1-h|N^\vee)}$, see 2) in remark 1.18. The final value of $aa[i]$ will be the exponent of ν in formula (1.22).

The corresponding value of formula (1.23) for a divisor D in the i^{th} class of our A_j is $chg[i]$.

Since $cc[i,p]$ is the number of divisors in the i^{th} class,

$$l[i] * invdtr[i] * chg[i] * cc[i,p]$$

is what we called $\#class(D)prod(class(D))$.

The final value of rr is the sum that we want to compute.

Just a final note: when we are using Maple we do not use the complex value of ν. Every time we have a polynomial $q(\nu)$ we use $rem(q(\nu), \sum_{i=0}^{p-1} \nu^i, \nu)$, where $rem(a, b, x)$ is a maple function that computes the residue of the polynomial a modulo b. The third parameter makes rem regard a and b as polynomials in the x variable. In some cases (for instance in **precharclass**()) we use the formula

$$\frac{1}{1-\nu} = \frac{1}{p}\sum_{j=0}^{p-2}(p-1-j)\nu^j \tag{A.2}$$

which is true for $\nu \neq 1$ such that $\nu^p = 1$.

trg

```
> trg:=proc(g,nfix,A,p,b)
> #trg computes the trace of an automorphism h of
> #order prime p on "H^0(Sym(b,X),K^n)",
> # b can be any positive integer.
> #We assume that all the powers of h belong to
> #the same conjugacy class;
> #b is the dimension of the symmetric product
> #where we are working,
> #g is the genus of the curve X where the automorphism is acting;
> #nfix is the number of fixed points
> #(p1...p[nfix]) of the automorphism in the curve;
> #A is a vector with nfix positive entries(a1,a2,...)
> #and h acts on the tangent space
> #of pi as nu^(ai),where nu represents a
> #pth-root of unity;
> local j,ld,mdim,val,gamma,ffff;
> mdim:=iquo(b,p):
> #gamma is the genus of the quotient curve, we
> #are using Riemann-Hurwitz
> #theorem to compute it.
> gamma:=(2*(g-1)-(p-1)*nfix)/2/p+1:
> val:=0:
> for j from 0 to mdim do
> val:=val+ptstheta(sum1(j,p,g,nfix,A,b),gamma)
> od;
> val:=collect(val,yyy);
> ffff:=unapply(val,yyy):
> #Here we are replacing yyy by nu^n as anounced in lefpts,
> #sometimes is better not to replace it
> #because it makes it easier to perform some algebraic operations.
> ffff(nu^n);
> end proc:
```

sum1

```
> sum1 := proc(dim,p,g,nfix,A,b)
> local a,B,sum0,gamma,intsum0;
> #p,g,nfix,A,b are as defined in trg
> #so
> #sum1 is a function of dim.
> a:=monadd(nfix,p,b-p*dim):
> if a = 0 then 0 else
> gamma:=(2*(g-1)-(p-1)*nfix)/2/p+1:
> B:=classpts(A,nfix,exponents(a,nfix),p):
> sum0:=charclass1(dim,p,gamma,g)*tdsdx(dim,gamma)*
> lefpts(classes(B,p),p,dim,b,g):
> intsum0:=coeff(sum0,t,dim);
> rem(intsum0,sum(nu^i,i=0..p),nu) fi;
> end proc:
```

lefpts

```
> lefpts:=proc(cc,h,p,dim,b,g)
> #leftpts first computes " \Sigma #class(D)prod(class(D))",
> # where the sum #"\Sigma" runs over
> #the classes of divisors in $A_{dim}$.
> # cc is a matrix of dimension hxp,
> # it is the matrix of classes of points, to be more
> # precise the first p-1
> #coordinates of each row of cc represent a class of
> # a divisor and the p-coordinate
> #is the number of divisors in that class,
> #h should be
> #the number of classes of divisors in  $A_{dim}$
> #(not the number of elements in #$A_{dim}$);
> #b is supposed to be the dimension of the space where
> #the v.b we are interested in is defined
> local i,j,l,pol,invdtr,ff,aa,chg,rr;
> rr:=0;
> pol:= sum(nu^i,i=0..p-1);
> ff:=unapply(invers1mnu(p),nu);
> #ff(nu) is the inverse of (1- nu) if nu^p=1;
> #this(combined with rem( ,pol,nu)) will be used
> #just to simplify nu where it is possible.
> for i from 1 by 1 to h do  l[i]:=1;
> invdtr[i]:=1/(p^dim);
> if p = 2 then aa[i]:=dim else aa[i]:=0 end if;
> for j from 1 by 1 to p-1 do
> l[i]:= rem(
> precharclass(dim,p,j)^(cc[i,j])*l[i],t^(dim+1),t);
> l[i]:=rem(l[i],pol,nu);
> invdtr[i]:=rem(ff(nu^(p-j))^(cc[i,j])*invdtr[i],pol,nu);
> aa[i]:=aa[i]-j*cc[i,j];
> end do;
```

```
> aa[i]:=modp(aa[i],p);
> chg[i]:=convert(
> series(exp(n*((g-1-b)*eta+p*theta)*t),t=0,dim+1),polynom
> )*yyy^aa[i];
> #at the end yyy will be replaced by nu^n;
> rr:=rr+l[i]*invdtr[i]*chg[i]*cc[i,p] od;
> rr:=rem(rr,pol,nu);
> rr:=rem(rr,yyy^p-1,yyy);
> rem(rr,t^(dim+1),t);
> end proc:
```

newnops

```
> newnops:= proc(a)
> #newnops returns the number of sumands of a;
> #we do not use the maple function nops because
> #if "a" is a single monomial
> #then nops returns the number of factors.
> if a=0 then 1:
> else nops(a+ZZZ)-1:  fi:
> end proc:
```

newop

```
> newop:=proc(i,a)
> #mewop returns the i^th summand of a.
> if newnops(a)=1 then a:  else op(i,a):  fi:
> end proc:
```

monadd

```
> monadd:=proc(m,p,d)
> #monadd computes a polynomial whose terms are all
> #the monomials of degree d in m variables;
> #and the degree of each variable is at most p-1;
> #The monomials in monadd correspond to divisors
> #supported on the fixed point set
> #m = number of fixed points;
> #p is the order of the automorphism;
> #d is the degree of the monomials;
> local a,j;
> a:=sum(x[i],i=1..m)^d;
> for j from 1 to m do a:=rem(a,x[j]^p,x[j]) od;
> return expand(a);
> end proc:
```

exponents

```
> exponents:= proc(a,m)
> #a must be a polynomial in the variables x[1],...,x[m].
> #a must be expanded when entered;
> local h,i,j,k;
> h:=newnops(a);
> k:=Matrix(h,m);
> for i from 1 to h do
> for j from 1 to m do
> k[i,j]:= degree(newop(i,a),x[j]) od od;
> return(k,h);
> end proc:
```

classpts

```
> classpts := proc(A,m,k,h,p)
> #this computes the class of a point;
> #A is a vector with m entries, where m=number of fixed points;
> #the automorphism g acts as v^(A[i]) on
> #the tangent space at the i-th point, where
> #v is a primitive p-root of unity;
> #p is the order of the automorphism;
> #k is an hxm matrix;
> # h = newnops(monadd(m,p,some d)) = number of fixed point
> components  in Sym^d(X)
> #each row of k represents a point;
> #there are h points;
> local i,j,l,B;
> B:=matrix(h,p-1);
> for i from 1 to h do
> for j from 1 to p-1 do
> B[i,j]:=0 od:
> for j from 1 to m  do
> if (k[i,j]>0) then
> for l from 1 to k[i,j] do
> B[i,modp(A[j]*l,p)]:=B[i,modp(A[j]*l,p)]+1;
> od fi od od:
> return(B,h);
> end
> proc:
```

classes

```
> classes := proc(B,h,m)
> #this counts the multiplicity of the
> #rows of the matrix B;
> #B is a matrix of dimension h x m-1;
> #m is the order of the automorphism;
> local c,cc,i,j,e,r,s,ss,t;
> for i from 1 to m-1 do c[1,i]:=B[1,i] od;
> for i from 1 to h do c[i,m]:=0 od;
> t:=1;
> for i from 1 to h
> do"do1";
> r:=1:s:=0:
> while((s = 0)and(r < (t+1))) do "do2";
> ss:=1:j:=1:
> while((ss=1)and(j < m))do "do3";
> if B[i,j]= c[r,j] then j:=j+1;
> else ss:=0 fi od;"end od3";
> #ss=0 means  the row  i of B[] is
> #different from the row r of c[]
> #(more precisely  least the m-1 entries of the r row of c[]);
> if (ss = 1) then s:=1;
> c[r,m]:=c[r,m]+1;
> #we put s=1 to break the while cycle,
> #s=0 means the row i of B[] is not equal to
> #any of the classes we have so far
> # i.e the first r rows(or sub rows of lenght m-1) of c[];
> else r:=r+1 fi od;"end do2";
> if s = 0 then t:=t+1;
> c[t,m]:=1;
> #Here since the i row of B is not one of
> #the classes we had, we register it.
> for j from 1 to m-1 do
> c[t,j]:=B[i,j] od fi od;"enddo1";
> cc:= matrix(t,m);
```

```
> for i from 1 to t do
> for j from 1 to m do
> cc[i,j]:=c[i,j] od od;
> #the last coordinate cc[i,m] represents the order of
> #the class i.  Remember m is the order of the automorphism;
> return(cc,t);
> end proc:
```

tdsdx

```
> tdsdx := proc(d,g)
> #tdsdx computes the todd class of a
> #d-symmetric product of a curve of genus g;
> local p1,p2,p3,p4,tau;
> if d=0 then 1 else
> tau:=convert(series((t*eta*exp(-eta*t)+exp(-eta*t)-1)/
> (t*eta*(1-exp(-eta*t))),t=0,(d+1)*(g+1)+1),polynom);
> p1:=sum((t*theta*tau)^j/(j!),j=0..g);
> p2:=convert(series((eta*t/(1-exp(-eta*t)))^(d-g+1),
> t=0,(d+1)*(1+g)+1),polynom);
> #notice we are using a very large order for
> #the expansion of the series,
> #namely (g+1)*(d+1)+1 rather than d+1,
> #because d+1 does not produce correct values
> #for tdsdx(1,g).
> #This makes the program too slow.
> #For instance for d=1, g=11,
> #the calculation lasted 2444.370 seconds.
> #The problem is the expansion of tau, I think.
> p3:=series(p2*p1,t=0,((g+1)*(d+1))+1);
> p4:=convert(p3,polynom);
> return(rem(p4,t^(d+1),t)) fi;
> end proc:
```

charclass1

```
> charclass1 := proc(m,n,gamma,g)
> #See  section "The computation" for
> #the definition of charclass1;
> #m = dimension of symmetric product;
> #n= order of automorphism(a prime number);
> #gamma = genus of quotient curve;
> #g genus of curve;
> local A,p,q,U1,U2,U3,U4;
> A:= m+(gamma - g)/(n-1);
> p:= unapply(sum(z^i,i=0..n-1),z);
> q:= unapply(-z*diff(p(z),z)/p(z),z);
> U1:= (n^A)*p(exp(-eta*t))^(-A);
> U2:=sum((t*theta* q(exp(-t*eta)))^j/(j!),j=0..gamma);
> U3:= series(U1*U2,t=0,(gamma+1)*m+n);
> #try to find the best value for the order in U3,
> #maybe (gamma+1)*m+n is not the best value,
> #it makes the program slow;
> U4:=convert(U3,polynom);
> return(rem(U4,t^(m+1),t));
> end proc:
```

invers1mnu

```
> invers1mnu :=proc(p)
> #computes the inverse of 1-nu, if nu^p=1.
> sum((p-1-j)*nu^(j),j=0..p-2)/p
> end proc:
```

precharclass

```
> precharclass := proc(d,p,pow)
> #See section "The calculation" for the
> #definition of this function.
> #d is the dimension of the symmetric product;
> #pow is the exponet of nu, i.e it nu^pow is the eigenvalue;
> local p1,p2,p3,p4,p5,f,pol,j;
> if pow = 0 then 1 else pol:=sum(nu^j,j=0..p-1);
> p1:=series((1-1/nu)/(1-exp(-eta*t)/nu),t=0,d+1);
> p2:=convert(p1,polynom);
> #factor(p2) is a polynomial in nu divided by (1-nu)^d;
> #the inverse of (1-nu)^d is p3:
> p3:=rem((invers1mnu(p))^d,pol,nu);
> #So multipliying p2 by 1:
> p4:=p3*factor((1-nu)^d*p2);
> p5:=rem(p4,pol,nu);
> f:=unapply(p5,nu);
> f(nu^pow) fi;
> end proc:
```

ptstheta

```
> ptstheta := proc(a,g)
> #g is the genus of the quotient curve;
> #a must be a polynomial in the variables theta,eta;
> #a must be  expanded when entered;
> #this program makes each term of the form
> #theta^r*eta*t to r!binomial(g,r)*t;
> local aa,b,i,j,k,ff;
> aa:=expand(a);
> k:=0;
> for i from 1 by 1 to newnops(aa) do
> k :=
> degree(newop(i,aa),theta)!
> *binomial(g,degree(newop(i,aa),theta))*newop(i,aa)+k od:
> ff:=unapply(k,theta,eta):
> return(ff(1,1));
> end proc:
```

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