Vector bundles on manifolds: the cohomology of projective algebraic varieties

Al-Ofi, Abdulaziz Slaim

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and

The Cohomology of Projective Algebraic Varieties

by

Abdulaziz Slaim Al-Ofi

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ABSTRACT

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This Thesis looks at different areas of mathematics which require the notion of a differentiable manifold. The first three chapters contain various materials such as differentiable manifolds, submanifolds, tangent bundles, differential forms, integrations, Stokes' theorem, and certain concepts of Riemannian geometry. In the last chapter, we show that the kernel of a given elliptic operator L is finite dimensional. We also show that on compact differentiable manifolds for each de Rham cohomology class σ there exists a unique harmonic form φ, showing that the de Rham cohomology is finite dimensional.

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INTRODUCTION

Chapter 1 is divided into two sections. The goal of Section 1 is to introduce the concept of a manifold and give examples. Section 2 introduces vector bundles, and we give some examples. We discuss in detail the tangent bundle of a manifold, which has some special features to make it a very particular type of manifold. We also discuss some of the classical results in differentiable vector bundles.

Chapter 2 is devoted to two different types of operations. In Section 1 we introduce derivations of geometrical objects. Section 2 discusses the reverse operations of derivations, namely the integrations. We also present Stokes' Formula which is the higher dimensional version of the fundamental theorem of calculus (Leibniz-Newton Formula.)

Chapter 3 is divided into four sections. Section 1 introduces differential forms in terms of tangent bundle and cotangent bundle. Sections 2, 3, and 4 are devoted to give basic definitions of the classical concepts of Riemannian such as metrics, connections, and curvatures. These are carried out in the context of differentiable $\boldsymbol{C}$-vector bundles over a differentiable manifold X. We also show that every vector bundle admits a Hermitian metric.

Finally, in Chapter 4, we describe the general theory of elliptic differential operators on differentiable manifolds. In Section 1 we introduce Sobolev spaces. We also give the fundamental Sobolev and Rellich theorems (see [1]), providing the proofs of these theorems in special cases. In Section 2 we discuss the basic structure of differential operators and their symbols. We state an important result of differential operators which shows that the adjoint operator L^* of a differential operator L exists. Moreover, the symbol of L^* is the adjoint (up to sign) of the symbol of L. Section 3 is to introduce generalizations of differential operators called pseudodifferential operators. In Section 4 we show that there exists an inverse for a given elliptic operator L. We also show that the kernel of L is finite dimensional. Moreover, we establish the fundamental decomposition theorem of Hodge for self-adjoint elliptic operators. In Section 5 we introduce elliptic complexes and give some examples. We also prove that there is a canonical isomorphism between the cohomology vector space of degree j of a given elliptic complex E and the vector space of $\mathrm{Ker}\Delta_j = \mathcal{H}(E_j) = \Delta_j$-harmonic sections, showing that the de Rham cohomology is finite dimensional.

CONTENTS

Chapter 1: Manifolds and Vector Bundles

This chapter is divided into two sections. In the first section, we will study the geometry of manifolds, the analysis of functions which are defined on manifolds, and the interaction of the geometry and analysis of manifolds. In the second section, we will study the concept of vector bundles on manifolds. We will develop the concept of the tangent bundle. Finally, we will discuss the continuous and C^∞ classification of vector bundles.

1. Manifolds

There are many classes of manifolds. In this section, we are interested in differentiable manifolds and complex manifolds. We will start with some basic definitions in which we will use the following notations. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the fields of real and complex numbers, respectively, with their usual topologies, and let K denote either of these fields.

If D is an open subset of K^n, we shall be concerned with the following function spaces on D:

(a) $K = \boldsymbol{R}$:

(1) $\mathcal{E}(D)$ will denote the real-valued differentiable functions on D, which we simply call C^∞ functions on D; i.e. $f \in \mathcal{E}(D)$ if and only if f is a real-valued function such that partial derivatives of all orders exist and are continuous at all points of D. Note that $\mathcal{E}(D)$ is often denoted by $C^\infty(D)$.

(2) $\mathcal{A}(D)$ will denote the real-valued real-analytic functions on D; i.e. $\mathcal{A}(D) \subset \mathcal{E}(D)$, and $f \in \mathcal{A}(D)$ if and only if the Taylor expansion of f converges to f in a neighbourhood of any point of D.

(b) $K = \boldsymbol{C}$:

$\mathcal{O}(D)$ will denote the complex-valued holomorphic functions on D; i.e. if $(z_1, \ldots, z_n)$ are coordinates in $\boldsymbol{C}^n$, then $f \in \mathcal{O}(D)$ if and only if near each point $z^0 \in D$, f can be represented by a convergent power series of the form

$$f(z) = f(z_1, \dots, z_n) = \sum_{\alpha_1, \dots, \alpha_n = 0}^{\infty} a_{\alpha_1, \dots, \alpha_n} (z_1 - z_1^0)^{\alpha_1} \dots (z_n - z_n^0)^{\alpha_n}.$$

These particular classes of functions will be used to define the particular classes of manifolds that we will be interested in.

Definition 1.1: Let M be a Hausdorff topological space. If h is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\boldsymbol{R}^n$, then h is called a coordinate map, and the pair (U, h) is called a coordinate system.

Definition 1.2: A topological n-manifold is a Hausdorff topological space with a countable basis which is locally homeomorphic to an open subset of $\boldsymbol{R}^n$. The integer n is called the topological dimension of the manifold. Note that the additional condition of a countable basis (countable at infinity) is important for doing analysis on manifolds.

Suppose that $\mathcal{S}$ is one of three classes of functions defined on the open subset of K^n, described above. We let $\mathcal{S}(D)$ denote the functions of $\mathcal{S}$ defined on D, an open set in K^n. That is, $\mathcal{S}(D)$ is either $\mathcal{E}(D), \mathcal{A}(D)$, or $\mathcal{O}(D)$.

Definition 1.3: An $\mathcal{S}$-structure, $\mathcal{S}_M$, on a K-manifold M is a family of K-valued continuous functions defined on the open sets of M such that :

(a) There exists an integer n with the property that, for every $p \in M$, there exists an open neighbourhood U of p and a homeomorphism $h : U \to U'$, where U' is open in K^n , such that for any open set $V \subset U$

$$f : V \longrightarrow K \in \mathcal{S}_M \text{ if and only if } f \circ h^{-1} \in \mathcal{S}(h(V)).$$

(b) If $f : U \to K$, where $U = \cup_i U_i$ and U_i is open in M, then

$$f \in \mathcal{S}_M \text{ if and only if } f|_{U_i} \in \mathcal{S}_M \text{ for each } i.$$

Remark 1.4: It follows clearly from (a) that

(1) If $K = \boldsymbol{R}$, then the dimension, k, of the topological manifold M is equal to n.

(2) If $K = \boldsymbol{C}$, then the dimension, k, of the topological manifold M is equal to $2n$.
However, in either case n will be called the K-dimension of M, denoted by $dim_K M = n$, which we shall call the real dimension for case (1), and the complex dimension for case (2).

(3) A manifold M with an $\mathcal{S}$-structure is called an $\mathcal{S}$-manifold, denoted by $(M, \mathcal{S}_M)$, and the elements of $\mathcal{S}_M$ are called $\mathcal{S}$-functions on M.

For our three classes of functions we have defined

(a) $\mathcal{S} = \mathcal{E}$: differentiable (or C^∞) manifold, and the functions in $\mathcal{E}_M$ are called C^∞ functions on open subsets of M.

(b) $\mathcal{S} = \mathcal{A}$: real-analytic manifold, and the functions in $\mathcal{A}_M$ are called real-analytic functions on open subsets of M.

(c) $\mathcal{S} = \mathcal{O}$: complex-analytic manifold, and the functions in $\mathcal{O}_M$ are called holomorphic on open subsets of M.

Definition 1.5: There exists an integer n with the property that, for every $p \in M$, there exists an open neighbourhood U of p and a homeomorphism $h : U \to U' \subset K^n$. The pair (U, h) is called a *coordinate chart* around p.

Definition 1.6: (a) An $\mathcal{S}$-morphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is a continuous map, $F : M \to N$, such that

$$f \in \mathcal{S}_N \text{ implies } f \circ F \in \mathcal{S}_M.$$

(b) An $\mathcal{S}$-isomorphism is an $\mathcal{S}$-mapping $F : (M, \mathcal{S}_M) \longrightarrow (N, \mathcal{S}_N)$ such that $F : M \longrightarrow N$ is a homeomorphism, and

$$F^{-1} : (N, \mathcal{S}_N) \longrightarrow (M, \mathcal{S}_M) \text{ is an } \mathcal{S}\text{-morphism.}$$

In our three classes of functions, the concept of an $\mathcal{S}$-morphism and $\mathcal{S}$-isomorphism have special names:
(a) $\mathcal{S} = \mathcal{E}$: differentiable mapping and diffeomorphism of M to N.
(b) $\mathcal{S} = \mathcal{A}$: real-analytic mapping and real-analytic isomorphism of M to N.
(c) $\mathcal{S} = \mathcal{O}$: holomorphic mapping and biholomorphism of M to N.

It follows from the above definitions that **a Differentiable Manifold** M, with a differentiable structure of class C^∞ is a Hausdorff space with a collection of coordinate systems $\{(U_\alpha, h_\alpha) : \alpha \in A\}$ satisfying the following three properties :
(a) $\cup_{\alpha \in A} U_\alpha = M$.
(b) If we have two coordinate systems $h_\alpha : U_\alpha \longrightarrow K^n$ and $h_\beta : U_\beta \longrightarrow K^n$ such that $U_\alpha \cap U_\beta \neq \emptyset$, then

$$h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \longrightarrow h_\beta(U_\alpha \cap U_\beta)$$

is a differentiable of class C^∞ for all $\alpha, \beta \in A$ and is an $\mathcal{S}$-isomorphism on open subsets of $(K^n, \mathcal{S}_{K^n})$.
(c) The collection $\{(U_\alpha, h_\alpha) : \alpha \in A\}$ is a maximal with respect to (b); that is, if (U, h) is a coordinate system such that $h \circ h_\alpha^{-1}$ and $h_\alpha \circ h^{-1}$ are C^∞ for all $\alpha \in A$, then $(U, h) \in \{(U_\alpha, h_\alpha) : \alpha \in A\}$.

Definition 1.7: Let N be a closed subset of an $\mathcal{S}$-manifold M, then N is called an $\mathcal{S}$-submanifold of M if for each point $x_0 \in N$, there is a coordinate system $h : U \longrightarrow U' \subset K^n$, where $x_0 \in U$, with the property that $h_{|U \cap N}$ is mapped onto $U' \cap K^k$, where $0 \leq k \leq n$. Here $K^k \subset K^n$ is the standard embedding of the linear subspace K^k into K^n, and k is called the K-dimension of N, and $n - k$ is called the K-codimension of N.

From the above definition , we immediately announce a proposition:

Proposition 1.8: An $\mathcal{S}$-submanifold of an $\mathcal{S}$-manifold is also an $\mathcal{S}$-manifold.

Proof : Let M be an $\mathcal{S}$-manifold of dimension n and N be an $\mathcal{S}$-submanifold of dimension k, where $k \leq n$. Let $\mathcal{S}_N = \{$ all continuous $f : U' \to K$ such that for every $x \in U'$ (U' is open in N), there exists $x \in U$ (U is open in M) such that $f_{|U \cap U'} = g_{|U \cap U'}$ for some $g : U \to K\}$. We have to show that an $\mathcal{S}$-submanifold N satisfies the two conditions of Definition 1.3.

First condition, ($\Rightarrow$): If $f \in \mathcal{S}_N(U')$ and $x \in U'$, we have to produce a coordinate map $h' : U' \to V' \subset K^k$, then we would like to show that $f \circ h'^{-1} \in \mathcal{S}(V')$. Let $h : U \to V \subset K^n$ such that $h' = h_{|N \cap U}$ with $V' = V \cap K^k$, where $N \cap U = U'$. For $x \in U'$ there exists U which is open in M such that $x \in U$ and there exists $g \in \mathcal{S}_M(U)$ such that $g_{|U' \cap U} = f_{|U' \cap U}$. Replace h and h' by the restriction to U and $U' \cap U$. Then we get $g \circ h^{-1} \in \mathcal{S}(h(U))$ and $g \circ h^{-1}{}_{|h(U' \cap U)} \in \mathcal{S}(h(U' \cap U))$. Thus $g \circ h^{-1}{}_{|h(U' \cap U)} = f \circ h'^{-1}{}_{|h(U' \cap U)}$, where $h(U' \cap U) \subseteq V'$. This holds for all x. Therefore, $f \circ h'^{-1} \in \mathcal{S}(V')$.
($\Leftarrow$): If $h' : U' \to V'$ and $h : U \to V$ such that $h' = h_{|U'}$ and $V' = V \cap K^k$. Let $\pi : K^n \to K^k$. Replace V with $\pi^{-1}(V') \cap V$ and U with $h^{-1}(V)$. Let $f : U' \to K$ such that $f \circ h'^{-1} \in \mathcal{S}(U')$. Because of the projection $\pi : V \to V'$ we have

$$f \circ h'^{-1} \circ \pi \in \mathcal{S}(V)\,.$$

Then

$$g = f \circ h'^{-1} \circ \pi \circ h \in \mathcal{S}_M(U)\,,$$

and $g_{|U'} = f$ since $h_{|U'} = h'$ and $\pi_{|V'} = \mathrm{id}_{V'}$.

Second condition, ($\Rightarrow$): Let $f \in \mathcal{S}_N(U')$. If $x \in U'_i \subset U'$, then there exists $g \in \mathcal{S}_M(U)$ with $x \in U$ such that

$$g_{|U \cap U'} = f_{|U \cap U'}\,.$$

Then

$$g_{|U \cap U'_i} = f_{|U \cap U'_i}\,.$$

($\Leftarrow$): Let $U'_i \subset U'$ and $f : U' \to K$. Let $f_{|U'_i} \in \mathcal{S}_N(U'_i)$ for all i. Since $x \in U'$ then $x \in U'_i$ for some i. Thus there exists $\tilde{g} \in \mathcal{S}_M(\tilde{U})$ with $x \in \tilde{U}$ and $f_{U'_i \cap \tilde{U}} = \tilde{g}_{U'_i \cap \tilde{U}}$. Since U'_i is open in N then there exists V open in M such that $V \cap N = U'_i$. Take $U = V \cap \tilde{U}$ and $g = \tilde{g}_{|U}$. Since $U'_i \cap U \subseteq U'_i \cap \tilde{U}$ then we get

$$f_{|U'_i \cap U} = \tilde{g}_{|U'_i \cap U} = g_{|U'_i \cap U}\,.$$

We will end this section by providing some examples of manifolds:

Example 1.9: The simplest example of a manifold is, of course, the Euclidean space $\boldsymbol{R}^n$ itself. For each $x \in \boldsymbol{R}^n$, we can take U to be all of $\boldsymbol{R}^n$ and h = identity. Then $\boldsymbol{R}$ becomes a real-analytic (hence, differentiable). Similarly, for $\boldsymbol{C}^n$ in which becomes a complex-analytic manifold.

Example 1.10: The second simplest example of a manifold is an open ball in $\boldsymbol{R}^n$. In this case we can take U to be the entire open ball. Since an open ball in $\boldsymbol{R}^n$ is homeomorphic to $\boldsymbol{R}^n$.

Example 1.11 : An open subset of a manifold is again a manifold. Let N be an open subset of a manifold M. If $p \in N \subset M$, and U is an open neighbourhood of p (contains some open set $U \cap N$ with $p \in U \cap N$) which is homeomorphic to $\boldsymbol{R}^n$ by a homeomorphism $h : U \longrightarrow \boldsymbol{R}^n$, then $h(U \cap N) \subset \boldsymbol{R}^n$ is open set containing $h(p)$.
Consequently, there is an open ball W with $h(p) \in W \subset h(U \cap N)$. Thus, $p \in h^{-1}(W) \subset U \cap N \subset U$. Since $h_{|U \cap N} : U \cap N \longrightarrow \boldsymbol{R}^n$ is continuous, then set $h^{-1}(W)$ is open in $U \cap N$, and thus open in N. It is of course homeomorphic to W. So it is connected and give a coordinate neighbourhood of p in N.

Example 1.12: The only connected 1-manifolds are $\boldsymbol{R}$ and the circle or 1-dimensional sphere, S^1, defined by :

$$S^1 = \{x \in \boldsymbol{R}^2 : d(x, 0) = 1\}$$

The function $f : (0, 2\pi) \longrightarrow S^1$ defined by $f(\theta) = (\cos(\theta), \sin(\theta))$ is a homeomorphism; it is even continuous, though not one-one on $[0, 2\pi]$. We denote the point $(\cos(\theta), \sin(\theta)) \in S^1$ simply by $\theta \in [0, 2\pi]$. The function $g : (-\pi, \pi) \longrightarrow S^1$, defined by the same formula, is also a homeomorphism; together with f it shows that S^1 is indeed a manifold.

2. Vector Bundles

In this section, we will carry on to use the notation $\mathcal{S}$ to denote one of three structures on manifolds which are $\mathcal{E}, \mathcal{A}$, or $\mathcal{O}$.

2.1. Basic Definitions :

Definition 2.1.1: A continuous map $\pi : E \longrightarrow X$ of one Hausdorff E, onto another Hausdorff X, is called a K-vector bundle of rank r , if the following conditions are satisfied :

(a) $E_p := \pi^{-1}(p)$, for $p \in X$, is a K-vector space of dimension r (E_p is called the fibre over p)

(b) For every $p \in X$, there is a neighbourhood U of p and a homeomorphism

$$h : \pi^{-1}(U) \longrightarrow U \times K^r$$

such that, for all q in U, $h(E_q) = \{q\} \times K^r$ and h^q, defined by the composition:

$$h^q : E_q \rightarrow \{q\} \times \boldsymbol{K}^r \overset{proj.}{\rightarrow} \boldsymbol{K}^r \,,$$

is a K-vector space isomorphism. The pair (U, h) is called a *local trivialization.*

(c) If (U_α, h_α) and (U_β, h_β) are two local trivializations with $U_\alpha \cap U_\beta \neq \emptyset$; then the map:

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times K^r \longrightarrow (U_\alpha \cap U_\beta) \times K^r$$

induces a map :

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(r, K)$$

where

$$g_{\alpha\beta}(q) = h_\alpha^q \circ (h_\beta^q)^{-1} : K^r \longrightarrow K^r,$$

is a linear isomorphism.

Remark 2.1.2 :

1. For a K-vector bundle $\pi : E \longrightarrow X, E$ is called the *total space* and X is called the *base space*; and we often say that E is a vector bundle over X.

2. The functions $g_{\alpha\beta}$ are called *transition functions* of the K-vector bundle $\pi : E \to X$ (with respect to the two local tivializations above).

The transitoin functions $g_{\alpha\beta}$ satisfy the condition:

$$(1.1) \qquad g_{\alpha\beta} \cdot g_{\gamma\beta}^{-1} \cdot g_{\gamma\alpha} = I_r \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma,$$

where the product is a matrix product and I_r is the identity matrix of rank r.

Definition 2.1.3: A K-vector bundle of rank r , $\pi : E \to X$ is said to be an $\mathcal{S}$-bundle if E and X are $\mathcal{S}$-manifolds, π is an $\mathcal{S}$-morphism, and the local trivializations are $\mathcal{S}$-isomorphisms.

Suppose that on M, we are given an open covering $\tilde{U} = \{U_\alpha\}$, and that to each ordered nonempty intersection $U_\alpha \cap U_\beta$ we have assigned an $\mathcal{S}$-function:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(r, K)$$

satisfying the compatibility condition (1.1). Now, we let

$$\tilde{E} = \textstyle\coprod_\alpha U_\alpha \times K^r \qquad \text{(disjoint union)}$$

equipped with the natural product topology and $\mathcal{S}$-structure.
We define an equivalence realation on $\tilde{E}$ by setting :

$$(x, v) \sim (y, w), \quad \text{for } (x, v) \in U_\beta \times K^n, (y, w) \in U_\alpha \times K^n$$

if and only if

$$x = y \quad \text{and} \quad w = g_{\alpha\beta}(x)v.$$

The fact that this is a well-defined equivalence relation is a consequence of the compatibility condition (1.1). We now let $E = \tilde{E}/\sim$, (the set of equivalence classes), equipped with quotient topology, and we let $\pi : E \to M$ be the mapping which sends a representative (x, v) of a point $p \in E$ to the first coordinate $x \in M$. It is clear from the construction that E carries an $\mathcal{S}$-structure and is an $\mathcal{S}$-bundle.

Example 2.1.4: (Trivial Bundle) : Let M be an $\mathcal{S}$-manifold. Then

$$\pi : M \times K^n \longrightarrow M,$$

where π is the natural projection, is an $\mathcal{S}$-bundle; called the *trivial bundle.*

2.2. The Tangent Bundle :

Definition 2.2.1 : Let M be a manifold, p a point of M. A smooth path through p is a smooth map $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\alpha(0) = p$.

Definition 2.2.2 : Let M^n be a differentiable manifold and p_0 a point in M. Two smooth paths $\alpha, \beta : (-\varepsilon, \varepsilon) \rightarrow M$ such that $\alpha(0) = \beta(0) = p_0$ are said to have a 1st order contact at p_0, if there exist local coordinates $(x) = (x^1, \ldots, x^n)$ near p_0 such that

$$\dot{x}_\alpha(0) = \dot{x}_\beta(0)$$

where $(x \circ \alpha(t)) = (x_\alpha(t)) = (x^1_\alpha(t), \cdots, x^n_\alpha(t))$ and $(x \circ \beta(t)) = (x_\beta(t)) = (x^1_\beta(t), \ldots, x^n_\beta(t))$. We write this as $\alpha \sim_1 \beta$.

Lemma 2.2.3 : $\sim_1$ is an equivalence relation.

Proof : We state a sketch of proof : $\sim_1$ is trivially reflexive and symmetric, so we only have to check the transitivity.
Let $\alpha \sim_1 \beta$ and $\beta \sim_1 \gamma$. Thus, there exist local coordinates $(x) = (x^1, \ldots, x^n)$ and $(y) = (y^1, \ldots, y^n)$ near p_0 such that $(\dot{x}_\alpha(0)) = (\dot{x}_\beta(0))$ and $(\dot{y}_\beta(0)) = (\dot{y}_\gamma(0))$. The transitivity follows from the equality

$$\dot{y}^i_\gamma(0) = \dot{y}^i_\beta(0) = \textstyle\sum_j \frac{\partial y^i}{\partial x^j} \dot{x}^j_\beta(0) = \sum_j \frac{\partial y^i}{\partial x^j} \dot{x}^j_\alpha(0) = \dot{y}^j_\alpha(0).$$

Definition 2.2.4 : Let M be a differentiable manifold. A *tangent vector* to M at p is an equivalence class of curves through p modulo the first order contact relation. The equivalence class of a smooth path $\alpha(t)$ through p will be denoted by $[\alpha(t)]$. The set of these equivalence classes is denoted by T_pM

and is called the *tangent space* to M at p.

Lemma 2.2.5 : T_pM has a natural structure of vector space.

Proof: Choose $(x^1, \ldots, x^n)$ local coordinates near p such that $x^i(p) = 0$, $\forall i$ and let $[\alpha], [\beta]$ be classes of curves through p. We define a vector space structure on T_pM by the formulae

$$\lambda[\alpha] = [x^{-1}(\lambda x_\alpha(t))]\,,$$

$$[\alpha] + [\beta] = [x^{-1}(x_\alpha(t) + x_\beta(t))]\,.$$

We first have to check that this definition is well defined. If $\alpha_1 \sim_1 \alpha_2$, then we have to show that $x^{-1}(\lambda x_{\alpha_1}(t)) \sim_1 x^{-1}(\lambda x_{\alpha_2})$. Since

$$\frac{\partial}{\partial t}(\lambda x_{\alpha_1}(t))_{|t=0} = \frac{\partial}{\partial t}(\lambda x_{\alpha_2}(t))_{|t=0}$$

$$\lambda\frac{\partial}{\partial t}(x_{\alpha_1}(t))_{|t=0} = \lambda\frac{\partial}{\partial t}(x_{\alpha_2}(t))_{|t=0}\,,$$

and since

$$\alpha_1 \sim_1 \alpha_2\,.$$

Then

$$x^{-1}(\lambda x_{\alpha_1}(t)) \sim_1 x^{-1}(\lambda x_{\alpha_2})\,.$$

Now, let $\alpha_1 \sim_1 \alpha_2$ and $\beta_1 \sim_1 \beta_2$. Since

$$\alpha_1 \sim_1 \alpha_2 \text{ and } \beta_1 \sim_1 \beta_2\,.$$

Then

$$\dot{x}_{\alpha_1}(0) = \dot{x}_{\alpha_2}(0) \text{ and } \dot{x}_{\beta_1}(0) = \dot{x}_{\beta_2}(0)\,,$$

which implies

$$\tfrac{\partial}{\partial t}(x_{\alpha_1}(t) + x_{\beta_1}(t))_{|t=0} = \tfrac{\partial}{\partial t}(x_{\alpha_2}(t) + x_{\beta_2}(t))_{|t=0}\,.$$

We now have to check that this definition is independent of local coordinates. Suppose that we have another local coordinates $(y^1, \ldots, y^n)$ near p such that $y^i(p) = 0$, $\forall i$. Since, for fixed α,

$$\lambda\dot{x}_\alpha(0) = \frac{\partial}{\partial t}(\lambda x_\alpha(t))_{|t=0}\,,$$

and since

$$x^{-1}(x_\alpha(t)) \sim_1 y^{-1}(y_\alpha(t)).$$

Then

$$\begin{aligned}\lambda \dot{x}_\alpha(0) &= \frac{\partial}{\partial t}(xy^{-1}(\lambda y_\alpha(t)))_{|t=0} \\ &= [d(xy^{-1})(\lambda \frac{\partial y_\alpha(t)}{\partial t})]_{|t=0} \\ &= \lambda d(xy^{-1})(\frac{\partial y_\alpha(t)}{\partial t})_{|t=0} \\ &= \lambda d(xy^{-1})\dot{y}_\alpha(0) \quad .\end{aligned}$$

Also, for fixed α and β,

$$\dot{x}_\alpha(0) + \dot{x}_\beta(0) = \frac{\partial}{\partial t}(x_\alpha(t) + x_\beta(t))_{|t=0},$$

and since

$$x^{-1}(x_\alpha(t) + x_\beta(t)) \sim_1 y^{-1}(y_\alpha(t) + y_\beta(t)).$$

Then

$$\begin{aligned}\dot{x}_\alpha(0) + \dot{x}_\beta(0) &= \frac{\partial}{\partial t}(xy^{-1}(y_\alpha(t) + y_\beta(t)))_{|t=0} \\ &= [d(xy^{-1})(\frac{\partial y_\alpha(t)}{\partial t} + \frac{\partial y_\beta(t)}{\partial t})]_{|t=0} \\ &= d(xy^{-1})(\frac{\partial y_\alpha(t)}{\partial t})_{|t=0} + d(xy^{-1})(\frac{\partial y_\beta(t)}{\partial t})_{|t=0} \\ &= d(xy^{-1})\dot{y}_\alpha(0) + d(xy^{-1})\dot{y}_\beta(0).\end{aligned}$$

Now, let $(x^1, \ldots, x^n)$ be coordinates near $p \in M$ such that $x^i(p) = 0$, for all i. Consider the curves

$$e_k(t) = (t\delta_k^1, \ldots, t\delta_k^n), \qquad k = 1, \ldots, n$$

where δ_j^i denotes Kronecker's delta symbol (the function of two variables that takes the value 1 when $i = j$ and is zero otherwise.) Equivalently, one can define the e_k's implicitly by $x^i = 0$, for all $i \neq k$. We set

$$\tfrac{\partial}{\partial x^k}(p) = [e_k(t)]$$

Note that these vectors depend on the local coordinates $(x^1, \ldots, x^n)$.

Lemma 2.2.6 : $(\frac{\partial}{\partial x^k}(p))_{1 \leq k \leq n}$ is a basis of T_pM.

Proof : This follows from the obvious fact that any curve through the origin in $\boldsymbol{R}^n$ has first order contact with a line $t \mapsto (a_1 t, \ldots, a_n t)$.

The tangent bundle : we are in a position to construct a vector bundle by taking the disjoint union of all tangent spaces (described above), to be called the *tangent bundle* TM of M. Let

$$TM = \coprod_{p \in M} T_pM,$$

as sets. (We will describe the topology on TM later.)
There is a natural surjection

$$\pi : TM \longrightarrow M, \quad \pi(v) = p \Longleftrightarrow v \in T_pM.$$

Any local coordinate system $x = (x^i)$ defined over an open set $U \subset M$ produces a natural basis $(\frac{\partial}{\partial x^i}(p))$ of T_pM for any $p \in U$. Thus, an element $v \in TU = \coprod_{p \in U} T_pM$ is completely determined if we know to which tangent space it belongs (i.e. $p = \pi(v)$) and we also know its coordinates in the basis $(\frac{\partial}{\partial x^i}(p))$:

$$v = \sum_i X^i(v)(\frac{\partial}{\partial x^i}(p))$$

We thus have a bijection

$$\psi_x : TU \longrightarrow U^x \times \boldsymbol{K}^n \subset \boldsymbol{K}^n \times \boldsymbol{K}^n,$$

where U^x is the image of U in $\boldsymbol{K}^n$ via coordinates (x^i).
Hence, $\psi_x(T_pM) \subset \{\bar{x}\} \times \boldsymbol{K}^n$, where $x(p) = \bar{x}$. (Note : T_pM is called the fibre over p.)

Now, we can transfer the topology on $U^x \times \boldsymbol{K}^n$ to TU via the map ψ_x. Again we have to make sure this topology is independent of local coordinates.

To see this, we pick a different coordinate system $y = (y^i)$ on U . The coordinate independence referred to above is equivalent to the statement that the transition map

$$\psi_y \circ \psi_x^{-1} : U^x \times \boldsymbol{K}^n \longrightarrow TU \longrightarrow U^y \times \boldsymbol{K}^n$$

is a homeomorphism. Let $A = (\bar{x}; X^1, \ldots, X^n) \in U^x \times \boldsymbol{K}^n$. Then $\psi_x^{-1}(A) = (p, [\alpha(t)])$, where $x(p) = \bar{x}$ and $\alpha(t) \subset U$ is a curve through p given in the coordinates x as

$$\alpha(t) = \bar{x} + tX \ ,$$

since $\alpha(0) = x(p) = \bar{x}$.

Denote by $F : U^x \longrightarrow U^y$ the transition map $x \mapsto y$ which is induced by the above map. Then

$$\psi_y \circ {\psi_x}^{-1}(A) = (F(\bar{x}); Y^1, \ldots, Y^n) \ ,$$

where $[\alpha(0)] = (\dot{y}^j_\alpha(0)) = \sum Y^j \frac{\partial}{\partial y^j}(p)$. $[(y_\alpha(t))$ defines the curve $\alpha(t)$ in the coordinates (y^i). $]$

Applying the chain rule we deduce

$$Y^j = \dot{y}^j_\alpha(0) = \sum_i \frac{\partial y^j}{\partial x^i} \dot{x}^i(0) = \sum_i \frac{\partial y^j}{\partial x^i} X^i \ .$$

This proves that $\psi_y \circ \psi_x^{-1}$ is actually an $\mathcal{S}$-isomorphism.

The natural topology of TM is obtained by patching together the topologies of TU_β, where $\{(U_\beta, h_\beta)\}$ are local trivializations of M. A set $W \subset TM$ is open if its intersection with every TU_β is open in TU_β. Thus, the above argument shows that TM is an $\mathcal{S}$-manifold using $\{(TU_\beta, \psi_\beta)\}$ as trivializations. Moreover, the natural projection $\pi : TM \rightarrow M$ is an $\mathcal{S}$-morphism.

Remark 2.2.7 : The fact that the local trivializations are $\mathcal{S}$-isomorphisms is equivalent to the fact that the transition maps are $\mathcal{S}$-morphisms.

Proposition 2.2.8 : A differentiable mapping of differentiable manifolds

$$f : M \longrightarrow N$$

induces a differentiable mapping

$$df : TM \longrightarrow TN$$

such that :
(a) $df(T_pM) \subset T_{f(p)}$, for all $p \in M$.
(b) The restriction to each tangent space

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

is linear.

Proof : Recall that T_pM is the space of tangent vectors to curves through p. If $\alpha(t)$ is such a curve, $\alpha(0) = p$, then $\beta(t) = f(\alpha(t))$ is a smooth curve through $q = f(p)$, and we define

$$df([\alpha(t)]) = [\beta(t)].$$

One checks easily that if $\alpha_1 \sim \alpha_2$ then $f(\alpha_1) \sim f(\alpha_2)$. Thus, df is well defined . To prove the map $df_p : T_pM \to T_qN$ is linear, it suffices to verify it in any local coordinates $(x^1, \ldots, x^m)$ near p and $(y^1, \ldots, y^n)$ near q, since any two choices differ by a linear substitution.

Therefore, we can regard f as a collection of maps

$$(x^1, \ldots, x^m) \longrightarrow (y^1(x^1, \ldots, x^m), \ldots, y^n(x^1, \ldots, x^m))$$

A basis in T_pM is given by $\{\frac{\partial}{\partial x^i}\}$ while a basis of T_qN is given by $\{\frac{\partial}{\partial y^j}\}$. Then df_p is the linear operator given in these bases by the matrix $(\frac{\partial y^j}{\partial x^i})_{1 \leq j \leq n, 1 \leq i \leq m}$. Classically, the mapping df_p is called the *Jacobian* of the differentiable map f. In particular, this map is a differentiable mapping.

Definition 2.2.9 : Let $\pi : E \to X$ be an $\mathcal{S}$-bundle and U an open subset of X . Then the restriction of E to U, denoted by $E|_U$ is the $\mathcal{S}$-bundle

$$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \longrightarrow U\,.$$

Definition 2.2.10 : (a) Let $\pi_E : E \to X$ and $\pi_F : F \to X$ be $\mathcal{S}$-bundles over X. Then a homomorphism of $\mathcal{S}$-bundles

$$f : E \longrightarrow F$$

is an $\mathcal{S}$-morphism of total spaces which preserves fibres and is K-linear on each fibre, i.e., f commutes with the projections

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ & X \end{tikzcd}$$

and is a K-linear mapping when restricted to fibres.

(b) An $\mathcal{S}$-bundle isomorphism is an $\mathcal{S}$-homorphism which is an $\mathcal{S}$-isomorphism on the total spaces and a K-vector space isomorphism on the fibres.

(c) Two $\mathcal{S}$-bundles over X are equivalent if there is some $\mathcal{S}$-bundle isomorphism between them. This clearly defines an equivalence relation on the $\mathcal{S}$-bundles over an $\mathcal{S}$-manifold X.

We have defined in the above definition a bundle homomorphism of two bundles over the same base space. We now would like to define a mapping between bundles over different base spaces.

Definition 2.2.11 : an $\mathcal{S}$-bundle morphism between two $\mathcal{S}$-bundles $\pi_E : E \to X$ and $\pi_F : F \to Y$ is an $\mathcal{S}$-morphism

$$f : E \longrightarrow F$$

which takes fibres of E linearly to fibres in F. An $\mathcal{S}$-bundle morphism $f : E \to F$ induces an $\mathcal{S}$-morphism $\tilde{f}(\pi_E(e)) = \pi_F(f(e))$, in other words, the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "\tilde{f}"'] & Y. \end{tikzcd}$$

Definition 2.2.12 : Let $f : X \to Y$ be an $\mathcal{S}$-morphism and let $\pi : E \to Y$ be an $\mathcal{S}$-bundle defined by an open cover (U_α) of Y and transition maps $(g_{\alpha\beta})$. The *pullback of E by f* is the $\mathcal{S}$-bundle

$$\pi' : E' \longrightarrow X$$

defined by the open cover $(f^{-1}(U_\alpha))$ of X and the transition maps $(g_{\alpha\beta} \circ f)$.

Proposition 2.2.13 : Given an $\mathcal{S}$-morphism $f : X \to Y$ and an $\mathcal{S}$-bundle $\pi : E \to Y$, there exists an $\mathcal{S}$-bundle $\pi' : E' \to X$ and an $\mathcal{S}$-bundle morphism f' such that the following diagram commutes

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Moreover, for every $\tilde{\pi}$ and $\tilde{f}$ such that the outer part of the below diagram commutes (i.e., $\pi \circ \tilde{f} = f \circ \tilde{\pi}$), there exists h so that the full diagram

$$\begin{tikzcd}
\tilde{E} \arrow[dr, dotted, "h"] \arrow[drr, bend left, "\tilde{f}"] \arrow[ddr, "\tilde{\pi}"'] & & \\
& E' \arrow[r, "f'"'] \arrow[d, "\pi'"] & E \arrow[d, "\pi"'] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes, and that E' is unique for this property. We denote E' by f^*E.

Proof : See R.O.Wells [8], pages 25-26.

Definition 2.2.14 : An $\mathcal{S}$-section of an $\mathcal{S}$-bundle $\pi : E \to X$ is an $\mathcal{S}$-morphism

$$s : X \longrightarrow E ,$$

such that $\pi \circ s = 1_x$, where 1_x is the identity on X. So s maps a point in the base space into the fibre over that point.

Remark 2.2.15: $\mathcal{S}(X, E)$ will denote the $\mathcal{S}$-sections of E over X, and $\mathcal{S}(U, E)$ will denote the $\mathcal{S}$-sections of $E|_U$ over $U \subset X$, i.e., $\mathcal{S}(U, E) = \mathcal{S}(U, E|_U)$.

In the following example , we will state an important result.

Example 2.2.16: Consider the trivial bundle $M \times \boldsymbol{R}$ over a differentiable manifold M. Then $\mathcal{E}(M, M \times \boldsymbol{R})$ can be identified in a natural way with $\mathcal{E}(M)$, the global real-valued functions on M . Similarly, $\mathcal{E}(M, M \times \boldsymbol{R}^n)$ can be identified with global differentiable mappings of M into $\boldsymbol{R}^n$ (i.e. vector-valued functions .)
Therefore, since vectors bundles are locally of the form $U \times \boldsymbol{R}^n$, then we can identify the sections of a vector bundle locally with vector-valued functions, where two different local trivilizations are related by the transition functions for the bundle. Thus, sections can be thought of as "twisted" vector-valued functions.

Definition 2.2.17: Let $\pi : E \to X$ be an $\mathcal{S}$-bundle. An $\mathcal{S}$-submanifold $F \subset E$ is said to be an $\mathcal{S}$-subbundle of E if

(a) $F \cap E_x$ is a vector subspace of E_x.

(b) $\pi_{|F} : F \to X$ has the structure of an $\mathcal{S}$-bundle induced by the $\mathcal{S}$-bundle structure of E, i.e., if there exist local trivializations for E and F which are compatible as in the following diagram:

$$\begin{array}{ccc} E_{|U} & \longrightarrow & U \times K^r \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle id \times j} \\ F_{|U} & \longrightarrow & U \times K^s \end{array}$$

where $s \leq r$, and the map j is the natural inclusion of K^s as subspace of K^r and i is the inclusion of F in E.

Suppose that $f : E \to F$ is a vector bundle homomorphism of K-vector bundles over a space X. We define

$$\text{Ker } f = \bigcup_{x \in X} \text{Ker } f_x,$$

$$\text{Im } f = \bigcup_{x \in X} \text{Im } f_x,$$

where $f_x = f_{|E_x}$. Moreover, we say that f has *constant rank* on X if rank f_x (as a K-linear mapping) is constant for $x \in X$.

Proposition 2.2.18: Let $f : E \to F$ be an $\mathcal{S}$-homomorphism of $\mathcal{S}$-bundles over X. If f has constant rank on X, then Ker f and Im f are $\mathcal{S}$-subbundles of E and F, respectively. In particular, f has constant rank if f is injective or surjective.

Example 2.2.19: A *short exact sequence* of vector bundles is a sequence of vector bundles (and vector bundle homomorphisms) of the following form

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0$$

which is exact at E', E, and E''. In particular, f is injective and g is surjective, and Im f = Ker g is a subbundle of E.

Chapter 2: Calculus on Manifolds

This chapter is divided into two sections. We will give in the first section the basic definitions of differential forms and compute some examples. In the second section, we will study the orientation and integration on $\boldsymbol{R}^n$. Moreover, we will state an important result in the theory of integration, namely, *Stokes' theorem.*

1. Derivations of $\Omega^*(\boldsymbol{R}^n)$

Let $x_1, \ldots, x_n$ be linear coordinates on $\boldsymbol{R}^n$. We define Ω^* to be the graded algebra over $\boldsymbol{R}$ generated by $dx_1, \ldots, dx_n$ with the relations

1. $dx_i \wedge dx_i = 0$.
2. $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $i \neq j$.

As a vector space over $\boldsymbol{R}$, Ω^* has basis

$$1, dx_i, \underbrace{dx_i \wedge dx_j}_{i<j}, \underbrace{dx_i \wedge dx_j \wedge dx_k}_{i<j<k}, \ldots, dx_1 \wedge \ldots \wedge dx_n \quad .$$

Definition 1.1: The C^∞ differential forms on $\boldsymbol{R}^n$ are elements of

$$\Omega^*(\boldsymbol{R}^n) = \{C^\infty \text{ functions on } \boldsymbol{R}^n\} \otimes_{\boldsymbol{R}} \Omega^*.$$

$\Omega^*(\boldsymbol{R}^n)$ is naturally graded as $\Omega^*(\boldsymbol{R}^n) = \oplus_{q=0}^n \Omega^q(\boldsymbol{R}^n)$. The elements of $\Omega^q(\boldsymbol{R}^n)$ are called $C^\infty q$-forms on $\boldsymbol{R}^n$, or differential forms of order q.
If w is a q-form, then w can be uniquely written as

$$w = \sum f_{i_1 \ldots i_q} dx_{i_1} \wedge \ldots \wedge dx_{i_q},$$

where the coefficients $f_{i_1 \ldots i_q}$ are C^∞ functions and $i_1 < \ldots < i_q$. We also write $w = \sum f_I dx_I$, where $I = (i_1, \ldots, i_q)$ and $1 \leq i_1 < i_2 < \ldots < i_q \leq n$.

Definition 1.2: There is a differential operator

$$d : \Omega^q(\boldsymbol{R}^n) \longrightarrow \Omega^{q+1}(\boldsymbol{R}^n),$$

defined as follows:

1. if $f \in \Omega^0(\boldsymbol{R}^n)$, then $df = \sum \frac{\partial f}{\partial x_i} dx_i \quad .$
2. if $w = \sum f_I dx_I$, then $dw = \sum df_I \wedge dx_I \quad .$

The operator d is called the *exterior differentiation.*

Example 1.3: If $w = xdy$, then $dw = dx \wedge dy$.

Example 1.4: **(For** $n = 3$**)**

1. Let $f \in C^\infty(\boldsymbol{R}^3)$, or $f \in \Omega^0(\boldsymbol{R}^3)$, then

$$df = \tfrac{\partial f}{\partial x}dx + \tfrac{\partial f}{\partial y}dy + \tfrac{\partial f}{\partial z}dz \quad .$$

   So df looks like the gradient of f.

2. Let $w = f_1 dx + f_2 dy + f_3 dz \quad \in \Omega^1(\boldsymbol{R}^3)$. Then

$$\begin{aligned} dw &= df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz \\ &= (\tfrac{\partial f_3}{\partial y} - \tfrac{\partial f_2}{\partial z})dy \wedge dz - (\tfrac{\partial f_1}{\partial z} - \tfrac{\partial f_3}{\partial x})dx \wedge dz + (\tfrac{\partial f_2}{\partial x} - \tfrac{\partial f_1}{\partial y})dx \wedge dy. \end{aligned}$$

   So dw looks like a curl.

3. Let $w = f_1 dy \wedge dz - f_2 dx \wedge dz + f_3 dx \wedge dy \quad \in \Omega^2(\boldsymbol{R}^3)$. Then

$$dw = (\tfrac{\partial f_1}{\partial x} + \tfrac{\partial f_2}{\partial y} + \tfrac{\partial f_3}{\partial z})dx \wedge dy \wedge dz.$$

   This looks very much like a divergence.

$\Omega^0(\boldsymbol{R}^3)$, and $\Omega^3(\boldsymbol{R}^3)$ are each free of rank one, and $\Omega^1(\boldsymbol{R}^3)$ and $\Omega^2(\boldsymbol{R}^3)$ are each free of rank three over the C^∞ functions.

Definition 1.5: The *wedge product* of two differential forms, written $t \wedge w$, is as follows:

$$\text{if} \quad t = \sum f_I dx_I \quad \text{and} \quad w = \sum g_J dx_J,$$

then

$$t \wedge w = \sum f_I g_J dx_I \wedge dx_J.$$

Note that $t \wedge w = (-1)^{deg(t) \times deg(w)} w \wedge t$, where $deg(t)$ and $deg(w)$ are the orders of t, and w, respectively.

Proposition 1.6: Let w_1, w_2, and w be q-forms. Then

1. $d(\lambda w_1 + w_2) = \lambda dw_1 + dw_2, \quad \text{if } \lambda \in \boldsymbol{R}$.
2. $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^q w_1 \wedge dw_2$
3. $d(dw) = 0$. Briefly, $d^2 = 0$.

Proof:

1. is clear.
2. By linearity it suffices to check this for

$$t = f_I dx_I, \quad w = g_J dx_J.$$

Then

$$\begin{aligned} d(t \wedge w) &= d(f_I g_J) dx_I \wedge dx_J = (df_I) g_J dx_I \wedge dx_J + f_I (dg_J) dx_I \wedge dx_J \\ &= (dt) \wedge w + (-1)^{deg(t)} t \wedge dw. \end{aligned}$$

where $d(f_I g_J) = (df_I) g_J + f_I (dg_J)$ is simply the ordinary product rule.

3. It clearly suffices to only consider q-forms of the form

$$w = f dx_I$$

then

$$dw = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I$$

so

$$d(dw) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I \right)$$

in this sum, the terms

$$\frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I$$

and

$$\frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I$$

cancel in pairs, for $i \neq j$. If $i = j$, then $(dx_i)^2 = 0$.

Definition 1.7: The complex $\Omega^*(\boldsymbol{R}^n)$ together with the differential operator d is called the *de Rham* complex on $\boldsymbol{R}^n$. It has the form

$$\Omega^0(\boldsymbol{R}^n) \xrightarrow{d} \Omega^1(\boldsymbol{R}^n) \xrightarrow{d} \cdots\cdots \xrightarrow{d} \Omega^n(\boldsymbol{R}^n)\,.$$

Moreover, the kernels of d are the *closed forms* and the images of d are the *exact forms.*

Remark 1.8: The exact forms are, by Proposition 1.6(3), closed.

The de Rham complex may be viewed as a good given set of differential equations, whose solutions are the closed forms. For instance, if we consider a closed 1-form $f dx + g dy$ on $\boldsymbol{R}^2$, then

$$\begin{gathered}
d(f dx + g dy) = 0\\
\frac{\partial g}{\partial x} dx \wedge dy + \frac{\partial f}{\partial y} dy \wedge dx = 0\\
\frac{\partial g}{\partial x} dx \wedge dy - \frac{\partial f}{\partial y} dx \wedge dy = 0.
\end{gathered}$$

Thus,

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0.$$

Hence, giving f, g on $\boldsymbol{R}^2$ with $d(f dx + g dy) = 0$ is tantamount to solving the differential equation $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$.

The Functor Ω^*

Definition 1.9: Let $y_1, \ldots, y_m$ and $x_1, \ldots, x_n$ be the standard coordinates on $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$, respectively. A differentiable map

$$g : \boldsymbol{R}^m \longrightarrow \boldsymbol{R}^n$$

induces a pullback map on C^∞ functions

$$g^* : \Omega^0(\boldsymbol{R}^n) \longrightarrow \Omega^0(\boldsymbol{R}^m)$$

$$g^*(f) = f \circ g.$$

We would like to extend this pullback map to a pullback map of C^∞-algebras

$$g^* : \Omega^*(\boldsymbol{R}^n) \longrightarrow \Omega^*(\boldsymbol{R}^m)$$

in such a way that it commutes with d. These conditions determine g^* uniquely:

$$g^*(\sum f_I dx_{i_1} \wedge \ldots \wedge dx_{i_q}) = \sum (f_I \circ g) dg_{i_1} \wedge \ldots \wedge dg_{i_q},$$

where $g_i = x_i \circ g$ is the i-th component of the function g.

Proposition 1.10: The pullback map g^* on forms commutes with d.

Proof: This is obviously an application of the chain rule. Since

$$\begin{aligned} dg^*(f_I dx_{i_1} \wedge \ldots \wedge dx_{i_q}) &= d((f_I \circ g) dg_{i_1} \wedge \ldots \wedge dg_{i_q}) \\ &= d(f_I \circ g) \wedge dg_{i_1} \wedge \ldots \wedge dg_{i_q} \end{aligned}$$

and

$$\begin{aligned} g^* d(f_I dx_{i_1} \wedge \ldots \wedge dxi_q) &= g^*(\textstyle\sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_q}) \\ &= \textstyle\sum_{i=1}^n ((\frac{\partial f_I}{\partial x_i} \circ g) dg_i) \wedge dg_{i_1} \wedge \ldots \wedge dg_{i_q} \\ &= d(f_I \circ g) \wedge dg_{i_1} \wedge \ldots \wedge dg_{i_q} \end{aligned}$$

the proposition is proved.

By applying this proposition with g^* corresponding to a change of coordinate system

$$(x_1, \ldots, x_n) = g(y_1, \ldots, y_n).$$

We see that we can calculate the exterior derivative in any coordinate system.

We will now consider some modifications in order to extend the functor Ω^* to differentiable manifolds. Recall that a differentiable structure on a manifold is given by a coordinate system (U_α, h_α), and the transition functions $g_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}$ are diffeomorphisms of open subsets of $\boldsymbol{R}^n$. Hence, if we let $y_1, \ldots, y_n$ be the standard coordinates on $\boldsymbol{R}^n$, then we can write $h_\alpha = (x_1, \ldots, x_n)$, where the $x_i = y_i \circ h_\alpha$ form a coordinate system on U_α. A function f on U_α is differentiable if and only if $f \circ h_\alpha^{-1}$ is differentiable on $\boldsymbol{R}^n$.

If f is a differentiable function on U_α, the partial derivative $\frac{\partial f}{\partial x_i}$ is defined to be the i-th partial derivative of the pullback function $f \circ h_\alpha^{-1}$ on $\boldsymbol{R}^n$:

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ h_\alpha^{-1})}{\partial y_i}(h_\alpha(p)),$$

where $p \in U_\alpha$. We also recall that the tangent space to a manifold M at p, written T_pM, is the vector space over $\boldsymbol{R}$ spanned by the operators $\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)$, and a smooth vector field on U_α is a linear combination $X_\alpha = \sum f_i \frac{\partial}{\partial x_i}$ where the f_i's are differentiable on U_α. Consider other coordinates $(y_1, \ldots, y_n)$. Then $X_\alpha = \sum g_j \frac{\partial}{\partial y_j}$ where $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_j}$ satisfy the chain rule:

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

A C^∞ vector field on M may be viewed as a collection of vector fields X_α on U_α which agree on the overlaps $U_\alpha \cap U_\beta$. Therefore, we can make the following definition.

A **differential form** w on M is a collection of forms w_U for U in the coordinate systems defining M, which are compatible in the following sense: If i and j are the inclusions

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[dr, "j"'] & U \\ & V \end{tikzcd}$$

then

$$i^* w_U = j^* w_V \text{ in } \Omega^*(U \cap V).$$

By Proposition 1.10, the exterior derivative and the wedge product extend to differential forms on a manifold.

By Proposition 1.10, a differentiable map of differentiable manifolds

$$f : M \longrightarrow N$$

induces in a natural way a pullback map on forms

$$f^* : \Omega^*(N) \longrightarrow \Omega^*(M).$$

2. Orientation and Integration

Let $x_1, \ldots, x_n$ be the standard coordinates on $\boldsymbol{R}^n$. Recall that the Riemann integral of a differentiable function f with compact support is

$$\int_{\boldsymbol{R}^n} f|dx_1 \wedge \ldots \wedge dx_n| = \lim_{\Delta x_i \to 0} \sum f \Delta x_1 \ldots \Delta x_n.$$

Definition 2.1: The integral of an n-form on $\boldsymbol{R}^n$ with compact support $w = f dx_1 \wedge \ldots \wedge dx_n$ is defined to be the Riemann integral $\int_{\boldsymbol{R}^n} f|dx_1 \wedge \ldots \wedge dx_n|$.

Remark 2.2: The order of $x_1, \ldots, x_n$ matters in a differential form, but it does not matter in a Riemann integral. If π is a permutation of $\{1, \ldots, n\}$, then

$$\int_{\boldsymbol{R}^n} f dx_{\pi(1)} \wedge \ldots \wedge dx_{\pi(n)} = (sgn \quad \pi) \int_{\boldsymbol{R}^n} f|dx_1 \wedge \ldots \wedge dx_n|,$$

but

$$\int_{\boldsymbol{R}^n} f|dx_{\pi(1)} \wedge \ldots \wedge dx_{\pi(n)}| = \int_{\boldsymbol{R}^n} f|dx_1 \wedge \ldots \wedge dx_n|.$$

However, where is no possibility of confusing we shall use the usual calculus notation $\int_{\boldsymbol{R}^n} f dx_1 \wedge \ldots \wedge dx_n$.

We now want to emphasize that the Definition 2.1 depends on the coordinates $x_1, \ldots, x_n$. So, we shall use the concept of a transformation. Thus, we see that there is a change of coordinates given by a diffeomorphism

$$g : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$$

with coordinates $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ respectively:

$$x_i = x_i \circ g(y_1, \ldots, y_n) = g_i(y_1, \ldots, y_n).$$

where $dg_1 \wedge \ldots \wedge dg_n = J(g) dy_1 \wedge \ldots \wedge dy_n$, and $J(g) = \det(\frac{\partial x_i}{\partial y_j})$ is the *Jacobian determinant* of g.

We now study how the integral $\int w$ transforms under such diffeomorphisms where $w = f dx_1 \wedge \ldots \wedge dx_n$. Hence,

$$\begin{aligned}\int_{\boldsymbol{R}^n} g^* w &= \int_{\boldsymbol{R}^n} (f \circ g) dg_1 \wedge \ldots \wedge dg_n \\ &= \int_{\boldsymbol{R}^n} (f \circ g) J(g) |dy_1 \wedge \ldots \wedge dy_n|\end{aligned}$$

relative to the coordinates $y_1, \ldots, y_n$. On the other hand, by the change of variables formula

$$\begin{aligned}\int_{\boldsymbol{R}^n} w &= \int_{\boldsymbol{R}^n} f(x_1, \ldots, x_n) |dx_1 \wedge \ldots \wedge dx_n| \\ &= \int_{\boldsymbol{R}^n} (f \circ g) |J(g)| |dy_1 \wedge \ldots \wedge dy_n|.\end{aligned}$$

Thus

$$\int_{\boldsymbol{R}^n} g^* w = \pm \int_{\boldsymbol{R}^n} w \qquad (*)$$

depending on whether the Jacobian determinant is positive or negative.

In general, if g is a diffeomorphism of open subsets of $\boldsymbol{R}^n$ and if the Jacobian determinant $J(g)$ is everywhere positive, then g is said to be *orientation preserving.* Note that the integral on $\boldsymbol{R}^n$ is not invariant under the whole group of diffeomorphisms of $\boldsymbol{R}^n$. It is only invariant under the subgroup of orientation preserving diffeomorphisms of $\boldsymbol{R}^n$.

Definition 2.3: A differentiable manifold M with its coordinate systems $\{(U_\alpha, h_\alpha) : \alpha \in A\}$ is said to be orientable if all the transition functions $g_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}$ are orientation preserving diffeomorphisms.

Now, we will introduce the concept of *partitions of unity* which will be used later on.

Definition 2.4: Let M be a differentiable manifold and $(U_\alpha)_{\alpha\in A}$ an open cover of M. A partition of unity subordinated to this cover is a family $(p_\gamma)_{\gamma\in G} \subset C^\infty(M)$ satisfying the following conditions:

1. $0 \le p_\gamma \le 1$, for all γ .
2. $\exists\phi : G \to A$ such that $\mathrm{supp}(p_\gamma) \subset U_{\phi(\gamma)}$.
3. The family $(\mathrm{supp}(p_\gamma))$ is locally finite, i.e., any point $x \in M$ admits an open neighbourhood intersecting only finitely many supports $\mathrm{supp}(p_\gamma)$.
4. $\sum_\gamma p_\gamma(x) = 1$, for all $x \in M$.

Theorem 2.5: **(Existence of Partitions of Unity)** Let M be a differentiable manifold and $\{U_\alpha : \alpha \in A\}$ an open cover of M. Then there exists a countable partition of unity $\{p_i : i = 1, 2, 3, \ldots\}$ subordinate to the cover $\{U_\alpha\}$ with $\mathrm{supp}(p_i)$ compact for each i. If one does not require compact supports, then there is a partition of unity $\{p_\alpha\}$ subordinate to the cover $\{U_\alpha\}$, i.e. $\mathrm{supp}(p_\alpha) \subset U_\alpha$, with at most countably many of the p_α not identically zero.

Proof: See Frank W. Warner [7], page 10.

Proposition 2.6: A manifold M of dimension n is orientable *if and only if* it has a global nowhere vanishing n-form.

Proof: Observe that a diffeomorphism $g : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is orientation preserving if and only if there exists a positive function λ such that $g^*(dx_1 \wedge \ldots \wedge dx_n) = \lambda dx_1 \wedge \ldots \wedge dx_n$ at every point.
($\Rightarrow$) Suppose that M is orientable with coordinate systems $\{(U_\alpha, h_\alpha) : \alpha \in A\}$. Then

$$(h_\beta h_\alpha^{-1})^*(dx_1 \wedge \ldots \wedge dx_n) = \lambda dx_1 \wedge \ldots \wedge dx_n$$

for some positive function λ. Thus

$$h_\beta^* dx_1 \wedge \ldots \wedge dx_n = (h_\alpha^* \lambda)(h_\alpha^* dx_1 \wedge \ldots \wedge dx_n). \qquad (**)$$

Consider $(p_\gamma)_{\gamma\in G} \subset C_0^\infty$, a partition of unity subordinated to the open cover $(U_\alpha)_{\alpha\in A}$, i.e., there exists a map $\phi : G \to A$ such that

$$\text{supp}(p_\gamma) \subset U_{\phi(\gamma)} \text{ for all } \gamma \in G.$$

We now let

$$w = \sum_\gamma p_\gamma w_{\phi(\gamma)}$$

where for all $\alpha \in A$ we define $w_\alpha = h_\alpha^* dx_1 \wedge \ldots \wedge dx_n$. Thus, we see that on $U_\alpha \cap U_\beta$,the $w_\beta = f(w_\alpha)$ where $f = h_\alpha^*(\lambda) = \lambda \circ h_\alpha$ is a positive function on $U_\alpha \cap U_\beta$.

At each point x in M, the all forms w_α, if $x \in U_\alpha$, are positive multiples of one another. Since $P_\gamma \geq 0$ and not all p_γ can vanish at a x, the form w is nowhere vanishing.

($\Leftarrow$) Suppose that M has a global nowhere vanishing n-form w.

Let $h_\alpha : U_\alpha \to \boldsymbol{R}^n$ be a coordinate map. Then, on U_α,

$$h_\alpha^* dx_1 \wedge \ldots \wedge dx_n = f_\alpha w$$

where f_α is a nowhere vanishing real-valued function on U_α. Thus, f_α is either everywhere positive or everywhere negative. We replace h_α by $\psi_\alpha = g \circ h_\alpha$ only when f_α is negative, where $g : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is the orientation reversing diffeomorphism

$$g(x_1, \ldots, x_n) = (-x_1, \ldots, x_n).$$

Since $\psi_\alpha^* dx_1 \wedge \ldots \wedge dx_n = h_\alpha^* g^* dx_1 \wedge \ldots \wedge dx_n = -h_\alpha^* dx_1 \wedge \ldots \wedge dx_n = (-f_\alpha)w$, we can assume f_α to be positive for all α. Hence, any transition function

$$h_\beta h_\alpha^{-1} = h_\alpha(U_\alpha \cap U_\beta) \longrightarrow h_\beta(U_\alpha \cap U_\beta)$$

will pull back $dx_1 \wedge \ldots \wedge dx_n$ to a positive multiple of itself as in (**).

Therefore, M is orientable with $\{(U_\alpha, h_\alpha) : \alpha \in A\}$.

Proposition 2.7: A differentiable manifold M is orientable *if and only if* there exists an open cover $(U_{\alpha_i})_{\alpha_i \in A} \{i = 1, 2, 3, \ldots\}$ and local coordinate systems $h_{\alpha_i} : U_{\alpha_i} \to \boldsymbol{R}^n$ such that

$$J(g_{\alpha_1\alpha_2}) > 0 \quad \text{on} \quad h_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2}). \tag{2.1}$$

for all the transition functions $g_{\alpha_1\alpha_2} = h_{\alpha_2} \circ h_{\alpha_1}^{-1}$.

Proof:
($\Leftarrow$) Assume that there exists an open cover with $J(g_{\alpha_1\alpha_2}) > 0$ on $h_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2})$.
Consider $(p_\gamma)_{\gamma\in G} \subset C_0^\infty(M)$ a partition of unity subordinated to the cover $(U_{\alpha_i})_{\alpha_i\in A}$. Thus

$$\mathrm{supp}(p_\gamma) \subset U_{\alpha_i} \qquad \text{for all } \gamma \in G$$

Consider

$$w = \sum_\gamma p_\gamma w_{\alpha_i}$$

where for all $\alpha_i \in A$ we define $w_{\alpha_i} = dx^1_{\alpha_i} \wedge \ldots \wedge dx^n_{\alpha_i}$.

Therefore, the form w is nowhere vanishing since condition (2.1) implies that on $U_{\alpha_1} \cap U_{\alpha_2} \cap \ldots \cap U_{\alpha_m}$ the forms $w_{\alpha_1}, w_{\alpha_2}, \ldots, w_{\alpha_m}$ differ by a positive multiplicative factor, e.g., $w_{\alpha_1} = \det(\frac{\partial x^i_{\alpha_1}}{\partial x^j_{\alpha_2}})w_{\alpha_2}$.

($\Rightarrow$) Let w be a global nowhere vanishing n-form on M and consider a local coordinate system $h_{\alpha_i} : U_{\alpha_i} \to \boldsymbol{R}^n$. Then

$$w|_{U_{\alpha_i}} = f_{\alpha_i} dx^1_{\alpha_i} \wedge \ldots \wedge dx^n_{\alpha_i},$$

where the differentiable functions are nowhere vanishing and on $U_{\alpha_1} \cap \ldots \cap U_{\alpha_m}$ they satisfy the gluing condition

$$\Delta_{\alpha_i\alpha_j} = \det(\frac{\partial x^r_{\alpha_i}}{\partial x^k_{\alpha_j}}) = \frac{f_{\alpha_j}}{f_{\alpha_i}}.$$

A permutation ϕ of the variables $x^1_{\alpha_i}, \ldots, x^n_{\alpha_i}$ will change $dx^1_{\alpha_i} \wedge \ldots \wedge dx^n_{\alpha_i}$ by a factor $\varepsilon(\phi)$ so we can always arrange these variables in such an order. Hence $f_{\alpha_i} > 0$, and this will insure the positivity condition

$$\Delta_{\alpha_i\alpha_j} > 0\,.$$

Remark 2.8:

1. Any two global nowhere vanishing n-forms w and w' on an orientable manifold M of dimension n are related by a nowhere vanishing function:

$$w = fw'.$$

2. If M is, in addition, connected, then f is either everywhere positive or everywhere negative.

3. We say that w and w' are *equivalent* if f is positive. Thus a connected orientable manifold M has two equivalence classes of nowhere vanishing n-forms. Either class is called an orientation on M, written $[M]$.

Example 2.9: The *canonical orientation* on $\boldsymbol{R}^n$ is the orientation given by $dx_1 \wedge \ldots \wedge dx_n$ where $x_1, \ldots, x_n$ are the canonical cartesian coordinates.

Definition 2.10: Let $[M]$ be a fixed orientation on M. Given a n-form $t \in \Omega^n(M)$, we define its integral as follows

$$\int_M t = \sum_\alpha \int_{U_\alpha} p_\alpha t$$

where $\{p_\alpha\}$ a partition of unity subordinated to an open cover $\{U_\alpha\}$ of M. $\int_{U_\alpha} p_\alpha t$ means $\int_{\boldsymbol{R}^n} (h_\alpha^{-1})^*(p_\alpha t)$ for some orientation preserving trivialization $h_\alpha : U_\alpha \to \boldsymbol{R}^n$, and $p_\alpha t$ has compact support.

Proposition 2.11: Let M be orientable manifold. Then the above integral $\int_M t$ is independent of the coordinate systems $\{(U_\alpha, h_\alpha) : \alpha \in A\}$ and the partition of unity $\{p_\alpha\}$.

Proof: Let $\{(V_\beta, h_\beta) : \beta \in B\}$ be another coordinate system and $\{q_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Since $\sum_\beta q_\beta = 1$, then

$$\sum_\alpha \int_{U_\alpha} p_\alpha t = \sum_{\alpha,\beta} \int_{U_\alpha} p_\alpha q_\beta t.$$

Now $p_\alpha q_\beta t$ has support in $U_\alpha \cap U_\beta$, so

$$\int_{U_\alpha} p_\alpha q_\beta t = \int_{V_\beta} p_\alpha q_\beta t. \quad by(*)$$

Therefore

$$\sum_\alpha \int_{U_\alpha} p_\alpha t = \sum_{\alpha,\beta} \int_{V_\beta} p_\alpha q_\beta t = \sum_\beta \int_{V_\beta} q_\beta t\,.$$

Stokes' Formula

Stokes' formula is the higher dimensional version of the fundamental theorem of calculus (Leibniz-Newton Formula)

$$\int_a^b df = f(b) - f(a)$$

where $f : [a,b] \to \boldsymbol{R}$ is a differentiable function and $df = f'(x)dx$.

In fact, as we shall see, the higher dimensional formula follows from the simpler 1-dimensional situation. We will first introduce the concept of *manifold with boundary* in order to achieve the correct formulation of the general version.

Definition 2.12: An orientable manifold M of dimension n with boundary is given by the coordinate systems $\{(U_\alpha, h_\alpha) : \alpha \in A\}$ where U_α is homeomorphic under h_α to either $\boldsymbol{R}^n$ or the upper half space $H_1^n = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n; x_1 \geq 0\}$.

Example 2.13: A closed interval $I = [a,b]$ is a differentiable 1-dimensional manifold with boundary $\partial I = \{a, b\}$.

Example 2.14: The closed unit ball $B^3 \subset \boldsymbol{R}^3$ is an orientable manifold with boundary $\partial B^3 = S^2$.

In the definition of manifold with boundary the half planes H^n play a role analogous to that of $\boldsymbol{R}^n$ for ordinary manifolds. Let $H_1^n = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n; x_1 \geq 0\}$ with the relative topology of $\boldsymbol{R}^n$, and denote by ∂H_1^n the subspace defined by $\partial H_1^n = \{(x_1, \ldots, x_n) \in H_1^n; x_1 = 0\}$. Then ∂H_1^n is the same space whether considered as a subspace of $\boldsymbol{R}^n$ or H_1^n. It is called the *boundary* of H_1^n. Of course all of these spaces carry the metric topology derived from the metric of $\boldsymbol{R}^n$, and ∂H_1^n is homeomorphic to $\boldsymbol{R}^{n-1}$ by the map:

$$(x_2, \ldots, x_n) \to (0, x_2, \ldots, x_n).$$

We will now introduce the notion of a diffeomorphism of H_1^n. Let U, V be open subsets of H_1^n. We say that U and V are diffeomorphic if there exists a one-to-one map $g : U \to V$ (onto) such that g and g^{-1} are both C^∞ maps. Moreover, if $U, V \subset \boldsymbol{R}^n - \partial H_1^n$, then U and V are actually open in $\boldsymbol{R}^n$ so that this definition coincides with our previous one. On the other hand, if $U \cap \partial H_1^n \neq \emptyset$, then we claim that $V \cap \partial H_1^n \neq \emptyset$ and that $g(U \cap \partial H_1^n) \subset V \cap \partial H_1^n$. Similarly, $g^{-1}(V \cap \partial H_1^n) \subset U \cap \partial H_1^n$; in other words, diffeomorphisms on open sets of H_1^n take boundary points to boundary points and interior points to interior points. We also notice that $U \cap \partial H_1^n$ and $V \cap \partial H_1^n$ are open subsets of ∂H_1^n, a submanifold of $\boldsymbol{R}^n$ diffeomorphic to $\boldsymbol{R}^{n-1}$; and g, g^{-1} restricted to these open sets in ∂H_1^n are diffeomorphisms.

We will prove in the following lemma that the boundary ∂M of M is an $(n-1)$-dimensional manifold. Moreover, the coordinate systems of M induces in a natural way an orientation $[\partial M]$ on ∂M with its coordinate systems.

Lemma 2.15: Let $g : H_1^n \to H_1^n$ be a diffeomorphism of the upper half space with everywhere positive Jacobian determinant $(J(g) > 0)$. Then g induces a map $\bar{g}$ of the boundary of H_1^n to itself. The induced map $\bar{g}$ is a diffeomorphism of $\boldsymbol{R}^{n-1}$ with positive Jacobian determinant everywhere $(J(\bar{g}) > 0)$.

Proof: By the *inverse function theorem*, an interior point of H_1^n must be the image of an interior point. Hence g maps the boundary to the boundary. We will now check that $\bar{g}$ has positive Jacobian determinant. We will verify this for the case $n = 2$:

Let g be given by

$$x_1 = g_1(y_1, y_2)$$
$$x_2 = g_2(y_1, y_2)\,.$$

Then $\bar{g}$ is given by

$$0 = g_1(0, y_2)$$
$$x_2 = g_2(0, y_2)\,.$$

By assumption

$$\begin{vmatrix} \frac{\partial g_1}{\partial y_1}(0, y_2) & \frac{\partial g_1}{\partial y_2}(0, y_2) \\ \frac{\partial g_2}{\partial y_1}(0, y_2) & \frac{\partial g_2}{\partial y_2}(0, y_2) \end{vmatrix} > 0\,.$$

Since $0 = g_1(0, y_2)$ for all y_2, $\frac{\partial g_1}{\partial y_2}(0, y_2) = 0$, and because g maps the upper half to itself,

$$\frac{\partial g_1}{\partial y_1}(0, y_2) > 0\,.$$

Therefore

$$J(\bar{g}) = \frac{\partial g_2}{\partial y_2}(0, y_2) > 0.$$

The general case is similar.

Let the upper half space $H_1^n = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n; x_1 \geq 0\}$ be given the canonical orientation $dx_1 \wedge \ldots \wedge dx_n$. Then the induced orientation on its boundary $\partial H_1^n = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n; x_1 = 0\}$ is given by the $(n-1)$-form $-dx_2 \wedge \ldots \wedge dx_n$. Note that the minus sign is needed to make Stokes' theorem sign-free as we will see later.

In general for an orientable manifold M with boundary, we define the induced orientation $[\partial M]$ on ∂M by the following:
If h is an orientation preserving diffeomorphism of some open set U in M into the upper half space H_1^n, then

$$h^*[\partial H_1^n] = [\partial M]|_{\partial U},$$

where $\partial U = (\partial M) \cap U$.

We are now in a position to state a basic result in the theory of integration:

Theorem 2.16: **(Stokes' theorem)** Let M be an orientable manifold of dimension n with boundary ∂M. If w is an $(n-1)$-form with compact support on M, then

$$\int_M dw = \int_{\partial M} w.$$

Proof: Using a partition of unity the verification is reduced to the following two situations.

- **Case 1.** w is a compactly supported $(n-1)$-form in $\boldsymbol{R}^n$. We have to show

$$\int_{\boldsymbol{R}^n} dw = 0.$$

It suffices to consider the special case

$$w = f(x) dx_2 \wedge \ldots \wedge dx_n,$$

where $f(x)$ is a compactly supported differentiable function. The general case is a linear combination of these special situations. We compute

$$\int_{R^n} dw = \int_{R^n} \frac{\partial f}{\partial x_1} dx_1 \wedge \ldots \wedge dx_n \,.$$

By Fubini's theorem,

$$= \int_{R^{n-1}} \left(\int_R \frac{\partial f}{\partial x_1} dx_1 \right) dx_2 \wedge \ldots \wedge dx_n = 0 \,.$$

But

$$\int_R \frac{\partial f}{\partial x_1} dx_1 = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_1} dx_1 = f(\infty, x_2, \ldots, x_n) - f(-\infty, x_2, \ldots, x_n) = 0$$

because f has compact support.

Case 2. w is a compactly supported $(n-1)$-form on H_1^n. Let

$$w = \sum_i f_i(x) dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n.$$

Then

$$dw = \left(\sum_i (-1)^{i+1} \frac{\partial f}{\partial x_i}\right) dx_1 \wedge \ldots \wedge dx_n.$$

One can see as above that

$$\int_{H_1^n} \frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n = 0 \qquad \forall i \neq 1$$

For $i = 1$ we have

$$\int_{H_1^n} \frac{\partial f}{\partial x_1} dx_1 \wedge \ldots \wedge dx_n = \int_{R^{n-1}} \left(\int_0^{\infty} \frac{\partial f}{\partial x_1} dx_1 \right) dx_2 \wedge \ldots \wedge dx_n$$

$$= \int_{R^{n-1}} (f(\infty, x_2, \ldots, x_n) - f(0, x_2, \ldots, x_n)) dx_2 \wedge \ldots \wedge dx_n$$

$$= -\int_{\boldsymbol{R}^{n-1}} f(0, x_2, \ldots, x_n) dx_2 \wedge \ldots \wedge dx_n$$

$$= \int_{\partial H_1^n} w.$$

The last equality follows from the fact that the induced orientation on ∂H_1^n is given by $-dx_2 \wedge \ldots \wedge dx_n$. This concludes the proof of Stokes' theorem.

Remark 2.17: Stokes' theorem shows that the integral $\int_M dw$ is independent of the behaviour of w inside M. It only depends on the behaviour of w on the boundary ∂M.

Example 2.18: Let $w = xdy \wedge dz$ and $dw = dx \wedge dy \wedge dz$, then

$$\int_{S^2} xdy \wedge dz = \int_{B^3} dx \wedge dy \wedge dz = \frac{4\pi}{3},$$

the volume of the unit ball $B^3 \subset \boldsymbol{R}^3$.

Chapter 3: Differential Geometry

In this chapter we want to develop some of the basic differential geometric concepts in the context of differentiable $\boldsymbol{C}$-vector bundles whereas we described vector bundles in chapter 1. We will denote by the term vector bundle a differentiable $\boldsymbol{C}$-vector bundle E over a differentiable manifold X. However, an analogous treatment holds for differentiable $\boldsymbol{R}$-vector bundles. Moreover, we will give the basic definitions of the classical concepts of Riemannian such as metrics, connections, and curvatures. Some of these properties are important in the next chapter.

1. Differential Forms

In this section we will develop the concept of differential forms in terms of tangent bundle and cotangent bundle.

To start this section we suppose that $E \to X$ is a vector bundle of rank r and that $f = (e_1, \ldots, e_r)$ is a frame at $x \in X$, i.e., there is a neighbourhood U of x and sections $\{e_1, \ldots, e_r\}, e_i \in \mathcal{E}(U, E)$, which are linearly independent at each point of U.

Suppose that $f = f_U$ is a given frame on the domain of definition U and that $g : U \to GL(r, \boldsymbol{C})$ is a differentiable mapping. Then there is an action of g on the set of all frames on the open set U defined by

$$f \mapsto fg\,,$$

where

$$(fg)(x) = (\sum_{p=1}^{r} g_{p1}(x)e_p(x), \ldots, \sum_{p=1}^{r} g_{pr}(x)e_p(x)) \qquad x \in U,$$

is a new frame, i.e., $(fg)(x) = f(x)g(x)$, and we have the usual matrix product at every point. It is clear that fg is defined on U. The mapping g is called the *mapping of a change of frame.* Moreover, for any two frames f and f' over U, we see that there exists a change of frame g defined over U such that $f' = fg$.

Definition 1.1: Let $E \to X$ be a vector bundle, and suppose that $\xi \in \mathcal{E}(U, E)$ for some U open in X. Let $f = (e_1, \ldots, e_r)$ be a frame over U for E. Then

$$\xi = \sum_{p=1}^{r} \xi^p(f) e_p \qquad (1.1)$$

where the $\xi^p(f) \in \mathcal{E}(U)$ are uniquely determined smooth functions on U.

The last equality induces a mapping

$$\ell_f : \mathcal{E}(U, E) \to \mathcal{E}(U)^r \equiv \mathcal{E}(U, U \times \boldsymbol{C}^r)$$

such that

$$\xi \mapsto \xi(f) = \begin{bmatrix} \xi^1(f) \\ \cdot \\ \cdot \\ \cdot \\ \xi^r(f) \end{bmatrix}$$

where the $\xi^p(f)$ are defined by (1.1).

Suppose that g is a change of frame over U. Then

$$\xi^p(fg) = \sum_{\sigma=1}^{r} g_{p\sigma}^{-1} \xi^\sigma(f),$$

which implies that

$$\xi(fg) = g^{-1}\xi(f)$$

or

$$g\xi(fg) = \xi(f) \qquad (1.2).$$

Remark 1.2:

1. The equality (1.1) gives a vector representation for sections $\xi \in \mathcal{E}(U, E)$.
2. The equality (1.2) shows the transformation of a section under a change of frame for the vector bundle E.

If E is a holomorphic vector bundle, then we also have holomorphic frames, i.e., $f = (e_1, \ldots, e_r), e_j \in \mathcal{O}(U, E)$ and $e_1 \wedge \ldots \wedge e_r(x) \neq 0$ for all $x \in U$. Moreover, all changes of frame are holomorphic, given by a holomorphic mapping

$$g : U \longrightarrow GL(r, \boldsymbol{C}).$$

Then with respect to a holomorphic frame we have the vector representation

$$\ell_f : \mathcal{O}(U, E) \longrightarrow \mathcal{O}(U)^r$$

given by $\xi \mapsto \xi(t)$ as before and the transformation rule for a holomorphic change of frame is still given by (1.2).

Now, let $\pi : E \to X$ be a vector bundle (as described in Definition 2.1.1, Chapter 1). We can derive from this vector bundle a new vector bundle which is the *dual bundle* $E^* \to X$ to be the vector bundle with fibre E_x^*; trivializations

$$h_U : E_U \longrightarrow U \times \boldsymbol{C}^r$$

(where $E_U = \pi^{-1}(U)$) then induce maps

$$h_U^* : E_U^* \longrightarrow U \times \boldsymbol{C}^{r^*} \cong U \times \boldsymbol{C}^r \,,$$

which give $E^* = \cup E_x^*$ the structure of a manifold. The construction is most easily expressed in terms of transition functions: if $E \to X$ has transition functions $\{g_{\alpha\beta}\}$, then $E^* \to X$ is just the vector bundle given by transition functions

$$j_{\alpha\beta}(x) = {}^t g_{\alpha\beta}(x)^{-1} \,.$$

Similarly, if $E \to X$, $F \to X$ are vector bundles of rank r and m with transition functions $\{g_{\alpha\beta}\}$ and $\{f_{\alpha\beta}\}$, respectively, then we can define bundles:

1. and $E \oplus F$, given by transition functions

$$j_{\alpha\beta}(x) = \begin{bmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & f_{\alpha\beta}(x) \end{bmatrix} \in GL(\boldsymbol{C}^r \oplus \boldsymbol{C}^m) \,.$$

2. $E \otimes F$, given by transition functions

$$j_{\alpha\beta}(x) = g_{\alpha\beta}(x) \otimes f_{\alpha\beta}(x) \in GL(\boldsymbol{C}^r \otimes \boldsymbol{C}^m) \,.$$

3. $\wedge^p E$, given by transition functions

$$j_{\alpha\beta}(x) = \wedge^p g_{\alpha\beta}(x) \in GL(\wedge^p \boldsymbol{C}^r)\,.$$

In particular, $\wedge^r E$ is a line bundle given by

$$j_{\alpha\beta}(x) = \det g_{\alpha\beta}(x) \in GL(1, \boldsymbol{C}) = \boldsymbol{C}^*\,,$$

called the *determinant* bundle of E.

Definition 1.3: The cotangent bundle is defined to be the *dual bundle* $T^*(X)$ of the tangent bundle $T(X)$.

Definition 1.4: We define

$$\wedge T(X) = \oplus_{p=0}^{n} \wedge^p T(X)$$

$$\wedge T^*(X) = \oplus_{p=0}^{n} \wedge^p T^*(X)$$

where n is the dimension of the manifold X.

Definition 1.5: Suppose that $E \to X$ is a vector bundle. Then we let

$$\mathcal{E}^p(X, E) = \mathcal{E}(X, \wedge^p T^*(X) \otimes_{\boldsymbol{C}} E)$$

be the differential forms of degree p on X with coefficients in E.

Lemma 1.6: Let E and E' be vector bundles over X. Then there is an isomorphism

$$\tau : \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E}(E') \longrightarrow \mathcal{E}(E \otimes E').$$

Proof: Given $\xi \in \mathcal{E}(U, E), \eta \in \mathcal{E}(U, E')$, define $\tau(\xi \otimes \eta) \in \mathcal{E}(U, E \otimes E')$ by

$$\tau(\xi \otimes \eta)(x) = \xi(x) \otimes \eta(x)\,.$$

We construct a converse as follows. Choose a locally finite covering $\{U_i\}$ of U such that over each U_i we have frames $\{e_{\sigma,i}\}$ for E, $\{e'_{p,i}\}$ for E'; and a partition of unity $\{\phi_i\}$ such that $\mathrm{supp}(\phi_i) \subset U_i$, and functions $\psi_i \in \mathcal{D}(X)$ such that $\mathrm{supp}(\psi_i) \subset U_i$ and $\psi_i \equiv 1$ on $\mathrm{supp}(\phi_i)$. Given $\gamma \in \mathcal{E}(U, E \otimes E')$, write

$$\gamma(x) = \textstyle\sum_{\sigma,p} \gamma_i^{\sigma p}(x) e_{\sigma,i}(x) \otimes e'_{p,i} \quad \text{for } x \in U_i \quad .$$

We can extend $\phi_i \gamma_i^{\sigma p}$ by zero to a function in $\mathcal{E}(U)$, and extend $\psi_i e_{\sigma,i}$ to a section in $\mathcal{E}(U, E)$, and $\psi_i e'_{p,i}$ to a section in $\mathcal{E}(U, E')$. Then

$$\sum_{\sigma,p} \sum_i (\phi_i \gamma_i^{\sigma p})(\psi_i e_{\sigma,i}) \otimes (\psi_i e'_{p,i}) \in \mathcal{E}(E) \otimes_{\mathcal{E}(X)} \mathcal{E}(E'),$$

and one verifies that τ maps this to γ, because $\sum_i \psi_i(x)^2 \phi_i(x) = 1$.

Corollary 1.7: Let E be a vector bundle over X. Then

$$\mathcal{E}^p \otimes_{\mathcal{E}} \mathcal{E}(X, E) \equiv \mathcal{E}^p(X, E).$$

Proof: $\mathcal{E}(X, \wedge^p T^*(x)) \otimes_{\mathcal{E}} \mathcal{E}(X, E)$

$$= \mathcal{E}(X, \wedge^p T^*(X) \otimes_C E)$$

$$= \mathcal{E}^p(X, E).$$

We shall denote the image of $\varphi \otimes \xi$ under the isomorphism in Corollary 1.7 by $\varphi.\xi \in \mathcal{E}^p(X, E)$ where $\varphi \in \mathcal{E}^p(X)$ and $\xi \in \mathcal{E}(X, E)$. Suppose that f is a frame for E over $U \subset X$. Then we have a local representation for $\xi \in \mathcal{E}^p(U, E)$ similar to (1.1) given by

$$\ell_f^p : \mathcal{E}^p(U, E) \longrightarrow [\mathcal{E}^p(U)]^q$$

such that

$$\xi \mapsto \begin{bmatrix} \xi^1(f) \\ \cdot \\ \cdot \\ \cdot \\ \xi^q(f) \end{bmatrix}$$

defined by the relation

$$\xi = \sum_{p=1}^{q} \xi^p(f) \cdot e_p.$$

where $\xi^p(f) \in \mathcal{E}^p(U)$. Moreover, if g is a change of frame over U, then we have the transformation law for the local representation of vector-valued differential forms

$$\xi(fg) = g^{-1} \xi(f), \quad \xi \in \mathcal{E}^p(X, E).$$

2. Metric on Vector Bundles

Definition 2.1: Let $E \to X$ be a vector bundle. A *Hermitian metric* h on E is an assignment of a Hermitian inner product $\langle \cdot\,, \cdot \rangle_x$ to each fibre E_x of E such that for any open $U \subset X$ and $\xi, \eta \in \mathcal{E}(U, E)$ the function

$$\langle \xi, \eta \rangle : U \longrightarrow \boldsymbol{C}$$

given by

$$\langle \xi, \eta \rangle(x) = \langle \xi(x), \eta(x) \rangle_x$$

is C^∞.

Definition 2.2: A vector bundle E equipped with a Hermitian metric h is called a *Hermitian vector bundle.*

Definition 2.3: Let E be a Hermitian vector bundle and suppose that $f = (e_1, \dots, e_r)$ is a frame for E over some open set U. Then

$$h(f)_{p\sigma} = \langle e_\sigma, e_p \rangle,$$

and we define $h(f) = [h(f)_{p\sigma}]$ as the $r \times r$ matrix of the C^∞ functions $\{h(f)_{p\sigma}\}$, where $r = \mathrm{rank} E$.

Remark 2.4: $h(f)$ is a positive definite Hermitian symmetric matrix and is a local representative for the Hermitian metric h with respect to the frame f.

Example 2.5: For any $\xi, \eta \in \mathcal{E}(U, E)$, we write

$$\begin{aligned}\langle \xi, \eta \rangle &= \langle \sum_p \xi^p(f) e_p, \sum_\sigma \eta^\sigma(f) e_\sigma \rangle \\ &= \sum_{p,\sigma} \overline{{}^t\eta^\sigma(f)} h_{p\sigma}(f) \xi^p(f),\end{aligned}$$

where the last product is a matrix multiplication. Moreover, if g is a change of frame over U, then

$$h(fg) = \overline{{}^tg} h(f) g,$$

which is the transformation law for local representations of the Hermitian metric.

Theorem 2.6: Every vector bundle $E \to X$ admits a Hermitian metric.

Proof: There exists a locally finite covering $\{U_\alpha\}$ of X and frames f_α defined on U_α. We define a Hermitian metric h_α on $E|_{U_\alpha}$ by setting, for any $\xi, \eta \in E_x, \quad x \in U_\alpha,$

$$\langle \xi, \eta \rangle_x^\alpha = \overline{{}^t\eta(f_\alpha)}(x) \cdot \xi(f_\alpha)(x).$$

We now use a partition of unity. Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to the covering $\{U_\alpha\}$. For $\xi, \eta \in E_x$, we let

$$\langle \xi, \eta \rangle_x = \sum_\alpha \rho_\alpha(x) \langle \xi, \eta \rangle_x^\alpha.$$

It is clear that if $\xi, \eta \in \mathcal{E}(U, E)$, then the function

$$\begin{aligned} x \mapsto \langle \xi(x), \eta(x) \rangle_x &= \sum_\alpha \rho_\alpha(x) \langle \xi(x), \eta(x)_x^\alpha \rangle \\ &= \sum_\alpha \rho_\alpha \overline{{}^t\eta(f_\alpha)}(x) \cdot \xi(f_\alpha)(x) \end{aligned}$$

is a C^∞ function U. Note that it is easy to verify that h is indeed a Hermitian inner product on each fibre of E, since $0 \leq \rho_\alpha(x) \leq 1$ for all α. Thus, $\langle \cdot, \cdot \rangle$ gives a Hermitian metric for $E \to X$.

3. Connection on Vector Bundles

Definition 3.1: Let $E \to X$ be a vector bundle. Then a *connection* D on $E \to X$ is a $\boldsymbol{C}$-linear mapping

$$D : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E)$$

which satisfies

$$D(\varphi\xi) = d\varphi \cdot \xi + \varphi D\xi, \qquad (3.1)$$

where $\varphi \in \mathcal{E}(X)$ and $\xi \in \mathcal{E}(X, E)$.

Remark 3.2:

1. (3.1) implies that D is a first order differential operator mapping $\mathcal{E}(X, E)$ to $\mathcal{E}^1(X, E) = \mathcal{E}(X, T^*(X) \otimes E)$, which will be defined in the next chapter.

2. In the case where E is the trivial line bundle, i.e. $E = X \times \boldsymbol{C}$, we may take the ordinary exterior differentiation

$$d : \mathcal{E}(X) \longrightarrow \mathcal{E}^1(X)$$

as a connection on E .

3. A connection is a generalization of exterior differentiation to vector-valued differential forms.

Definition 3.3: Let $f = (e_1, \ldots, e_r)$ be a frame over U for a vector bundle $E \to X$, equipped with a connection D. Then the *connection matrix* $\theta(D, f)$ associated with the connection D and the frame f is as

$$\theta(D, f) = [\theta_{p\sigma}(D, f)], \qquad \theta_{p\sigma} \in \mathcal{E}^1(U)$$

where

$$De_\sigma = \sum_{p=1}^{r} \theta_{p\sigma}(D, f) \cdot e_p.$$

For a fixed connection D we denote the matrix $\theta(D, f)$ more briefly by $\theta(f)$.

We can use the connection matrix to describe the action of D on sections of E. If $\xi \in \mathcal{E}(U, E)$ for a given frame f, then

$$\begin{aligned} D\xi &= D(\sum_p \xi^p(f) e_p) \\ &= \sum_\sigma d\xi^\sigma(f) e_\sigma + \sum_p \xi^p(f) De_p \\ &= \sum_\sigma [d\xi^\sigma(f) + \sum_p \xi^p \theta_{\sigma p}(f)] \cdot e_\sigma \, . \end{aligned}$$

Then

$$D\xi = \sum_{\sigma} [d\xi(f) + \theta(f) \wedge \xi(f)]^{\sigma} \cdot e_{\sigma} ,$$

where we set

$$d\xi(f) = \begin{bmatrix} d\xi^1(f) \\ \cdot \\ \cdot \\ \cdot \\ d\xi^r(f) \end{bmatrix}$$

and the wedge product is the ordinary matrix multiplication of matrices with differential form coefficients.

Thus we see that

$$\begin{aligned} D\xi(f) &= d\xi(f) + \theta(f) \wedge \xi(f) \\ &= [d + \theta(f)]\xi(f) \end{aligned}$$

thinking of $d + \theta(f)$ as being an operator acting on vector-valued functions.

Remark 3.4: In the case where E is the tangent bundle of X, i.e., $E = T(X)$, then the analogue of a connection on a $\boldsymbol{R}$-vector bundle as above defines an *affine connection* in the usual sense.
If $w = (w_1, \ldots, w_n)$ is a frame for $T^*(X)$ over U, then

$$\theta_{p\sigma} = \sum_{k=1}^{n} \Gamma^p_{\sigma k} w_k, \quad \Gamma^p_{\sigma k} \in \mathcal{E}(U).$$

In the classical case these are the *Schwarz-Christoffel* symbols associated with a given connection.

4. Curvature on Vector Bundles

Suppose that $E \longrightarrow X$ is a vector bundle equipped with a connection D. Let $\tilde{E} = \mathrm{Hom}(E, E)$ be the vector bundle whose fibres are $\mathrm{Hom}(E_x, E_x)$.

Definition 4.1: Let $E \to X$ be a vector bundle with connection D and let $\theta(f) = \theta(D, f)$ be the associated connection matrix and fix a frame f on $U \subset X$. Then we put

$$\Theta(D, f) = d\theta(f) + \theta(f) \wedge \theta(f)$$

which is an $r \times r$ matrix of 2-forms, i.e.,

$$\Theta_{p\sigma} = d\theta_{p\sigma} + \sum \theta_{pk} \wedge \theta_{k\sigma}.$$

We will call $\Theta(D, f)$ the *curvature matrix* associated with the connection matrix $\theta(f)$.

Lemma 4.2: Let g be a change of frame and define $\theta(f)$ and $\Theta(f)$ as above. Then

1. $dg + \theta(f)g = g\theta(fg)$
2. $\Theta(fg) = g^{-1}\Theta(f)g$

Proof:

1. If $fg = (\sum_{p=1}^{r} g_{p1}e_p, \ldots, \sum_{p=1}^{r} g_{pr}e_p) = (e'_1, \ldots, e'_r)$, then

$$\begin{aligned} D(e'_\sigma) &= \sum_{v=1}^{r} \theta_{v\sigma}(fg)e'_v \\ &= \sum_{v,p} \theta_{v\sigma}(fg)g_{pv}e_p \end{aligned}$$

and on the other hand

$$D(\sum_p g_{p\sigma}e_p) = \sum_p dg_{p\sigma}e_p + \sum_{p,\tau} g_{p\sigma}\theta_{\tau p}e_\tau.$$

By comparing coefficients, we obtain

$$g\theta(fg) = dg + \theta(f)g.$$

2. If we apply the exterior derivative to $g\theta(fg) = dg + \theta(f)g$, then we obtain

$$d\theta(f) \cdot g - \theta(f) \wedge dg = dg \wedge \theta(fg) + g \cdot d\theta(fg) \qquad (4.1)$$

and we have

$$\theta(fg) = g^{-1}dg + g^{-1}\theta(f)g \qquad (4.2)$$

By substituting (4.2) in (4.1) we obtain an algebraic expression for $g \cdot d\theta(fg)$ in terms of the quantities $d\theta(f), \theta(f), dg, g, g^{-1}$. Then we can write

$$g[d\theta(fg) + \theta(f) \wedge \theta(f)] = g\Theta(fg)$$

in terms of these same quantities. This is the same as

$$[d\theta(f) + \theta(f) \wedge \theta(f)]g = \Theta(f)g.$$

Therefore

$$g\Theta(fg) = \Theta(f)g$$
$$\Theta(fg) = g^{-1}\Theta(f)g.$$

Lemma 4.3: For all $\xi(f)$ in $\mathcal{E}(X, E)$:

$$[d + \theta(f)][d + \theta(f)]\xi(f) = \Theta(f)\xi(f)\,,$$

So that D^2 and Θ are the same operator $\mathcal{E}(X, E) \to \mathcal{E}^2(X, E)$.

Proof: $(d + \theta)(d + \theta)\xi = d^2\xi + \theta \cdot d\xi + d(\theta \cdot \xi) + \theta \wedge \theta \cdot \xi$

$$= \theta \cdot d\xi + d\theta - \theta \cdot d\xi + \theta \wedge \theta \cdot \xi$$
$$= d\theta \cdot \xi + \theta \wedge \theta \cdot \xi$$
$$= \Theta \cdot \xi\,.$$

Definition 4.4: Let D be a connection in a vector bundle $E \to X$. Then the *curvature* $\Theta_E(D)$ is defined to be that element $\Theta \in \mathcal{E}^2(X, \operatorname{Hom}(E, E))$ such that the $\mathcal{E}(X)$-linear mapping

$$\Theta : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^2(X, E)$$

has the representation with respect to a frame f locally

$$\Theta(f) = \Theta(D, f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

Note that $\Theta_E(D)$ is well defined, since $\theta(f)$ satisfies the transformation property in Lemma 4.2(2), which ensure that $\Theta(f)$ a global element in $\mathcal{E}^2(X, \mathrm{Hom}(E, E))$.

Remark 4.5: The curvature Θ is an $\mathcal{E}(X)$-linear mapping

$$\Theta : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^2(X, E),$$

but the connection D is not $\mathcal{E}(X)$-linear because the transformation formula of $\theta(f)$ involves derivatives of the change of frames.

Finally, we can extend the definition of D to higher order differential forms with values in a vector bundle E by setting

$$D\xi(f) = d\xi(f) + \theta(f) \wedge \xi(f).$$

Thus

$$D : \mathcal{E}^p(X, E) \longrightarrow \mathcal{E}^{p+1}(X, E)$$

where $\xi \in \mathcal{E}^p(X, E)$. This extension is known as **covariant differentiation.**

Chapter 4: Elliptic Operators and The Cohomology of Projective Algebraic Varieties

1. Sobolev Spaces

Let X be a compact differentiable manifold with a strictly positive smooth measure μ. This means that $d\mu$ is a volume element (or density) which can be expressed in local coordinates $(x_1, \dots, x_n)$ by

$$d\mu = p(x)dx_1 \cdots dx_n \,,$$

where the coefficients transform by

$$p(x)dx_1 \cdots dx_n = \tilde{p}(y)|\frac{\partial x_i}{\partial y_j}|dy_1 \cdots dy_n \,,$$

where $\tilde{p}(y)dy_1 \cdots dy_n$ is the representation with respect to the change of coordinates $x \mapsto y$ and $(\partial x_i / \partial y_j)$ is the corresponding Jacobian matrix. Such measures always exist. If X is orientable, then the volume element $d\mu$ can be chosen to be a positive differential form of degree n.

Definition 1.1: Let E be a differentiable vector bundle over X. Let $\mathcal{E}^k(X, E)$ be the k-th order differentiable sections of E over X, $0 \leq k < \infty$, and for $k = \infty$: $\mathcal{E}^\infty(X, E) = \mathcal{E}(X, E)$. Then we denote the compactly supported sections by

$$\mathcal{D}(X, E) \subset \mathcal{E}(X, E) \,,$$

and the compactly supported functions by

$$\mathcal{D}(X) \subset \mathcal{E}(X) \,.$$

Moreover, $\mathcal{D}^k \subset \mathcal{E}^k$ for $0 \leq k < \infty$.

Definition 1.2: If E is, in addition, Hermitian, then we define an inner product $\langle \cdot \,, \cdot \rangle$ on $\mathcal{E}(X, E)$ by setting

$$\langle \xi, \eta \rangle = \int_X \langle \xi(x), \eta(x) \rangle_E d\mu$$

where $\langle \cdot \,, \cdot \rangle_E$ is the Hermitian metric on E.

Let $\|\xi\|_0 = \langle \xi, \xi \rangle^{\frac{1}{2}}$ be the L^2-norm and let $W^0(X, E)$ be the completion of $\mathcal{E}(X, E)$. Let $\{U_\alpha, h_\alpha\}$ be a finite trivializing cover of E, then we have the following diagram

$$\begin{tikzcd} E_{|U_\alpha} \arrow[r, "h_\alpha"] \arrow[d] & \tilde{U}_\alpha \times \boldsymbol{C}^m \arrow[d] \\ U_\alpha \arrow[r, "\tilde{h}_\alpha"'] & \tilde{U}_\alpha , \end{tikzcd}$$

where h_α is a bundle isomorphism as described in Chapter 1 (Definition 2.1.1), and $\tilde{h}_\alpha : U_\alpha \to \tilde{U}_\alpha \subset \boldsymbol{R}^n$ are local coordinate systems for the manifold X. Then let

$$h_\alpha^* : \mathcal{E}(U_\alpha, E) \longrightarrow [\mathcal{E}(\tilde{U}_\alpha)]^m$$

be the induced map. Let $\{p_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$. We define, for $\xi \in \mathcal{E}(X, E)$,

$$\|\xi\|_{s,E} = \sum_\alpha \|h_\alpha^* p_\alpha \xi\|_{s,\boldsymbol{R}^n} ,$$

where $\| \quad \|_{s,\boldsymbol{R}^n}$ is the Sobolev norm for a compactly supported differentiable function

$$f : \boldsymbol{R}^n \longrightarrow \boldsymbol{C}^m ,$$

defined by

$$\|f\|_{s,\boldsymbol{R}^n}^2 = \int |\hat{f}(y)|^2 (1 + |y|^2)^s dy , \quad (1.1)$$

where

$$\hat{f}(y) = (2\pi)^{-n} \int e^{-i\langle x,y \rangle} f(x) dx \quad (1.1')$$

is the Fourier transform in $\boldsymbol{R}^n$. If $\|\xi\|_s < \infty$, for s a positive integer, then this means that ξ has s derivatives in L^2 since we know that the norm $\| \quad \|_{s,\boldsymbol{R}^n}$ is equivalent to the norm

$$\left[\sum_{|\alpha| \le s} \int_{\boldsymbol{R}^n} |D^\alpha f|^2 dx \right]^{\frac{1}{2}} , \qquad f \in \mathcal{D}(\boldsymbol{R}^n) ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex. (See, e.g., Hörmander [3], Chapter 1, Sec. 1.7.) This follows essentially from the basic fact about Fourier transforms that

$$\widehat{D^\alpha f}(y) = i^{|\alpha|} y^\alpha \hat{f}(y) ,$$

where $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_i = \frac{\partial}{\partial x_i}$, and $\|f\|_0 = (2\pi)^n \|\hat{f}\|_0$.

We let $W^s(X,E)$ be the completion of $\mathcal{E}(X,E)$ with respect to the norm $\| \quad \|_s$ defined on E. $W^s(X,E)$ depends on the choice of partition of unity and the local trivialization.

Theorem 1.3: (Sobolev): Let X be a compact differentiable manifold and let $\dim_{\boldsymbol{R}} X = n$. Suppose that $k < s - \frac{n}{2} - 1$. Then

$$W^s(X,E) \subset \mathcal{E}^k(X,E).$$

Proof: See R. A. Adams [1], page 97.

Theorem 1.4: (Rellich Kondrachov theorem): Let X be a compact differentiable manifold. Then the natural inclusion

$$j : W^s(X,E) \hookrightarrow W^r(X,E)$$

for $r < s$ is a completely continuous linear map.

Proof: See R. A. Adams [1], page 144.

We mean by completely continuous that the image of a closed ball is compact, i.e., j is a compact operator. We shall prove the above theorems in special cases.

Proposition 1.5: Let f be a measurable L^2 function in $\boldsymbol{R}^n$ with $\|f\|_{s,\boldsymbol{R}^n} < \infty$, and suppose k is a nonnegative integer such that $k < s - \frac{n}{2} - 1$ and $s \in \boldsymbol{N}$. Then $f \in C^k(\boldsymbol{R}^n)$ after a possible change on a set of measure zero.

Proof: $\|f\|_s < \infty$ means that

$$\int_{\boldsymbol{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty.$$

Let

$$\tilde{f}(x) = \int_{\boldsymbol{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi) d\xi$$

be the inverse Fourier transform, if the integral converges. It is known that if the inverse Fourier transform exists, then $\tilde{f}(x)$ agrees with $f(x)$ almost everywhere (see, e.g., Hörmander [3], Chapter 1, Sec. 1.7), and that $f \in C^0(\boldsymbol{R}^n)$ if this integral exists, making the appropriate change on a set of measure zero. Similarly, for some constant c,

$$D^\alpha f(x) = c \int e^{i\langle x,\xi\rangle} \xi^\alpha \hat{f}(\xi) d\xi$$

will be continuous derivatives of f if the integral converges. Therefore, we need to show that for $|\alpha| \leq k$ the integrals

$$\int e^{i\langle x,\xi\rangle} \xi^\alpha \hat{f}(\xi) d\xi$$

converge and it will follow that $f \in C^k(\boldsymbol{R}^n)$. But we have

$$\begin{aligned}\int |\hat{f}(\xi)||\xi|^{|\alpha|} d\xi &= \int |\hat{f}(\xi)|(1+|\xi|^2)^{\frac{s}{2}} \frac{|\xi|^{|\alpha|}}{(1+|\xi|^2)^{\frac{s}{2}}} d\xi \\ &\leq \|f\|_s \left(\int \frac{|\xi|^{2|\alpha|}}{(1+|\xi|^2)^s} d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

We have chosen s so that this last integral exists, and we have

$$\int |\hat{f}(\xi)||\xi|^{|\alpha|} d\xi < \infty .$$

Thus, the proposition is proved.

We will now prove a simple case of Rellich's theorem (Theorem 1.4). We shall write $K \subset\subset \boldsymbol{R}^n$ provided $\overline{K} \subset \boldsymbol{R}^n$ and $\overline{K}$ is compact.

Proposition 1.6: Suppose that we have a sequence $\{f_v\}$ in $W^s(\boldsymbol{R}^n)$ and that all f_v have compact support in $K \subset\subset \boldsymbol{R}^n$. Assume that $\|f_v\|_s \leq 1$ for all v. Then for any $r < s$ there exists a subsequence $\{f_{v_i}\}$ which converges in $W^r(\boldsymbol{R}^n)$.

Proof: We first observe that for $\xi, \eta \in \boldsymbol{R}^n, s \in \boldsymbol{Z}^+$,

$$(1+|\xi|^2)^{\frac{s}{2}} \leq 2^{\frac{s}{2}}(1+|\xi-\eta|^2)^{\frac{s}{2}}(1+|\eta|^2)^{\frac{s}{2}} . \qquad (1.2)$$

To see this we write
$1+|\zeta+\eta|^2 \leq 1+(|\zeta|+|\eta|)^2 \leq 1+2(|\zeta|^2+|\eta|^2) \leq 2(1+|\zeta|^2)(1+|\eta|^2)$.
If we put $\xi = \zeta + \eta$, then we get the above inequality (1.2).

Now, let $\varphi \in \mathcal{D}(\boldsymbol{R}^n)$ be chosen so that $\varphi \equiv 1$ on K. Then from a standard relation between the Fourier transform and *convolution* we have that

$$f_v = \varphi f_v \,.$$

This implies

$$\hat{f}_v(\xi) = (2\pi)^{-n} \int \hat{\varphi}(\xi - \eta) \hat{f}_v(\eta) d\eta \qquad (1.3)$$

(see, e.g., Nicolaescu [4], Lemma 9.2.9). Therefore we obtain from (1.2) and (1.3) that

$$(1+|\xi|^2)^{\frac{s}{2}} |\hat{f}_v(\xi)| \leq 2^{\frac{s}{2}} (2\pi)^{-n} \int (1+|\xi-\eta|^2)^{\frac{s}{2}} |\hat{\varphi}(\xi-\eta)| (1+|\eta|^2)^{\frac{s}{2}} |\hat{f}_v(\eta)| d\eta$$

$$\leq C_{s,\varphi} \, \|f_v\|_s \leq C_{s,\varphi} \,,$$

where $C_{s,\varphi}$ is a constant depending on s and φ. Thus, $|\hat{f}_v(\xi)|$ is uniformly bounded on compact subsets of $\boldsymbol{R}^n$.

Similarly, by differentiating (1.3) we obtain that all derivatives of $\hat{f}_v$ are uniformly bounded on compact subsets in the same manner. Therefore, by Arzela Ascoli theorem, there is a subsequence f_{v_i} such that $\hat{f}_{v_i}$ converges in the C^∞ topology to a C^∞ function on $\boldsymbol{R}^n$.

Let $\epsilon > 0$ be given. Suppose that $0 \leq r < s$. Then there is a ball B_ϵ such that

$$\frac{1}{(1+|\xi|^2)^{s-r}} < \epsilon$$

for ξ outside the ball B_ϵ. Then consider

$$\|f_{v_i} - f_{v_j}\|_r^2 = \int_{\boldsymbol{R}^n} \frac{|\hat{f}_{v_i}(\xi) - \hat{f}_{v_j}(\xi)|^2}{(1+|\xi|^2)^{s-r}} (1+|\xi|^2)^s d\xi$$

$$\leq \int_{B_\epsilon} |\hat{f}_{v_i}(\xi) - \hat{f}_{v_j}(\xi)|^2 (1+|\xi|^2)^r d\xi + \epsilon \int_{\boldsymbol{R}^n - B_\epsilon} |\hat{f}_{v_i}(\xi) - \hat{f}_{v_j}(\xi)|^2 (1+|\xi|^2)^s d\xi$$

$$\leq \int_{B_\epsilon} |\hat{f}_{v_i}(\xi) - \hat{f}_{v_j}(\xi)|^2 (1+|\xi|^2)^r d\xi + 2\epsilon \,,$$

where by assumption $\|f_{v_i}\|_s \leq 1$. Since we know that $\hat{f}_{v_i}$ converges on compact subsets we can choose v_i, v_j large enough so that the first integral is $< \epsilon$, and thus f_{v_i} is a Cauchy sequence in $W^r(\boldsymbol{R}^n)$.

2. Differential Operators

Definition 2.1: Let E and F be differentiable $\boldsymbol{C}$-vector bundles over a differentiable manifold X. Let

$$L : \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, F)$$

be a $\boldsymbol{C}$-linear map. We say that L is a *differential operator* of order at most k if, for all choices of local coordinates and local trivializations, the induced mapping $\tilde{L}$ given by the following diagram

$$\begin{tikzcd}
[\mathcal{E}(U)]^p \arrow[r, "\tilde{L}"] \arrow[d, "\equiv"'] & [\mathcal{E}(U)]^q \arrow[d, "\equiv"] \\
\mathcal{E}(U, U \times \boldsymbol{C}^p) & \mathcal{E}(U, U \times \boldsymbol{C}^q) \\
\mathcal{E}(U, E_{|U}) \arrow[r, "L"'] \arrow[u] & \mathcal{E}(U, F_{|U}) \arrow[u]
\end{tikzcd}$$

is a linear partial differential operator of order at most k, where we consider $U \subset \boldsymbol{R}^n$ and $\tilde{L}$ acting on restrictions of global sections. That is, for $f = (f_1, \ldots, f_p)$ in $[\mathcal{E}(U)]^p$

$$L(f)_i = \sum_{j=1,\ldots,p\,;|\alpha|\leq k} a_\alpha^{ij} D^\alpha f_j, \quad i = 1, \ldots, q,$$

where α is a multiindex.

Definition 2.2:

1. A differential operator is said to be of *order at most* k if there are no derivatives of *order* $\geq k+1$ appearing in any local representation.

2. We denote by $\mathrm{Diff}_k(E, F)$ the vector space of all differential operators of order at most k mapping $\mathcal{E}(X, E)$ to $\mathcal{E}(X, F)$. We define $\mathrm{Diff}(E, F)$ to

be the union of all $\mathrm{Diff}_k(E, F)$, i.e., $\mathrm{Diff}(E, F) = \cup_k \mathrm{Diff}_k(E, F)$.

Example 2.3: (The exterior derivative in $\boldsymbol{R}^n$) The differential operator

$$d : \Omega^q(\boldsymbol{R}^n) \longrightarrow \Omega^{q+1}(\boldsymbol{R}^n)$$

is of order 1.

We now need to discuss the concept of a *formal adjoint operator.*

Definition 2.4: Let E and F be Hermitian $\boldsymbol{C}$-vector bundles over a differentiable manifold X with a measure μ, and let L be in $\mathrm{Diff}(E, F)$. Then the operator $S \in \mathrm{Diff}(F, E)$ is said to be a formal adjoint of L if for all $f \in \mathcal{D}(X, E) \subset \mathcal{E}(X, E)$ and $g \in \mathcal{D}(X, F) \subset \mathcal{E}(X, F)$

$$\int_X \langle Lf \, , \, g\rangle_F d\mu = \int_X \langle f \, , \, Sg\rangle_E d\mu \, ,$$

where $d\mu$ is the volume element on X, and $\mathcal{D}$ is as described in Definition 1.1.

Example 2.5: Let U, V, and W be finite dimensional inner product spaces, and suppose that

$$U \xrightarrow{L} V \xrightarrow{S} W$$

is exact. Let $L^* : V \to U$ and $S^* : W \to V$ be adjoints of L and S respectively. Then the operator $S^*S + LL^*$ is an isomorphism on V. To show this, we let v be a nonzero element of V. We need to only show that $(S^*S + LL^*)(v) \neq 0$. Now

$$\langle (S^*S + LL^*)v, v\rangle = \langle Sv, Sv\rangle + \langle L^*v, L^*v\rangle \, .$$

If $Sv \neq 0$, then $(S^*S + LL^*)(v) \neq 0$. If $Sv = 0$, then v lies, by exactness, in the image of L. But L^* is injective on the image of L. Thus $L^*v \neq 0$, which implies that $(S^*S + LL^*)(v) \neq 0$.

Remark 2.6: The formal adjoint of a differential operator $L \in \mathrm{Diff}(E, F)$ is denoted by L^*, if it exists.

Lemma 2.7: Any $L \in \mathrm{Diff}(E, F)$ admits at most one formal adjoint.

Proof: Let S_1, S_2 be two formal adjoints of L. Then for all $g \in \mathcal{D}(X, F)$

$$\int_X \langle f\,,\,(S_1 - S_2)g\rangle_E d\mu = 0$$

for all $f \in \mathcal{D}(X, E)$. This implies $(S_1 - S_2)g = 0$ for all $g \in \mathcal{D}(X, F)$. If $g \in \mathcal{E}(X, F)$ is not necessarily compactly supported, then choosing $(p) \subset \mathcal{D}(X)$ a partition of unity, we conclude using the locality of $S = S_1 - S_2$ as described in Definition 2.1 that

$$Sg = \sum pS(pg) = 0\,.$$

Proposition 2.8: Suppose $L \in \text{Diff}(E, F)$ and $S \in \text{Diff}(F, G)$ admit formal adjoints $L^* \in \text{Diff}(F, E)$ and $S^* \in \text{Diff}(G, F)$, respectively. Then SL admits a formal adjoint, and

$$(SL)^* = L^* S^*\,.$$

Proof: For any $f \in \mathcal{D}(X, E)$, $g \in \mathcal{D}(X, F)$, and $h \in \mathcal{D}(X, G)$

$$\int_X \langle SLf\,,\,h\rangle_G\,d\mu = \int_X \langle Lf\,,\,S^*f\rangle_F\,d\mu$$

$$= \int_X \langle f\,,\,L^*S^*h\rangle_E\,d\mu\,.$$

We want to define the *symbol* of a differential operator. The symbol will be used for the classification of differential operators into various types. Recall that $T^*(X)$ is the cotangent bundle to a differentiable manifold X, let $T'(X)$ denote $T^*(X)$ with zero section deleted.

Definition 2.9: Let $\pi : T'(X) \to X$ denote the natural projection map. We set, for any $k \in \boldsymbol{Z}$,

$$\text{Smbl}_k(E, F) = \{\sigma \in \text{Hom}(\pi^*E, \pi^*F) \text{ such that } \sigma(x, pv) = p^k\sigma(x, v) \text{ for all } p > 0, (x, v) \in T'(X)\}\,,$$

where π^*E and π^*F denote (as in Chapter 1, Definition 2.2.12) the pullback of E and F by π to $T'(X)$.

Definition 2.10: Let $L \in \mathrm{Diff}_k(E,F)$. Then we have a linear map

$$\sigma_k : \mathrm{Diff}_k(E,F) \longrightarrow \mathrm{Smbl}_k(E,F) \ ,$$

where $\sigma_k(L)$ is called the k-symbol of differential operator L. Let $(x,v) \in T'(X)$ and $e \in E_x$ be given. Find $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X,E)$ such that $dg_x = v$, and $f(x) = e$. Then we define

$$\sigma_k(L)(x,v)e = L(\frac{1}{k!}(g - g(x))^k f)(x) \in F_x \,.$$

Thus $\sigma_k(L)(x,v)$ is a linear mapping from E_x to F_x which defines an element of $\mathrm{Smbl}_k(E,F)$.

Proposition 2.11: The symbol map σ_k gives rise to an exact sequence

$$\mathrm{Diff}_{k-1}(E,F) \xrightarrow{j} \mathrm{Diff}_k(E,F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(E,F) \ . \qquad (2.1)$$

where j is the natural inclusion.

Proof: We have to show that the k-symbol of a differentiable operator of order k has a certain form in local coordinates. Let L be a linear partial differential operator

$$L : [\mathcal{E}(U)]^p \longrightarrow [\mathcal{E}(U)]^q$$

where U is open in $\boldsymbol{R}^n$, and $p = \mathrm{rank}(E)$ and $q = \mathrm{rank}(F)$. If

$$L = \sum_{|\alpha| \le k} A_\alpha D^\alpha \,,$$

where $\{A_\alpha\}$ are $p \times q$ matrices of C^∞ functions on U, then we choose $g \in \mathcal{E}(U)$ such that $v = dg = \sum_{i=1}^n \xi_i dx_i$, i.e., $D_i g(x) = \xi_i$. Let $e \in \boldsymbol{C}^p$. Then we have

$$\sigma_k(L)(x,v) = \sum_{|\alpha| \le k} A_\alpha D^\alpha (\frac{1}{k!}(g - g(x))^k e)(x) \,.$$

Clearly, the evaluation at x of derivatives of order $\le k-1$ will equal zero, since there will be a factor of $[g - g(x)]|_x = 0$ remaining. Hence, the only nonzero part is

$$\sum_{|\alpha| = k} A_\alpha(x) \frac{k!}{k!} (D_1 g(x))^{\alpha_1} \cdots (D_n g(x))^{\alpha_n}$$

$$= \sum_{|\alpha|=k} A_\alpha(x)\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha .$$

For each fixed (x, v), $\sigma_k(L)(x, v)$ is a linear mapping from $x \times \boldsymbol{C}^p \mapsto x \times \boldsymbol{C}^q$, given by the usual multiplication of a vector in $\boldsymbol{C}^p$ by the matrix

$$\sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha .$$

It is easy to see that if $\sigma_k(L) = 0$, then the differential operator L has k-th order terms equal to zero, and thus L is a differential operator of order $k-1$.

The mapping σ_k in (2.1) is, by Definition 2.10, well defined. We now want to see that the kernel is contained in $\mathrm{Diff}_{k-1}(E, F)$, so it suffices to see that this is true for a local representation of the operator. This then follows from the local representation for the symbol $\sigma_k(L)(x, v) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha$.

Theorem 2.12: Let E, F, and G be vector bundles over a differentiable manifold X. If $L \in \mathrm{Diff}_k(E, F)$ and $S \in \mathrm{Diff}_m(F, G)$, then

$$SL = S \circ L \in \mathrm{Diff}_{k+m}(E, G) \quad .$$

Moreover, $\sigma_{k+m}(SL) = \sigma_m(S) \cdot \sigma_k(L)$, where the right hand product is the product of the linear mapping involved.

Proof: The relation $\sigma_{k+m}(SL) = \sigma_m(S) \cdot \sigma_k(L)$ is clear for calculations in local representations for the symbols of S and L as described in proof of Proposition 2.11.

We will end this section by stating an important property of the basic properties of differential operators which is the existence of a formal adjoint.

Theorem 2.13: Suppose X is a differentiable manifold, and let L be in $\mathrm{Diff}_k(E, F)$. Then the adjoint operator L^* exists, and $L^* \in \mathrm{Diff}_k(F, E)$. Moreover, $\sigma_k(L^*) = (-1)^k \sigma_k(L)^*$, where $\sigma_k(L)^*$ is the adjoint of the linear map

$$\sigma_k(L)(x, v) : E_x \longrightarrow F_x .$$

Proof: Suppose that μ is a strictly positive smooth measure on X, and that h_E and h_F are Hermitian metrics on E and F, respectively. The inner product for any $\xi, \eta \in \mathcal{D}(X, E)$ is (as in Definition 1.2) given by

$$\langle \xi \, , \, \eta \rangle = \int_X \langle \xi \, , \, \eta \rangle_E \, d\mu \, .$$

Recall that if ξ and η have compact support in a neighbourhood where E admits a local frame $f = (e_1, \ldots, e_q)$, then we have (as shown in Chapter 3, Sec.2)

$$\langle \xi \, , \, \eta \rangle = \int_{\boldsymbol{R}^n} \overline{{}^t\eta}(x) h_E(x) \xi(x) p(x) dx \, ,$$

where $p(x)$ is a density, and

$$\xi(x) = \xi(f)(x) = \begin{bmatrix} \xi^1(f)(x) \\ \cdot \\ \cdot \\ \cdot \\ \xi^q(f)(x) \end{bmatrix} ,$$

and the same for $\eta(x)$. Similarly, let F admit a local frame $g = (e_1, \ldots, e_p)$. Then for $\sigma, \tau \in \mathcal{D}(X, F)$, we have

$$\langle \sigma \, , \, \tau \rangle = \int_{\boldsymbol{R}^n} \overline{{}^t\tau}(x) h_F(x) \sigma(x) p(x) dx \, ,$$

where

$$\sigma(x) = \sigma(g)(x) = \begin{bmatrix} \sigma^1(g)(x) \\ \cdot \\ \cdot \\ \cdot \\ \sigma^p(g)(x) \end{bmatrix} ,$$

and the same for $\tau(x)$. Suppose that $L : \mathcal{D}(X, E) \to \mathcal{D}(X, F)$ is a linear differential operator of order k, and assume that the sections have support in a trivializing neighbourhood U which gives local coordinates for X near some point. Then we can write

$$\langle L\xi \, , \, \tau \rangle = \int_{\boldsymbol{R}^n} \overline{{}^t\tau}(x) h_F(x) (\tilde{L}(x, D) \xi(x)) p(x) dx \, ,$$

where

$$\tilde{L}(x, D) = \sum_{|\alpha| \le k} C_\alpha(x) D^\alpha$$

is a $p \times q$ matrix of partial differential operators, i.e., $C_\alpha(x)$ is a $p \times q$ matrix of C^∞ functions in $\boldsymbol{R}^n$. So we can write

$$\langle L\xi \, , \, \tau \rangle = \int_{\boldsymbol{R}^n} \sum_{|\alpha| \le k} \overline{{}^t\tau}(x) p(x) h_F(x) C_\alpha(x) D^\alpha \xi(x) dx \, .$$

By integrating by parts, since ξ and τ have compact support we obtain

$$\begin{aligned} \langle L\xi \, , \, \tau \rangle &= \sum_{|\alpha| \le k} (-1)^{|\alpha|} \int_{\boldsymbol{R}^n} D^\alpha(\overline{{}^t\tau}(x) p(x) h_F(x) C_\alpha(x)) \xi(x) dx \\ &= \int_{\boldsymbol{R}^n} \overline{{}^t(\sum_{|\alpha| \le k} \tilde{C}_\alpha(x) D^\alpha \tau(x))} h_E(x) \xi(x) p(x) dx \, , \end{aligned}$$

where $\tilde{C}_\alpha(x)$ are $q \times p$ matrices of C^∞ functions defined by the formula

$$\overline{{}^t(\sum_{|\alpha| \le k} \tilde{C}_\alpha(x) D^\alpha \tau(x))} = \sum_{|\alpha| \le k} (-1)^{|\alpha|} D^\alpha(\overline{{}^t\tau}(x) p(x) h_F(x) C_\alpha(x)) (h_E(x))^{-1} (p(x))^{-1} \, , \qquad (2.4)$$

and hence the $\tilde{C}_\alpha$ have derivatives of metric on F and of the density $p(x)$ on X. Thus, the formula (2.4) suffices to define a linear differential operator

$$L^* : \mathfrak{D}(X, F) \longrightarrow \mathfrak{D}(X, E)$$

which is the desired adjoint of L. Moreover, the symbol $\sigma_k(L^*)$ is given by the terms in (2.4) which only differentiate τ, since all other terms give lower order terms in the expression $\sum_{|\alpha| \le k} \tilde{C}_\alpha(x)$. It is easy to see that the symbol of L^* is the adjoint (up to sign) of the symbol of L by representing $\sigma_k(L)$ with respect to these local frames and computing its adjoint as a linear mapping.

3. Pseudodifferential Operators

In this section we will generalize the concept of differential operators in order to find a class of operators which will serve as almost inverses for elliptic partial differential operators as we will see in the next section. Such generalizations of differential operators are called *pseudodifferential operators.*

Recall that if U is an open set in $\boldsymbol{R}^n$, and if $p(x, \xi)$ is a polynomial in ξ of degree m with coefficients C^∞ functions in the variable $x \in U$, then we can obtain the most general linear partial differential operators in U by letting $P = p(x, D)$ be the differential operator obtained by replacing the vector $\xi = (\xi_1, \ldots, \xi_n)$ by $(-iD_1, \ldots, -iD_n)$, where we set $D_j = (\partial/\partial x_j)$ and $(-i)^{|\alpha|}D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ replaces $\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ in the polynomial $p(x, \xi)$. By using the Fourier transform, for $u \in \mathcal{D}(U)$, we can write

$$Pu(x) = \int_{\boldsymbol{R}^n} e^{i\langle x, \xi\rangle} p(x, \xi)\hat{u}(\xi)d\xi, \qquad (3.1)$$

where

$$\hat{u}(\xi) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{-i\langle x, \xi\rangle} u(x)dx$$

is the Fourier transform of u, and $x \in U$ for $u(x)$ (extending u by zero).

Remark 3.1:

1. The action of the operator P is defined by a polynomial $p(x, \xi)$ on functions in the domain U.

2. Since $\mathcal{D}(U)$ is dense in most interesting spaces it suffices to know if the action of P is continuous on such functions.

We shall now define classes of functions which possess several important properties of the polynomials considered above.

Definition 3.2: Let U be an open set in $\boldsymbol{R}^n$ and let m be any integer.

1. Let $\tilde{S}^m(U)$ be the class of C^∞ functions $p(x, \xi)$ defined on $U \times \boldsymbol{R}^n$, satisfing the following property: for any compact set $K \subset U$, and for any multiindices α, β, there exists a constant $C_{\alpha,\beta,K}$ depending on α, β, K, and $p(x, \xi)$ so that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|}, \quad x \in K, \xi \in \boldsymbol{R}^n .$$

2. Let $S^m(U)$ denote the set of $p \in \tilde{S}^m(U)$ such that

$$\lim_{\lambda \to \infty} \frac{p(x,\lambda\xi)}{\lambda^m} = \sigma_m(p)(x,\xi)$$

exists, and

$$p(x,\xi) - \rho(||\xi||) \cdot \sigma_m(p)(x,\xi) \in \tilde{S}^{m-1}(U)\,,$$

where $\rho : \boldsymbol{R} \geq 0 \to \boldsymbol{R}$ is a C^∞ function, $\rho \equiv 0$ around 0, and $\rho \equiv 1$ for $x \geq 1$. Note that $\sigma_m(p)(x,\xi)$ is the m-th order homogeneous part if $p(x,\xi)$ is a polynomial in ξ of degree m, because the lower order terms go to zero in the limit.

3. Let $\tilde{S}_0^m(U)$ denote the class of $p \in \tilde{S}^m(U)$ such that there is a compact set $K \subset U$, so that for any $\xi \in \boldsymbol{R}^n$, the function $p(x,\xi)$, considered as a function of $x \in U$, has compact support in K, i.e., $p(x,\xi)$ has uniform compact support in the x-variable. Let $S_0^m(U) = \tilde{S}_0^m(U) \cap S^m(U)$.

Note that property (1) expresses the growth in the ξ variable near ∞.

Lemma 3.3: Suppose that $p \in S^m(U)$. Then $\sigma_m(p)(x,\xi)$ is a C^∞ function on $U \times (\boldsymbol{R}^n - \{0\})$ and is homogeneous of degree m in ξ.

Proof: By Arzela Ascoli theorem, it suffices to show that for any compact subset of the form $K \times L$, where K and L are compacts in U and $(\boldsymbol{R}^n - \{0\})$, respectively, we have that the limit in Definition 3.2(2) converges uniformly on $K \times L$ and that all derivatives in x and ξ of $(p(x,\lambda\xi)/\lambda^m)$ are uniformly bounded on $K \times L$ for $\lambda \in (1,\infty)$. This follows from the estimates in Definition 3.2(1) since

$$D_x^\beta D_\xi^\alpha\left(\frac{p(x,\lambda\xi)}{\lambda^m}\right) = D_x^\beta D_\xi^\alpha p(x,\lambda\xi) \cdot \frac{\lambda^{|\alpha|}}{\lambda^m}\,,$$

and hence

$$\begin{aligned}
|D_x^\beta D_\xi^\alpha\left(\frac{p(x,\lambda\xi)}{\lambda^m}\right)| &\leq C_{\alpha,\beta,K}(1+\lambda|\xi|)^{m-|\alpha|} \cdot \frac{\lambda^{|\alpha|}}{\lambda^m} \\
&\leq C_{\alpha,\beta,K}\left(\left(\frac{1}{\lambda}\right)^{m-|\alpha|} + \cdots + |\xi|^{m-|\alpha|}\right) \\
&\leq C_{\alpha,\beta,K}(\lambda^{-1} + |\xi|)^{m-|\alpha|} \\
&\leq C_{\alpha,\beta,K} \sup_{\xi \in L}(1+|\xi|)^{m-|\alpha|} < \infty\,.
\end{aligned}$$

Therefore all derivatives are uniformly bounded. Particulary, the limit also is uniform. We still have to show the homogeneity. Let $\tau > 0$, then

$$\begin{aligned}
\sigma(p)(x,\tau\xi) &= \lim_{\lambda\to\infty} \frac{p(x,\lambda\tau\xi)}{\lambda^m} \\
&= \lim_{\lambda\to\infty} \frac{p(x,\lambda\tau\xi)}{(\tau\lambda)^m}\cdot\tau^m \\
&= \lim_{\lambda'\to\infty} \frac{p(x,\lambda'\xi)}{(\lambda')^m}\cdot\tau^m \qquad (\lambda'=\tau\lambda) \\
&= \sigma(p)(x,\xi)\cdot\tau^m .
\end{aligned}$$

Now, for $p(x,\xi) \in \tilde{S}^m(U)$, we define an operator $L(p)$ in analogy with (3.1):

$$L(p)u(x) = \int e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)d\xi\,, \qquad (3.2)$$

and we call $L(p)$ a *canonical pseudodifferential operator* of order m.

Lemma 3.4: $L(p)$ is a linear operator map

$$L(p) : \mathcal{D}(U) \longrightarrow \mathcal{E}(U)\,.$$

Proof: Linearity is clear. Let $u \in \mathcal{D}(U)$. Then we have, for any multi-index α,

$$(-i)^{|\alpha|}\xi^\alpha\hat{u}(\xi) = (2\pi)^{-n}\int e^{-i\langle x,\xi\rangle}D^\alpha u(x)dx\,,$$

and hence $|\xi^\alpha\hat{u}(\xi)|$ is bounded for any α since u has compact support. This implies that

$$|\hat{u}(\xi)| \leq C(1+|\xi|)^{-N}\,,$$

for any large N, i.e., $\hat{u}(\xi)$ goes to zero at ∞ faster than any polynomial. Thus

$$|D_x^\beta p(x,\xi)\hat{u}(\xi)| \leq C(1+|\xi|)^{m-N}$$

is the estimate for any derivatives of the integrand in (3.2). Obviously, the last equality implies that the integral in (3.2) converges nicely enough to differentiate under the integral sign. Hence $L(p)u(x) \in \mathcal{E}(U)$. It is clear from the same estimates that $L(p)$ is indeed a continuous linear mapping $\mathcal{D}(U)$ into $\mathcal{E}(U)$.

Definition 3.5: Suppose X is a differentiable manifold. We define $\mathrm{OP}_k(E, F)$ as the vector space of $\boldsymbol{C}$-linear mappings

$$T : \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, F)$$

such that there is a continuous extension of T

$$T_s : W^s(X, E) \longrightarrow W^{s-k}(X, F)$$

for all s.

Theorem 3.6: Suppose that $p \in \tilde{S}_0^m(U)$. Then $L(p)$ is in OP_m.

Proof: See R. O. Wells [8], page 126.

We now want to define pseudodifferential operators on a manifold X.

Definition 3.7: Let L be a linear map

$$L : \mathcal{D}(X) \longrightarrow \mathcal{E}(X)\,.$$

Then we say that L is a *pseudodifferential operator* of order m on X if and only if for any coordinate chart U and any open set $U' \subset\subset U$ there exists a $p \in S_0^m(U)$ (considering U as an open subset of $\boldsymbol{R}^n$) so that if $u \in \mathcal{D}(U')$, then (extending u by zero to be in $\mathcal{D}(X)$)

$$Lu|_U = L(p)u\,,$$

i.e., by restricting to the coordinate U, there is a function $p \in S_0^m(U)$ so that the operator is a canonical pseudodifferential operator as in (3.2).

Definition 3.8: Let E and F be vectors bundles over a differentiable manifold X, and let L be a linear map

$$L : \mathcal{D}(X, E) \longrightarrow \mathcal{E}(X, F)\,.$$

Then L is called a *pseudodifferential operator* of order m on X if and only if for any coordinate chart U with trivializations of E and F over U and for any open set $U' \subset\subset U$ there exists a $q \times p$ matrix (p^{ij}), $p^{ij} \in S_0^m(U)$, so that the induced map

$$L_U : \mathcal{D}(U')^p \longrightarrow \mathcal{E}(U)^q$$

is a matrix of canonical pseudodifferential operators $L(p^{ij})$, $i = 1, \ldots, q$, $j = 1, \ldots, p$ defined by (3.2), where $p = \operatorname{rank}(E)$ and $q = \operatorname{rank}(F)$.

Remark 3.9: The additional restriction in Definition 3.7 and 3.8 of the restricting the action of L_U to functions supported in $U' \subset\subset U$ is due to the fact that in general a pseudodifferential operator is not a local operator, i.e., it does not preserve supports in the sense that $\operatorname{supp}(Lu) \subset \operatorname{supp}(u)$, whereas differential operators do.

Definition 3.10: The local m-symbol of a pseudodifferential operator of order m

$$L : \mathcal{D}(X, E) \longrightarrow \mathcal{D}(X, F)$$

is the matrix

$$\sigma_m(L_U)(x, \xi) = [\sigma_m(p^{ij})(x, \xi)], \quad i = 1, \ldots, q\,, j = 1, \ldots, p$$

with respect to a coordinate chart U and trivializations of E and F over U, where the p^{ij} are in $S^m(U)$. Note that $\sigma_m(L_U)$ will also depend on $U' \subset\subset U$.

We will now study the behaviour of the local m-symbol under local diffeomorphisms in order to obtain a global m-symbol of L. Recall that if a differential operator is locally expressed as

$$L = \sum_{|\alpha| \leq m} C_\alpha(y) D_y^\alpha\,,$$

and we make a change of coordinates $x = F(y)$, then we can express the same operator in terms of the new coordinates

$$\tilde{L} = \sum_{|\alpha| \leq m} \tilde{C}_\alpha(x) D_x^\alpha$$

and

$$\tilde{L}(u(F(y))) = \sum_{|\alpha| \leq m} \tilde{C}_\alpha(F(y)) D_x^\alpha u(F(y))\,.$$

Theorem 3.11: Let $U \subset\subset \boldsymbol{R}^n$ and let $p \in S_0^m(U)$. Suppose that F is a diffeomorphism of U onto itself (in coordinates $x = F(y)$, $x, y \in \boldsymbol{R}^n$). Suppose that $U' \subset\subset U$ and define the linear map

$$\tilde{L} : \mathcal{D}(U') \longrightarrow \mathcal{E}(U)$$

by setting

$$(\tilde{L}(v))(y) = (L(p)(F^*v))(F(y)) .$$

Recall that $F^* : \mathcal{E}(U) \to \mathcal{E}(U)$ is given by $F^*v = v \circ F^{-1}$. Then there is a function $q \in S_0^m(U)$, so that $\tilde{L} = L(q)$. Moreover,

$$\sigma_m(q)(y, \eta) = \sigma_m(p)(F(y), [{}^t(\frac{\partial F}{\partial y})]^{-1}\eta) .$$

This theorem tells us that the set of pseudodifferential operators is invariant under local changes of coordinates and that the local symbols transform in a precise manner, depending on the Jacobian $(\partial F/\partial y)$ matrix.

Proof: Let $p \in S_0^m(U)$. Then we see easily from (3.2) and the Fourier inversion theorem that

$$L(p)(u(x)) = (2\pi)^{-n} \int\int e^{i\langle \xi, x - z\rangle} p(x,\xi) u(z) dz d\xi . \qquad (3.3)$$

So the purpose is to generalize this representation somewhat by allowing the function p to also depend on the variable z. Considering functions $q(x, \xi, z)$ on $\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n$, with compact support in the x-variable and z-variable, and satisfying the following conditions (as in Definition 3.2):

$$|D_\xi^\alpha D_x^\beta D_z^\gamma q(x, \xi, z)| \le C_{\alpha,\beta,\gamma}(1 + |\xi|)^{m-|\alpha|},$$

$$\lim_{\lambda\to\infty} \tfrac{q(x,\lambda\xi,x)}{\lambda^m} = \sigma_m(q)(x, \xi, x) \qquad (3.4)$$

exists, and

$$q(x, \xi, x) - \sigma_m(q)(x, \xi, x) \in \tilde{S}^{m-1}(U) .$$

We will need the next two propositions to be used in the proof of this theorem.

Proposition 3.12: Let $q(x, \xi, z)$ satisfy the conditions in (3.4) and let the operator Q be defined by

$$Qu(x) = \int e^{i\langle \xi, x-z \rangle} q(x, \xi, z) u(z) dz d\xi$$

for $u \in \mathcal{D}(\boldsymbol{R}^n)$. Then there exists a $p \in S_0^m(\boldsymbol{R}^n)$ such that $Q = L(p)$. Moreover,

$$\sigma_m(p)(x, \xi) = \lim_{\lambda \to \infty} \frac{q(x, \lambda\xi, x)}{\lambda^m}$$

and

$$p(x, \xi) = \sigma_m(p)(x, \xi) \in \tilde{S}^{m-1}(U) .$$

Proof: See R. O. Wells [8], page 130.

Suppose that we rewrite (3.3) formally as

$$L(p)(u(x)) = (2\pi)^{-n} \int \left[\int e^{i\langle \xi, x-z \rangle} p(x, \xi) d\xi \right] u(z) dz$$

and let

$$K(x, x-z) = \int e^{i\langle \xi, x-z \rangle} p(x, \xi) d\xi .$$

Then

Proposition 3.13: $K(x, w)$ is a C^∞ function of x and w provided that $w \neq 0$.

Proof: See R. O. Wells [8], page 132.

We now proceed to prove Theorem 3.11. We have that $p(x, \xi)$ has compact support in U in the x-variable. Let $\tilde{\Psi}(x) \in \mathcal{D}(U)$ be chosen so that $\tilde{\Psi} \equiv 1$ on $\text{supp}(p) \cup U'$ and we set $\Psi(y) = \tilde{\Psi}(F(y))$. We have, as in (3.3),

$$L(p)(u(x)) = (2\pi)^{-n} \int \int e^{i\langle \xi, x-z \rangle} p(x, \xi) u(z) dz d\xi ,$$

where $u \in \mathcal{D}(U')$. We let $z = F(w)$ and $v(w) = u(F(w))$, obtaining

$$L(p)(u(F(y))) = (2\pi)^{-n} \int\int e^{i\langle \xi, F(y)-F(w)\rangle} p(F(y),\xi) |\frac{\partial F}{\partial w}| v(w) dw d\xi \,,$$

where $|\partial F/\partial w|$ is the Jacobian determinant. By the mean value theorem we see that

$$F(y) - F(w) = H(y,w)\cdot(y-w)\,,$$

where $H(y,w)$ is a nonsingular matrix for w close to y and $H(w,w) = (\partial F/\partial w)w$.

Now, let $\chi_1(y,w)$ be a smooth nonnegative function such that $\chi_1(y,w) \equiv 1$ near the diagonal Δ in $U \times U$ and with support on a neighbourhood of Δ where $H(y,w)$ is invertible. Let $\chi_2 = 1 - \chi_1$. Thus we have

$$L(p)(u(F(y))) = (2\pi)^{-n} \int\int e^{i\langle \xi, H(y,w)\cdot(y-w)\rangle} p(F(y),\xi) |\frac{\partial F}{\partial y}| v(w) dw d\xi \,,$$

and by setting $\eta = {}^tH(y,w)\xi$, we obtain

$$\begin{aligned} L(p)(u(F(y))) &= (2\pi)^{-n}[\int\int e^{i\langle \eta, y-w\rangle} p(F(y), [{}^tH(y,w)]^{-1}\eta) \\ &\quad \times |\frac{\partial F}{\partial w}| \Psi(w) \frac{\chi_1(y,w)}{|H(y,w)|} v(w) dw d\eta + E(u(F(y)))\,] \\ &= (2\pi)^{-n}[\int\int e^{i\langle \eta, y-w\rangle} q_1(y,\eta,w) v(w) dw d\eta + E(u(F(y)))\,]\,, \end{aligned}$$

where $q_1(y,\eta,w) = p(F(y), [{}^tH(y,w)]^{-1}\eta)|\frac{\partial F}{\partial w}|\frac{\chi_1(y,w)}{|H(y,w)|}\Psi(w)$, and that $\Psi \in \mathcal{D}(U)$ is (as chosen above) identically 1 on a neighbourhood of $\mathrm{supp}(v(y))$. Thus $q_1(y,\eta,w)$ has compact support in the y-variable and w-variable. Moreover, it satisfies the conditions of (3.4) so we have

$$\begin{aligned} \sigma_m(q_1)(y,\eta,y) &= \lim_{\lambda\to\infty} \frac{q_1(y,\lambda\eta,y)}{\lambda^m} \\ &= \lim_{\lambda\to\infty} \frac{p(F(y), \lambda[{}^t(\frac{\partial F}{\partial y})]^{-1}\eta)}{\lambda^m} \\ &= \sigma_m(p)(F(y), [{}^t(\frac{\partial F}{\partial y})]^{-1}\eta)\,, \end{aligned}$$

and the desired growth of

$$\sigma_m(q_1)(y,\eta,y) - q_1(y,\eta,y)$$

follows from the hypothesized growth of

$$\sigma_m(p)(x,\eta) - p(x,\eta)\,.$$

We still have to worry about the term E which is corresponding to χ_2. Here we claim that E is an operator in which will give no contribution to the symbol. In fact, we have

$$\begin{aligned}
E(u(x)) &= \int\int e^{i\langle\xi,x-z\rangle}p(x,\xi)\chi_2(x,z)u(z)dzd\xi \\
&= \int\left[\int e^{i\langle\xi,x-z\rangle}p(x,\xi)\chi_2(x,z)d\xi\right]u(z)dz \\
&= \int \chi_2(x,z)K(x,x-z)u(z)dz\,,
\end{aligned}$$

where $K(x,x-z) = K(x,w) = \int e^{\langle\xi,w\rangle}p(x,\xi)d\xi$ is a (as in Proposition 3.13) C^∞ function of x and $w = x - z \neq 0$.

Also $\chi_2(x,z)$ identically vanishes near $x-z=0$, so the product $\chi_2(x,z)K(x,x-z)$ is a smooth on $U \times \boldsymbol{R}^n$. Let $y = F^{-1}(x)$ and $w = F^{-1}(z)$ be new coordinates, so we can rewrite the operator E in terms of y and w as

$$\begin{aligned}
E(u(F(y))) &= \int \chi_2(F(y),F(w))K(F(y),F(y)-F(w))u(F(w))|\frac{\partial F}{\partial w}|dw \\
&= \int W(y,w)(F^*)^{-1}u(w)dw\,,
\end{aligned}$$

where $W(y,w)$ is a C^∞ function on $U \times U$, which we rewrite as

$$= \int W(y,w)\Psi(w)(F^*)^{-1}u(w)dw\,,$$

where $\Psi \equiv 1$ on $\mathrm{supp}((F^*)^{-1}u)$. Then we have

$$E(u(F(y))) = \int M(y,w)v(w)dw\,,$$

where $v(w) = (F^*)^{-1}u(w)$ and

$$M(y,w) = \chi_2(F(y),F(w))K(F(y),F(y)-F(w))|\frac{\partial F}{\partial w}|\Psi(w).$$

Thus, $M(y,w)$ is a smooth function. Then the operator

$$E(u(F(y))) = \int M(y,w)v(w)dw$$

can be written as

$$E(u(F(y))) = \int e^{i\langle w,\xi\rangle} q_2(y,\xi)\hat{v}(\xi)d\xi,$$

where

$$q_2(y,\xi) = \int e^{i\langle y-w,\xi\rangle} M(y,w)dw,$$

and $q_2 \in \tilde{S}_0^r(U)$ for all r. To see this we rewrite q_2 as

$$q_2(y,\xi) = e^{i\langle y,\xi\rangle} \int e^{-i\langle w,\xi\rangle} M(y,w)dw.$$

Thus q_2 is (except for the factor $e^{i\langle y,\xi\rangle}$) for each fixed y the Fourier transform of a compactly supported function, and then it is easy to see (by integrating by parts) that $q_2(y,\xi)$, as a function of ξ, is rapidly decreasing at infinity, i.e.,

$$(1+|\xi|)^N |q_2(y,\xi)| \mapsto 0.$$

It immediately follows that $q_2 \in \tilde{S}_0^r(U)$for all r. Such an operator E is often referred to as a *smooth operator of order $-\infty$ with C^∞ kernel.* This implies easily that

$$\sigma_m(q_2)(y,\xi) \equiv 0.$$

Thus we can let $q = q_1 + q_2$, and we then have

$$\tilde{L} = L(q),$$

and the symbols behave correctly (here we let $q_1(y,\xi,w)$ be replaced by $q_1(y,\xi)$, as given by Proposition 3.12). The theorem is proved.

Definition 3.14: Let X be a differentiable manifold and let

$$L : \mathcal{D}(X) \longrightarrow \mathcal{E}(X)$$

be a pseudodifferential operator. Then L is said to be a *pseudodifferential operator* of order m on X if, for any choice of local coordinates chart $U \subset X$, the corresponding canonical pseudodifferential operator L_U is of order m, i.e.,

$$L_U = L(p)\,,$$

where $p \in S^m(U)$. We denote the all pseudodifferential operators of order m on X by $\mathrm{PDiff}_m(X)$.

Proposition 3.15: Suppose X is a compact differentiable manifold. If $L \in \mathrm{PDiff}_m(X)$, then $L \in \mathrm{OP}_m(X)$.

Proof: This is immediate from Theorem 3.6 and the definition of Sobolev norms on a compact manifold, using a finite covering of X by coordinate charts.

Remark 3.16: If $p \in S^m(U)$ for some $U \subset \boldsymbol{R}^n$, then $p \in S^{m+k}(U)$ for any positive k. Thus we have the natural inclusion

$$i : \mathrm{PDiff}_m(X) \hookrightarrow \mathrm{PDiff}_{m+k}(X), \quad k \geq 0\ .$$

Moreover, in this case, $\sigma_{m+k}(p) = 0$, $k > 0$.

We denote $\mathrm{Smbl}_m(X \times \boldsymbol{C}, X \times \boldsymbol{C})$ by $\mathrm{Smbl}_m(X)$ for simplicity.

Proposition 3.17: There exists a canonical linear map

$$\sigma_m : \mathrm{PDiff}_m(X) \longrightarrow \mathrm{Smbl}_m(X) \quad ,$$

which is locally defined in a coordinate chart $U \subset X$ by

$$\sigma_m(L_U)(x,\xi) = \sigma_m(p)(x,\xi)\,,$$

where $L_U = L(p)$ and where $(x,\xi) \in U \times (\boldsymbol{R}^n - \{0\})$ is a point in $T^*(U)$ expressed in the local coordinates of U.

Proof: We need to show that the local representation of $\sigma_m(L)$ transforms correctly so that it is indeed globally a homomorphism of $T^*(X) \times \boldsymbol{C}$ into itself, which is homogeneous in the cotangent vector variable of order m.

But this follows from the transformation formula for $\sigma_m(p)$, given in Theorem 3.11, under a local change of variables. The linearity of σ_m is clear.

Definition 3.18: Let E and F be vector bundles over a differentiable manifold X. We denote by $\mathrm{PDiff}_m(E,F)$ the space of pseudodifferential operators of order m mapping $\mathcal{D}(X,E)$ into $\mathcal{E}(X,F)$.

Proposition 3.19: Let E and F be as above. Then there exists a canonical linear map

$$\sigma_m : \mathrm{PDiff}_m(E,F) \longrightarrow \mathrm{Smbl}_m(E,F) \quad ,$$

which is locally defined in a coordinate chart $U \subset X$ by

$$\sigma_m(L_U)(x,\xi) = [\sigma_m(p^{ij})(x,\xi)] \, ,$$

where $L_U = [L(p^{ij})]$ is a matrix of canonical pseudodifferential operators, and where $(x,\xi) \in U \times (\boldsymbol{R}^n - \{0\})$ is a point in $T^*(U)$ expressed in the local coordinates of U.

We are now in a position to state one of the fundamental results of pseudodifferential operators on manifolds.

Theorem 3.20: Let E and F be as above. Then the following sequence is exact

$$0 \to \mathrm{K}_m(E,F) \xrightarrow{j} \mathrm{PDiff}_m(E,F) \xrightarrow{\sigma_m} \mathrm{Smbl}_m(E,F) \to 0 \quad ,$$

where σ_m is the canonical symbol map given by Proposition 3.19, and $\mathrm{K}_m(E,F)$ is the kernel of σ_m, and that j is the natural injection. Moreover, $\mathrm{K}_m(E,F) \subset \mathrm{OP}_{m-1}(E,F)$.

Proof: We need to show that σ_m is surjective and $\sigma_m(L) = 0$ implies that L is an operator of order $m-1$. We first prove the surjection of σ_m. Let $\{U_\alpha\}$ be a locally finite cover of X by coordinate charts U_α over which E and F are both trivializable. We also let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$, and let $\{\psi_\alpha\}$ be a family of functions such that $\psi_\alpha \in \mathcal{D}(U_\alpha)$, where $\psi_\alpha \equiv 1$ on $\mathrm{supp}(\phi_\alpha)$. We then let χ be a C^∞ function on $\boldsymbol{R}^n$ with $\chi \equiv 0$ near $0 \in \boldsymbol{R}^n$, and $\chi \equiv 1$ outside the unit ball.

Let $s \in \mathrm{Smbl}_m(E,F)$ be given , and write $s = \sum_\alpha \phi_\alpha s = \sum_\alpha s_\alpha$, where $s_\alpha = \phi_\alpha s$, and $\mathrm{supp}(s_\alpha) \subset \mathrm{supp}(\phi_\alpha) \subset U_\alpha$. Then by Proposition 3.19, we see that $s_\alpha = [s_\alpha^{ij}]$ is a matrix of homogeneous functions

$$s_\alpha^{ij} : U_\alpha \times (\boldsymbol{R}^n - \{0\}) \longrightarrow \boldsymbol{C},$$

and $s_\alpha^{ij}(x, r\xi) = r^m s_\alpha^{ij}$, for $r > 0$. We let $p_\alpha^{ij}(x,\xi) = \chi(\xi) s_\alpha^{ij}(x,\xi)$ so that $p_\alpha^{ij} \in S_0^m(U)$ and that

$$\sigma_m(p_\alpha^{ij}) = s_\alpha^{ij}. \qquad (*)$$

We now let

$$L_\alpha : \mathcal{D}(U_\alpha)^p \longrightarrow \mathcal{E}(U_\alpha)^q$$

be defined by $L_\alpha u = [L(p_\alpha^{ij})]u$ with the usual matrix (as in Proposition 3.19) acting on the vector u. We set $u_\alpha = \psi_\alpha u$, and we consider each u_α as a vector in $\mathcal{D}(U_\alpha)^p$ by trivializations. Thus we can define the operator L on the vector u as

$$Lu = \sum_\alpha \psi_\alpha(L_\alpha u_\alpha),$$

and it is clear that

$$L : \mathcal{D}(X,E) \longrightarrow \mathcal{E}(X,F)$$

is an element of $\mathrm{PDiff}_m(E,F)$, since locally it is represented by a matrix of canonical pseudodifferential operators of order m. Note that it is necessary to multiply by ψ_α in order to sum, since $L_\alpha u_\alpha$ is C^∞, where we consider $L_\alpha u_\alpha$ as an element of $\mathcal{E}(U_\alpha, F)$ in U_α, but it does not necessarily extend in a C^∞ manner to a C^∞ section of F over X. Therefore it remains to show that $\sigma_m(L) = s$. But this is simple, since $(\psi_\alpha L_\alpha) \in \mathrm{PDiff}_m(E,F)$ and

$$\begin{aligned} \sigma_m(\psi_\alpha L_\alpha)(x,\xi) &= \sigma_m(\psi_\alpha(x) p_\alpha^{ij}(x,\xi)) \\ &= \psi_\alpha(x) s_\alpha^{ij}(x,\xi) \quad \text{by } (*) \\ &= s_\alpha^{ij}, \end{aligned}$$

since (by assumption) $\psi_\alpha \equiv 1$ on $\mathrm{supp}(s_\alpha) \subset \mathrm{supp}(\phi_\alpha)$. It follows that

$$\sigma_m(\psi_\alpha L_\alpha) = s_\alpha,$$

and by linearity of the symbol map

$$\sigma_m(\sum_\alpha \psi_\alpha L_\alpha) = \sum_\alpha s_\alpha = s\,.$$

We have now to show that if $\sigma_m(L) = 0$, then L is an operator of order $m-1$. Note that $\sigma_\alpha(L) = 0$ for some $L \in \mathrm{PDiff}_m(E,F)$ means that in a local trivializing coordinate chart U, L has the representation $L_U = [L(p^{ij})]$, $p^{ij} \in S^m(U)$. Thus,

$$\sigma_m(L)_{|U} = \sigma_m(L_U) = [\sigma_m(p^{ij})] = 0\,.$$

By Definition 3.2(2), we see that

$$p^{ij} \in \tilde{S}^{m-1}(U)\,.$$

Hence L_U is, by Theorem 3.6, an operator of order $m-1$. Thus L will be an operator of order $m-1$.

Theorem 3.21: Let E, F, and G be vector bundles over a compact differentiable manifold X. If $L \in \mathrm{PDiff}_k(E,F)$ and $S \in \mathrm{PDiff}_m(F,G)$, then

$$SL = S \circ L \in \mathrm{PDiff}_{k+m}(E,G) \quad .$$

Moreover, $\sigma_{k+m}(SL) = \sigma_m(S) \cdot \sigma_k(L)$, where the right hand product is the composition product of the linear mapping involved.

Proof: See R.O.Wells [8], pages 138-139.

4. Elliptic Operators

Let E and F be vector bundles over a differentiable manifold X.

Definition 4.1: Let $s \in \mathrm{Smbl}_k(E,F)$. Then s is said to be *elliptic* if and only if for any $(x,\xi) \in T^*(X)$, the linear map

$$s(x,\xi) : E_x \longrightarrow F_x$$

is an isomorphism, where $\xi \neq 0$.

Note that, in particular, both E and F must have the same fibre dimension. We shall be interested in the case where $E = F$.

Definition 4.2: Let $L \in \mathrm{PDiff}_k(E, F)$. Then L is said to be *elliptic* of order k if and only if $\sigma_k(L)$ is an elliptic symbol.

Example 4.3: Let

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}^2 ,$$

where $D_{ij}^2 = \partial^2/\partial x_i \partial x_j$, the functions a_{ij} are defined on some bounded domain $M \subset \boldsymbol{R}^n$ and satisfy the condition $\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$ $\forall x \in M$ and $\forall \xi = (\xi_1, \ldots, \xi_n) \in \boldsymbol{R}^n$ for some $0 < \lambda \leq \Lambda$, and $a_{ij}(x) = a_{ji}(x)$.

Remark 4.4: If L is elliptic of order k, then L is also an operator of order $k+1$, but is not elliptic of order $k+1$ since $\sigma_{k+1}(L) = 0$.

We shall call any operator $L \in \mathrm{OP}_{-1}(E, F)$ a *smoothing operator.*

Definition 4.5: Let $\mathrm{PDiff}(E, F) = \cup_k \mathrm{PDiff}_k(E, F)$ and let $L \in \mathrm{PDiff}(E, F)$. Then $\tilde{L} \in \mathrm{PDiff}(F, E)$ is called a *pseudoinverse* for L if it has the following properties

$$\begin{aligned} &\text{(a)} \qquad L \circ \tilde{L} - I_F \in \mathrm{OP}_{-1}(F) \\ &\text{(b)} \qquad \tilde{L} \circ L - I_E \in \mathrm{OP}_{-1}(E) \quad , \end{aligned}$$

where I_E and I_F denote the identity operators on E and F, respectively.

Theorem 4.6: Let k be any integer and let $L \in \mathrm{PDiff}_k(E, F)$ be elliptic. Then there exists a pseudoinverse for L.

Proof: Let $s = \sigma_k(L)$. Then s^{-1} exists as a linear map

$$s^{-1}(x, \xi) : F_x \longrightarrow E_x$$

since s is invertible, and $s^{-1} \in \mathrm{Smbl}_{-k}(F, E)$. Now, let $\tilde{L}$ be any pseudodifferential operator in $\mathrm{PDiff}_{-k}(F, E)$ such that $\sigma_{-k}(\tilde{L}) = s^{-1}$, whose existence is guaranteed by Theorem 3.20. We have that

$$\sigma_0(L \circ \tilde{L} - I_F) = \sigma_0(L \circ \tilde{L}) - \sigma_0(I_F) ,$$

and, therefore, by Theorem 3.21 and letting $\sigma_0(I_F) = J_F$ the identity in $\mathrm{Smbl}_0(F)$

$$\begin{aligned}\sigma_0(L \circ \tilde{L}) - J_F &= \sigma_k(L) \cdot \sigma_{-k}(\tilde{L}) - J_F \\ &= J_F - J_F = 0\,.\end{aligned}$$

Thus, by Theorem 3.20, we see that

$$L \circ \tilde{L} - I_F \in \mathrm{OP}_{-1}(F) \quad .$$

Similarly, $\tilde{L} \circ L - I_E \in \mathrm{OP}_{-1}(E)$.

Definition 4.7: Let X be a differentiable manifold. Suppose $L \in \mathrm{OP}_k(E, F)$. Then we say that L is *compact* if for every s the extension

$$L_s : W^s(X, E) \longrightarrow W^{s-k}(X, F)$$

is a compact operator as a mapping of *Banach* spaces.

Proposition 4.8: Let X be a compact differentiable manifold, and let $S \in \mathrm{OP}_{-1}(E, E)$. Then S is compact as operator of order 0, i.e., an element of $\mathrm{OP}_0(E, E)$.

Proof: We have for any s the following commutative diagram

$$\begin{tikzcd} W^s(X,E) \arrow[r, "\tilde{S}_s"] \arrow[d, "S_s"'] & W^s(X,E) \\ W^{s+1}(X,E) \arrow[ur, "j"'] & \end{tikzcd} ,$$

where S_s is the extension of S, since $S \in \mathrm{OP}_{-1}(E, E)$, and we have from the fact that $\mathrm{OP}_{-1}(E, E) \subset \mathrm{OP}_0(E, E)$ that $\tilde{S}_s$ is the extension of S as $S \in \mathrm{OP}_0(E, E)$. Since j is a compact operator by Rellich's theorem (Theorem 1.4), then $\tilde{S}_s$ must also be compact. Thus, by Definition 4.7, S is compact as operator of order 0.

In the rest of this section we will let E and F be fixed Hermitian vector bundles over a compact differentiable manifold X.

Definition 4.9: Let $L \in \mathrm{PDiff}_k(E, F)$. Then we set

$$\mathcal{H}_L = \{\xi \in \mathcal{E}(X, E) : L\xi = 0\},$$

and we let

$$\mathcal{H}_L^{\perp} = \{\eta \in W^0(X, E) : \langle \xi \, , \, \eta \rangle = 0, \xi \in \mathcal{H}_L\}$$

denote the orthogonal complement in $W^0(X, E)$ of $\mathcal{H}_L$, restricted to C^∞ sections, where $\langle \xi \, , \, \eta \rangle$ is as in Definition 1.2.

Theorem 4.10: Let B be a Banach space and let S be a compact operator, $S : B \to B$. Let $T = I - S$, then

1. Ker T is finite dimensional.
2. The range $T(B)$ of T is closed, and Coker $T = B/T(B)$ is finite dimensional.

Proof: See Riesz and Sz.-Nagy [5], pages 179-182.

Definition 4.11: An operator T on a Banach space is called a *Fredholm operator* if T has finite dimensional kernel and cokernel.

Here in our applications the Banach spaces are the Sobolev spaces $W^s(X, E)$.

Theorem 4.12: Let $L \in \mathrm{PDiff}_k(E, F)$ be an elliptic pseudodifferential operator. Then there exists a pseudoinverse $\tilde{L}$ for L so that $L \circ \tilde{L}$ and $\tilde{L} \circ L$ have, for each integer s, continuous extensions as Fredholm operators: $W^s(X, F) \to W^s(X, F)$ and $W^s(X, E) \to W^s(X, E)$, respectively.

Proof: This is immediate from Theorem 4.6, Proposition 4.8, and Theorem 4.10.

We now have the important finiteness theorem for elliptic differential operators.

Theorem 4.13: Let k be any integer. Let $L \in \mathrm{PDiff}_k(E, F)$ be elliptic, and let

$$L_s : W^s(X, E) \longrightarrow W^{s-k}(X, F).$$

Then

1. $\mathcal{H}_{L_s} = \ker L_s \subset \mathcal{E}(X, E)$, i.e., $\mathcal{H}_{L_s} = \mathcal{H}_L$ for all s.
2. $\dim \mathcal{H}_{L_s} = \dim \mathcal{H}_L < \infty$.

Proof:

1. We suppose that $L \in \mathrm{PDiff}_k(E, F)$ is elliptic, and $\xi \in W^s(X, E)$ has the property that $L_s\xi = \sigma \in \mathcal{E}(X, F)$. We now want to show that $\xi \in \mathcal{E}(X, E)$ so that $\mathcal{H}_{L_s} \subset \mathcal{E}(X, E)$. If $\tilde{L}$ is a pseudoinverse for L, then $\tilde{L} \circ L - I = S \in \mathrm{OP}_{-1}(X, E)$. Now $L\xi \in \mathcal{E}(X, F)$ implies that $(\tilde{L} \circ L)\xi \in \mathcal{E}(X, E)$, and hence
$$\xi = I\xi = (\tilde{L} \circ L - S)\xi\,.$$
Since we assume that $\xi \in W^s(X, E)$ and since $(\tilde{L} \circ L)\xi \in \mathcal{E}(X, E)$, and that $S\xi \in W^{s+1}(X, E)$, it follows that $\xi \in W^{s+1}(X, E)$. Repeating this process, we see that $\xi \in W^{s+k}(X, E)$ for all $k > 0$. By Sobolev's theorem (Theorem 1.3), it follows that $\xi \in \mathcal{E}^l(X, E)$ for all $l > 0$, and hence $\xi \in \mathcal{E}(X, E)$. Thus $\mathcal{H}_{L_s} \subset \mathcal{E}(X, E)$, and $\mathcal{H}_{L_s} = \mathcal{H}_L$.

2. Let $\tilde{L}$ be a pseudoinverse for L, by Theorem 4.12, the map
$$(\tilde{L} \circ L)_s : W^s(X, E) \longrightarrow W^s(X, E)$$
has finite dimensional kernel. Moreover, we deduce from the following diagram of Banach spaces
$$\begin{tikzcd} W^s(X, E) \arrow[r, "(\tilde{L}\circ L)_s"] \arrow[d, "L_s"'] & W^s(X, E) \\ W^{s-k}(X, F)\,, \arrow[ur, "\tilde{L}_{s-k}"'] & \end{tikzcd}$$
that $\ker L_s \subset \ker(\tilde{L} \circ L)_s$. Therefore $\mathcal{H}_{L_s}$ is finite dimensional for all s, and hence $\dim \mathcal{H}_{L_s} = \dim \mathcal{H}_L < \infty$.

Note that the cokernel of L_s of this theorem is also finite dimensional, i.e., coker $L_s < \infty$. For the proof of this point we include this reference: (see, e.g., Nicolaescu [4], Theorem 9.4.7.)

Proposition 4.14: Let $L \in \mathrm{PDiff}_k(E, F)$. Then L is elliptic if and only if L^* is elliptic.

Proof: This follows immediately from Definition 4.2 and Theorem 2.13.

Theorem 4.15: Let $k \geq 0$ and let $L \in \mathrm{PDiff}_k(E, F)$ be elliptic, and suppose that $\tau \in \mathcal{H}_{L^*}^{\perp}$. Then there exists a unique $\xi \in W^k(X, E)$ such that $L\xi = \tau$ and such that ξ is orthogonal to $\mathcal{H}_L$ in $W^0(X, E)$, i.e., $\xi \in \mathcal{H}_L^{\perp}$. Moreover, if $\tau \in \mathcal{E}(X, F)$ as well, then $\xi \in \mathcal{E}(X, E)$ also.

Proof: First we will solve the equation $L\xi = \tau$, where $\xi \in W^0(X, E)$, and then it will follow from the proof of Theorem 4.13(1) that ξ is C^∞ since τ is C^∞. Consider the following diagram of Banach spaces:

$$\begin{tikzcd} W^k(X,E) \arrow[r, "L_k"] \arrow[d, leftrightarrow] & W^0(X,F) \arrow[d, leftrightarrow] \\ W^{-k}(X,E) & W^0(X,F) \arrow[l, "L^*{}_0"'] \end{tikzcd},$$

The vertical arrows indicate the duality relation between the Banach spaces indicated. Since $\mathcal{H}_{L^*}^{\perp}$ is the orthogonal complement of the kernel of the transpose, then the closure of the range is perpendicular to the kernel of the transpose. Thus $L_k(W^k(X, E))$ is dense in $\mathcal{H}_{L^*}^{\perp}$. Moreover, L_k has closed range which follows from Theorem 4.10(2). Hence the equation $L_k\xi = \tau$ has a solution $\xi \in W^k(X, E)$. By orthogonally projecting ξ onto the closed subspace $\mathrm{Ker}L_k = \mathcal{H}_L$ which is described in Theorem 4.13, we obtain a unique solution.

Definition 4.16: Let $L \in \mathrm{PDiff}_k(E) = \mathrm{PDiff}_k(E, E)$. Then we say that L is *self-adjoint* if $L = L^*$.

We now want to state the following fundamental decomposition theorem for self-adjoint elliptic operators.

Theorem 4.17: Let $k \geq 0$ and let $L \in \mathrm{PDiff}_k(E)$ be self-adjoint and elliptic. Then there exist linear maps H_L and G_L

$$H_L : \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, E)$$

$$G_L : \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, E)$$

so that

1. $H_L(\mathcal{E}(X,E)) = \mathcal{H}_L(E)$ and $\dim_C \mathcal{H}_L(E) < \infty$.

2. $L \circ G_L + H_L = G_L \circ L + H_L = I_E$, where $I_E =$ identity on E.

3. H_L and $G_L \in \mathrm{OP}_0(E)$, and, in particular, extend to bounded operators on $W^0(X,E)$.

4. $\mathcal{E}(X,E) = \mathcal{H}_L(X,E) \oplus G_L \circ L(\mathcal{E}(X,E)) = \mathcal{H}_L(X,E) \oplus L \circ G_L(\mathcal{E}(X,E))$, and this decomposition is orthogonal in $W^0(X,E)$ with respect to the inner product.

Proof: Let H_L be the orthogonal projection in $W^0(E)$ onto the closed subspace $\mathcal{H}_L(E)$, which is, by Theorem 4.13, finite dimensional. There is, as seen in the proof of Theorem 4.15, a bijective continuous map

$$L_k : W^k(X,E) \cap \mathcal{H}_L^{\perp} \longrightarrow \mathcal{H}_L^{\perp}\,.$$

By the Banach open mapping theorem, L_k has a continuous linear inverse which we denote by G_0:

$$G_0 : \mathcal{H}_L^{\perp} \longrightarrow W^k(X,E) \cap \mathcal{H}_L^{\perp}\,.$$

We extend G_0 to all of $W^0(X,E) = \mathcal{H}_L \oplus \mathcal{H}_L^{\perp}$ by letting $G_0(\xi) = 0$ if $\xi \in \mathcal{H}_L$, and noting that $W^k(X,E) \subseteq W^0(X,E)$, we see that

$$G_0 : W^0(X,E) \longrightarrow W^0(X,E)\,.$$

Moreover,

$$L_k \circ G_0 = I_E - H_L\,,$$

since $L_k \circ G_0 =$ identity on $\mathcal{H}_L^{\perp}$. Similarly,

$$G_0 \circ L_k = I_E - H_L$$

for the same reason. Since $G_0(\mathcal{E}(X,E)) \subset \mathcal{E}(X,E)$, by Theorem 4.15, we see that we can restrict the linear Banach space maps above to $\mathcal{E}(X,E)$. Also, $L_k|_{\mathcal{E}(X,E)} = L$ so $L : \mathcal{E}(X,E) \to \mathcal{E}(X,E)$ by definition. Let $G_L = G_{0|\mathcal{E}(X,E)}$, and it becomes clear that all of the conditions 1-4 are satisfied.

Remark 4.18:

1. We call the sections in $\mathcal{H}_L$, for L a self-adjoint elliptic operator, *L-harmonic sections.*

2. The operator G_L given by Theorem 4.17 is called the *Green's operator* associated to L.

5. The Cohomology of Projective Algebraic Varieties

Definition 5.1: Let $E_0, E_1, \ldots, E_N$ be a sequence of differentiable vector bundles defined over a compact differentiable manifold X. Suppose that there is a sequence of differential operators, of some fixed order k, $L_0, L_1, \ldots, L_{N-1}$ mapping as in the following sequence:

$$\mathcal{E}(E_0) \xrightarrow{L_0} \mathcal{E}(E_1) \xrightarrow{L_1} \cdots \xrightarrow{L_{N-1}} \mathcal{E}(E_N).$$

Associated with the above sequence is the associated symbol sequence (using the notation of Sec. 2)

$$0 \to \pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^* E_1 \xrightarrow{\sigma(L_1)} \cdots \xrightarrow{\sigma(L_{N-1})} \pi^* E_N \to 0,$$

where we denote by $\sigma(L_j)$ the k-symbol of the operator L_j.

Definition 5.2: The sequence of operators and vector bundles in Definition 5.1 is called a *complex* if $L_i \circ L_{i-1} = 0$, $i = 1, \ldots, N-1$. Such a complex is called an *elliptic complex* if the associated symbol sequence in Definition 5.1 is exact.

Example 5.3: A single elliptic operator $[L : \mathcal{E}(E) \to \mathcal{E}(F)] \in \mathrm{PDiff}_k(E, F)$ is a simple example of an elliptic complex.

Example 5.4: (The de Rham Complex) Suppose X is a differentiable manifold. Then the sequence of exterior differential operators and differential forms is as follows:

$$\mathcal{E}(X, \wedge^0 T^*) \xrightarrow{d} \mathcal{E}(X, \wedge^1 T^*) \xrightarrow{d} \cdots \xrightarrow{d} (X, \wedge^n T^*),$$

where $T^* = T^*(X) \otimes \boldsymbol{C}$, and the sequence of the associated 1-symbol maps

$$\wedge^0 T_x^* \overset{\sigma_1(d)(x,v)}{\longrightarrow} \wedge^1 T_x^* \overset{\sigma_1(d)(x,v)}{\longrightarrow} \cdots \overset{\sigma_1(d)(x,v)}{\longrightarrow} \wedge^n T_x^*$$

is exact, (see, e.g., Nicolaescu [4], Sections 7.1.3, and 9.4.3.) We call this elliptic complex by the *de Rham complex.*

Definition 5.5: Suppose that E is a complex as defined above. Then we let

$$H^q(E) = \frac{Z^q(E)}{B^q(E)}$$

be the *cohomology groups* (vector spaces) of the complex E, where $Z^q(E) = \mathrm{Ker} L_q : \mathcal{E}(E_q) \to \mathcal{E}(E_{q+1})$ and $B^q(E) = \mathrm{Im} L_{q-1} : \mathcal{E}(E_{q-1}) \to \mathcal{E}(E_q)$, $q = 0, 1, \ldots, N$. For this definition to make sense, we make the convention that $L_{-1} = L_N = E_{-1} = E_{N+1} = 0$, i.e., we make a trivial extension to a complex larger at both ends.

Let E be an elliptic complex, and equip each vector bundle E_j in E with a Hermitian metric and the corresponding Sobolev space structures as in Sec. 1. Associated with each operator $L_j : \mathcal{E}(E_j) \to \mathcal{E}(E_{j+1})$, we have the adjoint operator $L_j{}^* : \mathcal{E}(E_{j+1}) \to \mathcal{E}(E_j)$. Then we define the *Laplacian operators* of the elliptic complex E by

$$\Delta_j = L_j{}^* L_j + L_{j-1} L_{j-1}{}^* : \mathcal{E}(E_j) \longrightarrow \mathcal{E}(E_j), \qquad j = 0, 1, \ldots, N,$$

and these operators are well-defined elliptic operators of order $2k$. Moreover, each Δ_j is self-adjoint which follows easily from the fact that $(L_j{}^*)^* = L_j$ and that the adjoint operation is linear. Since each Δ_j is self-adjoint and elliptic we can, by Theorem 4.17, associate to each Laplacian operator a Green's operator G_{Δ_j}, which we shall denote by G_j. Moreover, we let

$$\mathcal{H}(E_j) = \mathcal{H}_{\Delta_j} = \mathrm{Ker}\Delta_j : \mathcal{E}(E_j) \to \mathcal{E}(E_j)$$

be the Δ_j-harmonic sections, and let

$$H_j : \mathcal{E}(E_j) \longrightarrow \mathcal{E}(E_j)$$

be the orthogonal projection onto the closed subspace $\mathcal{H}(E_j)$.

Definition 5.6: Let $\mathcal{E}(E) = \sum_{j=0}^{N} \oplus \mathcal{E}(E_j)$ denote the graded vector space. Then we define operators L, L^*, Δ, G, H on $\mathcal{E}(E)$, by letting

$$L(\xi) = L(\xi_0 + \cdots + \xi_N) = L_0\xi_0 + \cdots + L_N\xi_N ,$$

where $\xi = \xi_0 + \cdots + \xi_N$ is the decomposition of $\xi \in \mathcal{E}(E)$ into homogeneous components corresponding to the above grading. The other operators are defined similarly. We have the formal relations still holding:

$$\Delta = LL^* + L^*L$$
$$I = H + G\Delta = H + \Delta G ,$$

which follows from the identities in each of the graded components, coming from Theorem 4.17. Note that L and L^* are operators of degree $+1$ and -1, respectively, and that Δ, G, and H are operators of degree 0.

We extend the inner product on $\mathcal{E}(E_j)$ to $\mathcal{E}(E)$ in the usual Euclidean manner, i.e.,

$$\langle \xi, \eta \rangle_E = \sum_{j=0}^{N} \langle \xi_j, \eta_j \rangle_{E_j} .$$

We shall denote by $\mathcal{H}(E) = \sum_{j=0}^{N} \oplus \mathcal{H}(E_j)$ the total space of Δ-harmonic sections.

Theorem 5.7: Let $(\mathcal{E}(E), L)$ be an elliptic complex equipped with an inner product. Then

1. There is an orthogonal decomposition

$$\mathcal{E}(E) = \mathcal{H}(E) \oplus LL^*G\mathcal{E}(E) \oplus L^*LG\mathcal{E}(E) .$$

2. The following commutation relations are valid:

   (a) $I = H + \Delta G = H + G\Delta$.

   (b) $HG = GH = H\Delta = \Delta H = 0$.

   (c) $L\Delta = \Delta L$, $L^*\Delta = \Delta L^*$.

   (d) $LG = GL$, $L^*G = GL^*$.

3. $\dim \mathcal{H}(E) < \infty$, and there is a canonical isomorphism

$$\mathcal{H}(E_j) \cong H^j(E) .$$

Proof:

1. From Theorem 4.17(4) we obtain immediately the orthogonal decomposition
$$\begin{aligned}\mathcal{E}(E) &= \mathcal{H}(E) \oplus \Delta \circ G\mathcal{E}(E) \\ &= \mathcal{H}(E) \oplus (LL^* + L^*L)G\mathcal{E}(E)\,.\end{aligned}$$
We need to show that $LL^*G\mathcal{E}(E)$ and $L^*LG\mathcal{E}(E)$ are orthogonal. Suppose that $\xi, \eta \in \mathcal{E}(E)$. Then consider the inner product
$$\langle LL^*G\xi, L^*LG\eta\rangle = \langle L^2L^*G\xi, G\eta\rangle\,,$$
and the latter inner product vanishes since $L^2 = 0$.

2. (a) and (b) follow from Theorem 4.17. (c) follows from Definition 5.6. To show that (d) is valid we shall first have the following proposition.

Proposition 5.8: Let $\xi \in \mathcal{E}(E)$. Then $\Delta\xi = 0$ if and only if $L\xi = L^*\xi = 0$. Moreover, $LH = HL = L^*H = HL^* = 0$.

Proof: See R. O. Wells [8], page 153.

Using the above proposition and the construction of G, we observe that both L and G vanish on $\mathcal{H}(E)$. Therefore it suffices to show that $LG = GL$ on the orthogonal of $\mathcal{H}(E)$. It follows from the decomposition in Theorem 4.17(4) that any smooth ξ in the orthogonal of $\mathcal{H}(E)$ is of the form $\xi = \Delta\varphi$ for some φ in $\mathcal{E}(E)$. Therefore we must show that $LG\Delta\varphi = GL\Delta\varphi$ for all $\varphi \in \mathcal{E}(E)$. By using $I = H + G\Delta$, we then have
$$\begin{aligned}L\varphi &= HL\varphi + G\Delta L\varphi \\ &= HL\varphi + GL\Delta\varphi\,,\end{aligned}$$
since $L\Delta = \Delta L$. We also have
$$L\varphi = LH\varphi + LG\Delta\varphi\,.$$

Therefore we obtain
$$GL\Delta\varphi - LG\Delta\varphi = LH\varphi - HL\varphi$$

$$GL\Delta\varphi = LG\Delta\varphi\,,$$

since, by Proposition 5.8, we know that $LH\varphi = HL\varphi = 0$.

3. The finiteness follows from Theorem 4.17(1). To prove the desired isomorphism we recall that $H^q(E) = Z^q(E)/B^q(E)$, as in Definition 5.5, and let
$$\Psi : Z^q(E) \longrightarrow \mathcal{H}(E_q)$$
be defined by $\Psi(\xi) = H(\xi)$. It follows from Proposition 5.8 that Ψ is a surjective linear map. Then we have to show that $\mathrm{Ker}\Psi = B^q(E)$. Suppose that $\xi \in Z^q(E)$ and $H(\xi) = 0$. By the decomposition in part (1) of this theorem, we obtain
$$\xi = H\xi + LL^*G\xi + L^*LG\xi\,.$$
Since $H\xi = 0$ and $LG = GL$ we obtain $\xi = LL^*G\xi$, and hence $\xi \in B^q(E)$.

Example 5.9: Let $(\mathcal{E}^*(X), d)$ be the de Rham complex on a differentiable manifold X. Let us assume that X is compact and that there is a metric on X. This induces an inner product on $\wedge^p T^*(X)$ for each p, and hence $(\mathcal{E}^*(X), d)$ becomes an elliptic complex with an inner product. We denote the associated Laplacian by $\Delta = \Delta_d = dd^* + d^*d$. Let

$$\mathcal{H}^r(X) = \mathcal{H}_\Delta(\wedge^r T^*(X))$$

be the vector space of *harmonic forms* on X. By the de Rham's theorem, we have
$$H^r(X, \boldsymbol{C}) \equiv H^r(\mathcal{E}^*(X))\,,$$
(see, e.g., Warner [7], pages 205-207.) We shall let $H^r(X, \boldsymbol{C})$ denote the de Rham group when we work on a differentiable manifold, making the isomorphisms above an identification. We thus obtain by Theorem 5.7(3) that

$$H^r(X, \boldsymbol{C}) \equiv \mathcal{H}^r(X)\,.$$

This means that for each de Rham cohomology class $\sigma \in H^r(X, \boldsymbol{C})$ there exists a unique harmonic form φ, and hence the de Rham cohomology is finite dimensional.

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