Aspects of D-Branes as BPS monopoles

Barrett, Jessica K.

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Jessica K. Barrett

A Thesis presented for the degree of
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Centre for Particle Theory
Department of Mathematical Sciences
University of Durham
England

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Dedicated to

My parents, Barbara and Richard, my brother Jon, and my boyfriend Christos.

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Abstract

We investigate some of the properties of D-brane configurations which behave as BPS monopoles. The two D-brane configurations we will study are the enhançon and D-strings attached to D3-branes.

We will start by investigating D3-branes wrapped on a K3 manifold, which are known as enhançons. They look like regions of enhanced gauge symmetry in the directions transverse to the branes, and therefore behave as BPS monopoles. We calculate the metric on moduli space for n enhançons, following the methods used by Ferrell and Eardley for black holes. We expect the result to be the higher-dimensional generalisation of the Taub-NUT metric, which is the metric on moduli space for n BPS monopoles.

Next we will study D-strings attached to D3-branes; the ends of the D-strings behave as BPS monopoles of the worldvolume gauge theory living on the D3-branes. In fact the D-string/D3-brane system is a physical realisation of the ADHMN construction for BPS monopoles. We aim to test this correspondence by calculating the energy radiated during D-string scattering, working with the non-Abelian Born-Infeld action for D-strings. We will then compare our result to the equivalent monopole calculation of Manton and Samols.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, the University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the declaration or in the text. Chapter 1 contains a review of preliminary material. Chapters 2, 3 and 4 also contain some sections of review material, which are clearly marked as such in the text. The remainder of chapter 2 is based on collaboration with Prof. Clifford Johnson, and is published in [1]. The remainder of chapter 3 is based on collaboration with Dr. Peter Bowcock, and is published in ref. [2]. The remainder of chapter 4 is based on unpublished work in collaboration with Dr. Peter Bowcock.

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Chapter 1

Introduction

This thesis is based on the work we have carried out concerning D-brane configurations which behave like BPS monopoles. Since the discovery of D-branes in string theory, and the realisation that they can be alternatively interpreted as soliton solutions of supergravity, it has been apparent that our knowledge of branes of all types is crucial for our understanding of string theory and M-theory. Studying the properties of D-branes should deepen our understanding of string theory at strong coupling, and give us more insight into the nature of M-theory. Also, the construction of D-brane configurations allows us to construct multi-dimensional gauge theories in string theory. Therefore an understanding of the many and varied ways in which D-branes interact with one another is an essential step towards the goal of constructing a realistic model from string theory.

Research has shown that there are many instances in which a D-brane configuration behaves like a magnetic monopole. In particular we will be investigating here the enhançon, and D-strings attached to D3-branes. Since much is already known about magnetic monopoles, it is natural for us to use what we know to develop a deeper understanding of D-branes.

In this introductory chapter we will review some basic facts about the properties of D-branes and monopoles. In chapter 2 we focus on the case of the enhançon, and describe our calculation of the metric on moduli space for many enhançons. In chapters 3 and 4 we investigate the energy radiated during the scattering of D-strings stretched between D3-branes. We compare the result to the calculation of

Manton and Samols in ref. [80] for the energy radiated during monopole scattering.

1.1 Properties of D-Branes

We discuss here some of the basic aspects of D-branes; how they arise as BPS states in string theory, and their complementary description as soliton states of supergravity. All the material from this section is reviewed in refs. [3] and [4]. See also refs. [5] and [6] for more compact reviews of D-branes.

1.1.1 D-Branes in String Theory

A Dp-brane in string theory is defined to be a surface with p spacelike dimensions and one timelike dimension on which the ends of open strings are constrained to lie. Historically D-branes in string theory were discovered by the action of T-duality on the bosonic string (see ref. [7]). T-duality is a duality between string theory compactified on a circle of radius R and string theory compactified on a circle of radius $\tilde{R} = \alpha'/R$.[1] Let $X^\mu(\tau, \sigma)$ be the fields which describe the embedding of the string worldsheet, with parameters τ and σ, into the target space (for bosonic string theory $\mu = 0, \ldots, 25$). For a closed string, compactification on a circle corresponds to the identity

$$X^{25}(\sigma + 2\pi, \tau) \equiv X^{25}(\sigma, \tau) + 2\pi R m \ , \tag{1.1.1}$$

where m is an integer called the winding number; it is the number of times the closed string has wound around the extra dimension. The left-moving and right-moving momenta of the bosonic string, p_L^{25} and p_R^{25}, are discretised and contain extra components due to the winding number:

$$p_L^{25} = \frac{n}{R} + \frac{mR}{\alpha'} \ , \quad p_R^{25} = \frac{n}{R} - \frac{mR}{\alpha'} \ . \tag{1.1.2}$$

where $n \in \mathbb{Z}$ labels the momentum states. In the limit $R \to \infty$ the momentum states with $n < R$ become light, while the winding states become so massive as

[1]Here α' is the universal Regge parameter, which has dimension L^2. It sets the fundamental string tension $T_F = 1/2\pi\alpha'$.

to be irrelevant; we have a 26-dimensional theory, as we would expect. In the T-dual theory $R \to 0$, and the momentum states become massive, indicating the disappearance of the compactified dimension, but the winding states become light; a new uncompactified dimension has opened up. So we see that closed strings in 26 dimensions are T-dual to closed strings in 26 dimensions. However, open strings are unable to wind around the compactified dimension, and so there is no open string equivalent of the winding number m. So there is no uncompactified dimension which opens up in the limit $R \to 0$, and we see that open strings in 26 dimensions are T-dual to open strings in 25 dimensions. For a theory containing both open and closed strings there appears to be a discrepancy here; does the T-dual theory have 25 or 26 dimensions? This discrepancy is resolved by observing that under T-duality the usual Neumann boundary condition for the ends of the open string gets changed into a Dirichlet boundary condition of the form

$$\tilde{X}^{25}(\tau, \sigma = \pi) - \tilde{X}^{25}(\tau, \sigma = 0) = 2\pi l \tilde{R} \ , \tag{1.1.3}$$

where $\sigma = 0$ and $\sigma = \pi$ are the ends of the open string, $\tilde{X}^{\mu}$ denotes a T-dualised direction, and l is an integer. This means that the ends of the open string are fixed in the T-dualised direction, and so they must end on a surface with 24 spacelike dimensions, which we call a D24-brane. Similarly, T-dualising on a direction parallel to a Dp-brane produces a D$(p-1)$-brane, and T-dualising on a direction transverse to a Dp-brane produces a D$(p+1)$-brane. The D25-brane, which fills all the 26 dimensions of bosonic string theory, corresponds to a theory in which the open strings are unconstrained.

In the spectrum of open string theory there is a massless vector which has a $U(1)$ gauge invariance. This gauge invariance manifests itself as follows

$$|\phi\rangle \sim |\phi\rangle + \lambda|\psi\rangle \ , \tag{1.1.4}$$

where $|\phi\rangle$ is any open string state, and $|\psi\rangle$ is a 'spurious' state, which can be added without physical effect since it is orthogonal to all other states and null. In the worldvolume of a single D-brane the open string therefore corresponds to the massless gauge boson of a $U(1)$ field theory living on the brane. The string is massless because it can achieve vanishing length. Let us consider what happens when we allow

N D-branes to coincide. There is now a choice of D-branes for the open string to end on; there are N^2 choices for each string, and all are massless, corresponding to the gauge bosons of a $U(N)$ gauge theory (see refs. [8] and [9]). The gauge theory can be broken by separating the D-branes; the strings stretched between two separated branes become massive, thus breaking the gauge group. This is equivalent to the Higgs mechanism in $SU(N)$ gauge theory, where the gauge bosons acquire masses due to the non-zero expectation value of the Higgs field. We will comment further on how the Higgs mechanism works in the D-brane context later on. Since the gauge vector is constrained to the worldvolume of the D-brane, we will henceforth denote it by A_α, where α is an index corresponding to a D-brane direction, $\alpha = 0, \ldots, p$. We will denote the corresponding two-form gauge field strength by $F_{\alpha\beta}$, where $F = dA$.

There are other fields living on the worldvolume of a Dp-brane whose dynamics we must take into account. We can see that this must be the case by considering the T-duality transformation of a D25-brane. The gauge vector A_α of a D25-brane has 26 components, $\alpha = 0, \ldots, 25$. Under T-duality on the X^{25} direction a D25-brane transforms into a D24-brane, whose gauge field has 25 components, $\alpha = 0, \ldots, 24$. The extra component of the D25-brane gauge field, A_{25}, transforms into a scalar field, which we will denote Φ^{25}

$$\begin{pmatrix} A_0 \\ \vdots \\ A_{24} \\ A_{25} \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \\ \vdots \\ A_{24} \\ \Phi^{25} \end{pmatrix}, \tag{1.1.5}$$

If we T-dualise further we find that a Dp-brane has $(25-p)$ scalar fields living on its worldvolume, which we will denote Φ^i, $i = p+1, \ldots, 25$. Since these fields originate from components of A_α, they belong to the adjoint representation of the gauge group on the D-branes. To see the role the Φ^i play, let us consider N coincident Dp-branes; from the discussion of the previous paragraph the gauge group on these branes is $U(N)$. If we compactify the direction X^p on a circle of radius R, then we can include a constant gauge field

$$A_p = \mathrm{diag}\{\theta_1, \theta_2, \ldots, \theta_\mathrm{N}\}/2\pi\mathrm{R}\ . \tag{1.1.6}$$

This gauge field is in the diagonal subgroup of $U(N)$, namely $U(1)^N$. Locally it is pure gauge, but when we gauge it away the charged fields pick up a phase

$$\mathrm{diag}\{e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_N}\} . \tag{1.1.7}$$

An analysis of the open string mode expansion reveals that an open string which has charge +1 under $U(1)_i$ has its Dirichlet boundary condition (1.1.3) changed to

$$\tilde{X}^p(\pi) - \tilde{X}^p(0) = (2\pi l + \theta_i)\tilde{R} . \tag{1.1.8}$$

From equation (1.1.6) we can write, up to an arbitrary additive constant,

$$\tilde{X}^p = \theta_i \tilde{R} = 2\pi\alpha' A_{p,ii} . \tag{1.1.9}$$

So we see that the scalar field Φ^p is related to the position of the D-brane in the transverse direction $\tilde{X}^p$ as follows

$$\Phi^p = \frac{\tilde{X}^p}{2\pi\alpha'} . \tag{1.1.10}$$

More precisely, the N eigenvalues of the field Φ^j represent the positions of the N D-branes in the direction x^j as follows

$$X^j_a = 2\pi\alpha' \Phi^j_a , \tag{1.1.11}$$

where X^j_a is the position of the a^{th} D-brane in the x^j direction, and Φ^j_a denotes the a^{th} eigenvalue of the field Φ^j.

We can now see where the Higgs mechanism, which we mentioned above, comes from. From equation (1.1.10) we can see that separating D-branes in the X^i direction corresponds to giving a vacuum expectation value to the field Φ^i. Therefore it is Φ^i which plays the role of the Higgs field, breaking the gauge group when the branes are separated.

In our discussion so far we have considered the fields originating from the open strings, but not those originating from the closed strings. Closed strings can penetrate all of the 26 dimensions of the bosonic theory, and therefore they should be considered as background fields to the worldvolume of the D-brane. The massless part of the closed string spectrum contains the background metric, $G_{\mu\nu}$, the anti-symmetric Kalb-Ramond field, $B_{\mu\nu}$, and the dilaton Φ (not to be confused with the transverse fields Φ^j which we discussed above). The vacuum expectation value of the dilaton field, which we will denote Φ_0, sets the string coupling g_s as follows

$$g_s = e^{\Phi_0} \; . \tag{1.1.12}$$

From now on we will turn our attention away from the bosonic string theory to superstring theory in ten dimensions. The arguments used so far concerning D-branes in bosonic string theory all apply to the superstring theory as well. The fields G, B and Φ come from the massless part of the NS-NS sector of closed superstring theory. In addition there are fields coming from the massless part of the R-R sector. For type IIA superstring theory these are the R-R fields $C^{(1)}$, $C^{(3)}$ (where $C^{(p)}$ denotes a p-form field), and their Hodge duals $C^{(5)}$ and $C^{(7)}$. In type IIB string theory we have $C^{(0)}$, $C^{(2)}$ and their Hodge duals $C^{(6)}$, $C^{(8)}$, and also the self-dual field $C^{(4)}$. It was shown in ref. [10], using a tadpole calculation, that the Dp-brane is a source for the R-R field $C^{(p+1)}$ (i.e. the action containing $C^{(p+1)}$ is a generalistion of the term $e \int A_\mu v^\mu$ from electromagnetism, where e is the electric charge and v^μ is the velocity - see section 1.2 for more details). In type II string theory the field $C^{(7-p)}$ is related to the field $C^{(p+1)}$ by Hodge duality, $dC^{(7-p)} = F^{(8-p)} = *F^{(p+2)} = *dC^{(p+1)}$. We say that the D$p$-brane is a magnetic source for $C^{(7-p)}$. In type IIA string theory there exist Dp-branes for $p = 0, 2, 4, 6$ and 8. In type IIB string theory there exist Dp-branes for $p = -1, 1, 3, 5, 7, 9$. The D8-brane and D9-brane, for which there are no corresponding R-R fields, are special cases which we will not be considering here.

The actions describing the dynamics of the worldvolume fields are the Born-Infeld and Chern-Simons actions. We will review these actions both for the Abelian case and for the non-Abelian case in section 1.2.

Let us consider the tension τ_p and the Ramond-Ramond charge μ_p of a Dp-brane. These can be calculated using the vacuum cylinder diagram as follows. This diagram corresponds to the exchange of a closed string between two parallel D-branes (it looks like a cylinder joining the two D-branes, hence its name). Its amplitude can be calculated by a tree-level closed string diagram computation. But this diagram also corresponds to an open string going round in a loop. So its amplitude can also be calculated by a one-loop open string diagram computation. In fact the amplitude is zero, which is to be expected seeing as the open string calculation is a supersymmetric vacuum diagram. However, the closed string amplitude can be separated into terms describing the exchange of NS-NS states, and terms describing the exchange of R-R states. Equating coefficients with the corresponding terms in the open string amplitude leads to two equations. The equation relating to NS-NS terms gives an expression for τ_p, since the NS-NS fields of the closed string couple to τ_p. The expression is

$$\tau_p = g_s^{-1}(2\pi)^{-p}(\alpha')^{-\frac{p+1}{2}} . \tag{1.1.13}$$

The equation relating to R-R terms gives an expression for μ_p, which is

$$\mu_p = (2\pi)^{-p}(\alpha')^{-\frac{p+1}{2}} . \tag{1.1.14}$$

So the tension and charge of a Dp-brane are related as follows

$$\tau_p = g_s^{-1}\mu_p . \tag{1.1.15}$$

This equality between charge and tension resembles a BPS bound. In fact an analysis of the supersymmetries preserved by a D-brane state shows that exactly half of the supersymmetries present in the background are broken, and so it is indeed a BPS state. The amplitude of the vacuum cylinder diagram being zero is also indicative of a BPS state; the NS-NS diagrams, which correspond to the attractive gravity and scalar forces, cancel the R-R diagrams, which correspond to repulsive R-R forces, so that there is no net force between two parallel D-branes.

1.1.2 D-Branes as Supergravity Solitons

In the previous section we described how D-branes arise in string theory as surfaces on which open strings can end. In this section we describe the alternative formulation of D-branes, as soliton solutions of supergravity. These solutions can be considered as solitons in the sense that they are localised in the transverse directions. See ref. [11] for a review of D-branes from the supergravity point of view.

Supergravity arises from string theory by taking the low energy limit $\alpha' \to 0$ (i.e. the limit in which all massive modes become infinitely massive and can be neglected). The low energy limit of string theory is ten-dimensional supergravity, from ref. [12]. The type IIA string theory contains only even R-R forms; the bosonic part of the corresponding supergravity action in the string frame is

$$\begin{aligned} S_{IIA} &= \frac{1}{2\kappa_0^2}\int d^{10}x\,(-G)^{1/2}\left\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^{(3)})^2\right]\right. \\ &\left.-\frac{1}{4}(G^{(2)})^2-\frac{1}{48}(G^{(4)})^2\right\}-\frac{1}{4\kappa_0^2}\int B^{(2)}dC^{(3)}dC^{(3)}\,, \end{aligned} \tag{1.1.16}$$

where

$$2\kappa_0^2 g_s^2 = 16\pi G_N \tag{1.1.17}$$

sets the ten-dimensional Newton's constant G_N. Here $H^{(3)}$, $G^{(2)}$ and $G^{(4)}$ are defined in terms of the NS-NS and R-R field as follows

$$H^{(3)} = dB^{(2)}\,, \tag{1.1.18}$$

$$G^{(2)} = dC^{(1)}\,, \quad G^{(4)} = dC^{(3)} + H^{(3)}\wedge C^{(1)}\,. \tag{1.1.19}$$

The type IIB string theory contains only odd R-R forms; the equivalent action is

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_0^2}\int d^{10}x\,(-G)^{1/2}\left\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^{(3)})^2\right]\right. \\ &\left.-\frac{1}{12}(G^{(3)}+C^{(0)}H^{(3)})^2-\frac{1}{2}(dC^{(0)})^2-\frac{1}{480}(G^{(5)})^2\right\} \\ &-\frac{1}{4\kappa_0^2}\int\left(C^{(4)}+\frac{1}{2}B^{(2)}-C^{(2)}\right)G^{(3)}H^{(3)}\,, \end{aligned} \tag{1.1.20}$$

where $H^{(3)}$ is defined in (1.1.18) and

$$G^{(3)} = dC^{(2)}\,, \quad G^{(5)} = dC^{(4)} + H^{(3)}C^{(2)}\,, \tag{1.1.21}$$

and $C^{(0)}$ is the R-R scalar. We also need to impose the self-duality constraint on $F^{(5)} = dC^{(4)}$ by hand in the equations of motion

$$F^{(5)} = *F^{(5)}\,. \tag{1.1.22}$$

The solution to the supergravity equations of motion which corresponds to N coincident D-branes is, from ref. [13],

$$ds^2 = Z_p^{-1/2}\eta_{\alpha\beta}\,dx^\alpha dx^\beta + Z_p^{1/2}dx^i dx^i\,, \tag{1.1.23}$$

$$e^{2\Phi} = g_s^2\,Z_p^{\frac{3-p}{2}}\,, \tag{1.1.24}$$

$$C^{(p+1)} = (Z_p^{-1}-1)g_s^{-1}dx^0\wedge\cdots\wedge dx^p\,, \tag{1.1.25}$$

where, as before, $\alpha, \beta = 0, \ldots, p$ are the D-brane directions, and $i = p+1, \ldots, 9$ are the transverse directions. The function Z_p is a harmonic function given by

$$Z_p = 1 + \left(\frac{r_p}{r}\right)^{7-p} , \tag{1.1.26}$$

where

$$r^2 = \sum_i x^i x^i , \tag{1.1.27}$$

and

$$r_p^{7-p} = d_p (2\pi)^{p-2} g_s N \alpha'^{\frac{7-p}{2}} , \quad d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma\left(\frac{7-p}{2}\right) . \tag{1.1.28}$$

The above solution has the appropriate symmetries of (p+1)-dimensional Poincaré invariance in the brane directions and $SO(9-p)$ rotational symmetry in the transverse directions. In fact the solution we have described above is not the solution to the pure supergravity equations of motion; we must also include in the action the source terms for the tension and the R-R charge of the D-branes. These are given by the Dirac-Born-Infeld action and the Chern-Simons action respectively, which we will discuss in section 1.2.

The supergravity solution (1.1.23) - (1.1.25) is valid providing the lengthscale of the solution, given by r_p, is large compared to the string length $l_s = \sqrt{\alpha'}$ (recall that superstring theory reduces to supergravity in the limit $\alpha' \to 0$, i.e. $l_s \to 0$). From (1.1.28) we have

$$r_p^{7-p} \sim g_s N \alpha'^{(7-p)/2} , \tag{1.1.29}$$

and so the supergravity solution is valid as long as $g_s N \gg 1$. We also need the string coupling to be weak for the supergravity solution to be valid, and so we require $N \gg 1$. For open string perturbation theory each loop in an open string Feynman diagram carries a factor of $g_s N$ (the N comes from a trace over the gauge indices). So open string perturbation theory is valid when $g_s N \ll 1$. Therefore what we have are two descriptions of open string theory which are valid in complementary regions of the parameter space. In open string perturbation theory D-branes can be treated as objects in flat background, whereas in supergravity the curvature of spacetime must be taken into account.

1.2 D-Brane Worldvolume Actions

We devote this section to a description of the Dirac-Born-Infeld and Chern-Simons actions, which describe the dynamics of the fields living on the worldvolume of D-branes. A detailed knowledge of the properties of these actions will be necessary for the chapters which follow this one. We will begin in section 1.2.1 by describing the Abelian D-brane actions, and then in section 1.2.2 we will move on to the less familiar non-Abelian D-brane actions. In section 1.2.3 we will describe some corrections to the Dirac-Born-Infeld and Chern-Simons actions which will be necessary for our discussion of the enhançon mechanism in section 1.4.1. Most of the material from this section can be found in refs. [3] and [4].

1.2.1 Abelian D-Brane Actions

Here we will review the actions which describe the dynamics of a single D-brane. The gauge group on the brane is therefore the Abelian group $U(1)$, and the fields living on the brane, A_α and Φ^i, commute with one another. We will consider only the bosonic fields of the ten-dimensional type II superstring theory.

We start with the action describing the coupling of the D-brane fields to the NS-NS background fields, which will turn out to be the Dirac-Born-Infeld action. If we introduce the coordinates ξ^α on the brane, then the metric on the worldvolume of the brane is the background metric pulled-back to the brane's worldvolume. This is given by

$$P[G]_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta} G_{\mu\nu} \ , \tag{1.2.1}$$

where we have used the notation P to denote the pull-back. Recall from (1.1.11) that for the transverse fields Φ^i we have

$$X^i = 2\pi\alpha'\Phi^i \ , \tag{1.2.2}$$

and so the pull-back of the metric describes the coupling of the background metric to the transverse fields. The appropriate action for the metric is given by the higher-dimensional equivalent of the action for a point particle; it is the tension of the brane multiplied by the higher-dimensional 'volume' of the surface swept out by the

D-brane's worldvolume. This action is given by

$$-T_p \int d^p\xi \sqrt{-\det\left(P[G]\right)}\ . \tag{1.2.3}$$

We also need to introduce the appropriate couplings to the fields B, Φ and A into the action (1.2.3). We will discuss first the coupling to $F = dA$ (see ref. [14]). Consider a D2-brane in the directions X^1 and X^2, with a constant gauge field F_{12}. One can fix the gauge such that the only non-zero component of A is $A_2 = X^1 F_{12}$. T-dualising in the X^2 direction, and using the relation (1.1.9)

$$\tilde{X}^2 = 2\pi\alpha' X^1 F_{12}\ , \tag{1.2.4}$$

i.e. the result is a D1-brane which is tilted in the X^1-X^2 plane. The action for the D1-brane is

$$S \sim \int d\xi = \int dX^1 \sqrt{1 + (\partial_1 \tilde{X}^2)^2} = \int dX^1 \sqrt{1 + (2\pi\alpha' F_{12})^2}\ . \tag{1.2.5}$$

Generalising the above result, the appropriate factor of F in the action of a Dp-brane is

$$S_p = -T_p \int d^{p+1}\xi\ \sqrt{-\det\left(P[G]_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}\ . \tag{1.2.6}$$

The dependence of the D-brane action on B can be deduced from the combination of B and F in the string world-sheet action, which is given by

$$\frac{1}{2\pi\alpha'}\int_{\mathcal{M}} B + \int_{\partial\mathcal{M}} A\ . \tag{1.2.7}$$

It turns out that the combination of the fields B and F which is invariant under the gauge transformations of both B and A is $(B + 2\pi\alpha' F)$ (see ref. [15]). So the required action for a Dp-brane is

$$S^p_{DBI} = -T_p \int d^{p+1}\xi\, e^{-\Phi} \sqrt{-\det\left(P[G]_{\alpha\beta} + P[B]_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}\ , \tag{1.2.8}$$

which is the Dirac-Born-Infeld action. The factor of the dilaton in (1.2.8) gives the appropriate factor of the string coupling, $g_s = e^{\Phi_0}$, since the action is also the action for open string scattering at tree level. The tension T_p in (1.2.8) is related to the physical tension τ_p by

$$T_p = e^{\Phi_0}\tau_p = g_s \tau_p\ . \tag{1.2.9}$$

This action can also be deduced using the fact that the D-brane fields originated from the ends of open strings, whose action is a sigma model action (see ref. [18]).

The Dirac-Born-Infeld action (1.2.8) is correct to all orders in α', but it is only accurate for slowly varying field strengths, since derivatives of F have been neglected (see ref. [19]).

The T-duality rules for the NS-NS fields can be deduced from the action for an open string in a curved background, which is a sigma model action. They are given by

$$\begin{aligned}
\tilde{G}_{p,p} &= \frac{1}{G_{p,p}}\,, \quad \tilde{G}_{\mu p} = \frac{B_{\mu p}}{G_{p,p}}\,, \quad \tilde{G}_{\mu\nu} = \frac{G_{\mu\nu} - G_{\mu p}G_{\nu p} - B_{\mu p}B_{\nu p}}{G_{p,p}}\,, \\
\tilde{B}_{\mu p} &= \frac{G_{\mu p}}{G_{p,p}}\,, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu p}G_{\nu p} - G_{\mu p}B_{\nu p}}{G_{p,p}}\,, \\
e^{2\tilde{\Phi}} &= \frac{e^{2\Phi}}{G_{p,p}}\,,
\end{aligned} \tag{1.2.10}$$

where $\tilde{G}$, $\tilde{B}$ and $\tilde{\Phi}$ are the T-dualised fields. We assume that the fields in (1.2.10) are independent of the direction x^p, and so the metric G has an isometry corresponding to translations in the x^p direction. Because we will only consider T-duality in spacelike directions, the norm of the associated Killing vector is greater than zero, i.e. $G_{p,p} > 0$. It is possible to use the rules given in (1.2.10), along with (1.1.5) for the D-brane fields, to show that the Dirac-Born-Infeld action is indeed invariant under T-duality, as we would expect (see refs. [16] and [17]).

The low energy limit of string theory is $\alpha' \to 0$, when all the massive fields become infinitely massive, and can be ignored. Expanding the Abelian Dirac-Born-Infeld action (1.2.8) as a series in α', we find that the leading order term is given by

$$-\frac{T_p\,(2\pi\alpha')^2}{4}\int d^{p+1}\xi\, e^{-\Phi} F_{\alpha\beta}F^{\alpha\beta}\,, \tag{1.2.11}$$

which is Yang-Mills theory with the coupling

$$g^2_{YM,p} = g_s T_p^{-1}(2\pi\alpha')^{-2} = g_s(2\pi)^{p-2}\alpha'^{(p-3)/2}\,. \tag{1.2.12}$$

Next we consider the coupling of the D-branes to the R-R fields. Since the Dp-brane acts as a source for the $C^{(p+1)}$ R-R field, we must integrate this field over the Dp-brane's worldvolume. Again, since $C^{(p+1)}$ is a background field, it must be pulled back to the D-brane's worldvolume. So the coupling we require is

$$\mu_p \int_{\mathcal{M}_{p+1}} P[C^{(p+1)}] \ , \tag{1.2.13}$$

where $\mathcal{M}_{p+1}$ is the D-brane worldvolume (the term (1.2.13) is the higher-dimensional generalisation of the term $e\int A_\mu v^\mu$ in the action for electromagnetism, where e is the electric charge, and v^μ is the velocity). We must also consider the possibility of lower-dimensional R-R fields coupling to the antisymmetric fields B and F. Using T-duality arguments it is possible to show that the full coupling to the R-R fields is given by

$$S_{CS} = \mu_p \int_{\mathcal{M}_{p+1}} P\left[\sum C^{(n)} e^B\right] e^{2\pi\alpha' F} \ , \tag{1.2.14}$$

which is the Chern-Simons action (see refs. [20] and [21]). The combination of B and F in (1.2.14) can be understood using the same arguments as those we discussed for the Dirac-Born-Infeld action above. The summation sign in (1.2.14) means that we must include all possible combinations of $C^{(n)}$, B and F which have the form specified in (1.2.14), and which combine to give a $(p+1)$-form. In particular this means that a Dp-brane can couple to lower-dimensional R-R fields. When this happens the lower-dimensional brane becomes delocalised in its transverse directions which are parallel to the higher-dimensional brane; it is said to be 'smeared out' in the extra directions.

1.2.2 Non-Abelian D-Brane Actions

In this section we consider the worldvolume actions for N D-branes. The gauge group on the branes is now non-Abelian; for N coincident D-branes it is $U(N)$. Therefore the fields A_α and Φ^i, which live on the D-brane, are now non-commuting. We take the convention that the Φ^i and the A_α are hermitian matrices, so that the field strength on the branes is given by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta] \ . \tag{1.2.15}$$

We first outline some considerations which must be taken into account when extending the Abelian actions (1.2.8) and (1.2.14) to the non-Abelian case (see ref. [22]). For the actions to remain gauge invariant all partial derivatives must be converted into covariant derivatives in the non-Abelian case, i.e.

$$\partial_\alpha \Phi^i \to D_\alpha \Phi^i = \partial_\alpha \Phi^i + i[A_\alpha, \Phi^i] \, . \qquad (1.2.16)$$

Recall from equation (1.1.5) that under T-duality in the x^p direction we have $A_p \to \Phi^p$, and so

$$D_p \Phi^i \to i[\Phi^p, \Phi^i] \, . \qquad (1.2.17)$$

And so we may need to include extra commutators of the Φ^i in our non-Abelian actions, which would be zero in the Abelian case. Also, it is necessary to perform some sort of trace over the gauge indices.

We start by considering the non-Abelian extension of the Dirac-Born-Infeld action (1.2.8). The full non-Abelian Dirac-Born-Infeld action was found in ref. [22] by taking the Dirac-Born-Infeld action for a D9-brane and applying T-duality (see also ref. [23]). Since a D9-brane has no transverse directions in string theory, there are no Φ^i's, and so the non-Abelian generalisation of (1.2.8) with $p = 9$ does not contain any extra commutators. So the non-Abelian Born-Infeld action for the D9-brane is defined to be

$$S_9 = -T_9 \int d^{10}\xi \; \mathrm{STr} \left(\mathrm{e}^{-\Phi} \sqrt{-\det \left(\mathrm{G}_{\alpha\beta} + \mathrm{B}_{\alpha\beta} + 2\pi\alpha' \mathrm{F}_{\alpha\beta} \right)} \right) \, , \qquad (1.2.18)$$

where STr denotes the symmetrised trace, which we will explain below. In what follows we will assume static gauge, in which $\xi^\alpha = x^\alpha$ for the D-brane directions. Applying T-duality to the action (1.2.18) results in the non-Abelian Dirac-Born-Infeld action for a Dp-brane, which is

$$\begin{aligned} S^p_{DBI} = -T_p \int d^{p+1}\xi \mathrm{STr} \Big(e^{-\Phi} & \sqrt{-\det \left(P[E_{\alpha\beta} + E_{\alpha i}(Q^{-1} - \delta)^{ij} E_{j\beta}] + 2\pi\alpha' F_{\alpha\beta} \right)} \\ & \times \sqrt{\det(Q^i{}_j)} \Big) \, , \end{aligned} \qquad (1.2.19)$$

where

$$E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \, , \qquad (1.2.20)$$

and

$$Q^i{}_j = \delta^i{}_j + i2\pi\alpha' \, [\Phi^i, \Phi^k] \, E_{kj} \, . \qquad (1.2.21)$$

In the derivation of the action (1.2.19) it was assumed that $\det Q^i_{\ j} \neq 0$ (see ref. [22] for more details). The Abelian case is given by $Q^i_{\ j} = \delta^i_{\ j}$, when the action (1.2.19) reduces to the Abelian Born-Infeld action (1.2.8). In the low energy limit the leading order term in α' of (1.2.19) is the action for non-Abelian $U(N)$ super-Yang-Mills theory, as we would expect.

Next we consider the non-Abelian extension of the Chern-Simons action (1.2.14). Again, the form of this action can be deduced by requiring consistency with T-duality. The result is [22]

$$S^p_{CS} = \mu_p \int \mathrm{STr}\left(\mathrm{P}\left[\mathrm{e}^{\mathrm{i}2\pi\alpha'\mathbf{i}_\Phi\mathbf{i}_\Phi} \sum \left(\mathrm{C}^{(\mathrm{n})}\mathrm{e}^{\mathrm{B}}\right)\right] \mathrm{e}^{2\pi\alpha'\mathrm{F}}\right) . \qquad (1.2.22)$$

Here $\mathbf{i}_\Phi$ denotes the interior product by Φ^i regarded as a vector in the transverse space, e.g. for a two-form $C^{(2)}$

$$\begin{aligned} \mathbf{i}_\Phi C^{(2)} &= \Phi^i C^{(2)}_{i\mu} dx^\mu , \\ \mathbf{i}_\Phi\mathbf{i}_\Phi C^{(2)} &= \Phi^i\Phi^j C^{(2)}_{ij} = \frac{1}{2} C^{(2)}_{ij} \left[\Phi^j, \Phi^i\right] . \end{aligned} \qquad (1.2.23)$$

So the $\mathbf{i}_\Phi$ introduce dependence on the commutators of the Φ^i into the non-Abelian Chern-Simons action. As with the Abelian case, only forms of dimension $(p+1)$, and combinations of forms whose dimensions total $(p+1)$ contribute to the action (1.2.22), since they must be integrated over the worldvolume of the Dp-brane.

In the non-Abelian actions (1.2.19) and (1.2.22) a symmetrised trace over the worldvolume fields has been included. This notation indicates that we should symmetrise over all orderings of $F_{\alpha\beta}$, $D_\alpha\Phi^i$ and $[\Phi^i, \Phi^j]$ when we take the trace. Naively one might include an ordinary trace in the non-Abelian actions. However, this leads to an ambiguity over the ordering of the fields when calculating the determinant in the action (1.2.19). The symmetrised trace prescription was proposed by Tseytlin in refs. [24] and [25] as an alternative. The argument behind Tseytlin's conjecture was that, if we expand the Dirac-Born-Infeld action in powers of the field strength F, then the symmetrised trace prescription retains only even powers of F. This is encouraging because odd powers of F can be written in terms of derivatives of F, which, as we pointed out in section 1.2.1, are neglected in the Abelian Dirac-Born-Infeld action. Therefore in generalising the Abelian Dirac-Born-Infeld action, the symmetrised trace prescription appears to be the correct way to include a trace over the non-Abelian fields. In ref. [26] it was argued that the symmetrised trace is the correct trace operation to include because it is the only candidate which allows the existence of certain solitons, e.g. monopoles in the D3-brane action (on which the work in two chapters of this thesis will be based).

Having obtained candidates for the non-Abelian D-brane actions, (1.2.19) and (1.2.22), they can be tested using open string scattering calculations. Such calculations have received much attention in the literature; see for example ref. [27], and the references therein. It turns out that the symmetrised trace prescription is correct at order $(\alpha')^2$, but that there are higher order corrections at order $(\alpha')^3$ and $(\alpha')^4$. We will be working to leading order in α', and so these corrections need not concern us.

1.2.3 Couplings to the Background Curvature

It turns out that the Dirac-Born-Infeld action and the Chern-Simons action, which we discussed in the previous two sections, are not the whole story. When we wrap a D-brane on a non-trivial manifold we must also take into account the corrections to the action due to the background curvature. It will be sufficient for our purposes to work with the Abelian Dirac-Born-Infeld and Chern-Simons actions. We will discuss the case of a Dp-brane wrapped on a K3 manifold, $p \geq 4$, which will be relevant for our discussion of the enhançon mechanism in section 1.4.1. Recall that a K3 manifold is a four-dimensional, Ricci flat, simply connected, compact Kähler manifold with $SU(2)$ holonomy. It contains one four-cycle and 22 independent two-cycles. When we say that a D-brane is 'wrapped' on K3 we mean that four of the brane's dimensions take on the geometry of a K3 manifold, and the D-brane metric (1.1.23) is modified accordingly.

For the Dirac-Born-Infeld action the relevant corrections are given by [28]

$$\begin{aligned} S^p_{DBI} &= -\tau_p \int d^{p+1}\xi\, e^{-\Phi}\sqrt{-\det(P[G+B]_{\alpha\beta}+2\pi\alpha' F_{\alpha\beta})} \\ &\quad \times \left(1-\frac{(4\pi^2\alpha')^2}{768\pi^2}\left(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}-R^{ij\alpha\beta}R_{ij\alpha\beta}-2\hat{R}^{\alpha\beta}\hat{R}_{\alpha\beta}+2\hat{R}^{ij}\hat{R}_{ij}\right)\right. \\ &\quad \left. +O((\alpha')^4)\right)\,, \end{aligned} \tag{1.2.24}$$

where, again, the indices $\alpha, \beta, \gamma, \delta$ denote directions tangent to the worldvolume of the brane, and i, j denote transverse directions. $\hat{R}_{\alpha\beta}$ is constructed by pulling-back the Riemann tensor to the D-brane worldvolume, then contracting it. Similarly, $\hat{R}_{ij}$ is constructed by pulling-back the Riemann tensor to the space transverse to the D-brane worldvolume, and then contracting. For the case of a D-brane wrapped on K3, only the corrections involving D-brane directions contribute. Also, since K3 is Ricci flat, the only contribution from the D-brane directions is $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. After integrating over the K3 directions, the action (1.2.24) becomes

$$S^p_{DBI} = -\int d^{p-3}\xi\, e^{-\Phi}\,(\tau_p V_{K3} - \tau_{p-4})\sqrt{-\det(P[G+B]_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}\ , \qquad (1.2.25)$$

where V_{K3} is the volume of the K3 manifold. Note that the tension in (1.2.25) now has an extra component due to the curvature of the K3 manifold.

Including the curvature couplings in the Chern-Simons action, the action becomes

$$\mu_p \int_{\mathcal{M}_{p+1}} P\left[\sum_n C^{(n)} e^B\right] e^{2\pi\alpha' F}\sqrt{\hat{\mathcal{A}}(4\pi^2\alpha' R)}\ , \qquad (1.2.26)$$

where $\hat{\mathcal{A}}$ is the 'A-roof' or Dirac genus (see refs. [29] and [21]). $\hat{\mathcal{A}}$ is a generating function, which can be expanded in terms of the Pontryagin polynomials p_i as follows

$$\hat{\mathcal{A}}(F) = 1 - \frac{1}{24}p_1(F) + \frac{1}{5760}(7p_1(F)^2 - 4p_2(F)) + \cdots\ . \qquad (1.2.27)$$

Keeping just the first two terms in the expansion (1.2.27), we have

$$\hat{\mathcal{A}}(4\pi^2\alpha' R) = 1 + \frac{(4\pi^2\alpha')^2}{24}\left(\frac{1}{8\pi^8}\right)\mathrm{TrR}\wedge\mathrm{R} + \mathrm{O}(\alpha'^4)\ . \qquad (1.2.28)$$

Integrating over K3 (see ref. [29]), the additional term in (1.2.26) is

$$-\mu_{p-4}\int_{\mathcal{M}_{p-3}} C^{(p-3)}\ , \qquad (1.2.29)$$

where $\mathcal{M}_{p-3}$ is the manifold consisting of the unwrapped brane directions.

Taking (1.2.25) and (1.2.29) together, we deduce that wrapping a Dp-brane on a K3 manifold induces a unit negative charge of a D$(p-4)$-brane. Note that this is a different object to an anti-D2-brane; it has a negative charge, but it preserves the same unbroken supersymmetries as a D$(p-4)$-brane. From (1.2.25) we see that the overall tension is given by

$$\tau = \tau_p V_{K3} - \tau_{p-4}\ , \qquad (1.2.30)$$

where the volume of the K3 manifold, V_{K3}, may be allowed to depend on the directions transverse to the K3 manifold. The equation (1.2.30) has the potential to vanish and to become negative. But the enhançon mechanism will prevent this from happening, as we will see in section 1.4.1.

1.3 Magnetic Monopoles

In this section we will review some of the aspects of magnetic monopoles which we will be using later on. See refs. [30] and [31] for monopole reviews. The magnetic monopole was originally postulated in the 19^{th} century as a pointlike source for the magnetic field; it was studied by Dirac in ref. [32]. If we include magnetic source terms in Maxwell's equations then they take the form

$$\partial_\mu F^{\mu\nu} = j_e^\nu \ , \quad \partial_\mu {*F^{\mu\nu}} = j_m^\nu \ , \tag{1.3.1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength, and j_e^ν and j_e^ν are the electric and magnetic four-currents respectively. The equations (1.3.1) are invariant under the transformation

$$F^{\mu\nu} \leftrightarrow *F^{\mu\nu} \ , \quad j_e^\nu \leftrightarrow j_m^\nu \ , \tag{1.3.2}$$

which is called electromagnetic duality (in essence the electric and magnetic fields are exchanged, as well as the electric and magnetic currents). For a pointlike magnetic charge there is no single gauge potential which can describe the magnetic field everywhere; we must define two overlapping gauge potentials, one for the Northern half of a two-sphere surrounding the charge, and one for the Southern half. By requiring that physical quantities are continuous in the overlapping region it turns out that the electric charge is quantised as follows

$$eg = 2\pi n \ , \quad n \in \mathbb{Z} \ , \tag{1.3.3}$$

where e is the electric charge and g is the magnetic charge. This is the Wu-Yang derivation [33] of Dirac's quantisation condition (see ref. [32] for the original derivation). It is this property which makes the magnetic monopole so attractive to quantum field theorists.

In this thesis we will be dealing with a topological version of the monopole, which is called the 't Hooft-Polyakov monopole. In section 1.3.1 we will review some of the basic properties of the 't Hooft-Polyakov monopole, and the limit which results in the BPS monopole. In section 1.3.2 we will discuss ways of constructing BPS monopole solutions, in particular the ADHMN construction. In section 1.3.3 we will review monopole moduli spaces, and their bearing on monopole scattering.

1.3.1 The 't Hooft-Polyakov Monopole and BPS Monopoles

The 't Hooft-Polyakov monopole is a topological soliton solution of the Yang-Mills-Higgs theory. It was originally postulated independently by 't Hooft and Polykov in refs. [34] and [35] respectively. The appropriate Lagrangian is

$$\mathcal{L} = \frac{1}{g_{YM}^2}\left\{-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - V(\Phi)\right\} , \tag{1.3.4}$$

where g_{YM} is the Yang-Mills coupling. Here $\mu, \nu, \ldots$ denote spacetime indices, and $a, b, \ldots$ denote the gauge indices labelling the adjoint representation of the gauge group. We take the gauge group to be $SU(2)$ for simplicity at this stage. The gauge field is defined as follows

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \epsilon^{abc}A^b_\mu A^c_\nu , \tag{1.3.5}$$

and the covariant derivative is defined to be

$$D_\mu\Phi^a = \partial_\mu\Phi^a - \epsilon^{abc}A^b_\mu\Phi^c . \tag{1.3.6}$$

The Bianchi identity

$$D_\mu * F^{a\,\mu\nu} = 0 \tag{1.3.7}$$

follows from the definition of $F^{a\,\mu\nu}$. The potential $V(\Phi)$ is chosen such that the vacuum expectation value of the Higgs field Φ is non-zero. We will take

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^a\Phi^a - v^2\right)^2 . \tag{1.3.8}$$

V is minimised when $\Phi^a\Phi^a = v^2$; we say that the Higgs field Φ has a non-zero expectation value v. This breaks the symmetry of the gauge group from $SU(2)$ to

$U(1)$. Then the perturbative spectrum of the theory consists of a massless photon, massive spin one gauge bosons $W^{\pm}$ with mass

$$m_W = v \ , \tag{1.3.9}$$

and also the Higgs field.

We define

$$\begin{aligned} E_i^a &= F_{0i}^a \ , \\ B_i^a &= \frac{1}{2}\epsilon_{ijk}F^{a\,jk} \ , \end{aligned} \tag{1.3.10}$$

where $i, j, k \in \{1, 2, 3\}$. Then E_i^a and B_i^a are analogous to the electric and magnetic fields, respectively, of electromagnetism. Then the energy density corresponding to the Lagrangian (1.3.4) is

$$\mathcal{E} = \frac{1}{2}\left(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a + \Pi^a\Pi^a + \vec{D}\Phi^a \cdot \vec{D}\Phi^a\right) + V(\Phi) \ , \tag{1.3.11}$$

where Π^a is the momentum conjugate to Φ, $\Pi^a = D_0\Phi^a$. From the form of the energy density (1.3.11) we can see that $\mathcal{E}$ is positive, and $\mathcal{E} = 0$ if and only if the following conditions are met

$$F^{a\,\mu\nu} = D^\mu\Phi^a = V(\Phi) = 0 \ . \tag{1.3.12}$$

So the conditions (1.3.12) define the vacuum solution of the theory. For a solution to have finite energy it is necessary that the vacuum conditions are met on the surface of the sphere at spatial infinity, which we will denote S^2_∞. In particular, the Higgs field on the surface of S^2_∞ must minimise the potential V, with $V(\Phi) = 0$. We define M_H to be the set of all such configurations, then

$$M_H = \{\Phi^a : \Phi^a\Phi^a = v^2\} \ , \tag{1.3.13}$$

which has the topology of a two-sphere. Therefore, for finite energy, the Higgs field configuration at spatial infinity defines a map from one two-sphere to another

$$\Phi : S^2_\infty \to M_H \ . \tag{1.3.14}$$

Such a map has a topological quantity associated with it called a winding number, which we will denote by n. For the case considered here, i.e. $SU(2)$ broken down to

$U(1)$, the winding numbers are the integers, and are given by

$$n = \frac{1}{8\pi} \int \mathrm{Tr}(B_i D_i \Phi)\, d^3x \ , \quad n \in \mathbb{Z} \ . \tag{1.3.15}$$

In our discussion so far we have not explained the connection to magnetic monopoles. It turns out that for a solution with $n \neq 0$ to have finite energy there must be a non-zero gauge field to cancel a contribution from the angular part of $(\vec{\nabla}\Phi^a)^2$ at spatial infinity. The required gauge potential has an angular component which falls off as $1/r$ as $r \to \infty$, giving rise to a non-zero magnetic field at infinity, with magnetic charge given by

$$g = \frac{4\pi n}{g_{YM}} \ . \tag{1.3.16}$$

Equation (1.3.16) is once again Dirac's quantisation condition with $g_{YM} = 2e$ (the factor of 2 comes from the charge of field in the fundamental representation of $SU(2)$). So we deduce that the solutions we have been discussing, with winding number $n \neq 0$, are magnetic monopoles with magnetic charge g. The conservation of g comes about because it would take an infinite amount of energy to transform a solution with $n = n_1$ into a solution with $n = n_2 \neq n_1$.

Next we will look for a Bogomol'nyi bound on the energy for the monopole solutions (see ref. [36]). Consider a static solution with the electric field set to zero. Then the energy of the solution is given by

$$\begin{aligned} E &= \frac{1}{g_{YM}^2} \int d^3r \left(\frac{1}{2} \vec{B}^a \cdot \vec{B}^a + \frac{1}{2} \vec{D}\Phi^a \cdot \vec{D}\Phi^a \right) + V(\Phi) \\ &\geq \int d^3r \left(\frac{1}{2} \vec{B}^a \cdot \vec{B}^a + \frac{1}{2} \vec{D}\Phi^a \cdot \vec{D}\Phi^a \right) \\ &= \frac{1}{2g_{YM}^2} \int d^3r \, (\vec{B}^a - \vec{D}\Phi^a) \cdot (\vec{B}^a - \vec{D}\Phi^a) \\ &\quad + \frac{1}{g_{YM}^2} \int d^3r \, \vec{B}^a \cdot \vec{D}\Phi^a \ . \end{aligned} \tag{1.3.17}$$

The first term in (1.3.17) is positive. Integrating the second term by parts and using the Bianchi identity (1.3.7)

$$\int d^3r \, \vec{B}^a \cdot \vec{D}\Phi^a = vg \ . \tag{1.3.18}$$

where g is the magnetic charge. So we have

$$E \geq \frac{|vg|}{g_{YM}^2} \ , \tag{1.3.19}$$

where we have included the modulus in (1.3.19) to account for the case with negative magnetic charge. Equality holds in (1.3.19) if and only if $V(\Phi) = 0$ and

$$\vec{B}^a = \vec{D}\Phi^a \ . \tag{1.3.20}$$

Consider the condition $V(\Phi) = 0$. For the 't Hooft-Polyakov monopole of the Yang-Mills-Higgs theory (1.3.4) with potential given by (1.3.8), this condition is realised if we take the limit $\lambda \to 0$, while keeping fixed the boundary condition

$$|\Phi| \to v \quad \text{as} \quad r \to \infty \ . \tag{1.3.21}$$

This limit is called the BPS limit, and the monopole of this theory which satisfies the Bogomol'nyi equation is the BPS monopole (see ref. [37]). From (1.3.19), the mass of a BPS monopole is given by

$$m_{mon} = \frac{vg}{g_{YM}^2} \ . \tag{1.3.22}$$

The theories we will be dealing with in this thesis will be supersymmetric theories in which $V(\Phi) \equiv 0$. However, it still makes sense to impose the boundary condition (1.3.21) because to change from one value of the Higgs expectation value v to another in the theory would take an infinite amount of energy. Therefore the value of v has to stay fixed once it has been imposed, and so the theory has a well-defined Hilbert space.

It can be shown that for a 't Hooft-Polyakov monopole configuration, i.e. for the winding number given by (1.3.15) to be non-zero, the Higgs field must be zero at some point in space. At this point the unbroken $SU(2)$ gauge symmetry is restored.

In our discussion so far we have considered solutions which are purely magnetic. Solutions with combined electric and magnetic charge are called dyons (see ref. [38]). For dyons we have the Dirac-Zwanziger-Schwinger condition, which generalises the Dirac quantisation condition. For two dyons with electric charges e_1, e_2 and magnetic charges g_1, g_2 it is given by

$$e_1 g_2 - e_2 g_1 = 2\pi n \ . \tag{1.3.23}$$

By calculating the energy of a static configuration with non-zero electric and magnetic fields, one can also obtain the generalised Bogomol'nyi bound,

$$E \geq v\,(e^2 + g^2)^{1/2} \ , \tag{1.3.24}$$

for a dyon with magnetic charge g and electric charge e.

1.3.2 Constructing Monopole Solutions

In this section we consider the techniques available for constructing explicit solutions for BPS monopoles. We will take the gauge group to be $SU(2)$ throughout.

The charge one solution to the Bogomol'nyi equation (1.3.20) was first found by Prasad and Sommerfield in ref. [37]. It is given by

$$\begin{aligned} \Phi^a &= \frac{r^a}{g_{YM}r^2}H(vg_{YM}r) \ , \\ A_i^a &= \epsilon^a_{ij}\frac{r^j}{g_{YM}r^2}(1-K(vg_{YM}r)) \ , \end{aligned} \tag{1.3.25}$$

where

$$H(y) = y\coth y - 1 \ , \quad K(y) = \frac{y}{\sinh y} \ . \tag{1.3.26}$$

Note that the solution (1.3.25) is spherically symmetric. For monopoles with monopole number $n \geq 2$ there are no spherically symmetric solutions. But for $n = 2$ an axially symmetric solution has been calculated numerically in ref. [39]. Its charge is concentrated in a ring around the origin, so that it has the shape of a two-torus (i.e. a 'doughnut'). Also, for $n \geq 2$, multimonopole solutions can be constructed which are n static, well-separated versions of the charge one solution (1.3.25). As was shown in ref. [40], this can be done for BPS monopoles because the Higgs field is massless, and so the Higgs force is long-range. The scalar attraction due to the Higgs field cancels out the magnetic repulsion between monopoles so that they can remain static.

There are also some indirect methods which can be used to construct multimonopole solutions. One of these uses twistor methods, and was originally described by Ward in ref. [41]. Using this method, Ward also obtained the axially symmetric two-monopole configuration described in the previous paragraph. This approach was studied further by Hitchin in ref. [42]. In the rest of this section we will discuss another indirect technique which can, in principle, be used to construct all the monopole solutions for a given monopole number n. This technique was adapted from the ADHM instanton construction of ref. [43] by Nahm in refs. [44] and [45]; it is called the ADHMN construction. See ref. [46] for a review.

First we demonstrate that the Bogomol'nyi equation (1.3.20) is equivalent to the self-duality equation in four dimensions

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \ , \quad \mu,\nu,\rho,\sigma \in \{1,2,3,4\} \ , \tag{1.3.27}$$

when the fields are independent of the Euclidean time x_4. Setting

$$A_4 = \Phi \tag{1.3.28}$$

in (1.3.27) recovers the Bogomol'nyi equation (1.3.20).

The first step of the ADHMN construction is to find the Nahm data. These are $n \times n$ Hermitian matrices, which depend on the real parameter $\xi \in [0, 2]$, and which satisfy the following

1. Nahm's equations
$$\frac{dT_i}{d\xi} = \frac{i}{2}\epsilon_{ijk}[T_j, T_k] \ . \tag{1.3.29}$$

2. The T_i have simple poles at $\xi = 0$ and $\xi = 2$.

3. The matrix residues at the poles form an irreducible n-dimensional representation of $SU(2)$.

The second step of the ADHMN construction is to solve the construction equation, which is given by

$$\left(\mathbb{I}_{2n}\frac{d}{d\xi} + \mathbb{I}_n \otimes \frac{x_j\sigma_j}{2} - T_j \otimes \sigma_j\right)\vec{v} = 0 \ . \tag{1.3.30}$$

The equation (1.3.30) must be solved for the complex $2n$-vector $\vec{v}(\xi, \vec{x})$. For $SU(2)$ there are two linearly independent solutions for $\vec{v}$, which we call $\vec{v}_1$ and $\vec{v}_2$. These can be normalised so that

$$\int_0^2 d\xi \, v_p^\dagger v_q = \delta_{pq} \ , \quad p, q \in \{1,2\} \ . \tag{1.3.31}$$

In the third step of the ADHMN construction, the solutions for the Higgs field Φ and the gauge potential A_i can be calculated from the v_p. Assembling a $2n \times 2$ matrix $\mathbf{v}$ out of $\vec{v}_1$ and $\vec{v}_2$, the solutions are given by

$$\Phi = \int_0^2 (\xi - 1)\,\mathbf{v}^\dagger\mathbf{v}\,d\xi \ , \quad A_i = -i\int_0^2 \mathbf{v}^\dagger\partial_i\mathbf{v}\,d\xi \ . \tag{1.3.32}$$

For the general $SU(N)$ case there is a fundamental monopole associated with each factor of $U(1)$ which occurs when the gauge group is broken. These monopoles look like embeddings of the unit $SU(2)$ monopole. Then the Nahm data becomes more complicated; there is a set of Nahm data associated with each $U(1)$ factor. We will not discuss this in detail as it will not be necessary for the work described in this thesis; see ref. [47] for a review.

Returning to the Nahm data for the $SU(2)$ case, we can orientate the Nahm data for a two-monopole so that it has the form

$$T_i = -\frac{f_i}{2}\sigma_i \ , \quad i \in \{1,2,3\} \ , \tag{1.3.33}$$

where f_i are three real functions of ξ, and σ_i are the Pauli matrices. With this ansatz Nahm's equations reduce to

$$\frac{df_1}{d\xi} = f_2 f_3 \ , \quad \frac{df_2}{d\xi} = f_3 f_1 \ , \quad \frac{df_3}{d\xi} = f_1 f_2 \ . \tag{1.3.34}$$

The solution to (1.3.34) satisfying the appropriate boundary conditions is [48]

$$\begin{aligned} f_1(\xi,k) &= -\frac{K(k)}{\mathrm{sn}(\mathrm{K(k)}\xi,\mathrm{k})} \ , \quad f_2(\xi,k) = -\frac{K(k)\mathrm{dn}(\mathrm{K(k)}\xi,\mathrm{k})}{\mathrm{sn}(\mathrm{K(k)}\xi,\mathrm{k})} \ , \\ f_3(\xi,k) &= -\frac{K(k)\mathrm{cn}(\mathrm{K(k)}\xi,\mathrm{k})}{\mathrm{sn}(\mathrm{K(k)}\xi,\mathrm{k})} \ , \end{aligned} \tag{1.3.35}$$

where $K(k)$ is the complete elliptic integral of the first kind with parameter k, which is defined as follows

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\tau)^{-1/2}\, d\tau \ . \tag{1.3.36}$$

Also, $\mathrm{sn(x,k)}$, $\mathrm{cn(x,k)}$ and $\mathrm{dn(x,k)}$ are the Jacobi elliptic functions with argument x and parameter k. See ref. [49] for a review of the properties of these functions. The parameter k is a modulus of the solution with $0 < k < 1$. The functions $f_1(\xi,k)$ and $f_2(\xi,k)$ are symmetric about $\xi = 1$, while $f_3(\xi,k)$ is antisymmetric about $\xi = 1$ (this can be seen from the periodicity properties of the elliptic functions - see ref. [49]).

1.3.3 The Moduli Space and Monopole Scattering

In this section we will discuss the moduli space of monopole solutions. By the 'modulus' of a solution we mean a physical zero mode, i.e. a parameter of the solution which can be changed without changing the energy of the solution, or alternatively a parameter corresponding to a flat direction in the potential energy. The charge one monopole (1.3.25) has four moduli. Three of these are easy to spot - they correspond to the freedom to reposition the monopoles anywhere in $\mathbb{R}^3$. The fourth modulus is not so obvious. It can be obtained by deforming a BPS monopole configuration in such a way that the Bogomol'nyi equation (1.3.20) and the Gauss' law constraint, given by

$$D_i E_i + [\Phi, D_0 \Phi] = 0 \ , \tag{1.3.37}$$

are kept fixed. Working in $A_0 = 0$ gauge, there is a unique physical transformation which obeys these constraints; it is given by the following rigid gauge transformation

$$U = e^{\chi(t)\, \Phi} \ , \tag{1.3.38}$$

When $\dot{\chi} = 0$ in (1.3.38) χ is a physical zero mode because the transformed configuration still satisfies the Bogomol'nyi equation. The transformation (1.3.38) belongs to the global $U(1)$ group of electromagnetism, which is compact since it has been embedded in $SU(2)$, and so χ is periodic. The modulus χ is therefore called the phase of the monopole. For a charge n monopole there are $4n$ moduli, of which $3n$ represent the positions of the monopoles (at least when the monopoles are well-separated), and n represent the monopoles phases.

The moduli define a $4n$-dimensional manifold $\mathcal{M}$ called the moduli space, with each modulus corresponding to a dimension of $\mathcal{M}$. See ref. [50] for a detailed geometrical discussion of the properties of monopole moduli spaces. In ref. [51] Manton showed how it is possible to use the moduli space to describe slow motion monopole scattering. He argued that, for an initial configuration which is tangent to the moduli space, and in the limit of low velocity, the motion is constrained to min(V), where min(V) is the space of solutions which minimise the potential of the Yang-Mills theory, i.e. the moduli space. Then the motion is described by a geodesic in the moduli space, with the action

$$S = \frac{1}{2} \int dt\, g_{\kappa\lambda}\, \dot{z}^\kappa \dot{z}^\lambda \ , \tag{1.3.39}$$

where z^κ are the moduli with $\kappa, \lambda = 1, \ldots, 4n$, and $g_{\kappa\lambda}$ is the metric on moduli space. This action can be obtained from the standard action for a path in the configuration space of monopole solutions (see ref. [51] for details). The action (1.3.39) is the effective action in which terms of order $\dot{z}^3$ and higher have been ignored. For monopole motion this means that there are some effects, such as energy radiation, which have been ignored. We will focus more on the energy radiated during monopole scattering in chapters 3 and 4.

For a two-monopole solution the geometry of the moduli space is (see ref. [52]),

$$\mathbb{R}^3 \times \frac{S^1 \times M}{\mathbb{Z}_2} \, . \tag{1.3.40}$$

In (1.3.40) the $\mathbb{R}^3$ corresponds to the centre of mass position of the monopoles, the S^1 to the overall phase, and M is a four-dimensional manifold which specifies the relative position and phase. The quotient by $\mathbb{Z}_2$ is because the monopoles cannot be distinguished. The metric on moduli space for the centre of mass coordinate $\vec{X}$ and the overall phase χ is flat

$$d\vec{X} \cdot d\vec{X} + d\chi^2 \, . \tag{1.3.41}$$

The metric on moduli space for M is more interesting. It can be shown to be invariant under $SO(3)$ rotations, and hyper-Kähler. In ref. [52] these properties were used to deduce the asymptotic form of the metric in the limit that the monopoles are well-separated; it is the Taub-NUT metric with a negative mass parameter, which is given by

$$ds^2 = U(r)\, d\vec{r} \cdot d\vec{r} + 4\, U(r)^{-1}(d\psi + \vec{\omega} \cdot d\vec{r})^2 \, , \tag{1.3.42}$$

where $\vec{r} = (\vec{x}_1 - \vec{x}_2)$ is the relative position of the monopoles, and $\psi = (\chi_1 - \chi_2)$ is the relative phase. Here $U(r)$ is a function of $r = |\vec{r}|$, which is given by

$$U(r) = 1 - \frac{2}{r} \, . \tag{1.3.43}$$

Also ω is defined by

$$\vec{\nabla} \times \vec{\omega} = \frac{\vec{r}}{r^3} \, . \tag{1.3.44}$$

Note that, although the metric (1.3.42) contains a singularity at $r = 2$, this is not a problem because the singularity is outside the region in which (1.3.42) is valid. The unique metric with the properties of $SO(3)$ isometry and hyper-Kählerity, which has Taub-NUT as its asymptotic limit, and is also smooth, is the Atiyah-Hitchin metric. We will not discuss the details of this metric here - the Taub-NUT metric will be sufficient for our purposes.

For monopoles with monopole number n the metric on moduli space is not known accurately. It was calculated in the asymptotic limit by Gibbons and Manton in ref. [53]. They used the fact that allowing the phases to depend on time results in a non-zero electric field, so that the monopoles become dyons. By studying the dynamics of the dyons, and reinterpreting the electric charges as phases, they used (1.3.39) to deduce the asymptotic form of the metric. They obtained the generalisation of the Taub-NUT metric to higher dimensions. Defining $r_{rs} = |\vec{x}_r - \vec{x}_s|$ and

$$\nabla_r \times \vec{\omega}_{rs} = \nabla_r \left(\frac{1}{r_{rs}} \right) , \tag{1.3.45}$$

where $r, s = 1, \ldots, n$. The metric is given by

$$ds^2 = g_{rs}\, d\vec{x}_r \cdot d\vec{x}_s + (g^{-1})_{rs}(d\theta_r + \vec{W}_{ru} \cdot d\vec{x}_u)\,(d\theta_s + \vec{W}_{su} \cdot d\vec{x}_u) , \tag{1.3.46}$$

where

$$\begin{aligned}
g_{rr} &= 1 - \sum_{r \neq s} \frac{1}{r_{rs}} , \quad \text{(no sum over } r) , && (1.3.47)\\
g_{rs} &= \frac{1}{r_{rs}} , \quad (r \neq s) , && (1.3.48)\\
\vec{W}_{rr} &= -\sum_{r \neq s} \vec{\omega}_{rs} , \quad \text{(no sum over } r) , && (1.3.49)\\
\vec{W}_{rs} &= \vec{\omega}_{rs} , \quad (r \neq s) . && (1.3.50)
\end{aligned}$$

We next discuss a particular geodesic in the two-monopole moduli space which we will make use of in chapters 3 and 4 (see ref. [50]). Consider two monopoles approaching each other headlong along the x^1 axis. When they collide the monopoles form the axisymmetric 'doughnut' configuration in the x^1-x^2 plane. Therefore the motion corresponds to motion in a sub-manifold of the moduli space which is invariant under rotations of angle π about the x^3-axis. This submanifold is therefore isometric to $\mathbb{R}^2/\mathbb{Z}_2$, which is the cone with vertex angle $\pi/3$. Since the Atiyah-Hitchin manifold is smooth, we can think of the sub-manifold as this cone with a smoothed out vertex. The geodesic corresponding to headlong collision is the one which bounces back from the vertex of the cone. This geodesic corresponds to scattering by an angle of $\pi/2$ in $\mathbb{R}^3$.

1.4 D-Branes as BPS Monopoles

In sections 1.1 and 1.2 we have reviewed some of the basic properties of D-branes, and in section 1.3 we have reviewed some of the properties of BPS monopoles. In this section we will review some of the ideas on which the work in this thesis was based; how D-branes can act as BPS monopoles. We will concentrate on two specific examples; the enhançon, and D-strings (i.e. D1-branes) attached to D3-branes. There are many other examples of D-brane configurations which act as BPS monopoles, which we will not discuss here (some of them are related to the configurations we shall discuss by the various dualities of string theory). In section 1.4.1 we will review the enhançon, and in section 1.4.2 we will review D-strings attached to D3-branes.

1.4.1 The Enhançon Mechanism

In this section we will review what happens when a Dp-brane is wrapped on a K3 manifold for $p \geq 4$. Recall from section 1.2.3 that this induces a unit negative charge of a D$(p-4)$-brane. To be concrete we will discuss the case $p = 6$. Most of the material from this section can be found in the original enhançon paper, ref. [54], and in ref. [4].

The Supergravity Solution

Using the harmonic function rule for p-brane solutions (see refs. [55] and [56]), the supergravity solution for this object in the string frame is given by

$$\begin{aligned} ds^2 &= Z_2^{-1/2} Z_6^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + Z_2^{1/2} Z_6^{1/2} dx^i dx^i \\ &\quad + V^{1/2} Z_2^{1/2} Z_6^{-1/2} dS_{K3}^2 , && (1.4.1) \\ e^{2\Phi} &= g_s^2 Z_2^{1/2} Z_6^{-3/2} , && (1.4.2) \\ C^{(3)} &= -g_s^{-1}(1 - Z_2^{-1}) dx^0 \wedge dx^4 \wedge dx^5 , && (1.4.3) \\ C^{(7)} &= -V g_s^{-1}(1 - Z_6^{-1}) dx^0 \wedge dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9 , && (1.4.4) \end{aligned}$$

where α,β now represent the directions common to both branes, $\alpha, \beta \in \{0, 4, 5\}$, and as usual i,j represent directions transverse to the branes, $i, j \in \{1, 2, 3\}$. The harmonic functions Z_2 and Z_6 are given by

$$Z_2 = 1 + \frac{r_2}{r} \ , \quad Z_6 = 1 + \frac{r_6}{r} \ , \tag{1.4.5}$$

with $r = (x^i x^i)^{1/2}$ and

$$r_2 = -\frac{(2\pi)^4 g_s N \alpha'^{5/2}}{2V} \ , \quad r_6 = \frac{g_s N \alpha'^{1/2}}{2} \ . \tag{1.4.6}$$

N is the number of D6-branes, and also the number of induced D2-branes. The parameter V is the volume of K3, measured at spatial infinity. From the form of the metric (1.4.1) we see that the volume of K3 at radius r is given by

$$V_{K3}(r) = V \frac{Z_2(r)}{Z_6(r)} \ . \tag{1.4.7}$$

There is something obviously wrong with the supergravity solution we have presented above; it has a naked singularity at $r = -r_2$, when $Z_2 = 0$. So the region $r < |r_2|$ appears to be unphysical. The resolution to this problem is the enhançon mechanism, which was proposed in ref. [54].

A Brane-Probe Calculation

There is clearly a problem with the supergravity solution given by equations (1.4.1) - (1.4.6). In order to understand the geometry better we can probe it with a single D6-brane wrapped on K3, which we move in from spatial infinity. The effective action governing the motion of the brane probe is the Dirac-Born-Infeld action and the Chern-Simons action together, taking into account the curvature couplings discussed above. It is given by

$$\begin{aligned} S_{probe} &= -\int_{\mathcal{M}_2} d^3\xi \, e^{-\Phi} \left(\tau_6 V_{K3}(r) - \tau_2\right) \sqrt{-\det(P[G]_{\alpha\beta})} \\ &\quad + \mu_6 \int_{\mathcal{M}_2 \times K3} C^{(7)} - \mu_2 \int_{\mathcal{M}_2} C^{(3)} \ , \end{aligned} \tag{1.4.8}$$

where $\mathcal{M}_2$ is the unwrapped part of the D-branes' worldvolume, and P denotes the pullback to $\mathcal{M}_2$. The D-brane charges μ_2 and μ_6 were given in equation (1.1.14). We allow the D-brane probe to move with very slow velocity. As for the monopole case, which we discussed in section 1.3.3, the positions of the D-brane probe in the three transverse dimensions are moduli (there is also a phase, which we will discuss later).

Moving with slow velocity therefore corresponds to following the path of a geodesic in moduli space. Since D-branes are BPS states, this motion does not affect their potential energy, but they will have a small kinetic energy. We can calculate this by expanding the action (1.4.8) up to terms which are second order in the transverse velocity v. The resulting Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{probe} &= -\frac{\mu_6 V Z_2 - \mu_2 Z_6}{g_s Z_2 Z_6} + \frac{\mu_6 V}{g_s}\left(Z_6^{-1} - 1\right) - \frac{\mu_2}{g_s}\left(Z_2^{-1} - 1\right) \\ &\quad + \frac{1}{2g_s}\left(\mu_6 V Z_2 - \mu_2 Z_6\right) v^2 + O(v^4) \\ &= -\frac{1}{g_s}\left(\mu_6 V - \mu_2\right) + \frac{1}{2g_s}\left(\mu_6 V Z_2 - \mu_2 Z_6\right) v^2 \,. \end{aligned} \tag{1.4.9}$$

The first term in (1.4.9) is a constant potential energy, as expected, and the second term is the kinetic energy due to a non-trivial metric on the moduli space. The effective tension of the probe is

$$\tau_{eff} = \frac{1}{g_s}\left(\mu_6 V Z_2 - \mu_2 Z_6\right) \tag{1.4.10}$$

$$\quad = \frac{(2\pi)^{-2}\alpha'^{-3/2}}{g_s}\left(\frac{V}{V_*} - 1 - \frac{g_s N \alpha'^{1/2}}{r}\right) , \tag{1.4.11}$$

where we have defined $V_* = (2\pi)^4(\alpha')^2$. We will assume that $V > V_*$ so that the tension of the probe brane at spatial infinity is positive. The radius r_e is defined to be the value of r where the tension of the brane-probe is zero,

$$r_e = \frac{2V}{V - V_*}|r_2| \,. \tag{1.4.12}$$

Note that V_* is therefore the volume of K3 at this radius, $V_{K3}(r_e) = V_*$. For $r < r_e$ the effective tension is negative, and therefore unphysical. This tells us that it is unphysical to allow the brane probe to move inside the sphere with radius r_e. The interpretation is that the brane probe becomes delocalised as it approaches this sphere, and so the charge of the D-branes is smeared out to live on the surface of this sphere in the transverse directions. Inside the shell there are no sources, and so the geometry is flat, and is given by

$$\begin{aligned} ds^2 &= [Z_2(r_e)Z_6(r_e)]^{-1/2}\eta_{\alpha\beta}dx^\alpha dx^\beta + [Z_2(r_e)Z_6(r_e)]^{1/2}dx^i dx^i \,, \\ &\quad + V_*^{1/2} dS_{K3}^2 \end{aligned} \tag{1.4.13}$$

$$e^{2\Phi} = g_s^2 \frac{Z_2(r_e)}{Z_6(r_e)} , \quad C^{(3)} = C^{(7)} = 0 \,. \tag{1.4.14}$$

It was shown in ref. [57] that the solution given by (1.4.1) - (1.4.6) outside $r = r_e$, and (1.4.13) - (1.4.14) inside $r = r_e$ can be consistently matched onto one another.

The Enhançon as a BPS Monopole

Let us examine more closely what is happening at the radius r_e. Recall that the D6-brane couples magnetically to $C^{(1)}$, and the D2-brane to $C^{(5)}$. So the magnetic duals of the wrapped D6-branes are D4-branes wrapped on K3, with some induced D0-brane charge. Since $\mu_4/\mu_0 = \mu_6/\mu_2$, a wrapped D4-brane probe also becomes tensionless at the enhançon radius.

This scenario is suggestive of an enhanced gauge symmetry, with the wrapped D4-branes acting as the W-bosons of the theory. To see where the gauge symmetry has come from, consider the R-R fields wrapping the cycles of the K3 manifold. The unwrapped $C^{(1)}$ R-R potential is one $U(1)$, and the $C^{(5)}$ R-R potential wrapped on the overall four-cycle of the K3 is another. It is the diagonal combination of these two $U(1)$'s which provides the broken $U(1)$ gauge symmetry. The W-bosons becoming massless at the radius r_e is indicative of an enhanced gauge symmetry; it is enhanced from $U(1)$ to $SU(2)$. For this reason the wrapped D6-brane that we have been discussing is called an enhançon, and the radius r_e is the enhançon radius.

So the enhançon is a region of enhanced gauge symmetry in the transverse dimensions (there are three of them). Therefore it appears to be like the BPS monopoles that we were discussing in section 1.3 (see ref. [58]). This ties in with the enhançon being the magnetic dual of the W-boson of the enhanced gauge symmetry. There must also be the enhançon equivalent of a Higgs field - it is related to the running volume of the K3 manifold V_{K3}, because it is V_{K3} that sets the mass of the enhançon, via (1.2.30).

So we have seen that the enhançon behaves like a BPS monopole in a theory with gauge symmetry broken from $SU(2)$ down to $U(1)$.

The Field Theory Interpretation

Let us now consider the interpretation of the enhançon from the point of view of the field theory living on the branes. On the unwrapped brane worldvolume there is a (2+1)-dimensional $SU(N)$ gauge theory. Since wrapping the D6-brane on K3 breaks half of its supersymmetries, the theory has eight supercharges. The moduli space of supersymmetric vacua of the field theory is parametrised by the vacuum expectation values of the three transverse fields Φ^i (recall these fix the positions of the branes in the transverse space). Allowing the Φ^i to have vacuum expectation values breaks the gauge symmetry of the branes from $SU(N)$ to $U(1)^{N-1}$. Then the gauge fields living on the branes' worldvolume are Abelian, and can be dualised to give $N-1$ more scalars, which are also moduli (see below for the details of this calculation). Therefore the moduli space is $4(N-1)$-dimensional. In fact it is known that the moduli space of $(2+1)$-dimensional supersymmetric gauge theory with eight supercharges is hyper-Kähler (see ref. [59]), and so its dimension has to be a multiple of four. The full moduli space is identical to that of N BPS monopoles (see refs. [60], [61] and [62]).

The Enhançon Phase

The arguments discussed in the previous paragraph imply that the moduli space of the enhançon solution is identical to the moduli space of BPS monopoles. This supports the theory that the enhançon behaves like a BPS monopole. To make this argument more explicit, let us consider the moduli space of the D6-brane probe. To include the phase terms we must include the effects of the gauge field on the D-branes, $F = dA$. The Dirac-Born-Infeld action is modified as follows

$$-\det(G_{\alpha\beta}) \rightarrow -\det(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}) \ . \tag{1.4.15}$$

The D6-brane field also couples magnetically to the gauge field via the term

$$-2\pi\alpha' \int_{\mathcal{M}_2} C^{(1)} \wedge F \ . \tag{1.4.16}$$

The field strength $G^{(2)} = dC^{(1)}$ is Hodge dual to $G^{(8)} = dC^{(7)}$, so the field $C^{(1)}$ is given by

$$C^{(1)} = -\frac{r_6}{g_s} \cos\theta \, d\phi \ . \tag{1.4.17}$$

As we mentioned above, we can exchange the vector potential A_μ for a scalar s by introducing an auxiliary vector field v_μ which acts as a Lagrange multiplier for F. In the DBI part of the action we must replace $2\pi\alpha' F_{\alpha\beta}$ by $e^{2\Phi}(\mu_6 V_{K3}(r) - \mu_2)^{-2} v_\alpha v_\beta$, and add the term $2\pi\alpha' \int_{\mathcal{M}_2} F \wedge v$ overall (see ref. [63] for the details of this calculation). Integrating out v in this action gives us back the action we started with. However, we can also treat F as a Lagrange multiplier - it's equation of motion gives

$$\epsilon^{\alpha\beta\gamma}\partial_\beta \left(\mu_2 P[C^{(1)}]_\gamma + v_\gamma\right) = 0 \ . \qquad (1.4.18)$$

Integrating the above equation we find

$$\mu_2 P[C^{(1)}]_\alpha + v_\alpha = \partial_\alpha s \ , \qquad (1.4.19)$$

where the scalar s is the fourth modulus. Using (1.4.19) to eliminate v_α from the action, and using a static gauge, we can calculate the effective Lagrangian of the probe

$$\mathcal{L}_{eff} = F(r)\left(\dot{r}^2 + r^2\dot{\Omega}^2\right) + F(r)^{-1}\left(\frac{\dot{s}}{2} - \mu_2 C_\phi \frac{\dot{\phi}}{2}\right)^2 \ , \qquad (1.4.20)$$

where

$$F(r) = \frac{Z_6}{2g_s}\left(\mu_6 V_{K3}(r) - \mu_2\right) \ . \qquad (1.4.21)$$

Taking the low energy limit $\alpha' \to 0$, while holding the gauge theory coupling fixed

$$g_{YM}^2 = (2\pi)^4 g_s \alpha'^{3/2} V^{-1} \ , \qquad (1.4.22)$$

and the energy scale $U = r/\alpha'$, the metric on moduli space is given by

$$ds^2 = f(U)\left(dU^2 + U^2 d\Omega^2\right) + f(U)^{-1}\left(d\sigma - \frac{N}{4\pi^2} A_\phi d\phi\right)^2 \ , \qquad (1.4.23)$$

where

$$f(U) = \frac{1}{4\pi^2 g_{YM}^2}\left(1 - \frac{g_{YM}^2 N}{U}\right) \ , \qquad (1.4.24)$$

where $\sigma = \alpha' s$ and $A_\phi = \pm 1 - \cos\theta$. As we promised, the metric (1.4.23) is a Taub-NUT metric with negative mass parameter. By comparison with the monopole case, including the effects of instanton corrections should complete the metric (1.4.23) to the full Atiyah-Hitchin metric. However, this has not yet been shown explicitly.

1.4.2 D-Strings Attached to D3-Branes

As the second example of a D-brane configuration acting as a BPS monopole, we consider in this section a D-string attached to D3-branes.

To put this discussion into context, let us first consider a solution to the Abelian Dirac-Born-Infeld action for a Dp-brane, which was found by Callan and Maldacena in ref. [64]. They considered solutions with the electric field on the D-brane excited, and just one transverse field excited, X^9 say. The solution is given by

$$X^9 = \frac{c_p}{r^{p-2}} \ , \quad F_{0r} = \partial_r X^9 \ , \tag{1.4.25}$$

where r is the radial coordinate in the D-brane's worldvolume, and c_p is a constant. Recall that X^9 represents the position of the Dp-brane in the x^9 direction, and so the solution (1.4.25) looks like an infinitely long spike protruding from the worldvolume of the Dp-brane. For this reason this solution is called the BIon spike. Note that the gauge field in (1.4.25) is purely electric. Recall from section 1.1.1 that the gauge field on a D-brane results from the charge associated with the end of an open string. Therefore we expect this electric field to originate from the end of an open string. Indeed, a calculation of the energy density of the solution yields the energy of a semi-infinite fundamental string. This provides further evidence that this solution represents a fundamental string attached to the Dp-brane, and that the end of the string looks like a pointlike source of electric charge in the D-brane's worldvolume.

From now on we will restrict our attention to the case $p = 3$. Note that in the solution (1.4.25) the geometry of the D3-brane is bent into a spike in response to the tension exerted by the string. This phenomenon was studied in more detail in ref. [65]. The interpretation of the geometry of the solution (1.4.25), with a semi-infinite string attached to a D3-brane, is that the string is strong enough to pull the spike out of the D3-brane indefinitely; the local R-R charge of the spike is that of a D3-brane. In ref. [65] the electric-magnetic dual of the solution (1.4.25) was also studied. Under the duality, the fundamental string transforms into a D-string, the electric charge becomes a magnetic charge, and the D3-brane remains a D3-brane. So the dual picture consists of a D-string attached to a D3-brane, with the end of the D-string acting as a pointlike magnetic charge, i.e. a magnetic monopole. For a single D-string the monopole charge is one, and so the solution is the Prasad-Sommerfield solution given in equation (1.3.25).

As we have discussed above, a D-string should represent a magnetic monopole in the worldvolume of the D3-brane at the point where the string is attached. This configuration was studied in refs. [66] and [67], which we shall review next. Consider again the Dirac-Born-Infeld action with a single transverse field Φ^9 excited, but this time with a magnetic field excited, $B^i = 1/2\epsilon^{ijk}F_{jk}$. We substitute this ansatz into the Abelian Dirac-Born-Infeld action (1.2.8), setting all other fields to zero, and assuming flat background. We wish to find a static solution, so there is no kinetic energy, and the potential energy, which is the negative of the action we have obtained, is

$$\begin{aligned} E &= T_3 \int d^3\xi \sqrt{1 + (\alpha')^2|\vec{\nabla}\Phi^9|^2 + (\alpha')^2|\vec{B}|^2 + (\alpha')^4(\vec{B}\cdot\vec{\nabla}\Phi^9)^2} \\ &= T_3 \int d^3\xi \sqrt{(\alpha')^2|\vec{\nabla}\Phi^9 \pm \vec{B}|^2 + (1 \pm (\alpha')^2\vec{B}\cdot\vec{\nabla}\Phi^9)^2} \ . \end{aligned} \tag{1.4.26}$$

Using the Bianchi identity in the form $\vec{\nabla}\cdot\vec{B} = 0$, the second term under the square root is a total derivative. This term is topological since it only depends on the boundary values of the fields at infinity and near singular points. Since the first term under the square root in equation (1.4.26) is always positive, we can write

$$E \geq T_3 \int d^3\xi \left(1 \pm (\alpha')^2\vec{B}\cdot\vec{\nabla}\Phi^9\right) \ . \tag{1.4.27}$$

Equality holds in (1.4.27) when

$$\vec{\nabla}\Phi^9 = \pm\vec{B} \ , \tag{1.4.28}$$

which is the BPS condition for a magnetic monopole, with the transverse field Φ^9 acting as the Higgs field. The solution to (1.4.28) and the Bianchi identity, which corresponds to the solution (1.4.25) for the electric field, is

$$\Phi^9(r) = \frac{n}{2r} \ , \quad \vec{B}(\vec{r}) = \mp\frac{n}{2r^3}\vec{r} \ . \tag{1.4.29}$$

Substituting this solution into the energy (1.4.27) we find

$$E = T_3 \int d^3\xi + nT_1 \int_0^\infty d(\alpha'\Phi^9) \ , \tag{1.4.30}$$

which is the energy for a D3-brane with n infinitely long D-strings protruding from it, as we would expect.

The solution obtained above describes D-strings intersecting with a D3-brane. We should also be able to study this solution from the point of view of the fields living on the D-strings. Since we have n D-strings we should use the non-Abelian Born-Infeld action, which we discussed in section 1.2.2. We excite three of the transverse scalars Φ^i, $i \in \{1,2,3\}$ which will correspond to the D3-brane directions. The Φ^i are now $n \times n$ matrices transforming in the adjoint representation of the $U(n)$ worldvolume gauge group. We set all other fields to zero, and again assume flat background. We can use the non-Abelian Dirac-Born-Infeld action (1.2.19) to deduce the energy of this ansatz. Again, there is no kinetic energy, and so the total energy is given by

$$\begin{aligned} E &= T_1 \int d\sigma\, \mathrm{STr} \sqrt{1 + (\alpha')^2(\partial_\sigma \Phi^{\mathrm{i}})^2 - \frac{1}{2}(\alpha')^2[\Phi^{\mathrm{i}}, \Phi^{\mathrm{j}}]^2 - \frac{1}{4}(\alpha')^4 \left(\epsilon^{\mathrm{ijk}}\partial_\sigma \Phi^{\mathrm{i}}[\Phi^{\mathrm{j}}, \Phi^{\mathrm{k}}]\right)^2} \\ &= T_1 \int d\sigma\, \mathrm{STr} \sqrt{(\alpha')^2 \left(\partial_\sigma \Phi^{\mathrm{i}} \mp \frac{\mathrm{i}}{2}\epsilon^{\mathrm{ijk}}[\Phi^{\mathrm{j}}, \Phi^{\mathrm{k}}]\right)^2 + \left(1 \pm \frac{\mathrm{i}}{2}(\alpha')^2 \epsilon^{\mathrm{ijk}}\partial_\sigma \Phi^{\mathrm{i}}[\Phi^{\mathrm{j}}, \Phi^{\mathrm{k}}]\right)^2}\ . \end{aligned} \tag{1.4.31}$$

From (1.4.31) we can see that the first term under the square root is always positive, and so we can write

$$\begin{aligned} E &\geq T_1 \int d\sigma\, \mathrm{STr} \left(1 \pm \frac{\mathrm{i}}{2}(\alpha')^2 \epsilon^{\mathrm{ijk}}\partial_\sigma \Phi^{\mathrm{i}}[\Phi^{\mathrm{j}}, \Phi^{\mathrm{k}}]\right) \\ &= NT_1 \int d\sigma \pm \frac{i}{3}(\alpha')^2 T_1 \int d\sigma\, \partial_\sigma \mathrm{Tr}(\epsilon^{\mathrm{ijk}}\Phi^{\mathrm{i}}\Phi^{\mathrm{j}}\Phi^{\mathrm{k}})\ . \end{aligned} \tag{1.4.32}$$

We are allowed to do this because we are again just left with a topological term in (1.4.32). Equality in (1.4.32) holds when

$$\partial_\sigma \Phi^i = \pm \frac{i}{2}\epsilon^{ijk}[\Phi^j, \Phi^k]\ , \tag{1.4.33}$$

which are Nahm's equations.

So what we have is a nice physical realisation of the ADHMN construction of section 1.3.2, with the D3-brane point of view describing the (3+1)-dimensional gauge theory description of a monopole, and the D-string point of view describing the (1+1)-dimensional Nahm perspective. Nahm's equations in this context were also derived in ref. [68] using supersymmetry arguments.

The solution to Nahm's equations proposed in ref. [66] is

$$\Phi^i = \hat{r}(\sigma)\, \alpha^i \ , \tag{1.4.34}$$

where the α^i are the $n \times n$ matrix representation of the $SU(2)$ algebra, and

$$\hat{r}(\sigma) = \pm\frac{1}{2\sigma} \ . \tag{1.4.35}$$

which is again an infinitely long spike. The geometry of this solution is a noncommutative two-sphere whose physical radius is given by

$$r(\sigma) \sim \frac{\alpha'\sqrt{n^2-1}}{\sigma} \ . \tag{1.4.36}$$

where the factor of $\sqrt{n^2-1}$ in (1.4.36) is the Casimir of $SU(N)$ (see ref. [69]).

The range of validity of these solutions was discussed in ref. [66]. Recall that in the Born-Infeld action derivatives of the gauge field have been neglected. From the D3-brane perspective, the transverse fields are slowly varying as $r \longrightarrow \infty$, but they start to vary rapidly as $r \longrightarrow 0$. From the D-string perspective the reverse is true. Schematically we require $\sqrt{\alpha'}\partial^2\Phi^i \gg \partial\Phi^i$ in order to be able to ignore the contributions from the derivatives of the Φ^i. For the D3-brane solution (1.4.29) we have

$$r \gg \sqrt{\alpha'} \ , \quad \sigma \ll n\sqrt{\alpha'} \ , \tag{1.4.37}$$

and for the D-string solution (1.4.34) we have

$$\sigma \gg \sqrt{\alpha'} \ , \quad r \ll n l_s \ . \tag{1.4.38}$$

So the solutions are valid in two complementary regions of the space, which may overlap for large n.

We have discussed how Nahm's equations arise from the non-Abelian Dirac-Born-Infeld action for D-strings. But the ADHMN construction also specifies boundary conditions for the Nahm data. It was shown in refs. [70] and [71] how these arise in the D-brane picture. We will outline the argument from ref. [70], although we will not go into the details of the calculation, since it is beyond the scope of this thesis. The boundary conditions arise from considering the fundamental strings which connect the D3-brane to the D-strings. These are localised at the intersection of the two branes, and they prevent the ends of the D-strings from leaving the worldvolume of the D3-brane, since they would then become massive. In field theory language their degrees of freedom belong to the fundamental hypermultiplet of the supersymmetric gauge theory. Because they are localised, their terms in the low-energy effective Lagrangian come with a delta function $\delta(\sigma)$ attached. The equations of motion of this Lagrangian are Nahm's equations, with an extra term involving the delta function $\delta(\sigma)$. It is this term which leads to the pole in the solution at $\sigma = 0$.

Chapter 2

Enhançons as BPS Monopoles: the Moduli Space Perspective

In this chapter we describe the work carried out in our paper, ref. [1], in which we calculate the metric on moduli space for n enhançons. We have explained in section 1.4.1 how the enhançon behaves like a BPS monopole, and how the supersymmetric gauge theory on the branes forces the moduli space to be hyper-Kähler. Therefore we expect the metric on moduli space for the enhançons to be the Atiyah-Hitchin metric. Here we will work with the limit in which the enhançons are far apart from one another, and so we expect to obtain the generalised Taub-NUT metric (1.3.46) as our result. Although it has already been shown that the moduli space for a brane probe is Taub-NUT, it is worthwhile showing that this result does indeed generalise to the case of n enhançons.

We start, in section 2.1, by reviewing the procedure we will use to calculate the metric on moduli space; it was used in refs. [72], [73] and [74] to calculate the metric on moduli space for maximally charged black holes. In section 2.2 we dimensionally reduce over the brane directions to obtain a $(3+1)$-dimensional system. Then in section 2.3 we describe our calculation for the metric on moduli space, and in section 2.4 we summarise the chapter.

2.1 The Metric on Moduli Space of Maximally Charged Black Holes

We will use the procedure from refs. [72], [73] and [74] to calculate the metric on moduli space. In those papers the metric on moduli space was calculated for maximally charged Reissner-Nordstrom black holes in the Einstein-Maxwell system by applying the method of Manton from ref. [51] for the slow motion scattering of monopoles, which we reviewed in section 1.3.3. Since the black holes are maximally charged, their masses and charges saturate the Bogomol'nyi bound for black holes, $Q_a = G^{1/2}M_a$, where the index a labels the ath black hole. This means that, for stationary maximally charged black holes, the electromagnetic repulsion cancels the gravitational attraction, and so Manton's method is applicable. We will outline the steps of the calculation from these papers. We omit the details of the calculation, since they are very similar to those we will describe for the enhançons, in the rest of this chapter.

1. First allow the moduli, which are the positions of the black holes $\vec{x}_a$, to depend on time, and perturb the fields to take account of the time dependence.

2. Substitute the perturbed fields into the action which governs the system, neglecting all terms which are of order u_a^3 or higher, where u_a is the velocity of the ath black hole.

3. From this action, deduce the equations of motion for the field perturbations (these will be correct up to linear order in u_a).

4. Solve the equations of motion for the perturbations, and substitute them back into the action.

The resulting action will be an effective action, dependent only on the velocities $\vec{u}_a$ and the density of the static solution. From this action the metric on moduli space can be deduced, using (1.3.39).

The above calculation was generalised to the case of the Einstein-Maxwell-dilaton system in ref. [75]. The action for that system in four dimensions is given by

$$S_{EMD} \sim \int d^4x\,\sqrt{-g}\left(R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{4}e^{-a\Phi}(F^{(2)})^2\right) , \tag{2.1.1}$$

where R is the Ricci scalar, $F^{(2)}$ is the $U(1)$ gauge field strength, and Φ is the dilaton. We will take the dilaton constant coupling to be $a = 1$, which is the case for string theory. The solution to (2.1.1) for n black holes is given by

$$ds^2 = -F(\vec{x})^{-1}dt^2 + F(\vec{x})d\vec{x}^2 , \quad A = (1 - F(\vec{x})^{-1})dt , \tag{2.1.2}$$

$$e^{-2\Phi} = F(\vec{x}) , \tag{2.1.3}$$

where, if we include source terms,

$$F(\vec{x}) = 1 + \sum_{a=1}^{n} \frac{k_a}{|\vec{x} - \vec{x}_a|} . \tag{2.1.4}$$

In (2.1.4) k_a sets the mass of the ath black hole via $m_a = k_a/2$. This solution is very similar to the D-brane solution (1.1.23) - (1.1.25), if we generalise it to a higher-dimensional spacetime and allow the black holes to be higher-dimensional membrane objects. The calculation of Shiraishi was generalised to unwrapped branes in ref. [76]. Since these solutions are very similar to the enhançon solution, it makes sense for us to adopt the procedure used in these papers to calculate the metric on moduli space for the enhançons.

2.2 Dimensional Reduction to Four Dimensions

In order to apply Ferrell and Eardley's technique from refs. [72], [73] and [74], we must compactify over the spatial brane directions $x^4, \ldots, x^9$. This will leave us with the three-dimensional space transverse to the branes, as well as time, to make four dimensions in total. We will assume that the enhançons are far apart, so that they can be treated as pointlike objects in this four-dimensional spacetime. In this section we will describe the dimensional reduction of the brane directions, compactifying them on K3 × T^2. We follow the dimensional reduction procedure given in ref. [77]. We will allow the volumes of the K3 and the T^2 to vary; these will act as scalar fields in the dimensionally reduced version of the action. As in the enhançon supergravity solution (1.4.1) - (1.4.4) these scalar fields will be dependent only on the directions transverse to all branes, x^1, x^2 and x^3, and time.

Our dimensional reduction procedure applies to compactification on any Ricci flat six-dimensional manifold. We choose to focus on the case of $K3 \times T^2$ because, from what we know about the enhançon mechanism, we expect the moduli space to be Taub-NUT in this case. Also because compactification on K3 gives us $\mathcal{N} = 1$ supersymmetry for the $(2+1)$-dimensional gauge theory on the D2-branes, whose moduli space of vacua is known to be Taub-NUT.

2.2.1 The Type IIA Action in the Einstein Frame

We start with the type IIA ten-dimensional supergravity action in the string frame with a D6-brane field strength and a D2-brane field strength excited. From (1.1.16), this is

$$S^S_{IIA} = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-g^S} \left\{ e^{-2\Phi}[R^S + 4(\nabla^S\Phi)^2] - \frac{1}{2.4!}(F^{(4)}_S)^2 - \frac{1}{2.8!}(F^{(8)}_S)^2 \right\} . \tag{2.2.1}$$

It will be more convenient for us to work in the Einstein frame, so we convert from string frame to Einstein frame by setting

$$g^S_{MN} = e^{(\Phi-\Phi_0)/2} g^E_{MN} , \tag{2.2.2}$$

where the superscript S denotes the string frame, and the superscript E denotes the Einstein frame. Here M, N denote spacetime indices, $M, N = 0, \ldots, 9$. In (2.2.2) Φ_0 is the expectation value of the dilaton field Φ, and $g_s = e^{\Phi_0}$ is the string coupling. Let $\tilde{\Phi} = (\Phi - \Phi_0)$. Then

$$\sqrt{-g^S} = e^{\frac{10\tilde{\Phi}}{4}} \sqrt{-g^E} . \tag{2.2.3}$$

The formula for the Ricci scalar under a conformal transformation of the form (2.2.2) is

$$R^S = e^{-\tilde{\Phi}/2}\left(R^E - \frac{(D-1)}{2}((\nabla^E)^2\tilde{\Phi}) - \frac{(D-2)(D-1)}{2}(\nabla^E\tilde{\Phi})^2 \right) . \tag{2.2.4}$$

where D is the number of dimensions (see for example [78]). Since we are working in ten dimensions, this yields

$$R^S = e^{-\tilde{\Phi}/2}\left(R^E - \frac{9}{2}((\nabla^E)^2\tilde{\Phi}) - \frac{9}{16}(\nabla^E\tilde{\Phi})^2\right) . \qquad (2.2.5)$$

The kinetic terms in (2.2.1) transform as follows

$$(\nabla^S\Phi)^2 = e^{-\tilde{\Phi}/2}(\nabla^E\tilde{\Phi})^2 , \qquad (2.2.6)$$

$$(F_S^{(4)})^2 = g_s^{-2}e^{-2\tilde{\Phi}}(F_E^{(4)})^2 , \quad (F_S^{(8)})^2 = g_s^{-2}e^{-4\tilde{\Phi}}(F_E^{(8)})^2 , \qquad (2.2.7)$$

where we have defined

$$F_S^{(4)} = g_s^{-1}F_E^{(4)} , \quad F_S^{(8)} = g_s^{-1}F_E^{(8)} . \qquad (2.2.8)$$

Substituting (2.2.5) - (2.2.7) into (2.2.1) gives us the action in the Einstein frame

$$\begin{aligned} S_{IIA}^E &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g^E}\left\{R^E - \frac{1}{2}(\nabla^E\tilde{\Phi})^2 - \frac{1}{2.4!}e^{\frac{\tilde{\Phi}}{2}}(F_E^{(4)})^2\right. \\ &\quad \left. - \frac{1}{2.8!}e^{-\frac{3\tilde{\Phi}}{2}}(F_E^{(8)})^2\right\} , \end{aligned} \qquad (2.2.9)$$

For simplicity, in what follows we will relabel $\tilde{\Phi}$ as Φ.

2.2.2 Dimensional Reduction on K3

Here we dimensionally reduce the wrapped brane dimensions on $K3$. After the dimensional reduction it will be necessary to perform a conformal transformation so that the gravity part of the dimensionally reduced action is of the Einstein-Hilbert form.

Dimensional Reduction

We rewrite the action (2.2.9) as

$$S_{IIA} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}\left\{\hat{R} - \frac{1}{2}(\hat{\nabla}\Phi)^2 - \frac{1}{2.4!}e^{\frac{\Phi}{2}}(\hat{F}^{(4)})^2 - \frac{1}{2.8!}e^{-\frac{3\Phi}{2}}(\hat{F}^{(8)})^2\right\} , \qquad (2.2.10)$$

where we have relabelled all the fields with hats to indicate that they are ten-dimensional fields. The hat in $(\hat{F}^{(4)})^2$ also indicates that the metric $\hat{g}_{\mu\nu}$ is used to compute the square, i.e.

$$(\hat{F}^{(4)})^2 = \hat{g}^{\mu_1\nu_1}\hat{g}^{\mu_2\nu_2}\hat{g}^{\mu_3\nu_3}\hat{g}^{\mu_4\nu_4}\hat{F}^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\hat{F}^{(4)}_{\nu_1\nu_2\nu_3\nu_4} . \qquad (2.2.11)$$

$\hat{F}^{(4)}$ is a 4-form field strength with potential $\hat{F}^{(4)} = d\hat{C}^{(3)}$. Similarly $\hat{F}^{(8)} = d\hat{C}^{(7)}$. We wish to calculate the dimensionally reduced version of (2.2.10) when the directions x^6, x^7, x^8, x^9 are compactified on a K3 manifold.

We set

$$\hat{g}_{MN} = \begin{pmatrix} \bar{g}_{\rho\sigma} & 0 \\ 0 & V^{1/2}\, e^{\beta/2}\, g^{K3}_{mn} \end{pmatrix} , \tag{2.2.12}$$

where $M, N = 0, \ldots, 9$, $\rho, \sigma = 0, \ldots, 5$, $m, n = 6, \ldots, 9$ and $V_{K3} = Ve^{\beta}$ is the volume of the K3 manifold, with V being a constant of dimension L^4. The form of the metric which we have chosen in (2.2.12) is consistent with the enhançon metric (1.4.1).

We will assume that $\bar{g}_{\rho\sigma}$, β, Φ and $\hat{F}^{(8)}$ are independent of the compactified directions x^m. We also take the metric on $K3$, g^{K3}_{mn}, to depend only on the x^m. Again, these assumptions are consistent with the enhançon solution (1.4.1) - (1.4.4). Then a calculation, which we have given in appendix A, yields

$$\hat{R} = \bar{R} - \frac{5}{4}(\bar{\nabla}\beta)^2 - 2(\bar{\nabla}^2\beta) . \tag{2.2.13}$$

Also

$$\sqrt{-\hat{g}} = Ve^{\beta}\sqrt{-\bar{g}}\,\det(g^{K3}) . \tag{2.2.14}$$

We make the choice that each non-vanishing component of $\hat{C}^{(7)}$ contains the indices 6,7,8,9. This is consistent with the form of the enhançon solution for $C^{(7)}$ (1.4.4). We set

$$\bar{C}'^{(3)}_{\alpha_1\alpha_2\alpha_3} = V^{-1}\hat{C}^{(7)}_{\alpha_1\alpha_2\alpha_3 6789} , \tag{2.2.15}$$

where we have used a prime to distinguish the dimensionally reduced D6-brane potential $\bar{C}'^{(3)}$ from the D2-brane potential, which in the dimensionally reduced action we will denote $\bar{C}^{(3)}$. Then

$$(\hat{F}^{(8)})^2 = 8.7.6.5\, e^{-2\beta}(\bar{F}'^{(4)})^2 \tag{2.2.16}$$

Also

$$\hat{C}^{(3)}_{\alpha_1\alpha_2\alpha_3} = \bar{C}^{(3)}_{\alpha_1\alpha_2\alpha_3} , \tag{2.2.17}$$

since $\hat{C}^{(3)}$ does not have components in the $K3$ directions.

Substituting (2.2.13) - (2.2.17) into the action (2.2.10), we get the six-dimensional action

$$\begin{aligned} S_{IIA} &= \frac{1}{2\kappa^2}\int d^6x\, V e^{\beta}\sqrt{-\bar{g}}\left[\bar{R} - \frac{5}{4}(\bar{\nabla}\beta)^2 - 2(\bar{\nabla}^2\beta) - \frac{1}{2}(\bar{\nabla}\Phi)^2 \right. \\ & \left. -\frac{1}{2.4!}e^{\frac{\Phi}{2}}(\bar{F}^{(4)})^2 - \frac{1}{2.4!}e^{-\frac{3\Phi}{2}}e^{-2\beta}(\bar{F}'^{(4)})^2\right] . \end{aligned} \tag{2.2.18}$$

Conformal Transformation

The action (2.2.18) is not of the standard form since there is a factor of e^{β} multiplying $\bar{R}$. In order to remove this factor we perform the following conformal transformation:

$$\bar{g}_{\rho\sigma} = e^{-\beta/2}\tilde{g}_{\rho\sigma} . \tag{2.2.19}$$

Then, again using the formula (2.2.4) for the Ricci scalar under a conformal transformation, we find

$$\bar{R} = e^{\beta/2}\left[\tilde{R} + \frac{5}{2}\tilde{\nabla}^2\beta - \frac{5}{4}(\tilde{\nabla}\beta)^2\right] . \tag{2.2.20}$$

Also

$$\sqrt{-\bar{g}} = e^{-3\beta/2}\sqrt{-\tilde{g}} , \tag{2.2.21}$$

and

$$(\bar{\nabla}\beta)^2 = e^{\beta/2}(\tilde{\nabla}\beta)^2 , \quad (\bar{\nabla}\Phi)^2 = e^{\beta/2}(\tilde{\nabla}\Phi)^2 , \tag{2.2.22}$$

$$(\bar{\nabla}^2\beta) = e^{\beta/2}(\tilde{\nabla}^2\beta) - e^{\beta/2}(\tilde{\nabla}\beta)^2 . \tag{2.2.23}$$

For the gauge field terms,

$$(\bar{F}^{(4)})^2 = e^{2\beta}(\tilde{F}^{(4)})^2 , \quad (\bar{F}'^{(4)})^2 = e^{2\beta}(\tilde{F}'^{(4)})^2 . \tag{2.2.24}$$

Substituting (2.2.20) - (2.2.24) into the action (2.2.18), we get

$$\begin{aligned} S_{IIA} &= \frac{V}{2\kappa^2}\int d^6x\, \sqrt{-\tilde{g}}\left[\tilde{R} - \frac{1}{2}(\tilde{\nabla}\beta)^2 - \frac{1}{2}(\tilde{\nabla}\Phi)^2 \right. \\ & \left. -\frac{1}{2.4!}e^{\frac{\Phi}{2}}e^{3\beta/2}(\tilde{F}^{(4)})^2 - \frac{1}{2.4!}e^{-\frac{3\Phi}{2}}e^{-\beta/2}(\tilde{F}'^{(4)})^2\right] , \end{aligned} \tag{2.2.25}$$

where we have neglected the total derivative terms.

2.2.3 Compactifying on T^2

Here we dimensionally reduce the action (2.2.25) over the directions x^4, x^5, which we will compactify on a T^2.

Dimensional Reduction

We set

$$\tilde{g}_{\rho\sigma} = \begin{pmatrix} \bar{\bar{g}}_{\mu\nu} & 0 \\ 0 & e^{\rho}\,\delta_{rs} \end{pmatrix} , \tag{2.2.26}$$

where $\rho, \sigma = 0, \ldots, 5$, $\mu, \nu = 0, \ldots, 3$, $r, s = 4, 5$. As in section 2.2.2 we assume that $\bar{\bar{g}}^{\mu\nu}$, ρ, $\hat{\Phi}$, $\tilde{F}^{(4)}$ and $\tilde{F}'^{(4)}$ are independent of the compactified directions x^4, x^5. Then a similar calculation to that given in appendix A for the dimensionally reduced Ricci scalar yields

$$\tilde{R} = \bar{\bar{R}} - 2\bar{\bar{\nabla}}^2\rho - \frac{3}{2}(\bar{\bar{\nabla}}\rho)^2 . \tag{2.2.27}$$

Also

$$\sqrt{-\tilde{g}} = e^{\rho}\sqrt{-\bar{\bar{g}}} , \tag{2.2.28}$$

and

$$(\tilde{\nabla}\beta)^2 = (\bar{\bar{\nabla}}\beta)^2 , \quad (\tilde{\nabla}\Phi)^2 = (\bar{\bar{\nabla}}\Phi)^2 . \tag{2.2.29}$$

We make the choice that the non-vanishing components of $\tilde{C}^{(3)}$ and $\tilde{C}'^{(3)}$ contain the indices 0,4,5, which is consistent with the form of the enhançon solution for $C^{(3)}$ (1.4.3) and $C^{(7)}$ (1.4.4). We set

$$\bar{\bar{C}}^{(1)}_0 = \tilde{C}^{(3)}_{045} , \quad \bar{\bar{C}}'^{(1)}_0 = \tilde{C}'^{(3)}_{045} . \tag{2.2.30}$$

Then

$$(\tilde{F}^{(4)})^2 = 4.3.e^{-2\rho}\bar{\bar{F}}^2 , \quad (\tilde{F}'^{(4)})^2 = 4.3.e^{-2\rho}\bar{\bar{F}}'^2 . \tag{2.2.31}$$

Substituting (2.2.27) - (2.2.31) into (2.2.25) we get the four-dimensional action

$$\begin{aligned} S_{IIA} &= \frac{L^2 V}{16\pi\hat{G}} \int d^4x\, e^{\rho}\sqrt{-\bar{\bar{g}}} \left[\bar{\bar{R}} - 2\bar{\bar{\nabla}}^2\rho - \frac{3}{2}(\bar{\bar{\nabla}}\rho)^2 - \frac{1}{2}(\bar{\bar{\nabla}}\beta)^2 - \frac{1}{2}(\bar{\bar{\nabla}}\Phi)^2 \right. \\ &\left. -\frac{1}{4}e^{\frac{\Phi}{2}}e^{3\beta/2}e^{-2\rho}(\bar{\bar{F}}^{(2)})^2 - \frac{1}{4}e^{-\frac{3\Phi}{2}}e^{-\beta/2}e^{-2\rho}(\bar{\bar{F}}'^{(2)})^2 \right] , \end{aligned} \tag{2.2.32}$$

where L is a constant which will set the lengthscale of the compactified T^2 dimensions at spatial infinity.

Conformal Transformation

Again we wish to convert the action (2.2.32) to the standard form. We remove the factor of e^{ρ} multiplying $\bar{\bar{R}}$ in the action (2.2.32) by performing the following conformal transformation

$$\bar{\bar{g}}_{\mu\nu} = e^{-\rho} g_{\mu\nu} \ . \tag{2.2.33}$$

Then the Ricci scalar becomes

$$\bar{\bar{R}} = e^{\rho}\left[R + 3\nabla^2\rho - \frac{3}{2}(\nabla\rho)^2\right] , \tag{2.2.34}$$

and

$$\sqrt{-\bar{\bar{g}}} = e^{-2\rho}\sqrt{-g} \ . \tag{2.2.35}$$

The scalar fields transform as follows

$$(\bar{\bar{\nabla}}\rho)^2 = e^{\rho}(\nabla\rho)^2 \ , \quad (\bar{\bar{\nabla}}\beta)^2 = e^{\rho}(\nabla\beta)^2 \ , \quad (\bar{\bar{\nabla}}\Phi)^2 = e^{\rho}(\nabla\Phi)^2 \ , \tag{2.2.36}$$

and

$$(\bar{\bar{\nabla}}^2\rho) = e^{\rho}(\nabla^2\rho) - e^{\rho}(\nabla\rho)^2 \ . \tag{2.2.37}$$

For the gauge field terms

$$(\bar{\bar{F}}^{(2)})^2 = e^{2\rho}(\bar{\bar{F}}^{(2)})^2 \ , \quad (\bar{\bar{F}}'^{(2)})^2 = e^{2\rho}(\bar{\bar{F}}'^{(2)})^2 \ . \tag{2.2.38}$$

Substituting (2.2.34) - (2.2.38) into the action (2.2.32) and neglecting total derivative terms, we get the dimensionally reduced action

$$\begin{aligned} S_{IIA} &= \frac{L^2 V}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - (\nabla\rho)^2 - \frac{1}{2}(\nabla\beta)^2 - \frac{1}{2}(\nabla\Phi)^2\right. \\ &\left. -\frac{1}{4}e^{\frac{\Phi}{2}}e^{3\beta/2}e^{-\rho}(F^{(2)})^2 - \frac{1}{4}e^{-\frac{3\Phi}{2}}e^{-\beta/2}e^{-\rho}(F'^{(2)})^2\right] \ . \end{aligned} \tag{2.2.39}$$

This is the four-dimensional action of the compactified theory.

2.2.4 A Symmetry of the Action

Note that the six-dimensional action (2.2.25) is invariant under the transformation $\Phi \longrightarrow -\beta$, $\beta \longrightarrow -\Phi$ and $F^{(4)} \leftrightarrow F'^{(4)}$ and the four-dimensional action is invariant under a similar transformation with $F^{(2)} \leftrightarrow F'^{(2)}$. That is, the dilaton is exchanged with the volume of K3, while the D2-brane and D6-brane potentials are interchanged.

This symmetry is a consequence of the duality between type IIA strings compactified on K3 and heterotic strings compactified on T^4. Under this duality, the dilaton field e^{Φ}, which plays the role of the type IIA string coupling in the type IIA string action, becomes the volume of the T^4 in the heterotic string action. And vice versa, the field e^{β} plays the role of the volume of K3 in the type IIA action, and the role of the heterotic string coupling in the heterotic string action.

In terms of the field strengths, the $F^{(8)}$ field strength in the ten-dimensional type IIA action is Hodge dual to a $F^{(2)}$ field strength. The $F^{(2)}$ field strength is not wrapped in the six-dimensional type IIA theory. Under the heterotic-type IIA duality this $F^{(2)}$ field strength becomes an $F^{(2)}$ field strength in the heterotic theory, which is wrapped on the T^4 directions. Therefore in the ten-dimensional heterotic string theory the corresponding field strength is $F^{(6)}$, which is Hodge dual to $F^{(4)}$. So the $F^{(8)}$ in the ten-dimensional string theory becomes $F^{(4)}$ in the heterotic theory, and vice versa, as the transformation requires. In other words, the D6-brane charges in the type IIA string theory are transformed into Kaluza-Klein monopole charges in the heterotic string theory, and the D2-brane charges are transformed into H-monopole (wrapped NS5-brane) charges.

2.2.5 The Static Solution

We restate here the ten-dimensional solution which corresponds to the enhançon (at least outside the enhançon radius),

$$
\begin{aligned}
ds_S^2 &= Z_2^{-1/2} Z_6^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + Z_2^{1/2} Z_6^{1/2} dx^i dx^i + V^{1/2} Z_2^{1/2} Z_6^{-1/2} dS_{K3}^2 , \\
e^{2\Phi} &= g_s^2 Z_2^{1/2} Z_6^{-3/2} , \\
C^{(3)} &= g_s^{-1}(1 - Z_2^{-1}) dx^0 \wedge dx^4 \wedge dx^5 , \\
C^{(7)} &= V g_s^{-1}(1 - Z_6^{-1}) dx^0 \wedge dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9 .
\end{aligned}
$$

This is a solution to the ten-dimensional type IIA action, (2.2.1) with added source terms (which we will consider in section 2.3.3). We need to apply the compactification procedure from sections 2.2.2 and 2.2.3 to this solution in order to obtain the enhançon solution to the four-dimensional compactified action (2.2.39).

We convert the string frame solution to the Einstein frame using

$$\begin{aligned} g^E_{MN} &= e^{-(\Phi-\Phi_0)/2} g^S_{MN} \\ &= Z_2^{-1/8} Z_6^{3/8} g^S_{MN} , \end{aligned} \quad (2.2.40)$$

and we apply the formula (2.2.8) for the gauge field. Then the ten-dimensional solution in the Einstein frame is

$$\begin{aligned} ds_E^2 &= Z_2^{-5/8} Z_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + Z_2^{3/8} Z_6^{7/8} dx^i dx^i \\ &\quad + V^{1/2} Z_2^{3/8} Z_6^{-1/8} dS_{K3}^2 , \end{aligned} \quad (2.2.41)$$

$$e^{2\tilde{\Phi}} = Z_2^{1/2} Z_6^{-3/2} , \quad (2.2.42)$$

$$C^{(3)} = (1 - Z_2^{-1}) dx^0 \wedge dx^4 \wedge dx^5 , \quad (2.2.43)$$

$$C_E^{(7)} = V(1 - Z_6^{-1}) dx^0 \wedge dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9 . \quad (2.2.44)$$

Again we will relabel $\tilde{\Phi}$ as Φ. We wish to compactify this ten-dimensional solution following the same steps as in sections 2.2.2 and 2.2.3 to obtain the four-dimensional solution to the action (2.2.39).

Compactifying on K3

First we compactify the ten-dimensional solution (2.2.41) - (2.2.44) over the $K3$ directions x^6, x^7, x^8, x^9 as in section 2.2.2

We apply the dimensional reduction procedure from section 2.2.2, as described in equations (2.2.12), (2.2.15) and (2.2.17), to the enhançon solution (2.2.41) - (2.2.44). We thus obtain the six-dimensional enhançon solution to the action (2.2.18), which is

$$d\bar{s}^2 = Z_2^{-5/8} Z_6^{-1/8} \eta_{\alpha\beta} dx^\alpha dx^\beta + Z_2^{3/8} Z_6^{7/8} dx^i dx^i , \quad (2.2.45)$$

$$e^\beta = Z_2^{3/4} Z_6^{-1/4} , \quad e^{2\Phi} = Z_2^{1/2} Z_6^{-3/2} , \quad (2.2.46)$$

$$\bar{C}^{(3)} = (1 - Z_2^{-1}) dx^0 \wedge dx^4 \wedge dx^5 , \quad (2.2.47)$$

$$\bar{C}'^{(3)} = (1 - Z_6^{-1}) dx^0 \wedge dx^4 \wedge dx^5 . \quad (2.2.48)$$

We then perform the conformal transformation (2.2.19) which implies

$$\tilde{g}_{\rho\sigma} = Z_2^{3/8} Z_6^{-1/8} \bar{g}_{\rho\sigma} . \quad (2.2.49)$$

So the solution to the action (2.2.25) is

$$\tilde{ds}^2 = Z_2^{-1/4} Z_6^{-1/4} \eta_{\alpha\beta} dx^\alpha dx^\beta + Z_2^{3/4} Z_6^{3/4} dx^i dx^i \ , \tag{2.2.50}$$

$$e^\beta = Z_2^{3/4} Z_6^{-1/4} \ , \quad e^{2\Phi} = Z_2^{1/2} Z_6^{-3/2} \ , \tag{2.2.51}$$

$$\tilde{C}^{(3)} = (1 - Z_2^{-1}) dx^0 \wedge dx^4 \wedge dx^5 \ , \tag{2.2.52}$$

$$\tilde{C}'^{(3)} = (1 - Z_6^{-1}) dx^0 \wedge dx^4 \wedge dx^5 \ . \tag{2.2.53}$$

Compactifying on T^2

Now we wish to compactify the six-dimensional solution (2.2.50) - (2.2.53) over the T^2 directions x^4, x^5 following the same steps as in section 2.2.3.

We apply the dimensional reduction procedure from equations (2.2.26) and (2.2.30) of section 2.2.3 to obtain the four-dimensional enhançon solution to the action (2.2.32), which is

$$\bar{\bar{ds}}^2 = -Z_2^{-1/4} Z_6^{-1/4} dx^0 dx^0 + Z_2^{3/4} Z_6^{3/4} dx^i dx^i \ , \tag{2.2.54}$$

$$e^\rho = Z_2^{-1/4} Z_6^{-1/4} \ , \quad e^\beta = Z_2^{3/4} Z_6^{-1/4} \ , \quad e^{2\Phi} = Z_2^{1/2} Z_6^{-3/2} \ , \tag{2.2.55}$$

$$\bar{\bar{C}}^{(1)} = (1 - Z_2^{-1}) dx^0 \ , \quad \bar{\bar{C}}'^{(1)} = (1 - Z_6^{-1}) dx^0 \ . \tag{2.2.56}$$

We then perform the conformal transformation (2.2.33), which implies

$$g_{\mu\nu} = Z_2^{-1/4} Z_6^{-1/4} \bar{\bar{g}}_{\mu\nu} \ . \tag{2.2.57}$$

So the four-dimensional enhançon solution to the action (2.2.39) is

$$ds^2 = -Z_2^{-1/2} Z_6^{-1/2} dx^0 dx^0 + Z_2^{1/2} Z_6^{1/2} dx^i dx^i \ , \tag{2.2.58}$$

$$e^\rho = Z_2^{-1/4} Z_6^{-1/4} \ , \quad e^\beta = Z_2^{3/4} Z_6^{-1/4} \ , \quad e^{2\Phi} = Z_2^{1/2} Z_6^{-3/2} \ , \tag{2.2.59}$$

$$C^{(1)} = (1 - Z_2^{-1}) dx^0 \ , \quad C'^{(1)} = (1 - Z_6^{-1}) dx^0 \ . \tag{2.2.60}$$

2.3 The Multi-Enhançon Moduli Space

We observe that the dimensionally reduced action (2.2.39) is identical to the action for four-dimensional gravity, with two $U(1)$ gauge potentials, coupled to three scalar fields, as we would expect. As we discussed in section 2.1, the form of the four-dimensional enhançon solution (2.2.58) - (2.2.60) is very similar to that of Reissner-Nordstrom black holes in the Einstein-Maxwell-dilaton system.

In this section we follow the steps outlined in section 2.1 to calculate the metric on moduli space for the enhançons. We then describe how to augment this calculation to include the phase terms in the metric.

2.3.1 The Static Solution

We start with the static four-dimensional enhançon solution given by (2.2.58) - (2.2.60), which we restate here, relabelling $C'^{(1)}$ as $\tilde{C}^{(1)}$,

$$\begin{aligned} ds^2 &= -Z_2^{-1/2} Z_6^{-1/2} dx^0 dx^0 + Z_2^{1/2} Z_6^{1/2} dx^i dx^i \ , &(2.3.1)\\ e^\rho &= Z_2^{-1/4} Z_6^{-1/4} \ , \quad e^\beta = Z_2^{3/4} Z_6^{-1/4} \ , \quad e^{2\Phi} = Z_2^{1/2} Z_6^{-3/2} \ , &(2.3.2)\\ C^{(1)} &= (1 - Z_2^{-1}) dx^0 \ , \quad \tilde{C}^{(1)} = (1 - Z_6^{-1}) dx^0 \ . &(2.3.3) \end{aligned}$$

The solutions for ds^2, e^Φ, $C^{(1)}$ and $\tilde{C}^{(1)}$ are in agreement with the black hole solution (2.1.2) - (2.1.4) if we take $Z_2 = Z_6$. The main difference between the black hole solution and the enhançon solution is the extra scalar fields β and ρ in the enhançon solution.

Since we are taking the limit where the enhançons are a long distance apart, we can ignore their spatial extent, and assume that they are pointlike. Therefore the source terms for the $U(1)$ charges in the action have the form of δ-functions. Then the equations of motion for $C^{(1)}$ and $\tilde{C}^{(1)}$ imply that Z_2 and Z_6 obey

$$\nabla^2 Z_2 = -4\pi \sum_a (r_2)_a \, \delta^{(3)}(\vec{x} - \vec{x}_a) \ , \quad \nabla^2 Z_6 = -4\pi \sum_a (r_6)_a \, \delta^{(3)}(\vec{x} - \vec{x}_a) \ , \qquad (2.3.4)$$

where the positions of the enhançons are denoted $\vec{x}_a$, with $a = 1, \ldots, n$. As expected, the $\vec{x}_a$ are the moduli of the solution. We write

$$(r_2)_a = -\frac{(2\pi)^4 g_s Q_a \alpha'^{5/2}}{2V} \ , \quad (r_6)_a = \frac{g_s Q_a \alpha'^{1/2}}{2} \ , \qquad (2.3.5)$$

where Q_a is the number of D6-branes in the ath enhançon. Equations (2.3.4) have the asymptotically flat solutions

$$Z_2 = 1 + \sum_a \frac{(r_2)_a}{|\vec{x} - \vec{x}_a|} \ , \quad Z_6 = 1 + \sum_a \frac{(r_6)_a}{|\vec{x} - \vec{x}_a|} \ . \qquad (2.3.6)$$

The solutions (2.3.6) contain singularities at $\vec{x} = \vec{x}_a$, which lead to infinities in the calculation of the effective action. As in ref. [74] we will avoid these infinities by assuming a general charge distribution $\tilde{Q}(\vec{x})$. Then (2.3.4) become

$$\nabla^2 Z_2 = -\frac{(2\pi)^4 g_s\, \alpha'^{5/2}}{2V}\, \tilde{Q}(\vec{x})\ , \quad \nabla^2 Z_6 = \frac{g_s\, \alpha'^{1/2}}{2}\, \tilde{Q}(\vec{x})\ . \tag{2.3.7}$$

We will take the enhançon limit $\tilde{Q} \to \sum_a Q_a \delta^{(3)}(\vec{x} - \vec{x}_a)$ in the final stage of the calculation. It will turn out that the infinities will cancel to leave a finite answer for the effective action (i.e. we have to regularise the problem, but renormalisation is not necessary).

2.3.2 Perturbing the Static Solution

In the slow-velocity approximation we can make the static solutions time-dependent by allowing the moduli to depend on time $\vec{x}_a \to \vec{x}_a(t)$. We define $\vec{u}_a$ to be the velocity of the ath enhançon, so that $\vec{u}_a = \dot{\vec{x}}_a(t)$. For the general charge density we define $\vec{u} = \dot{\vec{x}}(t)$ to be the velocity of a charged particle of dust.

We perturb the enhançon solution (2.3.1) - (2.3.3) to take into account the effects of the time dependence. Since we are assuming that $u = |\vec{u}|$ is small we only need calculate the perturbed fields to $O(u)$. As in [74], we can expand fields which are even or odd under time reversal in even or odd powers respectively of u. Then a Taylor expansion of a Lagrangian about a static solution reveals that first-order perturbations in quantities which are even under time reversal vanish. Therefore the perturbed solution has the form

$$\begin{aligned} ds^2 &= -Z_2^{-1/2} Z_6^{-1/2} dt^2 + Z_2^{1/2} Z_6^{1/2} d\vec{x}^2 + 2\vec{N}\cdot d\vec{x}\ , && (2.3.8)\\ C^{(1)} &= (1 - (Z_2)^{-1})\, dt + \vec{A}\cdot d\vec{x}\ , && (2.3.9)\\ \tilde{C}^{(1)} &= (1 - (Z_6)^{-1})\, dt + \vec{\tilde{A}}\cdot d\vec{x}\ , && (2.3.10) \end{aligned}$$

where $\vec{N}$, $\vec{A}$ and $\vec{\tilde{A}}$ are perturbations which are of first-order in the velocity $\vec{u}$. The scalar fields Φ, β and ρ remain unperturbed. The perturbations $\vec{A}$ and $\vec{N}$ depend on time through $\vec{x}(t)$.

According to Manton's technique we can neglect radiation effects, because these effects are of higher order than u^2. Therefore we can assume that the energy in the system remains in the zero modes; the non-zero modes are not excited. This means that the motion takes the form of a geodesic in moduli space, as we explained in section 1.3.3.

2.3.3 The Action in the Slow Motion Limit

We wish to find the equations of motion for the perturbations $\vec{N}$, $\vec{A}$ and $\vec{\tilde{A}}$ in order to express these fields as functions of $\tilde{Q}$ and $\vec{u}$. We need expressions for $\vec{N}$, $\vec{A}$ and $\vec{\tilde{A}}$ to $O(u)$, so we must calculate the perturbed action to $O(u^2)$. Therefore we will substitute the perturbed solutions (2.3.8) - (2.3.10) into the action, neglecting terms $O(u^3)$, then we will derive equations of motion for the perturbations from the resulting approximate action.

In section 2.2 we found that the ten-dimensional type IIA supergravity action with four dimensions compactified on $K3$, and two dimensions compactified on T^2 reduces to the following four-dimensional action

$$S_{IIB} = S_{gravity} + S_{Maxwell} + S_{scalar} \ , \tag{2.3.11}$$

where

$$S_{gravity} = k \int d^4x \sqrt{-g}\, R \ , \tag{2.3.12}$$

$$S_{Maxwell} = k \int d^4x \sqrt{-g} \left(-\frac{1}{4} e^{\Phi/2} e^{3\beta/2} e^{-\rho} (F^{(2)})^2 - \frac{1}{4} e^{-3\Phi/2} e^{-\beta/2} e^{-\rho} (\tilde{F}^{(2)})^2 \right) , \tag{2.3.13}$$

$$S_{scalar} = k \int d^4x \sqrt{-g} \left(-(\nabla\rho)^2 - \frac{1}{2}(\nabla\beta)^2 - \frac{1}{2}(\nabla\Phi)^2 \right) , \tag{2.3.14}$$

where $k = L^2 V / 2\kappa^2$ is a constant.

Substituting the perturbed solutions (2.3.8) - (2.3.10) into the action (2.3.11), and integrating by parts several times, we find

$$\begin{aligned}
S_{IIB}^{approx} &= k \int d^4x \left\{ -\frac{1}{2}\frac{|\vec{\nabla} \times (\vec{A} + Z_2^{-1/2} Z_6^{1/2} \vec{N})|^2}{Z_2^{-1} Z_6} - \frac{1}{2}\frac{|\vec{\nabla} \times (\vec{\tilde{A}} + Z_2^{1/2} Z_6^{-1/2} \vec{N})|^2}{Z_2 Z_6^{-1}} \right. \\
&+ \frac{(\vec{\nabla} \times (\vec{A} + Z_2^{-1/2} Z_6^{1/2} \vec{N})) \cdot (\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \vec{N}))}{Z_6} \\
&+ \frac{(\vec{\nabla} \times (\vec{\tilde{A}} + Z_2^{1/2} Z_6^{-1/2} \vec{N})) \cdot (\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \vec{N}))}{Z_2} \\
&- \frac{1}{2}\frac{|\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \vec{N})|^2}{Z_2 Z_6} - \dot{Z}_2 \dot{Z}_6 - \vec{\nabla}(\dot{Z}_2) \cdot \left(\vec{A} + Z_2^{-1/2} Z_6^{1/2} \vec{N}\right) \\
&\left. - \vec{\nabla}(\dot{Z}_6) \cdot \left(\vec{\tilde{A}} + Z_2^{1/2} Z_6^{-1/2} \vec{N}\right) \right\} . \qquad (2.3.15)
\end{aligned}$$

We also need to include source terms in the action for the matter density and for the current. To find the matter source terms we need to dimensionally reduce the Born-Infeld action for the D6-branes and the Born-Infeld action for the D2-branes. From (1.2.8) of section 1.2.1 we see that these are given by

$$S_{matter} = - \int d^7\xi \, e^{-\Phi} \, T_6 \sqrt{-\det \hat{G}^S} + \int d^3\xi \, e^{-\Phi} T_2 \sqrt{-\det \bar{G}^S} \; , \qquad (2.3.16)$$

where $\hat{G}^S$ and $\bar{G}^S$ are the induced metrics on the D6-brane worldvolume and the D2-brane worldvolume respectively, and T_6 and $-T_2$ are the D6-brane tension and the (negative) D2-brane tension respectively (see equation (1.2.9)). We follow the same steps as in section 2.2 to reduce the ten-dimensional action (2.3.16) to a four-dimensional one; we convert to the Einstein frame, then compactify on $K3$, then compactify on T^2 to get

$$S_{matter} = -L^2 \int dt \, e^{-\Phi/4} \, e^{-3\beta/4} \, e^{\rho/2} (e^{\Phi} e^{\beta} V \tau_6 - \tau_2) \sqrt{-G_{00}} \; , \qquad (2.3.17)$$

where $\tau_p = T_p \, g_s^{-1}$ is the physical tension, which is given in equation (1.1.13). Also, G_{00} is the metric induced from the four-dimensional metric (2.3.8). Substituting the perturbed solutions (2.3.8) - (2.3.10) into the action (2.3.17), we find

$$S_{matter}^{approx} = -L^2 \int dt \, Z_2^{-1} (V Z_2 Z_6^{-1} \tau_6 - \tau_2) \left(1 - Z_2^{1/2} Z_6^{1/2} \vec{N} \cdot \vec{u} - \frac{1}{2} Z_2 Z_6 \vec{u}^2\right) . \qquad (2.3.18)$$

The BPS bounds (1.1.15) give

$$\tau_6 = \frac{1}{g_s} q_6 \, \mu_6 \; , \quad \tau_2 = \frac{1}{g_s} q_2 \, \mu_2 \; , \qquad (2.3.19)$$

where q_6 is the D6-brane charge and q_2 is the D2-brane charge and, from equation (1.1.14), $\mu_2 = (2\pi)^{-2}\alpha'^{-3/2}$ and $\mu_6 = (2\pi)^{-6}\alpha'^{-7/2}$. In terms of the current density $\tilde{Q}(\vec{x})$ we have

$$q_6 = -q_2 = \int d^3x\, \tilde{Q}(\vec{x}) \ . \tag{2.3.20}$$

So

$$S^{approx}_{matter} = -L^2 \int d^4x\, \frac{\tilde{Q}}{g_s} \left(V Z_6^{-1}\mu_6 - Z_2^{-1}\mu_2\right) \left(1 - Z_2^{1/2} Z_6^{1/2} \vec{N}\cdot\vec{u} - \frac{1}{2} Z_2 Z_6 \vec{u}^2\right) \ . \tag{2.3.21}$$

The source term in the action for $C^{(3)}$ is given by

$$S^{(3)}_{current} = \frac{\mu_2 q_2}{g_s} \int d^3\xi\, \frac{1}{3!}\, \epsilon^{\alpha_0\alpha_1\alpha_2}\, C^{(3)}_{\beta_0\beta_1\beta_2} \frac{\partial X^{\beta_0}}{\partial \xi^{\alpha_0}} \frac{\partial X^{\beta_1}}{\partial \xi^{\alpha_1}} \frac{\partial X^{\beta_2}}{\partial \xi^{\alpha_2}} \ , \tag{2.3.22}$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 = 0, 1, 2$. Similarly for $C^{(7)}$ we have

$$S^{(7)}_{current} = \frac{\mu_6 q_6}{g_s} \int d^7\xi\, \frac{1}{7!}\, \epsilon^{\kappa_0\kappa_1\ldots\kappa_6}\, C^{(7)}_{\lambda_0\lambda_1\ldots\lambda_6} \frac{\partial X^{\lambda_0}}{\partial \xi^{\kappa_0}} \cdots \frac{\partial X^{\lambda_6}}{\partial \xi^{\kappa_6}} \ , \tag{2.3.23}$$

where $\kappa_0, \ldots, \kappa_6, \lambda_0, \ldots, \lambda_6 = 0, \ldots, 6$. Compactifying (2.3.22) on T^2 and (2.3.23) on $T^2 \times K3$, then substituting in the perturbed solution (2.3.8) - (2.3.10) gives

$$\begin{aligned} S^{approx}_{current} &= L^2 \int d^4x \left((Z_2^{-1} - 1) + \vec{A}\cdot\vec{u} \right) \frac{\tilde{Q}\mu_2}{g_s} \\ &\quad + L^2 \int d^4x \left((Z_6^{-1} - 1) + \tilde{\vec{A}}\cdot\vec{u} \right) \frac{\tilde{Q} V \mu_6}{g_s} \ . \end{aligned} \tag{2.3.24}$$

Altogether we have

$$S_{approx} = S^{approx}_{IIB} + S^{approx}_{matter} + S^{approx}_{current} \ . \tag{2.3.25}$$

Substituting (2.3.15), (2.3.21) and (2.3.24) into (2.3.25) we get

$$
\begin{aligned}
S_{approx} &= k\int d^4x \Bigg\{ -\frac{1}{2}\frac{|\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})|^2}{Z_2^{-1}Z_6} - \frac{1}{2}\frac{|\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})|^2}{Z_2 Z_6^{-1}} \\
&+\frac{(\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N}))\cdot(\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N}))}{Z_6} \\
&+\frac{(\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N}))\cdot(\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N}))}{Z_2} \\
&-\frac{1}{2}\frac{|\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})|^2}{Z_2 Z_6} - \dot{Z}_2\dot{Z}_6 \\
&-\tilde{Q}\left(\frac{1}{g_s}q_6\mu_6 V + \frac{1}{g_s}q_2\mu_2\right) + \frac{1}{2}\tilde{Q}\left(\frac{V\mu_6 Z_2}{g_s} - \frac{\mu_2 Z_6}{g_s}\right)u^2 \\
&-\left(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N}\right)\cdot\left(\vec{\nabla}(\dot{Z}_2)+\tilde{Q}\frac{\mu_2}{g_s}\vec{u}\right) \\
&-\left(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N}\right)\cdot\left(\vec{\nabla}(\dot{Z}_6)-\tilde{Q}\frac{V\mu_6}{g_s}\vec{u}\right)\Bigg\} . \qquad (2.3.26)
\end{aligned}
$$

The dynamical variables in (2.3.26) are $\vec{N}$, $\vec{A}$ and $\vec{\tilde{A}}$ (Z_2 and Z_6 are not dynamical, since they are fixed by equation (2.3.6)).

2.3.4 Perturbation Equations of Motion

Since we have calculated S^{approx} up to $O(u^2)$, we can derive equations of motion from it which are correct to $O(u)$. The equations of motion for $\vec{A}$ and $\vec{\tilde{A}}$ are

$$
\begin{aligned}
-k\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_2^{-1}Z_6}\right) + k\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_6}\right) \\
-k\vec{\nabla}\dot{Z}_2 - \frac{\tilde{Q}\mu_2}{g_s}L^2\vec{u} = 0 , \qquad (2.3.27)
\end{aligned}
$$

$$
\begin{aligned}
-k\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2 Z_6^{-1}}\right) + k\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2}\right) \\
-k\vec{\nabla}\dot{Z}_6 - \frac{\tilde{Q}V\mu_6}{g_s}L^2\vec{u} = 0 , \qquad (2.3.28)
\end{aligned}
$$

and the equation of motion for $\vec{N}$ is

$$\begin{aligned}
&-kZ_2^{-1/2}Z_6^{1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_2^{-1}Z_6}\right)-kZ_2^{1/2}Z_6^{-1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2Z_6^{-1}}\right)\\
&+kZ_2^{-1/2}Z_6^{1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_6}\right)+kZ_2^{1/2}Z_6^{1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_6}\right)\\
&+kZ_2^{1/2}Z_6^{-1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2}\right)+kZ_2^{1/2}Z_6^{1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2}\right)\\
&-kZ_2^{1/2}Z_6^{1/2}\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2Z_6}\right)-Z_2^{-1/2}Z_6^{1/2}\left(k\vec{\nabla}\dot{Z}_2+\frac{\tilde{Q}\mu_2}{g_s}L^2\vec{u}\right)\\
&-Z_2^{1/2}Z_6^{-1/2}\left(k\vec{\nabla}\dot{Z}_6-\frac{\tilde{Q}V\mu_6}{g_s}L^2\vec{u}\right)=0\ .
\end{aligned}\tag{2.3.29}$$

If we define

$$\vec{K}=c\nabla^{-2}(\tilde{Q}\vec{u})\ ,\tag{2.3.30}$$

for some constant c, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{K})=-\frac{cV}{(2\pi)^5\alpha'^{5/2}g_s}\vec{\nabla}\dot{Z}_2-c\tilde{Q}\vec{u}\ ,\tag{2.3.31}$$

where we have used the expression for Z_2 (2.3.6) and also charge conservation in the form

$$\partial_0(\tilde{Q})+\vec{\nabla}\cdot(\tilde{Q}\vec{u})=0\ .\tag{2.3.32}$$

Comparing (2.3.31) to the equation of motion (2.3.27), we find that with $c=\mu_2L^2/g_sk$. We get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{K})=-\vec{\nabla}\dot{Z}_2-\frac{\tilde{Q}\mu_2}{g_sk}\vec{u}L^2\ ,\tag{2.3.33}$$

where we have used $2\kappa^2=(2\pi)^7\alpha'^4$. Similarly we can define

$$\vec{\tilde{K}}=-\tilde{c}\nabla^{-2}(\tilde{Q}\vec{u})\ ,\tag{2.3.34}$$

with $\tilde{c}=V\mu_6L^2/g_sk$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\tilde{K}})=-\vec{\nabla}\dot{Z}_6+\frac{V\tilde{Q}\mu_6}{g_sk}\vec{u}L^2\ .\tag{2.3.35}$$

Taking linear combinations of the equations of motion (2.3.27), (2.3.28) and

(2.3.29), and using (2.3.33) and (2.3.35) we get

$$\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_2^{-1}Z_6}-\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_6}\right)=\vec{\nabla}\times(\vec{\nabla}\times\vec{K})\ , \tag{2.3.36}$$

$$\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2Z_6^{-1}}-\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2}\right)=\vec{\nabla}\times(\vec{\nabla}\times\vec{\tilde{K}})\ , \tag{2.3.37}$$

$$\vec{\nabla}\times\left(\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_6}+\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2}\right.$$
$$\left.-\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2Z_6}\right)=0\ . \tag{2.3.38}$$

2.3.5 The Effective Action

We can integrate the equations of motion (2.3.36) - (2.3.38) to get

$$\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_2^{-1}Z_6}-\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_6}=\vec{\nabla}\times\vec{K}+\vec{\nabla}\alpha\ , \tag{2.3.39}$$

$$\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2Z_6^{-1}}-\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2}=\vec{\nabla}\times\vec{\tilde{K}}+\vec{\nabla}\tilde{\alpha}\ , \tag{2.3.40}$$

$$\frac{\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N})}{Z_6}+\frac{\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N})}{Z_2}$$
$$-\frac{\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N})}{Z_2Z_6}=\vec{\nabla}\nu\ , \tag{2.3.41}$$

where ν, α and $\tilde{\alpha}$ are functions of integration. Using the techniques of ref. [74] it is possible to show that the main contribution of the functions of integration to the effective action comes from the regions very near the enhançons. As in ref. [74], expanding the fields in powers of $r=|\vec{x}-\vec{x}_a|$ near the $a^{\rm th}$ enhançon, one finds that the functions of integration do not contribute to the effective action. We will therefore neglect them from now on.

Then linear combinations of (2.3.39) - (2.3.41) give

$$\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N}) = -\vec{\nabla}\times\vec{\tilde{K}}\ , \tag{2.3.42}$$

$$\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N}) = -\vec{\nabla}\times\vec{K}\ , \tag{2.3.43}$$

$$\vec{\nabla}\times(Z_2^{1/2}Z_6^{1/2}\vec{N}) = -Z_2\vec{\nabla}\times\vec{\tilde{K}}-Z_6\vec{\nabla}\times\vec{K}\ . \tag{2.3.44}$$

Substituting (2.3.42), (2.3.43) and (2.3.44) into the action (2.3.26) and integrating by parts gives

$$\begin{aligned}S_{approx} &= \int d^4x \left[k\left\{ -\dot{Z}_2\dot{Z}_6 + \frac{1}{2}(\vec{\nabla}\times(\vec{A}+Z_2^{-1/2}Z_6^{1/2}\vec{N}))\cdot(\vec{\nabla}\times\vec{K}) \right.\right.\\ &\left. +\frac{1}{2}(\vec{\nabla}\times(\vec{\tilde{A}}+Z_2^{1/2}Z_6^{-1/2}\vec{N}))\cdot(\vec{\nabla}\times\vec{\tilde{K}})\right\}\\ &\left. +L^2\left\{ -\frac{\tilde{Q}}{g_s}(\mu_6 V-\mu_2)+\frac{1}{2}\frac{\tilde{Q}}{g_s}(\mu_6 VZ_2-\mu_2 Z_6)\,u^2\right\}\right] . \end{aligned} \tag{2.3.45}$$

We now take the enhançon limit, in which the charge density becomes

$$\tilde{Q} = \sum_a Q_a \delta^{(3)}(\vec{x}-\vec{x}_a) , \tag{2.3.46}$$

where $\vec{x}_a$ and Q_a are the position and charge respectively of the ath black hole. Then Z_2 and Z_6 are given by the equations (2.3.6). Also the equations (2.3.30) and (2.3.34) for $\vec{K}$ and $\vec{\tilde{K}}$ have the solutions

$$\vec{K} = -\frac{1}{4\pi}\frac{\mu_2 L^2}{g_s k}\sum_a \frac{Q_a}{r_a}\vec{u}_a , \quad \vec{\tilde{K}} = -\frac{1}{4\pi}\frac{V\mu_6 L^2}{g_s k}\sum_a \frac{Q_a}{r_a}\vec{u}_a , \tag{2.3.47}$$

where $\vec{r}_a = \vec{x}-\vec{x}_a$.

Then the first term in the action (2.3.45) becomes

$$\dot{Z}_2\dot{Z}_6 = \sum_{a,b}\frac{(r_2)_a(r_6)_b}{r_a^3 r_b^3}\left\{(\vec{r}_a\cdot\vec{u}_a)(\vec{r}_b\cdot\vec{u}_b)\right\} , \tag{2.3.48}$$

and the second term becomes

$$(\vec{\nabla}\times\vec{K})\cdot(\vec{\nabla}\times\vec{\tilde{K}}) = -\frac{1}{(4\pi)^2}\frac{V\mu_2\mu_6 L^4}{g_s^2k^2}\sum_{a,b}\frac{Q_aQ_b}{r_a^3r_b^3}\left\{(\vec{r}_a\cdot\vec{r}_b)(\vec{u}_a\cdot\vec{u}_b)-(\vec{r}_a\cdot\vec{u}_b)(\vec{r}_b\cdot\vec{u}_a)\right\} . \tag{2.3.49}$$

Consider the fifth term in the action (2.3.45). Writing the delta function in $\tilde{Q}$ as

$$\delta^{(3)}(\vec{x}-\vec{x}_a) = -\frac{1}{4\pi}\vec{\nabla}^2\left(\frac{1}{r_a}\right) , \tag{2.3.50}$$

then integrating by parts, we find

$$\begin{aligned}\int d^3x\,\frac{L^2\tilde{Q}u^2}{2g_s}\left(V\mu_6 Z_2-\mu_2 Z_6\right) &= \sum_a \frac{L^2u_a^2}{2g_s}(V\mu_6-\mu_2)\,Q_a\\ &-\frac{1}{4\pi}\sum_{a,b}\int d^3x\,\frac{L^2u_a^2}{2\alpha'(2\pi)^2}\frac{Q_aQ_b}{r_a^3r_b^3}(\vec{r}_a\cdot\vec{r}_b) . \end{aligned} \tag{2.3.51}$$

Substituting (2.3.48), (2.3.49) and (2.3.51) into the action (2.3.45), and rearranging, we find

$$S_{eff} = \int L_{eff}\, dt \ , \tag{2.3.52}$$

where

$$\begin{aligned} L_{eff} = &- \frac{L^2}{g_s}(\mu_6 V - \mu_2) \sum_a Q_a + \frac{L^2}{g_s}(\mu_6 V - \mu_2) \sum_a \frac{Q_a u_a^2}{2} \\ &+ \frac{L^2}{4(2\pi)^3\alpha'} \int d^3x \sum_{a,b} \frac{Q_a Q_b}{r_a^3 r_b^3} \left\{ (\vec{r}_a \times \vec{r}_b)\cdot(\vec{u}_a \times \vec{u}_b) - \frac{1}{2}|\vec{u}_a - \vec{u}_b|^2 (\vec{r}_a \cdot \vec{r}_b) \right\} \ . \end{aligned} \tag{2.3.53}$$

2.3.6 The Metric on Moduli Space

For two enhançons (2.3.53) reduces to

$$\begin{aligned} L_{eff} \ = \ &-\frac{L^2}{g_s}(\mu_6 V - \mu_2)(Q_1 + Q_2) + \frac{L^2}{g_s}(\mu_6 V - \mu_2)\left(\frac{Q_1 u_1^2}{2} + \frac{Q_2 u_2^2}{2}\right) \\ &+ \frac{L^2}{4(2\pi)^3\alpha'} \int d^3x \, \frac{Q_1 Q_2}{r_1^3 r_2^3} \left\{ (\vec{r}_1 \times \vec{r}_2)\cdot(\vec{u}_1 \times \vec{u}_2) \right. \\ &\left. - \frac{1}{2}|\vec{u}_1 - \vec{u}_2|^2 (\vec{r}_1 \cdot \vec{r}_2) \right\} , \end{aligned} \tag{2.3.54}$$

where $r_1 = |\vec{x} - \vec{x}_1|$, and $r_2 = |\vec{x} - \vec{x}_2|$.

Consider the integral

$$I = \int d^3x \frac{(\vec{r}_1 \cdot \vec{r}_2)}{r_1^3 r_2^3} \ . \tag{2.3.55}$$

We can introduce a Feynman parameter ω using the formula

$$\frac{1}{A^\alpha B^\beta} = \int_0^1 d\omega \, \frac{\omega^{\alpha-1}(1-\omega)^{\beta-1}}{[\omega A + (1-\omega)B]^{\alpha+\beta}} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \ . \tag{2.3.56}$$

Then (2.3.55) becomes

$$I = \int d^3x \int_0^1 d\omega \frac{\omega^{1/2}(1-\omega)^{1/2}(\vec{r}_1 \cdot \vec{r}_2)}{[\omega(x^2 - 2\vec{x}\cdot\vec{x}_1 + x_1^2) + (1-\omega)(x^2 - 2\vec{x}\cdot\vec{x}_2 + x_2^2)]^3} \frac{\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \ . \tag{2.3.57}$$

Completing the square in the denominator in (2.3.57), and substituting $\vec{y} = \vec{x} - \omega\vec{x}_1 - (1-\omega)\vec{x}_2$ gives

$$\begin{aligned} I \ = \ &\frac{\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \int_0^1 d\omega \, \omega^{1/2}(1-\omega)^{1/2} \times \\ &\int d^3y \, \frac{y^2 + (2\omega - 1)\vec{y}\cdot(\vec{x}_1 - \vec{x}_2) - \omega(1-\omega)(\vec{x}_1 - \vec{x}_2)^2}{[y^2 + \omega(1-\omega)(\vec{x}_1 - \vec{x}_2)^2]^3} \ . \end{aligned} \tag{2.3.58}$$

Now

$$\int d^3y \, \frac{(2\omega-1)\vec{y}\cdot(\vec{x}_1-\vec{x}_2)}{[y^2+\omega(1-\omega)(\vec{x}_1-\vec{x}_2)^2]^3} = 0 \ , \tag{2.3.59}$$

since the integrand is the sum of odd functions of the y_i. Therefore we can write

$$I = \frac{\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \int_0^1 d\omega \, \omega^{1/2}(1-\omega)^{1/2} \int d\Omega_2 \, dy \frac{y^2(y^2-a^2)}{(y^2+a^2)^3} \ , \tag{2.3.60}$$

where $a^2 = \omega(1-\omega)(\vec{x}_1-\vec{x}_2)^2 > 0$. We can do the y integral using contour integration to get

$$\begin{aligned} I &= \frac{\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})} \int_0^1 d\omega \int d\Omega_2 \, \frac{\pi}{8|\vec{x}_1-\vec{x}_2|} \\ &= \frac{4\pi}{|\vec{x}_1-\vec{x}_2|} \ . \end{aligned} \tag{2.3.61}$$

Using Feynman parameters again we can show that

$$\int d^3x \, \frac{(\vec{r}_1\times\vec{r}_2)\cdot(\vec{u}_1\times\vec{u}_2)}{r_1^3 r_2^3} = 0 \ . \tag{2.3.62}$$

Substituting (2.3.61) and (2.3.62) into (2.3.54) we get

$$L_{eff} = \left(C - \frac{D}{|\vec{x}_1-\vec{x}_2|}\right)(\vec{u}_1^{\,2}+\vec{u}_2^{\,2}) + \frac{2D}{|\vec{x}_1-\vec{x}_2|}\vec{u}_1\cdot\vec{u}_2 \ , \tag{2.3.63}$$

where

$$C = \frac{2L^2}{g_s}(\mu_6 V - \mu_2) \ , \quad D = \frac{2L^2}{(4\pi)^2\alpha'} \ . \tag{2.3.64}$$

In order to compare our metric to the Taub-NUT metric, we can rewrite L_{eff} in terms of the overall position $\vec{X}$ and relative position $\vec{r}$ of the monopoles, which are given by

$$\vec{X} = (\vec{x}_1+\vec{x}_2) \ , \quad \vec{r} = (\vec{x}_1-\vec{x}_2) \ . \tag{2.3.65}$$

The overall velocity $\vec{U}$ and relative velocity $\vec{u}$ are then defined by $\vec{U} = \dot{\vec{X}}$ and $\vec{u} = \dot{\vec{r}}$. We find

$$L_{eff} = \frac{1}{2}C\,\vec{U}^2 + \frac{1}{2}\left(C - \frac{2D}{|\vec{r}|}\right)\vec{u}^2 \ . \tag{2.3.66}$$

From this we see that the metric for the overall position of the D-branes is flat, as we would expect from comparison with the monopole case, and the metric on moduli space for the relative position of the D-branes is given by

$$ds^2 = \frac{1}{2}\left(C - \frac{2D}{r}\right)(dr^2 + r^2 d\Omega_2^2) \ , \tag{2.3.67}$$

where $r = |\vec{r}|$ and $d\Omega_2^2$ is the metric on a two-sphere with coordinates (θ, ϕ). Note that this metric is identical to the position moduli part of the Taub-NUT metric with negative mass parameter. We will discuss how to incorporate the phase terms in section 2.3.7 below.

The metric (2.3.67) is the position moduli part of the metric on moduli space for two enhançons. From the form of the effective Lagrangian (2.3.54) we see that it consists of only two-body interactions. Therefore we can obtain the metric for the N enhançons by summing over the result for the two-body case. The resulting metric has the following form

$$ds^2 = g_{ab} d\vec{x}^a \cdot d\vec{x}^b \, , \tag{2.3.68}$$

where

$$\begin{aligned} g_{ab} &= \frac{l_e}{|\vec{x}_a - \vec{x}_b|} \, , \quad a \neq b \, , && (2.3.69) \\ g_{aa} &= 1 - \sum_{a \neq b} \frac{l_e}{|\vec{x}_a - \vec{x}_b|} \, , \quad \text{no sum on } a \, . && (2.3.70) \end{aligned}$$

In (2.3.69) and (2.3.70) we have removed the constant factor in front of the metric (2.3.67), and we have defined

$$l_e = \frac{D}{C} = \frac{\alpha' g_s (2\pi)^4 \alpha'^{5/2}}{4} = \frac{r_e}{4} \, . \tag{2.3.71}$$

The metric we have obtained is the position moduli part of the generalised Taub-NUT metric (1.3.46). In the next section we will discuss how to incorporate the phase terms into this metric.

2.3.7 Adding the phase terms

In the work we have described so far in this chapter we have found the metric on moduli space for the position moduli $\vec{x}^a$, which is given in equation (2.3.68). But there must be some moduli missing from that metric because, as we explained in section 1.4.1, the moduli space for n enhançons is identical to the moduli space of the three-dimensional supersymmetric gauge theory on the worldvolume of the D2-branes. This space is hyper-Kähler with $4n$ dimensions, whereas the moduli space in (2.3.68) has only $3n$ dimensions; there are n moduli missing. Comparing to the metric on moduli space for n BPS monopoles (1.3.46) we find that the missing moduli are those corresponding to the phases of the monopoles.

In ref. [53] Gibbons and Manton obtained the phase terms in the metric on moduli space for magnetic monopoles by endowing the monopoles with some electric charge, and then reinterpreting the electric charges as phases. For the enhançon case that we are studying here the missing part of the calculation is an electric coupling of the Ramond-Ramond field which is Hodge dual to $C^{(7)}$; we denote this field $C_D^{(1)}$. It couples to the gauge field living on the D-branes, $F_{\alpha\beta}$ in the following way:

$$-2\pi\alpha'\mu_2 \int \mathrm{Tr}\left[\mathrm{C_D^{(1)}} \wedge \mathrm{F}\right] . \tag{2.3.72}$$

We have defined the field $C_D^{(1)}$ to be the Hodge dual field to $C^{(7)}$, so

$$dC_D^{(1)} = e^{-3\Phi/2} * F^{(8)} = e^{-3\Phi/2} * dC^{(7)} . \tag{2.3.73}$$

After some algebra we find

$$\vec{\nabla}(Z_6) = \vec{\nabla} \times C_D^{(1)} . \tag{2.3.74}$$

The Born-Infeld actions are also modified as follows:

$$-\det(G_{\alpha\beta}) \quad \longrightarrow \quad -\det\left(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right) . \tag{2.3.75}$$

The field $F_{\alpha\beta}$ is the gauge field that lives on the unwrapped part of the branes' worldvolume. Since we are considering the Abelian case where the branes are far apart from one another, we can write $F_{\alpha\beta}$ as

$$F_{\alpha\beta} = \sum_a F^a_{\alpha\beta}\mathbf{t}^a , \tag{2.3.76}$$

where the index a labels the branes, $a = 1, \ldots n$, and $\mathbf{t}^a$ is the ath $U(1)$ generator. We choose the following basis for the $\mathbf{t}^a$

$$\mathbf{t}^a = \mathrm{diag}\{\ldots, 0, \ldots, 1, \ldots, 0, \ldots\} , \tag{2.3.77}$$

where the '1' occurs in the ath position. Then

$$\mathrm{Tr}\left(\mathbf{t}^{\mathrm{a}}\mathbf{t}^{\mathrm{b}}\right) = \delta^{\mathrm{ab}} . \tag{2.3.78}$$

We can calculate the charge Q^a of any object ζ with respect to the ath $U(1)$ as follows

$$Q^a(\zeta) = \mathrm{Tr}(\mathbf{t}^a\zeta) . \tag{2.3.79}$$

Let us consider the Dirac-Born-Infeld theory in the Yang-Mills limit $\alpha' \to 0$. In this limit the part of the action involving the determinant is given by

$$\begin{aligned} & \mathrm{Tr}\left((-\mathrm{detG})\left(1+\frac{1}{2}(2\pi\alpha')^2\mathrm{G}^{\alpha\beta}\mathrm{G}^{\gamma\delta}\mathrm{F}^{\mathrm{a}}_{\alpha\gamma}\mathrm{F}^{\mathrm{a}}_{\beta\delta}\mathbf{t}^{\mathrm{a}}\mathbf{t}^{\mathrm{b}}\right)\right)+\mathrm{O}((\alpha')^4) \\ = & \ (-\det G)\left(1+\frac{1}{2}(2\pi\alpha')^2G^{\alpha\beta}G^{\gamma\delta}F^a_{\alpha\gamma}F^a_{\beta\delta}\right)+O((\alpha')^4) . \end{aligned} \tag{2.3.80}$$

The prefactor to the kinetic term for the F's can be thought of as a metric $\tilde{g}_{ab}$ on the space of $U(1)$ generators

$$S_{YM} \sim \int d^3\xi \tilde{g}_{ab} F^a_{\alpha\beta} F^{b\,\alpha\beta} . \tag{2.3.81}$$

From (2.3.80) we find that $\tilde{g}_{ab}$ is apparently proportional to the identity matrix.

Let us take a step back for a moment, and consider the metric $\tilde{g}_{ab}$. This metric defines the space of $U(1)$ generators $\mathbf{t}^a$. However, from section 1.1.1, the $\mathbf{t}^a$ can be interpreted as the n positions of the enhançons, $\vec{x}^a$. We deduce that the metric $\tilde{g}_{ab}$ should be identical to the metric g_{ab} in equation (2.3.68). But this does not agree with the discussion above, which suggested that $\tilde{g}_{ab}$ is proportional to the identity matrix; what has gone wrong? In fact equation (2.3.80) is incorrect because the techniques we have used to obtain it are unable to handle the interactions between the $U(1)$ generators. We would need to apply some sort of regularisation procedure on the generators $\mathbf{t}^a$, similar to that we used for the $\vec{x}^a$. Unfortunately it is not currently known how to do this.

It turns out that we can manage without such extra mathematical tools. Since we know that $\tilde{g}_{ab} = g_{ab}$, we can build up the action involving F from scratch. The Yang-Mills term is given by

$$S_{YM} = \frac{1}{4g^2_{YM}} \int d^3\xi g_{ab} F^a_{\alpha\beta} F^{b\,\alpha\beta} . \tag{2.3.82}$$

where g_{YM} is the Yang-Mills coupling, which in this case is given by

$$g^2_{YM} = \frac{g_s}{(2\pi\alpha')^2}(\mu_6 V - \mu_2)^{-1} . \tag{2.3.83}$$

We must also include the coupling of F to the field $C_D^{(1)}$, which takes the form of the Chern-Simons coupling (2.3.72). We need to build up this term carefully. Consider for now the perspective of a single brane, brane a say. The magnetic field on brane a due to brane b, pulled back to brane a's worldvolume, is given by

$$(C_D^{(1)})_\alpha = \vec{\tilde{\omega}}_{ab} \cdot \partial_\alpha \vec{X}^a \mathbf{t}^b \ , \tag{2.3.84}$$

where

$$\vec{\nabla}_a \times \vec{\tilde{\omega}}_{ab} = \vec{\nabla}_a \left(\frac{1}{|\vec{x}_a - \vec{x}_b|} \right) (4\pi)^2 \alpha' \mu_2 \ . \tag{2.3.85}$$

The form of $\vec{\tilde{\omega}}$ in (2.3.85) can be deduced from the relation (2.3.74) between Z_6 and $C_D^{(1)}$, and the solution (2.3.6) for Z_6. The field (2.3.84) couples to the gauge field F, given in (2.3.76). So the expression we require is

$$\begin{aligned} S_{CS} &= -\frac{1}{8\pi} \int d^3\xi \, \mathrm{Tr} \left(\epsilon^{\alpha\beta\gamma} \mathbf{t}^c F^c_{\alpha\beta} \vec{\tilde{\omega}}_{ab} \cdot \partial_\gamma \vec{X}^a \mathbf{t}^b \right) \\ &= -\frac{1}{8\pi} \int d^3\xi \, \epsilon^{\alpha\beta\gamma} F^b_{\alpha\beta} \vec{\omega}_{ab} \cdot \partial_\gamma \vec{X}^a \ , \end{aligned} \tag{2.3.86}$$

where we have defined

$$\vec{\omega}_{ab} = \vec{\tilde{\omega}}_{ab} (4\pi)^2 \alpha' \mu_2 \ , \tag{2.3.87}$$

to obtain the correct coupling in (2.3.86). There is one more term we need to include in the action. We are working with $(2+1)$-dimensional gauge theory, for which the field F^a has a winding number $n^a \in \mathbb{Z}$ (i.e. a topological invariant, equivalent to the instanton number in $(3+1)$-dimensional gauge theory). It is given by

$$n^a = \int d^3\xi \, \epsilon^{\alpha\beta\gamma} \partial_\gamma F^a_{\alpha\beta} \ . \tag{2.3.88}$$

The n^a couple to phases σ_a, giving the following term in the action

$$S_{wind} = \int d^3\xi \, \epsilon^{\alpha\beta\gamma} \partial_\gamma F^a_{\alpha\beta} \sigma_a \ . \tag{2.3.89}$$

For the path integral e^{iS} to be single-valued, the phases σ_a must be 2π periodic. Then the total action dependent on F is given by

$$S_F = S_{YM} + S_{CS} + S_{wind} \ . \tag{2.3.90}$$

We can replace F in the action (2.3.90) by σ_a, using the procedure of ref. [63]. The equation of motion for $F^a_{\alpha\beta}$ is

$$g_{ab} F^{b\,\alpha\beta} - \epsilon^{\alpha\beta\gamma} \partial_\gamma \sigma_a + \epsilon^{\alpha\beta\gamma} \vec{\omega}_{ba} \cdot \partial_\gamma \vec{X}^b = 0 \ . \tag{2.3.91}$$

Inverting this equation, and substituting it back into the action (2.3.90), we find

$$S_F = \frac{1}{2}\int d^3\xi\, \frac{g_{YM}^2}{(4\pi)^2}\,(g^{-1})^{ab}(\partial_\alpha\sigma_a + \vec{\omega}_{ac}\cdot\partial_\alpha\vec{x}^c)(\partial^\alpha\sigma_b + \vec{\omega}_{bc}\cdot\partial_\alpha\vec{x}^c)\ . \qquad (2.3.92)$$

Compactifying the D2-brane spatial directions on a T^2 in the action (2.3.92) gives

$$S_F = \frac{1}{2}\int dt\, \frac{g_{YM}^2 L^2}{(4\pi)^2}\,(g^{-1})^{ab}(\partial_t\sigma_a + \vec{\omega}_{ac}\cdot\partial_t\vec{x}^c)(\partial^t\sigma_b + \vec{\omega}_{bc}\cdot\partial_t\vec{x}^c)\ . \qquad (2.3.93)$$

So, modulo some constant factors, the phase terms in the metric on moduli space are given by

$$(g^{-1})^{ab}(d\sigma_a + \vec{\omega}_{ac}\cdot d\vec{x}^c)(d\sigma_b + \vec{\omega}_{bc}\cdot d\vec{x}^c)\ . \qquad (2.3.94)$$

2.3.8 The Final Metric

Putting our result for the phases from the previous section together with our result for the position moduli, we find that the final metric on moduli space is given by

$$ds^2 = g_{ab}d\vec{x}^a\cdot d\vec{x}^b + (g^{-1})^{ab}(d\sigma_a + \vec{\omega}_{ac}\cdot d\vec{x}^c)(d\sigma_b + \vec{\omega}_{bc}\cdot d\vec{x}^c)\ , \qquad (2.3.95)$$

where g_{ab} is given by equations (2.3.69) and (2.3.70). This metric is identical to the generalised Taub-NUT metric (1.3.46).

2.4 Summary

We have calculated the metric on moduli space for n enhançons in the limit that they are far apart from one another. We followed the procedure from refs. [72], [73] and [74]. Our result was the generalised Taub-NUT metric with negative mass parameter, which is identical to the metric on moduli space for BPS monopoles in the same limit, as we had predicted. We expect that the full metric on moduli space will turn out to be the $4n$ dimensional hyper-Kähler manifold which generalises the Atiyah-Hitchin manifold.

In the process of our calculation we realised that the enhançon is dual to the bound state of a Kaluza-Klein monopole and an H-monopole in heterotic string theory compactified on T^4 (see ref. [79]). Therefore we have indirectly computed the metric on moduli space for n of these objects as well.

It is interesting to note that D6-branes wrapped on T^4 instead of K3 (or indeed any other Ricci flat manifold) also have position moduli whose metric is given by (2.3.68) (this was shown in ref. [76]). In order to calculate the phase terms in the K3 case we used the field theory living on the $(2+1)$-dimensional world-volume of the induced D2-branes. The equivalent calculation in the T^4 case would involve the $(2+1)$-dimensional field theory living on the unwrapped D6-brane directions. Recall from section 2.3.7 that in the K3 case this calculation relied on there being a term in the action like $\int C^{(1)} \wedge F$ (see (2.3.72)). But the action for the field theory in the T^4 case, which originates from a $(6+1)$-dimensional field theory, does not necessarily contain such a term. However, it is possible to generate a term of this form, and it would be interesting to investigate this further in the future.

Chapter 3

Using D-Strings to Describe Monopoles - Analytic Calculations

3.1 Introduction

In this chapter we will turn our attention to the second example from section 1.4 of a D-brane configuration acting as a BPS monopole, i.e. a D-string stretched between two D3-branes.

In all the work we reviewed in section 1.4.2 the solutions described a semi-infinite string attached to a D3-brane. In refs. [81], [82], a slightly different scenario was considered, where the string was truncated by placing the system in the background of a second D3-brane. The work we describe in this chapter will take a different approach - we will explicitly construct the solution for D-strings stretched between two D3-branes. It turns out that in order to do this we will need to take an ansatz for the positions of the D-string in the D3-branes which is not spherically symmetric (this is in contrast to the ansatz (1.4.34) which was used in ref. [66]).

As we explained in section 1.4.2, the D3-D1Brane configuration provides a physical realisation of the ADHMN construction. In order to test the correspondence further, it seems natural to ask whether there is any monopole calculation, whose result is known, which could be recalculated in the $(1 + 1)$ dimensional D-string picture. We could then compare the two results. One possibility is the calculation of the energy radiated during scattering of the D-strings, which has been calculated for the BPS magnetic monopole case by Manton and Samols in ref. [80]. They obtained the result $E_{\rm rad} \sim 1.35 m_{\rm mon} v_{-\infty}^5$, where $m_{\rm mon}$ is the mass of the monopoles, and $v_{-\infty}$ is their asymptotic velocity. We will aim to recalculate this result, this time working from the point of view of the D-strings. In this chapter we present our analytic calculations, which lead to equations of motion for the non-zero mode perturbations of the D-strings. We will describe our numerical calculations of the energy radiated, and present our results, in the next chapter.

The layout of this chapter is as follows. In section 3.2 we review the calculation of the energy radiated during monopole scattering from ref. [80]. In order to repeat this calculation from the D-string perspective we will need to perturb the static solution to analyse the dynamics of the scattering beyond Manton's geodesic approximation (which, as we pointed out in section 1.3.3, ignores the effects of the radiation). There is already some work in the literature concerning perturbations of the BIon spike - we will review these in section 3.3. The calculations from section 3.4 onwards will cover the material from our paper, ref. [2]. In section 3.4 we will analyse the non-Abelian Born-Infeld action for D-strings in flat background, and present the solution corresponding to two D-strings stretched between D3-branes. We will point out that it is necessary to be careful when taking the $\alpha' \to 0$ limit to ensure that the mass of the W-boson, and the mass of the monopole remain finite. We will also discuss the validity of the solutions, given the limitations of the Born-Infeld action, as described in section 1.4.2. In section 3.5 we will repeat the calculation from section 3.4, but this time working in a D3-brane background, instead of flat background. In section 3.6 we will describe the scattering of the two D-strings using the slow motion technique of Manton, which we reviewed in section 1.3.3. In section 3.7 we will examine the effects of perturbing the BPS solution, and calculate the equations of motion for the perturbation; these perturbations will embody the effects of the energy radiation. Finally, we will present a summary of the chapter in section 3.8.

3.2 The Energy Radiated During Monopole Scattering

In this section we review the work of Manton and Samols, ref. [80], where they calculate the energy radiated during the scattering of BPS magnetic monopoles in the slow velocity limit. We take the case here in which the monopoles' motion is restricted to the x^1-x^2 plane. They approach each other along the x^1-axis, then scatter at 90° to move away from each other along the x^2-axis (see section 1.3.3). The two monopoles are assumed to have the same charge.

Since the monopoles are moving in the slow velocity limit, a first approximation to the motion is the geodesic approximation, as described in section 1.3.3. In a multipole expansion of the fields the leading order contribution to the radiation then comes from the leading order multipole moment.

From the Bogomol'nyi equation for a BPS monopole (1.3.20) it can be seen that the magnetic and scalar multipoles are equal. A multipole expansion gives

$$\Phi(\vec{r}) = \frac{Q}{r} - \frac{d_i r_i}{r^3} + \frac{1}{2}\frac{Q_{ij} r_i r_j}{r^5} + \cdots , \tag{3.2.1}$$

where the multipole moments are

$$Q = \int d^3r' \, \rho(\vec{r}\,') , \tag{3.2.2}$$

$$d_i = \int d^3r' \, \rho(\vec{r}\,') r'_i , \tag{3.2.3}$$

$$Q_{ij} = \int d^3r' \, \rho(\vec{r}\,')(3r'_i r'_j - r^2\delta_{ij}) , \tag{3.2.4}$$

and where $\rho(\vec{r})$ is the magnetic charge distribution and $i, j = 1, 2, 3$.

We will take $v = 1$, where v is the Higgs expectation value, and $g_{YM} = 1$. Then from (1.3.22) the monopole charge is equal to the monopole mass, m_{mon}, and

$$Q = 2m_{mon} . \tag{3.2.5}$$

Since the two monopoles have equal charge, $d_i = 0$, and so the leading order contribution to the radiation comes from the quadrupole moment Q_{ij}. During the scattering the field configuration evolves smoothly from the asymptotic configuration, when the monopoles are far apart along the x^1-axis, to the axisymmetric 'doughnut' configuration at the point of scattering, to being far apart again along the x^2-axis. Therefore the configuration is always invariant under reflection in the x^1 and x^2 axes, i.e. $\rho(r'_1, r'_2, r'_3) = \rho(-r'_1, r'_2, r'_3) = \rho(r'_1, -r'_2, r'_3)$. Using this in the definition of the quadrupole moment, equation (3.2.4), reveals that Q_{ij} is diagonal,

$$Q_{ij} = Q_i \delta_{ij} \quad \text{(no sum over } i) \ , \quad Q_1 + Q_2 + Q_3 = 0 \ . \tag{3.2.6}$$

In the asymptotic limit, when the monopoles are far apart, they can be treated as pointlike objects. Then from the definition of the quadrupole moment, equation (3.2.4), we have

$$Q_1 = -m_{mon} r^2 \ , \quad Q_2 = \frac{1}{2} m_{mon} r^2 \ , \quad Q_3 = \frac{1}{2} m_{mon} r^2 \ , \tag{3.2.7}$$

for $t < 0$, and with Q_1 and Q_2 interchanged for $t > 0$. Here r is the distance between the monopoles, which can be calculated in the asymptotic limit by integrating the Taub-NUT metric (1.3.42) with respect to time to get

$$r(t) = 2vt + \ln 2vt + c + \frac{\ln 2vt}{2vt} + \frac{1}{4vt}(2c - 1) + \cdots \ , \tag{3.2.8}$$

where c is a constant with

$$c = 1 + \ln 2 \ .$$

A constant R is defined such that (3.2.7) is a good approximation to the quadrupole moments, providing $r > R$. A comparison with the Atiyah-Hitchin metric shows that it is safe to take $R \simeq 10$.

When the monopoles coincide the configuration is axisymmetric in the x^1-x^2 plane, and the far field on the x^3-axis is given by

$$\begin{aligned} \Phi &= \frac{2 r_3 m_{mon}}{r_3^2 + \pi^2/4} \\ &= \frac{2 m_{mon}}{r_3} \left(1 - \frac{\pi^2}{4 r_3^2} + \cdots \right) \ . \end{aligned} \tag{3.2.9}$$

Comparing this to the multipole expansion (3.2.1), and using the axisymmetry in the x^1-x^2 plane gives

$$Q_1 = Q_2 = -\frac{1}{2} m_{mon} \pi^2 \ , \quad Q_3 = m_{mon} \pi^2 \ . \tag{3.2.10}$$

The behaviour of the quadrupole moment during the scattering is some smooth interpolation between (3.2.7) and (3.2.10). The total energy radiated, expressed in terms of Q is given by the formula

$$E_{\rm rad} = \frac{1}{432\pi} \sum_i \int dt \; \dddot{Q}_i^2(t) \; .$$

which can be found in any standard electromagnetic textbook, such as ref. [83] or ref. [84].

Minimising this energy with respect to the unknown quadrupole moments, subject to the boundary conditions discussed above, and solving the resulting equations numerically, gives a lower bound for the energy radiated, which is

$$E_{\rm rad} \simeq 1.35 \, m_{\rm mon} \, v_{-\infty}^5 \; , \tag{3.2.11}$$

where $v_{-\infty}$ is the asymptotic velocity of the monopoles initially. Moreover, Manton and Samols argue that (3.2.11) is a good approximation to the energy radiated, not just a lower bound, because the minimising configuration they found exhibited all the important properties expected of the true evolution in time.

3.3 Perturbations of the BIon Spike Solution

In this section we will review the work that has been done in the literature on the dynamics of the spike soliton for the case of a string attached to a D3-brane.

In ref. [64], for an F-string attached to a D3-brane, the dynamics of small transverse fluctuations were studied (i.e. fluctuations which are transverse to all the brane directions). In that paper only S-wave fluctuations were considered. If η denotes the fluctuation coordinate, then the linearised small fluctuation expansion of the Born-Infeld action about the spike solution (1.4.25) is given by

$$-\left(1 + \frac{g_S^2 c^2}{r^4}\right) \partial_t^2 \eta + r^{-2} \partial_r (r^2 \partial_r \eta) = 0 \; , \tag{3.3.1}$$

where c is a constant. For $r \longrightarrow \infty$ (i.e. in the D3-brane region), the equation (3.3.1) looks like a (3+1)-dimensional wave equation. For $r \longrightarrow 0$, i.e. in the string region, a change of coordinates to $\sigma = g_s c/r$ results in a (1+1)-dimensional wave equation.

Restricting to a wave of frequency ω, (3.3.1) becomes

$$\left(\frac{d^2}{d\xi^2}+1+\frac{\kappa^2}{\xi^4}\right)\eta=0\ , \tag{3.3.2}$$

where

$$\xi=\frac{\omega g_s c}{r}\ , \quad \kappa=g_s c\omega^2\ . \tag{3.3.3}$$

We can change to a new coordinate x, which satisfies

$$\frac{dx}{d\xi}=\sqrt{1+\frac{\kappa^2}{\xi^4}}\ . \tag{3.3.4}$$

We also transform η to $\tilde{\eta}$ as follows:

$$\tilde{\eta}=\left(1+\frac{\kappa^2}{\xi^4}\right)^{1/4}\eta\ . \tag{3.3.5}$$

Then (3.3.2) becomes

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\tilde{\eta}=\tilde{\eta}\ , \tag{3.3.6}$$

where

$$V(x)=\frac{5\kappa^2}{(\xi(x)^2+\kappa^2/\xi(x)^2)^2}\ . \tag{3.3.7}$$

So the problem reduces to a scattering problem with potential $V(x)$, where from (3.3.4) we see that the coordinate x is defined on the real axis. In ref. [64] it was shown that, in the low energy limit, a potential of the form (3.3.7) is perfectly reflecting. This means that the end of the string is unable to leave the surface of the D-brane in the transverse directions, i.e. it has a Dirichlet boundary condition in those directions, as we would expect. In refs. [82] and [85] it was also shown that relative transverse fluctuations of the spike (i.e. fluctuations in one of the D3-brane directions) lead to the Neumann boundary condition. In ref. [86] the case of perturbations carrying non-zero angular momentum modes was considered. It was found that these perturbations could be transmitted down the spike. Therefore the 'F-string' appears to remain effectively three-dimensional all the way along the spike. This might have been expected for a infinite spike from the discussion of the shape of the BIon spike in section 1.4.2.

Everything we have described so far in this section has been done from the point of view of the D3-brane action. If we consider D-strings attached to a D3-brane we also have the option to study the problem from the point of view of the non-Abelian DBI action for the D-strings; this was done in ref. [66]. It was found that there was good agreement with the calculations which had been done from the D3-brane perspective, providing the angular momentum l was not too high. However, there was a difference for perturbations with large l; the D3-brane calculations found that perturbations with arbitrarily large l could be transmitted down the spike, whereas the D-string calculation showed that the spectrum of modes was truncated at $l_{max} = n - 1$ for n D-strings. This discrepancy could be due to the three-dimensional nature of the semi-infinite spike, all the way out to infinity.

The calculations we will describe in this chapter differ from the calculations we have reviewed above in that we will consider D-strings stretched between two D3-branes, rather than semi-infinite D-strings attached to a D3-brane. We will need to consider an ansatz for the D3-brane directions which is not spherically symmetric, therefore the perturbation equations will turn out to be more complicated in our case.

3.4 The Action for D-Strings in Flat Background

In this section we investigate the non-Abelian Dirac-Born-Infeld action for D-strings in flat background, and obtain the solution corresponding to two D-strings stretched between two D3-branes.

3.4.1 The Born-Infeld Action

The non-Abelian Dirac-Born-Infeld action for D-strings is given by equation (1.2.19) in section 1.2.2.

$$S = -T_1 \iint d\tau d\sigma \ \mathrm{STr} \left(\frac{e^{-\phi} \left(-D\right)^{1/2}}{\left(-\det\left(E^{ij}\right)\right)^{1/2}} \right) , \tag{3.4.1}$$

where

$$D = \det \begin{pmatrix} E_{ab} - E_{ai} E^{ij} E_{jb} + \alpha' F_{\alpha\beta} & E_{ak} E^{kj} + \alpha' D_\alpha \Phi^j \\ -E^{ik} E_{kb} - \alpha' D_\beta \Phi^i & E^{ij} + \alpha' \left[\Phi^i, \Phi^j\right] \end{pmatrix} , \tag{3.4.2}$$

with $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$ and where $\alpha, \beta = 0, 9$ are the D-string directions which we have taken as $x^0 = \tau = t$ and $x^9 = \sigma$. Here $i, j = 1, \ldots, 8$ are the transverse directions. We also take the dilaton Φ to be constant, $B_{\mu\nu} = 0$, and we fix the gauge such that $F_{\alpha\beta} = 0$. We excite the fields Φ^1, Φ^2, Φ^3, which will turn out to correspond to the D3-brane directions, and we set $\Phi^4 = \cdots = \Phi^8 = 0$, which is consistent with the equations of motion.

In flat background we have

$$E_{\mu\nu} = G_{\mu\nu} = \mathrm{diag}\,(-1, 1, \ldots, 1) ,$$

giving

$$\det(E^{\mu\nu}) = -1 ,$$

So

$$D = -\det \begin{pmatrix} -1 & 0 & \alpha'\dot{\Phi}_1 & \alpha'\dot{\Phi}_2 & \alpha'\dot{\Phi}_3 \\ 0 & 1 & \alpha'\Phi_1' & \alpha'\Phi_2' & \alpha'\Phi_3' \\ -\alpha'\dot{\Phi}_1 & -\alpha'\Phi_1' & 1 & \alpha'[\Phi_1, \Phi_2] & \alpha'[\Phi_1, \Phi_3] \\ -\alpha'\dot{\Phi}_2 & -\alpha'\Phi_2' & \alpha'[\Phi_2, \Phi_1] & 1 & \alpha'[\Phi_2, \Phi_3] \\ -\alpha'\dot{\Phi}_3 & -\alpha'\Phi_3' & \alpha'[\Phi_3, \Phi_1] & \alpha'[\Phi_3, \Phi_2] & 1 \end{pmatrix} , \qquad (3.4.3)$$

where a dot denotes differentiation with respect to t and a prime denotes differentiation with respect to σ (except in the case of α', which is a constant).

To describe the D-strings funnelling out into D3-branes we should take the transverse fields to belong to representations of the group $SU(2)$. In order to have two D-strings the appropriate representation to take is the (2×2) representation, i.e. the Pauli matrices σ_j. So we take the following ansatz, which will correspond to the $90°$ scattering of the D-strings,

$$\Phi_j = -\frac{1}{2}\phi_j\sigma_j \quad \text{(no summation over } j) , \qquad (3.4.4)$$

where $j = 1, 2, 3$, and the ϕ_j are real functions of t and σ (this ansatz is consistent with the Φ_j being anti-hermitian).

Substituting the ansatz (3.4.4) into the determinant (3.4.3), then evaluating the determinant and performing the symmetrised trace gives the action

$$\begin{aligned} S \;=\; -T_1 \textstyle\int_{-\infty}^{\infty} dt \int_0^L d\sigma & \left\{ \left(1+\frac{\alpha'^2}{4}\partial_\sigma(\phi_1\phi_2\phi_3)\right)^2 + \frac{\alpha'^2}{4}\left((\phi_1' - \phi_2\phi_3)^2 + \cdots\right) \right. \\ & -\frac{\alpha'^4}{16}\left(\partial_t(\phi_1\phi_2\phi_3)\right)^2 - \frac{\alpha'^2}{4}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2\right) \\ & \left. -\frac{\alpha'^4}{16}\left((\dot{\phi}_1\phi_2' - \dot{\phi}_2\phi_1')^2 + \cdots\right)\right\}^{1/2} , \end{aligned} \tag{3.4.5}$$

where L is the length of the string. In (3.4.5) $+\cdots$ denotes addition of cyclic permutations of ϕ_1, ϕ_2 and ϕ_3 - we will adopt this notation throughout the rest of this chapter.

3.4.2 Taking the Low Energy Limit

We will investigate what happens to the action (3.4.5) in the low energy limit $\alpha' \to 0$.

To ensure that the limit is taken in an appropriate manner we must make precise the dictionary between the monopole of Yang-Mills-Higgs theory, and the monopole described by a D-string stretched between two D3-branes. Recall from section 1.3.1 that the monopole of Yang-Mills-Higgs theory is described by the following action

$$S_{\mathrm{YM}} = \frac{1}{g_{\mathrm{YM}}^2} \int dt\, d^3x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi D^\mu\Phi\right) ,$$

As explained in that section, giving the Higgs field Φ an expectation value v results in a mass for the W-boson with

$$m_{\mathrm{W}} = v \; . \tag{3.4.6}$$

Recall from equation (1.3.22) that the monopole solution for the resulting action has mass

$$m_{\mathrm{mon}} = \frac{v}{g_{\mathrm{YM}}^2} \; . \tag{3.4.7}$$

We wish to compare the results (3.4.6) and (3.4.7) to those of the corresponding D-brane picture, in which the monopole is represented by a D-string stretched between two D3-branes. The $(3+1)$-dimensional picture, equivalent to the monopole picture described above, is described by the $(3+1)$-dimensional Born-Infeld action for the D3-branes. In the D3-brane picture we have $g_{\mathrm{YM}}^2 = g_s$ from equation (1.2.12). The Higgs field Φ is represented by an excited transverse field, X^9 say, which represents the position of the D3-brane in the corresponding direction x^9. Since v has the dimensions of 1/length, by dimensional analysis we have

$$v = \langle \Phi \rangle = \frac{\langle X^9 \rangle}{\alpha'} = \frac{L}{\alpha'} \ ,$$

where L represents the distance between the D3-branes in the x^9 direction, and therefore the length of the strings stretched between the D3-branes in that direction. The W-boson corresponds to a fundamental string stretched between the D3-branes, so we have

$$m_{\mathrm{W}} = T_{\mathrm{F}} L = \frac{L}{\alpha'} = v \ ,$$

where T_{F} is the fundamental string tension. The above result for m_W agrees with the monopole result (3.4.6). Also the monopole corresponds to a D-string stretched between the D3-branes, so, using equation (1.2.9) for the tension of a D-brane, we find

$$m_{\mathrm{mon}} = T_1 L = \frac{L}{g_S \alpha'} = \frac{v}{g_{\mathrm{YM}}^2} \ ,$$

which also agrees with the monopole result (3.4.7).

Returning to the D-string picture, in taking the limit $\alpha' \to 0$ the usual procedure is to take the string length $L = \alpha' v$ to be fixed, while the Higgs expectation value $v \to \infty$. Recall from the discussion in section 1.2.1 that applying this limit to the action (3.4.5), and keeping terms of order $(\alpha')^2$ results in the Yang-Mills action. In our case it is given by

$$\begin{aligned} S \ = -T_1 \int_{-\infty}^{\infty} dt \int_0^L d\sigma \Bigg(1 - \frac{\alpha'^2}{4} \partial_\sigma \left(\phi_1\phi_2\phi_3\right) - \frac{\alpha'^2}{8} \left((\phi_1' - \phi_2\phi_3)^2 + \cdots \right) \\ + \frac{\alpha'^2}{8} \left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2 \right) \Bigg) \ . \end{aligned} \tag{3.4.8}$$

The Bogomol'nyi equations for the action (3.4.8) are Nahm's equations (1.3.29). For our ansatz (3.4.4) they are

$$\phi_1' = \phi_2\phi_3 \ , \qquad \phi_2' = \phi_3\phi_1 \ , \qquad \phi_3' = \phi_1\phi_2 \ . \tag{3.4.9}$$

In general, the equations of motion derived from (3.4.8), which are also implied by

(3.4.9) in the static case, are

$$\ddot{\phi}_1 - \phi_1'' + \phi_1(\phi_2^2 + \phi_3^2) = 0 \,, \tag{3.4.10}$$

$$\ddot{\phi}_2 - \phi_2'' + \phi_2(\phi_3^2 + \phi_1^2) = 0 \,, \tag{3.4.11}$$

$$\ddot{\phi}_3 - \phi_3'' + \phi_3(\phi_1^2 + \phi_2^2) = 0 \,. \tag{3.4.12}$$

However in this limit, i.e. as $v \to \infty$, m_{W} (3.4.6) and m_{mon} (3.4.7) both blow up to infinity, which is clearly undesirable.

Instead of taking the limit $\alpha' \to 0$ as described above, we will take an alternative limit in which we keep v fixed, while allowing the string length $L \to 0$. This limit ensures that the mass of the W-boson (3.4.6), and the mass of the monopole (3.4.7) remain fixed and finite, unlike the previous limit we described. In the remainder of this section we will investigate the form of the action (3.4.5) under this alternative limit.

3.4.3 Rescaling the String Coordinate

Although we are taking the limit in which $L \to 0$, in order to make a connection with the discussion of section 1.3.2, we will work in coordinates in which the string length runs from 0 to 2. And so we perform the rescaling

$$\sigma \to \xi \equiv \frac{2}{\alpha' v}\sigma \,, \tag{3.4.13}$$

To ensure that the Bogomol'nyi equations of the rescaled action retain a familiar form, we must also rescale the ϕ_i as follows

$$\phi_i \to f_i \equiv \frac{\alpha' v}{2}\phi_i \,. \tag{3.4.14}$$

Under this rescaling the action (3.4.5) becomes

$$\begin{aligned} S = -T_1 \tfrac{\alpha' v}{2} \textstyle\int_{-\infty}^{\infty} dt \int_0^2 d\xi \Bigg\{ &\left(1 + \frac{4}{\alpha'^2 v^4}\partial_\xi(f_1 f_2 f_3)\right)^2 + \frac{4}{\alpha'^2 v^4}\left((f_1' - f_2 f_3)^2 + \cdots\right) \\ &- \frac{4}{\alpha'^2 v^6}\left(\partial_t(f_1 f_2 f_3)\right)^2 - \frac{1}{v^2}\left(\dot{f}_1^2 + \dot{f}_2^2 + \dot{f}_3^2\right) \\ &- \frac{4}{\alpha'^2 v^6}\left((\dot{f}_1 f_2' - \dot{f}_2 f_1')^2 + \cdots\right) \Bigg\}^{1/2} . \end{aligned} \tag{3.4.15}$$

3.4.4 The Bogomol'nyi Equations and the D3-Brane Solution

We define

$$\tilde{H} = 1 + \frac{4}{\alpha'^2 v^4} \partial_\xi (f_1 f_2 f_3) \ . \tag{3.4.16}$$

Then the action (3.4.15) is

$$\begin{aligned} S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \ \tilde{H} \Bigg\{ 1 &+ \frac{4}{\alpha'^2 v^4 \tilde{H}^2} \left((f_1' - f_2 f_3)^2 + \cdots \right) \\ &- \frac{4}{\alpha'^2 v^6 \tilde{H}^2} \left(\partial_t (f_1 f_2 f_3) \right)^2 - \frac{1}{v^2 \tilde{H}^2} \left(\dot{f}_1^2 + \dot{f}_2^2 + \dot{f}_3^2 \right) \\ &- \frac{4}{\alpha'^2 v^6 \tilde{H}^2} \left((\dot{f}_1 f_2' - \dot{f}_2 f_1')^2 + \cdots \right) \Bigg\}^{1/2} \ . \end{aligned} \tag{3.4.17}$$

For a static solution, $\dot{f}_1 = \dot{f}_2 = \dot{f}_3 = 0$, the action is minimised providing that

$$f_1' = f_2 f_3 \ , \qquad f_2' = f_3 f_1 \ , \qquad f_3' = f_1 f_2 \ . \tag{3.4.18}$$

When (3.4.18) holds, the Lagrangian density is equal to a total derivative term. So (3.4.18) are the Bogomol'nyi equations for the action (3.4.17), and are identical to the Bogomol'nyi equations found by taking the usual limit (3.4.9); again they are Nahm's equations. So for the purposes of finding the BPS solutions it does not matter which limit we take. However, since we will be interested here in perturbations of the BPS solutions, it is important that we use the correct limit for our calculations, i.e. we must use (3.4.17) instead of (3.4.8).

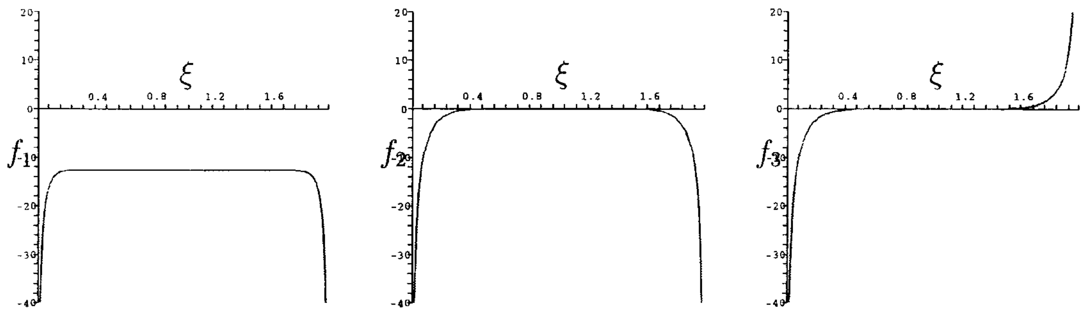
The appropriate solutions to Nahm's equations are those corresponding to a two-monopole. From equation (1.3.35) of section 1.3.2 we know that they are

$$f_1(\xi, k) = \frac{-K(k)}{\mathrm{sn}(K(k)\xi, k)} \ , \tag{3.4.19}$$

$$f_2(\xi, k) = \frac{-K(k)\mathrm{dn}(K(k)\xi, k)}{\mathrm{sn}(K(k)\xi, k)} \ , \tag{3.4.20}$$

$$f_3(\xi, k) = \frac{-K(k)\mathrm{cn}(K(k)\xi, k)}{\mathrm{sn}(K(k)\xi, k)} \ . \tag{3.4.21}$$

The solutions (3.4.19) - (3.4.21) have divergences at $\xi = 0$ and $\xi = 2$ because this was imposed as a condition on the Nahm data. In the D-string description these divergences correspond to the positions of the D3-branes along the D-strings, and



(a) (b) (c)

Figure 3.1: Plots of (a) $f_1(\xi, k)$ (b) $f_2(\xi, k)$, (c) $f_3(\xi, k)$ with $k = 0.9999999999$

originate from fundamental strings localised at the brane intersection, as described in section 1.4.2. Recall that the parameter k is a modulus which runs from $k = 0$ to $k = 1$. The limit $k \longrightarrow 1$ corresponds to the asymptotic limit in which the D-strings are far apart from one another. Figure 3.1 shows graphs of $f_1(\xi, k)$, $f_2(\xi, k)$ and $f_3(\xi, k)$ for $k = 0.999999999$. In this limit

$$f_1(\xi, k) \sim K(k) \ , \quad f_2(\xi, k) \sim f_3(\xi, k) \sim 0 \ , \tag{3.4.22}$$

except, of course, near the ends of the string, $\xi = 0$ and $\xi = 2$, where all three functions have poles. The approximations (3.4.22) become more exact as $k \longrightarrow 1$, since the width of the poles tends to zero in this limit.

At $k = 0$ we have $f_1(\xi, k) = f_2(\xi, k)$, and this configuration corresponds to the monopole configuration which is axisymmetric in the x^1–x^2 plane, where the two monopoles coincide (i.e. the 'doughnut' configuration, which we discussed in section 1.3.2). In this limit the Jacobian elliptic functions are trigonometric functions.

If we take the limit $\alpha' \longrightarrow 0$ in (3.4.17), keeping v fixed, then

$$\alpha'^2 \tilde{H} \longrightarrow \frac{4}{v^4} \partial_\xi (f_1 f_2 f_3) \ ,$$

and we can expand the square root in (3.4.17) to get

$$S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \left\{ \tilde{H} + \frac{1}{\partial_\xi (f_1 f_2 f_3)} \left[-\frac{1}{2} \left((f_1' - f_2 f_3)^2 + \cdots \right) \right.\right.$$
$$\left.\left. + \frac{1}{2v^2} \left(\partial_t (f_1 f_2 f_3) \right)^2 + \frac{1}{2v^2} \left((\dot{f}_1 f_2' - \dot{f}_2 f_1')^2 + \cdots \right) \right] \right\} . \tag{3.4.23}$$

We can calculate the metric on moduli space for the solutions (3.4.19) - (3.4.21) using Manton's technique, described in ref. [51] for slowly moving monopoles. The motion is described by a geodesic in the moduli space of parameters. So we must allow the modulus k to vary slowly with time, such that the Bogomol'nyi equations (3.4.18) continue to hold. Then the action reduces to

$$S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \left\{ \tilde{H} + \frac{1}{2v^2} (\dot{f}_1^2 + \dot{f}_2^2 + \dot{f}_3^2) \right\} ,$$

which gives the correct metric on moduli space, since $\tilde{H}$ is a total derivative.

3.4.5 Validity of the Born-Infeld Action

In what we have done we have been using the Born-Infeld action. However, as we discussed in section 1.4.2, there are limitations to the Born-Infeld action - it is not very accurate when the geometry is highly curved, because higher derivatives of the fields on the brane have been ignored. So we expect the action (3.4.23), which we obtained from the Born-Infeld action, to become inaccurate near the ends of the D-strings, where the geometry does become curved. This suggests that the limit we proposed in section 3.4.2, in which we take the string length to be small, may make the situation worse, since we are taking to be small the region in which the Born-Infeld action is accurate.

In the light of this discussion our calculation does not seem very promising. However, we may be redeemed by the fact that, in our approximation, we are working close to the BPS solutions (3.4.19) - (3.4.21). More precisely, we have taken

$$\dot{f}_i \sim \varepsilon , \tag{3.4.24}$$

and

$$(f_1' - f_2 f_3) \sim \varepsilon , \quad (f_2' - f_3 f_1) \sim \varepsilon , \quad (f_3' - f_1 f_2) \sim \varepsilon , \tag{3.4.25}$$

where ε is small. Since we expect the BPS solution to be a correct solution to the full equations of motion, it is reasonable to assume that motion in which the solutions remain close to the BPS solution is accurately described by the Born-Infeld action (3.4.23).

3.5 The Action for D-Strings in a D3-Brane Background

In the previous section we were working with D-strings in a flat background. In this section we will consider the same scenario, but this time in a D3-brane background in order to take into account the effects of the back-reaction of the D3-branes on the geometry. We will conjecture that the action remains unchanged when this back-reaction is taken into account.

Recall from section 1.1.2 that the supergravity solution for a Dp-brane becomes more accurate when we have a large number of D-branes present. So we will consider the case with D-strings stretched between two parallel stacks of D3-branes, each of which contains N_3 D3-branes. In this configuration we have a gauge group $SU(2N_3)$ on the D3-branes broken down to $SU(N_3) \times SU(N_3) \times U(1)$, according to the usual interpretation of gauge groups on branes.

3.5.1 The Born-Infeld Action

For now we concentrate on the Dirac-Born-Infeld action. We will investigate the contribution from the Chern-Simons action in section 3.5.3 below. We again use the non-Abelian Dirac-Born-Infeld action for D-strings (3.4.1), which we restate here

$$S_{\mathrm{DBI}} = -T_1 \iint d\tau d\sigma \ \mathrm{STr}\left(\frac{e^{-\phi}\left(-D\right)^{1/2}}{\left(-\det\left(E^{ij}\right)\right)^{1/2}}\right) . \tag{3.5.1}$$

As before we take $F_{\alpha\beta} = 0$ and we set $\Phi^4 = \cdots = \Phi^8 = 0$. However, this time we wish to include the effects of the D3-branes on the geometry. From equation (1.1.23) in section 1.1.2 we find that the background metric for D3-branes is given by

$$ds^2 = -\frac{dt^2}{\sqrt{\mathcal{H}}} + \sqrt{\mathcal{H}}d\sigma^2 + \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\sqrt{\mathcal{H}}} + \sqrt{\mathcal{H}}(dx^m)^2 , \tag{3.5.2}$$

where $m = 4, \ldots, 8$ and the harmonic function $\mathcal{H}$, which corresponds to N_3 D3-branes positioned at $\sigma = 0$, and N_3 positioned at $\sigma = L$ is given by

$$\mathcal{H} = 1 + \frac{gN_3}{\pi}\frac{\alpha'^2}{\sigma^4} + \frac{gN_3}{\pi}\frac{\alpha'^2}{|\sigma - L|^4} . \tag{3.5.3}$$

We also have $B_{\mu\nu} = 0$, and for D3-branes the dilaton Φ is constant.

Substituting (3.5.2) into the definition of D, equation (3.4.2), we find

$$D = \det \begin{pmatrix} -\frac{1}{\sqrt{\mathcal{H}}} & 0 & \alpha'\partial_0\Phi^1 & \alpha'\partial_0\Phi^2 & \alpha'\partial_0\Phi^3 \\ 0 & \sqrt{\mathcal{H}} & \alpha'\partial_1\Phi^1 & \alpha'\partial_1\Phi^2 & \alpha'\partial_1\Phi^3 \\ -\alpha'\partial_0\Phi^1 & -\alpha'\partial_1\Phi^1 & \sqrt{\mathcal{H}} & \alpha'[\Phi^1,\Phi^2] & \alpha'[\Phi^1,\Phi^3] \\ -\alpha'\partial_0\Phi^2 & -\alpha'\partial_1\Phi^2 & \alpha'[\Phi^2,\Phi^1] & \sqrt{\mathcal{H}} & \alpha'[\Phi^2,\Phi^3] \\ -\alpha'\partial_0\Phi^3 & -\alpha'\partial_1\Phi^3 & \alpha'[\Phi^3,\Phi^1] & \alpha'[\Phi^3,\Phi^2] & \sqrt{\mathcal{H}} \end{pmatrix} .$$

As in the previous section, to describe the D-strings funnelling out into D3-branes, we should take the fields Φ_i to be proportional to the Pauli matrices

$$\Phi_j = -\frac{1}{2}\phi_j\sigma_j \quad \text{(no summation over } j) , \tag{3.5.4}$$

where $j = 1, 2, 3$, and the ϕ_j are real functions of t and σ Then some calculation yields

$$\begin{aligned} S_{\rm DBI} = -T_1 \iint dt d\sigma \Bigg(\left(1 + \frac{\alpha'^2}{4\mathcal{H}}\partial_\sigma(\phi_1\phi_2\phi_3)\right)^2 - \frac{\alpha'^2}{4}(\dot{\phi}_1{}^2 + \dot{\phi}_2{}^2 + \dot{\phi}_3{}^2) \\ - \frac{\alpha'^4}{16\mathcal{H}}(\partial_t(\phi_1\phi_2\phi_3))^2 + \frac{\alpha'^2}{4\mathcal{H}}\left((\phi_1' - \phi_2\phi_3)^2 + \cdots\right) \\ - \frac{\alpha'^4}{16\mathcal{H}}\left((\dot{\phi}_1\phi_2' - \dot{\phi}_2\phi_1')^2 + \cdots\right)\Bigg)^{1/2} . \end{aligned} \tag{3.5.5}$$

3.5.2 Rescaling the Action

To compare with the flat background case we will need to rescale the coordinate σ to a new coordinate ξ so that one D3-brane is positioned at $\xi = 0$ and the other is at $\xi = 2$. So we need to calculate the coordinate distance between the two branes. In order to do this consider the action for a test D-string in the supergravity background (3.5.2). The Born-Infeld action for the D-string is of the form

$$S_{\rm DBI} = -T_1 \iint d\sigma dt \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu})} ,$$

where $g_{\mu\nu}$ is the metric for the background with two D3-branes that we had previously. We can take

$$X^0 = t \,, \quad X^9 = \sigma \,, \tag{3.5.6}$$

$$X^i = v^i t \,, \quad X^m = \omega^m t \,, \tag{3.5.7}$$

where $i = 1, 2, 3$ and $m = 4, \ldots, 8$. Since the string must end on the D3-brane, we have $\omega^m = 0$. So

$$S_{\mathrm{DBI}} = -T_1 \iint d\sigma dt \sqrt{-\det \begin{pmatrix} (1 - v^i v^i)\mathcal{H}^{-1/2} & 0 \\ 0 & \mathcal{H}^{1/2} \end{pmatrix}} \tag{3.5.8}$$

$$= -T_1 \iint d\sigma dt \sqrt{1 - v^i v^i} \tag{3.5.9}$$

$$= -LT_1 \int dt \sqrt{1 - v^i v^i} \,, \tag{3.5.10}$$

where L is the coordinate length along the string. Comparing this to the action for a relativistic particle

$$S = -m \int dt \sqrt{1 - (v^i)^2} \,,$$

we obtain

$$L = \frac{m}{T_1} = \alpha' v \,.$$

This is identical to the coordinate length along the string in the flat background case.

We rescale the coordinate σ and the fields ϕ_i to ξ and f_i respectively, so that the positions of the two D-branes become 0 and 2, and the f_i still obey Nahm's equations, as we did in section 3.4.3 for the flat background case. So

$$\sigma = \frac{\alpha' v}{2} \xi \,, \tag{3.5.11}$$

$$\phi_i = \frac{2}{\alpha' v} f_i \,. \tag{3.5.12}$$

Then the action becomes

$$S_{\mathrm{DBI}} = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \left[\left(1 + \frac{1}{\mathcal{H}} \frac{4}{\alpha'^2 v^4} \partial_\xi (f_1 f_2 f_3) \right)^2 - \frac{1}{v^2} (\dot{f_1}^2 + \dot{f_2}^2 + \dot{f_3}^2) \right.$$
$$- \frac{1}{\mathcal{H}} \frac{4}{\alpha'^2 v^6} (\partial_t (f_1 f_2 f_3))^2 + \frac{1}{\mathcal{H}} \frac{4}{\alpha'^2 v^4} \left((f_1' - f_2 f_3)^2 + \cdots \right)$$
$$\left. - \frac{1}{\mathcal{H}} \frac{4}{\alpha'^2 v^6} \left((\dot{f}_1 f_2' - \dot{f}_2 f_1')^2 + \cdots \right) \right]^{1/2} \,, \tag{3.5.13}$$

with

$$\mathcal{H} = 1 + \frac{16gN_3}{\pi}\frac{1}{\alpha'^2 v^4 \xi^4} + \frac{16gN_3}{\pi}\frac{1}{\alpha'^2 v^4 \left|\xi - 2\right|^4} , \tag{3.5.14}$$

where $\xi = 2$ is the location of the second D3-brane.

According to refs. [87] and [88], the Bogomol'nyi equations derived from (3.5.13) should be the same as those we calculated for the flat background case, which are given in equations (3.4.18). In order to verify this we will need to include the contribution from the Chern-Simons action, which we will discuss in the next section.

3.5.3 The Chern-Simons Action

Recall from section 1.2.2 that the non-Abelian Chern-Simons action for the D-strings takes the form

$$S_{\text{CS}} = \mu_1 \iint d\tau\, d\sigma\, \text{STr}\left(P\left[e^{i\alpha' i_\Phi i_\Phi} \sum C^{(n)}\right]\right) , \tag{3.5.15}$$

where P denotes the pullback to the D-strings' worldvolume, and i_Φ denotes the interior product by Φ^i regarded as a vector in the transverse space. In (3.5.15) we have set $F = B = 0$. Since we have two D-strings, the relation between the tension and the charge is given by

$$T_1 = 2\mu_1 . \tag{3.5.16}$$

Recall from equation (1.1.25) that the R-R field $C^{(4)}$, which acts as a source for the D3-brane, is given by

$$C^{(4)} = (\mathcal{H}^{-1} - 1)\, dt \wedge dx^1 \wedge dx^2 \wedge dx^3 , \tag{3.5.17}$$

Substituting (3.5.17) into the action (3.5.15) results in a non-trivial interaction term, which is

$$S_{\text{CS}} = i\, \alpha' \mu_1 \iint d\tau\, d\sigma\, \text{STr}\left(P\left[i_\Phi i_\Phi C^{(4)}\right]\right) .$$

Now

$$i_\Phi i_\Phi C^{(4)} = \frac{1}{2}\left[\Phi^j, \Phi^i\right] C^{(4)}_{ij\mu\nu}\, dx^\mu \wedge dx^\nu .$$

So we find

$$S_{\text{CS}} = i\frac{\alpha'^2}{2}\,\mu_1 \iint dt\, d\sigma\, \text{STr}\left(\partial_\sigma \Phi^i [\Phi^k, \Phi^j]\, C^{(4)}_{tijk}\right) .$$

Substituting in the ansatz (3.5.4) for the Φ^i, and the expression (3.5.17) for $C^{(4)}_{tijk}$, and performing the rescalings (3.4.13) and (3.4.14) we find

$$S_{\rm CS} = \frac{4}{\alpha' v^3}\,\mu_1 \iint dt\,d\xi\,\left(\mathcal{H}^{-1} - 1\right)\partial_\xi\left(f_1 f_2 f_3\right)\ .$$

Using the expression for $\tilde{H}$ (3.4.16), and ignoring the total derivative term and the constant terms, we find

$$S_{\rm CS} = \alpha' v \mu_1 \iint dt\,d\xi \left(\frac{\tilde{H}}{\mathcal{H}}\right)\ . \qquad (3.5.18)$$

And using the BPS condition (3.5.16), this becomes

$$S_{\rm CS} = \frac{\alpha' v}{2}\,T_1 \iint dt\,d\xi \left(\frac{\tilde{H}}{\mathcal{H}}\right)\ . \qquad (3.5.19)$$

3.5.4 The Bogomol'nyi Equations

In the D3-brane background the total action is given by

$$S = S_{\rm DBI} + S_{\rm CS}\ , \qquad (3.5.20)$$

where $S_{\rm DBI}$ and $S_{\rm CS}$ are given by (3.5.13) and (3.5.19) respectively. Let us consider a static configuration with $\dot{f}_1 = \dot{f}_2 = \dot{f}_3 = 0$, which also satisfies

$$f_1' = f_2 f_3\ , \qquad f_2' = f_3 f_1\ , \qquad f_3' = f_1 f_2\ , \qquad (3.5.21)$$

Substituting this ansatz into the Dirac-Born-Infeld action (3.5.13), we find that it reduces to

$$S_{\rm DBI} = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \left(1 + \frac{1}{\mathcal{H}}\frac{4}{\alpha'^2 v^4}\partial_\xi (f_1 f_2 f_3)\right)\ . \qquad (3.5.22)$$

Adding the Chern-Simons term (3.5.19) to this, we obtain the total action

$$S = -T_1 \frac{\alpha' v}{2} \iint dt\,d\xi \left(1 - \frac{1}{\mathcal{H}}\right)\ , \qquad (3.5.23)$$

which is a constant, since $\mathcal{H}$ is given by equation (3.5.14), which is independent of the f_i. We deduce that (3.5.21) are the Bogomol'nyi equations for the case with the D3-brane background. Note that they are identical to those we had in the flat background case (3.4.18), as we expected.

In fact, the Bogomol'nyi equations are given by (3.5.21), irrespective of the form of $\mathcal{H}$. This means that the Bogomol'nyi equations are not affected by the way in which we minimally break the $SU(2N_3)$ symmetry; for example a string stretched between a single D3-brane on one side and a large stack of D3-branes on the other would have the same Bogomol'nyi equations. This suggests that the D-strings may only be aware of one D3-brane at each end - it makes no difference to them how many other D3-branes we stack together.

Note that we do not need to consider here the jumping points of $SU(N)$ Nahm data, because we are dealing with the case of minimal symmetry breaking with only one interval of Nahm data; see ref. [47] for a discussion of how jumping points occur in the D-brane description.

3.5.5 The Supergravity Limit

In section 3.4.4, in the flat background case, we expanded out the square root in the Born-Infeld action by taking the low energy limit $\alpha' \to 0$. However, if we compare the action we used for section 3.4.4, which was (3.4.15), with the action (3.5.13) which we are currently dealing with, there is a significant difference. In (3.4.15) the leading order term in α' is the first term under the square root, which is of order $1/\alpha'^4$. However, in (3.5.13) all terms are of leading order α'^0, and so all contribute to the leading order term in α'. This would make an expansion in α' very messy for the action (3.5.13).

There is an alternative limit we can take; recall that for the supergravity solution to be accurate we need N_3 to be large. In this limit

$$\frac{1}{\mathcal{H}} \sim \frac{\pi}{16 g_s N_3} \alpha'^2 v^4 h(\xi) \ , \tag{3.5.24}$$

where $h(\xi)$ is a function of ξ given by

$$h(\xi) = \frac{1}{\xi^{-4} + |\xi - 2|^{-4}} \ .$$

We can expand the square root in (3.5.13) as a series in $1/N_3$. So defining

$$J \equiv 1 - \frac{1}{v^2} \left(\dot{f_1}^2 + \dot{f_2}^2 + \dot{f_3}^2 \right) \ ,$$

then

$$\begin{aligned}
S_{\text{DBI}} = -T_1\frac{\alpha' v}{2}\int_{-\infty}^{\infty} dt \int_0^2 d\xi\,\sqrt{J}\,\Bigg[1 &+ \frac{8h(\xi)}{JN_3}\,\partial_\xi(f_1f_2f_3) + \frac{16h(\xi)^2}{JN_3^2}\,(\partial_\xi(f_1f_2f_3))^2 \\
&-\frac{4h(\xi)}{JN_3v^2}\,(\partial_t(f_1f_2f_3))^2 + \frac{4h(\xi)}{JN_3}\,((f_1' - f_2f_3)^2 + \cdots) \\
&-\frac{4h(\xi)}{JN_3v^2}\left((\dot{f}_1f_2' - \dot{f}_2f_1')^2 + \cdots\right)\Bigg]^{1/2} \\
\simeq -T_1\frac{\alpha' v}{2}\int_{-\infty}^{\infty} dt \int_0^2 d\xi\,\sqrt{J}\,\Bigg(1 &+ \frac{4h(\xi)}{JN_3}\,\partial_\xi(f_1f_2f_3) - \frac{2h(\xi)}{JN_3v^2}\,(\partial_t(f_1f_2f_3))^2 \\
&+\frac{2h(\xi)}{JN_3}\,((f_1' - f_2f_3)^2 + \cdots) - \frac{2h(\xi)}{JN_3v^2}\left((\dot{f}_1f_2' - \dot{f}_2f_1') + \cdots\right)\Bigg)\,,
\end{aligned} \tag{3.5.25}$$

where we have substituted the expression for $\mathcal{H}$ (3.5.24) in (3.5.25) in order to make the dependence on N_3 explicit.

Since we are dealing with solutions close to the Bogomol'nyi bound we will assume that $\dot{f}_i \sim \varepsilon$, and all cyclic permutations of $(f_1' - f_2f_3) \sim \varepsilon$, where ε is small (see (3.4.24) and (3.4.25)). So we can expand the factors of $\sqrt{J}$ and $1/J$ in powers of ε to get

$$\begin{aligned}
S_{\text{DBI}} \simeq -T_1\frac{\alpha' v}{2}\int_{-\infty}^{\infty} dt \int_0^2 d\xi\,\Bigg[1 &+ \frac{1}{\mathcal{H}}\tilde{H} - \frac{1}{2v^2}\left(1 - \frac{1}{\mathcal{H}}\tilde{H}\right)(\dot{f}_1^2 + \dot{f}_2^2 + \dot{f}_3^2) \\
&-\frac{2}{\mathcal{H}\alpha'^2v^6}\,(\partial_t(f_1f_2f_3))^2 + \frac{2}{\mathcal{H}\alpha'^2v^4}\,((f_1' - f_2f_3)^2 + \cdots) \\
&-\frac{2}{v^2}\frac{1}{\mathcal{H}\alpha'^2v^4}\left((\dot{f}_1f_2' - \dot{f}_2f_1')^2 + \cdots\right)\Bigg]\,,
\end{aligned} \tag{3.5.26}$$

where

$$\tilde{H} \sim \frac{4}{\alpha'^2v^4}\partial_\xi(f_1f_2f_3)\,, \tag{3.5.27}$$

as we had in (3.4.16) in the flat background case.

Adding in the Chern-Simons term (3.5.19), we find that the total action is

$$\begin{aligned}
S_{\text{DBI}} \simeq -T_1\frac{\alpha' v}{2}\int_{-\infty}^{\infty} dt \int_0^2 d\xi\,\Bigg[1 &- \frac{1}{2v^2}\left(1 - \frac{1}{\mathcal{H}}\tilde{H}\right)(\dot{f}_1^2 + \dot{f}_2^2 + \dot{f}_3^2) \\
&-\frac{2}{\mathcal{H}\alpha'^2v^6}\,(\partial_t(f_1f_2f_3))^2 + \frac{2}{\mathcal{H}\alpha'^2v^4}\,((f_1' - f_2f_3)^2 + \cdots) \\
&-\frac{2}{\mathcal{H}\alpha'^2v^6}\left((\dot{f}_1f_2' - \dot{f}_2f_1')^2 + \cdots\right)\Bigg]\,.
\end{aligned} \tag{3.5.28}$$

3.5.6 Comparing with the Flat Background Case

We will compare (3.5.28) to the result we had for the flat background case, which was

$$S \simeq -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \left[\tilde{H} + \frac{2}{\tilde{H}\alpha'^2 v^4} \left((f_1' - f_2 f_3)^2 + \cdots \right) - \frac{2}{\tilde{H}\alpha'^2 v^6} \left(\partial_t (f_1 f_2 f_3) \right)^2 \right.$$
$$\left. - \frac{2}{\tilde{H}\alpha'^2 v^6} \left((\dot{f}_1 f_2' - \dot{f}_2 f_1')^2 + \cdots \right) \right] . \tag{3.5.29}$$

The two actions (3.5.28) and (3.5.29) agree up to a total derivative for solutions close to the Bogomol'nyi bound, providing that

$$\mathcal{H} = \tilde{H} . \tag{3.5.30}$$

Now $\mathcal{H}$ is a harmonic function, and $\tilde{H}$ is expressed in terms of elliptic functions, so clearly the above equality cannot hold exactly. But $\mathcal{H}$ and $\tilde{H}$ are qualitatively similar in that they are both symmetric about $\xi = 1$, and they have the same pole behaviour at $\xi = 0$ and $\xi = 2$. Recall that the Born-Infeld action is only an approximate action, so it is possible that when the full action is used instead, the equality above may be exact.

3.6 Scattering D-Strings

Here we consider how to make the static solutions (3.4.19) - (3.4.21) time dependent in order to describe the scattering of the D-strings.

3.6.1 Describing the Scattering

We wish to initiate the motion of the D-strings in the limit where the D-strings are very far apart. Recall from section 3.4.4 that the required limit is $k \longrightarrow 1$. In this limit we have $f_1 \approx K(k)$, $f_2 \approx 0$ and $f_3 \approx 0$. Recall that Φ_1, the field describing the positions of the D-strings in the x^1 direction, is given by $\Phi_1 = f_1 \sigma_1$, and σ_1 has eigenvalues ± 1. From the discussion around equation (1.1.11) in section 1.1.1 it follows that the D-strings' positions are at $\pm f_1$ in the x^1 direction, and 0 in the x^2 and x^3 directions.

We describe the D-string scattering by decreasing k to $k = 0$ as time increases. At $k = 0$ we have $f_1 = f_2$, and the D-strings can be thought of as being at a minimum distance apart from each other (although we have to be careful about what 'distance' actually means in this scenario, since we are dealing with noncommutative geometry). At this point we have the 'doughnut' configuration, described in section 1.3.2.

To conclude the scattering picture we swap the roles of f_1 and f_2 at $k = 0$. Therefore as time tends to infinity, the D2-strings grow further apart, but this time in the x^2-direction. So in this description the two D-strings scatter at 90°. Recall from section 1.3.3 that two monopoles also scatter at 90° when they approach one another headlong.

3.6.2 A Symmetry of the Solutions

We quote here a transformation of the Jacobian elliptic functions, which can be found in ref. [49]. If we transform k and ξ as follows

$$k \mapsto \tilde{k} = \frac{ik}{k'} , \qquad \xi \mapsto \tilde{\xi} = k'\xi ,$$

where

$$k' = \sqrt{1 - k^2} ,$$

then the Jacobian elliptic functions transform as

$$\mathrm{sn}(\tilde{\xi}, \tilde{k}) = k' \frac{\mathrm{sn}(\xi, k)}{\mathrm{dn}(\xi, k)} , \quad \mathrm{cn}(\tilde{\xi}, \tilde{k}) = \frac{\mathrm{cn}(\xi, k)}{\mathrm{dn}(\xi, k)} , \quad \mathrm{dn}(\tilde{\xi}, \tilde{k}) = \frac{1}{\mathrm{dn}(\xi, k)} .$$

Also the complete elliptic integral of the first kind $K(k)$ transforms as

$$K(\tilde{k}) = k'K(k) ,$$

Under this transformation the functions f_1, f_2 and f_3 transform as

$$f_1(\tilde{\xi}, \tilde{k}) = f_2(\xi, k) , \quad f_2(\tilde{\xi}, \tilde{k}) = f_1(\xi, k) , \quad f_3(\tilde{\xi}, \tilde{k}) = f_3(\xi, k) .$$

Therefore this transformation takes motion from before the scattering to motion after the scattering and vice versa (since at the point of scattering $k = 0$, f_1 and f_2 are interchanged).

Under this transformation k takes on imaginary values. Therefore we could think of the motion after scattering as being described by a modulus k continued to imaginary values. Equivalently we could think of the motion as being described by a modulus k^2, with $k^2 \to 1$ as the initial configuration, and $k^2 \to -\infty$ as the final configuration, and the point at which k^2 is zero as the 'minimum distance' configuration.

3.7 Perturbing the Fields

We wish to describe the scattering of the two D-strings, taking into account the possibility of energy radiation. Therefore it is not enough to allow the modulus k to depend on time in the solutions to the Bogomol'nyi equations (3.4.19) - (3.4.21); we also need to include perturbations to account for the energy in the non-zero modes. We perturb the ansatz (3.4.4) as follows

$$\Phi^i = \sum_i \left(-\frac{1}{2}\phi_i(\sigma, k) + \tilde{\varepsilon}_i(\sigma, \tau) \right) \sigma_i + \sum_{i \neq j} \hat{\varepsilon}_{ij}(\sigma, \tau)\, \sigma_j \ , \qquad (3.7.1)$$

where $\phi_i(\sigma, k)$ is the static solution to the Bogomol'nyi equations (3.4.9) (it is related to the static solution (3.4.19) - (3.4.21) by a rescaling of the ϕ_i given by equation (3.4.14), and a rescaling of σ given by (3.4.13)). The $\tilde{\varepsilon}_i(\sigma, \tau)$ can be thought of as 'diagonal' perturbations of the Φ^i, with the $\hat{\varepsilon}_{ij}(\sigma, \tau)$ as the 'off-diagonal' perturbations. As before, the σ_i are the Pauli matrices. Substituting the ansatz (3.7.1) into the non-Abelian Born-Infeld action (3.4.1), and applying the symmetrised trace prescription, we find that there are no terms in the action which are of linear order in the $\hat{\varepsilon}_{ij}$. This means that the equations of motion for the $\hat{\varepsilon}_{ij}$ are at least linear in $\hat{\varepsilon}_{ij}$, and so the $\hat{\varepsilon}_{ij}$ can be consistently set to zero. Furthermore, when we evolve a configuration with $\hat{\varepsilon}_{ij} = 0$ initially the $\hat{\varepsilon}_{ij}$ modes can never be excited. We will start the D-strings' motion with a configuration tangent to the static solution, i.e. $\Phi^i(\xi, t = 0) = -\phi_i(\sigma, k_0)\sigma_i/2$, where k_0 is some initial value of k, so we will have $\hat{\varepsilon}_{ij} = 0$ initially (see section 4.5.4 for more details of the initial conditions). Therefore, since we will have $\hat{\varepsilon}_{ij} = 0$ initially, we should take $\hat{\varepsilon}_{ij} = 0$ at all times.

We have shown that we can neglect the 'off-diagonal' perturbations of the Φ_i when we perturb the static solution. Therefore, instead of working from scratch with the non-Abelian Dirac-Born-Infeld action (3.4.1), we can just perturb the fields in the low energy, rescaled action (3.4.17). We will relabel the fields $f_i(\xi, t)$ in that action as $\varphi_i(\xi, t)$ in order to keep the notation f_i for the static solutions (3.4.19) - (3.4.21). Then we write

$$\varphi_i = f_i + \varepsilon_i \ , \tag{3.7.2}$$

We take the slow motion approximation, with $\dot{f}_i$ small, and therefore we can assume that the perturbations ε_i and their derivatives ε_i' and $\dot{\varepsilon}_i$ are also small and of the same order as ε_i.

Substituting (3.7.2) into the action (3.4.23), we get

$$S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \ \frac{1}{2} \left[\left\{ (\dot{f}_1 + \dot{\varepsilon}_1)^2 + (\dot{f}_2 + \dot{\varepsilon}_2)^2 + (\dot{f}_3 + \dot{\varepsilon}_3)^2 \right\} \right.$$
$$\left. - \frac{1}{H} \left\{ (\varepsilon_1' - f_2 \varepsilon_3 - f_3 \varepsilon_2)^2 + \cdots \right\} \right] \ , \tag{3.7.3}$$

where $+\cdots$ again denotes the addition of all cyclic permutations of the indices of the first term in the brackets, and we have defined

$$H = (f_1^2 f_2^2 + f_2^2 f_3^2 + f_3^2 f_1^2) \ .$$

We next define the row vectors O_i,

$$O_1 = \begin{pmatrix} \partial_\xi & -f_3 & -f_2 \end{pmatrix} \ , \tag{3.7.4}$$
$$O_2 = \begin{pmatrix} -f_3 & \partial_\xi & -f_1 \end{pmatrix} \ , \tag{3.7.5}$$
$$O_3 = \begin{pmatrix} -f_2 & -f_1 & \partial_\xi \end{pmatrix} \ . \tag{3.7.6}$$

Then we can rewrite the action (3.7.3) in the more compact form

$$S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_0^2 d\xi \left[\frac{1}{2} (\dot{\vec{f}} + \dot{\vec{\varepsilon}})^2 - \frac{1}{2H} (O_i \vec{\varepsilon}, O_i \vec{\varepsilon}) \right] \ . \tag{3.7.7}$$

If we had taken the alternative limit in section 3.4.2, i.e. $v \to \infty$ as $\alpha' \to 0$, which is the Yang-Mills limit, then we would have $H = 1$ in (3.7.7). Then the equations of motion with respect to $\vec{\varepsilon}$ would be

$$\ddot{\vec{f}} + \ddot{\vec{\varepsilon}} + O_i^\dagger O_i \vec{\varepsilon} = 0 \ ,$$

where

$$O_1^\dagger = \begin{pmatrix} -\partial_\xi \\ -f_3 \\ -f_2 \end{pmatrix} \ , \quad O_2^\dagger = \begin{pmatrix} -f_3 \\ -\partial_\xi \\ -f_1 \end{pmatrix} \ , \quad O_3^\dagger = \begin{pmatrix} -f_2 \\ -f_1 \\ -\partial_\xi \end{pmatrix} \ .$$

Now we include the factor of H in (3.7.7). Following the discussion from section 3.3, we rescale $\xi \mapsto x$ such that

$$\frac{dx}{d\xi} = \sqrt{H} \ , \tag{3.7.8}$$

and we make the following definitions

$$\begin{aligned}
F_i &= \frac{f_i}{\sqrt{H}} \ , && (3.7.9)\\
\Omega_1 &= \begin{pmatrix} \partial_x & -F_3 & -F_2 \end{pmatrix} \ , && (3.7.10)\\
\Omega_2 &= \begin{pmatrix} -F_3 & \partial_x & -F_1 \end{pmatrix} \ , && (3.7.11)\\
\Omega_3 &= \begin{pmatrix} -F_2 & -F_1 & \partial_x \end{pmatrix} \ . && (3.7.12)
\end{aligned}$$

Then substituting (3.7.8), (3.7.9), (3.7.10), (3.7.11) and (3.7.12) into (3.7.7) we get

$$S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{dx}{\sqrt{H}} \left[\frac{1}{2} (\sqrt{H}\vec{F} + \vec{\varepsilon})^2 - \frac{1}{2} (\Omega_i \vec{\varepsilon}, \Omega_i \vec{\varepsilon}) \right] \ . \tag{3.7.13}$$

Next we rescale

$$\varepsilon_i \mapsto \eta_i = \varepsilon_i H^{-1/4} \ , \tag{3.7.14}$$

and we redefine the Ω_i

$$\begin{aligned}
\Omega_1 &= \begin{pmatrix} \partial_x(\ln H^{1/4}) + \partial_x & -F_3 & -F_2 \end{pmatrix} \ , && (3.7.15)\\
\Omega_2 &= \begin{pmatrix} -F_3 & \partial_x(\ln H^{1/4}) + \partial_x & -F_1 \end{pmatrix} \ , && (3.7.16)\\
\Omega_3 &= \begin{pmatrix} -F_2 & -F_1 & \partial_x(\ln H^{1/4}) + \partial_x \end{pmatrix} \ . && (3.7.17)
\end{aligned}$$

Then substituting (3.7.14), (3.7.15), (3.7.16) and (3.7.17) into (3.7.13) we get

$$S = -T_1 \frac{\alpha' v}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (H^{1/4}\vec{F} + \vec{\eta})^2 - \frac{1}{2} (\Omega_i \vec{\eta}, \Omega_i \vec{\eta}) \right] \ . \tag{3.7.18}$$

Then the equation of motion for $\vec{\eta}$ is

$$\ddot{\vec{\eta}} + \Omega_i^\dagger \Omega_i \vec{\eta} = -H^{1/4} \ddot{\vec{F}} - \frac{1}{4H^{3/4}} \dot{H} \dot{\vec{F}} \ . \tag{3.7.19}$$

These equations are three coupled equations, with each of the form of a Laplacian with a potential given by the form of the $\Omega_i^\dagger \Omega_i$, and a forcing term given by the right-hand side of equation (3.7.19).

3.8 Conclusions

We have presented the solutions to Nahm's equations (3.4.19) - (3.4.21) which represent two D-strings stretched between D3-branes. We compared the flat background calculation to the calculation in a D3-brane background, and made a conjecture which would ensure that the two agree. The D3-brane background calculation also showed that, when considering two stacks of D3-branes, it doesn't matter how many branes are put in each stack - the shape of the D-strings remains the same. Therefore it appears that the D-string 'funnel' can only feel the presence of a single brane - it doesn't notice the rest of the branes in the stack. This was maybe to be expected from the monopole solutions for $SU(2N)$ broken down to $SU(N) \times SU(N) \times U(1)$, which transform trivially under the two groups, as we discussed in section 1.3.2. We have described the slow motion scattering of the D-strings by allowing the modular parameter k to depend on time. We have also derived the equations of motion (3.7.19) for the time evolution of perturbations on the D-strings.

Our object in these calculations has been to calculate the energy radiated during the scattering of D-strings stretched between D3-branes. The perturbations corresponding to the non-zero modes of the solutions describe the energy radiated. Therefore it remains to solve the equations of motion for the perturbations and to calculate the energy retained in them in order to find out the energy radiated. We have performed this calculation numerically, and we will discuss our numerical calculations in the next chapter.

Chapter 4

Using D-Strings to Describe Monopole Scattering - Numerical Calculations

4.1 Introduction

In the previous chapter we described the scattering of two D-strings stretched between D3-branes, and calculated linearised equations of motion for perturbations to the static solution. We can think of the D-strings' motion as being split into zero modes and non-zero modes. The zero modes describe the motion of the centres of mass of the two D-strings; it is the motion described by Manton's geodesic approximation from ref. [51]. The dynamics of the non-zero modes is described by the dynamics of the perturbations to the static solution. After the D-strings have scattered, the energy in the non-zero modes represents the energy radiated during scattering, as we will show explicitly in section 4.4.3 below.

Unfortunately we were unable to solve the equations of motion (3.7.19) analytically, and so we decided instead to use numerical techniques to solve the problem. We will present our numerical calculations in this chapter. The chapter is organised as follows. In section 4.2 we will explain how it was unfeasible to solve the equations of motion for the Born-Infeld action numerically, and how we solved the equations of motion for the Yang-Mills action instead. We will also discuss the differences between these two systems. In section 4.3 we will describe the motion of the D-strings in the asymptotic limit using analytic calculations. In section 4.4 we will show that the total energy of the system is conserved, and demonstrate the decoupling of the energy between zero modes and non-zero modes in the asymptotic limit. In section 4.5 we will describe the numerical techniques we have used to solve the equations of motion. In section 4.6 we will describe the numerical techniques we have used to calculate the energy radiated during scattering, and we will present the results of our calculations. In section 4.7 we will summarise the chapter, and discuss some conclusions.

4.2 Yang-Mills vs Born-Infeld

In this section we will describe how it is unfeasible to solve the equations of motion for the Born-Infeld action numerically, and we shall examine the Yang-Mills action which we shall use instead.

In order to solve numerically the equations of motion for the perturbations (3.7.19) we would have to evolve in time the modular parameter k (since the equations depend on the zero-mode solutions $f_1(\xi, k)$, $f_2(\xi, k)$ and $f_3(\xi, k)$). It turns out to be too complicated to do this numerically, the program taking too long to run because of the necessity to evolve $k(t)$ in time. As an alternative approach we could try to solve the full equations of motion for the fields φ_i. The equation of motion for φ_1, derived from the Born-Infeld action (3.4.23) is

$$
\begin{aligned}
&-\partial_\xi\left(-\frac{1}{2\partial_\xi(f_1f_2f_3)^2}f_2f_3\left[((f_1'-f_2f_3)^2+\cdots)+\frac{1}{v^2}\partial_t(f_1f_2f_3)^2\right.\right.\\
&\quad\left.+\frac{1}{v^2}\left((\dot{f}_1f_2'-\dot{f}_2f_1')^2+\cdots\right)\right]\\
&\quad\left.+\frac{1}{(\partial_\xi(f_1f_2f_3))}\left\{(f_1'-f_2f_3)-\frac{1}{v^2}(\dot{f}_1f_2'-\dot{f}_2f_1')\dot{f}_2-\frac{1}{v^2}(\dot{f}_1f_3'-\dot{f}_3f_1')\dot{f}_3\right\}\right)\\
&-\partial_t\left(\frac{1}{v^2\partial_\xi(f_1f_2f_3)}\left\{\partial_t(f_1f_2f_3)f_2f_3+(\dot{f}_1f_2'-\dot{f}_2f_1')f_2'+(\dot{f}_1f_3'-\dot{f}_3f_1')f_3'\right\}\right)\\
&+\left(-\frac{1}{(\partial_\xi(f_1f_2f_3))^2}(f_2'f_3+f_3'f_2)\left[((f_1'-f_2f_3)^2+\cdots)+\frac{1}{v^2}\partial_t(f_1f_2f_3)^2\right.\right.\\
&\quad\left.+\frac{1}{v^2}\left((\dot{f}_1f_2'-\dot{f}_2f_1')^2+\cdots\right)\right]\\
&\quad\left.+\frac{1}{\partial_\xi(f_1f_2f_3)}\left\{-f_3(f_2'-f_1f_3)-f_2(f_3'-f_1f_2)+\frac{1}{v^2}\partial_t(f_1f_2f_3)(\dot{f}_2f_3+\dot{f}_3f_2)\right\}\right)\\
&=0\,, \qquad (4.2.1)
\end{aligned}
$$

and the equations of motion for φ_2 and φ_3 can be obtained from cyclic permutations of (4.2.1). We found that these equations of motion were also too complicated to solve numerically.

We turned instead to the Yang-Mills equations of motion (3.4.10) - (3.4.12). These equations are much simpler than the Born-Infeld equations of motion, and so we were able to solve them numerically. We will consider here the differences between the two systems.

In section 3.4.2 we described how the low energy limit $\alpha' \to 0$ can be taken in two ways; keeping L fixed leads to the Yang-Mills action, whereas keeping v fixed leads to what we have called the Born-Infeld action. As we pointed out in that section, the Yang-Mills limit leads to monopoles and W-bosons with infinite mass. And so, when we solve the Yang-Mills equations numerically we will be describing the scattering of infinitely massive monopoles. This means that the energy radiated during scattering, given by equation (3.2.11), will also be infinite. To keep our calculations finite we will calculate the ratio of energy radiated to the total energy, since this quantity will not depend on the monopole mass.

To get an idea of the differences between the Yang-Mills action and the Born-Infeld action, let us consider the equations of motion for the perturbations derived from each action. We calculated the perturbation equation of motion for the Born-Infeld action in section 3.7. Using the definitions of the O_i (3.7.4) - (3.7.5), we find that these equations take the form

$$\ddot{\varepsilon}_i - \frac{\partial}{\partial \xi}\left(\frac{1}{H}\frac{\partial \varepsilon_i}{\partial \xi}\right) + V_i(\varepsilon_1, \varepsilon_2, \varepsilon_3) = -\ddot{f}_i \ , \tag{4.2.2}$$

where V_i are potentials which depend on ε_1, ε_2 and ε_3. We can put the equations (4.2.2) in the form of coupled wave equations by rescaling the coordinate ξ, as in [64]

$$\xi \to x \ , \quad \frac{dx}{d\xi} = \sqrt{H} \ . \tag{4.2.3}$$

Due to the poles in H at $\xi = 0$ and $\xi = 2$, the range of the new coordinate x is $-\infty < x < \infty$. We also need to redefine the fields

$$\varepsilon_i \to \eta_i = \varepsilon_i H^{-1/4} \ . \tag{4.2.4}$$

Then the equations of motion for the η_i are

$$\ddot{\eta}_i - \frac{\partial^2 \eta_i}{\partial x^2} + \mathcal{V}_i(\eta_1, \eta_2, \eta_3) = -H^{1/4}\ddot{F}_i - \frac{1}{4H^{3/4}}\dot{H}\dot{F}_i \ , \tag{4.2.5}$$

where we have defined

$$F_i = \frac{f_i}{\sqrt{H}} \ . \tag{4.2.6}$$

So the perturbation equations of motion for the Born-Infeld action take the form of three coupled wave equations, defined on the real line, with forcing terms given by the right hand side of (4.2.5). So for the Born-Infeld equations the energy radiated is the energy that escapes to $x = \pm\infty$ during scattering.

Next we consider the perturbation equations of motion for the Yang-Mills action. They are given by

$$\ddot{\epsilon}_i - \frac{\partial^2 \epsilon_i}{\partial \xi^2} + U_i(\epsilon_1, \epsilon_2, \epsilon_3) = -\ddot{f}_i \ , \tag{4.2.7}$$

where U_i are the Yang-Mills potentials. These equations already take the form of three coupled wave equations, without the need to redefine ξ. So the Yang-Mills perturbation equations are wave equations in a box, since $0 < \xi < 2$. It is not clear that any energy will be able to radiate in this case, because it may get trapped inside the box (i.e. the energy may be reflected back and forth within the box, without the possibility of escaping up the poles, which correspond to the D3-branes' spacetime).

4.3 The D-String Dynamics in the Asymptotic Limit

In this section we will present some more analytic calculations which will be useful for our numerical work.

4.3.1 The Static Solutions and their Asymptotic Expansions

We have shown in the previous chapter that the Dirac-Born-Infeld action for two D-strings with the ansatz (3.4.4) for the Φ^i yields the Bogmonl'nyi equations:

$$f_1' - f_2 f_3 = 0 \ , \quad f_2' - f_3 f_1 = 0 \ , \quad f_3' - f_1 f_2 = 0 \ , \tag{4.3.1}$$

where the f_i are defined in terms of the ϕ_i by the redefinitions (3.4.14). We have also shown that the solutions to (4.3.1) which correspond to D-strings stretched between two D3-branes are:

$$f_1(\xi, k) = \frac{-K(k)}{\mathrm{sn}(K(k)\xi, k)} \ , \tag{4.3.2}$$

$$f_2(\xi, k) = \frac{-K(k)\mathrm{dn}(K(k)\xi, k)}{\mathrm{sn}(K(k)\xi, k)} \ , \tag{4.3.3}$$

$$f_3(\xi, k) = \frac{-K(k)\mathrm{cn}(K(k)\xi, k)}{\mathrm{sn}(K(k)\xi, k)} \ . \tag{4.3.4}$$

where k is a modular parameter which runs from $k = 0$ to $k = 1$. Let us examine the functions f_1, f_2, and f_3 more closely in the limit $k \longrightarrow 1$. In the crudest approximation we have:

$$f_1 \sim K(k) \ , \quad f_2 \sim 0 \ , \quad f_3 \sim 0 \ , \tag{4.3.5}$$

with

$$K(k) \longrightarrow \infty \quad \text{as} \quad k \longrightarrow 1 \ , \tag{4.3.6}$$

(except near $\xi = 0$ and $\xi = 2$, where the f_i have poles). So in the limit $k \longrightarrow 1$ the monopoles are positioned far apart on the x^1 axis. We can get a better approximation for f_1, f_2 and f_3 in the far-field limit by expanding the elliptic functions in (4.3.2) - (4.3.4) as series in $k' \equiv \sqrt{1-k^2}$.

In order to obtain f_1, f_2 and f_3 as functions of k' we will use the following transformation of elliptic functions, which can be found in ref. [49]. Under the transformation

$$\xi \to \tilde{\xi} = -i\xi \ , \quad k \to \tilde{k} = k' = \sqrt{1-k^2} \ , \tag{4.3.7}$$

the elliptic functions transform as

$$\begin{aligned}
K(k) &\to \tilde{K}(k) = K'(k) = K(k') \ , \\
\mathrm{sn}(\xi,\mathrm{k}) &\to \mathrm{sn}(\tilde{\xi},\tilde{\mathrm{k}}) = -\mathrm{i}\frac{\mathrm{sn}(\xi,\mathrm{k})}{\mathrm{cn}(\xi,\mathrm{k})} \ , \quad \mathrm{cn}(\xi,\mathrm{k}) \to \mathrm{cn}(\tilde{\xi},\tilde{\mathrm{k}}) = \frac{1}{\mathrm{cn}(\xi,\mathrm{k})} \ , \\
\mathrm{dn}(\xi,\mathrm{k}) &\to \mathrm{dn}(\tilde{\xi},\tilde{\mathrm{k}}) = \frac{\mathrm{dn}(\xi,\mathrm{k})}{\mathrm{cn}(\xi,\mathrm{k})} \ .
\end{aligned} \tag{4.3.8}$$

Then f_1, f_2 and f_3 become

$$\begin{aligned}
f_1(\xi,k) &= -iK(k)\frac{\mathrm{cn}(\mathrm{iK(k)}\xi,\mathrm{k}')}{\mathrm{sn}(\mathrm{iK(k)}\xi,\mathrm{k}')} \ , \quad f_2(\xi,k) = -iK(k)\frac{\mathrm{dn}(\mathrm{iK(k)}\xi,\mathrm{k}')}{\mathrm{sn}(\mathrm{iK(k)}\xi,\mathrm{k}')} \ , \\
f_3(\xi,k) &= -iK(k)\frac{1}{\mathrm{sn}(\mathrm{iK(k)}\xi,\mathrm{k}')} \ .
\end{aligned} \tag{4.3.9}$$

We will also use from ref. [49] the expansions for $\mathrm{sn}(\xi,\mathrm{k})$, $\mathrm{cn}(\xi,\mathrm{k})$ and $\mathrm{dn}(\xi,\mathrm{k})$ for small k. These are

$$\begin{aligned}
\mathrm{sn}(\xi,\mathrm{k}) = &\ \frac{2\pi}{K(k)}\left(\frac{1}{4}\sin\left(\frac{\pi\xi}{2K(k)}\right) + \frac{5k^2}{64}\sin\left(\frac{\pi\xi}{2K(k)}\right) + \frac{k^2}{64}\sin\left(\frac{3\pi\xi}{2K(k)}\right)\right. \\
&\left. + \cdots\right) \ , \\
\mathrm{cn}(\xi,\mathrm{k}) = &\ \frac{2\pi}{K(k)}\left(\frac{1}{4}\cos\left(\frac{\pi\xi}{2K(k)}\right) + \frac{3k^2}{64}\cos\left(\frac{\pi\xi}{2K(k)}\right) + \frac{k^2}{64}\sin\left(\frac{3\pi\xi}{2K(k)}\right)\right. \\
&\left. + \cdots\right) \ , \\
\mathrm{dn}(\xi,\mathrm{k}) = &\ \frac{2\pi}{K(k)}\left(1 + \frac{k^2}{4}\cos\left(\frac{\pi\xi}{2K(k)}\right) + \cdots\right) \ .
\end{aligned} \tag{4.3.10}$$

Substituting these into the expressions (4.3.2) - (4.3.4) for f_1, f_2 and f_3 we find

$$\begin{aligned}
f_1(\xi,k) = -\frac{K(k)}{\sinh(\xi K(k))} &\left(\cosh(\xi K(k)) + \tfrac{1}{4}\tfrac{\xi K(k)}{\sinh(\xi K(k))}(k')^2 - \tfrac{1}{4}\cosh(\xi K(k))(k')^2\right. \\
&\left. + \cdots\right) \ ,
\end{aligned} \tag{4.3.11}$$

and

$$\begin{aligned}
f_2(\xi,k) = -\frac{K(k)}{\sinh(\xi K(k))} &\left(1 + \tfrac{1}{4}\tfrac{\xi K(k)\cosh(\xi K(k))}{\sinh(\xi K(k))}(k')^2 + \tfrac{1}{4}\cosh^2(\xi K(k))(k')^2\right. \\
&\left. - \tfrac{(k')^2}{2} + \cdots\right) \ ,
\end{aligned} \tag{4.3.12}$$

and

$$f_3(\xi,k) = -\frac{K(k)}{\sinh(\xi K(k))}\left(1+\tfrac{1}{4}\tfrac{\xi K(k)\cosh(\xi K(k))}{\sinh(\xi K(k))}(k')^2 - \tfrac{1}{4}\cosh^2(\xi K(k))(k')^2\right.$$
$$\left.+\cdots\right. . \qquad (4.3.13)$$

We can also expand $K(k)$ as a series in k',

$$K(k) = (\ln 4 - \ln k') + \frac{1}{4}(\ln 4 - \ln k' - 1)(k')^2 + O(\ln k'(k')^4)\ , \qquad (4.3.14)$$

which implies

$$e^{-2K(k)} = \frac{(k')^2}{16} + O(K(k)e^{-4K(k)})\ . \qquad (4.3.15)$$

Using (4.3.15) in (4.3.11), (4.3.12) and (4.3.13), we can write f_1, f_2, and f_3 as series in e^{-2K} to get

$$f_1(\xi,K) = -\frac{K}{\sinh(\xi K)}\left(\cosh(\xi K) + 4\left(\tfrac{\xi K}{\sinh(\xi K)} - \cosh(\xi K)\right)e^{-2K}\right.$$
$$\left.+O(K^2e^{-4K})\right)\ , \qquad (4.3.16)$$

and

$$f_2(\xi,K) = -\frac{K}{\sinh(\xi K)}\left(1 + 4\left(\tfrac{\xi K\cosh(\xi K)}{\sinh(\xi K)} + \cosh^2(\xi K) - 2\right)e^{-2K}\right.$$
$$\left.+O(K^2e^{-4K})\right)\ , \qquad (4.3.17)$$

and

$$f_3(\xi,K) = -\frac{K}{\sinh(\xi K)}\left(1 + 4\left(\tfrac{\xi K\cosh(\xi K)}{\sinh(\xi K)} - \cosh^2(\xi K)\right)e^{-2K}\right.$$
$$\left.+O(K^2e^{-4K})\right)\ . \qquad (4.3.18)$$

4.3.2 Introducing Time Dependence

We can introduce time dependence into the static solutions (4.3.2) - (4.3.4) by allowing k to depend on time, $k(t)$, and giving the solutions an initial velocity via a non-zero $\dot{k}(t)$. When the monopoles are far apart we can use the approximations (4.3.16) - (4.3.18), which depend on k only through $K(k)$. So we can discard the parameter $k(t)$ in favour of the parameter $K(t)$. This is a more natural parameter to use because $f_1 \sim K$, and so the velocity of the monopoles is $\dot{f}_1 \sim \dot{K}$.

We will describe here how to calculate expressions for $\dot{K}$ and $\ddot{K}$ using energy conservation (we will show explicitly in section 4.4.2 that the total energy of the system is conserved). Energy conservation takes the form:

$$\frac{1}{2}T\int_0^2 \left(\frac{df_1}{dt}\right)^2 + \left(\frac{df_2}{dt}\right)^2 + \left(\frac{df_3}{dt}\right)^2 d\xi = E \ , \tag{4.3.19}$$

where we have neglected the contribution of the potential energy because it is very small in the asymptotic limit. Here E is a constant which represents the initial energy of the D-strings, i.e. their energy when they are an infinite distance apart. Also T is a constant which is determined by the mass of the monopole. Recall from section 3.4.2 that in our Yang-Mills system the mass of the monopole is infinite. However, ultimately we will be interested in the ratio of the energy radiated to the total energy of the system, which does not depend on the monopole mass. In practise, in order to be able to handle the calculations numerically, we will just set the mass to be $m = 2$. But we should bear in mind that our results will only really make sense when considered as ratios of energies.

Writing (4.3.19) in terms of the parameter $K(t)$, and using that f_1, f_2 and f_3 only depend on time through K, we obtain

$$\frac{1}{2}\dot{K}^2\int_0^2 \left(\frac{df_1}{dK}\right)^2 + \left(\frac{df_2}{dK}\right)^2 + \left(\frac{df_3}{dK}\right)^2 d\xi = E \ . \tag{4.3.20}$$

We define

$$I_1 = \int_0^2 \left(\frac{df_1}{dK}\right)^2 + \left(\frac{df_2}{dK}\right)^2 + \left(\frac{df_3}{dK}\right)^2 d\xi \ . \tag{4.3.21}$$

Then the expression for $\dot{K}$ is given by

$$\dot{K} = \sqrt{\frac{2E}{I_1}} \ . \tag{4.3.22}$$

Using the leading order terms in the expansions (4.3.16) - (4.3.18) for f_1, f_2 and f_3, we find

$$I_1 = 2\left(1 - \frac{1}{K} + O(Ke^{-2K})\right) \ . \tag{4.3.23}$$

To obtain $\ddot{K}$ we differentiate (4.3.20) to get

$$\begin{aligned}\dot{K}^2\int_0^2 &\left(\frac{df_1}{dK}\right)\left(\frac{d^2f_1}{dK^2}\right) + \left(\frac{df_2}{dK}\right)\left(\frac{d^2f_2}{dK^2}\right) + \left(\frac{df_3}{dK}\right)\left(\frac{d^2f_3}{dK^2}\right) d\xi \\ &+\ddot{K}\int_0^2 \left(\frac{df_1}{dK}\right)^2 + \left(\frac{df_2}{dK}\right)^2 + \left(\frac{df_3}{dK}\right)^2 d\xi = 0 \ . \end{aligned} \tag{4.3.24}$$

So

$$\ddot{K} = -\dot{K}^2 \frac{I_2}{I_1} = -E \frac{I_2}{I_1^2} , \tag{4.3.25}$$

where

$$I_2 = \int_0^2 \left(\frac{df_1}{dK}\right)\left(\frac{d^2 f_1}{dK^2}\right) + \left(\frac{df_2}{dK}\right)\left(\frac{d^2 f_2}{dK^2}\right) + \left(\frac{df_3}{dK}\right)\left(\frac{d^2 f_3}{dK^2}\right) d\xi . \tag{4.3.26}$$

Again, using the leading order terms in (4.3.16) - (4.3.18), we find

$$I_2 = \frac{1}{K^2}\left(1 + O(K^3 e^{-2K})\right) . \tag{4.3.27}$$

4.3.3 The Asymptotic Expansion of K(t)

The leading order terms in equation (4.3.22) give

$$\dot{K} = E^{1/2}\left(1 - \frac{1}{K}\right)^{-1/2} . \tag{4.3.28}$$

We let $v_{-\infty}$ be the velocity of the D-strings in the asymptotic limit $t \to -\infty$. Then $E = v_{-\infty}^2$. We can separate the variables in (4.3.28) to get

$$\int \left(1 - \frac{1}{K}\right)^{1/2} dK = \int v_{-\infty}\, dt . \tag{4.3.29}$$

In the asymptotic limit K is large, and so we can expand out the square root in (4.3.29)

$$\int \left(1 - \frac{1}{2K} - \frac{1}{8K^2} + O\left(\frac{1}{K^3}\right)\right) dK = \int v_{-\infty}\, dt . \tag{4.3.30}$$

Integrating this expression gives

$$K - \frac{1}{2}\ln K + \frac{1}{8K} + O\left(\frac{1}{K^2}\right) = v_{-\infty}(t + T_0) , \tag{4.3.31}$$

where T_0 is a constant parameter corresponding to the freedom to translate the problem in time. To first order the solution to (4.3.31) is

$$K = v_{-\infty}(t + T_0) . \tag{4.3.32}$$

We can find higher order solutions by perturbing (4.3.32) and substituting it into (4.3.31). So we write

$$K = v_{-\infty}(t + T_0) + \epsilon(t) , \tag{4.3.33}$$

where $\epsilon \sim \ln v_{-\infty}(t+T_0)$. Then substituting this expression in (4.3.31) gives the following asymptotic expansion for $K(t)$

$$\begin{aligned} K &= v_{-\infty}(t+T_0) + \frac{1}{2}\ln(v_{-\infty}(t+T_0)) + \frac{\ln(v_{-\infty}(t+T_0))}{4v_{-\infty}(t+T_0)} - \frac{1}{8v_{-\infty}(t+T_0)} \\ &\quad + O\left(\frac{(\ln(v_{-\infty}t)^2}{(v_{-\infty}t)^2}\right) . \end{aligned} \tag{4.3.34}$$

Note that this agrees with the expression (3.2.8) for $r(t)$ in the three-dimensional calculation by Manton and Samols in [80], if we take $c = 2v_{-\infty}T_0 - \ln 2$ (this agreement was to be expected, since the transformation between the metric of the (3 + 1)-dimensional monopole description and the metric on Nahm data is known to be an isometry from ref. [89]).

4.3.4 Decoupling of Zero Modes and Non-zero Modes

In the introduction to this chapter we described how the D-strings' motion can be thought of as being split into two parts; the motion of the zero modes, i.e. the motion of the centres of mass of the D-strings, and the motion of the non-zero modes, which act as perturbations on top of the zero modes. (see equation (3.7.2) and the discussion above it). Energy can be transferred between the zero modes and the non-zero modes as a result of the interaction between the two D-strings. But when they are far apart, and the interaction can be neglected, we expect the zero modes and the non-zero modes to decouple. We will show here explicitly that this is the case.

We consider the D-strings' motion after scattering, when

$$f_1 \sim 0 \ , \quad f_2 \sim -K(t) \ , \quad f_3 \sim 0 \ , \tag{4.3.35}$$

with $\dot{K}(t)$ constant. The approximations (4.3.35) hold true for all ξ, except when ξ is very close to 0 or 2, where f_1, f_2 and f_3 all contain poles. For now we will work with the approximations (4.3.35); we will consider the effects of the poles later on in this section.

The linearised equation of motion for ϵ_1 from the Yang-Mills action is

$$\ddot{\epsilon_1} - \epsilon_1'' + \epsilon_1(f_2^2 + f_3^2) + 2f_1(f_2\epsilon_2 + f_3\epsilon_3) = 0 \ , \tag{4.3.36}$$

and the equations for ϵ_2 and ϵ_3 are given by cyclic permutations of (4.3.36). With the approximations (4.3.35) these equations of motion become

$$\ddot{\epsilon}_1 - \epsilon_1'' + K^2\epsilon_1 = 0 , \tag{4.3.37}$$

$$\ddot{\epsilon}_2 - \epsilon_2'' = 0 , \tag{4.3.38}$$

$$\ddot{\epsilon}_3 - \epsilon_3'' + K^2\epsilon_3 = 0 . \tag{4.3.39}$$

Since the D-strings are far apart, K is large, and (4.3.37) and (4.3.39) imply that

$$\epsilon_1 = 0 , \quad \text{and} \quad \epsilon_3 = 0 . \tag{4.3.40}$$

So it seems that in the asymptotic limit the energy in the non-zero modes is entirely contained in ϵ_2, which from (4.3.38) takes the form of a harmonic oscillator, and has completely decoupled from the zero mode motion.

The above analysis is accurate away from the boundaries $\xi = 0$ and $\xi = 2$. We now consider what happens at the boundary $\xi = 0$; the calculation for the boundary $\xi = 2$ works in a similar way due to the symmetries of the f_i about $\xi = 1$ that we pointed out in section 1.3.2. Consider the equations of motion for the full fields φ_i, which are

$$\ddot{\varphi}_1 - \varphi_1'' + \varphi_1(\varphi_2^2 + \varphi_3^2) = 0 , \quad \ddot{\varphi}_2 - \varphi_2'' + \varphi_2(\varphi_3^2 + \varphi_1^2) = 0 , \tag{4.3.41}$$

$$\ddot{\varphi}_3 - \varphi_3'' + \varphi_3(\varphi_1^2 + \varphi_2^2) = 0 . \tag{4.3.42}$$

Expanding the functions φ_1, φ_2 and φ_3 as series in ξ, and imposing the equations of motion (4.3.41) - (4.3.42) we find

$$\varphi_1 = -\frac{1}{\xi} + a_1\xi + O(\xi^3) , \quad \varphi_2 = -\frac{1}{\xi} + b_1\xi + O(\xi^3) , \tag{4.3.43}$$

$$\varphi_3 = -\frac{1}{\xi} + c_1\xi + O(\xi^3) , \tag{4.3.44}$$

where a_1, b_1 and c_1 are functions of time which satisfy

$$a_1 + b_1 + c_1 = 0 . \tag{4.3.45}$$

Now when we expand the solutions for the f_i (4.3.2) - (4.3.4) as series in small ξ we obtain

$$f_1 = -\frac{1}{\xi} + a_1^f\xi + O(\xi^3) , \quad f_2 = -\frac{1}{\xi} + b_1^f\xi + O(\xi^3) , \tag{4.3.46}$$

$$f_3 = -\frac{1}{\xi} + c_1^f\xi + O(\xi^3) , \tag{4.3.47}$$

where

$$a_1^f + b_1^f + c_1^f = 0 \ . \tag{4.3.48}$$

So we can deduce that the series for the $\epsilon_i(\xi, t)$ have the form

$$\epsilon_1 = a_1^\epsilon \xi + O(\xi^3) \ , \quad \epsilon_2 = b_1^\epsilon \xi + O(\xi^3) \ , \tag{4.3.49}$$

$$\epsilon_3 = c_1^\epsilon \xi + O(\xi^3) \ , \tag{4.3.50}$$

where

$$a_1^\epsilon + b_1^\epsilon + c_1^\epsilon = 0 \ . \tag{4.3.51}$$

So we can see that $\epsilon_i \longrightarrow 0$ as $\xi \longrightarrow 0$, which gives us the following boundary conditions for the ϵ_i

$$\epsilon_i(0, t) = \epsilon_i(2, t) = 0 \ . \tag{4.3.52}$$

So we deduce that the boundary conditions (4.3.52) are consistent with (4.3.40), and ϵ_2 being a harmonic oscillator.

4.4 Energy Considerations

In this section we outline some general points about the energy of the Yang-Mills system we wish to solve.

4.4.1 The Energy Densities

The kinetic and potential energy densities of the system governed by the Yang-Mills action (3.4.8) are given in terms of the functions φ_i by

$$\text{K.E. density} = \frac{1}{2}\left(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2\right) \ , \tag{4.4.1}$$

$$\text{P.E. density} = \frac{1}{2}\left(\varphi_1'^2 + \varphi_2'^2 + \varphi_3'^2 + \varphi_1^2\varphi_2^2 + \varphi_2^2\varphi_3^2 + \varphi_3^2\varphi_1^2\right) \ . \tag{4.4.2}$$

We can rewrite the potential energy density as

$$\text{P.E. density} = \frac{1}{2}\left((\varphi_1' - \varphi_2\varphi_3)^2 + (\varphi_2' - \varphi_1\varphi_3)^2 + (\varphi_3' - \varphi_1\varphi_2)^2\right) + \partial_\xi(\varphi_1\varphi_2\varphi_3) \ , \tag{4.4.3}$$

where $\partial_\xi(\varphi_1\varphi_2\varphi_3)$ contains singular terms which, when integrated, contribute an infinite constant to the potential energy. Neglecting this term we obtain

$$\text{P.E. density} = \frac{1}{2}\left((\varphi_1' - \varphi_2\varphi_3)^2 + (\varphi_2' - \varphi_1\varphi_3)^2 + (\varphi_3' - \varphi_1\varphi_2)^2\right) \ . \tag{4.4.4}$$

Recall that the Bogomol'nyi equations for the Yang-Mills action (3.4.8) are given by

$$\varphi_1' - \varphi_2\varphi_3 = 0 \ , \quad \varphi_2' - \varphi_3\varphi_1 = 0 \ , \quad \varphi_3' - \varphi_1\varphi_2 = 0 \ . \tag{4.4.5}$$

The potential energy density for a solution obeying (4.4.5) is therefore zero, as we would expect. Also, we can see that the potential energy density (4.4.4) for a general field configuration measures the deviation of the solution away from the BPS solution.

4.4.2 Energy Conservation

We will show here that the total energy remains conserved. We consider the Noether currents for time translation $t \longrightarrow t - a$, which are defined by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\partial_0\varphi_i - \mathcal{L}\,\delta_0^\mu \ .$$

Therefore the Noether currents are

$$\begin{aligned} j^0 &= \frac{1}{2}(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2) + \frac{1}{2}(\varphi_1'^2 + \varphi_2'^2 + \varphi_3'^2) + \frac{1}{2}(\varphi_1^2\varphi_2^2 + \varphi_2^2\varphi_3^2 + \varphi_3^2\varphi_1^2) \ , \\ j^1 &= \varphi_1'\dot\varphi_1 + \varphi_2'\dot\varphi_2 + \varphi_3'\dot\varphi_3 \ . \end{aligned}$$

As usual we have current conservation in the form

$$\partial_0 j^0 - \partial_1 j^1 = 0 \ .$$

We define the energy E to be the charge associated with the Noether current j^0,

$$E = \int_0^2 j^0 d\xi \ .$$

Therefore we have

$$\begin{aligned} \partial_t E &= \int_0^2 \partial_t j^0 d\xi \\ &= \int_0^2 \partial_\xi j^1 d\xi \\ &= [j^1]_0^2 \\ &= [\varphi_1'\dot\varphi_1 + \varphi_2'\dot\varphi_2 + \varphi_3'\dot\varphi_3]_0^2 \ , \end{aligned} \tag{4.4.6}$$

where we have used current conservation to get from the first line to the second.

Using the expansions for the φ_i (4.3.43) - (4.3.44) near $\xi = 0$, we have

$$\begin{aligned} \varphi_1'\dot{\varphi}_1 + \varphi_2'\dot{\varphi}_2 + \varphi_3'\dot{\varphi}_3 &= \frac{(\dot{a}_1 + \dot{b}_1 + \dot{c}_1)}{\xi} + O(\xi) \\ &= 0 \quad \text{at } \xi = 0 \ . \end{aligned} \tag{4.4.7}$$

Similarly we can show that the contribution to (4.4.6) at $\xi = 2$ is also zero. Therefore we have

$$\partial_t E = 0 \ ,$$

and so the total energy of the system is conserved.

4.4.3 Decoupling of the Energy

We have shown in section 4.3.4 that the motion of the zero modes and the motion of the non-zero modes decouple when the D-strings are far apart. Therefore we also expect the energy in the non-zero modes to decouple from the energy in the zero modes in this limit. We will show here explicitly that this is the case.

The kinetic energy density is given by equation (4.4.1). Substituting $\varphi_i = f_i + \epsilon_i$ we find that the coupling between the zero modes and the non-zero modes in the kinetic energy is generated by the terms:

$$\int_0^2 \dot{f}_i \dot{\epsilon}_i \, d\xi \ . \tag{4.4.8}$$

But we have shown in section 4.3.4 that the non-zero ϵ_i behave like harmonic oscillators in the asymptotic limit, and the $\dot{f}_i$ are approximately constant. Then, in this limit,

$$\int_0^2 \dot{f}_i \dot{\epsilon}_i \, d\xi = 0 \ , \tag{4.4.9}$$

and so the kinetic energy does indeed decouple. (The poles of the f_i at $\xi = 0$ and $\xi = 2$ are fixed, so that $\dot{f}_i = 0$ at $\xi = 0, 2$. Therefore we do not need to worry about the contribution of the poles to (4.4.8)).

Next consider the potential energy. Substituting $\varphi_i = f_i + \epsilon_i$ into the potential energy density, and keeping only terms which are quadratic in ϵ_i, we find that the potential energy is given by

$$\text{P.E.} = \frac{1}{2} \int_0^2 \left((\epsilon_1' - f_2 \epsilon_2 - f_3 \epsilon_3)^2 + \{\text{cyclic perms.}\} \right) d\xi \ . \tag{4.4.10}$$

Away from the poles the f_i are given by the approximations (4.3.35) in the asymptotic limit. We have also shown in section 4.3.4 that $\epsilon_1 = \epsilon_3 = 0$ in this limit. Then the potential energy is given by

$$\begin{aligned} \text{P.E.} &= \frac{1}{2}\int_0^{\delta} \left((\epsilon_2')^2 + (-f_3\epsilon_2)^2 + (-f_1\epsilon_2)^2\right) d\xi + \frac{1}{2}\int_{\delta}^{2-\delta} (\epsilon_2')^2 d\xi \\ &+\frac{1}{2}\int_{2-\delta}^{2} \left((\epsilon_2')^2 + (-f_3\epsilon_2)^2 + (-f_1\epsilon_2)^2\right) d\xi \ , \end{aligned} \tag{4.4.11}$$

where the first and third terms in (4.4.11) take into account the behaviour of the f_i near the boundaries. The series expansions (4.3.46) - (4.3.47) for the f_i and (4.3.49) - (4.3.50) for the ϵ_i imply that

$$f_i\epsilon_i = -\epsilon_i'(\xi)|_{\xi=0} + O(\xi^2) \ , \tag{4.4.12}$$

where we have used that $b_1^{\epsilon} = \epsilon_2(\xi)|_{\xi=0}$. So the contributions to the potential energy from the two boundary terms are given by

$$\frac{3}{2}\int_0^{\delta} (\epsilon_2')^2 \, d\xi + \frac{3}{2}\int_{2-\delta}^{2} (\epsilon_2')^2 \, d\xi \ . \tag{4.4.13}$$

Since these terms are finite, and $\delta \to 0$ as the D-strings get further apart, the contributions to the potential energy from (4.4.13) are negligible. So the potential energy is given by

$$\text{P.E.} = \frac{1}{2}\int_0^2 (\epsilon_2')^2 d\xi \ , \tag{4.4.14}$$

which has also decoupled from the zero mode motion.

We have shown that the energy in the non-zero modes after the D-strings have scattered has become decoupled from the energy in the zero modes. It is this energy, that has been transferred to the non-zero modes as a result of the D-string scattering, which represents the energy radiated during scattering.

4.5 Solving the Equations of Motion Numerically

In this section we will describe the numerical methods we used to solve the Yang-Mills equations of motion. The equations are

$$\ddot{\varphi}_1 - \varphi_1'' + \varphi_1(\varphi_2^2 + \varphi_3^2) = 0 \ , \quad \ddot{\varphi}_2 - \varphi_2'' + \varphi_2(\varphi_3^2 + \varphi_1^2) = 0 \ , \tag{4.5.1}$$

$$\ddot{\varphi}_3 - \varphi_3'' + \varphi_3(\varphi_1^2 + \varphi_2^2) = 0 \ , \tag{4.5.2}$$

where a dot denotes differentiation with respect to time, and a prime denotes differentiation with respect to ξ. We will describe in section 4.5.2 how we removed the poles in the solutions for the φ_i in order to evolve them numerically; we will find from the results presented in section 4.6.2 that this method will give us sufficient numerical stability.

4.5.1 The Range of ξ

We wish to solve the equations (4.5.1) - (4.5.2) for $\xi \in [0, 2]$. We will start the motion by taking the φ_i to be equal to the f_i for some chosen value of k, k_0 say, and giving the φ_i a velocity via a non-zero initial value of $\dot{k}$, $\dot{k}_0$ say. So initially we will set

$$\varphi_i(\xi, 0) = f_i(\xi, k_0) \ , \quad \dot{\varphi}_i(\xi, 0) = \dot{f}_i(\xi, k_0) = \dot{k}_0 \frac{\partial f_i}{\partial k}(\xi, k_0) \ . \tag{4.5.3}$$

Recall from section 1.3.2 that f_1 and f_2 are symmetric about $\xi = 1$, and f_3 is antisymmetric. From (4.5.3) we can see that the initial configuration for the φ_i and $\dot{\varphi}_i$ will also have these symmetries. By inspection of the equations of motion (4.5.1) - (4.5.2) we can see that the configurations will be evolved in time in such a way that these symmetries are preserved. Therefore it will be sufficient to solve the equations of motion for the φ_i with ξ in the range $\xi \in [0, 1]$, because we can then use these symmetries to deduce the solutions for the φ_i on the full range of ξ.

4.5.2 Removing the Singularities in the φ_i

Recall from the series solutions (4.3.43) - (4.3.44) that near $\xi = 0$ the φ_i take the following form

$$\varphi_i(\xi, t) = -\frac{1}{\xi} + O(\xi) \ . \tag{4.5.4}$$

So the φ_i have singularities at $\xi = 0$. A numerical program would not be able to handle this singularity, and so we must remove it by the following redefinition

$$g_i(\xi, t) \equiv \frac{1}{\xi} + \varphi_i(\xi, t) \ . \tag{4.5.5}$$

Note that

$$\dot{g}_i(\xi,t) = \dot{\varphi}_i(\xi,t) \ , \tag{4.5.6}$$

because the poles are fixed in time. The equations of motion for the fields g_i are

$$\ddot{g}_1 - g_1'' + g_1(g_2^2+g_3^2) - \frac{(g_2^2+g_3^2+2g_1g_2+2g_1g_3)}{\xi} + \frac{(2g_1+2g_2+2g_3)}{\xi^2} = 0 \ , \tag{4.5.7}$$

$$\ddot{g}_2 - g_2'' + g_2(g_3^2+g_1^2) - \frac{(g_3^2+g_1^2+2g_2g_3+2g_2g_1)}{\xi} + \frac{(2g_1+2g_2+2g_3)}{\xi^2} = 0 \ , \tag{4.5.8}$$

$$\ddot{g}_3 - g_3'' + g_3(g_1^2+g_2^2) - \frac{(g_1^2+g_2^2+2g_3g_1+2g_3g_2)}{\xi} + \frac{(2g_1+2g_2+2g_3)}{\xi^2} = 0 \ . \tag{4.5.9}$$

The third and fourth terms in these equations are apparently singular at $\xi = 0$. However, from the series solutions for the φ_i (4.3.43) - (4.3.44), we have

$$g_1(\xi,t) = a_1\xi + O(\xi^3) \ , \quad g_2(\xi,t) = b_1\xi + O(\xi^3) \ , \tag{4.5.10}$$

$$g_3(\xi,t) = c_1\xi + O(\xi^3) \ , \tag{4.5.11}$$

with

$$a_1 + b_1 + c_1 = 0 \ . \tag{4.5.12}$$

From this we can deduce that the terms which appear to be singular in (4.5.7) - (4.5.9) are in fact finite at $\xi = 0$.

4.5.3 Numerical Methods

We wish to solve the equations of motion (4.5.7) - (4.5.9) numerically, given initial configurations $g_i(\xi,0)$ and boundary conditions $g_i(0,t)$ and $g_i(1,t)$. We will discuss these in sections 4.5.4 and 4.5.5 respectively. To solve the equations we will use the numerical procedure from ref. [90], which we will briefly review here. The program we used to solve the equations was based on the program in ref. [90], with appropriate modifications. We also used the routines from ref. [91] to calculate the Jacobian elliptic functions numerically.

We model spacetime using a two-dimensional lattice. The spatial dimension is discretised by splitting it into N regular intervals of size $d\xi = \frac{1}{N}$, with the points on the lattice labelled as ξ_j for $j = 0, \ldots, N$. The time dimension is also discretised, with dt being the time lapse between successive points in time.

We take a configuration at time t, and evolve it to time $t + dt$ as follows.

1. First we calculate all spatial derivatives of the g_i that are present in the equations of motion for the configuration at time t. For the equations of motion for g_i (4.5.7) - (4.5.9) we require g_i''. We calculate g_i'' at point ξ_j using seven ξ-points

$$g_i''(\xi_j, t) = \frac{1}{d\xi^2}\left(\frac{1}{90}g_i(\xi_{j+3}, t) - \frac{3}{20}g_i(\xi_{j+2}, t) + \frac{3}{2}g_i(\xi_{j+1}, t) - \frac{49}{18}g_i(\xi_j, t)\right.$$
$$\left. + \frac{3}{2}g_i(\xi_{j-1}, t) - \frac{3}{20}g_i(\xi_{j-2}, t) + \frac{1}{90}g_i(\xi_{j-3}, t)\right) \tag{4.5.13}$$

   (for points close to the boundaries it may not always be possible to use seven points in the calculation of g_i'' - see section 4.5.5 for more details).

2. Using $g_i''(\xi_j, t)$ we can calculate $\ddot{g}_i(\xi_j, t)$ for all points ξ_j on the spatial lattice using the equations of motion. Therefore we effectively have $(N - 1)$ coupled equations to evolve in time, one for each spatial point (but neglecting the boundary points ξ_0 and ξ_{N+1}, which we will deal with later). So we can use a fourth-order Runge-Kutta procedure on each equation to evolve it from time t to time $t + dt$.

3. Finally we evolve the boundary points $\xi_0 = 0$ and $\xi_{N+1} = 1$ in time according to the specified boundary conditions.

This procedure evolves the solution at time t to a solution at time $t + dt$, and can be repeated many times until we reach the final configuration.

To obtain solutions with sufficient accuracy we used $d\xi = 0.0001$ and $dt = 0.00005$ in our program.

4.5.4 Initial Conditions

We wish to start the motion with the monopoles moving tangential to the static solutions. So we take

$$\varphi_i(\xi, 0) = f_i(\xi, k_0) \; , \tag{4.5.14}$$

where k_0 is chosen so that the D-strings are sufficiently far apart initially. In [80], Manton and Samols found that two monopoles cease to interact with one another for $r > 10$. So we must start our motion with $f_1 > 10$. We take $k_0 = 0.9999999999$, for which $K(k_0) = 12.55264624$.

Having specified initial configurations for the φ_i, we also need to specify initial configurations for the $\dot{\varphi}_i$. We want to start the motion by allowing k to depend on time in the initial configurations (4.5.14). So the initial configurations for the $\dot{\varphi}_i$ are

$$\dot{\varphi}_i(\xi, 0) = \dot{k}_0 \frac{df_i}{dk}\bigg|_{(\xi, k_0)} , \tag{4.5.15}$$

where $\dot{k}_0$ is a parameter we must choose to fix the initial velocity of the D-strings. We can fix $\dot{k}_0$ by choosing a value for $\dot{f}_i(1, k_0)$, which is the initial velocity of the centres of the D-strings. Since $f_1(\xi, k_0)$ is approximately constant in ξ initially, $\dot{f}_i(1, k_0)$ gives approximately the initial velocity of the D-strings,

$$v_{init} = \dot{f}_i(1, k_0) . \tag{4.5.16}$$

We choose v_{init} to be small initially; we will take two cases (a) $v_{init} = 0.05$ and (b) $v_{init} = 0.1$. $\dot{k}_0$ can be calculated using

$$\dot{k}_0 = \frac{v_{init}}{\frac{df_1}{dk}\big|_{(1, k_0)}} . \tag{4.5.17}$$

Then the initial conditions for the $\dot{\varphi}_i$ can be calculated using (4.5.17) in (4.5.15).

The initial conditions for the g_i and $\dot{g}_i$ can be deduced from the initial conditions for the φ_i and $\dot{\varphi}_i$ respectively using the definitions (4.5.5) and (4.5.6).

4.5.5 Boundary Conditions

We describe here the boundary conditions we used to fix $g_i(0, t)$ and $g_i(1, t)$. We will also describe how we calculated $g_i''(\xi, t)$ for points near the boundaries.

We first consider the left-hand border, $\xi = 0$. The series expansions (4.5.10) - (4.5.11) for the g_i near $\xi = 0$ imply

$$g_i(0, t) = \dot{g}_i(0, t) = 0 . \tag{4.5.18}$$

It is not necessary to calculate g_i'' at $\xi = 0$, because the solutions are fixed there by the boundary condition (4.5.18), and do not need to be evolved in time. For the point next to the left-hand border, ξ_1, the best we can do is to evolve the point using 3 points to calculate $g_i''(\xi_1, t)$

$$g_i''(\xi_1, t) = \frac{1}{d\xi^2}\left(g_i(\xi_0, t) + g_i(\xi_2, t) - 2g_i(\xi_1, t)\right) .$$

Similarly for the next point ξ_2, the best we can do is to use 5 points to calculate $g_i''(\xi_2, t)$

$$g_i''(\xi_2, t) = \frac{1}{d\xi^2}\left(-\frac{1}{12}g_i(\xi_0, t) + \frac{4}{3}g_i(\xi_1, t) - \frac{5}{2}g_i(\xi_2, t) + \frac{4}{3}g_i(\xi_3, t) - \frac{1}{12}g_i(\xi_4, t)\right) . \tag{4.5.19}$$

For ξ_j with $j > 2$ we have enough points to use the 7-point calculation for $g_i''(\xi_j, t)$ from equation (4.5.13), until $j > (N - 2)$, when we reach the right-hand border.

To fix the right-hand border we use the symmetry properties of the solutions about $\xi = 1$. These symmetries imply that

$$\varphi_1'(1, t) = \varphi_2'(1, t) = 0 , \quad \varphi_3(1, t) = 0 . \tag{4.5.20}$$

These imply for the g_i

$$g_1'(1, t) = g_2'(1, t) = -1 , \tag{4.5.21}$$

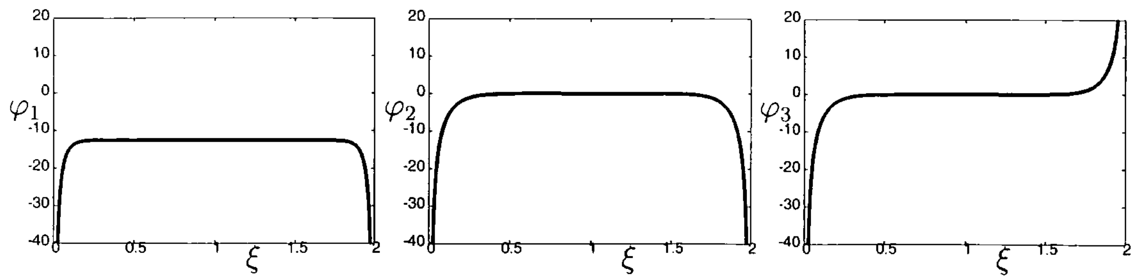
$$g_3(1, t) = 1 . \tag{4.5.22}$$

On the right-hand border, $\xi = \xi_{N+1} = 1$, g_3 is fixed by (4.5.22). So the boundary condition is

$$g_3(1, t) = 1 , \quad \dot{g}_3(1, t) = 0 . \tag{4.5.23}$$

But we cannot fix $g_1(1, t)$ and $g_2(1, t)$ in the same way, because their values may vary at $\xi = 1$; it is only their derivatives with respect to ξ which are fixed. Instead of fixing a boundary condition, we evolved g_1 and g_2 using the Runge-Kutta method, as described in section 4.5.3, using their symmetry about $\xi = 1$ to calculate g_1'' and g_2'' at ξ_{N+1} using five points. We obtained

$$g_1''(\xi_{N+1}, t) = \frac{1}{d\xi^2}\left(-\frac{1}{6}g_1(\xi_{N-1}, t) + \frac{8}{3}g_1(\xi_N, t) - \frac{5}{2}g_1(\xi_{N+1}, t) - \frac{7}{3}dx - \frac{4}{3}dx^3 + \frac{8}{3}dx^5\right) , \tag{4.5.24}$$



(a) φ_1 (b) φ_2 (c) φ_3

Figure 4.1: Plots of the numerical solutions for the φ_i at $t = 0$ with $v_{init} = 0.05$.

and similarly for g_2. We can also use the symmetries to calculate all the g_i'' at the point next to the right-hand border, $\xi = \xi_N = 1 - d\xi$. For g_1 we found

$$\begin{aligned} g_1''(\xi_N, t) =& \frac{1}{d\xi^2}\bigg(- \frac{1}{12}g_1(\xi_{N-2}, t) + \frac{4}{3}g_1(\xi_{N-1}, t) - \frac{31}{12}g_1(\xi_N, t) + \frac{4}{3}g_1(\xi_{N+1}, t) \\ &+ \frac{1}{6}dx + \frac{1}{6}dx^3 + \frac{1}{6}dx^5\bigg) , \end{aligned} \tag{4.5.25}$$

and similarly for g_2''. For g_3 we found

$$\begin{aligned} g_3''(\xi_N, t) =& \frac{1}{d\xi^2}\bigg(- \frac{1}{12}g_3(\xi_{N-2}, t) + \frac{4}{3}g_3(\xi_{N-1}, t) - \frac{29}{12}g_3(\xi_N, t) + \frac{4}{3}g_3(\xi_{N+1}, t) \\ &+ \frac{1}{6}\left(7 - dx^2 - dx^4\right)\bigg) . \end{aligned} \tag{4.5.26}$$

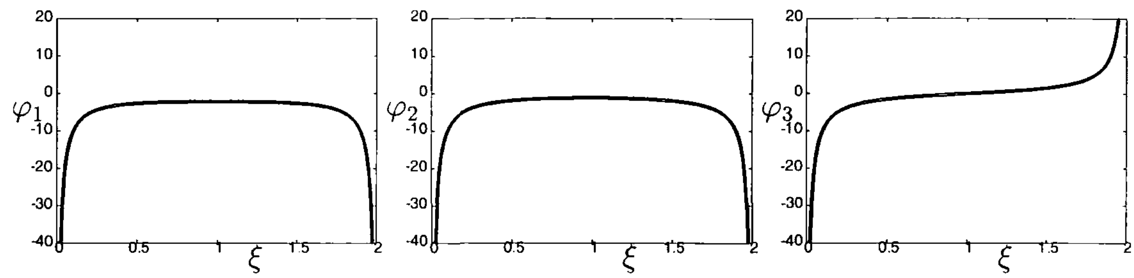
For the next point ξ_{N-1}, we find that it is sufficient to use the 5 point calculation as in (4.5.19).

4.5.6 Results

Figures 4.1, 4.2 and 4.3 show graphs of some of the solutions we obtained from our numerical program.

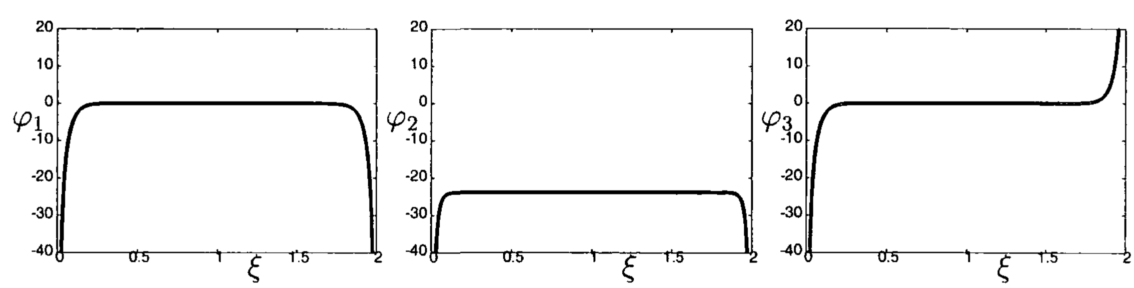
Figure 4.1 shows the initial configuration for φ_1, φ_2 and φ_3. As we explained in section 4.5.4, this configuration is given by $\varphi_i(\xi, 0) = f_i(\xi, k)$ with $k = 0.9999999999$. From these graphs we can see that $\varphi_1 \sim -K(0.9999999999) = -12.55264624$ and $\varphi_2 \sim 0$ and $\varphi_3 \sim 0$, except for the poles at $\xi = 0$ and $\xi = 2$, as we expected.

Figure 4.2 shows the solutions for φ_1, φ_2 and φ_3 at the point of scattering. Here $\varphi_1 = \varphi_2$, which corresponds to the axisymmetric monopole solution (the 'dough-



(a) φ_1 (b) φ_2 (c) φ_3

Figure 4.2: Plots of the solutions for the φ_i at $t = 200$ with $v_{init} = 0.05$.



(a) φ_1 (b) φ_2 (c) φ_3

Figure 4.3: Plots of the numerical solutions for the φ_i at $t = 650$ with $v_{init} = 0.05$.

nut'), which we discussed in section 1.3.2. The time evolution from the configurations in figure 4.1 to the configurations in figure 4.2 is a smooth deformation between the graphs in figure 4.1 and the graphs in 4.2.

Figure 4.3 shows the solutions for φ_1, φ_2 and φ_3 after scattering, with $t = 650$. Note that after scattering φ_1 and φ_2 have exchanged roles, as expected. This corresponds to the D-strings scattering at 90°.

4.6 Calculating the energy radiated

In this section we describe the techniques we have used to calculate the energy radiated during scattering, using the numerical solutions for the g_i from the program discussed in the previous section.

The energy densities in terms of the g_i are

$$\begin{aligned}
\text{K.E. density} &= \frac{1}{2}(\dot{g}_1^2 + \dot{g}_2^2 + \dot{g}_3^2) \ , && (4.6.1)\\
\text{P.E. density} &= \frac{1}{2}(g_1'^2 + g_2'^2 + g_3'^2) + \frac{1}{2}(g_1^2 g_2^2 + g_2^2 g_3^2 + g_3^2 g_1^2)\\
&\quad -\frac{1}{\xi}(g_1 g_2(g_1+g_2) + g_2 g_3(g_2+g_3) + g_3 g_1(g_3+g_1))\\
&\quad +\frac{1}{\xi^2}(g_1^2 + g_2^2 + g_3^2 + g_1 g_2 + g_2 g_3 + g_3 g_1)\\
&\quad -(g_1' g_2 g_3 + g_2' g_3 g_1 + g_3' g_3 g_1)\\
&\quad +\frac{1}{\xi}(g_1(g_2'+g_3') + g_2(g_3'+g_1') + g_3(g_1'+g_2')) \ . && (4.6.2)
\end{aligned}$$

Although there appear to be singularities in the potential energy density (4.6.2) at $\xi = 0$, all terms are in fact finite when we substitute in the series expansions for the g_i (4.5.10) - (4.5.11) (as was the case for the equations of motion (4.5.7) - (4.5.9)). We find

$$\text{P.E. density}(0,t) = \frac{3}{2}(a_1 + b_1 + c_1)^2 = 0 \ . \qquad (4.6.3)$$

4.6.1 Numerical methods

In this section we describe the numerical methods we have used to calculate the energies in our numerical solutions.

In order to calculate the potential energy density (4.6.2) we need to calculate g_i' for all ξ-points. For the interior ξ-points we used four points to calculate g_i'

$$g_i'(\xi_n, t) = \frac{1}{d\xi}\left(\frac{1}{12}g_i(\xi_{n-2}, t) + \frac{2}{3}g_i(\xi_{n-1}, t) - \frac{2}{3}g_i(\xi_{n+1}, t) - \frac{1}{12}g_i(\xi_{n+2}, t)\right) . \quad (4.6.4)$$

On the left-hand border, $\xi_0 = 0$, the potential energy density is zero (see (4.6.3)). For the point next to the left-hand border, ξ_1, we used two points to calculate g_i'

$$g_i'(\xi_1, t) = \frac{1}{d\xi}\left(\frac{1}{2}g_i(\xi_2, t) - \frac{1}{2}g_i(\xi_0, t)\right) . \quad (4.6.5)$$

On the right-hand border, ξ_{N+1}, we used the boundary conditions (4.5.21),

$$g_1'(\xi_{N+1}, t) = g_2'(\xi_{N+1}, t) = -1 . \quad (4.6.6)$$

To calculate g_3' at $\xi_{N+1} = 1$, we used the symmetries of the φ_i about $\xi = 1$, as we did in section 4.5.5 to calculate g_i''. Using four points we find

$$g_3'(\xi_{N+1}, t) = \frac{1}{d\xi}\left(\frac{1}{6}g_3(\xi_{N-1}, t) - \frac{4}{3}g_3(\xi_N, t) + \frac{7}{6} + \frac{2}{3}d\xi^2 - \frac{4}{3}d\xi^4\right) . \quad (4.6.7)$$

Similarly for the point next to the right-hand border, ξ_N, we used the symmetries of the φ_i about $\xi = 1$ to get

$$\begin{aligned} g_1'(\xi_N, t) =& \frac{1}{d\xi}\left(\frac{1}{12}g_1(\xi_{N-2}, t) - \frac{4}{3}g_1(\xi_{N-1}, t) - \frac{1}{12}g_1(\xi_N, t) + \frac{4}{3}g_1(\xi_{N+1}, t)\right. \\ &\left. + \frac{1}{6}d\xi + \frac{1}{6}d\xi^3\right) . \end{aligned} \quad (4.6.8)$$

$g_2'(\xi_N, t)$ has the same form as given in (4.6.8), whereas $g_3'(\xi_N, t)$ is given by

$$\begin{aligned} g_3'(\xi_N, t) =& \frac{1}{d\xi}\left(\frac{1}{12}g_3(\xi_{N-2}, t) - \frac{4}{3}g_3(\xi_{N-1}, t) - \frac{1}{12}g_3(\xi_N, t) + \frac{4}{3}g_3(\xi_{N+1}, t)\right. \\ &\left. - \frac{1}{6} - \frac{1}{6}d\xi^2 - \frac{1}{6}d\xi^4\right) . \end{aligned} \quad (4.6.9)$$

Having calculated the energy densities, we integrated them using Simpson's rule. Simpson's rule states that the integral

$$I = \int_{\xi_0}^{\xi_N} f \, d\xi \quad (4.6.10)$$

can be calculated numerically to order $d\xi^4$ using

$$I_{\text{approx}} = \frac{d\xi}{3}\left(f_0 + f_{N+1} + 4\sum_{\text{m odd}} f_m + 2\sum_{\text{m even}} f_m\right) , \quad (4.6.11)$$

where $f_0 = f(\xi_0)$, $f_N = f(\xi_N)$ and $f_m = f(\xi_m)$ for $0 < m < N$, with the interval $[\xi_0, \xi_N]$ being discretised to the $(N+1)$ points ξ_i with $0 \leq i \leq N$, which are distance $d\xi$ apart. See ref. [92] for a discussion of Simpson's rule.

4.6.2 Calculating the Energy in the φ_i

In this section we present the results of our energy calculations for the φ_i (i.e. the energy in the zero modes and the non-zero modes together).

First we present the total energy in the φ_i in table 4.1 for $v_{init} = 0.05$ and in table 4.2 for $v_{init} = 0.1$. Because we expect the total energy to be conserved (see section 4.4.2), the order of magnitude at which the total energy deviates from conservation gives us a measure of the numerical inaccuracy in our calculation.

a) From table 4.1, for $v_{init} = 0.05$, we can see that for $0 \leq t \leq 725$ the total energy is conserved up to order 10^{-11}. So we deduce that our calculations of the energy in the φ_i are correct to order around 10^{-11} (this error is slightly higher than that predicted from our numerical methods, but this was to be expected since part of our numerical routine involved the cancellation of numbers of similar orders of magnitude).

   For the later times $t = 750$ and $t = 775$ the total energy has started to increase slightly by around 3×10^{-11}, so there appear to be some extra numerical inaccuracies coming into effect at these later times.

b) Similarly from table 4.2, for $v_{init} = 0.1$, the numerical inaccuracy in the total energy is around 1×10^{-11} for $0 \leq t \leq 260$, and around 2×10^{-11} for $t > 260$.

Next we present the potential energy in the φ_i in figure 4.4(a) for $v_{init} = 0.05$, and in figure 4.4(b) for $v_{init} = 0.1$. As we pointed out in section 4.4.1, the potential energy measures the deviation of the solution from the solutions to the Bogomol'nyi equations, the f_i. So the potential energy originates entirely from the non-zero modes. In section 4.3.4 we found that the non-zero modes behave like harmonic oscillators when the D-strings are far apart. So at late times the kinetic energy is of the same order as the potential energy, and so the magnitude of the potential energy is approximately half the total energy in the non-zero modes.

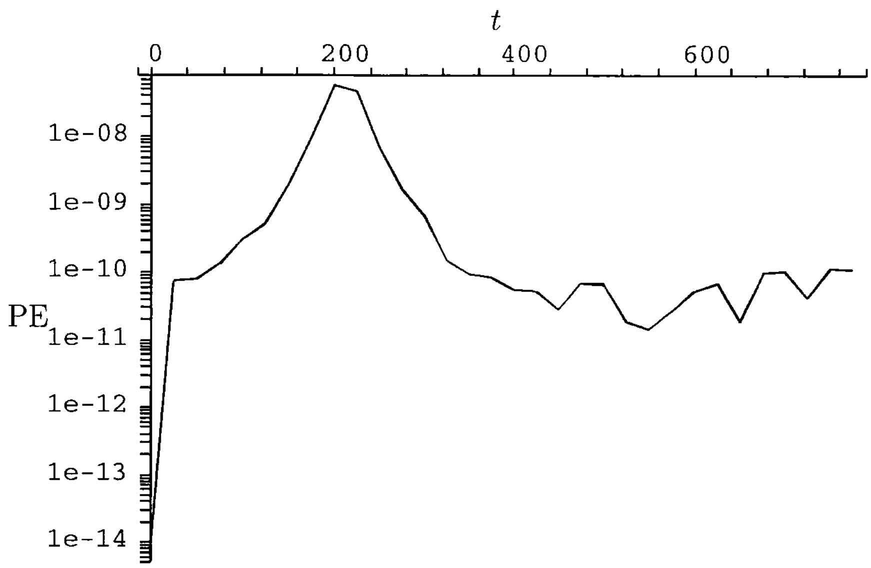
a) In the graph in figure 4.4(a), for $v_{init} = 0.05$, we can see that the potential energy increases up to order 10^{-8} around the point of scattering $t \approx 200$. After scattering the potential energy decreases back down to order 10^{-11}, which is

Time	Total energy in the φ_i	Time	Total energy in the φ_i
0	0.0023008388119039563	400	0.0023008388028718015
25	0.0023008388080045165	425	0.0023008388041421109
50	0.0023008388040325276	450	0.0023008388043416878
75	0.0023008388035914325	475	0.0023008388094051973
100	0.0023008388045730349	500	0.0023008388101757415
125	0.0023008388064388033	525	0.0023008388052310619
150	0.0023008388051433152	550	0.0023008387979028245
175	0.0023008388043897336	575	0.0023008388038038862
200	0.0023008388049627743	600	0.0023008388137977869
225	0.0023008388052134037	625	0.002300838810471318
250	0.0023008388054121024	650	0.002300838814647431
275	0.0023008388046028864	675	0.002300838803442902
300	0.0023008388045520022	700	0.0023008388033493934
325	0.0023008388058299747	725	0.0023008388157896432
350	0.0023008387998940645	750	0.0023008388221448599
375	0.0023008388018124133	775	0.0023008388304597495

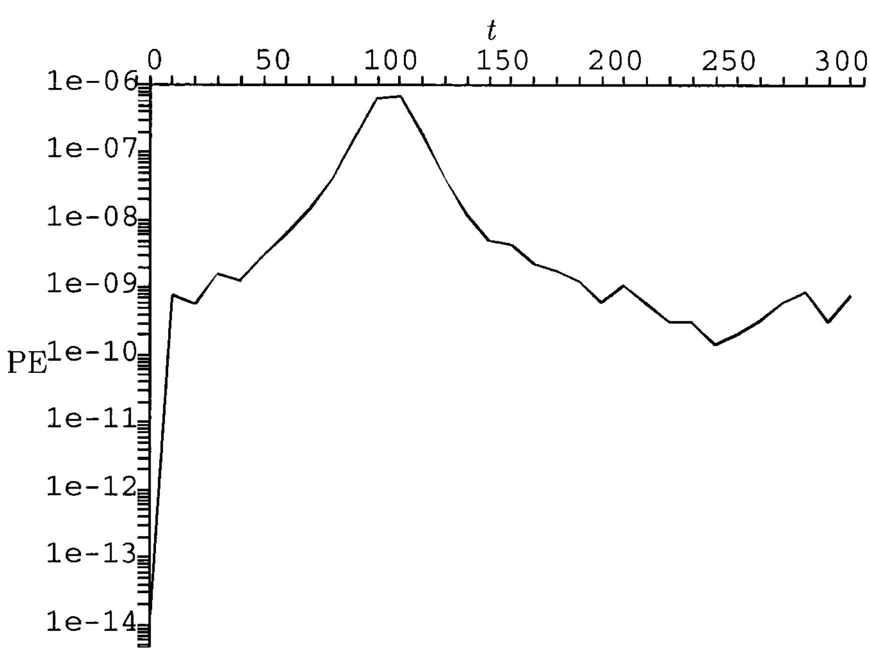
Table 4.1: Table showing the total energy in the numerical solutions φ_i at different times, with initial velocity $v_{init} = 0.05$.

Time	Total energy in the φ_i	Time	Total energy in the φ_i
0	0.009203355247594738	160	0.0092033552484803473
10	0.0092033552425785873	170	0.0092033552543866384
20	0.0092033552427108634	180	0.009203355255166067
30	0.0092033552452017824	190	0.0092033552560935126
40	0.0092033552483386689	200	0.0092033552553977428
50	0.009203355246432756	210	0.0092033552608578265
60	0.0092033552471731343	220	0.0092033552573199656
70	0.0092033552516308723	230	0.0092033552536499603
80	0.0092033552505522004	240	0.0092033552486662056
90	0.0092033552513087272	250	0.009203355252019858
100	0.0092033552511301756	260	0.0092033552427458355
110	0.0092033552512737656	270	0.0092033552315330305
120	0.0092033552511940186	280	0.0092033552317870183
130	0.0092033552524599157	290	0.0092033552332674035
140	0.00920335525272108	300	0.0092033552319790869
150	0.0092033552508128894	310	0.0092033552417399717

Table 4.2: Table showing the total energy in the numerical solutions φ_i at different times, with initial velocity $v_{init} = 0.1$.



(a) $v_{init} = 0.05$



(b) $v_{init} = 0.1$

Figure 4.4: Logarithmic plot of the potential energy in the φ_i against time for $v_{init} = 0.05$

the order of the numerical inaccuracies in this calculation. This suggests that all the energy has been transferred back into the zero modes after scattering, and therefore no energy has been radiated.

b) Similarly for $v_{init} = 0.1$, in the graph in figure 4.4(b), we find the potential energy increases up to the order of 10^{-6} around the point of scattering at $t \approx 100$. Then it decreases back down to the order of 10^{-10} after scattering. Although this is slightly higher than the order of numerical inaccuracy, it is still much lower than we would expect from Manton's prediction, which would give $E_{rad} \sim 10^{-5}$.

4.6.3 Separating the non-zero modes from the zero modes

In the previous section we deduced the energy radiated from the potential energy of the full numerical solution g_i. In the next section we will calculate the energy in the non-zero modes ϵ_i directly. In order to do this we need to separate out the non-zero modes ϵ_i from the full solutions φ_i. In this section we will describe the method we have used to separate them out. Recall that

$$\varphi_i(\xi, t) = f_i(\xi, k(t)) + \epsilon_i(\xi, t) \ , \tag{4.6.12}$$

where $f_i(\xi, k(t))$ are the 'static' solutions (4.3.2) - (4.3.4), and $\epsilon_i(\xi, t)$ are the non-zero modes. The value of $k(t)$ at time t completely specifies the functions $f_i(\xi, k(t))$. So, given $k(t)$, we can calculate the ϵ_i by subtracting $f_i(\xi, k(t))$ from our numerical solutions for the φ_i.

Finding $k(t)$ is difficult because the f_i are complicated functions. However, if we work in the asymptotic limit, when the D-strings are far apart, we can use the leading order terms in the series (4.3.16) - (4.3.18) for the f_i, which are (after scattering):

$$f_1 = -K\frac{1}{\sinh(\xi K)} \ , \quad f_2 = -K\frac{\cosh(\xi K)}{\sinh(\xi K)} \ , \tag{4.6.13}$$

$$f_3 = -K\frac{1}{\sinh(\xi K)} \ . \tag{4.6.14}$$

Then we can work with the parameter $K(t)$ instead of $k(t)$.

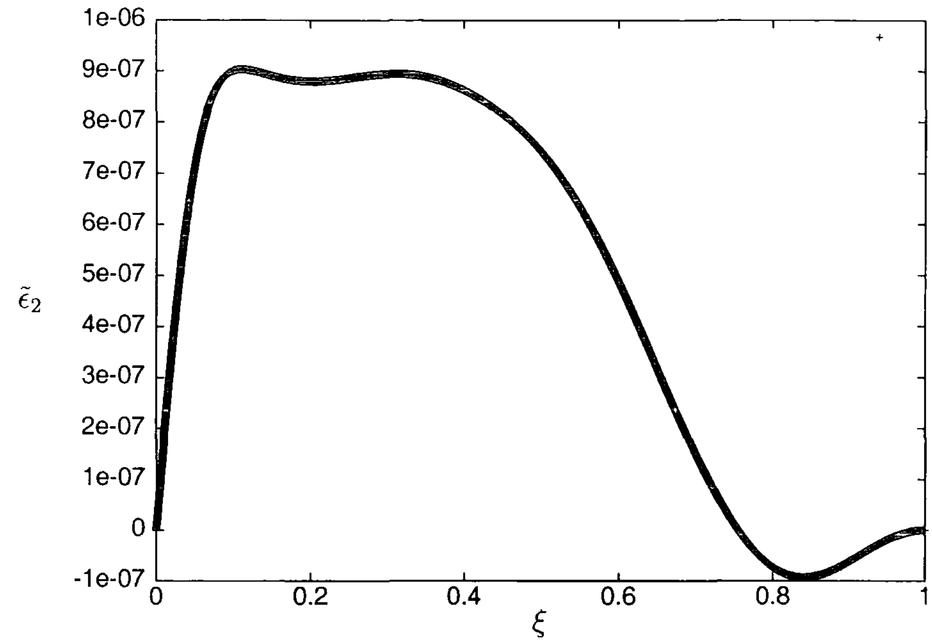


Figure 4.5: Graph showing $\tilde{\epsilon}_2(\xi, t)$ for $t = 700$ and $v_{init} = 0.05$

At $\xi = 1$ the approximations (4.6.13) - (4.6.14) can be expanded as series in e^{-2K} to give

$$f_1(\xi = 1, K) = O(Ke^{-2K}) , \quad f_2(\xi = 1, K) = -K + O(Ke^{-2K}) , \tag{4.6.15}$$

$$f_3(\xi = 1, K) = O(Ke^{-2K}) . \tag{4.6.16}$$

Substituting these into (4.6.12), we find for φ_2

$$\varphi_2(\xi = 1, t) = -K(t) + \epsilon_2(\xi = 1, t) . \tag{4.6.17}$$

We find a first approximation for K, call it $\tilde{K}$, by taking $\epsilon_2(\xi = 1, t) = 0$. Then $K(t)$ is just the value of $-\varphi_2$ at $\xi = 1$ at time t. Using $K(k(t)) = \tilde{K}$ in (4.6.12) we can calculate our first approximation for ϵ_2, call it $\tilde{\epsilon}_2$,

$$\tilde{\epsilon}_2(\xi, t) = \varphi_2(\xi, t) - f_2(\xi, \tilde{K}) . \tag{4.6.18}$$

The graph in figure 4.5 shows $\tilde{\epsilon}_2(\xi, t)$ for $t = 650$ with $v_{init} = 0.05$.

Recall from section 4.3.4 that we expect $\epsilon_2(\xi, t)$ to take the form of a harmonic oscillator in the asymptotic limit. So we expect

$$\int_0^2 \epsilon_2(\xi, t)\, d\xi = 0 . \tag{4.6.19}$$

This is clearly not the case for $\epsilon_2 = \tilde{\epsilon}_2$ for the graph in figure 4.5. This is because we took $\epsilon_2(1,t) = 0$ in calculating our estimate $\tilde{K}$. We can improve our estimate for ϵ_2 by taking

$$\epsilon_2(1,t) = \delta \; , \tag{4.6.20}$$

in (4.6.17), where δ is chosen such that (4.6.19) is true. If we assume that f_2 is constant then

$$\delta \approx \int_0^2 \tilde{\epsilon}_2(\xi, t)\, d\xi \; . \tag{4.6.21}$$

So we can calculate an improved estimate for K, call it $\hat{K}$, using (4.6.17),

$$\varphi_2(1,t) = -\hat{K} + \delta \; . \tag{4.6.22}$$

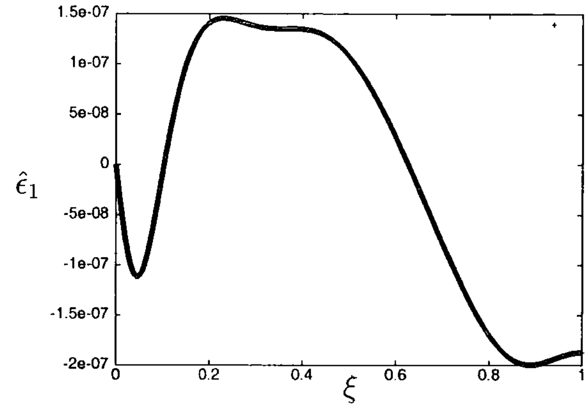
And we can calculate an improved estimate for the ϵ_i, call them $\hat{\epsilon}_i$ using

$$\hat{\epsilon}_i(\xi,t) = \varphi_i(\xi,t) - f_i(\xi,\hat{K}) \; . \tag{4.6.23}$$

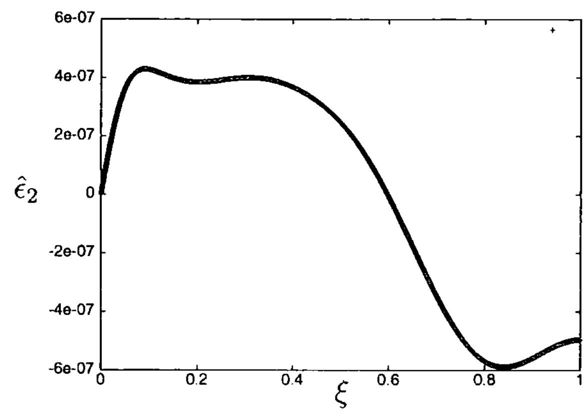
We can calculate an approximation for $\dot{K}$ and the $\dot{\epsilon}_i$ using a similar procedure to that described above.

The results of this procedure for the ϵ_i and $\dot{\epsilon}_i$ at $t = 700$ for $v_{init} = 0.05$ are given in the graphs in figures 4.6 and 4.7 respectively. The graph for ϵ_2 does seem to take the form of a harmonic oscillator, as we expected from section 4.3.4. However, the graphs for ϵ_1 and ϵ_3, which we expect to be zero in the asymptotic limit from section 4.3.4, are not much smaller than ϵ_2. This is because, in that section, in order to obtain $\epsilon_1 = \epsilon_3 = 0$ we neglected terms of order $1/K^2$ in the equations of motion for ϵ_1 (4.3.37) and ϵ_3 (4.3.39). At $t = 700$ $K \approx 24$ and $1/K^2 \approx 1/600$, so these neglected terms are still reasonably large. We conclude that at this time the zero modes and non-zero modes have not completely decoupled, leading to errors in our approximation procedure.

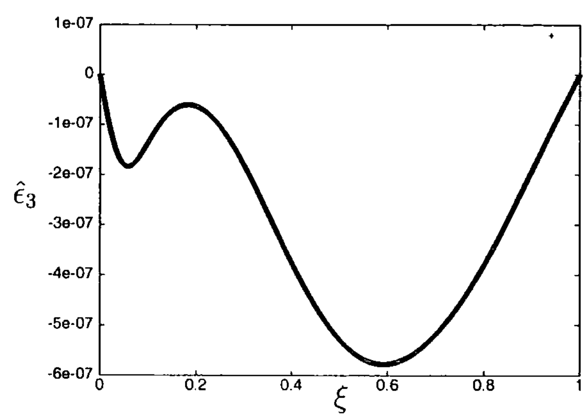
The results for the approximations for K and $\dot{K}$ from the method described above are given in table 4.3 for $v_{init} = 0.05$ and in table 4.4 for $v_{init} = 0.1$. For $v_{init} = 0.05$ we have $\epsilon_i \sim 10^{-7}$ at $t = 650$ from the graphs in figure 4.6. So our result for K at $t = 650$ is correct, at least to order 10^{-7}. So it is worth proceeding to use these results to calculate the energy in the non-zero modes, in spite of the inaccuracies in our method which we discussed above.



(a) $\hat{\epsilon}_1$

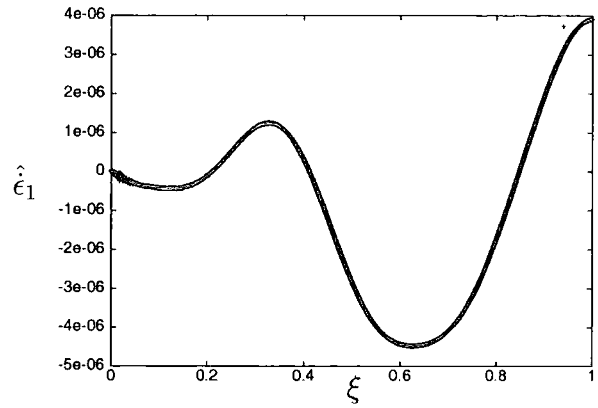


(b) $\hat{\epsilon}_2$

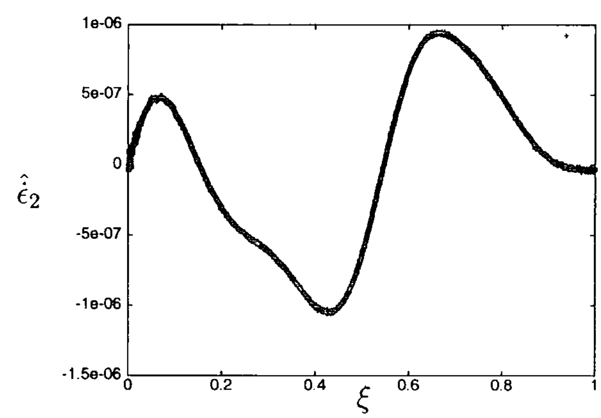


(c) $\hat{\epsilon}_3$

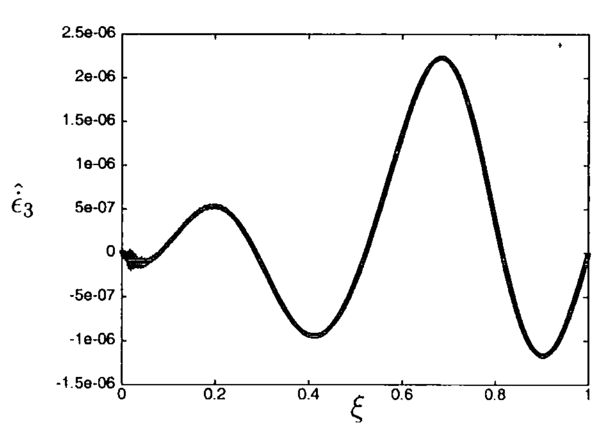
Figure 4.6: Graphs showing the $\hat{\epsilon}_i(\xi, t)$ for $t = 700$ and $v_{init} = 0.05$



(a) $\hat{\dot\epsilon}_1$



(b) $\hat{\dot\epsilon}_2$



(c) $\hat{\dot\epsilon}_3$

Figure 4.7: Graphs showing the $\hat{\dot\epsilon}_i(\xi, t)$ for $t = 700$ and $v_{init} = 0.05$

$Time$	K	$\dot{K}$
325	7.680119995022257	0.051431773724365769
350	8.9587692742531644	0.050890837263380694
375	10.225917788445766	0.050499674421269791
400	11.484537657012579	0.050202380890253015
425	12.736568653767913	0.049968841747149655
450	13.983349590185279	0.049780127466368805
475	15.22584516271003	0.049624395287705729
500	16.464774138455645	0.049493741101254325
525	17.700686939799251	0.049382094290754054
550	18.934014703559839	0.049286216460765905
575	20.165101601234785	0.049202360879168044
600	21.394226868661601	0.049129188080153978
625	22.621620253924849	0.049063762833720333
650	23.847473133479966	0.049005543236356822
675	25.071946612115951	0.048953323932775197
700	26.295177622390725	0.0489060136694019
725	27.517283598259574	0.048863102166153673
750	28.738365938824291	0.048824095413400083
775	29.958512884936482	0.048788264692838888

Table 4.3: Table showing the approximate values for K and $\dot{K}$, calculated using the method described in section 3.6.3, for $v_{init} = 0.05$.

Time	K	$\dot{K}$
130	4.2528185487278103	0.10955490088679257
140	5.3310367773774052	0.10642112694577098
150	6.3847076708880319	0.10446144578589162
160	7.4222831652072534	0.10313046001030289
170	8.4485637959300757	0.10216922583118802
180	9.4664709163334937	0.10144251134346198
190	10.477911667493794	0.10086772807437738
200	11.484199801359248	0.10040412922644423
210	12.486282292425347	0.10002230200495914
220	13.484863263207957	0.099702905926008153
230	14.480482559027372	0.099429881851804125
240	15.473562599582436	0.099191582006006154
250	16.464440746205028	0.098986270927379813
260	17.453390705432682	0.098807028106371125
270	18.440637838834185	0.098646325983717795
280	19.426369398126088	0.098502604562491852
290	20.410743946579903	0.098374461405384403
300	21.393895600174691	0.098258158780689761
310	22.375939554267504	0.098152194222099989

Table 4.4: Table showing the approximate values for K and $\dot{K}$, calculated using the method described in section 3.6.3, for $v_{init} = 0.1$.

4.6.4 Calculating the energy in the non-zero modes directly

In this section we present the results of calculating the energy in the ϵ_i, having used the techniques of the previous section to separate out the ϵ_i from the φ_i. In order to calculate the energy in the ϵ_i we have to assume that the zero modes and non-zero modes have decoupled from one another. Then the kinetic energy density and potential energy density for the ϵ_i are given by

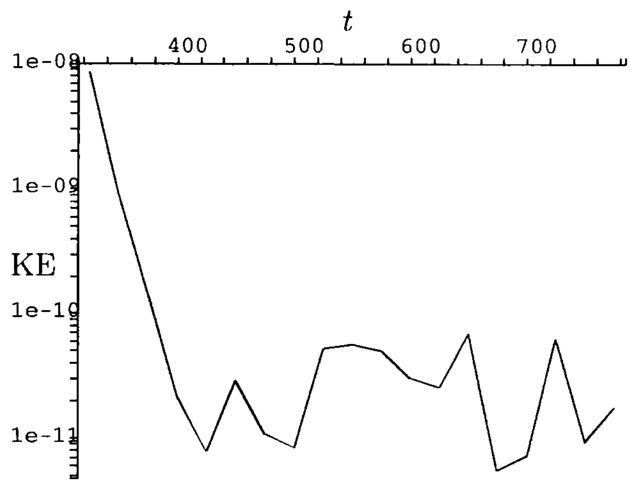
$$
\begin{aligned}
\text{K.E. density} &= \frac{1}{2}(\dot{\epsilon}_1^2 + \dot{\epsilon}_2^2 + \dot{\epsilon}_3^2) \,, && (4.6.24)\\
\text{P.E. density} &= \frac{1}{2}(\epsilon_1'^2 + \epsilon_2'^2 + \epsilon_3'^2) \\
&\quad + \frac{1}{\xi^2}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1) \\
&\quad + \frac{1}{\xi}(\epsilon_1(\epsilon_2' + \epsilon_3') + \epsilon_2(\epsilon_3' + \epsilon_1') + \epsilon_3(\epsilon_1' + \epsilon_2')) \,, && (4.6.25)
\end{aligned}
$$

where we have neglected all terms of order ϵ^3 and higher in the potential energy density (4.6.25).

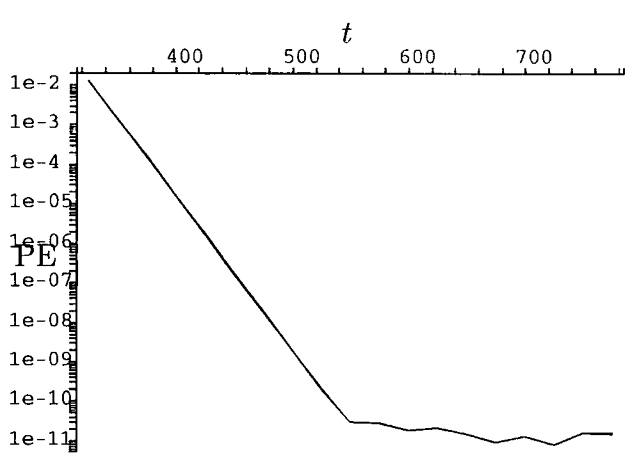
The graphs in figures 4.8 and 4.9 show the potential and kinetic energies calculated for $v_{init} = 0.05$ and $v_{init} = 0.1$ respectively.

a) In figure 4.8, for $v_{init} = 0.05$, the potential energies are much higher than those found in section 4.6.2 for $325 < t < 525$. This could be because the zero modes have not decoupled sufficiently from the zero modes at these earlier times, leading to inaccuracies in the energy densities (4.6.24) and (4.6.25). For $t \geq 550$ the potential energies are of the order 10^{-11}, which agrees with the results presented in section 4.6.2. As we claimed in that section, the kinetic energy in the ϵ_i is of the same order as the potential energy in the ϵ_i at later times.

b) Similarly for $v_{init} = 0.1$, in figure 4.9, the potential energies are higher than expected for $130 \leq t < 250$. For $t \geq 260$ the potential energies are of the order 10^{-10}, which agrees with the results presented in section 4.6.2. Again, the kinetic energy in the ϵ_i is of the same order as the potential energy in the ϵ_i at later times.

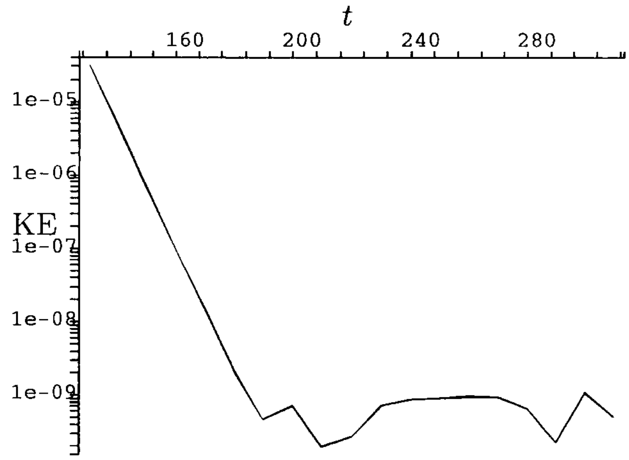


(a) kinetic energy

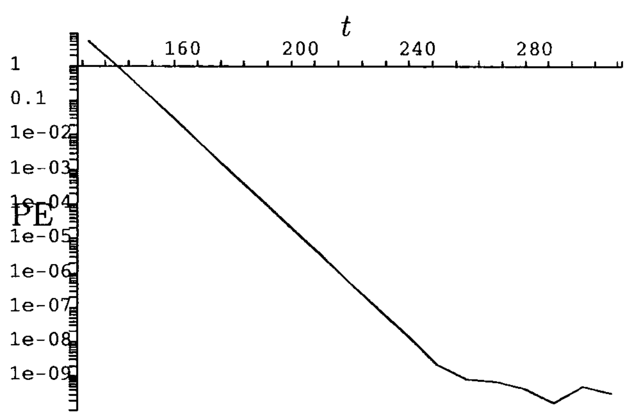


(b) potential energy

Figure 4.8: Graphs showing the kinetic and potential energy densities in the ϵ_i, calculated using the method described in section 3.6.3, for $v_{init} = 0.05$.



(a) kinetic energy



(b) potential energy

Figure 4.9: Graphs showing the kinetic and potential energy densities in the ϵ_i, calculated using the method described in section 3.6.3, for $v_{init} = 0.1$.

4.7 Conclusions

We have described in this chapter our numerical calculations for the energy radiated during D-string scattering. We found that the Born-Infeld equations of motion were too complicated to solve numerically, and so we solved the Yang-Mills equations of motion instead.

Our numerical results reproduce the 90° scattering which we expected from the comparison to monopole scattering. It is encouraging that this follows directly from the equations of motion, rather than having to be put in by hand, as we did in section 3.6.

Our calculations for the energy radiated during D-string scattering indicate that no energy is radiated, in contrast to the prediction of Manton and Samols in ref. [80], which we reviewed in section 3.2. However, it may be that, by making the approximation to the Yang-Mills system, we prevented it from being able to radiate energy. It would be nice to be able to support the conclusion we have reached here with further evidence from the Born-Infeld action. We will discuss possible approaches for doing this in our conclusions in chapter 5.

Chapter 5

Conclusions

At the time of writing, string theory is the only candidate we have as a theory of quantum gravity. However, so far, all attempts to use the known properties of string theory to build a realistic four-dimensional model have failed. In the light of this, we should seek to understand string theory better in the hope that we may uncover some properties of the theory which could lead us to a realistic model. We have seen how D-branes are necessary for superstring theory to be consistent; they act as sources for the Ramond-Ramond fields, which are necessary to fill out the supergravity multiplets. D-branes also exist outside of string theory as objects in their own right; they are soliton solutions of supergravity. Although much is known about the properties of D-branes, there are still many features which are currently not well understood. It is therefore important that we continue to explore the unknown properties of D-branes in the hope of finding new ways of proceeding.

We have seen that there are many ways in which a D-brane configuration can be regarded as a BPS monopole. Since BPS monopoles have been studied extensively since the 1970's, it makes sense to use what we already know about monopoles to teach us more about D-branes. This was the motivation on which the work in this thesis was based.

One example of a D-brane configuration behaving as a BPS monopole is the enhançon mechanism. When we wrap a D6-brane on a K3 manifold there is some negative D2-brane charge induced. To avoid having a negative tension, the D6/D2-brane charge is smeared out on a spherical shell in the transverse dimensions. The

W-boson of the theory, which is the D4/D0-brane object Hodge dual to the D6/D2-brane one, become massless at the enhançon radius, indicating the presence of a region of enhanced gauge symmetry. For this reason the D6/D2-brane object is called the enhançon. We deduce that it has the behaviour of a BPS monopole, which also looks like a place of enhanced gauge symmetry.

These arguments imply that the metric on moduli space for enhançons, in the limit that they are far apart, will be the generalised Taub-NUT metric. We have shown explicitly in chapter 2 that this is the case. Obtaining the position terms in the Taub-NUT metric was relatively straightforward, following Ferrell and Eardley's calculation for black holes from refs. [72], [73] and [74]. Obtaining the phase terms in the Taub-NUT metric was more difficult; our calculation highlights once again the inability of supergravity techniques to handle situations where stringy physics is important. Since the phase terms in the enhançon case arise from the coupling of the fields to the gauge field F living on the branes, we were able to deduce them by building the action for F from scratch, and using the method of ref. [63].

Having seen some of the things that monopoles can teach us about the enhançon, it's also worth asking the following question: can the enhançon teach us anything about monopoles? We know that the enhançon is composed of tensionless objects. This was discovered by moving a brane probe in from infinity, and observing that it becomes tensionless at the enhançon radius. But the enhançon is not massless; its mass is related to the vacuum expectation value of the Higgs field at infinity, as is the case for a BPS monopole. For the enhançon this is set by the volume of the K3 manifold at infinity. This seems strange; it would be interesting to see if a similar effect happens for monopoles by studying the behaviour of a monopole probe in the background of a monopole with large charge.

A second example of a D-brane configuration behaving as a BPS monopole is D-strings attached to a D3-brane; the ends of the D-strings look like monopoles in the worldvolume gauge theory of the D3-brane. Minimising the D3-brane action with the appropriate fields excited leads to the Bogomol'nyi equation for a BPS monopole. And minimising the D-strings action with the appropriate fields excited leads to Nahm's equations. So the D-string/D3-brane construction provides a physical realisation of the ADHMN construction for a BPS monopole.

Again, we may be able to use what we know about monopoles to learn about this D-brane configuration. In chapters 3 and 4 we aimed to calculate the energy radiated during the scattering of two D-strings stretched between two D3-branes, working from the D-string perspective. We could then compare our result with the calculation of Manton and Samols for the energy radiated during monopole scattering from ref. [80]. By taking the appropriate limit of the non-Abelian Born-Infeld action for D-strings, we obtained the action to describe D-string scattering in the low-energy limit. However, we were unable to solve the equations of motion for this action analytically. We tried instead to solve it numerically. But the equations of motion were too complicated to solve numerically, so we approximated the theory to a Yang-Mills theory, and solved the equations of motion resulting from the Yang-Mills action instead. Our results showed that there was no energy radiated during D-string scattering, in contrast with the prediction of Manton and Samols, which was $E_{rad} \sim 1.35 m_{mon} v_{\infty}^5$, where E_{rad} is the energy radiated, m_{mon} is the mass of the monopole, and v_∞ is the initial velocity of the monopoles. However, our result could have been affected by approximating to the Yang-Mills action, because this confines the theory to a finite box.

It would be nice to be able to confirm our numerical result from a calculation using the Born-Infeld action. It may be possible to do this in the future by expanding out the perturbations in terms of their angular momentum modes. An expansion of this kind has been done by Constable, Myers and Tafjord in ref. [66] for the case of semi-infinite D-strings attached to a D3-brane (although in that calculation the authors worked with a different limit of the Born-Infeld action). It may also be nice to investigate the effects on our calculation of the higher order corrections to the non-Abelian Born-Infeld action.

In conclusion, a lot of work has already been done which has uncovered many beautiful properties of D-branes. However, there are many situations where we cannot describe the physics accurately because our hands are tied by our inability to understand some important aspects of D-brane behaviour. In particular, we must seek to understand better the behaviour of non-Abelian fields in the Dirac-Born-

Infeld and Chern-Simons actions, and also in supergravity. We should also seek to understand the string theory actions and the supergravity action in the sectors of the theory in which they are currently ill-defined. The examples of D-brane configurations which have discussed in this thesis highlight some of these problems. For example, in the case of the enhançon, we would like to understand better the behaviour of the non-Abelian fields, which must be present where the gauge symmetry is enhanced. Some progress has already been made in this direction in ref. [93]. This would also be necessary for us to be able to construct explicitly the corrections to the Taub-NUT metric for the enhançons; we expect the full metric to be the higher-dimensional generalisation of the Atiyah-Hitchin metric, by comparison with the monopole case (see the discussion in ref. [58]). See also ref. [94] for a more recent discussion of these issues. In our other D-brane configuration, D-strings stretched between D3-branes, we require the full version of the non-Abelian action in order to show that the solutions presented here are solutions to the equations of motion of the full theory; we expect this to be the case by comparison with the S-dual picture, a fundamental string stretched between D3-branes. We would also like to be able to understand the solutions in the region where the geometry becomes highly curved. We conclude that more progress is necessary in these areas before we can claim to fully understand the D-brane configurations we have discussed. We look forward to following the progress of the theory in the future.

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Appendix A

Calculating the Dimensionally Reduced Ricci Scalar

In this appendix we outline in more detail the calculation from section 2.2 to calculate the Ricci scalar of the dimensionally reduced theories.

We wish to calculate the Ricci scalar for a metric of the form

$$G_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} & 0 \\ 0 & V^{1/2} e^{\rho/2} \gamma_{ij} \end{pmatrix} ,$$

where G_{MN} is a D-dimensional metric, $\bar{g}_{\mu\nu}$ is a $(D-d)$-dimensional metric, and γ_{ij} is a d-dimensional metric, which is Ricci flat. So $M, N = 0, \ldots, (D-1)$, $\mu, \nu = 0, \ldots, (D-d-1)$, and $i, j = (D-d), \ldots, (D-1)$. We take $\bar{g}_{\mu\nu}$ and ρ to be independent of the x^i and γ_{ij} to be independent of the x^μ.

We make the conformal transformation

$$G_{MN} = e^{\rho/2} H_{MN} ,$$

so that

$$H_{MN} = \begin{pmatrix} e^{-\rho/2} \bar{g}_{\mu\nu} & 0 \\ 0 & V^{1/2} \gamma_{ij} \end{pmatrix} .$$

Then the formula for the Ricci scalar under a conformal transformation (2.2.4) gives

$$R(G) = e^{-\rho/2} \Big(R(H) - \frac{D-1}{2} (\nabla^2_{(H)} \rho) - \frac{(D-1)(D-2)}{16} (\nabla_{(H)} \rho)^2 \Big) , \qquad \text{(A.0.1)}$$

where the subscript H in the second and third terms of (A.0.1) indicate that the metric H is used to raise and lower indices in these terms.

Since $H_{\mu\nu}$ is independent of the x^i and H_{ij} is independent of the x^μ, we have

$$\begin{aligned} R(H) &= R(e^{-\rho/2}\bar{g}) + R(V^{1/2}\gamma) \\ &= e^{\rho/2}\left(R(\bar{g}) + \frac{(D-d-1)}{2}(\bar{\nabla}^2\rho) - \frac{(D-d-2)(D-d-1)}{16}(\bar{\nabla}\rho)^2 \right) \end{aligned} \tag{A.0.2}$$

where again we have used the formula for the Ricci scalar under a conformal transformation and we have also used that γ_{ij} is Ricci flat.

We can calculate the terms $(\nabla_{(H)}\rho)^2$ and $(\nabla^2_{(H)}\rho)$ in (A.0.1) - we obtain

$$(\nabla_{(H)}\rho)^2 = e^{\rho/2}(\bar{\nabla}\rho)^2 \ , \tag{A.0.3}$$

and

$$(\nabla^2_{(H)}\rho) = H^{\mu\nu}\partial_\mu(\partial_\nu\rho) - H^{MN}\Gamma(H)^\lambda_{MN}(\partial_\lambda\rho) \ . \tag{A.0.4}$$

By definition

$$\Gamma(H)^R_{MN} = \frac{1}{2}H^{RS}\left(H_{SM,N} + H_{SN,M} - H_{MN,S} \right) \ ,$$

(see ref. [78]). So we get

$$\Gamma(H)^\lambda_{\mu\nu} = -\frac{1}{4}\left(\rho_{,\nu}\delta^\lambda_\mu + \rho_{,\mu}\delta^\lambda_\nu - \rho_{,\sigma}G^{\lambda\sigma}G_{\mu\nu} \right) + \bar{\Gamma}^\lambda_{\mu\nu} \ , \tag{A.0.5}$$

and

$$\Gamma(H)^\lambda_{ij} = 0 \ . \tag{A.0.6}$$

Substituting (A.0.5) and (A.0.6) into (A.0.4) we get

$$(\nabla^2_H\rho) = e^{\rho/2}(\bar{\nabla}^2\rho) + \frac{(2-(D-d))}{4}e^{\rho/2}(\bar{\nabla}\rho)^2 \ . \tag{A.0.7}$$

Substituting (A.0.2), (A.0.3) and (A.0.7) into (A.0.1),we get

$$R(G) = R(\bar{g}) - \frac{d}{2}(\bar{\nabla}^2\rho) - \frac{d(d+1)}{2}(\bar{\nabla}\rho)^2 \ ,$$

as required.