Non-static brane probes, topological charges and calibrations

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A Thesis presented for the degree of
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Abstract

In this thesis we consider probe branes in 10- and 11-dimensional supergravity backgrounds. Firstly, we consider probing a class of 11-dimensional backgrounds with giant gravitons. These backgrounds arise from lifting solutions of 4-dimensional $U(1)^4$ and 7-dimensional $U(1)^2$ gauged supergravities. We find that giant gravitons degenerate to massless particles exist in arbitrary lifted backgrounds, and furthermore both these objects are degenerate to massive charged particles probing the associated lower-dimensional gauged supergravity solutions. We then move on to consider superalgebras for M2- and M5-brane probes in general 11-dimensional supersymmetric backgrounds. We derive the form of the topological charges which appear in the super-translation part of the algebra. These charges are given by the integral (over the spatial world-volume of the brane) of certain closed forms constructed from Killing spinors and background fields. The super-translation algebra allows us to derive BPS bounds on the energy/momentum of probe branes in these general supersymmetric backgrounds. These bounds can be interpreted as generalized calibration bounds for these branes. We then use a similar procedure in type IIB supergravity to construct a calibration bound for a giant graviton in $AdS_5 \times S^5$. As a by-product of this construction, we find a number of differential and algebraic relations satisfied by p-forms constructed from Killing spinors in type IIB supergravity. These relations are valid for the most general supersymmetric backgrounds. We then show that the calibration bound which we have constructed is saturated by a large class of general giant gravitons in $AdS_5 \times S^5$, which are defined via holomorphic surfaces in $\mathbb{C}^{1,2} \times \mathbb{C}^3$. Moreover, dual giant gravitons also saturate the calibration bound. We find that both these branes minimize "energy minus momentum" in their homology class.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapters 1 and 2 of this thesis contain a review of the necessary background material. Chapter 3 is based on joint work with my supervisor, Dr Douglas Smith [1]. Chapter 4 is a review of background material. Chapter 5 contains some review sections, but § 5.1.3 onwards is original work done in collaboration with my supervisor and Dr David Page [2]. Chapter 6 is based on joint work with my supervisor [3]. Chapter 7 contains some initial review, but § 7.2 onwards is original work done in collaboration with my supervisor [3].

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Chapter 1

Introduction

The search for the theory of quantum gravity is one of the outstanding problems facing theoretical physics. The most promising theory so far, which incorporates quantum mechanics as well as general relativity in a mathematically consistent frame-work, is String Theory. String theory has been studied since the 1960s. It was originally studied under the guise of the so-called "dual models" for the strong interactions which were proposed before the advent of QCD. These models were suggested because they could reproduce the observed relation between the mass and spin of hadrons, namely $m^2 = J/\alpha'$, where α' is a constant called the Regge slope. However, in the early 1970s it was realized that these dual models, which were later realized to be theories of 1-dimensional strings, could only give an approximate qualitative description of hadrons. In particular, they failed to produce the observed behaviour of scattering amplitudes at high energies. Consequently, in 1974 these dual models – or various versions of string theory – were dismissed for the strong interactions in favour of QCD.

However, at the same time it was realized that string theory is a good candidate for the theory of quantum gravity. The reasons for this are as follows. Firstly, in the low energy regime, string theory reduces to a gravitational theory which contains Einstein's general relativity. Moreover, in the high energy regime, string theory gives well-behaved expressions for scattering amplitudes involving gravitons. In the 1960s and 70s it was a well-established problem that quantum field theories involving gravitons were non-renormalizable, i.e. there were serious ultra-violet divergences in scattering amplitudes that could not be removed using the methods of renormalization. However, string theory circumvents this problem by "smoothing out" point-like interaction vertices. In particular, Feynman diagrams describing scattering processes are no longer composed of lines and points, but rather they are smooth 2-dimensional surfaces. This means that the infinities which arise from zero size interaction vertices in ordinary field theory calculations involving gravitons do not occur in string theory, and the scattering amplitudes are

UV finite.

We now give some details of the different types of strings and their excitations. Further details can be found in the following text-books: Refs. [4–6]. There are two basic types of strings: open strings which have end-points, and closed strings which do not. There is one input parameter in string theory, namely the tension of the string, which is given by

$$T = \frac{1}{2\pi l_s^2} = \frac{1}{2\pi\alpha'}$$

where l_s is the characteristic string length and $\alpha' = l_s^2$. It is widely accepted that the string length is tiny compared to the scales probed by the most energetic particle accelerators. Motivated by the ideas of quantum gravity, the string length is often taken to coincide with the Planck length, 10^{-33} cm, but it is not necessarily the same. From the field theory point of view each string contains an infinite number of particles. In particular, when one quantizes a string, a number of massless fields and an infinite tower of massive excitations are produced. The massive excitations of a string have mass of the order l_s^{-1}, and so are too massive to be seen at any accelerators. The only accessible string excitations are the massless fields, which are therefore relevant to phenomenology. For example, the massless excitations of a closed string include a massless spin 2 particle, which we identify with the graviton, g_{mn}, and a scalar field – the dilaton – which we denote by ϕ. The dilaton is interesting as its value determines the strength of string-string interactions. Therefore, string theory determines its own coupling strength dynamically through this excitation. This is clearly very different to the situation in ordinary quantum field theories, where the coupling strength is an input parameter. As well as the graviton and dilaton, other massless excitations of strings include Ramond-Ramond anti-symmetric tensor fields, Abelian and non-Abelian gauge fields, and various scalars and fermions. The Ramond-Ramond tensor fields will be of particular interest later on because the sources of these fields are solitonic extended objects called D-branes.

An interesting feature of string theory is that it puts certain demands on the properties of the background space-time in which the strings propagate. In particular, the dimension of the space-time must either be 26 (for bosonic strings) or 10 (for superstrings). These dimensions arise from requiring that the theories are anomaly free. We will generally be interested in the superstring theories, not the bosonic string theories. The fact that string theory exists in backgrounds with such large numbers of dimensions is a difficult issue since we only observe 4 dimensions (3 spatial and 1 time) in our universe. It has been a long-standing problem to try to connect 10-dimensional string theory backgrounds with our 4-dimensional world. There has been much progress on this problem in recent years. One of the most successful ideas is that the 6 extra dimensions are very small, and curled up on a manifold. Since these dimensions are small, we would not see evidence of them at current accelerator scales. Often the manifolds used in these compactifications are 6-dimensional Calabi-Yau manifolds. These manifolds have the correct properties to ensure that the 4-dimensional backgrounds obtained after the compactification preserve supersymmetry. One then might hope to obtain the minimally supersymmetric standard model in 4 dimensions, and then have some mechanism (ideally with a stringy origin) for supersymmetry breaking. This is the subject of much current research, see for example Refs. [7–13].

An alternative solution to compactifying the extra dimensions is the idea of a brane-world [14–17]. This scenario allows the extra dimensions to be large, and in some cases infinite[1]. However, in general, observable matter and gauge fields must be confined to 3+1 dimensions (to agree with strict experimental bounds), and it is only gravity, and perhaps exotic matter such as the dilaton, which are allowed to permeate the extra dimensions. Fortunately, string theory contains extended objects, called branes, which have the right properties to realize this kind of set-up, as we now describe. Since the early 1990s, it has been known that in some superstring theories there exist solitonic extended objects called Dirichlet p-branes, or Dp-branes. These objects are non-perturbative objects in string theory. In detail, Dp-branes are p-dimensional hypersurfaces, with a $(p+1)$-dimensional Minkowskian world-volume, which arise as the surfaces on which open strings can end [2]. The dimension p can take various values depending on the particular string theory (As we will describe in a moment, there are 5 different superstring theories.). In fact, D-branes are not just surfaces, but they are dynamical objects. For example, they have gravitational interactions and they can move. Furthermore, D-branes carry charges corresponding to the Ramond-Ramond anti-symmetric tensor fields. In particular, a Dp-brane will carry charge associated to a $(p+1)$-form Ramond-Ramond potential, $A^{(p+1)}$. These tensor potentials arise in the quantization of strings, however, the strings themselves cannot source these fields. It is interesting that an extended object, namely a brane, is needed to source this type of charge.

Returning to brane-worlds, the idea is that perhaps our observable universe is a brane embedded in a higher dimensional space-time, for example a D3-brane in a 10-dimensional string theory background. Now, as described above, we require that all observable matter and fields are confined to 3+1 dimensions, i.e. confined to our brane. This idea becomes more plausible when we combine it with the fact that branes have fields that exist only on their world-volume, and not in the embedding space. In some situations we can identify

[1]Experimental tests of gravity have only probed to sizes of order 1 millimetre, so extra dimensions of size less than 1 mm would not be detected. However, it is also possible to have much larger, or infinite, extra dimensions if one uses a mechanism [17] for localizing the graviton wave function in the extra dimensions, so that gravity is effectively 4-dimensional.

[2]Dirichlet or "D" refers to the boundary conditions on the open strings.

these fields with observable matter and gauge fields. The idea that our universe is a brane embedded in a higher-dimensional space-time has many interesting implications. For example, the assumption that only gravity can permeate the extra dimensions has been proposed as a possible reason why gravity is such a weak force compared with the other forces. The study of brane-worlds is the subject of much current research in string theory and cosmology, see for example Ref. [18] and the references within it.

It has been known for some time that there are 5 consistent superstring theories, all in 10 dimensions. These theories all contain gravity and they are type IIA, type IIB, type I, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic string theory. For a few years it was assumed that one of these 5 theories must be selected by nature, although it was not clear why. This also seemed to contradict the expectation that the theory of quantum gravity is unique, since all 5 superstring theories appeared to have the right properties. However, in the mid-1990s it was realized that the 5 superstring theories are connected by a web of strong/weak coupling dualities [19–23]. These dualities strongly suggest the existence of a new theory – known as M-theory – whose low energy limit is 11-dimensional supergravity. The high energy version of M-theory is not known, but it does not appear to involve strings. In fact, the dynamical degrees of freedom of M-theory are unknown. However, there are great hopes that this theory may be the unique theory of quantum gravity, with the 5 different string theories appearing as effective theories in different regions of the parameter space. One way of studying M-theory is to consider the low energy theory, i.e. 11-dimensional supergravity. However, due to the web of dualities connecting the string theories, it is also important to study the 10-dimensional supergravities, which arise in the low energy limit of the 5 string theories. In this thesis we will study many features of these supergravity theories.

As well as giving us clues about M-theory, the study of supergravity theories is important since solutions of these theories can be used as backgrounds for compactifications of string/M-theory. Furthermore, the study of branes in supergravity has led to an interesting duality between string theory and certain gauge theories. This duality is known as the AdS/CFT correspondence [24] and it relates two mathematical theories which are naively completely different, and actually exist in different numbers of dimensions. A practical application of the AdS/CFT correspondence is that it allows one to obtain non-perturbative information about a gauge theory by considering a perturbative expansion of the supergravity theory. Understanding more about this conjecture, and using it to obtain information about different field theories, is a key reason why it is important to investigate solutions of supergravity theories. We will be particularly interested in finding energy minimizing configurations of probe branes within fixed supergravity solutions. As we will see for the case of giant gravitons in $AdS_5 \times S^5$, these brane configurations can have interesting interpretations in the dual gauge theory.

In the next section we will review 11-dimensional supergravity and 10-dimensional type IIB supergravity. We will discuss bosonic solutions of these theories, and the requirements for these solutions to be supersymmetric. Solutions of both of these theories will be considered in later chapters of this thesis. In § 1.2 we will describe branes in 11-dimensional and type IIB supergravity. We will give examples of brane solutions, and we will discuss the notion of a probe brane. In particular, we will discuss the idea of a probe brane calculation, which will be especially important in the forthcoming chapters. In § 1.3 we give a brief description of the AdS/CFT correspondence. In § 1.4 we give the full outline of this thesis.

1.1 Supergravity

In this section we review 11-dimensional supergravity and 10-dimensional type IIB supergravity. These theories arise in the low energy limit of M-theory and type IIB string theory respectively. We will consider bosonic solutions of these theories and discuss the requirements for the solutions to be supersymmetric.

1.1.1 11-dimensional supergravity

There is a unique minimal[3] supergravity theory in 11 dimensions which is referred to as 11-dimensional supergravity. This theory has the following field content: a metric, $ds^2 = g_{mn}dx^m dx^n$, a 3-form gauge potential, $A^{(3)}$, and a fermionic field called the gravitino, ψ (with components ψ^α_m, where α is a spinor index). We will usually be interested in situations where the gravitino is set to zero, and only the bosonic fields, ds^2 and $A^{(3)}$, are non-zero. The action for the bosonic fields is given by

$$S = \frac{1}{2\kappa^2}\int d^{11}x\,\sqrt{-g}\,R - \int\left(\frac{1}{2}\,F^{(4)}\wedge *F^{(4)} + \frac{1}{6}\,A^{(3)}\wedge F^{(4)}\wedge F^{(4)}\right) \qquad (1.1)$$

where $F^{(4)} = dA^{(3)}$ is the 4-form field strength associated to $A^{(3)}$, g is the determinant of the metric and R is the 11-dimensional Ricci scalar. The quantity κ is related to Newton's constant in 11 dimensions by

$$2\kappa^2 = 16\pi G_{11}$$

[3]In fact this theory is also maximal as it has 32 supercharges, which is the highest possible number for a physical theory.

The operations of $\wedge$ and $*$ are defined in Appendix A. The equations of motion corresponding to this action are given by,

$$R_{mn} - \frac{1}{12}\left(F^{(4)}_{mr_1r_2r_3}F^{(4)\ r_1r_2r_3}_{n} - \frac{1}{12}g_{mn}(F^{(4)})^2\right) = 0 \tag{1.2}$$

$$d * F^{(4)} + \frac{1}{2}F^{(4)} \wedge F^{(4)} = 0 \tag{1.3}$$

where the indices $m, n, \cdots = 0, 1, \ldots, 9, \natural$ (As explained in Appendix A, we are using the symbol $\natural$ for the 10th spatial direction.). A solution of 11-dimensional supergravity consists of a metric, ds^2, and a closed 4-form field strength, $F^{(4)}$, which together solve these equations of motion (the closure property of $F^{(4)}$ arises since $F^{(4)} = dA^{(3)}$). We denote bosonic solutions of 11-dimensional supergravity by $(ds^2, F^{(4)})$. Note that the second equation above can be interpreted as a (generalized) Bianchi identity for the dual 7-form field strength, $F^{(7)} = *F^{(4)}$. This field strength can be associated to a 6-form gauge potential, $A^{(6)}$, where

$$dA^{(6)} = *F^{(4)} + \frac{1}{2}A^{(3)} \wedge F^{(4)}$$

so that $d^2A^{(6)} = 0$ reproduces the second equation above. Note that if we want to consider brane solutions of 11-dimensional supergravity, then we should also include appropriate source terms in the field equations. For example, for an M2-brane solution, a source term J_{M2} should be included on the right hand side of Eq.(1.3). While for an M5-brane, the Bianchi identity for $F^{(4)}$ becomes $dF^{(4)} = J_{M5}$, where J_{M5} is the source term.

We should also remark that the low-energy effective action for M-theory will contain an infinite number of higher order corrections in addition to the 11-dimensional supergravity action above. The form of most of these corrections is unknown, apart from a few exceptions, but it is known that the corrections will affect the equations of motion and supersymmetry variations. Some consequences of the known corrections are described in Ref. [25]. We will ignore them from now on and assume that they are not important at the low energies we consider.

A solution of supergravity can have the additional property that it is supersymmetric. For a bosonic 11-dimensional supergravity solution, $(ds^2, F^{(4)})$, this means that the solution admits a 32-component Killing spinor. In eleven dimensions, irreducible spinors have 32 real components (Majorana) and they form a representation of the group $Spin(1,10)$. A Killing spinor, ϵ, satisfies the following equation for each value of $m = 0, 1 \ldots, \natural$,

$$\nabla_m\epsilon + \frac{1}{288}\left[\Gamma_m{}^{n_1n_2n_3n_4} - 8\delta_m^{n_1}\Gamma^{n_2n_3n_4}\right]F^{(4)}_{n_1n_2n_3n_4}\epsilon = 0 \tag{1.4}$$

where Γ_m are the Dirac matrices in 11 dimensions with conventions in Appendix A. The covariant derivative of the spinor is defined by

$$\nabla_m \epsilon = \partial_m \epsilon + \frac{1}{4} \omega^{\hat{n}}{}_{\hat{p}\ m} \Gamma_{\hat{n}}{}^{\hat{p}}\ \epsilon \tag{1.5}$$

where $\omega^{\hat{n}}{}_{\hat{p}\ m}$ are the components of the connection 1-forms for the metric and the hats denote tangent space indices, so $\Gamma_{\hat{n}}$ are tangent space Γ-matrices. The equation (1.4) is known as the Killing spinor equation and it arises because we are considering purely bosonic solutions of supergravity, so all fermionic fields are set to zero, i.e. the gravitino ψ is zero. This means that the supersymmetry variation of all bosonic fields vanishes, since these variations involve fermionic fields. Therefore, for a bosonic solution to be supersymmetric, we simply need to ensure that the supersymmetry variation of the fermionic field, ψ, is zero. The equation one obtains from this is the Killing spinor equation, Eq. (1.4), where ϵ corresponds to a supersymmetry transformation parameter. The number of independent Killing spinors, ϵ, satisfying Eq. (1.4) corresponds to the amount of supersymmetry the solution preserves, i.e. a solution which possesses N independent Killing spinors preserves N out of a possible 32 supersymmetries.

Note that the name "Killing" for these spinors is appropriate since Killing spinors can always be used to construct Killing vectors. Essentially the idea is to sandwich a Dirac matrix between a commuting Killing spinor, ϵ, and its conjugate, $\bar{\epsilon}$, as follows: $K^m = \bar{\epsilon}\Gamma^m\epsilon$, where all spinor indices are contracted. We will prove that K^m are components of a Killing vector in Chapter 5.

A problem which has generated much interest in the past few years is to try to classify the bosonic supersymmetric solutions of supergravity theories in various dimensions. Some important progress on this problem has been made, and it now known that there are 4 maximally supersymmetric solutions in 11 dimensions [26,27]. These solutions possess 32 Killing spinors. Specifically, they are: flat space, the $AdS_4 \times S^7$ solution, the $AdS_7 \times S^4$ solution and the pp-wave background. The most general supersymmetric solutions in 11 dimensions have also been classified in some recent work by Gauntlett and collaborators [28,29]. These solutions possess at least one Killing spinor. The approach used in these papers was to construct p-forms of different degrees from one Killing spinor of the background, in an analogous construction to K above. These p-forms satisfy a set of differential and algebraic relations. Moreover, these forms define a mathematical structure known as a G-structure, which is the reduction of the $Spin(1,10)$ frame bundle to a G-sub-bundle. For general supersymmetric solutions there are only two possibilities[4]

[4]That is, if one Killing spinor is used to construct the forms. There will be more possibilities for the G-structure if we consider forms constructed from 2 or more Killing spinors.

for the G-structure, according to whether the Killing vector K is time-like or null: in the time-like case the G-structure group is $SU(5)$, and in the null case the G-structure is $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Knowledge of the G-structure, together with the differential and algebraic equations satisfied by the p-forms, allows some of the metric and 4-form field strength components of the corresponding solutions to be determined. General solutions corresponding to time-like K were classified in Ref. [28] and the null case was discussed in Ref. [29]. Similar techniques have also been used [30–39] to (partially) classify supersymmetric solutions in various lower-dimensional supergravity theories.

1.1.2 Type IIB supergravity

We now describe one of the 10-dimensional supergravity theories, namely type IIB supergravity. This theory is a chiral theory and it arises in the low energy limit of type IIB string theory. The bosonic fields in this theory are the metric, $ds^2 = g_{mn}dx^m dx^n$, the dilaton, ϕ, three independent Ramond-Ramond gauge potentials $C^{(0)}$, $C^{(2)}$ and $C^{(4)}$, and a Neveu-Schwarz-Neveu-Schwarz (NS-NS) 2-form gauge potential, $B^{(2)}$. The fermionic fields are the gravitino, ψ_m^α, and the axino-dilatino, λ^α. We will generally be interested in cases where the fermionic fields are set to zero.

We now define the field strengths associated to the many gauge potentials in this theory. Firstly, the 2-form potential, $B^{(2)}$, has an associated 3-form field strength, $H^{(3)} = dB^{(2)}$. For the Ramond-Ramond gauge potentials we construct composite field strengths G, where

$$G^{(2i+1)} = dC^{(2i)} - H^{(3)} \wedge C^{(2i-2)}$$

and $i = 0, 1, 2, 3, 4$. For the case $i = 0$ we take $C^{(2i-2)} = C^{(-2)} \equiv 0$. Note that the higher dimensional potentials, $C^{(6)}$ and $C^{(8)}$ are not independent, but are related to $C^{(0)}$, $C^{(2)}$ and $C^{(4)}$ via following duality relations between the field strengths: $G^{(9)} = *G^{(1)}$ and $G^{(7)} = - * G^{(3)}$. We also have the condition that the 5-form field strength, $G^{(5)}$, is self-dual, i.e. $G^{(5)} = *G^{(5)}$. This condition is difficult to enforce from the action for type IIB supergravity, and generally it must be included as an additional constraint on solutions of the equations of motion.

In the string frame the action for the bosonic fields in type IIB supergravity is

$$\begin{aligned} S &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\Big[e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}(H^{(3)})^2\right) \\ &\quad -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left((G^{(1)})^2 - \frac{1}{3!}(G^{(3)})^2 - \frac{1}{2\cdot 5!}(G^{(5)})^2\right) \\ &\quad +\frac{1}{4\kappa_{10}^2}\int\left(C^{(4)} + \frac{1}{2}B\wedge C^{(2)}\right)\wedge G^{(3)}\wedge H^{(3)} \end{aligned}$$

where κ_{10} is related to Newton's constant, G_{10}, in 10 dimensions as follows:

$$2\kappa_{10}^2 = 16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8 \tag{1.6}$$

and g_s is set by the asymptotic value of the dilaton at infinity: $g_s = e^{\phi_0}$. The equations of motion for type IIB supergravity can be derived from this action. However, one must always impose the constraint $G^{(5)} = *G^{(5)}$ by hand on any solution of the equations of motion.

Given a bosonic solution of type IIB supergravity (which consists of a metric, dilaton and field strengths $G^{(2i+1)}$ and $H^{(3)}$, which together solve the equations of motion), this solution can have the additional property that it is supersymmetric. Then the solution will possess Killing spinors. Type IIB supergravity has $\mathcal{N} = 2$ supersymmetry and it is a chiral theory. This means that the supersymmetry transformations involve two spinors, ϵ^1 and ϵ^2, which have the same chirality. In particular, ϵ^1 and ϵ^2 are 32-component real spinors which satisfy $\Gamma_{11}\epsilon^i = \epsilon^i$, where Γ_{11} is the chirality matrix, given explicitly by $\Gamma_{11} = \Gamma^{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}\hat{7}\hat{8}\hat{9}}$ (the hats denote tangent space indices). We will often combine the spinors ϵ^1 and ϵ^2 into a 64-component real spinor, $\epsilon = (\epsilon^1, \epsilon^2)^T$. Note that due to the chirality condition on the spinors, each ϵ^i has only 16 non-zero components, and so correspondingly ϵ has 32 non-zero components. In type IIB supergravity, Killing spinors obey two types of equations. These equations arise from requiring that the supersymmetry variation of the gravitino and the axino-dilatino vanishes. The gravitino Killing spinor equation is a differential equation $D_m\epsilon = 0$, $m = 0, 1, \ldots, 9$. The precise form of the differential operator D_m will be given in Chapter 6. The axino-dilatino Killing spinor equation is an algebraic equation, given by $\mathcal{P}\epsilon = 0$. Here $\mathcal{P}$ is a projection matrix which does not involve derivatives but it does contain the background fields. Again, we will defer the precise form of $\mathcal{P}$ to Chapter 6.

As an aside we note that type IIB supergravity has the interesting feature that it is invariant under the group $SL(2,\mathbb{R})$ (See Ref. [5] for the type IIB supergravity action rewritten in a manifestly $SL(2,\mathbb{R})$ invariant fashion.). In the full string theory, this symmetry group is restricted to only involve integers, and the group becomes $SL(2,\mathbb{Z})$. This group is generated by $\tau \to -\tau^{-1}$ and $\tau \to \tau + 1$, where $\tau = C^{(0)} + ie^{-\phi}$. Moreover, the fields $B^{(2)}$ and $C^{(2)}$ transform as a doublet of $SL(2,\mathbb{Z})$. The invariance of the full type IIB string theory under $SL(2,\mathbb{Z})$ is known as S-duality. S-duality also relates type I string theory to $SO(32)$ heterotic string theory.

1.2 Branes in supergravity

As described in the introduction, string theory contains extended objects called D-branes. A useful way to describe these D-branes is to consider them from the point of view of 10-dimensional supergravity theories, since these theories are obtained in the low energy limit of string theory. It turns out that there are also branes in 11-dimensional supergravity. This implies that M-theory also contains branes, although we cannot see this directly from the high energy theory (since we don't have a complete formulation of M-theory). From the supergravity point of view, branes in 10 and 11 dimensions are very similar. In both cases they are p-dimensional extended objects which are charged with respect to a $(p+1)$-form gauge potential. Of course, branes in 11 dimensions cannot correspond to the surfaces where open strings can end (since there are no strings in M-theory), so they are not D-branes. We refer to them as M-branes. We will discuss branes in 11-dimensional supergravity and type IIB supergravity in this section.

The branes we consider will be BPS objects, which means that their tension and charge are equal. It turns out that the BPS condition also means that the branes preserve $\frac{1}{2}$ supersymmetry. Furthermore, these properties imply that branes reside in a short representation of the supersymmetry algebra. This means that many of the properties of these branes should not change as we go to high energies (for example, the tension of the brane and the spectrum of excitations should not change), since objects in short representations of supersymmetry are protected against higher order corrections. In this case the corrections could have a quantum or a stringy/M-theory origin.

In this section we will consider branes in two different ways. The first way will be to consider branes sourcing a supergravity background, i.e. we will discuss the supergravity solutions which arise from branes warping the geometry of space-time around them. The second way will be to consider branes as test objects (i.e. probe branes) in a fixed background. This will involve the world-volume description of branes.

1.2.1 Brane solutions

We begin by describing supergravity solutions corresponding to branes in 11-dimensional supergravity. Recall that in 11-dimensional supergravity the only field strengths are $F^{(4)}$ and its dual $F^{(7)} = *F^{(4)}$. So the only possible branes are an M2-brane and an M5-brane, which are electrically charged with respect to $F^{(4)}$ and $F^{(7)}$. Alternatively, we can think of the M5-brane as being magnetically charged with respect to $F^{(4)}$.

The supergravity solutions corresponding to a stack of N parallel coincident M2- or

M5-branes are given by

$$\begin{aligned} ds^2 &= H^{\frac{p-8}{9}} dx^2_{(1,p)} + H^{\frac{p+1}{9}} dx^2_{(10-p)} \\ F^{(p+2)} &= \mp d(H^{-1}) \wedge \epsilon_{1,p} \\ H &= 1 + \frac{c_p N}{r^{8-p}} \end{aligned} \qquad (1.7)$$

where $p = 2, 5$ and c_p is a constant, whose precise value will not be needed. The $\mp$ in $F^{(p+2)}$ corresponds to whether we are considering branes or anti-branes. Here $dx^2_{(1,p)}$ is the Minkowski metric in $(p+1)$ dimensions, given by

$$dx^2_{(1,p)} = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^p)^2$$

and $\epsilon_{1,p}$ is the volume form on this space. The metric $dx^2_{(10-p)}$ is given by

$$dx^2_{(10-p)} = (dx^{p+1})^2 + \cdots + (dx^\natural)^2 = dr^2 + r^2 d\Omega^2_{(9-p)}$$

where in the second equality we have rewritten the metric in terms of the radial coordinate, r, where

$$r^2 = \sum_{i=p+1}^{\natural} (x^i)^2$$

and $d\Omega^2_{(9-p)}$ is the usual metric on a $(9-p)$-dimensional sphere. It is easy to check that the metric and 4-form field strength in Eq. (1.7) gives a solution of the 11-dimensional supergravity field equations. For example, the equation of motion for $F^{(p+2)}$ is satisfied (with an appropriate source term at $r = 0$) because H is a harmonic function of r. The interpretation of the above solutions is that the N coincident p-branes are situated at $r = 0$ (hence the source term at $r = 0$), and they have world-volume coordinates $x^0, \ldots, x^p$. The coordinate r gives the radial distance away from the branes. Note that the $(p+1)$-dimensional world-volume of the branes is Poincaré invariant. Moreover, the $(10-p)$-dimensional space transverse to the branes has rotational invariance. These are precisely the symmetries one would expect for a stack of coincident p-branes.

In type IIB supergravity there are many more types of branes than in 11 dimensions. This is due to the large number of different gauge potentials in this theory. Firstly, there are Dp-branes which are charged with respect to the Ramond-Ramond potentials and have NS-NS 3-form, $H^{(3)}$, identically zero. Here p can take the values $1, 3, 5, 7, 9$. The supergravity solutions corresponding to a stack of N coincident Dp-branes are given by

$$\begin{aligned} ds^2 &= H^{-1/2}dx^2_{(1,p)} + H^{1/2}dx^2_{(9-p)} \\ G^{(p+2)} &= \mp d(H^{-1}) \wedge \epsilon_{1,p} \\ e^{\phi} &= H^{\frac{3-p}{4}} \\ H &= 1 + \frac{c_p N}{r^{7-p}} \end{aligned} \tag{1.8}$$

Again, the interpretation is that the branes are situated at $r = 0$ and have world-volume coordinates $x^0, \ldots x^p$. The coordinate r gives the radial distance away from the stack of branes. The main difference between these D-branes and the M-branes above is that here the dilaton is excited (apart from the case $p = 3$) , whereas there is no dilaton in 11-dimensional supergravity.

In type IIB supergravity there are also branes which are charged with respect to the NS-NS 3-form $H^{(3)}$. There are two branes of this type, namely a 5-brane – the NS5-brane – and a fundamental string. These are not D-branes, since they do not correspond to surfaces on which open strings can end. We will not be particularly interested in these types of branes in this thesis, so we do not present the solutions here. However, solutions corresponding to stacks of these branes can be found in Ref. [40].

The solutions we have presented here are for simple configurations of coincident parallel branes. However, more complicated configurations of branes are possible, and the corresponding supergravity solutions can be constructed. For example, it is possible to construct orthogonal and non-orthogonal intersecting brane configurations, and configurations of branes ending on other branes. Examples of these types of configurations and the procedures for obtaining the associated supergravity solutions are given in Ref. [40].

The brane solutions given above all preserve $\frac{1}{2}$ supersymmetry, i.e. each solution possesses 16 Killing spinors. We could obtain the Killing spinors for each background by substituting the metric and field strength(s) into the 11-dimensional or type IIB Killing spinor equations. In both cases these equations can be solved and the 16 independent Killing spinors are given by $\epsilon = (-g_{00})^{1/4}\epsilon_0$, where ϵ_0 is a 32-component constant spinor which satisfies the following projection condition[5]: $\Gamma\epsilon_0 = \epsilon_0$, where Γ is a projection matrix which we now define. For the 11-dimensional brane solutions, the projection matrix Γ is given by

$$\Gamma = \pm\Gamma_{\hat{0}\hat{1}\ldots\hat{p}} \tag{1.9}$$

where $\Gamma_{\hat{0}}, \ldots, \Gamma_{\hat{p}}$ are tangent space Γ-matrices in the directions $x^0, \ldots x^p$. The $\pm$ refers whether this is a brane or anti-brane solution (and matches the $\mp$ in the expression for

[5]Since ϵ and ϵ_0 are simply related by a scale, we can consider the projectors acting on either ϵ_0 or ϵ.

$F^{(p+2)}$). For the type IIB solutions, we have the following expression for the projection matrix for a Dp-brane,

$$\Gamma = \pm i(\sigma_3)^{\frac{p+1}{2}}\sigma_2 \otimes \Gamma_{\hat{0}\hat{1}\ldots\hat{p}} \tag{1.10}$$

Here the $\{\sigma_1, \sigma_2, \sigma_3\}$ are the usual 2×2 Pauli matrices. The matrix Γ in Eq. (1.10) is 64-dimensional and acts on the 64-component constant spinor $\epsilon_0 = (\epsilon_0^1, \epsilon_0^2)^T$, where ϵ_0^1 and ϵ_0^2 are 32-component constant spinors with the same chirality, i.e. satsifying $\Gamma_{11}\epsilon_0^i = \epsilon_0^i$.

In both the 10- and 11-dimensional cases there are 16 independent Killing spinors which satisfy the projection condition $\Gamma\epsilon = \epsilon$. This is because for each allowed choice of p, the matrix Γ squares to the identity. Moreover, Γ is symmetric and traceless. Therefore, this matrix can be diagonalized as $diag(+1, \ldots, +1, -1, \ldots, -1)$ where there are equal numbers of $+1$ and -1. In the 11-dimensional case, Γ is 32-dimensional, and so there are 16 independent ϵ_0 which satisfy the projection condition. In the 10-dimensional case, Γ is 64-dimensional, which naively would suggest that there are 32 Killing spinors. However, since the spinors obey the chirality projection condition $\Gamma_{11}\epsilon_0^i = \epsilon_0^i$, which commutes with the matrix Γ, the number of Killing spinors is reduced by a factor of $\frac{1}{2}$. Therefore, the 10-dimensional brane backgrounds also possess 16 Killing spinors. Thus all the brane backgrounds given above preserve $\frac{1}{2}$ supersymmetry, i.e. 16 out of a possible 32 supersymmetries.

So far we have only considered supergravity solutions corresponding to brane backgrounds. However, it is generally true that the Killing spinors for any supersymmetric background can be expressed in terms of a set of projection conditions together with a scale for ϵ. The number of independent projection conditions corresponds to the number of Killing spinors the solution admits (and hence the number of preserved supersymmetries). In particular, each successive projection condition reduces the number of Killing spinors by a factor of $\frac{1}{2}$.

1.2.2 Probe calculations

Given a supergravity background one is often interested in finding energy minimising embeddings of branes. One way of doing this is to perform a probe calculation. The idea is to place a "test" brane in a fixed supergravity background and then to examine the dynamics of this brane. The aim is to minimize the brane's energy. If the energy can be minimized, then the probe brane can exist in this supergravity background. Note that in probe calculations we always make the approximation that the back reaction of the probe brane on the supergravity background is negligible. This is an approximation to the real physics, but in many situations it is well justified, since the back reaction can be shown to be small. The key ingredient used in a probe calculation is the brane's world-volume action, which we now introduce.

The minimal action for a p-brane without dilatonic coupling is given by

$$S = -T_p \left\{ \int d^{p+1}\sigma \sqrt{-\gamma} - \int \mathcal{P}(A^{(p+1)}) \right\} \tag{1.11}$$

where T_p is the tension of the brane[6] and γ is the determinant of the induced $(p+1)$-dimensional metric on the brane world-volume, whose components are given by

$$\gamma_{ab} = g_{mn} \frac{dx^m}{d\sigma^a} \frac{dx^n}{d\sigma^b} \tag{1.12}$$

Here g_{mn} are the components of the metric for the background supergravity solution and σ^a, $a = 0, 1, \ldots, p$, are coordinates on the world-volume of the brane. The term involving $\mathcal{P}(A^{(p+1)})$ in the action is the called the Wess-Zumino term. Here $A^{(p+1)}$ is a $(p+1)$-form gauge potential for the background supergravity solution and the quantity $\mathcal{P}(A^{(p+1)})$ is the pull-back of this potential to the brane world-volume, i.e.

$$\mathcal{P}(A^{(p+1)}) = \frac{1}{(p+1)!} \partial_{a_0} x^{m_0} \ldots \partial_{a_p} x^{m_p} A_{m_0 \ldots m_p} \, d\sigma^{a_0} \wedge \cdots \wedge d\sigma^{a_p}$$

In a probe calculation the first step is to compute the action Eq. (1.11). We then compute the Hamiltonian for the probe brane, and attempt to minimize its energy. If the energy can be minimized, then typically the position of the brane at this minimum will be specified in some of the directions transverse to the brane world-volume. However, often there will be freedom in the brane's position in the other transverse directions. We will see many examples of probe calculations in Chapters 2 and 3.

In fact, we can also consider supersymmetry for branes from this world-volume perspective. The idea is to consider embedding the probe brane in superspace. In the cases we are interested in (i.e. branes in type IIB or 11-dimensional supergravity), superspace is parameterized by 10 or 11 bosonic coordinates, x^m, and 32 fermionic coordinates, θ^α. Embedding the brane in superspace results in a supersymmetric action for the probe brane, i.e. the probe action above, Eq. (1.11), is augmented to include fermions in a supersymmetric way. For world-volume supersymmetry one must also have equal numbers of on-shell bosonic and fermionic degrees of freedom. This requires a fermionic symmetry on the brane world-volume called κ-symmetry which projects out half of the fermionic degrees of freedom. The κ-symmetry transformations are very similar to the supersymmetry transformations on the brane. If we fix the κ-symmetry (i.e. we choose which

[6]For a D-brane, T_p can be calculated from a 1-loop open string amplitude, which gives $T_p = (2\pi)^{-p} g_s^{-1} l_s^{-p-1}$. For an M-brane, $T_p = (2\pi)^{-p} l_p^{-p-1}$ where l_p is the Planck length.

components of the fermionic fields to project out) then the condition for preservation of world-volume supersymmetry is

$$\Gamma\epsilon = \epsilon \tag{1.13}$$

where the matrix Γ is given in Eqs. (1.9) and (1.10) for M- and D-brane probes aligned along the $x^0, \ldots, x^p$ directions. More generally, we can write the projection matrix for a p-brane in terms of the matrix $\Sigma_{(p+1)}$ where

$$\Sigma_{(p+1)} = \frac{1}{(p+1)!\sqrt{-\gamma}}\, \epsilon^{a_0 \ldots a_p} \partial_{a_0} x^{m_0} \ldots \partial_{a_p} x^{m_p} \Gamma_{m_0 \ldots m_p} \tag{1.14}$$

and $\epsilon^{012\ldots} = -1$ in our conventions (see Appendix A). Then for p-branes in 11 dimensions, $\Gamma = \pm\Sigma_{(p+1)}$, and for D$p$-branes in type IIB supergravity

$$\Gamma = \pm i (\sigma_3)^{\frac{p+1}{2}} \sigma_2 \otimes \Sigma_{(p+1)}$$

i.e. in each case we have replaced $\Gamma_{\hat{0}\ldots\hat{p}}$ by the more general expression $\Sigma_{(p+1)}$. Note that the supersymmetry projection condition, Eq. (1.13), is a local condition, i.e. it must be satisfied at each point on the world-volume of the probe brane. In general, the projection condition is different at each point on the world-volume, and so typically all supersymmetry is broken. Of course in special cases the conditions are the same at each point and a non-zero fraction of supersymmetry is preserved.

Now that we have the supersymmetry projection conditions for a probe brane, it is not always necessary to perform a probe calculation to establish whether a probe brane can be embedded in a supersymmetric background. In particular, if one can embed a probe brane so that some of the supersymmetry of the background is still retained, then this brane will not experience any force from the background, and it will be an energy minimizing configuration. The simplest example of this comes from considering a background generated by N coincident branes. If we add a probe brane of the same type and orientation as the background branes then this probe will not experience any force from the other branes. This can be seen from a probe calculation, but it is implied because both the background branes and the probe brane have the same supersymmetry projection condition, and so the whole configuration preserves $\frac{1}{2}$ supersymmetry. The probe brane can be placed anywhere in the transverse space parallel to the background branes. In fact, we could also add a probe brane which is not parallel to the background branes, provided that some of the supersymmetry is still retained. Supersymmetric embeddings of branes will be discussed further in Chapter 4.

In fact the action given in Eq. (1.11) is not the most general action for a probe brane. We can also allow non-zero gauge fields on the world-volume of Dp-branes and the M5-

brane. For a Dp-brane in type IIB supergravity the most general bosonic action we consider[7] is given by

$$S = -T_p \left\{ \int d^{p+1}\sigma e^{-\phi} \sqrt{-\det(\gamma_{ab} + \mathcal{F}_{ab})} - \int \sum_n \mathcal{P}(C^{(n)}) \wedge e^{\mathcal{F}} \right\}$$

Here $\mathcal{F} = 2\pi l_s^2 F - \mathcal{P}(B)$, where B is the space-time NS-NS 2-form potential and F is a 2-form field strength associated to a U(1) gauge potential on the world-volume of the brane. In the second term, n can take the values $0, 2, 4, 6, 8$, but the integral is restricted to only include $(p + 1)$-forms. This means that the Dp-brane can couple to other Ramond-Ramond potentials, as well as the usual coupling to $C^{(p+1)}$. We can also have world-volume gauge fields on the M5-brane. However, the form of this action is quite complicated, as it involves a 2-form gauge potential whose associated 3-form field strength satisfies a non-linear self-duality condition. The full covariant form of this action can be found in Ref. [41].

We now show that at low energies, the world-volume action for a D-brane reduces precisely to a supersymmetric Yang-Mills action. To see this we consider the simple case of a D3-brane probe in Minkowski space. Since in flat space all background field strengths are zero, the action for this brane is simply

$$S = -T_3 \int d^4\sigma \sqrt{-\det(\gamma_{ab} + 2\pi l_s^2 F_{ab})} \tag{1.15}$$

where $\gamma_{ab} = \partial_a x^m \partial_b x^n \eta_{mn}$, and η_{mn} are the components of the flat metric. We now fix the reparameterization invariance of the world-volume by setting $\sigma^0 = x^0, \sigma^1 = x^1, \sigma^2 = x^2$ and $\sigma^3 = x^3$. Then the directions transverse to the brane are $x^4, \ldots x^9$, and we will interpret these coordinates as scalar fields in the world-volume action. We write these 6 coordinates more suggestively as

$$\Phi^i(\sigma^a) = \frac{1}{2\pi l_s^2}\, x^i(\sigma^a) \tag{1.16}$$

where $i = 4, \ldots, 9$. We then expand the determinant in Eq. (1.15) around $l_s = 0$. Taking $l_s \to 0$ corresponds to taking the low energy limit and it serves to decouple gravitational interactions from the brane world-volume theory. The leading order term in this expansion is

$$S_0 = -4\pi^2 l_s^4 T_3 \int d^4\sigma \left(\frac{1}{2}\partial_a \Phi^i \partial^a \Phi^i + \frac{1}{4} F_{ab} F^{ab} \right) \tag{1.17}$$

This is simply the bosonic part of the maximally supersymmetric Yang-Mills action in 4

[7] One could also include curvature corrections to the action, but we will not consider this possibility.

dimensions with gauge group $U(1)$. Notice that the gauge coupling here is

$$g_{YM}^2 = \frac{1}{4\pi^2 l_s^4 T_3}$$

Using the formula for the tension of a Dp-brane, $T_p = (2\pi)^{-p} g_s^{-1} l_s^{-p-1}$, this gives $g_{YM}^2 = 2\pi g_s$.

Now if we consider N coincident D3-branes then the world-volume action should be a non-Abelian version of the action in Eq. (1.15). The precise form of this non-Abelian action is not known. However, it is known that in the limit $l_s \to 0$, the dynamics is described by a maximally supersymmetric Yang-Mills theory with gauge group $U(N)$. In fact, one factor of $U(1)$ simply describes the centre of mass motion of the branes, and it decouples, so effectively the gauge group is $SU(N)$.

1.3 The AdS/CFT correspondence

In this section we describe an interesting duality between string theory and gauge theories – namely the AdS/CFT correspondence. This correspondence was proposed by Maldacena [24], and the precise details were elucidated by Witten [42], and Gubser, Klebanov and Polyakov [43]. Note that while this duality has not been proved, it has passed many tests, and there is a large body of evidence (see Ref. [44] and the references within it) which supports this conjecture. We will consider one example of the correspondence which involves N coincident D3-branes. This is the best understood example of the AdS/CFT correspondence.

Recall from the previous section that the low energy world-volume theory on N coincident D3-branes is maximally supersymmetric (i.e. $\mathcal{N} = 4$) Yang-Mills theory in 4 dimensions with gauge group $SU(N)$. In fact this theory is actually a conformal field theory, i.e. it is invariant under conformal transformations, hence the "CFT" in the name of this conjecture. We now give some motivation for why this theory is "dual" to type IIB supergravity on the space $AdS_5 \times S^5$. (In fact in its strongest form, the AdS/CFT conjecture says that this gauge theory is equivalent to the full type IIB string theory on $AdS_5 \times S^5$. However, we will only consider the supergravity limit of the conjecture.) To do this we will use the supergravity solution which is sourced by the stack of N coincident D3-branes. We then take the limit $l_s \to 0$ in an appropriate way so that we consider the same limit as for the gauge theory. On the supergravity side this amounts to considering the near-horizon limit of the solution (i.e. the region close to the D-branes) and scaling this region up.

From Eq. (1.8), the metric around a stack of N coincident D3-branes is given by

$$ds^2 = H^{-1/2}(-(dx^0)^2 + (dx^1)^2 + \dots (dx^3)^2) + H^{1/2}(dr^2 + r^2 d\Omega_5^2)$$

where

$$H = 1 + \frac{c_3 N}{r^4}$$

and the constant $c_3 = 4\pi g_s l_s^4$. We want to take the limit $l_s \longrightarrow 0$, but in such a way that the Yang-Mills coupling $g_{YM} = \sqrt{2\pi g_s}$ remains fixed and the gauge theory masses and vacuum expectation values remain fixed. The correct quantity to keep fixed is $U = r/l_s^2$, while we take the limit $l_s \longrightarrow 0$ (therefore we are also taking $r \longrightarrow 0$ at the same time, which means we are considering the near-horizon limit). To see that we should keep U fixed, consider separating one of the branes a distance r from the stack. Then we obtain massive states in the gauge theory from strings stretched between the separated brane and the stack. These states are W-bosons in the gauge theory[8], and their mass is given by the string tension multiplied by the separation, i.e. $m \sim r/l_s^2$. Therefore, to keep the W-boson masses constant we must keep $U = r/l_s^2$ fixed. In terms of U, the function H is given by

$$H = 1 + \frac{4\pi g_s N}{U^4 l_s^4}$$

Now if we take the limit $l_s \longrightarrow 0$,

$$H \longrightarrow \frac{4\pi g_s N}{U^4 l_s^4}$$

In this limit the metric becomes

$$\frac{1}{l_s^2} ds^2 = \frac{U^2}{L^2} ds^2_{(1,3)} + L^2 \frac{dU^2}{U^2} + L^2 d\Omega_5^2 \tag{1.18}$$

where $L^2 \equiv \sqrt{4\pi g_s N} = \sqrt{2 g_{YM}^2 N}$. This is the metric for $AdS_5 \times S^5$ where the radius of both AdS_5 and S^5 is L. The supergravity description of this background is valid as long as the curvatures are small. We can ensure that the curvatures are small by making L large. This amounts to taking the effective 't Hooft coupling, $\lambda \equiv g_{YM}^2 N$, large (In fact, we must also have N large so that we can ignore stringy corrections, i.e. we want $g_s N$ large but g_s small, which means N is large.). By contrast, the gauge theory description is valid when the 't Hooft coupling is small. Therefore, we have a duality between the two theories, with non-overlapping regions of validity. Effectively, this means that we can make predictions about the non-perturbative nature of the gauge theory by considering the supergravity description in its perturbative regime.

[8]In fact, separating one of the branes from the stack corresponds to Higgsing part of the gauge group

In practice, we must have a concrete way of associating quantities in the gauge theory with objects in the supergravity theory. The prescription [42, 43] is that operators in the gauge theory are associated to states in the supergravity theory, and the association takes place on the boundary[9] of AdS_5. Note that the boundary of AdS_5 is a 4-dimensional Minkowksi space, and it is interpreted as the space on which the gauge theory lives. To associate a field in the supergravity theory with an operator on the boundary, the requirement is that this field couples to the operator in a way that respects the symmetries of the problem. This is the basis for all state/operator associations in the AdS/CFT correspondence. We will see some examples of this association in Chapter 2 when we consider giant gravitons.

1.4 Outline of thesis

We begin this thesis by considering a particular class of branes, known as giant gravitons, which exist in $AdS \times S$ backgrounds. Giant gravitons have a spherical topology and they are expanded within the spherical part of the space-time. From our point of view these branes are interesting as they are non-static. In particular, they must move to couple to the background flux which prevents them from collapsing. Historically, the interest in giant gravitons arose from the AdS/CFT correspondence. This is because these branes play an important role in resolving a paradox within this correspondence known as the "stringy exclusion principle". We will introduce giant gravitons and the stringy exclusion principle in Chapter 2. In this chapter we will also discuss dual giant gravitons, which are spherical branes similar to giant gravitons, but they are expanded within the AdS part of the space-time. Then we will describe the conformal field theory interpretation of giants and dual giants, and the resolution of the stringy exclusion principle. In Chapter 3 we continue discussing giant gravitons. However, we will consider these branes in more general backgrounds than $AdS \times S$. In particular, we consider backgrounds that are lifts of 4-dimensional $U(1)^4$ and 7-dimensional $U(1)^2$ gauged supergravity theories. Using probe calculations we show that giant gravitons exist in all these lifted backgrounds, which typically are not supersymmetric. Moreover, these branes are equivalent to massive charged particles probing the lower-dimensional gauged supergravity backgrounds.

In Chapter 4 we discuss a geometrical technique for finding energy minimizing embeddings of branes. This is the method of calibrations. This method involves a calibration form, which is a p-form, ϕ, which satisfies some special conditions. We can use ϕ to find surfaces which have minimal volume in their homology class. These surfaces are called

[9] We always mean the conformal boundary of AdS.

"calibrated". Moreover, these surfaces correspond (at least in some cases) to minimal energy cycles for branes to wrap. We will see that in supersymmetric backgrounds it is always possible to construct a form which has the right properties to be a calibration. Therefore, in supersymmetric backgrounds we can always use calibrations to find minimal energy surfaces for branes to wrap.

In Chapter 5 we consider the underlying structure of supersymmetric backgrounds by considering their superalgebras. A superalgebra comprises a set of commutators and anti-commutators of operators that generate the symmetries of the background. We will be interested in the modifications to superalgebras that arise from placing probe branes in supersymmetric backgrounds. It is well known that the super-translation part of the algebra is modified when one includes a probe brane in the background. However, the exact form of this modification is known only for some specific classes of backgrounds. Our result is to find the precise modification to the super-translation algebra for a probe brane in a general 11-dimensional supersymmetric background. The technique we use is to construct a set of differential forms of different degrees from the Killing spinors of the background. These forms obey a set of differential equations which can be manipulated to construct a closed 2-form and a closed 5-form. We argue that these closed forms are the topological charges which appear in the super-translation algebra for a probe M2- and M5-brane in a general supersymmetric background.

Using the form of the superalgebra derived in § 5.2 we can then derive a BPS bound (or "calibration bound") on the energy/momentum of the probe brane. For each type of brane, the BPS bound involves some calibrating form(s). Therefore, the super-translation algebra allows us to derive the correct calibrating form(s) for the particular probe brane under consideration. Moreover, the BPS bound tells us what quantity a calibrated brane will minimize.

In Chapters 6 and 7 we work towards formulating a calibration bound for giant gravitons in $AdS_5 \times S^5$. This is an interesting problem to work on as most previous work on calibrations has focussed on static brane configurations. Our construction involves a number of steps. Firstly, in Chapter 6 we construct a set of p-forms of different degrees from a Killing spinor of type IIB supergravity. From the Killing spinor equations and Fierz identities we derive a number of differential and algebraic identities satisfied by these forms. In Chapter 7 we describe an interesting construction of giant gravitons [45] via holomorphic surfaces in $\mathbb{C}^{1,2} \times \mathbb{C}^3$, which is a (complex) embedding space for $AdS_5 \times S^5$. These giant gravitons are more general than the original example considered in Chapter 2. Moreover, the construction via holomorphic surfaces makes the supersymmetry projection conditions for these branes very simple. Our aim is to show that these general giant gravitons are calibrated. To do this we manipulate the differential equations derived in Chapter 6 to find the closed form which appears as a topological charge in the super-translation algebra for a giant graviton. We then use this algebra to find the calibration bound satisfied by these branes. We then show that this bound is saturated by holomorphic giant gravitons. Interestingly, we find that the dual giant graviton introduced in Chapter 2 also saturates this bound. Moreover, we find that both the holomorphic giants and dual giants minimize "energy minus momentum" in their homology class.

Chapter 2

Giant gravitons

In this chapter we introduce giant gravitons. Giant gravitons are spherical branes which exist in $AdS \times S$ spaces. These branes were first discovered by McGreevy, Susskind and Toumbas [46] as a solution to the following paradox in the AdS/CFT correspondence. In $\mathcal{N} = 4$ SU(N) super-Yang Mills theory there exists a family of chiral primary operators consisting of single traces of scalar fields. These operators are dual to massless single particle states carrying momentum on the spherical part of $AdS_5 \times S^5$. The important feature of these operators is that they have bounded R-charge. This implies a cutoff on the momentum of the massless states propagating on the S^5. However, it is difficult to understand how this cutoff on the momentum arises. Naively, one would expect the momentum of the particle to be allowed to get arbitrarily large. This paradox is referred to as the "stringy exclusion principle".

The idea of McGreevy, Susskind and Toumbas was to find an alternative description of the massless particles in terms of spherical branes. These spherical branes, or giant gravitons, are expanded D3-branes which wrap an S^3 within the S^5 part of the geometry. They carry angular momentum on the sphere and are degenerate to the massless single particle states. The important feature of giant gravitons is that their radius grows with increasing angular momentum. However, since the giant graviton radius cannot exceed the radius of the S^5, there is a natural cutoff on the angular momentum of these states. Moreover, the cutoff obtained from the giant graviton description precisely matches the bound on the R-charge of the single trace operators[1].

Since the original work on giant gravitons, there has been much interest in these objects. For example, more general giant graviton configurations have been found in a variety of supergravity backgrounds, such as the pp-wave background [48–52]. Giant

[1]Later it was realized that the single trace operator description of giant gravitons was not quite correct, but rather the dual operators are a family of sub-determinants and determinants of scalar fields [47]. These operators have the same cut-off on their R-charge. We will discuss these operators in § 2.3.

gravitons have also been found in backgrounds which are not supersymmetric [1, 53, 54]. Progress has also been made on understanding the microscopical description of giant gravitons in $AdS \times S$ backgrounds [55–57]. Furthermore, it has been conjectured [58] that distributions of giant gravitons might source the naked singularities in superstar solutions.

In this chapter we will describe simple D3-brane giant gravitons in $AdS_5 \times S^5$. To begin, we will perform a probe calculation to show that such objects exist, they are degenerate to massless particles, and their radius grows with increasing angular momentum. In § 2.2 we will describe a related brane known as a dual giant graviton. This is a D3-brane which is expanded within the AdS part of the space-time. Then in § 2.3 we will discuss the field theory interpretation of giant gravitons and dual giant gravitons.

2.1 The simple giant graviton

We consider an example of a giant graviton in $AdS_5 \times S^5$. In this case the giant graviton is a D3-brane, which wraps an S^3 within the S^5 and carries angular momentum on S^5. We begin by describing the $AdS_5 \times S^5$ solution of type IIB supergravity.

The metric on this space-time is given by $ds^2 = ds^2_{AdS} + ds^2_S$, where[2]

$$ds^2_{AdS} = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_3^2 \tag{2.1}$$

where L is the radius of curvature of AdS_5 (and S^5) and $d\Omega_3^2$ is the usual metric on a unit 3-sphere:

$$d\Omega_3^2 = d\alpha_1^2 + \sin^2 \alpha_1 d\alpha_2^2 + \sin^2 \alpha_1 \sin^2 \alpha_2 d\alpha_3^2$$

where $0 \leq \alpha_1, \alpha_2 \leq \pi$ and $0 \leq \alpha_3 \leq 2\pi$. The metric on S^5, which also has radius L, is

$$ds^2_S = L^2 \sum_{i=1}^{3} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) \tag{2.2}$$

where $\mu_i \geq 0$ satisfy $\sum_{i=1}^{3} \mu_i^2 = 1$ and $0 \leq \phi_i \leq 2\pi$. We can parameterize the μ_i by

$$\mu_1 = \cos\theta_1, \quad \mu_2 = \sin\theta_1 \cos\theta_2, \quad \mu_3 = \sin\theta_1 \sin\theta_2 \tag{2.3}$$

where $0 \leq \theta_i \leq \pi/2$. In these coordinates the metric on the sphere becomes

$$ds^2_S = L^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cos^2\theta_1 d\phi_1^2 + \sin^2\theta_1 \cos^2\theta_2 d\phi_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\phi_3^2\right) \tag{2.4}$$

[2]Here we write the AdS metric in different coordinates to the version in Eq. (1.18) of Chapter 1.

The only non-zero field strength for this solution is the 5-form, $G^{(5)}$, which is given by

$$G^{(5)} = -\frac{4}{L}\left(vol(S^5) + vol(AdS_5)\right) \tag{2.5}$$

where $vol(S^5)$ and $vol(AdS_5)$ are volume forms on the 5-dimensional spaces, given explicitly by

$$vol(S^5) = L^5 \sin^3\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \; d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \tag{2.6}$$

$$vol(AdS_5) = r^3 \sin^2\alpha_1 \sin\alpha_2 \; dt \wedge dr \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \tag{2.7}$$

The 5-form field strength, $G^{(5)}$, is related to a 4-form gauge potential, $C^{(4)}$, by $G^{(5)} = dC^{(4)}$ (this agrees with the definitions in Chapter 1 since $H^{(3)}$ is zero for this background). A giant graviton probe in $AdS_5 \times S^5$ will couple to $C^{(4)}$ to prevent it collapsing under gravity. Now, $G^{(5)}$ is closed, but not exact. Therefore, $C^{(4)}$ can only be determined locally. We will be interested in giant gravitons at fixed θ_1, and we will assume that $\cos\theta_1 \neq 0$ (i.e. $\theta_1 \neq \pi/2$). Therefore, we integrate $G^{(5)}$ with respect to θ_1 to obtain the following component of $C^{(4)}$ that couples to the giant graviton world-volume,

$$C^{(4)} = -L^4 \sin^4\theta_1 \sin\theta_2 \cos\theta_2 d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 + \ldots \tag{2.8}$$

Note that because the giant graviton is expanded in the S^5 part of the geometry, the AdS piece of $C^{(4)}$ will not contribute to the brane action.

The giant graviton we consider is a D3-brane which wraps a 3-sphere parameterized by θ_2, ϕ_2, ϕ_3 at fixed $\theta_1 \neq \pi/2$. We take the brane to move rigidly in the direction ϕ_1. For simplicity, we consider the giant graviton to be at $r = 0$ in the AdS space (although the brane could travel along any time-like trajectory in AdS_5, see chapter 3 for details). The action for this probe brane is given by

$$S = -T_3 \left\{ \int dt d\theta_2 d\phi_2 d\phi_3 \sqrt{-\gamma} - \int \mathcal{P}(C^{(4)}) \right\} \tag{2.9}$$

where T_3 is the D3-brane tension. Here γ is the determinant of the induced metric and the second term is the pull-back of the 4-form potential to the brane world-volume. The induced metric is obtained by pulling back the $AdS_5 \times S^5$ metric to the brane world-volume using the formula in Eq. (1.12). We obtain

$$ds^2_{g.g.} = \left(-1 + L^2 \cos^2\theta_1 \dot{\phi}_1^2\right) dt^2 + L^2 \sin^2\theta_1 \left(d\theta_2^2 + \cos^2\theta_2 d\phi_2^2 + \sin^2\theta_2 d\phi_3^2\right) \tag{2.10}$$

From this metric it is clear that the radius of the S^3 wrapped by the giant graviton is

$r_{g.g.} = L\sin\theta_1$. The determinant of this metric is

$$\gamma = -L^6 \sin^6\theta_1 \sin^2\theta_2 \cos^2\theta_2 \ (1 - L^2\cos^2\theta_1\dot{\phi}_1^2) \tag{2.11}$$

Moreover, from Eq. (2.8), the Wess-Zumino term in the action is

$$\mathcal{P}(C^{(4)}) = \dot{\phi}_1 L^4 \sin^4\theta_1 \sin\theta_2\cos\theta_2 \ dt\wedge d\theta_2 \wedge d\phi_2 \wedge d\phi_3 \tag{2.12}$$

Therefore, we obtain the following Lagrangian for the probe brane,

$$\mathcal{L} = -A\ \sqrt{1 - L^2\cos^2\theta_1\dot{\phi}_1^2} + AL\sin\theta_1\dot{\phi}_1 \tag{2.13}$$

where $A \equiv T_3 L^3 \sin^3\theta_1 \sin\theta_2\cos\theta_2$. The momentum conjugate to ϕ_1 is

$$P_{\phi_1} = \frac{AL^2\cos^2\theta_1\dot{\phi}_1}{\sqrt{1 - L^2\cos^2\theta_1\dot{\phi}_1^2}} + AL\sin\theta_1 \tag{2.14}$$

This is a conserved quantity since the Lagrangian does not contain any explicit dependence on ϕ_1. Using this expression for P_{ϕ_1}, we obtain the following Hamiltonian,

$$\mathcal{H} = P_{\phi_1}\dot{\phi}_1 - \mathcal{L} \ = \ \sqrt{\frac{(P_{\phi_1} - AL\sin\theta_1)^2}{L^2\cos^2\theta_1} + A^2}$$

We can re-write this Hamiltonian using trigonometric identities as

$$\mathcal{H} = \frac{1}{L}\sqrt{P_{\phi_1}^2 + \tan^2\theta_1\left(P_{\phi_1} - \frac{N}{V_3}\sin^2\theta_1\sin\theta_2\cos\theta_2\right)^2} \tag{2.15}$$

where we have substituted $N = V_3 T_3 L^4$, where $V_3 = 2\pi^2$ is the surface area of a unit S^3. Note that N is actually an integer corresponding to the quantized flux of $G^{(5)}$ through the S^5. We now want to minimise the energy with respect to θ_1. Since $\mathcal{H}$ involves a sum of squares, the minimisation is straight-forward. The first minimum occurs when

$$P_{\phi_1} = \frac{1}{V_3}\ P_1 \sin\theta_2\cos\theta_2$$

where $P_1 = N\sin^2\theta_1$ is constant. This minimum corresponds to the expanded brane solution (i.e. the giant graviton solution). Note that this definition of P_1 means that the total momentum, $\int d\theta_2 d\phi_2 d\phi_3\ P_{\phi_1}$, is simply equal to P_1 for this configuration. We can also calculate the energy at this minimum as follows,

$$E = \int d\theta_2 d\phi_2 d\phi_3 \; \mathcal{H} = \int \frac{P_{\phi_1}}{L} = \frac{P_1}{L}$$

i.e. this is a BPS minimum. Moreover, if we compare the expression for P_1 with the radius of the giant graviton, $r_{g.g.} = L \sin\theta_1$, we find

$$P_1 = \frac{N}{L^2} r_{g.g.}^2$$

This means that the radius of the wrapped sphere grows with increasing angular momentum. Since $r_{g.g.} \leq L$, the angular momentum of the giant graviton is bounded, $P_1 \leq N$. Moreover, the maximum value of angular momentum, N, occurs when the brane has maximal radius, $r_{g.g.} = L$.

There is also another minimum of $\mathcal{H}$ at $\theta_1 = 0$. This corresponds to the massless particle solution because the radius ($L \sin\theta_1$) is zero at this point. The massless particle carries angular momentum on the S^5 and is a BPS object, with energy $E = \frac{P_1}{L}$. Therefore, both the giant graviton and the massless particle have the same energy. For large angular momentum (of order N), the giant graviton description is more reliable than the point-like particle description. This is because the particle has a huge energy concentrated at a point ($N^{3/4}$) and is subject to very large quantum corrections (see Ref. [46] for an estimate of these corrections). By contrast, the giant graviton has the same energy spread out over the surface of the brane. In Ref. [46] the authors suggested that the singular point-like solution is resolved by blowing up into a brane as the momentum on the sphere increases. Roughly speaking, the massless particle becomes less and less point-like as its momentum increases. Moreover, there is no change in energy as this change occurs, as both objects are BPS. This is analogous to the Myers' effect [59] in type IIA supergravity where a system of D0-branes blows up into a 2-sphere in the presence of a 4-form flux. In the Myers' effect this spherical configuration is interpreted as a bound D0-D2-brane state.

In Ref. [60] it was shown that the massless particle and the giant graviton preserve the same supersymmetries. This gives further support to the idea that one should associate giant gravitons with massless particles. We now see that there is a further D3-brane configuration, degenerate to both the massless particle and the giant graviton, which is expanded in the AdS_5 part of the space-time. This configuration also preserves the same supersymmetries [60].

2.2 Dual giant gravitons

We have seen that giant gravitons carrying the same quantum numbers as massless single particles exist in $AdS_5 \times S^5$. It was noticed in Refs. [60, 61] that one can also consider brane expansion in the AdS_5 part of the space-time. In particular, there exist D3-branes which wrap a 3-sphere in the AdS_5 part of the geometry and carry angular momentum on the S^5. These branes are known as dual giant gravitons, and they carry the same quantum numbers as ordinary giant gravitons and massless particles. To see that these objects exist we perform a probe calculation.

Recall from Eq. (2.1) that the metric on AdS_5 is given by,

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2(d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3^2) \qquad (2.16)$$

We consider a D3-brane which wraps a 3-sphere parameterised by $\alpha_1, \alpha_2, \alpha_3$ at fixed r. We take the motion of the brane on S^5 to be in the direction ϕ_1 at fixed $\theta_1 = 0$. The action for this probe brane is

$$S = -T_3\left\{\int dt d\alpha_1 d\alpha_2 d\alpha_3 \sqrt{-\gamma} - \int \mathcal{P}(C^{(4)})\right\}$$

where T_3 is the D3-brane tension, γ is the determinant of the induced metric and the second term is the pull-back of the 4-form potential to the brane world-volume. This term prevents the brane collapsing under gravity even though it wraps a topologically trivial cycle (an S^3). The induced metric on the dual giant world-volume is

$$ds^2 = \left(-1-\frac{r^2}{L^2} + L^2\dot{\phi}_1^2\right) dt^2 + r^2(d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3^2) \qquad (2.17)$$

where we have pulled back the $AdS_5 \times S^5$ metric to the brane world-volume. From this metric the radius of the dual giant is r. We now compute the first term in the action,

$$\sqrt{-\gamma} = r^3 \sin^2\alpha_1 \sin\alpha_2 \sqrt{1+\frac{r^2}{L^2} - L^2\dot{\phi}_1^2}$$

For the second term we note that the relevant part of the 5-form field strength, $G^{(5)}$, is

$$-\frac{4}{L} vol(AdS_5) = -\frac{4}{L} r^3 \sin^2\alpha_1 \sin\alpha_2 dt \wedge dr \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \qquad (2.18)$$

Since the brane has fixed r, we integrate this term with respect to r to obtain the following term which couples to the brane,

$$C^{(4)} = \frac{1}{L} r^4 \sin^2 \alpha_1 \sin \alpha_2 dt \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \tag{2.19}$$

Hence, we obtain the following Lagrangian for the dual giant graviton,

$$\mathcal{L} = A \left(-r^3 \sqrt{1 + \frac{r^2}{L^2} - L^2 \dot{\phi}_1^2} + \frac{r^4}{L} \right) \tag{2.20}$$

where $A = T_3 \sin^2 \alpha_1 \sin \alpha_2$. The momentum conjugate to ϕ_1 is thus

$$P_{\phi_1} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \frac{A r^3 L^2 \dot{\phi}_1}{\sqrt{1 + \frac{r^2}{L^2} - L^2 \dot{\phi}_1^2}}$$

This is a conserved quantity since the Lagrangian contains no explicit dependence on ϕ_1. We can use this to calculate the Hamiltonian. After some algebra we obtain,

$$\mathcal{H} = P_{\phi_1} \dot{\phi}_1 - \mathcal{L} = \frac{N}{L} \left(\sqrt{\left(1 + \frac{r^2}{L^2}\right) \left(\frac{P_{\phi_1}^2}{N^2} + \frac{r^6}{L^6} \right)} - \frac{r^4}{L^4} \right) \tag{2.21}$$

where we have integrated over the compact brane coordinates, α_1, α_2 and α_3, and $N = V_3 T_3 L^4$ as before. We now want to minimise the energy with respect to r. Solving $\partial \mathcal{H} / \partial r = 0$ we find two minima,

$$r = 0, \qquad P_{\phi_1} = \frac{N r^2}{L^2} \tag{2.22}$$

The energy at both critical points is $E = P_1 / L$, where $P_1 = \int P_{\phi_1}$ is the total momentum. The first minimum corresponds to the point-like particle. The second minimum corresponds to an expanded brane configuration, i.e. the dual giant graviton. From Eq. (2.22), clearly the radius of the dual giant, r, grows with increasing momentum. However, here there is no bound on the angular momentum since the range of r is $0 < r < \infty$. So the dual giant graviton is another configuration which is degenerate to the massless particle, but it does not have the correct properties to solve the stringy exclusion principle as its angular momentum is not bounded. In Ref. [60] it was shown that the dual giant graviton preserves exactly the same supersymmetries as the massless particle and the giant graviton. Therefore, there are two expanded brane configurations which are degenerate to the massless particle and preserve the same supersymmetries. We will discuss the field theory interpretation of all of these objects in the next section.

Giant gravitons and dual giants degenerate to massless particles are also found in

$AdS_4 \times S^7$ and $AdS_7 \times S^4$. In $AdS_4 \times S^7$ the giant graviton is an M5-brane which wraps an S^5 within the S^7. The dual giant graviton is an M2-brane which wraps an S^2 within the AdS_4 part of the space-time. In $AdS_7 \times S^4$ the situation is reversed and the giant graviton is a spherical M2-brane in S^4 and the dual giant is a spherical M5-brane in AdS_7. In both space-times the probe calculations for these branes are analogous to the calculations presented in § 2.1–2.2, and the results are qualitatively very similar. In this chapter, however, we will focus only on the $AdS_5 \times S^5$ case because the dual conformal field theory for this this space-time is much better understood than the other cases. We now discuss the conformal field theory interpretation of these branes.

2.3 Conformal field theory interpretation

As discussed in § 1.3, the AdS/CFT conjecture proposes that in the large N limit type IIB supergravity on $AdS_5 \times S^5$ is dual to an $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory in four dimensions with gauge group $SU(N)$. This conjecture is now very well tested and so we assume that it holds. We now give some details of the 4-dimenionsal gauge theory, first in $\mathcal{N} = 4$ language and then in terms of a $\mathcal{N} = 1$ subgroup of the supersymmetry algebra.

The 4-dimensional SYM theory has 6 adjoint scalar fields, X^i, $(i = 1, \dots 6)$ and 4 adjoint fermions λ^A_α, $(A = 1, \dots 4)$. The theory has an R-symmetry group $SU(4)$ which is the cover of $SO(6)$. The scalar fields form a vector of $SO(6)$ and the adjoint fermions form a positive chirality spinor of $SO(6)$. It is useful to describe this theory from the point of view of a $\mathcal{N} = 1$ subalgebra of the $\mathcal{N} = 4$ supersymmetry algebra. In particular, this means we combine the scalars and fermions into 3 chiral superfields, Φ^μ, $\mu = 1, 2, 3$. In this setting the $SU(4)$ R-symmetry is partially hidden. However, the "visible" part of the group comprises an $SU(3)$ which rotates the chiral superfields Φ^μ and a $U(1)$ which acts on one of the Φ^μ (the $SU(3)$ symmetry means that it doesn't matter which Φ^μ is chosen). The charge of an operator under the $U(1)$ symmetry is referred to as its R-charge. One way of forming gauge invariant operators is to take traces of the chiral superfields, e.g.

$$T^{\mu_1 \dots \mu_l} = \mathrm{Tr}\left(\Phi^{(\mu_1}\Phi^{\mu_2}\dots\Phi^{\mu_l)}\right) \tag{2.23}$$

where the $SU(3)$ indices are symmetrized over to ensure the operator is irreducible. This operator is gauge invariant due to the cyclicity of the trace and the fact that Φ transforms in the adjoint of $SU(N)$. The R-charge of this operator is equal to l times the R-charge of Φ^μ. Taking the free field limit one finds that the R-charge is precisely l and it remains l for all values of the coupling (which in practical terms means we can take the R-charge of Φ^μ to be 1). An important fact about these operators is that they have a bound on their R-charge: $l \leq N$. This arises from the properties of traces of matrices in the adjoint of $SU(N)$. In particular, if one takes $l > N$ in $T^{\mu_1 \cdots \mu_l}$ the operator decomposes into a sum of products of lower R-charge operators.

The original idea [46] was to associate a giant graviton carrying angular momentum l with $T^{\mu_1 \cdots \mu_l}$, where the $SU(3)$ structure of the operator corresponds to the plane of rotation of the brane in S^5. There are two reasons that this association was believed to be correct. Firstly, the bound on the R-charge of the operators $T^{\mu_1 \cdots \mu_l}$ precisely matches the bound on the angular momentum for a giant graviton (derived in § 2.1). Secondly, for low R-charge, the number of traces in an operator counts the number of particles. Since in the stringy exclusion principle giant gravitons are associated to single point-like particles carrying angular momentum on the S^5, one would expect that the operator dual to a giant graviton would contain a single trace.

However, it was realized in Ref. [47] that the single trace description of giant gravitons is not quite correct. The reason for this is that operators corresponding to states with different numbers of giant gravitons are not orthogonal when the R-charge is of order N. In particular, at order N the correspondence between the number of traces and particle number breaks down. The fact that operators containing different numbers of giant gravitons are not orthogonal is in contradiction with the semi-classical description, where there is a clear distinction between these states. In particular, in the semi-classical description states containing different numbers of giant gravitons only weakly interact with each other and transitions are suppressed.

To address these problems, the authors of Ref. [47] proposed that giant gravitons should instead be associated to a family of sub-determinant and determinant operators as follows. A giant graviton with angular momentum l should be associated to

$$\mathcal{O}_l = \det{}_l \Phi = \frac{1}{l!} \epsilon_{i_1 \ldots i_l k_1 \ldots k_{N-l}} \epsilon^{j_1 \ldots j_l k_1 \ldots k_{N-l}} \Phi^{i_1}_{j_1} \ldots \Phi^{i_l}_{j_l} \tag{2.24}$$

where i, j, k are $SU(N)$ indices and we have suppressed the $SU(3)$ indices on Φ. Like the single trace operators, these operators are gauge invariant and by definition they have maximum R-charge N. At maximum R-charge, which corresponds to maximum angular momentum, N, for the giant graviton, the sub-determinant becomes the usual determinant of the matrix Φ^i_j, i.e.

$$\mathcal{O}_N = \frac{1}{N!} \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} \Phi^{i_1}_{j_1} \ldots \Phi^{i_N}_{j_N} = \det \Phi$$

The motivation for using this family of operators to describe giant gravitons came from considering wrapped D3-branes on related geometries [62–66], where the dual operators were known to involve determinants of chiral fields. The important feature of this new family of operators is that the overlap between operators corresponding to different numbers of giant gravitons is exponentially suppressed. Therefore, the transition probability between states containing different numbers of giant gravitons is very small, in agreement with the semi-classical description.

The remaining question is what do the dual giants, and indeed the point-like gravitons, correspond to in the dual description? We now have a sensible set of operators for giant gravitons, but we haven't yet discussed the field theory dual of point-like gravitons and dual giant gravitons. The field theory interpretation of these objects has been discussed in Refs. [67, 68]. Firstly, in Ref. [67] it was argued that the point-like graviton configuration is unphysical whenever a giant graviton configuration carrying the same angular momentum is allowed, i.e. whenever the radius of the giant graviton exceeds the string scale. Therefore, one does not expect to find a field theory operator corresponding to a point-like graviton except in a very restricted range of momenta where the giant graviton description does not make sense. The second point in Ref. [67] is that the dual giant configuration has a completely different particle interpretation to an ordinary sphere giant. In particular, a dual giant carrying angular momentum l arises from l point-like gravitons, each carrying angular momentum 1, which form a bound state and expand into a brane via quantum effects. This differs completely from an ordinary giant graviton which arises from one point-like graviton, which carries angular momentum l, blowing up into a brane. These two scenarios can be neatly described via one Young Tableau [68]. The consequence for the field theory is that an operator with R-charge l can have two possible supergravity duals - either in terms of giant gravitons or dual giants - but the two supergravity configurations have non-overlapping regimes of validity. Therefore, for a given set of parameters, the field theory operator has only one sensible supergravity dual. However, to determine precisely which supergravity dual corresponds to a given operator for particular parameters, one would require a better understanding of the auxiliary theory[3], which has not yet been given.

[3]The auxiliary theory lives on l coincident gravitons in the $AdS_5 \times S^5$ background. This theory has been discussed in Refs. [57, 67]. In this picture the giant gravitons, point-like gravitons and dual giants correspond to vacuua of this theory.

Chapter 3

Giant gravitons in non-supersymmetric backgrounds

In this chapter we consider probing a family of 11-dimensional supergravity solutions with giant gravitons. These solutions arise from lifting arbitrary solutions of 4-dimensional $U(1)^4$ and 7-dimensional $U(1)^2$ gauged supergravities. Typically, these solutions will not preserve supersymmetry. Our main result is to show that giant gravitons in the 11-dimensional lifted geometries are equivalent to massive charged particles in the corresponding lower-dimensional gauged supergravity background. Furthermore, both these objects are equivalent to massless particles in 11 dimensions, which carry momentum on the internal part of the lifted geometry.

We begin by studying the case of 4-dimensional $U(1)^4$ gauged supergravity. This theory arises from the reduction of 11-dimensional supergravity on an S^7. The correspondence between the 4-dimensional and 11-dimensional solutions is given in Ref. [69], and we discuss this in § 3.1.1. Roughly speaking, the 11-dimensional lifted backgrounds are composed of a product of the 4-dimensional background with an internal 7-dimensional space which has the topology of a 7-sphere, but an unusual metric (one can think of this as a "squashed" 7-sphere). For this class of 11-dimensional supergravity solutions, the giant graviton probe is an M5-brane which wraps a 5-sphere in the internal space. It is supported from collapse by coupling magnetically to the 4-form field strength, $F^{(4)}$. Before we embark on the giant graviton probe calculation, we first consider a massless particle carrying angular momentum on the internal space of an arbitrary 11-dimensional lifted geometry. We show that the action of this particle reduces to that of a massive charged particle in four dimensions. Then we perform a brane probe calculation to show that the same massive charged particle has yet another description in terms of an M5-brane giant graviton which also carries momentum on the internal 7-sphere. In particular, we show that a massive charged particle probing *any* solution of the gauged supergravity is equivalent to a massless particle or M5-brane probing the 11-dimensional lift, with specific embeddings in the internal space. This extends the results of [54] for the closely related case of 5-dimensional $\mathrm{U}(1)^3$ gauged supergravity[1]. Moreover, our results agree with the calculations in pure $AdS \times S$ geometries, discussed in § 2. However, here the geometries are much more complicated (although the pure $AdS \times S$ geometries arise as a special case where all the gauge fields and scalar fields are set to zero).

In § 3.2 we repeat the above calculations for lifted solutions of 7-dimensional $\mathrm{U}(1)^2$ gauged supergravity. In this case the giant graviton probe is an M2-brane wrapping a 2-sphere in the internal 4-dimensional space. Qualitatively, the results are exactly the same, i.e. giant gravitons and massless particles probing an 11-dimensional lifted solution are equivalent to massive charged particles probing the corresponding lower-dimensional background.

In § 3.3 we apply our results to probe superstar geometries. These backgrounds arise as the extremal limit of charged black holes [70–72], and they are conjectured to be sourced by distributions of giant gravitons [58, 73, 74]. Instead of performing full giant graviton probe calculations, we can work entirely within the simpler lower-dimensional setting, using massive charged particles to probe the geometry. The original results in this chapter are published in Ref. [1].

3.1 Probing lifted 4-d $U(1)^4$ solutions

In this section we introduce the 4-dimensional $U(1)^4$ theory and discuss the lift ansatze for the metric and the 4-form field strength. Then we probe a general lifted geometry with a massless particle and an M5-brane giant graviton. We find that these objects are both equivalent to a massive charged particle probing the associated 4-dimensional background.

3.1.1 4-d gauged supergravity and lift ansatze

The compactification of 11-dimensional supergravity on S^7 leads to gauged $\mathcal{N} = 8$ supergravity in four dimensions with gauge group SO(8). This theory arises from consistently truncating the massive Kaluza-Klein modes of the compactified 11-dimensional supergravity. Consequently, all solutions of this 4-dimensional supergravity theory correspond to solutions of the 11-dimensional theory. In practice, however, the relationship between solutions of the two theories is complicated and highly implicit. To provide a concrete realization of this relationship one can consistently truncate the 4-dimensional $\mathcal{N} = 8$

[1]In this case the 5-dimensional solutions lift to solutions of type IIB supergravity.

theory to a $\mathcal{N} = 2$ theory. This corresponds to truncating the full gauge group SO(8) to its Cartan subgroup, $U(1)^4$. The explicit relationship [69] between solutions of the $\mathcal{N} = 2$ $U(1)^4$ theory and 11-dimensional supergravity is shown in the following.

The 4-dimensional $\mathcal{N} = 2$ supergravity theory has a bosonic sector consisting of the metric, four commuting U(1) gauge fields, three dilatons and three axions. We will be interested in cases where the axions are set to zero. While this is not completely consistent (since terms of the form $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ will source axions) it suffices for the present purposes since we will only consider electrically charged solutions. The Lagrangian for this theory is given by

$$\frac{1}{\sqrt{-g}}\,\mathcal{L}_4 = R - \frac{1}{2}(\partial\vec{\varphi})^2 + \frac{8}{L^2}(\cosh\varphi_1 + \cosh\varphi_2 + \cosh\varphi_3) - \frac{1}{4}\sum_{i=1}^{4} e^{\vec{a_i}\cdot\vec{\varphi}}(F^i_{(2)})^2 \qquad (3.1)$$

where g is the determinant of the 4-dimensional metric, $ds^2_{(1,3)} = g_{\mu\nu}dx^\mu dx^\nu$. Here $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ are the three dilaton fields, the quantities $F^i_{(2)} = dA^i_{(1)}$, $i = 1, \ldots 4$, are the four U(1) field strength tensors, and the 3-vectors $\vec{a_i}$, $i = 1, \ldots, 4$, satisfy

$$M_{ij} = \vec{a_i}\cdot\vec{a_j} = 4\delta_{ij} - 1 \qquad (3.2)$$

The three dilaton fields can be conveniently parameterized in terms of four scalar quantities X_i, $i = 1, \ldots 4$, where

$$X_i = e^{-\frac{1}{2}\vec{a_i}\cdot\vec{\varphi}} \qquad (3.3)$$

The X_i satisfy the constraint $X_1X_2X_3X_4 = 1$. The Lagrangian (3.1) leads to the following equations of motion,

$$\begin{aligned} d *_{(1,3)} d\log(X_i) &= \frac{1}{4}\sum_j M_{ij}X_j^{-2} *_{(1,3)} F^j_{(2)} \wedge F^j_{(2)} + \frac{1}{L^2}\sum_{j,k} M_{ij}X_jX_k\epsilon_{(1,3)} \\ &\quad -\frac{1}{L^2}\sum_j M_{ij}X_j^2\,\epsilon_{(1,3)} \qquad (3.4) \\ d(X_i^{-2} *_{(1,3)} F^i_{(2)}) &= 0 \qquad (3.5) \end{aligned}$$

together with the 4-dimensional Einstein-Maxwell equations coupled to scalars X_i. Here $*_{(1,3)}$ means dualizing with respect to the 4-dimensional metric $ds^2_{(1,3)}$, and $\epsilon_{(1,3)}$ is the volume form on this space. Solutions of this 4-dimensional theory can be "lifted" to solutions of 11-dimensional supergravity as follows [69],

$$ds^2_{11} = \Delta^{2/3}ds^2_{(1,3)} + \Delta^{-1/3}\sum_{i=1}^{4}\left(L^2X_i^{-1}d\mu_i^2 + X_i^{-1}\mu_i^2(Ld\phi_i + A^i_{(1)})^2\right) \qquad (3.6)$$

where $\Delta \equiv \sum_{i=1}^4 X_i \mu_i^2$. The lift ansatz for the 4-form field strength tensor is

$$F^{(4)} = \frac{2U}{L} \epsilon_{(1,3)} + \frac{L}{2} \sum_i X_i^{-1} *_{(1,3)} dX_i \wedge d(\mu_i^2) + \frac{L}{2} \sum_i X_i^{-2} d(\mu_i^2) \wedge (Ld\phi_i + A^i_{(1)}) \wedge *_{(1,3)} F^i_{(2)} \tag{3.7}$$

where $U \equiv \sum_{i=1}^4 (X_i^2 \mu_i^2 - \Delta X_i)$. The four ϕ_i satisfy $0 \leq \phi_i \leq 2\pi$ and the coordinates μ_i define a unit 3-sphere, $S : \sum_{i=1}^4 \mu_i^2 = 1$. They can be parameterized as

$$\mu_1 = \cos\theta_1 \qquad \mu_2 = \sin\theta_1 \cos\theta_2 \qquad \mu_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \qquad \mu_4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \tag{3.8}$$

where $0 \leq \theta_1, \theta_2, \theta_3 \leq \pi/2$. The compact coordinates $\theta_1, \theta_2, \theta_3$ and $\phi_1, \ldots \phi_4$ parameterize the internal 7-dimensional space of the lifted solution. This internal space has the topology of a 7-sphere but its metric is not the usual 7-sphere metric (except in the special case where the gauge fields and dilatons are set to zero and we recover the $AdS_4 \times S^7$ metric and its associated 4-form field strength). It is important to note that the metric and 4-form field strength given in Eqs. (3.6)-(3.7) solve the 11-dimensional supergravity field equations *provided* $(ds^2_{(1,3)}, X_i, A^i_{(1)})$ is a solution of the 4-dimensional theory given above. In particular, it is easy to check that $dF^{(4)} = 0$ if the 4-dimensional equations of motion are satisfied.

We will be interested in probing the 11-dimensional lifted solutions in Eqs. (3.6)-(3.7) with massless particles and giant gravitons. The giant gravitons are M5-branes which wrap an S^5 in the 7-dimensional internal space. These 5-branes are prevented from collapse by coupling to the 6-form potential, $A^{(6)}$, which is related to the 4-form field strength via the dual field strength, $F^{(7)} = *_{(11)} F^{(4)} = dA^{(6)}$. To obtain the relevant piece of $A^{(6)}$ we must first dualize the 4-form field strength in Eq. (3.7), and then integrate it. This procedure involves a number of tricks and intermediate results analogous to those obtained in Ref. [54]. We perform the calculation in detail in § B.1 and present the results here. We find that the dual 7-form field strength is given by,

$$\begin{aligned} F^{(7)} = *_{(11)} F^{(4)} &= -\frac{2L^2 U}{\Delta^2} W \bigwedge_k \mu_k (Ld\phi_k + A^k_{(1)}) \\ &\quad - \frac{L^2}{2\Delta^2} \sum_{ij} X_j dX_i \wedge \mu_i \mu_j Z_{ij} \bigwedge_k \mu_k (Ld\phi_k + A^k_{(1)}) \\ &\quad + \frac{L^2}{2\Delta} \sum_{ij} F^i_{(2)} \wedge Z_{ij} X_j \mu_j \bigwedge_{k \neq i} \mu_k (Ld\phi_k + A^k_{(1)}) \end{aligned} \tag{3.9}$$

where $i, j, k, \cdots = 1, \ldots, 4$ and we use the following notation,

$$\bigwedge_{m \neq i} d\phi_m = \frac{1}{3!} \sum_{j,k,l} \epsilon_{ijkl}\, d\phi_j \wedge d\phi_k \wedge d\phi_l$$

where $\epsilon_{1234} = +1$. Here W is the usual volume form on the 3-sphere, S,

$$W = \frac{1}{6}\, \epsilon_{ijkl}\, \mu_i d\mu_j \wedge d\mu_k \wedge d\mu_l = \sin^2\theta_1 \sin\theta_2 d\theta_1 \wedge d\theta_2 \wedge d\theta_3$$

and Z_{ij} are 2-forms on S, given by

$$Z_{ij} \equiv \sum_{k,l} \epsilon_{ijkl}\, d\mu_k \wedge d\mu_l$$

It is reasonably straighforward to check that the Bianchi identity, $dF^{(7)} = 0$, holds for $F^{(7)}$ given in Eq. (3.9). This does not require use of the equations of motion and it provides a check on the relative signs in Eq. (3.9). The only subtlety is that one must take $F^i_{(2)} \wedge F^j_{(2)} = 0$ for this to hold, which corresponds to neglecting the axions, as already discussed.

Since $dF^{(7)}$ vanishes identically, the 6-form potential, $A^{(6)}$, associated to $F^{(7)}$ must exist at least locally. In fact, it is not possible to determine $A^{(6)}$ globally, but it can be found locally as we show explicitly in § B.1.2. The local region we consider is where $\mu_1 \neq 0$. As we will see, this is appropriate for the giant graviton probe in § 3.1.3, as we will take the probe to be at fixed $\mu_1 \neq 0$. In this region, $A^{(6)}$ is given by

$$A^{(6)} = -\frac{L^2}{2\mu_1\Delta} \sum_i X_i \mu_i Z_{i1} \bigwedge_j \mu_j (L d\phi_j + A^j_{(1)}) + \frac{L^2}{2}\, \epsilon_{1jkl}\, \mu_k^2 \mu_l d\mu_l \wedge F^j_{(2)} \bigwedge_{m \neq j} (L d\phi_m + A^m_{(1)}) \tag{3.10}$$

where a sum over j, k, l is implicit in the second term. In § 3.1.3 we will see that the first term in Eq. (3.10) contributes to the action for the probe giant graviton, while the second term does not couple to the brane. Note that from now on we will drop the subscripts (1) and (2) from the gauge potentials, A^i, and associated 2-form field strengths, F^i.

3.1.2 A massless particle probe

As a warm-up to the brane probe calculation, we consider a massless particle moving in a general 11-dimensional lifted geometry, Eqs. (3.6)–(3.7), carrying some conserved angular momentum on the compact internal 7-dimensional space (the "7-sphere"). We are interested in how this particle appears in the associated 4-dimensional space-time. Clearly if the particle is stationary on the 7-sphere it will simply appear as a massless particle in four dimensions. However, if the particle carries some angular momentum on the internal space we expect that it will behave as a massive charged particle in the 4-dimensional space-time. Here we show that this is indeed the case by performing a probe calculation and minimizing the energy of the particle in the compact directions.

To simplify the calculation we begin by considering the action for a *massive* particle moving in eleven dimensions. Then we will move to the Hamiltonian formulation and take the mass to zero. The action is given by

$$S = -m \int dt \sqrt{-g_{mn}\dot{x}^m \dot{x}^n} \tag{3.11}$$

where g_{mn} is the 11-dimensional metric from Eq. (3.6) $(m, n = 0, 1 \ldots, 9, \natural)$, $x^0 = t$ and $\dot{x}^m = dx^m/dt$. We assume that the motion on the 7-sphere is only in the ϕ_i directions, and that the particle is stationary in the μ_i directions. Therefore, the Lagrangian is given explicitly by

$$\mathcal{L} = -m \left(-\Delta^{2/3} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \Delta^{-1/3} \sum_{i=1}^{4} X_i^{-1}\mu_i^2 (L\dot{\phi}_i + A^i_\mu \dot{x}^\mu)^2 \right)^{1/2} \tag{3.12}$$

where $g_{\mu\nu}$ are the components of the 4-dimensional metric. The momentum conjugate to ϕ_i can be easily computed for each $i = 1, \ldots 4$. One obtains,

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = -\frac{m^2 L}{\Delta^{1/3}\mathcal{L}} X_i^{-1}\mu_i^2 (L\dot{\phi}_i + A^i_\mu \dot{x}^\mu) \tag{3.13}$$

Since the Lagrangian contains no explicit dependence on ϕ_i these momenta are time-independent. We want to rearrange Eq. (3.13) to write $\dot{\phi}_i$ in terms of the momenta, P_j. A few lines of algebra yields

$$\dot{\phi}_i = \frac{\Delta^{1/3} P_i X_i}{L^2 \mu_i^2} \left(\frac{-\Delta\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\kappa + m^2 \Delta^{1/3}} \right)^{1/2} - \frac{1}{L} A^i_\mu \dot{x}^\mu \tag{3.14}$$

where

$$\kappa \equiv \sum_{j=1}^{4} \frac{\Delta^{2/3}\, P_j^2 X_j}{L^2 \mu_j^2} \tag{3.15}$$

We now construct the Hamiltonian (or Routhian[2]) by conjugating the $\dot{\phi}_i$ variables,

$$\mathcal{R} = \sum_i P_i \dot{\phi}_i - \mathcal{L} \tag{3.16}$$

$$= \sum_i \frac{\Delta^{1/3} P_i^2 X_i}{L^2 \mu_i^2} \left(\frac{-\Delta \ g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\kappa + m^2 \Delta^{1/3}} \right)^{1/2} - \frac{1}{L} \sum_i P_i A^i_\mu \dot{x}^\mu - \mathcal{L} \tag{3.17}$$

In the limit $m \to 0$ this becomes

$$\mathcal{R} = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \left(\sum_i \frac{\Delta \ P_i^2 X_i}{L^2 \mu_i^2} \right)^{1/2} - \frac{1}{L} \sum_i P_i A^i_\mu \dot{x}^\mu \tag{3.18}$$

We want to minimize this energy function with respect to the sphere coordinates μ_i. This can be achieved by defining two 4-vectors $\mathbf{U}$ and $\mathbf{V}$ with components

$$U_i = \sqrt{\frac{X_i}{\mu_i^2} \frac{P_i}{L}} \qquad V_i = \sqrt{X_i \mu_i^2} \tag{3.19}$$

and recognizing that the quantity in brackets in Eq. (3.18) can be written as

$$\left(\sum_i \frac{\Delta \ P_i^2 X_i}{L^2 \mu_i^2} \right)^{1/2} = |\mathbf{U}||\mathbf{V}| \tag{3.20}$$

By the Cauchy-Schwarz inequality the minimum value of this expression is $\mathbf{U} \cdot \mathbf{V} = \sum_i X_i P_i / L$ which occurs when $\mathbf{U}$ and $\mathbf{V}$ are parallel. The constraint $\sum_i \mu_i^2 = 1$ determines the constant of proportionality relating $\mathbf{U}$ and $\mathbf{V}$ when they are parallel. We obtain,

$$\mathbf{U} = \left(\sum_i \frac{P_i}{L} \right) \mathbf{V}$$

which implies that the minimal energy configuration occurs when $\mu_i^2 = P_i / \sum_j P_j$. Therefore, after minimizing the energy in the compact directions we obtain the following Routhian,

$$\mathcal{R} = \frac{1}{L} \sum_i P_i X_i \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} - \frac{1}{L} \sum_i P_i A^i_\mu \dot{x}^\mu \tag{3.21}$$

This is just the Lagrangian for a massive charged particle with scalar coupling moving in a 4-dimensional space-time with metric $ds^2_{(1,3)} = g_{\mu\nu} dx^\mu dx^\nu$, i.e. the massless particle action in 11 dimensions reduces to the action for a 4-dimensional massive charged particle in the associated gauged supergravity background. It is important to note that we have

[2]More precisely, this quantity is called a Routhian, since we are not conjugating the $\dot{x}^\mu$ variables

not assumed any special form for the 4-dimensional solution, i.e. this calculation is valid for arbitrary lifted solutions of the gauged supergravity.

3.1.3 Brane probe calculation

We now consider probing an arbitrary lifted solution, Eqs. (3.6)–(3.7), with a giant graviton. In this case the giant graviton is an M5-brane which wraps an S^5 within the internal 7-dimensional space. We take the wrapped S^5 to be parameterized by $\sigma^i = \{\theta_2, \theta_3, \phi_2, \phi_3, \phi_4\}$ and we assume that the brane moves rigidly in the ϕ_1 direction at fixed θ_1, with arbitrary rigid motion in the 4-dimensional space. The action for this brane is,

$$S_5 = -T_5 \int dt d\theta_2 d\theta_3 d\phi_2 d\phi_3 d\phi_4 \left[\sqrt{-\gamma} - \dot{x}^\mu A^{(6)}_{\mu\theta_2\theta_3\phi_2\phi_3\phi_4} - \dot{\phi}_1 A^{(6)}_{\phi_1\theta_2\theta_3\phi_2\phi_3\phi_4}\right] \tag{3.22}$$

where γ is the determinant of the induced metric on the brane world-volume, $\gamma = \det(\gamma_{ab})$, and the last two terms arise from the pull-back of the 6-form gauge potential to the brane. To calculate the induced metric we use the formula from Chapter 1, namely

$$\gamma_{ab} = g_{mn} \frac{\partial x^m}{\partial \sigma^a} \frac{\partial x^n}{\partial \sigma^b} \tag{3.23}$$

Here g_{mn} is the 11-dimensional lifted metric from Eq. (3.6), x^m are embedding coordinates for the brane in this background and $\sigma^a = \{t, \sigma^i\}$ $(a, b = 0, \ldots, 5)$ are the brane's world-volume coordinates. The 6-dimensional induced metric, γ_{ab}, is slightly messy to write down, but it has non-zero entries along the diagonal and in the (t, ϕ_i) positions. Evaluating the determinant of this metric gives

$$\gamma = \frac{L^{10} X_1^2 \alpha}{\Delta} \sin^{10}\theta_1 \cos^2\theta_2 \sin^6\theta_2 \cos^2\theta_3 \sin^2\theta_3 \left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{\cos^2\theta_1}{X_1 \Delta}\dot{\Phi}^2\right) \tag{3.24}$$

where $\dot{\Phi} \equiv L\dot{\phi}_1 + A^1_\mu \dot{x}^\mu$, and

$$\alpha \equiv X_2 \cos^2\theta_2 + X_3 \sin^2\theta_2 \cos^2\theta_3 + X_4 \sin^2\theta_2 \sin^2\theta_3$$

Note that the 1-form A^1 appears in this determinant because we are considering motion in the ϕ_1 direction and the metric contains the combination $Ld\phi_1 + A^1$. Now, the Wess-Zumino terms in the action can be determined simply by reading off the relevant components of $A^{(6)}$ from Eq. (3.10). Using the parameterization for the μ_i coordinates given in Eq. (3.8), one finds

$$\dot{x}^\mu A^{(6)}_{\mu\theta_2\theta_3\phi_2\phi_3\phi_4} + \dot{\phi}_1 A^{(6)}_{\phi_1\theta_2\theta_3\phi_2\phi_3\phi_4} = \frac{L^5}{\Delta}\sin^6\theta_1 \sin^3\theta_2\cos\theta_2\cos\theta_3\sin\theta_3 \;\alpha\; \dot{\Phi} \tag{3.25}$$

Thus, the action for the giant graviton is given by

$$\begin{aligned} S_5 &= -T_5 L^5 \int dt d\theta_2 d\theta_3 d\phi_2 d\phi_3 d\phi_4 \frac{\sin^5\theta_1}{\sqrt{\Delta}}\cos\theta_2\sin^3\theta_2\cos\theta_3\sin\theta_3 \left\{ -\frac{\sin\theta_1}{\sqrt{\Delta}}\;\alpha\;\dot{\Phi} \right. \\ &\left. + \sqrt{X_1^2\alpha\left(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{\cos^2\theta_1}{X_1\Delta}\dot{\Phi}^2\right)}\right\} \end{aligned} \tag{3.26}$$

This action contains no explicit dependence on ϕ_1. Thus, the momentum conjugate to $\dot{\phi}_1$, which we denote by P_{ϕ_1}, is time independent. Conjugating the variable $\dot{\phi}_1$ we obtain the following Routhian, $\mathcal{R} = P_{\phi_1}\dot{\phi}_1 - \mathcal{L}$, where $\mathcal{L}$ is the Lagrangian corresponding to the action above,

$$\begin{aligned} \mathcal{R} &= \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\left(\frac{X_1\Delta}{\cos^2\theta_1}\left(P_{\phi_1} - \frac{N\alpha}{V_5\Delta}\sin^6\theta_1\cos\theta_2\sin^3\theta_2\cos\theta_3\sin\theta_3\right)^2\right. \\ &\left. + \frac{N^2X_1^2\alpha}{V_5^2\Delta}\sin^{10}\theta_1\cos^2\theta_2\sin^6\theta_2\cos^2\theta_3\sin^2\theta_3\right)^{1/2} - \frac{P_{\phi_1}\dot{A}^1}{L} \end{aligned} \tag{3.27}$$

where $N \equiv T_5 V_5 L^6$ and $V_5 = \pi^3$ is the surface area of a unit 5-sphere in flat space. The terms inside the square root above can be rewritten as the following sum of squares:

$$X_1^2 P_{\phi_1}^2 + X_1\alpha\tan^2\theta_1\left(P_{\phi_1} - \frac{N}{V_5}\sin^4\theta_1\cos\theta_2\sin^3\theta_2\cos\theta_3\sin\theta_3\right)^2 \tag{3.28}$$

Then the Routhian becomes,

$$\begin{aligned} \mathcal{R} &= \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\sqrt{X_1^2P_{\phi_1}^2 + X_1\alpha\tan^2\theta_1\left(P_{\phi_1} - N\sin^4\theta_1\cos\theta_2\sin^3\theta_2\cos\theta_3\sin\theta_3/V_5\right)^2} \\ &- \frac{P_{\phi_1}\dot{A}^1}{L} \end{aligned}$$

This rearrangement makes it easy to minimize the energy over θ_1. There are two minima of $\mathcal{R}$ which occur at $\theta_1 = 0$ and

$$P_{\phi_1} = \frac{P_1}{V_5}\cos\theta_2\sin^3\theta_2\cos\theta_3\sin\theta_3$$

where P_1 is constant given by $P_1 = N\sin^4\theta_1$. The minimum at $\theta_1 = 0$ corresponds classically to a massless particle, rather than a brane expanded on S^5. This solution is singular with respect to the gravitational field equations because it represents a huge amount of energy concentrated at a point, which leads to uncontrolled quantum corrections [46]. However, the second minimum corresponds to an expanded giant graviton. At this expanded minimum the Routhian reduces to

$$\mathcal{R} = \left(\frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}X_1P_1 - \frac{1}{L}P_1A^1_\mu\dot{x}^\mu\right)\frac{1}{V_5}\,\cos\theta_2\sin^3\theta_2\cos\theta_3\sin\theta_3 \tag{3.29}$$

Note that P_1 is the centre of mass momentum of the brane, and $P_1 = N\sin^4\theta_1$ agrees with the result obtained in Ref. [60] for giant gravitons in $AdS_4 \times S^7$. Integrating $\mathcal{R}$ over the spatial coordinates of the brane, $\sigma^i = \{\theta_2, \theta_3, \phi_2, \phi_3, \phi_4\}$, we obtain

$$\mathcal{R} = \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\;P_1X_1 - \frac{1}{L}\;P_1\dot{A}^1_\mu\dot{x}^\mu \tag{3.30}$$

This is just the Lagrangian for a massive charged particle with scalar coupling moving in a 4-dimensional space-time with metric $ds_{(1,3)} = g_{\mu\nu}dx^\mu dx^\nu$. Note that this particle is BPS as both the mass and charge are equal to P_1/L. Equivalently we could have chosen the probe brane to move in any of the four ϕ_i directions. Then minimizing the energy over the remaining compact coordinates would give

$$E_i = \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\;P_iX_i - \frac{1}{L}P_iA^i_\mu\dot{x}^\mu \tag{3.31}$$

for the energy of that brane. So we find that by minimizing the energy in the compact directions, the giant graviton action reduces to that of a 4-dimensional massive charged particle coupled to a scalar field. This means that probing an 11-dimensional lifted solution with a giant graviton is equivalent to probing the corresponding 4-dimensional solution with a charged particle. Note that the energy obtained above precisely agrees with the Routhian obtained from the massless particle probe calculation in Eq. (3.21) if we set all but one of the particle momenta, P_i, to zero. As in the massless particle probe calculation, we haven't specified a particular form for the 4-dimensional solution, so this result is valid for arbitrary lifted solutions of the 4-dimensional gauged supergravity.

Note that the results we obtain certainly agree with the calculations in pure $AdS_4 \times S^7$ where giant gravitons are associated to single particle states. However, it is somewhat surprising that giant gravitons degenerate to massless particles also exist in this much more general class of backgrounds, which are generically not supersymmetric. Technically, our result depends on the fact that the quantity under the square root in Eq. (3.27) could be rearranged as a sum of squares. If this did not happen the minimization would be much more complicated and probably not produce such a simple result. It seems that the lift ansatze in Eqs. (3.6)-(3.7) have precisely the right properties to allow this to happen for these giant graviton probes.

3.2 Probing lifted 7-d $U(1)^2$ solutions

In this section we introduce 7-dimensional $U(1)^2$ gauged supergravity. This theory arises from the compactification of 11-dimensional supergravity on S^4. Solutions of the 7-dimensional theory can be lifted to solutions of 11-dimensional supergravity, and we give the ansatze for this lift [69]. Then we perform a giant graviton probe calculation in an arbitrary lifted background. In analogy to the calculation in § 3.1.3, it is possible to solve for the embedding in the internal space, and the giant graviton action then reduces to that of a massive charged particle probing the associated 7-dimensional gauged supergravity background.

3.2.1 Supergravity reduction on S^4 and lift ansatze

The Kaluza-Klein reduction of 11-dimensional supergravity on S^4 leads to $\mathcal{N}=4$ supergravity in 7 dimensions with gauge group SO(5). As in the previous case, this $\mathcal{N}=4$ theory can be consistently truncated to a $\mathcal{N}=2$ supergravity theory coupled to a vector multiplet. The vector multiplet consists of the 7-dimensional metric, a 2-form potential, four vector potentials and four scalars. We are interested in a further truncation of the 7-dimensional theory where only the metric, two vector potentials and two scalars are retained in the bosonic sector. That is, the only gauge fields retained are those corresponding to the $U(1)^2$ Cartan subgroup of SO(5). Like the previous case, where we neglected axions, this further truncation is not completely consistent. However, it is consistent for solutions which satisfy $F^1 \wedge F^2 = 0$ (where F^1 and F^2 are the $U(1)$ field strengths in the 7-dimensional theory). These solutions can be lifted to solutions of 11-dimensional supergravity, and we describe the lift ansatze to in the following.

The Lagrangian for the 7-dimensional truncated $\mathcal{N}=2$ theory is given by

$$\begin{aligned}\frac{1}{\sqrt{-g}}\mathcal{L}_7 &= R - \frac{1}{2}\sum_{i=1}^{2}(\partial \ln X_i)^2 - \frac{1}{4}(\partial \ln X_0)^2 + \frac{1}{L^2}(4X_1X_2 + 2X_0X_1 + 2X_0X_2 - \frac{1}{2}X_0^2) \\ &\quad -\frac{1}{4}\sum_{i=1}^{2}X_i^{-2}(F^i)^2 \end{aligned} \tag{3.32}$$

Here g is the determinant of the 7-dimensional metric, $ds^2_{(1,6)} = g_{\mu\nu}dx^\mu dx^\nu$, R is the 7-dimensional Ricci scalar, and the quantities X_0, X_1, X_2 parameterize the two scalar fields and they satisfy the constraint $X_0 = (X_1X_2)^{-2}$. We denote the two $U(1)$ gauge potentials by A^i, $i = 1, 2$, and $F^i = dA^i$ are the corresponding field strengths. The 7-dimensional equations of motion can be deduced from the above Lagrangian. We obtain,

$$\begin{aligned} d *_{(1,6)} d\log(X_i) &= -\frac{4X_0^{-1/2}}{L}\,\epsilon_{(1,6)} - \frac{2X_0X_i}{L^2}\,\epsilon_{(1,6)} - X_i^{-2}F^i \wedge *_{(1,6)}F^i - 2\lambda \\ d *_{(1,6)} d\log(X_0) &= -\frac{4X_0}{L^2}\sum_{i=1}^{2} X_i\,\epsilon_{(1,6)} + \frac{2X_0^2}{L^2}\,\epsilon_{(1,6)} - 2\lambda \\ d\left(X_i^{-2} *_{(1,6)} F^i\right) &= 0 \end{aligned}$$

together with the Einstein-Maxwell equations coupled to the scalar fields. Here $*_{(1,6)}$ refers to dualizing within the 7-dimensional space and $\epsilon_{(1,6)}$ is the volume form on this space. The quantity λ is defined by

$$\lambda = \frac{1}{5L^2}\left(-8X_0^{-1/2} - 4X_0\sum_{i=1}^{2}X_i + X_0^2\right)\epsilon_{(1,6)} - \frac{1}{5}\sum_{i=1}^{2}X_i^{-2}F^i \wedge *_{(1,6)}F^i \tag{3.33}$$

Note that we use the following convention for indices in this section: $i, j, \cdots = 1, 2$ and $a, b, \cdots = 0, 1, 2$. Solutions of the above equations of motion can be lifted to solutions of 11-dimensional supergravity via the lift ansatze [69]:

$$ds_{11}^2 = \tilde{\Delta}^{1/3}ds_{(1,6)}^2 + \tilde{\Delta}^{-2/3}\left(L^2\sum_{a=0}^{2}X_a^{-1}d\mu_a^2 + \sum_{i=1}^{2}X_i^{-1}\mu_i^2(Ld\phi_i + A^i)^2\right) \tag{3.34}$$

$$\begin{aligned} F^{(7)} &= -\frac{2U}{L}\,\epsilon_{(1,6)} - \frac{1}{L}\,\tilde{\Delta}X_0\,\epsilon_{(1,6)} - \frac{L}{2}\sum_{a=0}^{2}X_a^{-1} *_{(1,6)} dX_a \wedge d(\mu_a^2) \\ &- \frac{L}{2}\sum_{i=1}^{2}X_i^{-2}d(\mu_i^2)\wedge(Ld\phi_i + A^i)\wedge *_{(1,6)}F^i \end{aligned} \tag{3.35}$$

where $0 \le \phi_1, \phi_2 \le 2\pi$ and the quantities $\tilde{\Delta}$ and U are defined by

$$\tilde{\Delta} = \sum_{a=0}^{2}X_a\mu_a^2, \qquad U = \sum_{a=0}^{2}(X_a^2\mu_a^2 - \tilde{\Delta}X_a)$$

The variables μ_a, $a = 0, 1, 2$, define a unit 2-sphere, $\tilde{S} : \sum_a \mu_a^2 = 1$. They can be parameterized by

$$\mu_0 = \sin\theta_1\cos\theta_2, \qquad \mu_1 = \cos\theta_1, \qquad \mu_2 = \sin\theta_1\sin\theta_2, \tag{3.36}$$

where $0 \le \theta_1 \le \pi/2$, $0 \le \theta_2 \le \pi$. The polar coordinates $\theta_1, \theta_2, \phi_1, \phi_2$ parameterize the internal 4-dimensional space of this lifted solution. This space has the topology of an S^4, but its metric is not the usual 4-sphere metric except in the special case where $X_i = 1$

and $A^i = 0$. In this case we recover $AdS_7 \times S^4$. In general, the metric and 7-form field strength (where $F^{(7)}$ is related to the usual 4-form field strength by $F^{(4)} = - *_{(11)} F^{(7)}$) given in Eqs. (3.34)-(3.35) will be a solution of 11-dimensional supergravity provided that $(ds^2_{(1,6)}, X_i, A^i)$ is a solution of the 7-dimensional theory described above.

3.2.2 Obtaining the 3-form potential $A^{(3)}$

We want to consider probing the lifted 11-dimensional supergravity solutions, Eqs. (3.34)–(3.35), with giant gravitons. These giant gravitons are M2-branes with an S^2 topology. They will be supported from collapse by coupling to a 3-form potential, $A^{(3)}$, which is related to the 7-form field strength, $F^{(7)}$, via the dual 4-form field strength $F^{(4)} = - *_{(11)} F^{(7)} = dA^{(3)}$. Therefore, to find $A^{(3)}$ explicitly we must first dualize $F^{(7)}$, given in Eq. (3.35), and then integrate the resulting 4-form. In many ways this is similar to the previous case (§ 3.1.1 with details in Appendix B.1) where we dualized $F^{(4)}$ and then integrated the resulting 7-form to obtain the 6-form potential, $A^{(6)}$. The main differences in these calculations arise in the intermediate steps because here the sphere is even-dimensional, and thus parameterized slightly differently compared to the S^7. In this section we simply present the results of the calculation for $A^{(3)}$, and the full details are given in Appendix B.2.

If we dualize $F^{(7)}$, given in Eq. (3.35), we obtain the following 4-form field strength,

$$\begin{aligned} F^{(4)} = - *_{(11)} F^{(7)} &= -\frac{2LU}{\tilde{\Delta}^2} \tilde{W} \bigwedge_i \mu_i(Ld\phi_i + A^i) - \frac{LX_0}{\tilde{\Delta}} \tilde{W} \bigwedge_i \mu_i(Ld\phi_i + A^i) \\ &- \frac{L}{\tilde{\Delta}^2} \sum_{a,b} \mu_a dX_a \wedge \tilde{Z}_{ab}\, \mu_b X_b \bigwedge_i \mu_i(Ld\phi_i + A^i) \\ &- \frac{L}{\tilde{\Delta}} \sum_{a,i} F^i \wedge \tilde{Z}_{ia}\, \mu_a X_a \bigwedge_{j\neq i} \mu_j(Ld\phi_j + A^j) \end{aligned} \tag{3.37}$$

where $\tilde{W}$ is the following volume form on the 2-sphere $\tilde{S}$,

$$\tilde{W} = \frac{1}{2}\sum_{a,b,c} \epsilon_{abc}\mu_a d\mu_b \wedge d\mu_c = -\sin\theta_1 d\theta_1 \wedge d\theta_2$$

and we use the convention that $\epsilon_{012} = +1$. The quantities $\tilde{Z}_{ab}$ are 1-forms on $\tilde{S}$ defined by

$$\tilde{Z}_{ab} = \sum_c \epsilon_{abc} d\mu_c$$

Note that in $F^{(4)}$ we use the following shorthand notation:

$$\bigwedge_{j\neq i} d\phi_j \equiv \sum_j \epsilon_{ij} d\phi_j$$

where $\epsilon_{12} = 1$. Using some identities which we derive in Eqs. (B.31)-(B.33), one can show that the 4-form field strength given above obeys $dF^{(4)} = 0$. This means that $F^{(4)}$ can be integrated at least locally. As in the previous case for $F^{(7)}$, it is not possible to write $F^{(4)} = dA^{(3)}$ with $A^{(3)}$ well-defined over the whole space-time. However, $A^{(3)}$ can be found *locally* everywhere. For example, in the region where $\mu_1 \neq 0$, $A^{(3)}$ is given by

$$A^{(3)} = -\frac{L}{\mu_1 \tilde{\Delta}} \sum_a \mu_a X_a \tilde{Z}_{a1} \bigwedge_i \mu_i (L d\phi_i + A^i) - L\mu_0 F^2 \wedge (L d\phi_1 + A^1) \tag{3.38}$$

In the next section we will consider giant graviton probes moving in arbitrary lifted backgrounds at fixed $\mu_1 \neq 0$. The above form for $A^{(3)}$ will allow the coupling of the probe brane to the 3-form potential to be determined explicitly.

3.2.3 Brane probe calculation

We now consider probing an arbitrary lifted solution, given in Eqs. (3.34)-(3.35), with a giant graviton. As in § 3.1.2 we could also consider probing these solutions with a massless particle which carries angular momentum on the internal 4-sphere. However, this calculation is entirely analogous to the calculation in § 3.1.2 for lifted 4-dimensional geometries, so we will not include it here. The result from this calculation is that a massless particle probe is equivalent to a massive charged particle coupled to scalars probing the associated 7-dimensional space-time. We now see that a giant graviton probe is also equivalent to this 7-dimensional particle.

In this case the giant graviton is an M2-brane which wraps an S^2 within the internal 4-sphere. We take the brane world-volume to be parameterized by the coordinates t, θ_2, ϕ_2, where θ_2, ϕ_2 are coordinates on the wrapped S^2. We consider rigid motion of the brane in the ϕ_1 direction at fixed $\theta_1 \neq \pi/2$ (which corresponds to fixed $\mu_1 \neq 0$). The motion in the non-compact 7-dimensional space is arbitrary, but is assumed to be independent of the coordinates θ_2, ϕ_2, so that only rigid motion of the brane is considered. The action for this brane is given by

$$S_2 = -T_2 \int dt d\theta_2 d\phi_2 \left[\sqrt{-\gamma} - \dot{x}^\mu A^{(3)}_{\mu\theta_2\phi_2} - \dot{\phi}_1 A^{(3)}_{\phi_1\theta_2\phi_2}\right] \tag{3.39}$$

Here γ is the determinant of the induced metric on the (3-dimensional) brane world-volume and the last two terms arise from the pull-back of the 3-form potential to the brane. The induced metric can be calculated readily by pulling back the 11-dimensional metric, Eq. (3.34), to the brane. Evaluating the determinant of this metric gives

$$\gamma = \tilde{\Delta}^{-1} L^4 X_1^2 \sin^4\theta_1 \sin^2\theta_2 \alpha \left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{\cos^2\theta_1}{X_1\tilde{\Delta}}\dot{\Phi}^2 \right) \tag{3.40}$$

where $\alpha \equiv X_0 \cos^2\theta_2 + X_2 \sin^2\theta_2$, $\dot{\Phi} \equiv L\dot{\phi}_1 + A^1_\mu \dot{x}^\mu$ and $g_{\mu\nu}$ are the components of the 7-dimensional metric, $ds^2_{(1,6)} = g_{\mu\nu}dx^\mu dx^\nu$. The components of $A^{(3)}$ which couple to the brane can be read off from Eq. (3.38), using the parameterization for μ_a given in Eq. (3.36). We obtain,

$$\dot{x}^\mu A^{(3)}_{\mu\theta_2\phi_2} + \dot{\phi}_1 A^{(3)}_{\phi_1\theta_2\phi_2} = \frac{L^2}{\tilde{\Delta}} \sin^3\theta_1 \sin\theta_2 \ \alpha \ \dot{\Phi} \tag{3.41}$$

Thus, we obtain the following Lagrangian for the giant graviton,

$$\mathcal{L} = -T_2 L^2 \left\{ \frac{X_1}{\sqrt{\tilde{\Delta}}} \sin^2\theta_1 \sin\theta_2 \alpha^{1/2} \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{\cos^2\theta_1}{X_1\tilde{\Delta}}\dot{\Phi}^2} - \frac{1}{\tilde{\Delta}} \sin^3\theta_1 \sin\theta_2 \alpha \dot{\Phi} \right\} \tag{3.42}$$

As in the previous case, there is no explicit dependence on ϕ_1 in the Lagrangian and so the momentum conjugate to $\dot{\phi}_1$, which we denote by P_{ϕ_1}, is time independent. We use P_{ϕ_1} to construct the Routhian, $\mathcal{R} = P_{\phi_1}\dot{\phi}_1 - \mathcal{L}$,

$$\begin{aligned} \mathcal{R} \ &= \ \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \left(\frac{X_1\tilde{\Delta}}{\cos^2\theta_1} \left(P_{\phi_1} - \frac{N}{\tilde{\Delta}V_2} \sin^3\theta_1 \sin\theta_2 \alpha \right)^2 + \frac{N^2 X_1^2 \alpha}{\tilde{\Delta} V_2^2} \sin^4\theta_1 \sin^2\theta_2 \right)^{1/2} \\ &\quad -\frac{1}{L} P_{\phi_1} A^1_\mu \dot{x}^\mu \end{aligned} \tag{3.43}$$

where $N = T_2 L^3 V_2$ and $V_2 = 4\pi$ is the surface area of a unit 2-sphere. As before, the quantity in the square root can be rearranged as a sum of squares to give

$$\mathcal{R} = \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \left(X_1^2 P_{\phi_1}^2 + X_1 \alpha \tan^2\theta_1 \left(P_{\phi_1} - \frac{N}{V_2} \sin\theta_1 \sin\theta_2 \right)^2 \right)^{1/2} - \frac{1}{L} P_{\phi_1} A^1_\mu \dot{x}^\mu \tag{3.44}$$

It is now simple to minimize the energy over θ_1. There are two minima: $\theta_1 = 0$ and $P_{\phi_1} = P_1 \sin\theta_2 / V_2$, where $P_1 = N \sin\theta_1$ is constant. Like the previous case, the minimum at $\theta_1 = 0$ is singular as it corresponds to the point-like particle solution and represents a huge energy concentrated at a point. From now on we consider the second minimum, which corresponds to the giant graviton. At this minimum the Routhian becomes

$$\mathcal{R} = \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \ X_1 P_1 - \frac{1}{L} P_1 A^1_\mu \dot{x}^\mu \tag{3.45}$$

where we have integrated over the spatial coordinates of the brane, θ_2, ϕ_2, and hence the factors of V_2 cancel. Note that P_1 is the centre of mass momentum for the brane, and $P_1 = N \sin\theta_1$ agrees with the result in Ref. [60] for giant gravitons in $AdS_7 \times S^4$. The above Routhian is the Lagrangian for a massive charged BPS particle in 7 dimensions with scalar coupling. Equivalently, we could have chosen the brane to move in the ϕ_2 direction and wrap a different S^2. This would produce an entirely analogous result. Therefore, after minimizing the energy in the compact directions, the energy of a probe brane moving in the ϕ_i direction is given by

$$E_i = \frac{1}{L}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\; X_i P_i - \frac{1}{L} P_i A^i_\mu \dot{x}^\mu \tag{3.46}$$

where $i = 1, 2$. Therefore, probing the 11-dimensional lifted solutions, Eqs. (3.34)–(3.35), with a giant graviton is equivalent to probing the related 7-dimensional geometry with a massive charged particle. Again, this result depends on the fact that the quantity in the square root in Eq. (3.43) can be rearranged as a sum of squares to simplify the minimization procedure. As in § 3.1.3, we have not assumed any special form for the 7-dimensional solution, so this result is valid for arbitrary lifted solutions of the gauged supergravity.

3.3 Probing superstars with giant gravitons

In this section we use the giant graviton probe calculations of § 3.1.3 and § 3.2.3 to probe a specific class of 11-dimensional lifted solutions, namely the superstar geometries. Superstars are solutions in 10 and 11 dimensions that are lifts of certain gauged supergravity solutions which contain naked singularities. These lower-dimensional solutions arise by taking the supersymmetric limit of a family of black hole solutions. In this limit the horizon disappears[3], and the space-time is left with a naked singularity. The corresponding lifted solutions are supersymmetric and they also inherit the naked singularity from lower dimensions. It is thought that these superstar solutions may be sourced by giant gravitons, with the naked singularity interpreted physically as a collection of giant gravitons in the internal space. Evidence for this was first given in Ref. [58] where the authors considered type IIB superstar geometries and they argued that the dipole field which is excited in the 5-form field strength near the singularity corresponded to the dipole field excited by a distribution of giant gravitons. Moreover, they showed that this distribution

[3]In the multiply charged cases, the horizon disappears before we reach the supersymmetric limit. However, in the singly charged cases, the horizon area shrinks to zero size precisely at the supersymmetric limit.

of giant gravitons produced the correct mass and internal momentum for the superstar.

Further investigations of superstars were made in Ref. [74], where giant graviton probe calculations were performed in singly charged 11-dimensional superstar geometries. These calculations gave further evidence to support the conjecture that giant gravitons source these geometries.

In this section we will consider two types of superstar solutions. Firstly, we consider superstars which are lifts of particular 4-dimensional gauged supergravity solutions. Then we will consider superstars which are lifts of particular 7-dimensional solutions. The idea is to consider probing these superstar geometries with giant gravitons to establish whether the naked singularity has a physical interpretation in terms of a distribution of giant gravitons. If it does, then a giant graviton probe of the same type as the background should have minimal energy at the position of the naked singularity. However, due to the general results obtained in § 3.1.3 and § 3.2.3, probing a lifted solution with a giant graviton is equivalent to probing the corresponding lower-dimensional solution with a charged particle. Therefore, we will be able to perform these probe calculations very simply, with reference only to the associated lower-dimensional solution.

3.3.1 Superstar backgrounds from 4-d solutions

In this section we consider 11-dimensional superstar solutions which are lifts of 4-d $U(1)^4$ gauged supergravity solutions. The relevant lift ansatze are given in Eqs. (3.6)-(3.7). The 4-dimensional solutions of interest are the following 4-charge AdS black hole solutions,

$$ds_4^2 = -\frac{f}{(H_1H_2H_3H_4)^{1/2}}dt^2 + (H_1H_2H_3H_4)^{1/2}(f^{-1}dr^2 + r^2d\Omega_2^2) \tag{3.47}$$

where $d\Omega_2^2$ is the usual metric on a 2-sphere and

$$f = 1 - \frac{\mu}{r} + \frac{4r^2}{L^2}H_1H_2H_3H_4 \tag{3.48}$$

$$H_i = 1 + \frac{q_i}{r} \tag{3.49}$$

$$A^i = (H_i^{-1} - 1)dt \tag{3.50}$$

$$X_i = \frac{(H_1H_2H_3H_4)^{1/4}}{H_i} \tag{3.51}$$

These solutions are parameterized by the four $U(1)$ charges, q_i, $i = 1, \dots, 4$, and the non-extremality parameter, μ. We assume without loss of generality that $q_1 \geq q_2 \geq q_3 \geq q_4 \geq 0$. In the extremal limit, $\mu = 0$, these solutions become supersymmetric and a naked singularity appears at $r = -q_4$. The apparent singularity in the metric at $r = 0$ is a removable coordinate singularity (unless $q_4 = 0$, in which case $r = 0$ coincides with the naked singularity). In the extremal case, $\mu = 0$, we choose a new coordinate $\rho = r + q_4$ and extend the space-time past the coordinate singularity to $\rho = 0$. This gives the following set of supersymmetric solutions which have a naked singularity[4] at $\rho = 0$:

$$ds_4^2 = -\frac{\tilde{f}}{(\tilde{H}_1\tilde{H}_2\tilde{H}_3\tilde{H}_4)^{1/2}}dt^2 + (\tilde{H}_1\tilde{H}_2\tilde{H}_3\tilde{H}_4)^{1/2}(\tilde{f}^{-1}d\rho^2 + d\Omega_2^2) \tag{3.52}$$

where

$$\tilde{f} = (\rho - q_4)^2 + \frac{4}{L^2}\tilde{H}_1\tilde{H}_2\tilde{H}_3\tilde{H}_4 \tag{3.53}$$

$$\tilde{H}_i = \rho + q_i - q_4 \tag{3.54}$$

$$A^i = -\frac{q_i}{\tilde{H}_i}dt \tag{3.55}$$

$$X_i = \frac{(\tilde{H}_1\tilde{H}_2\tilde{H}_3\tilde{H}_4)^{1/4}}{\tilde{H}_i} \tag{3.56}$$

Now if we lift these 4-dimensional solutions to 11 dimensions, the corresponding superstar solutions will inherit the naked singularity at $\rho = 0$ (This is clear from the form of the lift ansatz for the metric given in Eq. (3.6).). These superstar solutions are supersymmetric and satisfy the BPS condition $M = \sum_i q_i$. This means that the background should have a simple physical interpretation in terms of fundamental degrees of freedom. In particular, there should be zero binding energy between the fundamental constituents. The conjecture is that these fundamental degrees of freedom are giant gravitons. To test this idea we will probe each superstar geometry with another giant graviton which carries the same type of charge as one of the constituents of the background. Note that a $U(1)$ charge, q_i, corresponds to a giant graviton with momentum in the ϕ_i direction. This is because the lifted metric contains the $U(1)$ gauge potentials in the combinations $(Ld\phi_i + A^i)$. If the conjecture is correct, one expects that a probe carrying the same type of charge as one of the non-zero background charges will have minimal energy at the naked singularity, $\rho = 0$[5]. Physically, this corresponds to being able to consistently place another giant graviton of the same type as the background at the position of the naked singularity (which should be allowed as the conjectured distribution of giant gravitons in the background preserves supersymmetry). We also expect $\rho = 0$ to be a BPS minimum for the probe, i.e. for a probe carrying momentum in the ϕ_i direction in a background with $q_i \neq 0$, we should find $E_i = P_i/L$. This is because the solutions we consider are BPS.

[4]Note that in the case that all $q_i = 0$ we obtain the AdS_4 solution which lifts to $AdS_4 \times S^7$. In this special case there is no naked singularity.

[5]For example, for a background with $q_1, q_2 \neq 0$ we will probe with a giant graviton carrying momentum in the ϕ_1 direction and then with another giant graviton carrying momentum in the ϕ_2 direction. We expect the energy of both probes to be minimized at $\rho = 0$.

We consider a giant graviton probe carrying momentum in the ϕ_i direction and wrapping an S^5 within the internal 7-sphere of the superstar background. We look for solutions which are stationary in the 4-dimensional space, i.e. $\dot{x}^\nu = 0$ except for $\nu = 0$. From the probe calculation in § 3.1.3 (particularly Eq. (3.30)) we obtain the following energy function for such a probe brane,

$$E_i = \frac{1}{L}\sqrt{-g_{00}}\; P_i X_i - \frac{P_i A_0^i}{L} = \frac{P_i}{L}\frac{\tilde{f}^{1/2} + q_i}{\tilde{H}_i} \tag{3.57}$$

where the energy of the brane in the compact directions has already been minimized, and now it just depends on the details in the 4-dimensional space. There are five distinct cases for the charge of the superstar solution, and we will consider each background in turn. In each case we will evaluate the energy function for a probe giant graviton and then calculate whether it has a BPS minimum at the position of the naked singularity, $\rho = 0$.

1. Background has all $q_i = 0$, i.e. $AdS_4 \times S^7$. There is a BPS minimum for all 4 types of probe (i.e. moving in each ϕ_i direction, $i = 1, 2, 3, 4$) at $\rho = 0$, as expected.

2. $q_1 \neq 0$, all other $q_i = 0$. The probe coupling to A^1 (i.e. moving in ϕ_1 direction) has a BPS minimum at $\rho = 0$. The energy of probes coupling to A^2, A^3, A^4 saturates the BPS bound at $\rho = 0$, but the gradient of the potential is non-zero at $\rho = 0$.

3. $q_1, q_2 \neq 0$, all other $q_i = 0$. The energy of probes coupling to A^1, A^2 saturates the BPS bound at $\rho = 0$, but the gradient of the potential is non-zero. Probes coupling to A^3, A^4 neither saturate the BPS bound, nor have a minimum at $\rho = 0$.

4. $q_1, q_2, q_3 \neq 0$, $q_4 = 0$. The energy of probes coupling to A^1, A^2, A^3 saturates the BPS bound at $\rho = 0$, but the gradient of the potential is infinite. Energy of probe coupling to A^4 diverges as $\rho \to 0$.

5. $q_1, q_2, q_3, q_4 \neq 0$. Energy of probe coupling to A^4 diverges as $\rho \to 0$. The gradient of the potential for probes coupling to A^1, A^2, A^3 is non-zero at $\rho = 0$.

Note that these cases are distinct since the limits $q_i \to 0$ are not generally smooth, i.e. it is not possible to work out the energies and gradients at $\rho = 0$ for case 5 (where all $q_i \neq 0$) and then take smooth limits $q_i \to 0$ to obtain the other cases.

From the above information we see that our results certainly support the conjecture in the singly charged case, since in this case the probe which carries the same type of charge as the background has a BPS minimum at $\rho = 0$. In the doubly and triply charged cases, the gradients of the potentials are non-zero at $\rho = 0$, but the BPS bounds are saturated by the relevant probes. However, the fact that the gradients of the potentials are non-zero is perhaps not important, since $\rho = 0$ is at the edge of the space-time and the energy is minimized at this point. Therefore, we are cautiously optimistic that our results support the conjecture in the doubly and triply charged cases. However, we do note that in the triply charged case the gradient of the potential is infinite at $\rho = 0$, which seems rather unusual. Another slightly unusual feature is that in the doubly charged case the energy of a probe brane coupling to A^3 is not minimized, although the energy of this probe is minimized in the triply charged background. This indicates that it is not a smooth procedure to build up backgrounds with different types of charges (in contrast to the situation in flat space, or in brane backgrounds). Clearly, for the quadruply charged case our results no longer support the conjecture because the energy of the probe coupling to A^4 becomes infinite at $\rho = 0$. In this case the background preserves the least amount of supersymmetry, and so it is possible that higher order curvature corrections to the background and brane action become important, which might modify the results.

3.3.2 Superstar backgrounds from 7-d solutions

In this section we consider 11-dimensional superstar solutions which are lifts of certain 7-dimensional $U(1)^2$ gauged supergravity solutions. The 7-dimensional solutions of interest are the following family of black hole solutions,

$$ds_7^2 = -\frac{f\, r^{2/5}}{(H_1 H_2)^{4/5}} dt^2 + (H_1 H_2\, r^2)^{1/5} (f^{-1} r^4 dr^2 + d\Omega_5^2) \tag{3.58}$$

where $d\Omega_5^2$ is the usual metric on a unit 5-sphere and

$$\begin{aligned} f &= r^6 - \mu r^2 + \frac{1}{4L^2} H_1 H_2 && (3.59)\\ H_i &= r^4 + q_i && (3.60)\\ A^i &= -q_i H_i^{-1} dt && (3.61)\\ X_i &= \frac{(H_1 H_2)^{2/5}}{H_i} && (3.62)\end{aligned}$$

Here the index $i = 1, 2$. The parameters for these black hole solutions are the two $U(1)$ charges, q_i, and the non-extremality parameter, μ. We will assume, without loss of generality, that $q_1 \geq q_2 \geq 0$. In the extremal case, where $\mu = 0$, these solutions become supersymmetric and there is a naked singularity at $r = 0$. If we lift to 11 dimensions, this naked singularity is inherited by the 11-dimensional solution. As in the previous case, we want to understand whether this singularity can be interpreted as a distribution of giant gravitons in the internal 4-sphere. We will use the results of § 3.2.3 to probe these 11-dimensional superstar solutions with giant gravitons. We want to see whether the energy of a giant graviton probe, carrying the same charge as one of the constituents of the background, is minimized at $r = 0$, and whether it is a BPS minimum ($E_i = P_i/L$).

We consider an M2-brane giant graviton probe which carries momentum in the ϕ_i direction, and wraps a 2-sphere. Furthermore, we take the brane to be stationary in the 7-dimensional space, i.e. $\dot{x}^\nu = 0$ except for $\nu = 0$. From Eq. (3.46), the energy of such a probe is given by

$$E_i = \frac{1}{L}\sqrt{-g_{00}}\, X_i P_i - \frac{1}{L} P_i A^i_0 = \frac{P_i}{L} \frac{f^{1/2} r + q_i}{H_i} \tag{3.63}$$

There are three distinct cases of background charge to consider, and we evaluate the energy function for the probe branes in each case.

1. All $q_i = 0$, i.e. $AdS_7 \times S^4$. BPS minimum at $r = 0$ for both types of probe (i.e. moving in both ϕ_i directions), as expected.

2. $q_1 \neq 0$, $q_2 = 0$. Probe coupling to A^1 (i.e. moving in ϕ_1 direction) has a BPS minimum at $r = 0$. Energy of probe coupling to A^2 diverges as $r \to 0$.

3. $q_1, q_2 \neq 0$. The energy of both probes saturates the BPS bound at $r = 0$, but the gradient of the potential is non-zero at $r = 0$.

From these results we see that it is certainly sensible to interpret the singly charged background as being sourced by giant gravitons, since in this case a probe carrying the same type of charge as the background has a BPS minimum at $r = 0$. In the doubly charged case, the gradients of the potentials are non-zero, but the BPS bounds are saturated by both probes. As in the previous case, since $r = 0$ is at the edge of the space-time, the fact that the gradients are non-zero might not be important. Therefore, we are cautiously optimistic that our results support the conjecture in the doubly charged case also. However, we note that it is slightly unusual that in the singly charged background the energy of the probe coupling to A^2 diverges at $r = 0$, while in the doubly charged background this probe has a BPS minimum. Again, this indicates that backgrounds with different charges cannot be built up smoothly by adding more branes.

To summarize, we have found evidence to support the conjecture that both types of superstars (i.e. 4-d and 7-d lifted) are sourced by giant gravitons. However, there is some uncertainty in how to interpret the results for the quadruply charged superstars in the 4-dimensional case, since here the energy of one of the relevant probes goes to infinity at $\rho = 0$. Moreover, it is rather strange that in some cases the energy of a probe giant graviton has a BPS minimum at the naked singularity even though the gradient of the potential is non-zero. It would be interesting to try to understand this by performing further probe calculations, perhaps incorporating curvature corrections. Nevertheless, these superstar backgrounds have given us a specific example where the giant graviton/charged particle relationship is extremely helpful. In particular, this relationship has allowed us to investigate the nature of singularities without performing the rather involved brane probe calculations.

Chapter 4

Calibrations

In this chapter we introduce the method of calibrations. This a geometrical technique which allows one to find minimal energy configurations for probe branes in various backgrounds. This method involves a *calibration*, which is a p-form, ϕ, which satisfies some particular conditions. As we will see, we can define surfaces which are "calibrated" with respect to ϕ. These surfaces have the property that they have minimal volume in their homology class. For a static probe brane wrapping one of these surfaces, this translates into minimal energy for the brane. Therefore, the problem of finding minimal energy brane configurations translates into finding calibrated surfaces. For backgrounds that admit calibrations, this method is preferable to performing a probe calculation, since it does not require the same level of guess work. For example, in a probe calculation one must first guess what surface the brane will wrap, and then one performs the calculation to determine whether the energy is minimized. However, if the background admits a calibration, then these surfaces can be determined more systematically by finding the calibrated surfaces of ϕ. Often this turns out to be quite simple, as we will see.

We begin by defining a calibration, ϕ, for a manifold and the associated calibrated surfaces. Then we show that static probe branes wrapping calibrated surfaces have minimal energy. We then introduce a class of manifolds which naturally admit calibrations. These are the manifolds of special holonomy. Many of these manifolds can be used to construct supersymmetric supergravity solutions with vanishing flux[1]. In these backgrounds the calibrated surfaces give brane configurations which are supersymmetric, as well as energy minimizing[2]. Moreover, the calibrations can be constructed from the Killing spinors of the background.

[1]In this chapter we will be interested in the 11-dimensional solutions that can be constructed, although much of the discussion could be easily extended to other dimensions.

[2]This has been taken a step further in Refs. [75–81], where the authors constructed full back-reacted geometries corresponding to branes wrapping calibrated cycles in special holonomy manifolds. These solutions give new examples of AdS/CFT.

In fact, for a general supersymmetric solution it is always possible to construct a calibration from the Killing spinors of the background. In supersymmetric backgrounds with non-vanishing flux (i.e. $F^{(4)} \neq 0$ for the 11-dimensional case) the forms we construct from Killing spinors are generalized calibrations, which are slightly different to standard calibrations. However, generalized calibrations can still be used to find energy minimizing configurations for probe branes in these backgrounds. We will discuss generalized calibrations in the context of 11-dimensional backgrounds with $F^{(4)} \neq 0$ in § 4.4.

4.1 Standard calibrations

We begin by giving the mathematical definition of a calibration. Then we define the calibrated submanifolds and show that these surfaces have minimal volume in their homology class.

Consider a d-dimensional manifold $(\mathcal{M}, g)$. A calibration is a p-form $\phi \in \Lambda^p \mathcal{M}$ which satisfies two properties. Firstly, ϕ is closed, i.e.

$$d\phi = 0 \tag{4.1}$$

Secondly, for any tangent p-plane[3], ξ, the pull-back of ϕ to ξ is less than or equal to the volume form on ξ, i.e.

$$\phi\Big|_{\xi} \leq vol\Big|_{\xi} \tag{4.2}$$

where the volume form is induced from the metric. A p-dimensional oriented submanifold, $\mathcal{N} \subset \mathcal{M}$, is called *calibrated* if at every point on $\mathcal{N}$ the bound in Eq. (4.2) is saturated. More precisely, for each tangent space $T_x\mathcal{N}$,

$$\phi\Big|_{T_x\mathcal{N}} = vol\Big|_{T_x\mathcal{N}}$$

An important property of calibrated submanifolds is that they have minimal volume compared to other submanifolds in the same homology class. This can easily be seen by considering two p-submanifolds, $\mathcal{N}, \mathcal{L} \subset \mathcal{M}$, in the same homology class. This means that we can write $\mathcal{N} = \mathcal{L} + \partial\Xi$ (and hence $\mathcal{N}$ and $\mathcal{L}$ share the same boundary, i.e. $\partial\mathcal{L} = \partial\mathcal{N}$), where $\partial\Xi$ is the boundary of a $(p+1)$-dimensional submanifold $\Xi \subset \mathcal{M}$. Now we assume that $\mathcal{N}$ is a calibrated submanifold. Therefore, the total volume of $\mathcal{N}$ is given by,

$$Vol(\mathcal{N}) = \int_{x\in\mathcal{N}} vol\Big|_{T_x\mathcal{N}} = \int_{\mathcal{N}} \phi = \int_{\mathcal{L}+\partial\Xi} \phi = \int_{\mathcal{L}} \phi + \int_{\Xi} d\phi \tag{4.3}$$

[3]Precisely, a tangent p-plane is a vector subspace, ξ, of some tangent space $T_x\mathcal{M}$ to $\mathcal{M}$. The dimension of ξ is p, and we assume that ξ is equipped with an orientation.

where we have used Stoke's theorem in the last equality. Now we use the fact that ϕ is a calibration to write the expression on the right hand side of Eq. (4.3) as

$$\int_{\mathcal{L}} \phi + \int_{\Xi} d\phi = \int_{\mathcal{L}} \phi \leq \int_{y \in \mathcal{L}} vol\Big|_{T_y \mathcal{L}} = Vol(\mathcal{L}) \tag{4.4}$$

Therefore, from Eqs. (4.3) and (4.4) we have $Vol(\mathcal{N}) \leq Vol(\mathcal{L})$, i.e. the calibrated manifold, $\mathcal{N}$, has minimal volume compared to an arbitrary manifold, $\mathcal{L}$, in the same homology class. Thus, given a calibration, ϕ, on a manifold, we can look for surfaces calibrated by ϕ, and these surfaces will have minimal volume. Surfaces of minimal volume are interesting from the point of view of supergravity because in some cases they correspond to minimal energy surfaces for branes to wrap, as we now explain.

Consider a static 11-dimensional supergravity background with metric given by

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j \tag{4.5}$$

where $i, j = 1, \ldots, 9, \natural$. We assume that this metric solves the 11-dimensional supergravity field equations with $F^{(4)} = 0$. Suppose we now include a static M2-brane in this background, with world-volume coordinates t, σ^1, σ^2 (where we have fixed some of the reparameterisation invariance of the brane action by choosing the time-like coordinate to be $\sigma^0 = t$). The Lagrangian for this probe M2-brane is given by

$$\mathcal{L} = -T_2 \int d^2\sigma \sqrt{-\gamma}$$

where T_2 is the tension of the brane and γ is the determinant of the induced metric on the brane world-volume. Due to the form of the background metric, together with the fact that the M2-brane is static, the only non-zero components of the induced metric are

$$\gamma_{tt} = -1, \qquad \gamma_{ab} = g_{ij} \frac{\partial x^i}{\partial \sigma^a} \frac{\partial x^j}{\partial \sigma^b}$$

where $a, b = 1, 2$. Therefore, the determinant of the induced metric is $\gamma = -\det(\gamma_{ab})$, i.e. it is simply related to the determinant of the metric on the 2-dimensional spatial world-volume. Using this result and moving to the Hamiltonian formalism ($\mathcal{H} = -\mathcal{L}$ in this case), we see that the M2-brane minimizes the following energy functional,

$$E = T_2 \int d^2\sigma \sqrt{\det(\gamma_{ab})} = T_2 \times Vol$$

where $Vol = \int d^2\sigma \sqrt{\det(\gamma_{ab})}$ is the volume of the brane. Thus the energy and volume of the brane are equivalent, up to a (constant) factor of the tension. This calculation also works for probe M5-branes provided there are no fields excited on the world-volume. To summarize, we have shown that for a static probe brane to have minimal energy it must wrap a minimal volume submanifold. Thus, branes wrapping calibrated cycles have minimal energy. Similar results could also be obtained for branes in other supergravity theories.

We now consider a class of manifolds which naturally admit calibrating forms. These are the manifolds of special holonomy. We will see that in many cases these manifolds can be used to construct supergravity solutions, and the calibrations naturally defined on them can be used to find minimal energy cycles that branes can wrap in these backgrounds.

4.2 Special holonomy and calibrations

We begin this section by defining the notion of holonomy. We then introduce the idea of a manifold with special holonomy. Manifolds with special holonomy are interesting because they naturally have calibrating forms associated to them. Moreover, in several cases, these manifolds admit covariantly constant spinors. We will see that in these cases the special holonomy manifolds can be used to construct supersymmetric supergravity solutions, with the constant spinors corresponding to Killing spinors for the background. The calibrating forms defined on the special holonomy manifolds can be then used to find energy minimizing embeddings of branes in these backgrounds. Furthermore, we will see that these minimal energy embeddings are also supersymmetric embeddings, i.e. probe branes wrapping these cycles will preserve some of the background supersymmetry.

4.2.1 Holonomy

Suppose we have a d-dimensional manifold, $\mathcal{M}$, equipped with a Riemannian metric, g, and associated Levi-Civita connection, ∇. The connection ∇ allows us to define parallel transport on the manifold. For example, given a vector $v \in T_x\mathcal{M}$ we can parallel transport this vector around a closed loop, C, which begins and ends at x. This procedure will generally change the direction of v, but it will not change its length (since we are using the Levi-Civita connection). After transporting around C, the resulting vector is related to the original vector v by an $SO(d)$ rotation, i.e.

$$v' = A_C v$$

where A_C is an $SO(d)$ matrix, and the subscript C indicates that this matrix depends on the path taken. We can now consider the collection of matrices $\{A_C\}$ acting on an arbitrary vector $v \in T_x\mathcal{M}$, with C any closed loop through x. This set of matrices forms a subgroup of $SO(d)$ (not necessarily a proper subgroup), called the holonomy group of $\mathcal{M}$ at x, denoted $H_x(\mathcal{M})$. Now suppose we consider another point $y \in \mathcal{M}$. Then assuming $\mathcal{M}$ is connected, y can be connected to x via a piece-wise smooth path γ. Then the holonomy groups at x and y are related by $H_y(\mathcal{M}) = P_\gamma H_x(\mathcal{M}) P_\gamma^{-1}$, where $P_\gamma : T_x\mathcal{M} \to T_y\mathcal{M}$ is a linear transformation associated to the path γ. Therefore, we define the holonomy group of the manifold, denoted H, to be the subgroup $H_x(\mathcal{M})$ defined up to conjugation by elements of $SO(d)$. This definition means that H is independent of the choice of base-point x.

Now the manifold $\mathcal{M}$ is said to have *special holonomy* if H is a proper subgroup of $SO(d)$. Manifolds with special holonomy are characterized by the existence of invariant forms. We will see that these forms can be used as calibrations. The possible Riemannian holonomy subgroups, H, of a d-dimensional manifold, $\mathcal{M}$, have been classified by Berger [82]. This classification is based on the classification of Lie groups, and it uses the fact that the holonomy group strongly restricts the curvature tensor, R_{mnpq}, of the manifold. We briefly discuss each possibility in Berger's classification in turn, and give the invariant forms for each case. We will see that several of the special holonomy manifolds have the right properties to allow them to be used in the construction of supersymmetric supergravity solutions.

4.2.2 Kähler manifolds

The first possibility for the special holonomy group is $H = U(m) \subset SO(2m)$, where $d = 2m$ is the dimension of the manifold, $\mathcal{M}$. Here $m \geq 2$ is an integer, so this possibility can only occur when the manifold is even-dimensional. Manifolds with $H = U(m)$ are called Kähler manifolds. These manifolds are characterized by admitting an invariant 2-form, ω, known as the Kähler 2-form, which obeys $\nabla\omega = 0$. Kähler manifolds automatically admit a complex structure, i.e. they are complex manifolds. This means that there is a complex structure tensor I_i^j, with $I^2 = -1$ $(i, j = 1, \ldots 2m)$, which satisfies certain properties. In fact, the Kähler 2-form is related to the complex structure tensor and the metric as follows,

$$\omega_{ij} = I_i^k \, g_{kj}$$

where ω_{ij} are the components of ω and g_{kj} are the components of the metric on $\mathcal{M}$. Now we can choose an orthonormal basis of 1-forms, $\{e^1, e^2, \ldots e^{2m}\}$, where $e^{2i} = I \cdot e^{2i-1}$ and the dot denotes the action of I on 1-forms. The Kähler 2-form is then

$$\omega = e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m}$$

If we introduce complex coordinates on $\mathcal{M}$, $\{z^p, z^{\bar{q}}\}$, where $p, q = 1, \ldots, m$, then ω can also be written as,

$$\omega = \frac{i}{2}\, g_{p\bar{q}}\, dz^p \wedge dz^{\bar{q}}$$

where $g_{p\bar{q}}$ are the coordinates of the metric with respect to the complex coordinates.

Now, since the Kähler form is covariantly constant, it is also closed, i.e. $d\omega = 0$. Thus ω satisfies the first property required for a calibration. Moreover, ω also satisfies the second property for a calibration, Eq. (4.2) [83]. Therefore, we can use ω as a calibration. In fact, for any Kähler manifold (with $d = 2m$) we have the following set of calibrating forms,

$$\phi = \frac{1}{p!}\, \omega^p$$

where $p = 1, \ldots, m$. The Wirtinger theorem states that the calibrated submanifolds of ϕ are the complex submanifolds, i.e. submanifolds specified by the zeros of a set of holomorphic functions, $f_n(z^1, \ldots, z^m)$. Therefore, the calibrated submanifolds in a Kähler manifold are the complex submanifolds, and these submanifolds have minimal volume in their homology class.

4.2.3 Calabi-Yau manifolds

The second possibility for the holonomy group is $H = SU(m) \subset SO(2m)$ where $d = 2m$ is the dimension of the manifold, $\mathcal{M}$. Again, $m \geq 2$ is an integer, so this requires the manifold to be even-dimensional. These manifolds are called Calabi-Yau, denoted $CY(m)$, and they are a special case of a Kähler manifold. Therefore, like the Kähler case, these manifolds are complex, with complex structure, I. Calabi-Yau manifolds possess two independent invariant forms; ω, the Kähler 2-form, and Ω, the holomorphic $(m, 0)$-form. In an orthonormal frame, $\{e^1, e^2, \ldots e^{2m}\}$, these invariant forms are given by

$$\omega = e^1 \wedge e^2 + \cdots + e^{2m-1} \wedge e^{2m} \qquad (4.6)$$

$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge \cdots \wedge (e^{2m-1} + ie^{2m}) \qquad (4.7)$$

where $e^{2i} = I \cdot e^{2i-1}$, as in the Kähler case. Both ω and Ω are invariant under the action of the holonomy group, $SU(m)$, which means that $\nabla\omega = \nabla\Omega = 0$. Therefore, ω and Ω are closed. We can generically construct two types[4] of calibrations from these forms as

[4] Actually for $m = 4$ there is an additional calibrating form, namely $\frac{1}{2}\omega^2 + Re(e^{i\theta}\Omega)$. This is actually the Cayley 4-form (see § 4.2.5) since $CY(4)$ can be viewed as a special case of a $Spin(7)$ manifold. The calibrated submanifolds are Cayley 4-cycles.

follows,

$$\phi_1 = \frac{1}{p!}\,\omega^p \tag{4.8}$$
$$\phi_2 = Re(e^{i\theta}\Omega) \tag{4.9}$$

where θ is an arbitrary constant phase. Clearly ϕ_1 and ϕ_2 are closed, and they also satisfy the second property for a calibration, Eq. (4.2), as proved in Ref. [84]. The calibrated submanifolds are the complex submanifolds, which are calibrated by ϕ_1 (just as in the Kahler case), and the special Lagrangian submanifolds, which are calibrated by ϕ_2. Special Lagrangian submanifolds have been studied extensively, for example in Ref. [85].

Calabi-Yau manifolds have two interesting properties that general Kahler manifolds do not possess. Firstly, these manifolds admit covariantly constant spinors, i.e. for $CY(m)$ there exist spinors in $d = 2m$ which satisfy $\nabla\rho = 0$, where ∇ is the spin connection associated to the usual Levi-Civita connection. In general, the manifold $CY(m)$ admits two covariantly constant chiral spinors, which are related by complex conjugation. These spinors can be used to construct the forms ω and Ω as follows,

$$\omega_{ij} = -i\rho^\dagger\Gamma_{ij}\,\rho \tag{4.10}$$
$$\Omega_{i_1\dots i_m} = \rho^T\Gamma_{i_1\dots i_m}\rho \tag{4.11}$$

where Γ_i are the Dirac matrices in $d = 2m$, with indices $i, j, \dots = 1, \dots 2m$, and ρ and ρ^* are the covariantly constant (commuting) spinors. Clearly, for the above components of ω and Ω to match the forms given in Eqs. (4.6)-(4.7), the spinors must obey a particular set of projection conditions, as we will see explicitly later. The second interesting property of Calabi-Yau manifolds is that they are Ricci flat, i.e. $R_{ij} = 0$. We will now see that these two properties mean that Calabi-Yau manifolds can be used to construct supersymmetric supergravity solutions. Our discussion will focus on the 11-dimensional supersymmetric supergravity solutions that can be constructed from Calabi-Yau manifolds (and later, from other special holonomy manifolds). However, these ideas carry over to supergravity theories in other dimensions, where similar supersymmetric backgrounds can be constructed from Calabi-Yau manifolds and from some other special holonomy manifolds.

Recall that supersymmetric supergravity solutions are characterized by admitting Killing spinors. The Killing spinor equation in 11 dimensions is schematically given by $\nabla\epsilon + F\cdot\Gamma\epsilon = 0$, where F is the 4-form background field strength and ϵ is a 32-component Majorana spinor. If we set $F = 0$, the Killing spinor equation reduces to $\nabla\epsilon = 0$, i.e. we obtain the covariantly constant spinor equation. Now, Calabi-Yau manifolds have covariantly constant spinors ρ, ρ^*. Therefore, geometries of the form $\mathbb{R}^{1,10-2m} \times CY(m)$

Background	Dimension of $\mathcal{M}$	Holonomy	No. of supersymmetries
$\mathbb{R}^{1,10-d} \times T^d$	d	1	32
$\mathbb{R}^{1,6} \times CY(2)$	4	$SU(2)$	16
$\mathbb{R}^{1,4} \times CY(3)$	6	$SU(3)$	8
$\mathbb{R}^{1,2} \times CY(4)$	8	$SU(4)$	4
$\mathbb{R} \times CY(5)$	10	$SU(5)$	2
$\mathbb{R} \times CY(3) \times CY(2)$	10	$SU(3) \times SU(2)$	4
$\mathbb{R}^{1,2} \times CY(2) \times CY(2)$	8	$SU(2) \times SU(2)$	8
$\mathbb{R}^{1,2} \times HK_2$	8	$Sp(2)$	6
$\mathbb{R}^{1,2} \times Spin(7)$	8	$Spin(7)$	2
$\mathbb{R}^{1,3} \times G_2$	7	G_2	4

Table 4.1: 11-dimensional supersymmetric supergravity backgrounds of the form $\mathbb{R}^{1,10-d} \times \mathcal{M}_d$, together with number of preserved supersymmetries (out of a possible 32). Note that T^d is the d-dimensional torus.

will automatically possess Killing spinors, ϵ, which are obtained from the direct product of a constant spinor in $\mathbb{R}^{1,10-2m}$ with ρ or ρ^*. Therefore, backgrounds constructed from the direct product $\mathbb{R}^{1,10-2m} \times CY(m)$ preserve a non-zero fraction of the supersymmetry. The amount of supersymmetry preserved by these backgrounds is given in Table 4.1.

The backgrounds $\mathbb{R}^{1,10-2m} \times CY(m)$ also satisfy the 11-dimensional supergravity equations of motion (which of course we require for a supergravity background). This is because in the absence of background fields, the only non-trivial supergravity field equation is $R_{ij} = 0$. Calabi-Yau manifolds satisfy this equation as they are Ricci flat, so backgrounds of the form $\mathbb{R}^{1,10-2m} \times CY(m)$ will also satisfy this equation. Therefore, backgrounds constructed from Calabi-Yau manifolds are supersymmetric and satisfy the 11-dimensional supergravity field equations with $F^{(4)} = 0$.

Note that the key properties of Ricci flatness and covariantly constant spinors meant that Calabi-Yau manifolds were suitable to be used in the construction of supersymmetric supergravity backgrounds. However, these properties are actually much more generally found for special holonomy manifolds. In fact, the following three classes of manifolds with special holonomy, which we will discuss in § 4.2.4–4.2.5, admit covariantly constant spinors and are Ricci flat. First, however, we consider the role of the calibrations, ϕ_1 and ϕ_2, in the supergravity backgrounds $\mathbb{R}^{1,10-2m} \times CY(m)$.

Calibrated cycles and supersymmetry

We consider backgrounds of the form $\mathbb{R}^{1,10-2m} \times CY(m)$. We have argued that these background are supersymmetric solutions of 11-dimensional supergravity with $F^{(4)} = 0$. Moreover, these backgrounds have calibrating forms defined on them, which are inherited from the calibrations ϕ_1 and ϕ_2 on $CY(m)$. As usual, the submanifolds calibrated by ϕ_1 and ϕ_2 have minimal volume. Therefore, if we consider wrapping a static probe brane on a calibrated cycle then this brane will have minimal volume in its homology class. As explained in § 4.1, this translates into minimal energy for the brane provided there are no gauge fields excited on its world-volume. In supersymmetric backgrounds, such as $\mathbb{R}^{1,10-2m} \times CY(m)$, not only do branes wrapping calibrated cycles have minimal energy, but they also preserve a non-zero fraction of the supersymmetry.

To see this, let's consider the example of the background $\mathbb{R} \times CY(5)$. This is a solution of 11-dimensional supergravity with metric given by

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j$$

where g_{ij} is a metric with $SU(5)$ holonomy. This background inherits two types of calibration, ϕ_1 and ϕ_2, from the $CY(5)$. We will be interested in the calibrated cycles of ϕ_1. Now $\phi_1 = \omega^p/p!$, and we take ω to be given by

$$\omega = e^1 \wedge e^2 + \cdots + e^9 \wedge e^\natural$$

Here $\{e^1, \ldots, e^9, e^\natural\}$ is an orthonormal basis for the $CY(5)$ metric (which has components g_{ij}) and we have chosen the complex structure so that $e^2 = I \cdot e^1$, and so on. We now consider wrapping a static M2-brane probe on a cycle calibrated by ω. From the form of ω we can have calibrated M2-branes which wrap the following 2-cycles: $\{e^1, e^2\}$, $\{e^3, e^4\}$, $\ldots \{e^9, e^\natural\}$. This is because the volume of these branes is given by the integral of $e^i \wedge e^{i+1}$, and the pull-back of ω to the surface of these branes is also $e^i \wedge e^{i+1}$. Since they are calibrated, these branes will have minimal energy, and we now see that they preserve supersymmetry.

From Table 4.1, the background $\mathbb{R} \times CY(5)$ admits two Killing spinors, ϵ (These spinors are related to the 2 covariantly constant $d = 10$ spinors, ρ and ρ^*, on $CY(5)$, but we will deal with the $d = 11$ spinors, ϵ, from now on.). Due to the choice of complex structure, the projection conditions satisfied by these spinors are [86],

$$\Gamma_{12}\epsilon = \Gamma_{34}\epsilon = \Gamma_{56}\epsilon = \Gamma_{78}\epsilon = \Gamma_{9\natural}\epsilon$$

where the indices on the Γ matrices refer to the orthonormal basis above. We can rewrite these projections using the identity $\Gamma_{0123456789\natural} = 1$ to obtain,

$$\Gamma_{012}\epsilon = \Gamma_{034}\epsilon = \Gamma_{056}\epsilon = \Gamma_{078}\epsilon = \Gamma_{09\natural}\epsilon = \epsilon \tag{4.12}$$

From Chapter 1 we know that these are exactly the supersymmetry projection conditions expected for M2-branes wrapping the cycles $\{e^1, e^2\}$, $\{e^3, e^4\}$, $\ldots \{e^9, e^{10}\}$, i.e. these branes are supersymmetric. Moreover, the conditions in Eq. (4.12) are already satisfied by the Killing spinors on $CY(5)$, so including an M2-brane wrapping any of these calibrated cycles will not break any more supersymmetry, i.e. we can wrap the brane on any of these 2-cycles for free. This is not always the case; usually adding a probe brane to a supersymmetric background reduces the amount of preserved supersymmetry.

Note that the projections in Eq. (4.12) imply that ω can be expressed in the following form,

$$\omega = \frac{1}{2}\epsilon^T \Gamma_{0ij}\epsilon \; e^i \wedge e^j \tag{4.13}$$

where ϵ is either of the two Killing spinors, and we have normalized $\epsilon^T\epsilon = 1$. This is analogous to the expression for ω in terms of ρ and $\rho^\dagger$ given in Eq. (4.10). Note that ϵ is a commuting spinor, and in general the p-forms we construct will involve commuting Killing spinors. In fact, the calibrating forms ϕ_1 and ϕ_2 for this background can all be constructed from Killing spinors in a similar way to Eq. (4.13). In § 4.3 we will show that calibrations constructed from Killing spinors always give calibrated surfaces which correspond to supersymmetric branes. First, however, we discuss the remaining special holonomy groups.

4.2.4 Hyper-Kahler manifolds

The third possibility for the holonomy group is $H = Sp(n) \subset SO(4n)$, where $d = 4n$ is the dimension of the manifold, $\mathcal{M}$. Here $n \geq 2$ is an integer[5], so the dimension of $\mathcal{M}$ must be divisible by 4 in this case. These manifolds are called Hyper-Kahler, denoted HK_n, and they are very similar to Calabi-Yau manifolds. However, Hyper-Kahler manifolds have several complex structures (the number of complex structures is divisible by 3, and each set of 3 satisfies the algebra of imaginary quaternions). For example, an HK_2 manifold, i.e. an 8-dimensional manifold with holonomy group $Sp(2)$, has three complex structures, I^i, $i = 1, 2, 3$, which satisfy the following algebra,

$$I^i \cdot I^j = -\delta^{ij} + \epsilon^{ijk} I^k$$

This is the algebra of imaginary quaternions. These complex structures correspond to 3 independent Kahler 2-forms, ω^i, and 3 holomorphic (4,0)-forms, Ω^i, which are not independent of the Kahler 2-forms, but can be written as linear combinations of $\omega^i \wedge \omega^j$. As before, the forms ω^i and Ω^i are closed, and they can be used to construct calibrating forms exactly as in the Calabi-Yau case (see Eqs. (4.8) and (4.9)).

[5]Note that $Sp(1) \approx SU(2)$ so we do not include the case $n = 1$.

Like the Calabi-Yau case, Hyper-Kahler manifolds possess covariantly constant spinors. For example, HK_2 manifolds admit 3 covariantly constant spinors, from which ω^i and Ω^i can be constructed. Hyper-Kahler manifolds also have the property that they are Ricci flat. Therefore, using the same arguments as in the Calabi-Yau case, these manifolds can be used to construct supersymmetric supergravity backgrounds with vanishing flux. For 11 dimensions, the only possibility is the background $\mathbb{R}^{1,2} \times HK_2$ with $F^{(4)} = 0$. The amount of supersymmetry preserved by this background is 3/16. Again, the idea is that we can use the calibrating forms defined on HK_2 to find energy minimizing cycles for probe branes to wrap in this background. Moreover, since these calibrations admit a Killing spinor construction, the calibrated branes will be supersymmetric (see § 4.3 for a proof of this).

4.2.5 Exceptional holonomy groups

The two remaining holonomy groups of interest are referred to as *exceptional*. This is because these groups only occur for one dimension, d, of the manifold. The first exceptional holonomy group is $H = Spin(7) \subset SO(8)$ which is possible for $d = 8$ manifolds. The second exceptional holonomy group is $G_2 \subset SO(7)$ which can occur when $d = 7$. Both these groups give rise to 8- and 7-dimensional manifolds which are Ricci flat and possess covariantly constant spinors. Therefore, like the Calabi-Yau and Hyper-Kahler cases, these manifolds can be used to construct supersymmetric supergravity solutions with vanishing flux. In 11 dimensions these are $\mathbb{R}^{1,2} \times Spin(7)$ and $\mathbb{R}^{1,3} \times G_2$. The amount of supersymmetry preserved by these backgrounds is given in Table 4.1. Both $Spin(7)$- and G_2-manifolds possess invariant forms, which can be used as calibrations. Like the Calabi-Yau and Hyper-Kahler cases, these calibrations are inherited by the supergravity backgrounds $\mathbb{R}^{1,2} \times Spin(7)$ and $\mathbb{R}^{1,3} \times G_2$ where they can be used to find energy minimizing, supersymmetric embeddings for branes in these backgrounds. We now briefly describe some features of $Spin(7)$- and G_2-manifolds and give the associated calibrating forms.

First, we consider the case where $H = Spin(7)$ and the manifold is 8-dimensional. In this case the manifold possesses a $Spin(7)$ invariant 4-form, ψ, which in an orthonormal basis $\{e^1, \ldots, e^8\}$ can be written as

$$\begin{aligned} \psi &= e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + e^{1357} \\ &\quad - e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457} + e^{2468} \end{aligned} \tag{4.14}$$

where $e^{1234} \equiv e^1 \wedge e^2 \wedge e^3 \wedge e^4$ etc. This form is known as the Cayley 4-form and it satisfies $\nabla\psi = 0$. Thus ψ is also closed. In fact, ψ also satisfies the property in Eq. (4.2) [84], so

ψ is a calibration. Manifolds with $Spin(7)$ holonomy admit one covariantly constant real chiral spinor, ρ. The chirality condition on ρ is $\Gamma_{1...8}\ \rho = \rho$, where the Γ-matrices are real and the indices refer to the orthonormal basis introduced above. The Cayley 4-form can be constructed from ρ as follows,

$$\psi_{ijkl} = -\rho^T \Gamma_{ijkl}\ \rho \tag{4.15}$$

where the indices $i, j, k, l = 1, \ldots, 8$. In the background $\mathbb{R}^{1,2} \times Spin(7)$ the spinor ρ will lift to 2 covariantly constant $d = 11$ spinors. The 4-form ψ can also be constructed using these $d = 11$ spinors, in a similar way to Eq. (4.15).

The second exceptional holonomy group is G_2, which can occur for 7-dimensional manifolds only. Manifolds with G_2 holonomy possess an invariant 3-form, ϕ. In an orthonormal frame $\{e^1, \ldots, e^7\}$ this 3-form can be written as

$$\phi = e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567}$$

This 3-form has the property that $\nabla\phi = 0$, which is equivalent to the two conditions $d\phi = d * \phi = 0$, where $*\phi$ is the dual 4-form to ϕ. Therefore, these manifolds have a closed 3-form, ϕ, and closed 4-form, $*\phi$, associated to them. Both these forms can be used as calibrations as they satisfy the property in Eq. (4.2) [84]. Moreover, manifolds with G_2 holonomy possess a single covariantly constant spinor, ρ. This spinor can be used to construct the 3-form ϕ as follows,

$$\phi_{ijk} = -i\rho^T \Gamma_{ijk} \rho$$

where here the Γ-matrices are purely imaginary, and $i, j, k = 1, \ldots, 7$ in this case. Note that in the background $\mathbb{R}^{1,3} \times G_2$, there are 4 covariantly constant $d = 11$ spinors which are derived from ρ. Both the calibrations ϕ and $*\phi$ admit a construction in terms of these $d = 11$ spinors.

Note that there are also some other possibilities for the special holonomy of a Riemannian manifold that we have not discussed. Firstly, there is the group $H = Sp(m) \times Sp(1)$ where $m = d/4$ and $m \geq 2$. This case is not interesting from the point of view of supergravity solutions, as manifolds which possess this holonomy are not Ricci flat and do not possess covariantly constant spinors. Secondly, Berger's list also includes locally symmetric spaces. These are spaces that are locally isomorphic to G/H for Lie groups G and H. Some simple examples of locally symmetric spaces include $\mathbb{R}^n$ with the Euclidean metric, S^n with the usual sphere metric, and $\mathbb{CP}^n$ with the Fubini-Study metric.

One further note is that we have only considered holonomy groups for Riemannian manifolds. However, Lorentzian holonomy groups are also possible, and these groups have also been classified [87]. These groups can be used to construct supersymmetric supergravity solutions which are non-static [88].

To summarize, we have discussed the possible special holonomy groups for a Riemannian manifold. All these holonomy groups are associated to invariant p-forms, which can be used as calibrations. In the Calabi-Yau, Hyper-Kahler and exceptional cases, the corresponding manifolds are Ricci flat and possess covariantly constant spinors. These two properties mean that these manifolds can be used to construct supersymmetric supergravity backgrounds with vanishing flux. These backgrounds take the form $\mathbb{R}^{1,10-d} \times \mathcal{M}_d$ where $\mathcal{M}_d$ is a d-dimensional manifold with holonomy group either $SU(d/2)$, $Sp(d/4)$, $Spin(7)$ (for $d = 8$ only), G_2 (for $d = 7$ only) or some reducible combination of these groups, e.g. $SU(2) \times SU(2)$ for $d = 8$. The amount of supersymmetry preserved by these backgrounds is given in Table 4.1. We have discussed how the calibrating forms naturally defined on $\mathcal{M}_d$ give minimal energy embeddings for branes in these backgrounds. Moreover, we have seen that all the calibrating forms inherited from M_d (where M_d is one of the choices above) can be constructed from the Killing spinors of the background. We now show that this means that branes wrapping calibrated cycles are supersymmetric, as well as having minimal energy. (We saw this for a particular example of a supersymmetric background earlier. We now show that this is true in general.)

4.3 Calibrations from Killing spinors

In this section we consider 11-dimensional supersymmetric backgrounds with $F^{(4)} = 0$. We show that in these backgrounds the Killing spinors can be used to construct differential forms, and these forms satisfy the properties required for a calibration. Moreover, the calibrated submanifolds are supersymmetric as well as energy minimizing, i.e. if we wrap a static probe brane on a calibrated submanifold, then this brane will be supersymmetric.

Consider an 11-dimensional supergravity solution with $F^{(4)} = 0$ and metric given by[6]

$$ds^2 = -(dx^0)^2 + g_{ij}dx^i dx^j \tag{4.16}$$

where $i, j = 1, \ldots, 10$. We assume this background is supersymmetric, i.e. it admits at least one Killing spinor, ϵ (which is always a commuting spinor for our purposes). Since the background has $F^{(4)} = 0$, ϵ is covariantly constant, i.e. $\nabla\epsilon = 0$. We now construct a p-form from ϵ as follows,

$$\phi = \frac{1}{p!}\bar{\epsilon}\Gamma_{i_1 \ldots i_p}\epsilon \; dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{4.17}$$

[6]Note that this is a special form for the metric, but it is valid for the backgrounds $\mathbb{R}^{1,10-d} \times \mathcal{M}_d$.

where $\bar{\epsilon} = \epsilon^T \Gamma^0$. In fact, the p-form ϕ will vanish unless $p = 1, 2 \bmod 4$. This is due to the symmetry properties of the product of Γ-matrices sandwiched between the spinors (see Appendix A).

We now show that ϕ satisfies the two properties required for a calibration. Firstly, since ϵ is covariantly constant, ϕ is automatically closed. Secondly, if we pull back ϕ to a tangent p-plane, ξ, with coordinates $\sigma^1, \ldots, \sigma^p$, we obtain,

$$\phi\Big|_{\xi} = \epsilon^T \Gamma_{\xi} \epsilon \sqrt{\tilde{\gamma}}\, d^p\sigma \tag{4.18}$$

where the matrix Γ_ξ is given by

$$\Gamma_\xi = -\frac{1}{p!\sqrt{\tilde{\gamma}}}\, \epsilon^{a_1 \ldots a_p} \partial_{a_1} x^{i_1} \ldots \partial_{a_p} x^{i_p} \Gamma_{0 i_1 \ldots i_p} \tag{4.19}$$

and $a_1, \ldots, a_p$ refer to the p-plane coordinates $\sigma^1, \ldots, \sigma^p$. Here we have introduced factors of $\tilde{\gamma}$, which is the determinant of the induced metric on the p-plane (hence $\sqrt{\tilde{\gamma}}\, d^p\sigma$ is the volume form on ξ). For $p = 1, 2 \bmod 4$, the matrix in Eq. (4.19) satisfies $\Gamma_\xi^2 = 1$ and $\Gamma_\xi^T = \Gamma_\xi$. This means that $\frac{1}{2}(1 - \Gamma_\xi)$ is a Hermitian projector, and so

$$\epsilon^T \frac{(1-\Gamma_\xi)}{2} \epsilon = \epsilon^T \frac{(1-\Gamma_\xi)}{2} \frac{(1-\Gamma_\xi)}{2} \epsilon = \left|\left| \frac{(1-\Gamma_\xi)}{2} \epsilon \right|\right|^2 \geq 0$$

That is,

$$\epsilon^T \Gamma_\xi \epsilon \leq \epsilon^T \epsilon = 1 \tag{4.20}$$

where we have chosen to normalize the Killing spinor by $\epsilon^T \epsilon = 1$. Substituting the above inequality in Eq. (4.18) gives

$$\phi\Big|_{\xi} \leq \sqrt{\tilde{\gamma}}\, d^p\sigma = vol\Big|_{\xi} \tag{4.21}$$

Therefore, ϕ satisfies the second property required for a calibration. From Eq. (4.20) it is clear that the calibration bound, Eq. (4.21), is saturated only when $\Gamma_\xi \epsilon = \epsilon$. However, this is exactly the supersymmetry projection condition for a static p-brane wrapping ξ, since the matrix Γ_ξ matches the p-brane projector in Eq. (1.14) in Chapter 1 (if we restrict to the case we are considering here where the brane has $\sigma^0 = t$). Therefore, the calibrated cycles of ϕ are supersymmetric cycles for static p-branes to wrap.

Hence, we have shown that ϕ constructed from Killing spinors as in Eq. (4.17) is a calibration. Furthermore, static probe branes wrapping calibrated cycles of ϕ are supersymmetric (as well as energy minimizing). Note that in the backgrounds $\mathbb{R}^{1,10-d} \times \mathcal{M}_d$, where M_d is one of the special holonomy manifolds discussed in the last paragraph of § 4.2.5, all calibrations inherited from M_d can be constructed from Killing spinors in this way. In more general supersymmetric backgrounds, e.g. when the metric takes a more general form to Eq. (4.16), it is also possible to construct calibrating forms in this way. In fact, even when $F^{(4)} \neq 0$, we can construct generalized calibrations from the Killing spinors of the background, as we will see in the next section.

4.4 Backgrounds with non-zero flux

We now consider 11-dimensional solutions which have a non-zero 4-form field strength, $F^{(4)} \neq 0$. To begin with, we will not assume that these backgrounds are supersymmetric, although we will specialize to the supersymmetric case at the end of § 4.4.2. We wish to find calibrations for branes in these backgrounds. To do this we will need to relax one of the requirements for a calibration, namely the condition that ϕ is closed. However, we will still require the second condition, Eq. (4.2), to hold. In this case ϕ will be referred to as a generalized calibration. We make this modification because in backgrounds with $F^{(4)} \neq 0$ the energy and volume of a probe brane are not equivalent. Rather, the energy of a probe brane is given schematically by $E = Vol + WZ$, where WZ is the Wess-Zumino term for the brane, which arises from the coupling of the brane to the background gauge potential. We will see that in certain circumstances the modification $d\phi \neq 0$ is exactly what is required for calibrated branes to have minimal energy, rather than minimal volume. Roughly speaking, the non-zero derivative of ϕ takes account of the Wess-Zumino term in the energy of the brane.

Generalized calibrations were first introduced in Ref. [89] in the context of anti-de Sitter backgrounds, and were considered for more general backgrounds in Ref. [90]. For branes with non-zero world-volume fields, there are other types of generalized calibrations, which were discussed in Ref. [91].

We begin our discussion by considering the consequences of relaxing the requirement that ϕ is closed. Then we find the condition on $d\phi$ which allows one to associate the quantity minimized by a calibrated cycle with the energy of a brane. In a supersymmetric background we will see that this condition is automatically satisfied by a calibration constructed from Killing spinors. Furthermore, in this case the calibrated cycles are supersymmetric.

4.4.1 Generalized calibrations

We define a generalized calibration, ϕ, to be a p-form which is not necessarily closed, but it satisfies the condition in Eq. (4.2). That is, given any tangent p-plane, ξ, the pull-back of ϕ to ξ is less than or equal to the volume form on ξ, i.e.

$$\phi\Big|_\xi \leq vol\Big|_\xi \tag{4.22}$$

where the volume form is induced from the metric. As before, submanifolds which saturate this calibration bound at every point are referred to as *calibrated*, but these submanifolds will no longer have minimal volume. To see this, consider two p-submanifolds $\mathcal{N}$ and $\mathcal{L}$ in the same homology class. This means we can write $\mathcal{N} = \mathcal{L} + \partial\Xi$, where Ξ is a $(p+1)$-dimensional submanifold. We take $\mathcal{N}$ to be calibrated by ϕ. Therefore,

$$Vol(\mathcal{N}) = \int_{x\in\mathcal{N}} vol\Big|_{T_x\mathcal{N}} = \int_{\mathcal{N}} \phi = \int_{\mathcal{L}+\partial\Xi} \phi \le \int_{x\in\mathcal{L}} vol\Big|_{T_x\mathcal{L}} + \int_{\partial\Xi} \phi \tag{4.23}$$

where, in the last step, we have used the fact that ϕ satisfies Eq. (4.22). We now introduce a reference p-submanifold, $\mathcal{W}$, in the same homology class as $\mathcal{N}$ and $\mathcal{L}$. This means that we can write $\mathcal{N} - \mathcal{W} = \partial\Lambda_1$ and $\mathcal{L} - \mathcal{W} = \partial\Lambda_2$, where Λ_1 and Λ_2 are $(p+1)$-dimensional submanifolds. Using the relationship $\mathcal{N} = \mathcal{L} + \partial\Xi$ we find that $\partial\Xi = \partial\Lambda_1 - \partial\Lambda_2$. Therefore, the inequality in Eq. (4.23) becomes,

$$Vol(\mathcal{N}) \le Vol(\mathcal{L}) + \int_{\partial\Lambda_1} \phi - \int_{\partial\Lambda_2} \phi$$

Using Stoke's theorem, this implies,

$$Vol(\mathcal{N}) - \int_{\Lambda_1} d\phi \le Vol(\mathcal{L}) - \int_{\Lambda_2} d\phi \tag{4.24}$$

Therefore, the calibrated manifold, $\mathcal{N}$, minimizes the quantity

$$Vol(\mathcal{N}) - \int_{\Lambda_1} d\phi \tag{4.25}$$

in its homology class. Note that if ϕ is closed, the calibrated manifold has minimal volume, as before. When ϕ is not closed, the minimized quantity can, under certain circumstances, be associated to the energy of a probe p-brane in a supergravity background with $F^{(4)} \neq 0$. We now describe how this works for M2-brane probes[7]. In this case ϕ is a 2-form and $\mathcal{N}, \mathcal{L}$ are 2-dimensional submanifolds.

4.4.2 Generalized calibrations and energy

Consider an 11-dimensional static supergravity background with $F^{(4)} \neq 0$. This background possesses a time-like Killing vector, K, which we take to be $K = \partial/\partial t$, and we denote the norm of k by $\nu = \sqrt{-K^2}$. We now consider probing this background with

[7]In 11-dimensions we have M2-branes and M5-branes only. The M2-brane case is simpler, so we focus on it here.

a static $M2$-brane with world-volume coordinates t, σ^1, σ^2. The energy functional minimized by this brane is [90]

$$E = T_2 \int d^2\sigma [\sqrt{-\gamma} + \iota_K A^{(3)}]$$

where γ is the determinant of the induced metric on the world-volume and $A^{(3)}$ is a 3-form gauge potential for $F^{(4)}$. Note that $\iota_K A^{(3)}$ is a 2-form constructed from $A^{(3)}$ and K, and this form is evaluated on the brane world-volume. From now on we will drop the indices (3) and (4) on the gauge potential, $A^{(3)}$, and field strength, $F^{(4)}$, for convenience. Now, since the M2-brane is static, the determinant of the induced metric, γ, decomposes as $\gamma = -\nu^2 det(\gamma_{ab})$, where γ_{ab} is the metric induced on the spatial world-volume of the brane, and $a, b = 1, 2$ refer to the coordinates σ^1, σ^2. Therefore, the brane minimizes

$$E = \int d^2\sigma [\nu \sqrt{\det(\gamma_{ab})} + \iota_K A] \tag{4.26}$$

where we have left out the constant overall factor of T_2 since it is not important here. The first term in this expression corresponds to the volume of the brane (ν corresponds to a red-shift factor), and the second term corresponds to the electrostatic energy. If we compare this to Eq. (4.25) we see that the first terms in the two expressions match if we identify $\mathcal{N}$ with the 2-d surface wrapped by the brane. To identify the second terms we require $d\phi = -d\iota_K A$, i.e. ϕ is not closed, but its derivative is specified by the background fields. In this case the second term in Eq. (4.25) becomes,

$$-\int_{\Lambda_1} d\phi = \int_{\Lambda_1} d\iota_K A = \int_{\partial\Lambda_1} \iota_K A = \int_{\mathcal{N}} \iota_K A - c$$

where we have used Stoke's theorem in the second equality and we have rewritten $\partial\Lambda_1 = \mathcal{N} - \mathcal{W}$. Here $c = \int_{\mathcal{W}} \iota_K A$ is an arbitrary constant (since $\mathcal{W}$ is an arbitrary manifold), which cancels from both sides of the inequality in Eq. (4.24). Therefore, provided ϕ satisfies $d\phi = -d\iota_K A$ and we identify $\mathcal{N}$ with the surface of the M2-brane, then the quantity minimized in Eq. (4.25) is equal to the energy of a static M2-brane probe, given in Eq. (4.26). Thus, the calibrated submanifolds of ϕ are minimal energy cycles for M2-branes to wrap in these backgrounds.

If the background is supersymmetric, there is a natural construction of ϕ using Killing spinors. Formally this construction is the same as in Eq. (4.17), except now the constituent Killing spinors are not covariantly constant. Rather, they satisfy the full Killing spinor equation, Eq. (5.1), with non-zero F. Schematically, this is given by $\nabla\epsilon + F \cdot \Gamma\epsilon = 0$. We can construct a calibration 2-form from a Killing spinor, ϵ, as follows,

$$\phi = \frac{1}{2}\bar{\epsilon}\Gamma_{ij}\epsilon dx^i \wedge dx^j \tag{4.27}$$

where $\bar{\epsilon} = \epsilon^T\Gamma^{\hat{0}}$ where $\Gamma^{\hat{0}}$ is a tangent space Γ matrix, i.e. $\Gamma^{\hat{0}} = \nu\Gamma^0$ (In Eq. (4.17) there was no need to distinguish between these two Γ-matrices as $\nu = 1$ for that case.). The derivative of ϕ in Eq. (4.27) can be calculated by replacing derivatives of ϵ using the Killing spinor equation. This procedure gives $d\phi = \iota_K F$. For a certain gauge choice of A, one finds $\iota_K F = -d\iota_K A$, and so $d\phi = -d\iota_K A$ as required. We will prove that $d\phi = \iota_K F = -d\iota_K A$ from the Killing spinor equation explicitly in Chapter 5. Therefore, if we construct ϕ from Killing spinors, then ϕ satisfies the correct condition for the calibrated cycles to correspond to minimal energy submanifolds for branes to wrap. Furthermore, branes wrapping calibrated cycles are supersymmetric. The argument for this follows exactly the same route as in § 4.3.

Note that so far we have only dealt with generalized calibrations for M2-branes. The M5-brane case is more involved as there can be non-zero world-volume gauge fields which will contribute to the energy of the brane. Despite this complication, the calibration bound for an M5-brane in flat space has been derived [91], and the extension to general non-flat backgrounds has been considered in our paper [2]. We will discuss this in § 5.2.

4.4.3 An example: M2-brane background

We now consider a particular example of a background with $F \neq 0$, namely the background sourced by N coincident M2-branes. This background is supersymmetric, and thus possesses Killing spinors. We will show how a 2-form generalized calibration, ϕ, can be constructed from the Killing spinors. We will use this calibration to find supersymmetric embeddings of further M2-branes in this background.

From § 1.2, the metric and 4-form field strength for this background are given by

$$\begin{aligned} ds^2 &= H^{-2/3}(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2) + H^{1/3}((dx^3)^2 + \dots (dx^\natural)^2) && (4.28) \\ F &= dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 && (4.29) \end{aligned}$$

where the M2-branes which source this background are aligned along the x^0, x^1, x^2 directions. Here H is a harmonic function of r, the radial distance away from the branes, where

$$r^2 = \sum_{I=3}^{\natural}(x^I)^2$$

It is straight-forward to check that the vector $K = \partial/\partial x^0$ is a time-like Killing vector for this background, i.e. $\nabla_{(m}K_{n)} = 0$. The vielbein for this metric is given by $e^i = H^{-1/3}dx^i$, for $i = 0, 1, 2$, and $e^I = H^{1/6}dx^I$, for $I = 3, \ldots, \natural$.

This background has 16 Killing spinors, $\epsilon = H^{-1/6}\epsilon_0$, where ϵ_0 are constant spinors (normalized by $\epsilon_0^T\epsilon_0 = 1$) which satisfy the usual M2-brane projection condition:

$$\Gamma_{012}\epsilon_0 = \epsilon_0$$

where the indices on the Γ-matrices refer to the orthonormal basis, $\{e^i, e^I\}$. We now make some additional projections, compatible with the above condition, to select one of the 16 Killing spinors. A possible choice for these additional projections is

$$\Gamma_{034}\epsilon_0 = \Gamma_{056}\epsilon_0 = \Gamma_{078}\epsilon_0 = \Gamma_{09\natural}\epsilon_0 = \epsilon_0$$

$$\Gamma_{13579}\epsilon_0 = \epsilon_0 \tag{4.30}$$

We now use this selected Killing spinor to construct ϕ as in Eq. (4.27). We obtain,

$$\phi = -H^{-1}dx^1 \wedge dx^2 - dx^3 \wedge dx^4 - dx^5 \wedge dx^6 - dx^7 \wedge dx^8 - dx^9 \wedge dx^\natural \tag{4.31}$$

where we have used the orthonormal basis specified above, and the norm of ϵ is $\epsilon^T\epsilon = H^{-1/3}\epsilon_0^T\epsilon_0 = H^{-1/3}$. Since ϕ is constructed from a Killing spinor it will automatically satisfy the calibration bound Eq. (4.22) (we proved this in § 4.3). We now calculate the exterior derivative of ϕ to show that it also satisfies $d\phi = -d\iota_K A$, which means that ϕ will calibrate minimal energy probe M2-branes in this background. The exterior derivative of ϕ is

$$d\phi = -dH^{-1} \wedge dx^1 \wedge dx^2$$

By integrating F we obtain the following 3-form potential,

$$A = H^{-1}dx^0 \wedge dx^1 \wedge dx^2$$

Therefore, $\iota_K A = H^{-1}dx^1 \wedge dx^2$ and so

$$d\iota_K A = dH^{-1} \wedge dx^1 \wedge dx^2 = -d\phi$$

Therefore, ϕ given in Eq. (4.31) satisfies the right property to calibrate minimal energy supersymmetric probe M2-branes in this background. From the form of ϕ in Eq. (4.31), we see that an M2-brane can wrap one of the following 2-cycles and be calibrated: $\{x^1, x^2\}$, $\{x^3, x^4\}$, …, $\{x^9, x^\natural\}$. Including one of these probe branes in the background will reduce the supersymmetry of the configuration by a factor of $\frac{1}{2}$ except if the brane wraps the

$\{x^1, x^2\}$ cycle, in which case the probe M2-brane is parallel to the background M2-branes which source the geometry, and no supersymmetry is broken. Notice that the cycles $\{x^3, x^4\}$, ..., $\{x^9, x^\natural\}$ are selected due to the particular projection conditions chosen in Eq. (4.30). Clearly, we could have chosen these additional projections in a different way, and this would have led to different supersymmetric cycles for probe M2-branes to wrap.

In the next chapter we will consider superalgebras for general supersymmetric backgrounds. In particular, we will be interested in the modifications to the super-translation part of the algebra which arise when probe branes are placed in supersymmetric backgrounds. We will see that the properties of generalized calibrations arise naturally from the modified super-translation algebra. In particular, we will derive the calibration bound for an M5-brane in a general supersymmetric background with non-zero gauge fields on its world-volume.

Chapter 5

Topological charges for branes

In this chapter we will consider supersymmetry algebras in eleven dimensions. In general, a supersymmetry algebra consists of a set of commutators and anti-commutators between the momentum operators P_m, the supersymmetry generators, Q_α, and the Lorentz generators, M_{np} [1]. We will be particularly interested in a sub-algebra of the full supersymmetry algebra which is generated by the operators P_m and Q_α (which we will refer to as the super-translation algebra). In flat space, this sub-algebra is given by the following set of commutators and anti-commutators,

$$[P_m, P_n] = 0, \qquad [P_m, Q_\alpha] = 0$$

$$\{Q_\alpha, Q_\beta\} = (C\Gamma_m)_{\alpha\beta} P^m$$

where C is the charge conjugation matrix. It has been known for some time that if one includes a probe brane in flat space, there is a modification to the supersymmetry algebra. In particular, the anti-commutator $\{Q, Q\}$ acquires an additional term on the right hand side. For example, if one places a probe M2-brane in flat space, one finds the following anti-commutator for the Qs [92],

$$\{Q_\alpha, Q_\beta\} = (C\Gamma_m)_{\alpha\beta} P^m + \frac{1}{2}(C\Gamma_{mn})_{\alpha\beta} Z^{mn} \tag{5.1}$$

where

$$Z^{mn} = \pm \int dx^m \wedge dx^n$$

and the integration is taken over the spatial world-volume of the M2-brane, and the $\pm$ in Z^{mn} corresponds to whether this is a brane or anti-brane [93]. This additional term

[1] If additional charges appear in the algebra, then the full supersymmetry algebra will also include commutation relations amongst these charges, and relations between the charges and the operators P_m, Q_α and M_{np}.

in the algebra is only found when one considers probe M2-branes in the background. However, strictly speaking one should think of the new term as always being present in the algebra, but conventional states (i.e. point particles) are not charged under Z^{mn}, and so for these states the supertranslation algebra reduces to $\{Q_\alpha, Q_\beta\} = (C\Gamma_m)_{\alpha\beta}P^m$. Note that this new term in the algebra is a topological charge for the M2-brane. The charge is topological because it involves the integral of a *closed form*, and therefore it depends only on the homology class of the probe brane configuration.

Similarly, for an M5-brane probe in flat space, the anti-commutator $\{Q, Q\}$ acquires a topological charge given by the integral of a closed form over the 5-dimensional spatial world-volume of the brane (Again, strictly speaking, this charge is always present in the algebra but it is only excited by M5-branes.). In both the M2- and M5-brane cases, the new topological charges in the algebra have trivial commutation relations with the operators P_m and Q_α (at least in the flat space case).

The aim in this chapter is to find the form of the anti-commutator $\{Q, Q\}$ for probe M2- and M5-branes in arbitrary supersymmetric backgrounds. Motivated by the flat space case, we will look for topological charges for the probe branes which arise from integrals of closed forms. We begin by considering the Killing spinors of a general supersymmetric background. These spinors obey a set of differential equations (the Killing spinor equations) which involve the metric and background fields. Following Ref. [28] we can use the Killing spinors of a supersymmetric background to construct a set of p-forms. These forms obey a number of algebraic and differential conditions which will be described in § 5.1.1 and § 5.1.2. In § 5.1.3 we show that the differential conditions satisfied by the forms can be manipulated to produce a closed 2-form and a closed 5-form. In § 5.2 we will argue that these closed forms appear in the super-translation algebra for probe M2-/M5-branes in arbitrary supersymmetric backgrounds. Moreover, we will show how the algebra for the M5-brane can be modified to allow for arbitrary world-volume fields. The super-translation algebras we propose agree with the algebras found for probe branes in various specific backgrounds in Refs. [90–92, 94, 95]. In § 5.3 we will present an example of a non-trivial supersymmetric background. We will explicitly find the expressions for the closed forms and we will use these expressions to obtain the supersymmetry algebras for M2- and M5-branes probes in this background. The original results discussed in this chapter have been published in Ref. [2].

5.1 General 11-d supersymmetric backgrounds

In this section we discuss bosonic supersymmetric solutions (ds^2, F) of 11-dimensional supergravity[2]. Recall from Chapter 1 that these backgrounds are characterized by admitting Killing spinors. The Killing spinor equation is

$$\nabla_m \epsilon + \frac{1}{288}\left[\Gamma_m{}^{n_1 n_2 n_3 n_4} - 8\delta_m^{n_1}\Gamma^{n_2 n_3 n_4}\right] F_{n_1 n_2 n_3 n_4}\epsilon = 0 \tag{5.1}$$

where ϵ is a 32-component real Majorana spinor which transforms under the group $Spin(1, 10)$. Here F is the 4-form field strength for the background, Γ_m are the Dirac matrices in 11 dimensions (with conventions in Appendix A), and the covariant derivative of ϵ is defined by

$$\nabla_m \epsilon = \partial_m \epsilon + \frac{1}{4}\omega^{\hat{n}}{}_{\hat{p}\ m}\Gamma_{\hat{n}}{}^{\hat{p}}\ \epsilon$$

where $\omega^{\hat{n}}{}_{\hat{p}\ m}$ are the components of the connection 1-forms for the metric (the hats denote tangent space indices). For a background to admit a Killing spinor, Eq. (5.1) must be satisfied for each $m = 0, 1, \ldots, \natural$ (so the Killing spinor equation is really 11 equations). The number of independent Killing spinors satisfying these equations corresponds to the number of supersymmetries preserved by the background.

Note that the Killing spinor equations are first order, while the Einstein equations are second order. Therefore, it is generally be easier to solve the Killing spinor equations to find supersymmetric solutions of supergravity. The existence of a Killing spinor guarantees that some components of the Einstein equations are satisfied. In some cases, for example where the metric is diagonal [96], the Killing spinor equations ensure that all components of the Einstein equations are satisfied, and one only needs to check that the equations of motion and Bianchi identities for the field strengths are satisfied.

It is helpful to translate the condition for supersymmetry, Eq. (5.1), into several conditions on p-forms of various degrees that can be constructed from Killing spinors. The idea is to start with a set of commuting Killing spinors, $\{\epsilon^i, i = 1, \ldots N\}$, and construct p-forms with components given by,

$$K_m^{ij} = \bar{\epsilon}^i \Gamma_m\ \epsilon^j \tag{5.2}$$

$$\omega_{mn}^{ij} = \bar{\epsilon}^i \Gamma_{mn}\ \epsilon^j \tag{5.3}$$

$$\Sigma_{mnpqr}^{ij} = \bar{\epsilon}^i \Gamma_{mnpqr}\ \epsilon^j \tag{5.4}$$

where $\bar{\epsilon}^i \equiv (\epsilon^i)^T \Gamma^{\hat{0}}$ is the conjugate spinor ($\Gamma^{\hat{0}}$ acts as the charge conjugation matrix,

[2]In this chapter we drop the indices (4) and (3) from the 4-form field strength, F, and its associated 3-form gauge potential, A.

C), and all spinor indices are contracted. These forms are symmetric in i and j. This is because in each case the product of Γ matrices between the spinors is a symmetric matrix (see Appendix A), and the components of the spinors commute. There is also a set of forms which are anti-symmetric in i and j. Their components are given by,

$$X^{ij} = \bar{\epsilon}^i \epsilon^j \tag{5.5}$$

$$Y^{ij}_{mnp} = \bar{\epsilon}^i \Gamma_{mnp} \epsilon^j \tag{5.6}$$

$$Z^{ij}_{mnpq} = \bar{\epsilon}^i \Gamma_{mnpq} \epsilon^j \tag{5.7}$$

One could also consider constructing p-forms with $p > 6$, but these forms will simply be dual to the lower-dimensional forms. The forms introduced in Eqs. (5.2)-(5.7) obey a number of algebraic and differential conditions [28]. The algebraic conditions arise from Fierz identities, while the differential conditions follow from the Killing spinor equation. We will describe these conditions in § 5.1.1 and § 5.1.2 respectively. However, from now on we will concentrate on the case where the forms are constructed from one (commuting) Killing spinor, ϵ. This reduces the number of possible forms, since X, Y and Z are automatically zero (due to anti-symmetry in i and j) and there is just one 1-form, K, one 2-form, ω, and one 5-form, Σ, to consider. Obviously, since we are just dealing with just one Killing spinor, our results will hold for the most general supersymmetric solutions.

5.1.1 Algebraic conditions

We begin by discussing the algebraic conditions satisfied by the forms. These conditions can be derived from Fierz identities. In particular, the following Fierz identity re-expresses the product $M_{\alpha\beta}N_{\gamma\delta}$ (where M and N are real 32×32 matrices, and $\alpha, \beta, \ldots$ are spinor indices) in terms of a basis of Γ matrices,

$$\begin{aligned} M_{\alpha\beta}N_{\gamma\delta} = \frac{1}{32}\Big[& (NM)_{\alpha\delta}\delta_{\gamma\beta} + (N\Gamma^m M)_{\alpha\delta}(\Gamma_m)_{\gamma\beta} - \frac{1}{2!}(N\Gamma^{mn}M)_{\alpha\delta}(\Gamma_{mn})_{\gamma\beta} \\ & -\frac{1}{3!}(N\Gamma^{mnp}M)_{\alpha\delta}(\Gamma_{mnp})_{\gamma\beta} + \frac{1}{4!}(N\Gamma^{mnpq}M)_{\alpha\delta}(\Gamma_{mnpq})_{\gamma\beta} \\ & +\frac{1}{5!}(N\Gamma^{mnpqr}M)_{\alpha\delta}(\Gamma_{mnpqr})_{\gamma\beta}\Big] \end{aligned} \tag{5.8}$$

If we take $M = \Gamma_{n_1 \ldots n_p}$ and $N = \Gamma^{n_1 \cdots n_p}$, for $p = 1, 2, 5$, then the following identities between the forms K, ω and Σ can be derived [28],

$$\Sigma^2 = -6K^2 \tag{5.9}$$

$$\omega^2 = -5K^2 \tag{5.10}$$

where for a p-form, α, we define

$$\alpha^2 = \frac{1}{p!}\, \alpha_{n_1 \dots n_p}\, \alpha^{n_1 \dots n_p}$$

The identities in Eqs. (5.9) and (5.10) suggest that there is only one independent Lorentz scalar (which we can take to be K^2) that can be constructed from K, ω and Σ. In Ref. [87] this was proved to be the case and it was also shown that $K^2 \leq 0$, i.e. K is either time-like or null. Since K^2 is a Lorentz scalar it will remain fixed under Lorentz transformations of the constituent spinor, which correspond to transformations of the spinor under the group $Spin(1,10)$. Therefore, we can label the orbits of $Spin(1,10)$ by the value of K^2. Since $K^2 = 0$ or $K^2 < 0$, there are only two orbits of the group because the spinors with $K^2 < 0$ can always be re-scaled to give the same value of K^2, e.g. $K^2 = -1$. Moreover, $Spin(1,10)$ acts transitively on the level sets of K^2 [87], so different spinors in the same orbit are related by a Lorentz transformation.

For the two cases, $K^2 = 0$ and $K^2 < 0$, we can choose a convenient set of projection conditions which define the constituent spinor (up to scale). For $K^2 < 0$ a possible set of projection conditions is

$$\Gamma_{012}\epsilon = \Gamma_{034}\epsilon = \Gamma_{056}\epsilon = \Gamma_{078}\epsilon = \Gamma_{09\natural}\epsilon = \epsilon$$

$$\Gamma_{013579}\epsilon = \epsilon \tag{5.11}$$

where here the Γ matrices all have tangent space indices, i.e. $(\Gamma_m)^2 = \pm 1$. In fact, one of the above conditions is not independent, since $\Gamma_{0123456789\natural} \equiv 1$. The forms K, ω and Σ corresponding to this set of projection conditions are given by,

$$K = \Delta e^0 \tag{5.12}$$

$$\omega = -\Delta(e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 + e^7 \wedge e^8 + e^9 \wedge e^\natural) \tag{5.13}$$

$$\Sigma = -\frac{1}{2}\Delta^{-2} K \wedge \omega \wedge \omega - \Delta Re(\Omega) \tag{5.14}$$

where Ω is the complex 5-form,

$$\Omega = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \wedge (e^7 + ie^8) \wedge (e^9 + ie^\natural)$$

and $\Delta = \epsilon^T \epsilon$ is the norm of the spinor. Note that clearly K in Eq. (5.12) is timelike. Moreover, it is easy to check that the forms obey the algebraic identities in Eqs. (5.9) and (5.10). Now, since $Spin(1,10)$ acts transitively on the level sets of K^2, this means that the projection conditions for *any* spinor with $K^2 < 0$ can be brought into this particular form (5.11) by an appropriate choice of vielbein (choosing a frame is the same as performing a

$Spin(1,10)$ rotation on the spinor).

The above p-forms define an $SU(5)$ structure on the underlying manifold. This structure group corresponds to the stability group of the spinor used to construct the forms. In general, this group is referred to as the G-structure. Information about the G-structure has been used in Refs. [28] to classify the form of supergravity solutions which possess Killing spinors. For example, in Ref. [28] the $SU(5)$ structure corresponding to timelike K was used to specify properties of the metric and 4-form field strength for solutions which possess such a Killing spinor.

We now consider the case where $K^2 = 0$. In this case a possible set of projection conditions for the constituent spinor is

$$\Gamma_{01}\epsilon = -\epsilon$$

$$\Gamma_{2345}\epsilon = \Gamma_{2367}\epsilon = \Gamma_{2389}\epsilon = \Gamma_{2468}\epsilon = -\epsilon \tag{5.15}$$

These conditions together with $\Gamma_{0123456789\natural} = 1$ give an additional equation

$$\Gamma_{\natural}\epsilon = -\epsilon \tag{5.16}$$

where all Γ matrices in the above equations have tangent space indices. The forms corresponding to the above projection conditions are

$$K = \Delta(e^0 + e^1) \tag{5.17}$$

$$\omega = -K \wedge e^{\natural} \tag{5.18}$$

$$\Sigma = -K \wedge \phi \tag{5.19}$$

where ϕ is the Cayley 4-form,

$$\begin{aligned}\phi = {} & e^{2345} + e^{6789} + e^{2367} - e^{2569} - e^{3478} + e^{2468} + e^{3579} \\ & + e^{4589} + e^{4567} - e^{3469} + e^{2389} - e^{2578} - e^{2479} - e^{3568}\end{aligned} \tag{5.20}$$

and we are using the short-hand notation $e^{2345} \equiv e^2 \wedge e^3 \wedge e^4 \wedge e^5$ etc. Note that K in Eq. (5.17) is clearly null and the forms obey the algebraic identities in Eqs. (5.9) and (5.10). Again the idea is that the projection conditions for any Killing spinor with null K can be put into the above form by an appropriate choice of vielbein. In this case the forms define a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. This corresponds to the stability group of the spinor used to construct the forms. The form of supergravity solutions which possess this structure have been classified in Ref. [29].

It is important to note that the situation is much more complicated when we consider forms constructed from more than one Killing spinor. In these cases, typically more than one independent Lorentz scalar can be constructed. Consequently, there will be more orbits of $Spin(1,10)$ to consider, and correspondingly more sets of projection conditions to be defined. These backgrounds have not yet been considered in general, and it would be very interesting to investigate them and to try to formulate a classification for the corresponding solutions.

So far, we have only dealt with the consequences of the algebraic relations in Eqs.(5.9)–(5.10). However, there are many other algebraic relations that can be derived from the Fierz identity in Eq. (5.8). For example, the following identities can be derived [28],

$$\iota_K\omega = 0, \qquad \iota_K\Sigma = \frac{1}{2}\,\omega\wedge\omega, \tag{5.21}$$

$$\omega\wedge\Sigma = \frac{1}{2K^2}K\wedge\omega\wedge\omega\wedge\omega \tag{5.22}$$

where the components of the 1-form $\iota_K\omega$ are given by $(\iota_K\omega)_m \equiv K^n\omega_{nm}$ (see Appendix A for the full definition of the interior product of forms). These identities can also be derived very simply using the explicit p-forms obtained for time-like and null K in Eqs. (5.12)–(5.14) and Eqs. (5.17)–(5.19).

5.1.2 Differential conditions

We now derive the differential equations satisfied by the forms following Ref. [28]. These equations arise because the forms are constructed from spinors which satisfy the Killing spinor equation, Eq. (5.1). To illustrate this we first consider a general p-form, ϕ, constructed from a Killing spinor, ϵ. The components of ϕ are

$$\phi_{n_1\dots n_p} = \bar{\epsilon}\Gamma_{n_1\dots n_p}\epsilon$$

We can calculate the covariant derivative of ϕ as follows,

$$\begin{aligned}\nabla_m\phi_{n_1\dots n_p} &= \nabla_m\left(\bar{\epsilon}\Gamma_{n_1\dots n_p}\epsilon\right)\\ &= \overline{(\nabla_m\epsilon)}\Gamma_{n_1\dots n_p}\epsilon + \bar{\epsilon}\Gamma_{n_1\dots n_p}\nabla_m\epsilon\end{aligned}$$

where $\overline{(\nabla_m\epsilon)} = (\nabla_m\epsilon)^T\Gamma^{\hat{0}}$ and the Killing spinor equation is used to make the following replacements,

$$\begin{aligned}\nabla_m\epsilon &= -\frac{1}{288}\left[\Gamma_m{}^{r_1r_2r_3r_4} - 8\delta_m^{r_1}\Gamma^{r_2r_3r_4}\right]F_{r_1r_2r_3r_4}\epsilon\\ \overline{(\nabla_m\epsilon)} &= \frac{1}{288}\,\bar{\epsilon}\left[\Gamma_m{}^{r_1r_2r_3r_4} + 8\delta_m^{r_1}\Gamma^{r_2r_3r_4}\right]F_{r_1r_2r_3r_4}\end{aligned}$$

Using this technique to calculate the covariant derivatives of K, ω and Σ one finds [28],

$$\nabla_m K_n = \frac{1}{6}\,\omega^{r_1r_2}F_{r_1r_2mn} + \frac{1}{6!}\,\Sigma^{\,r_1r_2r_3r_4r_5} * F_{r_1r_2r_3r_4r_5mn} \tag{5.23}$$

$$\begin{aligned}\nabla_m\omega_{np} &= \frac{1}{3\cdot 4!}\,g_{m[n}\Sigma_{p]}{}^{r_1r_2r_3r_4}F_{r_1r_2r_3r_4} + \frac{1}{3\cdot 3!}\,\Sigma_{np}{}^{r_1r_2r_3}F_{mr_1r_2r_3}\\ &- \frac{1}{3\cdot 3!}\,\Sigma_{m[n}{}^{r_1r_2r_3}F_{p]r_1r_2r_3} + \frac{1}{3}\,K^rF_{rmnp}\end{aligned} \tag{5.24}$$

$$\begin{aligned}\nabla_m\Sigma_{npqrs} &= \frac{1}{6}K^r * F_{rmnpqrs} - \frac{10}{3}F_{m[npq}\omega_{rs]} - \frac{5}{6}F_{[npqr}\omega_{s]m}\\ &- \frac{10}{3}g_{m[n}\omega_p{}^tF_{|t|qrs]} + \frac{5}{6}\,F_{m[n|r_1r_2|}(*\Sigma)^{r_1r_2}{}_{pqrs]}\\ &+ \frac{5}{6}\,F_{[np|r_1r_2|}(*\Sigma)^{r_1r_2}{}_{qrs]m} - \frac{5}{9}g_{m[n}F_{p|r_1r_2r_3|}(*\Sigma)^{r_1r_2r_3}{}_{qrs]}\end{aligned} \tag{5.25}$$

Anti-symmetrizing over all free indices in the equations above, we obtain the following equations for the exterior derivatives of the forms,

$$dK = \frac{2}{3}\,\iota_\omega F + \frac{1}{3}\,\iota_\Sigma * F \tag{5.26}$$

$$d\omega = \iota_K F \tag{5.27}$$

$$d\Sigma = \iota_K * F - \omega\wedge F \tag{5.28}$$

Moreover, symmetrizing over the indices m, n in Eq. (5.23) we find,

$$\nabla_{(m}K_{n)} = 0$$

i.e. K is a Killing vector. This is equivalent to the statement that $\mathcal{L}_K g_{mn} = 0$, where $\mathcal{L}$ is the Lie derivative. This means K is a symmetry of the metric. In fact, K will be a symmetry of the full solution if additionally $\mathcal{L}_K F = 0$. To calculate $\mathcal{L}_K F$ we use the following definition of the Lie derivative of a general p-form, ψ, along a vector, X,

$$\mathcal{L}_X\psi = d(\iota_X\psi) + \iota_X d\psi \tag{5.29}$$

Applying this rule to calculate $\mathcal{L}_K F$ we find,

$$\mathcal{L}_K F = \iota_K dF + d\iota_K F = d(d\omega) = 0 \tag{5.30}$$

where we have used the Bianchi identity for F together with Eq. (5.27) in the last step. Therefore, K is a bosonic symmetry of the supersymmetric solution (ds^2, F, ϵ). This fact will be useful when we construct closed forms in the next section.

5.1.3 Constructing the closed forms

In this section we construct a closed 2-form and a closed 5-form for a general supersymmetric solution (ds^2, F, ϵ). This will involve adding pieces involving background fields to the forms ω and Σ to cancel the terms on the right hand side of Eqs. (5.27) and (5.28). In § 5.2 we will argue that the closed 2-form and 5-form that we construct appear as topological charges in the superalgebras for M2- and M5-brane probes in general supersymmetric backgrounds.

Recall from the last section that the 1-form K is a symmetry of a general supersymmetric solution, i.e. $\mathcal{L}_K g = \mathcal{L}_K F = 0$. Now, the 4-form field strength F is related to a 3-form gauge potential, A, via $F = dA$. Therefore, we can choose a gauge for A which respects the symmetry generated by K, i.e. $\mathcal{L}_K A = 0$. To see that this gauge choice is always possible, we use the fact that the following commutator acting on an arbitrary form, ϕ, vanishes:

$$[\mathcal{L}_K, d]\ \phi = 0 \tag{5.31}$$

This commutation relation is easily proved using the expression for the Lie derivative given in Eq. (5.29). Applying this commutator to A we find that $d(\mathcal{L}_K A) = 0$ since $\mathcal{L}_K F = 0$. Therefore, locally we can write $\mathcal{L}_K A = d\Omega$, where Ω is a 2-form. Now if we make a gauge transformation on A, i.e. $A \longrightarrow A + d\Lambda$, this relation becomes

$$\mathcal{L}_K A \ = \ d\Omega - \mathcal{L}_K(d\Lambda) = d(\Omega - \mathcal{L}_K\Lambda)$$

where in the second equality we have used Eq. (5.31) again. Now we can choose Λ appropriately so that $\mathcal{L}_K\Lambda = \Omega$, and we have $\mathcal{L}_K A = 0$ in this particular gauge. Therefore, we have shown that if $\mathcal{L}_K F = 0$ then we can always choose a gauge for A which satisfies $\mathcal{L}_K A = 0$. Note that this proof works for field strengths of any dimension and their related gauge potentials.

We now discuss the consequences of choosing this gauge for A. Using the definition of the Lie derivative given in Eq. (5.29) we have

$$\mathcal{L}_K A = d(\iota_K A) + \iota_K dA = 0 \tag{5.32}$$

i.e. $d(\iota_K A) = -\iota_K F$. Using this relation together with Eq. (5.27) we find that the 2-form $\omega + \iota_K A$ is closed:

$$d(\omega + \iota_K A) = \iota_K F - \iota_K F = 0$$

We will see that this closed 2-form appears naturally in the supersymmetry algebra for an M2-brane probe in a supersymmetric background. In particular, it gives rise to a topological charge for the brane.

Similarly, we can find a closed 5-form. Firstly, since $\mathcal{L}_K F = \mathcal{L}_K g = 0$ this implies that $\mathcal{L}_K * F = 0$ also. Now $*F$ is a 7-form field strength which satisfies the following equation of motion,

$$d * F + \frac{1}{2} F \wedge F = 0$$

We can introduce a 6-form gauge potential, C, for $*F$ which satisfies,

$$dC = *F + \frac{1}{2} A \wedge F$$

Then $d^2 C = 0$ gives the equation of motion for $*F$. Since $\mathcal{L}_K * F = \mathcal{L}_K F = \mathcal{L}_K A = 0$ we can choose a gauge for C which respects this symmetry, i.e. $\mathcal{L}_K C = 0$ (the proof for this is exactly the same as above). If we choose the gauge in this way then the 5-form, $\Sigma + \iota_K C + A \wedge (\omega + \frac{1}{2}\iota_K A)$, is closed:

$$\begin{aligned} d\left(\Sigma + \iota_K C + A \wedge (\omega + \frac{1}{2}\iota_K A)\right) &= \iota_K * F - \omega \wedge F + \mathcal{L}_K C - \iota_K \left(*F + \frac{1}{2} A \wedge F\right) \\ &\quad + F \wedge (\omega + \frac{1}{2}\iota_K A) - A \wedge \left(\iota_K F + \frac{1}{2}(\mathcal{L}_K A - \iota_K F)\right) \\ &= 0 \end{aligned}$$

where we have used the gauge choice $\mathcal{L}_K A = 0$ and the fact that $\iota_K (A \wedge F) = \iota_K A \wedge F - A \wedge \iota_K F$ to show that these terms sum to zero. This closed 5-form will appear in the superalgebra for an M5-brane probe in an arbitrary supersymmetric background, as we will soon see.

5.2 Supersymmetry algebras

We begin this section by introducing the full supersymmetry algebra for conventional states in 11-dimensional flat Minkowski space. This is given by the following set of commutators and anti-commutators:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma_m) P^m \tag{5.33}$$

$$\begin{gathered} [P_m, Q_\alpha] = 0, \qquad [P_m, P_n] = 0 \\ [M_{mn}, M_{pq}] = \eta_{m[p} M_{q]n} - \eta_{n[p} M_{q]m}, \\ [M_{mn}, Q_\alpha] = -\frac{1}{4} (\Gamma_{mn} Q)_\alpha, \qquad [P_m, M_{np}] = \eta_{m[n} P_{p]} \end{gathered} \tag{5.34}$$

Here Q_α is a 32-component Majorana spinor, P_m generates translations and M_{mn} are the Lorentz generators. The matrix C is the charge conjugation matrix, which we will take to be $\Gamma^{\hat{0}}$ from now on. We will be mostly interested in the part of the algebra which involves the anti-commutator of the Qs (the super-translation algebra). We can rewrite this anti-commutator by introducing a constant commuting Majorana spinor parameter, ϵ^α (which is a Killing spinor for flat space – the set of Killing spinors in flat space is simply the set of all constant spinors). Then the anti-commutator becomes,

$$\{\epsilon^\alpha Q_\alpha, \epsilon^\beta Q_\beta\} = (\epsilon^T C \Gamma_m \epsilon) P^m \tag{5.35}$$

Notice that the term in round brackets in Eq. (5.35) is simply $\bar{\epsilon}\Gamma_m\epsilon \equiv K_m$, where K is the 1-form we defined in § 5.1. In this case the components of K are constant, since the constituent Killing spinor is constant. We can write this anti-commutator in the following short-hand notation

$$2(\epsilon Q)^2 = K^m P_m \tag{5.36}$$

This is equivalent to the original anti-commutator, Eq. (5.33), if we demand that the above equation holds for arbitrary constant ϵ (i.e. we demand it to hold for an arbitrary Killing spinor ϵ of the background).

We now move on to consider general supersymmetric backgrounds. Such backgrounds possess Killing spinors, $\epsilon = \epsilon(x^m)$, which satisfy the Killing spinor equation, Eq. (5.1). These spinors are generally not constant, but depend on the 11-dimensional coordinates, x^m. Each Killing spinor corresponds to a preserved supercharge, ϵQ, for the solution. The algebra of these supercharges is given by Eq. (5.36), where K is now a field (i.e. not constant) constructed from the relevant Killing spinor.

Since ϵQ are supercharges, they correspond to fermionic symmetries of the background. Therefore, we expect that $K^m P_m$ is a bosonic charge and corresponds to a bosonic symmetry of the solution (since it arises from the anti-commutator of fermionic charges). In general, a bosonic symmetry is associated to an infinitesimal coordinate transformation which leaves the solution invariant (i.e. an "infinitesimal diffeomorphism"). Such a coordinate transformation is associated to a vector field which acts by the Lie derivative. The quantity $K^m P_m$ is associated to the vector field K, and therefore it acts on supergravity fields by the operator $\mathcal{L}_K$. However, in § 5.1.2 we proved that $\mathcal{L}_K g = \mathcal{L}_K F = 0$, for $K = \bar{\epsilon}\Gamma\epsilon$. That is, the background fields are invariant under this action. Hence $K^m P_m$ is a bosonic charge for the supersymmetric solution (ds^2, F, ϵ) and it corresponds to a bosonic symmetry of the solution. The algebra satisfied by the bosonic charges is

$$[K \cdot P, J \cdot P] = (\mathcal{L}_K J) \cdot P$$

where K and J are Killing vectors constructed from Killing spinors. As expected, these bosonic charges act on each other by the Lie derivative. Note that because K and J are both Killing vectors the Lie derivative satisfies $\mathcal{L}_K J = -\mathcal{L}_J K$, as required for the commutator. We can also consider the mixed bosonic-fermionic commutator. This is given by

$$[K \cdot P, \epsilon Q] = (\mathcal{L}_K \epsilon)^\alpha Q_\alpha$$

where $\mathcal{L}_K$ here is the spinorial Lie derivative, which is defined only along Killing vector fields.

So far we have considered the part of the supersymmetry algebra which is generated by the Killing spinors of the background. However, the background may possess other bosonic symmetries which are generated by vector fields which do not take the form $K_m = \bar{\epsilon}\Gamma_m \epsilon$. In fact, it is generally not possible to construct all Killing vector fields for a background from the Killing spinors. For example, in flat space there are Killing vectors corresponding to the rotational symmetry of the background which are not constructible from the flat space Killing spinors. In general, the supersymmetry algebra for a background is determined by the Killing spinors up to purely bosonic factors [97,98].

We now consider the addition of branes to general supersymmetric backgrounds. We will be interested in the modifications to the anti-commutator of fermionic charges, Eq. (5.36). We will find that the branes induce additional topological charges in this part of the algebra.

5.2.1 Supersymmetry algebra for M2-branes

In this section we consider adding a probe M2-brane to a general supersymmetric background. However, before we consider the situation for a general background, we first review the algebra for flat space coupled to an M2-brane probe. In particular, we will be interested in the anti-commutator of the supersymmetry generators, Q. This was first considered in Ref. [92].

Recall that the Lagrangian for a probe M2-brane is schematically given by

$$\mathcal{L}_{M2} = -T_2(\sqrt{-g} + \mathcal{P}(A))$$

where g is the determinant of the induced metric on the brane's world-volume and $\mathcal{P}(A)$ is the 3-form gauge potential pulled back to the brane. We could write this Lagrangian in the full super-space formalism, i.e. with fermions as well as bosons induced on the brane world-volume. We could then perform a supersymmetry transformation on all supergravity fields. Under such a transformation, one finds that the Lagrangian is not manifestly invariant; rather, it changes by a total derivative term. This induces a modification to the fermionic charges, Q, and the anti-commutator changes as follows,

$$\{Q_\alpha, Q_\beta\} = (C\Gamma_m)_{\alpha\beta} P^m + \frac{1}{2}\,(C\Gamma_{mn})_{\alpha\beta} Z^{mn} \tag{5.37}$$

where

$$Z^{mn} = \pm \int dx^m \wedge dx^n \tag{5.38}$$

and the integration is taken over the spatial world-volume of the M2-brane and the $\pm$ corresponds to brane/anti-brane. If we introduce coordinates (σ^1, σ^2) on the spatial world-volume of the brane then Z^{mn} is given explicitly by

$$Z^{mn} = \pm \int \epsilon^{ij} \frac{\partial x^m}{\partial \sigma^i} \frac{\partial x^n}{\partial \sigma^j} d^2\sigma$$

As in the previous section, we can rewrite this algebra by introducing a constant commuting spinor ϵ^α. Then the anti-commutator becomes,

$$2(\epsilon Q)^2 = K_m P^m \pm \frac{1}{2}\,\omega_{mn} Z^{mn} \tag{5.39}$$

where ω_{mn} are components of the 2-form ω constructed in § 5.1 from Killing spinors (which are constant spinors in flat space). If we substitute the integral for Z^{mn} into Eq. (5.39) and rewrite the momentum, P^m, as an integral of the momentum density, $p^m(\sigma)$, over the spatial world-volume of the brane, then we obtain

$$2(\epsilon Q)^2 = \int d^2\sigma K_m p^m(\sigma) \pm \int \omega \tag{5.40}$$

where we have brought the constant coefficients K_m and ω_{mn} inside the integrals. In particular, the second term combines nicely to give the integral of the 2-form ω. The above expression is valid for a probe M2-brane in flat space.

We now consider the case of a probe M2-brane in a general supersymmetric background. We propose that the generalization to the super-translation algebra is given by

$$2(\epsilon Q)^2 = \int d^2\sigma K_m p^m(\sigma) \pm \int (\omega + \iota_K A) \tag{5.41}$$

where ϵ is a Killing spinor for the background and K and ω are fields constructed from ϵ, as described in § 5.1. Now, as shown in § 5.1.3, the 2-form $\omega + \iota_K A$ is closed. This means that the extension to the algebra in Eq. (5.41) is topological. This is a property which is generally expected for extensions to supersymmetry algebras [90]. Moreover, this generalization agrees with the algebra for M2-branes in curved backgrounds for timelike K, which was presented in Ref. [90]. However, here we do not require K to be time-like.

Note also that the combination $\int(K_m p^m + \iota_K A)$ is very natural, since it generalizes the replacement of p_m with $p_m + A_m$ for a charged particle in an electromagnetic field. Here the M2-brane is electrically charged with respect to the 3-form potential A.

Since $(\epsilon Q)^2 \geq 0$, the super-translation algebra, Eq. (5.41), gives rise to a BPS bound on the energy/momentum of the M2-brane. We find,

$$\int d^2\sigma K_m p^m(\sigma) \geq \mp \int (\omega + \iota_K A) \tag{5.42}$$

where the left hand side is the energy/momentum of the M2-brane, and the right hand side is a topological charge for the brane. The $\mp$ in this bound can be chosen to make the topological charge term positive, so that the bound is of the type $E \geq |Q|$. Note that the topological charge term is only defined up to the addition of closed forms, i.e. we are free to add $d\Omega$ to the integrand on the right hand side, where Ω is any 1-form.

We now discuss the connection between the BPS bound given above and the calibrations and generalized calibrations that were discussed in Chapter 4. Firstly, we consider a static probe M2-brane in flat space. We take K to be the time-like 1-form dx^0. Since we are in flat space $\iota_K A = 0$, and from the choice of K we have $K \cdot p = -p_0$. We can identify $-p_0$ with the Hamiltonian density $\mathcal{H}$ [90]. However, for static probes in backgrounds with $F = 0$, $\mathcal{H}$ is simply equal to the volume density (as we saw in Chapter 4). Therefore, $K \cdot p = vol$ and the above bound becomes

$$\int d^2\sigma \; vol \geq \mp \int \omega$$

This is (the integrated) ordinary calibration bound in Eq. (4.2), where ω is the calibration form, and the $\mp$ refers to the orientation of the brane. Moreover, since we are in flat space, we also have $d\omega = 0$, as required for these calibrations. Therefore, essentially we find the ordinary calibration conditions of § 4.1 from the BPS bound in Eq. (5.42). Note that this argument would also follow through for 11-dimensional backgrounds constructed from Calabi-Yau and Hyper-Kahler manifolds (these backgrounds possess 2-form calibrations and they are supersymmetric).

The second case we consider is where the probe M2-brane is static, but the background has $F \neq 0$. However, we assume that the background possesses a time-like Killing vector, which we identify with K. Then the left hand side of the BPS bound becomes $-p_0 = \mathcal{H}$, and the right hand side is $\omega + \iota_K A$. Rearranging the bound (and choosing a definite orientation for the brane) then gives

$$\int d^2\sigma(\mathcal{H} - \iota_K A) \geq \int \omega$$

where $\iota_K A$ is evaluated on the brane world-volume. Now, recall from § 4.4 that in this case the Hamiltonian density is simply given by the volume density plus $\iota_K A$, i.e. $\mathcal{H} = vol + \iota_K A$. Therefore, the bound above simply reduces to $\int d^2\sigma\ vol \geq \int \omega$, as expected from § 4.4. Moreover, we have seen that the Killing spinor equations imply $d\omega = \iota_K F$. Now if we choose the gauge $\mathcal{L}_K A = 0$ then $\iota_K F = -d(\iota_K A)$ and hence $d\omega = -d(\iota_K A)$, as required for the generalized calibrations in § 4.4. Therefore, we have seen that both types of calibrations considered in Chapter 4 arise from the super-translation algebra above. However, the BPS bound (or calibration bound) we have constructed in Eq. (5.42) is applicable to more general situations, for example where K is null, or where the brane is non-static. We will consider these cases when we discuss generalized calibrations for giant gravitons in Chapter 7.

5.2.2 Supersymmetry algebra for M5-branes

We now consider the super-translation algebra for a probe M5-brane in an arbitrary supersymmetric background. As in the M2-brane case, we begin by considering a probe M5-brane coupled to flat space. In this case, the anti-commutator of the Qs is

$$\{Q_\alpha, Q_\beta\} = (C\Gamma_m)_{\alpha\beta} P^m + \frac{1}{5!}\,(C\Gamma_{mnpqr})_{\alpha\beta} Z^{mnpqr}$$

where

$$Z^{mnpqr} = \pm \int dx^m \wedge dx^n \wedge dx^p \wedge dx^q \wedge dx^r$$

and the integration is taken to be over the spatial world-volume of the M5-brane. As in the previous case, the $\pm$ simply refers to whether the probe is a brane or anti-brane. In analogy with the M2-brane case, we can introduce a constant commuting Majorana spinor, ϵ^α, and rewrite this as

$$2(\epsilon Q)^2 = K_m P^m \pm \frac{1}{5!}\Sigma_{mnpqr} Z^{mnpqr}$$

where K and Σ are the 1-form and 5-form constructed in § 5.1 from Killing spinors (which are constant spinors in flat space). We now substitute the expression for Z^{mnpqr}, and write P^m as an integral of the momentum density, $p^m(\sigma)$, over the spatial world-volume of the brane. Then the algebra becomes

$$2(\epsilon Q)^2 = \int d^5\sigma K_m p^m(\sigma) \pm \int \Sigma$$

where both integrations are over the 5-dimensional spatial world-volume of the brane, and $\sigma^1, \ldots \sigma^5$ are coordinates on this space.

Our proposal for a general supersymmetric background is to replace the integral of Σ by the integral of the closed form $\Sigma + \iota_K C + A \wedge (\omega + \frac{1}{2}\iota_K A)$ which we constructed in § 5.1.3 (Recall that we will need to impose the gauge choices $\mathcal{L}_K A = \mathcal{L}_K C = 0$ to ensure this 5-form is closed.). That is, the anti-commutator becomes

$$2(\epsilon Q)^2 = \int d^5\sigma K_m p^m(\sigma) \pm \int \left(\Sigma + \iota_K C + A \wedge (\omega + \frac{1}{2}\iota_K A)\right)$$

This extension is topological because it consists of an integral of a closed form. In fact, the M5-brane probe can also have a non-zero 2-form gauge field, B, on its world-volume. This gauge field is related to a 3-form field strength, dB. We can construct an additional closed 5-form involving this world-volume gauge field as follows:

$$dB \wedge (\omega + \iota_K A) \tag{5.43}$$

This 5-form is closed since the 2-form $\omega + \iota_K A$ is closed. We can include this 5-form in the above anti-commutator to allow for non-zero world-volume fields on the probe M5-brane:

$$2(\epsilon Q)^2 = \int d^5\sigma K_m p^m(\sigma) \pm \int \left(\Sigma + \iota_K C + (A + dB) \wedge (\omega + \iota_K A) - \frac{1}{2} A \wedge \iota_K A\right) \tag{5.44}$$

where the relative normalization of the new term, Eq. (5.43), comes from comparing with Ref. [91] in the flat space limit. The new M5-brane algebra, Eq. (5.44) extends the results of Refs. [91, 95] and agrees with them in the appropriate limits. As in the M2-brane case, the extended algebra, Eq. (5.44), gives rise to the following BPS bound on the energy/momentum of the M5-brane,

$$\int d^5\sigma K_m p^m(\sigma) \geq \mp \int \left(\Sigma + \iota_K C + (A + dB) \wedge (\omega + \iota_K A) - \frac{1}{2} A \wedge \iota_K A\right) \tag{5.45}$$

where the right hand side is a topological charge for the probe brane. Again the $\mp$ refers to the fact that the bound is of the type $E \geq |Q|$, i.e. we can choose the sign appropriately to make the topological charge term positive. Note that again the integrand is only defined up to the addition of a closed forms.

As in the M2-brane case, we can consider this bound for special classes of backgrounds to obtain known calibration conditions for M5-branes. However, this bound also allows for situations that have not been considered in detail, for example non-static probe M5-branes with non-zero world-volume gauge fields.

5.3 Example of brane in non-flat background

In this section we consider a particular supersymmetric background, namely the background corresponding to N coincident M5-branes. This background preserves $\frac{1}{2}$ supersymmetry. The 16 Killing spinors can be found explicitly and we will choose one of them to construct the forms K, ω and Σ. These forms can then be used to find a closed 2-form and a closed 5-form as shown in § 5.1.3. We will give the explicit expressions for the closed forms and show how they appear in the super-translation algebra for M2- and M5-brane probes. We will also give the relevant BPS bounds on the energy/momentum of probe branes in this background.

Recall from § 1.2 that the supergravity solution corresponding to N coincident M5-branes is given by

$$\begin{aligned} ds^2 &= H^{-1/3}(-(dx^0)^2 + (dx^1)^2 + \ldots + (dx^5)^2) + H^{2/3}((dx^6)^2 + \ldots + (dx^\natural)^2) && (5.46) \\ *F &= -dH^{-1} \wedge dx^0 \wedge dx^1 \wedge \ldots \wedge dx^5 && (5.47) \end{aligned}$$

Here the background M5-branes are aligned along the 012345 directions, and H is a harmonic function of r, the radial distance away from the branes, where

$$r^2 = \sum_{i=6}^{\natural} (x^i)^2$$

We can dualize $*F$ to obtain the 4-form field strength, $F = - * (*F)$. We obtain,

$$F = \frac{1}{r}\frac{\partial H}{\partial r}\frac{1}{4!}\epsilon_{ijklm} x^i dx^j \wedge dx^k \wedge dx^l \wedge dx^m$$

where $i, j, \ldots = 6, 7, \ldots, \natural$ and $\epsilon_{6\ldots\natural} = +1$. Therefore, the non-zero components of F are all transverse to the background M5-branes. Recall from § 1.2 that the Killing spinors for this background are given by $\epsilon = H^{-1/12}\epsilon_0$, where ϵ_0 is a constant spinor satisfying

$$\Gamma_{012345}\epsilon_0 = \epsilon_0 \tag{5.48}$$

and we can normalize the constant spinor by $(\epsilon_0)^T\epsilon_0 = 1$. Since there is just one projection condition, this background possesses 16 Killing spinors, and preserves $\frac{1}{2}$ supersymmetry. In Chapter 1 we showed that there are two ways of obtaining the projection condition above. Firstly, one can simply substitute the background supergravity solution into the Killing spinor equation and solve. The second way comes from requiring the brane world-volume to be supersymmetric. The projection condition then arises from fixing the κ-symmetry on the brane (see § 1.2 for details).

We now construct the forms K, ω and Σ. To do this we must select one of the 16 Killing spinors of the background. This can be achieved by making further projections on the spinor, consistent with Eq. (5.48), as follows,

$$\Gamma_{01}\epsilon_0 = -\epsilon_0$$

$$\Gamma_{2345}\epsilon_0 = \Gamma_{2367}\epsilon_0 = \Gamma_{2389}\epsilon_0 = \Gamma_{2468}\epsilon_0 = -\epsilon_0$$

where again these Γ matrices have tangent space indices. These projection conditions give K, ω and Σ as follows:

$$K = \Delta(e^0 + e^1) = H^{-1/3}(dx^0 + dx^1) \tag{5.49}$$

$$\omega = -K \wedge e^{\natural} = -(dx^0 + dx^1) \wedge dx^{\natural} \tag{5.50}$$

$$\Sigma = -K \wedge \phi \tag{5.51}$$

where $\Delta = H^{-1/6}$ and ϕ is the Cayley 4-form, given by

$$\phi = H^{-2/3}dx^{2345} + H^{4/3}dx^{6789} + H^{1/3}\Big[dx^{2367} - dx^{3478} + dx^{2468} + dx^{3579} - dx^{2569} + dx^{4589} + dx^{4567} - dx^{3469} + dx^{2389} - dx^{2578} - dx^{2479} - dx^{3568}\Big]$$

and we use the short-hand notation $dx^{2345} \equiv dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5$, etc.

Now that we have explicit expressions for the forms, we can verify that the equations for dK, $d\omega$ and $d\Sigma$, given in Eqs. (5.26)-(5.28), are satisfied. The simplest equation to check is Eq. (5.27) for $d\omega$. Clearly, from the explicit expression for ω in Eq. (5.50), we have $d\omega = 0$. Moreover, since the 4-form field strength, F, has non-zero components only in the transverse directions (i.e. in the $6, 7, 8, 9, \natural$ directions), it is clear that $\iota_K F = 0$. Therefore,

$$d\omega = 0 = \iota_K F \ ,$$

so the equation for $d\omega$, Eq. (5.27), is satisfied. Similarly, we can check that the equation for $d\Sigma$, Eq. (5.28), is satisfied. From the explicit expression for Σ we find,

$$d\Sigma = dH \wedge (dx^0 + dx^1) \wedge (H^{-2}dx^{2345} - dx^{6789})$$

We can also easily work out the terms which appear on the right hand side of Eq. (5.28). We find,

$$\iota_K * F = -dH^{-1} \wedge (dx^0 + dx^1) \wedge dx^{2345}$$

$$\omega \wedge F = dH \wedge (dx^0 + dx^1) \wedge dx^{6789}$$

So clearly,

$$d\Sigma = \iota_K * F - \omega \wedge F$$

as required by Eq. (5.28). Similarly, the equation for dK, Eq. (5.26), is straight-forward to verify, but we do not present the details here as it is slightly messy.

We now work out the closed forms in the M2- and M5-brane super-translation algebras. The closed 2-form, which appears in the M2-brane algebra, is given by $\omega + \iota_K A$. The first step is to choose a gauge for A which satisfies $\mathcal{L}_K A = 0$. Now, this means that

$$\mathcal{L}_K A = d\iota_K A + \iota_K F = d\iota_K A = 0$$

where we have used the fact that in this case $\iota_K F = 0$. So we can consistently choose a gauge for A such that $\iota_K A = 0$. Then the closed 2-form is simply,

$$\omega + \iota_K A = \omega = -(dx^0 + dx^1) \wedge dx^{\natural}$$

The gauge choice $\iota_K A = 0$ also simplifies the expression for the closed 5-form defined in § 5.1.3. It becomes,

$$\Sigma + \iota_K C + (A + dB) \wedge \omega \tag{5.52}$$

Now, the 6-form potential C satisfies $dC = *F + \frac{1}{2} A \wedge F$. However, in this background $A \wedge F \equiv 0$. Therefore, to find C we simply integrate $*F$. We obtain,

$$C = -(H^{-1} - 1) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5$$

It is easy to verify that this gauge for C satisfies $\mathcal{L}_K C = 0$. We can now compute $\iota_K C$ which appears in the closed 5-form,

$$\iota_K C = (H^{-1} - 1)(dx^0 + dx^1) \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \tag{5.53}$$

The remaining subtlety is how to define $A \wedge \omega$, since A is a magnetic potential for this background, and so it is not globally well-defined. The natural solution is to define the integral of $A \wedge \omega$ over the spatial 5-dimensional world-volume of the brane as the integral of $F \wedge \omega$ over a 6-dimensional surface whose boundary is the 5-brane surface. The quantity $F \wedge \omega$ is well-defined, and is given explicitly by

$$F \wedge \omega = (dx^0 + dx^1) \wedge dH \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9$$

We can integrate $F \wedge \omega$ once to give the quantity that we will identify with $A \wedge \omega$ up to the addition of a closed form (Recall that all topological terms are defined up to the addition of closed forms.),

$$-(H-1)(dx^0+dx^1)\wedge dx^6\wedge dx^7\wedge dx^8\wedge dx^9 \tag{5.54}$$

Therefore, adding Eqs. (5.51),(5.53) and (5.54), we obtain the following expression for the closed 5-form,

$$\iota_K C+\Sigma+(A+dB)\wedge\omega=(dx^0+dx^1)\wedge\phi_f-dB\wedge(dx^0+dx^1)\wedge dx^\natural \tag{5.55}$$

where the terms have combined such that ϕ_f is the Cayley 4-form on flat space:

$$\begin{aligned}\phi_f &= dx^{2345}+dx^{6789}+dx^{2367}-dx^{2569}-dx^{3478}\\ &+ dx^{2468}+dx^{3579}+dx^{4589}+dx^{4567}-dx^{3469}\\ &+ dx^{2389}-dx^{2578}-dx^{2479}-dx^{3568}\end{aligned}$$

Clearly the 5-form in Eq. (5.55) is closed.

Now that we have the expressions for the closed 2-form and closed 5-form we can write down the extended super-translation algebra for probe branes in the M5-brane background. For the M2-brane the anti-commutator is given by,

$$2(\epsilon Q)^2=\int_{M2}K\cdot p\mp\int_{M2}(dx^0+dx^1)\wedge dx^\natural$$

and the corresponding BPS bound on the energy/momentum of the probe M2-brane is,

$$\int_{M2}(-p_0+p_1)\geq\pm\int_{M2}(dx^0+dx^1)\wedge dx^\natural$$

where we have used the explicit form for K, given in Eq. (5.49), to rewrite the term $K\cdot p$. Note that the indices on p_m are coordinate space indices. As before, $-p_0$ can be associated to the Hamiltonian for the brane, so the integral $\int -p_0$ gives the energy of the brane.

For an M5-brane probe the super-translation algebra is,

$$2(\epsilon Q)^2=\int_{M5}K\cdot p\pm\int_{M5}\left(-(dx^0+dx^1)\wedge\phi_f+dB\wedge\omega\right)$$

and the corresponding BPS bound on the energy/momentum of this probe is given by,

$$\int_{M5}(-p_0+p_1)\geq\pm\int_{M5}\left((dx^0+dx^1)\wedge\phi_f-dB\wedge\omega\right)$$

For a static probe M5-brane, which is parallel to the background M5-branes, this bound becomes,

$$\int_{M5} \mathcal{H} \geq \int_{M5} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \tag{5.56}$$

This is the same as the calibration bound one would obtain for adding an M5-brane probe to flat space. Therefore, it is consistent to add a probe M5-brane parallel to the background branes. Moreover, ϕ_f contains many other terms, so there are many other possibilities for adding M5-brane probes to the background without breaking all the supersymmetry.

Chapter 6

Type IIB supersymmetric backgrounds

In this chapter we consider the Killing spinors associated to type IIB supersymmetric supergravity backgrounds. In particular, we will be interested in using these spinors to construct p-forms of different degrees. These p-forms are analogous to the forms which were constructed from $d = 11$ Killing spinors in Chapter 5. Like the 11-dimensional case, we will use the type IIB Killing spinor equations and some Fierz identities to derive differential equations and some algebraic relations for these forms. Our motivation is to use these equations to obtain generalized calibrations for branes in type IIB supersymmetric backgrounds. In particular, in Chapter 7 we will use the results obtained here to find a generalized calibration for giant gravitons in $AdS_5 \times S^5$.

In contrast to the 11-dimensional case, type IIB supergravity has two types of Killing spinor equations. The first equation is differential, and it arises from requiring that the supersymmetry variation of the gravitino vanishes. In § 6.1 we will use this equation to derive a number of differential equations satisfied by the p-forms in a general supersymmetric background. Note that all our equations will be valid for the most general supersymmetric backgrounds, i.e. backgrounds which possess at least one Killing spinor and have background field strengths $H^{(3)}$, $G^{(1)}$, $G^{(3)}$ and $G^{(5)}$ non-zero. The second type of Killing spinor equation is an algebraic equation, which arises from the variation of the axino-dilatino. In § 6.2 we will use this equation to derive some algebraic relations between the forms and the background field strengths. We will also use Fierz identities to derive another set of algebraic relations between the forms.

The results we derive here extend the partial results of Refs. [37–39] which have been obtained for backgrounds which preserve 4-dimensional Poincaré invariance. In these papers the focus was on using the p-forms to find the G-structure for backgrounds with this symmetry, and then using the G-structure to classify the form of the corresponding solutions. We anticipate that both the differential and algebraic relations we derive here will play an important role in the full classification of supersymmetric type IIB backgrounds, as our equations are valid for the most general supersymmetric backgrounds. However, we will not attempt to make this classification here. The results described in this chapter are reported in Ref. [3]

6.1 Differential equations for the p-forms

In this section we construct p-forms of different degrees from a Killing spinor of type IIB supergravity. We then use the gravitino Killing spinor equation to compute derivatives of these forms.

Following Ref. [99], the gravitino Killing spinor equation in the string frame is $D_m\epsilon = 0$, where $m = 0, 1, \ldots, 9$ and ϵ is a 64-component spinor, with two chiral components:

$$\epsilon = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$

In particular, the spinors ϵ^i, $i = 1, 2$, are 32-dimensional, and satisfy the chirality projection condition $\Gamma_{11}\epsilon^i = \epsilon^i$ (This condition reduces the number of non-zero components of each ϵ^i to 16, which means that ϵ has only 32 non-zero components.). The derivative operator acting on ϵ is given by

$$D_m = \nabla_m + \frac{1}{8}H^{(3)}_{mr_1r_2}\Gamma^{r_1r_2} \otimes \sigma_3 + \frac{1}{16}e^{\phi}\sum_{a=1}^{5}\frac{(-1)^{a-1}}{(2a-1)!}G^{(2a-1)}_{r_1\ldots r_{2a-1}}\Gamma^{r_1\ldots r_{2a-1}}\Gamma_m \otimes \lambda_a \tag{6.1}$$

where ϕ is the dilaton and ∇ is the usual spin connection defined in Eq. (1.5) of Chapter 1. The matrices λ_a are defined as follows

$$\lambda_a = \begin{cases} \sigma_1 & \text{if } a \text{ even,} \\ i\sigma_2 & \text{if } a \text{ odd.} \end{cases} \tag{6.2}$$

where σ_1, σ_2 and σ_3 are the usual Pauli matrices. Recall from Chapter 1 that the NS-NS field strength, $H^{(3)}$, is defined by $H^{(3)} = dB^{(2)}$ (we will often omit the index (3) on $H^{(3)}$), and the field strengths $G^{(2a+1)}$ are defined by

$$G^{(2a+1)} = dC^{(2a)} - H^{(3)} \wedge C^{(2a-2)}$$

where $C^{(2a)}$ are Ramond-Ramond gauge potentials. These field strengths are not all independent, but $G^{(7)} = - * G^{(3)}$, $G^{(9)} = *G^{(1)}$ and $G^{(5)}$ is self-dual ($G^{(5)} = *G^{(5)}$).

We now construct p-forms of different degrees from a single Killing spinor ϵ. However, due to the chirality of the spinors, many p-forms that we construct are automatically zero. For example, suppose we construct the following 0-form: $\bar{\epsilon}^1\epsilon^2$, where $\bar{\epsilon}^i \equiv (\epsilon^i)^T\Gamma^{\hat{0}}$. Then

$$\bar{\epsilon}^1\epsilon^2 = (\epsilon^1)^T\Gamma^{\hat{0}}\epsilon^2 = \frac{1}{4}(\epsilon^1)^T(1+\Gamma_{11}^T)\Gamma^{\hat{0}}(1+\Gamma_{11})\epsilon^2 = 0 \tag{6.3}$$

where we have used the following facts: $\Gamma_{11}\epsilon^i = \epsilon^i$, $\Gamma_{11}^T = \Gamma_{11}$ and the fact that Γ_{11} anti-commutes with each $\Gamma^{\hat{m}}$. Extending this logic to forms of other degrees, it is clear that the only non-zero p-forms are those with p odd. The components of a generic odd p-form, ω^{ij}, are given by

$$\omega^{ij}_{m_1\ldots m_p} = \bar{\epsilon}^i\Gamma_{m_1\ldots m_p}\epsilon^j \tag{6.4}$$

where $i, j = 1, 2$. In general, the construction in Eq. (6.4) will produce 2×2 matrices of forms for each odd degree p, however not all of these forms are non-zero. The possible non-zero forms are denoted as follows,

- 1-forms, with components:

$$K^{ij}_m = \bar{\epsilon}^i\Gamma_m\epsilon^j$$

- 3-forms, with components:

$$\Phi^{ij}_{mnp} = \bar{\epsilon}^i\Gamma_{mnp}\epsilon^j \qquad \text{for } i \neq j,$$

- 5-forms, with components:

$$\Sigma^{ij}_{mnpqr} = \bar{\epsilon}^i\Gamma_{mnpqr}\epsilon^j$$

It is also possible to construct some higher-dimensional forms (two 7-forms, Π^{ij}, for $i \neq j$, and four 9-forms, Ω^{kl}). However, these forms are simply dual to the lower-dimensional forms as follows; $\Pi^{ij} = - * \Phi^{ij}$, $\Omega^{kl} = *K^{kl}$. Note also that the 5-forms, Σ^{ij}, are all self-dual. Moreover, there exist relations between the "off-diagonal" forms, namely:

$$K^{12} = K^{21}, \qquad \Phi^{12} = -\Phi^{21}, \qquad \Sigma^{12} = \Sigma^{21} \tag{6.5}$$

These relations can be easily proved by computing the transpose of the components of each form. This means that there are only 7 "independent" forms to consider: K^{11}, K^{22}, K^{12}, Φ^{12}, Σ^{11}, Σ^{22} and Σ^{12}. Actually these forms are not independent, since they obey complicated algebraic relations (some of which we will derive in the next section).

We now compute the covariant derivatives of these forms. For each p-form, ω^{ij}, whose components are given in Eq. (6.4), we will compute

$$\begin{aligned} \nabla_n \omega^{ij}_{m_1 \dots m_p} &= \nabla_n (\bar{\epsilon}^i \Gamma_{m_1 \dots m_p} \epsilon^j) \\ &= (\overline{\nabla_n \epsilon^i}) \Gamma_{m_1 \dots m_p} \epsilon^j + \bar{\epsilon}^i \Gamma_{m_1 \dots m_p} (\nabla_n \epsilon^j) \end{aligned} \tag{6.6}$$

where $\overline{\nabla_n \epsilon^i} \equiv (\nabla_n \epsilon^i)^T \Gamma^{\hat{0}}$. The idea is to use the gravitino Killing spinor equation, $D_m \epsilon = 0$, to replace the covariant derivatives of ϵ with terms involving the fields strengths, metric and dilaton. The second step is to antisymmetrize over the indices $n, m_1, \dots m_p$ to obtain the ordinary derivative of ω^{ij}, i.e. $d\omega^{ij}$. This is entirely analogous to the procedure in Chapter 5 for computing the derivatives of the forms constructed from $d = 11$ Killing spinors. However, here the computations are messier due to the large number of terms in the type IIB Killing spinor equation. Therefore, we will not present all the details of these calculations here. However, we will show one of the simpler calculations in Appendix C, namely the computation for dK^{12}. The other calculations will be similar to this, but with more indices.

We now present the results for the ordinary derivatives of the forms. While the equations look complicated, they are valid for the most general supersymmetric backgrounds which have non-zero field strengths, H, $G^{(1)}$, $G^{(3)}$ and $G^{(5)}$. Starting with the 1-forms, K^{ij}, we have

$$dK^{11} = -\iota_{K^{11}} H + \frac{e^{\phi}}{2} \Big(- \iota_{G^{(1)}} \Phi^{12} + \iota_{G^{(3)}} \Sigma^{12} + \iota_{K^{12}} G^{(3)} - \iota_{\Phi^{12}} G^{(5)} \Big) \tag{6.7}$$

The equation for K^{22} can be obtained from Eq. (6.7) by replacing

$$dK^{11} \to dK^{22}, \qquad \iota_{K^{11}} H \to -\iota_{K^{22}} H,$$

with all other terms remaining the same. For K^{12} we obtain,

$$(dK^{12})_{mn} = \frac{1}{2} H^{\ r_1 r_2}{}_{[m} \Phi^{12}_{n] r_1 r_2} + \frac{e^{\phi}}{4} \Big(\iota_{G^{(3)}} (\Sigma^{11} + \Sigma^{22}) + \iota_{(K^{11} + K^{22})} G^{(3)} \Big)_{mn} \tag{6.8}$$

The equation for K^{21} is exactly the same as above since $K^{21} = K^{12}$. For the 1-forms it is also interesting to calculate $\nabla_{(m} K^{ij}_{n)}$, i.e. symmetrizing over the indices. If this quantity vanishes then K^{ij} corresponds to a Killing vector. In fact, we find that only the combination $K^{11} + K^{22}$ is Killing, i.e.

$$\nabla_{(m} (K^{11} + K^{22})_{n)} = 0$$

We will see in Chapter 7 that the combination $K^{11} + K^{22}$ appears naturally in the calibration bound for D3-branes. This is not surprising as in § 4.4 we saw that a Killing vector, K, was involved in the generalized calibration conditions. For D3-branes in type IIB supergravity we will see that $K^{11}+K^{22}$ plays the role of K.

The 3-form Φ^{ij} is non-zero only when $i\neq j$. The ordinary derivative of Φ^{12} is

$$\begin{aligned}(d\Phi^{12})_{mnpq} = {} & H^{\ r_1r_2}{}_{[m}\Sigma^{12}_{npq]r_1r_2}+\frac{3}{2}K^{12}\wedge H \\ + {} & \frac{e^{\phi}}{2}\Big(\iota_{G^{(1)}}(\Sigma^{11}+\Sigma^{22})-\iota_{(K^{11}+K^{22})}G^{(5)}+\frac{1}{2}(K^{11}-K^{22})\wedge G^{(3)} \\ & +G^{(3)}_{r_1r_2[m}(\Sigma^{22}-\Sigma^{11})_{npq]}{}^{r_1r_2}\Big)\end{aligned} \tag{6.9}$$

where the omitted indices are understood to be ${}_{[mnpq]}$. Since $\Phi^{21}=-\Phi^{12}$ we do not need to work out the equation for Φ^{21} separately. We now consider the 5-forms Σ^{ij}. For Σ^{11} we obtain the following differential equation

$$\begin{aligned}(d\Sigma^{11})_{mnpqrs} = {} & -\frac{15}{2}H^{t}{}_{[mn}\Sigma^{11}_{pqrs]t} \\ & +\frac{e^{\phi}}{2}\Big\{2K^{12}\wedge G^{(5)}+2G^{(1)}\wedge\Sigma^{12}-3\iota_{G^{(1)}}\Pi^{12}+3\iota_{G^{(3)}}\Omega^{12} \\ & -15G^{(3)}_{t[mn}\Sigma^{12}{}_{pqrs]}{}^{t}+15\Phi^{12}{}_{t[mn}G^{(5)}_{pqrs]}{}^{t}-12G^{(3)}_{t_1t_2[m}\Pi^{12}{}_{npqrs]}{}^{t_1t_2}\Big\}\end{aligned} \tag{6.10}$$

where in this equation and in Eqs. (6.11)-(6.12) the omitted indices are understood to be ${}_{[mnpqrs]}$. The ordinary derivative of Σ^{22} is

$$\begin{aligned}(d\Sigma^{22})_{mnpqrs} = {} & \frac{15}{2}H^{t}{}_{[mn}\Sigma^{22}_{pqrs]t} \\ & +\frac{e^{\phi}}{2}\Big\{-2K^{12}\wedge G^{(5)}-2G^{(1)}\wedge\Sigma^{12}-3\iota_{G^{(1)}}\Pi^{12}+3\iota_{G^{(3)}}\Omega^{12} \\ & -15G^{(3)}_{t[mn}\Sigma^{12}{}_{pqrs]}{}^{t}+15\Phi^{12}{}_{t[mn}G^{(5)}_{pqrs]}{}^{t}+12G^{(3)}_{t_1t_2[m}\Pi^{12}{}_{npqrs]}{}^{t_1t_2}\Big\}\end{aligned} \tag{6.11}$$

and the equation for Σ^{12} is

$$\begin{aligned}(d\Sigma^{12})_{mnpqrs} = {} & \frac{3}{2}H^{t_1t_2}{}_{[m}\Pi^{12}{}_{npqrs]t_1t_2}-\frac{3}{2}H\wedge\Phi^{12} \\ & +\frac{e^{\phi}}{4}\Big\{2(K^{22}-K^{11})\wedge G^{(5)}+2G^{(1)}\wedge(\Sigma^{22}-\Sigma^{11}) \\ & +3\iota_{G^{(3)}}(\Omega^{11}+\Omega^{22})-15G^{(3)}_{t[mn}(\Sigma^{22}+\Sigma^{11})_{pqrs]}{}^{t}\Big\}\end{aligned} \tag{6.12}$$

The equation for Σ^{21} is exactly the same as above since $\Sigma^{21}=\Sigma^{12}$.

6.2 Algebraic relations for the p-forms

There are two ways to obtain algebraic relations between the forms. The first way is to use the algebraic Killing spinor equation. This Killing spinor equation arises from requiring that the supersymmetry variation of the axino-dilatino vanishes. As we will see, this equation gives relations between different products of p-forms with background field strengths. The second way is to use Fierz identities, which relate products of different numbers of Γ-matrices to each other. We will see that these identities give relations between the p-forms, without involving the background field strengths.

The algebraic Killing spinor equation is given by $\delta\lambda = \mathcal{P}\epsilon = 0$ where [100]

$$\mathcal{P} = \Gamma^m \partial_m \phi + \frac{1}{12} H^{(3)}_{m_1 m_2 m_3} \Gamma^{m_1 m_2 m_3} \otimes \sigma_3 + \frac{e^\phi}{4} \sum_{a=1}^{5} \frac{(-1)^{a-1}(a-3)}{(2a-1)!} G^{(2a-1)}_{m_1 \ldots m_{2a-1}} \Gamma^{m_1 \ldots m_{2a-1}} \otimes \lambda_a \tag{6.13}$$

Here ϕ is the dilaton, σ_3 is the third Pauli matrix and λ_a are the 2×2 matrices given in Eq. (6.2). Algebraic identities can be obtained from this equation by constructing $\bar{\epsilon}^i \Gamma_{m_1 \ldots m_p} (\mathcal{P}\epsilon)^j = 0$, where p can take values from $0, \ldots, 10$. For $p = 0$ we obtain the following set of identities,

$$\bar{\epsilon}^1 (\mathcal{P}\epsilon)^1 = K^{11} \cdot d\phi - e^\phi K^{12} \cdot G^{(1)} + \frac{e^\phi}{2} G^{(3)} \cdot \Phi^{12} = 0 \tag{6.14}$$

$$\bar{\epsilon}^2 (\mathcal{P}\epsilon)^1 = K^{21} \cdot d\phi - e^\phi K^{22} \cdot G^{(1)} + \frac{1}{2} H \cdot \Phi^{21} = 0 \tag{6.15}$$

$$\bar{\epsilon}^1 (\mathcal{P}\epsilon)^2 = K^{12} \cdot d\phi + e^\phi K^{11} \cdot G^{(1)} - \frac{1}{2} H \cdot \Phi^{12} = 0 \tag{6.16}$$

$$\bar{\epsilon}^2 (\mathcal{P}\epsilon)^2 = K^{22} \cdot d\phi + e^\phi K^{21} \cdot G^{(1)} + \frac{e^\phi}{2} G^{(3)} \cdot \Phi^{21} = 0 \tag{6.17}$$

where

$$G^{(3)} \cdot \Phi^{12} = \frac{1}{3!} G^{(3)}_{m_1 m_2 m_3} (\Phi^{12})^{m_1 m_2 m_3}$$

and the dot products of other p-forms are similarly defined (for the full definition see Appendix A). Now, if we consider the case $p = 1$ in our algebraic identity, we find that all terms in $\bar{\epsilon}^i \Gamma_m (\mathcal{P}\epsilon)^j$ automatically vanish (this follows from the fact that the only non-zero forms constructed from Killing spinors are 1-,3-,5-,7- and 9-forms), and we obtain no identities from this case. However, for $p = 2$ we obtain another set of four identities given by,

$$\begin{aligned}
0=\bar{\epsilon}^1\Gamma_{mn}(\mathcal{P}\epsilon)^1 &= \left(K^{11}\wedge d\phi - e^{\phi}K^{12}\wedge G^{(1)} - e^{\phi}\iota_{G^{(1)}}\Phi^{12}\right)_{mn} + \frac{e^{\phi}}{2}\Phi^{12}_{r_1r_2[m}G^{(3)}_{n]}{}^{r_1r_2} \\
&+\frac{1}{2}\left(\iota_H\Sigma^{11} - \iota_{K^{11}}H + e^{\phi}\iota_{G^{(3)}}\Sigma^{12} - e^{\phi}\iota_{K^{12}}G^{(3)}\right)_{mn} \qquad (6.18)
\end{aligned}$$

$$\begin{aligned}
0=\bar{\epsilon}^2\Gamma_{mn}(\mathcal{P}\epsilon)^1 &= \left(K^{21}\wedge d\phi - e^{\phi}K^{22}\wedge G^{(1)} + \iota_{d\phi}\Phi^{21}\right)_{mn} + \frac{1}{2}\Phi^{21}_{r_1r_2[m}H_{n]}{}^{r_1r_2} \\
&+\frac{1}{2}\left(\iota_H\Sigma^{21} - \iota_{K^{21}}H + e^{\phi}\iota_{G^{(3)}}\Sigma^{22} - e^{\phi}\iota_{K^{22}}G^{(3)}\right)_{mn} \qquad (6.19)
\end{aligned}$$

$$\begin{aligned}
0=\bar{\epsilon}^1\Gamma_{mn}(\mathcal{P}\epsilon)^2 &= \left(K^{12}\wedge d\phi + e^{\phi}K^{11}\wedge G^{(1)} + \iota_{d\phi}\Phi^{12}\right)_{mn} - \frac{1}{2}\Phi^{12}_{r_1r_2[m}H_{n]}{}^{r_1r_2} \\
&+\frac{1}{2}\left(-\iota_H\Sigma^{12} + \iota_{K^{12}}H + e^{\phi}\iota_{G^{(3)}}\Sigma^{11} - e^{\phi}\iota_{K^{11}}G^{(3)}\right)_{mn} \qquad (6.20)
\end{aligned}$$

$$\begin{aligned}
0=\bar{\epsilon}^2\Gamma_{mn}(\mathcal{P}\epsilon)^2 &= \left(K^{22}\wedge d\phi + e^{\phi}K^{21}\wedge G^{(1)} + e^{\phi}\iota_{G^{(1)}}\Phi^{21}\right)_{mn} + \frac{e^{\phi}}{2}\Phi^{21}_{r_1r_2[m}G^{(3)}_{n]}{}^{r_1r_2} \\
&+\frac{1}{2}\left(-\iota_H\Sigma^{22} + \iota_{K^{22}}H + e^{\phi}\iota_{G^{(3)}}\Sigma^{21} - e^{\phi}\iota_{K^{21}}G^{(3)}\right)_{mn} \qquad (6.21)
\end{aligned}$$

These identities can be combined using the relations $K^{12} = K^{21}$, $\Phi^{12} = -\Phi^{21}$ and $\Sigma^{12} = \Sigma^{21}$. However, generally this doesn't make the expressions much simpler (although it eliminates some terms, so it might be useful in some situations). The final set of four identities comes from $p = 4$. For example,

$$\begin{aligned}
0=\bar{\epsilon}^1\Gamma_{mnpq}(\mathcal{P}\epsilon)^1 &= \left(\iota_{d\phi}\Sigma^{11} - e^{\phi}\iota_{G^{(1)}}\Sigma^{12} + e^{\phi}G^{(1)}\wedge\Phi^{12}\right)_{mnpq} \\
&-\frac{1}{2}\left(K^{11}\wedge H + e^{\phi}K^{12}\wedge G^{(3)} - e^{\phi}\iota_{G^{(3)}}\Pi^{12}\right)_{mnpq} \\
&-H^{r_1r_2}{}_{[m}\Sigma^{11}_{npq]r_1r_2} - e^{\phi}G^{(3)}_{r_1r_2[m}\Sigma^{12}_{npq]}{}^{r_1r_2} \qquad (6.22)
\end{aligned}$$

Again, there are three other similar identities for $p = 4$, given by

$$\begin{aligned}
0=\bar{\epsilon}^2\Gamma_{mnpq}(\mathcal{P}\epsilon)^1 &= \left(\iota_{d\phi}\Sigma^{21} - e^{\phi}\iota_{G^{(1)}}\Sigma^{22} - d\phi\wedge\Phi^{21}\right)_{mnpq} \\
&-\frac{1}{2}\left(K^{21}\wedge H + e^{\phi}K^{22}\wedge G^{(3)} - \iota_H\Pi^{21}\right)_{mnpq} \\
&-H^{r_1r_2}{}_{[m}\Sigma^{21}_{npq]r_1r_2} - e^{\phi}G^{(3)}_{r_1r_2[m}\Sigma^{22}_{npq]}{}^{r_1r_2} \qquad (6.23)
\end{aligned}$$

$$\begin{aligned}
0=\bar{\epsilon}^1\Gamma_{mnpq}(\mathcal{P}\epsilon)^2 &= \left(\iota_{d\phi}\Sigma^{12} + e^{\phi}\iota_{G^{(1)}}\Sigma^{11} - d\phi\wedge\Phi^{12}\right)_{mnpq} \\
&+\frac{1}{2}\left(K^{12}\wedge H - e^{\phi}K^{11}\wedge G^{(3)} - \iota_H\Pi^{12}\right)_{mnpq} \\
&+H^{r_1r_2}{}_{[m}\Sigma^{12}_{npq]r_1r_2} - e^{\phi}G^{(3)}_{r_1r_2[m}\Sigma^{11}_{npq]}{}^{r_1r_2} \qquad (6.24)
\end{aligned}$$

$$\begin{aligned} 0 = \bar{\epsilon}^2\Gamma_{mnpq}(\mathcal{P}\epsilon)^2 &= \left(\iota_{d\phi}\Sigma^{22} + e^{\phi}\iota_{G^{(1)}}\Sigma^{21} - e^{\phi}G^{(1)} \wedge \Phi^{21}\right)_{mnpq} \\ &+\frac{1}{2}\left(K^{22} \wedge H - e^{\phi}K^{21} \wedge G^{(3)} + e^{\phi}\iota_{G^{(3)}}\Pi^{21}\right)_{mnpq} \\ &+H^{r_1r_2}{}_{[m}\Sigma^{22}_{npq]r_1r_2} - e^{\phi}G^{(3)}_{r_1r_2[m}\Sigma^{21}_{npq]}{}^{r_1r_2} \end{aligned} \tag{6.25}$$

If we take $p > 4$ in $\bar{\epsilon}^i\Gamma_{m_1\ldots m_p}(\mathcal{P}\epsilon)^j = 0$, we obtain identities which are simply the duals of those obtained for $p < 4$. Therefore, Eqs. (6.14)-(6.25) give the full set of independent identities that can be derived from the algebraic Killing spinor equation.

The second way to obtain algebraic identities between the forms is to use Fierz identities. There are many possible Fierz identities for the Dirac matrices in 10 dimensions. However, here we will consider one particular class of identities given by [101]

$$\left(\Gamma^{(l)\ m_1\ldots m_l}\right)_{\alpha\beta}\left(\Gamma^{(l)}_{m_1\ldots m_l}\right)_{\gamma\delta} = \sum_{k=0}^{10} A_{(lk)}\left(\Gamma^{(k)\ n_1\ldots n_k}\right)_{\alpha\delta}\left(\Gamma^{(k)}_{n_1\ldots n_k}\right)_{\gamma\beta} \tag{6.26}$$

where $\alpha, \beta, \gamma, \delta$ are spinor indices and the coefficients $A_{(lk)}$ are given explicitly by

$$A_{(lk)} = \frac{l!}{16\cdot k!}(-1)^{\frac{(l+k)^2-l-k}{2}} \sum_{p=\max\{0,l+k-10\}}^{\min\{k,l\}} (-1)^p \begin{pmatrix} 10-k \\ l-p \end{pmatrix}\begin{pmatrix} k \\ p \end{pmatrix}$$

These identities allow us to find relationships between $K^{ij}\cdot K^{kl}$, $\Phi^{ij}\cdot\Phi^{kl}$ and $\Sigma^{ij}\cdot\Sigma^{kl}$, where $i,j,k,l \in \{1,2\}$. In fact, somewhat surprisingly, these Fierz identities give

$$K^{ij}\cdot K^{kl} = 0, \qquad \Phi^{ij}\cdot\Phi^{kl} = 0, \qquad \Sigma^{ij}\cdot\Sigma^{kl} = 0 \tag{6.27}$$

which implies that $\Pi^{ij}\cdot\Pi^{kl}$ and $\Omega^{ij}\cdot\Omega^{kl}$ also. This is different to the 11-dimensional case. In 11 dimensions the Killing vector K can be time-like or null [28,87]. However, our results show that the Killing vector for type IIB supergravity, namely $K^{11}+K^{22}$, can only be null. Moreover, since each K^{ij} is null and all scalar products vanish, this means that all K^{ij} are proportional to the same null vector, i.e. $K^{ij} = c^{ij}\tilde{K}$, where c^{ij} are constants. Note that Φ^{ij} and Σ^{ij} are also null, and all scalar products of these forms vanish too. We find the same results using the Γ-matrix algebra package GAMMA [102]. Presumably there are other non-trivial algebraic relations which could be obtained by considering other types of Fierz identities (e.g. $K \wedge \Phi$ and $\iota_K\Sigma$ might be related). However, we will not investigate this here as we will not need any algebraic relations between the forms to construct generalized calibrations, which is the main focus of the next chapter.

To summarize, in this chapter we have constructed p-forms from Killing spinors of type IIB supergravity. We find that non-zero 1-,3-,5-,7- and 9-forms can be constructed. Using the gravitino Killing spinor equation we have derived differential equations that the forms satisfy in a general supersymmetric background. In analogy to the 11-dimensional case, one combination of the 1-forms is Killing, namely $K^{11} + K^{22}$. We have also derived some algebraic identities for the forms using the algebraic Killing spinor equation and Fierz identities. These differential and algebraic relations could now be used for classifying general supersymmetric type IIB backgrounds using the ideas of G-structures. However, one complication in 10 dimensions is that there are four independent background field strengths, so classifying the most general supersymmetric backgrounds might be more difficult than the 11-dimensional case, where there is only one independent field strength. In the next chapter we will use the differential equations derived here to construct generalized calibrations for D3-branes in type IIB supersymmetric backgrounds. In particular, we will be interested in finding a calibrating form for giant gravitons in $AdS_5 \times S^5$.

Chapter 7

Holomorphic giant gravitons and calibrations

In this chapter we will investigate generalized calibrations for branes in supersymmetric type IIB supergravity backgrounds. Our aim is to find a generalized calibration and calibration bound for giant gravitons in $AdS_5 \times S^5$. However, much of the discussion will be applicable to more general situations. The approach we will use is to derive the calibration bound from the super-translation algebra for a probe D3-brane in a supersymmetric type IIB background. This algebra can be found using the methods of Chapter 5 adapted to type IIB supergravity. Now, recall that giant gravitons are non-static spherical branes. The fact that they are non-static makes them an interesting example to consider from the point of view of calibrations, as most previous work on calibrations has involved static probe branes (for example, in Ref. [81] generalized calibrations for static 5-branes in particular type IIB backgrounds were discussed). Here we aim to understand this particular example of a non-static brane using calibrations.

An interesting construction of giant gravitons has been proposed by Mikhailov [45]. In this construction the space $AdS_5 \times S^5$ is embedded in $\mathbb{C}^{1,2} \times \mathbb{C}^3$. The spatial worldvolume of the giant graviton then arises from the intersection of a holomorphic surface in $\mathbb{C}^3$ with the embedded S^5. The motion of the giant graviton is also specified in this construction. Giant gravitons constructed in this way are supersymmetric. Moreover, these configurations are much more general than the original example of a giant graviton found in Ref. [46] (which we presented in § 2.1). The calibration bound that we derive will allow us to prove that these general giant gravitons are calibrated.

The outline of this chapter is as follows. In § 7.1 we review the Mikhailov construction of giant gravitons in $AdS_5 \times S^5$ in detail. In particular, we give the supersymmetry projection conditions for these branes. These projection conditions are then used to find a set of p-forms relevant to these branes, using the method in Chapter 6. In § 7.2 we show that the p-forms obey the correct differential and algebraic relations. Then in § 7.3 we consider the super-translation algebra for D3-branes in backgrounds where the dilaton and the field strengths H, $G^{(1)}$ and $G^{(3)}$ are zero. This class of backgrounds includes $AdS_5 \times S^5$, as well as more general backgrounds. We use the supersymmetry algebra to derive a calibration bound for general D3-branes in these backgrounds. Then we specialize to the case of giant gravitons, and we find the calibration bound which should be saturated by these branes. It turns out that the speed of the giant graviton is specified precisely by requiring that the calibration bound is saturated. In § 7.4 we consider a dual giant graviton and show that it saturates the same calibration bound. Interestingly, we find that these calibrated branes all minimize "energy minus momentum" in their homology class, rather than just the energy. The original results in this chapter are reported in Ref. [3].

7.1 Giant gravitons in $AdS_5 \times S^5$ from holomorphic surfaces

In this section we review the Mikhailov construction of giant gravitons in $AdS_5 \times S^5$ via holomorphic surfaces [45]. This construction gives a large class of giant graviton configurations, generalizing the example given in § 2.1. We begin our discussion by defining the embedding of $AdS_5 \times S^5$ in $\mathbb{C}^{1,2} \times \mathbb{C}^3$. We then use the complex structure of the embedding space to define the spatial world-volume of a giant graviton and its motion on S^5. We discuss the supersymmetry projection conditions for these branes in § 7.1.3, and in § 7.1.4 we show how a specific choice of holomorphic surface reproduces the simple giant graviton of § 2.1.

7.1.1 The complex structure of $AdS_5 \times S^5$

We begin by embedding the S^5 part of the geometry in flat $\mathbb{C}^3$, which has complex coordinates Z_i $(i = 1, 2, 3)$, which can be written in terms of 6 real polar coordinates as $Z_i = \mu_i e^{i\phi_i}$, where $0 \leq \phi_i \leq 2\pi$, $\mu_i \geq 0$. The metric on $\mathbb{C}^3$ is given by

$$ds^2 = |dZ_1|^2 + |dZ_2|^2 + |dZ_3|^2 = \sum_{i=1}^{3} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) \tag{7.1}$$

$\mathbb{C}^3$ has a complex structure, I, which acts on the basis 1-forms as follows,

$$I : dZ_i \longrightarrow -idZ_i$$

This is equivalent to the following transformations of the real 1-forms: $d\mu_i \longrightarrow \mu_i d\phi_i$ and $\mu_i d\phi_i \longrightarrow -d\mu_i$. The sphere is defined in $\mathbb{C}^3$ by

$$S^5 : |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 = 1 \tag{7.2}$$

where we have set the radius to 1 for convenience. Note that this means that the radius of curvature of AdS_5 is also 1 (If we compare with expressions in Chapter 2, we should set $L = 1$ everywhere.). The metric on S^5 is given by the metric on $\mathbb{C}^3$, Eq. (7.1), restricted to the sphere. The embedding of S^5 in $\mathbb{C}^3$ allows us to define a radial 1-form, $e^r \in T^*\mathbb{C}^3$, which is orthogonal to the sphere at every point. Explicitly, e^r is given by

$$e^r = \mu_1 d\mu_1 + \mu_2 d\mu_2 + \mu_3 d\mu_3$$

Note that e^r does not belong to the 10-dimensional space-time, and it has no physical meaning. We can act with the complex structure on e^r to produce a new 1-form $e^{||} = I \cdot e^r$, which is given explicitly by

$$e^{||} = \mu_1^2 d\phi_1 + \mu_2^2 d\phi_2 + \mu_3^2 d\phi_3 \tag{7.3}$$

This 1-form does belong to $AdS_5 \times S^5$ and it gives a preferred direction on S^5. Note that $e^{||}$ has unit length on S^5. We will see later that $e^{||}$ is the direction of motion for giant gravitons in this construction. Now, calculating the derivative of $e^{||}$, we obtain

$$de^{||} = 2(\mu_1 d\mu_1 \wedge d\phi_1 + \mu_2 d\mu_2 \wedge d\phi_2 + \mu_3 d\mu_3 \wedge d\phi_3) \equiv 2\omega \tag{7.4}$$

where ω is the Kähler 2-form on $\mathbb{C}^3$. We can also write ω in another orthogonal basis as follows,

$$\omega = \mathcal{N} e^r \wedge e^{||} + e^{I_1} \wedge e^{J_1} + e^{I_2} \wedge e^{J_2} \tag{7.5}$$

Here $\{e^{I_1}, e^{J_1}, e^{I_2}, e^{J_2}\}$ are unit 1-forms on $\mathbb{C}^3$, where

$$e^{J_k} = I \cdot e^{I_k}, \qquad k = 1, 2$$

These 1-forms are orthogonal to each other and to $\{e^{||}, e^r\}$. The factor of $\mathcal{N}$ in Eq. (7.5) ensures that e^r and $e^{||}$ are normalised everywhere on $\mathbb{C}^3$. Explicitly, $\mathcal{N} = (\mu_1^2+\mu_2^2+\mu_3^2)^{-1}$. Since e^I and e^J are non-zero 1-forms on S^5, the restriction of ω to the sphere is simply

$$\omega\Big|_{S^5} = \left(e^{I_1} \wedge e^{J_1} + e^{I_2} \wedge e^{J_2}\right)\Big|_{S^5} \tag{7.6}$$

This restricted Kähler 2-form will appear later when we construct p-forms relevant to supersymmetric giant gravitons.

It is also possible to define a complex structure for AdS_5. In particular, we embed AdS_5 in flat $\mathbb{C}^{1,2}$, which has complex coordinates $W_a = u_a + iv_a$ ($a = 0, 1, 2$). The flat metric on $\mathbb{C}^{1,2}$ is given by

$$ds^2 = -|dW_0|^2 + |dW_1|^2 + |dW_2|^2$$

$\mathbb{C}^{1,2}$ has a complex structure, $\tilde{I}$, which acts on the basis 1-forms as

$$\tilde{I} : dW_a \longrightarrow -idW_a$$

i.e. $du_a \longrightarrow dv_a$ and $dv_a \longrightarrow -du_a$. The embedding of AdS_5 in $\mathbb{C}^{1,2}$ is given by

$$|W_0|^2 - |W_1|^2 - |W_2|^2 = (u_0)^2 + (v_0)^2 - (u_1)^2 - (v_1)^2 - (u_2)^2 - (v_2)^2 = 1 \qquad (7.7)$$

The metric on AdS_5 is given by the flat metric on $\mathbb{C}^{1,2}$ restricted to this surface. In a similar way to the S^5, we can define a radial 1-form, $e^{\perp}$, which is orthogonal to AdS_5 at every point. Explicitly, $e^{\perp}$ is given by

$$e^{\perp} = u_0 du_0 + v_0 dv_0 - u_1 du_1 - v_1 dv_1 - u_2 du_2 - v_2 dv_2$$

We can act with the complex structure on $e^{\perp}$ to obtain a time-like direction, $e^0 = \tilde{I} \cdot e^{\perp}$, which belongs to the cotangent space of AdS_5:

$$e^0 = u_0 dv_0 - v_0 du_0 - u_1 dv_1 + v_1 du_1 - u_2 dv_2 + v_2 du_2 \qquad (7.8)$$

This is a preferred timelike direction on AdS_5, and it will appear later in the supersymmetry projection conditions for giant gravitons. The derivative of e^0 is related to the Kähler form on $\mathbb{C}^{1,2}$, denoted $\tilde{\omega}$, by

$$de^0 = 2(du_0 \wedge dv_0 - du_1 \wedge dv_1 - du_2 \wedge dv_2) \equiv -2\tilde{\omega} \qquad (7.9)$$

In a local region close to the sphere (such that $e^{\perp}$ and e^0 remain time-like), $\tilde{\omega}$ can be written in a different basis as

$$\tilde{\omega} = -\tilde{\mathcal{N}} e^{\perp} \wedge e^0 + e^{a_1} \wedge e^{b_1} + e^{a_2} \wedge e^{b_2} \qquad (7.10)$$

where $e^{b_k} = \tilde{I} \cdot e^{a_k}$, $k = 1, 2$, are unit spacelike 1-forms and $\tilde{\mathcal{N}}$ normalizes $e^{\perp}$ and e^0 in this region. The above form for $\tilde{\omega}$ restricts conveniently to AdS_5 as

$$\tilde{\omega}\Big|_{AdS} = (e^{a_1} \wedge e^{b_1} + e^{a_2} \wedge e^{b_2})\Big|_{AdS} \tag{7.11}$$

This 2-form will appear later when we construct p-forms for supersymmetric giant gravitons.

Later it will be useful to parameterise AdS_5 with "polar" coordinates. In particular, we can take

$$W_0 = \cosh\rho \, e^{it}, \qquad W_1 = \sinh\rho \, (\Omega_1 + i\Omega_2), \qquad W_2 = \sinh\rho \, (\Omega_3 + i\Omega_4)$$

where $\sum_{i=1}^4 \Omega_i^2 = 1$. With this parametrization, the embedding condition for AdS_5, given in Eq. (7.7), is automatically satisfied. Moreover, the metric on AdS_5 becomes

$$ds^2_{AdS} = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \sum_{i=1}^4 d\Omega_i^2 \tag{7.12}$$

supplemented with the condition that $\sum_{i=1}^4 \Omega_i^2 = 1$. In the next sections we will also need the expression for e^0 in these coordinates:

$$e^0 = \cosh^2\rho \, dt - \sinh^2\rho \, (\Omega_1 d\Omega_2 - \Omega_2 d\Omega_1 + \Omega_3 d\Omega_4 - \Omega_4 d\Omega_3) \tag{7.13}$$

Note that the metric Eq. (7.12) becomes the usual AdS_5 metric of § 2.1 if we take $r = \sinh\rho$ and we write the coordinates Ω_i in terms of angles $\alpha_1, \alpha_2, \alpha_3$ as follows,

$$\begin{aligned} \Omega_1 &= \cos\alpha_1 & \Omega_2 &= \sin\alpha_1 \cos\alpha_2 \\ \Omega_3 &= \sin\alpha_1 \sin\alpha_2 \cos\alpha_3 & \Omega_4 &= \sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \end{aligned}$$

So in this case we obtain the following metric on AdS_5,

$$ds^2 = -\left(1 + r^2\right) \, dt^2 + \frac{dr^2}{1 + r^2} + r^2(d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3^2) \tag{7.14}$$

7.1.2 Giant graviton construction

Giant gravitons in $AdS_5 \times S^5$ are D3-branes which have their spatial world-volume entirely contained in the S^5 part of the geometry. In this construction, the spatial world-volume of the brane is defined by the intersection of a holomorphic surface in $\mathbb{C}^3$ with the S^5. In particular, we consider the class of holomorphic surfaces, $C \subset \mathbb{C}^3$, which have complex dimension 2 (4 real dimensions). These surfaces are specified by a single equation,

$$F(Z_1, Z_2, Z_3) = 0$$

Here F depends only on the holomorphic coordinates Z_i (i.e. it does not depend on the $\bar{Z}_i s$). The intersection of C with S^5 is a 3-dimensional surface, Σ, which we take to be the spatial world-volume of the giant graviton at time $t = 0$.

Giant gravitons have a non-trivial motion on the S^5. In this construction they are defined to move with the speed of light (c=1 in our units) in the direction $e^{||}$. Typically the surface of the giant graviton Σ will not be orthogonal to $e^{||}$ (in fact the construction would break down if the brane was completely orthogonal to $e^{||}$ at any point). Therefore, at each point on the brane, $e^{||}$ can be decomposed into a component normal to the brane, denoted e^{ϕ}, and a component parallel to the brane, denoted e^{ψ}, i.e.

$$e^{||} = -ve^{\phi} - \sqrt{1 - v^2} e^{\psi} \tag{7.15}$$

where $0 < v < 1$. In fact, v turns out to be the speed of the giant graviton in the direction e^{ϕ}. This association arises from requiring the brane to be supersymmetric [45], and we will see in § 7.3 that this condition is also encoded in the calibration bound for giant gravitons. Since $v < 1$, it means that the surface elements of the brane move at less than the speed of light, even though the centre of mass of the brane (which does not lie on the brane) moves with the speed of light. Now, due to the holomorphic construction of Σ, the directions wrapped by the brane are e^{ψ}, e^K, e^L, where $\{e^K, e^L\}$ are unit 1-forms which define a complex 2-cycle orthogonal to $e^{||}$, i.e. $e^L = I \cdot e^K$ and $e^{||} \cdot e^K = e^{||} \cdot e^L = 0$ [45].

We can actually define the full world-volume of the giant graviton using the holomorphic function F. Due to the form of $e^{||}$ given in Eq. (7.3), the full world-volume of the giant graviton is given by the intersection of S^5 with the following surface [45]

$$F(e^{it} Z_1, e^{it} Z_2, e^{it} Z_3) = 0$$

The above equation describes the holomorphic surface C translated in the direction $e^{||}$ at the speed of light (We can think of this as $F(Z_1(t), Z_2(t), Z_3(t)) = 0$, where $Z_i(t) = e^{it} Z_i$ are comoving coordinates). We will see in § 7.1.4 that the giant graviton introduced in § 2.1 is a simple case of this construction; one takes the holomorphic surface to be $F = Z_1 - d$, where d is a constant. However, since any holomorphic surface can be used, more complicated giant gravitons are also included in this description. Mikhailov proves that all giant gravitons in this construction preserve at least $\frac{1}{8}$ supersymmetry. In the next section we will discuss the supersymmetry projection conditions for these branes.

Note that the Mikhailov construction does not specify the AdS trajectory of the giant graviton. However, if t in the above expression is identified with the time coordinate in the AdS_5 metric in Eq. (7.12), then we are implicitly assuming that the trajectory of the giant graviton is $\rho = 0$. However, it is known that giant gravitons behave as free massive particles in AdS_5 [54], so they can move along any time-like geodesic. The trajectory $\rho = 0$ is one particular time-like geodesic in AdS_5 (where the particle is stationary at $\rho = 0$), and it can be related to any other time-like geodesic in AdS_5 by an appropriate change of coordinates [103]. Therefore, without loss of generality, we will consider giant gravitons sitting at $\rho = 0$, since other AdS trajectories can be easily related to this.

One further note is that the Mikhailov construction for giant gravitons can be applied to more general backgrounds than $AdS_5 \times S^5$. For example, it is also possible to construct holomorphic giant gravitons in $AdS_5 \times T^{1,1}$, where $T^{1,1}$ is embedded into the conifold. (Here the holomorphic surfaces C are defined with reference to the complex structure of the conifold. The spatial world-volume of a giant graviton then arises from the intersection of one of these holomorphic surfaces with $T^{1,1}$. For more details see Ref. [104].)

7.1.3 Giant gravitons and supersymmetry

This construction of giant gravitons via holomorphic surfaces in the 12-dimensional complex space $\mathbb{C}^{1,2} \times \mathbb{C}^3$ means that they preserve supersymmetry. Moreover, the supersymmetry projection conditions can be written down in a very simple way. This is due in part to the fact that Killing spinors in $AdS_5 \times S^5$ become covariantly constant spinors in the 12-dimensional space[1], so everything simplifies in the higher-dimensional setting. In particular, Mikhailov finds the supersymmetry conditions with reference to the $d = 12$ covariantly constant spinors, and then projects these conditions down to spinors in 10 dimensions. We give the conditions on the $d = 10$ spinors here. The amount of supersymmetry preserved by a particular giant graviton depends on the function F which defines the holomorphic surface C (and hence the brane surface Σ). If F depends on (1, 2, 3) of the complex coordinates[2] then the resulting giant graviton configuration will preserve ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$) of the supersymmetry respectively. The projection conditions satisfied by the most general configurations, which preserve $\frac{1}{8}$ supersymmetry, are given by [45]

$$\begin{aligned} \Gamma^0\Gamma^{||}\epsilon^1 &= \epsilon^1 \\ \Gamma^{I_k}\Gamma^{J_k}\epsilon^1 &= -\epsilon^2, \qquad k = 1,2 \end{aligned} \tag{7.16}$$

[1]This interesting fact was first realized in Ref. [105], and was highlighted in Ref. [106].

[2]Up to linear holomorphic redefinitions of Z_i which do not alter the amount of supersymmetry preserved.

Here ϵ^1, ϵ^2 are the chiral spinors of type IIB supergravity, which form a Killing spinor $\epsilon = (\epsilon^1, \epsilon^2)^T$ of $AdS_5 \times S^5$ (so ϵ satisfies $D_m \epsilon = 0$ and $\mathcal{P}\epsilon = 0$ for this background, where these operators were defined in Chapter 6). Moreover,

$$\Gamma^0 = \Gamma(e^0), \qquad \Gamma^{||} = \Gamma(e^{||}), \qquad \Gamma^{I_k} = \Gamma(e^{I_k}), \qquad \Gamma^{J_k} = \Gamma(e^{J_k})$$

where the 1-forms $e^0, e^{||}, e^{I_k}, e^{J_k}$ are defined in § 7.1.1 and here they are all evaluated on $AdS_5 \times S^5$ (N.B. while these projections are made with reference to the complex structure of $\mathbb{C}^{1,2} \times \mathbb{C}^3$, everything now is in 10 dimensions, so the forms must be evaluated on the lower-dimensional space). Note that since the Γ-matrices in the projection conditions correspond to unit 1-forms in $AdS_5 \times S^5$, they all square to ± 1 (i.e. they are tangent space Γ-matrices). In finding these projection conditions, Mikhailov shows that v, defined in Eq. (7.15), must be associated with the physical speed of the giant graviton (see Ref. [45] for details). In § 7.2 we will use the projection conditions given in Eq. (7.16) to explicitly construct the differential forms, K^{ij}, Φ^{ij} and Σ^{ij}, relevant to these branes. First, however, we give a simple example of the Mikhailov construction, where the giant graviton of § 2.1 is reproduced.

7.1.4 A simple example of the construction

In this section we consider a particular holomorphic function in $\mathbb{C}^3$, namely $F = Z_1 - d$, where d is a constant. We construct the giant graviton corresponding to this function and show that the original giant graviton of Ref. [46], which we presented in § 2.1, arises from the Mikhailov construction for this choice of F.

The giant graviton is specified at $t = 0$ by $F = Z_1 - d = 0$. Following Mikhailov's prescription this means that the world-volume of the giant graviton is given by the intersection of the following surfaces

$$\begin{cases} e^{it} Z_1 - d = 0 \\ |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1 \end{cases}$$

In the AdS space, we take the giant graviton to sit at $\rho = 0$. Writing $Z_i = \mu_i e^{i\phi_i}$, the first equation becomes

$$\mu_1 e^{i(\phi_1 + t)} - d = 0$$

We can solve this by taking $\mu_1 = |d|$ and $\phi_1 = -t + const$, i.e. $\dot{\phi}_1 = -1$ and so the brane moves in the ϕ_1 direction. Recall that the equation for S^5, given in Eq. (7.2), is

$\sum_{i=1}^{3}\mu_i^2 = 1$, where $\mu_i \geq 0$. We can parametrize μ_i by two angles as follows,

$$\mu_1 = \cos\theta_1, \qquad \mu_2 = \sin\theta_1\cos\theta_2, \qquad \mu_3 = \sin\theta_1\sin\theta_2$$

where $0 \leq \theta_1, \theta_2 \leq \pi/2$. In these coordinates, the metric on S^5 is given by

$$ds^2_{S^5} = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cos^2\theta_1 d\phi_1^2 + \sin^2\theta_1\cos^2\theta_2 d\phi_2^2 + \sin^2\theta_1\sin^2\theta_2 d\phi_3^2$$

Now, since the giant graviton surface has $\mu_1 = |d|$, this translates into $\theta_1 = const.$ Moreover, the giant graviton moves in the ϕ_1 direction, so it must wrap the remaining sphere coordinates: θ_2, ϕ_2, ϕ_3, which define an S^3. Therefore, the induced metric on the giant graviton world-volume is

$$ds^2_{g.g.} = (-1 + \cos^2\theta_1\dot{\phi}_1^2)dt^2 + \sin^2\theta_1[d\theta_2^2 + \cos^2\theta_2 d\phi_2^2 + \sin^2\theta_2 d\phi_3^2] \tag{7.17}$$

where the term $-dt^2$ comes from pulling back the AdS_5 metric to $\rho = 0$. This metric agrees with the induced metric, Eq. (2.10), for the giant graviton in § 2.1 if we set $L = 1$. Note that the terms in square brackets in Eq. (7.17) correspond to the usual metric on S^3. Therefore, the radius of this spherical giant graviton is $\sin\theta_1$, as in Chapter 2.

Finally, we calculate the speed of this brane according to the Mikhailov construction. From Eqs. (7.3) and (7.15) we have

$$e^{||} = \mu_1^2 d\phi_1 + \mu_2^2 d\phi_2 + \mu_3^2 d\phi_3 = -ve^{\phi} - \sqrt{1-v^2}e^{\psi}$$

where e^{ϕ} is a unit 1-form corresponding to the physical direction of motion of the brane. We know that the giant graviton moves in the direction ϕ_1, and from the metric on S^5 a unit 1-form in this direction is $e^{\phi} = \mu_1 d\phi_1$. Therefore, comparing with the equation above we find $v = -\mu_1 = -\cos\theta_1$ for the giant graviton's speed (Note that this agrees with $\dot{\phi}_1 = -1$ found earlier, since $v = de^{\phi}/dt = \mu_1\dot{\phi}_1 = -\mu_1$).

In summary, the holomorphic surface $F = Z_1 - d$ produces a giant graviton which wraps an S^3 of radius $\sin\theta_1$, and moves in the ϕ_1 direction with $\dot{\phi}_1 = -1$. This is precisely the original giant graviton of Ref. [46] which was presented in § 2.1. Note that because F only depends on Z_1, this brane preserves $\frac{1}{2}$ supersymmetry. This agrees with the supersymmetry calculations of Ref. [60] for this particular brane configuration.

7.2 Differential forms for giant gravitons

In this section we use the projection conditions given in Eq. (7.16) to construct the forms K^{ij}, Φ^{ij} and Σ^{ij} relevant to holomorphic giant gravitons. These forms were introduced in

Chapter 6, and they are constructed from one Killing spinor of the background, which in this case is $AdS_5 \times S^5$. We will check explicitly that the differential and algebraic relations derived in Chapter 6 are satisfied by these forms.

To begin our construction of these forms, we must make some additional projections which are compatible with Eq. (7.16) to select one Killing spinor which we will use to construct the forms. The projection conditions in Eq. (7.16) admit 4 independent Killing spinors, so we need to make another two projections to reduce this number to 1 (because each projection reduces the number of allowed spinors by $\frac{1}{2}$). The obvious way to make compatible projections is to treat the complex structure of AdS_5 in a similar way to the complex structure of S^5. Therefore, one set of possible projections is

$$\Gamma^{a_k}\Gamma^{b_k}\epsilon^1 = -\epsilon^2 \qquad k = 1,2 \tag{7.18}$$

where

$$\Gamma^{a_k} = \Gamma(e^{a_k}), \qquad \Gamma^{b_k} = \Gamma(e^{b_k})$$

and e^{a_k}, e^{b_k}, defined in § 7.1.1, are non-zero unit 1-forms which we evaluate on AdS_5. These 1-forms are orthogonal to $\{e^0, e^{||}, e^{I_k}, e^{J_k}\}$, so the above projections commute with the existing projections in Eq. (7.16). Note that again the matrices in the above projection conditions are tangent space Γ-matrices, since they are associated to unit 1-forms. Therefore, the full set of projection conditions is

$$\Gamma^0\Gamma^{||}\epsilon^1 = \epsilon^1,$$

$$\Gamma^{I_k}\Gamma^{J_k}\epsilon^1 = -\epsilon^2, \qquad \Gamma^{a_k}\Gamma^{b_k}\epsilon^1 = -\epsilon^2 \qquad k = 1,2 \tag{7.19}$$

Note that in this basis the chirality condition, $\Gamma^{\hat{0}\dots\hat{9}}\epsilon^i = \epsilon^i$, becomes

$$\Gamma^{0a_1b_1a_2b_2I_1J_1I_2J_2||}\epsilon^i = \epsilon^i$$

where all these Γ-matrices are defined above.

Using the projection conditions given in Eq. (7.19) we can now compute all the p-forms which were defined in § 6.1. This will give us the set of p-forms relevant to giant gravitons. Firstly, the 1-forms, K^{ij}, which have components $K^{ij}_m = \bar{\epsilon}^i\Gamma_m\epsilon^j$, are given by

$$\begin{aligned} K^{11} &= K^{22} = \Delta(e^0 + e^{||}) \\ K^{12} &= K^{21} = 0 \end{aligned} \tag{7.20}$$

where Δ is the normalisation of the spinors, $\Delta \equiv (\epsilon^1)^T\epsilon^1 = (\epsilon^2)^T\epsilon^2$, and $\bar{\epsilon}^i = (\epsilon^i)^T\Gamma^0$ (N.B. $\Gamma^0 = \Gamma(e^0)$). The 3-forms Φ^{ij} are non-zero only for $i \neq j$. Their components are given by $\Phi^{ij}_{mnp} = \bar{\epsilon}^i \Gamma_{mnp} \epsilon^j$. In this case we obtain

$$\Phi^{12} = -\Phi^{21} = \Delta(e^0 + e^{||}) \wedge (\omega_S + \tilde{\omega}_{AdS}) \tag{7.21}$$

where ω_S and $\tilde{\omega}_{AdS}$ are the restricted Kähler 2-forms on S^5 and AdS_5 respectively. These 2-forms were defined precisely in Eqs. (7.6) and (7.11). The 5-forms, Σ^{ij}, have components given by $\Sigma^{ij}_{mnpqr} = \bar{\epsilon}^i \Gamma_{mnpqr} \epsilon^j$. In this case we find that these forms are given by

$$\begin{aligned} \Sigma^{11} &= \Sigma^{22} = \Delta(e^0 + e^{||}) \wedge \left(-\frac{1}{2}\omega_S \wedge \omega_S - \frac{1}{2}\tilde{\omega}_{AdS} \wedge \tilde{\omega}_{AdS} - \omega_S \wedge \tilde{\omega}_{AdS} \right) \\ \Sigma^{12} &= \Sigma^{21} = 0. \end{aligned} \tag{7.22}$$

We now calculate the derivatives of the forms K, Φ and Σ, and show that they obey the differential equations derived in § 6.1. Recall that these equations relate derivatives of forms to terms involving the background field strengths. In $AdS_5 \times S^5$ the only non-zero field strength is $G^{(5)}$. Explicitly, $G^{(5)}$ is given by $G^{(5)} = -4\{vol(AdS_5) + vol(S^5)\}$. In our basis this is

$$G^{(5)} = -2(e^0 \wedge \tilde{\omega}_{AdS} \wedge \tilde{\omega}_{AdS} + e^{||} \wedge \omega_S \wedge \omega_S) \tag{7.23}$$

To calculate the derivatives of the forms we will need the following results,

$$\begin{aligned} de^0 &= -2\tilde{\omega}_{AdS}, \\ de^{||} &= 2\omega_S, \\ d\Delta &= 0 \end{aligned} \tag{7.24}$$

The first two equations follow from Eqs. (7.4) and (7.9), together with the fact that e^0 and $e^{||}$ are evaluated on $AdS_5 \times S^5$. The third equation can be derived by writing $\Delta = \bar{\epsilon}^1 \Gamma_0 \epsilon^1$ and calculating $d\Delta$ using the Killing spinor equation. We will not go into the details of this calculation here, as it is analogous to the calculations in Chapter 6. Note that the third equation allows us to set $\Delta = 1$, which we do in the following.

We first consider the differential equation for K^{11}. From Eqs. (7.20) and (7.24) the derivative is given by

$$dK^{11} = de^0 + de^{||} = 2(-\tilde{\omega}_{AdS} + \omega_S) \tag{7.25}$$

From Eq. (6.7) in Chapter 6, this should be related to $\iota_{\Phi^{12}} G^{(5)}$, which we compute using the expression for Φ^{12} given in Eq. (7.21):

$$\iota_{\Phi^{12}} G^{(5)} = -4(-\tilde{\omega}_{AdS} + \omega_S)$$

Therefore,

$$dK^{11} = -\frac{1}{2}\iota_{\Phi^{12}}G^{(5)}$$

This is precisely what we expect from Eq. (6.7) since the dilaton is constant for $AdS_5 \times S^5$. The equation for K^{22} works in the same way as above. The differential equations for K^{12} and K^{21}, given in Eq. (6.8) of Chapter 6, are trivially satisfied as both left and right hand sides are identically zero.

We now consider the differential equation for Φ^{12}. Firstly, from Eqs. (7.21) and (7.24) we have

$$\begin{aligned} d\Phi^{12} &= d\left[(e^0 + e^{||}) \wedge (\omega_S + \tilde{\omega}_{AdS})\right] \\ &= 2(\omega_S \wedge \omega_S - \tilde{\omega}_{AdS} \wedge \tilde{\omega}_{AdS}) \end{aligned} \tag{7.26}$$

From Eq. (6.9) this should be related to $\iota_{(K^{11}+K^{22})}G^{(5)}$ which we can compute:

$$\iota_{(K^{11}+K^{22})}G^{(5)} = 4(\tilde{\omega}_{AdS} \wedge \tilde{\omega}_{AdS} - \omega_S \wedge \omega_S) \tag{7.27}$$

Therefore, from Eqs. (7.26) and (7.27), we have

$$d\Phi^{12} = -\frac{1}{2}\iota_{(K^{11}+K^{22})}G^{(5)} \tag{7.28}$$

as required. The equation for Φ^{21} works in the same way. Note that Eq. (7.28) is the type of condition we expect for a generalized calibration, as we saw in § 4.4. In the next section we will see precisely how Φ is related to a generalized calibration for D3-branes in supersymmetric type IIB backgrounds.

Using Eqs. (7.22) and (7.24), we can calculate the derivative of the 5-form Σ^{11}. We find

$$d\Sigma^{11} = -\tilde{\omega}_{AdS} \wedge \omega_S \wedge \omega_S + \omega_S \wedge \tilde{\omega}_{AdS} \wedge \tilde{\omega}_{AdS} \tag{7.29}$$

From Eq. (6.10), the components $(d\Sigma^{11})_{mnpqrs}$ should be equal to

$$\frac{15}{2}\Phi^{12}{}_{t[mn}G^{(5)}{}_{pqrs]}{}^{t}$$

since $K^{12} = K^{21} = 0$ and the dilaton is zero. By considering different combinations of the indices, one finds that

$$\frac{15}{2}\Phi^{12}{}_{t[mn}G^{(5)}{}_{pqrs]}{}^{t} = \left(-\omega_S \wedge \omega_S \wedge \tilde{\omega}_{AdS} + \omega_S \wedge \tilde{\omega}_{AdS} \wedge \tilde{\omega}_{AdS}\right)_{mnpqrs}$$

and hence

$$(d\Sigma^{11})_{mnpqrs} = \frac{15}{2}\Phi^{12}{}_{t[mn}G^{(5)}{}_{pqrs]}{}^{t} \tag{7.30}$$

as required. The equation for Σ^{22}, given in Eq. (6.11), works in the same way since $\Sigma^{11} = \Sigma^{22}$ in this case. The equations for Σ^{12} and Σ^{21}, given in Eq. (6.12), are trivially satisfied, since all terms in these equations are identically zero.

The algebraic relations derived in Chapter 6 are easy to verify. Firstly, since $G^{(5)}$ is the only non-zero background field for the $AdS_5 \times S^5$ solution, the algebraic identities derived from the Killing spinor equation in Eqs. (6.14)-(6.25) are all automatically satisfied, as all terms in these equations vanish. Secondly, from the form of K, Φ and Σ given here, we have $K^{ij} \cdot K^{kl} = \Phi^{ij} \cdot \Phi^{kl} = \Sigma^{ij} \cdot \Sigma^{kl} = 0$, i.e. they satisfy the algebraic relations derived from Fierz identities in Eq. (6.27) as required.

7.3 Calibrations for giant gravitons

In this section we consider constructing a generalized calibration for giant gravitons. Our approach is to first consider the super-translation algebra for D3-branes in flat 10-dimensional space. We then extend the flat space algebra to allow for backgrounds with non-zero $G^{(5)}$. This is analogous to the extensions of $d = 11$ supersymmetry algebras which were discussed in Chapter 5. The extended algebra will allow us to find a calibration bound for a giant graviton in $AdS_5 \times S^5$, and we will see that the bound involves the 3-forms Φ^{ij}. Using the form for Φ^{ij} given in the previous section we will see that all giant gravitons constructed from holomorphic surfaces are calibrated. Furthermore, we will see that the quantity minimized by these calibrated branes is not simply the energy. Rather, the giant gravitons minimize "energy minus momentum" in their homology class.

7.3.1 The super-translation algebra and calibration bound

The super-translation algebra for D3-branes in flat space is given by [107]

$$\{Q_{i\alpha}, Q_{j\beta}\} = \delta_{ij}(C\Gamma_m)_{\alpha\beta}P^m + (i\sigma_2)_{ij}\frac{1}{3!}\,(C\Gamma_{mnp})_{\alpha\beta}Z^{mnp} \tag{7.31}$$

where

$$Z^{mnp} = \int dx^m \wedge dx^n \wedge dx^p \tag{7.32}$$

and the integral is taken over the spatial world-volume of the brane. The indices $i, j \in \{1, 2\}$ and α, β are spinor indices. The matrix C is the charge conjugation matrix, which we will always take to be $\Gamma^0 \equiv \Gamma(e^0)$. The quantity P^m is the total 10-momentum of the brane. The term involving Z is a topological charge for the D3-brane. The fact that this term is topological is clear from Eq. (7.32), since Z is defined as the integral of a closed form over the spatial world-volume of the brane.

We now introduce a constant commuting spinor, $\epsilon = (\epsilon^1, \epsilon^2)^T$, and contract all indices in Eq. (7.31) with the indices of ϵ to obtain

$$2(Q\epsilon)^2 = (K^{11} + K^{22}) \cdot P + \int (\Phi^{12} - \Phi^{21}) \tag{7.33}$$

where $Q\epsilon = Q_1\epsilon^1 + Q_2\epsilon^2$, and the spinor indices are also contracted. We can rewrite the first term in Eq. (7.33) as an integral over the spatial world-volume of the brane, to obtain

$$2(Q\epsilon)^2 = \int (K^{11} + K^{22}) \cdot p + \int 2\Phi^{12} \tag{7.34}$$

where p_M is the momentum density of the brane, and we have used the fact that $\Phi^{12} = -\Phi^{21}$ to rewrite the second term. Note that the integrand Φ^{12} is closed. This is because the Killing spinor, ϵ, used to build the forms is constant (In flat space, Killing spinors and constant spinors are equivalent.).

We now want to consider the super-translation algebra for a curved background with non-zero $G^{(5)}$, but with all other field strengths and the dilaton zero. This will allow us to consider the case we are interested in, namely D3-brane giant gravitons in $AdS_5 \times S^5$. Using the method of Chapter 5, we can find the curved space super-translation algebra by modifying Eq. (7.34) as follows. First we promote the constant spinor ϵ to a Killing spinor of the background. This means that the forms K^{11}, K^{22} and Φ^{12} are no longer constant, but become fields. Secondly, we replace Φ^{12} by a closed 3-form, since for non-zero $G^{(5)}$ this form is not closed. In particular,

$$d\Phi^{12} = -\frac{1}{2}\iota_{(K^{11}+K^{22})}G^{(5)} = -\frac{1}{2}\iota_K G^{(5)} \tag{7.35}$$

where $K \equiv K^{11} + K^{22}$. However, we can construct a closed 3-form from Φ^{12} by manipulating this equation. The starting point is to compute the Lie derivative of $G^{(5)}$ along the direction K. Using the expression for the Lie derivative given in Eq. (A.8) of Appendix A, this is given by

$$\mathcal{L}_K G^{(5)} = d(\iota_K G^{(5)}) + \iota_K dG^{(5)}$$

Differentiating Eq. (7.35) and using the fact that $dG^{(5)} = 0$ for this background, it is easy to see that the two terms here vanish independently and $\mathcal{L}_K G^{(5)} = 0$. This means we can choose a gauge for the 4-form Ramond-Ramond potential $C^{(4)}$ such that $\mathcal{L}_K C^{(4)} = 0$ also. In that case

$$d\left(2\Phi^{12} - \iota_K C^{(4)}\right) = -\iota_K G^{(5)} - d\iota_K C^{(4)} = -\mathcal{L}_K C^{(4)} = 0$$

Therefore, we propose that for backgrounds with non-zero $G^{(5)}$, the 3-form $2\Phi^{12}$ should be replaced by

$$2\Phi^{12} - \iota_K C$$

in the super-translation algebra (from now on we will drop the index (4) on $C^{(4)}$). So, the algebra becomes

$$2(Q\epsilon)^2 = \int K \cdot p + \int \left(2\Phi^{12} - \iota_K C \right) \tag{7.36}$$

Clearly this reduces to the original flat space algebra if we set the 4-form potential, C, to zero. We now use the fact that $(Q\epsilon)^2 \geq 0$ to obtain the following calibration bound:

$$\int (K \cdot p - \iota_K C) \geq -\int 2\Phi^{12} \tag{7.37}$$

where the integrals are over the spatial world-volume of the brane (and so $\iota_K C$ and Φ^{12} are understood to be pulled back to the brane). This bound is valid for all D3-branes in supersymmetric backgrounds which have field strengths $G^{(1)}$, $G^{(3)}$, H zero and the dilaton also zero. In particular, we will see in § 7.3.2 that holomorphic giant gravitons in $AdS_5 \times S^5$ saturate this bound, i.e. they are calibrated. Moreover, in § 7.4 we will see that the dual giant graviton of Ref. [60] is also calibrated.

First, however, we show that any brane which saturates the bound Eq. (7.37) (i.e. a calibrated brane) minimises the quantity $\int K \cdot p$ in its homology class. To prove this, consider two 3-dimensional manifolds U and V in the same homology class. Moreover, we assume that the manifold U is calibrated, i.e.

$$\int_U (K \cdot p - \iota_K C) = -\int_U 2\Phi^{12} \tag{7.38}$$

Now since U and V are in the same homology class, we can write $U = V + \partial\Xi$ where $\partial\Xi$ is the boundary of a 4-dimensional manifold Ξ. Therefore,

$$\int_U (K \cdot p - \iota_K C) = -\int_{V+\partial\Xi} 2\Phi^{12} \tag{7.39}$$

Now using Stoke's theorem together with Eq. (7.35) we have

$$-\int_{V+\partial\Xi} 2\Phi^{12} = -\int_V 2\Phi^{12} + \int_\Xi \iota_K G^{(5)}$$

Since we have chosen a gauge where $\mathcal{L}_K C = 0$, it follows that $\iota_K G^{(5)} = -d\iota_K C$, and therefore,

$$\int_\Xi \iota_K G^{(5)} = -\int_{\partial\Xi} \iota_K C = -\int_U \iota_K C + \int_V \iota_K C$$

where we have used Stoke's law again, and rewritten $\partial\Xi = U - V$ in the last step.

Therefore, using the above two equations we see that Eq. (7.39) becomes

$$\begin{aligned} \int_U (K\cdot p - \iota_K C) &= -\int_V 2\Phi^{12} - \int_U \iota_K C + \int_V \iota_K C \\ &\leq \int_V (K\cdot p - \iota_K C) - \int_U \iota_K C + \int_V \iota_K C \end{aligned}$$

where we have used the calibration bound Eq. (7.37) for V in the second line. Rearranging, this is just,

$$\int_U K\cdot p \leq \int_V K\cdot p \tag{7.40}$$

i.e. U has minimal $\int K\cdot p$ compared to all other manifolds in the same homology class.

To get an idea of what this means, we can consider the case where K is simply the timelike vector e^0. Then $K\cdot p = -p_0$ and $-p_0$ can be identified with the Hamiltonian density for the brane [90]. Therefore, in this case, the quantity minimized by a calibrated brane is its energy. However, as shown in § 7.2, giant gravitons have null K. This means that the quantity minimized by calibrated giant gravitons is "Energy minus momentum", as we now see.

7.3.2 Holomorphic giant gravitons

We now specialise to the case of giant gravitons in $AdS_5 \times S^5$. That is, we consider the calibration bound Eq. (7.37) with K and Φ relevant to holomorphic giant gravitons. Using the expressions for K^{ij} and Φ^{ij} given in Eqs. (7.20) and (7.21), we obtain the following expression for the bound,

$$\int \left(-p_0 + p_{||} - \iota_0 C - \iota_{||} C\right) \geq \int -(e^0 + e^{||}) \wedge (\omega_S + \tilde{\omega}_{AdS}) \tag{7.41}$$

where the integrals are over the spatial world-volume of the brane. Since the spatial world-volume of a giant graviton is entirely contained in the S^5 part of the geometry, this inequality reduces to

$$\int \left(\mathcal{H} + p_{||} - \iota_0 C - \iota_{||} C\right) \geq \int -e^{||} \wedge \omega_S \tag{7.42}$$

where we have identified $-p_0$ with the Hamiltonian density $\mathcal{H}$. From the previous section we know that calibrated branes minimise $\int K\cdot p$, which in this case is

$$\int K\cdot p = 2\int \left(-p_0 + p_{||}\right) \propto \int \left(\mathcal{H} + p_{||}\right)$$

Now, recall that the physical motion of the giant graviton is in the direction e^{ϕ}, where $e^{||} = -ve^{\phi} - \sqrt{1-v^2}e^{\psi}$. There is no physical momentum in the direction e^{ψ}, so the quantity minimized by a calibrated giant graviton is actually

$$\int (\mathcal{H} + p_{||}) = \int (\mathcal{H} - vp_{\phi})$$

i.e. calibrated giant gravitons minimise the total energy minus the total physical momentum, $J = \int vp_{\phi}$, which is a conserved charge. Note that this agrees with Ref. [108] where the generator of time translations for giant gravitons is $E - J$ (N.B. our definition of the direction ϕ is different to the definition in Ref. [108]). We will now see that giant gravitons constructed from holomorphic surfaces saturate the bound in Eq. (7.42) and hence have minimal energy minus momentum in their homology class. This indicates that the Mikhailov construction is indeed correct. Moreover, we will see that a brane which wraps the same surface as a holomorphic giant graviton, but travels at the wrong speed (i.e. at a different speed to that specified in the Mikhailov construction), does not saturate the bound.

We begin by evaluating the quantities $\mathcal{H}$ and $p_{||}$ which appear on the left hand side of the bound Eq. (7.42). To do this we must first calculate the giant graviton Lagrangian. Schematically, this is given by

$$\mathcal{L} = -\sqrt{-\gamma} + \mathcal{P}(C) \tag{7.43}$$

where γ is the determinant of the induced metric on the brane, and $\mathcal{P}(C)$ is the pull-back of the 4-form gauge potential to the giant graviton world-volume. To calculate the induced metric, we rewrite the metric on S^5 in a basis which is related to the giant graviton world-volume:

$$ds^2_{S^5} = (e^{\phi})^2 + (e^n)^2 + d\Sigma^2 \tag{7.44}$$

Here e^{ϕ}, defined in Eq. (7.15), is the physical direction of motion of the brane, and e^n is a unit 1-form on S^5 which is orthogonal to e^{ϕ} and to the brane surface, Σ. The 3-dimensional metric $d\Sigma^2$ is the metric on the spatial world-volume of the giant graviton. This rewriting allows us to calculate the induced metric very easily. We obtain,

$$ds^2_{g.g.} = (-1 + \dot{\phi}^2)dt^2 + d\Sigma^2 \tag{7.45}$$

where $\dot{\phi} = de^{\phi}/dt$ and t is a time-like coordinate for the brane, which coincides with the AdS_5 time (i.e. we are choosing static gauge). The term $-dt^2$ comes from the pull-back of AdS_5 metric to the trajectory $\rho = 0$ (Recall that we consider giant gravitons sitting at $\rho = 0$ in the AdS space.). Now we evaluate the quantity $\mathcal{P}(C)$. Since the giant graviton moves in the e^ϕ direction, this is simply given by

$$\mathcal{P}(C) = \left(C_{t\sigma^1\sigma^2\sigma^3} + \dot{\phi}C_{\phi\ \sigma^1\sigma^2\sigma^3}\right) dt \wedge d^3\sigma$$

where σ^i $(i = 1, 2, 3)$ are the coordinates on the spatial world-volume of the brane. Therefore, we obtain the following Lagrangian density for the giant graviton,

$$\mathcal{L} = -\sqrt{(1-\dot{\phi}^2)\Sigma} + C_{t\sigma^1\sigma^2\sigma^3} + \dot{\phi}C_{\phi\ \sigma^1\sigma^2\sigma^3} \tag{7.46}$$

where Σ is the determinant of the metric $d\Sigma^2$. From this Lagrangian we can calculate the momentum conjugate to ϕ. We obtain,

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\dot{\phi}\,\sqrt{\Sigma}}{\sqrt{1-\dot{\phi}^2}} + C_{\phi\ \sigma^1\sigma^2\sigma^3} \tag{7.47}$$

Therefore, the Hamiltonian is

$$\mathcal{H} = p_\phi\dot{\phi} - \mathcal{L} = \frac{\sqrt{\Sigma}}{\sqrt{1-\dot{\phi}^2}} - C_{t\sigma^1\sigma^2\sigma^3} \tag{7.48}$$

We could, of course, rewrite $\mathcal{H}$ in terms of p_ϕ rather than $\dot{\phi}$. However, it will be more convenient to leave $\mathcal{H}$ in this form for calculating the calibration bound. We now calculate the remaining quantities on the left hand side of the bound Eq. (7.42). Firstly, $p_{||} = -vp_\phi$, so the momentum in the direction $e^{||}$ is

$$p_{||} = -\frac{v\dot{\phi}\,\sqrt{\Sigma}}{\sqrt{1-\dot{\phi}^2}} - vC_{\phi\ \sigma^1\sigma^2\sigma^3}$$

Moreover,

$$\int \left(\iota_0 C + \iota_{||}C\right) = \int \left(-C_{t\sigma^1\sigma^2\sigma^3} - v\ C_{\phi\sigma^1\sigma^2\sigma^3}\right)\ d^3\sigma$$

where we have used the fact that $e^0 = dt$ on the giant graviton trajectory together with the decomposition $e^{||} = -ve^\phi - \sqrt{1-v^2}e^\psi$, given in Eq. (7.15). So the left hand side of the calibration bound Eq. (7.42) becomes

$$\int \left(\mathcal{H} + p_{||} - \iota_0 C - \iota_{||}C\right) = \int \frac{\sqrt{\Sigma}(1-v\dot{\phi})}{\sqrt{1-\dot{\phi}^2}}\, d^3\sigma \tag{7.49}$$

Note that in the Mikhailov construction the speed of the giant graviton in the direction e^ϕ is v, i.e. $\dot{\phi} = v$. However, one could also consider a brane which wraps the same surface

Σ, but has a different speed, i.e. $\dot{\phi} \neq v$. We will see that these branes are not calibrated, so we leave v and $\dot{\phi}$ as distinct quantities for the moment.

We now calculate the right hand side of the calibration bound Eq. (7.42) for a general holomorphic giant graviton. Using the decomposition of $e^{||}$ given in Eq. (7.15) we have

$$\int -e^{||} \wedge \omega_S = \int \sqrt{1-v^2}\, e^{\psi} \wedge \omega_S$$

Now, recall that the directions wrapped by the brane surface Σ are e^{ψ}, e^K, e^L where $\{e^K, e^L\}$ define a complex 2-cycle. Therefore, the pull-back of $e^{\psi} \wedge \omega_S$ to the brane is simply the spatial world-volume of the brane, i.e.

$$\int e^{\psi} \wedge \omega_S = \int \sqrt{\Sigma}\, d^3\sigma$$

Therefore, the right hand side of Eq. (7.42) is

$$\int -e^{||} \wedge \omega_S = \int \sqrt{(1-v^2)\Sigma}\, d^3\sigma \tag{7.50}$$

Clearly, the left and right hand sides of the calibration bound, Eqs. (7.49) and (7.50), are equal when $\dot{\phi} = v$, which is the speed specified by Mikhailov. For a giant graviton moving at the "wrong speed", i.e. $\dot{\phi} \neq v$, then

$$\frac{\sqrt{\Sigma}(1-v\dot{\phi})}{\sqrt{1-\dot{\phi}^2}} > \sqrt{(1-v^2)\Sigma}$$

i.e. the brane is not calibrated, but it does satisfy the correct inequality in Eq. (7.42). Therefore, in this section we have proved that holomorphic giant gravitons constructed using the Mikhailov construction are calibrated branes. Hence they have minimal energy minus momentum in their homology class. Moreover, a brane wrapping the same surface as a holomorphic giant graviton but traveling at the wrong speed is not calibrated.

7.4 Dual giant gravitons and calibrations

We now consider dual giant gravitons from the point of view of calibrations. Recall that dual giant gravitons are D3-branes which wrap a 3-dimensional surface in AdS_5 and have non-trivial motion on the S^5 part of the geometry. In this section we show that the dual giant graviton introduced in § 2.2 of Chapter 2 (originally in Ref. [60]) saturates the calibration bound Eq. (7.37). That is, we will show that

$$\int (K \cdot p - \iota_K C) = - \int 2\Phi^{12}$$

for this configuration. Now, recall from Chapter 2 that the dual giant graviton in § 2.2 preserves the same supersymmetries as the ordinary giant graviton introduced in § 2.1. Both these branes preserve one half of the background supersymmetry and the condition on the Killing spinors is $\Gamma^{0||}\epsilon^i = \epsilon^i$, $i = 1, 2$. Since the preserved supersymmetries are the same for both branes, we can make the same additional projections on the Killing spinors as in § 7.2. This means that the p-forms K, Φ and Σ will be exactly the same for the dual giant configuration as for holomorphic giant gravitons. These p-forms are given explicitly in Eqs. (7.20)–(7.22). Therefore, the calibration bound for the dual giant graviton is

$$\int (\mathcal{H} + p_{||} - \iota_0 C - \iota_{||} C) \geq \int -(e^0 + e^{||}) \wedge (\omega_S + \tilde{\omega}_{AdS}) \tag{7.51}$$

exactly as for giant gravitons. However, because dual giants wrap three AdS directions, the only term on the right hand side that contributes is $\int -e^0 \wedge \tilde{\omega}_{AdS}$. Therefore, the bound becomes

$$\int (\mathcal{H} + p_{||} - \iota_0 C - \iota_{||} C) \geq \int -e^0 \wedge \tilde{\omega}_{AdS} \tag{7.52}$$

We now show that the dual giant configuration of § 2.2 saturates this bound.

Recall that the $AdS_5 \times S^5$ metric is given by $ds^2 = ds^2_{AdS} + ds^2_S$ where

$$ds^2_{AdS} = -(1 + r^2)\, dt^2 + \frac{dr^2}{1 + r^2} + r^2(d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3^2)$$

and

$$ds^2_S = \sum_{i=1}^{3} (d\mu_i^2 + \mu_i^2 d\phi_i^2)$$

with the condition that $\sum_i \mu_i^2 = 1$. Note that both the radius of AdS_5 and S^5 is 1. In these coordinates the preferred time-like direction, e^0, defined in Eq. (7.13), becomes

$$e^0 = (1 + r^2)\, dt - r^2(\cos\alpha_2 d\alpha_1 - \cos\alpha_1 \sin\alpha_1 \sin\alpha_2 d\alpha_2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3) \tag{7.53}$$

The dual giant graviton we consider wraps a 3-sphere parameterised by $\alpha_1, \alpha_2, \alpha_3$ at fixed r. We denote the coordinates on the world-volume of this brane by σ^μ ($\mu = 0, 1, 2, 3$) and here $\sigma^0 = t$ (i.e. static gauge) and $\sigma^i = \alpha_i$. The dual giant graviton moves on the surface of S^5 along any equator. For concreteness, we take the motion on the sphere to be in the direction ϕ_1 with μ_i fixed to the values $\mu_1 = 1$, $\mu_2, \mu_3 = 0$.

We now calculate the quantities on the left hand side of the calibration bound Eq. (7.52).

To do this we must first calculate the Lagrangian for the dual giant graviton. As before, this is schematically given by

$$\mathcal{L} = -\sqrt{-\gamma} + \mathcal{P}(C)$$

where γ is the determinant of the induced metric on the brane world-volume. Computing this metric we obtain

$$ds^2 = (-1 - r^2 + \dot{\phi}_1^2)\, dt^2 + r^2(d\alpha_1^2 + \sin^2\alpha_1 d\alpha_2^2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3^2) \tag{7.54}$$

Therefore,

$$\sqrt{-\gamma} = \sqrt{1 + r^2 - \dot{\phi}_1^2}\, r^3 \sin^2\alpha_1 \sin\alpha_2$$

The pull-back of the 4-form potential is

$$\mathcal{P}(C) = (C_{t\alpha_1\alpha_2\alpha_3} + \dot{\phi}_1 C_{\phi_1\alpha_1\alpha_2\alpha_3}) dt \wedge d^3\alpha \tag{7.55}$$

Hence, we obtain the following Lagrangian for the dual giant,

$$\mathcal{L} = -\sqrt{1 + r^2 - \dot{\phi}_1^2}\, r^3 \sin^2\alpha_1 \sin\alpha_2 + C_{t\alpha_1\alpha_2\alpha_3} + \dot{\phi}_1 C_{\phi_1\alpha_1\alpha_2\alpha_3} \tag{7.56}$$

We can use this to calculate the momentum conjugate to ϕ_1. We obtain,

$$p_{\phi_1} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = \frac{r^3 \sin^2\alpha_1 \sin\alpha_2\ \dot{\phi}_1}{\sqrt{1 + r^2 - \dot{\phi}_1^2}} + C_{\phi_1\alpha_1\alpha_2\alpha_3}$$

Therefore, the Hamiltonian is

$$\mathcal{H} = p_{\phi_1}\dot{\phi}_1 - \mathcal{L} = \frac{r^3 \sin^2\alpha_1 \sin\alpha_2\ (1 + r^2)}{\sqrt{1 + r^2 - \dot{\phi}_1^2}} - C_{t\alpha_1\alpha_2\alpha_3} \tag{7.57}$$

Recall that $e^{||} = \sum_i \mu_i^2 d\phi_i$, which on the dual giant trajectory becomes $e^{||} = d\phi_1$. Therefore, $p_{||} = p_{\phi_1}$ and hence

$$\mathcal{H} + p_{||} = r^3 \sin^2\alpha_1 \sin\alpha_2 \frac{1 + r^2 + \dot{\phi}_1}{\sqrt{1 + r^2 - \dot{\phi}_1^2}} - C_{t\alpha_1\alpha_2\alpha_3} + C_{\phi_1\alpha_1\alpha_2\alpha_3} \tag{7.58}$$

We now need to calculate $\int \left(\iota_0 C + \iota_{||} C\right)$. From the form of e^0 given in Eq. (7.53) together with the fact that $e^{||} = d\phi_1$ on the trajectory, we obtain

$$\int \left(\iota_0 C + \iota_{||} C\right) = \int \left(-C_{t\alpha_1\alpha_2\alpha_3} + C_{\phi_1\alpha_1\alpha_2\alpha_3}\right) d^3\alpha$$

Hence,

$$\int (\mathcal{H} + p_{||} - \iota_0 C - \iota_{||} C) = \int r^3 \sin^2\alpha_1 \sin\alpha_2 \frac{1 + r^2 + \dot{\phi}_1}{\sqrt{1 + r^2 - \dot{\phi}_1^2}} \, d^3\alpha \tag{7.59}$$

which gives the left hand side of the bound.

The right hand side of the bound Eq. (7.52) is given by

$$\int -e^0 \wedge \tilde{\omega}_{AdS} \tag{7.60}$$

We can calculate $\tilde{\omega}_{AdS}$ easily since $de^0 = -2\tilde{\omega}_{AdS}$. Using the expression for e^0 given in Eq. (7.53) we obtain,

$$\begin{aligned} \tilde{\omega}_{AdS} &= -rdr \wedge dt + r^2 \sin\alpha_1 \cos\alpha_1 \sin^2\alpha_2 \, d\alpha_1 \wedge d\alpha_3 \\ &+ r^2 \sin^2\alpha_1 \sin\alpha_2 (d\alpha_1 \wedge d\alpha_2 + \cos\alpha_2 \, d\alpha_2 \wedge d\alpha_3) \\ &+ rdr \wedge (\cos\alpha_2 d\alpha_1 - \sin\alpha_1 \cos\alpha_1 \sin\alpha_2 d\alpha_2 + \sin^2\alpha_1 \sin^2\alpha_2 d\alpha_3) \end{aligned}$$

If we now calculate $e^0 \wedge \tilde{\omega}_{AdS}$ and use the fact that the spatial world-volume of the dual giant is parameterized by $\alpha_1, \alpha_2, \alpha_3$, the right hand side of the bound is given by,

$$\int -e^0 \wedge \tilde{\omega}_{AdS} = \int r^4 \sin^2\alpha_1 \sin\alpha_2 \, d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \tag{7.61}$$

Now if we compare Eqs. (7.59) and (7.61) we see that the left and right hand sides of the calibration bound, Eq. (7.52), are equal when $\dot{\phi}_1 = -1$. In the case where $\dot{\phi}_1 \neq -1$, we find

$$\frac{1 + r^2 + \dot{\phi}_1}{\sqrt{1 + r^2 - \dot{\phi}_1^2}} > r$$

which means that the brane is not calibrated, but the inequality in Eq. (7.52) is satisfied. In fact, for a brane wrapping $\alpha_1, \alpha_2, \alpha_3$ the calibration bound is saturated if and only if $\dot{\phi}_1 = -1$. Note that the speed $\dot{\phi}_1 = -1$ agrees with the speed one obtains from the probe calculation in § 2.2 (To see this one should equate the expression for p_{ϕ_1} in § 2.2 with Nr^2/L^2 which is the value of the momentum at the critical point.). Therefore, the dual giant graviton of § 2.2 saturates the calibration bound, and thus minimizes energy minus momentum in its homology class.

It is easy to show that like giant gravitons, the centre of mass of the dual giant moves along a null trajectory. This can be seen by evaluating the $AdS_5 \times S^5$ metric on the trajectory $\mu_1 = 1, \dot{\phi}_1 = -1$ with $r = 0$, which corresponds to the centre of mass of the brane. Moreover, like the giant graviton, the surface elements of the brane move at less than the speed of light. The time-like trajectory taken by a surface element is simply

$ds^2 = -r^2 dt^2$.

A future direction for this research is to establish whether there is a holomorphic description of dual giants. The presence of $\tilde{\omega}_{AdS}$ in the calibration bound indicates that such a description might well exist. However, so far our attempts at such a description have not led to any new configurations of dual giant gravitons.

To summarize, in this chapter we have constructed a calibration bound for giant gravitons. This bound was derived from the super-translation algebra, which we found in § 7.3. We showed that giant gravitons constructed from holomorphic surfaces saturate this bound. Moreover, branes wrapping the same surfaces but traveling at the wrong speed do not saturate the bound. The calibrated giant gravitons have minimal energy minus momentum in their homology class. Here the momentum is a conserved charge, corresponding to the R-charge in the dual field theory. We also saw that the dual giant of § 2.2 saturates the same calibration bound as the holomorphic giants. This brane also minimizes energy minus momentum in its homology class.

Chapter 8

Conclusion

In this thesis we have investigated aspects of supergravity theories in 10 and 11 dimensions. In particular, we have considered the problem of finding energy minimizing configurations of probe branes in various supergravity backgrounds. We began this discussion by introducing giant gravitons and dual giant gravitons in $AdS_5 \times S^5$. These are interesting branes to consider as they are spherical and non-static – in fact they must move to prevent collapse. In Chapter 3 we considered giant gravitons in more general backgrounds. In particular, we performed giant graviton probe calculations in two classes of 11-dimensional lifted geometries. We found that giant gravitons degenerate to massless particles exist in arbitrary lifted backgrounds. Moreover, these objects are both equivalent to massive charged particles probing the associated lower-dimensional gauged supergravity solution. We applied our results to probe superstar geometries. These geometries are conjectured to be sourced by giant gravitons. We tested this conjecture by performing giant graviton probe calculations to see if these branes had a BPS minimum at the position of the naked singularity. Our results supported the conjecture in most cases. However, there were some unusual features of our results which we were not able to fully understand. For example, the results for the quadruply charged superstar solutions did not agree with the expectations of the conjecture. This may indicate that quadruply charged superstars are not sourced by giant gravitons, that the singularities in these backgrounds are not physical, or that the reduced supersymmetry means that we should consider higher order curvature corrections to our probe calculations. It would be interesting to try to resolve this issue with further probe calculations.

In Chapter 4 we introduced the method of calibrations. This is a more geometrical way of finding energy minimizing brane configurations in supergravity backgrounds. Primarily this method is useful for backgrounds which preserve supersymmetry. We gave some specific examples of calibrations and showed how they could be used to find supersymmetric embeddings of branes in these backgrounds. In Chapter 5 we continued investigations of supersymmetric backgrounds by considering the superalgebras for these backgrounds. In particular, we found the form of the super-translation algebra for probe M2-/M5-branes in general 11-dimensional supersymmetric backgrounds. Previously, these algebras were known only for some specific classes of backgrounds. The technique we used was to construct p-forms of different degrees from the Killing spinors of the background. These forms obey a set of differential equations which can be manipulated to construct a closed 2-form and a closed 5-form. We argued that these closed forms are the topological charges which appear in the super-translation algebra for probe M2- and M5-branes in general supersymmetric backgrounds. The super-translation algebras we derived could then be used to find a BPS bound on the energy/momentum of a probe brane in a general supersymmetric background. These BPS bounds give us the relevant calibrating form(s) for a probe brane. Moreover, they tell us what quantity a calibrated brane will minimize.

In Chapters 6 and 7 we combined the ideas of non-static branes and calibrations to work on finding a generalized calibration for giant gravitons in $AdS_5 \times S^5$. We used the techniques of Chapter 5 to construct the super-translation algebra for a D3-brane in a type IIB supersymmetric background. We then used this algebra to find a calibration bound on the energy/momentum of the branes. As a by-product of this construction we derived a number of differential and algebraic identities for p-forms constructed from type IIB Killing spinors. These equations are valid in the most general supersymmetric backgrounds. This extends previous work [37–39] where equations of this type have been derived for specific classes of type IIB backgrounds.

To test the calibration bound on giant gravitons, we introduced a class of giant gravitons in $AdS_5 \times S^5$ which generalize the original example of Ref. [46]. In particular, we considered the Mikhailov construction [45] of giant gravitons via holomorphic surfaces in $\mathbb{C}^{1,2} \times \mathbb{C}^3$, which is an embedding space for $AdS_5 \times S^5$. Using this construction we showed that these general giant gravitons saturate the calibration bound. Moreover, these branes minimize energy minus momentum in their homology class. We also showed that the dual giant graviton configuration of Ref. [60] saturates the calibration bound and minimizes the same quantity as the ordinary giants.

While we have made some progress in understanding calibrations for one type of non-static brane, there is still much work to be done. For example, it would be interesting to try to understand other types of non-static branes using these techniques. Some work on this has already begun - for example in Ref. [109], where our method was followed to formulate a calibration bound for supertubes [110, 111] in type IIA supergravity. However, there are also other interesting non-static branes, such as supercurves [112, 113], and giant gravitons in other backgrounds. It would be interesting to find calibration bounds for these objects. This formalism might well allow us to find new configurations of these branes. Another future direction for this research is to try to understand the essential characteristics of backgrounds which admit non-static branes. This is a more difficult question to address, but the geometrical formalism of calibrations might well provide the required insight. Related to the work in this thesis is the classification problem for supersymmetric solutions of supergravity. Although there has been much important work on this subject, an outstanding issue is how to classify supersymmetric solutions of 10- and 11-dimensional supergravity which preserve more than minimal, but less than maximal, supersymmetry. The approach one could use is to construct more differential forms from the additional Killing spinors, and then try to classify the corresponding G-structure groups. While this probably would be very difficult for every fraction of preserved supersymmetry, it might be tractable for small amounts of preserved supersymmetry, i.e. for fractions such as $\frac{1}{16}$, $\frac{3}{32}$.

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Appendix A

Conventions

Throughout this thesis all Lorentzian metrics have the signature $(-,+,+,\ldots)$, while Riemannian metrics have signature $(+,\ldots,+)$. We will denote the components of the background metrics by g_{mn}, where the indices $m,n,p,q,\ldots$ are coordinate space indices. In 11 dimensions these indices run over $0,1,2,\ldots,8,9,\natural$, with the $\natural$ symbol used to avoid the confusion of "10" with 1, 0. In 10 dimensions the indices run over $0,1,\ldots,9$. We will use early greek letters $\alpha,\beta,\ldots$ for spinor indices only. The components of induced metrics on p-branes will generally be denoted by γ_{ab}, where a,b run over the $p+1$ coordinates on the brane world-volume.

We will consider the Dirac matrices (or "Γ-matrices") in 10 and 11 dimensions. In both cases these matrices are 32-dimensional and satisfy

$$\{\Gamma_m,\Gamma_n\} = 2g_{mn} \tag{A.1}$$

where $m,n = 0,\ldots,\natural$ in the 11-dimensional case, and $m,n = 0,\ldots 9$ in 10 dimensions. In both cases the Γ matrices can be taken to be real (i.e. one can choose the Majorana representation). We will use the following notation for the anti-symmetrized product of p Γ matrices:

$$\Gamma_{m_1\ldots m_p} = \Gamma_{[m_1}\ldots\Gamma_{m_p]} = \frac{1}{p!}\left(\Gamma_{m_1\ldots m_p} + \ldots\ldots\right)$$

where the sum contains all permutations of $m_1,\ldots,m_p$ weighted by an appropriate ±1. Note that here we have used square brackets to denote anti-symmetrizing over a set of indices. Similarly, we use round brackets to denote symmetrizing over the indices. We will also use the notation $|m|$ to indicate that the index m is not included in the (anti-)symmetrization. In the above equations we have used coordinate space Γ matrices, however, in some cases we will need to use the tangent space Γ matrices, $\Gamma_{\hat{m}}$. These are related to the coordinate space matrices by the vielbein as follows: $\Gamma_m = e_m^{\hat{m}}\Gamma_{\hat{m}}$, where the vielbein is defined by $g_{mn} = e_m^{\hat{m}}e_n^{\hat{n}}\eta_{\hat{m}\hat{n}}$ and $\eta_{\hat{m}\hat{n}}$ are the components of the flat metric.

The tangent space Γ matrices satisfy

$$\{\Gamma_{\hat{m}}, \Gamma_{\hat{n}}\} = 2\eta_{\hat{m}\hat{n}}$$

We will always use tangent space matrices in any explicit supersymmetry projection conditions so that we avoid factors of the vielbein appearing in these conditions. We now discuss some specific properties of the 10- and 11-dimensional Γ matrices in turn.

In 11 dimensions the Γ matrices satisfy the following relation: $\Gamma_{\hat{0}\hat{1}\hat{2}\hat{3}\ldots\hat{9}\hat{\natural}} = 1$. Therefore, we have duality relations between anti-symmetrized products of Γ matrices such as $\Gamma_{\hat{2}\hat{3}\ldots\hat{\natural}} = \Gamma_{\hat{0}\hat{1}}$ etc. In particular, each product of $p > 5$ Dirac matrices can be related to a product of 5 or less matrices. The Dirac matrices have a natural action on spinors. In 11 dimensions irreducible spinors have 32 real components (Majorana) and they form a representation of the group $Spin(1, 10)$. Given a spinor ϵ, its conjugate is defined by $\bar{\epsilon} = \epsilon^T C$, where C is the charge conjugation matrix in 11 dimensions. The matrix C must satisfy $C^T = -C$ and $C^2 = -1$. In the Majorana representation we can always choose $C = \Gamma^{\hat{0}}$. This matrix can be used to find the transpose of Dirac matrices as follows,

$$(\Gamma_m)^T = C\Gamma_m C, \qquad (\Gamma_{\hat{m}})^T = C\Gamma_{\hat{m}} C$$

In Chapter 5 we will need to know the symmetry properties of the following products of Dirac matrices: $C\Gamma_{\hat{m}_1\ldots\hat{m}_p}$. Using the above relations, it is easy to prove that for $p = 1, 2, 5$ the matrix $C\Gamma_{\hat{m}_1\ldots\hat{m}_p}$ is symmetric, while for $p = 0, 3, 4$ it is anti-symmetric. For example, for $p = 1$ we have

$$(C\Gamma_{\hat{m}})^T = -(\Gamma_{\hat{m}})^T C = -C\Gamma_{\hat{m}}C^2 = C\Gamma_{\hat{m}}$$

as expected. Note that for $p > 5$ the products $C\Gamma_{\hat{m}_1\ldots\hat{m}_p}$ are simply dual to the lower dimensional cases using the duality relation described above.

In 10 dimensions the Dirac matrices are 32-dimensional. They are also real and satisfy the algebra in Eq. (A.1). Type IIB supergravity is a chiral theory with $\mathcal{N} = 2$ supersymmetry. Therefore, the supersymmetry transformations involve two spinors, ϵ^i, $i = 1, 2$, which have the same chirality. The spinors ϵ^i are 32-dimensional and both obey $\Gamma_{11}\epsilon^i = \epsilon^i$ where $\Gamma_{11} = \Gamma^{\hat{0}\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}\hat{7}\hat{8}\hat{9}}$. Due to the chirality condition on the spinors, each ϵ^i has only 16 non-zero components. We define the conjugate spinors in type IIB by $\bar{\epsilon}^i = (\epsilon^i)^T C$, and we take $C = \Gamma^{\hat{0}}$ as in the 11-dimensional case. The Killing spinor equations in type IIB supergravity involve the Pauli matrices. These are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We now give our conventions for differential forms. A p-form ω is defined in terms of its components as follows,

$$\omega = \frac{1}{p!}\omega_{m_1\ldots m_p} dx^{m_1} \wedge \cdots \wedge dx^{m_p} \tag{A.2}$$

The set of all p-forms on a manifold $\mathcal{M}$ is denoted $\Lambda^p(\mathcal{M})$. For a d-dimensional manifold we will generally express p-forms in terms of one of the following bases of 1-forms: either we will use the coordinate basis $\{dx^0, \ldots, dx^{d-1}\}$, or the orthonormal basis $\{e^0, \ldots, e^{d-1}\}$.

The wedge product of a p-form, ω, with a q-form, v, is defined (in components) by

$$(\omega \wedge v)_{m_1\ldots m_{p+q}} = \frac{(p+q)!}{p!q!}\, \omega_{[m_1\ldots m_p} v_{m_{p+1}\ldots m_{p+q}]} \tag{A.3}$$

The Hodge dual of a p-form, ω, dualized within a d-dimensional space, is a $(d-p)$-form, $*\omega$, with components,

$$(*\omega)_{m_1\ldots m_{d-p}} = \frac{1}{p!}\sqrt{|g|}\; \epsilon_{m_1\ldots m_{d-p} n_1\ldots n_p}\; \omega^{n_1\ldots n_p} \tag{A.4}$$

where $|g|$ is the modulus of the determinant of the metric on the d-dimensional space. In a Lorentzian space-time we use the convention that $\epsilon_{012\ldots} = +1$ and $\epsilon^{012\ldots} = -1$ (so ϵ is not a tensor; it is just a symbol). In a Riemannian space we will use $\epsilon_{123\ldots} = +1$ and $\epsilon^{123\ldots} = +1$. In a Lorentzian space-time one finds

$$* * w = (-1)^{p(d-p)+1} w$$

For Riemannian spaces this relation differs by an overall factor of -1.

In Chapter 4 onwards, we will often come across the interior product of forms. The definition of the interior product of a q-form, v, with a p-form, ω, where $q > p$, is

$$(\iota_\omega v)_{n_1\ldots n_{q-p}} = \frac{1}{p!}\omega^{m_1\ldots m_p}\; v_{m_1\ldots m_p n_1\ldots n_{q-p}} \tag{A.5}$$

A useful result which helps to simplify several expressions in Chapter 6 and Appendix C is the following: given a q-form, v, and p-form, w, where $q > p$,

$$\iota_{*v} * w = (-1)^{p(q-p)+1} \iota_w v \tag{A.6}$$

Again this relation differs by an overall factor of -1 if we dualize within a Riemannian space. We define the dot product of p-forms w and v by

$$w \cdot v = \frac{1}{p!} w_{m_1 \ldots m_p} v^{m_1 \ldots m_p} \tag{A.7}$$

and the square of w is $w^2 = w \cdot w$. Moreover, the Lie derivative of a p-form ω along the direction specified by the vector X is defined by

$$\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega \tag{A.8}$$

Appendix B

Dualizing and Integrating forms in lifted backgrounds

B.1 Lifted backgrounds of the 4-d theory

In this section we will consider 11-dimensional backgrounds which are obtained by lifting solutions of 4-dimensional $U(1)^4$ gauged supergravity (as explained in § 3.1.1). We will calculate the 6-form potential, $A^{(6)}$, for a general lifted solution. This potential couples to M5-brane giant gravitons and prevents them from collapsing. The calculation for $A^{(6)}$ requires two steps. Firstly, in § B.1.1 we will dualize the 4-form field strength, $F^{(4)}$, to obtain the dual 7-form field strength, $F^{(7)}$. Then in § B.1.2 we will integrate $F^{(7)}$ locally to obtain the 6-form potential, $A^{(6)}$, which couples to the brane.

B.1.1 Dualizing $F^{(4)}$

Recall from Eq. (3.6) that the lift ansatz for the 11-dimensional metric is

$$ds_{11}^2 = \Delta^{2/3} ds_{(1,3)}^2 + \Delta^{-1/3} ds_7^2$$

where $ds_{(1,3)}^2$ is the metric for the 4-dimensional gauged supergravity solution and ds_7^2 is the metric on the internal 7-dimensional space. Moreover, from Eq. (3.7) the 4-form field strength for this background is given by

$$F^{(4)} = \frac{2U}{L}\,\epsilon_{(1,3)} + \frac{L}{2}\sum_i X_i^{-1} *_{(1,3)} dX_i \wedge d(\mu_i^2) + \frac{L}{2}\sum_i X_i^{-2} d(\mu_i^2) \wedge (L d\phi_i + A_{(1)}^i) \wedge *_{(1,3)} F_{(2)}^i \tag{B.1}$$

We now derive some results which will allow us to find the Hodge dual of this form. To begin, we consider dualizing a $(p+q)$-form of type $\alpha^{(p)} \wedge \beta^{(q)}$ in this background. Here $\alpha^{(p)}$

is a p-form in the 4-dimensional space and $\beta^{(q)}$ is a q-form in the internal 7-dimensional space. We obtain the following result,

$$*_{(11)}(\alpha^{(p)} \wedge \beta^{(q)}) = (-1)^{p(7-q)}\Delta^{(1-4p+2q)/6}(*_{(1,3)}\alpha^{(p)} \wedge *_{(7)}\beta^{(q)}) \tag{B.2}$$

where $*_{(1,3)}$ and $*_{(7)}$ refer to dualizing within the 4- and 7-dimensional spaces which have metrics $ds^2_{(1,3)}$ and ds^2_7 respectively. Our conventions for dualizing forms are given in Appendix A. In fact, from Eq. (3.6) we see that the metric on the internal 7-dimensional space splits further into two parts;

$$ds_7^2 = L^2 \sum_i X_i^{-1} d\mu_i^2 + \sum_i X_i^{-1}\mu_i^2(Ld\phi_i + A^i)^2 \tag{B.3}$$

where the μ_i define a 3-sphere, S. Therefore, a result similar to Eq. (B.2) holds for dualizing forms in seven dimensions, namely

$$*_{(7)}(\alpha^{(r)} \wedge \beta^{(s)}) = (-)^{r(4-s)}L^{3-2r}(*_{(3)}\alpha^{(r)} \wedge *_{(4)}\beta^{(s)}) \tag{B.4}$$

where $\alpha^{(r)}$ is an r-form in the μ_i directions and $\beta^{(s)}$ is an s-form in the ϕ_i part of the 7-dimensional space. Here $*_{(4)}$ refers to the metric $ds_4^2 = \sum_i X_i^{-1}\mu_i^2(Ld\phi_i + A^i)^2$ and $*_{(3)}$ refers to the metric

$$d\tilde{s}_4^2 = \sum_{i=1}^{4} X_i^{-1} d\mu_i^2 \tag{B.5}$$

restricted to the 3-sphere S: $\sum_{i=1}^4 \mu_i^2 = 1$. Due to this constraint on μ_i, dualizing forms on S is not completely straightforward. Therefore, we will need the following result,

$$*_{(3)}\alpha^{(r)} = (-1)^{r+1}\ \tilde{*}_{(4)}(e_4 \wedge \alpha^{(r)}) \tag{B.6}$$

where $\alpha^{(r)}$ is an arbitrary r-form on S. Here $\tilde{*}_{(4)}$ refers to the metric $d\tilde{s}_4^2$ on $\mathbb{R}^4$ and $e_4 = \Delta^{-1/2}\sum_i \mu_i d\mu_i$ is a unit 1-form in $\Lambda^1(\mathbb{R}^4)$ which is normal to S. Essentially, we are using the embedding of S in $\mathbb{R}^4$ to dualize the forms on S. With the results in Eqs. (B.2)-(B.6) we are now almost ready to dualize $F^{(4)}$ in eleven dimensions, but first we will derive a few intermediate results to simplify the calculation.

We define the following 2-forms on S,

$$Z_i \equiv \sum_{j,k,l} \epsilon_{ijkl}\ \mu_j d\mu_k \wedge d\mu_l \tag{B.7}$$

$$Z_{ij} \equiv \sum_{k,l} \epsilon_{ijkl}\ d\mu_k \wedge d\mu_l \tag{B.8}$$

where $i, j, k, l = 1, \ldots 4$. The volume form on S is given by

$$W = \frac{1}{6} \sum_{i,j,k,l} \epsilon_{ijkl}\ \mu_i d\mu_j \wedge d\mu_k \wedge d\mu_l$$

Due to the constraint on μ_i it can easily be shown that the 2-forms Z_i and Z_{ij} satisfy the following three identities:

$$\begin{aligned} dZ_i &= 6\mu_i W && \text{(B.9)} \\ Z_i \wedge d\mu_j &= -2(\delta_{ij} - \mu_i\mu_j)W && \text{(B.10)} \\ \sum_j X_j\mu_j Z_j \mu_i &= \sum_j X_j \mu_j Z_{ji} + \Delta Z_i && \text{(B.11)} \end{aligned}$$

We can use these identities to obtain two further results which will be used to dualize $F^{(4)}$. Firstly, using the relation in Eq. (B.6) for dualizing forms on S, we have

$$\begin{aligned} *_{(3)}\ 1 = -\ *_{(4)} \left(\Delta^{-1/2} \sum_i \mu_i d\mu_i\right) &= -\frac{\Delta^{-1/2}}{6} \sum_{ij} X_j \mu_j d\mu_i \wedge Z_{ij} \\ &= -\frac{\Delta^{1/2}}{6} \sum_i Z_i \wedge d\mu_i + \frac{\Delta^{-1/2}}{6} \sum_{ij} \mu_i\mu_j X_j Z_j \wedge d\mu_i \\ &= \frac{\Delta^{1/2}}{3} \sum_i (1-\mu_i^2) W - \frac{\Delta^{-1/2}}{3} \sum_{ij} \mu_i\mu_j X_j (\delta_{ij} - \mu_i\mu_j) W \\ \Rightarrow \qquad *_{(3)}\ 1 &= \Delta^{1/2} W && \text{(B.12)} \end{aligned}$$

where we have used the identities Eqs. (B.11) and (B.10) in the second and third steps respectively. Secondly, we evaluate the 2-form $*_{(3)} d(\mu_i^2)$:

$$*_{(3)} d(\mu_i^2) = *_{(4)} \left(\Delta^{-1/2} \sum_j \mu_j d\mu_j \wedge 2\mu_i d\mu_i\right) = -\Delta^{-1/2} \sum_j X_i X_j \mu_i \mu_j Z_{ij} \qquad \text{(B.13)}$$

where again we have used Eq. (B.6) in the first step.

We can now dualize the first term in $F^{(4)}$, given in Eq. (B.1), using the results from Eqs. (B.2), (B.4) and (B.12). We obtain,

$$\begin{aligned} *_{(11)} \frac{2U}{L} \epsilon_{(1,3)} &= -\frac{2U}{L}\ \Delta^{-5/2}\ *_{(7)}\ 1 \\ &= -2UL^2\ \Delta^{-5/2}\ *_{(3)}\ 1 \wedge *_{(4)} 1 \\ &= -\frac{2UL^2}{\Delta^2}\ W \bigwedge_k \mu_k (L d\phi_k + A^k) && \text{(B.14)} \end{aligned}$$

Similarly, we can use the results in Eqs. (B.2)-(B.13) to dualize the two other terms in $F^{(4)}$. For the second term in $F^{(4)}$ one finds,

$$*_{(11)}\left(\frac{L}{2}\sum_i X_i^{-1} *_{(1,3)} dX_i \wedge d(\mu_i^2)\right) = -\frac{L^2}{2\Delta^2}\sum_{ij} X_j dX_i \wedge \mu_i\mu_j Z_{ij} \bigwedge_k \mu_k(Ld\phi_k + A^k) \tag{B.15}$$

and for the third term we obtain

$$*_{(11)}\left(\frac{L}{2}\sum_i X_i^{-2} d(\mu_i^2) \wedge (Ld\phi_i + A^i) \wedge *_{(1,3)}F^i\right) = \frac{L^2}{2\Delta}\sum_{ij} F^i \wedge Z_{ij}X_j\mu_j \bigwedge_{k\neq i} \mu_k(Ld\phi_k + A^k) \tag{B.16}$$

Therefore, the dual 7-form field strength is

$$\begin{aligned} F^{(7)} = *_{(11)}F^{(4)} &= -\frac{2L^2U}{\Delta^2} W \bigwedge_k \mu_k(Ld\phi_k + A^k) - \frac{L^2}{2\Delta^2}\sum_{ij} X_j dX_i \wedge \mu_i\mu_j Z_{ij} \bigwedge_k \mu_k(Ld\phi_k + A^k) \\ &\quad + \frac{L^2}{2\Delta}\sum_{ij} F^i \wedge Z_{ij}X_j\mu_j \bigwedge_{k\neq i} \mu_k(Ld\phi_k + A^k) \end{aligned} \tag{B.17}$$

We have checked that this 7-form satisfies $dF^{(7)} = 0$, which is the Bianchi identity for this solution. This calculation is straight-forward, but it is quite messy, so we do not present the details here. However, one must take $F^i \wedge F^j = 0$ for $dF^{(7)} = 0$ to hold. This corresponds to neglecting the axions, which was discussed in § 3.1.1.

B.1.2 Integrating $F^{(7)}$

We now wish to integrate $F^{(7)}$, obtained in Eq. (B.17), to determine the 6-form potential, $A^{(6)}$, which will couple to the M5-brane giant graviton. Since $dF^{(7)}$ vanishes identically, such an $A^{(6)}$ must exist, at least locally. In fact we will find that it is not possible to determine $A^{(6)}$ globally, but it can be found locally.

The first step is to use the identity in Eq. (B.11) to rewrite the following 3-form which appears in the second term of $F^{(7)}$,

$$\begin{aligned} \Delta^{-2}\sum_{ij} X_j dX_i \wedge \mu_i\mu_j Z_{ij} &= \Delta^{-1}\sum_i \mu_i dX_i \wedge Z_i - \Delta^{-2}\sum_{ij} X_j dX_i \wedge \mu_i^2\mu_j Z_j \\ &= \sum_i \partial_\mu\left(\frac{X_i\mu_i}{\Delta}\right) dx^\mu \wedge Z_i \end{aligned}$$

Using this result, the second term in $F^{(7)}$, given in Eq. (B.17), can be rewritten as follows,

$$-\frac{L^2}{2\Delta^2}\sum_{ij} X_j dX_i \wedge \mu_i \mu_j Z_{ij} \bigwedge_l \mu_l (Ld\phi_l + A^l) = -\frac{L^2}{2}\sum_i \partial_\mu \left(\frac{X_i \mu_i}{\Delta}\right) dx^\mu \wedge Z_i \bigwedge_l \mu_l (Ld\phi_l + A^l) \tag{B.18}$$

Thus we postulate that the 6-form potential, $A^{(6)}$, contains the following term,

$$\tilde{A}^{(6)} = -\frac{L^2}{2}\sum_i \frac{X_i \mu_i}{\Delta} Z_i \bigwedge_l \mu_l (Ld\phi_l + A^l) \tag{B.19}$$

Evaluating $d\tilde{A}^{(6)}$ gives

$$\begin{aligned} d\tilde{A}^{(6)} &= -\frac{2L^2 U}{\Delta^2} W \bigwedge_l \mu_l (Ld\phi_l + A^l) - \frac{L^2}{2}\sum_i \partial_\mu \left(\frac{X_i \mu_i}{\Delta}\right) dx^\mu \wedge Z_i \bigwedge_l \mu_l (Ld\phi_l + A^l) \\ &\quad -\frac{L^2}{2}\sum_j Z_j \wedge F^j \bigwedge_{l\neq j} \mu_l (Ld\phi_l + A^l) + \frac{L^2}{2\Delta}\sum_{ij} F^i \wedge Z_{ij} X_j \mu_j \bigwedge_{l\neq i} \mu_l (Ld\phi_l + A^l) \\ &\quad -6L^2 W \bigwedge_l \mu_l (Ld\phi_l + A^l) \end{aligned} \tag{B.20}$$

where we have used the identities in Eqs. (B.9)-(B.10) and recall from § 3.1.1 that $U \equiv \sum_i (X_i^2 \mu_i^2 - \Delta X_i)$. Therefore, comparing with $F^{(7)}$ in Eq. (B.17) we have,

$$F^{(7)} = d\tilde{A}^{(6)} + \frac{L^2}{2}\sum_j Z_j \wedge F^j \bigwedge_{l\neq j} \mu_l (Ld\phi_l + A^l) + 6L^2 W \bigwedge_l \mu_l (Ld\phi_l + A^l) \tag{B.21}$$

The sum of the last two terms in this expression is closed but not exact. For $\mu_1 \neq 0$ we can integrate them to see that they are equal to

$$d\left(\frac{L^2}{2}\frac{Z_1}{\mu_1}\bigwedge_j \mu_j (Ld\phi_j + A^j) + \frac{L^2}{2}\,\epsilon_{1jkl}\,\mu_k^2 \mu_l d\mu_l \wedge F^j \bigwedge_{m\neq j} (Ld\phi_m + A^m)\right) \tag{B.22}$$

Therefore, in the region where $\mu_1 \neq 0$, $A^{(6)}$ is given by

$$A^{(6)} = -\frac{L^2}{2\mu_1 \Delta}\sum_i X_i \mu_i Z_{i1} \bigwedge_j \mu_j (Ld\phi_j + A^j) + \frac{L^2}{2}\,\epsilon_{1jkl}\,\mu_k^2 \mu_l d\mu_l \wedge F^j \bigwedge_{m\neq j} (Ld\phi_m + A^m) \tag{B.23}$$

where we have used the identity in Eq. (B.11) to replace the two terms involving Z_i with one term involving Z_{i1}.

B.2 Lifted backgrounds of the 7-d theory

In this section we will consider 11-dimensional supergravity solutions which are obtained by lifting solutions of 7-dimensional $U(1)^2$ gauged supergravity, as shown in § 3.2. We will derive the 3-form potential, $A^{(3)}$, for these backgrounds. This potential couples to M2-brane giant gravitons and prevents them from collapsing. As in the previous section, this calculation involves two steps. Firstly, we dualize the 7-form field strength, $F^{(7)}$, given in Eq. (3.35), to obtain the dual 4-form field strength, $F^{(4)} = - *_{(11)} F^{(7)}$. Then we integrate $F^{(4)}$ locally to obtain $A^{(3)}$. The steps involved in this calculation will be broadly similar to those in the previous section.

B.2.1 Dualizing $F^{(7)}$

Recall from Eq. (3.34) that the metric for the 11-dimensional lifted solution takes the form,

$$ds_{11}^2 = \tilde{\Delta}^{1/3} ds_{(1,6)}^2 + \tilde{\Delta}^{-2/3} ds_4^2$$

Moreover, from Eq. (3.35) the 7-form field strength is given by

$$\begin{aligned} F^{(7)} \;=\; & -\frac{2U}{L}\,\epsilon_{(1,6)} - \frac{1}{L}\,\tilde{\Delta} X_0\,\epsilon_{(1,6)} - \frac{L}{2}\sum_{a=0}^{2} X_a^{-1} *_{(1,6)} dX_a \wedge d(\mu_a^2) \\ & - \frac{L}{2}\sum_{i=1}^{2} X_i^{-2} d(\mu_i^2) \wedge (L d\phi_i + A^i) \wedge *_{(1,6)} F^i \end{aligned} \tag{B.24}$$

In analogy with the previous case, we first need a result for dualizing $(p+q)$-forms which split into a product of a p-form, $\alpha^{(p)}$, which lies in the $(6+1)$-dimensional space and a q-form, $\beta^{(q)}$, which lies in the internal 4-dimensional space. We find

$$*_{(11)}(\alpha^{(p)} \wedge \beta^{(q)}) = (-1)^{p(4-q)} \tilde{\Delta}^{(-2p+4q-1)/6} (*_{(1,6)}\alpha^{(p)} \wedge *_{(4)}\beta^{(q)}) \tag{B.25}$$

The metric on S^4 also splits into two parts:

$$ds_4^2 = L^2 \sum_{a=0}^{2} X_a^{-1} d\mu_a^2 + \sum_{i=1}^{2} X_i^{-1} \mu_i^2 (L d\phi_i + A^i)^2$$

where as in § 3.2 we take the indices $a, b, \cdots = 0, 1, 2$ and $i, j, \cdots = 1, 2$. Since the 4-dimensional metric splits in this way, we have a result similar to Eq. (B.25) for dualizing forms within the S^4, namely

$$*_{(4)}(\alpha^{(r)} \wedge \beta^{(s)}) = (-1)^{r(2-s)} L^{2-2r} (\tilde{*}_{(2)}\alpha^{(r)} \wedge *_{(2)}\beta^{(s)}) \tag{B.26}$$

where $\alpha^{(r)}$ is an r-form in the μ_a directions and $\beta^{(s)}$ is an s-form in the ϕ_i directions. Here $*_{(2)}$ means dualizing with respect to $ds_2^2 = \sum_i X_i^{-1}\mu_i^2(Ld\phi_i + A^i)^2$, whereas $\tilde{*}_{(2)}$ refers to the metric

$$ds_3^2 = \sum_{a=0}^{2} X_a^{-1} d\mu_a^2 \tag{B.27}$$

restricted to the 2-sphere $\tilde{S} : \sum_a \mu_a^2 = 1$. Dualizing forms on $\tilde{S}$ requires the following result [54],

$$\tilde{*}_{(2)}\alpha^{(r)} = (-1)^r *_{(3)} (e_3 \wedge \alpha^{(r)}) \tag{B.28}$$

where $*_{(3)}$ refers to the metric ds_3^2 on $\mathbb{R}^3$ (i.e. not restricted to $\tilde{S}$) and $e_3 = \tilde{\Delta}^{-1/2}\sum_a \mu_a d\mu_a$ is a unit 1-form on $\Lambda^1(\mathbb{R}^3)$ which is orthogonal to $\tilde{S}$. We now have all the necessary tools to dualize $F^{(7)}$ in eleven dimensions, but we will first derive a few intermediate results.

It is useful to define the following 1-forms [54],

$$\tilde{Z}_{ab} = \sum_c \epsilon_{abc} d\mu_c \tag{B.29}$$

$$\tilde{Z}_a = \sum_{b,c} \epsilon_{abc}\mu_b d\mu_c \tag{B.30}$$

Due to the constraint $\sum_a \mu_a^2 = 1$, there are three identities connecting these 1-forms [54]:

$$d\tilde{Z}_a = 2\mu_a \tilde{W} \tag{B.31}$$

$$\tilde{Z}_a \wedge d\mu_b = (\delta_{ab} - \mu_a\mu_b)\tilde{W} \tag{B.32}$$

$$\sum_a X_a\mu_a\tilde{Z}_{ab} = \sum_a X_a\mu_a\tilde{Z}_a\mu_b - \tilde{\Delta}\tilde{Z}_b \tag{B.33}$$

where $\tilde{W} = \frac{1}{2}\epsilon_{abc}\mu_a d\mu_b \wedge d\mu_c$ is the volume form on $\tilde{S}$. Using these identities together with Eq. (B.28) we can obtain the following intermediate results,

$$\tilde{*}_{(2)} d\mu_a = -\tilde{*}_{(3)}(\tilde{\Delta}^{-1/2}\sum_b \mu_b d\mu_b \wedge d\mu_a) = X_0^{-1/4}\tilde{\Delta}^{-1/2}\sum_b X_a\tilde{Z}_{ab}\mu_b X_b \tag{B.34}$$

$$\tilde{*}_{(2)} 1 = \tilde{*}_{(3)}(\tilde{\Delta}^{-1/2}\sum_a \mu_a d\mu_a) = X_0^{-1/4}\tilde{\Delta}^{1/2}\tilde{W} \tag{B.35}$$

The factors of X_0 arise from the relation $X_0 = (X_1X_2)^{-2}$, which means that the determinant of the metric ds_3^2 is $X_0^{-1/2}$. However, these factors will cancel out when we dualize terms in $F^{(7)}$. For example, using Eqs. (B.25), (B.26) and (B.35), we can dualize the first term in $F^{(7)}$, given in Eq. (B.24), as follows,

$$\begin{aligned} *_{(11)}\left(-\frac{2U}{L}\epsilon_{(1,6)}\right) &= \frac{2U}{L}\tilde{\Delta}^{-5/2} *_{(4)} 1 \\ &= 2UL\tilde{\Delta}^{-5/2}\tilde{*}_{(2)}1 \wedge *_{(2)}1 \\ &= 2UL\tilde{\Delta}^{-2}X_0^{-1/4}\tilde{W}\bigwedge_i X_i^{-1/2}\mu_i(Ld\phi_i + A^i) \\ &= 2UL\tilde{\Delta}^{-2}\tilde{W}\bigwedge_i \mu_i(Ld\phi_i + A^i) \end{aligned} \tag{B.36}$$

where we have used the constraint $X_0 = (X_1X_2)^{-2}$ in the last line. Similarly we can dualize the other terms in $F^{(7)}$ using the results Eqs. (B.25)-(B.35). We find

$$\begin{aligned} F^{(4)} = - *_{(11)} F^{(7)} &= -\frac{2LU}{\tilde{\Delta}^2}\tilde{W}\bigwedge_i \mu_i(Ld\phi_i + A^i) - \frac{LX_0}{\tilde{\Delta}}\tilde{W}\bigwedge_i \mu_i(Ld\phi_i + A^i) \\ &- \frac{L}{\tilde{\Delta}^2}\sum_{a,b}\mu_a dX_a \wedge \tilde{Z}_{ab}\mu_b X_b \bigwedge_i \mu_i(Ld\phi_i + A^i) \\ &- \frac{L}{\tilde{\Delta}}\sum_{i,b} F^i \wedge \tilde{Z}_{ib}\mu_b X_b \bigwedge_{j\neq i} \mu_j(Ld\phi_j + A^j) \end{aligned} \tag{B.37}$$

It is straightforward to show that $dF^{(4)} = 0$, using the identities given in Eqs. (B.31)-(B.33) (this works provided $F^1 \wedge F^2 = 0$, which we assume. See discussion in § 3.2). This means that $F^{(4)}$ can be integrated at least locally.

B.2.2 Integrating $F^{(4)}$

The procedure for integrating $F^{(4)}$ is very similar to the method in § B.1.2 for integrating $F^{(7)}$. Essentially, we rewrite some of the terms in $F^{(4)}$ using the identities in Eqs. (B.31)-(B.33). Then we are able to guess some of the terms which appear in $A^{(3)}$, and we integrate the remainder. Due to the similarity with § B.1.2 we will not show the calculation for $A^{(3)}$ but just give the final results. As before, it is not possible to write $F^{(4)} = dA^{(3)}$ with $A^{(3)}$ well-defined over the whole space-time. However, $A^{(3)}$ can be found locally everywhere. For example, in the region where $\mu_1 \neq 0$, $A^{(3)}$ is given by

$$A^{(3)} = -\frac{L}{\mu_1\tilde{\Delta}}\sum_\alpha \mu_\alpha X_\alpha \tilde{Z}_{\alpha 1}\bigwedge_i \mu_i(Ld\phi_i + A^i) - L\mu_0 F^2 \wedge (Ld\phi_1 + A^1) \tag{B.38}$$

Appendix C

The derivative of K^{12}

In this appendix we give the detailed calculation for one of the results presented in Chapter 6, namely the ordinary derivative of K^{12} given in Eq. (6.8). Recall that K^{12} is constructed from the 16-dimensional constituent spinors, ϵ^1 and ϵ^2, of a single Killing spinor, ϵ, as follows:

$$K^{12}_n = \bar{\epsilon}^1 \Gamma_n \epsilon^2$$

where $n = 0, 1, \ldots, 9$. To calculate dK^{12} we will first calculate the covariant derivative of K^{12}. Then we will use the Killing spinor equation to replace derivatives of ϵ^1 and ϵ^2 with terms involving the metric, dilaton and background field strengths. Finally, we will anti-symmetrize over the free indices.

The covariant derivative of K^{12} is given by,

$$\nabla_m K^{12}_n = \overline{(\nabla_m \epsilon^1)} \Gamma_n \epsilon^2 + \bar{\epsilon}^1 \Gamma_n \nabla_m \epsilon^2 \tag{C.1}$$

where $\overline{(\nabla_m \epsilon^1)} = (\nabla_m \epsilon^1)^T \Gamma^{\hat{0}}$. Now we use the Killing spinor equation to replace the covariant derivatives of ϵ^1 and ϵ^2. Recall that the gravitino Killing spinor equation for type IIB supergravity is

$$\nabla_m \epsilon + \frac{1}{8} H^{(3)}_{m r_1 r_2} \Gamma^{r_1 r_2} \otimes \sigma_3 \epsilon + \frac{e^{\phi}}{16} \sum_{a=1}^{5} \frac{(-1)^{a-1}}{(2a-1)!} G^{(2a-1)}_{r_1 \ldots r_{2a-1}} \Gamma^{r_1 \ldots r_{2a-1}} \Gamma_m \otimes \lambda_a \epsilon = 0 \tag{C.2}$$

where $\epsilon = (\epsilon^1, \epsilon^2)^T$ and the matrices λ_a are defined by

$$\lambda_a = \begin{cases} \sigma_1 & \text{if } a \text{ even}, \\ i\sigma_2 & \text{if } a \text{ odd}. \end{cases}$$

and σ_1, σ_2, σ_3 are the usual Pauli matrices. From Eq. (C.2) we can read off the following expression for $\nabla_m \epsilon^1$,

$$\begin{aligned}\nabla_m \epsilon^1 &= -\frac{1}{8} H_{mr_1r_2}\Gamma^{r_1r_2}\epsilon^1 - \frac{e^\phi}{16}\Big(G^{(1)}_r \Gamma^r \Gamma_m \epsilon^2 - \frac{1}{3!} G^{(3)}_{r_1r_2r_3}\Gamma^{r_1r_2r_3}\Gamma_m\epsilon^2 + \frac{1}{5!}G^{(5)}_{r_1\ldots r_5}\Gamma^{r_1\ldots r_5}\Gamma_m\epsilon^2 \\ &\quad -\frac{1}{7!}G^{(7)}_{r_1\ldots r_7}\Gamma^{r_1\ldots r_7}\Gamma_m\epsilon^2 + \frac{1}{9!}G^{(9)}_{r_1\ldots r_9}\Gamma^{r_1\ldots r_9}\Gamma_m\epsilon^2\Big)\end{aligned}$$

If we calculate the transpose of this equation and multiply by $\Gamma^{\hat{0}}$, we obtain

$$\begin{aligned}\overline{(\nabla_m \epsilon^1)} &= \frac{1}{8} H_{mr_1r_2}\bar{\epsilon}^1\Gamma^{r_1r_2} - \frac{e^\phi}{16}\Big(G^{(1)}_r \bar{\epsilon}^2 \Gamma_m \Gamma^r + \frac{1}{3!} G^{(3)}_{r_1r_2r_3}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1r_2r_3} + \frac{1}{5!}G^{(5)}_{r_1\ldots r_5}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1\ldots r_5} \\ &\quad +\frac{1}{7!}G^{(7)}_{r_1\ldots r_7}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1\ldots r_7} + \frac{1}{9!}G^{(9)}_{r_1\ldots r_9}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1\ldots r_9}\Big)\end{aligned}$$

where we have used the property that $(\Gamma^m)^T = \Gamma^{\hat{0}}\Gamma^m\Gamma^{\hat{0}}$. Therefore, the first term in the expression for $\nabla_m K^{12}_n$, given in Eq. (C.1), is

$$\begin{aligned}\overline{(\nabla_m \epsilon^1)}\Gamma_n\epsilon^2 = \frac{1}{8} H_{mr_1r_2}\bar{\epsilon}^1\Gamma^{r_1r_2}\Gamma_n\epsilon^2 &- \frac{e^\phi}{16}\Big(G^{(1)}_r \bar{\epsilon}^2 \Gamma_m \Gamma^r\Gamma_n\epsilon^2 + \frac{1}{3!} G^{(3)}_{r_1r_2r_3}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1r_2r_3}\Gamma_n\epsilon^2 \\ &+\frac{1}{5!}G^{(5)}_{r_1\ldots r_5}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1\ldots r_5}\Gamma_n\epsilon^2 + \frac{1}{7!}G^{(7)}_{r_1\ldots r_7}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1\ldots r_7}\Gamma_n\epsilon^2 \\ &+\frac{1}{9!}G^{(9)}_{r_1\ldots r_9}\bar{\epsilon}^2\Gamma_m\Gamma^{r_1\ldots r_9}\Gamma_n\epsilon^2\Big)\end{aligned} \quad \text{(C.3)}$$

We now compare each term in this expression to the possible non-zero forms which can be constructed from the spinors. For example, the first term in Eq. (C.3) involves $\bar{\epsilon}^1\Gamma^{r_1r_2}\Gamma_n\epsilon^2$. However, this must be equal to the following combination of the 3-form Φ^{12} and the 1-form K^{12}:

$$\bar{\epsilon}^1\Gamma^{r_1r_2}\Gamma_n\epsilon^2 = (\Phi^{12})^{r_1r_2}{}_n + \delta^{r_2}_n(K^{12})^{r_1} - \delta^{r_1}_n(K^{12})^{r_2}$$

Contracting this expression with $H_{mr_1r_2}$ we obtain

$$\frac{1}{8}H_{mr_1r_2}\bar{\epsilon}^1\Gamma^{r_1r_2}\Gamma_n\epsilon^2 = \frac{1}{8}H^{r_1r_2}{}_m(\Phi^{12})_{nr_1r_2} - \frac{1}{4}(\iota_{K^{12}}H)_{mn} \quad \text{(C.4)}$$

for the first term in Eq. (C.3). Treating each term in Eq. (C.3) in this way, and neglecting any symmetric combinations of the indices m, n we obtain,

$$\begin{aligned}\overline{(\nabla_{[m}\epsilon^1)}\Gamma_{n]}\epsilon^2 &= \frac{1}{8}H^{r_1r_2}{}_{[m}\Phi^{12}_{n]r_1r_2} - \frac{1}{4}(\iota_{K^{12}}H)_{mn} \\ &\quad +\frac{e^\phi}{16}\left(\iota_{K^{22}}G^{(3)} + \iota_{G^{(3)}}\Sigma^{22} + \iota_{G^{(7)}}\Omega^{22} + \iota_{\Sigma^{22}}G^{(7)}\right)_{mn}\end{aligned} \quad \text{(C.5)}$$

Now we can use the identity in Eq. (A.6) of Appendix A to simplify $\iota_{G^{(7)}}\Omega$ as follows,

$$\iota_{G^{(7)}}\Omega^{22} = -\iota_{*G^{(3)}} * K^{22} = \iota_{K^{22}}G^{(3)}$$

where in the first step we have used the fact that $G^{(7)} = -*G^{(3)}$ and $\Omega = *K$, as explained in Chapter 6. Similarly, we use the identity (A.6) to rewrite $\iota_{\Sigma^{22}} G^{(7)}$ as

$$\iota_{\Sigma^{22}} G^{(7)} = -\iota_{*\Sigma^{22}} * G^{(3)} = \iota_{G^{(3)}} \Sigma^{22}$$

where we have used the fact that Σ^{22} is self-dual in the first step. Therefore, Eq. (C.5) becomes

$$\overline{(\nabla_{[m}\epsilon^1)}\Gamma_{n]}\epsilon^2 = \frac{1}{8} H^{r_1 r_2}{}_{[m}\Phi^{12}_{n]r_1 r_2} - \frac{1}{4}(\iota_{K^{12}} H)_{mn} + \frac{e^{\phi}}{8}\left(\iota_{K^{22}} G^{(3)} + \iota_{G^{(3)}}\Sigma^{22}\right)_{mn} \qquad \text{(C.6)}$$

From Eq. (C.1) we see that this is the first term in the expression for $\nabla_{[m} K^{12}_{n]}$. The second term in $\nabla_{[m} K^{12}_{n]}$ comes from the second term in Eq. (C.1). This term can be calculated in a completely analogous way to the first term, so we do not show this here. Overall we obtain,

$$\nabla_{[m} K^{12}_{n]} = \frac{1}{4} H^{r_1 r_2}{}_{[m}\Phi^{12}_{n]r_1 r_2} + \frac{e^{\phi}}{8}\left(\iota_{(K^{22}+K^{11})} G^{(3)} + \iota_{G^{(3)}}(\Sigma^{22} + \Sigma^{11})\right)_{mn} \qquad \text{(C.7)}$$

Now the ordinary derivative of K^{12} is related to this by $(dK^{12})_{mn} = 2\nabla_{[m} K^{12}_{n]}$, so we obtain

$$(dK^{12})_{mn} = \frac{1}{2} H^{r_1 r_2}{}_{[m}\Phi^{12}_{n]r_1 r_2} + \frac{e^{\phi}}{4}\left(\iota_{(K^{22}+K^{11})} G^{(3)} + \iota_{G^{(3)}}(\Sigma^{22} + \Sigma^{11})\right)_{mn} \qquad \text{(C.8)}$$

as stated in Eq. (6.8).