Aspects of model building in noncommutaive quantum field theories

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Aspects of Model Building in Noncommutative Quantum Field Theories

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Abstract

We review quantum field theories on noncommutative Minkowski spaces (NCQFTs), concentrating on the mixing between ultra-violet and infra-red degrees of freedom in such theories.

We use background field perturbation theory at the one-loop level to calculate the three and four point functions in supersymmetric NCQFT. We use the results of this calculation to show that the infra-red logarithmic divergences that arise as a result of the UV/IR mixing can be reproduced by an explicitly gauge-invariant low-energy effective action expressed in terms of Wilson lines.

We present a noncommutative gauge theory that has the ordinary Standard Model as its low-energy limit. The model is based on the gauge group $U(4) \times U(3) \times U(2)$ and satisfies all the key constraints that are imposed by noncommutativity: the UV/IR mixing effects, restrictions on representations and charges of matter fields and the cancellation of noncommutative gauge anomalies. At energies well below the noncommutative mass scale our model flows to the commutative Standard model plus additional free $U(1)$ degrees of freedom which decouple due to the UV/IR mixing. Our model also predicts the values of the hypercharges of the Standard Model fields. We find that in order to accomodate the matter content of the Standard Model it is necessary to introduce extra, as yet undetected, matter fields.

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Declaration

I declare that no material presented in this thesis has previously been submitted for a degree at this or any other university.

The research described in this thesis has been carried out in collaboration with Dr. Valentin V. Khoze and Dr. Gabriele Travaglini and has been published as follows:

- Effective Actions, Wilson Lines and the IR/UV Mixing in Noncommutative Supersymmetric Quantum Field Theories,
  Jonathan Levell and Gabriele Travaglini, JHEP 0403:021,2004
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- Noncommutative Standard Modelling,
  Jonathan Levell and Valentin V. Khoze, JHEP 09 (2004) 019
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There is a theory that if anyone ever discovers exactly what the Universe is for and why it is here it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory that states that this has already happened,

Douglas Adams, "The Hitchhiker's Guide to the Galaxy"

Contents

Chapter 1

Introduction

IT'S BECAUSE OF THE UNCERTAINITY PRINCIPLE.

'What's that?'

I'M NOT SURE.

Death in "The Fifth Elephant" by Terry Pratchett

Over the last fifty years, huge strides have been made in our understanding of nature on sub-atomic scales. We have evolved and then tested what has become known as the *Standard Model* of particle physics and we can make (and test!) predictions about the interactions of particles so small that they appear point-like on a scale of 10^{-18}m.

The Standard Model is a quantum field theory defined on a commutative Minkowski space (a commutative quantum field theory for brevity) which means it assumes that, in principle, spatial positions of a particle can be measured to an arbitrarily high accuracy (as long as the momentum of the particle being measured is not also required[1]). Because the predictions of

[1]Heisenberg's uncertainty principle precludes knowing both position and momentum of a particle to an infinite accuracy.

the Standard Model are in agreement with experiment, this is obviously not an unreasonable assumption, at least on the scales we access with modern day accelerators. What about scales smaller than we can yet measure? Is it reasonable to expect that positions can be measured to infinite precision given arbitrarily good equipment? For more than fifty years people have been investigating theories in which measurements of position along one axis necessarily affect positions with respect to other axes (an uncertainty relation for co-ordinates).

In 1930 Heisenberg wrote a letter [1] discussing the idea of introducing uncertainty relations for positions to remove the infinities in his calculation of the self-energy of the electron. Other physicists including Pauli [2], Oppenheimer, Snyder [3] and C.N. Yang [4] worked on such models and then in 1948 Moyal introduced what is now known as the Moyal star product in order to discuss the mathematical structure of quantum mechanics on a noncommutative space-time [5].

However another idea, *renormalisation*, solved the problem of the electron self-energy that was Heisenberg's original motivation for introducing the idea of noncommutativity and research in noncommutative quantum physics died out for roughly forty years.

Meanwhile, physicists have been looking for a grand unified theory; a theory that would have a unified description of all the forces of nature. The Standard Model, for all its successes has drawbacks, principal amongst these is that it neglects gravity. Although gravity is unmeasurably weak on a sub-atomic scale, a complete description of nature would include all known forces. So far, attempts to include gravity into a renormalisable quantum field theory have met with little success and some physicists have turned their attention to *string theory* which elegantly contains equations for gauge field theories and gravity.

In contrast with commutative field theory, string theory is non-local. In particular string theory introduces a new dimensionful quantity; the fundamental string length $l_s = \frac{1}{\sqrt{\alpha'}}$. The scale l_s can be viewed as the distance scale below which the classical properties of space are dramatically modified. One famous consequence of this is the UV-finiteness of string theory and the quantum gravity that follows from it. Another consequence of this is that space can become noncommutative below a certain, string-inspired scale[2].

Connes (and collaborators) had been using noncommutative ideas in pure mathematics during the 1980s [6] and when Connes, Douglas and Schwarz showed that noncommutative quantum field theory was a possible low energy limit of string theory [7] there was a revival of interest in the subject. Seiberg and Witten extended these ideas [8] and since then there have been more than a thousand papers on the subject.

Aside from their connection with string theory there are other reasons to be hopeful for the prospects of noncommutative models. As we probe smaller and smaller structures in space-time we require a probe with an increasingly short wavelength, i.e. a higher-energy. Energy is a source of gravitation and at extremely small scales space-time will be deformed by the act of measurement. Above a critical scale the space-time would collapse, forming a black hole and it would seem that at such energies, if not lower, a description of the universe would have to be non-local (i.e. not assume measurements can be made with infinite accuracy). Noncommutative quantum field theo-

[2]This scale does not have to be of the order of l_s, it can be a combination of l_s and various background fields in string-brane theory [7,8]. In particular the noncommutative length scale can be $\gg l_s$.

ries provide an intriguing theoretical laboratory, allowing the study of non-local interactions in a quantum field theory context. New effects, specific to noncommutative theories, such as ultra-violet/infra-red mixing (where the physics at high-energy affects the physics at much lower energy) have been studied but it not yet known whether the real world is noncommutative.

Noncommutative models incorporating the Standard Model have been constructed previously in the literature and in particular in [9, 10] but none of these models have been phenomenologically viable (because they ignore, implicitly or explicitly, the important effects of ultra-violet/infra-red mixing). This thesis will present work that led up to the inception of a new model (presented in Chapter 5) that meets all the key constraints on building noncommutative models and has the Standard Model as its low energy limit.

1.1 Thesis Outline

The next chapter will introduce noncommutative quantum field theories focusing on their differences from usual commutative theories. The key difference is the mixing between the high-energy and low-energy degrees of freedom that means that physics in the ultra-violet regime of the theory affects the physics in the infra-red (the so-called UV/IR mixing). In some theories this leads to a new class of infra-red quadratic divergences which render the theory unphysical but in a large class of theories (including supersymmetric theories) these divergences cancel leaving only logarithmic divergences. We review the classic calculation that showed that the effect of the UV/IR mixing in these theories is to cause the trace $U(1)$ part of the high-energy $U(N)$ to decouple and become unobservable, leaving an $SU(N)$ group as found in the Standard Model.

The subsequent chapter will look at low energy actions of noncommutative quantum theories. It will show, with explicit calculations that low energy actions can be written down which model the logarithmic divergences previously discussed. This serves to confirm that the $U(N)$ gauge group has not been broken at the level of the action and therefore the breaking is, in fact, dynamical.

Chapter 4 contains a summary of the ordinary Standard Model, outlining the forces and the fields that are included. It gives a brief overview of the Higgs mechanism and the anomaly cancellation that constrains the values of the hyper-charges in the model. The chapter ends with a brief discussion of the limitations of the model.

In Chapter 5, a model is introduced that flows to the Standard Model at low energies. We discuss how the correct values of the hyper-charges arise naturally in the model and outline the Higgs potential and Yukawa terms that are necessary to make the model a phenomenologically viable candidate for a theory of nature.

Chapter 6 then provides some concluding remarks for the thesis, reviewing what has been done and discusses possible future work.

Chapter 2

An Overview of Noncommutative Quantum Field Theory

"Y'know," he said, "It's very hard to talk quantum in a language originally designed to tell other monkeys where the ripe fruit is."

The Sweeper in "The Night Watch" by Terry Pratchett

In this chapter we will briefly review some details of field theories on noncommutative spaces. More detailed reviews exist (see for example [11, 12]) and have been used in compiling this summary. However, this chapter will provide the necessary and (hopefully) sufficient material for reading the subsequent chapters.

Although other types of noncommutative field theories have been studied in the literature this thesis will be restricted to field theories on space-times where the usual space-time co-ordinates x^μ are promoted to Hermitian operators which obey:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{2.1}$$

where $\theta^{\mu\nu}$ is a constant, real anti-symmetric matrix of c-numbers with dimensions of length squared. Note that this explicitly breaks Lorentz invariance.

2.1 The Star Product

Field theories with operators for space-time co-ordinates as described are equivalent to theories which have normal, commuting numbers as co-ordinates but have the ordinary product of fields replaced with the star product, $*$

$$(\phi_1 * \phi_2)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^y\partial_\nu^z}\phi_1(y)\phi_2(z)|_{y=z=x} \tag{2.2}$$

It can quickly be checked that the above definition of the star-product gives (2.1) when $\phi_1(x) \to x^\mu$ and $\phi_2(x) \to x^\nu$. It can be shown [5] that $*$-product is uniquely defined given that we require associativity to all orders in θ.

The key property of the star product is that it differs from the usual product of two fields only by the addition of a total derivative:

$$f(x) * g(x) = f(x)g(x) + \partial_\mu \alpha(x) \tag{2.3}$$

where

$$\alpha(x)_\mu = \frac{i}{2}\theta_{\mu\nu}f(x)\partial^\nu g(x) - \frac{1}{8}\theta_{\mu\rho}\theta_{\nu\sigma}\partial_\nu f(x)\partial_\rho\partial_\sigma g(x) + \ldots \tag{2.4}$$

This means that the Moyal commutator of two fields will be a total derivative which will be zero if integrated over all space so therefore:

$$\int d^4x\, f(x) * g(x) = \int d^4x\, g(x) * f(x) = \int d^4x\, f(x)g(x) \quad . \tag{2.5}$$

Because of this property, terms in the Lagrangian containing only two fields (kinetic and mass terms) will be unaffected by using $*$-products and so propagators will be unaffected.

Products of more than two fields are affected by the use of the $*$-product and so the Feynman rules for vertices are altered. Although the star product is by definition noncommutative, under integration over all space it is possible to cyclically permute the fields:

$$\int d^4x\, f_1 * f_2 \ldots * f_{n-1} * f_n = \int d^4x\, f_n * f_1 * f_2 \ldots * f_{n-1} \quad , \tag{2.6}$$

so that in noncommutative theories, integration acts like a trace.

When multiplying Fourier transforms of fields, a very useful formula is:

$$e^{ik_1 x} * \ldots * e^{ik_n x} = e^{-\frac{i}{2}\sum_{i<j}(k_i \tilde{k}_j)} e^{i(k_1 + \ldots + k_n)x} \tag{2.7}$$

where we have adopted the notation that is used throughout the thesis of $\tilde{k}_\mu = \theta_{\mu\nu}k_\nu$

We can write down the noncommutative counterpart to a commutative Lagrangian by the simple replacement of the usual product with star products. e.g. the Euclidean action for noncommutative, four dimensional ϕ^4 theory is:

$$S = \int d^4x \left(\frac{1}{2}(\partial_\mu \phi * \partial_\mu \phi) + \frac{1}{2}m^2 \phi * \phi + \frac{g}{4!}\phi * \phi * \phi * \phi \right) \tag{2.8}$$

It would then be possible to go on to derive the Feynman rules in the usual fashion which we will not do here as our calculations are performed in the background field method which is discussed in section 2.4.

2.2 Space-Space Noncommutativity

If $\theta^{\mu\nu}$ takes its most general form then there will be a commutation relation between position and time. Time has special role in quantum mechanics and it has been shown that having a noncommutative time causes a breakdown in unitary and the optical theorem no longer holds, [13–15].

In order to avoid this, we restrict our attention to theories with space non-commutativity as opposed to spacetime noncommutativity, i.e. $\theta^{0i} = 0$. In this case $\theta^{\mu\nu}$ can be a generic rank 3, anti-symmetric matrix. It is always possible to work in a co-ordinate system where:

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.9}$$

2.3 Gauge Theory

In a gauge theory, when local gauge transformations are made to the fields in the theory the changes are cancelled by changes in a field, the gauge field. We start with the Lagrangian for a gauge field $A_\mu \equiv A_\mu^a t^a$.

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\left(F_{\mu\nu} * F^{\mu\nu}\right) \tag{2.10}$$

We adopt (for this and the subsequent chapter) the convention that the generators of our gauge groups t^a are anti-Hermitian and normalised such that $\mathrm{Tr}(T^aT^b) = \frac{\delta^{ab}}{2}$. Notice the overall factor of $1/g^2$, compared to the usual conventions where such a factor does not appear we have made the redefinition: $gA_\mu \rightarrow A_\mu$. Also note that the coupling does not appear in either of the following:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_*, \\ \delta A_\mu &= \partial_\mu \alpha + [A_\mu, \alpha]_*. \end{aligned} \tag{2.11}$$

In the above equation, the δA^a_μ refers to the change under an infinitesimal gauge transformation and under a finite gauge transformation the coupling still does not appear:

$$A_\mu \rightarrow U * (A_\mu + \partial_\mu) * U^{-1} \tag{2.12}$$

In a commutative gauge theory, a large variety of groups are used for model building, e.g. $SU(N)$, $SO(N)$, and the exceptional groups. In a noncommutative setting the only allowed groups are the unitary groups (as discussed later). In a commutative theory the unitary symmetry groups used are either $U(1)$ or an $SU(N)$ group. Commutative $U(1)$ is a special class of theory called Abelian, it has a single generator, t^0, so the commutator of two gauge fields,

$$[A_\mu(x)t^0, A_\nu(x)t^0] \quad , \tag{2.13}$$

will vanish. This significantly changes the physics of the theory; it is the commutator term in the field-strength tensor which causes the self-interactions of gauge bosons in non-Abelian theories. In noncommutative $U(1)$, (2.13) will not vanish, although the single generator commutes with itself, the fields, like any functions of space, do not commute and so we have a non-abelian theory.

From a naive point of view, $SU(N)$ is not a valid gauge group in noncommutative theories, the commutator in the field strength becomes:

$$[A_\mu, A_\nu]_* = A^a_\mu * A^b_\nu t^a t^b - A^b_\nu * A^a_\mu t^b t^a \tag{2.14}$$

And we have:

$$t^a t^b = \frac{1}{2} f^{abc} t^c - \frac{\delta^{ab}\mathbb{1}}{2N} - \frac{i}{2} d^{abc} t^c \tag{2.15}$$

and so a component of the field strength tensor will in general, be along the identity. In order to resolve this problem, it is usual to work with $U(N)$ groups rather than $SU(N)$, where there is a generator proportional to the identity and this is not a problem [16, 17].

Some authors [18, 19] have defined noncommutative gauge theories in groups other than U(N) but such constructions generally involve perturbatively expanding in the noncommutativity parameter $\theta^{\mu\nu}$. As we shall see in section 2.5, doing such an expansion misses some of the crucial physics of noncommutative theories and we restrict ourselves to $U(N)$ groups.

Because the noncommutative $U(1)$ is non-Abelian, it differs in important respects from its commutative counterpart. Critically, as in commutative non-abelian theories, the charge of a particle is determined entirely by its representation. The only noncommutatively-allowed representations are fundamental, anti-fundamental, bi-fundamental and adjoint. Other representations can be formed by attaching semi-infinite lines to the particles [20, 21] but this seems unnatural and will be avoided in this thesis.

If $U \equiv e^{\alpha^a(x)t^a} \in U(N)$ then a fundamental field, ϕ transforming as: $\phi \to U*\phi$ will have charge $+1$. An anti-fundamental field transforms as $\phi \to \phi * U^{-1}$ and will have charge -1. An adjoint field has a charge of 0 and transforms as $\phi \to U * \phi * U^{-1}$.

To see this, consider, for example, a Dirac fermion. The action is:

$$S_f = \int d^d x \, \bar{\psi} * \gamma^\mu D_\mu * \psi \qquad . \tag{2.16}$$

In order to keep this gauge invariant, the covariant derivative needs to be:

$$\begin{aligned} D_\mu^{fund} * \psi(x) &= \partial_\mu \psi(x) + A_\mu * \psi(x) \\ D_\mu^{anti} * \psi(x) &= \partial_\mu \psi(x) - \psi(x) * A_\mu \\ D_\mu^{adj} * \psi(x) &= \partial_\mu \psi(x) + [A_\mu, \psi(x)]_* \end{aligned} \tag{2.17}$$

which forces the representations to have the charges stated [22].

More complex representations are not allowed because, in a noncommutative theory [23–25], the gauge rotations does not commute. For example, consider a rank-2 tensor representation, t^{ij} of $U(N)$

$$t^{ij} \to U^i_{i'} * U^j_{j'} * t^{i'j'}. \tag{2.18}$$

Because of the noncommutativity, this breaks the closure property, $(t^U)^V = t^{U*V}$. The only way we can add an extra gauge index to a field already transforming in the fundamental representation is if the new gauge rotations act on the right, hence bi-fundamental (and adjoint) representations are allowed.

2.4 The Background Field Method

In this section we will discuss the background field method which is used in all the subsequent calculations in this chapter and the subsequent one. The background field method is useful as it allows the high-energy degrees of freedom to be integrated out while still preserving gauge invariance. The commutative case is reviewed in [26] and our treatment closely follows that text and [27] which developed the method in a noncommutative setting. Following the treatment of [27] we perform our calculations in Euclidean space.

We split the gauge field into a classical, slowly varying (but still noncommutative) background field B_μ and a fluctuating quantum field N_μ:

$$A_\mu = B_\mu + N_\mu \tag{2.19}$$

We are now going to integrate out the fluctuating field leaving an effective action for the background field, schematically:

$$\exp(-S_{eff}) = \int \mathcal{D}N \exp(-S_{YM}) \tag{2.20}$$

The Lagrangian (equation (2.10)) can be rewritten respecting this decomposition of the gauge field. Writing the covariant derivative with respect to the background field on an adjoint field, ϕ as: $D(B)_\mu * \phi \equiv \partial_\mu \phi + [B^a_\mu t^a, \phi]_*$. Then:

$$F_{\mu\nu} \rightarrow F^B_{\mu\nu} + D(B)_\mu * N_\nu - D(B)_\nu * N_\mu + [N_\mu, N_\nu]_* \tag{2.21}$$

where

$$F^B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]_* \quad . \tag{2.22}$$

When the background field is considered fixed then the Langrangian has a local gauge symmetry for our fluctuating field N_μ :

$$N_\mu \rightarrow N_\mu + D(B)_\mu * \alpha + [N_\mu, \alpha]_* \tag{2.23}$$

Following [27] we introduce as our gauge-fixing term:

$$S_{g.f.} = \frac{1}{g^2} \int d^4x \, \mathrm{Tr}((D_\mu(B) * N_\nu) * \mathrm{Tr}((D_\mu(B) * N_\nu) \tag{2.24}$$

and corresponding ghost term:

$$S_{ghost} = 2 \int d^4x \, \mathrm{Tr}(\bar{c} * D_\mu(B) * D_\mu(B+N) * c) \tag{2.25}$$

Rewriting the gauge, gauge-fixing and ghost terms, keeping terms which either do not depend on the fluctuating field or are quadratic in it, we find:

$$\begin{aligned} S_{YM} + S_{GF} + S_{ghost} = & -\frac{1}{2g^2} \int d^4x \, \mathrm{Tr}(F^B_{\mu\nu} * F^B_{\mu\nu}) - \frac{1}{g^2} \int d^4x \, \mathrm{Tr}(N_\mu * [M_{\mu\nu}, N_\nu]_*) \\ & - 2 \int d^4x \, \mathrm{Tr}(\bar{c} * D^2(B) * c) \quad , \end{aligned} \tag{2.26}$$

where

$$M_{\mu\nu} = -D^2(B)\delta_{\mu\nu} - 2F^B_{\mu\nu} \quad . \tag{2.27}$$

We also add terms for fermions[1] and scalars to the action,

$$S_{matter} = -2 \int d^4x \, \mathrm{Tr}(\bar{\psi} * \not{D} * \psi) + \mathrm{Tr}\left(\overline{(D_\mu * \phi)} * (D_\mu * \phi)\right) \quad . \tag{2.28}$$

Where the covariant derivative for the adjoint and fundamental fields are

[1] Although we use Dirac fermions here we will split them into Weyl fermions later

respectively:

$$D_\mu^{fund}(B) * \phi = \partial_\mu \phi + B_\mu * \phi$$
$$D_\mu^{adj}(B) * \phi = \partial_\mu \phi + [B_\mu, \phi]_* \tag{2.29}$$

We want to write these terms in a form similar to (2.26) so:

$$\begin{aligned}(\not{D}\,)^2 &= \gamma^\mu \gamma^\nu D_\mu * D_\nu \\ &= \frac{1}{2}(\{\gamma_\mu, \gamma_\nu\} + [\gamma_\mu, \gamma_\nu])D_\mu * D_\nu \\ &= D^2(B)_\mu - 2iS^{\mu\nu} D_\mu * D_\nu\end{aligned} \tag{2.30}$$

where $S^{\mu\nu}$ is the generator of the Euclidean Lorentz group for spinors. We can then summarise all these terms into an action functional that describes a spin-j field in the background of B_μ

$$S[\phi] = -\int d^4x\, \phi_{m,a} * \Delta^{ab}_{mn} * \phi_{n,b} \tag{2.31}$$

where a and b are gauge-indices,m and n are spin-indices and

$$\Delta^{ab}_{mn} * \phi_{n,b} \equiv \left(-D^2(B)\delta_{mn}\delta^{ab} + 2i(F^B_{\mu\nu})^{ab}\frac{1}{2}J^{\mu\nu}_{mn}\right) * \phi_{n,b} \tag{2.32}$$

with $F^{ab} \equiv \sum_A F^A t^A_{ab}$ and the J generators for the various spins are:

$$\begin{aligned} J &= 0 && \text{for scalars} \\ J &= \frac{i}{2}[\sigma^{\mu\nu}]_{\alpha\beta} && \text{for Weyl fermions} \\ J &= i(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma) && \text{for vectors} \end{aligned} \tag{2.33}$$

In order to integrate out the fluctuating fields we start with the partition function for a microscopic scalar field ϕ:

$$Z = N \int \mathcal{D}\phi \, e^{-S[\phi]} \quad , \tag{2.34}$$

where N is a normalisation constant. We the expand the action in terms of the slowly-varying background field, $C(x)$ (So $\phi(x) = C(x) + \delta\phi$). We assume that these fluctuations to not extend out to infinity, so as $|x| \to \infty$ then $\phi \to C$ and $\delta\phi \to 0$. We can then write:

$$S[\phi] = S[C] + \left.\frac{\delta S}{\delta \phi}\right|_{\phi=C} \delta\phi + \frac{1}{2}\delta\phi \left.\frac{\delta^2 S}{\delta \phi^2}\right|_{\phi=C} \delta\phi + ... \tag{2.35}$$

We can ignore the second term on the right-hand side of (2.35), as it contributes only to tadpole diagrams [26, 27]. The third term will become a functional determinant. Higher order terms are ignored as we will work to one-loop order.

Using the above equation we can write:

$$Z = Ne^{-S[C]} \int \mathcal{D}\delta\phi \, e^{-\frac{1}{2}\delta\phi.\Delta.\delta\phi} \tag{2.36}$$

We then introduce a complete, ortho-normal set of basis states for $\phi(x)$: $\{\phi_n\}$ such that:

$$\Delta\phi_n(x) = \lambda_n \phi_n(x) \tag{2.37}$$

Because of the completeness of this set of states we can then write:

$$\delta\phi(x) = \sum_n c_n \phi_n(x) \qquad \text{and} \qquad \int \mathcal{D}\delta\phi(x) = \prod_n \int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}} \tag{2.38}$$

We can then write the partition function as:

$$Z = Ne^{-S[C]} \prod_n \int \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_n \lambda_n c_n^2} \tag{2.39}$$

because

$$\begin{aligned} \delta\phi.\Delta.\delta\phi &= \sum_n \sum_m c_m c_n \phi_m.\lambda_n \phi_n \\ &= \sum_n \lambda_n c_n^2 \end{aligned} \tag{2.40}$$

due to the orthonormality of the set $\{\phi_n\}$.

We can then use the result:

$$\int_{-\infty}^{\infty} \frac{dc_n}{\sqrt{2\pi}} e^{\frac{1}{2}\lambda_n c_n^2} = \frac{1}{\sqrt{\lambda_n}} \quad . \tag{2.41}$$

Because $\det \Delta = \prod_n \lambda_n$ we can arrive at:

$$Z = Ne^{-S[C]} \det^{-\frac{1}{2}} \Delta \tag{2.42}$$

Had we used a spinor field rather than a scalar we would have arrived at an answer of the same form except that the determinant would have had a different power [26].

When we take all fields into consideration, including the term that describes the background field itself, once we integrate out the fluctuations our action becomes, at one-loop:

$$S_{eff}[B] = -\frac{1}{2g^2} \int d^4x \, \mathrm{Tr} F^B_{\mu\nu} * F^B_{\mu\nu} - \sum \alpha_j \log \det \Delta \tag{2.43}$$

Here the sum extends over all the fields in the theory and the possible values for α_j are listed in table 2.1:

	ghost	real scalar	Weyl fermion	gauge field
α_j	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$d(j)$	1	1	2	4
$C(j)$	0	0	$\frac{1}{2}$	2

Table 2.1: Constants for the various fields in the theory

We then define a new function $\mathcal{K}$ as the d'Alembertian (from equation (2.32)) with the derivative piece removed, i.e. $\Delta = -\partial^2 + \mathcal{K}$ and we can then write:

$$\begin{aligned}\log\,\det\Delta &= \log\,\det(-\partial^2 + \mathcal{K}) \\ &= \log\,\det(-\partial^2) + \log\,\det(1 + (-\partial^2)^{-1}\mathcal{K}) \\ &= \log\,\det(-\partial^2) + \mathrm{tr}\,\log(1 + (-\partial^2)^{-1}\mathcal{K})\,. \end{aligned} \tag{2.44}$$

The first term in the above equation corresponds only to vacuum bubbles and is ignored. The second term has an expansion in terms of Feynman diagrams that will be used throughout the first half of this thesis.

In the next chapter we will perform calculations using a $U(1)$ gauge group as well as $U(N)$ so the Feynman rules for both are included here. Because of the relative simplicity we start with the rules for $U(1)$.

We concentrate on fields in the adjoint representation because it is only this representation that is involved in the ultra-violet/infra-red mixing effects that we are working towards examining [22, 27].

2.4.1 Feynman rules for a U(1) group

To begin with, we will look at a generic field, ϕ, in the adjoint representation, using 2.32 and the covariant derivative from equation (2.29) we can write Δ acting on ϕ as:

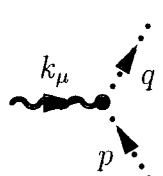
$$\Delta * \phi = -\partial^2 - [(\partial_\mu B_\mu), \phi]_* - 2[B_\mu \partial_\mu, \phi]_* - [B_\mu, [B_\mu, \phi]_*]_* + 2i\left(\frac{1}{2}J^{\mu\nu}[F^B_{\mu\nu}, \phi]_*\right) \tag{2.45}$$

Aside from the first term, each term on the right hand side of the above equation will contribute to a vertex in our expansion of the logarithm in our action (equation (2.43)). Note that if we were working in a commutative $U(1)$ then each commutator in equation (2.45) would vanish and we have no vertices, this is expected, in such a theory the adjoint representation is completely equivalent to the singlet representation.

We now proceed to calculate these vertices, starting with the second and third terms on the right-hand side of the above equation:

$$-\int d^4x\, \bar{\phi} * [(\partial_\mu B_\mu) + 2B_\mu \partial_\mu, \phi]_* \quad . \tag{2.46}$$

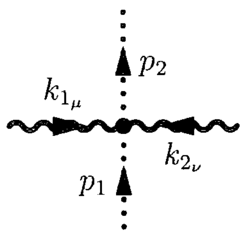
Using (2.7) we find that the $\phi - B - \phi$ vertex is:



$$= -2(p+q)_\mu \sin\frac{1}{2}k\tilde{p}$$

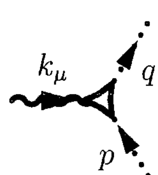
Vertex factors containing complex exponentials of momenta are a generic feature of noncommutative quantum field theories but in the case of the adjoint representation of $U(1)$, they are arranged into sin functions and, as expected, in the commutative limit, $\theta^{\mu\nu} \longrightarrow 0$ then $\tilde{p} \longrightarrow 0$ and the vertex disappears.

The other vertices follow in a similar fashion. The fourth term on the right-hand side of equation (2.45) gives:



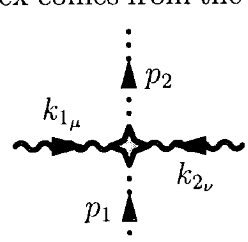
$$= -4\delta_{\mu\nu}\sin\frac{1}{2}p_2\tilde{k}_1 \sin\frac{1}{2}k_2\tilde{p}_1$$

So far we have derived the standard vertices for an adjoint scalar coupled to noncommutative $U(1)$ theory but there are still two more vertices coming from the last term in (2.45). These extra vertices are not present just because it is a noncommutative theory, instead they are due to the background field method (c.f. [26]). The first vertex comes from the part of the field strength tensor (equation (2.22)) proportional to a single power of the background field:



$$= 4iJ^{\alpha\mu}k_\alpha\sin(k\tilde{p})$$

The final vertex comes from the commutator part of the field strength tensor.



$$= 4iJ^{\mu\nu}\sin\frac{1}{2}k_1\tilde{k}_2 \sin\frac{1}{2}p_1\tilde{p}_2$$

2.4.2 The Feynman Rules for $U(N)$

Because noncommutative $U(1)$ is nonabelian, the structure of the vertices is very similar. The main difference is that now, for two adjoint fields, ϕ_1 and

ϕ_2:

$$[\phi_1, \phi_2]_* = \frac{1}{2}\left(\{\phi_1^A, \phi_2^B\}_* f^{ABC} - i[\phi_1^A, \phi_2^B]_* d^{ABC}\right) t^C \ . \tag{2.47}$$

where f^{ABC} (d^{ABC}) is completely antisymmetric (symmetric) in its indices. If we let $A = (0, a)$ where $a = 1...N^2-1$ label the SU(N) generators then f^{abc}, d^{abc} are the same as in $SU(N)$, and $f^{0bc} = 0$, $d^{0BC} = \sqrt{2/N}\ \delta^{BC}$, $d^{00a} = 0$, $d^{000} = \sqrt{2/N}$.

We also have $t^0 = (\frac{1}{i\sqrt{2N}})\mathbb{1}_N$ and:

$$\mathrm{Tr}(t^A t^B) = -\frac{\delta^{AB}}{2} \tag{2.48}$$

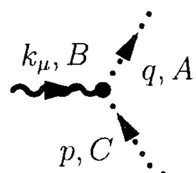
Using this, we can calculate the Feynman rules as before [28] and they are shown in figure 2.1

We should note that the 4-point J-vertex (the fourth vertex in 2.1) will play no part in our calculation of the two point function, where $k_1 + k_2 = p_1 + p_2 = 0$

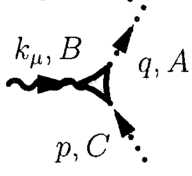
2.5 UV/IR Mixing

Now that the background field method formalism has been introduced we turn our attention to the focal point for this work; the ultra-violet/infra-red mixing [29, 30].

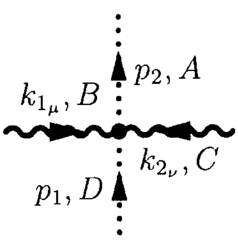
As we have seen, the vertices in the theory at hand contain factors of the form: $e^{ik\tilde{p}}$ where p is a loop momentum and k is an external momentum. When we calculate the divergences (for example in the next section we will look at the polarisation tensor for gauge bosons), we find two types of in-



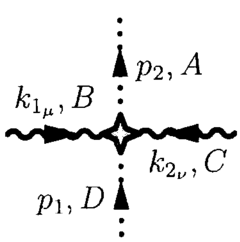
$$= i\,(2p+k)_\mu \left(-d^{ABC}\sin\left(\frac{k\tilde{p}}{2}\right) + f^{ABC}\cos\left(\frac{k\tilde{p}}{2}\right)\right)$$



$$= 2J^{\alpha\mu}k_\alpha\left(-d^{ABC}\sin\left(\frac{k\tilde{p}}{2}\right) + f^{ABC}\cos\left(\frac{k\tilde{p}}{2}\right)\right)$$



$$= \delta^{\mu\nu}\left(-d^{BHA}\sin\left(\frac{p_2\tilde{k}_1}{2}\right) + f^{BHA}\cos\left(\frac{p_2\tilde{k}_1}{2}\right)\right) \cdot \left(f^{BHA}\cos\left(\frac{k_2\tilde{p}_2}{2}\right) + d^{BHA}\sin\left(\frac{k_2\tilde{p}_2}{2}\right)\right)$$



$$= -iJ^{\mu\nu}\left(-d^{BCE}\sin\left(\frac{k_1\tilde{k}_2}{2}\right) + f^{BCE}\cos\left(\frac{k_1\tilde{k}_2}{2}\right)\right) \cdot \left(d^{EDA}\sin\left(\frac{p_1\tilde{p}_2}{2}\right) + f^{EDA}\cos\left(\frac{p_1\tilde{p}_2}{2}\right)\right)$$

Figure 2.1: *Feynman rules for $U(N)$. The wavy lines represent the background fields, the dotted lines the high-virtuality fields.*

tegral contribute. Firstly in some terms these vertex factors cancel, leaving no dependence on the noncommutativity factor $\theta^{\mu\nu}$. This first type of integrals are referred to as planar because in the double-line formulation of the Feynman rules (not used in this thesis) they arise from planar diagrams. These planar integrals are identical to their commutative counter-parts and introduce ultra-violet divergences that have to be renormalised in the same manner as in a commutative theory. In the second type of integral, referred to as non-planar, these vertex factors do not cancel. At high-energies, rapid oscillations of these phase factors will cause the integral to converge meaning it no longer causes an ultra-violet divergence but at low energies, where $k \longrightarrow 0$ then the phase-factor becomes irrelevant and the divergence reappears but now as an infra-red divergence.

The new infrared divergences have a profound effect on the theory, for large classes of theories, new quadratic divergences will violently change the structure of the theory, altering the dispersion relation for the photon dramatically and ensuring the theory is unphysical. In certain types of theory (in particular supersymmetric theories[2]) these quadratic divergences cancel leaving only logarithmic divergences. These logarithmic divergences affect the running of the couplings and can cause a dynamical breaking of gauge symmetry by causing $U(1)$ factors of the gauge group to decouple [27, 28].

We now go on to demonstrate this with explicit calculation; the one-loop correction to the polarisation tensor of the gauge boson in a $U(N)$ group.

2.5.1 One-loop calculation of the effective action

We define the *Wilsonian polarization tensor*, $\Pi^{AB}_{\mu\nu}(k)$ so that the term in our effective action that is quadratic in the background field is:

$$2\int \frac{d^4k}{(2\pi)^4} B^A_\mu(k) B^B_\nu(-k) \Pi^{AB}_{\mu\nu}(k) \tag{2.49}$$

This tensor will have the structure:

$$\Pi_{\mu\nu}(k) = \Pi_1(k^2, \tilde{k}^2)(k^2\delta_{\mu\nu} - k_\mu k_\nu) + \Pi_2(k^2, \tilde{k}^2)\left(\frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^4}\right) \quad . \tag{2.50}$$

In a commutative theory, the second term would not exist and the first term contains the only suitable transverse tensor. In a noncommutative theory the second term appears and leads to a power-like singularity in the infrared.

[2]indeed only fields in the adjoint representation need to have partners

As we shall see in a large class of theories, including supersymmetric theories this second term will be absent.

Once we have performed this calculation we will be able to determine the running of the various couplings in theory:

$$\left[\frac{1}{g_{eff}^2(k)}\right]^{AB} = \frac{\delta^{AB}}{g_{micro}^2} + 4\Pi_1^{AB}(k) \tag{2.51}$$

2.5.2 Feynman diagrams

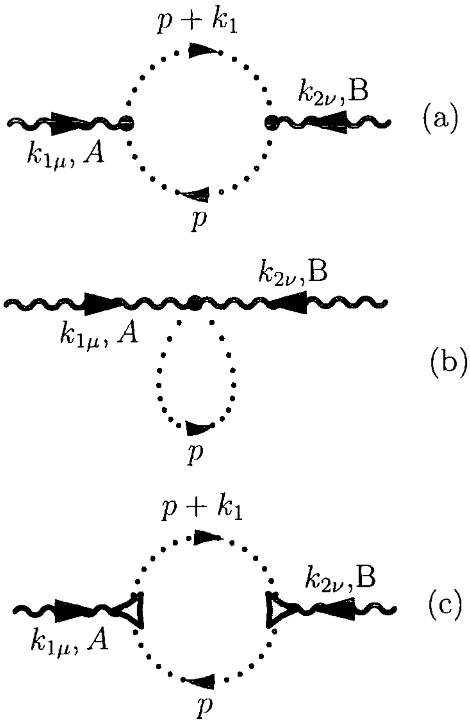


Figure 2.2: *1-loop corrections to the polarisation tensor*

The only Feynman diagrams that contribute are shown in figure 2.2. Using the $U(N)$ Feynman rules from the previous section, the first diagram (2.2a) gives:

$$-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}B_\mu^A(k)B_\nu^B(-k)\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\,\frac{-(2p+k)_\mu(2p+k)_\nu M^{AB}(k,p)}{(p^2+m^2)[(p+k)^2+m^2]} \tag{2.52}$$

where we define

$$M^{AB}(k,p)\equiv\left(-d^{ALM}\sin\frac{k\tilde{p}}{2}+f^{ALM}\cos\frac{k\tilde{p}}{2}\right)\left(d^{BML}\sin\frac{k\tilde{p}}{2}+f^{BML}\cos\frac{k\tilde{p}}{2}\right) \tag{2.53}$$

Notice that the integral over the loop momentum in equation (2.52) is done over d-dimensional space rather than 4 dimensions as might naively be expected. The integrals involved will turn out to diverge, by performing the integral in d dimensions and then setting $d = 4-\epsilon$ where ϵ is infinitesimally small will isolate the infinities into $\frac{1}{\epsilon}$ poles which we can remove with renormalisation. This method of regulating the integrals preserves both gauge and Lorentz symmetry.

The second diagram (2.2b) gives:

$$\int\frac{d^4k}{(2\pi)^4}B_\mu^A(k)B_\nu^B(-k)\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\frac{-\delta_{\mu\nu}M^{AB}(k,p)}{p^2+m^2} \tag{2.54}$$

and the final diagram (2.2c) gives:

$$-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}B_\mu^A(k)B_\nu^B(-k)\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\frac{-4J^{\mu\rho}J^{\nu\lambda}k_\lambda k_\rho M^{AB}(k,p)}{(p^2+m^2)[(p+k)^2+m^2]} \tag{2.55}$$

The traces in equations (2.52), (2.54) and (2.55) are over spin indices and we can evaluate them given that (using the constants in table 2.1):

$$\mathrm{Tr}\mathbb{1}_j\equiv d(j) \tag{2.56}$$

and

$$\mathrm{Tr}(J^{\mu\rho}J^{\nu\lambda})_j = C(j)(\delta^{\mu\nu}\delta^{\rho\lambda} - \delta^{\mu\lambda}\delta^{\nu\rho}) \tag{2.57}$$

We can then extract the polarisation tensor:

$$\begin{aligned}\Pi^{AB}_{\mu\nu}(k) = -\sum_j \alpha_j \int \frac{d^dp}{(2\pi)^d} \Bigg[d(j)M^{AB}(k,p)\left(\frac{1}{4}\frac{(2p+k)_\mu(2p+k)_\nu}{(p^2+m_j^2)[(p+k)^2+m_j^2]} - \frac{1}{2}\frac{\delta^{\mu\nu}}{p^2+m_j^2}\right) \\ +C(j)M^{AB}(k,p)\frac{k^2\delta^{\mu\nu}-k_\mu k_\nu}{(p^2+m_j^2)[(p+k)^2+m_j^2]}\Bigg]\end{aligned} \tag{2.58}$$

We can also use the equations [31]:

$$\begin{aligned} f^{ALM}f^{BML} &= -Nc_A\delta_{AB} \\ d^{ALM}d^{BML} &= Nd_A\delta_{AB} \\ f^{ALM}d^{BML} &= 0 \end{aligned} \tag{2.59}$$

where $c_A = \delta_{0A}$ and $d_A = 2 - c_A$. This allows equation (2.53) to be rewritten as:

$$M^{AB}(k,p) = -N\delta^{AB}(1 - \delta^{0A}\mathrm{cos}k\tilde{p}) \tag{2.60}$$

Loop integrals involving the first term in equation (2.60) give rise to the planar contribution and are analogous to their commutative counterparts. Integrals involving the second term in equation (2.60) give the non-planar contribution and cause the UV/IR mixing. Equation (2.60) already shows that it is exclusively degrees of freedom associated with the generator $t^0 \propto \mathbb{1}$ that will exhibit the UV/IR mixing.

In order to proceed further we need to evaluate some integrals and we pause our discussion while the techniques to do this are introduced.

2.5.3 Loop Integrals

The first integral we require is:

$$\int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2+m^2)[(p+k)^2+m^2]} \tag{2.61}$$

In order to simplify this we use *Schwinger parameters*[3] to combine the propagators. This is the observation that for any propagator, A:

$$\frac{1}{A} = \int_0^\infty dx\, e^{-xA} \tag{2.62}$$

So we can rewrite equation (2.61) as:

$$\begin{aligned} &\int \frac{d^dp}{(2\pi)^d} \int_0^\infty d\alpha_1 d\alpha_2 \exp(-\alpha_1(p^2+m^2) - \alpha_2[(p+k)^2+m^2]) \\ =& \int \frac{d^dl}{(2\pi)^d} \int_0^\infty d\alpha_1 d\alpha_2 \exp\left(-\alpha m^2 - \alpha l^2 - \frac{\alpha_1\alpha_2}{\alpha}k^2\right) \end{aligned} \tag{2.63}$$

where in the second line $\alpha = \alpha_1 + \alpha_2$, $l = p + \frac{\alpha_2}{\alpha}k$. We then change variables of integration again, integrating over a d-dimensional sphere, Ω_d:

$$\int \frac{d^dl}{(2\pi)^d} \to \int \frac{\Omega_d}{(2\pi)^d} \int_0^\infty dl\, l^{d-1} \qquad \text{where} \qquad \int \Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{2.64}$$

We proceed by using the formula

$$\int e^{-bx^2} x^a dx = \frac{1}{2}\Gamma\left(\frac{1+a}{2}\right) b^{\frac{1}{2}(-1-a)} \quad a > -1 \quad b > 0 \tag{2.65}$$

to perform the integration over momenta. We also introduce a new variable λ, by adding $\delta(\lambda - \alpha)$ to our equation and then integrating λ from 0 to

[3] It would have been possible to do the integral in other ways, for example using Feynman parameters but when we turn to more complicated integrals with factors like $e^{ik\tilde{p}}$ in the numerator the method described here will prove to be simpler

infinity. We arrive at:

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\int_0^\infty d\lambda\int_0^1 d\xi_1 d\xi_2\lambda^{1-\frac{d}{2}}\exp(-\lambda[m^2+\xi_1\xi_2p^2])\delta(1-\xi_1-\xi_2) \tag{2.66}$$

where $\xi_i = \frac{\alpha_i}{\lambda}$ we then use

$$\int x^a e^{-bx}dx = \Gamma(1+a)b^{-1-a} \tag{2.67}$$

which gives:

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\Gamma\left(2-\frac{d}{2}\right)\Delta^{\frac{d}{2}-2} \tag{2.68}$$

where $\Delta = m^2+x(1-x)p^2$. At this point we make use of the fact that we have been using integrals over d-dimensions rather than doing the calculations direct in $d=4$. We write d as $4-\epsilon$ where ϵ is infinitesimally small. Some care has to be taken because we will sometimes be working with supersymmetric theories, we do not want to break supersymmetry with our regularisation and this is an issue because the supersymmetric multiplets have different contents in different dimensions. However such problems can be avoided by employing the $\overline{DR}$ scheme [32].We can then Taylor expand in ϵ:

$$(4\pi)^{\frac{\epsilon}{2}} = e^{\frac{\epsilon}{2}\ln(4\pi)} = 1+\epsilon\frac{1}{2}\ln(4\pi)+\mathcal{O}(\epsilon^2) \tag{2.69}$$

and

$$\Gamma\left(2-\frac{d}{2}\right) = \frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon) \tag{2.70}$$

where γ_E is the Euler-Mascheroni constant and we finally arrive at,

$$\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2+m^2)[(p+k)^2+m^2]} = \frac{1}{(4\pi)^2}\int_0^1 dx\,\frac{2}{\epsilon}-\ln\Delta-\gamma_E-\ln(4\pi)+\mathcal{O}(\epsilon) \tag{2.71}$$

This integral then requires renormalisation to remove the $\frac{1}{\epsilon}$ pole. It has been shown that noncommutative gauge theories with a $U(N)$ gauge group can be renormalised in an equivalent way to commutative theories [31]. The pole can be removed by reparameterising the couplings and fields involved. For instance we rewrite the background field that appeared in the Lagrangian and we have been working with as:

$$B^\mu = z_3^{\frac{1}{2}} B^\mu_{phys} \tag{2.72}$$

Where B^μ_{phys} will be the physical background field and z_3 is an infinite constant. The reparameterisation will add counter-terms which can be used to cancel the infinity in this integral. However it does introduce an arbitrary scale called the renormalisation scale, μ. So our finite answer is eventually:

$$-\frac{1}{(4\pi)^2}\int_0^1 dx \ln\left(\frac{\Delta}{4\pi\mu^2}\right) \tag{2.73}$$

The other integrals we require can now be calculated using a similar method. The non-planar equivalent of this first integral, has a factor of $e^{ik\tilde{p}}$ in the numerator. Because we used Schwinger parameters the extra exponential does not pose a problem and instead of equation (2.66) we have instead:

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\int_0^\infty d\lambda \int_0^1 d\xi_1 d\xi_2 \lambda^{1-\frac{d}{2}} \exp\left(-\lambda[m^2+\xi_1\xi_2 p^2] - \frac{\tilde{p}^2}{4\lambda}\right)\delta(1-\xi_1-\xi_2) \tag{2.74}$$

and we can use the integral:

$$\int_0^\infty dx\, x^a e^{-bx} e^{\frac{-c}{x}} = 2c^{\frac{1}{2}(a+1)} b^{-\frac{1}{2}(a+1)} K_{a+1}(2\sqrt{bc}) \tag{2.75}$$

where $K_n(x)$ are the modified Bessel functions of the second kind. We find:

$$\int \frac{d^4p}{(2\pi)^d} \frac{e^{ik\tilde{p}}}{(p^2+m^2)[(p+k)^2+m^2]} = \frac{2}{(4\pi)^2} \int_0^1 dx\, K_0(\sqrt{\Delta \tilde{p}}) \tag{2.76}$$

Comparing equations (2.76) and (2.73) already demonstrates the crux of the UV/IR mixing. The oscillations of the exponential in the numerator means that the integral requires no regularisation or renormalisation to make it finite. On the other hand the small argument expansion of $K_0(z)$: $K_0(z) = -\ln(\frac{z}{2})....$ means that, in the low-energy limit where $p \to 0$ we have a logarithmic divergence in the infra-red that would not occur in the equivalent commutative theory.

The other integrals required have the momentum in the numerator, using the same techniques discussed with the extra formulae (remembering that we are using Euclidean space):

$$\begin{aligned} \int \frac{d^dl}{(2\pi)^d} l_\mu e^{-\alpha l^2} &= 0 \\ \int \frac{d^dl}{(2\pi)^d} l_\mu l_\nu e^{-\alpha l^2} &= \frac{1}{d} \int \frac{d^dl}{(2\pi)^d} l^2 \delta^{\mu\nu} e^{-\alpha l^2} \end{aligned} \tag{2.77}$$

2.5.4 The Polarisation Tensor

We can now use the results of our integration, developed in the last section and insert them into equation (2.58) to write down the polarisation tensor:

$$\begin{aligned}\Pi^{AB}_{\mu\nu}(k) =& -\frac{N}{4(4\pi)^2}(k^2\delta^{\mu\nu} - k_\mu k_\nu) \\ &\times \sum_j \alpha_j \int_0^1 dx\, \left(4C(j) - d(j)(1-2x^2)\right) \left[\delta^{AB}\ln\left(\frac{\Delta}{4\pi\mu^2}\right) + 2\delta^{A0}\delta^{B0}K_0(\sqrt{\Delta}|\tilde{k}|)\right] \\ &+ \frac{N}{8(4\pi)^2}\delta^{A0}\delta^{B0}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\sum_j \alpha_j d(j) \int_0^1 dx\, \Delta K_2(\sqrt{\Delta}|\tilde{k}|) \end{aligned} \tag{2.78}$$

where $\Delta = m^2 + x(1-x)k^2$ Using the constants shown in table (2.1) we can rewrite this as:

$$\Pi^{AB}_{\mu\nu}(k) = (k^2\delta_{\mu\nu} - k_\mu k_\nu)(\delta^{AB}\Pi_{1a}(k) + \delta^{A0}\delta^{B0}\Pi_{1b}(k)) + \frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\delta^{A0}\delta^{B0}\Pi_{2a}(k) \tag{2.79}$$

Comparing (2.79) with (2.50), we have just removed the gauge structure from $\Pi^{AB}_1(k)$ and $\Pi^{AB}_2(k)$ to give scalar functions:

$$\begin{aligned}\Pi_{1a}(k) =& \frac{N}{4(4\pi)^2}\int_0^1 dx\, \left((4-(1-2x)^2)\ln\left(\frac{\Delta_v}{4\pi\mu^2}\right) - (1-(1-2x)^2)\sum_f \ln\left(\frac{\Delta_f}{4\pi\mu^2}\right)\right. \\ &\left. -\frac{1}{2}(1-2x)^2\sum_s \ln\left(\frac{\Delta_s}{4\pi\mu^2}\right)\right)\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}\Pi_{1b}(k) =& \frac{N}{2(4\pi)^2}\int_0^1 dx\, \left((4-(1-2x)^2)K_0(\sqrt{\Delta_v}|\tilde{k}|) - (1-(1-2x)^2)\sum_f K_0(\sqrt{\Delta_f}|\tilde{k}|)\right. \\ &\left. -\frac{1}{2}(1-2x)^2\sum_s K_0(\sqrt{\Delta_s}|\tilde{k}|)\right)\end{aligned} \tag{2.81}$$

and

$$\Pi_{2a}(k) = -\frac{N}{8(4\pi)^2}\int_0^1 dx \left[\Delta_v K_2(\sqrt{\Delta_v}|\tilde{k}|) - \sum_f \Delta_f K_2(\sqrt{\Delta_f}|\tilde{k}|) + \frac{1}{2}\sum_s \Delta_s K_2(\sqrt{\Delta_s}|\tilde{k}|)\right] \quad (2.82)$$

The indices v, f and s are used to distinguish between contributions from the gauge bosons (and ghosts), fermions and scalars respectively. The mass appearing in Δ_v will be zero but the masses appearing in the sums over fermions and scalars are general, for example there is no requirement that all the fermions have the same mass. The sum for the scalars is in terms of real scalars, for each complex scalar in the theory there will be two terms in the sum.

2.5.5 The Running of the Coupling

Using the expression that we have arrived at for the polarisation tensor, we can calculate the running of the coupling using equation (2.51) which we reproduce below:

$$\left[\frac{1}{g^2_{eff}(k)}\right]^{AB} = \frac{\delta^{AB}}{g^2_{micro}} + 4\Pi_1^{AB}(k) \quad (2.83)$$

In a commutative theory the polarisation tensor would contain only $\Pi_{1a}(k)$ (equation (2.80)). and indeed in noncommutative theories, at high energy scales (compared to the energy scale of noncommutativity $M_{NC} \sim \sqrt{\frac{1}{\theta}}$) the coupling runs in an equivalent fashion to a commutative theory. At low energies, $\Pi_{1b}(k)$ (equation (2.81)) dominates, however it only affects the

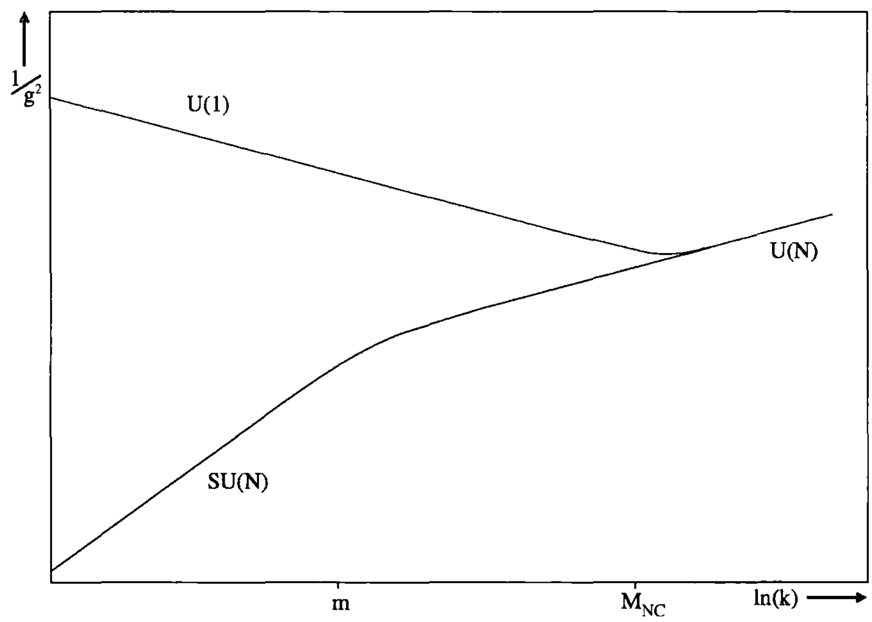


Figure 2.3: *The running of the couplings in a softly-broken N=1 theory with one chiral multiplet*

$U(1)$ degrees of freedom (see equation (2.79)). $\Pi_{1b}(k)$ encapsulates the new logarithmic divergences ($K_0(z) = -\ln(\frac{z}{2})$....) and it causes the affected gauge bosons to decouple [27, 28].

Figure (2.3) shows how the coupling varies in a softly-broken $\mathcal{N} = 1$ theory with one chiral multiplet, where m denotes a typical mass for the adjoint matter. As can be seen from the graph, in the infrared regime of the theory, the gauge bosons associated with the U(1) have decoupled. This a central point of this calculation and the rest of this thesis relies upon it; logarithmic divergences caused by UV/IR mixing cause a dynamical breaking of the $U(N)$ gauge invariance, down to $SU(N)$. It will aid us in model building, allowing the Standard Model gauge group to arise naturally from our noncommutative models. This is the extremely useful effect of the logarithmic divergences but we continue our investigation, examining the quadratic divergences that might spoil this elegant picture.

Note that although the running of the $SU(N)$ coupling is not shown intercepting the x-axis in the infra-red regime of figure 2.3, in a generic theory it will do. Having the coupling go to infinity like this does not signal that the theory is unphysical, it indicates a region of strong coupling where our perturbative calculations fail. In some theories, for instance the electroweak theory in the Standard Model, the gauge bosons acquire a mass (via the Higgs mechanism described in chapter 4) at an energy above the strong coupling regime and the coupling freezes out at this scale.

2.5.6 Quadratic Divergences

The quadratic divergences are encapsulated in $\Pi_{2a}(k)$ (equation (2.82)) and in order to study their effects we use our expression for the vacuum polarisation tensor which corresponds to the sum of all 1PI insertions into the propagator to calculate the full propagator for the gauge bosons which will be the infinite sum over all possible numbers of insertions of the vacuum polarisation function. This section has not been written anywhere in the literature for the $U(N)$ case, however it is a trivial extension of the case of $U(1)$ theories, described in [33, 34].

If we introduce the following projectors:

$$
\begin{aligned}
P^{AB}_{\mu\nu} &= \delta^{AB}(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \\
Q^{AB}_{\mu\nu} &= \delta^{A0}\delta^{B0}(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \\
R^{AB}_{\mu\nu} &= \delta^{A0}\delta^{B0}(\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2})
\end{aligned}
\tag{2.84}
$$

then with a slight abuse of matrix notation we can write:

$$
\mathbf{P}^n = P^{AB}_{\alpha\beta} P^{BC}_{\beta\gamma} ... P^{GH}_{\lambda\mu} P^{HI}_{\mu\nu} = P^{AI}_{\alpha\nu} = \mathbf{P} \tag{2.85}
$$

and similarly,

$$
\mathbf{Q}^n = \mathbf{Q} \qquad \mathbf{R}^n = \mathbf{R} \qquad \mathbf{PR} = \mathbf{RP} = \mathbf{QR} = \mathbf{RQ} = \mathbf{R} \quad \text{and} \quad \mathbf{PQ} = \mathbf{QP} = \mathbf{Q}\,. \tag{2.86}
$$

Using these projectors we can write (2.79) as:

$$
\mathbf{\Pi}(k) = k^2 \Pi_{1a}(k)\mathbf{P} + k^2 \Pi_{1b}(k)\mathbf{Q} + \Pi_{2a}(k)\mathbf{R} \tag{2.87}
$$

and thus:

$$
\begin{aligned}
\mathbf{\Pi}(k)^n &= (k^2\Pi_{1a}(k))^n \, \mathbf{P} + (k^2)^n[(\Pi_{1a}(k) + \Pi_{1b}(k))^n - \Pi_{1b}(k)^n]\mathbf{Q} \\
&+ \left[(k^2\Pi_{1a}(k) + k^2\Pi_{1b}(k) + \Pi_{2a}(k))^n - (k^2\Pi_{1a}(k) + k^2\Pi_{1b}(k))^n\right] \mathbf{R}\,.
\end{aligned}
\tag{2.88}
$$

The full propagator is:

$$G^{AB}_{\mu\nu}(k) = \frac{ig_0^2}{k^2}\left[1 + \frac{-g_0^2}{k^2}\mathbf{\Pi}(k) + \left(\frac{-g_0^2}{k^2}\right)^2 \mathbf{\Pi}(k)^2 \ldots\right]^{AB}_{\mu\nu} \tag{2.89}$$

So, summing the series we find:

$$G^{AB}_{\mu\nu}(k) = \frac{ig_0^2 k_\mu k_\nu}{k^2}\delta^{AB} + (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\left[\frac{ig_0^2(\delta^{AB} - \delta^{A0}\delta^{B0})}{k^2(1 + g_0^2\Pi_{1a}(k))} + \frac{ig_0^2\delta^{A0}\delta^{B0}}{k^2(1 + g_0^2(\Pi_{1a}(k) + \Pi_{1b}(k))}\right]$$
$$+ig_0^2\delta^{A0}\delta^{B0}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\left[\frac{1}{1 + g_0^2\left(\Pi_{1a}(k) + \Pi_{1b}(k) + \frac{\Pi_{2a}(k)}{k^2}\right)} - \frac{1}{1 + g_0^2\left(\Pi_{1a}(k) + \Pi_{1b}(k)\right)}\right] \tag{2.90}$$

The pole conditions that arise from this propagator are dependent on whether we are considering the U(1) degrees of freedom (which correspond to $G^{00}_{\mu\nu}$). For such gauge bosons the pole conditions are:

$$k^2\left(\frac{1}{g_0^2} + \Pi_{1a}(k) + \Pi_{1b}(k)\right) = 0 \quad \text{and} \quad k^2\left(\frac{1}{g_0^2} + \Pi_{1a}(k) + \Pi_{1b}(k)\right) + \Pi_{2a}(k) = 0 \tag{2.91}$$

which are (allowing for the change in notation) the same conditions that were found for the gauge boson in a U(1) theory [33, 34].

However the other $N^2 - 1$ degrees of freedom have a simpler condition:

$$k^2\left(\frac{1}{g_0^2} + \Pi_1(k)\right) = 0 \tag{2.92}$$

We see from equation (2.92) that the gauge bosons which interact at low energies are massless however the gauge boson associated with U(1) has acquired a mass. In order to determine whether the mass is tachyonic, we use the small argument expansion, $K_2(z) = 2z^{-2} - \frac{1}{2} + \ldots.$ and we can write

$\Pi_{2a}(k)$ as:

$$\Pi_{2a}(k) = \frac{N}{8(4\pi)^2}\left[(-1+n_f-\frac{1}{2}n_s)(\frac{2}{\tilde{k}^2}-\frac{k^2}{12})-\frac{1}{2}(\sum_f m_f^2-\frac{1}{2}\sum_s m_s^2)\right] \tag{2.93}$$

The dispersion relations for physical particles are obtained by rotating into Minkowski space. We are considering only space-space noncommutativity so that then we can write:

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.94}$$

and rotation into Minkowski space then requires:

$$p^2 \to -p^2 \qquad \text{and} \qquad \tilde{p}^2 \to \theta^2(p_1^2+p_2^2) \tag{2.95}$$

In theories with supersymmetry or softly broken supersymmetry then $-1+n_f-\frac{1}{2}n_s = 0$ which eliminates the infra-red quadratic divergences and in order to avoid tachyons we find that the masses must satisfy the following condition[4]:

$$\sum_f m_f^2 - \frac{1}{2}\sum_s m_s^2 \leqslant 0 \tag{2.96}$$

As long as equation (2.96) is satisfied then we find, as already outlined that our microscopic, noncommutative U(N) looks, in the infrared, like a commutative SU(N) theory, the gauge boson degrees of freedom associated with

[4]When we consider broken gauge groups the tree level masses of the gauge bosons will contribute with the same sign as the masses of the scalars

the U(1) have become massive but they have decoupled from the rest of the theory and are unobservable.

2.5.7 More General Theories

Our calculation has been performed in a theory with only adjoint matter, in order to consider more realistic theories, it is necessary to generalise the results and we present a summary of the consequences here.

First, one can include N_f flavours of matter fields transforming in the fundamental representation[5] of the gauge group, and, second, the gauge group $U(N)$ can be spontaneously broken to $U(n)$ at a scale m, so that some of the gauge bosons and matter fields become massive.

In the UV region, the theory is a noncommutative $U(N)$ and there is a single coupling constant,

$$\frac{(4\pi)^2}{g^2_{U(N)}(k)} \to b_0^{U(N)\,N_f} \log k^2 , \qquad \text{as } k^2 \to \infty . \tag{2.97}$$

Here $b_0^{U(N)\,N_f}$ is the 1-loop coefficient of the beta function of the microscopic $U(N)$ theory with N_f fundamental flavours.[6] In the IR region, two things happen: the trace $U(1)$ factor decouples below the noncommutative mass, and also, all the massive degrees of freedom freeze at momentum scales below

[5] We recall that the only representations allowed in a noncommutative gauge theory are adjoint, (anti-)fundamental and bi-fundamental ones. As we have already included adjoint representations, and since bi- and anti-fundamental representations are essentially the same as fundamental ones, to cover the general case it is sufficient to add just fundamental representations.

[6] $b_0^{U(N)\,N_f}$ takes the same value as in the corresponding commutative $SU(N)$ theory.

their masses,

$$\frac{1}{g^2_{U(N)}(k)} \to \frac{1}{g^2_{U(1)}(k)} \oplus \frac{1}{g^2_{SU(n)}(k)} \mathbb{1}_{[n^2-1]\times[n^2-1]} \tag{2.98}$$

where

$$\frac{(4\pi)^2}{g^2_{SU(n)}(k)} \to b_0^{SU(n)} \log k^2 , \tag{2.99}$$

$$\frac{(4\pi)^2}{g^2_{U(1)}(k)} \to -(2b_0^{U(N)\,N_f=0} - b_0^{U(N)\,N_f}) \log k^2 , \qquad \text{as } k^2 \to 0 . \tag{2.100}$$

The UV/IR mixing affects only the $U(1)$ coupling and, hence, the first equation (2.99) takes the standard commutative and recognisable form. However, the $U(1)$ coupling is affected by the UV/IR mixing and leads to the slope in the IR given by $-2b_0^{U(N)\,N_f=0}+b_0^{U(N)\,N_f}$ as follows from (2.100). This expression for the slope follows the fact that the θ-dependent phase factors cancel in Feynman diagrams involving fundamental fields propagating in the loop [22] and do not cancel for adjoint fields in the loop.

Running couplings $\frac{1}{g^2(k)}$ of noncommutative $U(1)$ (supersymmetric) theories were first derived and plotted over the full range of the momentum scale k in [27]. Our expressions in (2.97),(2.100) are in agreement with those results in the asymptotic regions $k^2 \to \infty$ and $k^2 \to 0$. It should be noted that expressions such as (2.100) are valid in the extreme infrared, at finite values of k^2 comparable to various mass scales in the theory, the coupling changes slope.

Adding fundamental matter to the model will not affect the conclusions concerning non-logarithmic UV/IR mixing effects which are controlled by matter in the adjoint representation. These effects are rendered harmless as long as two conditions are met. Firstly, for each fermion in the adjoint representation, there is a gauge field or complex scalar that is also in the adjoint representation (and vice-versa). This condition removes quadratic IR divergences from the $U(1)$ polarisation tensor. Secondly the sum of the mass squared for the adjoint fermions must be less than or equal to the sum of the mass squared for the complex scalars and the gauge bosons. If this condition is satisfied, there are no tachyons in the decoupled $U(1)$ gauge sector.

2.5.8 Origins of UV/IR mixing

Following [29] we consider whether there is an intuitive reason that the introduction of noncommutativity should cause low-energy physics to be affected by high energy degrees of freedom. In order to make the calculations simpler we will temporarily abandon gauge theories and turn instead to the simpler case of ϕ^3 theory (and discuss the implication for gauge theories afterwards). Our Euclidean action is then:

$$S = \int d^dx \, \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi * \phi * \phi \tag{2.101}$$

The root cause of the mixing is that the star product causes high energy, localised wave packets to interact with each other even when well separated, i.e. they have long distance effects.

The star product (2.2) of two functions of position, z can be rewritten as [35]:

$$(\phi_1 * \phi_2)(z) = \int d^dz_1 d^dz_2 \phi_1(z_1)\phi_2(z_2)K(z_1, z_2, z) \tag{2.102}$$

where

$$K(z_1, z_2, z) = \frac{1}{\det(\theta)} e^{2i(z-z_1)^\mu \theta^{-1}_{\mu\nu}(z-z_2)^\nu} \tag{2.103}$$

$|K(z_1, z_2, z)|$ does not depend on z_1, z_2 or z, if oscillations of the phase of K did not damp the integral the $*$-product would be infinitely non-local. In order to investigate further, we use two dimensions, $z^\mu = (x, y)$ so $[x, y] = i\theta$.

If we assume that ϕ_i is slowly varying over a region of size $\Delta_{x_i} \times \Delta_{y_i}$ then the integral over x_1 will be proportional to:

$$\int dx_1\, \phi_1(x_1, y_1) e^{\frac{-ix_1(y-y_2)}{\theta}} \quad , \tag{2.104}$$

and will therefore be suppressed if

$$\frac{\Delta_{x_1}|y - y_2|}{\theta} \gg 1 \tag{2.105}$$

i.e. the x_1 integral will be non-zero at points at a distance, δy_2 less than or equal to $\frac{\theta}{\Delta_{x_1}}$ away from the non-zero region of ϕ_2 in the y-direction. For all the integrals to be non-zero we therefore require that:

$$\delta y_2 \Delta_{x_1} \approx \delta y_1 \Delta_{x_2} \approx \delta x_2 \Delta_{y_1} \approx \delta x_1 \Delta_{y_2} \approx \theta \tag{2.106}$$

Figure 2.4 shows two localised fields, ϕ_1 and ϕ_2 which do not overlap, the star product of these fields is non-zero in a region between the two fields.

In the case where $\phi_1 = \phi_2 = \phi$ and ϕ has widths Δ_x, Δ_y the widths δ_x and δ_y of $\phi * \phi$ are:

$$\delta_x \approx \max\left(\Delta_x, \frac{\theta}{\Delta_y}\right) \text{and} \delta_y \approx \max\left(\Delta_y, \frac{\theta}{\Delta_x}\right) \tag{2.107}$$

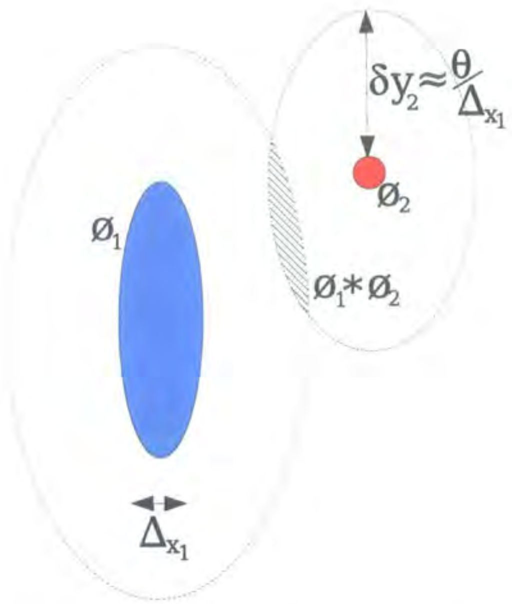


Figure 2.4: *The hatched region shows where $\phi_1 * \phi_2$ is non-zero if the fields are non-zero in the shaded regions*

So that if ϕ is nonzero in a region much smaller than the noncommutative scale, $\sqrt{\theta}$ then $\phi * \phi$ will be nonzero over a much bigger region, larger than the noncommutative scale.

The classical equation of motion for ϕ^3 theory is:

$$(\partial^2 - m^2)\phi(x) = \frac{g}{2}(\phi * \phi)(x) \quad . \tag{2.108}$$

and the perturbative solution is:

$$\phi(x) = \phi_0(x) - \frac{g}{2}\int d^d y G(x-y)(\phi_0 * \phi_0)(y) + ... \tag{2.109}$$

where $\phi_0(x)$ is the solution for the free field and $G(x)$ is a Green's function.

So that when a wave packet which was confined to a very small region of size $\delta \ll \theta$ interacts it is spread over a region of size $\frac{\theta}{\delta} \gg \theta$.

If we now consider particles running in virtual loops, a particle of energy

$E \gg \frac{1}{\sqrt{\theta}}$ (size $\frac{1}{E} \ll \sqrt{\theta}$) spreads to a size θE (energy $\frac{1}{\theta E}$). Therefore if we add a UV cut-off, removing particles with $E > \Lambda$ then we will have implicitly created an infra-red cut-off $\Lambda_{IR} = \frac{1}{\theta\Lambda}$.

This is a useful intuitive picture but the generalisations to gauge theory is not straight-forward. Firstly the regions that our fields are non-zero in will alter under gauge transformation although this can be circumvented by choosing a particular gauge and performing the calculation in that. It is also impossible to construct local operators as we will see in the next section (although creating operators spread over a small region will be possible). Also our theory may become strongly coupled in the infra-red and therefore we would need to consider different degrees of freedom. Despite these complications the "thumb-nail" sketch we have drawn in this section provides an intuitive way of viewing UV/IR mixing, the core idea of which translates into gauge theories.

2.6 Local Operators and Wilson lines

As might be expected in a noncommutative theory, writing down local operators is not possible. In commutative theories, local gauge invariant operators can be constructed by taking the trace of a gauge covariant object, but in a noncommutative theory, the cyclic property of a trace is not applicable unless the fields are integrated over all space.For example,

$$\mathrm{tr} \int d^d x F_{\mu\nu} * F_{\mu\nu} \tag{2.110}$$

would no longer be gauge invariant without the space time integral.

In order to circumvent this problem it is possible to use Wilson lines [37, 39]. Wilson lines are comparators, they can be used to compare fields at different points by cancelling out the difference in gauge rotations between the two points. In a commutative theory, only closed Wilson loops are gauge-invariant. In a noncommutative theory the picture is somewhat different.

The formula for a straight, open Wilson line that runs from x to $x + \tilde{p}$ is:

$$W(x) = P_* e^{\int_0^1 d\sigma \tilde{p}^\mu A_\mu(x+\tilde{p}\sigma)} \tag{2.111}$$

where the P_* denotes path-ordering so that

$$W(x) = 1 + \sum_{n=1}^{\infty} \int_0^1 d\sigma_1 \int_{\sigma_1}^1 d\sigma_2 ... \int_{\sigma_{n-1}}^1 d\sigma_n \tilde{p}^{\mu_1} ... \tilde{p}^{\mu_n} A_{\mu_1}(x+\tilde{p}\sigma_1) * ... * A_{\mu_n}(x+\tilde{p}\sigma_n) \tag{2.112}$$

Under a gauge transformation, the Wilson line transforms as:

$$W(x) \rightarrow U(x) * W(x) * U(x+\tilde{p})^\dagger \tag{2.113}$$

If we Fourier transform this operator:

$$W(p) = \int d^4x \, Tr W(p) * e^{ipx} \quad , \tag{2.114}$$

then under a gauge transformation we have:

$$W(p) \rightarrow \int d^4x \, U(x) * W(x) * U(x+\tilde{p})^\dagger * e^{ipx} \quad . \tag{2.115}$$

We can now use the rather surprising result that in noncommutative theories,

e^{ipx} is a translation operator, i.e.:

$$e^{ipx} * f(x) = f(x + \tilde{p}) * e^{ipx} \tag{2.116}$$

Although equation (2.116) might seems surprising, we can prove it for an arbitrary plane wave, e^{ikx} directly from (2.7). This result means that in non-commutative quantum field theories, spacetime translations are a subgroup of the gauge transformation and it has also been shown that rotations are a subgroup of gauge transformations as well [38].

Using (2.116) gives:

$$W(k) \rightarrow \int d^4x \, U(x) * W(x) * e^{ipx} * U(x + \tilde{p} - \tilde{p})^\dagger = W(k) \qquad , \tag{2.117}$$

and we see that a straight, open Wilson line of length $\tilde{p}$ is, in fact, a gauge invariant object. If we had not constrained the shape of the Wilson lines we would have found that any shaped Wilson line with its end points separated by a distance $\tilde{p}$ would have be gauge invariant. In the commutative limit, $\theta \rightarrow 0$ the separations between the end-points goes to zero and we would have found that there are no gauge invariant Wilson lines except Wilson loops.

We can now build "semi-local" operators by attaching them to one end of a straight Wilson line of length $\tilde{p}$. It was pointed out in [39] that the set of all local operators attached to all shapes of Wilson lines is an over-complete set and that, for straight Wilson lines we get the same operator irrespective of where on the line the operator is attached.

2.7 Summary of the chapter

We have now completed our brief overview of noncommutative field theories and have seen that they display many novel properties which we summarise here for easy reference.

- Firstly, in noncommutative quantum field theories, we can not work with $SU(N)$ groups, in order for the group to close we need to add an extra generator and have instead, $U(N)$.
- We have also seen that there are restrictions on the allowed representations and charges of the fields. The only allowed representations are fundamental, anti-fundamental, bi-fundamental and adjoint. Fields transforming under a noncommutative $U(1)$ group cannot have arbitrary charges. Only 0 and ± 1 are allowed.
- Although the propagators in a noncommutative theory are the same as their commutative counter parts, the vertices acquire extra factors of the form $e^{ik\tilde{p}}$ where k is an external momenta and p is a loop momentum. These extra factors cause a mixing between the physics at high-energies and the physics at low energies (UV/IR) that is unique to noncommutative theories.
- The UV/IR mixing effects can cause a new class of quadratic divergences in the infra-red, these can be avoided if there are the same number of fermionic and bosonic degrees of freedom in the adjoint representation. For example, supersymmetric noncommutative gauge theories do not have these divergences. Tachyonic masses can also arise

but these can be avoided if the masses of fields in the adjoint representation satisfy an inequality (specified in equation (2.96))

- The UV/IR mixing also causes logarithmic divergences to arise in the infra-red of noncommutative theories. These divergences arise even in theories where the quadratic divergences cancel. If the gauge group of a noncommutative theory is $U(1)$, these new divergences will cause the theory to decouple in the infra-red. If the gauge group is U(N) then at low energies the group will be broken to $SU(N)$ and a decoupled $U(1)$.

- Whilst local operators can not be written down, we have shown that operators do not have to extend over all space, if open Wilson lines (with a length depending on the momentum of the operator) are attached, then "semi-local" operators can be constructed.

In the next chapter these open Wilson lines will be employed, as we turn our attention to effective actions and write down an explicitly gauge invariant action which models the logarithmic divergences that we have seen arise as a result of the UV/IR mixing.

Chapter 3

Effective Actions

He didn't meddle with the fabric of time and space, and they kept out of his greenhouses. The way [Modo] saw it, it was a partnership.

"The Hog Father" by Terry Pratchett

3.1 Introduction

In the previous chapter we discussed the ultra-violet/infra-red mixing and saw that in $U(N)$ theories where the induced quadratic divergences were not present the effect of the logarithmic divergences was to alter the running of the coupling of the trace $U(1)$ so that it decouples in the infra-red (see figure 2.3).

In the infra-red, the leading order terms in the derivative expansion of the Wilsonian effective action are:

$$S_{\text{eff}} \ni \frac{1}{4g_1^2(k)} \int d^4x\, F_{\mu\nu}^{U(1)} F_{\mu\nu}^{U(1)} + \frac{1}{4g_N^2(k)} \int d^4x\, \mathrm{Tr}(F_{\mu\nu}^{SU(N)} F_{\mu\nu}^{SU(N)}) , \quad (3.1)$$

In this chapter we will show that low energy actions can be constructed that contains these terms dressed with higher-order terms so that the full $U(N)$ gauge invariance is present.

Given that the couplings for the $SU(N)$ and the $U(1)$ pieces run so differently, allowing the gauge bosons to be easily distinguished in experiment, it might be considered surprising that a fully $U(N)$ gauge invariant noncommutative action can be written down as we will show. Surprising as it is, such an action can be written down. Instead of the symmetry being broken at the level of the action there is a dynamical breaking of symmetry. Prior to the calculation outlined in this chapter there was already a growing body of evidence that this is the case.

In the special case of $N = 4$ supersymmetry it was shown [40] (see also [41–43]) that one-loop effective actions were gauge invariant. The large number of symmetries in the theory make the calculations more tractable but the essential feature, UV/IR mixing is still present.

The authors of refs. [44,45] showed that the quadratic divergences that arise in noncommutative quantum theories with a generic particle content could be reproduced in an effective action using terms that contained open Wilson lines. In particular, section (3.1) of [45] introduced an interesting and simple method of evaluating the quadratically divergent contributions of generic n-point functions in a U(1) non-supersymmetric theory but it seems difficult to extend such methods to the logarithmic divergences.

It is the logarithmic divergences that will interest us; any phenomenologically acceptable model needs to be arranged such that the quadratic divergences are cancelled. It was conjectured in [45] that the logarithmic divergences can be reproduced by the following $U(N)$ gauge invariant term:

$$S_{\rm eff}^{(1)} = -\frac{\mathcal{C}}{4} \int \frac{d^4p}{(2\pi)^4} \mathcal{O}^{\mu\nu}(-p)\, T(p)\, \mathcal{O}_{\mu\nu}(p) \ , \tag{3.2}$$

where the gauge-invariant operator $\mathcal{O}_{\mu\nu}$ is defined by

$$\mathcal{O}_{\mu\nu}(p) = {\rm Tr} \int d^4x \ P_* \left(F_{\mu\nu}(x)\, e^{\int_0^1 d\sigma\, \tilde{p}^\mu A_\mu(x+\tilde{p}\,\sigma)} \right) * e^{ipx} \ , \tag{3.3}$$

and P_* stands for integration along the open Wilson line together with path ordering with respect to the star-product [46]. The matching of (3.2) with the analytic results [27] for the $U(1)$ effective coupling constant $1/g_1^2(k)$ of (3.5) determines the function $T(p)$ and the numerical constant $\mathcal{C}$ appearing in (3.2). One easily finds that

$$T(p) = \frac{2}{(4\pi)^2} \int_0^1 dx\, K_0(\sqrt{x(1-x)}\, |p||\tilde{p}|) \ , \tag{3.4}$$

where K_0 is a Bessel function, with $K_0(z) \longrightarrow -\log(z/2)$ as $z \longrightarrow 0$. Moreover, $\mathcal{C} = 2\alpha_0/N$, where α_0 is defined by the asymptotic behaviour of the running of the $U(1)$:

$$\frac{1}{g_1^2(k)} \longrightarrow \pm \frac{\alpha_0}{(4\pi)^2} \log k^2 \ , \tag{3.5}$$

where the plus (minus) sign corresponds to $k^2 \longrightarrow \infty$ $(k^2 \longrightarrow 0)$.

In this chapter we would like confirm the interesting conjecture of [45] with an explicit field theory calculation. We restrict our attention to supersymmetric theories to simplify the calculations. This simplification will not prevent our conclusions from applying to any physical noncommutative quantum field theory. As we saw in the second chapter, for each fermionic degree of freedom relevant to UV/IR mixing (i.e. in the adjoint representation) there must be an equivalent bosonic degree of freedom or the theory becomes unphysical.

To start probing the presence of the Wilson line operator in (3.3) we need to calculate an n-point function of gauge fields with $n \geq 3$; for this reason, we will concentrate on the cases of three- and four-point functions of background fields. Our formalism is, however, general, and allows in principle to calculate generic n-point correlators. We will focus our attention on a generic $\mathcal{N} = 1$ supersymmetric field theory with N_f adjoint chiral multiplets and make use of the background field method. The case $N_\mathrm{f} = 0$ corresponds to pure $\mathcal{N} = 1$ super Yang-Mills, whereas for $N_\mathrm{f} = 1, 3$ we have the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories, respectively. The results of our computations confirm the presence of the term (3.2) in the effective action. However, our results also show that we need to include another term $S^{(2)}_{\mathrm{eff}}$ in the effective action, which can be written as

$$S^{(2)}_{\mathrm{eff}} = \frac{\mathcal{C}}{2} \int \frac{d^4 p}{(2\pi)^4} \, \mathcal{O}_{F^2}(p) \; \mathcal{W}'(-p) \, T(p) \; , \tag{3.6}$$

where

$$\mathcal{O}_{F^2}(p) = \mathrm{Tr} \int d^4 x \; P_* \left(F_{\mu\nu}(x) F^{\mu\nu}(x) e^{\int_0^1 d\sigma \, \tilde{p}^\mu A_\mu (x + \tilde{p}\sigma)} * e^{ipx} \right) \quad , \tag{3.7}$$

and $\mathcal{W}'(p)$ denotes the open Wilson line operator $\mathcal{W}(p)$ with the $\mathcal{O}(A^0)$ term subtracted, where, in our conventions

$$\mathcal{W}(p) = \mathrm{Tr} \int d^4 x \; P_* \left(e^{\int_0^1 d\sigma \, \tilde{p}^\mu A_\mu (x + \tilde{p}\,\sigma)} \right) * e^{ipx} \; . \tag{3.8}$$

So that $\mathcal{W}'(p)$ is:

$$\mathcal{W}'(p) = \mathrm{Tr} \int d^4 x \; P_* \left(e^{\int_0^1 d\sigma \, \tilde{p}^\mu A_\mu (x + \tilde{p}\,\sigma)} - 1 \right) * e^{ipx} \; . \tag{3.9}$$

$S^{(2)}_{\rm eff}$ is manifestly gauge invariant, and again contains open Wilson lines.

The appearance of the term $S^{(2)}_{\rm eff}$ in (3.6) is not entirely unexpected. Indeed, similar extra contributions were predicted for the quadratic divergence in [47], where a Wilsonian calculation of the effective action was performed using the matrix model approach to noncommutative gauge theories.

Similar results were also obtained in [48] using the bosonic world-line approach. It would be interesting to apply the bosonic worldline approach, used in [48] for non-supersymmetric theories, to the case of supersymmetric theories considered here, and see if that formalism would lead to more tractable expressions than those obtained using conventional background perturbation theory.

In the first version of the paper [49] that this chapter is based upon, the four-point calculation had not yet been performed and we proposed to use the following expression, instead of (3.6):

$$\tilde{S}^{(2)}_{\rm eff} = -i\,\frac{C}{4}\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\,\mathcal{Q}(-q_1,-q_2)\;\theta^{\mu\nu}\mathcal{O}_{\mu\nu}(q_1+q_2)\,T(q_1+q_2)\ , \quad (3.10)$$

where

$$\mathcal{Q}(p,q) = {\rm Tr}\int d^4x\; L_*\left(F_{\mu\nu}(x)F^{\mu\nu}(x)e^{\int_0^1 d\sigma\,\tilde{p}^\mu A_\mu(x+\tilde{p}\sigma)} * e^{ipx}\right)\,\frac{\sin(q\tilde{p}/2)}{q\tilde{p}}\ . \quad (3.11)$$

The interaction term in (3.10), though manifestly gauge invariant, is not satisfactory as it stands, since $\theta^{\mu\nu}$ appears in (3.10) not only inside the Wilson lines but also explicitly. This, in turn, would render its interpretation from the D-brane perspective very difficult. It is easy to check that, once we expand the Wilson line in (3.6) up to $\mathcal{O}(A)$, this term contributes to the three-point function in the same way as the term in (3.10) does. Indeed, we will show that (3.6) and (3.10) both produce a contribution to the three-point function which is in precise agreement with the direct calculation in the microscopic theory. Of course, at the level of four-point functions (3.6) and (3.10) start producing contributions which are different. By comparing the perturbative result for a *four*-point function to the corresponding result derived from the effective action, we will be able in the next sections to confirm that (3.6) is the correct expression to be incorporated in the effective action (rather than (3.10)), as also suggested by D-brane physics [47].

The plan of the rest of this chapter is as follows. In the next section we will obtain the contributions to the three- and four-point functions of gauge fields from the terms $S_{\text{eff}}^{(1)}$ and $S_{\text{eff}}^{(2)}$ in the effective action, Eqs. (3.2) and (3.6), respectively. Using the background field method, we will then go on to calculate the three- and four-point functions of background fields. We wll then finally compare the perturbative results we derived to the result obtained from the effective action, finding agreement.

3.2 Three- and four-point functions from an effective action with Wilson lines

3.2.1 The three-point function

We begin by calculating the contribution from the effective interaction in (3.2) to the three-point function

$$\Gamma^{ABC}_{\mu\nu\rho}(k_1, k_2, k_3) := \int \prod_{i=1}^{3} d^4x_i \; e^{i\sum_{i=1}^{3} k_i x_i} \, \langle \, A^A_\mu(x_1) \, A^B_\nu(x_2) \, A^C_\rho(x_3) \, \rangle \; . \quad (3.12)$$

In order to calculate the contribution from (3.2) we need only to expand the expression for $\mathcal{O}_{\mu\nu}(p)$ in (3.3) up to order A^2. We then Fourier transform and use (2.47) and (2.2), to get:

$$\mathcal{O}_{\mu\nu}(p) = l_{\mu\nu}(p) + q^{(1)}_{\mu\nu}(p) + q^{(2)}_{\mu\nu}(p) + \mathcal{O}(A^3) \ , \tag{3.13}$$

where

$$\begin{aligned} l_{\mu\nu}(p) &= -\sqrt{\frac{N}{2}} \ \left[p_\mu A^0_\nu(p) - p_\nu A^0_\mu(p)\right] \ , \\ q^{(1)}_{\mu\nu}(p) &= i \int \frac{d^4q}{(2\pi)^4} \, A^A_\mu(q) A^A_\nu(p-q) \ \sin\frac{q\tilde{p}}{2} \ , \\ q^{(2)}_{\mu\nu}(p) &= i \int \frac{d^4q}{(2\pi)^4} \ \left[q_\mu A^A_\nu(q) - q_\nu A^A_\mu(q)\right] (\tilde{p}\cdot A^A(p-q)) \ \frac{\sin(q\tilde{p}/2)}{q\tilde{p}} \ . \end{aligned} \tag{3.14}$$

Using (3.13), we get the following contribution to the three-point function from the effective action $S^{(1)}_{\text{eff}}$ of (3.2):

$$\begin{aligned} \Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3)\big|_{S^{(1)}_{\text{eff}}} &= i\mathcal{C}\sqrt{\frac{N}{2}} \frac{\sin(k_2\tilde{k}_3/2)}{k_2\tilde{k}_3} \ (2\pi)^4 \, \delta^{(4)}(k_1+k_2+k_3) \\ & \cdot \Big\{ T(k_1) \ \delta^{A0}\delta^{BC} \left[\tilde{k}_{1\nu}P^{13}_{\rho\mu} + \tilde{k}_{1\rho}P^{12}_{\nu\mu} + (k_2\tilde{k}_3)\, Q^1_{\mu,\nu\rho}\right] \\ & +T(k_2) \ \delta^{B0}\delta^{CA} \left[\tilde{k}_{2\mu}P^{23}_{\rho\nu} + \tilde{k}_{2\rho}P^{12}_{\nu\mu} + (k_2\tilde{k}_3)\, Q^2_{\nu,\rho\mu}\right] \\ & +T(k_3) \ \delta^{C0}\delta^{AB} \left[\tilde{k}_{3\mu}P^{23}_{\rho\nu} + \tilde{k}_{3\nu}P^{13}_{\rho\mu} + (k_2\tilde{k}_3)\, Q^3_{\rho,\mu\nu}\right] \Big\} \ , \end{aligned} \tag{3.15}$$

where we have defined

$$P^{ij}_{\mu\nu} \ := \ \delta_{\mu\nu}(k^i\cdot k^j) \ - \ k^i_\mu k^j_\nu \ , \qquad Q^i_{\rho,\mu\nu} \ := \ \delta_{\rho\mu}\, k^i_\nu \ - \ \delta_{\rho\nu}\, k^i_\mu \ . \tag{3.16}$$

In a similar way we can calculate the contribution to the three-point function from the term $S_{\text{eff}}^{(2)}$ in (3.6), obtaining:

$$\Gamma_{\mu\nu\rho}^{ABC}(k_1,k_2,k_3)\big|_{S_{\text{eff}}^{(2)}} = iC\sqrt{\frac{N}{2}}\,\frac{\sin(k_2\tilde{k}_3/2)}{k_2\tilde{k}_3}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2+k_3) \\ \cdot\left\{\delta^{A0}\delta^{BC}\,T(k_1)\,\tilde{k}_{1\mu}P_{\rho\nu}^{23} + \delta^{B0}\delta^{AC}\,T(k_2)\,\tilde{k}_{2\nu}P_{\mu\rho}^{31} + \delta^{C0}\delta^{AB}\,T(k_3)\,\tilde{k}_{3\rho}P_{\nu\mu}^{12}\right\} . \tag{3.17}$$

3.2.2 The four-point function

In this section we compute the contributions to the four-point function obtained from the effective action $S_{\text{eff}}^{(1)}$ and $S_{\text{eff}}^{(2)}$ given in (3.2), (3.6), respectively. For the sake of simplicity we will restrict ourselves to the case of noncommutative $U(1)$ gauge group, and compute the four-point function

$$\Gamma_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4) := \int\prod_{i=1}^{4} d^4x_i\; e^{i\sum_{i=1}^{4}k_ix_i}\,\langle\, A_\mu(x_1)\,A_\nu(x_2)\,A_\rho(x_3)\,A_\sigma(x_4)\,\rangle\ . \tag{3.18}$$

The result for $\Gamma_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)$ is better expressed in terms of the quantities $J_2(k_1,k_2)$ and $J_3(k_1,k_2,k_3)$ introduced in [46] (see Appendix B of that paper), where

$$J_2(k_1,k_2) = \frac{\sin\frac{k_1\tilde{k}_2}{2}}{\frac{k_1\tilde{k}_2}{2}}\ , \tag{3.19}$$

$$J_3(k_1,k_2,k_3) = \frac{\sin\frac{k_2\tilde{k}_3}{2}\,\sin\frac{k_1(\tilde{k}_2+\tilde{k}_3)}{2}}{\frac{(k_1+k_2)\tilde{k}_3}{2}\;\frac{k_1(\tilde{k}_2+\tilde{k}_3)}{2}} + \frac{\sin\frac{k_1\tilde{k}_3}{2}\,\sin\frac{k_2(\tilde{k}_1+\tilde{k}_3)}{2}}{\frac{(k_1+k_2)\tilde{k}_3}{2}\;\frac{k_2(\tilde{k}_1+\tilde{k}_3)}{2}}\ . \tag{3.20}$$

Not surprisingly, the functions $J_2(k_1,k_2)$ and $J_3(k_1,k_2,k_3)$ [46] arise in the context of noncommutative effective action for the one-loop F^4 term in $\mathcal{N}=4$ super Yang-Mills.

In the same way as it was done for the three-point function, one finds that the contribution to the four-point function generated by the term (3.2) is given by the following expression:

$$\Gamma_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)|_{S^{(1)}_{\rm eff}} = (2\pi)^4\,\delta^{(4)}(k_1+\cdots+k_4)\Big\{\,2\,T(k_4) \tag{3.21}$$

$$\cdot\Big[\tilde k^4_\rho Q^4_{\sigma,\nu\mu}\sin\frac{k_1\tilde k_2}{2}\,J_2(k_3,k_4)+\tilde k^4_\nu Q^4_{\sigma,\rho\mu}\sin\frac{k_1\tilde k_3}{2}\,J_2(k_2,k_4)+\tilde k^4_\mu Q^4_{\sigma,\rho\nu}\sin\frac{k_2\tilde k_3}{2}\,J_2(k_1,k_4)\Big]$$

$$-T(k_4)\,J_3(k_1,k_2,k_3)\Big[P^{41}_{\mu\sigma}\,\tilde k^4_\nu\,\tilde k^4_\rho\;+\;P^{42}_{\nu\sigma}\,\tilde k^4_\mu\,\tilde k^4_\rho\;+\;P^{43}_{\rho\sigma}\,\tilde k^4_\mu\,\tilde k^4_\nu\Big]$$

$$-T(k_1+k_2)\Big[\delta_{\mu\rho}\delta_{\nu\sigma}\sin\frac{k_1\tilde k_2}{2}\sin\frac{k_3\tilde k_4}{2}+\frac{1}{2}P^{13}_{\rho\mu}(\tilde k_1+\tilde k_2)_\nu(\tilde k_3+\tilde k_4)_\sigma\,J_2(k_1,k_2)J_2(k_3,k_4)$$

$$+\frac{1}{2}Q^3_{\mu,\nu\rho}\sin\frac{k_1\tilde k_2}{2}(\tilde k_3+\tilde k_4)_\sigma\,J_2(k_3,k_4)\;+\;\frac{1}{2}\,Q^1_{\rho,\sigma\mu}\sin\frac{k_3\tilde k_4}{2}(\tilde k_1+\tilde k_2)_\nu\,J_2(k_1,k_2)\Big]$$

$$+\;\text{permutations}\Big\}\,.$$

Notice the appearance of the function $J_3(k_1,k_2,k_3)$ in the previous expression (3.21).

We now compute the contribution to the four-point function derived from the term (3.6). After some straightforward calculations, one gets:

$$\Gamma_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)|_{S^{(2)}_{\rm eff}} = (2\pi)^4\,\delta^{(4)}(k_1+\cdots+k_4)\Big\{\,2T(k_4)\,\tilde k^4_\sigma \tag{3.22}$$

$$\cdot\Big[-Q^1_{\mu,\rho\nu}\frac{k_2\tilde k_3}{2}J_2(k_2,k_3)\cos\frac{k_1\tilde k_4}{2}-Q^2_{\nu,\rho\mu}\frac{k_1\tilde k_3}{2}J_2(k_1,k_3)\cos\frac{k_2\tilde k_4}{2}-Q^3_{\rho,\mu\nu}\frac{k_2\tilde k_1}{2}J_2(k_1,k_2)\cos\frac{k_3\tilde k_4}{2}\Big]$$

$$+\frac{1}{2}\,P^{12}_{\nu\mu}\,\tilde k^4_\rho\,J_2(k_3,k_4)\,\cos\frac{k_1\tilde k_2}{2}\;+\;\frac{1}{2}\,P^{13}_{\rho\mu}\,\tilde k^4_\nu\,J_2(k_2,k_4)\,\cos\frac{k_1\tilde k_3}{2}\;+\;\frac{1}{2}\,P^{23}_{\rho\nu}\,\tilde k^4_\mu\,J_2(k_1,k_4)\,\cos\frac{k_2\tilde k_3}{2}\Big]$$

$$-\frac{1}{2}\Big[T(k_1+k_2)J_2(k_1,k_2)\cos\frac{k_3\tilde{k}_4}{2}(\tilde{k}_1+\tilde{k}_2)_\mu(\tilde{k}_1+\tilde{k}_2)_\nu P^{34}_{\sigma\rho}+$$

$$T(k_1+k_3)J_2(k_1,k_3)\cos\frac{k_2\tilde{k}_4}{2}(\tilde{k}_1+\tilde{k}_3)_\mu(\tilde{k}_1+\tilde{k}_3)_\rho P^{24}_{\sigma\nu}+$$

$$T(k_1+k_4)J_2(k_1,k_4)\cos\frac{k_1\tilde{k}_4}{2}(\tilde{k}_1+\tilde{k}_4)_\mu(\tilde{k}_1+\tilde{k}_4)_\sigma P^{32}_{\nu\rho}\Big] + \text{ permutations }\Big\}\,.$$

3.3 Perturbative calculations in the microscopic theory

We now move on to the background field method computation of Green's functions in the microscopic theory. For convenience, we present the three-point function and four-point function calculations separately.

3.3.1 The three-point function of background fields

We start with the calculation of the three-point function of background gauge fields $\Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3)$ defined in (3.12). To this end, we will need to expand the logarithm in (2.43) up to three powers of the background field. The resulting Feynman diagrams are shown in figures 3.1-3.4 (where we do not draw permutations of the diagrams).

The Feynman diagrams can be conveniently classified according to the number of J-vertices they contain. Diagrams with no J-vertices, represented in figure 3.1, give a vanishing contribution to the correlator. This is because each of these diagrams gets a factor of $\mathrm{Tr}\,\mathbb{1}_j \equiv d(j)$ from the trace over spin

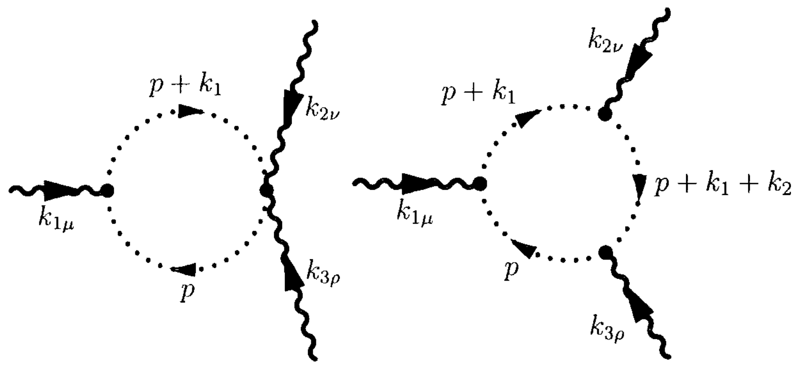


Figure 3.1: *Feynman diagrams with no J-vertices.*

indices, where $d(j)$ is the number of spin components of the field, $d(j) = 1$ for scalars, 2 for Weyl fermions and 4 for gauge fields, respectively. We focus only on supersymmetric theories, where the cancellation between fermionic and bosonic degrees of freedom implies that

$$\sum_j \alpha_j \, d(j) = 0 \; . \tag{3.23}$$

Therefore, each diagram which contains no J-vertices vanishes separately when it is summed over all the fields in the theory. Similarly, diagrams with exactly one insertion of the J-vertices (figure 3.2) vanish, since the trace over spin indices gives $\mathrm{Tr}\, J^{\mu\nu} = 0$.

With these simplifications, we are left with the diagrams of figures 3.3 and 3.4, which we now compute. We will calculate these diagrams in a low-energy approximation where the background fields have a much smaller momentum than the cut-off for the fluctuating fields, so that $k_i k_j \to 0$, while we keep $k_i \tilde{k}_j$ finite [40,42,43]. This low-energy approximation has the great advantage that all of the integrals over the loop momentum can be performed explicitly [41, 42] (see also the discussion after (3.33)).

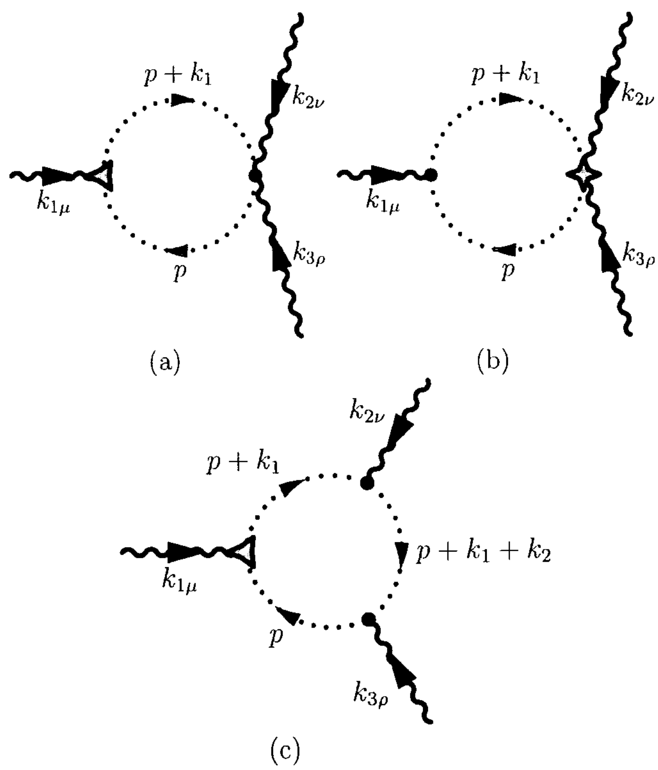


Figure 3.2: *Feynman diagrams with a single insertion of J-vertices (denoted by a triangle and a star).*

We first consider diagrams with two J-vertices, represented in figure 3.3. The contribution to the correlator $\Gamma^{ABC}_{\mu\nu\rho}(k_1, k_2, k_3)$ from diagram (3.3a) is:

$$-2i\Big(\sum_j \alpha_j \operatorname{Tr}\left(J^{\nu\rho}J^{\alpha\mu}\right)_j\Big)\ (2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\int\frac{d^4p}{(2\pi)^4}\,k_{1\alpha}\frac{\delta^{DF}}{p^2}\frac{\delta^{EG}}{(p+k_1)^2}$$
$$\left[-d^{EAD}\sin\Big(\frac{k_1\tilde{p}}{2}\Big)+f^{EAD}\cos\Big(\frac{k_1\tilde{p}}{2}\Big)\right]\left[-d^{CBH}\sin\Big(\frac{k_2\tilde{k}_3}{2}\Big)+f^{CBH}\cos\Big(\frac{k_2\tilde{k}_3}{2}\Big)\right]$$
$$\left[d^{HGF}\sin\Big(\frac{(p+k_1)\tilde{p}}{2}\Big)+f^{HGF}\cos\Big(\frac{(p+k_1)\tilde{p}}{2}(p+k_1)\tilde{p}\Big)\right], \tag{3.24}$$

where the sum is over all the fields in the theory. We can simplify the products of d's and f's in (3.24) by using the relations derived in (2.8)–(2.11) of [31]. In addition, the product of J's can be rewritten using

$$\operatorname{Tr}(J^{\mu\rho}J^{\nu\lambda})_j \;=\; C(j)\,(\delta^{\mu\nu}\delta^{\rho\lambda}-\delta^{\mu\lambda}\delta^{\nu\rho})\;, \tag{3.25}$$

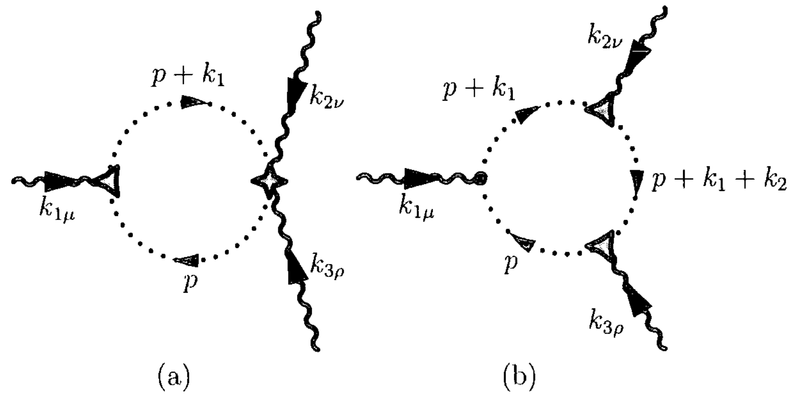


Figure 3.3: *Feynman diagrams with two J-vertices.*

where

$$C(j) \equiv \quad 0 \quad \text{for scalars,} \qquad \frac{1}{2} \quad \text{for Weyl fermions,} \qquad 2 \quad \text{for vectors.} \tag{3.26}$$

The remaining integrals can then be evaluated by first writing the sines and cosines in terms of exponentials, and then using

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\tilde{k}}}{p^2(p+k)^2} = \frac{2}{(4\pi)^2} \int_0^1 dx \; K_0(\sqrt{A}|k||\tilde{k}|) \equiv T(k) \quad , \tag{3.27}$$

where $A = x(1-x)$ and the function T is the same as in (3.4). In this way, the contribution to the three-point function from diagram (3.3a) (and its permutations) becomes:

$$[\Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3)]_{4a} = 4i \left(\sum_j \alpha_j \, C(j) \right) \sqrt{2N}\, (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3) \sin\!\left(\frac{k_2\tilde{k}_3}{2} \right) \tag{3.28}$$

$$\left[T(k_1)\, Q^1_{\mu,\rho\nu}\, \delta^{A0}\delta^{BC} + T(k_2)\, Q^2_{\nu,\mu\rho}\, \delta^{B0}\delta^{AC} + T(k_3)\, Q^3_{\rho,\nu\mu} \delta^{C0}\delta^{AB} \right] .$$

Diagram (3.3b) contributes to the correlator $\Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3)$ as:

$$4i\left(\sum_j \alpha_j \operatorname{Tr}\left(J^{\alpha\nu}J^{\beta\rho}\right)_j\right)(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)$$
$$\int\frac{d^4p}{(2\pi)^4}k_{2\alpha}k_{3\beta}(2p+k_1)_\mu\frac{\delta^{EF}}{p^2}\frac{\delta^{GH}}{(p+k_1)^2}\frac{\delta^{IJ}}{(p+k_1+k_2)^2}$$
$$\left[-d^{EAJ}\sin\Big(\frac{k_1\tilde{p}}{2}\Big)+f^{EAJ}\cos\Big(\frac{k_1\tilde{p}}{2}\Big)\right]\left[-d^{GBF}\sin\Big(\frac{k_2(\tilde{p}+\tilde{k}_1)}{2}\Big)+f^{GBF}\cos\Big(\frac{k_2(\tilde{p}+\tilde{k}_1)}{2}\Big)\right]$$
$$\left[-d^{ICH}\sin\Big(\frac{k_3(\tilde{p}+\tilde{k}_1+\tilde{k}_2)}{2}\Big)+f^{ICH}\cos\Big(\frac{k_2(\tilde{p}+\tilde{k}_1+\tilde{k}_2)}{2}\Big)\right]. \tag{3.29}$$

First, we rewrite sines and cosines in terms of exponentials in the same way as for diagram (3.3a). We will then need to evaluate integrals of the form

$$\mathcal{L}(\sigma,\beta,\gamma)=\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\tilde{\sigma}}}{p^2(p+\beta)^2(p+\beta+\gamma)^2}. \tag{3.30}$$

In the background field method, we integrate out highly fluctuating momenta; here, it will be extremely convenient to integrate momenta above an infrared scale μ. Effectively, this amounts to introducing a small mass term μ^2 in each propagator, so that (3.30) is turned into

$$L(\sigma,\beta,\gamma)=\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\tilde{\sigma}}}{(p^2+\mu^2)[(p+\beta)^2+\mu^2][(p+\beta+\gamma)^2+\mu^2]}. \tag{3.31}$$

Introducing Schwinger parameters, we can recast this integral as

$$\begin{aligned}L(\sigma,\beta,\gamma) &= \int\frac{d^4p}{(2\pi)^4}\int_0^\infty d\alpha_i\, e^{ip\tilde{\sigma}}e^{-\alpha_1(p^2+\mu^2)-\alpha_2((p+\beta)^2+\mu^2)-\alpha_3((p+\beta+\gamma)^2+\mu^2)} \qquad (3.32)\\ &= \int\frac{d^4p}{(2\pi)^4}\int_0^\infty d\alpha_i\,\exp(-\alpha\mu^2)\exp\Big[i\Big(l-\frac{1}{\alpha}(\beta\alpha_2+\alpha_3(\beta+\gamma))\Big)\tilde{\sigma}\Big]\\ &\quad\exp\Big[-\alpha l^2+\frac{1}{\alpha}\Big(-\alpha_1(\alpha_2+\alpha_3)\beta^2-\alpha_3(\alpha_1+\alpha_2)\gamma^2-2\alpha_1\alpha_3\beta\gamma\Big)\Big],\end{aligned}$$

where $\alpha=\alpha_1+\alpha_2+\alpha_3$ and $l=p+\frac{1}{\alpha}\left[\beta\alpha_2+\alpha_3(\beta+\gamma)\right]$. Following [42], we change variables to $\xi_i = \alpha_i/\alpha$ and add a new integration variable λ with a delta function, $\delta\left(\lambda - \sum_i \alpha_i\right)$. After performing the loop momentum integration, we obtain:

$$L(\sigma,\beta,\gamma) = \int_0^\infty \frac{d\lambda}{(4\pi)^2}\, e^{-\tilde{\sigma}^2/(4\lambda)-\lambda\mu^2} \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_3 \;\; e^{-i[(1-\xi_1)\beta+\gamma\xi_3]\tilde{\sigma}} \cdot e^{-\lambda\left[\xi_1(1-\xi_1)\beta^2+\xi_3(1-\xi_3)\gamma^2-2\xi_1\xi_3\beta\gamma\right]} . \quad (3.33)$$

In the low-energy approximation we are considering, where $k_i k_j \to 0$ while $k_i\tilde{k}_j$ is kept finite, the integration becomes feasible [41, 42], and the results for the required cases are:

$$L(\sigma,\sigma,\gamma) = M(\mu|\tilde{\sigma}|)\left(\frac{1-e^{-i\gamma\tilde{\sigma}}}{(\gamma\tilde{\sigma})^2} - \frac{i}{\gamma\tilde{\sigma}}\right) , \quad (3.34)$$

$$L(\sigma,\beta,\sigma) = M(\mu|\tilde{\sigma}|)\left(i\frac{e^{-i\beta\tilde{\sigma}}}{\beta\tilde{\sigma}} + \frac{e^{i\beta\tilde{\sigma}}-1}{(\beta\tilde{\sigma})^2}\right) , \quad (3.35)$$

and the case where $\sigma \neq \beta \neq \gamma$ but $\sigma+\beta+\gamma=0$:

$$\left[L(\sigma,\beta,\gamma)\right]_{\sigma+\beta+\gamma=0} = M(\mu|\tilde{\sigma}|)\left(\frac{i}{\gamma\tilde{\sigma}} + \frac{e^{-i\beta\tilde{\sigma}}-1}{(\gamma\tilde{\sigma})(\beta\tilde{\sigma})}\right) , \quad (3.36)$$

where we have defined

$$M(\mu|\tilde{\sigma}|) := \int_0^\infty \frac{d\lambda}{(4\pi)^2}\, e^{-\tilde{\sigma}^2/(4\lambda)-\lambda\mu^2} . \quad (3.37)$$

We also need a variant of the L integral with an extra power of p_μ in the numerator. We calculate this by noting that

$$\int \frac{d^4p}{(2\pi)^4} \frac{p_\mu \, e^{ip\tilde{\sigma}}}{p^2(p+\beta)^2(p+\beta+\gamma)^2} = -i\frac{d}{d\tilde{\sigma}^\mu} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\tilde{\sigma}}}{p^2(p+\beta)^2(p+\beta+\gamma)^2} \, . \tag{3.38}$$

After some algebra, the contributions of diagram (3.3b) and its permutations to the three-point function becomes, in the low-energy approximation we are considering:

$$\begin{aligned}[\Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3)]_{4\mathrm{b}} \;=\;& 8i\sqrt{2N}\Big(\sum_j \alpha_j \, C(j)\Big)\frac{\sin(k_2\tilde{k}_3/2)}{k_2\tilde{k}_3}(2\pi)^4\delta^{(4)}(k_1+k_2+k_3) \\ & \Big\{P^{23}_{\rho\nu}\,\Big(\dot{M}_\mu(\mu|\tilde{k}_1|)\;\delta^{A0}\delta^{BC}+\dot{M}_\mu(\mu|\tilde{k}_2|)\;\delta^{B0}\delta^{AC}+\dot{M}_\mu(\mu|\tilde{k}_3|)\;\delta^{C0}\delta^{AB}\Big) \\ & P^{13}_{\rho\mu}\,\Big(\dot{M}_\nu(\mu|\tilde{k}_1|)\;\delta^{A0}\delta^{BC}+\dot{M}_\nu(\mu|\tilde{k}_2|)\;\delta^{B0}\delta^{AC}+\dot{M}_\nu(\mu|\tilde{k}_3|)\;\delta^{C0}\delta^{AB}\Big) \\ & P^{12}_{\nu\mu}\Big(\dot{M}_\rho(\mu|\tilde{k}_1|)\;\delta^{A0}\delta^{BC}+\dot{M}_\rho(\mu|\tilde{k}_2|)\;\delta^{B0}\delta^{AC}+\dot{M}_\rho(\mu|\tilde{k}_3|)\;\delta^{C0}\delta^{AB}\Big)\Big\}\,,\end{aligned} \tag{3.39}$$

where $\dot{M}_\mu(z) := (dM/dz^\mu)(z)$. Since $\dot{M}_\mu(z) \;=\; (z_\mu/2)\,S(z)$, where

$$S(z) \;=\; \frac{2}{(4\pi)^2}\,K_0(z)\;, \tag{3.40}$$

we can finally recast (3.39) as

$$\begin{aligned}[\Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3)]_{4\mathrm{b}} \;=\;& 4i\sqrt{2N}\Big(\sum_j \alpha_j \, C(j)\Big)\frac{\sin(k_2\tilde{k}_3/2)}{k_2\tilde{k}_3}(2\pi)^4\delta^{(4)}(k_1+k_2+k_3) \\ & \Big\{P^{23}_{\rho\nu}\,\Big(\tilde{k}_{1\mu}S(\mu|\tilde{k}_1|)\;\delta^{A0}\delta^{BC}+\tilde{k}_{2\mu}S(\mu|\tilde{k}_2|)\;\delta^{B0}\delta^{AC}+\tilde{k}_{3\mu}S(\mu|\tilde{k}_3|)\;\delta^{C0}\delta^{AB}\Big) \\ & P^{13}_{\rho\mu}\,\Big(\tilde{k}_{1\nu}S(\mu|\tilde{k}_1|)\;\delta^{A0}\delta^{BC}+\tilde{k}_{2\nu}S(\mu|\tilde{k}_2|)\;\delta^{B0}\delta^{AC}+\tilde{k}_{3\nu}S(\mu|\tilde{k}_3|)\;\delta^{C0}\delta^{AB}\Big) \\ & P^{12}_{\nu\mu}\Big(\tilde{k}_{1\rho}S(\mu|\tilde{k}_1|)\;\delta^{A0}\delta^{BC}+\tilde{k}_{2\rho}S(\mu|\tilde{k}_2|)\;\delta^{B0}\delta^{AC}+\tilde{k}_{3\rho}S(\mu|\tilde{k}_3|)\;\delta^{C0}\delta^{AB}\Big)\Big\}\,.\end{aligned} \tag{3.41}$$

The last diagram to compute is shown in figure 3.4. It is easily seen from the Feynman rule of the "triangle" J-vertex that this diagram gives a sub-

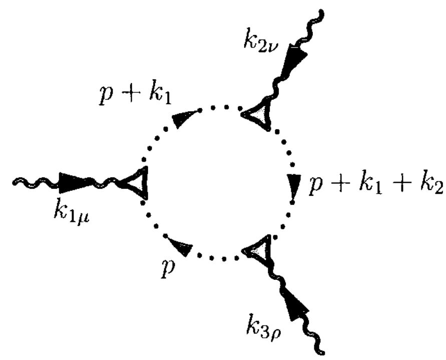


Figure 3.4: *This diagram contains three insertions of the triangle J-vertex.*

leading contribution in the low-energy approximation $k_i k_j \to 0$ and $k_i \tilde{k}_j$ finite, when compared to the diagrams (3.3a), (3.3b) computed so far, hence we will discard its contribution. Summarising, the full three-point function is obtained by adding up the results (3.28) and (3.41).

3.3.2 The four-point function of background fields

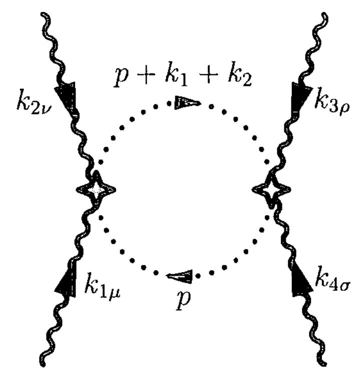
In this section we present the calculation of the four-point function of background fields in the microscopic theory. The computation proceeds in much the same way as that of the three-point function presented in the previous section. The calculations are of course more lengthy, each diagram contains a product of four factors of the form (see the Feynman rules in section 2.4.2):

$$d^{ABC}\sin(p_1\tilde{p}_2) + f^{ABC}\cos(p_1\tilde{p}_2) \qquad (3.42)$$

Therefore there are sixteen terms, each a product of four trigonometric functions, each such term corresponds to sixteen integrals and each integral contributes a number of terms to the answer. Such a calculation would be best performed with a computer but to avoid such a step, we will limit ourselves to the case of gauge group $U(1)$ (the Feynman rules are given in section

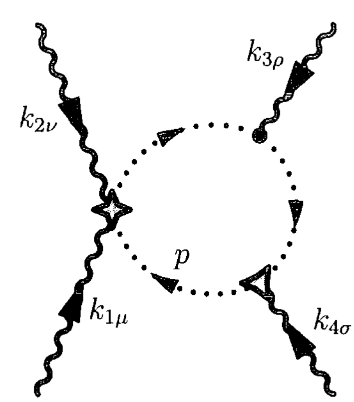
2.4.1).

As in the three-point function case, only diagrams with at least two insertions of J-vertices give a non-vanishing contribution (in the supersymmetric theories we are interested in). Furthermore, terms in the effective action



(a)

Figure 3.5: *This diagram contains two insertions of the star J-vertex.*



(b)

Figure 3.6: *Feynman diagram containing a single insertion of the star and of the triangle J-vertex.*

expressions without the functions P or Q (defined in (3.16)) must arise from diagrams containing no powers of external momenta in their vertices. The only such candidates are therefore the diagram shown in figure 3.5 and its permutations. The expression for this diagram is proportional to:

$$4(\delta^{\mu\sigma}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\sigma}) \sin\Big(\frac{1}{2}k_1\tilde{k}_2\Big) \sin\Big(\frac{1}{2}k_3\tilde{k}_4\Big)\, T(k_1+k_2)\ . \tag{3.43}$$

We now consider the Feynman diagram in figure 3.6 (and its permutations). This diagram gives a contribution to the correlator $\Gamma_{\mu\nu\rho\sigma}(k_1,k_2,k_3,k_4)$ which is proportional to:

$$\begin{aligned} &Q^4_{\sigma,\rho\nu}\sin\Big(\frac{1}{2}k_2\tilde{k}_3\Big)J_2(k_1,k_4)\dot{M}_\mu(\mu|\tilde{k}_4|) + Q^4_{\sigma,\rho\mu}\sin\Big(\frac{1}{2}k_1\tilde{k}_3\Big)J_2(k_2,k_4)\dot{M}_\nu(\mu|\tilde{k}_4|) \\ &+Q^4_{\sigma,\nu\mu}\sin\Big(\frac{1}{2}k_1\tilde{k}_2\Big)J_2(k_3,k_4)\dot{M}_\rho(\mu|\tilde{k}_4|) + Q^3_{\rho,\nu\mu}\sin\Big(\frac{1}{2}k_1\tilde{k}_2\Big)J_2(k_3,k_4)\dot{M}_\sigma(\mu|\tilde{k}_4|) \\ &+Q^2_{\nu,\rho\mu}\sin\Big(\frac{1}{2}k_1\tilde{k}_3\Big)J_2(k_2,k_4)\dot{M}_\sigma(\mu|\tilde{k}_4|) + Q^1_{\mu,\rho\nu}\sin\Big(\frac{1}{2}k_2\tilde{k}_3\Big)J_2(k_1,k_4)\dot{M}_\sigma(\mu|\tilde{k}_4|)\ . \end{aligned} \tag{3.44}$$

Finally, the remaining diagrams give rise to terms which are proportional to the functions P^{ij} defined in (3.16). In order to calculate these contributions, we need the expressions for a few new integrals. Firstly, we need to consider the integral $L(\sigma,\beta,\gamma)$, defined in (3.33), for the case where $\sigma \neq \beta \neq \gamma$ but $\sigma+\beta+\gamma \neq 0$. We find that:

$$[\,L\,(\sigma,\beta,\gamma)\,]_{\sigma+\beta+\gamma\neq 0} = \frac{M\,(\mu|\tilde{\sigma}|)}{\gamma\tilde{\sigma}}\left(\frac{e^{-i\beta\tilde{\sigma}}-1}{\beta\tilde{\sigma}} + \frac{e^{-i(\beta+\gamma)\tilde{\sigma}}-1}{(\beta+\gamma)\,\tilde{\sigma}}\right)\ . \tag{3.45}$$

It is also necessary to calculate several integrals containing four insertions of propagators. These integrals can be evaluated in a similar way to that used in the the calculation of the integrals appearing in the three-point function calculation. For example, one needs to evaluate, for $\sigma + \beta + \gamma + \delta = 0$,

$$L\,(\sigma,\beta,\gamma,\delta) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\tilde{\sigma}}}{(p^2+\mu^2)[(p+\beta)^2+\mu^2][(p+\beta+\gamma)^2+\mu^2][(p+\beta+\gamma+\delta)^2+\mu^2]}$$

$$\xrightarrow{k_ik_j \to 0} \frac{N(\tilde{\sigma})}{\tilde{\sigma}(\gamma+\delta)}\left[\frac{1-e^{-i\tilde{\sigma}(\beta+\gamma)}}{i\tilde{\sigma}(\beta+\gamma)}\left(\frac{1}{\tilde{\sigma}\delta}+\frac{1}{\tilde{\sigma}\gamma}\right)-\frac{1}{\tilde{\sigma}\delta}-\frac{1-e^{-i\tilde{\sigma}\beta}}{i(\tilde{\sigma}\gamma)(\tilde{\sigma}\beta)}\right],$$

(3.46)

where

$$N(\tilde{\sigma}) = \frac{1}{16\pi^2}\int_0^\infty d\lambda\,\lambda\, e^{-\lambda\mu^2-\frac{\tilde{\sigma}^2}{4\lambda}}\,, \tag{3.47}$$

and in the last line we have used the low-energy approximation $k_ik_j \to 0$. Using such integrals, one sees the emergence of terms proportional to the J_2-

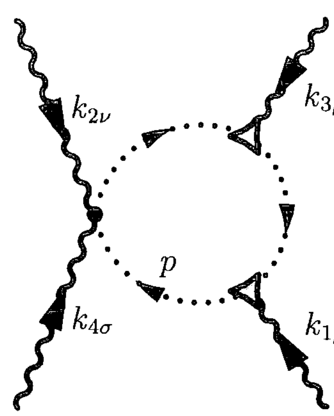


Figure 3.7: *This diagram contains two insertions of the triangle J-vertex.*

and J_3-functions defined in (3.19) and (3.20), respectively. We skip the details of the calculation, which is rather lengthy but, for example, the diagram shown in figure 3.7 produces a term containing $M(\tilde{k}_4)$ and which turns out to be proportional to the expressions

$$\frac{2}{k_1\tilde{k}_4}\left[\frac{\cos(\frac{1}{2}(k_1\tilde{k}_4-k_2\tilde{k}_3))-\cos(\frac{1}{2}(k_1\tilde{k}_3-k_2\tilde{k}_4))}{k_3\tilde{k}_4}+\frac{\cos(\frac{1}{2}(k_1\tilde{k}_3+k_2\tilde{k}_4))-\cos(\frac{1}{2}(k_1\tilde{k}_3-k_2\tilde{k}_4))}{k_2\tilde{k}_4}\right] . \tag{3.48}$$

The previous expression (3.48) is precisely equal to $J_3(k_1, k_2, k_3)$ defined in (3.20) after imposing $k_4 = -(k_1 + k_2 + k_3)$.

3.4 Comparison to the result from the effective action

We are now ready to compare our perturbative results with the expressions for the three- and four-point functions obtained from the effective action $S_{\text{eff}} = S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)}$, where $S_{\text{eff}}^{(1)}$ and $S_{\text{eff}}^{(2)}$ are given in (3.2) and (3.6), respectively.

We begin by considering the three-point function of background gauge fields. In this case, the full perturbative result is obtained by summing up (3.28) with (3.41). We elaborate further these expressions by first performing the sum over the spin j. For definiteness, we consider an $\mathcal{N} = 1$ supersymmetric theory with N_{f} adjoint chiral superfields, for which

$$\Big(\sum_j \alpha_j\, C(j)\Big) = -\frac{1}{4}(3 - N_{\text{f}}) . \tag{3.49}$$

The case $N_{\text{f}} = 0$ corresponds to pure $\mathcal{N} = 1$ super Yang-Mills; for $N_{\text{f}} = 1, 3$ we have the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories, respectively. Notice that, in the latter case, the contribution to the three-point function vanishes. Secondly, we observe that (3.41) was derived in the low-energy approximation $k_i k_j \to 0$, with $k_i\tilde{k}_j$ fixed and finite. We also introduced a small infrared regulating mass

μ. In order to compute the corresponding limit of (3.28), we note that this amounts to perform the following modification on the function T of (3.27):

$$T(k) \xrightarrow{\mu} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\tilde{k}}}{(p^2+\mu^2)[(p+k)^2+\mu^2]} \overset{k^2\to 0}{\longrightarrow} \frac{2}{(4\pi)^2} K_0(\mu|\tilde{k}|) \equiv S(\mu|\tilde{k}|) \quad (3.50)$$

where the first arrow stands for equality after introducing the regulator μ in the expression for T, and the second means equality in the limit $k^2 \to 0$ (at fixed $|k||\tilde{k}|$).

Taking these observations into account, the one-loop perturbative expression for the three-point function, in the low-energy regime $k_i k_j \to 0$ and $k_i\tilde{k}_j$ finite, is given by:

$$\begin{aligned}\Gamma^{ABC}_{\mu\nu\rho}(k_1,k_2,k_3) &= i\sqrt{2N}(N_{\rm f}-3)\,\frac{\sin(k_2\tilde{k}_3/2)}{k_2\tilde{k}_3}\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3) \qquad (3.51)\\ &\Big\{\delta^{A0}\delta^{BC}S(\mu|\tilde{k}_1|)\left[\tilde{k}_{1\mu}P^{23}_{\rho\nu}+\tilde{k}_{1\nu}P^{13}_{\rho\mu}+\tilde{k}_{1\rho}P^{12}_{\nu\mu}+(k_2\tilde{k}_3)\,Q^1_{\mu,\nu\rho}\right]\\ &+\delta^{B0}\delta^{CA}S(\mu|\tilde{k}_2|)\left[\tilde{k}_{2\mu}P^{23}_{\rho\nu}+\tilde{k}_{2\nu}P^{13}_{\rho\mu}+\tilde{k}_{2\rho}P^{12}_{\nu\mu}+(k_2\tilde{k}_3)\,Q^2_{\nu,\rho\mu}\right]\\ &+\delta^{C0}\delta^{AB}S(\mu|\tilde{k}_3|)\left[\tilde{k}_{3\mu}P^{23}_{\rho\nu}+\tilde{k}_{3\nu}P^{13}_{\rho\mu}+\tilde{k}_{3\rho}P^{12}_{\nu\mu}+(k_2\tilde{k}_3)\,Q^3_{\rho,\mu\nu}\right]\,.\end{aligned}$$

This perturbative result (3.51) should be contrasted with the result (3.15) (with $C = 2\alpha_0/N$) obtained from the original expression (3.2) for the effective action, where, from the results of [27, 28], it follows that

$$\alpha_0 = -4\Big(\sum_j \alpha_j C_j\Big)N = (3-N_{\rm f})N\,, \qquad (3.52)$$

the sum over j being extended only to fields in the adjoint. The expressions (3.51) and (3.2) differ in two respect. First, (3.2) contains the function T, whereas the perturbative result (3.51) contains the function S. This is easily explained by remembering (3.50), i.e. that at low energy $T \to S$. Second, and more importantly, the perturbative expression (3.51) contains, in addition to the terms in (3.15), also a contribution proportional to

$$\delta^{A0}\delta^{BC}\, S(\mu|\tilde{k}_1|)\, \tilde{k}_{1\mu} P^{23}_{\rho\nu} + \delta^{B0}\delta^{AC}\, S(\mu|\tilde{k}_2|)\, \tilde{k}_{2\nu} P^{13}_{\rho\mu} + \delta^{C0}\delta^{AB}\, S(\mu|\tilde{k}_3|)\, \tilde{k}_{3\rho} P^{12}_{\nu\mu}\,. \tag{3.53}$$

This new contribution does not arise from the originally conjectured action $S^{(1)}_{\text{eff}}$ in (3.2). However, as we have discussed in the introduction, this term (3.53) is precisely reproduced by adding the contribution $S^{(2)}_{\text{eff}}$ in (3.6) to the original effective action term in (3.2).

Finally, we consider now the matching of the four-point function obtained from the effective action, Eqs. (3.21) and (3.22), against the perturbative calculation presented in the previous section. We will find that the perturbative calculation is precisely reproduced by the effective action $S_{\text{eff}} = S^{(1)}_{\text{eff}} + S^{(2)}_{\text{eff}}$, where $S^{(1)}_{\text{eff}}$ and $S^{(2)}_{\text{eff}}$ are the expressions in (3.2) and (3.6).[1]

Feynman diagrams in figure 3.5 (and its permutations) generate the contribution (3.43), which precisely matches the terms in our expression (3.21) which contains $\delta_{\mu\rho}\delta_{\nu\sigma}$ and no insertions of P and Q functions. Similarly, the first three terms in (3.44) precisely reproduce the terms generated by $S^{(1)}_{\text{eff}}$ containing both the $T(k_4)$ and Q functions. The remaining terms in (3.44) correspond to terms produced by $S^{(2)}_{\text{eff}}$ (see (3.22)), when we consider the low-energy limit $\cos(\frac{1}{2}k_i\tilde{k}_j) \to 1$ and $\sin(\frac{1}{2}k_i\tilde{k}_j) \to \frac{1}{2}k_i\tilde{k}_j$. Finally, as anticipated in the previous section, combinations of the remaining Feynman diagrams reproduce those terms in S_{eff} that contain the P function.

Summarising, we have a complete agreement between the low-energy limit

[1] In particular, the four-point function calculation discriminates between the terms (3.6) and (3.10).

of the perturbative calculation of three- and four-point functions in the microscopic theory, and the corresponding result obtained from the low-energy effective action $S_{\text{eff}} = S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)}$.

Chapter 4

The Standard Model

The universe, they said, depended for its operation on the balance of four forces which they identified as charm, persuasion, uncertainty and bloody-mindedness.

"The Light Fantastic" by Terry Pratchett

The Standard Model is a mundane name for a beautiful theory of nature. Although it is not perfect it is an outstanding achievement, describing the interaction of particles on sub-atomic scales. This chapter will outline the Standard Model in enough detail that the reader will be able to follow the following chapter where a version of the Standard Model based on noncommutative spaces will be introduced. Many more thorough reviews exist and many books have been devoted to the subject, for much more detail and an extensive list of references see [26, 50].

The Standard Model is a gauge theory that describes the interactions of sub-atomic particles known as quarks and leptons. Our experiments have probed scales as small as 10^{-18}m and as far as we can determine both quarks and leptons are elementary, i.e. they are not composed of smaller particles. All matter is composed of these tiny particles[1]. These quarks and leptons feel four forces, the strong force (QCD), the electro-weak force, hyper-charge and gravity. The Standard Model describes only the first three; gravity is unmeasurably weak on the sub-atomic scale and particle physicists often neglect it (also consistent, renormalisable quantum theories of gravity have proved hard to formulate). At low energies, the electro-weak and hyper-charge forces are spontaneously-broken to electromagnetism by the Higgs mechanism.

The nuclei of each atom is composed of protons and neutrons and each of these is composed of 3 valence quarks[2].

Each valence quark in a proton or neutron is one of two "flavours" of quark; an up quark with electric charge $+\frac{2}{3}$ (on a scale where the electric charge of the proton is $+1$) and a down quark with charge $-\frac{1}{3}$. A proton contains two up valence quarks and a down valence quark and a neutron contains two down valence quarks and an up valence quark.

As well as up and down quarks, there are two other pairs of quark, charm and strange and top and bottom. Each pair of quarks is more massive than the preceding pair but aside from that, they interact with all the forces identically. Each pair of quarks is called a generation or family and only the first generation (up and down) is required to make all the matter that we observe in nature.

Like quarks, leptons come in three generations, the first family consists of

[1]Cosmologists predict a more exotic type of matter called dark matter which may or may not consist of the super-partners to the particles outlined in this chapter

[2]other quarks and anti-quarks can spontaneously appear out of the vacuum (sea quarks) and the actual fraction of the momentum of a proton that is carried by particular species of quark are given by complicated parton distribution functions

the electron, e^- (charge -1) and a (so far) unmeasurably light particle called an electron neutrino ν_e which is electrically neutral. The other families consist of the muon with the muon neutrino and the tau with its associated neutrino. Aside from the Higgs Boson (discussed later) these are the only matter particles in the Standard Model. We now turn our attention to the forces (and the gauge bosons that mediate them).

We will proceed by introducing the strong force in section 4.1, before covering the electro-weak force (section 4.2). We will then introduce the Higgs' mechanism and summarise the particle content before discussing anomaly cancellation and briefly mentioning some of the problems of the model.

4.1 The Strong Force

The only elementary matter particles that feel this force more properly called Quantum Chromo-Dynamics (QCD) are quarks. Each quark can be one of three "colours" and there is a local symmetry between these colours. The gauge bosons that mediate this force (called gluons) also carry colour charge and can interact with other gluons. The gauge group that describes this force is SU(3).

In order to write down the Lagrangian we first introduce ψ, a triplet of Dirac fermions (each corresponding to a colour of quark):

$$\psi(x) = \begin{pmatrix} q_R(x) \\ q_G(x) \\ q_B(x) \end{pmatrix} \tag{4.1}$$

which transforms in the fundamental representation:

$$\psi(x) \to U(x)\psi(x) \quad \text{where} \quad U(x) = \exp(i\alpha^a(x)\frac{t^a}{2}) \in SU(3) \tag{4.2}$$

The t^a are the generators of SU(3) and the $\alpha^a(x)$ are arbitrary functions of x. Note that we have changed conventions from the anti-Hermitian generators used in the first part of this thesis to Hermitian generators which are more usually used when discussing the Standard Model. We will use these Hermitian generators throughout the remainder of this thesis.

We also introduce the covariant derivative associated with the local $SU(3)$ symmetry:

$$D_\mu = \partial_\mu - igG_\mu^a t^a \tag{4.3}$$

We will write $\not{D} = \gamma^\mu D_\mu$ and $G_\mu^{1...8}(x)$ are the gluon fields. We introduce a kinetic term for the gluons using the field-strength tensor,$F_{\mu\nu}^a$:

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^a t^a \tag{4.4}$$

So we have, writing the generators explicitly,

$$F_{\mu\nu}^a t^a = \partial_\mu G_\nu^a t^a - \partial_\nu G_\mu^a t^a - ig[A_\mu^a t^a, A_\mu^b t^b] \tag{4.5}$$

The Lagrangian is:

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu}^a)^2 \tag{4.6}$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$ i.e. the Hermitian conjugate multiplied by the first of the Dirac gamma matrices (which we will denote by γ through-out this chapter).

Note that there is no mass term for the quarks $m\bar{\psi}\psi$, in the above equation.

It would be gauge-invariant in QCD but as we shall see shortly such a term will be forbidden in a chiral theory.

4.2 The Weinberg-Salam Model

As well as QCD, the Standard Model Lagrangian contains terms that describe both weak isospin and weak hypercharge. The gauge group of the former is $SU(2)$ and the latter is $U(1)$.

The gauge group for weak isospin is usually denoted $SU(2)_L$ where the subscript L denotes the fact that only left-handed particles feel this force. The left-handed component of a Dirac fermion is projected out by:

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi \quad \text{where} \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.7}$$

We arrange the left-handed particles into doublets, whereas the right-handed particles remain singlets. For example:

$$l_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L , \quad e_R^-, u_R, d_R \tag{4.8}$$

We can then add to the Lagrangian terms describing the electroweak interaction for each generation. For the first generation we add,

$$\mathcal{L} = \bar{l}_L(i\not{D})l_L + \bar{e}_R(i\not{D})e_R + \bar{q}_L(i\not{D})q_L + \bar{u}_R(i\not{D})u_R + \bar{d}_R(i\not{D})d_R \tag{4.9}$$

where the covariant derivative in the above equation is:

$$D_\mu = \partial_\mu - igW^a_\mu t^a - \frac{i}{2}g'YB_\mu \qquad (4.10)$$

In equation (4.10) the t^a are the generators of the $SU(2)_L$ group. For the left-handed doublets these are usually taken to be $t^a = \frac{1}{2}\sigma^a$ and σ^a $(a = 1..3)$ are the Pauli matrices. For the right handed singlets this term will disappear. The Y in (4.10) is the hyper-charge of the particle (listed in table 4.1) and at this stage could be completely arbitrary although the electric charges of the particles will depend on the assignments and so the observed electric charges will fix these numbers. In section 4.4 we'll see that cancellation of gauge anomalies puts constraints on the possible hypercharges. In the Noncommutative Model introduced in the next section these charges will be a prediction of the theory and their agreement with the observed electric charges is an elegant coup for the model.

We also need to add to the Lagrangian kinetic terms for the gauge bosons, of the groups. The term for the $SU(2)$ bosons, denoted by W^a_μ is $-\frac{1}{4}(W^a_{\mu\nu})^2$, where

$$W^a_{\mu\nu}t^a = \partial_\mu W^a_\nu t^a - \partial_\nu W^a_\mu t^a - ig[W^a_\mu t^a, W^b_\mu t^b] \quad . \qquad (4.11)$$

The field-strength tensor for the $U(1)$ gauge boson is slightly different, $B^{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ there is no term containing the commutator of two gauge fields because $U(1)$ has only a single generator (which was not written explicitly) and so, in a commutative theory, two gauge fields will commute. The commutator term causes the self-interaction of the gauge bosons and a theory with no such interactions is called *Abelian* (More properly an Abelian theory is a theory where the gauge rotations commute, there are no Abelian noncommutative theories because the functions of space as well as the generators do not commute).

4.2.1 Symmetry Breaking

At low energies the $SU(2) \times U(1)$ symmetry is spontaneously broken to the $U(1)$ symmetry of electromagnetism. We do this by introducing a new doublet of complex scalar fields with hypercharge $Y = 1$ and adding corresponding terms to the Lagrangian:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{with} \quad \mathcal{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \tag{4.12}$$

where $\mu^2 < 0$, $\lambda > 0$ and the covariant derivative is defined by equation (4.10). In the vacuum we expect the field ϕ to minimise the potential, and we pick for our vacuum expectation value,

$$\phi = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with} \quad v^2 \equiv -\frac{\mu^2}{\lambda} \tag{4.13}$$

Expanding around the vacuum requires only a single, real scalar field called the Higgs' field, $h(x)$:

$$\phi(x) = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} . \tag{4.14}$$

We can gauge away the other 3 real degrees of freedom using an $SU(2)$ rotation. The choice of vacuum has broken the gauge group, but although it has non-zero quantum numbers; $T = \frac{1}{2}$, $T^3 = -\frac{1}{2}$ and $Y = 1$, the generator,

$Q = T^3 + \frac{Y}{2}$ leaves the vacuum invariant, i.e.

$$\phi_0 \to e^{i\alpha(x)Q}\phi_0 = \phi_0 \quad \forall\alpha(x) \tag{4.15}$$

As we will see in the next section, our Higgs field has given masses to all the gauge bosons except the photon and the gauge group has been broken from $SU(2) \times U(1)$ to a gauge group with single generator Q which is the U(1) of electromagnetism.

4.2.2 Masses for the gauge bosons

To calculate the masses for the gauge bosons, we insert the vacuum expectation value (v.e.v.) for the Higgs field into the $|D^\mu\phi|^2$ term in the Lagrangian where we use the notation $|x|^2 \equiv (x)^\dagger(x)$ and each SU(2) generator is $\frac{1}{2}\times$ a Pauli matrix:

$$\left|\left(-i\frac{g_2}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W^3 \end{pmatrix} - i\frac{g_1}{2}B_\mu\right)\sqrt{\frac{1}{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2$$
$$= \frac{v^2}{8}\left(g_2^2\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] + \left[g_1 B_\mu - g_2 W_\mu^3\right]^2\right) \tag{4.16}$$

We have three mass terms, corresponding to three massive gauge bosons.

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \quad \text{and} \quad Z_\mu^0 = \frac{1}{\sqrt{g_2^2 + g_1^2}}\left(g_2 W_\mu^3 - g_1 B_\mu\right) \tag{4.17}$$

where the W bosons both have the same mass, M_W and the Z boson has a mass of M_Z where

$$M_W = \frac{g_2 v}{2} \quad \text{and} \quad M_Z = \sqrt{g_2^2 + g_1^2}\frac{v}{2}. \tag{4.18}$$

The key point is that the field orthogonal to M_Z has not acquired a mass, and this field, which we denote by A_μ:

$$A_\mu = \frac{1}{\sqrt{g_2^2 + g_1^2}}(g_2 B_\mu + g_1 W_\mu^3) \quad , \tag{4.19}$$

is associated with the unbroken generator $Q = T^3 + \frac{Y}{2}$ and is the photon of electromagnetism.

4.2.3 Weak Mixing angle

In the literature, it is usual to specify the relationship between the gauge bosons in the before symmetry breaking and after symmetry breaking basis in terms of an angle. As we will use this device in the next chapter we introduce it here.

We define the *weak mixing angle* such that,

$$\begin{pmatrix} A_\mu \\ Z_\mu^0 \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} \quad . \tag{4.20}$$

Looking at the mass terms involving W_μ^3 and B_μ which we can read off from the second term on the right-hand side of (4.16):

$$\frac{v^2}{8} \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \begin{pmatrix} g_2^2 & -g_1 g_2 \\ -g_1 g_2 & g_1^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \tag{4.21}$$

The eigen-vectors are:

$$\text{massless} : \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad \text{massive} : \begin{pmatrix} g_2 \\ -g_1 \end{pmatrix} \tag{4.22}$$

So we find:

$$\cos\theta_w = \frac{g_2}{\sqrt{g_1^2+g_2^2}} \quad , \quad \sin\theta_w = \frac{g_1}{\sqrt{g_1^2+g_2^2}} \quad , \quad \tan\theta_w = \frac{g_2}{g_1} \quad . \tag{4.23}$$

Comparing this result with equation (4.18) we find

$$M_W = M_Z \cos\theta_w \tag{4.24}$$

and when this result is compared to experiment a good agreement is found, supporting the model.

4.2.4 Yukawa terms

When discussing the QCD lagrangian it would have been possible (although we refrained) to add a mass term for the quarks. In a chiral theory, such as the Weinberg-Glashow model, such mass terms would not be possible. Attempting to write down a mass for the electron for example:

$$-m_e\bar{e}e = -m_e\bar{e}(\frac{1}{2}(1-\gamma^5)+\frac{1}{2}(1-\gamma^5))e \tag{4.25}$$

$$= -m_e(\bar{e}_R e_L + \bar{e}_L e_R) \tag{4.26}$$

This problem can be circumvented using the versatile Higgs field. We can add to the Lagrangian, terms involving the Higgs that will lead to mass terms for the electron and for the down quark:

$$\mathcal{L}_m = -\lambda_e\bar{l}_L\cdot\phi e_R - \lambda_d\bar{q}_L\cdot\phi d_R \tag{4.27}$$

When we insert the vacuum expectation value for the Higgs into the above equation we will get masses:

$$m_e = \frac{1}{\sqrt{2}}\lambda_e v \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\lambda_d v \quad . \tag{4.28}$$

Notice that if the couplings (which are free parameters) were of order unity then we would expect masses of order the electroweak scale, that the electron is so light is something of a mystery.

These mass terms are gauge invariant under both $SU(2)$ and $U(1)$ gauge groups as they have to be to preserve the gauge invariance but this requirement makes it harder to write down a mass term for the up quarks. It is however possible. Writing the SU(2) indices explicitly (where we have previously been suppressing them) we can write:

$$\mathcal{L}_u = -\lambda_u \epsilon^{ab} \bar{q}_{La} \phi_b^\dagger u_R \quad . \tag{4.29}$$

This will give a mass term to the up-quark but there is a catch, the anti-symmetric tensor with two indices, ϵ^{ab} is only invariant under the group $SU(2)$, if we had a larger symmetry group (as we will do in the next chapter) such a term would not be possible.

4.3 Summary of Particles

We have introduced the gauge-groups (and therefore the forces) at the heart of the model but as of yet the spectrum of matter, the quarks and leptons has been alluded to rather than outlined and we rectify that in this section.

The matter in the Standard Model is divided into three groups called generations, the fields in the first generation are shown in 4.1[3]. The second and third generations have an equivalent particle content to the first except that the masses of the particles are much heavier. Particles in the heavier generations decay (via the weak interaction) into particles of lighter generations so all the known stable matter in the universe consists of particles in the lightest generation.

Field	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	Hypercharge
e_R			$\square$	-2
ν_R				0
$l_L = \begin{pmatrix}\nu_L & e_L\end{pmatrix}$		$\square$	$\square$	-1
u_R	$\square$		$\square$	$+\frac{4}{3}$
d_R	$\square$		$\square$	$-\frac{2}{3}$
$q_L = \begin{pmatrix}u_L & d_L\end{pmatrix}$	$\square$	$\square$	$\square$	$+\frac{1}{3}$
G_μ	$\square\bar{\square}$			0
W_μ		$\square\bar{\square}$		0
B_μ			$\square\bar{\square}$	0
ϕ		$\square$	$\square$	1

Table 4.1: Representations for various fields in the theory

4.4 Anomaly Cancellation

So far we have managed to write a Lagrangian in which the right and left handed particles transform in different representations of the gauge group (a *chiral* theory). It is not entirely straight-forward however, loop diagrams can cause symmetries which are conserved classically, to be broken at the

[3]the table also includes the Higgs fields and gauge bosons which are not part of a generation; there is only a single copy of each

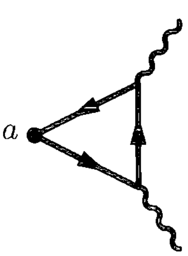


Figure 4.1: *A loop diagram that can cause currents to be non-conserved*

quantum level.

In chiral theories, diagrams such as the one shown in figure 4.1 can cause a current to diverge (i.e. it is no longer conserved). The dotted vertex in the diagram corresponds to the gauge symmetry current, which in a theory with a left handed fermion, ψ would be:

$$j^{\mu a} = \bar{\psi}\gamma_\mu \left(\frac{1-\gamma^5}{2}\right) t^a \psi \tag{4.30}$$

The two external lines correspond to gauge bosons, $A^b_\nu t^b$ and $A^c_\lambda t^c$. Such diagrams lead to a contribution to the divergence proportional to:

$$\partial_\mu j^{\mu a} \propto \mathrm{tr}[t^a\{t^b, t^c\}] \tag{4.31}$$

If the gauge currents are not conserved, the Ward identity no longer holds, unphysical states in the theory are not necessarily cancelled and unitary is no longer conserved.

In order for the Standard Model to be consistent, all contributions of the form of (4.31) must vanish, and indeed they do. Because the gauge bosons couple to the chiral current we can equivalently consider diagrams where a third gauge boson is coupled to the dotted vertex and the diagram is a one-loop correction to the three-gauge boson vertex.

When the bosons are all in $SU(3)$, we have a non-chiral theory (QCD) and contributions from left and right handed particles have opposite signs and cancel out. If the generators are all $SU(2)$ (i.e. the Pauli matrices) then because $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$ the contribution will vanish. Similarly because the trace of a single generator of either $SU(2)$ or $SU(3)$ is zero, if only one boson is from either group there will be zero contribution. The only possible remaining anomalies are therefore:

Two SU(3) bosons, 1 U(1) boson: This is proportional to $\text{tr}[t^a t^b Y]$, i.e. the sum of the hypercharges for all the fermions charged under both SU(3) and U(1); the left and right handed quark (different chiralities contributing with opposite signs). We have, using the hypercharges from table 4.1:

$$+\frac{4}{3} - \frac{2}{3} - 2 \times \frac{1}{3} = 0 \tag{4.32}$$

Two SU(2) bosons, 1 U(1) boson: Similarly to the last case, the contribution is proportional to the hypercharge of the particles involved, this time, the left-handed fermions and we have

$$(-1) + (-1) + 3 \times \frac{1}{3} + 3 \times \frac{1}{3} = 0 \tag{4.33}$$

(the factor of three is because both the up quark and down quarks come in three colours).

Three U(1) bosons: Each generator involved is the Y of hypercharge so we sum the hypercharge cubed for each particle (left and right handed with opposite signs) that has nonzero hypercharge:

$$(-2)^3 - 2 \times (-1)^3 + 3 \times [(\frac{4}{3})^3 + (-\frac{2}{3})^3 + 2 \times (\frac{1}{3})^3] = 0 \quad , \qquad (4.34)$$

(the factors of 2 and 3 are for number of flavour and colour respectively).

The gravitational anomaly: This anomaly is slightly different, and only applies if gravity is considered. If the Standard Model was to be coupled to gravity, it was shown in [51] that we would require the sum of the hypercharges in each generation to be zero, and indeed it is.

These anomaly cancellation conditions constrain the allowed hypercharges for the particles in the model; it seems almost miraculous that the hypercharges required to get the correct electric charges fit these very tight constraints. In the model introduced in the next chapter, anomalies are much more of a problem and treated differently but the hypercharges will arise naturally with their correct values

4.5 Problems of the Model

The Standard Model of Particle Physics has been dramatically successful, and there is no major experimental data (with the partial exception of neutrino oscillations, discussed below) that contradicts the model. However it is not perfect and outlined here are some of the open issues.

- Recent experiments show that neutrinos oscillate between species. Such behaviour requires the neutrinos to have different masses. A right-handed neutrino, singlet under all the gauge groups needs to be added to the Model. The extremely light-scale of the masses seems unnatural, although the see-saw mechanism [52] can be used to explain this.

- Precision measurements of electroweak observables indicate that the mass of the Higgs boson is of the order of the mass of the W boson, however loop corrections to the Higgs boson mass cause quadratic divergences. If we assume that the Standard Model is valid up to large energies then in order to keep the Higgs mass light, either the parameters in the Lagrangian must be fine tuned or the quadratic divergences must cancel. The spin of the particle running in the loop determines the sign of the contribution; fermions and bosons give and opposite contribution. In a supersymmetric theory, where for each fermion there is an equivalent boson and vice-versa, the divergences cancel. This is a good argument for TeV scale supersymmetry but the jury will remain out until the Large Hadron Collider.

- The tiny, non-zero scale of the cosmological constant presents another problem. The cosmological constant corresponds to the energy density of the vacuum. In a completely supersymmetric model, this would be zero. In a non-supersymmetric model, the prediction would be of order of a typical energy scale of the theory to the power of 4. The energy scale of QCD, Λ_{QCD} would lead to an estimate out by tens of orders of magnitude.

- The Standard Model includes no description of gravity. On the quantum scale gravity is negligibly weak but a complete theory of the laws of nature would include such a description. String theory includes a quantum description of gravity but has yet to find experiment verification (and such a test seems a long way off).

- The gauge couplings are not the only quantities that run in the Standard Model, the Yukawa couplings and, λ, the quartic self-coupling of the Higgs also run. These couplings are not asymptotically free; at some large energy scale they will become infinite and so, even neglecting gravity, the Standard Model can not be a truly microscopic theory of our universe.

- The Standard Model lends itself to more speculative questions; why are there three generations? why is the gauge group $SU(3) \times SU(2) \times U(1)$? Why do we have the particular spectrum of masses (especially when most of the particles have masses well below the electro-weak scale)? Such questions are beyond the scope of the model itself.

Listing the shortcomings of the model as we have above, does not detract from the fact that that this model, outlined in a few pages fits our experiments at high-energies with incredible accuracy. The model that will be introduced in the next chapter does not claim to solve these problems, it is an attempt to include the essential features of this successful model, the forces and its particle spectrum into a noncommutative context.

Chapter 5

The Noncommutative Standard Model

The goal was the Theory of Everything, but Ponder would settle for the Theory of Something, and, late at night, when Hex appeared to be sulking, he despaired of even a Theory of Anything.

"The Last Continent" by Terry Pratchett

5.1 Introduction

We have now completed our survey of the ordinary Standard Model and it is now time to marry that with the noncommutative field theories discussed earlier and try to write down a Noncommutative Standard Model (NCSM). The motivations for trying to write down such a theory are manifold. An explosion of research in the field was triggered when Seiberg and Witten showed that the type of theories considered in this thesis could be obtained from open string field theories with constant B-field [8]. If it could be shown that the Standard Model could not be embedded into a noncommutative theory it would provide us with new information about string theory. Aside from string theory, noncommutative theories have new and novel features and an NCSM might provide us with new solutions to problems in the Standard Model and beyond. Finally, noncommutative field theories provide many constraints on model building that provide an interesting technical challenge.

The constraints that are imposed on the noncommutative model builder are:

1. UV/IR mixing can cause quadratic divergences (see section 2.5.6) and decoupling of $U(1)$ degrees of freedom (section 2.5.5);
2. the gauge groups are restricted to $U(N)$ groups (section 2.3);
3. fields can transform only in (anti-)fundamental, bi-fundamental and adjoint representations of the gauge groups (section 2.3);
4. the charges of matter fields are restricted to 0 and ± 1, and this makes it difficult to get fractional electric charges for quarks (section 2.3)
5. gauge anomalies cannot be cancelled in a chiral noncommutative theory, hence the anomaly-free theory must be vector-like (compare section 5.2.3 with section 4.4).

Prior to the work presented in this thesis, many authors published models which have satisfied one or more of the above constraints. A large body of work has consisted of examining models where the exponential factors in the vertices ($e^{ik\tilde{p}}$ where k is an external momenta and p is a loop momenta) are Taylor expanded. Such models neglect the UV/IR effects caused by such factors.

The authors of Ref. [9] made an important step in noncommutative model building by proposing a noncommutative model which satisfies criteria 2, 3 and 4. Their model has the noncommutative gauge group $U(3) \times U(2) \times U(1)$ with matter fields transforming only in (bi-)fundamental representations, and remarkably, it predicts the hypercharges of the Standard Model. In many respects their model is similar to the bottom-up approach of [53] to the string embedding of the Standard Model in purely commutative settings. Unfortunately, the noncommutative $U(3) \times U(2) \times U(1)$ model of [9] is affected by the UV/IR mixing which causes the $U(1)$ hypercharge sector to decouple. In this version of an NCSM, UV/IR effects are again explicitly ignored and a new type of particle called the 'Higgsac' was introduced with turned out to break noncommutative gauge invariance, causing effects like a violation [54] of unitarity[1]. The plan for this chapter is to construct a noncommutative embedding of the Standard Model which satisfies all the requirements listed above. The model is based on the gauge group $U(4) \times U(3) \times U(2)$ with matter fields transforming in noncommutatively allowed representations. In the infrared the gauge group is spontaneously broken to the Standard Model group by a Higgs mechanism. We need a larger gauge group than the authors of [9] in order to incorporate the UV/IR mixing effects, yet remarkably we still find the correct values of the hypercharges for all the fields of the Standard Model, we avoid the introduction of Higgsac field but will find ourselves forced to introduce extra matter fields.

The conclusion one can draw from this is that it is conceivable to embed a commutative $SU(N)$ theory, such as e.g. QCD or the weak sector of the Standard Model into a supersymmetric noncommutative theory in the UV,

[1]Very recently a second version of this model [55] has been proposed by the authors, in this updated version the Higgsac is dressed with semi-infinite Wilson lines.

but some extra care should be taken with the QED $U(1)$ sector. It is, in fact, pretty clear that the UV/IR mixing makes it impossible to interpret a noncommutative $U(1)$ theory as an ultraviolet embedding of ordinary QED. The low-energy theory emerging from the noncommutative $U(1)$ theory will become free in the IR (rather than just weakly coupled, it will not freeze out at any mass scale) and in addition will have other pathologies. When supersymmetric theories are softly-broken down to $\mathcal{N} = 0$ non-logarithmic IR divergences can re-appear. Models with the U(1) gauge group have been analysed [33,34,56] and tachyons can only be avoided if the model has $\mathcal{N} = 4$ supersymmetry; even in this case the tachyons are avoided at the expense of giving a mass to the photon and fine tuning is required to keep this below experimental limits. The prospects for phenomenologically acceptable versions of such models looks bleak.

It is becoming pretty clear that the only realistic way to embed QED into noncommutative settings is to recover the electromagnetic $U(1)$ from a *traceless* diagonal generator of some higher $U(N)$ gauge theory. The trace-$U(1)$ part of this theory will decouple in the IR due to IR/UV mixing effects, and a traceless diagonal generator can give $U(1)$ as well as some non-Abelian $U(n)$ factors in favourable settings. So it seems that in order to embed QED into a noncommutative theory one should learn how to embed the whole Standard Model.

In the following section we will show how the UV/IR mixing leads to the decoupling of the overall $U(1)$ factors from the gauge groups. In section 3 we will introduce the model, calculate the hypercharges, and discuss the gauge-, the fermion- and the Higgs-sectors. We will also outline how to cancel all the gauge anomalies by extending the model.

The model presented in this chapter is one example of how the Standard Model can be embedded into a microscopic noncommutative gauge theory. One particularly interesting future direction would be to find a realistic supersymmetric version which would exhibit a dynamical supersymmetry breaking. This is motivated by the UV/IR-decoupled $U(1)$ degrees of freedom which provide a natural candidate for the hidden sector of dynamical supersymmetry breaking, as explained in [57].

In the model that we outline in the next section, we can satisfy both these conditions hence the only consequence of the UV/IR mixing is that the U(1) degrees of freedom decouple and become unobservable at low-energies. Due to this fact, in much of what will follow the overall $U(1)$ factors of all three $U(N)$ gauge groups considered below can be safely dropped at energies much below the noncommutative scale relevant to the commutative Standard Model.

5.2 The Noncommutative Standard Model

As was mentioned earlier, all fields in a Noncommutative Gauge Theory must transform in the adjoint, fundamental, anti-fundamental or bi-fundamental representations. We assign the fields to the representations shown in table (5.1), note that, unlike in the Standard Model, no field is charged under more than two groups. All the matter fermion fields come in three generations (which is not indicated explicitly in the table), and furthermore, the table will be extended in section 5.2.3.

As can been seen from the table we have introduced three more Higgs fields, $\phi_{A\bar{C}}$, $\phi_{C\bar{B}}$, $\phi_{B\bar{A}}$ and ϕ_B compared to the Standard Model's single Higgs. The

Field	$U_C(4)$	$U_B(3)$	$U_A(2)$	Hypercharge
e_R			$\square$	-2
ν_R				0
$l_L = (\nu_L \quad e_L)$		$\bar{\square}$	$\square$	-1
u_R	$\square$		$\bar{\square}$	$+\frac{4}{3}$
d_R	$\square$			$-\frac{2}{3}$
$q_L = (u_L \quad d_L)$	$\square$	$\bar{\square}$		$+\frac{1}{3}$
C_μ	$\square\bar{\square}$			0
B_μ		$\square\bar{\square}$		0
A_μ			$\square\bar{\square}$	0
ϕ_B		$\square$		1
$\phi_{C\bar{B}}$	$\square$	$\bar{\square}$		$+\frac{1}{3}$
$\phi_{B\bar{A}}$		$\square$	$\bar{\square}$	$+\frac{1}{3}$
$\phi_{A\bar{C}}$	$\bar{\square}$		$\square$	

Table 5.1: Representations for various fields in the theory

scalar potential (discussed in section 5.2.5) will induce the following VEV structure:

$$\langle\phi_{C\bar{B}}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \quad \langle\phi_{B\bar{A}}\rangle = \begin{pmatrix} \tilde{v} & 0 \\ 0 & 0 \\ 0 & b \end{pmatrix} \quad \langle\phi_{A\bar{C}}\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \quad \langle\phi_B\rangle = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \tag{5.1}$$

The scalar potential will mean that the VEVs a, b c are much larger that v and $\tilde{v}$ which will turn out to be the electroweak breaking scale.

The gauge bosons for the groups $U_C(4)$, $U_C(3)$ and $U_C(2)$ are respectively:

C^p ($p = 0..15$), B^q ($q = 0..8$) and A^r ($r = 0..3$). So, for example, ϕ_B which transforms as

$$\phi_B \to U * \phi_B \qquad U \in U_B(3) \tag{5.2}$$

will have a covariant derivative:

$$D_\mu * \phi_B = \partial_\mu * \phi_B + i g_B B^q_\mu t^q * \phi_B \tag{5.3}$$

and its Hermitian conjugate which transforms in the anti-fundamental, i.e.

$$\phi_B^\dagger \to \phi_B^\dagger * U^{-1} \qquad U \in U_B(3) \tag{5.4}$$

will have a covariant derivative:

$$D_\mu * \phi_B^\dagger = \partial_\mu * \phi_B^\dagger + i g_B \phi_B^\dagger * B^q_\mu t^q \tag{5.5}$$

In the following discussion we will neglect the generator of the trace U(1) of each group (C^0, B^0 and A^0) for simplicity. These generators will decouple at low-energies and would merely complicate the analysis, the results in the infra-red region of the theory are unchanged.

5.2.1 The Gauge Sector

The vacuum expectation value for $\phi_{C\bar{B}}$ will partially break the gauge group. The covariant derivative is:

$$D_\mu \langle \phi_{C\bar{B}} \rangle = \partial_\mu \langle \phi_{C\bar{B}} \rangle + \frac{i}{2} g_C C^p_\mu T^p * \langle \phi_{C\bar{B}} \rangle - \frac{i}{2} g_B \langle \phi_{C\bar{B}} \rangle * B^q_\mu \lambda^q \tag{5.6}$$

where the $SU(4)$ generators $T^{1\ldots15}$ are listed in Appendix B and the $SU(3)$

generators $\lambda^{1..8}$ are taken to be the Gell-Mann matrices. The $(D_\mu\langle\phi_{C\bar{B}}\rangle)^\dagger (D_\mu\langle\phi_{C\bar{B}}\rangle)$ term in the Lagrangian will contain diagonal mass-terms:

$$\frac{a^2}{8}\left(\begin{array}{r} g_3^2\left((C_\mu^9)^2+(C_\mu^{10})^2+(C_\mu^{11})^2+(C_\mu^{11})^2+(C_\mu^{13})^2+(C_\mu^{14})^2\right) \\ +g_2^2\left((B_\mu^4)^2+(B_\mu^5)^2+(B_\mu^6)^2+(B_\mu^7)^2\right)\end{array}\right) \tag{5.7}$$

and non-diagonal mass-terms:

$$\frac{a^2}{8}\left(\frac{3}{2}g_3^2(C_\mu^{15})^2+\frac{4}{3}g_2^2(B_\mu^8)^2-2\sqrt{(}2)g_2g_3B_\mu^8C_\mu^{15}\right) \tag{5.8}$$

If we rotate to a new basis:

$$\begin{pmatrix} M_\mu^1 \\ M_\mu^2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{CB} & \sin\theta_{CB} \\ -\sin\theta_{CB} & \cos\theta_{CB} \end{pmatrix} \begin{pmatrix} C_\mu^{15} \\ B_\mu^8 \end{pmatrix} \tag{5.9}$$

where

$$\cos\theta_{CB} = \frac{\sqrt{2}g_2}{\sqrt{2g_2^2+\frac{9}{4}g_3^2}} \qquad \sin\theta_{CB} = \frac{\frac{3}{2}g_3}{\sqrt{2g_2^2+\frac{9}{4}g_3^2}} \tag{5.10}$$

Then M_μ^1 will be massless but M_μ^2 will acquire a mass so, out of the $U(4)\times U(3)$ that we start with, the following gauge bosons are still massless: $C_\mu^{1..8}$ (which we will identify with the $SU(3)_C$ of the Standard Model, $B_\mu^{1..3}$ (which will will identify with $SU(2)_L$) and M_μ^1.

The covariant derivative for $\phi_{B\bar{A}}$ will lead to a term involving its vacuum expectation value. Because $\tilde{v} << b$ (in equation (5.1)) we will temporarily set $\tilde{v} \to 0$

$$D_\mu\langle\phi_{B\bar{A}}\rangle = \partial_\mu\langle\phi_{B\bar{A}}\rangle + \frac{i}{2}g_B B^q\gamma^q * \langle\phi_{B\bar{A}}\rangle - \frac{i}{2}g_A\langle\phi_{C\bar{B}}\rangle * A^r\sigma^r \tag{5.11}$$

where the $SU(2)$ generators σ^r are the usual Paul matrices. However, $B^8_\mu = \sin\theta_{CB} M^1_\mu + \cos\theta_{CB} M^2_\mu$ so ignoring the massive gauge bosons we have:

$$D_\mu\langle\phi_{B\bar{A}}\rangle = \partial_\mu\langle\phi_{B\bar{A}}\rangle + \frac{i}{2} g_B B^q_\mu \gamma^q * \langle\phi_{B\bar{A}}\rangle + \frac{i}{2} g_0 M^1_\mu \gamma^8 * \langle\phi_{B\bar{A}}\rangle - \frac{i}{2} g_A \langle\phi_{C\bar{B}}\rangle * A^r_\mu \lambda^r \tag{5.12}$$

where $g_0 = g_B \sin\theta_{CB}$ and now $q, r = 1..3$.

The resulting diagonal mass-terms will be:

$$\frac{b^2}{8}\left(g_1^2\left((A^1_\mu)^2 + (A^2_\mu)^2\right)\right) \tag{5.13}$$

and the remaining mass-terms are:

$$\frac{b^2}{8}\left(g_1^2(A^3_\mu)^2 + \frac{4}{3}g_0^2(M^1_\mu)^2 - \frac{4}{\sqrt{3}}g_0 g_1 (A^3_\mu)(M^1_\mu)\right) \tag{5.14}$$

We can diagonalise these by writing:

$$\begin{pmatrix} Y_\mu \\ M^3_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{BA} & \sin\theta_{BA} \\ -\sin\theta_{BA} & \cos\theta_{BA} \end{pmatrix} \begin{pmatrix} C^{15}_\mu \\ B^8_\mu \end{pmatrix} \tag{5.15}$$

where

$$\cos\theta_{BA} = \frac{\sqrt{3}g_1}{\sqrt{3g_1^2 + 4g_0^2}} \qquad \sin\theta_{BA} = \frac{\frac{3}{2}g_3}{\sqrt{3g_1^2 + 4g_0^2}} \tag{5.16}$$

The field labelled Y_μ is the gauge boson for a massless $U(1)$ and will be identified with the hypercharge whilst the M^3_μ field has acquired a mass.

If we now calculate which gauge degrees of freedom are given a mass by $\phi_{A\bar{C}}$ it will turn out that no massless degrees of freedom acquire a mass; the gauge group is broken no further. In particular the Y_μ field remains unchanged.

To summarise, the microscopic group $U_C(4)$ has been broken to $SU(3)\times U(1)$, $U_B(3)$ has been broken to $SU(2)_L \times U(1)$ and $U_A(2)$ has been broken to $U(1)$, Only one linear combination of the three U(1) factors remains massless.

5.2.2 Hypercharges

The hypercharge for each particle is determined by the representation of the particle under the microscopic gauge groups. The ideas in this section follow [9, 53] but because of our unusual gauge-group $(U(4) \times U(3) \times U(2))$ the details differ.

The coupling of the right handed electron (c.f table 5.1) to the hypercharge is determined by:

$$\begin{aligned}\bar{e}_R\gamma^\mu D_\mu e_R =&\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{i}{2}g_1\bar{e}_R\gamma^\mu e_R A^3_\mu\sigma^3 + \frac{i}{2}g_1\bar{e}_R\gamma^\mu[e_R, A^3_\mu]\sigma^3 \\ =&\bar{e}_R\gamma^\mu\partial_\mu e_R + \frac{i}{2}g_1\bar{e}_R\gamma^\mu e_R\sin\theta_{BA}Y_\mu\sigma^3 \end{aligned} \tag{5.17}$$

We have ignored the $[e_R, A^3_\mu]$ term as we are considering scales well below the noncommutative scale. The coupling between Y_μ and the particle in the first row of the U(2) doublet is therefore $g_A\sin\theta_{BA}$. This should be proportional to the hyper-charge, $-2g'$ where we define $g' \equiv \frac{1}{2}g_A\sin\theta_{BA}$ to be the coupling to the hypercharge. With this definition, the hyper-charge of all the other particles in the model is now fixed.

The right-handed down quark transforms in the fundamental of $U_C(4)$:

$$\bar{d}_R\gamma_\mu D_\mu d_R = \bar{d}_R\gamma_\mu\partial_\mu d_R + \frac{i}{2}g_C\bar{d}_R\gamma_\mu d_R T_{15}C^{15}_\mu + ... \tag{5.18}$$

Writing $C^{15}_\mu = \cos\theta_{CB} M^1_\mu - \sin\theta_{CB} M^2_\mu$ and then $M^1_\mu = \cos\theta_{BA} Y_\mu - \sin\theta_{BA} N^3_\mu$ then the term that will determine the coupling is:

$$\frac{i}{2} g_C \bar{d}_R \gamma^\mu d_R T_{15} \cos\theta_{CB} \cos\theta_{BA} Y_\mu \tag{5.19}$$

So the coupling for the right-handed down quark (for all except the fourth particle in the multiplet) is:

$$Y_{d_R} = \frac{g_C}{\sqrt{6}} \cos\theta_{CB} \cos\theta_{BA} \tag{5.20}$$

Using equations (5.10) and (5.16) we find $\frac{Y_{d_R}}{g'} = \frac{-2}{3}$ i.e. the Standard Model value.

We can calculate the hypercharges of the other particles in an analogous fashion. For example the multiplet of left-handed leptons has a term which can be written (ignoring massive fields):

$$\begin{aligned} &= \frac{i}{2} \bar{\psi}^l_L \gamma_\mu \psi^l_L (g_A \sigma^3 A^3_\mu - g_B B^8_\mu \lambda^8) \\ &= \frac{i}{2} \bar{\psi}^l_L \gamma_\mu \psi^l_L (g_A \sigma^3 \sin\theta_{BA} Y_\mu - g_B \lambda^8 \sin\theta_{CB} M^1_\mu) \\ &= \frac{i}{2} \bar{\psi}^l_L \gamma_\mu \psi^l_L (g_A \sigma^3 \tan\theta_{BA} \cos\theta_{BA} - g_B \lambda^8 \sin\theta_{CB} \cos\theta_{BA}) Y_\mu \end{aligned} \tag{5.21}$$

So the hypercharge of the left-handed leptons will be:

$$\begin{aligned} \frac{Y_{\psi^l_L}}{g'} &= \frac{\left(g_A \tan\theta_{BA} - \frac{g_B}{\sqrt{3}} \sin\theta_{CB}\right) \cos\theta_{BA}}{\frac{1}{2} g_A \cos\theta_{BA} \tan\theta_{BA}} \\ &= -1 \end{aligned} \tag{5.22}$$

The hypercharges of the other fields is listed in table (5.1) and each agrees with its Standard Model value.

5.2.3 Anomalies and Extra Fields

Anomalies have been thoroughly studied in a noncommutative context [22, 23, 58, 59]. Unlike the commutative anomaly cancellation conditions (section 4.4) noncommutative theories have a much more stringent condition. The contribution from a chiral fermion is proportional to:

$$\mathrm{Tr}(t^a t^b t^c) \equiv \frac{1}{2}\mathrm{Tr}(t^a\{t^b, t^c\}) + \frac{1}{2}\mathrm{Tr}(t^a[t^b, t^c]) \tag{5.23}$$

where right and left handed fermions give a contribution of opposite sign. Notice that the first term on the right-hand side of equation (5.23) looks like the commutative piece but the second term is new. The generally accepted conclusion is that in order for a noncommutative theory to be free of chiral anomalies, the theory must be vector-like.

The matter content introduced so far is chiral, as it must be in order to match the Standard Model matter content at low energies; we have left-handed (but no right-handed) fermions under the $U_B(3)$ gauge group that will become the $SU(2)_L$ group in the low-energy limit of the theory. To fix the problem we introduce three extra heavy generations, one for each observed generation. Each particle in these heavy generations must have the opposite chirality to their Standard Model counterpart.

Although these extra generations circumvent the problems with anomalies, this might also be possible by adding fewer fields to the theory. However, in the next section we will see that these extra heavy generations are essential when writing down the necessary Yukawa terms for the theory.

We must also add three more fields to the model. As discussed earlier there are two conditions that need to be met in order to prevent the IR/UV effects rendering the theory unphysical. The first condition [2] eliminates quadratic divergences which in the polarisation tensor of the decoupled U(1) gauge bosons; for each fermion in the adjoint representation of a gauge group, there is a gauge field or complex scalar that is also in the adjoint representation (and vice-versa). We have adjoint gauge fields but no adjoint matter so we add massive adjoint fermion fields, one per microscopic gauge group; λ^A, λ^B and λ^C and note that a supersymmetric theory, the role of these fields would neatly be filled by the gaugino. Table 5.2 summarises the extra matter we have needed to add to the theory.

Field	$U_C(4)$	$U_B(3)$	$U_A(2)$	Hypercharge
E_L			$\square$	-2
N_L				0
$L_R = \begin{pmatrix} N_R & E_R \end{pmatrix}$		$\bar{\square}$	$\square$	-1
U_L	$\square$		$\bar{\square}$	$+\frac{4}{3}$
D_L	$\square$			$-\frac{2}{3}$
$Q_R = \begin{pmatrix} U_R & D_R \end{pmatrix}$	$\square$	$\bar{\square}$		$+\frac{1}{3}$
λ^C	$\square\bar{\square}$			0
λ^B		$\square\bar{\square}$		0
λ^A			$\square\bar{\square}$	0

Table 5.2: Representations for the "extra" fields added to the theory

[2] the second condition prevents tachyons and puts a constraint on the spectrum of masses for the adjoint particles (equation (2.96)) but no further matter needs to be introduced

5.2.4 Yukawa Couplings

Unlike the Standard Model, multiple Higgs fields are required in order give mass to all the particles. The Yukawa terms can be arranged into two categories. Firstly, there are terms that involve fields from the same generation. Secondly, because we have generations with opposite chirality, we have novel terms involving fields from different generations. Additionally, as in the Standard Model, there can be the usual mixing between the generations but we neglect these here for simplicity.

Yukawa terms of the first type are (for one light generation):

$$\nu_R^\dagger \phi_{B\bar{A}} l^L_{A\bar{B}} + e^R_A \phi^\dagger_{\bar{B}} l^\dagger_{LB\bar{A}} + q^L_{C\bar{B}} \phi_{B\bar{A}} u^\dagger_{RA\bar{C}} + d^R_C \phi^\dagger_{\bar{B}} q^\dagger_{LB\bar{C}} + \text{h.c} \tag{5.24}$$

and for a heavy generation:

$$N_L^\dagger \phi_{B\bar{A}} L^R_{A\bar{B}} + E^L_A \phi^\dagger_{\bar{B}} L^\dagger_{RB\bar{A}} + Q^R_{C\bar{B}} \phi_{B\bar{A}} U^\dagger_{LA\bar{C}} + D^L_C \phi^\dagger_{\bar{B}} Q^\dagger_{RB\bar{C}} + \text{h.c} \tag{5.25}$$

These terms on their own are not sufficient to give large masses to all those particles which are not observed at low energies, for example there is a fourth "colour" of quark that would not interact with the strong force, as the gauge group has been broken from $SU(4)$ to $SU(3)$ but would still interact electromagnetically.

The extra three generations in which each particle has the opposite chirality to its Standard Model equivalent (as introduced in section 5.2.3 to cure the problems with anomalies) also cures the problem here. The possible terms that mix a light generation with a heavy generation are:

$$E^{\dagger}_{L\bar{A}}\phi_{A\bar{C}}d^{R}_{C} + L^{\dagger}_{RB\bar{A}}\phi_{A\bar{C}}q^{L}_{C\bar{B}} + N^{\dagger}_{L}\phi_{A\bar{C}}u^{R}_{C\bar{A}} + l^{L}_{A\bar{B}}\phi^{\dagger}_{B\bar{C}}U^{\dagger}_{LC\bar{A}} + q^{L}_{C\bar{B}}\phi^{\dagger}_{B\bar{C}}N^{L} + \text{h.c.} \tag{5.26}$$

Notice that in the above generation-mixing terms (which violate baryon and lepton number) neither of the Higgs with an electroweak scale vacuum expectation appear, so leptoquark would only occur at a high energy scale, characterised by the a, b and c vacuum expectation values.

When all possible such Yukawa terms are included, the particle content of the model at low energies agrees with the observed spectrum of particles. Moreover the form of the coupling gives a natural explanation for the extremely small mass of the left-handed neutrino in the three light generations, the see-saw effect will naturally suppress their masses to be of order $\tilde{v}^2/a$ although there are enough parameters to keep the neutrinos in the three extra generations above the experimental bounds.

5.2.5 The Higgs Potential

The pattern of symmetry breaking and mass splittings in the preceding sections was dependent on a particular pattern (5.1) of vacuum expectation values for the Higgs fields. We now will construct a simple example of the scalar potential which generates the vev structure in Eq. (5.1).

First, using gauge transformations $SU_B(3)$, we put ϕ_B in the canonical form:

$$\langle\phi_B\rangle = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} . \tag{5.27}$$

Next, we require that $\langle \phi^{\dagger}_{\bar{B}} \phi_{B\bar{A}} \rangle = 0$ and further use the $SU_A(2)$ and $SU_B(2)$ transformations[3] to diagonalise $\phi_{B\bar{A}}$:

$$\langle \phi_{B\bar{A}} \rangle = \begin{pmatrix} \tilde{v} & 0 \\ 0 & 0 \\ 0 & b \end{pmatrix} . \tag{5.28}$$

Next we turn to $\phi_{C\bar{B}}$ and require that $\langle \phi_{C\bar{B}} \phi_B \rangle = 0$. We also use $SU_C(4)$ to simplify $\phi_{C\bar{B}}$ further, such that:

$$\langle \phi_{C\bar{B}} \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f \\ g & 0 & a \end{pmatrix} . \tag{5.29}$$

At this stage to achieve relatively simple expressions in (5.27),(5.28),(5.29), we have used all the available gauge symmetry and the orthogonality conditions which follow from the potential:

$$\lambda_1 |\phi^{\dagger}_{\bar{B}} \phi_{B\bar{A}}|^2 + \lambda_2 |\phi_{C\bar{B}} \phi_B|^2 . \tag{5.30}$$

This essentially leaves the third bi-fundamental Higgs unrestricted at this stage,

$$\langle \phi_{A\bar{C}} \rangle = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ z_5 & z_6 & z_7 & c \end{pmatrix} . \tag{5.31}$$

Before imposing restrictions on $\phi_{A\bar{C}}$, we would like to first further simplify the expressions (5.27),(5.28),(5.29).

[3] $SU_B(2)$ is the subgroup of $SU_B(3)$ which leaves (5.27) invariant.

We introduce another term in the scalar potential,

$$\lambda_3 \left(\mathcal{D}_{B\bar{B}} - \mu_1^2 \mathbb{1}_{B\bar{B}}\right)^2 \,, \tag{5.32}$$

where $\mathcal{D}_{B\bar{B}}$ is a bilinear combination of Higgs fields:

$$\mathcal{D}_{B\bar{B}} \equiv \phi_{B\bar{A}}\phi^{\dagger}_{A\bar{B}} - \phi^{\dagger}_{B\bar{C}}\phi_{C\bar{B}} + \phi_B \phi^{\dagger}_{\bar{B}} \,. \tag{5.33}$$

On the right hand side of (5.33) the indices $\bar{A}, A$ and $\bar{C}, C$ are summed over, but not the indices $\bar{B}, B$ which are left free, so that $\mathcal{D}_{B\bar{B}}$ transforms in the adjoint of $SU_B(3)$. The scalar potential (5.32) contains a trace over gauge indices, hence $\bar{B}, B$ are finally summed over, and the Higgs potential (5.32) is a gauge singlet.

At the minimum of the potential (5.32) we have,

$$\begin{pmatrix} |\tilde{v}|^2 - |g|^2 & 0 & -g^{\dagger}a \\ 0 & |v|^2 & 0 \\ -ga^{\dagger} & 0 & |b|^2 - |a|^2 - |f|^2 \end{pmatrix} = \begin{pmatrix} \mu_1^2 & 0 & 0 \\ 0 & \mu_1^2 & 0 \\ 0 & 0 & \mu_1^2 \end{pmatrix} , \tag{5.34}$$

and the vacuum solution is:

$$g = 0 \,, \quad |\tilde{v}|^2 = \mu_1^2 = |v|^2 \,, \quad |b|^2 - |a|^2 - |f|^2 = \mu_1^2 \,. \tag{5.35}$$

We continue reducing the number of free parameters in the vev structure in a similar way to the considerations above and introduce another term in the scalar potential:

$$\lambda_4 \left(\mathcal{E}_{C\bar{C}} - \mu_2^2 \mathbb{1}_{C\bar{C}}\right)^2 \,, \tag{5.36}$$

where $\mathcal{E}_{C\bar{C}}$ is defined as

$$\mathcal{E}_{C\bar{C}} \equiv \phi_{C\bar{B}}\phi^{\dagger}_{B\bar{C}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & |f|^2 & fa^{\dagger} \\ 0 & 0 & f^{\dagger}a & |a|^2 \end{pmatrix} . \tag{5.37}$$

The potential (5.36) is minimal at

$$a = \mu_2 \, , \qquad f = 0 \, , \tag{5.38}$$

which complements the configuration (5.35).

We now return to the so far unconstrained Higgs field (5.31) and write down new terms in the scalar potential

$$\lambda_5 \left(\mathcal{G}_{A\bar{A}} - \mu_1^2 \mathbb{1}_{A\bar{A}}\right)^2 + \lambda_6 \left(\mathcal{K}_{C\bar{C}}\right)^2 \, , \tag{5.39}$$

where

$$\mathcal{G}_{A\bar{A}} \equiv -\phi_{A\bar{C}}\phi^{\dagger}_{C\bar{A}} + \phi^{\dagger}_{A\bar{B}}\phi_{B\bar{A}} \, , \quad \mathcal{K}_{C\bar{C}} \equiv \phi_{C\bar{B}}\phi^{\dagger}_{B\bar{C}} - \phi^{\dagger}_{C\bar{A}}\phi_{A\bar{C}} \, . \tag{5.40}$$

The minimum of (5.39) is

$$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = 0 \, , \quad |c|^2 = |a|^2 = \mu_2^2 \, , \quad f = 0 \, . \tag{5.41}$$

The combined vacuum configuration gives

$$\langle\phi_{C\bar{B}}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \quad \langle\phi_{B\bar{A}}\rangle = \begin{pmatrix} \mu_1 & 0 \\ 0 & 0 \\ 0 & \sqrt{\mu_2^2+\mu_1^2} \end{pmatrix} \quad \langle\phi_{A\bar{C}}\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix} \quad \langle\phi_B\rangle = \begin{pmatrix} 0 \\ \mu_1 \\ 0 \end{pmatrix}$$

which reproduces (5.1).

5.3 Summary of the Chapter

The model that has been introduced in this chapter is constructed to fulfil the constraints placed on a noncommutative model as explained in the introduction to this chapter.

In order to meet these requirements we have been forced to introduce new matter fields. Because of the UV/IR mixing we have used the gauge group $U(4) \times U(3) \times U(2)$. In order for this group to be broken to the Standard Model it has been necessary to introduced three new Higgs fields in addition to the Standard Model Higgs. Because the matter fields are charged under larger groups they have extra "flavours" (e.g. $U(4)$ rather than $SU(3)$ of colour). These extra fields will naturally become heavy if Yukawa couplings are introduced which mix the three observed light generations of matter with three heavy generations each field having the opposite chirality to its light generation equivalent. These extra heavy generations also cure the problems of noncommutative gauge anomalies which require the theory to be vector-like. Lastly, in order to cause the quadratic divergences to cancel we have introduced three heavy, adjoint fermions, one charged under each $U(N)$ gauge group.

Although we have been forced to introduce many new fields, it seems that this is the minimum number if the world is indeed noncommutative. As the introduction to this thesis argued, noncommutative theories are well motivated. Firstly they are a possible low-energy limit of string theory and more generally we would expect any quantum theory of gravity to include a ultra-violet cut-off which prevents positions of particles from being measured with an arbitrarily high accuracy.

More work will be needed (as set out in the next chapter) before we can tell whether the world is really noncommutative or not.

Chapter 6

Conclusions

> *"At least I know I'm bewildered about the really fundamental and important facts of the universe." Treatle nodded. "I hadn't looked at it like that," he said, "But you're absolutely right. He's really pushed back the boundaries of ignorance."*
>
> Discworld Scientists in "Equal Rites" by Terry Pratchett

As the earlier chapters have shown, in some respects noncommutative quantum field theories are very different to their commutative counter-parts. What is, perhaps, more surprising is that in many ways they are very similar. Given the radical change to the core principles of the theory caused by the introduction of non-locality it might have been expected that constructing models which, at least at low energies, resemble our commutative Standard Model would prove insurmountable.

Our review of noncommutative quantum field theories, carried out in chapter 2, introduced many ways in which such theories are novel. We have seen that $U(N)$ groups (rather than the $SU(N)$ groups used in the Standard Model) are the natural choice to create renormalisable, noncommutative quantum field theories with. We saw that the the allowed representations were the fundamental, anti-fundamental, bi-fundamental and adjoint groups. A $U(1)$ group is non-abelian in noncommutative theories, and as such the charge of particles is fixed to be 0 or ± 1. Critically, the ultra-violet/infra-red mixing causes the coupling of a $U(1)$ gauge group to go to zero in the infra-red. This decoupling also applies to the trace $U(1)$ of a $U(N)$ gauge group, causing it to be dynamically broken into $SU(N)$ (and the decoupled $U(1)$). Quadratic divergences were also considered. In order to remove the quadratic divergences it was found that there needed to be the same number of fermionic and bosonic degrees of freedom in the adjoint representation. Ultra-violet/infra-red mixing effects could still generate a tachyonic mass for the decoupled trace $U(1)$ gauge boson however tachyonic masses could be avoided if the masses of fields in the adjoint representation fulfilled a certain requirement (see (2.96)).

We saw in chapter 3 that the logarithmic divergences which cause the decoupling can be modelled with effective actions that contain open Wilson lines. These actions are explicitly $U(N)$ gauge invariant but they are dressed with higher-derivative operators that become irrelevant in the infra-red.

Chapter 5 introduced the Noncommutative Standard Model. The model meets all the key constraints imposed by noncommutativity. In particular the model was based on $U(4)\times U(3)\times U(2)$ rather than $SU(3)\times SU(2)\times U(1)$. The gauge-group is broken to the Standard Model at low energies but the larger gauge group is necessary in order to prevent the ultra-violet/infra-red effects decoupling the $U(1)$ of hypercharge. These effects have been ignored in earlier models, making them unviable as models of the real world. It was found that in order to replicate the field content of the Standard Model new fields have to be introduced. Firstly a fermionic "partner" needs to be introduced for each gauge boson in order to ensure the model contains no quadratic divergences. In addition it was necessary to add three generations of right-handed matter. These extra generations made it possible for the model to be free of noncommutative anomalies which require that the matter content be vector not chiral. These extra generations also allow Yukawa terms which cause the extra flavours (caused by the larger gauge group) to be given a large mass.

Now such a model has been constructed, the obvious question is: what direction should future related research take? There seem to be two related avenues of research, both of which would be highly interesting. Firstly, the decoupled $U(1)$ degrees of freedom provide the possibility of creating a dynamically broken supersymmetric version of the model presented in this thesis [57]. If realistic mass spectrums could be generated by adding Fayet-Illiopoulos terms to the decoupled $U(1)$s, eliminating the need for "soft" supersymmetry breaking terms it would be a massive step forward.

It would also be interesting to study the phenomenology of the model presented here. It might naively be expected that because of the UV/IR mixing, low energy experiment should be able to set very stringent limits on noncommutativity, even at very high energies. However because the actual effects cause the affected degrees of freedom to become unobservable, it is not so straight-forward. However a detailed study has not been performed. It may be possible to set tight limits or even rule out the model.

In conclusion, the model presented here may be just the beginning and will hopefully generate significant engaging new research in the future. Being extremely optimistic, the new matter predicted by the model may be discovered by the LHC and therefore we look forward to the future with interest.

Appendix A

Generators of SU(4)

In the last chapter of this thesis we take the generators of SU(4) to be $t^a = \frac{1}{2}T^a$ where $a = 1..15$ and the T^a are:

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$T_{10} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad T_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad T_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$T_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad T_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad T_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

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