The Configuration Space of Two Particles Moving on a Graph

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A Thesis presented for the degree of
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Dedicated to

My Father

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Abstract

In this thesis we study the configuration space, $F(\Gamma, 2)$, of two particles moving without collisions on a graph Γ with a view to calculating the Betti numbers of this space. We develop an intersection theory for cycles in graphs inspired by the classical intersection theory for cycles in manifolds and we use this to develop an algorithm to calculate the second Betti number of $F(\Gamma, 2)$ for any graph Γ. We also use this intersection theory to provide a complete description of the cohomology algebra $H^*(F(\Gamma, 2), \mathbb{Q})$ for any planar graph Γ and to calculate explicit formulae for the Betti numbers of $F(\Gamma, 2)$ when Γ is a complete graph or a complete bipartite graph. We also investigate the generators of group $H_2(F(\Gamma, 2), \mathbb{Z})$ and show that for any planar graph this group is entirely generated by tori induced by disjoint cycles in the graph. For non-planar graphs the situation is more complicated and we show that there can exist a generator of $H_2(F(\Gamma, 2), \mathbb{Z})$ which is not the fundamental class of a surface embedded in the space $F(\Gamma, 2)$.

Declaration

The work in this thesis is based on research carried out at the Pure mathematics group, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

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I must also thank all those postgraduate students with whom I have shared offices, laughs, and sometimes tears, over the last four years; Femke, Ruth, Tan and many more too numerous to mention here. Special thanks must go to Joey Oliver who has provided invaluble support, friendship (and cricket commentary!) throughout the three years we shared an office. None of my acheivements would be possible without the love and support of my parents who I would like to thank for giving me the confidence to pursue this PhD.

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Chapter 1

Introduction

The configuration space of n distinct points, ordered or unordered, lying in a topological space X plays an important role in modern topology and has been extensively studied. Such spaces were introduced by Fadell and Neuwirth (for the case $X = \mathbb{R}^n$) in 1962 in [14] and since then many important results have been obtained. These configuration spaces have strong links with the study of braid groups; the fundamental group of the configuration space of n distinct, unordered points on the Euclidean plane is isomorphic to the classical braid group on n strings [14]. Configuration spaces have been used in the calculation of many important results on the subject of braid groups, for example by Arnol'd [4], Cohen [8] and Vassiliev [34]. The case where X is an algebraic variety or an orientable manifold was studied by Totaro in [33], where he introduces a spectral sequence which describes the cohomology algebra of these configuration spaces.

This thesis explores the topological properties of the configuration space of 2 distinct points lying on a finite graph. Spaces of n points moving without collision on a finite graph arise in the study of topological robotics and were first studied in this context by Ghrist in [21] and Ghrist and Koditschek in [22]. Such studies are motivated by problems in engineering involving the scheduling of a number of Automated Guided Vehicals (AGVs) operating without collisions on a pre-defined network of paths, i.e. a planar graph [22]. These spaces have also been studied by Farber in [15, 16] where the *topological complexity* ([16] Definition 4.6) of the configuration space of

n ordered points on a tree is calculated and this configuration space is explicitly described for the case $n = 2$ ([15] Theorem 10). There has also been some recent work on calculating the fundamental group of configuration spaces of n unordered particles moving on a tree, this was studied by Sabalka and Farley in [19] and by Abrams in [1]. The homology and cohomology of such spaces have also been studied by Farley in [17, 18] and by Sabalka and Farley in [20].

The main aim of this thesis is to study the configuration space of *two* ordered points on a graph, where the graph in question is not a tree, i.e. it contains non-trivial cycles. This thesis attempts to describe the homology and cohomology of such configuration spaces, concentrating on calculating the Betti numbers of the space. Such spaces have been studied as topological spaces in their own right, not linked to problems in engineering, under the name *deleted product spaces*. A survery of previous work on these spaces is given in Section 5.4.

We will now describe the structure of this thesis. Chapter 2 begins by describing some notation and concepts from graph theory which will be used throughout this work. We then introduce the configuration space, $F(\Gamma, n)$, the space of n particles moving on a finite graph without collisions, and the discretized version of this space, $D(\Gamma, n)$, which can be given the structure of a CW-complex that we will exploit throughout this thesis. We give examples of these spaces and, for $n = 2$, show that $D(\Gamma, 2)$ is homotopy equivalent to $F(\Gamma, 2)$ in all non-trivial cases. This chapter ends by stating a conjecture which describes the generators of the group $H_2(F(\Gamma, 2), \mathbb{Z})$. The conjecture states that the generators of this group come from surfaces embedded in the space $F(\Gamma, 2)$ which correspond to disjoint cycles and copies of the Kuratowski graphs, K_5 and $K_{3,3}$, embedded in the graph Γ. Finally we prove a lemma showing that the group $H_2(F(\Gamma, 2), \mathbb{Z})$ is isomorphic to the 2-dimensional oriented bordism group of $F(\Gamma, 2)$. The material in this chapter is not orignal and constitutes an overview of results and examples previously obtained in the subject.

In Chapter 3 we introduce the tools used to obtain the main results of this thesis. Our approach is based on studying the intersections of cycles in graphs in a similar spirit to the classical intersection theory for cycles on manifolds. The work in this chapter is an expansion on the first two sections of the joint paper [6] by my supervisor and myself. We begin by decomposing the space $\Gamma \times \Gamma$, introducing the extended diagonal N_Γ, whose structure is described using a distance function defined on the space $\Gamma \times \Gamma$. We then introduce the main tool used in this thesis, the *intersection form*,

$$I_\Gamma : H_1(\Gamma) \otimes H_1(\Gamma) \to H_2(N_\Gamma, \partial N_\Gamma)$$

and prove that group $H_2(F(\Gamma, 2), \mathbb{Z})$ is isomorphic to the kernel of the intersection form and the group $H_1(F(\Gamma, 2), \mathbb{Z})$ is isomorphic to the direct sum of the cokernel of the intersection form with two copies of the group $H_1(\Gamma, \mathbb{Z})$. Throughout this thesis all homology groups will be taken to have integer coefficients unless otherwise stated. In the last two sections of the chapter we discuss a method for expressing the intersection of two cycles in a graph as elements of a direct sum of 'local homology groups', each of which describes the properties of small subgraphs within the graph. This allows us to develop an algorithm to calculate the second Betti number of $F(\Gamma, 2)$ for any simple graph Γ.

The main result of this thesis is given at the beginning of Chapter 4. We calculate the second Betti number of $F(\Gamma, 2)$ for any planar graph Γ, proving that the group $H_2(F(\Gamma, 2), \mathbb{Z})$ is generated by tori given by products of disjoint cycles in the graph. In the remainder of this chapter we investigate the first Betti number of the space $F(\Gamma, 2)$ and calculate simple formulas for the first and second Betti number of $F(\Gamma, 2)$ for a large class of planar graphs which we call *regular planar graphs*. Finally, we describe the cohomology algebra with rational coefficients of $F(\Gamma, 2)$ for any regular planar graph Γ. The results in this chapter were published in the joint paper by myself and my supervisor [6].

In the final chapter of this thesis we turn our attention to non-planar graphs and consider the question; for which graphs is the intersection form epimorphic? The material in this chapter is original and has not been published elsewhere in the literature. We show that for all complete graphs and complete bipartite graphs the intersection form is epimorphic and this allows us to compute simple formulas for the Betti numbers of $F(\Gamma, 2)$ where Γ is a complete graph or a complete bipartite graph.

The results of this chapter also have the interesting consequence of showing that the second homology group of $F(\Gamma, 2)$ can have generators which are not induced by disjoint cycles or copies of the Kuratowski graphs embedded in the graph. The thesis ends with a discussion of previous work published concerning configuration spaces of two particles moving on a graph, and how the results of this thesis fit in to the context of this work.

Chapter 2

Configuration Spaces of Particles Moving on a Graph

We begin this chapter by introducing the main objects of study in this thesis. In Section 1 we intoduce the concepts and notations from graph theory which will be used throught this work. We then define the configuration space of n particles moving without collision on a graph and give some examples of these spaces. In Section 3 we discuss the main object of study for this thesis, the configuration space of *two* particles moving on a graph, and give some important basic results about this space. We end the chapter by describing a conjecture about the structure of the second homology group of such spaces and show that this homology group is isomorphic to the 2-dimensional oriented bordism group of the graph.

2.1 Definitions and Notation from Graph Theory

Throughout this thesis our main objects of study will be finite graphs.

Definition 2.1.1 A *finite graph* Γ is a one-dimensional CW-complex with finitely many cells, the zero-cells of Γ are called *vertices* and the one-cells are *edges.*

Unless otherwise stated, we will consider only connected finite graphs throughout this work.

A graph can be given an orientation by chosing a direction along each edge. If an edge e is oriented as in Figure 2.1 then the boundary of e, ∂e, viewing e as a one-dimensional cellular chain, is equal to the zero-dimensional chain $v_2 - v_1$. Throughout this thesis we will fix an orientation on all graphs we consider. Unless indicated otherwise, an edge e will refer to the closure of the one-cell e, i.e. the union of the cell e and its boundary vertices. We say that two edges are *incident* if

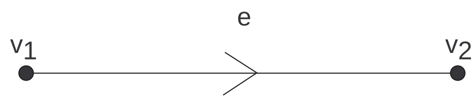


Figure 2.1: An oriented edge.

they have a common boundary vertex and that two vertices v_1 and v_2 are *joined* by an edge e if the boundary of e is equal to $\pm(v_2 - v_1)$.

Many of our results concern simple graphs,

Definition 2.1.2 A graph Γ is said to be *simple* if it contains no loops or multiple edges i.e., the boundary of every edge of Γ is the union of exactly two vertices, and no two vertices of Γ are joined by more than one edge.

In Chapter 4 we give some results concerning planar graphs.

Definition 2.1.3 A graph Γ is said to be *planar* if there exists a combinatorial embedding of Γ into the plane $\mathbb{R}^2$.

A *combinatorial embedding* is a map from the graph Γ into the plane $\mathbb{R}^2$ which maps the set of vertices of Γ into the plane in such a way that no two vertices occupy the same point of the plane and maps the edges of Γ in such a way that no two edges occupy the same point of the plane except at a vertex, this map also defines a topological embedding. Such an embedding is called a *plane drawing* of Γ.

The two *Kuratowski graphs*, K_5 and $K_{3,3}$, play an important role in this thesis. The graph K_5 is the complete graph on 5 vertices, that is the graph with 5 vertices such that every vertex is joined to every other vertex by exactly one edge. The graph $K_{3,3}$ is a complete bipartite graph. This graph has 6 vertices divided into two sets $\mathbb{X}$ and $\mathbb{Y}$ each containing three vertices. The set of edges of $K_{3,3}$ contains 9 edges so that each vertex of $\mathbb{X}$ is joined to every vertex of $\mathbb{Y}$ by exactly one edge.

There is a well known result of Kuratowski which states that a graph cannot be embedded in the plane $\mathbb{R}^2$ if and only if it contains a subdivision of either K_5 or $K_{3,3}$ as a subgraph.

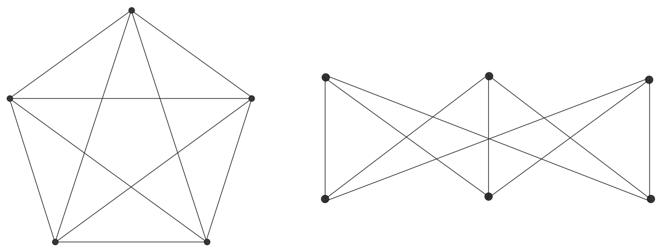


Figure 2.2: The Kuratowski graphs (left) K_5 (right) $K_{3,3}$

2.2 The Space $F(\Gamma, n)$

We begin by describing the configuration space of n particles moving without collisions on a graph. This is constructed by taking the n-fold product of Γ with itself and removing the diagonal,

$$\Delta = \{(x_1, \cdots, x_n) \in \Gamma \times \cdots \times \Gamma;\ x_i = x_j \text{ for some } i \neq j\}.$$

Definition 2.2.1 The *n-point configuration space* of a graph Γ, $F(\Gamma, n)$, is defined as,

$$F(\Gamma, n) = \{(x_1, x_2, \cdots, x_n) \in \Gamma \times \Gamma \times \cdots \times \Gamma; x_i \neq x_j \text{ for } i \neq j\} \tag{2.1}$$

The n-fold product of Γ is an n-dimensional CW-complex with m-cells constructed by taking the product of m edges with $n-m$ vertices. However, the space $F(\Gamma, 2)$ is not compact and cannot be given the structure of a finite cell-complex. We construct another space which does admit such a structure.

Definition 2.2.2 The *discrete n-point configuration space* of a graph Γ, $D(\Gamma, n)$ is defined as the union,

$$D(\Gamma, n) = \bigcup (\bar{\sigma_1} \times \bar{\sigma_2} \times \cdots \times \bar{\sigma_n}) \tag{2.2}$$

of all possible products

$$(\bar{\sigma_1} \times \bar{\sigma_2} \times \cdots \times \bar{\sigma_n})$$

where σ_i is a cell of Γ and $\bar{\sigma_i} \cap \bar{\sigma_j} = \emptyset$ for $i \neq j$ with $i, j = 1, \cdots, n$.

The space $D(\Gamma, n)$ is the space of configurations where the n particles lie on n cells whose closures are disjoint. So if one particle is lying on an edge, e, of Γ then no other particle can lie on the edge e, on either of the boundary vertices of e, or on any edge incident to e. Then $D(\Gamma, 2)$ admits an n-dimensional CW-complex structure with an m-cell described by a product of m disjoint edges of Γ with $n - m$ distinct vertices of Γ which are not boundary vertices of any of the m edges.

Example 2.2.1 We will describe the spaces $F(\Gamma, 2)$ and $D(\Gamma, 2)$ where Γ is the 'Y-graph'. That is the tree with three edges incident to one central vertex shown in Figure 2.3. This example is also described in the paper of Ghrist and Abrams, [3]. In this case $\Gamma \times \Gamma$ is made up of nine two cells. Six of them are the product of

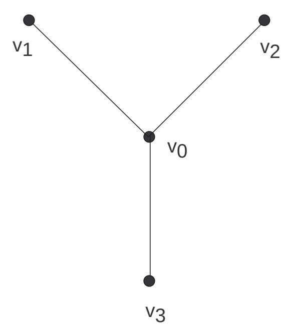


Figure 2.3: The Y-graph.

two distinct edges of Γ and by identifying common boundaries one can see that these cells form a disc with central point the zero-cell $v_0 v_0$. This cell represents the configuration where both particles occupy the vertex v_0 and must be removed in the construction of $F(\Gamma, 2)$ to obtain a punctured disc.

The remaining three two-cells represent configurations where both particles lie on the same edge. In constructing $F(\Gamma, 2)$ each of these cells is 'cut'into two triangular two-cells by removing the diagonal line which represents configurations where both particles occupy the same point of the edge. Attaching these cells to the punctured disc by identifying common boundaries we obtain the space shown in Figure 2.4 which has the homotopy type of a punctured disc.

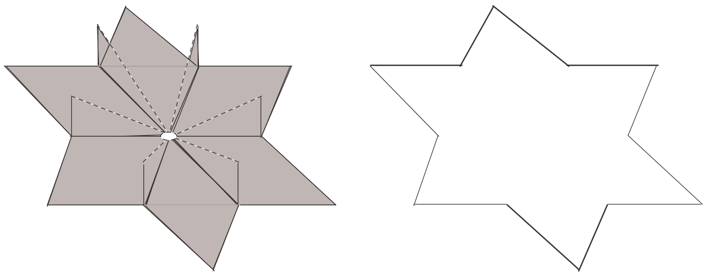


Figure 2.4: (left) The configuration space $F(\Gamma, 2)$ and (right) the space $D(\Gamma, 2)$, where Γ is the Y-graph. Dotted lines represent the diagonal, Δ.

To construct $D(\Gamma, 2)$ we take only cells of $\Gamma \times \Gamma$ which represent two particles lying on cells of Γ with disjoint closures. Since all edges of Γ are incident, $D(\Gamma, 2)$ must be one-dimensional. We see that $D(\Gamma, 2)$ can be identified with the boundary circle of the punctured disc $F(\Gamma, 2)$ since this represents configurations where one particle lies on a vertex of valence one, $v_i, i = 1, 2, 3$, and the other lies on an edge, e_j where $j \neq i$.

Note that the circle $D(\Gamma, 2)$ is a deformation retract of the punctured disc $F(\Gamma, 2)$.

Example 2.2.2 Now consider a graph Γ with five edges which is homeomorphic to the circle S^1.

Then $\Gamma \times \Gamma$ is equal to the torus $T^2 = S^1 \times S^1$. This can be given a CW-complex structure with 25 square two-cells each representing the product of a pair of edges in Γ. The line representing the diagonal Δ_Γ runs through two-cells formed from the

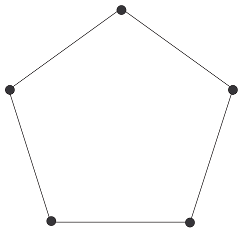


Figure 2.5: A graph homeomorphic to S^1.

product of an edge with itself along the meridional circle of the torus. Then $F(\Gamma, 2)$ is a torus with a meridional circle removed and hence is homeomorphic to an open cylinder.

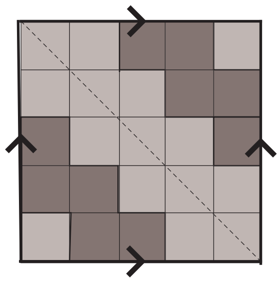


Figure 2.6: The space $F(\Gamma, 2)$ represented as a complex of 25 2-cells. The space $D(\Gamma, 2)$ is represented by the darker cells, and the diagonal Δ is shown as a dotted line. The boundary of the square is identified according to the arrows.

The space $D(\Gamma, 2)$ is a complex of ten two-cells given by products of disjoint edges in Γ. This is shown in dark grey in Figure 2.6. The ends of this complex are identified so it has the homotopy type of a circle.

Again we see that $D(\Gamma, 2)$ is a deformation retract of $F(\Gamma, 2)$.

2.3 The Case of Two Particles

Throughout the rest of this thesis we will study configuration spaces of *two* particles moving without collision on a graph, Γ. In this case, one can define an involution, τ on the space $\Gamma \times \Gamma$ given by interchanging the two particles.

Definition 2.3.1 Let the *involution map*, τ, be given by

$$\tau : \Gamma \times \Gamma \to \Gamma \times \Gamma \qquad (x, y) \mapsto (y, x). \tag{2.3}$$

This involution restricts to a map on the configuration spaces $F(\Gamma, 2)$ and $D(\Gamma, 2)$, and the symmetry it provides will be exploited throughout this thesis.

In Examples 2.2.1 and 2.2.2 we noted that $D(\Gamma, 2)$ was a deformation retract of $F(\Gamma, 2)$. It was shown by Abrams in his thesis [1] that this is true for any simple graph Γ when considering the case of two particles.

Theorem 2.3.2 *Let Γ be a simple graph. Then there exists a deformation retraction of $F(\Gamma, 2)$ onto $D(\Gamma, 2)$ which is equivariant with respect to the involution τ.*

Proof We take the deformation retraction described in Theorem 2.4 of [1]. This is performed in two steps. Let

$$X = F(\Gamma, 2) \setminus \bigcup_{e \in E(\Gamma)} (\text{Int } e \times \text{Int } e).$$

Step 1: We first retract $F(\Gamma, 2)$ onto X. If (x, y) is in $F(\Gamma, 2) \setminus X$ then the two particles x and y lie on distinct points of the same edge e in Γ. To perform the retraction we move particles x and y apart at a constant speed until at least one of the particles reaches a vertex. The resulting configuration is in X. Note that this map is continuous and that since Γ contains no loops it does not give rise to collisions. Also, applying τ to interchange the two particles and then performing the retraction will give the same configuration as performing the retraction and then interchanging the particles.

Step 2: We now retract X onto $D(\Gamma, 2)$. A configuration (x, y) is in $X \setminus D(\Gamma, 2)$ if (i) it is in the interior of a two-cell $e_1 \times e_2$ where e_1 and e_2 have one vertex, v, in common, or (ii) it is in the interior of a one-cell $v \times e$ or $e \times v$ where v is a boundary vertex of e.

In both cases we slide the particles away from the vertex v at a speed proportional to their original distance from v. This proportionality is necessary to make the map continuous. In case (i) we keep moving the particles until at least one particle lies on a vertex. In case (ii) we move the particles until the particle which began in the interior of e lies on a vertex. Note that the other particle in this case began at the vertex v and so does not move under this map.

This retraction is continuous and does not result in collisions. Finally we see that this retraction also commutes with the involution τ. $\square$

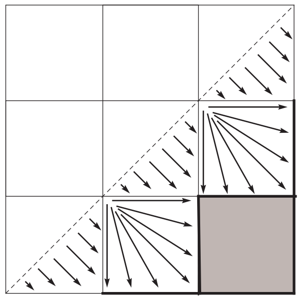


Figure 2.7: A deformation retraction of $F(\Gamma, 2)$ onto $D(\Gamma, 2)$.

Remarks Figure 2.7 illustrates the deformation retraction described in Theorem 2.3.2. The arrows show the direction of the retraction away from the dotted line representing the the diagonal Δ, to the space $D(\Gamma, 2)$ bounded by the thicker line.

Theorem 2.3.2 shows that to study any homotopy invariant property of $F(\Gamma, 2)$ it is enough to study the corresponding property of the cell-complex $D(\Gamma, 2)$.

It is shown by Patty in [27] that $F(\Gamma, 2)$, and hence by Theorem 2.3.2 $D(\Gamma, 2)$, is always path connected unless Γ is homeomorphic to the closed interval $[0, 1]$. This follows from Theorem 3 in Eilenberg's paper, [13]. In his paper Patty also proves that $F(\Gamma, 2)$ is aspherical, that is all the homotopy groups of $F(\Gamma, 2)$ are trivial except for the fundamental group. This result is stated for completeness, we will not make use of this fact in this thesis.

2.4 A Conjecture on the Generators of $H_2(F(\Gamma, 2), \mathbb{Z})$

In subsequent chapters we attempt to describe the homology groups of $F(\Gamma, 2)$ for various classes of graphs. A conjecture has been proposed about the generators of $H_2(F(\Gamma,2),\mathbb{Z})$ which is well known to people working with these configuration spaces although it has never been formally stated in the literature. The conjecture was brought to my attention by Henry Glover at the 2006 conference on Topological Robotics in Zurich.

The conjecture claims that the second dimensional homology of $F(\Gamma,2)$ is generated by surfaces embedded in the space, coming from copies of the Kuratowski graphs K_5 and $K_{3,3}$ and pairs of disjoint cycles embedded in the graph Γ. Before stating the conjecture we will describe the space $D(\Gamma,2)$ for Γ equal to K_5 or $K_{3,3}$.

Example 2.4.1 We show that $D(K_5,2)$ is homeomorphic to an orientable surface of genus 6. This example follows the method found in Ghrist's paper [3].

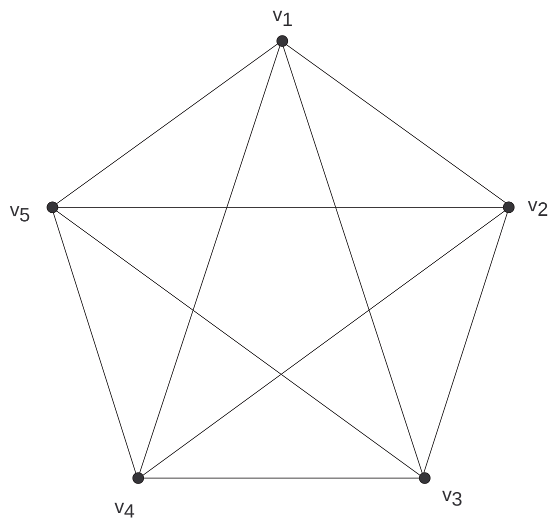


Figure 2.8: The graph K_5.

First we count the number of cells in $D(K_5,2)$. The zero-cells correspond to pairs of disjoint vertices of K_5, there are $5\times 4=20$ such pairs. The one-cells are given by pairs ev and ve where e is an edge of K_5 and v is a vertex of K_5 such that v is not in the boundary of e. There are $2\times 10\times 3=60$ such pairs since K_5 has 10 edges each of which is disjoint from the 3 remaining vertices of K_5 which are not in the boundary of the edge. The factor of 2 appears since we count ordered pairs. Finally, the two-cells of $D(K_5,2)$ are given by ordered pairs of disjoint edges. Each edge of K_5 is incident to six other edges of the graph, hence each edge is disjoint from 3 other edges, so we have $10 \times 3 = 30$ two-cells.

To see how these cells fit together, consider a zero cell v_1v_2. There are three edges in K_5 which are incident to v_1 but disjoint from v_2 and similarly there are three edges which are incident to v_2 but disjoint from v_1. So there are six one-cells attached to the zero-cell v_1v_2. For a one-cell, ev_1 or v_1e, to lie in the boundary of a two-cell there must be an edge e' disjoint from e which is incident to v_1. Now e is incident to v_2, so e' must be an edge disjoint from v_2 but incident to v_1. There are three such edges as discussed before, however one of these edges must be incident to the other boundary vertex of e since K_5 only has five vertices, so we find there are exactly two possibilities for e'. Hence every one-cell attached to v_1v_2 lies in exactly two two-cells and we get a locally Euclidean complex of six two-cells around the zero-cell v_1v_2 as shown in Figure 2.9.

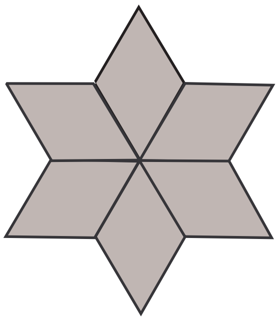


Figure 2.9: A locally Euclidean complex of six 2-cells around a zero cell in $D(K_5, 2)$.

This argument applies to every zero-cell and one-cell in $D(K_5, 2)$, so we have shown that $D(K_5, 2)$ is a two dimensional, locally Euclidean space; a surface. We also note that it is possible to move from any configuration in $D(K_5, 2)$ to any other, so $D(K_5, 2)$ is connected and further, it is possible give each cell of $D(K_5, 2)$ an orientation which induces a consistent orientation on the whole complex.

Then we can use the Euler characteristic of the connected orientable surface $D(K_5, 2)$ to calculate its genus. We have

$$\chi(D(K_5, 2)) = \#0\text{-cells} - \#1\text{-cells} + \#2\text{-cells} = 20 - 60 + 30 = -10$$

So the genus of $D(K_5, 2)$ is equal to $1 - \frac{1}{2}\chi = 6$.

Example 2.4.2 We can do a similar analysis to show that $D(K_{3,3}, 2)$ is homeomorphic to an orientable surface of genus 4.

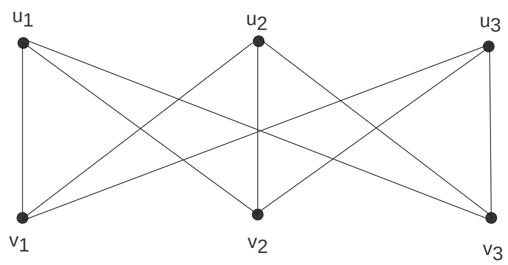


Figure 2.10: The graph $K_{3,3}$.

In this case the number of zero-cells is equal to $6 \times 5 = 30$, the number of one-cells is equal to $2 \times 9 \times 4 = 72$ and the number of two-cells is equal to $9 \times 4 = 36$.

A similar analysis to that above shows that we obtain two possible configurations of cells around the zero-cells of $D(K_{3,3}, 2)$. There are four one-cells incident to every zero-cell of the form v_iu_j or u_jv_i and around such cells we obtain a locally Euclidean complex of four two-cells. Around zero-cells of the form u_iu_j or v_iv_j we obtain a locally Euclidean complex of six two-cells.

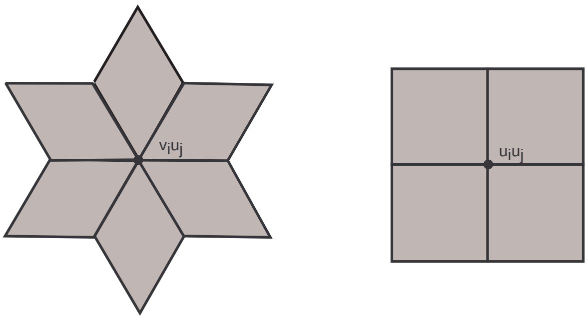


Figure 2.11: (right) The complex of four 2-cells around the 0-cell v_iu_j and (left) the complex of six 2-cells around the 0-cell u_iu_j.

Again, we can find a consistant orientation of the complex $D(K_{3,3}, 2)$ to obtain a connected orientable surface. In this case the Euler characteristic is given by $\chi(D(K_{3,3}, 2)) = 30 - 72 + 36 = -6$ and we find that the genus of $D(K_{3,3}, 2)$ is 4.

Remarks

1. The above examples show that

$$H_2(F(K_5, 2), \mathbb{Z}) = H_2(F(K_{3,3}, 2), \mathbb{Z}) = \mathbb{Z}$$

   In his thesis [1] Abrams shows that the Kuratowski graphs K_5 and $K_{3,3}$ are the only connected, simple graphs for which $F(\Gamma, 2)$ is homeomorphic to an orientable surface.

2. Suppose that we have a graph Γ which is homeomorphic to $S^1 \sqcup S^1$, we will denote one copy of S^1 by C_1 and the other by C_2. Then $D(\Gamma, 2)$ has four connected components; two disjoint copies of the torus T^2, one coming from the product $C_1 \otimes C_2$ and one from $C_2 \otimes C_1$, and two disjoint copies of the circle S^1, one coming from configurations where both particles lie on C_1 and one from configurations where the particles lie on C_2. This implies that the second homology group of $D(\Gamma, 2)$, $H_2(F(\Gamma, 2), \mathbb{Z})$, is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

We may now state the conjecture.

Conjecture 2.4.1 *Let K be a graph isomorphic to either $S^1 \sqcup S^1$, K_5 or $K_{3,3}$ and let Γ be a finite graph. Suppose there exists a topological embedding $i : K \to \Gamma$, then there also exists an embedding $j : F(K, 2) \to F(\Gamma, 2)$ which induces a map on the homology $j_* : H_2(F(K, 2), \mathbb{Z}) \to H_2(F(\Gamma, 2), \mathbb{Z})$.*

Let $\{i^p\}_{p \in P}$ be the set of all possible embeddings of K into Γ, for $K = K_5$ or $K_{3,3}$, and let $\{i^q\}_{q \in Q}$ be the set of all possible embeddings of $L \cong S_1 \sqcup S_1$ into Γ. Choose a generator, z_k for $H_2(F(K, 2), \mathbb{Z}) = \mathbb{Z}$ and a pair of generators y_l and x_l for $H_2(F(L, 2), \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Then the group $H_2(F(\Gamma, 2), \mathbb{Z})$ is generated by the homology classes $\{j^p_*(z_k)\}_{p \in P}$, $\{j^q_*(y_l)\}_{q \in Q}$, and $\{j^q_*(x_l)\}_{q \in Q}$.

2.5 Surfaces

The conjecture stated in the last section claims that all elements of $H_2(F(\Gamma, 2), \mathbb{Z})$ come from embeddings of orientable surfaces of certain genus into $F(\Gamma, 2)$. We now prove a result which makes a small step towards proving this statement by showing that all elements of $H_2(F(\Gamma,2),\mathbb{Z})$ correspond to maps of orientable surfaces in to $F(\Gamma,2)$. Though the maps in this case are not necessarily embeddings and no distinction is made as to the genera of the surfaces.

We denote by $\Omega_n(X)$ the n-dimensional oriented bordism group of the space X as described by Conner and Floyd in section 1.4 of [9] and by Ω_n the n-dimensional Thom bordism group as described in section 1.2 of [9].

Lemma 2.5.1 *Let X be a two-dimensional CW-complex. Then the group $H_2(X,\mathbb{Z})$ is isomorphic to $\Omega_2(X)$.*

Proof To prove this statment we will use the bordism spectral sequence as described by Conner and Floyd in [9] which is a particular case of the Atiyah-Hirzebruch spectral sequence first appearing in [5].

We will follow the notation used by Conner and Floyd in section 1.7 of [9]. Here it is stated that, given an n-dimensional CW-complex X, there is a spectral sequence $\{E^r_{p,q}\}$ with $E^2_{p,q} = H_q(X,\Omega_p)$ which converges to $\Omega_*(X)$. We will apply this spectral sequence to the case where X is two-dimensional.

We will describe the E^2 page of the spectral sequence. The space $E^2_{p,q}$ is equal to the group $H_q(X,\Omega_p)$ and since X is two-dimensional, $H_q(X,\Omega_p) = 0$ for $q > 2$. The Thom bordism groups Ω_n are completely described, see [24]. For low dimensions we have

$$\Omega_0 = \mathbb{Z}, \quad \Omega_1 = \Omega_2 = \Omega_3 = 0, \quad \Omega_4 = \mathbb{Z}, \quad \Omega_5 = \mathbb{Z}_2.$$

This implies that the spaces $E^2_{0,q}$ are equal to the homology groups $H_q(X,\mathbb{Z})$ and spaces $E^2_{p,q}$ are all trivial for $p = 1,2,3$ and so the E^2 page has no non-zero differentials d_2. The first possible non-zero differential is

$$d_5 : E^2_{5,0} \to E^2_{0,4},$$

so we have $E^2_{p,q} = \cdots = E^5_{p,q}$. However the groups $E^2_{p,q}$ are trivial for all $q \geq 3$ so all the differentials must be zero and the spectral sequence collapses at the E^2 page, i.e.

$$E^\infty_{p,q} = E^2_{p,q}.$$

Then since this sequence converges to $\Omega_*(X)$ we have

$$\Omega_*(X) = \bigoplus_{p+q=0}^{\infty} E^{\infty}_{p,q},$$

and

$$\Omega_2(X) \simeq \bigoplus_{p+q=2} E^2_{p,q} = E^2_{0,2} = H_2(X, \Omega_0) = H_2(X, \mathbb{Z})$$

as required. $\square$

Corollary 2.5.2 *For any graph Γ the group $H_2(F(\Gamma, 2), \mathbb{Z})$ is isomorphic to the oriented bordism group $\Omega_2(F(\Gamma, 2))$.*

Proof Lemma 2.5.1 implies that the group $H_2(D(\Gamma, 2), \mathbb{Z})$ is isomorphic to the bordism group $\Omega_2(D(\Gamma, 2))$ since $D(\Gamma, 2)$ has the structure of a CW-complex. The corollary then follows directly from Theorem 2.3.2. $\square$

Chapter 3

Intersections of Cycles in Graphs

In this chapter we describe in detail an intersection theory for cycles in graphs and its links to the homology of the configuration space $F(\Gamma, 2)$. The work was inspired by the intersection theory for cycles in manifolds which gives rise to results such as the Poincare duality theorem. Sections 3.1, 3.2 and 3.3 are an expansion of sections 1 and 2 of the paper [6] written by Michael Farber and myself. In Section 3.1 an explicit formula for the Euler Characteristic of $F(\Gamma, 2)$ is given and in the following two sections we introduce the *intersection form* and show how the properties of this map relate to the homology of $F(\Gamma, 2)$. In the final two sections we discuss a method for expressing the intersection of two cycles as elements of a direct sum of 'local homology groups', each of which describes the properties of small subgraphs within the graph. This allows the explicit calculation of generators for $H_2(F(\Gamma, 2))$. The work in these final sections is an extension of unpublished work begun by Michael Farber.

3.1 The Euler Characteristic of $F(\Gamma, 2)$

The remainder of this thesis is devoted to describing the homology groups of $F(\Gamma, 2)$. To this end, we present a formula for the Euler characteristic of $F(\Gamma, 2)$ made up of easily calculated invariants of the graph Γ.

Theorem 3.1.1 *Let Γ be a simple graph, let $V(\Gamma)$ denote the set of vertices of Γ and let $\mu(v)$ be the valence of the vertex $v \in \Gamma$. Then the Euler characteristic of the space $F(\Gamma, 2)$ is given by*

$$\chi(F(\Gamma, 2)) = \chi(\Gamma)^2 + \chi(\Gamma) - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2) \tag{3.1}$$

Proof It is enough to calculate the Euler characteristic of the discrete space $D(\Gamma, 2)$, since the Euler characteristic is a homotopy invariant of a space and by Lemma 2.3.2 $D(\Gamma, 2)$ is homotopy equivalent to $F(\Gamma, 2)$. We calculate this Euler characteristic by counting the number of cells of $D(\Gamma, 2)$.

The zero-cells of $D(\Gamma, 2)$ are given by ordered pairs, vu, of disjoint vertices of Γ. It is easy to see that there are $V^2 - V$ such pairs, where $V = |V(\Gamma)|$ is the number of vertices in Γ. The one-cells of $D(\Gamma, 2)$ are given by pairs of edges and vertices, ve and ev, such that e is an edge not incident to v. The number of edges not incident to a vertex $v \in \Gamma$ is equal to $E - \mu(v)$, where E is the number of edges in Γ. So the total number of one-cells in $D(\Gamma, 2)$ is equal to

$$2 \sum_{v \in V(\Gamma)} (E - \mu(v)). \tag{3.2}$$

Since,

$$\sum_{v \in V(\Gamma)} \mu(v) = 2E, \tag{3.3}$$

formula (3.2) can be rewritten as

$$2EV - 4E. \tag{3.4}$$

Finally, the two-cells of $D(\Gamma, 2)$ are given by pairs, ee', of disjoint edges of Γ. The number of such cells is given by

$$E^2 - E - \sum_{v \in V(\Gamma)} \mu(v)(\mu(v) - 1) \tag{3.5}$$

Here E^2 is the total number of pairs of edges of Γ. The second term removes the E two-cells of the form ee, and finally the last term removes the two-cells of the form ee' where e and e' have one common boundary vertex. Formula (3.5) can also be rewritten using formula (3.3) to give

$$E^2 + E - \sum_{v \in V(\Gamma)} \mu(v)^2. \tag{3.6}$$

Hence $\chi(D(\Gamma, 2)) = \chi(F(\Gamma, 2))$ is equal to

$$\begin{aligned}
&(V^2 - V) - (2EV - 4E) + (E^2 + E - \sum_{v \in V(\Gamma)} \mu(v)^2) \\
&= (V^2 + E^2 - 2EV) + (V - E) - (2V - 6E + \sum_{v \in V(\Gamma)} \mu(v)^2) \\
&= \chi(\Gamma)^2 + \chi(\Gamma) - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2).
\end{aligned}$$

□

Theorem 3.1 allows us to determine completely the Betti numbers of $F(\Gamma, 2)$ by calculating either $b_1(F(\Gamma, 2))$ or $b_2(F(\Gamma, 2))$, as long as $F(\Gamma, 2)$ is connected. This formula also follows from a more general theorem by Gal given in Corollary 2.7 of [16] which decsribes the Euler charateristic of $F(X, n)$ where X is any polyhedron.

3.2 The Intersection Form

In order to calculate the second Betti number of $F(\Gamma, 2)$ we will examine relations between the intersections of pairs of cycles in the graph Γ.

As described in Definition 2.1.1, a graph Γ can be thought of as a one-dimensional CW-complex, so we can consider the cellular chain complex of Γ

$$0 \to C_1(\Gamma) \xrightarrow{d} C_0(\Gamma) \to 0.$$

Here $C_1(\Gamma)$ is the one-dimensional chain group of Γ, that is the abelian group generated by all oriented edges of Γ with coefficients in $\mathbb{Z}$. Similarly $C_0(\Gamma)$ is the zero-dimensional chain group, the abelian group generated by all vertices of Γ with coefficients in $\mathbb{Z}$. The map d is the boundary map which maps an oriented edge e to the ordered pair $\pm(v-u)$ of its boundary vertices. We can then give the following defintion.

Definition 3.2.1 A *cycle* of a graph Γ is an element $C = \sum \alpha_i e_i$, $\alpha_i \in \mathbb{Z}$, of the chain group $C_1(\Gamma)$ such that $dC = 0$ in the group $C_0(\Gamma)$.

Remarks Since the two-dimensional chain group $C_2(\Gamma)$ is empty, cycles of Γ are identical to one-dimensional homology classes and $H_1(\Gamma)$ is isomorphic to the kernel of the boundary map d. Any cycle of a graph is a linear combination of the simple cycles in the graph, i.e. cycles homeomorphic to S^1.

We now state a classical result which allows one to easily calculate the rank of the group $H_1(\Gamma)$ for any finite graph Γ, proof of this result can be found in proposition 1A.2 of [23].

Definition 3.2.2 A *maximal spanning tree* of a graph Γ is a subtree of Γ, i.e. a subgraph containing no cycles, with vertex set equal to the vertex set of Γ.

Theorem 3.2.3 *Let Γ be a finite graph and let T be a maximal spanning tree for Γ. Denote by E_T the set of edges in $\Gamma - T$. Then the rank of $H_1(\Gamma)$ is equal to $|E_T|$, the cardinality of the set E_T.*

In order to examine the intersections of pairs of cycles in Γ we define a neighbourhood, N, of the diagonal $\Delta \in \Gamma \times \Gamma$ by,

$$N = N_\Gamma = \overline{\Gamma \times \Gamma - D(\Gamma, 2)}. \tag{3.7}$$

The space N inherits a cell structure from $\Gamma \times \Gamma$. Before describing this structure we first introduce a distance function between vertices of simple graph Γ which will be based on the combinatorial lengths of paths in Γ.

Definition 3.2.4 Let Γ be a simple graph. Then a *combinatorial path*, p from vertex $u \in \Gamma$ to vertex $v \in \Gamma$ is defined as a sequence of closed edges,

$$p = e_1, e_2, \cdots, e_k,$$

such that u is in the boundary of e_1, v is in the boundary of e_k and $e_i \cap e_{i+1} = v_i$, a vertex of Γ, for all $i = 1, \cdots, k-1$ such that $v_i \neq v_j$ for $i \neq j$ and $v_i \neq u, v$ for all $i = 1, \cdots, k-1$.

The *length of a path* p will be denoted by l_p and is defined as the number of edges in the path.

Denote by $\mathbb{P}(u, v)$ the set of all paths p from vertex u to vertex v in Γ and by $\mathbb{P}(u)$ the set of all paths p from a vertex u to any other vertex v in Γ.

We can then define a distance function on the set of vertices of Γ.

Definition 3.2.5 The *distance function* on a simple graph Γ is defined as

$$d : V(\Gamma) \times V(\Gamma) \to \mathbb{Z}; \qquad d(u, v) = \min_{p \in \mathbb{P}(u,v)} \{l_p\} \tag{3.8}$$

Remarks The map d defines the distance between two vertices of the graph to be the length of the shortest path between them. We can use this function to define the distance between an edge and a vertex of the graph. Suppose e is an edge of Γ with boundary vertices u_1 and u_2, and v is any vertex of Γ, then let $d(v, e) = d(e, v) = \min\{d(u_1, v), d(u_2, v)\}$.

Before continuing our discussion of the space N we introduce some definitions based on the function d which will be useful later in this chapter. In the following three defintions, Γ will denote a simple graph.

Definition 3.2.6 The *ball of radius* r *around the vertex* v in Γ, B_v^r, is defined to be the union of all paths $p \in \mathbb{P}(v)$ such that $l_p \leq r$.

Definition 3.2.7 The *ball,* B_e^r*, of radius* r *around the edge* e in Γ joining vertices v and u, is defined to be the union of the edge e with all paths $p \in \mathbb{P}(u)$ such that $l_p \leq r$ and all paths $q \in \mathbb{P}(v)$ such that $l_q \leq r$.

Definition 3.2.8 The *annulus of radii* r *and* s*, where* $r < s$*, around the vertex* v in Γ, $S_v^{r,s}$, is defined as,

$$S_v^{r,s} = \overline{B_v^s - B_v^r}.$$

We can now describe the cell structure of the space N, see (3.7). Removing the cells of $D(\Gamma, 2)$ from $\Gamma \times \Gamma$ removes all cells $\sigma_i \times \sigma_j$ such that $\bar{\sigma}_i \cap \bar{\sigma}_j = \emptyset$. So the space $\Gamma \times \Gamma - D(\Gamma, 2)$ will have zero-cells of the form vv, where v is a vertex of Γ, one-cells ve and ev where v is in the boundary of e and two-cells of the form ee or ee' where e and e' have one common boundary vertex.

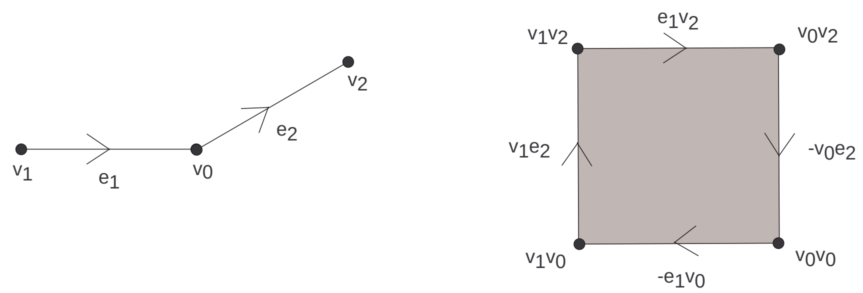


Figure 3.1: *Left* Two oriented edges, e_1 and e_2, in Γ. *Right* The two-cell $e_1 \otimes e_2$ in $\Gamma \times \Gamma$ with labelled boundary.

To obtain the space N, take the closure of all cells of $\Gamma \times \Gamma - D(\Gamma, 2)$, by including the boundary of every such cell into N. The boundary of a two cell $ee' \in \Gamma \times \Gamma - D(\Gamma, 2)$ includes one-cells of the form ve and ev where $d(e, v) = 1$, and the boundary of such a one-cell contains zero-cells of the form uv where $d(u, v) = 1$ or 2, as pictured in Figure 3.1. The boundaries of all other cells of $\Gamma \times \Gamma - D(\Gamma, 2)$ are contained in $\Gamma \times \Gamma - D(\Gamma, 2)$.

We obtain the following cell structure for N,

- Zero-cells - ordered pairs of vertices uv such that $0 \leq d(u, v) \leq 2$.
- One-cells - pairs ev and ve of edges and vertices such that $d(v, e) = 0$ or 1.
- Two-cells - ordered pairs of edges ee or ee' where e and e' have a common boundary vertex.

We also define another subcomplex of $\Gamma \times \Gamma$ called the *boundary* of N, let

$$\partial N = \partial N_\Gamma = N_\Gamma \cap D(\Gamma, 2) \tag{3.9}$$

This space is one-dimensional and also has an obvious cell structure. Its zero-cells are given by ordered pairs of vertices vu such that $d(v,u) = 1$ or 2, and its one-cells are given by pairs, ve and ev, of edges and vertices such that $d(e,v) = 1$. This is the space of configurations of two particles moving on the graph maintaining a distance of one or two edges between them.

We are now in a position to define the *intersection form* which is central to the work in the rest of this thesis.

Definition 3.2.9 The *intersection form* is a map

$$I = I_\Gamma : H_1(\Gamma) \otimes H_1(\Gamma) \to H_2(N, \partial N), \tag{3.10}$$

defined as the homomorphism induced on the homology by the inclusion $j : \Gamma \times \Gamma \to (\Gamma \times \Gamma, D(\Gamma, 2))$, where we identify $H_2(\Gamma \times \Gamma)$ with $H_1(\Gamma) \otimes H_1(\Gamma)$ by the Kunneth isomorphism, and identify $H_2(\Gamma \times \Gamma, D(\Gamma, 2))$ with $H_2(N, \partial N)$ by excision.

The link between the intersection form and the homology groups of $F(\Gamma, 2)$ is described in the following theorem.

Theorem 3.2.10 *Let Γ be a finite connected graph, not homeomorphic to the circle S^1. Then the following statements hold.*

1. *The group $H_2(F(\Gamma, 2))$ is isomorphic to the kernel of the intersection form I_Γ.*
2. *The group $H_1(F(\Gamma, 2))$ is isomorphic to the direct sum,*

$$\mathit{coker}(I_\Gamma) \oplus H_1(\Gamma) \oplus H_1(\Gamma)$$

The proof of Theorem 3.2.10 will require the following lemma.

Lemma 3.2.11 *Let Γ be a finite, connected graph not homeomorphic to the circle S^1. Then the map*

$$\alpha_* : H_1(F(\Gamma, 2)) \to H_1(\Gamma \times \Gamma), \tag{3.11}$$

induced by the inclusion of $F(\Gamma, 2)$ into $\Gamma \times \Gamma$, is an epimorphism.

Proof We can think of every cycle, γ, in $\Gamma \times \Gamma$ as a pair of cycles, (γ_1, γ_2), in Γ, where γ_i is a map from the circle S^1 into the graph Γ. To show that α_* is an epimorphism we need to show that any pair of cycles (γ_1, γ_2) in Γ is homotopic to a pair of cycles (γ_1', γ_2') which are disjoint at any moment of time in Γ and hence lie in $F(\Gamma, 2)$, i.e. if x is a point of the circle S^1 then $\gamma_1'(x) \neq \gamma_2'(x)$.

There is an isomorphism between $H_1(\Gamma \times \Gamma)$ and $H_1(x_0 \times \Gamma) \oplus H_1(\Gamma \times x_0)$ where x_0 is some point of Γ. This isomorphism together with the fact that $H_1(\Gamma)$ is generated by simple cycles implies that $H_1(\Gamma \times \Gamma)$ is generated by pairs of cycles (γ, γ') where one of the pair is a simple cycle and the other is a constant cycle. Suppose that γ' is the constant cycle at the point $x_0 \in \Gamma$, then if $x_0 \notin \gamma$, γ and γ' are disjoint and we are done.

Suppose that $x_0 \in \gamma$. Let v denote a vertex of γ with valence greater than or equal to 3. This is possible since Γ is connected and not homeomorphic to the circle. Then choose a vertex u which is joined to v but is not contained in the cycle γ. Then, since Γ is connected, we can deform γ' through a continuous homotopy along γ to v and then to the constant cycle at u, $\tilde{\gamma}'$. We then obtain a pair of disjoint cycles $(\gamma, \tilde{\gamma}')$ as required. □

Proof of Theorem 3.2.10. To prove this theorem we consider the long exact homology sequence of the pair $(\Gamma \times \Gamma, F(\Gamma, 2))$,

$$\begin{aligned} 0 \to H_2(F(\Gamma,2)) &\xrightarrow{\alpha_*} H_2(\Gamma \times \Gamma) \to H_2(\Gamma \times \Gamma, F(\Gamma, 2)) \\ &\xrightarrow{\partial} H_1(F(\Gamma,2)) \xrightarrow{\alpha_*} H_1(\Gamma \times \Gamma) \to \cdots \end{aligned} \tag{3.12}$$

Where α_* is the map induced by the inclusion of $F(\Gamma, 2)$ into $\Gamma \times \Gamma$. As was already mentioned in the definition of the intersection form (3.10), $H_2(\Gamma \times \Gamma)$ is isomorphic to $H_1(\Gamma) \otimes H_1(\Gamma)$ by the Kunneth isomorphism and $H_2(\Gamma \times \Gamma, F(\Gamma, 2))$ is isomorphic to $H_2(N, \partial N)$ by excision. Also Lemma 3.2.11 shows that the map α_* between the one dimensional homology groups is an epimorphism, so we obtain the sequence,

$$0 \to H_2(F(\Gamma,2)) \xrightarrow{\alpha_*} H_1(\Gamma) \otimes H_1(\Gamma) \xrightarrow{I_\Gamma} H_2(N, \partial N) \tag{3.13}$$

$$\xrightarrow{\partial} H_1(F(\Gamma,2)) \xrightarrow{\alpha_*} H_1(\Gamma) \oplus H_1(\Gamma) \to 0.$$

Statement 1 of the theorem then follows directly from this exact sequence. Statement 2 can be proved using the following short exact sequence, induced by exact sequence (3.13),

$$0 \to \operatorname{coker}(I_\Gamma) \to H_1(F(\Gamma,2)) \to H_1(\Gamma) \oplus H_1(\Gamma) \to 0. \qquad (3.14)$$

The group $H_1(\Gamma)\oplus H_1(\Gamma)$ is free, so this sequence splits and we obtain that $H_1(F(\Gamma,2))$ is isomorphic to the direct sum, $\operatorname{coker}(I_\Gamma) \oplus H_1(\Gamma) \oplus H_1(\Gamma)$. □

We have now reduced the problem of calculating the Betti numbers of $F(\Gamma,2)$ to calculating the dimension of the kernel and cokernel of the map I_Γ. To end this section, we state two important properties of the intersection form which follow directly from its defintion.

Lemma 3.2.12 *Let $z, z' \in H_1(\Gamma)$ be two homology classes which can be realised by disjoint cycles in Γ. Then*

$$I(z \otimes z') = I(z' \otimes z) = 0.$$

Proof This clearly follows from the definition of the intersection form (3.10). □

Lemma 3.2.13 *Consider homology classes $z, z' \in H_1(\Gamma)$, then*

$$I(z' \otimes z) = -\tau_*(I(z \otimes z'))$$

Where $\tau : (N, \partial N) \to (N, \partial N)$ is the restriction of involution (2.3) on $\Gamma \times \Gamma$ to $(N, \partial N)$.

Proof This statement follows from the definition of the cross product of two homology groups which satisfies the following commutativity relation,

$$T_* : H_k(X,\mathbb{Z}) \otimes H_l(Y,\mathbb{Z}) \to H_l(Y,\mathbb{Z}) \otimes H_k(X,\mathbb{Z}), \quad T_*(z \otimes z') = (-1)^{kl}(z' \otimes z)$$

induced by the involution

$$T : X \times Y \to Y \times X, \quad T(x,y) = (y,x),$$

see [23] Section 3B. In this case the involution τ (2.3) is equivalent to the map T and we obtain

$$I(z' \otimes z) = I(-\tau_*(z \otimes z')).$$

The map τ_* is a chain map and so commutes with the intersection form I to obtain

$$I(-\tau_*(z \otimes z')) = -\tau_*(I(z \otimes z')).$$

□

3.3 Consequences of the Intersection Form

In this section we discuss some useful consequences of the intersection form. First we can use exact sequence (3.13) to describe the Euler Characteristic of the space $(N, \partial N)$.

Lemma 3.3.1 *Let Γ be a simple graph, not homeomorphic to the circle S^1 or to the unit interval $[0,1]$, then the Euler Characteristic of the space $(N_\Gamma, \partial N_\Gamma)$ is given by,*

$$\chi(N_\Gamma, \partial N_\Gamma) = 1 - \chi(\Gamma) + \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2) \tag{3.15}$$

Proof If we apply the Euler-Poincare Theorem to the exact sequence (3.13) we obtain

$$b_2(F(\Gamma,2)) - b_1(\Gamma)^2 + \mathrm{rank}(H_2(N, \partial N)) - b_1(F(\Gamma,2)) + 2b_1(\Gamma) = 0$$

where $b_i(X)$ is the i^{th} Betti number of the space X. This implies that the rank of the group $H_2(N, \partial N)$, i.e. the second Betti number of $(N, \partial N)$, is given by

$$b_2(N, \partial N) = -b_2(F(\Gamma,2)) + b_1(F(\Gamma,2)) + b_1(\Gamma)^2 - 2b_1(\Gamma)$$

$$= \chi(\Gamma)^2 - \chi(F(\Gamma,2))$$

Note that $b_1(N, \partial N) = 0$, which is a consequence of Lemma 3.2.11 and the fact $H_1(N, \partial N)$ is isomorphic to $H_1(\Gamma \times \Gamma, F(\Gamma,2))$. Rewriting $\chi(F(\Gamma,2))$ using formula

(3.1) we obtain the following expression for the Euler Characteristic of $(N, \partial N)$,

$$1 - \chi(\Gamma) + \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2).$$

□

Remark Formula (3.15) will be useful in later calculations, especially those in Chapter 5, in which we address the question of when the intersection form is epimorphic. We can already show that the intersection form is epimorphic for the Kuratowski graphs K_5 and $K_{3,3}$. Examples 2.4.1 and 2.4.2 showed that $F(K_5, 2)$ and $F(K_{3,3}, 2)$ have the homotopy type of an orientable surface, therefore we have $H_2(F(K_5, 2)) = H_2(F(K_{3,3}, 2)) = \mathbb{Z}$. The surface $F(K_5, 2)$ had genus 6 so $H_1(F(K_5, 2))$ is a free abelian group of rank 12 and similarly, the surface $F(K_{3,3}, 2)$ has genus 4, so $H_1(F(K_{3,3}, 2))$ is a free abelian group of rank 8. Using the spanning tree method of Lemma 3.2.3 we can calculate that $H_1(K_5)$ has rank 6 and $H_1(K_{3,3})$ has rank 4, therefore since, $H_1(F(\Gamma, 2)) = \mathrm{coker}(I_\Gamma) \oplus H_1(\Gamma) \oplus H_1(\Gamma)$, in both these cases the cokernel of the intersection form, I, must be empty and hence I must be an epimorphism.

Lemma 3.3.2 *Let Γ be a simple graph. Suppose the intersection form I_Γ is an epimorphism. Then*

$$b_2(F(\Gamma, 2)) = b_1(\Gamma)^2 - b_1(\Gamma) + 1 - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2) \tag{3.16}$$

$$b_1(F(\Gamma, 2)) = 2b_1(\Gamma) \tag{3.17}$$

Proof This statement follows directly from exact sequence (3.13). If the intersection form I_Γ is epimorphic this exact sequence implies that the rank of $H_1(\Gamma) \otimes H_1(\Gamma)$ is equal to $b_2(F(\Gamma, 2))$ plus the rank of $H_2(N, \partial N)$. Using formula (3.15) we obtain the first statement of the lemma. The second follows from the fact that the cokernel of an epimorphic map is empty and statement 2 of Theorem 3.2.10. □

Finally, we cover some exceptional examples.

Lemma 3.3.3 *Let T be a tree. Then $H_2(F(T, 2))$ is the trival group and $H_1(F(T, 2))$*

is isomorphic to $H_2(N_T, \partial N_T)$ *and so*

$$b_1(F(T,2)) = \sum_{v \in V(T)} (\mu(v)-1)(\mu(v)-2) - 1.$$

.

Proof A tree, T, is contractible, therefore $H_1(T)$ is the trivial group. This means that in the case of a tree the exact sequence (3.13) is reduced to the sequence,

$$0 \to H_2(N_T, \partial N_T) \xrightarrow{\partial} H_1(F(T,2)) \to 0.$$

This proves the first part of the statement. From the calculation of the Euler Characteristic of $(N, \partial N)$ in Lemma 3.3.1 we can calculate the rank of $H_2(N_T, \partial N_T)$ which is equal to $b_1(F(T,2))$. Since T is contractible $b_1(T) = 0$, and we obtain $b_1(F(T,2)) = \sum_{v \in V(T)}(\mu(v)-1)(\mu(v)-2) - 1$ as required. □

Remark It has been shown by Farber in [15] that the space $F(T,2)$ for any tree T has the homotopy type of a wedge of $\sum_{v \in V(T)}(\mu(v)-1)(\mu(v)-2) - 1$ circles.

Example 3.3.1 Finally we consider the degenerate examples of a graph homeomorphic to the circle S^1 and the graph homeomorphic to a closed interval.

The case of a graph homeomorphic to the circle S^1 was discussed in Example 2.2.2, where we showed that $F(S^1,2)$ is homeomorphic to an open cylinder and $D(S^1,2)$ has the homotopy type of a circle. This means that $H_1(F(S^1,2)) = \mathbb{Z}$, but since the graph Γ is also homeomorphic to a circle, the group $H_1(\Gamma) = \mathbb{Z}$, hence $H_1(F(S^1,2))$ cannot be isomorphic to $\mathrm{coker}(I_\Gamma) \oplus H_1(\Gamma) \oplus H_1(\Gamma)$. This is due to the fact that the map

$$\alpha_* : H_1(F(\Gamma,2)) \to H_1(\Gamma \times \Gamma) \tag{3.18}$$

is not an epimorphism when Γ is homeomorphic to the circle S^1. To see this consider the space $(N, \partial N)$ for this example. The space N is represented by the light grey two-cells in Figure 3.2 and so has the homotopy type of an open cylinder with its boundary ∂N given by the two bounding circles of the cylinder. Hence the space $N/\partial N$ is constructed by collapsing the two boundary circles of the cylinder to a point and so is homeomorphic to the quotient space S^1/S^0 which is homotopy equivalent to the wedge sum $S^2 \vee S^1$. This implies that the homolgy groups of $N/\partial N$ are equal to $\mathbb{Z}$ in dimensions 1 and 2. Significantly, $H_1(N, \partial N) \neq \emptyset$ and so exact sequence 3.13 implies that the map α_* cannot be epimorphic.

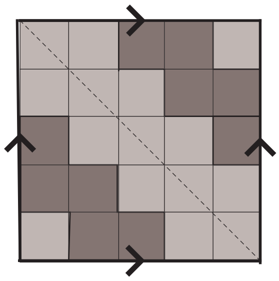


Figure 3.2: The space $F(S^1, 2)$, where S^1 is triangulated as a pentagon, represented as a complex of 25 2-cells. The space $D(S^1, 2)$ is represented by the darker cells, and the diagonal Δ is shown as a dotted line. The boundary of the square is identified according to the arrows.

Finally, we consider a graph Γ homeomorphic to the closed interval $[0, 1]$. In this case the space $D(\Gamma, 2)$ is equal to two points and $F(\Gamma, 2)$ has the homotopy type of two closed intervals. So the discrete configuration space is still a deformation retraction of $F(\Gamma, 2)$, but this space is not connected. Here the space N is equal to the whole space $\Gamma \times \Gamma$ and ∂N is equal to the two points of $D(\Gamma, 2)$ on the boundary of the disc. Hence $N/\partial N$ has the homotopy type of a circle. In this case the only non-trivial group in sequence (3.13) is $H_1(N, \partial N)$. So again Theorem 3.2.10 does not hold.

Figure 3.3: *Left* A graph Γ homeomorphic to the closed interval. *Right* The space $\Gamma \times \Gamma$, the dotted line represents the diagonal, Δ.

3.4 Calculating the Intersections of Cycles

In this section we describe an explicit formula for expressing the intersection of two cycles as an element of $H_2(N, \partial N)$. First we describe the conditions under which the extended diagonal, N_Γ, is homotopy equivalent to the graph Γ.

Lemma 3.4.1 *Let Γ be a simple, finite graph with every cycle of Γ having length at least 5. Then the extended diagonal, $N_\Gamma \subset \Gamma \times \Gamma$, is homotopy equivalent to Γ.*

Proof We show that the inclusion of Γ into N_Γ and the projection of N_Γ onto the first coordinate are homotopy equivalences. We denote the inclusion by

$$i : \Gamma \to N_\Gamma,\ x \mapsto (x, x) \tag{3.19}$$

and the projection by,

$$\pi : N_\Gamma \to \Gamma,\ (x, y) \mapsto x. \tag{3.20}$$

Clearly the map $\pi \circ i$ is equal to the identity map on Γ, so it remains to show that $i \circ \pi$ is homotopic to the identity map on N_Γ.

Consider the subset of N_Γ given by the preimage under π of a point $x \in \Gamma$, $\pi^{-1}(x)$. If $x = v$, a vertex of Γ, then $\pi^{-1}(v)$ is homeomorphic to the ball of radius 2 around v as described in Definition 3.2.6. If x is a point in the interior of an edge $e \in \Gamma$, then $\pi^{-1}(x)$ is homeomorphic to the ball of radius 1 around the edge e. Since Γ contains no cycles of length 3 or 4, any ball of radius 1 or 2 around an edge or a vertex in Γ must be a tree and hence contractible.

Fix a contraction

$$\rho_t^x : \pi^{-1}(x) \to \pi^{-1}(x), \quad t \in [0, 1] \tag{3.21}$$

of $\pi^{-1}(x)$ onto the point $(x, x) \in \pi^{-1}(x)$. So, ρ_0^x is the identity map on $\pi^{-1}(x)$, and ρ_1^x is the constant map at the point (x, x).

We combine these contractions to construct a homotopy between $i \circ \pi$ and id_N, the identity map on N.

$$R_t : N \to N, \quad t \in [0, 1] \tag{3.22}$$

is given by,

$$R_t(x,y) = \rho^x_{1-t}.$$

We then see that $R_0(x,y) = (x,x) = i \circ \pi(x,y)$ and $R_1(x,y) = (x,y) = \mathrm{id}_N(x,y)$. The continuity of this homotopy follows from the fact that each contraction, ρ^x_t, is continuous on the closed set $\pi^{-1}(x) \subset N$ and the union of the sets $\pi^{-1}(x)$ is equal to the space N. $\square$

Remark Lemma 3.4.1 does not hold if Γ contains cycles of length 3 or 4. Let $C = e_1 + e_2 + e_3$ be a cycle of length three in Γ. Then $C \otimes C = \sum_{i,j=1}^{3} e_i e_j$ is a non-trivial two-dimensional cycle in N, therefore N cannot have the same homotopy type as the one-dimensional graph Γ. Now consider a cycle of length 4 in Γ, $C' = e_1 + e_2 + e_3 + e_4$. Then the one-cycle, $v \times (e_1 + e_2 + e_3 + e_4)$, where v is a vertex of C', has length 4 but is not equal to the boundary of any 2-cell in N, and so is non-trivial in N. However it is mapped to the trivial cycle v by the projection π, therefore π cannot be a homotopy equivalence in this case.

We use the homotopy equivalence π to prove the following theorem.

Theorem 3.4.2 *Let Γ be a simple graph containing no cycles of length 3 or 4, and let B_e denote the ball of radius 1 around the edge e. Define ∂B_e, the* boundary *of B_e, to be the set of vertices $v \in \Gamma$ such that $d(v,e) = 1$. Then there exists an injective map,*

$$g : H_2(N_\Gamma, \partial N_\Gamma) \to \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e).$$

Proof Let $\pi : N \to \Gamma$ be the projection onto the first factor (3.20), and let A be the pre-image under π of all vertices of Γ,

$$A = \pi^{-1}(V(\Gamma)). \tag{3.23}$$

This is equal to the following union of disjoint sets

$$A = \bigcup_{v \in V(\Gamma)} v \times B_v$$

where B_v denotes the ball of radius 2 around the vertex $v \in \Gamma$. Note that $v \times B_v$ is homeomorphic to B_v, a subgraph of Γ, and that since Γ contains no cycles of length 3 or 4, B_v is contractible for every vertex $v \in \Gamma$.

We also define the boundary of A as $\partial A = A \cap \partial N$. Then ∂A is described by the following disjoint union

$$\partial A = \bigcup_{v \in V(\Gamma)} v \times S_v$$

where S_v denotes the annulus of radii 1 and 2 around the vertex v. Again we note that $v \times S_v$ is homeomorphic to the subgraph $S_v \subset \Gamma$ and that S_v has $\mu(v)$ contractible components.

Figure 3.4: *Left* A subgraph B_v *Right* The corresponding subgraph S_v

Consider the exact sequence of the triple $\partial N \subset A \cup \partial N \subset N$,

$$\cdots \to H_i(A \cup \partial N, \partial N) \to H_i(N, \partial N) \to H_i(N, A \cup \partial N) \xrightarrow{d} H_{i-1}(A \cup \partial N, \partial N) \to \cdots \tag{3.24}$$

We analyse the groups in this sequence. First note that $H_i(A \cup \partial N, \partial N)$ isomorphic to $H_i(A, \partial A)$ by excision, since $A \cap \partial N$ is defined to be ∂A. The space $A/\partial A$ has the homotopy type of the disjoint union $\bigsqcup_{v \in V(\Gamma)} B_v/S_v$. Hence we have

$$H_i(A, \partial A) \simeq \bigoplus_{v \in V(\Gamma)} H_i(B_v, S_v)$$

Each space B_v/S_v is a one-dimensional connected graph, so its only non-trivial reduced homology group will be $H_1(B_v, S_v)$. The quotient space B_v/S_v can be thought of as a graph with two vertices and $\mu(v)$ edges joining the two vertices, where $\mu(v)$ is the valence of v. Suppose we label the $\mu(v)$ edges incident to v as $e_0, \cdots, e_k$, $k = \mu(v) - 1$. Then we can describe a basis for $H_1(B_v, S_v)$ by taking all pairs of edges $e_0 \pm e_i$, $i = 1, \cdots, k$ with the parity of e_i chosen so that the boundary of $e_0 \pm e_i$ is lies in S_v. We obtain that $H_1(B_v, S_v) = \mathbb{Z}^{\mu(v)-1}$.

Now consider the space S_v, it is made up of $\mu(v)$ connected, contractible components, one for each vertex v_i joined to v, where v_i is the other boundary vertex of the edge e_i. By labelling each of the $\mu(v)$ contractible components of S_v by its corresponding vertex v_i we can describe a basis for the reduced homology group $\tilde{H}_0(S_v)$ by taking all pairs $v_0 - v_i$, $i = 1, \cdots, k$. Then, applying the cellular boundary map to the basis for $H_1(B_v, S_v)$ described above, we see that

$$\partial(e_0 \pm e_i) = \pm(v_0 - v_i) \tag{3.25}$$

and the boundary map is an isomorphism between $H_1(B_v, S_v)$ and $\tilde{H}_0(S_v) = \mathbb{Z}^{\mu(v)-1}$. So we have an isomorphism,

$$q : H_1(A, \partial A) \to \bigoplus_{v \in V(\Gamma)} \tilde{H}_0(S_v)$$

and $H_i(A, \partial A)$ is trivial for $i \neq 1$.

Now consider the group $H_i(N, A \cup \partial N)$. Let B_e denote the ball of radius one around the edge e, and let ∂B_e, the *boundary* of B_e, denote the set of vertices $v \in \Gamma$ such that $d(v, e) = 1$. One can think of N as the union of all subspaces $e \times B_e \in \Gamma \times \Gamma$, where e is an edge of Γ. The space $A \cup \partial N$ is the union of the one-skeletons of all such subspaces which can be written as the union $\partial e \times B_e \cup e \times \partial B_e$. The space $e \times B_e / \partial e \times B_e \cup e \times \partial B_e$ is a wedge of spheres and so has free homology groups. Hence we find that the group $H_i(N, A \cup \partial N)$ is isomorphic to the direct sum

$$H_i(N, A \cup \partial N) \simeq \bigoplus_{e \in E(\Gamma)} H_i(e \times B_e, \partial e \times B_e \cup e \times \partial B_e), \quad i > 0$$

Figure 3.5: *Left* The ball of radius one around the edge e, B_e *Right* The set of 5 vertices making up the boundary ∂B_e

For each group $H_i(e \times B_e, \partial e \times B_e \cup e \times \partial B_e)$, projection onto the second coordinate induces an isomorphism between $H_i(e \times B_e, \partial e \times B_e \cup e \times \partial B_e)$ and $H_{i-1}(B_e, \partial B_e)$. Consider an edge e with boundary vertices v and u, then the space $B_e/\partial B_e$ is a graph with three vertices, u, v and w; u and v are joined by the single edge, e, the pair u and w are joined by $\mu(u)$ edges and the pair v and w are joined by $\mu(v)$ edges. Applying Theorem 3.2.3 we obtain that $H_1(B_e, \partial B_e) = \mathbb{Z}^{\mu(u)+\mu(v)-3}$. Consider the group $\tilde{H}_0(\partial B_e)$, by choosing a basis for the group $H_1(B_e, \partial B_e)$ we can show that the celluar boundary map defines an isomorphism between $H_1(B_e, \partial B_e)$ and the reduced homology group $\tilde{H}_0(\partial B_e)$. Suppose that $\partial e = u - v$, choose an edge e_0 incident to e at vertex u. The basis consists of all pairs of edges $e_0 \pm e_i$ for all edges e_i incident to e at the vertex u and all triples $e_0 \pm e \pm e_j$ for all edges e_j incident to e at the vertex v. The parity of the edges in these basis elements is chosen by considering the orientations of edges in B_e and noting that in order for these basis elements to have boundary in ∂B_e, vertices u and v must appear with zero coefficient in the boundaris of these elements. Examining all possible orientations for the edges of B_e we see that this implies that the boundary map injectively maps this basis to all pairs $v_0 - v_i$ and $v_0 - v_j$ where v_0 is the other boundary vertex of e_0, v_i is the other boundary vertex of e_i and similarly for v_j. This evidently describes a basis for $\tilde{H}_0(\partial B_e)$ and so the boundary map does induce an isomorphism between $H_1(B_e, \partial B_e)$ and $\tilde{H}_0(\partial B_e)$. We obtain an isomorphism,

$$p : H_2(N, A \cup \partial N) \to \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e)$$

and $H_i(N, A \cup \partial N) = 0$ for $i \neq 2$.

We now have the following diagram,

$$\begin{array}{ccccccccc} 0 & \to & H_2(N, \partial N) & \xrightarrow{j_*} & H_2(N, A \cup \partial N) & \xrightarrow{d} & H_1(A \cup \partial N, \partial N) & \to & 0 \\ & & \downarrow & & p \downarrow & & q \downarrow & & . \\ 0 & \to & H_2(N, \partial N) & \xrightarrow{g} & \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e) & \xrightarrow{d_1} & \bigoplus_{v \in V(\Gamma)} \tilde{H}_0(S_v) & \to & 0 \end{array} \tag{3.26}$$

To complete the proof it remains to define the maps g and d_1 so that diagram 3.26 commutes and the second sequence is exact.

To make the first square commute we simply define the map g to be the composition $p \circ j_*$. To find the definition of the map d_1 we consider what happens to an element of $H_2(N, A \cup \partial N)$ under the maps p and $q \circ d$. First we must describe the structure of a homology class in $H_2(N, A \cup \partial N)$.

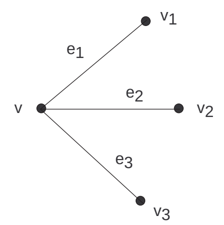


Figure 3.6: Three oriented edges incident to vertex v

Since $(N, A \cup \partial N)$ is two-dimensional, homology classes in $H_2(N, A \cup \partial N)$ are identical to cycles in $C_2(N, A \cup \partial N)$. Such cycles are represented by formal sums of two-cells ee' such that $e \cap e' \neq \emptyset$, the boundaries of such sums must lie in $A \cup \partial N$ and so must consist of one-cells ev with $d(e, v) = 1$ and $v'e'$ with $d(e', v') = 0, 1$. Consider a pair of edges e_1 and e_2 which are incident at the vertex v. For the two-cell e_1e_2 to appear in a cycle of $H_2(N, A \cup \partial N)$ with non-zero ceofficient the cycle must contain another two-cell e_1e_3 with non-zero coefficient, where e_3 is an edge also incident to the vertex v, see Figure 3.6. Then the sum

$$e_1e_2 \pm e_1e_3 \tag{3.27}$$

has boundary

$$(\delta_{v_1}v_1 - \delta_{v_1}v)e_2 - e_1(\delta_{v_2}v_2 - \delta_{v_2}v) \pm (\delta_{v_1}v_1 - \delta_{v_1}v)e_3 \mp e_1(\delta_{v_3}v_3 - \delta_{v_3}v), \tag{3.28}$$

where $\delta_{v_i} = \pm 1$ is the parity of the vertex v_i in the boundary of edge e_i. In order for this sum to have boundary in $A \cup \partial N$, we must choose the parity of the two-cell e_1e_3 so that the one-cell e_1v in the boundary of both e_1e_2 and e_1e_3 has coefficient zero in the boundary of $e_1e_2 \pm e_1e_3$. There are two cases, if $\delta_{v_2} = \delta_{v_3}$ then we must take the sum $e_1e_2 - e_1e_3$ and if $\delta_{v_2} \neq \delta_{v_3}$ we have the sum $e_1e_2 + e_1e_3$. Cycles in $H_2(N, A \cup \partial N)$ can also contain two-cells formed by a product of an edge with itself. Suppose a cycle contains e_4e_4 with non-zero coefficient, then it must contain a sum of the form

$$e_4e_4 \pm e_4e_5 \pm e_4e_6, \tag{3.29}$$

see Figure 3.7. This sum has boundary

$$(\delta_{v_4}v_4 - \delta_{v_4}v)e_4 - e_4(\delta_{v_4} - \delta_{v_4}v) \pm ((\delta_{v_4}v_4 - \delta_{v_4}v)e_5 - e_4(\delta_{v_5}v_5 - \delta_{v_5}v_4)) \tag{3.30}$$
$$\pm((\delta_{v_4}v_4 - \delta_{v_4}v)e_6 - e_4(\delta_{v_6}v_6 - \delta_{v_6}v)).$$

We see that there are four cases in which the boundary of this sum does not contain one-cells e_4v_4 or e_4v with non-zero coefficient, these are listed below;

- $\delta_{v_4} = \delta_{v_5}$ and $\delta_{v_4} = \delta_{v_6} \implies e_4e_4 + e_4e_5 - e_4e_6$
- $\delta_{v_4} = \delta_{v_5}$ and $\delta_{v_4} \neq \delta_{v_6} \implies e_4e_4 + e_4e_5 + e_4e_6$
- $\delta_{v_4} \neq \delta_{v_5}$ and $\delta_{v_4} = \delta_{v_6} \implies e_4e_4 - e_4e_5 - e_4e_6$
- $\delta_{v_4} \neq \delta_{v_5}$ and $\delta_{v_4} \neq \delta_{v_6} \implies e_4e_4 - e_4e_5 + e_4e_6$.

Any cycle in $C_2(N, A \cup \partial N)$ can be written as a linear combination of sums of the form 3.27 and 3.29, hence it is enough to show that the maps $d_1 \circ p$ and $q \circ d$ commute when applied to these sums.

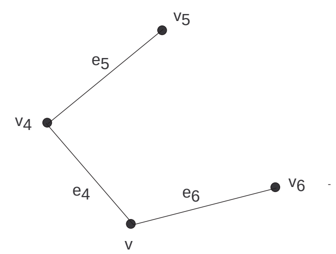


Figure 3.7: Edges incident to e_4

First we apply the map $q \circ d$ to 3.27, we will consider the case where $\delta_{v_2} = \delta_{v_3}$. The map d is the boundary map of a long exact sequence and so has the same effect as applying the cellular boundary map to 3.27,

$$d(e_1e_2 - e_1e_3) = (\delta_{v_1}v_1 - \delta_{v_1}v)e_2 - \delta_{v_2}e_1v_2 - (\delta_{v_1}v_1 - \delta_{v_1}v)e_3 + \delta_{v_3}e_1v_3 \tag{3.31}$$

The isomorphism q between $H_1(A \cup \partial N, \partial N)$ and $\bigoplus_{v \in V(\Gamma)} \tilde{H}_0(S_v)$ was described above as a composition of three maps, the first, which we will label a, maps $H_1(A \cup$

$\partial N, \partial N)$ to $H_1(A, \partial A)$ by exision. Hence we excise 1-cells lying only in $H_1(\partial N)$,

$$a \circ d(e_1e_2 - e_1e_3) = (\delta_{v_1}v_1 - \delta_{v_1}v)e_2 - (\delta_{v_1}v_1 - \delta_{v_1}v)e_3. \tag{3.32}$$

The second map, b, identifies $H_1(A, \partial A)$ with the direct sum of groups $\bigoplus_{v\in V(\Gamma)} H_1(B_v, S_v)$ by sorting the one-cells by the vertex in the first factor,

$$b \circ a \circ d(e_1e_2 - e_1e_3) = \delta_{v_1}v_1(e_2 - e_3) \oplus \delta_{v_1}v(e_3 - e_2) \tag{3.33}$$

Finally we apply the cellular boundary map, ∂, which was shown above to induce an isomorphism between $\bigoplus_{v\in V(\Gamma)} H_1(B_v, S_v)$ and $\bigoplus_{v\in V(\Gamma)} \tilde{H}_0(S_v)$. We have

$$\partial\circ b\circ a\circ d(e_1e_2 - e_1e_3) = \delta_{v_1}\begin{bmatrix} v & v_1 \\ \delta_{v_3}v_3 - \delta_{v_3}v + \delta_{v_2}v - \delta_{v_2}v_2 & \delta_{v_2}v_2 - \delta_{v_2}v + \delta_{v_3}v - \delta_{v_3}v_3 \end{bmatrix} \tag{3.34}$$

$$= \delta_{v_1}\begin{bmatrix} v & v_1 \\ \delta_{v_3}v_3 - \delta_{v_2}v_2 & \delta_{v_2}v_2 - \delta_{v_3}v_3 = 0 \end{bmatrix}. \tag{3.35}$$

This notation implies that the cycle is equal to the direct sum of cycle $\delta_{v_3}v_3 - \delta_{v_2}v_2$, in summand $\tilde{H}_0(S_v)$ and $\delta_{v_2}v_2 - \delta_{v_3}v_3 = 0$ in summand $\tilde{H}_0(S_{v_1})$, where we use the basis for spaces S_{v_i} described above. Note that $\delta_{v_3}v_3 - \delta_{v_2}v_2$ really does define a cycle since $\delta_{v_2} = \delta_{v_3}$. In the case $\delta_{v_2} \neq \delta_{v_3}$ a similar argument shows that

$$\partial \circ b \circ a \circ d(e_1e_2 + e_1e_3) = \delta_{v_1}\begin{bmatrix} v & v_1 \\ \delta_{v_3}v_3 + \delta_{v_2}v_2 & \delta_{v_2}v_2 + \delta_{v_3}v_3 = 0 \end{bmatrix}. \tag{3.36}$$

We can also apply the composition $\partial\circ b\circ a\circ d$ to sum 3.29, in case $\delta_{v_4} \neq \delta_{v_5}$ and $\delta_{v_4} = \delta_{v_6}$ we obtain the following cycle in $\bigoplus_{v\in V(\Gamma)} \tilde{H}_0(S_v)$,

$$\partial \circ b \circ a \circ d(e_4e_4 - e_4e_5 - e_4e_6) = \delta_{v_4}\begin{bmatrix} v & v_4 \\ \delta_{v_5}v_5 + \delta_{v_6}v_6 & -\delta_{v_5}v_5 - \delta_{v_6}v_6 \end{bmatrix}. \tag{3.37}$$

Note that again the pair $\delta_{v_5}v_5 + \delta_{v_6}v_6$ does represent a cycle in $\tilde{H}_0(S_v)$ since the conditions on the coefficients δ_{v_i} imply that $\delta_{v_5} \neq \delta_{v_6}$. Applying the map $\partial \circ b \circ a \circ d$ to sum 3.29 in the other three cases we always obtain a cycle of the form

$$\delta_{v_4}\begin{bmatrix} v & v_4 \\ \pm(v_5 - v_6) & \mp(v_5 - v_6) \end{bmatrix} \in \bigoplus_{v\in V(\Gamma)} \tilde{H}_0(S_v). \tag{3.38}$$

Now we apply the isomorphism p to 3.27 and 3.29. This map was also described above as the composition of three maps. The first map, α, identifies $H_2(N, A \cup \partial N)$ with $\bigoplus_{e \in E(\Gamma)} H_2(e \times B_e, \partial e \times B_e \cup e \times \partial B_e)$ by sorting the two-cells in a cycle of $H_2(N, A \cup \partial N)$ by the edge in the first factor. The second map, β, is the projection onto the second factor which induces an isomoprphism between $H_2(e \times B_e, \partial e \times B_e \cup e \times \partial B_e)$ and $\bigoplus_{e \in E(\Gamma)} H_1(B_e, \partial B_e)$. Finally, we apply the cellular boundary map, ∂, which was shown to induce an isomorphism between $\bigoplus_{e \in E(\Gamma)} H_1(B_e, \partial B_e)$ and $\bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e)$. Taking the cases $\delta_{v_2} = \delta_{v_3}$ and $\delta_{v_4} \neq \delta_{v_5}$, $\delta_{v_4} = \delta_{v_6}$ discussed above we obtain,

$$\partial \circ \beta \circ \alpha(e_1e_2 - e_1e_3) = \begin{bmatrix} e_1 \\ \delta_{v_2}v_2 - \delta_{v_3}v_3 \end{bmatrix} \tag{3.39}$$

$$\partial \circ \beta \circ \alpha(e_4e_4 - e_4e_5 - e_4e_6) = \begin{bmatrix} e_4 \\ \delta_{v_6}v_6 - \delta_{v_5}v_5 \end{bmatrix}. \tag{3.40}$$

This notation follows a similar pattern to that above with 3.39 consisting of the class $\delta_{v_2}v_2 - \delta_{v_3}v_3$ in the summand $\tilde{H}_0(\partial B_{e_1}) \in \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e)$. In all the other cases applying p to 3.27 and 3.29 always produces cycles of the form,

$$\partial \circ \beta \circ \alpha(e_1e_2 - e_1e_3) = \begin{bmatrix} e_1 \\ \pm(v_2 - v_3) \end{bmatrix} \tag{3.41}$$

$$\partial \circ \beta \circ \alpha(e_4e_4 - e_4e_5 - e_4e_6) = \begin{bmatrix} e_4 \\ \pm(v_6 - v_5) \end{bmatrix}, \tag{3.42}$$

respectively.

We must now define the map d_1 so that the maps $d_1 \circ \gamma \circ \beta \circ \alpha$ and $c \circ b \circ a \circ d$ commute. The map d_1 can be described as a matrix of maps. The rows of the matrix correspond to the vertices of Γ and the columns to the edges. In each column there are two non-zero maps, one for each boundary vertex of the corresponding edge. So if we consider the edge e_1, which has boundary $\delta v_1 v_1 - \delta_{v_1} v$ then in the entry in column e_1 and row v is the map $i_* : \tilde{H}_0(\partial B_{e_1}) \to \tilde{H}_0(S_v)$ induced by multiplication by $-\delta_{v_1}$ followed by the inclusion of ∂B_{e_1} into S_v. The other map in column e_1 lies in row v_1 and is the map $j_* : \tilde{H}_0(\partial B_e) \to \tilde{H}_0(S_v)$ induced by the map $j : \partial B_{e_1} \to S_{v_1}$ which multiplies element of ∂B_{e_1} by δ_{v_1} and includes them into S_{v_1}.

Applying the map d_1 to 3.39 we see it maps $\delta_{v_2}v_2 - \delta_{v_3}v_3 \in \tilde{H}_0(\partial B_{e_1})$ to the direct sum of $\delta_{v_1}(\delta_{v_3}v_3 - \delta_{v_2}v_2) \in \tilde{H}_0(S_v)$ and $\delta_{v_1}(\delta_{v_2}v_2 - \delta_{v_3}v_3) = 0 \in \tilde{H}_0(S_{v_1})$ as required. A similar analysis shows that d_1 also maps 3.40 and all other cases to the required elements of $\bigoplus_{v \in V(\Gamma)} \tilde{H}_0(S_v)$.

It remains to show that second sequence in diagram 3.26 is exact. This can be shown directly but also follows algebraically from the fact that diagram 3.26 commutes. Consider a class $z \in H_2(N, \partial N)$, then by the commutativity of diagram 3.26, $q \circ d \circ j_*(z) = d_1 \circ g(z) = d_1 \circ p \circ j_*(z)$. Also, $d \circ j_*(z) = 0$ so we have $d_1 \circ g(z) = 0$ as required. $\square$

We now describe an explicit method for calculating $g \circ I(x)$, for any elements $x \in H_1(\Gamma) \otimes H_1(\Gamma)$.

Choose a set of oriented, simple cycles in Γ, $\mathfrak{C} = \{z_i\}_{i=1}^n$ which form a free basis for the group $H_1(\Gamma)$. Then the kernel of I_Γ is generated by all linear combinations

$$x = \sum_{i,j=1}^{n} \alpha_{ij}(z_i \otimes z_j)$$

of simple tensors of elements of $\mathfrak{C}$ such that $I_\Gamma(x) = 0$ which is equivalent to $g \circ I(x) = 0$.

Definition 3.4.3 Define the *intersection* of two cycles $z, z' \in \mathfrak{C}$ to be the unique element $g \circ I(z \otimes z') \in \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e)$.

We now give an explicit description of the intersection of any pair of cycles $z, z' \in \mathfrak{C}$.

Consider a subgraph B_{e_i} in Γ. We label the vertices of ∂B_{e_i} by $v_0^i, v_1^i, \cdots v_{k_i}^i$. Where ∂B_{e_i} consists of k_i vertices.

Definition 3.4.4 For every edge $e_i \in \Gamma$, define a family of maps

$$f_{e_i}^{v_j^i} : H_2(N, \partial N) \to \mathbb{Z}$$

for every vertex $v_j^i \in \partial B_{e_i}$ by;

$$f_{e_i}^{v_j^i}(e_r \times e_s) = \begin{cases} 1 & \text{if } e_r = e_i \text{ and } v_j^i \in \overline{e}_s \\ 0 & \text{otherwise} \end{cases}$$

which can be extended linearly to an element of $H_2(N, \partial N)$. Note that since the space $(N, \partial N)$ is two-dimensional, homology classes in $H_2(N, \partial N)$ are identical to cycles in $C_2(N, \partial N)$ so these maps are well defined. Then define the *scalar intersection forms* to be the maps

$$I_{e_i}^{v_j^i} = f_{e_i}^{v_j^i} \circ I : H_1(\Gamma) \otimes H_1(\Gamma) \to \mathbb{Z}$$

Lemma 3.4.5 *Let $z = \sum_{p=1}^{l} \epsilon_p e_p$ and $z' = \sum_{q=1}^{l'} \epsilon'_q e'_q$ be two oriented, simple cycles representing homology classes in $\mathfrak{C}$, where $\epsilon_p, \epsilon'_q = \pm 1$ depending on the orientations of the edges e_p and e'_q and the chosen orientations for z and z'. Then*

$$I_{e_i}^{v_j^i}(z \otimes z') = \epsilon_i \epsilon'_j$$

if $z \otimes z'$ contains a summand of the form $\epsilon_i \epsilon'_j (e_i \otimes e'_j)$ where $v_j^i \in \overline{e}'_j$. Otherwise, $I_{e_i}^{v_j^i}(z \otimes z') = 0$.

Furthermore, let δ_j^i be the sign of v_j^i in the boundary of the edge $e_j \in B_{e_i}$, then $g \circ I(z \otimes z')$ is equal to the element

$$\bigoplus_{e_i \in E(\Gamma)} \sum_{j=0}^{k_i} (I_{e_i}^{v_j^i}(z \otimes z') \delta_j^i v_j^i) \in ker(d_1) \subset \bigoplus_{e_i \in E(\Gamma)} \tilde{H}_0(\partial B_{e_i})$$

Proof The first statement of this lemma follows directly from Definition 3.4.4 of the scalar intersection forms. To prove the second statement first note that if the two cycles z and z' do not intersect, the value of every scaler intersection form $I_{e_i}^{v_j^i}$ will be zero when applied to $z \otimes z'$, and $g \circ I(z \otimes z')$ will be equal to zero as required.

Now suppose that z and z' do intersect in the graph. Since z and z' are simple cycles, each point of intersection between the two cycles can have one of two forms; either some number of edges or a vertex. We will treat these two cases seperately, they are pictured in Figure 3.8.

In case one the cycles z and z' intersect at the vertex v. Since the space N has no three dimensional cells, the image under the intersection form I of $z \otimes z'$ must

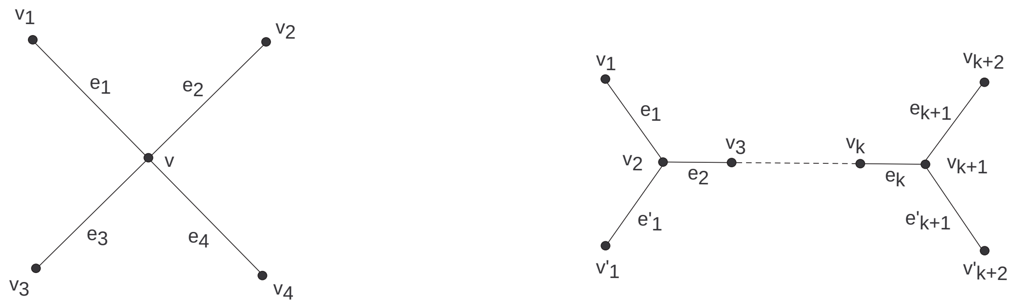


Figure 3.8: *Left* Case 1: Two cycles intersecting at the vertex v. *Right* Case 2: Two cycles intersecting along the edges $e_2, \cdots, e_k$.

lie in the chain group $C_2(N, \partial N)$, that is the abelian group of chains of two-cells in N with boundary in ∂N. The intersection form is induced by the inclusion of $\Gamma \times \Gamma$ into the pair $(\Gamma \times \Gamma, F(\Gamma, 2))$ followed by the excision map, therefore to form $I(z \otimes z')$ we take all simple tensors of edges $e_i \otimes e_j \in z \otimes z'$ such that $e_i \cap e_j \neq \emptyset$ and obtain,

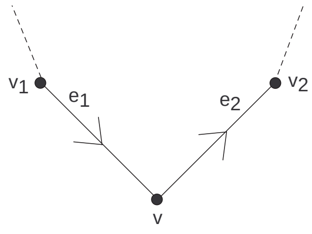
$$\epsilon_1\epsilon_3(e_1 \otimes e_3) + \epsilon_1\epsilon_4(e_1 \otimes e_4) + \epsilon_2\epsilon_3(e_2 \otimes e_3) + \epsilon_2\epsilon_4(e_2 \otimes e_4). \tag{3.43}$$

To show that 3.43 really does lie in $C_2(N, \partial N)$ we need to show that when we apply the boundary map we obtain an element of $C_1(\partial N)$. In order to do this we need to prove the following claim which will be useful throughout this proof.

Claim 1 Let e_1 and e_2 be two edges of a simple cycle $z \in H_1(\Gamma)$ such that $e_1 \cap e_2 = v$, a vertex. Let v_1 be the other boundary vertex of e_1 and v_2 the other boundary vertex of e_2. Let ϵ_i be the coefficient of edge $e_i \in z$, let δ_{v_i} be the sign of the vertex v_i in the boundary of the edge e_i and let δ_v^i be the sign of the vertex v in the edge e_i, $i = 1, 2$, then

1. $\epsilon_1\delta_{v_1} \neq \epsilon_2\delta_{v_2}$,
2. $\epsilon_1\delta_v^1 \neq \epsilon_2\delta_v^2$,
3. $\epsilon_1\delta_v^1 = \epsilon_2\delta_{v_2}$.

To prove this claim consider the two cases $\epsilon_1 = \epsilon_2$ and $\epsilon_1 \neq \epsilon_2$, shown in Figure 3.9. In the first case the two arrows on the edges e_1 and e_2 must point in the same direction and we see that $\delta_{v_1} \neq \delta_{v_2}$, also $\delta_v^1 \neq \delta_v^2$ and finally $\delta_v^1 = \delta_{v_2}$. In the second case the two arrows point in opposite directions and we see that the opposite statements hold, i.e. $\delta_{v_1} = \delta_{v_2}$ etc. This proves the claim.



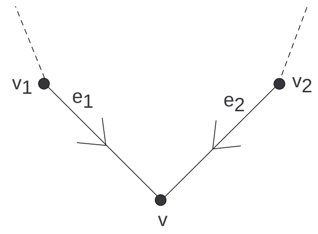


Figure 3.9: *Left* The case $\epsilon_1 = \epsilon_2$. *Right* The case $\epsilon_1 \neq \epsilon_2$

Applying the boundary map to expression 3.43 we obtain the expression,

$$\epsilon_1\epsilon_3[e_1(\epsilon_{v_3}v_3 - \epsilon_{v_3}v) - (\epsilon_{v_1}v_1 - \epsilon_{v_1}v)e_3] \tag{3.44}$$

$$\epsilon_1\epsilon_4[e_1(\epsilon_{v_4}v_4 - \epsilon_{v_4}v) - (\epsilon_{v_1}v_1 - \epsilon_{v_1}v)e_4]$$

$$\epsilon_2\epsilon_3[e_2(\epsilon_{v_3}v_3 - \epsilon_{v_3}v) - (\epsilon_{v_2}v_2 - \epsilon_{v_2}v)e_3]$$

$$\epsilon_2\epsilon_4[e_2(\epsilon_{v_4}v_4 - \epsilon_{v_4}v) - (\epsilon_{v_2}v_2 - \epsilon_{v_2}v)e_4],$$

where $\epsilon_{v_i} = \pm 1$ is the coefficent of the vertex v_i in the boundary of the edge e_i. To show that 3.44 is an element of $C_1(\partial N)$ we must show that the coefficient of every one-cell of the form e_iv or ve_i, $i = 1, \cdots, 4$ is equal to 0, since the vertex v is in the boundary of all the edges e_i. A one-cell e_iv has coefficient

$$-\epsilon_i(\epsilon_3\epsilon_{v_3} + \epsilon_4\epsilon_{v_4}) \tag{3.45}$$

in 3.44 and a one-cell of the form ve_i has coefficient

$$+\epsilon_i(\epsilon_1\epsilon_{v_1} + \epsilon_2\epsilon_{v_2}). \tag{3.46}$$

Since edges e_1 and e_2 and edges e_3 and e_4 satisfy the conditions of Claim 1 above, part 1 of this claim shows that $\epsilon_3\epsilon_{v_3} \neq \epsilon_4\epsilon_{v_4}$ and similarly $\epsilon_1\epsilon_{v_1} \neq \epsilon_2\epsilon_{v_2}$ hence cells of the form e_iv and ve_i have zero coefficient in 3.44 as required.

In exact sequence (3.24) the group $H_2(N, \partial N)$ is mapped to $H_2(N, A \cup \partial N)$ by the map induced by the quotient map collapsing A to a point, but since A is one-dimensional this map is equivalent to the inclusion map. So to map (3.43) to an element $g \circ I(z) \in \bigoplus_{e_i \in E(\Gamma)} \tilde{H}_0(\partial B_{e_i})$ we simply follow the sequence of isomorphisms from $H_2(N, A \cup \partial N)$ to $\bigoplus_{e_i \in E(\Gamma)} \tilde{H}_0(\partial B_{e_i})$ described in Theorem 3.4.2.

$$H_2(N, A \cup \partial N) \xrightarrow{\alpha} \bigoplus_{e \in E(\Gamma)} H_2(e \times B_e, \partial e \times B_e \cup e \times \partial B_e) \tag{3.47}$$

$$\xrightarrow{\beta} \bigoplus_{e \in E(\Gamma)} H_1(B_e, \partial B_e) \xrightarrow{\partial} \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e).$$

The map α collects all terms with the same edge as the first factor, we obtain,

$$\epsilon_1(\epsilon_3(e_1 \otimes e_3) + \epsilon_4(e_1 \otimes e_4)) \oplus \epsilon_2(\epsilon_3(e_2 \otimes e_3) + \epsilon_4(e_2 \otimes e_4)).$$

As described in Theorem (3.4.2), the map β is induced by the projection onto the second factor,

$$\epsilon_1(\epsilon_3 e_3 + \epsilon_4 e_4) \oplus \epsilon_2(\epsilon_3 e_3 + \epsilon_4 e_4).$$

Finally, the map ∂ is identical to the cellular boundary map. Applying this boundary map to the expression above we obtain,

$$\epsilon_1(\epsilon_3(\delta_{v_3} v_3 - \delta_{v_3} v) + \epsilon_4(\delta_{v_4} v_4 - \delta_{v_4} v)) \oplus \epsilon_2(\epsilon_3(\delta_{v_3} v_3 - \delta_{v_3} v) + \epsilon_4(\delta_{v_4} v_4 - \delta_{v_4} v)). \tag{3.48}$$

Here, δ_{v_i} is the sign of the vertex v_i in the boundary of the edge e_i.

As before, part 1 of Claim 1 shows that, $\epsilon_3 \delta_{v_3} \neq \epsilon_4 \delta_{v_4}$ so formula (3.48) is equal to the following element of $\bigoplus_{e_i \in E(\Gamma)} \tilde{H}_0(\partial B_{e_i})$,

$$\epsilon_1(\epsilon_3 \delta_{v_3} v_3 + \epsilon_4 \delta_{v_4} v_4) \oplus \epsilon_2(\epsilon_3 \delta_{v_3} v_3 + \epsilon_4 \delta_{v_4} v_4) \tag{3.49}$$

with the first summand an element of the group $\tilde{H}_0(\partial B_{e_1})$ and the second summand an element of $\tilde{H}_0(\partial B_{e_2})$.

We now show that (3.49) lies in the kernel of the map d_1.

Recall the description of the map d_1 given in Theorem 3.4.2. This description shows that d_1 maps by inclusion and then multiplication by δ_{v_1} the entry in $\tilde{H}_0(\partial B_{e_1})$

into $\tilde{H}_0(S_{v_1})$. Since v_3 and v_4 lie in the same component of S_{v_1} this maps to zero. Similarly the entry in $\tilde{H}_0(\partial B_{e_2})$ is mapped to zero in $\tilde{H}_0(S_{v_2})$. Both summands are mapped to $\tilde{H}_0(S_v)$ since both edges e_1 and e_2 are incident to v, noting that the summands are equal and applying statement 2 of the claim shows that we also obtain zero under this map.

Finally, we note that $I^{v_3}_{e_i}$ and $I^{v_4}_{e_i}$, $i = 1, 2$, are the only non-zero scalar intersection forms when applied to $z \otimes z'$ and that since $I^{v_3}_{e_i}(z \otimes z') = \epsilon_i\epsilon_3$ and $I^{v_4}_{e_i}(z \otimes z') = \epsilon_i\epsilon_4$, direct sum (3.49) is of the form given in the statement of this lemma.

We now consider case 2, as pictured in Figure 3.8. Here, $I(z\otimes z')$ can be represented by the following element of $C_2(N, \partial N)$,

$$\epsilon_1\epsilon'_1(e_1 \times e'_1) + \epsilon_1\epsilon_2(e_1 \times e_2) + \epsilon_2\epsilon'_1(e_2 \times e'_1) + \epsilon_2\epsilon_2(e_2 \times e_2) + \epsilon_2\epsilon_3(e_2 \times e_3)+ \tag{3.50}$$

$$\cdots + \epsilon_i\epsilon_{i-1}(e_i \times e_{i-1}) + \epsilon_i\epsilon_i(e_i \times e_i) + \epsilon_i\epsilon_{i+1}(e_i \times e_{i+1}) + \cdots$$

$$+\epsilon_k\epsilon_{k-1}(e_k\times e_{k-1})+\epsilon_k\epsilon_k(e_k\times e_k)+\epsilon_k\epsilon'_{k+1}(e_k\times e'_{k+1})+\epsilon_{k+1}\epsilon_k(e_{k+1}\times e_k)+\epsilon_{k+1}\epsilon'_{k+1}(e_{k+1}\times e'_{k+1}).$$

Applying the three maps α, β and ∂ of (3.47) gives the following direct sum,

$$\epsilon_1(\epsilon'_1(\delta'_1 v_1 - \delta'_1 v_2) + \epsilon_2(\delta_2 v_2 - \delta_2 v_3)) \tag{3.51}$$

$$\oplus\epsilon_2(\epsilon'_1(\delta'_1 v_1 - \delta'_1 v_2) + \epsilon_2(\delta_2 v_2 - \delta_2 v_3) + \epsilon_3(\delta_3 v_3 - \delta_3 v_4)) \oplus \cdots$$

$$\oplus\epsilon_i(\epsilon_{i-1}(\delta_{i-1} v_{i-1} - \delta_{i-1} v_i) + \epsilon_i(\delta_i v_i - \delta_i v_{i+1}) + \epsilon_{i+1}(\delta_{i+1} v_{i+1} - \delta_{i+1} v_{i+2})) \oplus \cdots$$

$$\oplus\epsilon_k(\epsilon_{k-1}(\delta_{k-1} v_{k-1} - \delta_{k-1} v_k) + \epsilon_k(\delta_k v_k - \delta_k v_{k+1}) + \epsilon'_{k+1}(\delta_{k+1} v_{k+1} - \delta_{k+1} v'_{k+2}))$$

$$\oplus\epsilon_{k+1}(\epsilon_k(\delta_k v_k - \delta_k v_{k+1}) + \epsilon'_{k+1}(\delta_{k+1} v_{k+1} - \delta_{k+1} v'_{k+2})).$$

Here, δ_j represents the sign of vertex v_j in the boundary of edge e_j, note that the sign of the other vertex in this boundary is equal to $-\delta_j$.

Applying statement 3 of the claim above shows that, for example, $\epsilon_{i-1}\delta_{i-1} = \epsilon_i\delta_i$ and similarly $\epsilon_i\delta_i = \epsilon_{i+1}\delta_{i+1}$. Applying this statement to every summand of (3.51) we obtain the following element,

$$\epsilon_1(\epsilon'_1\delta'_1 v_1 - \epsilon_2\delta_2 v_3) \oplus \epsilon_2(\epsilon'_1\delta'_1 v_1 - \epsilon_3\delta_3 v_4) \oplus \cdots \tag{3.52}$$

$$\oplus\epsilon_i(\epsilon_{i-1}\delta_{i-1}v_{i-1} - \epsilon_{i+1}\delta_{i+1}v_{i+2}) \oplus \cdots \oplus \epsilon_k(\epsilon_{k-1}\delta_{k-1}v_{k-1} - \epsilon'_{k+1}\delta_{k+1}v'_{k+2})$$
$$\oplus\epsilon_{k+1}(\epsilon_k\delta_k v_k - \epsilon'_{k+1}\delta_{k+1}v'_{k+2}).$$

We see that this lies in $\bigoplus_{e_i\in E(\Gamma)} \tilde{H}_0(\partial B_{e_i})$ since for any i, $\epsilon_{i-1}\delta_{i-1} = \epsilon_{i+1}\delta_{i+1}$. This can be seen using a similar argument to that used to prove the claim and is illustrated in Figure 3.10.

Figure 3.10: Three oriented edges e_{i-1}, e_i and e_{i+1} in a cycle. *Left* The case $\epsilon_{i-1} = \epsilon_{i+1}$. *Right* The case $\epsilon_{i-1} \neq \epsilon_{i+1}$.

We can show that (3.52) lies in the kernel of the map d_1 using a similar argument to that for case one. For example, the entries in $\tilde{H}_0(\partial B_{e_{i-1}})$ and $\tilde{H}_0(\partial B_{e_i})$, shown below, are mapped into S_{v_i} by d for all $i = 1, \cdots, k$.

$$\epsilon_{i-1}(\epsilon_{i-2}\delta_{i-2}v_{i-2} - \epsilon_i\delta_i v_{i+1}) \oplus \epsilon_i(\epsilon_{i-1}\delta_{i-1}v_{i-1} - \epsilon_{i+1}\delta_{i+1}v_{i+2}).$$

Note that the vertices v_{i-2} and v_{i-1} lie in the same component of S_{v_i}, as do vertices v_{i+1} and v_{i+2}. Applying statement 3 of the claim we see that $\epsilon_{i-2}\delta_{i-2} = \epsilon_{i-1}\delta_{i-1}$ and $\epsilon_i\delta_i = \epsilon_{i+1}\delta_{i+1}$. Then recalling the description of the map d_1 given in the remarks after Theorem 3.4.2, we apply statement 2 of the claim to show that $\epsilon_{i-1}\delta_i^{i-1} \neq \epsilon_i\delta_i^i$ where δ_r^s is the sign of the vertex v_r in the boundary of the edge e_s, and see that

$$d_1(\epsilon_{i-1}(\epsilon_{i-2}\delta_{i-2}v_{i-2} - \epsilon_i\delta_i v_{i+1}) \oplus \epsilon_i(\epsilon_{i-1}\delta_{i-1}v_{i-1} - \epsilon_{i+1}\delta_{i+1}v_{i+2})) = 0 \in \tilde{H}_0(S_{v_i}).$$

Finally, by examining which scalar intersection forms are non-zero when applied to $z\otimes z'$ we see that (3.52) has the form described in the statement of this lemma.

There is one final case to be considered, that of the intersection of a cycle, z, with itself. This case follows directly from arguments given in case 2 after the observation that, for each $e_i \in z$, $g \circ I(z \otimes z)$ has the following entry in $\tilde{H}_0(\partial B_{e_i})$

$$\epsilon_i(\epsilon_{i-1}(\delta_{i-1}v_{i-1} - \delta_{i-1}v_i) + \epsilon_i(\delta_i v_i - \delta_i v_{i+1}) + \epsilon_{i+1}(\delta_{i+1}v_{i+1} - \delta_{i+1}v_{i+2})).$$

□

3.5 An Algorithm to Calculate the Second Betti Number of $F(\Gamma, 2)$

We can combine the previous results of this chapter to give the following algorithm for calculating the second Betti number of $F(\Gamma, 2)$ for any simple graph Γ.

1. Take the first barycentric subdivision of the graph, to eliminate any cycles of length 3 or 4.

2. Choose a basis for $H_1(\Gamma)$, $\mathfrak{C} = \{z_i\}_{i=1}^k$, perhaps using the spanning tree method of Theorem 3.2.3.

3. Calculate

$$g \circ I(z_p \otimes z_q) = \bigoplus_{e_i \in E(\Gamma)} \sum_{j=0}^{k_i} (I_{e_i}^{v_j^i}(z_p \otimes z_q) \delta_j^i v_j^i), \in \bigoplus_{e_i \in E(\Gamma)} \tilde{H}_0(\partial B_{e_i}),$$

   for all pairs of cycles $z_p, z_q \in \mathfrak{C}$, $p, q = 1, \cdots, k$.

4. Use the images $\{g \circ I(z_p \otimes z_q)\}_{p,q=1}^k$ to construct the matrix form of the linear map $g \circ I$ between the vector spaces $H_1(\Gamma) \otimes H_1(\Gamma)$ and $\bigoplus_{e_i \in E(\Gamma)} \tilde{H}_0(\partial B_{e_i})$. The matrix will have $b_1(\Gamma)^2$ rows, each indexed by a pair of cycles in $\mathfrak{C}$, and $\sum_{v \in V(\Gamma)} \mu(v)(\mu(v) - \frac{3}{2})$ columns, indexed by the basis elements of the groups $\tilde{H}_0(\partial B_{e_i})$. The entries in the matrix should be coefficients in $\mathbb{Z}$ such that the linear combination of basis elements for each group $\tilde{H}_0(\partial B_{e_i})$ in row $z_p \otimes z_q$ is equal to the sum $\sum_{j=0}^{k_i} (I_{e_i}^{v_j^i}(z_p \otimes z_q) \delta_j^i v_j^i)$. Use a standard algorithm to calculate the rank, R, of this matrix. Then, by the rank-nullity theorem, the dimension of the kernel of $g \circ I$, i.e. the second Betti number of the group $H_2(F(\Gamma, 2))$, is equal to

$$b_1(\Gamma)^2 - R.$$

The above algorithm will always yield an accurate answer for the second Betti number of $F(\Gamma, 2)$ and each step uses methods of calculation which can be easily turned into a programmable algorithm. However, for any general graph the calculation could become quite large and it does not give any direct information as to the construction of the homology classes which generate $H_2(F(\Gamma, 2))$. In the remainder of this thesis we explore how to use the results of this chapter to describe the second homology group of $F(\Gamma, 2)$ more simply for certain classes of graphs. To close this chapter we apply the results of Section 3.4 to construct generators for $H_2(F(\Gamma, 2))$ for Γ equal to the Kuratowski graphs K_5 and $K_{3,3}$.

Example 3.5.1 In Example 2.4.1 we showed that $F(K_5, 2)$ has the homotopy type of an orientable surface of genus 6 and hence $H_2(F(K_5, 2), \mathbb{Z}) = \mathbb{Z}$. We will now calculate the generator of $H_2(F(K_5, 2), \mathbb{Z})$ as an element of $H_1(K_5) \otimes H_1(K_5)$.

First label the vertices of K_5 as v_1, v_2, v_3, v_4 and v_5 and orient each edge of the graph e^i_j from v_j to v_i where $i > j$. Choose a maximal spanning tree, T, for K_5 consisting of all edges emanating from the vertex v_5.

Then we have a basis for $H_1(K_5)$ consisting of 6 cycles C^i_j with $i, j = 1, \cdots, 4$ and $i > j$. However, K_5 contains cycles of length 3 so to apply Lemma 3.4.5 we must take the first barycentric subdivision of the graph. This introduces ten new vertices, one subdividing each edge of the graph. We label, u^i_j the vertex subdividing the edge e^i_j, and label, E^i_j, the edge joining u^i_j to v_i and label e^i_j the edge joining v_j to u^i_j. All the vertices u^i_j must be contained in any maximal spanning tree of K_5 so we expand T to include the edges E^i_j, $i, j = 1, \cdots, 4$, $i > j$, to obtain the spanning tree shown in Figure 3.11.

Then each of the six cycles generating $H_1(K_5)$ have the form

$$C^i_j = +(E^5_i + e^5_i) - (e^5_j + E^5_j) - (e^i_j + E^i_j), \qquad i, j = 1, \cdots, 4, \qquad i > j. \quad (3.53)$$

To calculate the generator of $H_2(F(K_5, 2))$, consider the image under the intersection form of tensors $C^i_j \otimes C^k_l$ where i, j, k, and l are all distinct, $i > j$, $k > l$. Then the two cycles intersect only at the vertex v_5 and we obtain the following element of $C_2(N, \partial N)$

$$I_{K_5}(C^i_j \otimes C^k_l) = +e^5_i \times e^5_k - e^5_i \times e^5_l - e^5_j \times e^5_k + e^5_j \times e^5_l. \quad (3.54)$$

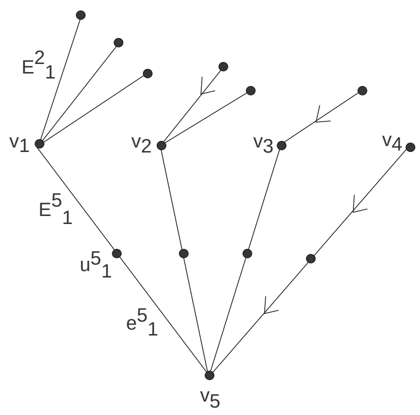


Figure 3.11: A maximal spanning tree for the first barycentric subdivision of K_5.

Then apply the map g to $I_{K_5}(C_j^i \otimes C_l^k)$ to obtain an element of the direct sum $\bigoplus_{e \in E(K_5)} \tilde{H}_0(\partial B_e)$ using the sequence of isomorphisms $p \circ j_*$ described in the proof of Theorem 3.4.2, we obtain,

$$g \circ I(C_j^i \otimes C_l^k) = \begin{bmatrix} e_i^5 & e_j^5 \\ u_l^5 - u_k^5 & u_k^5 - u_l^5 \end{bmatrix}. \tag{3.55}$$

This notation is the same as that used in Theorem 3.4.2 and shows that $g \circ I(C_j^i \otimes C_l^k)$ is the direct sum of the element $u_l^5 - u_k^5$ in the group $\tilde{H}_0(\partial B_{e_i^5})$ and the element $u_k^5 - u_l^5$ in the group $\tilde{H}_0(\partial B_{e_j^5})$.

Consider the group $\tilde{H}_0(\partial B_{e_1^5})$, a non-trivial element of this group appears as a summand of the image under $g \circ I$ of $C_2^1 \otimes C_4^3$, $C_3^1 \otimes C_4^2$ and $C_4^1 \otimes C_3^2$. We see that $g \circ I(C_2^1 \otimes C_4^3 - C_3^1 \otimes C_4^2 + C_4^1 \otimes C_3^2)$ has zero entry in the summand $\tilde{H}_0(\partial B_{e_1^5})$. A similar analysis for the three other edges e_2^5, e_3^5 and e_4^5 shows that the following linear combination of tensors has trivial image under the map $g \circ I$,

$$C_2^1 \otimes C_4^3 - C_3^1 \otimes C_4^2 + C_4^1 \otimes C_3^2 - C_4^3 \otimes C_2^1 + C_4^2 \otimes C_3^1 - C_3^2 \otimes C_4^1 \tag{3.56}$$

and hence corresponds to the generator of $H_2(F(K_5, 2))$. Note that this element can be written as the sum,

$$\sum_{ijkl} \epsilon_{(ijkl)} C_j^i \otimes C_l^k \tag{3.57}$$

where $(ijkl)$ runs over all permutations of indices $1, 2, 3, 4$ such that $i > j$, $k > l$ and $\epsilon_{(ijkl)}$ is the sign of the permutation.

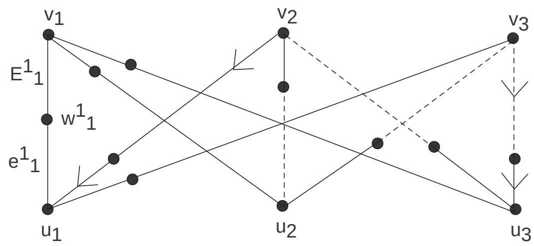


Figure 3.12: A maximal spanning tree for the first barycentric subdivision of $K_{3,3}$. The dotted edges are not included in the tree.

Example 3.5.2 In Example 2.4.2 it was shown that $F(K_{3,3}, 2)$ is also an orientable surface and so $H_2(F(K_{3,3}, 2), \mathbb{Z}) = \mathbb{Z}$. We can follow a similar process to that in Example 3.5.1 to find the generator of $H_2(F(K_{3,3}, 2))$ as an element of $H_1(K_{3,3}) \otimes H_1(K_{3,3})$. We take the first barycentric subdivision of the graph since $K_{3,3}$ contains cycles of length 4 and then construct a spanning tree for the graph as shown in Figure 3.12. We include in the spanning tree all subdivided edges of $K_{3,3}$ which are incident to vertices u_1 or v_1, and then extend the tree to include the vertices subdividing the remaining edges of the graph. All edges are oriented from vertex v_i to vertex u_j, we label w^i_j the vertex subdividing the edge joining vertex v_i to u_j, the edge joining v_i to w^i_j is labelled E^i_j and the edge joining vertex w^i_j to u_j is labelled e^i_j.

There are four edges of the subdivided graph which are not in the spanning tree, e^2_2, e^3_3, E^2_3 and E^3_2. These edges correspond to four cycles in the graph of length 8 which generate $H_1(K_{3,3})$. Label these cycles D^2_2, D^3_3, D^2_3 and D^3_2.

To calculate the generator of $H_2(F(K_{3,3}, 2), \mathbb{Z})$ first consider the image of $D^2_2 \otimes D^3_3$ under the intersection form $I_{K_{3,3}}$,

$$I(D^2_2 \otimes D^3_3) = E^1_2 \times (E^1_3 - E^1_1) + E^1_1 \times (-E^1_3 + E^1_1 + e^1_1) + e^1_1 \times (E^1_1 + e^1_1 - e^1_3) + e^1_2 \times (-e^1_1 + e^1_3). \tag{3.58}$$

Applying the map g to $I(B^2_2 \otimes B^3_3)$ we obtain,

$$g \circ I(D^2_2 \otimes D^3_3) = \begin{bmatrix} E^1_2 & E^1_1 & e^1_1 & e^2_1 \\ w^1_3 - w^1_1 & u_1 - w^1_3 & w^3_1 - v_1 & w^1_1 - w^3_1 \end{bmatrix}. \tag{3.59}$$

The generator we are looking for must be symmetric with respect to the involution τ (2.3), since for any $x \in H_1(K_{3,3}) \otimes H_1(K_{3,3})$ such that $g \circ I(x) = 0$, the image of $g \circ I(\tau_* x)$ is also zero, but $H_2(F(K_{3,3}, 2), \mathbb{Z}) = \mathbb{Z}$ must have exactly one generator. So such a generator must also contain $D_3^3 \otimes D_2^2$. Lemma 3.2.13 implies that $I(D_3^3 \otimes D_2^2) = -\tau_*(I(D_2^2 \otimes D_3^3))$, applying this to (3.58) yields the following,

$$g \circ I(D_3^3 \otimes D_2^2) = \begin{bmatrix} E_3^1 & E_1^1 & e_1^1 & e_1^3 \\ w_2^1 - w_1^1 & u_1 - w_2^1 & w_1^2 - v_1 & w_1^1 - w_1^2 \end{bmatrix}. \tag{3.60}$$

Examining the image under $g \circ I$ of all other simple tensors of the generating cycles of $H_1(K_{3,3})$ we find that,

$$g \circ I(D_3^2 \otimes D_2^3) = \begin{bmatrix} E_3^1 & E_1^1 & e_1^1 & e_1^2 \\ w_2^1 - w_1^1 & u_1 - w_2^1 & v_1 - w_1^3 & w_1^3 - w_1^1 \end{bmatrix} \tag{3.61}$$

and,

$$g \circ I(D_2^3 \otimes D_3^2) = \begin{bmatrix} E_2^1 & E_1^1 & e_1^1 & e_1^3 \\ w_3^1 - w_1^1 & u_1 - w_3^1 & v_1 - w_1^2 & w_1^2 - w_1^1 \end{bmatrix}. \tag{3.62}$$

Taking the direct sum of the four elements of $\bigoplus_{e \in E(K_{3,3})} \tilde{H}_0(\partial B_e)$ calculated above, we see that the combination

$$D_2^2 \otimes D_3^3 + D_3^3 \otimes D_2^2 + D_3^2 \otimes D_2^3 + D_2^3 \otimes D_3^2 \tag{3.63}$$

is mapped to zero by $g \circ I$ and is therefore the generator of $H_2(F(K_{3,3}, 2), \mathbb{Z})$.

Chapter 4

Planar Graphs

In this chapter we use the results of Chapter 3 to describe the homology and cohomology of $F(\Gamma, 2)$ where Γ is a planar graph. Our main result calculates the second Betti number of $F(\Gamma, 2)$ and describes the generators of the second homology group of this space. In the rest of the chapter we calculate simple formulas for the first and second Betti numbers of $F(\Gamma, 2)$ for a large class of planar graphs which we call regular planar graphs. We also describe a relationship between the first homology group of $F(\mathbb{R}^2, 2)$, the space of two particles moving without collisions on the plane, and the first homology group of $F(\Gamma, 2)$ for regular planar graphs. In the last section we describe the cup product for this space and calculate the cohomology algebra $H^*(F(\Gamma, 2), \mathbb{Q})$ where Γ is a regular planar graph. The material in this chapter covers the main results published in the joint paper [6] written by myself and Michael Farber, though with some modifications in the method of proof to incorporate the ideas developed in Section 3.4.

4.1 The Second Betti Number of $F(\Gamma, 2)$

The theorem below uses the theory developed in Chapter 3 to calculate the second Betti number of the configuration space $F(\Gamma, 2)$ for any planar graph Γ.

Theorem 4.1.1 *Let Γ be a planar graph. Consider an embedding of Γ into $\mathbb{R}^2$. Let*

$U_0, U_1, \cdots, U_r$ denote the faces *of Γ, that is the connected components of $\mathbb{R}^2 - \Gamma$, with U_0 denoting the unbounded face. Then the second Betti number of $F(\Gamma, 2)$ is equal to the number of ordered pairs (i, j) with $i, j \in \{0, 1, \cdots, r\}$ such that,*

$$\overline{U}_i \cap \overline{U}_j = \emptyset.$$

Furthermore, for each such pair (i, j) consider the torus T^2_{ij} in $F(\Gamma, 2)$ formed by all configurations where the first particle lies on the boundary of U_i and the second on the boundary of U_j. The fundamental classes $[T^2_{ij}] \in H_2(F(\Gamma, 2))$ of these tori freely generate $H_2(F(\Gamma, 2))$.

Proof Let z_i denote the homology class in $H_1(\Gamma)$ of the cycle represented by the boundary of the domain U_i passed in the anticlockwise direction. The classes $z_1, z_2, \cdots, z_r$ freely generate the group $H_1(\Gamma)$. The homology class, z_0, of the bounding cycle of the face U_0, is equal to the sum $z_1 + z_2 + \cdots + z_r$.

Consider an element $x \in H_1(\Gamma) \otimes H_1(\Gamma)$ such that $I_\Gamma(x) = 0$. We examine the structure of such an x. It can be written as the sum,

$$x = \sum_{i,j=1}^{r} x_{ij} z_i \otimes z_j, \qquad x_{ij} \in \mathbb{Z}.$$

The theorem can be proved by showing that the following claim is true.

The element x can be uniquely expressed as a linear combination of tensors of the form

$$\gamma_{ij} = z_i \otimes z_j, \tag{4.1}$$

such that $i, j \in \{1, \cdots, r\}$ and $\overline{U}_i \cap \overline{U}_j = \emptyset$, and of the form

$$\alpha_i = z_i \otimes z_0 \quad \text{and} \quad \beta_i = z_0 \otimes z_i \tag{4.2}$$

such that $\overline{U}_i \cap \overline{U}_0 = \emptyset$.

By Lemma 3.2.12 tensors (4.1) and (4.2) clearly lie in the kernel of I_Γ. The claim above can be reformulated as follows:

If a tensor $x \in H_1(\Gamma) \otimes H_1(\Gamma)$ is such that $I_\Gamma(x) = 0$ then there exist unique integers

$$a_1, a_2, \cdots, a_r, \qquad b_1, b_2, \cdots, b_r \in \mathbb{Z},$$

called left and right weights, such that

$$x_{ij} = a_i + b_j \tag{4.3}$$

for any pair (i, j) satisfying $\overline{U}_i \cap \overline{U}_j \neq \emptyset$. Moreover we require that

$$a_i = 0 = b_i \tag{4.4}$$

for any $i = 1, \cdots, r$ such that $\overline{U}_i \cap \overline{U}_0 \neq \emptyset$.

If such weights exist then the linear combination

$$\sum_{i=1}^{r} a_i z_i \otimes z_0 + \sum_{j=1}^{r} b_j z_0 \otimes z_j$$

has coefficient x_{ij} in front of each tensor $z_i \otimes z_j$ with $\overline{U}_i \cap \overline{U}_j \neq \emptyset$ and therefore x is a linear combination of tensors of the form (4.1) and (4.2).

It is enough to find the weight a_i, since we can find b_i using the formula

$$x_{ii} = a_i + b_i.$$

To prove this statement we show that having chosen one weight a_1 we can consistently transport this weight across edges to calculate other weights a_j. Let U_1 and U_2 be two faces of the graph with common edge e_1, see Figure 4.1. Suppose the weight a_1 is given, then we can calculate the weight a_2 from the following system of equations,

$$\begin{aligned} x_{11} &= a_1 + b_1 \\ x_{12} &= a_1 + b_2 \\ x_{21} &= a_2 + b_1 \\ x_{22} &= a_2 + b_2. \end{aligned}$$

We obtain the following solution to the system for a_2,

$$a_2 = x_{21} - x_{11} + a_1 = x_{22} - x_{12} + a_1$$

subject to the condition

$$x_{11} + x_{22} = x_{21} + x_{12}. \tag{4.5}$$

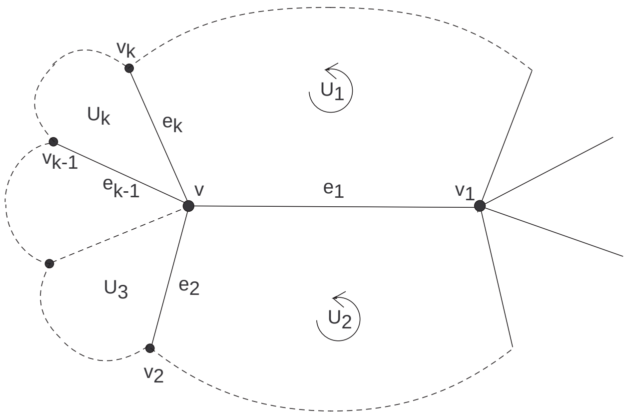


Figure 4.1: The ball B_{e_1} and corresponding faces $U_1, \cdots, U_k$.

To show that condition (4.5) holds we consider $g \circ I(x) \in \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e)$. Since $I(x) = 0$ we also have $g \circ I(x) = 0$. Consider B_{e_1} as a subgraph of Γ, since Γ is a planar graph, we assume that each edge of B_{e_1}, and hence each vertex of ∂B_{e_1}, lies in the boundary of exactly two faces of the graph. Consider all edges of B_{e_1} incident to vertex $v \in \partial e_1$, following the notation of Figure 4.1 and applying Lemma 3.4.5, we see that, for each $i \in \{i, \cdots, k\}$, the entry in the group $\tilde{H}_0(\partial B_{e_i})$ contains the sum,

$$\sum_{j=1, j \neq i}^{k} [x_{ij} I_{e_i}^{v_j}(z_i \otimes z_j) + x_{i+1j} I_{e_i}^{v_j}(z_{i+1} \otimes z_j) + x_{ij+1} I_{e_i}^{v_j}(z_i \otimes z_{j+1}) + x_{i+1j+1} I_{e_i}^{v_j}(z_{i+1} \otimes z_{j+1})] \delta_j v_j, \tag{4.6}$$

and that every occurrence of the vertices $v_1, \cdots, v_k$ in the group $\tilde{H}_0(\partial B_{e_i})$ is counted in this sum.

Let ϵ_r^s denote the orientation, ± 1, of edge e_r in the cycle z_s, then sum (4.6) above is equal to,

$$\sum_{j=1, j \neq i}^{k} [x_{ij} \epsilon_i^i \epsilon_j^j + x_{i+1j} \epsilon_i^{i+1} \epsilon_j^j + x_{ij+1} \epsilon_i^i \epsilon_j^{j+1} + x_{i+1j+1} \epsilon_i^{i+1} \epsilon_j^{j+1}] \delta_j v_j. \tag{4.7}$$

For the entry in the group $\tilde{H}_0(\partial B_{e_i})$ to be zero, the coefficient of each vertex v_j, $j \neq i$, $j \in \{1, \cdots, k\}$ must be equal to zero. Since every cycle z_i is oriented in the anticlockwise direction we have that $\epsilon_i^i \neq \epsilon_i^{i+1}$ for all $i \in \{1, \cdots, k\}$, together with (4.7) this implies

$$x_{ij} - x_{i+1,j} - x_{i,j+1} + x_{i+1,j+1} = 0 \quad \text{for all } j \neq i \in \{1, \cdots, k\}. \tag{4.8}$$

Here we assume without loss of generality that $\epsilon_i^i \epsilon_j^j \delta_j = +1$.

Since the argument above applies to all edges e_i incident to the vertex v equation (4.8) is also valid for all $i \in \{1, \cdots, k\}$. We can transform equation (4.8) into,

$$x_{ij} - x_{i,j+1} = x_{i+1,j} - x_{i+1,j+1}. \tag{4.9}$$

Then by induction we have that

$$x_{ij} - x_{i,j+1} = x_{i+1,p} - x_{i+1,p+1}, \quad \text{for all } i \neq p \in \{1, \cdots, k\}. \tag{4.10}$$

Taking $i = k$ and $j = 1$ in equation (4.9) above we obtain $x_{k1} - x_{k2} = x_{11} - x_{12}$ and taking $i = 2$ and $j = 1$ we obtain $x_{21} - x_{22} = x_{31} - x_{32}$. Then since, by (4.10), $x_{k1} - x_{k2} = x_{31} - x_{32}$ we obtain $x_{11} - x_{12} = x_{21} - x_{22}$ which is equivalent to condition (4.5).

Hence we have shown that the weight a_1 can be transported across an edge to find the weight a_2. Furthermore, (4.8) also implies that we can consistently export the weight a_1 around the vertex v. Rewrite (4.8) as

$$x_{i+1,j} - x_{ij} = x_{i+1,j+1} - x_{i,j+1}, \tag{4.11}$$

then consider the coefficients $x_{ij} = a_i + b_j$ for all $i, j \in \{1, \cdots, k\}$, $i \neq j$ and $i - j \neq \pm 1$. Introduce the equation x_{jj} to cancel the dependence on b_j to obtain

$$x_{jj} - x_{ij} = a_j - a_i. \tag{4.12}$$

Solving the system of equations for the a_i, $i = 1, \cdots, k$ we obtain,

for $j > i$,

$$a_j - a_i = x_{jj} - x_{ij} = \sum_{p=i}^{j-1} [x_{p+1,p} - x_{pp}] \tag{4.13}$$

and for $i > j$

$$a_i - a_j = x_{ij} - x_{jj} = \sum_{p=j}^{i-1} [x_{p+1,p} - x_{pp}]. \tag{4.14}$$

Equation (4.11) implies by induction that

$$x_{i+1,j} - x_{ij} = x_{i+1,q} - x_{iq} \quad \text{for all } i, j, q \in \{1, \cdots, k\}. \tag{4.15}$$

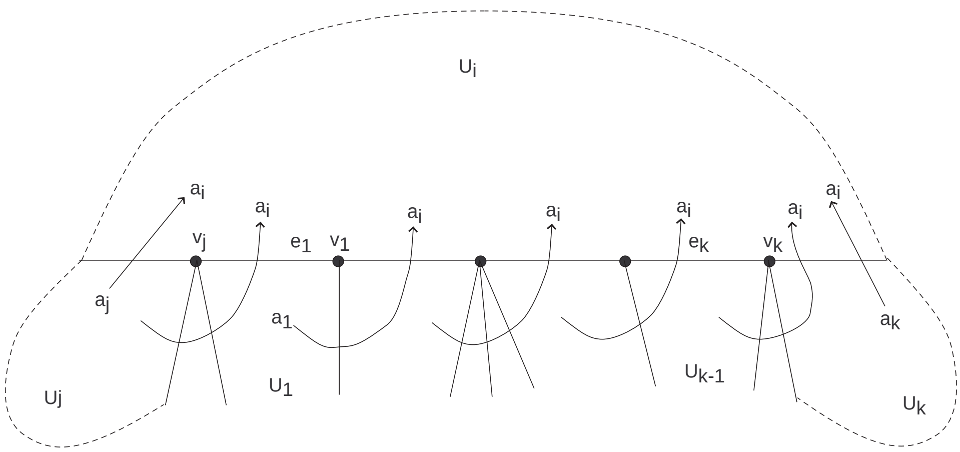


Figure 4.2: The face U_i and its bounding faces. Each arrow represents the exporting of the weight of a bounding face to the face U_i, each giving the same value for the weight a_i.

Then if we add $x_{p+1,j} - x_{p,j}$ for $p = i, \cdots j - 1$ to the right hand side of (4.13) and $x_{p+1,i} - x_{p,i}$ to the right hand side of (4.14), equation (4.15) implies (4.13) and (4.14).

We have shown that one can consistently export the weights a_i around a vertex. It remains to show that if we choose two different paths of edges in Γ between a face U_i and a face U_j and export the weight a_i along these paths to U_j, then both paths produce the same weight a_j. It is enough to show that if one considers a face U_i and two other faces U_j and U_k such that the boundaries of U_j and U_i contain a common edge and the boundaries of U_k and U_i contain a common edge, then calculating the weight a_i in two ways, by transporting the weight a_j across the boundary and by transporting the weight a_k across the boundary, one obtains the same value for a_i. The situation is illustrated in Figure 4.2. Consider the shortest combinatorial path, $e_1 e_2 \cdots e_k$, in the boundary of U_i between a vertex, v_j, also in the bounary of U_j, and a vertex, v_k, also in the boundary of U_k. Then one can export the weight a_j to calculate the weight a_i by passing the weight around the vertex v_j or passing straight from U_j to U_i. Since we can consistently export weights around a vertex both these methods will produce the same value for a_i. Exporting the weight around vertex v_j will produce the same value for a_i as exporting the weight a_1 from face

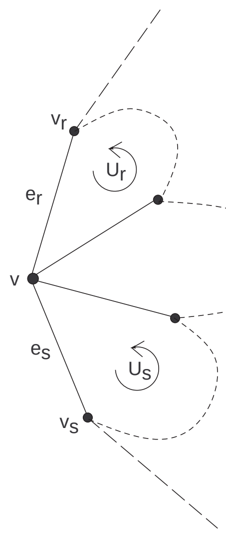


Figure 4.3: The faces of Γ whose boundaries contain vertex v , with v lying on the cycle z_0.

U_1 to U_i where the boundary of U_1 and U_i intersect along edge e_1. Again, this is equivalent to exporting the weight a_1 around the vertex, $v_1 = e_1 \cap e_2$, to calculate a_i. Continuing in this way we see that the transitivity of this relation implies that calculating the weight a_i from the weight a_k by passing from U_k to U_i is equivalent to calculating a_i by passing from U_j to U_i. This is illustrated in Figure 4.2.

We can also show that we can consistently export the weights around the graph by noting that the weights a_i form a flat line bundle over the space constructed by removing the vertices of the graph Γ from the sphere, a compactification of $\mathbb{R}^2$. We have shown that there is no local monodromy in the system of weights as one travels a path around a vertex, and since the fundamental group of the punctured sphere is generated by the loops around the punctures, which correspond to vertices, there can be no global monodromy either.

Now suppose that v lies in the cycle z_0, see Figure 4.3. Then the two edges of z_0 which intersect at v, e_r and e_s, each lie in only one generating cycle of $H_1(\Gamma)$, z_r and z_s respectively. Suppose that the weights a_r and b_r are given and $a_r = b_r = 0$. We show that transporting these weights around the vertex v gives $a_i = b_i = 0$ for all faces U_i whose boundaries contain v. Consider the summand of $g \circ I(x)$ in the

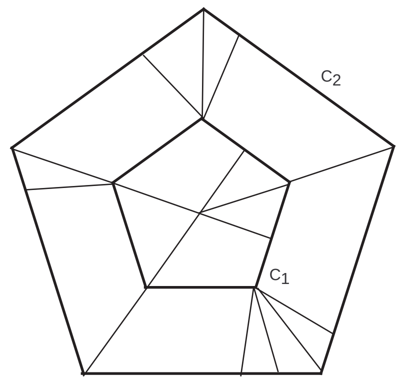


Figure 4.4: The annulus formed by cycles C_1 and C_2

group $\tilde{H}_0(\partial B_{e_r})$, it can contain only one copy of vertex v_s with sign given by

$$I_{e_r}^{v_s}(z_r \otimes z_s)\delta_s = \epsilon_r^r \epsilon_s^s \delta_s.$$

Hence we have that $x_{rs} = 0$, which implies that $a_r + b_s = 0$, then since $a_r = 0$ we have $b_s = 0$ also. The analogous argument shows that x_{sr} must also equal zero, hence $a_s + b_r = 0$ and $a_s = 0$ as required. An equation similar to (4.8) applies for all other faces U_i whose boundaries contain v, this along with the conditions $a_r = b_r = 0$ and $a_s = b_s = 0$ shows that $a_i = b_i = 0$ for all faces U_i incident to v. This completes the proof. □

Remarks

1. This theorem shows that in the case of planar graphs Conjecture 2.4.1 holds since the Kuratowski Theorem states that planar graphs contain no subgraphs isomorphic to K_5 or $K_{3,3}$ and so the group $H_2(F(\Gamma, 2))$ should be entirely generated by embeddings of disjoint cycles in Γ. The theorem also shows that the second Betti number of $F(\Gamma, 2)$ will always be even since the torus T_{ij}^2 is distinct from T_{ji}^2 in the space $F(\Gamma, 2)$.

2. We can explicitly describe elements $x \in H_1(\Gamma) \otimes H_1(\Gamma)$ such that $I_\Gamma(x) = 0$ but which contain simple tensors $z_i \otimes z_j$, with non-zero coefficients, where z_i and z_j are cycles given by passing anticlockwise around the boundaries of faces U_i and U_j of Γ such that $\overline{U}_i \cap \overline{U}_j \neq \emptyset$. Consider two disjoint cycles, C_1 and C_2 in Γ, with C_1 lying in the interior of C_2 so that C_1 and C_2 form the boundary

of an annulus in $\mathbb{R}^2$. Then $I_\Gamma(C_1 \otimes C_2) = 0$ by Lemma 3.2.12 and both C_1 and C_2 can be written as sums of faces of the graph which lie in the interior of the cycle C_2. Since C_1 and C_2 form an annulus, the closure of at least one face appearing as a summand of C_2 must intersect the closure of a face appearing as a summand of C_1, see Figure 4.4.

4.2 The First Betti Number of $F(\Gamma, 2)$

In this section we investigate the group $H_1(F(\Gamma, 2))$ and calculate the first Betti number of $F(\Gamma, 2)$ for a large class of planar graphs. First we show that the cokernel of the intersection form I is always non-empty for a planar graph.

Lemma 4.2.1 *For any connected planar graph $\Gamma \subset \mathbb{R}^2$ having at least one vertex of valence 3 or more, the cokernel of the intersection form I_Γ has rank at least* 1.

Proof We describe a two-cycle in $C_2(N, \partial N)$ and show that it cannot lie in the

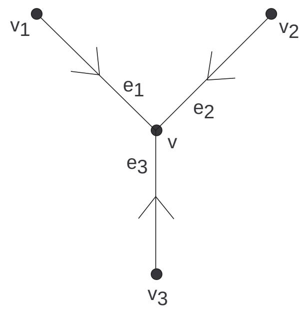


Figure 4.5: Three edges incident to the vertex v.

image of the intersection form I_Γ. Consider a vertex $v \in \Gamma$ with three edges, e_1, e_2 and e_3, incident to it, as shown in Figure 4.5. Then the following element of $C_2(N)$

$$e_1(e_2 - e_3) + e_2(e_3 - e_1) + e_3(e_1 - e_2),$$

has boundary

$$\partial y = \begin{array}{rcrcr} + & v_1(e_3 - e_2) & + & (e_1 - e_3)v_2 \\ + & v_3(e_2 - e_1) & + & (e_3 - e_2)v_1 \\ + & v_2(e_1 - e_3) & + & (e_2 - e_1)v_3 \end{array} \tag{4.16}$$

which is clearly a 1-dimensional cycle in $C_*(\partial N)$ since $d(v_i, e_j) = 1$ for all v_i and e_j such that $i \neq j$, $i, j = 1, 2, 3$. So y is a relative cycle in $C_2(N, \partial N)$. To show that y does not lie in the image of the intersection form, consider the map

$$\psi : (\Gamma \times \Gamma, D(\Gamma, 2)) \to (\mathbb{R}^2, \mathbb{R}^2 - \{0\})$$

given by

$$\psi(x, y) = x - y.$$

The image of ∂y under the map ψ is a closed curve in the punctured plane $\mathbb{R}^2 - \{0\}$ making one full twist around the origin. This is based on the observation that the angle made by the ray from the first to second point is always increasing. Therefore $\psi_*(\partial y)$ is a generator of $H_1(\mathbb{R}^2 - \{0\})$. This implies that $\psi_*(\partial y)$ and hence ∂y has infinite order. Consider the following exact sequence which follows directly from exact sequence (3.13) by replacing the space $F(\Gamma, 2)$ with the discrete space $D(\Gamma, 2)$.

$$\begin{aligned} 0 \to H_2(D(\Gamma, 2)) &\xrightarrow{\alpha_*} H_1(\Gamma) \otimes H_1(\Gamma) \xrightarrow{I_\Gamma} H_2(N, \partial N) \\ &\xrightarrow{\partial} H_1(D(\Gamma, 2)) \xrightarrow{\alpha_*} H_1(\Gamma) \oplus H_1(\Gamma) \to 0. \end{aligned} \tag{4.17}$$

We then see that y passes to a non-trivial element of the cokernel of the intersection form I, for if it was an element of the image of I then ∂y would be the trivial element by the exactness of the sequence (3.13) and could not have infinite order. $\square$

For a large class of planar graphs, which we will call *regular planar graphs*, we can find an explicit formula for the first and second Betti numbers of $F(\Gamma, 2)$.

Definition 4.2.2 Let $\Gamma \subset \mathbb{R}^2$ be a planar graph and denote by $U_0, U_1, \cdots, U_r$, the faces of Γ where U_0 denotes the unbounded face and $r = b_1(\Gamma)$. Such a graph is called *regular* if it has the following properties,

1. every vertex of Γ has valance greater than or equal to 3,

2. the closure of every face $\overline{U}_i$, with $i = 1, \cdots, r$ is contractible and the boundary of the face U_0 is homotopy equivalent to the circle S^1,

3. for every pair $i, j \in \{0, 1, \cdots, r\}$, $i \neq j$ the intersection $\overline{U}_i \cap \overline{U}_j$ is connected.

Theorem 4.2.3 *Let $\Gamma \subset \mathbb{R}^2$ be a regular planar graph with faces $U_0, U_1, \cdots, U_r$ where U_0 denotes the unbounded face of Γ. Then*

$$b_1(F(\Gamma, 2)) = 2b_1(\Gamma) + 1 \tag{4.18}$$

and

$$b_2(F(\Gamma, 2)) = b_1(\Gamma)^2 - b_1(\Gamma) + 2 - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2). \tag{4.19}$$

Proof By Theorem 4.1.1 the second Betti number of $F(\Gamma, 2)$ is equal to twice the number of pairs of disjoint faces of Γ, including the unbounded face U_0. The total number of pairs (U_i, U_j), $i \neq j$ of faces of Γ is equal to $r(r+1)$ or, $b_1(b_1+1) = b_1^2 + b_1$. By property 3, every pair of intersecting faces intersect either at a vertex or a single edge. By properties 2 and 3, each edge of Γ lies in the boundary of exactly two faces, hence there are $2E$ pairs of faces intersecting along an edge. Here $E = |E(\Gamma)|$, is the cardinality of the set of edges of Γ. The number of pairs of faces intersecting at a vertex is given by the formula

$$\sum_{v \in V(\Gamma)} \mu(v)(\mu(v) - 3).$$

To explain this formula consider a vertex $v \in \Gamma$ with $\mu(v)$ edges incident to it. Properties 1 and 2 imply that since the closure of every face of the graph must be contractible, and there can be no vertices of valence 1, each such face must be homeomophic to a disc and therefore its boundary must be homeormorphic to the circle S^1. There are therefore $\mu(v)$ distinct faces of the graph intersecting the vertex v, else Γ would contain a face whose boundary contained a self intersection at the vertex v. For each such face, U_i, there are $\mu(v) - 3$ other faces U_j such that $U_i \cap U_j = v$. Hence by Theorem 4.1.1 we obtain the following formula for $b_2(F(\Gamma, 2))$

$$b_2(F(\Gamma, 2)) = b_1^2 + b_1 - 2E - \sum_{v \in V(\Gamma)} \mu(v)(\mu(v) - 3). \tag{4.20}$$

By the Euler-Poincare Theorem,

$$\chi(\Gamma) = V - E = 1 - b_1(\Gamma). \tag{4.21}$$

Using this fact and applying some elementary transformations to the expression above we obtain formula (4.19).

Finally, to show that $b_1(F(\Gamma, 2)) = 2b_1(\Gamma) + 1$ note that the Euler-Poincare theorem implies that $\chi(F(\Gamma, 2)) = 1 - b_1(F(\Gamma, 2)) + b_2(F(\Gamma, 2))$. Substituting equation (4.19) for $b_2(F(\Gamma, 2))$ and equation (3.1) for the Euler characteristic of $F(\Gamma, 2)$ into this expression, then performing some transformations using equation (4.21) we obtain equation (4.18). □

Remarks

1. Note that the theorem above describes the first Betti number for a graph satisfying the appropriate conditions, but the theorem doesn't show that the group $H_1(F(\Gamma, 2), \mathbb{Z})$ is torsion free. Currently there is no known example of a graph Γ such that $H_1(F(\Gamma, 2), \mathbb{Z})$ contains torsion elements, but it has not been proven that such a graph cannot exist. We can remove this consideration by taking rational instead of integer coefficients.

2. We can describe $2b_1(\Gamma)+1$ explicit generators for the group $H_1(F(\Gamma, 2), \mathbb{Q})$. Let $c_1, c_2, \cdots c_r$ be cycles in Γ given by traversing the boundary of each face of Γ, U_i, in the anticlockwise direction. As discussed previously the homology classes of these cycles freely generate $H_1(\Gamma, \mathbb{Q}) \cong H_1(\Gamma, \mathbb{Z})$, so there are $b_1(\Gamma) = r$ such cycles. For each cycle c_i choose a vertex v_i which is disjoint from c_i in Γ. Then the homology classes of the following 1-cycles in $F(\Gamma, 2)$ give $2b_1(\Gamma)$ generators of $H_1(F(\Gamma, 2), \mathbb{Q})$,

$$y_1 = v_1c_1, \cdots, y_r = v_rc_r \text{ and } z_1 = c_1v_1, \cdots, z_r = c_rv_r.$$

To describe the remaining generator consider a vertex v with three edges e_1, e_2 and e_3 emenating from it, as pictured in Figure 4.5. Then the homology class of the following 1-cycle gives the final generator of $H_1(F(\Gamma, 2), \mathbb{Q})$,

$$y_0 = v_1(e_3 - e_2) + (e_1 - e_3)v_2 + v_3(e_2 - e_1) + (e_3 - e_2)v_1 + v_2(e_1 - e_3) + (e_2 - e_1)v_3.$$

Lemma 4.2.4 *Let Γ be a regular planar graph. Then the $2b_1(\Gamma)+1$ homology classes $\{y_0\}, \{y_1\}, \cdots, \{y_r\}$ and $\{z_1\}, \cdots, \{z_r\}$ freely generate the group $H_1(F(\Gamma, 2), \mathbb{Q})$.*

Proof First we show that the homology classes y_i and z_i are independent of the choice of vertex v_i, i.e. we show that $v_i c_i$ and $v_j c_i$ are homologous for any vertices v_i and v_j disjoint from c_i. This follows after showing that for every face boundary c_i, the space $\Gamma - c_i$ is path connected. If we choose a combinatorial path $p \in \Gamma$ connecting v_i and v_j then the cycle $v_i c_i - v_j c_i$ is equal to the boundary of the two-chain $p \times c_i$.

Assume $\Gamma - c_i$ is not path connected for some c_i and suppose v_i and v_j lie in seperate connected components of $\Gamma - c_i$. This implies that there exists an arc C in the unbounded face of Γ, U_0 such that the boundary of C equals two points $C \cap c_0$, where c_0 is the face boundary associated to U_0, and such that the two points v_i and v_j lie in different connected components of $\mathbb{R}^2 - (C \cup c_0)$. This implies that the intersection $c_i \cap c_0$ is disconnected, contradicting the fact that Γ is a regular planar graph.

To show that these $2r$ homology classes are generators of $H_1(F(\Gamma, 2), \mathbb{Q})$ recall from Theorem 3.2.10 that

$$H_1(F(\Gamma, 2), \mathbb{Q}) \cong H_1(\Gamma, \mathbb{Q}) \oplus H_1(\Gamma, \mathbb{Q}) \oplus \mathrm{coker} I_\Gamma.$$

Under this isomorphism the homology classes $\{\{y_i\}, \{z_i\}\}_{i=1}^r$ are mapped to a generating set of $H_1(\Gamma, \mathbb{Q}) \oplus H_1(\Gamma, \mathbb{Q})$ as follows from the proof of Lemma 3.2.11. Finally, it follows from the proof of Lemma 4.2.1 that the class $\{y_0\}$ lies in the cokernel of I_Γ, and Theorem 4.2.3 implies that therefore $\{y_0\}$ must generate the cokernel of I_Γ. $\square$

For regular planar graphs there is a relationship between the homology of the configuration space $F(\Gamma, 2)$ and the homology of the configuration space of two particles moving without collision on the plane, $\mathbb{R}^2$. We will denote this configuration space by $F(\mathbb{R}^2, 2)$.

Corollary 4.2.5 *For a regular planar graph $\Gamma \subset \mathbb{R}^2$, the map*

$$\beta : F(\Gamma, 2) \to F(\mathbb{R}^2, 2) \times \Gamma \times \Gamma$$

given by

$$(x, y) \mapsto ((x, y), x, y), \quad x, y \in \Gamma, \quad x \neq y$$

induces an isomorphism

$$\beta_* : H_1(F(\Gamma, 2), \mathbb{Q}) \to H_1(F(\mathbb{R}^2, 2) \times \Gamma \times \Gamma, \mathbb{Q}).$$

Proof The space $F(\mathbb{R}^2, 2)$ is constructed by removing from $\mathbb{R}^4$ the plane defined by the equations $x_1 = y_1$ and $x_2 = y_2$ where (x_1, x_2) and (y_1, y_2) are points of $\mathbb{R}^2$, this space has the homotopy type of the circle S^1. Therefore $H_1(F(\mathbb{R}^2, 2), \mathbb{Z}) = \mathbb{Z}$ and $H_1(F(\mathbb{R}^2, 2), \mathbb{Q}) = \mathbb{Q}$. This implies that $H_1(F(\mathbb{R}^2, 2) \times \Gamma \times \Gamma, \mathbb{Q})$ is has rank $2b_1(\Gamma) + 1$. By Theorem 4.2.3, $H_1(F(\Gamma, 2), \mathbb{Q})$ also has rank $2b_1(\Gamma) + 1$ so it remains to prove that β_* is an epimorphism.

We prove that β_* is an epimorphism for any planar graph with at least one vertex of valence 3 or more. In Lemma 4.2.1 we introduced a one-cycle ∂y (4.16). This cycle was shown in Lemma 4.2.4 to be a generator of the homology $H_1(F(\Gamma, 2), \mathbb{Q})$ and Lemma 4.2.1 showed that the image of ∂y under the map

$$\psi_* : H_1(F(\Gamma, 2)) \to H_1(\mathbb{R}^2 - \{0\})$$

induced by

$$\psi : F(\Gamma, 2) \to \{\mathbb{R}^2 - \{0\}\}; \qquad (x, y) \mapsto x - y$$

is the generator of $H_1(\mathbb{R}^2 - \{0\}, \mathbb{Q}) = \mathbb{Q}$.

Note that, in the case of planar graphs $F(\Gamma, 2) \subset F(\mathbb{R}^2, 2)$ since Γ can be embedded in $\mathbb{R}^2$. Expanding the domain of the map ψ to cover the space $F(\mathbb{R}^2, 2)$ induces an isomorphism

$$\psi_* : H_1(F(\mathbb{R}^2, 2)) \to H_1(\mathbb{R}^2 - \{0\}).$$

Since this map is an isomorphism the homology class of the pre-image ∂y under the map ψ must be a generator of $H_1(F(\mathbb{R}^2, 2))$, hence the inclusion map

$$F(\Gamma, 2) \to F(\mathbb{R}^2, 2)$$

maps a generator of $H_1(F(\Gamma, 2))$ to the generator of $H_1(F(\mathbb{R}^2, 2))$ and so is epimorphic. The fact that β_* is epimorphic then follows from Lemma 3.2.11 which states that the map induced by inclusion $\alpha_* : H_1(F(\Gamma, 2)) \to H_1(\Gamma \times \Gamma)$ is epimorphic. $\square$

Before finishing this section we present an example to show that Theorem 4.2.3 does not hold for planar graphs which are not regular.

Example 4.2.1 Consider the three graphs in Figure 4.6. Graph Γ_1 does not have property 2 of Definition 4.2.2 since the closures of two of its faces are not contractible. Graph Γ_2 does satisty property 2 but does not satisfy property 3 since the intersection of the boundaries of some pairs of faces consist of two disjoint edges. The third graph Γ_3 satisfies all conditions of Definition 4.2.2.

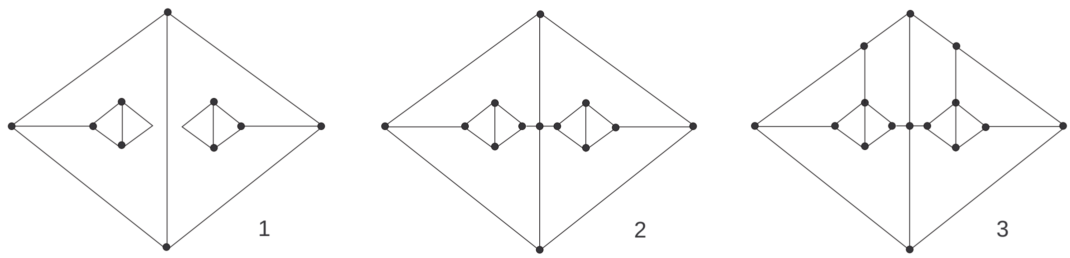


Figure 4.6: Three planar graphs. *Left* Γ_1 is not regular, *centre* Γ_2 is not regular, *right* Γ_3 is regular.

We show that for Γ_1 and Γ_2, the first Betti number of the configuration space $F(\Gamma_i, 2)$, $i = 1, 2$ is not equal to $2b_1(\Gamma_i) + 1$. For Γ_1, $b_1(\Gamma) = 6$ and so $\chi(\Gamma_1) = -5$. Using formula (3.1) we can calculate the Euler characteristic of $F(\Gamma_1, 2)$ as $\chi(F(\Gamma_1, 2)) = 0$. By Theorem 4.1.1 we know that the second Betti number of $F(\Gamma_1, 2)$ is equal to twice the number of pairs of disjoint faces of the graph, in this case there are 24 such pairs. Then using the Euler-Poincare theorem we calculate that $b_1(F(\Gamma_1, 2)) = 1 + 24 - 0 = 25$, whereas $2b_1(\Gamma_1) + 1 = 13$.

For Γ_2 we have that $b_1(\Gamma_2) = 8$ so $\chi(\Gamma_2) = -7$. Then formula (3.1) gives $\chi(F(\Gamma_2, 2)) = 12$. In this case, the number of ordered pairs of disjoint faces of the graph is 32 and so we calculate $b_1(F(\Gamma_2, 2)) = 1 + 32 - 12 = 21$ and $2b_1(\Gamma_2) + 1 = 17$.

Finally, we see that Theorem 4.2.3 does hold for the regular planar graph Γ_3. Here, $b_1(\Gamma_3) = 10$ and $\chi(\Gamma_3) = -9$. Then, $\chi(F(\Gamma_3, 2)) = 30$ and the number of ordered pairs of disjoint face boundaries equals 50. Hence we obtain that $b_1(F(\Gamma_3, 2)) = 51 - 30 = 21 = 2b_1(\Gamma_3) + 1$. We can also check that formula (4.19) for the second Betti number of $F(\Gamma_3, 2)$ holds. We have

$$\begin{aligned} b_2(F(\Gamma_3, 2)) &= b_1(\Gamma_3)^2 - b_1(\Gamma_3) + 2 - \sum_{v \in V(\Gamma_3)} (\mu(v) - 1)(\mu(v) - 2) \\ &= 100 - 10 + 2 - (12 \times 2) - (3 \times 3 \times 2) = 92 - 42 = 50 \end{aligned}$$

as required.

4.3 The Cup-Product and the Cohomology Algebra of $F(\Gamma, 2)$

In this section we describe the cohomology groups of $F(\Gamma, 2)$ for a planar graph Γ and consider the action of the cup-product

$$\cup : H^1(F(\Gamma, 2), \mathbb{Q}) \times H^1(F(\Gamma, 2), \mathbb{Q}) \to H^2(F(\Gamma, 2), \mathbb{Q})).$$

Let Γ be a planar graph containing at least one vertex of valance 3 or more. Let $U_0, U_1, \cdots, U_r$ denote the faces of Γ with U_0 denoting the unbounded face and $r = b_1(\Gamma)$. Let z_i denote the homology class in $H_1(\Gamma)$ of the bounding cycle of face U_i traversed in the anticlockwise direction. Then the classes $z_1, \cdots, z_r$ generate $H_1(\Gamma)$. Let

$$J(\Gamma) = \{(i, j) : \overline{U}_i \cap \overline{U}_j = \emptyset, i, j \in 0, \cdots, r\}.$$

By Theorem 4.1.1 we know that the group $H_2(F(\Gamma, 2), \mathbb{Q})$ is generated by the homology classes of all tori $T^2_{ij} \in F(\Gamma, 2)$ representing configurations where the first particle lies on the boundary of the face U_i and the second on the boundary of face U_j, with $(i, j) \in J(\Gamma)$. Let $\eta_{ij} \in H^2(F(\Gamma, 2), \mathbb{Q})$ be the cohomology class dual to the class $[T^2_{ij}] \in H_2(F(\Gamma, 2), \mathbb{Q})$, i.e.

$$\langle \eta_{ij}, [T^2_{kl}] \rangle = \begin{cases} 1, & \text{if } (i, j) = (k, l) \\ 0, & \text{otherwise.} \end{cases}$$

Then the set of cohomology classes,

$$\eta_{ij}, \qquad (i,j) \in J(\Gamma)$$

generate the vector space $H^2(F(\Gamma,2),\mathbb{Q})$.

Recall the map $\alpha_* : H_1(F(\Gamma,2)) \to H_1(\Gamma \times \Gamma)$ induced by inclusion. In Lemma 3.2.11 this map was shown to be an epimorphism. Therefore the map induced on cohomology

$$\alpha^* : H^1(\Gamma \times \Gamma, \mathbb{Q}) \to H^1(F(\Gamma,2),\mathbb{Q}) \tag{4.22}$$

is a monomorphism. We now describe the action of the cup-product on one-dimensional cohomology classes which lie in the image of α^*.

Theorem 4.3.1 *Let Γ be a planar graph and let $u_1, \cdots, u_r$ be a basis for $H^1(\Gamma,\mathbb{Q})$ dual to the basis $z_1, \cdots z_r$ for $H_1(\Gamma,\mathbb{Q})$ where z_i is the bounding cycle of the face U_i. Denote*

$$\xi_i = \alpha^*(u_i \times 1), \qquad \phi_i = \alpha^*(1 \times u_i). \tag{4.23}$$

Then the following statements hold,

1. $\xi_i \cup \xi_j = \phi_i \cup \phi_j = 0$ *for all* $i, j = 1, \cdots, r$.
2. $\phi_i \cup \xi_j = \epsilon_{ij}\eta_{ij} + \epsilon_{0j}\eta_{0j} + \epsilon_{i0}\eta_{i0} = -\xi_i \cup \phi_j$

where

$$\epsilon_{ij} = \begin{cases} 1, & (i,j) \in J(\Gamma) \\ 0, & \textit{otherwise.} \end{cases}$$

Proof To prove statement 1 we calculate,

$$\xi_i \cup \xi_j = \alpha^*(u_i \times 1) \cup \alpha^*(u_j \times 1) = \alpha^*(u_i \times 1 \cup u_j \times 1) = 0$$

since $H^2(\Gamma,\mathbb{Q})$ is trivial. For statment two we have

$$\phi_i \cup \xi_j = \alpha^*(1 \times u_i) \cup \alpha^*(u_j \times 1) = \alpha^*(1 \times u_i \cup u_j \times 1) = \alpha^*(-(u_j \times u_i)).$$

Then evaluating this cup-product on a homology class $[T^2_{pq}] \in H_2(F(\Gamma,2),\mathbb{Q})$ for some $(p, q) \in J(\Gamma)$ we obtain,

$$\begin{aligned}\langle \phi_i \cup \xi_j, [T^2_{pq}] \rangle &= \langle \alpha^*(-(u_j \times u_i)), [T^2_{pq}] \rangle \\ &= \langle -(u_j \times u_i), \alpha_*[T^2_{pq}] \rangle \\ &= \langle -(u_j \times u_i), z_p \times z_q \rangle \\ &= \langle u_j, z_p \rangle \langle u_i, z_q \rangle.\end{aligned}$$

This implies,

$$\phi_i \cup \xi_j = \sum_{(p,q) \in J(\Gamma)} [\langle u_j, z_p \rangle \langle u_i, z_q \rangle] \eta_{pq}.$$

However, since u_i is dual to z_i for $i = 1, \cdots, r$ this expression can be simplified further. Consider all cohomology classes η_{pq} such that p and q do not equal 0. Then we see that the coefficient of the sum is zero unless $p = j$ and $q = i$. Now suppose that $p = 0$. Then the coefficient becomes

$$\langle u_j, z_0 \rangle \langle u_i, z_q \rangle = \langle u_j, z_1 + \cdots + z_r \rangle \langle u_i, z_q \rangle.$$

The evaluation of u_j on z_0 is always 1 and so this coefficient is non-zero if and only if $q = i$. Applying a similar argument to the case $q = 0$ we obtain,

$$\phi_i \cup \xi_j = \eta_{ij} + \eta_{i0} + \eta_{0j}$$

which proves statement 2. □

Definition 4.3.2 A cohomology class $\xi \in H^1(F(\Gamma, 2), \mathbb{Q})$ is said to be *acyclic* if the evaluation $\langle \xi, vc \rangle = \langle \xi, cv \rangle = 0$, where vc and cv are 1-cycles in $F(\Gamma, 2)$ constructed by taking the product of any cycle c in the graph Γ with any vertex $v \in \Gamma$ which is disjoint from c.

We have the following theorem for acyclic cohomology classes.

Theorem 4.3.3 *Let Γ be a planar graph and let $\xi \in H^1(F(\Gamma, 2), \mathbb{Q})$ be an acyclic cohomology class. Then $\xi \cup \eta = \eta \cup \xi = 0$ for any cohomology class $\eta \in H^1(F(\Gamma, 2), \mathbb{Q})$.*

Proof Consider a torus $T^2_{ij} \in F(\Gamma, 2)$ for some pair $(i, j) \in J(\Gamma)$. Then given cohomology classes $\eta, \xi \in H^1(F(\Gamma, 2), \mathbb{Q})$ let $\xi' = \xi | T^2_{ij}$ and $\eta' = \eta | T^2_{ij}$ denote the restrictions of these classes to the torus so that $\xi', \eta' \in H^1(T^2_{ij}, \mathbb{Q})$. Then,

$$\langle \xi \cup \eta, [T^2_{ij}] \rangle = \langle \xi' \cup \eta', s_{ij} \rangle$$

where s_{ij} is the fundamental class of the torus T^2_{ij}. So it is enough to show that $\xi' = 0$ for any acyclic cohomology class ξ.

Consider two points, v_i in the boundary of face U_i and v_j in the boundary of face U_j. Then since U_i and U_j are disjoint the 1-cycles formed by taking the procuct of v_i with the boundary of U_j and the product of v_j with the boundary of U_i lie in $F(\Gamma, 2)$. Since ξ is acyclic, it must evaluate trivially on these cycles, however these cycles generate $H_1(T^2_{ij}, \mathbb{Q})$ and so $\xi' = 0$. □

Finally, we can summarise the results of this section to completely describe the cohomology algebra of regular planar graphs.

Theorem 4.3.4 *Let $\Gamma \subset \mathbb{R}^2$ be a regular planar graph with faces $U_0, U_1, \cdots, U_r$ where U_0 denotes the unbounded face of Γ and $b_1(\Gamma) = r$. Then there exists a unique acyclic cohomology class $\eta \in H^1(F(\Gamma, 2), \mathbb{Q})$ such that every cohomology class in $H^1(F(\Gamma, 2), \mathbb{Q})$ can be expressed uniquely in the form*

$$\xi = \alpha^*(u_i \times 1 + 1 \times u_j) + \lambda\eta$$

where $u_i, u_j \in H^1(\Gamma, \mathbb{Q})$ and $\lambda \in \mathbb{Q}$.

Proof In Lemma 4.2.4 we described $2b_1(\Gamma) + 1$ generators for $H_1(F(\Gamma, 2), \mathbb{Q})$ for a regular planar graph Γ. These were labelled $y_0, \cdots, y_r, z_1, \cdots, z_r$. Consider a basis for $H^1(F(\Gamma, 2), \mathbb{Q})$ dual to these generators. Then the dual generators $y_1*, \cdots, y_r*$ and $z_1*, \cdots z_r*$ generate the image of the monomorphism $\alpha^* : H^1(\Gamma \times \Gamma, \mathbb{Q}) \to H^1(F(\Gamma, 2), \mathbb{Q})$, since they are the dual of classes represented by 1-cycles of the form $v_i c_i$ and $c_i v_i$, where c_i is the bounding cycle of the face U_i and v_i is a vertex not contained in $\overline{U}_i$, which generate the group $H_1(\Gamma \times \Gamma, \mathbb{Q})$. Clearly, these classes are not acyclic. The final generator y_0 is constructed from a subtree of the graph Γ and is therefore acyclic and represents the unique acyclic cohomology class η. □

Remark Bringing together all the results of this section we can describe the action of the cup product for regular planar graphs. Consider two 1-dimensional cohomology classes $\xi, \xi' \in H^1(F(\Gamma, 2)\mathbb{Q})$. By Theorem 4.3.4 we know these must take the form

$$\xi = \alpha^*(u_i \times 1 + 1 \times u_j) + \lambda\eta \qquad \xi' = \alpha^*(u'_i \times 1 + 1 \times u'_j) + \lambda'\eta.$$

Applying Theorem 4.3.3 we see that the only non-zero part of $\xi \cup \xi'$ is given by

$$\alpha^*(u_i \times 1 + 1 \times u_j) \cup \alpha^*(u_i' \times 1 + 1 \times u_j').$$

The elements $u_i, u_j, u_i', u_j' \in H^1(\Gamma, \mathbb{Q})$ can be written as linear combinations with coefficients in $\mathbb{Q}$ of cohomology classes dual to homology classes in $H_1(\Gamma, \mathbb{Q})$ represented by the bounding cycles of faces of Γ. Following the notation of Theorem 4.3.1 we write

$$u_i = a_1 u_1 + \cdots + a_r u_r, \quad u_j = b_1 u_1 + \cdots + b_r u_r,$$

$$u_i' = a_1' u_1 + \cdots + a_r' u_r, \quad u_j' = b_1' u_1 + \cdots + b_r' u_r.$$

Then applying Theorem 4.3.1 we find that the cup product of ξ and ξ' is equal to

$$\sum_{i,j=1}^{r} [a_i b_j' - a_i' b_j] \epsilon_{ij} \eta_{ij} + \sum_{i=1}^{r} \left[\sum_{j=1}^{r} [a_i b_j' - a_i' b_j] \right] \epsilon_{i0} \eta_{i0} + \sum_{j=1}^{r} \left[\sum_{i=1}^{r} [a_i b_j' - a_i' b_j] \right] \epsilon_{0j} \eta_{0j}$$

where η_{ij} is the cohomology class dual to the torus $[T_{ij}^2] \in H_2(F(\Gamma, 2), \mathbb{Q})$.

Chapter 5

Non-Planar Graphs

In this chapter we examine how one can use the intersection form to calculate the Betti numbers of $F(\Gamma, 2)$ when Γ is a non-planar graph. We explore the question of which graphs have epimorphic intersection forms. This question arises naturally from Chapter 4 where we showed that, for planar graphs, the cokernel of the intersection form has rank 1, and it was discussed in the remarks following Lemma 3.3.1 that, for the two Kuratowski graphs, K_5 and $K_{3,3}$, the intersection form is an epimorphism. This suggests that for non-planar graphs, the cokernel of the intersection form could be empty.

In the first section of this chapter we examine the structure of the image group for the intersection form, $H_2(N, \partial N)$, splitting this group into its symmetric and anti-symmetric parts with respect to the involution τ (2.3). In section two we make use of this structure to show that the intersection form is an epimorphism for any complete bipartite graph of the form $K_{3,m}$. This result has the interesting consequence of showing Conjecture 2.4.1, which inspired the work in this thesis, to be false. In the remainder of this section we use the techniques developed in Chapter 3 to prove that the intersection form is epimorphic for all complete bipartite graphs. In the final section we turn our attention to complete graphs and prove that the intersection form is also epimorphic for these graphs. We finish by giving an example of a non-planar graph for which the intersection form has non-empty cokernel.

5.1 The Structure of the Group $H_2(N, \partial N)$

In order to show whether the intersection form is epimorphic for a particular graph it is necessary to consider the structure of the image group of the intersection form, $H_2(N, \partial N)$. To this end we consider the action of the involution τ on the space $(N, \partial N)$, given by the restriction of the involution on $\Gamma \times \Gamma$ which permutes the two factors. This involution map induces an involution τ_* on the homology group $H_2(N, \partial N; \mathbb{Q})$ which allows this group to be written as the direct sum,

$$H_2(N, \partial N; \mathbb{Q}) = H_2(N, \partial N; \mathbb{Q})^+ \oplus H_2(N, \partial N; \mathbb{Q})^-,$$

of the subgroup $H_2(N, \partial N; \mathbb{Q})^+ \subseteq H_2(N, \partial N; \mathbb{Q})$ generated by cycles symmetric with respect to the action of τ_*, and the subgroup $H_2(N, \partial N; \mathbb{Q})^- \subseteq H_2(N, \partial N; \mathbb{Q})$ generated by cycles anti-symmetric with respect to this action.

Lemma 5.1.1 *For any graph Γ, the symmetric part of the group $H_2(N_\Gamma, \partial N_\Gamma; \mathbb{Q})$ has rank*

$$rank H_2(N_\Gamma, \partial N_\Gamma; \mathbb{Q})^+ = \frac{1}{2} \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2), \tag{5.1}$$

and the anti-symmetric part of the group $H_2(N_\Gamma, \partial N_\Gamma; \mathbb{Q})$ has rank

$$rank H_2(N_\Gamma, \partial N_\Gamma; \mathbb{Q})^- = -\chi(\Gamma) + \frac{1}{2} \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2). \tag{5.2}$$

Proof Define the equivariant Euler characteristic of any subcomplex $X \subseteq \Gamma \times \Gamma$ by

$$\chi_\tau(X) = \sum_{i=0}^{2} (-1)^i \mathrm{trace}(\tau_* : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})).$$

Then, by the definition of the trace of a map, we have

$$\chi_\tau(N, \partial N) = \mathrm{rank} H_2(N, \partial N)^+ - \mathrm{rank} H_2(N, \partial N)^-$$

since $H_i(N, \partial N) = 0$ for $i = 0, 1$. The equivariant Euler characteristic, χ_τ, has an additivity property analogous to that of the usual Euler characteristic so,

$$\chi_\tau(N, \partial N) = \chi_\tau(N) - \chi_\tau(\partial N).$$

We now assume without loss of generality that Γ has no cycles of length less than 5. Lemma 3.4.1 then implies that the space N is homotopy equivalent to the graph Γ under the projection map π which projects N onto its first coordinate, hence the involution τ on N is homotopic to the identity map on Γ. The trace of the identity map on the group $H_1(\Gamma)$ is given by the rank of the group, $b_1(\Gamma)$, and the trace of the identity map on the group $H_0(\Gamma)$ is 1, the rank of $H_0(\Gamma)$. So

$$\chi_\tau(N) = 1 - (1 - \chi(\Gamma)) = \chi(\Gamma).$$

Now consider the one-dimensional space ∂N. The involution τ acts freely on the space ∂N on the chain level; τ maps a zero-cell uv to the zero-cell vu and it maps the one-cell ve to the one-cell ev and vice versa. Hence the involution τ_* has no fixed points and $\chi_\tau(\partial N) = 0$. Combining these results implies that

$$\chi_\tau(N, \partial N) = \chi(\Gamma).$$

so we obtain,

$$\chi_\tau(\Gamma) = \text{rank} H_2(N, \partial N)^+ - \text{rank} H_2(N, \partial N)^-.$$

Using the fact that

$$\text{rank} H_2(N, \partial N) = \text{rank} H_2(N, \partial N)^- + \text{rank} H_2(N, \partial N)^+ \tag{5.3}$$

and formula (3.15) for the Euler characteristic of $(N, \partial N)$ we find that,

$$2 \times \text{rank} H_2(N, \partial N)^- = -2\chi(\Gamma) + \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2).$$

Dividing this expression by 2 gives formula (5.2) and, finally, equation (5.3) implies formula (5.1). □

We will now examine the structure of the symmetric part of the group $H_2(N, \partial N)$. First we introduce a definition which will be used throughout this chapter.

Definition 5.1.2 Let Γ be a finite graph and let

$$y = \sum_{i=1}^{k} \alpha_i (e_i e_i') \quad \alpha_i \in \mathbb{Q}, \quad \alpha_i \neq 0$$

be a cellular two-chain in the space $\Gamma \times \Gamma$. Then the *support* of the chain y is defined to be the subgraph of Γ with edge set equal to the union of all the edges appearing in the formal sum representing y,

$$E(\mathrm{supp}(y)) = \bigcup_{i=1}^{k} \{e_i, e'_i\}$$

and vertex set equal to the union of the boundary vertices in Γ of the edges appearing in the formal sum representing y,

$$V(\mathrm{supp}(y)) = \bigcup_{i=1}^{k} \{\partial e_i, \partial e'_i\}.$$

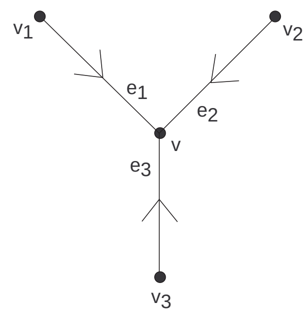


Figure 5.1: The Y-graph

Consider the graph consisting of a central vertex v and three edges, e_1, e_2, e_3 emanating from it. We will call this graph the *Y-graph.* Consider the homology class in $H_2(N_Y, \partial N_Y)$ represented by the relative cycle,

$$y = e_1e_2 - e_2e_1 + e_3e_1 - e_1e_3 + e_2e_3 - e_3e_2. \tag{5.4}$$

Note that this cycle was introduced in Lemma 4.2.1 where it was shown that for a planar graph, the homology class represented by this cycle lies in the cokernel of the intersection form. The cycle y is symmetric under the action of the involution τ and it generates the group $H_2(N_Y, \partial N_Y)$ since formula (5.1) implies that the dimension of the symmetric part of $H_2(N_Y, \partial N_Y)$ is 1 and the dimension of the anti-symmetric part is 0. We now show that for any graph Γ, the symmetric part of $H_2(N_\Gamma, \partial N_\Gamma)$ is generated by cycles analogous to y with supports given by embeddings of the Y-graph into Γ.

Theorem 5.1.3 *Let Y denote the Y-graph and let Γ be any finite, simple graph. Consider all possible combinatorial embeddings $h : Y \to \Gamma$. Each embedding induces a map on the homology,*

$$h_* : H_2(N_Y, \partial N_Y) \to H_2(N_\Gamma, \partial N_\Gamma)$$

which maps the generating class $y \in H_2(N_Y, \partial N_Y)$ to a homology class $y_h \in H_2(N_\Gamma, \partial N_\Gamma)$.

For each vertex $v \in \Gamma$ choose an edge e_v incident to v. Consider the set of all embeddings

$$h(e_v) : Y \to \Gamma$$

such that edge e_1 in Y is always mapped to edge e_v in Γ. Then the set of homology classes

$$G = \bigcup_{v \in V(\Gamma)} \{\{y_{h(e_v)}\}\}$$

freely generates the symmetric part of the group $H_2(N_\Gamma, \partial N_\Gamma)$.

Proof The set G contains

$$\sum_{v \in V(\Gamma)} \binom{\mu(v) - 1}{2} = \sum_{v \in V(\Gamma)} \frac{1}{2}(\mu(v) - 1)(\mu(v) - 2)$$

homology classes. Formula (5.1) confirms that the cardinality of G is equal to the rank of the symmetric part of $H_2(N_\Gamma, \partial N_\Gamma)$, so to prove the statement it is sufficient to show that the elements of G are linearly independent.

First consider the cycle y, (5.4). Apply Lemma 3.4.5 to write this cycle as an element of $\bigoplus_{e \in E(Y)} \tilde{H}_0(\partial B_e)$ to obtain

$$g \circ I_Y(y) = \begin{bmatrix} e_1 & e_2 & e_3 \\ v_2 - v_3 & v_3 - v_1 & v_1 - v_2 \end{bmatrix},$$

where element $v_2 - v_3$ lies in summand $\tilde{H}_0(\partial B_{e_1})$ etc.

Now consider a general graph Γ, let Φ_{e_v} denote the set of images $g \circ I_\Gamma(y_{h(e_v)}) \in \bigoplus_{e \in E(\Gamma)} \tilde{H}_0(\partial B_e)$ of the homology classes $y_{h(e_v)}$ where h runs over all embeddings mapping edge e_1 in Y to edge e_v in Γ. Label the remaining edges incident to v as

$e'_1, \cdots, e'_k$ where $k = \mu(v) - 1$. Then Φ_{e_v} contains $\frac{1}{2}(\mu(v)-1)(\mu(v)-2)$ elements of the form,

$$Y_{ij} = \pm \begin{bmatrix} e_v & e'_i & e'_j \\ v_i - v_j & v_j - v' & v' - v_i \end{bmatrix}$$

where $i, j \in \{1, \cdots, k\}$ and v' is the other boundary vertex of edge e_v. We call such elements *Y-cycles*. The parity of a Y-cycle depends on the orientation of the three edges making up the support of the cycle. There exist no relations between the Y-cycles $Y_{ij} \in \Phi_{e_v}$ since the vertex v_i appears in the summand $\tilde{H}_0(\partial B_{e_j})$ in exactly one Y-cycle, Y_{ij}, for all $i, j \in \{1, \cdots, k\}$.

Finally, note that there can be no relations between the elements of different sets Φ_{e_v}, and Φ_{e_u} because the balls of radius 1 around two vertices v and u can have at most one edge in common. $\square$

Remark The anti-symmetric part of the group $H_2(N, \partial N)$ for a general graph Γ is not simple to describe. Throughout the rest of this chapter we introduce some anti-symmetric cycles which can lie in the group $H_2(N, \partial N)$ and describe the structure of the anti-symmetric part of the group $H_2(N, \partial N)$ for certain classes of graphs.

5.2 Complete Bipartite Graphs

In this section we examine the properties of the intersection form for complete bipartite graphs. The vertex set of a complete bipartite graph, $K_{n,m}$, consists of two separated sets of vertices, a set $\mathbb{X}$ of n vertices and a set $\mathbb{Y}$ of m vertices. The edge set of the graph contains nm edges, one joining each vertex from set $\mathbb{X}$ to each vertex from set $\mathbb{Y}$. First we consider complete bipartite graphs of the form $K_{3,m}$.

Theorem 5.2.1 *Let $K_{3,m}$ be a complete bipartite graph with $m \geq 3$. Then the intersection form*

$$I_{K_{3,m}} : H_1(K_{3,m}, \mathbb{Q}) \otimes H_1(K_{3,m}, \mathbb{Q}) \to H_2(N_{K_{3,m}}, \partial N_{K_{3,m}}; \mathbb{Q})$$

is an epimorphism.

Proof To prove this theorem we will use the fact that the intersection form for the graph $K_{3,3}$ is epimorphic as discussed in the remarks following Lemma 3.3.1. We will show that $H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})$ is generated by cycles whose support lies in a subgraph of $K_{3,m}$ isomorphic to $K_{3,3}$. The intersection form $I_{K_{3,m}}$, when restricted to a subgraph isomorphic to $K_{3,3}$, is equal to the intersection form $I_{K_{3,3}}$ which we know to be epimorphic, hence any such formal sum with support in a copy of $K_{3,3}$ must be equal to the intersection of cycles in $H_1(K_{3,3}) \otimes H_1(K_{3,3}) \subset H_1(K_{3,m}) \otimes H_1(K_{3,m})$.

First we describe a set of independent generators for $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$. The graph $K_{3,3}$ consists of two separated sets of 3 vertices, $\mathbb{X}$ and $\mathbb{Y}$. Label the vertices in $\mathbb{X}$, v_1, v_2 and v_3 and those in $\mathbb{Y}$, u_1, u_2 and u_3. We orient each edge, $(v_i u_j)$, of $K_{3,3}$ from v_i to u_j and label the edge e^i_j.

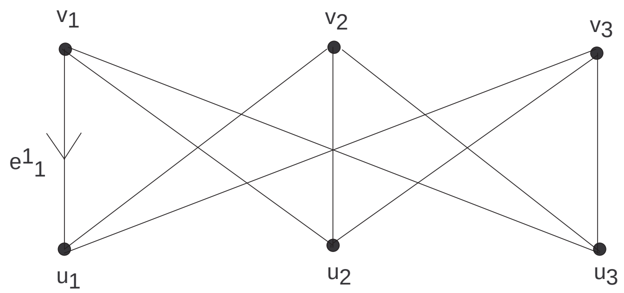


Figure 5.2: The labelled graph $K_{3,3}$.

Using Lemma 3.3.1 the rank of $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$ can be calculated to be 15. By Theorem 5.1.3 we know that the symmetric part of $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$ is generated by *Y-cycles*, such generators have the form,

$$[e^i_1 e^i_2]^+ + [e^i_3 e^i_1]^+ + [e^i_2 e^i_3]^+ \tag{5.5}$$

where $[e^i_j e^i_k]^+$ is equal to the symmetric sum $e^i_j e^i_k - e^i_k e^i_j$, $i, j, k \in \{1, 2, 3\}$ and the three edges e^i_1, e^i_2 and e^i_3 emanate from a single vertex v_i, $i = 1, 2$ or 3. Similar generators can be described at vertices u_i, $i = 1, 2, 3$. In $K_{3,3}$ there are six Y-cycles, one for each vertex of the graph, and they are all independent since each pair of edges occurs in exactly one Y-cycle.

To describe the anti-symmetric part of $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$ we construct an anti-symmetric cycle in $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$. Choose four vertices of the graph, v_i and v_j from $\mathbb{X}$, and u_p and u_q from $\mathbb{Y}$. Then we obtain the following anti-symmetric cycle,

$$S^{i,j}_{p,q} = [e^i_p e^i_q]^- + [e^j_p e^j_q]^- + [e^i_p e^j_p]^- + [e^i_q e^j_q]^- - e^i_p e^i_p - e^i_q e^i_q - e^j_p e^j_p - e^j_q e^j_q \tag{5.6}$$

where $[e^i_p e^i_q]^-$ represents the anti-symmetric sum $e^i_p e^i_q + e^i_q e^i_p$. We will call cycles of this form *S-cycles*. Counting the number of such cycles we find there are, $\binom{3}{2}\binom{3}{2} = 9$ S-cycles, which is equal to the rank of the anti-symmetric part of $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$.

To show that there are no relations between the S-cycles consider S^{ij}_{pq}, this contains the sums $[e^i_p e^i_q]^-$ and $[e^j_p e^j_q]^-$. Each of the sums $[e^i_p e^i_q]^-$ and $[e^j_p e^j_q]^-$ lie in exactly one other S-cycle, S^{ik}_{pq} and S^{kj}_{pq} respectively, so to cancel the sums $[e^i_p e^i_q]^-$ and $[e^j_p e^j_q]^-$ any relation must contain the linear combination

$$\pm a[S^{ij}_{pq} - S^{ik}_{pq} - S^{kj}_{pq}], \qquad a \in \mathbb{Q}.$$

However, this linear combination contains $\pm 2a[e^k_p e^k_q]^-$ and the sum $[e^k_p e^k_q]^-$ occurs only in the S-cycles S^{ik}_{pq} and S^{kj}_{pq} which already appear and cannot be added to the relation without re-introducing the sums $[e^j_p e^j_q]^-$ or $[e^i_p e^i_q]^-$. Hence no relation can exist between the S-cycles, and these cycles generate the anti-symmetric part of $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$. So we have found a set of fifteen independent generators for the group $H_2(N_{K_{3,3}}, \partial N_{K_{3,3}})$.

Now consider complete bipartite graphs of the form $K_{3,m}$. We label graphs of the form $K_{3,m}$ in a similar way to $K_{3,3}$. The graph has two sets of separated vertices $\mathbb{X}$ and $\mathbb{Y}$ with m vertices,$v_1, \cdots, v_m$, in $\mathbb{X}$ and 3 vertices, u_1, u_2, u_3, in $\mathbb{Y}$. Each edge e^i_j, $i = 1, \cdots, m$, $j = 1, 2, 3$ is oriented from v_i to u_j.

For a graph of the form $K_{3,m}$ we can use formulas (5.1) and (5.2) from Lemma 5.1.1 to calculate the ranks of the symmetric and anti-symmetric parts of the group $H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})$. We find,

$$\text{rank} H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})^+ = \frac{1}{2}[3(m-1)(m-2) + 2m] \tag{5.7}$$

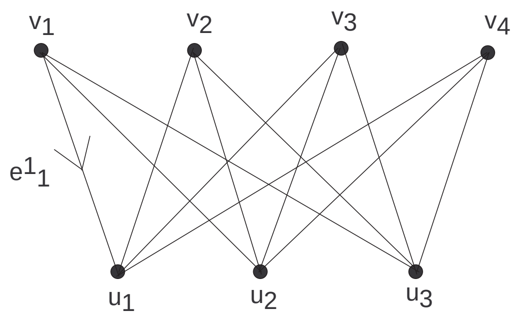


Figure 5.3: The labelled graph $K_{4,3}$.

$$= \frac{1}{2}[m^2 - 7m + 6],$$

and

$$\text{rank} H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})^- = 2m - 3 + \frac{1}{2}[3(m-1)(m-2) + 2m] \tag{5.8}$$
$$= \frac{3}{2}[m^2 - m].$$

From Theorem 5.1.3 we know that the symmetric part of the group $H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})$ is generated by

$$3\binom{m-1}{2} + m = \frac{1}{2}[m^2 - 7m + 6]$$

Y-cycles. We now show that the anti-symmetric part of the group is entirely generated by S-cycles. Each S-cycle is defined by choosing two vertices from $\mathbb{X}$ and two from $\mathbb{Y}$ so there are

$$\binom{3}{2}\binom{m}{2} = \frac{3}{2}[m^2 - m]$$

possible S-cycles. In view of (5.8) this implies that the anti-symmetric part of $H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})$ is entirely generated by S-cycles if there exist no relations between them.

Choosing two vertices v_i and v_j from $\mathbb{X}$ and u_p, u_q from $\mathbb{Y}$ consider the S-cycle S^{ij}_{pq}, this contains the sums $[e^i_p e^i_q]^-$ and $[e^j_p e^j_q]^-$ each of which lie in only one other S-cycle, S^{ik}_{pq} and S^{kj}_{pq} respectively. The independence of the S-cycles now follows using the argument which showed that the S-cycles in $K_{3,3}$ were independent.

Hence we have that the group $H_2(N_{K_{3,m}}, \partial N_{K_{3,m}})$ is generated by Y-cycles and S-cycles each of which lies in a subgraph isomorphic to $K_{3,3}$ so the intersection form

$I_{K_{3,m}}$ must be epimorphic. □

Corollary 5.2.2 *For any complete bipartite graph of the form, $K_{3,m}$, the Betti numbers of the space $F(K_{3,m}, 2)$ are,*

$$b_2(F(K_{3,m}, 2)) = m^2 - 3m + 1 \tag{5.9}$$

$$b_1(F(K_{3,m}, 2)) = 4(m - 1). \tag{5.10}$$

Proof This follows directly from Theorem 5.2.1 above and Lemma 3.3.2. Applying Lemma 3.3.2 we see that

$$\begin{aligned} b_2(F(K_{3,m}, 2)) &= 4(m-1)^2 - 2(m-1) + 1 - 3(m-1)(m-2) - 2m \qquad (5.11) \\ &= m^2 - 3m + 1 \end{aligned}$$

and

$$b_1(F(K_{3,m}, 2)) = 2 \times 2(m - 1) = 4(m - 1). \tag{5.12}$$

□

Corollary 5.2.3 *Conjecture 2.4.1 is false.*

Proof Conjecture 2.4.1 suggests that all the generators of $H_2(F(K_{3,m}, 2))$ should be induced by embeddings of K_5, $K_{3,3}$ or disjoint cycles into $K_{3,m}$. First we show that graphs of the form $K_{3,m}$ contain no disjoint cycles and no subgraphs isomorphic to K_5. Since graphs of the form $K_{3,m}$ are complete bipartite graphs all simple cycles in such a graph must contain at least 4 edges and at least two vertices from set $\mathbb{X}$ and at least two from set $\mathbb{Y}$. Since the set $\mathbb{Y}$ contains only 3 vertices this implies that every pair of simple cycles in $K_{3,m}$ must have at least one vertex in common and there can be no disjoint cycles. If $K_{3,m}$ contained a subgraph isomorphic to a subdivision of K_5 it would have to contain at least 5 vertices of valence at least 4. However $K_{3,m}$ can contain at most 3 vertices of valence greater than or equal to 4.

Then if Conjecture 2.4.1 is true, the group $H_2(F(K_{3,m}, 2))$ should be generated by surfaces of genus 4 each corresponding to a subgraph of $K_{3,m}$ isomorphic to $K_{3,3}$.

However, consider the graph $K_{3,4}$, this contains $\binom{4}{3} = 4$ subgraphs isomorphic to $K_{3,3}$ but equation (5.9) shows that

$$b_2(F(K_{3,4}, 2)) = 16 - 12 + 1 = 5.$$

Simlarly, for $m = 5$, the graph $K_{3,5}$ contains $\binom{5}{3} = 10$ subgraphs isomorphic to $K_{3,3}$ but equation (5.24) shows that $b_2(F(K_{3,4}, 2)) = 11$. Therefore for $K_{3,4}$ and $K_{3,5}$ Conjecture 2.4.1 is not true in its present form, indeed there must be at least one element in $H_2(F(K_{3,m}, 2), \mathbb{Z})$, $m = 4, 5$, which does not arise from an embedding of K_5, $K_{3,3}$ or a pair of disjoint cycles in the graph. For $m \geq 6$, however, the number of copies of $K_{3,3}$ in $K_{3,m}$ exceeds the second Betti number of $F(K_{3,m}, 2)$, so for these graphs it is possible that the second homology group is somehow generated by embeddings of $K_{3,3}$ in the graph. $\square$

Example 5.2.1 In Section 3.5 we described an algorithm for calculating the second Betti number of $F(\Gamma, 2)$ for any graph Γ using the results of Chapter 3. We can apply this algorithm to graphs $K_{3,m}$ to calculate a free generating set in $H_1(K_{3,m}) \otimes H_1(K_{3,m})$ for the kernel of the intersection form.

First we must take the first barycentric subdivision of $K_{3,m}$ since graphs of this form contain cycles of length 4. Label the graph according the labelling scheme in Lemma 5.2.1, i.e. with one set of m separated vertices labelled $v_1, \cdots, v_m$ and the other set of three vertices labelled u_1, u_2, and u_3. We label w_j^i, the vertex which bisects edge e_j^i and then label the oriented edge from v_i to w_j^i as E_j^i and the oriented edge from w_j^i to u_j by e_j^i. Choose a spanning tree, T, for the graph, which includes all edges emanating from vertices v_1 and u_1 along with every edge E_j^i with $i = 2, \cdots, m$ and $j = 2, 3$, see Figure 5.4. Then the graph $K_{3,m}$ contains $2(m - 1)$ edges which are not included in T, these edges define $2(m - 1)$ cycles which generate $H_1(K_{3,m})$, we denote the cycle containing edge e_j^i by C_j^i.

We must consider $4(m - 1)^2$ cycle intersections. Using Lemma 3.4.5 we can write these intersections as elements in the image of the map $g \circ I_{K_{3,m}}$ in the direct sum $\bigoplus_{e \in E(K_{3,m})} \tilde{H}_0(\partial B_e)$. Then to find the generators of the kernel of the intersection form we must find linear combinations of these intersections which sum to zero in the group $\bigoplus_{e\in E(K_{3,m})} \tilde{H}_0(\partial B_e)$, i.e. we must find relations between these intersections. Then the linear combinations of cycles in $H_1(K_{3,m}) \otimes H_1(K_{3,m})$ corresponding to these relations lie in the kernel of the intersection form. Here g is the injective map defined in Lemma 3.4.2,

$$g : H_2(N_\Gamma, \partial N_\Gamma) \to \bigoplus_{e\in E(\Gamma)} \tilde{H}_0(\partial B_e).$$

For ease of notation we will refer to the intersection, $g \circ I_{K_{3,m}}(C \otimes C')$, of two cycles C and C' simply as $C \otimes C'$. Consider cycle intersections of the form $C^i_j \otimes C^p_q$ where $i = p$ or $j = q$. We have,

$$C^i_j \otimes C^i_q = \pm \begin{bmatrix} E^1_j & E^1_1 & e^1_1 & e^i_1 & E^i_1 & E^i_j \\ w^1_1 - w^1_q & w^1_q - u_1 & v_1 - w^i_1 & v_2 - w^1_1 & w^i_q - u_1 & w^i_1 - w^i_q \end{bmatrix},$$

and

$$C^i_j \otimes C^p_j = \pm \begin{bmatrix} e^i_j & e^1_j & E^1_j & E^1_1 & e^1_1 & E^i_1 \\ w^1_p - w^p_j & w^p_j - v_1 & u_j - w^1_1 & u_1 - w^1_j & w^p_1 - v_1 & w^1_1 - w^p_1 \end{bmatrix}.$$

Since the vertex w^p_q, $p = 2, \cdots, m$, $q = 2, 3$ lies in exactly one generating cycle, C^p_q, and the edges e^i_j and E^i_j lie only in the cycle C^i_j we see that the vertex w^p_q appears in the summand $\tilde{H}_0(\partial B_{e^i_j})$ in exactly one cycle intersection, $C^i_j \otimes C^p_q$, where $i = p$ or $q = j$. Hence such intersections can not appear in any relations.

Consider intersections of the form $C^i_j \otimes C^p_q$ where $i \neq p$ and $j \neq q$. These intersections

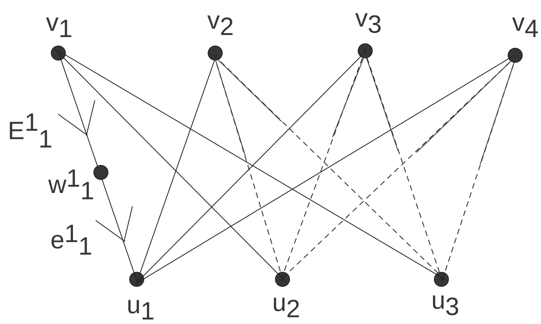


Figure 5.4: The subdivided graph $K_{4,3}$. The spanning tree T is shown in solid lines.

take two possible forms,

$$C_2^i \otimes C_3^p = \pm \begin{bmatrix} E_2^1 & E_1^1 & e_1^1 & e_1^i \\ w_3^1 - w_1^1 & u_1 - w_2^1 & w_1^p - v_1 & w_1^1 - w_1^p \end{bmatrix},$$

and

$$C_3^i \otimes C_2^p = \pm \begin{bmatrix} E_3^1 & E_1^1 & e_1^1 & e_1^i \\ w_3^1 - w_1^1 & u_1 - w_2^1 & w_1^p - v_1 & w_1^1 - w_1^p \end{bmatrix},$$

with $i, p = 2, \cdots, m$. Note that these are the only two intersections which contain the vertex $\pm w_1^i$ in the summand $\tilde{H}_0(\partial B_{e_1^i})$, so every linear combination which lies in the kernel of $I_{K_{3,m}}$ must be made up of pairs of the form $\pm[C_2^i \otimes C_3^p - C_3^i \otimes C_2^p]$. Any two of these pairs form a relation, for example,

$$g \circ I_{K_{3,m}} \left(\pm[C_2^i \otimes C_3^p - C_3^i \otimes C_2^p - C_2^j \otimes C_3^q + C_3^j \otimes C_2^q] \right)$$

$$= \begin{bmatrix} & E_2^1 & E_3^1 & E_1^1 & e_1^1 & e_1^i & e_1^j \\ + & w_3^1 - w_1^1 & 0 & u_1 - w_3^1 & w_1^p - v_1 & w_1^1 - w_1^p & 0 \\ + & 0 & w_1^1 - w_2^1 & w_2^1 - u_1 & v_1 - w_1^p & w_1^p - w_1^1 & 0 \\ + & w_1^1 - w_3^1 & 0 & w_3^1 - u_1 & v_1 - w_1^p & 0 & w_1^q - w_1^1 \\ + & 0 & w_2^1 - w_1^1 & u_1 - w_2^1 & w_1^q - v_1 & 0 & w_1^1 - w_1^q \end{bmatrix}.$$

Each column in this matrix corresponds to a group $\tilde{H}_0(\partial B_e)$, note that the sum of the entries in each column is zero, therefore the image of this combination is trivial in $\bigoplus_{e \in E(K_{3,m})} \tilde{H}_0(\partial B_e)$.

Fix an ordered pair of indices $i, p \in 2, \cdots, m$, and consider all linear combinations of the form

$$\pm[C_2^i \otimes C_3^p - C_3^i \otimes C_2^p - C_2^j \otimes C_3^q + C_3^j \otimes C_2^q]$$

where the index pairs (j, q) run over all ordered pairs chosen from $\{2, \cdots, m\}$ such that $(j, q) \neq (i, p)$, there are $2 \times \binom{m-1}{2} - 1$ such pairs. Denote this set of linear combination by G. Each intersection $C_2^j \otimes C_3^q$ and $C_3^j \otimes C_2^q$ appears in exactly one element of G, therefore the elements of G are linearly independent. Corrolary 5.2.2 implies that the second Betti number of $F(K_{3,m}, 2)$ is equal to

$$m^2 - 3m + 1 = 2 \times \binom{m-1}{2} - 1.$$

Therefore the set of linear combinations, G, of intersections freely generates the kernel of the intersection form $I_{K_{3,m}}$.

Consider an element of G,

$$\pm[C_2^i \otimes C_3^p - C_3^i \otimes C_2^p - C_2^j \otimes C_3^q + C_3^j \otimes C_2^q]$$

with (j, q) distinct from (i, p). Then the support of this linear combination, i.e. the union of all edges contained in the cycles forming the combination, contains five vertices, v_1, v_i, v_p, v_j and v_q as well as the three vertices u_1, u_2 and u_3. In fact the support is equal to the union of edges in $K_{3,m}$ isomorphic to a subgraph of the form $K_{5,3}$ containing the five vertices described. Similarly, if we consider an element of G with three distinct indices, with $j = i$ say, then the support of such a linear combination is isomorphic to a subgraph of the form $K_{3,4}$ containing the vertices v_1, v_i, v_p, v_q as well as the vertices u_1, u_2 and u_3.

Consider such a subgraph isomorphic to $K_{3,4}$. In Corollary 5.2.3 above it was shown that the second Betti number of this graph is five, yet it contains only four different subgraphs isomorphic to $K_{3,3}$. Each of these subgraphs induces an orientable surface of genus 4 embeded in the discrete configuration space $D(K_{3,4}, 2)$, and therefore corresponds to a homology class in the group $H_2(F(K_{3,4}, 2), \mathbb{Z})$. Such classes are represented by linear combinations of pairs of cycles in $H_1(K_{3,4}) \otimes H_1(K_{3,4})$ which lie in the kernel of the intersection map $I_{K_{3,4}}$. Following the discussion above we see that the kernel of $I_{K_{3,4}}$ can be generated by the linear combinations,

$$\begin{array}{ccccccc}
C_2^2 \otimes C_3^3 & - & C_3^2 \otimes C_2^3 & + & C_2^3 \otimes C_3^2 & - & C_3^3 \otimes C_2^2, \\
C_2^2 \otimes C_3^3 & - & C_3^2 \otimes C_2^3 & - & C_2^2 \otimes C_3^4 & + & C_3^2 \otimes C_2^4, \\
C_2^2 \otimes C_3^3 & - & C_3^2 \otimes C_2^3 & - & C_2^3 \otimes C_3^4 & + & C_3^3 \otimes C_2^4, \\
C_2^2 \otimes C_3^3 & - & C_3^2 \otimes C_2^3 & + & C_2^4 \otimes C_3^2 & - & C_3^4 \otimes C_2^2, \\
C_2^2 \otimes C_3^3 & - & C_3^2 \otimes C_2^3 & + & C_2^4 \otimes C_3^3 & - & C_3^4 \otimes C_2^3.
\end{array} \tag{5.13}$$

Four of these combinations have the full graph $K_{3,4}$ as their support. We ask if these combinations can be written as the sum of combinations representing the fundamental classes of the four surfaces of genus 4 embeded in $D(K_{3,4}, 2)$ as classes in

$H_2(F(K_{3,4},2),\mathbb{Z})$. Three of these classes are represented by the linear combinations,

$$\begin{array}{ccccccc} C_2^2 \otimes C_3^3 & - & C_3^2 \otimes C_2^3 & + & C_2^3 \otimes C_3^2 & - & C_3^3 \otimes C_2^2, \\ C_3^2 \otimes C_2^4 & - & C_2^2 \otimes C_3^4 & - & C_3^4 \otimes C_2^2 & + & C_2^4 \otimes C_3^2, \\ C_3^3 \otimes C_2^4 & - & C_2^3 \otimes C_3^4 & + & C_3^4 \otimes C_2^3 & - & C_2^4 \otimes C_3^3. \end{array} \tag{5.14}$$

These combinations represent the the fundamental classes of the surfaces corresponding to the subgraphs Γ_{123}, Γ_{124} and Γ_{134}, where Γ_{ijk} is the subgraph of $K_{3,4}$ isomorphic to $K_{3,3}$ containing the vertices v_i, v_j and v_k. The fundamental class of the final surface, corresponding the subgraph Γ_{234}, is represented by the sum of the three combinations (5.14).

We see that every pair $\pm[C_3^i \otimes C_2^j - C_2^i \otimes C_3^j]$ lies in exactly one of the combinations (5.14) corresponding to the subgraphs Γ_{123}, Γ_{124} and Γ_{134}, hence no sum of these combinations can be equal to any generator (5.13) of the kernel of $I_{K_{3,4}}$.

The space $D(K_{3,4},2)$ is not an orientable surface. This follows from a theorem by Abrams in [1] but can also be seen directly. If $D(K_{3,4},2)$ was a surface, then every one-cell would have to lie in the boundary of exactly two two-cells. Consider the one-cell $e_j^i u_k$, then this lies in the boundary of any two-cell $e_j^i e_k^q$ with $q \neq i$, there are 3 such edges e_k^q so this one-cell lies in the boundary of three two-cells. Hence we have a cycle in $H_2(F(K_{3,m},2),\mathbb{Z})$ which does not arise as a cycle in the homology of a surface embeded in the space $D(K_{3,4},2)$. However Corollary 2.5.2 shows that the group $H_2(F(K_{3,m},2),\mathbb{Z})$ is isomorphic to the 2-dimensional oriented bordism group of $F(K_{3,m},2)$, this implies that these generators must correspond to surfaces mapped into $F(K_{3,4},2)$ but the maps must be more complicated than simple embeddings.

In the following theorem we extend the techniques used in Example 5.2.1 to find the generators of $H_2(F(K_{n,m},2))$, for all complete bipartite graphs $K_{n,m}$.

Theorem 5.2.4 *Let $K_{n,m}$ be a complete bipartite graph, with $n \geq 3$ and $m \geq 3$. Then the intersection form*

$$I_{K_{n,m}} : H_1(K_{n,m},\mathbb{Z}) \otimes H_1(K_{n,m},\mathbb{Z}) \to H_2(N_{K_{n,m}}, \partial N_{K_{n,m}};\mathbb{Z})$$

is an epimorphism.

Proof To prove this theorem we will use the algorithm described in Section 3.5 to calculate the rank of the kernel of the intersection form $I_{K_{n,m}}$ and hence the second Betti number of $H_2(F(K_{n,m}, 2), \mathbb{Z})$.

To do this, take the first barycentric subdivision of the graph. We label the subdivided graph as follows; label the n vertices in the separated set, $\mathbb{X}$ as $v_1, \cdots, v_n$, and label the vertices in set $\mathbb{Y}$ as $u_1, \cdots, u_m$. Label the vertex which subdivides the edge joining vertex v_i to u_j as w_j^i, then label the oriented edge from vertex v_i to w_j^i as E_j^i and label the oriented edge from w_j^i to u_j as e_j^i. Next, choose a spanning tree, T for the graph. Include in T all edges emanating from the vertices u_1 and v_1 as well as all edges E_j^i with $i = 2, \cdots, n$ and $j = 2, \cdots, m$, see Figure 5.5.

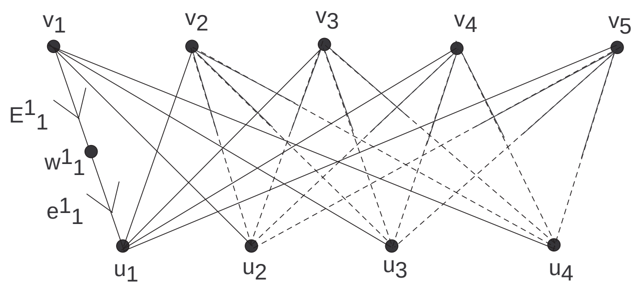


Figure 5.5: The subdivided graph $K_{5,4}$. The spanning tree T is shown in solid lines.

There are $(n-1)(m-1)$ edges of the form e_j^i with $i = 2, \cdots, n$ and $j = 2, \cdots, m$ which do not lie in the tree T. Each such edge defines a generating cycle of $H_1(K_{n,m})$ which will be denoted C_j^i. Therefore there are $(n-1)^2(m-1)^2$ cycle intersections and to calculate the second Betti number of $F(K_{n,m}, 2)$ we must first find all possible relations between these intersections in the group $\bigoplus_{e \in E(K_{n,m})} \tilde{H}_0(\partial B_e)$.

As in Example 5.2.1 we will refer to the intersection $g \circ I_{K_{n,m}}(C \otimes C')$ as $C \otimes C'$ to shorten the notation. Consider an intersection of the form $C_j^i \otimes C_q^p$ where $i = p$ or $j = q$. We have,

$$C_j^i \otimes C_q^i = \pm \begin{bmatrix} E_j^1 & E_1^1 & e_1^1 & e_1^i & E_1^i & E_j^i \\ w_1^1 - w_q^1 & w_q^1 - u_1 & v_1 - w_1^i & v_2 - w_1^1 & w_q^i - u_1 & w_1^i - w_q^i \end{bmatrix},$$

and

$$C^i_j \otimes C^p_j = \pm \left[\begin{array}{cccccc} e^i_j & e^1_j & E^1_j & E^1_1 & e^1_1 & E^i_1 \\ w^1_p - w^p_j & w^p_j - v_1 & u_j - w^1_1 & u_1 - w^1_j & w^p_1 - v_1 & w^1_1 - w^p_1 \end{array} \right].$$

Since the vertex w^p_q, $p = 2, \cdots, n$, $q = 2, \cdots, m$ lies in exactly one generating cycle, C^p_q, and the edges e^i_j and E^i_j lie only in the cycle C^i_j we see that the vertex w^p_q appears in the summand $\tilde{H}_0(\partial B_{e^i_j})$ in exactly one cycle intersection, $C^i_j \otimes C^p_q$, where $i = p$ or $q = j$. Hence such intersections cannot appear in any relations.

Similarly consider the intersection of a cycle C^i_j with itself. We have

$$C^i_j \otimes C^i_j =$$

$$\pm \left[\begin{array}{cccccccc} e^i_j & e^1_j & E^1_j & E^1_1 & e^1_1 & e^i_1 & E^i_1 & E^i_j \\ w^1_j - v_i & w^i_j - v_1 & u_j - w^1_1 & u_1 - w^1_j & w^i_1 - v_1 & w^1_1 - v_i & u_1 - w^i_j & u_3 - w^i_1 \end{array} \right].$$

The vertex v_i appears in the summand $\tilde{H}_0(\partial B_{e^i_j})$ only in this cycle intersection, so an intersection of a cycle with itself cannot appear in any relation.

This implies that the only intersections which can appear in a relation are of the form $C^i_j \otimes C^p_q$ with $i \neq p$ and $j \neq q$. We have,

$$C^i_j \otimes C^p_q = \pm \left[\begin{array}{cccc} E^1_j & E^1_1 & e^1_1 & e^i_1 \\ w^1_q - w^1_1 & u_1 - w^1_q & w^p_1 - v_1 & w^1_1 - w^p_1 \end{array} \right].$$

The choice of plus or minus sign arises since $C^i_j \otimes C^p_q = -C^p_q \otimes C^i_j$, see the proof of Lemma 3.2.13. We will assume without loss of generality that every intersection appears with positive sign.

We now show the kernel of the intersection form $I_{K_{n,m}}$ is generated by linear combinations of the form

$$+C^i_j \otimes C^p_q - C^i_x \otimes C^p_y - C^r_j \otimes C^s_q + C^r_x \otimes C^s_y \tag{5.15}$$

with $i \neq p$, $j \neq q$, $r \neq s$ and $x \neq y$. Note that the vertex w^1_q appears in the summand $\tilde{H}_0(\partial B_{E^1_j})$ only in the intersection of a cycle containing vertex u_j with a cycle containing vertex u_q, i.e. an intersection of the form $C^{i_1}_j \otimes C^{p_1}_q$ with $i_1 \neq p_1 \in \{1, \cdots, m\}$. Similarly, vertex w^p_1 appears in summand $\tilde{H}_0(\partial B_{e^i_1})$ only in the intersections of a cycle containing vertex v_i with a cycle containing v_p, i.e. an intersection of the form $C^i_{j_1} \otimes C^p_{q_1}$ with $j_1 \neq q_1 \in \{1, \cdots, n\}$. So any relation containing intersection $C^i_j \otimes C^p_q$ must contain a linear combination of the form

$$a[C^i_j \otimes C^p_q - C^i_{j_1} \otimes C^p_{q_1} - C^{i_1}_j \otimes C^{p_1}_q] \quad a \in \mathbb{Z} \tag{5.16}$$

with $j_1 \neq q_1$ and $i_1 \neq p_1$. Note that the intersections $C^i_j \otimes C^p_q$ and $C^i_{j_1} \otimes C^p_{q_1}$ also have the same pair of vertices, $u_1 - w^1_q$, in summand $\tilde{H}_0(\partial B_{E^1_1})$ and intersections $C^i_j \otimes C^p_q$ and $C^{i_1}_j \otimes C^{p_1}_q$ have the same pair of vertices in summand $\tilde{H}_0(\partial B_{e^1_1})$. The image under the map $g \circ I_{K_{n,m}}$ of the linear combination (5.16), when taken as an element of $H_1(K_{n,m}) \otimes H_1(K_{n,m})$, is equal to

$$\begin{bmatrix} E^1_{j_1} & E^1_1 & e^1_1 & e^{i_1}_1 \\ w^1_{q_1} - w^1_1 & u_1 - w^1_{q_1} & w^{p_1}_1 - v_1 & w^1_1 - w^{p_1}_1 \end{bmatrix}$$

which is equal to the intersection $C^{i_1}_{j_1} \otimes C^{p_1}_{q_1}$.

So we see that relations between these intersections must take the general form,

$$C^{i_0}_{j_0} \otimes C^{p_0}_{q_0} + \sum_{k=1}^{l} (-1)^k [C^{i_{k-1}}_{j_k} \otimes C^{p_{k-1}}_{q_k} + C^{i_k}_{j_{k-1}} \otimes C^{p_k}_{q_{k-1}}] + (-1)^{l+1} C^{i_l}_{j_l} \otimes C^{p_l}_{q_l} \tag{5.17}$$

with $i_k \neq p_k$ and $j_k \neq q_k$ for $k = 1, \cdots l$. Adding the sum

$$\sum_{k=1}^{l-1} C^{i_k}_{j_k} \otimes C^{p_k}_{q_k} - C^{i_k}_{j_k} \otimes C^{p_k}_{q_k} = 0 \tag{5.18}$$

to the relation transforms it into the sum

$$\sum_{k=1}^{l} C^{i_{k-1}}_{j_{k-1}} \otimes C^{p_{k-1}}_{q_{k-1}} - C^{i_{k-1}}_{j_k} \otimes C^{p_{k-1}}_{q_k} - C^{i_k}_{j_{k-1}} \otimes C^{p_k}_{q_{k-1}} + C^{i_k}_{j_k} \otimes C^{p_k}_{q_k}. \tag{5.19}$$

Each summand is a linear combination of form (5.15), as required.

We now construct a collection, G, of relations which are linearly independent and generate all relations of form (5.15). Choose an ordered pair of indices, (i, p) from the set $\{2, \cdots, n\}$ and another ordered pair of indices (j, q) from the set $\{2, \cdots, m\}$, then let G contain all relations of the form,

$$\pm[C^i_j \otimes C^p_q - C^i_x \otimes C^p_y - C^r_j \otimes C^s_q + C^r_x \otimes C^s_y] \tag{5.20}$$

where indices (r,s) range over all ordered pairs from $\{2,\cdots,n\}$ with $(r,s)\neq(i,p)$ and indices (x,y) range over all ordered pairs from $\{2,\cdots,m\}$ with $(x,y)\neq(j,q)$. Then for each choice of ordered pairs (r,s) from $\{2,\cdots,n\}$ and (x,y) from $\{2,\cdots,m\}$ the intersection $C_x^r\otimes C_y^s$ appears in exactly one relation in G, therefore the relations in G are linearly independent.

Finally consider a general relation of form (5.15),

$$\pm[C_{b_1}^{a_1}\otimes C_{b_2}^{a_2}-C_{b_3}^{a_1}\otimes C_{b_4}^{a_2}-C_{b_1}^{a_3}\otimes C_{b_2}^{a_4}+C_{b_3}^{a_3}\otimes C_{b_4}^{a_4}] \tag{5.21}$$

where $a_1,\cdots,a_4\in\{2,\cdots,n\}$ and $b_1,\cdots,b_4\in\{2,\cdots,m\}$. This relation is equal to the following sum of elements of G

$$\begin{array}{cl} + & [C_j^i\otimes C_q^p-C_{b_1}^i\otimes C_{b_2}^p-C_j^{a_1}\otimes C_q^{a_2}+C_{b_1}^{a_1}\otimes C_{b_2}^{a_2}] \\ - & [C_j^i\otimes C_q^p-C_{b_3}^i\otimes C_{b_4}^p-C_j^{a_1}\otimes C_q^{a_2}+C_{b_3}^{a_1}\otimes C_{b_4}^{a_2}] \\ - & [C_j^i\otimes C_q^p-C_{b_1}^i\otimes C_{b_2}^p-C_j^{a_3}\otimes C_q^{a_4}+C_{b_1}^{a_3}\otimes C_{b_2}^{a_4}] \\ + & [C_j^i\otimes C_q^p-C_{b_3}^i\otimes C_{b_4}^p-C_j^{a_3}\otimes C_q^{a_4}+C_{b_3}^{a_3}\otimes C_{b_4}^{a_4}]. \end{array}$$

Hence the relations in G generate all relations between the cycle intersections, i.e. the elements of $H_1(K_{n,m})\otimes H_1(K_{n,m})$ in G generate the kernel of the intersection form $I_{K_{n,m}}$.

The set G contains

$$({}^{(m-1)}P_2-1)({}^{(n-1)}P_2-1)=((n-1)(n-2)-1)((m-1)(m-2)-1)$$

relations and so the second Betti number of $F(K_{n,m},2)$ is equal to

$$((n-1)(n-2)-1)((m-1)(m-2)-1).$$

Finally, we show that this implies that the intersection form $I_{K_{n,m}}$ is an epimorphism. Exact sequence (3.13) implies that $I_{K_{n,m}}$ is epimorphic if and only if the second Betti number of $F(K_{n,m},2)$ is equal to the difference of the rank of $H_1(K_{n,m})\otimes H_1(K_{n,m})$ and the rank of $H_2(N_{K_{n,m}},\partial N_{K_{n,m}})$. We have,

$$\mathrm{rank}H_1(K_{n,m})\otimes H_1(K_{n,m})-\mathrm{rank}H_2(N_{K_{n,m}},\partial N_{K_{n,m}}) \tag{5.22}$$

$$=(n-1)^2(m-1)^2-(n-1)(m-1)+1-n(m-1)(m-2)-m(n-1)(n-2)$$

using Lemma 3.15 to calculate the rank of $H_2(N_{K_{n,m}}, \partial N_{K_{n,m}})$. Multiplying out the expression above we obtain

$$n^2m^2 - 3n^2m - 3nm^2 + 9nm + n^2 + m^2 - 3n - 3m + 1 \tag{5.23}$$

$$= ((n-1)(n-2)-1)((m-1)(m-2)-1)$$

as required. □

Corollary 5.2.5 *For any complete bipartite graph, $K_{n,m}$, the Betti numbers of the space $F(K_{n,m}, 2)$ are,*

$$b_2(F(K_{n,m}, 2)) = ((n-1)(n-2)-1)((m-1)(m-2)-1) \tag{5.24}$$

$$b_1(F(K_{3,m}, 2)) = 2(n-1)(m-1). \tag{5.25}$$

Proof This follows directly from Theorem 5.2.4 and Lemma 3.3.2. □

5.3 Complete Graphs

In this section we apply similar techniques to those applied in the previous section in the case of complete bipartite graphs to examine the properties of the intersection form for complete graphs. Let K_n denote the complete graph on n vertices, that is the graph with n vertices and $\binom{n}{2}$ edges, one edge joining each vertex to every other vertex of the graph.

Theorem 5.3.1 *Let K_n be the complete graph with n vertices, $n \geq 5$. Then the intersection form*

$$I_{K_n} : H_1(K_n, \mathbb{Q}) \otimes H_1(K_n, \mathbb{Q}) \to H_2(N_{K_n}, \partial N_{K_n}; \mathbb{Q})$$

is an epimorphism.

Proof The proof of this theorem uses a similar method to that employed to prove Theorem 5.2.1. We base the proof on the fact that the intersection form for K_5, I_{K_5}, is epimorphic, as was shown in the remarks following Lemma 3.3.1. We will show that the group $H_2(N_{K_n}, \partial N_{K_n})$ is generated by cycles with supports lying in subgraphs of K_n isomorphic to K_5. Since the intersection form I_{K_n} is equal to the intersection form I_{K_5} when restricted to a subgroup $H_1(K_5) \otimes H_1(K_5) \subseteq H_1(K_n) \otimes H_1(K_n)$, any cycle in the group $H_2(N_{K_n}, \partial N_{K_n})$, with support in a subgraph isomorphic to K_5 is equal to the intersection of cycles in $H_1(K_5) \otimes H_1(K_5) \subseteq H_1(K_n) \otimes H_1(K_n)$.

We consider the symmetric and anti-symmetric parts of $H_2(N_{K_n}, \partial N_{K_n})$ seperately. Theorem 5.1.3 shows that the symmetric part of the group $H_2(N_{K_n}, \partial N_{K_n})$ is generated by Y-cycles. The support of a Y-cycle is equal to the union of three edges emanating from a single vertex, i.e. a subgraph isomorphic to the Y-graph. All such Y-graphs embeded in K_n clearly lie in subgraphs of K_n isomorphic to K_5. Hence it remains to show that the anti-symmetric part of the group $H_2(N_{K_n}, \partial N_{K_n})$ is also generated by cycles whose supports lie in subgraphs of K_n isomorphic to K_5.

First we label the graph K_n. Label the n vertices of K_n as $v_1, \cdots, v_n$. Orient the edge of K_n joining vertex v_i to v_j from v_i to v_j where $i < j$ and label this edge as e^i_j, see Figure 5.6.

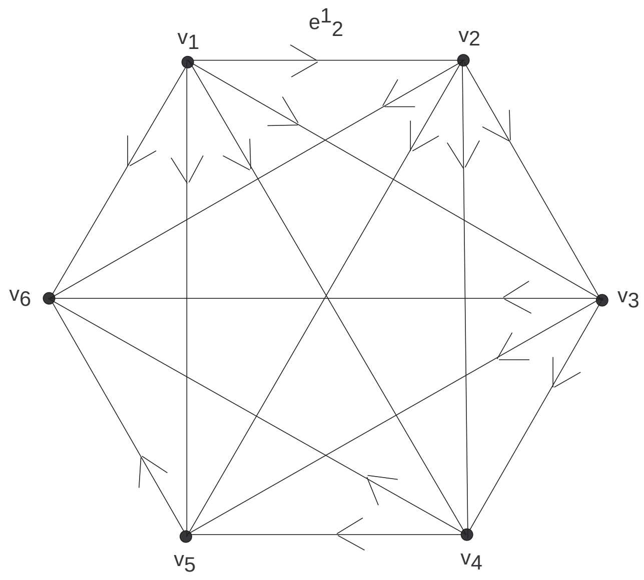


Figure 5.6: The labelled and oriented graph K_6.

We will show that the anti-symmetric part of $H_2(N_{K_n}, \partial N_{K_n})$ is generated by two types of anti-symmetric cycles. First consider a triangle in the graph K_n, i.e. the union of the edges of a cycle of length three in the graph. Then the product of the triangle with itself is an anti-symmetric cycle in the group $H_2(N_{K_n}, \partial N_{K_n})$ which we will call a T-cycle. The support of a T-cycle is a triangle in K_n, which clearly lies in a subgraph of K_n isomorphic to K_5. Consider the triangle in K_n containing vertices v_i, v_j and v_k where $i < j < k$, see Figure 5.7, then the T-cycle corresponding to this triangle is given by

$$T_{ijk} = [e^i_j e^j_k]^- - [e^i_j e^i_k]^- + [e^i_k e^j_k]^- - e^i_j e^i_j - e^i_k e^i_k - e^j_k e^j_k$$

where, as in Theorem 5.2.1,

$$[e^i_j e^j_k]^- = e^i_j e^j_k + e^j_k e^i_j.$$

There are $\binom{n}{3} = \frac{1}{6}n(n-1)(n-2)$ T-cycles in the group $H_2(N_{K_n}, \partial N_{K_n})$ and no relations exist between them since each pair of edges which appears in the cycle T_{ijk} has boundary involving all three vertices v_i, v_j and v_k and so can appear only in the T-cycle T_{ijk}.

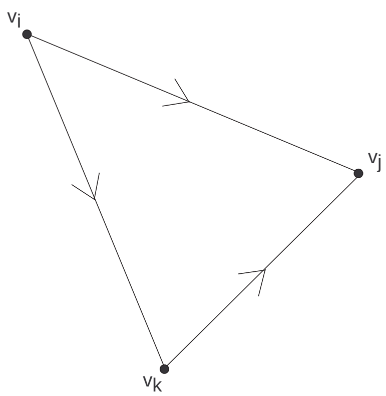


Figure 5.7: The triangle $v_1 v_j v_k$.

Now consider a graph equal to the union of four edges emanating from a central vertex, call this graph the X-graph. We describe an anti-symmetric cycle with support equal to the X-graph which we shall call an X-cycle, see Figure 5.8. Fix a vertex in the graph K_n, say v_1, to be the central vertex and choose two pairs of edges incident to v_1, (e_i^1, e_j^1) and (e_p^1, e_q^1). Construct the following cycle in $H_2(N_{K_n}, \partial N_{K_n})$,

$$X = e_i^1 e_p^1 - e_i^1 e_q^1 - e_j^1 e_p^2 - e_j^1 e_q^1.$$

Then to obtain an anti-symmetric cycle we take $X - \tau(X)$, where τ is the involution map (2.3), to obtain the X-cycle,

$${}_1X_{pq}^{ij} = [e_i^1 e_p^1]^- - [e_i^1 e_q^1]^- - [e_p^1 e_j^1]^- + [e_q^1 e_j^1]^-.$$

Chosing a vertex other than v_1 means the coefficients of the terms in the X-cycle may have different parity depending on the orientation of the edges of the graph. Each X-graph embeded in K_n is the support of three different X-cycles, it is clear that there are no relations between X-cycles whose supports have different central vertices, however there are relations between X-cycles with supports having the same central vertex.

Consider the graph, Γ, consisting of a central vertex, v_0 with m edges emanating from it. We label the edges of Γ as $e_1, \cdots, e_m$ and orient edge e_i from v_0 to v_i where v_i is the other boundary vertex of e_i, see Figure 5.9. We will show that all anti-symmetric cycles in the group $H_2(N_\Gamma, \partial N_\Gamma)$ are equal to linear combinations of X-cycles. Recall that a cycle in $H_2(N_\Gamma, \partial N_\Gamma)$ is given by a chain of two-cells in N_Γ, i.e. a formal sum of pairs of edges ee' such that $e \cap e' \neq \emptyset$, with boundary given by a chain of one-cells in ∂N_Γ. The space ∂N_Γ is made up of all pairs ve and ev of

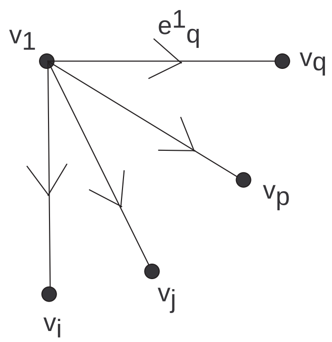


Figure 5.8: An X-graph with central vertex v_1.

closed edges and vertices such that $d(v, e) = 1$ or 2, where d represents the distance function (3.2.5) defined in Chapter 3.

First, we show that no cycle in the group $H_2(N_\Gamma, \partial N_\Gamma)$ can contain a summand of the form $e_i e_i$ with non-zero coefficient. The boundary of such a pair of edges is given by,

$$\partial(e_i e_i) = e_i(v_i - v_0) - (v_i - v_0)e_i.$$

This boundary is disjoint from ∂N_Γ since $d(e_i, v_0) = d(e_i, v_i) = 0$ for all edges $e_i \in \Gamma$. There exists no other edge in Γ incident to v_i, hence the boundary elements $e_i v_i$ and $-v_i e_i$ cannot be cancelled by adding any other pair of edges to $e_i e_i$ besides $-e_i e_i$ so such a product of an edge with itself cannot have non-zero coefficient in a cycle in $H_2(N_\Gamma, \partial N_\Gamma)$.

We now construct a general anti-symmetric cycle in the group $H_2(N_\Gamma, \partial N_\Gamma)$. Consider the sum $[e_{i_0} e_{j_0}]^-$ with $i_0 \neq j_0 \in \{1, \cdots, m\}$. The boundary of this sum contains the one-cells $-e_{i_0} v_0$ and $v_0 e_{j_0}$, these one-cells cannot lie in the boundary of a relative cycle in $H_2(N_\Gamma, \partial N_\Gamma)$ since $d(e_i, v_0) = 0$ for all edges of Γ. In order to cancel these boundary elements, any anti-symmetric cycle in $H_2(N_\Gamma, \partial N_\Gamma)$ must contain linear combinations of the form

$$\alpha([e_{i_0} e_{j_0}]^- - [e_{i_0} e_{j_1}]^- - [e_{i_1} e_{j_0}]^-).$$

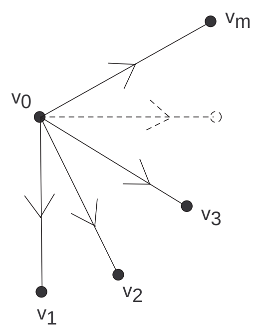


Figure 5.9: The graph Γ with m edges incident to a central vertex v_0.

Assuming $i_1 \neq j_1$, adding the sum $\alpha[e_{i_1}e_{j_1}]^-$ to this linear combination produces an X-cycle. We see that in general anti-symmetric cycles take the form,

$$\alpha[e_{i_0}e_{j_0}]^- + (-1)^{l+1}\alpha[e_{i_l}e_{j_l}]^- + \alpha\sum_{k=1}^{l}(-1)^k[e_{i_{k-1}}e_{j_k}]^- + [e_{i_k}e_{j_{k-1}}]^-. \tag{5.1}$$

Here we assume that $i_0 \neq j_0$ and $i_l \neq j_l$. If we assume that $i_k \neq j_k$ for all $k = 0, \cdots, l$ we may add to this expression the sum

$$\alpha\sum_{k=1}^{l-1}[e_{i_k}e_{j_k}]^- - [e_{i_k}e_{j_k}]^- = 0 \tag{5.2}$$

to obtain the the following sum of X-cycles,

$$\alpha\sum_{k=1}^{l}(-1)^k([e_{i_{k-1}}e_{j_{k-1}}]^- - [e_{i_{k-1}}e_{j_k}]^- - [e_{i_k}e_{j_{k-1}}]^- + [e_{i_k}e_{j_k}]^-). \tag{5.3}$$

Now suppose that $i_k = j_k$ for some $k \in \{1, \cdots, l-1\}$, no cycle in the group $H_2(N_\Gamma, \partial N_\Gamma)$ can contain the sum $\pm[e_{i_k}e_{i_k}]^-$, however adding $+[e_{i_{k+1}}e_{j_{k-1}}]^- - [e_{i_{k+1}}e_{j_{k-1}}]^-$ to the sum (5.1) in place of $\pm[e_{i_k}e_{i_k}]^i$ still results in a sum of X-cycles. The two X-cycles,

$$[e_{i_{k-1}}e_{j_{k-1}}]^- - [e_{i_{k-1}}e_{j_k}]^- - [e_{i_k}e_{j_{k-1}}]^- + [e_{i_k}e_{j_k}]^-$$

$$[e_{i_k}e_{j_k}]^- - [e_{i_k}e_{j_{k+1}}]^- - [e_{i_{k+1}}e_{j_k}]^- + [e_{i_{k+1}}e_{j_{k+1}}]^-$$

are replaced by

$$[e_{i_{k-1}}e_{j_k}]^- - [e_{i_{k-1}}e_{j_{k-1}}]^- - [e_{i_{k+1}}e_{j_k}]^- + [e_{i_{k+1}}e_{j_{k-1}}]^-$$

and

$$[e_{i_k}e_{j_{k-1}}]^- - [e_{i_k}e_{j_{k+1}}]^- - [e_{i_{k+1}}e_{j_{k-1}}]^- + [e_{i_{k+1}}e_{j_{k+1}}]^-$$

in sum (5.3). Similar arguments can be made when $i_k = j_k$ for more than one value of k. So we see that the anti-symmetric part of the group $H_2(N_\Gamma, \partial N_\Gamma)$ is generated by X-cycles.

Lemma 5.1.1 implies that the rank of the anti-symmetric part of the group $H_2(N_\Gamma, \partial N_\Gamma)$ is equal to

$$\frac{1}{2}(m-1)(m-2) - 1.$$

Then, considering all possible embeddings of Γ into the complete graph K_n, we see that there exists an anti-symmetric free subgroup of $H_2(N_{K_n}, \partial N_{K_n})$ containing all possible X-cycles, generated by $n[\frac{1}{2}(n-2)(n-3)-1]$ X-cycles.

We now describe relations between the X-cycles and T-cycles in $H_2(N_{K_n}, \partial N_{K_n})$. Consider the index set $\{1, \cdots, n\}$, choose three pairs of indices from this set, $(ij)(pq)(rs)$, with i, j, p, q, r, s all distinct, call such a choice an *index triple*. For each such index triple we obtain a relation between the X-cycles and T-cycles containing $T_{\alpha,\beta,\gamma}$ for all (α, β, γ) with α an index from the first pair of the triple, β from the second pair and γ from the third pair, and one X-cycle with central vertex v_α for α equal to each of the six indices and the other indices of the X-cycle given by the two pairs in the index triple not containing α. For example taking index triple (12)(34)(56) we obtain the relation,

$$T_{135} - T_{136} - T_{145} + T_{146} - T_{235} + T_{236} + T_{245} - T_{246} \tag{5.4}$$

$$+_1 X^{34}_{56} -_2 X^{34}_{56} -_3 X^{12}_{56} +_4 X^{12}_{56} -_5 X^{12}_{34} +_6 X^{12}_{34} = 0.$$

The choice of index triple completely determines the relation and we will refer to such a relation as an *index triple relation.* Note that the coefficients of the terms in an index triple relation do not follow the same pattern for every choice of index triple.

Let G be the subgroup of $H_2(N_{K_n}, \partial N_{K_n})$ with generating set equal to the union of the set of all T-cycles and a set of $n[\frac{1}{2}(n-2)(n-3)-1]$ X-cycles which generate the subgroup of all X-cycles in the group $H_2(N_{K_n}, \partial N_{K_n})$. The set of relations for G will be the set of all index triple relations.

Lemma 5.1.1 implies that the anti-symmetric part of the group $H_2(N_{K_n}, \partial N_{K_n})$ has rank

$$\frac{1}{2}n(n-1) - n + \frac{1}{2}n(n-2)(n-3). \tag{5.5}$$

The generating set of the subgroup G has cardinality,

$$\frac{1}{3}n(n-1)(n-2) + \frac{1}{2}n(n-2)(n-3) - n, \tag{5.6}$$

and the difference between these two expressions is

$$\begin{aligned}&\frac{1}{3}n(n-1)(n-2)-\frac{1}{2}n(n-1) \\ &=\binom{n}{3}-\binom{n}{2}.\end{aligned} \tag{5.7}$$

In order to prove the statment of the theorem we will transform the subgroup G using the Tietze Transformations until we obtain a presentation for G with $\frac{1}{2}n(n-1)-n+\frac{1}{2}n(n-2)(n-3)$ generators and no relations. This will show that G is a free group with rank equal to the rank of the anti-symmetric part of $H_2(N_{K_n}, \partial N_{K_n})$, therefore G must be isomorphic to the anti-symmetric part of $H_2(N_{K_n}, \partial N_{K_n})$ and so this group can be generated by X-cycles and T-cycles.

We will use the index triple relations between the X-cycles and T-cycles to eliminate $\binom{n}{3}-\binom{n}{2}$ T-cycles from the generating set for G. To do this we choose an ordered set of T-cycles and relations such that each T-cycle lies in its corresponding relation but does not lie in any subsequent relation. Consider the T-cycle T_{123}, this appears in the relation given by the index triple $(14)(25)(36)$ and so we can use this relation to remove T_{123} from the generating set of G, then the index triple relation $(14)(25)(36)$ is removed from the set of relations for G and T_{123} is replaced in all other index triple relations by $(14)(25)(36)-T_{123}$. Then consider T_{124}, this appears in the relation given by $(13)(25)(46)$ which does not contain T_{123} so we can also remove T_{124} from the generating set of G. Continuing in this way, we can remove $T_{12\alpha}$ from the generating set of G for $\alpha = 5, \cdots n-1$ using the relation $(1, \alpha-2)(2, \alpha-1)(\alpha, \alpha+1)$, since this relation does not contain $T_{12\beta}$ for any $\beta < \alpha \in \{1, \cdots, n\}$. Having removed these T-cycles from the generating set of G we cannot also remove T_{12n} since any index triple $(1, \alpha)(2, \beta)(n, \gamma)$ gives rise to a relation containing $T_{12\gamma}$ where $\gamma < n$ which has previously been removed from the generating set.

Following a similar pattern we can also remove all T-cycles $T_{13\alpha}$ with $\alpha = 4, \cdots, n-1$ using relations given by $(12)(3, \alpha-1)(\alpha, \alpha+1)$ for $\alpha = 5, \cdots, n-1$ and the relation given by $(12)(35)(46)$ for $\alpha = 4$. None of these relations contain any of the previously removed T-cycles as the first pair of indices is (12) in all of the index triples.

Following this pattern we can remove, in dictionary order, all T-cycles $T_{1\alpha\beta}$ with $\alpha = 2, \cdots, n-3$ and $\beta = \alpha+1, \cdots, n-1$. To remove $T_{1\alpha\beta}$ with $\beta - \alpha > 1$ we use the relation given by the index triple $(1, \alpha-1)(\alpha, \beta-1)(\beta, \beta+1)$ and to remove $T_{1\alpha\beta}$ with $\beta - \alpha = 1$ use relation given by the index triple $(1, \alpha-1)(\alpha, \beta+1)(\beta, \beta+2)$. Fixing α we see that the relation given by $(1, \alpha-1)(\alpha, \beta-1)(\beta, \beta+1)$ does not contain T-cycles $T_{1,\alpha,\gamma}$ for $\gamma < \beta$, similarly this relation does not contain any T-cycles $T_{1,\gamma,\beta}$ with $\gamma < \alpha$. In this way we remove

$$(n-3) + (n-4) + (n-5) + \cdots + (n-(n-2))$$
$$= \frac{1}{2}[4(n-4) + (n-4)(n-5)]$$

T-cycles from the generating set of G. It is not possible to remove any more T-cycles containing vertex v_1 from the generating set of G since it is not possible to choose an index triple which contains 1 but does not correspond to a relation containing a T-cycle $T_{1\alpha\beta}$ which has already been removed.

Then we must remove further T-cycles from the generating set of G using relations corresponding to index triples chosen from the index set $\{2, \cdots, n\}$. With some relabelling we can then apply the process of removing T-cycles containing vertex v_1 described above to remove

$$(n-4) + (n-5) + \cdots + (n-(n-2)) =$$
$$\frac{1}{2}[4(n-5) + (n-5)(n-6)]$$

T-cycles containing vertex v_2. Continuing by induction we then remove

$$(n-5) + \cdots + (n-(n-2)) = \frac{1}{2}[4(n-6) + (n-6)(n-7)]$$

T-cycles containing vertex v_3 from the generating set of G and so on until we have removed

$$\frac{1}{2}[4(n-4) + (n-4)(n-5)] + \frac{1}{2}[4(n-5) + (n-5)(n-6)] \qquad (5.8)$$
$$+ \cdots + \frac{1}{2}[4(n-(n-2)) + (n-(n-2))(n-(n-1))]$$

T-cycles from the generating set of G. Note that this series terminates when we consider an index set of less than 6 vertices, since every index triple relation contains at least 6 vertices. The final term in this series is always equal to 5, which is equal to $\binom{6}{3} - \binom{6}{2}$. The sum (5.8) can be written as the following sum of finite series,

$$2\sum_{i=1}^{n-2}(n-i) + \frac{1}{2}\sum_{i=4}^{n-2}(n-i)(n-(i+1)) \tag{5.9}$$
$$= 2\sum_{i=1}^{n-2}(n-i) + \frac{1}{2}\sum_{i=1}^{n-2}n^2 - \frac{1}{2}n\sum_{i=1}^{n-2}i$$
$$-\frac{1}{2}n\sum_{i=1}^{n-2}(i+1) + \frac{1}{2}\sum_{i=1}^{n-2}i^2 + \frac{1}{2}\sum_{i=1}^{n-2}i.$$

Taking the sums of these series using the formula for the sum of a finite arithmetic progression and the formula for the sum of the first k square numbers, we obtain the following expression in n,

$$2(n-5) + (n-5)(n-4) + \frac{1}{2}(n-5)n^2 - \frac{1}{4}n[4(n-5) + (n-5)(n-2)] \tag{5.10}$$
$$-\frac{1}{4}n[5(n-5)+(n-5)(n-1)]+\frac{1}{12}[(n-2)(n-1)(2n-3)]-7+\frac{1}{4}[4(n-5)+(n-5)(n-2)].$$

Simplifying expression (5.10) we obtain.

$$\frac{1}{6}n^3 - n^2 + \frac{5}{6}n = \frac{1}{6}n(n-1)(n-2) - \frac{1}{2}n(n-1) \tag{5.11}$$
$$= \binom{n}{3} - \binom{n}{2}.$$

Hence we have removed $\binom{n}{3} - \binom{n}{2}$ T-cycles from the generating set of G so, by (5.7), the cardinality of this generating set is now equal to the dimension of the anti-symmetric part of the group $H_2(N_{K_n}, \partial N_{K_n})$.

Finally we show that there are no relations between the remaining X-cycles and T-cycles in the generating set of G. The remaining T-cycles in the generating set of G are

$$\begin{array}{lll} T_{\alpha\beta n}, & \alpha < \beta & \alpha \in \{1, \cdots, n-3\}, \quad \beta \in \{\alpha+1, \cdots, n-1\} \\ T_{\alpha,n-2,n-1}, & \alpha \in \{1, \cdots, n-3\} & \\ T_{\alpha,n-2,n}, & \alpha \in \{1, \cdots, n-3\} & \\ T_{n-2,n-1,n}. & & \end{array}$$

Consider a T-cycle of the form

$$T_{\alpha\beta n}, \quad \alpha < \beta, \quad \alpha \in \{1, \cdots, n-3\}, \quad \beta \in \{\alpha+1, \cdots, n-1\}.$$

Then since every pair of incident edges in the graph lie in exactly one T-cycle, any relation containing $T_{\alpha\beta n}$ must also contain an X-cycle of the form ${}_{\alpha}X^{\beta\gamma}_{n\delta}$ to cancel edge pair $e^{\alpha}_{\beta}e^{\alpha}_{n}$. Since their are no relations between the X-cycles in the generating set of G, this in turn implies that the relation must contain the T-cycle $T_{\alpha,\gamma,\delta}$ to cancel edge pair $e^{\alpha}_{\gamma}e^{\alpha}_{\delta}$ but the only T-cycle of this form which remains in the generating set of G is $T_{\alpha,n-2,n-1}$ so the X-cycle must have the form ${}_{\alpha}X^{\beta,n-2}_{n,n-1}$. This implies that the relation contains $T_{\alpha\beta,n-1}$ but this T-cycle was removed from the generating set of G. Hence there can be no relation between the remaining elements of the generating set of G containing a T-cycle of the form $T_{\alpha\beta n}$.

Similar arguments show that the other remaining T-cycles, $T_{\alpha,n-2,n-1}, T_{\alpha,n-2,n}$ and $T_{n-2,n-1,n}$ cannot lie in any relation between the remaining generating cycles of G. □

Corollary 5.3.2 *For any complete graph K_n, the Betti numbers of the space $F(K_n, 2)$ are given by the following expressions,*

$$b_2(F(K_n,2)) = \frac{1}{4}(n^4 - 10n^3 + 31n^2 - 30n) + 1, \tag{5.12}$$

$$b_1(F(K_n,2)) = (n-1)(n-2). \tag{5.13}$$

Proof This follows from Theorems 5.3.1 and 3.3.2. Theorem 3.3.2 implies that since the intersection form I_{K_n} is epimorphic,

$$b_2(F(K_n,2)) = b_1(K_n)^2 - b_1(K_n) + 1 - \sum_{v\in V(K_n)} (\mu(v)-1)(\mu(v)-2), \tag{5.14}$$

and

$$b_1(F(K_n,2)) = 2b_1(K_n). \tag{5.15}$$

The first Betti number of a complete graph K_n is equal to $\binom{n-1}{2} = \frac{1}{2}(n-1)(n-2)$. Substituting this expression into 5.14 and 5.15 implies that

$$b_2(F(K_n,2)) = \frac{1}{4}(n-1)^2(n-2)^2 - \frac{1}{2}(n-1)(n-2) - n(n-2)(n-3) + 1, \tag{5.16}$$

and

$$b_1(F(K_n,2)) = (n-1)(n-2). \tag{5.17}$$

Simplifying 5.16 we obtain 5.12 to complete the proof. □

Remarks

1. Corrolary 5.3.2 confirms a result of Copeland and Patty in their paper [11]. In Theorem 6.2 of [11] the authors prove a formula for calculating the second Betti number of $F(K_n, 2)$, which corresponds exactly to 5.12, using an iterative method which calculates the second Betti number for a sequence of subgraphs of K_n.

2. Theorems 5.3.1 and 5.2.4 show that the intersection form is epimorphic for all complete and complete bipartite graphs, however the methods used in the proofs of these two Theorems take two different approaches. In Theorem 5.2.4 we show that the intersection form is epimorphic by directly calculating the second Betti number of $F(K_{n,m}, 2)$ using the methods set out in chapter 3. In Theorem 5.3.1 however, we take a less direct route by showing that the image group of the intersection form, $H_2(N, \partial N)$, is generated by cycles whose supports lie in subgraphs of the complete graph isomorphic to K_5 and then use the fact that the intersection form I_{K_5} is known to be epimorphic. A similar approach is also taken in Theorem 5.2.1 for complete bipartite graphs of the form $K_{3,m}$.

   In general I believe that, for all graphs, the group $H_2(N, \partial N)$ is generated by 'small cycles 'i.e. cycles whose support is the union of a small number of edges of the graph. Theorem 5.1.3 shows that the symmetric part of the group $H_2(N, \partial N)$ is generated by Y-cycles whose support is given by the union of just three edges of the graph. I believe it could be shown that, for all graphs, the anti-symmetric part of this group is generated by T-cycles, X-cycles and S-cycles all of which have supports small enough so that the support of each such cycle lies in a subgraph isomorphic to K_5 or $K_{3,3}$ in any complete or complete bipartite graph. Such a Theorem would directy imply that the intersection form should be epimorphic for all complete and complete bipartite graphs.

Lemma 4.2.1 shows that the intersection form is never an epimorphism for any planar graph, the theorems in this chapter show that for all complete graphs and complete bipartite graphs the intersection form is epimorphic. Therefore, it is natural to ask the question whether the intersection form is epimorphic for all non-planar graphs. In the next example we consider a graph which is non-planar but not complete and show that for this graph the intersection form is not epimorphic.

Example 5.3.1 We consider the graph Γ_1 shown in Figure 5.10. The graph Γ_1 is clearly not a complete graph nor a complete bipartite graph, however it is non-planar since it contains a copy of K_5 as a subgraph.

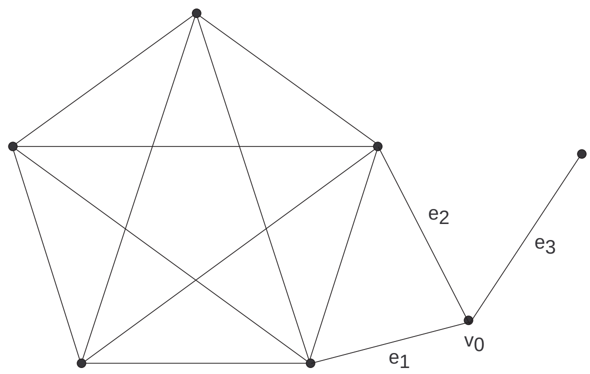


Figure 5.10: The graph Γ_1.

We show that the intersection form for this graph, I_{Γ_1}, is not epimorphic. Consider the Y-graph embeded in Γ_1 consisting of the union of the three edges, e_1, e_2, and e_3 which are incident to the vertex v_0. Then by Theorem 5.1.3, the generating set of the symmetric part of the group $H_2(N_{\Gamma_1}, \partial N_{\Gamma_1})$ contains one Y-cycle whose support is equal to this Y-graph.

This Y-cycle is clearly not equal to the intersection of any cycles in Γ_1 since the edge e_3 does not lie in any cycle of the graph. Hence the cokernel of the intersection form I_{Γ_1} is non-zero and I_{Γ_1} is not epimorphic.

5.4 Discussion

In this final section we discuss how the work in this thesis fits in with previous work on deleted product spaces of graphs. These spaces have been studied as important topological objects in their own right in papers by Patty [28], by Sarkaria [30], by Copeland [10] and by Copeland and Patty [11]. In these papers the authors attempt to describe the homology groups of such spaces, unfortunately however some of these papers contain some serious errors. In [28] Theorem 4.2 is incorrect, it underestimates the second Betti number of $F(\Gamma, 2)$. Similarly in [10] section two claims wrongly that the second homology group of a deleted product space is generated only by tori corresponding to products of disjoint cycles in the graph. This is disproved by Example 2.4.1 in Chapter 2 of this thesis which shows that the configuration space $F(K_5, 2)$, of the complete graph K_5, has the homotopy type of an orientable surface and therefore the second homology group of this space is isomorphic to the group of integers, however K_5 contains no disjoint cycles. The misconceptions in papers [28] and [10] are addressed in the joint paper [11], the abstact of this paper states *"the two-dimensional Betti numbers of the deleted product space are larger than they were originally thought to be"*. The paper goes on to calculate upper and lower bounds for the second Betti number of $F(\Gamma, 2)$ by decomposing the graph, adding one edge at a time. The paper makes use of the Kuratowski graphs in its analysis, and the last section uses the decomposition method to calculate the second Betti number of $F(\Gamma, 2)$ where Γ is any complete graph. Corollary 5.3.2 confirms this result.

The paper by Sarkaria [30] also acknowledges the importance of the Kuratowsik graphs K_5 and $K_{3,3}$ in calculating the Betti number of deleted product spaces of graphs. The paper claims without proof in statement 3.4.1 that, with $\mathbb{Z}_2$ coefficients, the second homology group of the deleted product space must be generated by tori and by orientable surfaces of genus 4 and 6 corresponding respectively to embeded copies of the two Kuratowski graphs, K_5 and $K_{3,3}$ in the graph. However statement 3.4.2 of this paper wrongly describes the dimension of this second homology group. Conjecture 2.4.1, which claims that the generators of the second homology group of

$F(\Gamma, 2)$ come from copies of the Kuratowski graphs and disjoint cycles embeded in the graph, is similar to statement 3.4.1 of [30]. However in section 3 of this chapter we proved Theorem 5.2.1 which showed this conjecture to be false.

The work in this thesis provides a method for calculating the second Betti number of the configuration space $F(\Gamma, 2)$ for any simple graph Γ as described in Section 3.5. The formula for the Euler characteristic of $F(\Gamma, 2)$ given in Lemma 3.1.1 then allows the calculation of the first Betti number. The results in Chapters 4 and 5 give simple formulas for the calculation of these Betti numbers for certain classes of graph. However the structure of the generators of the second homology group of $F(\Gamma, 2)$ for a general graph Γ remains unclear. For planar graphs we showed that this group is generated by pairs of disjoint cycles in the graph, however for graphs which cannot be embedded in the plane the situation is not well understood and is more complicated than was first thought. Theorem 5.2.1 shows that for graphs containing copies of the complete bipartite graphs $K_{3,4}$ or $K_{3,5}$, the group $H_2(F(\Gamma, 2), \mathbb{Z})$ may have generators which come from surfaces mapped into the space $F(\Gamma, 2)$ but not embedded in the space, as was suggested in Conjecture 2.4.1. Corollary 2.5.2 shows that the group $H_2(F(\Gamma, 2), \mathbb{Z})$ is isomorphic to $\Omega_2(F(\Gamma, 2))$, the two-dimensional oriented bordism group of $F(\Gamma, 2)$. This group consists of bordism classes of orientable surfaces mapped into the space $F(\Gamma, 2)$ but it places no restriction on the maps from the surfaces to the space $F(\Gamma, 2)$. Together these results suggest that more complicated generators of the group $H_2(F(\Gamma, 2), \mathbb{Z})$ may be possible.

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