One Loop Phenomenology of Type II String Theory: Intersecting D-Branes and Noncommutativity

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Abstract

We examine one loop amplitudes for open and closed strings in certain D-brane configurations, and investigate the consequences for phenomenology.

Initially we consider open strings at D6-brane intersections. We develop techniques for one-loop diagrams.The one-loop propagator of chiral intersection states is calculated exactly and its finiteness is shown to be guaranteed by RR tadpole cancellation. The result is used to demonstrate the expected softening of power law running of Yukawa couplings at the string scale. We also develop methods to calculate arbitrary N-point functions at one-loop, including those without gauge bosons in the loop. These techniques are also applicable to heterotic orbifold models.

One issue of the intersecting D6-brane models is that the Yukawa couplings of the simpler models suffer from the so-called "rank one" problem – there is only a single non-zero mass and no mixing. We consider the one-loop contribution of E2-instantons to Yukawa couplings on intersecting D6-branes, and show that they can provide a solution. In addition they have the potential to provide a geometric explanation for the hierarchies observed in the Yukawa couplings. In order to do this we provide the necessary quantities for instanton calculus in this class of background.

We then explore how the IR pathologies of noncommutative field theory are resolved when the theory is realized as open strings in background B-fields: essentially, since the IR singularities are induced by UV/IR mixing, string theory brings them under control in much the same way as it does the UV singularities. We show that at intermediate scales (where the Seiberg-Witten limit is a good approximation) the theory reproduces the noncommutative field theory with all the (un)usual features such as UV/IR mixing, but that outside this regime, in the deep infra-red, the theory flows continuously to the commutative theory and normal Wilsonian behaviour is restored. The resulting low energy physics resembles normal commutative physics, but with additional suppressed Lorentz violating operators. We also show that the phenomenon of UV/IR mixing occurs for the graviton as well, with the result that, in configurations where Planck's constant receives a significant one-loop correction (for example brane-induced gravity), the distance scale below which gravity becomes non-Newtonian can be much greater than any compact dimensions.

Declaration

I declare that no part of this thesis has been submitted elsewhere for any other degree or qualification. The work in this thesis is based on research carried out at the Centre for Particle theory, Department of Mathematical Sciences, Durham University, England, and is the result of collaboration between the author and collaborators Steven Abel and Chong-Sun Chu, published in

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Contents

List of Figures

Chapter 1

Introduction

String theory is a research programme built upon the almost Aristotelian attempt to deduce the fundamental laws of nature through reasoning; it starts from a few basic premises and arrives at a very general and beautiful theory with many startling properties. However, in the modern scientific world beauty is not the arbiter of the success of a theory: only experiment has that privilege, and thus the goal of string theorists is to connect it to the Standard Model, the most successful and accurate theory ever created.

The Standard Model of particle physics is built on Quantum Field Theory, in which every form of matter is associated with a field operator, and is at once a wave and a point-like particle. Infinities in calculations which must be absorbed into renormalisation constants are interpreted as signalling the breakdown of the theory at some currently inaccessible energy: it is built in that it is an incomplete description of nature, even without gravity. When we consider gravity, the scale at which the theory must break down becomes apparent; through considering the constants of the theories of particles and gravitation we find the Planck scale, $1.22 \times 10^{19} GeV$, to be significant. Since Quantum Field Theory is incompatible with the theory of gravity, General Relativity, we assume that this must be the energy at which we would certainly see evidence of the theory that provides a true description of both. We have a clear theoretical signal that the theories need to be modified, although essentially no signal as to how this should be accomplished, and unfortunately the energy at which we would definitely find the new theory is prohibitively beyond current experimental technology.

Since Quantum Field Theory is a general framework, its unification with General Relativity is not the only demand that we place upon our new theory. We require it to explain or predict at least some of the properties of the Standard Model, which consists of a curious pattern of fields: three families of fermions with a hierarchy of masses; each family having a pair of quarks and a pair of leptons, one of which is virtually massless; the bosons carrying the gauge forces which bind matter together with the strong and electroweak coupling, and the Higgs, the field whose vacuum expectation value breaks the electroweak symmetry and provides (almost) everything with mass. We also demand superpartners for all of these fields and a mechanism of supersymmetry breaking, in order to account for the disparity in hierarchy between electroweak and Planck energy scales (or possibly some other explanation). A prediction of these properties would also illuminate some of the cosmological problems, such as the existence of Dark Matter (widely assumed to be the lightest superpartner of the Standard Model fields), although we would also desire an explanation of Dark Energy and the apparent non-vanishing cosmological constant.

String theory provides a tantalising route to progress on all of the above problems. By assuming that the fundamental objects are strings rather than particles, it is possible to unify Quantum Field Theory with General Relativity; the appearance of the Einstein-Hilbert action as a coherent state of gravitons is a beautiful and surprising result. However, without supersymmetry we find that string theory is inconsistent: purely bosonic string theory contains tachyons[1], and whether these condense (and what they condense to) is an open problem. Thus the programme is intimately tied in with a supersymmetric explanation of the Hierarchy Problem.

The analogue in string theory of the choice of fields of the Standard Model is the *background* of string compactification. The natural setting for string theory is flat ten-dimensional spacetime, and thus we must assume instead that (usually) six dimensions are compact and small enough to be currently undetected. The properties of the compact dimensions then dictate the spectrum of fields observed by macro-

[1]Although bosonic string theory does provide a useful toy framework, a fact relied upon in this thesis.

scopic observers. Initially it was hoped that consistency would dictate an essentially unique theory; however, without further theoretical constraints the current understanding is of a presumably infinite landscape of possible backgrounds. This can be regarded as similar to the possibilities of fields and gauge groups in the framework of Quantum Field Theory. A particular non-trivial enterprise in string theory is the construction of backgrounds which reproduce the Standard Model (or supersymmetric extensions thereof) at low energies, and although it is no longer expected that a unique example will be distinguished from the many other examples, it is valuable to find the features of such edifices that account for the various mysteries of the Standard Model. Generally features of the Standard Model have simple geometric interpretations in string constructions, although some are rather subtle. It is then hoped that the model-building techniques will reveal the necessary theoretical input to provide an ultimate theory.

The current understanding of String Theory is that it is one theory with several formulations; in ten dimensions it is described by type I, IIA, IIB and heterotic $SO(32)$ and $E_8 \times E_8$:

- Heterotic string theory was long considered the main candidate for constructing realistic models. It consists of closed strings where the oscillations propagating in opposite directions around the loop experience different dimensions of spacetime with different amounts of supersymmetry; while the right moving modes experience supersymmetry amongst the bosonic and fermionic degrees of freedom and propagate in ten dimensions, the left moving modes have no supersymmetry and feel twenty-six dimensions. Ten of the left-moving dimensions are considered to be bosonic spacetime coordinates, while the remaining sixteen are converted into thirty-two real fermionic degrees of freedom, accounting for the internal symmetry $SO(32)$ or $E_8 \times E_8$ (these specific groups are required for the cancellation of anomalies). Starting from such a large symmetry group, model building involves specifying a fibration of the internal degrees of freedom over the six compact spacial dimensions in such a way as to break the symmetry, to obtain grand unified or standard model gauge groups and flavours. This was an attractive proposal, as it offers a "top down" approach where the fundamental theory is specified and "all" that is required is a manifold. There is still much work in this area, but we shall not consider it further.

- Type I strings are *unoriented*, in that they are invariant under exchange of left- and right-moving modes of closed strings, and contain open strings whose endpoints are indistinguishable. The endpoints of open strings do, however, must have gauge degrees of freedom, which anomaly cancellation determines to be SO(32).

- Type II strings were originally purely closed strings, with supersymmetry for both left- and right-moving modes, and hence propagating in ten dimensions with no additional internal degrees of freedom. This made them initially unattractive for model building until the advent of D-branes.

All of the above theories are related by dualities, and the overarching theory is M-theory, which exists in eleven dimensions and yields the other theories upon compactification; however, its formulation is unclear or difficult to work with (the *Matrix Theory* proposal is a subject of much interest, but great difficulty for model building). Hence string phenomenology predominantly focuses on the ten-dimensional theories. They are all defined in terms of a perturbative expansion, and the non-perturbative definition is not known. However, several non-perturbative objects have been found. Upon a transformation of the theory known as *T-duality*, D-branes (so called because they are hypersurfaces with Dirichlet boundary conditions for the strings) appear, the excitations of which are string states, and as such provide important tools in model building. By stacking several branes together, the string endpoints obtain gauge degrees of freedom, and thus type II strings attached to D-branes have become subject to significant recent interest; indeed this thesis shall focus upon that framework. Related to these are NS-5-branes, six-dimensional hypersurfaces upon which D-branes may end, and *Orientifold Planes*, which are non-dynamical hypersurfaces left invariant under an orientation reversal of the strings coupled to a reflection in space; the latter of these two shall also be relevant to this work.

D-branes offer an elegant way to restrict matter fields to a four-dimensional hypersurface, and hence have provided an important tool in model building. However, since gravity is mediated by closed strings not restricted to the branes, and since we observe four-dimensional gravity and no gravitinos, string models generally compactify six dimensions and break (at least some of) the supersymmetry in them. Hence some approaches involve just a D3-brane (3 labelling the number of spatial dimensions) in either a complicated *Calabi-Yau* manifold (a manifold with vanishing Ricci tensor and having SU(3) holonomy, necessary and sufficient to be a string background with N=1 supersymmetry) or a simple subset thereof, an *Orbifold*: a quotient of flat space by a subgroup of its symmetries. We shall have these models in mind in chapter 5, but will not refer to the specific details of them there. There are other approaches, where our four dimensions are considered to be the intersection of two D-branes for some fields. Specifically, a very popular construction involves several sets of D6-branes, where chiral superfields exist at the intersections and vector superfields are found on the full volume of the branes. These shall be of particular relevance in chapters 3 and 4. Since in these models we must always compactify the extra dimensions, and we wish to break supersymmetry for gravity, the same techniques involving orbifolds or other manifolds apply in each case. Indeed, they can usually be related by dualities, and the D3-brane models are generally augmented by the introduction of D7-branes, often required for consistency. However, the case of a toroidal orientifold background (essentially flat space with every extra dimension periodic, and a choice of orbifold and orientifold group) has attracted much interest since it allows many quantities to be calculated exactly while having sufficiently realistic properties to provide useful toy models. It is these models that we shall consider. There has been considerable interest in further embellishing these models by adding certain types of closed string field backgrounds (so called *three-form fluxes*) in order to stabilise the size and shape of the compact dimensions against deformation whilst breaking supersymmetry. However, upon introduction of these fluxes most quantities become (currently) incalculable in full string theory and it is only possible to apply supergravity or field theory analysis. Hence we shall neglect these models.

There remain several challenges for string phenomenology. Overall the aim is to provide a fully realistic model of the real world with a convincing method of supersymmetry breaking; it would then be hoped that there would be some predictive relation between the parameters. Working toward this within the sub-field of D-brane constructions, the low energy properties are still not completely understood; we consider $N = 1$ supersymmetric models, for which the perturbative superpotential receives no corrections at one loop, but the remaining properties (such as running of the couplings) are determined by the Kähler potential, which is poorly understood. The work in chapter 3 adds to the effort to understand it. It investigates the appearance of divergences that would render the theory inconsistent, and their cancellation. It also outlines new techniques for the calculation of diagrams necessary for the determination of the Kähler potential, and determination of the running of Yukawa couplings.

Although the superpotential is dominated by a perturbative component that is determined completely at tree level in these models, there are also non-perturbative contributions involving both tree and one-loop amplitudes, and they have not previously been calculated. The work of chapter 4 provides the essential tools for the calculation of these instanton contributions, and discusses how they can solve certain phenomenological problems in such frameworks.

Another challenge is to relate cosmological data or possible experiments to our particle models. In the study of D-branes it was discovered that string theory, under certain circumstances, gives rise to *noncommutative geometry*. This is a phenomenon where the operators corresponding to different coordinate directions have a non-zero commutator. It is an area of considerable work at present, with some hoping that it will provide a new method to unify field theory and gravity. From the phenomenologist's perspective, it provides many intriguing new behaviour in field theory, such as *UV/IR mixing*, where low energy interactions actually probe the high energy behaviour of the theory. One occasion that it occurs in string theory is the relative motion of two D-branes, and can be viewed as an uncertainty relation; an analogue of the Heisenberg relation. In this way, it is supposed by many to play a key role in the ultimate theory, and thus merits considerable current research. Its appearance would have many consequences for phenomenology, and would give clear signals in experiments; Chapter 5 investigates some of these in a framework where it appears naturally in string theory, including hitherto unexplored consequences for gravity.

We turn now to chapter 2, which provides an introduction to many of the concepts required for the rest of the thesis, including the formulation of type II string theory and a discussion of D-branes.

Chapter 2

Some Aspects of String Theory

This chapter briefly reviews the aspects of string theory relevant to the subsequent chapters, providing necessary references. For more material on the subject the reader should consult [1, 2].

2.1 Worldsheets, Actions and Backgrounds

2.1.1 Preliminaries

In the path-integral formulation of field theory, the fundamental objects (representing particles) are fields which may take any value at any point in space, and the amplitude for transition from one point in space (or momentum space) to another is given by integrating over all paths with a weight given by the action for that path or worldline. In string theory, the fundamental objects are one-dimensional and thus sweep out a worldsheet, so the path integral is weighted by an action depending on two variables. In bosonic string theory, the fields in the theory are then the *spacetime* coordinates, while in the superstring we also introduce fermionic coordinates for each spacetime dimension. We have an additional choice to make, that is whether the worldsheet has a boundary - this determines whether the fundamental objects are closed or open strings.

To study the theory, we must begin with an action. Without boundary terms (which we shall return to later), the purely bosonic part of the action for a general spacetime is taken to be a nonlinear sigma model and can be written

$$S_X = \frac{1}{4\pi\alpha'}\int_M d^2\sigma h^{1/2}\left[\left(h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\right] \tag{2.1.1}$$

where h^{ab} is the *Euclidean* worldsheet metric (in two dimensions, even with gravity, the Wick rotation from a Minkowski worldsheet is well justified) and $\{\sigma_1, \sigma_2\}$ are the worldsheet coordinates - generally we consider σ_1 to parametrise the direction along the string, and σ_2 to be the (Euclidean) time; X^μ are the target space coordinates; $G_{\mu\nu}$ is the *Minkowski* spacetime metric; $B_{\mu\nu}$ is an antisymmetric tensor field; R is the curvature *of the worldsheet* and Φ is the dilaton field. The dimensionful constant α' is proportional to the string length squared, and sets the scale of string interactions: as we take $\alpha' \to 0$, we expect to obtain particle behaviour. The label M in the above action denotes the worldsheet geometry, which we must specify.

The fields G, B and Φ are all considered to be (potentially running) couplings in the quantum theory - in order for string theory to be consistent on a given background spacetime it must be stable to quantum corrections of these fields, i.e. their beta-functions must vanish. It is reassuring to note that we can take flat space to be a consistent background; the general conditions are irrelevant for our purposes, but the backgrounds we shall be interested in are Ricci-flat ($R_{ab} = 0$) and thus we may consider six of the dimensions to be a compact manifold provided we maintain Ricci-flatness. The backgrounds of interest to us will have constant (or zero) antisymmetric tensor, which plays an important role when we consider open strings (worldsheets with boundaries). They also have a constant dilaton Φ_0, which plays a vital role: it becomes the coupling constant of the theory, since the term $\int R = \chi$ is topological and so assigns a coupling to worldsheets of different Euler characteristic χ. The path integral is then split into a sum over Euler characteristics, which is interpreted as a sum over numbers of loops.

2.1.2 Gauge Fixing

The action (2.1.1) was chosen to satisfy the symmetries that we expect: those of the spacetime background, and diffeomorphisms of the worldsheet. However, it possesses an additional worldsheet symmetry: Weyl invariance. It is invariant under local rescaling of the worldsheet metric

$$h^{ab} \longrightarrow e^{\omega(\sigma_1,\sigma_2)} h^{ab} \tag{2.1.2}$$

and thus is (at least so far classically) a conformal field theory. In creating a quantum theory of strings, we write a path integral using the above action, and we integrate over all possible worldsheets and embeddings into the spacetime coordinates. However, since we have diffeomorphism and also Weyl invariance of the worldsheet, we will have an inconsistent path integral unless we divide the measure by the volume of this symmetry group. This can be achieved using light-cone coordinates, but the covariant approach is to use the Faddeev-Poppov procedure and introduce ghost fields to fix the gauge. We write the Polyakov path integral

$$\langle ... \rangle = \sum_{\chi} \int \frac{[\mathrm{d}X \mathrm{d}h]}{Vol(G)} (...) \exp[-S - \Phi_0 \chi] \tag{2.1.3}$$

where G is the product of diffeomorphism and Weyl symmetry groups, and ... represents an insertion of operators; and insert the Faddeev-Poppov measure in the usual manner. The result is that the integration over metrics is reduced to an integral over moduli (parameters of the metric which can not be removed through diffeomorphism and Weyl transformations) and we can fix the worldsheet metric to be conformally flat:

$$\hat{h}^{ab} = e^{2\omega(\sigma_1,\sigma_2)} \delta^{ab} \tag{2.1.4}$$

but the price that we pay is the inclusion of anticommuting ghost fields b, c and $\tilde{b}, \tilde{c}$. Each modulus of the worldsheet metric contributes a b-ghost insertion; for our purposes, we shall only need consider the topologies of the disk, torus and annulus. The disk has no moduli, the annulus has one, and the torus has two. The ghost insertions for the annulus and torus are respectively $2\pi i b(0)$ and $-4\pi b(0)\tilde{b}(0)$.

With the new flat metric, we choose complex coordinates

$$w = \sigma_1 + i\sigma_2, \qquad \bar{w} = \sigma_1 - i\sigma_2 \tag{2.1.5}$$

and write the ghost action as

$$S_g = \frac{1}{2\pi} \int \mathrm{d}^2 w \left(b\bar{\partial} c + \tilde{b} \partial \tilde{c} \right) \tag{2.1.6}$$

In addition, there are still symmetries not fixed by specifying the metric, that must fixed by the Faddeev-Poppov procedure: these comprise the *conformal Killing group* (CKG), which is a subgroup of G. The generators of this group are called the *conformal Killing vectors*(CKVs), and for each of these we must fix a coordinate of an inserted operator (consistent with the CKG) and insert a c-ghost. The ghost insertions are independent of the target space geometry - they depend only on the class of worldsheet - and so the ghost portion of the path integral is universal to all string scattering amplitudes. Of relevance here is the result for the annulus: the determinant of the ghost insertions is given by $|\eta(it)|^2$, where t is the modulus of the annulus such that the fundamental domain is taken to be $[0, 1/2] \times [0, it]$ and η is the Dedekind Eta function.

2.1.3 Supersymmetry

We now wish to consider adding fermions to the theory. This is accomplished by implementing worldsheet supersymmetry. For a general background, the non-linear sigma model is complicated, so we shall specialise to the case of flat space. Then we can write the action, in conformal gauge, as

$$S_m = \frac{1}{4\pi} \int \mathrm{d}^2 w \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \psi_\mu \right) \tag{2.1.7}$$

where ψ^μ and $\tilde{\psi}^\mu$ are Grassmann (anticommuting) fields. We may write the stress-energy tensor associated with variation of the worldsheet metric

$$\begin{aligned} T_{w\bar{w}} &= 0 \\ T_{ww} &= -\frac{1}{\alpha'} \partial X^\mu \partial X_\mu - \frac{1}{2} \psi^\mu \partial \psi_\mu \\ T_{\bar{w}\bar{w}} &= -\frac{1}{\alpha'} \bar{\partial} X^\mu \bar{\partial} X_\mu - \frac{1}{2} \tilde{\psi}^\mu \bar{\partial} \tilde{\psi}_\mu \end{aligned} \tag{2.1.8}$$

which is conserved classically, translating to $T_{ww} \equiv T^B$ being holomorphic, and $T_{\bar{w}\bar{w}} \equiv \tilde{T}^B$ being antiholomorphic. Similarly, we have a tensor associated with superconformal transformations, with respectively holomorphic and antiholomorphic components $T^F(w)$ and $\tilde{T}^F(\bar{w})$:

$$\begin{aligned} T^F &= i\left(\frac{2}{\alpha'}\right)^{1/2} \psi^\mu \partial X_\mu \\ \tilde{T}_F &= i\left(\frac{2}{\alpha'}\right)^{1/2} \tilde{\psi}^\mu \bar{\partial} \tilde{X}_\mu \end{aligned} \tag{2.1.9}$$

If we look for the classical equations of motion from the above action, we first vary X and obtain

$$0 = \delta S_m = -\frac{1}{2\pi\alpha'}\int_M \mathrm{d}^2 w \delta X_\mu \partial\bar{\partial} X^\mu + \frac{i}{2\pi\alpha'}\oint_{\partial M} (\delta X_\mu \bar{\partial} X^\mu)\mathrm{d}\bar{w} - (\delta X_\mu \partial X^\mu)\mathrm{d}w \tag{2.1.10}$$

where ∂M is the boundary of the worldsheet M. This leads to the equation of motion:

$$\partial\bar{\partial} X^\mu = 0 \tag{2.1.11}$$

and thus classically X^μ splits into holomorphic and antiholomorphic parts. It also gives us the boundary conditions: if we consider the string to be a closed loop, then the worldsheet has no boundary (and clearly we must consider it periodic around the "length" direction σ_1), but if it is an open string then, since we consider the σ_2 direction to be the (Euclidean) time, the lowest genus worldsheet is an infinite strip, which we shall take to have boundaries at $\Re(w) = 0$ and $\Re(w) = 1/2$, and a possible boundary condition is

$$(\partial + \bar{\partial}) X^\mu = 0, \qquad \Re(w) = 0, 1/2\,. \tag{2.1.12}$$

These are Neumann boundary conditions. The alternative is to consider the string endpoints to be fixed at the boundaries:

$$\delta X_\mu = 0, \qquad \Re(w) = 0, 1/2 \tag{2.1.13}$$

which corresponds to

$$(\partial - \bar{\partial}) X = 0, \qquad \Re(w) = 0, 1/2 \tag{2.1.14}$$

since this is the derivative along the "time" direction. These are Dirichlet boundary conditions, the significance of which we shall return to later.

Considering now variations of the fermionic coordinates, we obtain

$$0 = \delta S = \frac{1}{2\pi}\int_M \mathrm{d}^2 w \delta\psi_\mu \bar{\partial}\psi^\mu + \delta\tilde{\psi}_\mu \partial\tilde{\psi}^\mu + \frac{i}{4\pi}\oint_{\partial M}(\tilde{\psi}_\mu \delta\tilde{\psi}^\mu)\mathrm{d}\bar{w} - (\psi_\mu \delta\psi^\mu)\mathrm{d}w. \quad (2.1.15)$$

The equations of motion are therefore

$$\bar{\partial}\psi^\mu = \partial\tilde{\psi}^\mu = 0 \quad (2.1.16)$$

and thus ∂X^μ and ψ^μ are holomorphic fields, and $\bar{\partial}X^\mu$ and $\tilde{\psi}^\mu$ are antiholomorphic. The boundary conditions for open string fermions on the infinite strip are

$$(\tilde{\psi}_\mu \delta\tilde{\psi}^\mu) - (\psi_\mu \delta\psi^\mu) = 0, \qquad \Re(w) = 0, 1/2 \quad (2.1.17)$$

which leads to

$$\begin{aligned} \psi^\mu(w) &= \tilde{\psi}^\mu(\bar{w}), \qquad \Re(w) = 0 \\ \psi^\mu(w) &= \pm\tilde{\psi}^\mu(\bar{w}), \qquad \Re(w) = 1/2. \end{aligned} \quad (2.1.18)$$

We can use the "doubling trick" to combine these fields into one holomorphic field on an infinite cylinder, of domain $[-1/2, 1/2] \times [-\infty, \infty]$:

$$\psi^\mu(w) = \begin{cases} \psi^\mu(w) & \Re(w) > 0 \\ \tilde{\psi}^\mu(-\bar{w}) & \Re(w) < 0 \end{cases}. \quad (2.1.19)$$

This converts the boundary conditions to a periodicity: $\psi^\mu(w+1) = \pm\psi^\mu(w) = e^{2\pi i v}\psi^\mu(w)$. The choice of a positive sign ($v = 0$) is called Ramond (R) boundary conditions, and a negative one ($v = 1/2$) Neveu-Schwarz (NS). Closed string fermions may also be periodic or antiperiodic, but the ψ^μ and $\tilde{\psi}^\mu$ fields have this condition separately and thus have four sectors: R-R,R-NS,NS-R and NS-NS (in principle, the bosons could also be antiperiodic, but this would be inconsistent with the symmetry of the target space).

Having established the equations of motion and boundary conditions, we can write down mode expansions for the fields as free waves. We shall be primarily interested in open strings, for which we can write

$$\begin{aligned} \partial X^\mu(w) &= -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{k=-\infty}^{\infty}\alpha_k^\mu e^{-iw(k+1)} \\ \psi^\mu(w) &= i^{-1/2}\sum_{k=-\infty}^{\infty}\psi_{k+v}^\mu e^{iw(k+v)}. \end{aligned} \quad (2.1.20)$$

When we quantise the theory, the coefficients become operators; the normalisation of the above is a choice that gives the (anti)commutators of the coefficients integer values, given by

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] &= m\eta^{\mu\nu}\delta_{m+n} \\ \{\psi^\mu_r, \psi^\nu_s\} &= \eta^{\mu\nu}\delta_{r+s}. \end{aligned} \tag{2.1.21}$$

Note that the bosonic and Ramond fields have zero-modes; the bosonic ones correspond to the centre-of-mass momentum, $\alpha^\mu_p = (2\alpha')^{1/2}p^\mu$. The Ramond zero modes will allow for spacetime fermions.

Finally in this section, having introduced the superpartners of the bosonic fields, in gauge-fixing the action we also require the superpartners of the ghost fields. Since the original ghosts are Grassmann fields, we find that we have commuting fields $\beta, \gamma, \tilde{\beta}$ and $\tilde{\gamma}$, with action

$$S_{sg} = \frac{1}{2\pi}\int \mathrm{d}^2 w \beta\bar{\partial}\gamma \tag{2.1.22}$$

and conjugate for the antiholomorphic $\tilde{\beta}$ and $\tilde{\gamma}$; for open string the antiholomorphic fields can be combined with the holomorphic ones using the doubling trick as elsewhere.

2.2 Conformal Field Theory

We now consider the quantum theory based on the actions considered in the previous section.

2.2.1 The Operator Product Expansion

If we have a set of local operators $\Phi_i(z,\bar{z})$ in a quantum theory, then the operator product expansion states that we can express the product of two of these operators as a sum over operators at a point:

$$\Phi_i(z,\bar{z})\Phi_j(w,\bar{w}) = \sum_k c^k_{ij}(z-w)\Phi_k(w,\bar{w}). \tag{2.2.1}$$

As an operator equation, this holds inside a general correlator provided that $|z-w|$ is smaller than the distance to any other operator insertions. Since the string action has conformal symmetry, we expect this to affect the operators of the theory, and indeed we find operators of interest have definite weights under global transformations:

$$\Phi_i'(\lambda z, \lambda \bar{z}) = \lambda^{-h_i} \bar{\lambda}^{-\tilde{h}_i} \Phi_i(z, \bar{z}) \tag{2.2.2}$$

where λ is a complex number and $(h_i, \tilde{h}_i)$ are the *conformal weights* of Φ_i. This property is actually required for unitarity of the theory, and restricts the OPE to

$$\Phi_i(z, \bar{z})\Phi_j(w, \bar{w}) = \sum_k (z-w)^{h_k - h_i - h_j} (\bar{z} - \bar{w})^{\tilde{h}_k - \tilde{h}_i - \tilde{h}_j} c_{ij}^k \Phi_k(w, \bar{w}). \tag{2.2.3}$$

so that now the c_{ij}^k are constants. In fact, all the properties of the theory can be extracted from those of the *primary fields*, which transform as

$$\Phi'(z', \bar{z}') = (\partial_z z')^{-h} (\partial_{\bar{z}} \bar{z}')^{-\tilde{h}} \Phi(z, \bar{z}). \tag{2.2.4}$$

The OPEs of primary fields generally have only finitely many singular terms, which determine most of the properties. For the string action, we can determine the OPEs by examining the path integral [1] or by considering the correlator of two fields. The result is

$$\begin{array}{llll} \partial X^\mu(z)\partial X^\nu(w) & \sim \frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2} & b(z)c(w) & \sim \frac{1}{(z-w)} \\ \bar{\partial} X^\mu(\bar{z})\bar{\partial} X^\nu(\bar{w}) & \sim \frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(\bar{z}-\bar{w})^2} & \tilde{b}(\bar{z})\tilde{c}(\bar{w}) & \sim \frac{1}{(\bar{z}-\bar{w})} \\ \psi^\mu(z)\psi^\nu(w) & \sim \frac{\eta^{\mu\nu}}{z-w} & \beta(z)\gamma(w) & \sim \frac{-1}{(z-w)} \\ \tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(\bar{w}) & \sim \frac{\eta^{\mu\nu}}{\bar{z}-\bar{w}} & \tilde{\beta}(\bar{z})\tilde{\gamma}(\bar{w}) & \sim \frac{-1}{(\bar{z}-\bar{w})}, \end{array} \tag{2.2.5}$$

where $\sim$ denotes only the singular terms in the OPE. From these, we can define a *conformal normal ordered* product of pairs of operators $: \Phi_i(z)\Phi_j(w) :$ by subtracting the singular terms in the OPE, and thus have a consistent definition of coincident operators - also giving a product which obeys the classical equations of motion. Thus the stress-energy tensor is written

$$\begin{aligned} T^B(z) &= \lim_{w \to z} \left(-\frac{1}{\alpha'} \partial X_\mu(w) \partial X^\mu(z) - \psi_\mu(w)\partial\psi^\mu(z) + \frac{3D}{2}\frac{1}{(z-w)^2} \right) \\ &\equiv -\frac{1}{\alpha'} : \partial X_\mu(z)\partial X^\mu(z) : - : \psi_\mu(z)\partial\psi^\mu(z) : \end{aligned} \tag{2.2.6}$$

where D is the number of spacetime dimensions. As we noted, the stress-energy tensor is the Noether current associated with variations of the metric, and thus is associated with conformal transformations. In the quantum theory, this becomes a Ward identity, with an *infinite* number of currents. The Ward identity gives the transformation of operators when we make a conformal transformation, and determines the OPE of the stress-energy tensor with a primary field $\Phi(w,\bar{w})$ to be

$$T^B(z)\Phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{(z-w)}\partial\Phi(w,\bar{w}). \tag{2.2.7}$$

From the action we can see that the weights of ∂X^μ, $\bar{\partial}X^\mu$, ψ^μ and $\tilde{\psi}^\mu$ must be $(1,0)$, $(0,1)$, $(1/2,0)$ and $(0,1/2)$ respectively, and this can also be seen from the OPEs, the definition of the stress-energy tensor and the above equation.

We also have a stress-energy tensor for the ghosts and superconformal ghosts, with components $T_g(z)$, T_{sg} given by

$$\begin{aligned} T_g^B(z) &= \; : b(z)\partial c(z) : -2\partial(: b(z)c(z)) : \\ T_{sg}^B(z) &= \; : \partial\beta(z)\gamma : -\frac{3}{2}\partial(: \beta(z)\gamma(z) :) \end{aligned} \tag{2.2.8}$$

and $\tilde{T}_g^B(\bar{z})$, $\tilde{T}_{sg}^B(\bar{z})$ given by the conjugate. The coefficients 2 and $\frac{3}{2}$ in the second terms are determined by the Faddeev-Poppov procedure (the above should be compared with the ψ^μ stress-energy tensor) which fixes the conformal weights to be $(2,0)$, $(-1,0)$, $(0,2)$ and $(0,-1)$ for $b, c, \tilde{b}$ and $\tilde{c}$ respectively, and $(3/2,0)$, $(-1/2,0)$, $(0,3/2)$ and $(0,-1/2)$ for $\beta,\gamma,\tilde{\beta}$ and $\tilde{\gamma}$.

If we now consider the OPE of the stress-energy tensor with itself, we find that in general it is not a primary field:

$$T^B(z)T^B(w) \sim \frac{c}{2(z-w)^4)} + \frac{2}{(z-w)^2}T^B(w) + \frac{1}{(z-w)}\partial T^B(w) \tag{2.2.9}$$

where c is the *central charge*, which for our case after summing over the stress-energy tensors for ∂X, ψ, (b,c) and (β,γ) is given by

$$c^{Tot} = \frac{3}{2}D - 15. \tag{2.2.10}$$

It transpires that this must be zero to have a consistent BRST operator, and for the Weyl symmetry to not be anomalous in the path integral: we thus conclude that there must be 10 flat spacetime dimensions.

2.2.2 The State-Operator Mapping

Having established the properties of the operators in our conformal field theory, we now wish to make the connection with states in our string Hilbert space. This is done via equation (2.1.20). We first consider fields on the infinite cylinder, and make a conformal transformation to turn this to the complex plane, using the variable $z = -e^{-i\pi w}$. This then translates the infinite past $\Im(z) = -\infty$ to the origin, and thus the asymptotic state of a field on the cylinder is found by considering it at the origin. We define

$$|\Phi_{in}\rangle = \lim_{z\to 0} \Phi(z,\bar{z})|0\rangle \tag{2.2.11}$$

and thus if we expand Φ as a mode expansion, the positive frequency modes must annihilate the vacuum:

$$\begin{aligned} \Phi(z) &= \sum_{m,n} z^{-m-h}\bar{z}^{-n-\tilde{h}}\Phi_{m,n} \\ \Phi_{m,n}|0\rangle &= 0, \qquad m > -h, n > -\tilde{h}. \end{aligned} \tag{2.2.12}$$

If we take the Hermitian conjugate of a field, then, since we are using Euclidean time, we must make the replacement $z \to 1/\bar{z}$, and demanding that the asymptotic *out* state has a well-defined inner product with the *in* state requires

$$\Phi^\dagger(z,\bar{z}) = \bar{z}^{-2h}z^{-2\tilde{h}}\Phi(1/\bar{z}, 1/z) \tag{2.2.13}$$

which translates to the condition on the mode operators

$$(\Phi^\mu_{m,n})^\dagger = \Phi^\mu_{-m,-n}, \tag{2.2.14}$$

and hence the negative modes are considered to be creation operators.

Note that in the variable z, we have a Laurent expansion, and so we can perform a contour integral to invert it:

$$\Phi_{m,n} = \frac{1}{2\pi i}\oint \mathrm{d}z z^{m+h-1}\frac{1}{2\pi i}\oint \mathrm{d}\bar{z}\bar{z}^{m+\tilde{h}-1}\Phi(z,\bar{z}). \tag{2.2.15}$$

So we have

$$\alpha^\mu_{-m}|0\rangle_{in} \leftrightarrow \left(\frac{2}{\alpha'}\right)^{1/2}\frac{i}{(m-1)!}\partial^m X^\mu(0), \qquad m > 0. \tag{2.2.16}$$

This is the *State-Operator Mapping.* For the bosonic fields we find that there is no additional operator associated with the vacuum - it is represented by the unit operator. However, in general we find that we need an additional operator for the vacuum.

2.2.3 The String Spectrum

In the operator formalism, we have the concept of *conformal normal ordering*, but in the canonical quantisation formalism, we also have the concept of normal ordering: the annihilation operators are placed to the right of the creation operators. It turns out that these are related. If we consider the mode expansion of the stress-energy tensor,

$$T^B(z) = \sum_{m=-\infty}^{\infty} z^{-m-2} L_m \tag{2.2.17}$$

(the L_m are called the *Virasoro* generators), then from equations (2.2.6) and (2.2.8) we can write the L_m in terms of normal ordered pairs of mode operators, except with a constant associated with the zero mode. We find

$$\begin{aligned}
L_m^X &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{\mu\, n} \\
L_m^\psi &= \frac{1}{4}\sum_{k=-\infty}^{\infty} (2k+2v-m)\psi^\mu_{m-k-v}\psi_{\mu\, k+v} + \frac{D}{16}(1-4v^2)\delta_m \\
L_m^g &= \sum_{n=-\infty}^{\infty} (m+n) b_{m-n} c_n - \delta_m \\
L_m^{sg} &= \frac{1}{2}\sum_{k=-\infty}^{\infty} (m+k+v)\beta_{m-k-v}\gamma_{k+v} + \delta_m \frac{1}{8}(1+2v)(3-2v)
\end{aligned} \tag{2.2.18}$$

where the above are assumed to be creation-annihilation ordered. These Virasoro generators all separately generate the Virasoro algebra, which has a central charge that cancels when all sectors of the theory are summed. The L_0 operator is the Hamiltonian of the system, and all physical states must satisfy the constraint

$$L_0^{Tot}|Phys\rangle = 0. \tag{2.2.19}$$

In the *Old Covariant Quantisation* scheme, this arises by requiring that the equation of motion $T_{ab}^{Tot} = 0$ holds as an operator equation; in BRST quantisation, it arises from insisting that physical states are annihilated by the BRST operator constructed from gauge-fixing the metric, and the additional requirement that they are annihilated by the zero modes of the b-ghosts. This requirement determines the string spectrum, since $\alpha_0^\mu = (2\alpha')^{1/2}k^\mu$ (where k^μ is the centre of mass momentum) we have an equation relating the rest mass of the state to the number of excitations.

Notably, we find that the NS vacuum state does not have zero rest mass:

$$
\begin{aligned}
(2\alpha' k^2 - 1/2)|0\rangle &= 0, \qquad NS \\
2\alpha' k^2|0\rangle &= 0, \qquad R,
\end{aligned}
\tag{2.2.20}
$$

and thus we find that the lowest NS state is a tachyon. Another feature is that if we excite the vacuum with the operator ψ_0^μ in the R sector, we do not change the rest mass. This second property actually tells us that the Ramond states are fermionic: the ψ_0^μ operators take on the role of gamma-matrices, and form a representation of the Clifford algebra in 10 dimensions. It is the **32** representation, which admits a decomposition by chirality into $\mathbf{16} + \mathbf{16}'$. We can choose a helicity basis by writing

$$
\begin{aligned}
\Gamma^{0\pm} &= \frac{1}{\sqrt{2}}(\psi_0^1 \pm \psi_0^0) \\
\Gamma^{a\pm} &= \frac{1}{\sqrt{2}}(\psi_0^{2a} \pm i\psi_0^{2a+1}), \qquad a = 1...4
\end{aligned}
\tag{2.2.21}
$$

and defining a state $|\chi_0\rangle$ such that $\Gamma^{n-}|\chi_0\rangle = 0$, so that we can build all other states as

$$
|s_0, s_1, s_2, s_3, s_4\rangle = \left(\prod_{n=4}^{0}(\Gamma^{n+})^{s_n+1/2}\right)|\chi_0\rangle, \tag{2.2.22}
$$

where $s_n = \pm 1/2$. Similarly we can complexify the fermion operators $\{\psi^\mu\} \rightarrow \{\Psi^n, \bar{\Psi}^n\}$.

The above represent basis spinors; a physical state requires a wavefunction built using these, so we can write, for example, $u^\alpha S_\alpha$ for a left-handed fermion where S_α consists only of left-handed basis spinors. Similarly, the massless NS states are given by $\psi_{1/2}^\mu|0\rangle, \mu = 0..9$, and so we write a wavefunction as $e_\mu \psi_{1/2}^\mu|0\rangle$. The equations of motion for the wavefunctions are determined by conditions similar to the Virasoro constraints above: they are derived from the physical state conditions arising from the superconformal current. Writing

$$
T_F(z) = \sum_{k=-\infty}^{\infty} G_{k+v} z^{-k-v-3/2} \tag{2.2.23}
$$

we impose the constraints

$$
G_r^{Tot}|Phys\rangle = 0, \qquad r \geq 0. \tag{2.2.24}
$$

This gives the Dirac equation $k \cdot \Gamma u = 0$ in the Ramond sector, and the transversality condition $k \cdot e$ of a gauge boson in the Neveu-Schwarz.

2.2.4 Bosonisation, Fermions and the GSO Projection

Through an interesting property of conformal field theory, it is actually possible to represent the fermionic operators in terms of bosonic ones. By considering a bosonic field $H(z)$, with OPE

$$H(z)H(w) \sim -\log(z-w) \tag{2.2.25}$$

we can write

$$\begin{aligned} \Psi^n(z) &\simeq \; :e^{iH_n(z)}: \\ \bar{\Psi}^n(z) &\simeq \; :e^{-iH_n(z)}: \\ :\Psi^n\bar{\Psi}^n(z): &\simeq i\partial H_n(z) \\ T_B^\psi &\simeq \frac{1}{2}:\partial H_n \partial H_n(z): \end{aligned} \tag{2.2.26}$$

since the OPEs are the same. This is *bosonisation*, and we can find a similar relationship for the superconformal ghosts that appear in amplitudes using a field $\phi(z)$ with "wrong" sign OPE $\phi(z)\phi(w) \sim \log(z-w)$. If we examine equation (2.2.18), we find that the correct weight for the vacuum state is only possible if we include an operator $e^{l\phi}$, with $l = -1/2$ for the Ramond ground state, and $l = -1$ for the Neveu-Schwarz. l is referred to as the *ϕ-charge* of the state. In addition, from the above equations we can see that we can write the Ramond ground state under the state-operator mapping as

$$|s_0, s_1, s_2, s_3, s_4\rangle \simeq e^{-\phi/2} \exp\left[i\sum_{n=0}^{4} s_i H_i\right]. \tag{2.2.27}$$

The coefficients s_n now become denoted *H-charge.* For general (non ground state, massive) Ramond operators, we will vary the s_n by integer coefficients by including various Ψ^n or $\bar{\Psi}^n$.

We previously noted that the theory has a tachyon, and we have massless fermions of both chiralities (yet we would like a chiral theory). Both problems can be solved at once, by the *GSO projection.* This involves defining an operator F to represent the world-sheet spinor number, and projecting out states according to their eigenvalue under $\exp(\pi i F)$. A formal definition of F can be made in terms of the spinor number currents $:\Psi_n\bar{\Psi}^n(z):$ and $:\beta\gamma(z):$ and requiring mutual locality (integer-moded OPE) with the BRST operator, but we shall simply note that for our purposes F is given by

$$F \equiv l + \sum_{n=0}^{4} s_n. \tag{2.2.28}$$

Since the NS vacuum is tachyonic, we wish to exclude it, and so we insist that $\exp(\pi i F)_{NS} = 1$ for a consistent theory. However, in the Ramond sector, we can choose either value. Indeed, we note that the action on the massless states is actually a chirality projection, and so we obtain a chiral theory with $N = 1$ supersymmetry in 10 dimensions (it is possible to obtain the spacetime supercharges in terms of the worldsheet quantities to demonstrate that the fermions and bosons are indeed superpartners). For closed strings, we can even choose different projections for holomorphic and antiholomorphic modes, and indeed type IIB string theory has $\exp(\pi i F) = 1$ for all sectors, while IIA has $\exp(\pi i F) = -1$ only for the antiholomorphic Ramond sector. The choice of GSO projection also distinguishes between type I and type I' open string theories.

2.2.5 Partition Functions

Having established the string spectrum, we can calculate a trace over it weighted by the Hamiltonian called the partition function. In statistical mechanics the partition function determines many of the thermodynamic properties of a system; in string theory it plays a similarly central role in determining many of the properties of the theory at one loop, even before we consider adding external states. For open strings, it is defined as

$$Z = \int_0^\infty \frac{\mathrm{d}t}{2t} \langle \mathbb{A}_{Vac} \rangle_1, \tag{2.2.29}$$

where the subscript 1 denotes the one-loop worldsheet, in this case the annulus, which has one modulus, t. $\mathbb{A}_{Vac}$ is the vacuum operator: in the NS sector it is merely $b_0 c_0$ (zero modes of the ghosts), while in the R sector it must include a spin field. To calculate the above, we could perform a path integral; however, the result is also given by the Coleman-Weinberg formula

$$Z = \int_0^\infty \frac{\mathrm{d}t}{2t} \mathrm{tr}\Big((-1)^{\boldsymbol{F}} \exp[-2\pi t L_0] \Big) , \tag{2.2.30}$$

where $\boldsymbol{F}$ is the *space-time* fermion number. Since we have separate NS and R sectors to consider, we might expect two partition functions; however, since we must impose the GSO projection it is more convenient to separate into four as follows:

$$\begin{aligned}
Z_1(t) &\equiv \mathrm{tr}_R((-1)^{\boldsymbol{F}}\exp[-2\pi t L_0]) \\
Z_2(t) &\equiv \mathrm{tr}_R(\exp[-2\pi t L_0]) \\
Z_3(t) &\equiv \mathrm{tr}_{NS}(\exp[-2\pi t L_0]) \\
Z_4(t) &\equiv \mathrm{tr}_{NS}((-1)^F\exp[-2\pi t L_0]).
\end{aligned} \tag{2.2.31}$$

where F is the *world-sheet* fermion number as before, and then write

$$Z = \int_0^\infty \frac{\mathrm{d}t}{2t}\sum_\nu \delta_\nu Z_\nu(t), \tag{2.2.32}$$

with phases $\delta_\nu = \{1,-1,1,-1\}$. Finally, it is worth noting that the annulus affords dual interpretations: either as open strings with the endpoints having specified boundary conditions, or as closed strings propagating as a cylinder. In the latter case, it is convenient to work in the variable $l = 1/t$ where high energy open strings are interpreted as low energy closed strings propagating for long distances.

2.2.6 The String S-Matrix

As previously mentioned, expectation values of groups of operators are calculated from a gauge-fixed path integral, and a single-string initial state on a tree-level world-sheet can be mapped to an operator at the origin. However, we wish to calculate amplitudes with asymptotic states including several strings. This is done by sewing tree-level worldsheets containing the separate string states onto the arbitrary-loop worldsheet. Since tree-level worldsheets have no moduli, and each contains only an operator at its origin, its size is irrelevant and we can shrink it to a point. However, for each worldsheet we sew in, we add a boundary for an open string, or a loop for a closed string, albeit at infinity and of vanishing size. These change the Euler characteristic of the surface, so although we do not increase the number of moduli we must add a factor of e^{Φ_0} for an open string, and $e^{2\Phi_0}$ for a closed string; these play the role of the coupling, thus we label $g_s \equiv e^{2\Phi_0}$, and g_o for the open string factor. The only remaining degree of freedom is to integrate the operators over the worldsheet boundaries (for open strings) or area (for closed strings), which we must do to ensure consistency, so finally expectation values that we calculate consist of operators from the state-operator mapping with an appropriate coupling factor each integrated over the worldsheet: the complete operator for a physical state in the string S-matrix is known as a *vertex operator*.

2.2.7 Pictures

Although we integrate vertex operators over the worldsheet, due to the gauge-fixing we must fix one coordinate and include a c-ghost for each CKV. Once we have worldsheet supersymmetry we promote the Riemann surfaces in the integrals to Super-Riemann surfaces, and thus must add superconformal ghost operators. The number of superghost insertions required depends upon the genus just as the number of c-insertions does; recalling that the superghost insertions are given by the $e^{l\phi}$ terms, we require a total ϕ-charge of $2g-2$, applying separately to holomorphic and antiholomorphic sectors for closed strings. Hence, if we want to have an arbitrary number of vertex operators, we must insert additional operators to give the correct total ϕ-charge; these are *Picture Changing Operators* (PCOs). The vertex operators from the state-operator mapping are regarded as having the superpartners of the coordinates fixed, while the picture-changed operators are equivalent to having unfixed coordinates. The picture-changed operator is given (for the purposes of this thesis) by

$$V^{l+1}(w) = \lim_{z \to 0} e^{\phi(z)} T_F^{X+\psi}(z) V^l(w). \tag{2.2.33}$$

Note that we label vertex operators by their ϕ-charge.

2.3 D-Branes and Orientifolds

2.3.1 R-R Charges

As described in the previous section, it is consistent for open strings to have their ends fixed on a hypersurface. However, for different theories the surfaces may only have certain dimensions. A T-duality transformation, whereby a dimension is compactified on a circle whose radius is taken to zero, is a legitimate transformation of the theory. It is equivalent to reversing the sign of antiholomorphic modes on that dimension; for closed strings, the modes winding round the circle exchange roles with the propagating modes as the circle shrinks, generating an effective non-compact dimension. The effect on the $R-R$ sector is to change the chirality; it interpolates between IIA and IIB theories. The massless states in this sector are tensor products of fermions, and hence bosonic. Thus if we write the holomorphic and antiholomorphic basis spinors S_α and S'_β, the wavefunction of the total state is $u^\alpha v^\beta S_\alpha S'_\beta$, which can be decomposed as

$$u^\alpha v^\beta S_\alpha S'_\beta = \sum_{n=0}^{9} C_{\mu_1,\dots\mu_n} S_\alpha (\hat{C}\Gamma^{\mu_1\cdots\mu_n})^{\alpha\beta} S'_\beta \tag{2.3.1}$$

where $\Gamma^{\mu_1\cdots\mu_n}$ is antisymmetrised on its indices, and $\hat{C}$ is the charge conjugation operator. The functions $C_n \equiv C_{\mu_1,\dots\mu_n}$ are antisymmetric forms (with C_0 a scalar). The equations of motion for them can be derived from constraints on the spectrum. Since the basis spinors have definite chirality, we find that different dimensions of forms are allowed for the different type-II string theories:

$$\begin{aligned} IIA: &\quad C_1, C_3, C_5, C_7, C_9 \\ IIB: &\quad C_0, C_2, C_4, C_6, C_8, C_{10}. \end{aligned} \tag{2.3.2}$$

T-duality changes between the above forms. We expect them to couple to objects in the effective action which carry $R-R$ charge, and indeed it can be shown that they are D-branes and orientifold planes. D-branes are the hypersurfaces to which open strings may fix their end-points. For open strings, which have no winding modes, the T-duality transformation exchanges Neumann and Dirichlet boundary conditions and thus changes the dimensions of any Dirichlet hypersurfaces - D-branes - present. The terms in the effective action induced by the branes are proportional to

$$\int C_{p+1} \tag{2.3.3}$$

where the volume of integration is the $(p+1)$-dimensional volume of the Dp-brane, p specifying the number of spatial dimensions. The coupling of the forms to Dp-branes tells us that only certain dimensions of branes are allowed:

$$\begin{aligned} IIA: \quad p = & \quad 0,2,4,6,8 \\ IIB: \quad p = & \quad -1,1,3,5,7,9 \end{aligned} \tag{2.3.4}$$

The $D(-1)$-brane is a special case, with even Dirichlet boundary conditions on the time direction, and this corresponds to an instanton. In fact, however, if we choose Dirichlet boundary conditions along the time direction for a brane with Neumann conditions along $p+1$ spatial directions, we have an extended instantonic object denoted an Ep-brane; we shall discuss these further in section 2.4.5 and chapter 4.

Note that the term in the effective action is a tadpole, which for compact dimensions will give rise to infinities in scattering amplitudes if it is not cancelled. This can be done using anti-D-branes, but to preserve supersymmetry we require Orientifold planes. Orientifold planes are a special object which carries negative R-R charge, and are defined as a hypersurface invariant under a reflection and worldsheet orientation reversal:

$$X^\mu(z,\bar{z}) \to R^\mu_\nu X^\nu(\bar{z},z) \tag{2.3.5}$$

where $\mathrm{Det}(R) = -1$. Orientifold planes arise from T-duality of unoriented strings, and their properties can be deduced in that way.

2.3.2 Gauge Theory: Chan-Paton Factors

For open strings with both ends attached to a D-brane, the massless states in the NS sector form a vector $\psi^\mu_{1/2}|0\rangle, \mu = 0..p$, which is interpreted as a gauge boson. As remarked in the previous section, its wavefunction has gauge degrees of freedom. For just one brane it is a $U(1)$ field. If, however, we introduce a stack of branes occupying the same hypersurface, we can attach the ends to different branes, and supply the additional degrees of freedom necessary; for N coincident branes, we have a $U(N)$ gauge boson. We can write the vertex operator with a *Chan-Paton Factor* λ, an $N \times N$ matrix representing the end-point states:

$$V^{-1}(\lambda, A_\mu, k_\mu, z) = \lambda A_\mu \psi^\mu e^{ik\cdot X}. \tag{2.3.6}$$

If we consider amplitudes involving strings with these factors, along each boundary we must have continuous Chan-Paton index, and thus the amplitude will include a factor of $tr(\lambda^1...\lambda^n)$ for n vertex operators on a given boundary. Hence amplitudes are invariant under $\lambda \to U\lambda U^\dagger$ for unitary U, and so we have operators in the adjoint representation.

When we have orientifolds (and orbifolds) we may include a projection on the Chan-Paton matrices in addition to the worldsheet and spacetime generators. These are manifest through a matrix γ_O for an operator O, and we must then insert them into amplitudes; a boundary is considered closed up to projection by γ_O, and thus the trace in the amplitudes may include them: $tr(\gamma_O\lambda^1...\lambda^n)$.

Note finally that we must also include Chan-Paton factors for the gauginos, and states stretched between stacks of branes, which will have the different ends transforming under different gauge groups.

2.4 Intersecting Brane Worlds

2.4.1 Introduction

Intersecting Brane Worlds are a class of toy models in type IIA string theory which have proven popular due to their ability to reproduce many features of the standard model while being simple compared to many other constructions. Recent reviews are given in [3,4], and some examples of model building are given in [5–14]. They consist of D6 branes wrapping $\mathbb{R}_4 \times \mathbb{T}_2 \times \mathbb{T}_2 \times \mathbb{T}_2$ in type IIA string theory, where the properties of the low energy theory are determined by the wrapping cycles of the branes on the compact six-dimensional manifold; as in other formulations of string theory, the parameters of the field theory have geometrical interpretations, but in this case they are particularly simple and many quantities can be calculated exactly.

In these models, each $\mathbb{T}_2$ is a manifold parametrised by two quantities: the complex and Kähler structures. In complex coordinates they are Riemann surfaces identified under $X^\kappa \sim X^\kappa + \sqrt{2}\pi i R_1^\kappa e^{-i\alpha^\kappa}$, $X^\kappa \sim X^\kappa + \sqrt{2}\pi i R_2^\kappa$, where R_1^κ and R_2^κ are radii and α^κ is a tilt angle, which is restricted in orientifold models to the discrete values $\pi/2$ ("untilted") or $\frac{R_1^\kappa}{R_2^\kappa}\cos\alpha^\kappa = 1/2$ ("tilted"). In terms of these quantities we can write the complex structure of the tori:

$$U^\kappa = \frac{R_2^\kappa}{R_1^\kappa} e^{i\alpha^\kappa}, \tag{2.4.1}$$

and Kähler modulus

$$T^\kappa = iR_1^\kappa R_2^\kappa \sin \alpha^\kappa = iT_2^\kappa. \tag{2.4.2}$$

In general there are restrictions placed upon the complex structure by the requirement of supersymmetry, but the Kähler structure is unrestricted and must be stabilised to eliminated the associated massless fields: however, we shall not treat this here, although it is a topic of ongoing interest in the literature.

The identifications above are associated with homology cycles $[\tilde{A}_i]$ and $[\tilde{B}_i]$ respectively. We can then write the homology cycle of brane a as

$$\Pi_a = (n_a^1[\tilde{A}_1] + m_a^1[\tilde{B}_1]) \times (n_a^2[\tilde{A}_2] + m_a^2[\tilde{B}_2]) \times (n_a^3[\tilde{A}_3] + m_a^3[\tilde{B}_3]). \tag{2.4.3}$$

Then n_a^i and m_a^i are the number of times the brane wraps cycles closed under the first and second of those identifications respectively, and together with a position coordinate contain all of the information about the wrapping of the brane. The volume of the brane can be conveniently written as $L_a = \prod_{\kappa=1}^3 L_a^\kappa$, with

$$L_a^\kappa = 2\pi\sqrt{(n_a^\kappa R_1^\kappa)^2 + (m_a^\kappa R_2^\kappa)^2 + 2n_a^\kappa m_a^\kappa R_1^\kappa R_2^\kappa \cos\alpha^\kappa}. \tag{2.4.4}$$

However, it is often more convenient to work in terms of the canonical $[A_i]$ and $[B_i]$ basis that represent the paths

$$\begin{aligned}
[A_i] &= [X, X + \sqrt{2}\pi R_1 \sin\alpha^i] \\
[B_i] &= [X, X + \sqrt{2}\pi i R_2]
\end{aligned} \tag{2.4.5}$$

and thus satisfy $[A_i] \cdot [B_j] = \delta_{ij}, [B_i] \cdot [A_j] = -\delta_{ij}, [A_i] \cdot [A_j] = 0, [B_i] \cdot [B_j] = 0$. We then have

$$\begin{aligned}
[\tilde{A}_i] &= [A_i] + [B_i]\frac{R_1}{R_2}\cos\alpha^i \\
[\tilde{B}_i] &= [B_i].
\end{aligned} \tag{2.4.6}$$

We can then define $\tilde{m}_a^i = m_a^i + n_a^i \frac{R_1}{R_2}\cos\alpha^i$ to write the three-cycle wrapped by a D6-brane a as

$$\Pi_a = \prod_{i=1}^{3}(n_a^i[A_i] + \tilde{m}_a^i[B_i]). \tag{2.4.7}$$

2.4.2 Mode Expansions and Quantisation

It is simple to quantise the string in such intersecting brane worlds, and thus obtain the vertex operators. We refer the reader to [15, 16]. We first consider two non-compact complex dimensions X_1 and X_2 with two branes meeting at an angle $\pi\theta^1$, and the infinite strip worldsheet with coordinate w, $0 \leq \Re(w) \leq 1/2, -\infty < \Im(w) < \infty$. The string mode expansion is just a free wave with different Dirichlet boundary conditions at the ends. Using complex dimensions $X = \frac{1}{\sqrt{2}}(X_1 + iX_2)$, we can write the mode expansion

$$\begin{aligned} \partial X(w) &= \sum_k \alpha_{k-\theta} e^{-i\pi w(-k+\theta-1)}, \qquad & \bar{\partial} X(\bar{w}) &= \textstyle\sum_k \tilde{\alpha}_{k-\theta} e^{i\pi\bar{w}(-k+\theta-1)} \\ \partial \bar{X}(w) &= \sum_k \bar{\alpha}_{k+\theta} e^{-i\pi w(-k-\theta-1)}, \qquad & \bar{\partial} \bar{X}(\bar{w}) &= \textstyle\sum_k \tilde{\bar{\alpha}}_{k+\theta} e^{i\pi\bar{w}(-k-\theta-1)} \ . \end{aligned} \tag{2.4.8}$$

We then combine the fields using the "doubling trick", giving a theory on the infinite cylinder:

$$\partial X(w) = \begin{cases} \partial X(w) & \Re(w) \geq 0 \\ -\bar{\partial}\bar{X}(\bar{w}) & \Re(w) < 0 \end{cases}, \tag{2.4.9}$$

and similarly for $\partial\bar{X}(w)$. The boundary conditions thus become a quasi-periodicity on the cylinder:

$$\begin{aligned} \partial X(w-1) &= e^{2\pi i\theta} \partial X(w) \\ \partial \bar{X}(w-1) &= e^{-2\pi i\theta} \partial \bar{X}(w) \end{aligned} \tag{2.4.10}$$

which is identical to the conditions for the holomorphic sector of closed strings on an orbifold - except with a continuous angle. This correspondence allows much of the technology to be adapted from orbifold calculations to intersecting brane calculations. Since the correspondence is to half the amplitude, it may appear that the calculations are simpler - however, this turns out not to be the case, as complications arise due to the non-rationality of the angle, the brane wrapping in compact spaces, and for loop diagrams the amplitudes are no longer modular forms.

[1]Throughout, in common with the literature, we shall use labels corresponding to angles with π factored out, and often refer to the label as an angle.

The above boundary conditions and mode expansions are valid for an *anti-clockwise* rotation angle θ from the brane on the boundary $\Re(w) = 0$ to that at $\Re(w) = 1/2$. However, we could have chosen a *clockwise* rotation, in which case we would replace $\theta \to 1 - \theta$. A similar substitution will occur if we swap the branes at each boundary. We shall persist with our choice in this section, but the reader should note that in appendix (A.3) in the interests of generality no choice is specified.

Having quantised the string with these boundary conditions, we find that the vacuum (which is annihilated by all positive frequency modes) does not correspond to the identity operator in the state-operator mapping. Indeed, in sowing the above worldsheet into a scattering amplitude, the emission or absorption of a string with ends on different branes necessarily changes the boundary conditions on the scattering worldsheet: we need to insert a boundary changing operator $\sigma_\theta(w)$. We determine its OPEs to be

$$\begin{aligned} \partial X(z)\sigma_\theta(w) &\sim (z-w)^{\theta-1}\tau_\theta(w) \\ \partial \bar{X}(z)\sigma_\theta(w) &\sim (z-w)^{-\theta}\tau'_\theta(w) \end{aligned} \tag{2.4.11}$$

where $\tau(w)$ and $\tau'(w)$ are "excited twist fields". The conformal weight is $h_{\sigma_\theta} = \frac{1}{2}\theta(1-\theta)$.

Having discussed the bosonic contribution, for the superstring we must consider the worldsheet fermions. Worldsheet supersymmetry determines that the periodicity of the fermions on the infinite cylinder must be complementary to the bosons, so for (post-doubling-trick) complex fermions we have

$$\begin{aligned} \Psi(w-1) &= e^{2\pi i v}e^{-2\pi i\theta}\Psi(w) \\ \bar{\Psi}(w-1) &= e^{-2\pi i v}e^{2\pi i\theta}\bar{\Psi}(w). \end{aligned} \tag{2.4.12}$$

Thus the mode expansions are simply shifted by an amount θ. To obtain vertex operators for the fermions, we can bosonise as usual, giving $\Psi(w) \sim e^{iH(w)}$ and $\bar{\Psi}(w) \sim e^{-iH(w)}$, and the requirement that the vacuum is annihilated by positive frequency modes determines the vacuum operator to be

$$A_v = e^{i(\theta+v-1/2)H(w)} \tag{2.4.13}$$

with conformal weight $\frac{1}{2}(\theta + v - 1/2)^2$. Notice that the degeneracy of the Ramond sector is broken: spinors will only have components in the parallel dimensions.

2.4.3 Phenomenology

We may now consider the spectrum of strings in intersecting brane worlds: we include three compact complex dimensions and consider intersections in each. For a given pair of branes, we label the intersection angles in each dimension θ_κ, $\kappa = 1, 2, 3$. Including the contributions from ghosts, we find that the Ramond ground state is still massless, but the energy of the Neveu-Schwarz ground state is determined by the intersection angles. Thus, they determine the amount of supersymmetry preserved by the branes. To preserve some supersymmetry they must be related by an $SU(3)$ rotation - (in the same way that the space group of an orbifold must be a discrete subgroup of $SU(3)$ to preserve some supersymmetry) - for parallel branes, there will be an $N = 4$ supermultiplet, while for one angle zero and the other two complementary there will be an $N = 2$ hypermultiplet, and finally for $\pi(\theta_1 + \theta_2 + \theta_3) = n\pi$ there will be an $N = 1$ chiral multiplet. These conditions are satisfied provided that we preserve supersymmetry in the closed string sector: the D-branes must wrap *special lagrangian* (sLag) cycles of the manifold. This is achieved by ensuring that the angles made by a brane relative to the orientifold plane sum to zero $\mod 2\pi$. While we would ultimately like a theory with broken supersymmetry, directly breaking by adjusting the angles involves, for most models, fine tuning, and moreover will generically lead to NS sector tadpoles signalling that the true vacuum is actually a supersymmetric state. Hence we require a different method of supersymmetry breaking, or at least a way to stabilise the branes at their non-supersymmetric angles; we shall thus consider only supersymmetric configurations.

As previously mentioned, each stack of coincident branes has an associated gauge group dependent on the number of branes in the stack. This leads to gauge groups of the form $U(N)$. However, since $U(N) = SU(N) \times U(1)$, the $U(1)$ factors are anomalous, and acquire a string-scale mass, except for certain linear combinations (see for example [17]). The nett result is that the hypercharge is a composite of the $U(1)$ fields from several stacks of branes.

Family replication occurs naturally in these setups: for a given non-parallel pair of branes on a compact manifold, they will generically intersect more than once as they both wrap around their homology cycles. For the toroidal background, we can write the number of intersections between two branes a and b as

$$I_{ab} = \prod_{\kappa=1}^{3} I_{ab}^{\kappa} = \prod_{\kappa=1}^{3} (n_a^{\kappa}\tilde{m}_a^{\kappa} - \tilde{m}_a^{\kappa} n_b^{\kappa}) \tag{2.4.14}$$

This then determines the number of generations for a given pair of gauge groups.

The Higgs fields are localised at the intersection between a stack of branes corresponding to the $SU(2)$ and one (or several) corresponding to the $U(1)_Y$. The potential can be realised by giving the scalar a tachyonic mass [18,19], by adjusting the intersection angles (the configuration will then condense by brane recombination, corresponding to the Higgs acquiring a vacuum expectation value). The tree-level Yukawa couplings for these models are determined then by the interaction of this field and fields at other intersections. The amplitude is dominated by worldsheet instantons - i.e. the classical action corresponding to stretching the worldsheet between the three intersections as a conformal map, which corresponds to the area of the triangle formed between them (and all possible wrappings around the tori). These were calculated in [15,19,20], and the results are also summarised in appendix A.1. Thus these models provide a simple geometric interpretation for the Yukawa couplings, which aids model building; the coupling is proportional to $e^{-Area/2\pi\alpha'}$, where $Area$ is the area of the triangle formed in each sub-torus between the intersections of the branes associated to the three fields coupled.

It would superficially appear that such a framework easily explains the Yukawa coupling structure: consider three stacks of branes a, b and c, with three branes in stack a, two in b and one in c. Then the H_u field is localised at the intersection between b and c; Q_L^i is between a and b; and $(U_R^{\alpha})^c$ is between a and c (α is a CP index counting the generation, and the superscript c denotes the conjugate field). We then have $I_{ab} = I_{ac} = 3$, $I_{bc} = 1$ to provide the correct number of generations, and the superpotential term $Y_{\alpha\beta}Q_L^{\alpha}H_u(U_R^{\beta})^c$ is suppressed by the exponential of the areas of triangles between these. However, as established in [19], since we have a "left brane" and a "right brane", we have a product form of the Yukawa coupling: due to model-building constraints, we have $I_{ab}^k = 3, I_{ab}^l = 1, I_{ab}^m = 1$ and $I_{ac}^k = 1, I_{ac}^l = 3, I_{ac}^m = 1$, for $k \neq l \neq m$, and thus the exponential factors arise in different tori, giving

$$Y_{\alpha\beta} \propto A_\alpha B_\beta \tag{2.4.15}$$

for vectors A_α, B_β. This matrix has rank one, and therefore only allows one non-zero eigenvalue; this is a problem for model-building in toroidal intersecting brane models, and we shall present a possible solution thereof in chapter 4.

2.4.4 Tadpole Cancellation

As remarked in the previous section, the R-R charges of the branes must be cancelled for the theory to be consistent (there are other constraints, but R-R charge cancellation is the most relevant for this thesis). The condition for this is a simple geometric one: writing μ_6 for the charge of a $D6$-brane; κ^2 for the 10-dimensional gravitational coupling; N_a for the number of branes a in a stack on homology cycle Π_a on the compact dimensions; and a' denoting the images of brane a under any orbifold, the relevant terms in the low energy effective action are

$$\begin{aligned} S \supset & -\frac{1}{4\kappa^2}\int_{\mathbb{R}^4\times T^6} dC_7 \wedge \star dC_7 \\ & + \mu_6 \left(\sum_a N_a \int_{\mathbb{R}^4\times\Pi_a} C_7 + \sum_a N_{a'} \int_{\mathbb{R}^4\times\Pi_{a'}} C_7 - 4\int_{\mathbb{R}^4\times\Pi_{O6}} C_7 \right). \end{aligned} \tag{2.4.16}$$

The equations of motion are

$$\frac{1}{\kappa^2} d\star(dC_7) = \mu_6 \left(-4\delta(\Pi_{O6}) + \sum_a N_a(\delta(\Pi_a) + \delta(\Pi_{a'})) \right). \tag{2.4.17}$$

Since the left hand side of the above is exact, it integrates to zero, and the tadpoles are thus cancelled if the overall three-cycle wrapped by the branes and orientifold planes is trivial in homology:

$$\sum_a N_a(\Pi_a + \Pi_{a'}) - 4\Pi_{O6} = 0. \tag{2.4.18}$$

It is worth noting that homology does not completely describe the R-R charges of the branes; since the branes carry gauge fields on them, there are additional K-theory constraints. For the cases of interest, these are not as restrictive as the above equations, and we refer the reader to [21–23].

2.4.5 Instanton Calculus

This subsection reviews some of the framework outlined in [24] and the references therein, the most relevant of which are [25, 26].

The non-renormalisation theorem ensures that for $N = 1$ supersymmetric intersecting brane worlds, the low energy field theory will have a perturbative superpotential that receives no corrections beyond tree level. However, there may be nonperturbatively generated terms which receive corrections up to one loop. In the field theory these are produced by instantons; in string theory, instantons are manifest in the form of the Euclidean branes mentioned earlier, which have Dirichlet boundary conditions along the time direction. For finite energy instantons (that we expect should contribute to physical processes) any Neumann directions must be a subset of those in the compact space; the Ep brane appears as a point in four-dimensional Minkowski space. The way that the Ep brane is embedded in the compact manifold then determines the properties of the instanton.

It is expected that Ep branes contribute to type IIA and IIB models; however, for the specific case of type IIA intersecting D6 brane models with six compact dimensions, we can only have contributions from E0, E2 and E4 instantons, and of these the most relevant are the E2 branes, since the toroidal manifold has no continuous one or five cycles. It is possible that E4 instantons with fluxes are relevant, but we shall not consider them in this work. Moreover, we shall only consider E2 instantons wrapping special lagrangian cycles of the compact manifold, as we expect only these to contribute to the superpotential

Strings attached to D-branes can be interpreted as fluctuations of the branes, and massless strings as moduli. For example, gauge multiplets (with both ends attached to the same brane) may be interpreted as moduli parametrising the deformation and translation of the brane. For E2 branes, the attached strings cannot carry momentum, and so it is not meaningful to discuss massless modes. However, the moduli of the instanton consist of modes which satisfy the physical state condition, equation (2.2.19). These may either have both ends attached to the E2 brane, or an end on the E2 brane and one on a D6 brane of the model. These are known as charged and uncharged zero modes respectively (reflecting their charge under the gauge groups). In string computations, only the charged zero mode vertex operators may carry a factor of the string coupling (equal to $\sqrt{g_s}$, g_s being the closed string coupling) while the uncharged zero modes and chiral superfields may not.

In this work only E2 brane configurations for which there are no bosonic zero modes are relevant; they would cause the amplitude to vanish in the cases of interest to us, but in cases that they do not the formalism is treated in [27], which appeared after the work of chapter 4 and [28]. For the fermionic zero modes, there are always uncharged present; the presence of charged zero modes depends on whether the E2 brane intersects some of the D6 branes of the model. For left-handed uncharged fermionic zero modes, there are two types: λ_a^i and $\bar{\lambda}_a^i$, determined by the sign of the intersection number $I_{E2,a}$ with positive for λ_a^i. This determines the angle present in the vertex operator (whether σ_ϕ or $\sigma_{1-\phi}$). The superscript index labels the charge under the gauge group: there is a different zero mode for each brane in the stack a.

The superpotential is holomorphic in the chiral superfields, and thus if we construct a string amplitude holomorphic in the chiral superfields, we will be guaranteed to calculate a contribution to only the superpotential in the effective lagrangian. If we perform such a calculation, it will in general depend upon the zero modes of the instanton, and so for our physical superpotential we must integrate these out. In a field theory calculation, the integration over zero modes is accompanied by a measure determined from the field configurations; in a string theory calculation, the measure actually appears when we include all possible contributing string diagrams. When a euclidean brane is present, we must include in an amplitude every possible worldsheet that contains a boundary on the E2 brane: we must sum an infinite number of disconnected worldsheets.

Another consequence of holomorphy is that the superpotential may only depend upon the string coupling through a term of the form $O(e^{-8\pi^2/g_{E2}^2})$, where g_{E2} is an effective coupling of order $\sqrt{g_s}$. Since disk worldsheets have an attached factor of g_s^{-1}, then the origin of this in the light of the above discussion is clear: it arises from including the sum of products of any number of disk diagrams with no vertex operators inserted, divided by a combinatoric factor. Moreover, since one loop partition functions contain no coupling factor, we must include every such worldsheet with one boundary on the E2 brane (when both boundaries lie on the E2 brane, the contribution vanishes by supersymmetry). However, we must also explicitly remove the zero mode contribution to the partition function, since they are to be integrated separately (or equivalently, we could state that they have factors of the string coupling present that render them inadmissible). We shall elucidate the calculation of these in section 4.2.2, but note that the total contribution to the measure is

$$\exp\left[Z'^M(E2,O6)+\sum_a Z'^A(D6_a,E2)+Z'^A(D6'_a,E2)\right]\equiv\frac{\mathrm{Pfaff}'(D_F)}{\sqrt{\det'(D_B)}} \tag{2.4.19}$$

where the right hand side is the field theory interpretation in terms of pfaffians and determinants, the primes indicate omission of the zero modes, and we have an orientifold so must include image branes $D6'_a$ and the Möbius contribution between the E2 brane and its image given by $Z'^M(E2,O6)$.

Finally, then, denoting uncharged zero modes by θ_i, the contribution to a superpotential term $\prod_n \Phi_n$ will be given by

$$W_{np}\supset\int \mathrm{d}^4x\prod_i \mathrm{d}\theta_i\prod_a\prod_j \mathrm{d}\lambda_a^j\prod_k \mathrm{d}\bar{\lambda}_a^k\ \exp(-8\pi^2/g_{E2}^2)\ \frac{\mathrm{Pfaff}'(D_F)}{\sqrt{\det'(D_B)}}\ \langle\prod_n\Phi_n\rangle. \tag{2.4.20}$$

The superfields in $\langle\prod_n\Phi_n\rangle$ can be split between disk and annulus worldsheets; every possible configuration must be included. For the disks, to ensure no further g_s dependence, two charged fermionic zero modes must be included. Additionally, clearly every zero mode in the model must be saturated by one in $\langle\prod_n\Phi_n\rangle$, since they are Grassmann variables. Arranging these such that they are non-zero then proves to be quite restrictive, and will be examined further for certain models in chapter 4.

2.5 Non-Commutative Geometry

So far we have considered the background $B_{\mu\nu}=0$. However, if we now give the antisymmetric tensor a constant value, we still have a consistent background for string theory - but some interesting consequences. The matter action in conformal gauge becomes

$$S_B=S_m+\alpha'\int \mathrm{d}^2z B_{\mu\nu}\left(-\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X^\nu+\frac{1}{2}\psi^\mu\bar{\partial}\psi^\nu-\frac{1}{2}\bar{\psi}^\mu\partial\bar{\psi}^\nu\right), \tag{2.5.1}$$

where S_m is the action (2.1.7). We also make the substitution $\eta^{\mu\nu} \to g^{\mu\nu}$ in S_m, where $g^{\mu\nu}$ still contains constant components but may involve a change of scale. In the presence of a worldsheet with boundaries (i.e. we consider open strings on a brane) this changes the boundary conditions to

$$\begin{aligned} g_{\mu\nu}\partial_n X^\nu + 2\pi i\alpha' B_{\mu\nu}\partial_t X^\nu &= 0 \\ (g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})\psi^\nu \mathrm{d}z &= \pm \mathrm{d}\bar{z}(g_{\mu\nu} - 2\pi\alpha' B_{\mu\nu})\tilde{\psi}^\nu, \end{aligned} \tag{2.5.2}$$

where ∂_n and ∂_t are derivatives normal and tangential to a boundary respectively, and the above are understood to be evaluated only at boundaries; the differentials $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ are along the boundary. Note that it is also possible to add boundary terms to the action (giving a vacuum expectation value to the gauge fields on the brane), the effect of which is to modify the field strength $B \to \mathcal{F} = B - \mathrm{d}A$.

The above action results in noncommutativity between the target space coordinates (and non-anticommutativity between the fermionic coordinates). To see this, if we compute the correlator of two target space fields on the disk (i.e. compute the *Greens' Function*) mapped to the upper half plane by insisting that it respects the (quantum) equation of motion and boundary conditions we obtain [29]

$$\langle X^\mu(z)X^\nu(w)\rangle = -\alpha'\left[g^{\mu\nu}\log\left|\frac{z-w}{z-\bar{w}}\right| + G^{\mu\nu}\log|z-w|^2 + \frac{\theta^{\mu\nu}}{2\pi\alpha'}\log\frac{z-\bar{w}}{\bar{z}-w}\right], \tag{2.5.3}$$

where, writing $g \equiv g_{\mu\nu}$, $B \equiv B_{\mu\nu}$,

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{g+2\pi\alpha' B}g\frac{1}{1-2\pi\alpha' B} \\ \theta^{\mu\nu} &= -(2\pi\alpha')^2\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}, \end{aligned} \tag{2.5.4}$$

and $G_{\mu\nu} = G^{-1}$. Restricting the fields to the boundary $\Im(z) = 0$, we obtain

$$\langle X^\mu(x)X^\nu(y)\rangle = -\alpha' G^{\mu\nu}\log(x-y)^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(x-y), \tag{2.5.5}$$

where

$$\epsilon(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}. \tag{2.5.6}$$

If we then reduce to a field theory by taking the limit $\alpha' \to 0$ with a transformation of the metric:

$$\begin{aligned} \alpha' &\sim \epsilon^{1/2} \to 0 \\ g &\sim \epsilon \end{aligned} \tag{2.5.7}$$

then the logarithmic term vanishes from the above correlator, but we obtain a field theory where the commutator of the spacetime coordinates is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}. \tag{2.5.8}$$

This allows us to obtain a noncommutative field theory from string theory. Note that in the above limit, gravity is decoupled. If, instead of taking the limit $\epsilon \to 0$ we merely consider it small (i.e. consider the effect of a high string scale, consistent with observation), then there will be an effect on gravity. In chapter 5 we shall investigate some of the consequences of this.

Chapter 3

Intersecting Brane Worlds at One Loop

This chapter investigates several aspects of calculations at one loop involving states at D6-brane intersections. The initial technology for such calculations was presented in [30], but here we refine and extend that to new types of string diagrams. The motivation is the discovery of a divergence in the two-point diagram; we consider the consequences of this. We also consider some consequences for phenomenology of the running of the Yukawa couplings in such models. This work is published in [31].

3.1 Introduction

As outlined in section 2.4, the perturbative superpotential for toroidal intersecting brane models is well known, since it results from a disk calculation. However, the effective field theory of the models is also specified by the Kähler potential, which is renormalised (in principle) to all orders. Currently the Kähler potential in *chiral matter* superfields is known to only quadratic terms and only at tree level [32,33] (one loop corrections to the moduli sectors of Kähler potentials in IIB models have been calculated in [34]). For a multitude of phenomenological reasons it is something we would like to understand better, especially its quantum corrections. In this chapter we go a step further in this direction with an analysis of interactions of chiral matter fields at one-loop.

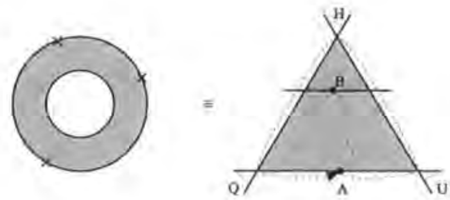

Figure 3.1: An annulus contribution to 3 point functions.

Figure 3.1 shows the physical principle of calculating a one-loop annulus for the example of a 3 point coupling, discussed in ref. [35]. Take a string stretched between two branes as shown and keep one end (B) fixed on a particular brane, whilst the opposing end (A) sweeps out a triangle (or an N-sided polygon for N-point functions). Chiral states are deposited at each vertex as the endpoint A switches from one brane to the next. As the branes are at angles and hence the open string endpoints free to move in different directions, the states have "twisted" boundary conditions, reflected in the vertex operators by the inclusion of so-called twist operators. Working out the CFT of these objects is usually the most arduous part of calculations on intersecting branes. The corresponding worldsheet diagram is then the annulus with 3 (N) vertex operators. In the presence of orientifolds there will be Möbius strip diagrams as well. There is no constraint on the relative positioning of the B brane (although the usual rule that the action goes as the square of the brane separation will continue to be obeyed), and it may be one of the other branes. (The case that brane B is at angles to the three other branes was not previously possible to calculate: we shall treat it in detail in this chapter.)

Finding the Kähler potential means extracting two point functions, but because the theory is defined on-shell we have to take an indirect approach; complete answers can be obtained only by factorising down from at least 4 point functions. To do this we first develop the general formalism for N-point functions in full. However it would be extremely tedious if we had to factorise down the full amplitude including the classical instanton action for every N-point diagram. Fortunately the OPE rules for the chiral states at intersections allow a short cut: first we use the fact that the superpotential is protected by the non-renormalisation theorem which has a stringy incarnation derived explicitly in [30]. In supersymmetric theories, the interesting diagrams are therefore the field renormalisation diagrams which we can get in the field theory limit where two vertices come together. A consistent procedure therefore is to use the OPE rules to factor pairs of *external* states onto a single state times the appropriate tree-level Yukawa coupling. In this way one extracts the *off-shell* two-point function.

This procedure allows us to consider various aspects of one-loop processes. For example, in $N = 1$ supergravity the non-renormalisation theorem does not protect the Kähler potential, and only particular forms of Kähler potential (essentially those with $\log\det(K_{ij}) = 0$ where K_{ij} is the Kähler metric) do not have quadratic divergences at one-loop order and higher (for a recent discussion see [36]). It is natural to wonder therefore how string theory ensures that such divergences are absent. Here we show that the conditions for cancellation of the divergences is identical to the Ramond-Ramond tadpole cancellation condition (essentially because in performing the calculation one factors down onto the twisted partition function). (At the level of the effective field theory there are two well known forms of tree-level Kähler metric that are consistent with the one-loop cancellation of divergences, the "Heisenberg" logarithmic form, and the "canonical" quadratic form). As a second application, we consider the subsequent determination of field renormalisation; we find agreement with the power law running that one deduces from the effective field theory. However we also see that as we approach the string scale the power law running dies away as the string theory tames the UV divergences.

3.2 One-Loop Scalar Propagator

3.2.1 Four-Point Amplitudes

String theory is defined only on-shell, so in order to calculate the energy dependence of physical couplings we must calculate physical diagrams and probe their behaviour as the external momenta are varied. To obtain the running of Yukawa couplings in intersecting brane models, we should in principle consider the full four-point amplitudes, since they are the simplest diagrams with non-trivial Mandelstam variables. However here we shall take the more efficient approach outlined in the introduction: due to the existence of a consistent off-shell extension of these amplitudes, it is possible to calculate three-point diagrams to obtain the Yukawa couplings. This was done in ref. [30] for the amplitude in certain limits (albeit with some inevitable ambiguities). There it was shown that for supersymmetric amplitudes the non-renormalisation theorem held as expected, and the low-energy behaviour is entirely dominated by the renormalisation of propagators. The latter *can* be consistently calculated in full by factorising down from the full four-point functions, and this is what we shall do here.

We shall focus on four-fermion amplitudes, since they factorise onto the lower-vertex amplitudes of interest - the Yukawa renormalisation amplitude and the two scalar amplitude. In intersecting brane models, the allowed amplitudes are constrained by the necessity of the total boundary rotation being an integer; supersymmetry then dictates the allowed chirality of the vertex operators via the GSO projection. In the case of $N = 1$ supersymmetry, the requirement of integer rotation results in two pairs of opposite chirality fermions being required. In the case of $N = 2$ or 4 supersymmetry, we are also allowed to have four fermions of the same chirality. The correlator for the non-compact dimensions in the case of $N = 1$ supersymmetry was calculated in ref. [37], in order to legitimise their calculations on orbifolds. We have checked that the $N = 2$-relevant amplitude gives the same limit upon factorisation. Thus, we can simply use their result to justify using the OPE behaviour of the vertex operators in order to obtain the low-energy limit of the four-point function, and we can proceed with the calculation of the scalar propagator to obtain the corrections to the Yukawa couplings. In the next subsection we outline some of the technology required, and in the ensuing subsections we extract the information about the running of the couplings.

3.2.2 Vertex Operators

To begin, we review the technology, assembled in ref. [30], for the calculation of loop amplitudes involving massless chiral superfields. The appropriate vertex operator to use for incoming states at an intersection depends on the angles in each subtorus. There are several possible conditions for supersymmetry (we will focus on

N=1 supersymmetric models) of which we will use the most straightforward: for *intersection* angles θ^κ (where κ runs over the complex compact dimensions 1 to 3) we have $\sum_\kappa \theta^\kappa = 1$ or 2. We have two possible conditions because each intersection supports one chiral and one antichiral superfield, with complimentary angles. With this condition the GSO projection correlates the chirality of the fermions with that of the rotation, and we obtain vertex operators as in [15]:

$$\begin{aligned} V_{-1}^{(ab)}(k,\theta^\kappa,z) &= \lambda^{(ab)} e^{-\phi} e^{ik\cdot X} \prod_\kappa e^{i\theta^\kappa H^\kappa} \sigma_{\theta^\kappa}^{(ab)}(z) \\ V_{-1/2}^{(ab)}(u,k,\theta^\kappa,z) &= \lambda^{(ab)} e^{-\frac{\phi}{2}} u^{\dot\alpha} \tilde S_{\dot\alpha} e^{ik\cdot X} \prod_{\kappa=1}^{3} e^{i(\theta^\kappa-1/2)H^\kappa} \sigma_{\theta^\kappa}^{(ab)}(z) \end{aligned} \tag{3.2.1}$$

for $\sum_\kappa \theta^\kappa = 1$, and for the "antiparticle":

$$\begin{aligned} V_{-1}^{(ab)}(k,1-\theta^\kappa,z) &= \lambda^{(ab)} e^{-\phi} e^{ik\cdot X} \prod_\kappa e^{-i\theta^\kappa H^\kappa} \sigma_{1-\theta^\kappa}^{(ab)}(z) \\ V_{-1/2}^{(ab)}(u,k,1-\theta^\kappa,z) &= \lambda^{(ab)} e^{-\frac{\phi}{2}} u^{\beta} S_{\beta} e^{ik\cdot X} \prod_{\kappa=1}^{3} e^{-i(\theta^\kappa-1/2)H^\kappa} \sigma_{1-\theta^\kappa}^{(ab)}(z) \end{aligned} \tag{3.2.2}$$

The worldsheet fermions have been written in bosonised form, where the coefficient α in $e^{i\alpha H}$ shall be referred to as the "H-charge". The spacetime Weyl spinor fields are the left-handed $\tilde S_\alpha = e^{\pm\frac{1}{2}(\mathcal{H}_0-\mathcal{H}_1)}$ and right handed $S_\beta = e^{\pm\frac{1}{2}(\mathcal{H}_0+\mathcal{H}_1)}$. The operators $\sigma_\theta^{(ab)}$ are boundary-changing operators (here between branes a and b), whose properties are discussed in Appendix A.1.

$\lambda^{(ab)}$ is the appropriate Chan-Paton factor for the vertex. We shall not require the specific properties of these, but in the amplitudes we consider they are accompanied by model-dependent matrices γ_i^a which encode the orientifold projections. These matrices have been described for many models (e.g. [5,6,38], but we will only need the results given in [39] for $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds:

$$\begin{aligned} \gamma_1^a &= \mathbf{1}_{N_a} \\ \mathrm{tr}\gamma^a_{\hat\theta^{N/2}} = \mathrm{tr}\gamma^a_{\hat\omega^{M/2}} = \mathrm{tr}\gamma^a_{\hat\theta^{N/2}\hat\omega^{M/2}} &= 0 \\ (\gamma^{\Omega R\hat\Theta^{k,l}a}_{\Omega R\hat\Theta^{k,l}})^* \gamma^a_{\Omega R\hat\Theta^{k,l}} &= \rho_{\Omega R\hat\Theta^{k,l}} \mathbf{1}_{N_a} \end{aligned} \tag{3.2.3}$$

where $\hat\theta^k$ is a $\mathbb{Z}_N$ twist, $\hat\omega^k$ is also present in $\mathbb{Z}_N \times \mathbb{Z}_M$, and $\rho_{\Omega R\hat\theta^k} = \pm 1$. $k(l)$ runs from 0 to $N-1$ $(M-1)$ for $\hat\theta^k$ $(\hat\omega^l)$, where $\hat\theta^0 = \hat\omega^0 = 1$. In the last expression we have used the notation $\hat{\Theta}^{k,l} = \hat{\theta}^k \hat{\omega}^l$, and so for example in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold we have $\rho_{\Omega R \hat{\Theta}^{1,0}} = \rho_{\Omega R \hat{\Theta}^{0,1}} = \rho_{\Omega R \hat{\Theta}^{1,1}} = -1$ and $\rho_{\Omega R \hat{\Theta}^{0,0}} = 1$. The massless four-fermion amplitude that we shall consider is given by

$$\mathcal{A}_4^1 = \int_0^\infty \mathrm{d}t \prod_{i=1}^{4} \int \mathrm{d}z_i$$
$$\langle V_{+1/2}^{(ca)}(u_4^{\alpha_4}, k_4, z_4) V_{-1/2}^{(ab)}(u_1^{\alpha_1}, k_1, z_1) V_{-1/2}^{(ba^{\|})}(u_2^{\dot{\alpha}_2}, k_2, z_2) V_{+1/2}^{(a^{\|}c)}(u_3^{\dot{\alpha}_3}, k_3, z_3) \rangle_{cc} \quad (3.2.4)$$

where brane $a^{\|}$ is parallel to brane a (or a itself in the simplest case).

3.2.3 Field Theory Behaviour

The behaviour of the amplitude in the field-theory limit is found by considering the momenta to be small. When we do this we find that, due to the OPEs of the vertex operators, the amplitude is dominated by poles where the operators are contracted together to leave a scalar propagator. From the previous section and the results of Appendix A.1, we find that two fermion vertices factorise as

$$V_{-1/2}^{(bd)}(u^{\dot{\alpha}_2}, k_2, 1-\nu^\kappa, z_2) V_{+1/2}^{(dc)}(u^{\dot{\alpha}_3}, k_3, , 1-\lambda^\kappa, z_3)$$
$$\sim (u_2 u_3)(z_2 - z_3)^{2\alpha' k_2 \cdot k_3 - \sum_\kappa \frac{\nu^\kappa + \lambda^\kappa}{2}} V_0(k_2 + k_3, z_2) g_o \prod_\kappa C_{\nu^\kappa, \lambda^\kappa}^{(bdc)1-\nu^\kappa-\lambda^\kappa} \quad (3.2.5)$$

for $\sum_\kappa \nu^\kappa = \sum_\kappa \lambda^\kappa = 1$, and $C_{\nu^\kappa, \lambda^\kappa}^{(bdc)1-\nu^\kappa-\lambda^\kappa}$ are the OPE coefficients, given in equation (A.1.19). The calculation involves a factorisation of the tree-level four-point function on first the gauge exchange and then the Higgs exchange, and a comparison with the field theory result [15].

Note that for consistency the classical instanton contribution should be included in the OPE coefficients as well as the quantum part. This means that as we go on to consider higher order loop diagrams the tree-level Yukawa couplings (including classical contributions) should appear in the relevant field theory limits. In the factorisation limit these two fermions yield the required pole of order one around z_2; we will obtain a similar pole for the other two fermion vertices. If we were to then integrate the amplitude over z_3 we would obtain a propagator $\frac{1}{2\alpha' k_2 \cdot k_3}$ preceding a three-point amplitude, and performing the integration over z_4, say, we would obtain

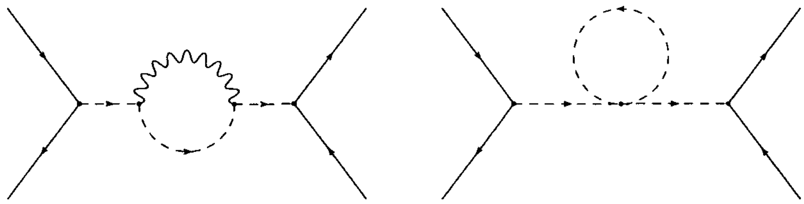

Figure 3.2: Feynman diagrams in the field theory equivalent to our limit; we have factorised onto the scalar propagator, and consider only gauge and self-couplings.

another propagator $\frac{1}{2\alpha' k_1 \cdot k_4}$ $(= \frac{1}{2\alpha' k_2 \cdot k_3})$, and have reduced the amplitude to a two-scalar amplitude.

In this manner the four-point function can be factorised onto the two point function and reduces to

$$\mathcal{A}_4^1 = \frac{1}{4(\alpha')^2 (k_1 \cdot k_4)(k_2 \cdot k_3)} (u_1 u_4)(u_2 u_3) Y^{(cab)} Y^{(ba\|c)}$$
$$G^{C_{bc}\bar{C}_{cb}} \int_0^\infty \mathrm{d}t \int_0^{it} \mathrm{d}q \langle V_0^{cb}(k, \theta^\kappa, 0) V_0^{bc}(-k, 1-\theta^\kappa, q) \rangle_{cc} \quad (3.2.6)$$

plus permutations. The factors $Y^{(cab)}$ and $Y^{(ba\|c)}$ are defined in the Appendix (A.1.20) - they are the Yukawa couplings, derived entirely from tree-level correlators in the desired basis. $G^{C_{bc}\bar{C}_{cb}}$ is the inverse of the Kähler metric $G_{C_{bc}\bar{C}_{cb}}$ for the chiral fields C_{bc} in the chosen basis; note that we do not require its specific form, which was calculated in [32, 33]. Thus we have reproduced the two Yukawa vertices in the field-theory diagrams (figure 3.2), coupled with a propagator which contains the interesting information about the running of the coupling. Note that if we had taken the limit $z_1 \to z_2$, $z_4 \to z_3$ then we would factorise onto a gauge propagator.

In the field theory the tree-level equivalent would have magnitude

$$\mathcal{A}_4^0 = (u_1 u_4)(u_2 u_3) \frac{1}{2k_1 \cdot k_4} Y^{(cab)} Y^{(ba\|c)} G^{C_{bc}\bar{C}_{cb}} \quad (3.2.7)$$

while the one-loop diagram yields

$$\mathcal{A}_4^1 = (u_1 u_4)(u_2 u_3) \frac{1}{u^2} \Pi(u) Y^{(cab)} Y^{(ba\|c)} (G^{C_{bc}\bar{C}_{cb}})^2 \quad (3.2.8)$$

where we put $u = 2k_1 \cdot k_4$ as the usual Mandelstam variable, and $\Pi(u)$ is the one-loop scalar propagator which we have yet to calculate (N.B. $\Pi(u) \sim G_{C_{bc}\bar{C}_{cb}}$). Thus if we want the renormalised Yukawa couplings in some basis (for example the basis where the fields are canonically normalised), we simply set $a^{\parallel} = a$ and write

$$(Y_R^{(cab)})^2 = (Y_0^{(cab)})^2(1 + \frac{1}{u}\Pi(u)G^{C_{bc}\bar{C}_{cb}} + \text{permutations}) \tag{3.2.9}$$

where "permutations" accounts for the equivalent factors coming from the renormalisations of the fermion legs. Alternatively (and more precisely) since the superpotential receives no loop corrections, the above can be considered as a renormalisation of the Kähler potential

$$G_{R,C_{bc}\bar{C}_{cb}} = G_{C_{bc}\bar{C}_{cb}}(1 + \frac{1}{u}\Pi(u)G^{C_{bc}\bar{C}_{cb}}) \tag{3.2.10}$$

3.2.4 The Scalar Propagator

The object that remains to be calculated is of course the scalar propagator $\Pi(u)$ itself, which we can get from the following one-loop amplitude:

$$\Pi(k^2) = \int_0^\infty \mathrm{d}t \int_0^{it} \mathrm{d}q \langle V_0^{ab}(k,\theta^\kappa,0)V_0^{ba}(-k,1-\theta^\kappa,q)\rangle \tag{3.2.11}$$

which represents wave-function renormalisation of the scalars in the theory since as we have seen four-point chiral fermion amplitudes will always factorise onto scalar two-point functions in the field-theory limit. We fix both vertices to the same boundary along the imaginary axis, and one of these we can choose to be at zero by conformal gauge-fixing. For the present, we shall specialise to the following amplitude

$$\mathcal{A}_{2(a)}^{(ab)} = \int_0^\infty \mathrm{d}t \int_0^{it} \mathrm{d}q \langle V_0^{ab}(k,\theta^\kappa,0)V_0^{ba}(-k,1-\theta^\kappa,q)\rangle_{a^{\parallel}a} \tag{3.2.12}$$

where the world sheet geometry is an annulus, and where one string end is always fixed to brane $a^{\parallel}$, and the other is for some portion of the cycle on brane a. In the latter region the propagating open string has untwisted boundary conditions so that in these diagrams the loop contains intermediate gauge bosons/gauginos. It is therefore these diagrams that will give Kaluza-Klein (KK) mode contributions to the beta functions and power law running. The alternative (where the states *never* have untwisted boundary conditions) corresponds to only chiral matter fields in the loops, is harder to calculate and will be treated later and in Appendix C.

We have allowed the fixed end of the string to be on a brane parallel to brane a (rather than just a itself), separated by a perpendicular distance y^κ in each sub-torus. Diagrams where $y^\kappa \neq 0$ correspond to heavy stretched modes propagating in the loop and would in any case be extremely suppressed, but for the sake of generality we will retain y^κ.

The diagram we are concentrating on here is present for any intersecting brane model, but in general $\Pi(k^2)$ receives contributions from other diagrams as well. For orientifolds, the full expression is

$$\Pi(k^2) = \sum_c \mathcal{A}^{(ab)}_{2(c)} + \sum_{k,l}\left(M^{(ab)}_{a,\Omega R\Theta^{k,l}a} + M^{(ab)}_{b,\Omega R\Theta^{k,l}b}\right) \tag{3.2.13}$$

where $M^{(ab)}_{a,\Omega R\Theta^{k,l}a}$ is a Möbius strip contribution, which we shall discuss later. Using the techniques described in Appendix A.2, we find the amplitude

$$\mathcal{A}^{(ab)}_{2(a)} = \int_0^\infty \mathrm{d}t \int_0^{it} \mathrm{d}q A^{(ab)}_{2(a)}(q,t) \tag{3.2.14}$$

where

$$\begin{aligned} A^{(ab)}_{2(a)} &= 2\alpha' g_o^2 k^2 \mathrm{tr}(\gamma^a)\mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)})(8\pi^2\alpha' t)^{-2}\left[\frac{\theta_1(q)}{\theta_1'(0)}e^{-\frac{\pi}{t}(\Im(q))^2}\right]^{-2\alpha' k^2} \\ &\sum_{\nu=1}^{4} N_\nu \delta_\nu \frac{\theta_\nu(0)}{\theta_1(q)}\prod_{\kappa=1}^{3-d'}\theta_1(q)^{-\theta_\kappa}\theta_\nu(\theta^\kappa q)(L^\kappa T^\kappa)^{-1/2} \\ &\sum_{n_1^\kappa}\sum_{n_2^\kappa}\exp[-\frac{(D_A^\kappa(n_A^\kappa))^2 L^\kappa}{4\pi\alpha' T^\kappa}]\exp[-\frac{(D_B^\kappa(n_B^\kappa))^2 T^\kappa}{4\pi\alpha' L^\kappa}] \end{aligned} \tag{3.2.15}$$

where the notation is as follows. The $\delta_\nu = \{1,-1,1,-1\}$ are the usual coefficients for the spin-structure sum. The θ_ν are the standard Jacobi theta functions (see Appendix A.4) with modular parameter it as on the worldsheet (so that we denote $\theta_\nu(z) \equiv \theta_\nu(z, it)$ as usual). D_A^κ and D_B^κ are the lengths of one-cycles, determined in the Appendix to be

$$\begin{aligned} D_A^\kappa(n_A^\kappa) &= \frac{1}{\sqrt{2}} n_A^\kappa L_a^\kappa \\ D_B^\kappa(n_B^\kappa) &= n_B^\kappa \sqrt{2}\frac{4\pi^2 T_2^\kappa}{L_a^\kappa} + y^\kappa \end{aligned} \tag{3.2.16}$$

where L_a^κ is the wrapping cycle length of brane a in sub-torus κ, and T_2^κ is the Kähler modulus for the sub-torus, given by $R_1^\kappa R_2^\kappa \sin\alpha^\kappa$, R_1^κ and R_2^κ are the radii of the torus, α^κ is the tilting parameter, and generally

$$L_a^\kappa = 2\pi\sqrt{(n_a^\kappa R_1^\kappa)^2 + (m_a^\kappa R_2^\kappa)^2 + 2n_a^\kappa m_a^\kappa R_1^\kappa R_2^\kappa \cos\alpha^\kappa}.$$

N_ν are normalisation factors that we will determine in the next subsection. Finally the classical (instanton) sum depends on two functions L^κ and T^κ

$$\begin{aligned} L^\kappa(q,\theta^\kappa) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}z\omega_1(z) \\ T^\kappa(q,\theta^\kappa) &= \int_{-\frac{1}{2}+it}^{-\frac{1}{2}} \mathrm{d}z\omega_1(z) \end{aligned} \tag{3.2.17}$$

where

$$\omega_1(z) = \left(\frac{\theta_1(z)}{\theta_1(z-q)}\right)^{\theta^\kappa-1} \frac{\theta_1(z-\theta^\kappa q)}{\theta_1(z-q)} \tag{3.2.18}$$

and where the contour for L^κ is understood to pass under the branch cut between 0 and q. Note that this contrasts with the four independent functions that we might expect from the equivalent closed string orbifold calculation.

3.2.5 Factorisation on Partition Function

The normalisation factors for the amplitude N_ν must still be determined by factorising on the partition function (i.e. bringing the remaining two vertices together to eliminate the branch cuts entirely) and using the OPE coefficients of the various CFTs. We should obtain

$$A_{2(a)}^{(ab)} \sim q^{-1-\sum_\kappa \theta^\kappa} 2\alpha' k^2 g_o^2 \mathrm{tr}(\gamma^a)\mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)}) Z_{aa}(it) \prod_{\kappa=1}^{3} C_{\theta^\kappa,1-\theta^\kappa}^{(aba)0} \tag{3.2.19}$$

for $\sum_\kappa \theta^\kappa \leq 1$, where g_o is the open string coupling, $C_{\theta^\kappa,1-\theta^\kappa}^{(aba)0}$ is the OPE coefficient of the boundary-changing operators determined in Appendix A.1 and $Z_{aa}(it)$ is the partition function for brane a. It is given by

$$Z_{aa}(it) = (8\pi^2\alpha' t)^{-2}\eta(it)^{-12} \sum_{\nu=1}^{4} \delta_\nu \theta_\nu^4(0) \prod_{\kappa=1}^{3} Z_a^\kappa \tag{3.2.20}$$

where

$$Z_a^\kappa(t, T_2^\kappa, L_a^\kappa) = \sum_{r^\kappa, s^\kappa} \exp\left[\frac{-8\pi^3\alpha' t}{(L_a^\kappa)^2}|r^\kappa + \frac{iT_2^\kappa s^\kappa}{\alpha'}|^2\right] \tag{3.2.21}$$

is the bosonic sum over winding and Kaluza-Klein modes. The factorisation occurs for $q \to 0$, when $L^\kappa \to 1$ and $T^\kappa \to it$, giving

$$N_\nu = \frac{e^\Phi}{\alpha' g_o^2}(2\pi)^3(\sqrt{2})^{-3+d'}\left(\prod_{\iota=1}^{d'}\theta_\nu(0)Z_a^\iota\right)\eta^{-6}(it)G_{C_{ab}\bar{C}_{ba}} \tag{3.2.22}$$

3.3 Divergences

With our normalised amplitude, we are finally able to probe its behaviour. The important limits are $q \to 0$ and $q \to it$ where there are poles; in the former case the pole is cancelled because of the underlying $N = 4$ structure of the gauge sector, but in the latter it is not, and it dominates the amplitude. We should comment here about a subtlety with these calculations which does not apply for many other string amplitudes: due to the branch cuts on the worldsheet, the amplitude is not periodic on $q \to q + it$. This causes the usual prescription of averaging over the positions of fixed vertices to give a symmetric expression to break down (this was an ambiguity in [30]): the gauge-fixing procedure asserts that one vertex *must* be fixed, which, to remain invariant as t changes, is placed at zero. The non-periodic nature of the amplitude is entirely expected from the boundary conformal field theory perspective: due to the existence of a non-trivial homological cycle on the worldsheet, we have two OPEs for the boundary changing operators, depending on how (i.e. whether) we combine the operators to eliminate one boundary. We have already used the expected behaviour in the limit $q \to 0$ to normalise the amplitude, but $q \to it$ yields new information, namely the partition function for string stretched between different branes: in this limit we obtain

$$A_{2(a)}^{(ab)} \sim (it-q)^{-1-\sum_\kappa \theta^\kappa - 2\alpha' k^2} 2\alpha' k^2 g_o^2 \mathrm{tr}(\gamma^a)\mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)})Z_{ab}(it)\prod_{\kappa=1}^{3} C_{\theta^\kappa,1-\theta^\kappa}^{(bab)0}, \tag{3.3.1}$$

where $Z_{ab}(it)$ is the partition function for string stretched between branes a and b, given by

$$Z_{ab}(it) = (8\pi^2\alpha' t)^{-2}\left(\prod_{\iota=1}^{d'} Z_a^\iota\right)\sum_\nu \delta_\nu\left(\frac{\theta_\nu(0)}{\eta^3(it)}\right)^{1+d'}\prod_{\kappa=1}^{3-d'} I_{ab}^\kappa\frac{\theta_\nu(i\theta^\kappa t)}{\theta_1(i\theta^\kappa t)} \tag{3.3.2}$$

For supersymmetric configurations, however, this partition function is zero, and hence we were required to calculate our correlator to find the behaviour in this limit. It is straightforward to show that our calculation gives this behaviour: as $q \to it$, the functions L^κ diverge logarithmically, and so we perform a Poisson resummation over n_B^κ - which reduces the instanton sums to unity. Simple complex analysis gives

$$T^\kappa = \int_{it}^{q} \mathrm{d}z\omega_1(z) \to B(\theta^\kappa, 1-\theta^\kappa)\exp[-\pi\theta^\kappa t]\frac{\theta_1(\theta^\kappa it)}{\theta_1'(0)} \tag{3.3.3}$$

and we also require the identity [39]:

$$\sin\pi\theta^\kappa = \frac{4\pi^2 T_2^\kappa I_{ab}^\kappa}{L_a^\kappa L_b^\kappa} \tag{3.3.4}$$

so that, for the $N=1$ supersymmetric choice of angles we obtain (after Riemann summation):

$$A_{2(a)}^{(ab)}(q,t) \sim (it-q)^{-1-2\alpha' k^2} 2k^2 e^\Phi \mathrm{tr}(\gamma^a)\mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)})\frac{I_{ab}}{L_b}2\pi(2\pi\sqrt{\alpha'})^3 G_{C_{ab}\bar{C}_{ba}}(8\pi^2\alpha' t)^{-2} \tag{3.3.5}$$

The above is clearly singular in the limit $k^2 \to 0$, where the integral over q is dominated by the behaviour at $q = it$, since at $q = 0$ the effective $N = 4$ SUSY of the gauge bosons cancels the pole. Using the usual prescription for these integrals we obtain the exact result for the amplitude, and find it is divergent:

$$\int_0^{it} \mathrm{d}q A_{2(a)}^{(ab)}(q,t) = \frac{e^\Phi}{4(\alpha')^{3/2}} G_{C_{ab}\bar{C}_{ba}} \mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)})\frac{N_a I_{ab}}{L_b}\int_0^\infty \mathrm{d}t t^{-2}. \tag{3.3.6}$$

This divergence is effectively due to RR-charge exchange between the branes, and so we look to contributions from other diagrams (as given in (3.2.13)) to cancel it. These consist of other annulus diagrams where one end resides on a brane other than "a" or "b", and Möbius strip diagrams.

Fortunately we do not need to calculate the amplitude of these additional contributions in their entirety; indeed we can obtain all that we need from knowledge of the partition functions and the properties of our boundary changing operators, along with a straightforward conjecture about the behaviour of the amplitude. For an annulus diagram where one string end remains on a brane "c", while the other end contains the vertex operators and thus is attached to brane "a" or "b", we obtain two contributions: one from each limit. From the OPEs of the boundary-changing operators, we expect to obtain

$$A^{(ab)}_{2(c)}(x,t) = 2\alpha' g_0^2 k^2 \mathrm{tr}(\gamma^c)\mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)})(x)^{-2-2\alpha' k^2}\left(C^{(aba)}Z_{ac} + C^{(bab)}Z_{bc}\right) \quad (3.3.7)$$

where x now denotes q or $it-q$ in the appropriate limits, and $C^{(aba)}$ is understood to be the product of the OPE coefficients for each dimension. Considering the partition functions (3.3.2) and the behaviour of the expression (3.3.5), then we propose that the effect of the division by zero in the above is to cancel one factor of x with a factor of $\theta_1(0)$. Hence, if we write equation (3.3.6) as $\mathcal{A}_0\frac{N_a I_{ab}}{L_b}$, we thus obtain for the divergence,

$$\mathcal{A}^{(ab)}_{2(c)} = \mathcal{A}_0\left(\frac{N_c I_{ac}}{L_a} + \frac{N_c I_{bc}}{L_b}\right) \quad (3.3.8)$$

Note that c is allowed to be any brane in the theory, including images under reflection and orbifold elements.

We must now also consider the contribution from Möbius strip diagrams, which, with the same conjectured behaviour as above, give us

$$M^{(ab)}_{a,\Omega R\Theta^{k,l}} = 2\alpha' g_0^2 k^2 tr(\lambda_1\lambda_1^{\dagger}(\gamma^{\Omega R\Theta^{k,l}a}_{\Omega R\Theta^{k,l}})^{*}\gamma^{a}_{\Omega R\Theta^{k,l}})q^{-2-2\alpha'}$$
$$\left(C^{(aba)}M_{a,\Omega R\hat{\Theta}^{k,l}a} + C^{(bab)}M_{b,\Omega R\hat{\Theta}^{k,l}b}\right) \quad (3.3.9)$$

where $M_{a,\Omega R\Theta^{k,l}a}$ denotes the Möbius diagram between brane a and its image under the orientifold group $\Omega R\hat{\Theta}^{k,l}a$, supplemented by a twist insertion $\hat{\Theta}^{k,l}$ to give a twist-invariant amplitude, given by [39]

$$M_{a,\Omega R\Theta^{k,l}a} = -(8\pi^2\alpha' t)^{-2}\left(\prod_{\iota=1}^{d^{\parallel}} n^{\iota}_{O6_{k,l}} L_i(t, T_2^{\iota}, L^{\iota}_{O6_{k,l}})\right)$$
$$\sum_{\nu}\delta_{\nu}\left(\frac{\theta_{\nu}(0, it+\frac{1}{2})}{\eta^3(it+\frac{1}{2})}\right)^{1+d^{\parallel}}\prod_{\kappa=1}^{3-d^{\parallel}} 2^{\delta^{\kappa}} I^{\kappa}_{a,O6_{k,l}}\frac{\theta_{\nu}(2\theta^{\kappa}it, it+\frac{1}{2})}{\theta_1(2\theta^{\kappa}it, it+\frac{1}{2})}, \quad (3.3.10)$$

where δ^{κ} [5] is zero for a orthogonal to the $O6_{k,l}$-plane in sub-torus κ, and 1 otherwise; $d^{\parallel}$ is the number of sub-tori in which brane a lies on the $O6_{k,l}$-plane, and $n^{\iota}_{O6_{k,l}}$ is the number of times the plane wraps the torus with cycle length $L^{\iota}_{O6_{k,l}}$. Here θ^{κ} is the angle between brane a and the $O6_{k,l}$-plane. We also define $I^{\kappa}_{a,O6_{k,l}}$ to be the total number of intersections of brane a with the $O6$-planes of the *class* $[O6_{k,l}]$ in sub-torus κ. For $d^{\parallel}$ non-zero we have zero modes:

$$L_i(t, T_2^{\iota}, L^{\iota}_{O6_{k,l}}) = \sum_{r^{\iota}, s^{\iota}} \exp -\frac{8\pi^3\alpha' t}{T_2^{\iota} L^{\iota}_{O6_{k,l}}} |r^{\iota} + \frac{i2^{\mu} T_2^{\iota} s^{\iota}}{\alpha'}|^2 \tag{3.3.11}$$

where $\mu = 0$ for untilted tori, and 1 for tilted tori. To obtain the divergence, the same procedure as before can be applied once we have taken into account the relative scaling of the modular parameter between the Möbius and annulus diagrams. Since the divergence occurs in the ultraviolet and is due to closed string exchange, to do this we transform to the closed string channel: we simply replace t by $1/l$ for the annulus, and $1/(4l)$ for the Möbius strip. This results in an extra factor of 4 preceding the Möbius strip divergences relative to those of the annulus diagrams, which are due to the charges of the $O6$-planes being 4 times those of the $D6$-branes. We thus obtain the total divergence

$$\mathcal{A}_2^1 = \frac{e^{\Phi}}{4(\alpha')^{3/2}} G_{C_{ab}\bar{C}_{ba}} \mathrm{tr}(\lambda^{(ab)}\lambda^{(ba)}) \int_0^{\infty} dl \Bigg\{ \frac{1}{L_a} \left(\sum_{c,c'} N_c I_{ac} - 4 \sum_{k,l} \rho_{\Omega R \hat{\Theta}^{k,l}} I_{a,O6_{k,l}} \right) + \frac{1}{L_b} \left(\sum_{c,c'} N_c I_{bc} - 4 \sum_{k,l} \rho_{\Omega R \hat{\Theta}^{k,l}} I_{b,O6_{k,l}} \right) \Bigg\} \tag{3.3.12}$$

Note that the total divergence is of the same form as that found in gauge coupling renormalisation. We recognise the terms in brackets as the standard expression for cancellation of anomalies, derived from the RR-tadpole cancellation conditions [6]:

$$[\Pi_a] \cdot \left(\sum_{c,c'} N_c [\Pi_c] - 4 \sum_{k,l} [\Pi_{O6^{k,l}}] \right) = 0 \tag{3.3.13}$$

where $[\Pi_a]$ is the homology cycle of a etc; note that the phases $\rho_{\Omega R \hat{\Theta}^{k,l}}$ will be the same as the sign of the homology cycles of the orientifold planes. Hence, we have shown that cancellation of RR-charges implies that in the limit that $k^2 \to 0$, the total two-point amplitude is zero.

3.4 Running Yukawas up to the String Scale

Having demonstrated the mechanism for cancellation of divergences in the two point function, we may now analyse its behaviour and expect to obtain finite results. As mentioned earlier, by examining the amplitude at small, rather than zero, k^2, we obtain the running of the coupling as appearing in four-point and higher diagrams where all Mandelstam variables are not necessarily zero. Unfortunately, we are now faced with two problems: we have only calculated the whole amplitude for one contribution; and an exact integration of the whole amplitude is not possible, due to the complexity of the expressions and lack of poles. However, this does not prevent us from extracting the field-theory behaviour and even the running near the string scale, but necessarily involves some approximations.

Focusing on our amplitude (3.2.15), which in the field theory limit comprises the scalar propagator with a self-coupling loop and a gauge-coupling loop, we wish to extract the dependence of the entire amplitude upon the momentum. Schematically, according to equation (3.2.9), we expect to obtain for Yukawa coupling Y, as $k^2 \to 0$,

$$Y_R - Y_0 \sim A + B + \frac{g^2\beta}{8\pi^2}\ln k^2 + \Delta + O(k^2) \tag{3.4.1}$$

where A represents the divergent term, the third term is the standard beta-function running, and the fourth term comprises all of the threshold corrections. This is the most interesting term: as discussed in [30] it contains power-law running terms, but with our complete expressions here we are able to see how the running is softened at the string scale. The term denoted B is a possible finite piece that is zero in supersymmetric configurations but that might appear in non-supersymmetric ones: in the field theory it would be proportional to the cutoff squared while in the (non-supersymmetric) string theory it would be finite. This term would give rise to the hierarchy problem. Had we found such a term in a supersymmetric model it would have been inconsistent with our expectations about the tree-level Kähler potential - in principle, it could only appear if there were not total cancellation among the divergent contributions.

The power-law running in the present case corresponds to both fermions and Higgs fields being localised at intersections, but gauge fields having KK modes for the three extra dimensions on the wrapped D6 branes [40]. For completeness we write the expression for KK modes on brane a with three different KK thresholds $\mu_{0,1,2}$ (one for each complex dimension - note that in principle we have different values for each brane a):

$$\Delta = \frac{g_a^2}{8\pi^2}\left((\beta-\hat{\beta})\ln\frac{\Lambda}{\mu_0}+\hat{\beta}\sum_{\delta=1}^{3}\frac{X_\delta}{\delta}\left[\left(\frac{\mu_\delta}{\mu_{\delta-1}}\right)^{\delta}-1\right]\prod_{i<\delta}\left(\frac{\mu_i}{\mu_{i-1}}\right)^{i}\right)+\Delta_S \quad (3.4.2)$$

where $\{\mu_\delta\} = \{(L_a^{\delta+1})^{-1}, \Lambda|\ \mu_0 < \mu_1 < \mu_2 < \Lambda\}$, and Λ is the string cutoff which should be $\mathcal{O}(1)$ for our calculation. $\{X_\delta\} = \{2, \pi, 4\pi/3\}$ is the correction factor for the sum, and Δ_S is the string-level correction. β and $\hat{\beta}$ are the beta-coefficients for the standard logarithmic running and power-law running respectively. Note that the above is found from an integral over the Schwinger parameter t' where the integrand varies as $(t')^{-\delta/2-1}$ in each region; t' is equivalent to the string modular parameter t, modulo a (dimensionful) proportionality constant.

To extract the above behaviour, while eliminating the divergent term, without calculating all the additional diagrams, we could impose a cutoff in our q integral (as in [30]); however the physical meaning of such a cut-off is obscure in the present case, so we shall not do that here. As in [30], we shall make the assumption that the classical sums are well approximated by those of the partition function for the gauge boson. However, we then subtract a term from the factors preceding the classical sum, which reproduces the pole term with no sub-leading behaviour in k^2; since the region in q where the Poisson resummation is required is very small, this is a good way to regulate the pole that we have in the integrand when we set k^2 to zero. To extract the Δ terms, we can set k^2 to zero in the integrand: this is valid except for the logarithmic running down to zero energy (i.e. large t), where the k^2 factors in the $\left[\frac{\theta_1(q)}{\theta_1'(0)}e^{-\frac{\pi}{t}(\Im(q))^2}\right]^{-2\alpha' k^2}$ term regulate the remaining t^{-1} behaviour of the integrand when the t integral is performed. This behaviour is then modified by powers of $(8\pi^2\alpha' t/(L_a^\kappa)^2)^{1/2}$ multiplying the classical sums after each Poisson-resummation, as t crosses each cutoff threshold, yielding power-law running as expected; this was obtained in [30], and so we shall not reproduce it here.

For $\Lambda^{-2} > t \sim 1$, we have an intermediate stage where the KK modes are all excited, but we have not yet excited the winding modes, represented by the sums over n_B^κ. As a first approximation, the integral, with our regulator term, is

$$\mathcal{A}_{2(a)}^{(ab)} \sim e^\Phi \frac{k^2}{8\pi(\alpha')^2\sqrt{2}} G_{C_{ab}\bar{C}_{ba}} \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}\lambda t^{-7/2}\eta^{-6}(it)$$
$$\left(\frac{\theta_1(\theta^1 i\lambda)\theta_1(\theta^2 i\lambda)\theta_1(\theta^3 i\lambda)}{\theta_1(i\lambda)} + \frac{\theta_1(\theta^1 it)\theta_1(\theta^2 it)\theta_1(\theta^3 it)}{\theta_1'(it)}\left(\frac{1}{(t-\lambda)} - \frac{1}{t}\ln t\right)\right) \quad (3.4.3)$$

where we have used $L^\kappa(i\lambda, it)T^\kappa(i\lambda, it) \approx it$. If we had not set k^2 to zero in the integrand, the second term in brackets would be

$$i^{-2\alpha' k^2}\left((t-\lambda)^{-1-2\alpha' k^2} + \frac{t^{-2\alpha' k^2} - 1}{2\alpha' k^2 t}\right).$$

The above can then be integrated numerically to give

$$\mathcal{A}_{2(a)}^{(ab)} \sim e^\Phi \frac{k^2}{8\pi(\alpha')^2\sqrt{2}} G_{C_{ab}\bar{C}_{ba}} \int_0^\infty \mathrm{d}t P(t) \quad (3.4.4)$$

where $P(t)$ is plotted in figure 3.3. It indicates how the threshold corrections are changing with energy scale probed. The figure clearly demonstrates the softening of the running near the string scale. Alas, we also find that the amplitude still diverges (negatively) as $t \to 0$, and so we conclude that the subtraction of the leading poles, rather than fully including the remaining pieces (i.e. the other annulus diagrams and the Möbius strip diagrams), was not enough to render a finite result. However we believe that the condition in eq.(3.3.13) ensures cancellation of the remaining divergences as well.

Despite this, the procedure of naive pole cancellation still enhances our understanding of the running up to the string scale, because the divergent pieces (depending as they do on much heavier modes) rapidly die away as t increases. This allows us to focus on the behaviour of the KK contribution. The quenching of UV divergent KK contributions is well documented at tree-level but has not been discussed in detail at one-loop. At tree-level the nett effect of the string theory is to introduce a physical Gaussian cut-off in the infinite sum over modes. For example a tree level s-channel exchange of gauge fields is typically proportional to [41, 42]

$$\sum_{\underline{n}} \frac{\prod_\kappa e^{-\beta^\kappa M_{n^\kappa}^2 \alpha'}}{s - M_{\underline{n}}^2}, \quad (3.4.5)$$

where $\beta^\kappa = (2\psi(1) - \psi(\theta^\kappa) - \psi(1-\theta^\kappa))$ and where $M_{\underline{n}}^2 = \sum_\kappa M_{n^\kappa}^2$ is the mass-squared of the KK mode. Note that the above is similar to results in [43, 44]. The

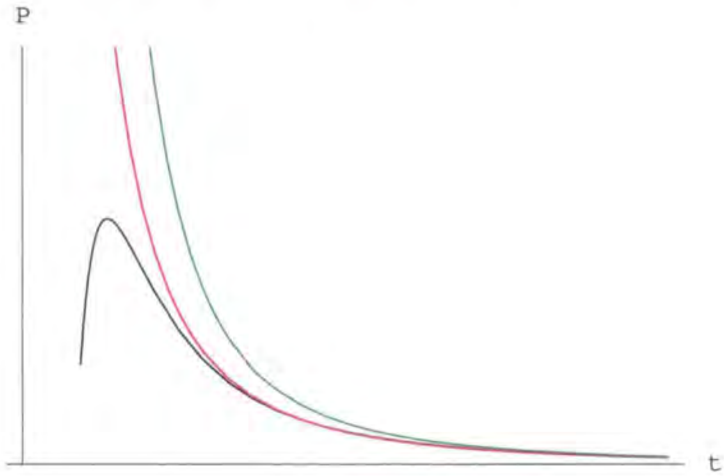

Figure 3.3: A linear plot of the "running" Yukawa coupling in modular parameter t, lower graph. The peak is very close to $t = 1$, i.e. the string scale. The top graph is the standard power-law behaviour continued. The middle graph is the field theory approximation using string improved propagators as defined in the text.

physical interpretation of this expression is that the D-branes have finite thickness of order $\sqrt{\alpha'}$. Consequently they are unable to excite modes with a shorter Compton wavelength than this which translates into a cut-off on modes whose mass is greater than the string mass. This suggests the possibility of "string improved propagators" for the field theory; for example scalars of mass m would have propagators of the form

$$\Delta(k) = \frac{\prod_\kappa e^{-\beta^\kappa (m^\kappa)^2 \alpha'}}{k^2 - m^2}, \tag{3.4.6}$$

where $m^2 = \sum_\kappa (m^\kappa)^2$, and we are neglecting gauge invariance. To test this expression we can follow through the field theory analysis in ref. [40] (which also neglects gauge invariance): the power law running is modified:

$$\frac{1}{t} t^{-\frac{\delta}{2}} \to \frac{1}{t} \prod_\kappa (t + \frac{\beta^\kappa}{\pi})^{-\frac{1}{2}}. \tag{3.4.7}$$

For comparison this curve is also included in figure 3.3. Clearly it provides a better approximation to the string theory up to near the string scale, justifying our propagators. (Of course the usual logarithmic divergence in the field theory remains once the KK modes are all quenched.) Also interesting is that this behaviour appears in our first approximation; we expect it to appear due to $T^\kappa(i\lambda, it)$ and $L^\kappa(i\lambda, it)$

differing from the values we have assumed, where we would also have to take into account the Poisson resummation required near $\lambda \to t$. We could thus attempt an improved approximation by modifying these quantities: however, the integrals rapidly become computationally intensive, and we shall not pursue this further here.

3.5 Further Amplitudes

Having discussed the one-loop scalar propagator and its derivation from N-point correlators of boundary-changing operators, we may wish to consider calculating the full N-point amplitudes themselves. The general procedure was outlined in [30] where three-point functions were considered. The complete procedure for general N-point functions is presented in Appendix A.2, the main new result being equation (A.2.12).

Furthermore, we may also wish to calculate diagrams in which there are only chiral matter fields in the loop; although we were able to use the OPEs to obtain the necessary information pertaining to the divergences, we would need these diagrams to study the full running of the scalar propagator, for example. We have constructed a new procedure for calculating them, and describe it in full in Appendix A.3; the crucial step is to modify the basis of cut differentials.

In calculating the contribution to the scalar propagator from diagrams with no internal gauge bosons, we find some intriguing behaviour. In the field theory, the diagrams which correspond to these (other than the tadpole) involve two Yukawa vertices, and hence we should expect the result to be proportional to products of Yukawa couplings; we expect to find the angular factors and the instanton sum. We already see from the CFT analysis that the divergences are not proportional to these which is perhaps not surprising. However when we calculate the finite piece of diagrams, although we do obtain the expected instanton terms in the classical action it is not immediately obvious that we obtain the Yukawa couplings here either. Yukawas attached to external legs will always appear by factorisation thanks to the OPE, but this is not the case for internal Yukawas. Hence, it seems likely that the field theory behaviour of composition of amplitudes is only approximately exhibited at low energy, and near the string scale this breaks down. This will be a source of flavour changing in models that at tree-level have flavour diagonal couplings. Unfortunately it cannot solve the "rank 1" problem of the simplest constructions however, because canonically normalising the fields (i.e. making the Kähler metric flavour diagonal) does not change the rank of the Yukawa couplings. We leave the full analysis of this to future work.

3.6 Conclusions

We have presented the procedure for calculating N-point diagrams in intersecting brane models at one-loop. We have shown that the cancellation of leading divergences in the scalar propagator for self and gauge couplings for supersymmetric configurations is guaranteed by RR-tadpole cancellation. The one-loop correction to the propagator is consistent with a canonical form of the Kähler potential in the field theory, or one of the no-scale variety, where there are no divergences in the field theory; had there been a constant term remaining, this would have corresponded to a divergence in the field-theory proportional to the UV cut-off squared, which would have been consistent with alternative forms of the Kähler potential. However, other than the expected corrections from power-law running, we cannot make specific assertions for corrections to the Kähler potential from the string theory.

When we investigated the energy dependence of the scalar propagator (in the off-shell extension, i.e. as relevant for the four-point and higher diagrams) we found that there still remained divergences, which can only be cancelled by a full calculation of all the diagrams in the theory. However information could be obtained about the intermediate energy regime where KK modes are active and affecting the running. We find the tree level behaviour whereby the string theory quenches the KK modes remains in the one-loop diagrams, and we proposed a string improved propagator that can take account of this in the field theory.

We also developed the new technology necessary for calculating annulus diagrams without internal gauge bosons, and mention some interesting new features. At present, however, the technology for calculating the Möbius strip diagrams does not exist, and this is left for future work.

Chapter 4

Realistic Yukawa Couplings through Instantons in Intersecting Brane Worlds

This chapter studies some non-perturbative aspects of intersecting brane worlds, and how they can resolve the "rank-one Yukawa problem" outlined in section 2.4.3. This work is published in [28].

4.1 Introduction

Recall from chapter 2 that the perturbative superpotential in intersecting brane models is given by a tree-level calculation (receiving no perturbative corrections) to be of the form

$$Y^0_{\alpha\beta} \sim A_\alpha B_\beta,$$

and is thus of rank one, giving rise to only one massive generation. There have been numerous subsequent attempts to solve this problem as well as other related analyses of questions regarding flavour (e.g. refs. [35, 42, 45–51] , but many of these lost the original link with geometry. In parallel there developed techniques for calculating both tree level [15, 16, 20, 32, 52] , and loop [30, 31, 34] amplitudes involving chiral (intersection) states on networks of intersecting D-branes.

The most recent development on the calculational side, whose consequences will be the subject of this chapter, has been the incorporation of the effect of instantons in ref. [24] (for related work see also [53–55] . This work laid out in detail how the contributions of so-called E2-instantons (i.e. branes with 3 Neumann boundary conditions in the compact dimensions and Dirichlet boundary conditions everywhere else) to the superpotential could be calculated. It also pointed out a number of expected phenomenological consequences of these objects. Thus far attention has mostly been paid to the fact that instantons do not necessarily respect the global symmetries of the effective theory and so are able to generate terms that may otherwise be disallowed. In particular they can be charged under the parent anomalous U(1)'s due to the Green-Schwarz anomaly cancellation mechanism. (Alternatively, these charges can be associated with states at the intersection of the E2 and D6 branes.) For example this can induce Majorana mass terms for neutrinos which are of the form

$$e^{-\frac{8\pi^2}{g_{E2}^2}} M_s \nu_R \nu_R \,,$$

and which would otherwise be forbidden. In this equation g_{E2} is an effective coupling strength which depends on the world-volume of the E2 instanton. These need not be equal to the gauge couplings of the MSSM and the Majorana mass-terms can be of the right order to generate the observed neutrino masses [24, 53]. Similarly interesting contributions occur for the μ-term of the MSSM, yielding a possible solution to the μ-problem.

In this chapter we reassess Yukawa couplings in the light of instanton contributions. In particular we claim that one-loop diagrams with E2 branes can solve the rank-one problem and lead to a Yukawa structure which is hierarchical. All the Yukawa couplings have by assumption the same charges under all symmetries, so the extra terms are induced by E2 instantons which do not intersect the D6-branes. The tree and one-loop contributions to Yukawa couplings are shown in figure 4.1; the tree level diagram consists of the usual disc diagram with three vertices, the one loop diagram is an annulus with the inside boundary being an E2 brane and three vertices on the D6-brane boundary. By explicit computation of these one-loop instanton contributions we show that the corrected Yukawas are of the form

$$Y_{\alpha\beta} = Y^0_{\alpha\beta} + Y^{np}_{\alpha\beta} ,$$

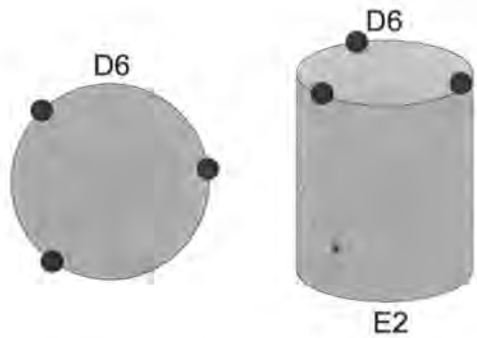

Figure 4.1: Contributions to 3 point functions: tree level and one-loop instanton.

where the non-perturbative contribution has rank 3. Moreover as for the neutrino Majorana masses, these contributions are exponentially suppressed by the instanton volume. Hence the 1^{st} and 2^{nd} generation masses are hierarchically smaller than the first.

In addition there is the possibility of making an interesting connection between the Yukawa hierarchies and the Majorana neutrino masses. The suppression of the latter with respect to the string scale should be similar to the suppression of 2^{nd} generation masses with respect to third generation ones. In general one sees that a direct connection between compactification geometry and Yukawa couplings would be manifest. The rest of this chapter is devoted to proving this result. In the next section we outline the techniques of instanton calculus and compute the necessary general results: these include the multiplying factors of disconnected tree-level and one-loop diagrams with no vertex operators, which must appear in every such process. The section that follows provides the annulus correlator, and in particular shows that the leading contribution yields instanton contributions to the Yukawa couplings of the advertised form.

4.2 Instanton Calculus

The general framework for calculating E2 instanton corrections to the superpotential in string backgrounds was proposed in [24]. This section elucidates the technical details of the calculus of E2 instanton corrections applied to toroidal orientifold intersecting brane worlds.

4.2.1 Tree Level Contributions

The non-perturbative $O(e^{-8\pi^2/g_{E2}^2})$ factor appearing in instanton contributions to the superpotential is given by the product of all possible disk diagrams whose boundary lies along the instanton and with either no vertex operators or an RR-tadpole operator. We shall label this contribution $e^{-S_{E2}^0}$. It is given by

$$S_{E2}^0 = \frac{2\pi}{g_s\sqrt{(\alpha')^3}} L_{E2} \tag{4.2.1}$$

where $L_{E2} = \prod_{\kappa=1}^3 L_{E2}^\kappa$ is the volume of the instanton. The above can be given more conveniently in terms of the gauge coupling on a reference brane a as

$$S_{E2}^0 = 8\pi^2 \frac{L_{E2}}{L_a} g_a^{-2} \tag{4.2.2}$$

4.2.2 Pfaffians and Determinants

To perform any calculation in an E2 instanton background requires the knowledge of the reduced Pfaffian and Determinant factor given by the exponential of the total partition function of states intersecting the instanton with the zero modes removed. In toroidal backgrounds, there are two classes of contributions to this: the first arises when the instanton is parallel to a $D6$-brane in one torus, and the second appears when there is no parallel direction.

One Parallel Direction

In this case, there is no zero mode associated provided that the branes do not intersect, i.e. brane a is separated from the E2 brane by some distance $y_{a,E2}$ in a torus i. In this case the partition function is given by

$$Z(E2, D6_a) = N_a I^j_{E2,a} I^k_{E2,a} \int \frac{\mathrm{d}t}{t} \sum_\nu \sum_{r^i,s^i} \delta_\nu \frac{\theta_\nu^2(\frac{it}{2})}{\theta_1^2(\frac{it}{2})} \frac{\theta_\nu(\phi_{E2,a} it)\theta_\nu(-\phi_{E2,a} it)}{\theta_1(\phi_{E2,a} it)\theta_1(-\phi_{E2,a} it)}$$
$$\times \exp\left[\frac{-8\pi^3\alpha' t}{(L^i_{E2})^2}|r^i + \frac{iT^i_2 s^i}{\alpha'} + \frac{i y_{a,E2} L^i_{E2}}{4\pi^2\alpha'}|^2\right] \quad (4.2.3)$$

After performing the sum over spin structures we obtain an expression that would also be found in gauge threshold correction calculations

$$Z(E2, D6_a) = N_a I^j_{E2,a} I^k_{E2,a} \int \frac{\mathrm{d}t}{t} \sum_{r^i,s^i} \exp\left[\frac{-8\pi^3\alpha' t}{(L^i_{E2})^2}|r^i + \frac{iT^i_2 s^i}{\alpha'} + \frac{i y_{a,E2} L^i_{E2}}{4\pi^2\alpha'}|^2\right]. \quad (4.2.4)$$

This expression can be integrated. When all $y_{a,E2} \to 0$ (where there are fermionic zero modes that must be regulated) it gives [39]

$$Z(E2, D6_a)_{y=0} = -N_a I^j_{E2,a} I^k_{E2,a} \left[\ln \frac{(L^i_{E2})^2}{\alpha'} + \ln|\eta(\frac{T^i}{\alpha'})|^4 + \kappa\right], \quad (4.2.5)$$

where $\kappa = \gamma_E - \log 4\pi$ is a moduli-independent constant. For non-zero $y_{a,E2}$, the expression differs from those found previously; first we must Poisson-resum the expression on r and s to obtain

$$Z(E2, D6_a) = N_a I^j_{E2,a} I^k_{E2,a} \frac{(L^i_{E2})^2}{T^i_2} \frac{1}{8\pi^2} \quad (4.2.6)$$
$$\times \int_0^\infty \mathrm{d}l \sum_{m,n} \exp\left[-\frac{l (L^i_{E2})^2}{8\pi}\left(\frac{1}{\alpha'}m^2 + \frac{\alpha'}{(T^i_2)^2}n^2\right) - 2in\frac{y_{a,E2} L^i_{E2}}{2\pi T^i_2}\right].$$

We then note the divergence for the piece $m, n = 0$ as $l \to \infty$ (and note that the separation between the branes will regulate any $l \to 0$ infra-red divergence). This arises in the NS sector and is cancelled for consistent models according to the condition [39]

$$\sum_{a=\{a,a'\}} \frac{(L^i_{E2})^2}{T^i_2} N_a I^j_{E2,a} I^k_{E2,a} - 4\sum_{\hat{k}} \frac{(L^i_{E2})^2}{T^i_2} I^j_{E2,O6_{\hat{k}}} I^k_{E2,O6_{\hat{k}}} = 0. \quad (4.2.7)$$

This is sufficient if there are no intersections between the $E2$-brane and the $D6$-branes of the model; if there are branes for which there are no parallel dimensions then the cancellation condition will include those.

We now rescale the variables and perform the integral:

$$Z(E2, D6_a) = N_a I^j_{E2,a} I^k_{E2,a} \frac{1}{\pi} \sum_{m,n\neq 0} \frac{\alpha'}{T^i_2} \frac{\exp\left[-2\pi i n \frac{y_{a,E2} L^i_{E2}}{2\pi^2 T^i_2}\right]}{|m + i\frac{(\alpha')}{T^i_2} n|^2}. \quad (4.2.8)$$

This can be recognised as appearing in Kronecker's second limit formula:

$$(2\Im(z))^s \sum_{(m,n)\neq(0,0)} \exp\left[2\pi i(mu+nv)\right]|m+nz|^{-2s} =$$
$$-4\pi\log|e^{-\pi u(v-uz)}\frac{\theta_1(v-uz,z)}{\eta(z)}| + O(s-1) \quad (4.2.9)$$

and thus we obtain

$$Z(E2,D6_a) = -N_a I^j_{E2,a} I^k_{E2,a}\left[\log\left|\frac{\theta_1(\frac{iy_{a,E2}L^i_{E2}}{2\pi^2\alpha'},\frac{iT^i_2}{\alpha'})}{\eta(\frac{iT^i_2}{\alpha'})}\right|^2 - \frac{y^2_{a,E2}}{2\pi^3\alpha'}\frac{(L^i_{E2})^2}{T^i_2}\right]. \quad (4.2.10)$$

This is a new result that also applies to gauge threshold corrections (as mentioned earlier). It is noteworthy that the IR regularisation by separation of the $E2$ brane from brane a does not commute with that used for $y_{a,E2} = 0$. This reflects the qualitative difference between E2 branes which wrap cycles where $b_1(\Pi_{E2}) = 0$ and those where the Betti number is non-zero; in the latter case (i.e. when the instanton does not pass through a fixed point) it acquires additional uncharged fermionic zero modes which must be integrated over, but when the separation between D6 and E2 branes reduces to zero there arise additional charged fermionic zero modes. We only consider configurations with no additional zero modes in this chapter, and thus we require the E2 branes to pass through fixed points and D6 branes to be separated from these by some small distances. This is a common feature of many models, although it is incompatible with the models of, for example, [14,56]. However, since the extra uncharged modes carry no coupling, it is actually possible for E2 instantons with six additional uncharged zero modes to contribute to the superpotential term considered in this chapter generated at one loop, and thus we expect our conclusions to extend to those models as well. The following analysis would be complicated by the plethora of vertex operators, and thus we do not consider them.

No Parallel Directions

Here we have the raw expression

$$Z(E2,D6_a) = N_a I_{E2,a}\int\frac{dt}{t}\sum_\nu \delta_\nu \frac{\theta^2_\nu(\frac{it}{2})}{\theta^2_1(\frac{it}{2})}\frac{\eta^3(it)}{\theta_\nu(0)}\prod_{\kappa=1}^3\frac{\theta_\nu(\phi^\kappa_{E2,a}it)}{\theta_1(\phi^\kappa_{E2,a}it)}. \quad (4.2.11)$$

However, this amplitude is not defined in the $\nu = 1$ sector, and additionally contains fermionic zero modes which must be regularised. The first issue is straightforward to resolve: provided that the model satisfies RR-charge cancellation, the $\nu = 1$ sector does not contribute. To address the second issue we must decide how to remove the zero mode from the trace over states: this is trivial once we express the amplitude in the open string channel. Since the integral over the modular parameter and the exponentiation of the partition function translates the sum over states into a product, it is clear that we must simply expand the partition function as a series in $e^{-\pi t}$ and remove the $O(1)$ term in the $\nu = 1, 2$ sectors. The piece subtracted turns out to be cancelled over all the contributions by the RR cancellation condition, and so we can ignore it.

We must now perform the sum over spin structures. To this end, we use the identity

$$\left(\frac{\theta_\nu(z)\theta_1'(0)}{\theta_1(z)\theta_\nu(0)}\right)^2 = \frac{\theta_\nu''(0)}{\theta_\nu(0)} - \partial^2 \log \theta_1(z) \tag{4.2.12}$$

to write the partition function as

$$Z(E2, D6_a) = N_a I_{E2,a} \int \frac{dt}{t} \sum_{\nu\neq 1} \delta_\nu \left[\theta_\nu''(0) - \theta_\nu(0)\partial^2 \log \theta_1(\frac{it}{2})\right] \times \frac{1}{2\pi\theta_1'(0)} \prod_{\kappa=1}^{3} \frac{\theta_\nu(\phi^\kappa_{E2,a} it)}{\theta_1(\phi^\kappa_{E2,a} it)}. \tag{4.2.13}$$

It is straightforward to show that, if we were to include the $\nu = 1$ term to the above, it would not contribute, and hence we can perform the spin structure sum using the usual Riemann theta identities. The term proportional to $\theta_\nu(0)$ vanishes, and we have remaining an expression identical to that found in gauge threshold corrections. The calculation is then that of [39], including the cancellation of divergences arising in the NS sector. The result is then

$$\frac{\mathrm{Pfaff}'(D_F)}{\sqrt{\det'(D_B)}} = \prod_k \left(\frac{\Gamma(1-2\phi^1_{E2,k})\Gamma(1-2\phi^2_{E2,k})\Gamma(1+2\phi^1_{E2,k}+2\phi^2_{E2,k})}{\Gamma(1+2\phi^1_{E2,k})\Gamma(1+2\phi^2_{E2,k})\Gamma(1-2\phi^1_{E2,k}-2\phi^2_{E2,k})}\right)^{-4I_{E2,O6_k}} \prod_{a=\{a,a'\}} \left(\frac{\Gamma(1-\phi^1_{E2,a})\Gamma(1-\phi^2_{E2,a})\Gamma(1+\phi^1_{E2,a}+\phi^2_{E2,a})}{\Gamma(1+\phi^1_{E2,a})\Gamma(1+\phi^2_{E2,a})\Gamma(1-\phi^1_{E2,a}-\phi^2_{E2,a})}\right)^{N_a I_{E2,a}}. \tag{4.2.14}$$

4.2.3 Vertex Operators and Zero Modes

Massless strings with at least one end on the E2 brane are the zero modes of the instanton. The most important of these (in that they are always present) are the fermionic modes with both ends on the E2 brane; these are the modes associated with the two broken supersymmetries in the spacetime dimensions. The vertex operator for toroidal models is thus

$$V_{\theta}^{-1/2} = \theta_i \tilde{S}^i \; e^{-\frac{\phi}{2}} \; \theta^I_{\alpha} \Theta^{\alpha} \tag{4.2.15}$$

where the internal spin field is $\Theta^{\alpha} = \prod_{\kappa=1}^{3} e^{\pm\frac{i}{2}H^{\kappa}}$; it is the spectral flow operator for the internal dimensions, or alternatively the internal part of the supercharge. This generically has four components after the GSO projection, and thus we have the eight modes of (broken) $N = 4$, but only the two modes that preserve the same $N = 1$ supersymmetry as the branes and orientifolds in the model will contribute. In practice this involves choosing the correct H-charges to transform a fermion into a boson in the internal dimensions, and without loss of generality we shall take this to be $\prod_{\kappa=1}^{3} e^{-\frac{i}{2}H^{\kappa}}$.

Massless fermions at an intersection between the E2 brane and a D6 brane of the model are internal zero modes of the instanton. They will not be relevant for the following analysis, but are in general vitally important for calculations; we list their vertex operators here for completeness:

$$\begin{aligned} V_{\lambda_a} &= \lambda^i_{a,I} e^{-\frac{\phi}{2}} \Delta \prod_{\kappa=1}^{3} e^{i(\phi^{\kappa}_{aE2}-1/2)H^{\kappa}} \sigma_{\phi^{\kappa}_{aE2}} \\ V_{\bar{\lambda}_b} &= \bar{\lambda}^i_{b,I'} e^{-\frac{\phi}{2}} \bar{\Delta} \prod_{\kappa=1}^{3} e^{i(\phi^{\kappa}_{E2b}-1/2)H^{\kappa}} \sigma_{\phi^{\kappa}_{E2b}} \end{aligned} \tag{4.2.16}$$

where I is the intersection number running from $1, ..., [\Pi_{E2} \cap \Pi_a]^+$, and I' runs from $1, ..., [\Pi_{E2} \cap \Pi_b]^-$ (note that $\phi_{E2b} - 1/2 = -(\phi_{bE2} - 1/2)$); $i = 1, .., N_a(N_b)$ is the Chan-Paton index. Δ and $\bar{\Delta}$ are the boundary-changing operators in the four non-compact dimensions which interpolate between Dirichlet and Neumann boundary conditions; their OPE is

$$\Delta(z)\bar{\Delta}(w) \sim (z - w)^{-1/2}. \tag{4.2.17}$$

4.3 Yukawa Coupling Corrections

A straightforward analysis of the possible diagrams in the instanton calculus shows that in the presence of fermionic zero modes an $E2$-instanton cannot contribute to a Yukawa term in the superpotential for the quarks. However, if there exists one or more sLag cycles for which there are no intersections with any branes in the model, then a contribution to this term is possible through a one-loop annulus diagram. Typically this involves the separation of the $E2$-instanton from each D6 brane in one sub-torus only, so that were the $E2$ brane replaced by a D6 brane wrapping the same three-cycle and the separations reduced to zero, this brane would preserve different $N = 2$ supersymmetries with each of the branes in the model. This possibility arises generically but not in every case in model-building, so implies a new moderate constraint, in order to benefit from the consequences of these instantons.

The superpotential term generated by these instantons can thus be determined from section 4.2. It involves three superfields on an annulus diagram, and thus two fermions and one boson together with two fermionic supersymmetry zero modes. We can then write

$$W_{np} = \int \mathrm{d}^4x \mathrm{d}^2\theta \langle V^0_{\phi_{ab}} V^{1/2}_{\psi_{bc}} V^{1/2}_{\psi_{ca}} V^{-1/2}_{\theta} V^{-1/2}_{\theta} \rangle_{a,E2} \frac{\mathrm{Pfaff}'(D_F)}{\sqrt{\det'(D_B)}} e^{-S^0_{E2}}. \tag{4.3.1}$$

Since there are now three picture-changing operators in the above amplitude, to obtain a non-zero result we must apply each operator to a different sub-torus direction; this is since each internal fermionic correlator must have zero nett charge, and thus the charges introduced by the supersymmetry zero modes must be cancelled by those of the PCOs. The amplitude will then have no momentum prefactors, and thus will not factorise onto a scalar propagator; this is explicitly shown in appendix B.2.

Having determined the Pfaffian and Tree-level factors in section 4.2, we must now determine the annulus correlator. The total amplitude can be written as

$$\langle V^0_{\phi_{ab}} V^{1/2}_{\psi_{bc}} V^{1/2}_{\psi_{ca}} V_\theta^{-1/2} V_\theta^{-1/2}\rangle_{a,E2} = \phi_{(ab)}\psi_{(bc)\alpha}C^{\alpha\beta}\psi_{(ca)\beta}\ \theta_1\theta_2$$
$$\times \int \mathrm{d}t\ it \prod_{i=2}^{3}\int_0^{it} \mathrm{d}z_i \prod_{j=1}^{2}\int_{1/2}^{1/2+it} \mathrm{d}w_j \langle e^{\frac{\phi}{2}(z_2)} e^{\frac{\phi}{2}(z_3)} e^{-\frac{\phi}{2}(w_1)} e^{-\frac{\phi}{2}(w_2)}\rangle \lim_{x_1\to z_1}\lim_{x_2\to z_2}\lim_{x_3\to z_3}$$
$$\times (x_1-z_1)(x_2-z_2)^{1/2}(x_3-z_3)^{1/2}\langle \tilde{S}^1(z_2)\tilde{S}^1(z_3)\tilde{S}^2(w_1)\tilde{S}^2(w_2)\rangle\langle \prod_{i=1}^{3} e^{ik_i\cdot X(z_i)}\rangle$$
$$\sum_{\{y_1,y_2,y_3\}=P(x_1,x_2,x_3)} \cdot \prod_{\kappa=1}^{3}\sqrt{\frac{2}{\alpha'}}\langle \partial\bar{X}^\kappa(y_\kappa)\sigma_{\phi^\kappa_{ab}}(z_1)\sigma_{\phi^\kappa_{bc}}(z_2)\sigma_{\phi^\kappa_{ca}}(z_3)\rangle$$
$$\times \langle e^{iH^\kappa(y_\kappa)} e^{i(\phi^\kappa_{(ab)}-1)H^\kappa(z_1)} e^{i(\phi^\kappa_{(bc)}-1/2)H^\kappa(z_2)} e^{i(\phi^\kappa_{(ca)}-1/2)H^\kappa(z_3)} e^{-\frac{i}{2}H^\kappa(w_1)} e^{-\frac{i}{2}H^\kappa(w_2)}\rangle \tag{4.3.2}$$

where z_1 has been fixed to 0, and the angles ϕ^κ_{ab}, ϕ^κ_{bc} and ϕ^κ_{ca} are external (hence $\phi^\kappa_{ab}+\phi^\kappa_{bc}+\phi^\kappa_{bc}=2$ and $\sum_{\kappa=1}^{3}\phi^\kappa=2$). $C^{\alpha\beta}=i(\Gamma^1\Gamma^2)^{\alpha\beta}$ is the charge conjugation operator. The above amplitude can be evaluated using the techniques outlined in appendix A.2 (and [30,31]). The perhaps unexpected form of the spin-field correlator is explained in appendix B.1. The most non-trivial part of the above is that involving the boundary-changing operators, which are dominated by worldsheet-instanton effects. We split $\partial X = \partial X_{qu} + \partial X_{cl}$, for which ∂X_{qu} has boundary conditions such that all vertices have no displacements, whereas ∂X_{cl} absorbs the displacements between the vertices. We have

$$\langle \partial\bar{X}_{qu}\prod_{i=1}^{N}\sigma_{\phi_i}\rangle = 0 \tag{4.3.3}$$

and thus the amplitude is dominated by worldsheet instanton effects. To show the above, we consider

$$\begin{aligned}\frac{\langle \partial\bar{X}(w)\prod_{i=1}^{N}\sigma_{\phi_i}(z_i)\rangle}{\langle\prod_{i=1}^{N}\sigma_{\phi_i}(z_i)\rangle} &\sim (w-z_i)^{-\phi_i} \qquad w\to z_i\\ \frac{\langle \bar\partial\bar{X}(\bar w)\prod_{i=1}^{N}\sigma_{\phi_i}(z_i)\rangle}{\langle\prod_{i=1}^{N}\sigma_{\phi_i}(z_i)\rangle} &\sim (\bar w-\bar z_i)^{\phi_i-1} \qquad \bar w\to \bar z_i\end{aligned} \tag{4.3.4}$$

and construct a set of differentials satisfying the above local monodromies and periodicity of the worldsheet; these are the differentials given in [30,31,57]. We then apply the global monodromy

$$\int_{\gamma_a} \mathrm{d}w\partial\bar{X} + \mathrm{d}\bar{w}\bar\partial\bar{X} = \bar{v}_a \tag{4.3.5}$$

where γ_a are a set of N paths on the worldsheet and v_a are the displacements between the vertices in the target space. There are N independent differentials that comprise $\partial\bar{X}$ and $\bar{\partial}\bar{X}$, and so the global monodromies determine the coefficients by linear algebra. Since the paths are independent, the equations are non-degenerate and for $v_a = 0$ we must set all of the coefficients to zero, establishing the claim above. Defining the $N \times N$ matrix W (i' runs through the set $\{z_1, z_2, ..., z_{N-2}\}$ and i'' denotes the complementary set $\{z_{N-1}, z_N\}$):

$$\begin{aligned} W_a^{i'} &= \int_{\gamma_a} \mathrm{d}z\omega^{i'}(z) \\ W_a^{i''} &= \int_{\gamma_a} \mathrm{d}\bar{z}\bar{\omega}^{i''}(\bar{z}) \end{aligned} \tag{4.3.6}$$

in terms of the N cut differentials $\{\omega^{i'}, \omega^{i''}\}$, we can then write (after applying the doubling trick to relate ∂X to $\bar{\partial}\bar{X}$)

$$\begin{aligned} \langle \partial\bar{X}(x) \prod_{i=1}^{N} \sigma_{\phi_i}(z_i)\rangle &= -\bar{v}_a(\bar{W}^{-1})^a_{i''}\omega^{i''}(x)\langle\prod_{i=1}^{N} \sigma_{\phi_i}(z_i)\rangle \\ \langle \bar{\partial}\bar{X}(\bar{x}) \prod_{i=1}^{N} \sigma_{\phi_i}(z_i)\rangle &= -\bar{v}_a(\bar{W}^{-1})^a_{i'}\bar{\omega}^{i'}(-\bar{x})\langle\prod_{i=1}^{N} \sigma_{\phi_i}(z_i)\rangle. \end{aligned} \tag{4.3.7}$$

The correlator is thus directly proportional to the displacements. Specialising now to the specific three-point function and using the prescription of [31] we have cycles $\{\gamma_a\} = \{A, B, C_2\}$ where A and B are the canonical cycles of the torus, and C_2 is the path passing between two vertices on the worldsheet. For this diagram, in each sub-torus there is one brane parallel to the $E2$ brane, and this represents a periodic cycle on the worldsheet. The prescription for the amplitude requires that we permute the vertices cyclically so that the periodic cycle passes along the real axis; writing $\{a^\kappa, b^\kappa, c^\kappa\}$ for the cyclic permutation of branes $\{a, b, c\}$ such that brane a^κ is parallel to the E2 brane in torus κ, we have

$$\begin{aligned} v_A &= \frac{1}{\sqrt{2}} n_A L_{a^\kappa} \\ v_B &= i\sqrt{2}(n_B \frac{4\pi^2 T_2^\kappa}{L_{a^\kappa}^\kappa} + y^\kappa) \\ v_{C_2} &= \frac{1}{\sqrt{2}} e^{i\phi^\kappa_{a^\kappa c^\kappa}}(n_C L^\kappa_{c^\kappa} + \Delta^\kappa) \end{aligned} \tag{4.3.8}$$

where Δ^κ is the shortest distance between the target-space intersections $b^\kappa c^\kappa$ and $c^\kappa a^\kappa$ along the brane c^κ, and y^κ is the distance between a^κ and the E2-brane. The phase $e^{i\phi^{\kappa}_{a^{\kappa}c^{\kappa}}}$ in the last line appears due to the orientation of brane a^{κ} relative to c^{κ}.

The exponential of the worldsheet instanton action depends upon the same displacements, and the amplitude can then be schematically written as

$$\langle V^{0}_{\phi_{ab}} V^{1/2}_{\psi_{bc}} V^{1/2}_{\psi_{ca}} V_{\theta}^{-1/2} V_{\theta}^{-1/2} \rangle_{a,E2} = \phi_{(ab)} \psi_{(bc)\alpha} C^{\alpha\beta} \psi_{(ca)\beta} \big(\theta_1\theta_2 - \theta_2\theta_1\big)$$
$$\int \mathrm{d}t \int \mathrm{d}z_2 \mathrm{d}z_3 \sum_{i=1}^{6} \prod_{\kappa=1}^{3} \Bigg(\sum_{n^{\kappa}_A, n^{\kappa}_B, n^{\kappa}_C} \frac{(f^{\kappa}_i)^a \bar{v}^{\kappa}_a}{\sqrt{\alpha'}} e^{-\frac{S^{\kappa}}{2\pi\alpha'}} \Bigg). \tag{4.3.9}$$

Note that the action which appears here is the one-loop action as derived from the monodromy conditions, and depends on the integration variables. The crucial part is that the functions f_i arise from the different permutations of applications of the picture-changing operators. Each choice of f_i corresponds to a different contribution that is separately factorisable across the tori (and different from the perturbative Yukawa term owing to the $\bar{v}^{\kappa}_a$ factors). Factorisability is in general lost upon performing the integral since there are no poles. However, since the functions are different, even if the integrals were dominated by a particular region of the moduli space, we would have Yukawa matrix corrections of the form

$$Y_{\alpha\beta} \supset \sum_{i=1}^{6} A^i_{\alpha} B^i_{\beta} \,. \tag{4.3.10}$$

This is a sum of six independent rank one matrices, giving a rank three Yukawa matrix as advertised in the introduction to this chapter. Note that the correction terms are suppressed relative to the perturbative superpotential by approximately the factor $e^{-S^0_{E2}}$; this provides not only rank 3 couplings but an explanation for the hierarchy in masses between the top quark and the others. At the level of this analysis there is no obvious explanation for the hierarchies between the 1^{st} and 2^{nd} generation; this could yet arise from the non-factorisation of the worldsheet instanton contribution. We leave this issue for future work.

Chapter 5

Noncommutativity from the string perspective: modification of gravity at a *mm* without *mm* sized extra dimensions

In this chapter we focus on type IIB string theory, using bosonic string theory as a toy model, and consider the noncommutative field theory that arises when a magnetic flux coupling to open strings on a brane is manifest in the macroscopic dimensions. We examine the phenomenon of UV/IR mixing in this framework. We then consider the effect on gravity, since the running of the closed string couplings is affected by the open string theory, feeling the effect of the noncommutativity. The work appears in [58].

5.1 Introduction

Gauge theories in which the coordinates are noncommuting,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{5.1.1}$$

are interesting candidates for particle physics, with curious properties (for general reviews of noncommutative gauge theories see refs. [29, 59, 60]). One whose consequences we would like to understand a little better is ultra-violet(UV)/infra-red(IR) mixing [61,62]. This is a phenomenon which gives rise to various pathologies in the field theory, making it, at best, difficult to understand. In this chapter we set about examining UV/IR mixing from the point of view of string theory with a background antisymmetric tensor ($B^{\mu\nu}$) field, which provides a convenient UV (and hence IR) completion. Along the way, as well as seeing how the pathological behaviour is smoothed out, we will outline the characteristic phenomenology of this general class of theories in the deep IR (i.e. at energy scales lower than those where noncommutative field theory is a good description): they resemble the $B = 0$ theories but with Lorentz violating operators which can be taken parametrically and continuously to zero by reducing the VEV of $B^{\mu\nu}$. As a bi-product we also show that the UV/IR mixing phenomenon extends to the gravitational sector (although a field theoretical interpretation for UV/IR mixing in gravity is difficult to obtain). This allows the curious possibility that gravity may be non-Newtonian on much longer length scales than those associated with the compact dimensions.

Because UV/IR mixing, and the particular problems and phenomena to which it gives rise, are rather subtle, we begin now with a detailed discussion of exactly what questions we would like the string theory to answer, after which we restate our findings in more precise terms. UV/IR mixing has its origin in the fact that the commutation relations intertwine large and small scales. At the simplest level, in a gedanken experiment where x_1 and x_2 do not commute, the uncertainty relation $\Delta x_1 \Delta x_2 \sim i\theta^{12}$ together with the usual Heisenberg uncertainty $\Delta x_1 \Delta p_1 \sim i$ imply $\Delta x_2 \sim -\theta^{12} \Delta p_1$: short distances in the 1 direction are connected to small momenta in the 2 direction and vice versa. At the field theory level, this intertwining of UV and IR leads to the infamous phenomenon of UV/IR mixing in the non-planar Feynman diagrams: nonplanar diagrams are regulated in the UV but diverge in the IR. Essentially, contrary to the standard picture of the Wilsonian effective action, heavy modes do not decouple in the IR so that, for example, trace U(1) factors of the gauge group run to a free field theory in the IR even if there are no massless excitations [63–67].

The agent responsible for these unusual and challenging features of noncommutative gauge field theories is the Moyal star product,

$$(\phi * \varphi)(x) \equiv \phi(x)\ e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}\ \varphi(x), \tag{5.1.2}$$

used in their definition. It induces a phase factor $\exp \frac{i}{2}k.\theta.q$ in the vertices, where k is an external momentum and q is a loop-momentum. This oscillating phase regulates the nonplanar diagrams in the UV, which can most easily be expressed using Schwinger integrals: for example the one-loop contribution to vacuum polarisation takes the form (c.f. [63–66, 68, 69])

$$\Pi_{\mu\nu}(k) \sim \int \frac{dt}{t} e^{-\frac{\tilde{k}^2}{4t}} \ldots \tag{5.1.3}$$

where $\tilde{k}^\mu = \theta^{\mu\nu}k_\nu$ and the ellipsis stands for factors independent of $\tilde{k}$. The exponential factor in the integrand is a regulator at $t \sim \tilde{k}^2 \sim k^2/M_{NC}^4$, where we define the generic noncommutativity scale by $\theta^{\mu\nu} = \mathcal{O}(M_{NC}^{-2})$. Thus the diagram, which without this factor would be UV divergent, is regulated but only so long as $\tilde{k} \neq 0$. The result is that the UV divergences of the planar diagrams reappear as IR poles in $\tilde{k}$ in the nonplanar diagrams.

These divergences are problematic. First they signal a discontinuity because the $\tilde{k} \to 0$ limit of the integrals is not uniformly convergent: physics in the limit $\theta \to 0$ does not tend continuously to the commutative theory. Moreover they lead to alarming violations of Lorentz invariance. For example, the lightcone is generally modified to a lightwedge [68, 70]. This is in sharp disagreement with observation. Furthermore in noncommutative gauge theory, the trace U(1) photon has a polarisation tensor given by [62]

$$\Pi_{\mu\nu} = \Pi_1(k^2, \tilde{k}^2)\ \left(k^2 g_{\mu\nu} - k_\mu k_\nu\right) + \Pi_2(k^2, \tilde{k}^2)\, \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^4}\,, \tag{5.1.4}$$

where the additional term $\sim \Pi_2$ is multiplied by a Lorentz violating tensor structure. It is absent in supersymmetric theories [62], but since supersymmetry is broken, we expect it to be at least of order M_{SUSY}^2 times by some factor logarithmic in $\tilde{k}$ (where M_{SUSY} is a measure of the supersymmetry breaking). The result is a mass of order M_{SUSY} for certain polarisations of the trace-U(1) photon while other polarisations remain massless [71]. Gymnastics are then required to prevent this trace U(1) photon mixing with the physical photon.

Clearly then, the outlook from the perspective of field theory is gloomy; because the IR singularities are a reflection of the fact that field theory is UV divergent, any attempt to resolve them without modifying the UV behaviour of the field theory is doomed. With this understanding, the general expectation is for a more encouraging picture in a theory with a UV completion, such as string theory. A more precise argument is the following. First it is easy to appreciate that, without an explicit UV completion, noncommutative field theory is unable to describe physics in the IR limit. As noted in ref. [72] and in the specific context of string theory in ref [73, 74], UV/IR mixing imposes a IR cut-off given by $|k| > \Lambda_{IR} = \frac{M_{NC}^2}{\Lambda_{UV}}$. Inside the range

$$\Lambda_{IR}^{ij} \sim \frac{1}{|\theta_{ij}|\Lambda_{UV}} < |k| < \Lambda_{UV}, \tag{5.1.5}$$

the field theory behaves in a Wilsonian manner, in the sense that modes with masses greater than the UV cut-off do not (up to small corrections) affect the physics there. However outside this range the Wilsonian approach breaks down because modes above Λ_{UV} affect the physics below Λ_{IR}. Indeed this inequality makes it impossible to make statements about either the $\theta^{ij} \to 0$ limit or the $\tilde{k} \to 0$ limit within field theory. *In other words, a UV completion is needed not only to describe physics above Λ_{UV} but also physics below* Λ_{IR}, and in particular to discuss the existence or otherwise of discontinuities there. The picture is most obvious in the context of running of gauge couplings. Between Λ_{IR} and Λ_{UV} the effective action accurately describes the running of the trace U(1) gauge coupling regardless of what happens above Λ_{UV}. Below Λ_{IR}, UV physics intervenes. For example a period of power law running due to KK thresholds in the UV is mirrored by the "inverse" power law running in the IR. Now, the precise UV completion may take various forms, but suppose for example that it acts like a simple exponential cut-off, $e^{-\frac{\Lambda_{UV}^2}{4t}}$, in the Schwinger integral. The planar diagrams are regulated in the usual manner, but the nett effect of the noncommutativity for the nonplanar diagrams is that the UV cut-off Λ_{UV}^2 is replaced by $\Lambda_{eff}^2 = 1/(\tilde{k}^2 + \Lambda_{UV}^{-2})$ [61]. In this case when $\tilde{k} \ll \Lambda_{UV}^{-1}$ (i.e. when we are below the IR cut-off) we would have

$$\Pi_{\mu\nu} \approx \Pi_1(k^2, \Lambda_{UV}^{-2}) \left(k^2 g_{\mu\nu} - k_\mu k_\nu\right) + \Pi_2(k^2, \Lambda_{UV}^{-2})\, \Lambda_{UV}^4 \tilde{k}_\mu \tilde{k}_\nu\,, \tag{5.1.6}$$

and normal Wilsonian behaviour would be restored, with the couplings matching those at the UV cut-off scale. Of course there is no reason to suppose that such a cut-off in any way resembles what actually happens in string theory, and to discuss the nature of the theory below Λ_{IR} requires full knowledge of the real UV completion. *What then are our general expectations for physics below Λ_{IR}? Does it correspond to an effective field theory? If so, what happens to the Lorentz violating divergences in the IR?*

These are the precise questions we would like to explore, using a framework in which the noncommutative gauge theory is realised as a low-energy effective theory on D-branes [29, 75–77]. Our arguments are based on the two point function as calculated on D-branes in the background of a non-zero B-field [73,74,78,79]. In such a theory, taking the zero slope limit in a particular way [29] ($\alpha' \to 0$ with $g_{\mu\nu} \sim \alpha'^2$) yields a noncommutative field theory in which the role of the noncommutativity parameter is played by the gauge invariant Born-Infeld field strength: indeed in this limit the open string metric and the noncommutativity parameter are given by [29]

$$G^{\mu\nu} = \left(\frac{1}{g-F} g \frac{1}{g+F} \right)^{\mu\nu} \tag{5.1.7}$$

with $F_{\mu\nu} = 2\pi\alpha' B_{\mu\nu}$, $B_{\mu\nu}$ being the (magnetic) field strength, and

$$\theta^{\mu\nu} = -2\pi\alpha' \left(\frac{1}{g-F} F \frac{1}{g+F} \right)^{\mu\nu} \tag{5.1.8}$$

respectively (we will henceforth restrict ourselves to noncommutativity in the space directions which we will label ij). The theory at finite α' provides a convenient UV completion of the noncommutative gauge theory. The UV "cut-off" acquires a physical meaning: it is the scale above which the noncommutative field theory description is invalid and string modes become accessible, and is of order

$$\Lambda_{UV} = 1/\sqrt{\alpha'}. \tag{5.1.9}$$

The IR "cut-off" is accordingly given by

$$\Lambda_{IR} = \sqrt{\alpha'} M_{NC}^2, \tag{5.1.10}$$

and, likewise, physics below this scale is best understood by performing a string calculation. We will rather loosely continue referring to the scale Λ_{IR} as the IR cut-off although of course we are chiefly interested in exploring the effective theory below it.

What we will show in this chapter is that *the one-loop effective theory in the $k \to 0$ limit (including any threshold contributions) is the same as the commutative $\theta = 0$ theory*, and in particular there are no IR divergences. Below Λ_{IR}, physics differs from the $\theta = 0$ physics only by nonsingular residual effects that are calculable in any specific model, and we will estimate their magnitude. In addition we point out that the two point function of the graviton also gets stringy contributions at one-loop which can modify gravity right down to Λ_{IR}: if for example $M_{NC} \sim 1\text{TeV}$ and $M_s \sim M_{Pl}$, then gravity is modified at a *mm* even when there are no large extra dimensions. This is an effect equivalent to the one described for the gauge theory however there is no simple effective field theory description and it is difficult to understand in terms of "planar" and "nonplanar".

The rest of the chapter is organised as follows. In the next section, we will discuss and determine the general form of UV/IR mixing in noncommutative field theory which is embedded in string theory. In section 3 and 4, the mentioned general characteristics of UV/IR mixing will be justified with explicit amplitude calculations based on bosonic and superstring models. In section 5, we will analyse how the graviton two point function is modified by noncommutativity. In section 6, we will discuss how noncommutativity in string theory may lead to a modification in the IR property of gravity. We will also discuss its phenomenological implications.

5.2 General remarks on UV/IR mixing in string theory

Assuming that the string theory amplitudes are finite (as issue to which we return in due course), it is natural that the IR singularities should be cured in much the same way as UV singularities are, since they are intimately related: they are essentially the same singularities. It is also natural that string theory should cure discontinuities afflicting the field theory; we certainly expect a string amplitude calculated at non-zero F, which is after all a rather mild background, to tend continuously to the one calculated at $F = 0$. What is more striking is that in a non-supersymmetric theory the Lorentz violating Π_2 term also tends to zero as $\tilde{k}^2/\alpha'$ below the IR cut-off, reminiscent of the field theory behaviour with the naive Schwinger cut-off.

Consider nonplanar annulus amplitudes in bosonic string theory on a Dp-brane. As we shall see, the general structure of a one loop diagram can be very heuristically written as

$$A_{NP} \sim \int \frac{dt}{t} t^{-\frac{(p+1)}{2}} e^{-\tilde{k}^2/4t} f(t). \tag{5.2.1}$$

The function $f(t)$ includes kinematic factors as well as sums over all the open string states in the loop. The integration parameter t is the parameter describing the annulus. In the field theory limit $\alpha' \to 0$ we recover the expected nonplanar field theory contribution, with t playing the role of a Schwinger parameter. In addition all but the massless open strings (and in this case the tachyon whose contribution we discard) do not contribute in this limit. In the present discussion we are of course not interested in taking the field theory limit but will instead keep α' finite. The crucial feature of the amplitudes governing the IR behaviour is that the nonplanar integrands always come with a factor $e^{-\tilde{k}^2/4t}$ irrespective of whether we are above or below Λ_{IR}. When $\tilde{k}^2 \gg \alpha'$ the integrand is killed everywhere in the stringy region $t < \alpha'$ and the amplitude is close to the field theoretical result. Indeed one may make a large t expansion rendering the amplitude identical to the field theoretical one. On the other hand in the area of most interest below the IR cut-off we have $\tilde{k}^2 \ll \alpha'$ and hence stringy $t < \alpha'$ regions also contribute to the integral. If the integrand is finite and free of singularities then in the limit as $k \to 0$ the amplitudes clearly tend continuously to their commutative equivalents. Thus the finiteness of the string amplitudes immediately guarantees that the $k \to 0$ limits and the $\theta \to 0$ limits give the same physics. Moreover in this limit we may expand the $e^{-\tilde{k}^2/4t}$ factor inside the integral. The nett result is that far below the IR cut-off one-loop amplitudes may be written as

$$A(\theta, k) \sim A(0, k)(1 + \lambda \frac{\tilde{k}^2}{\alpha'} + \ldots), \tag{5.2.2}$$

where λ is a factor including loop suppression and gauge couplings and the second piece is the leading term in the small $\tilde{k}^2/\alpha'$ expansion of the exponential factor. Note that the $A(0,k)$ prefactor includes the usual one-loop contributions of the commutative theory and hence all stringy threshold corrections. Thus although various compactification scenarios may result in vastly different threshold corrections, the leading effect of non-zero B field will always be of this form. (Extension to N-point amplitudes is trivial.)

Based on this generic expression for the amplitudes, phenomenology below Λ_{IR} takes on a characteristic form. First from the low energy point of view the nett effect of the non-zero B field is simply to take the non-planar contribution to thresholds of gauge couplings and move them down to the IR cut-off, inserting between Λ_{IR} and Λ_{UV} a region approximating conventional noncommutative field theory. Below Λ_{IR}, the leading deviation from the commutative theory (including all its stringy thresholds) has a factor $\tilde{k}^2/\alpha'$, with the dimensionality being made up by powers of α'.

Thus for example the Π_2 term is of the form

$$\Pi_2(k^2,\tilde{k}^2) \sim \lambda(\tilde{k}^2\alpha')^2 \tag{5.2.3}$$

in a non-supersymmetric theory and

$$\Pi_2(k^2,\tilde{k}^2) \sim \lambda(\tilde{k}^2\alpha')^2 \frac{M_{SUSY}^2}{\alpha'} \tag{5.2.4}$$

in a theory with supersymmetry softly-broken at a scale M_{SUSY}. (Note that the factor of $(\tilde{k}^2\alpha')^2$ is simply to undo the power of $\tilde{k}^{-4}$ in the above definition of Π_2.) This introduces a birefringence into the trace-U(1) photon, a polarisation dependent velocity shift of order

$$\Delta v \sim c\frac{\lambda M_{SUSY}^2 M_s^2}{M_{NC}^4}. \tag{5.2.5}$$

This effect is much milder than the naive expectation and can be made phenomenologically acceptable with a large M_{NC} even if the physical photon is predominantly made of trace U(1) photon as described in ref. [72]. The model dependent issue here which we will expand upon in the following sections is the coefficient λ which encapsulates the strength of the one-loop contributions (i.e. threshold corrections to couplings) relative to the tree level ones.

If the physical photon is decoupled from the trace U(1) photon (see for example [80] where the trace U(1) photon becomes weakly coupled in the IR and forms part of a hidden sector to break and mediate supersymmetry), then there can be interesting implications for gravity. Consider a theory where the physically observed Planck scale receives significant one-loop threshold corrections from the open string sector. This contribution can be computed from the two point function of the gravitons with the open string modes running in the loop. To gain some more intuition on what effects of noncommutativity might be, we turn to an effective field theory description. A reasonable (but, as it turns out, incorrect) guess for the effective field theory coupling the open string modes to the graviton is a lagrangian of the form

$$\mathcal{L} = \int d^4x \sqrt{-g} g^{\mu\mu'} g^{\nu\nu'} \frac{F_{\mu\nu} * F_{\mu'\nu'}}{4g^2}, \tag{5.2.6}$$

where there is, note, no star product between the "closed string metric" or in its determinant. The desired contribution can be computed from the two point function of the gravitons with the gauge bosons running in the loop, and thus our effective field theory above would generate "planar" and "non-planar" diagrams exactly as in the pure gauge case, the crucial point being the presence of a Moyal phase coming from the vertices. Thus one might expect that in string theory, turning on a B-field would separate planar and non-planar contributions to the graviton two point function, in much the same way as for the photon. Thanks to UV/IR mixing the nonplanar contributions would change all the way down to Λ_{IR} below which they would asymptote to the values of the commutative theory. There, the leading deviation in Planck's constant from that of the purely commutative theory should be precisely as described above for the gauge couplings. As we will see in the section 5 the true picture is actually more subtle than this [1]. Nevertheless the effect we described persists; namely that sub-leading $\tilde{k}^2/\alpha'$ suppressed corrections in the two point function of the graviton lead to a modification of gravity at energy scales higher than Λ_{IR}.

[1]In the field theory limit, an effective vertex involving a graviton and two photons exists (and indeed we compute it), but there is no such simple Lagrangian from which it could be derived.

5.3 UV/IR mixing in the bosonic string

We will first look at the 2-point function on the annulus for pure QED, equivalent to the noncommutative Yang-Mills action

$$S = -\int \frac{1}{4} F_{\mu\nu} * F^{\mu\nu} \tag{5.3.1}$$

The contributions to the 2-point amplitudes on Dp-branes in a non-compact 26-dimensional volume requires open string vertex operators

$$V = g_{D_p} \epsilon_\mu \partial X^\mu e^{ik\cdot x}. \tag{5.3.2}$$

which have been appropriately normalised ($g_{D_p}^2 = (2\pi)^{p-2} g_c (\alpha')^{\frac{p-3}{2}}$). This gives the amplitude

$$\begin{aligned} A_2(k,-k) &= -2\alpha' g_{D_p}^2 V_p \int_0^\infty \mathrm{d}t\, (8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} \eta(it)^{-24} \times \\ & \int_0^t dx\; e^{-2\alpha' k.G(x,x').k} \left(\varepsilon_1.G_{xx'}.\varepsilon_2 - 2\alpha'(\varepsilon_1.G_x.k)(\varepsilon_2.G_{x'}.k)\right)\Big|_{x'=0}. \end{aligned} \tag{5.3.3}$$

Here x, t play the role of *dimensionless* Feynman and Schwinger parameters respectively. At this point, we should comment that throughout this chapter we shall take the fundamental domain of the annulus to be $[0, 1/2] \times [0, it]$.

Note that we write the measure with integration over all of the vertices, and then use the annulus' translation invariance to fix one vertex, including a volume factor of t. The one-loop Green's functions required depend on whether the diagram is planar or nonplanar, and are given by [73, 78, 81]

$$G^{\alpha\beta}(x,x') = I_0 \delta^{\alpha\beta} + J\frac{(\theta^2)^{\alpha\beta}}{\alpha'^2} + K\frac{\theta^{\alpha\beta}}{\alpha'}, \tag{5.3.4}$$

where, for the planar case,

$$I_0^P(x-x') = \log|t\frac{\theta_1(\frac{x-x'}{t},\frac{i}{t})}{\eta^3(i/t)}|, \qquad J^P = 0, \qquad K^P(x-x') = -\frac{i}{4}\epsilon(x-x'), \tag{5.3.5}$$

and for the nonplanar case,

$$I_0^{NP}(x-x') = \log t\frac{\theta_4(\frac{x-x'}{t},\frac{i}{t})}{\eta^3(i/t)}, \qquad J^{NP} = \frac{-1}{8\pi t}, \qquad K^{NP}(x+x') = \pm\frac{\pi}{t}(x+x'), \tag{5.3.6}$$

where the $+(-)$ in K^{NP} applies for the outer (inner) boundary. The feature of these expressions which ensures the regularisation of the nonplanar diagram is the contraction $k.G.k$ appearing in the exponent of the integrand. We find

$$-2\alpha' k.G^P.k = -2\alpha' k^2 I_0^P, \tag{5.3.7}$$

$$-2\alpha' k.G^{NP}.k = -2\alpha' k^2 I_0^{NP} - \frac{\tilde{k}^2}{8\pi\alpha' t}. \tag{5.3.8}$$

Having established the Green's functions, we can perform the integration by parts in equation (5.3.4) to extract the kinematics. Defining (for ease of notation) $\hat{\Pi}_2 = \Pi_2 \tilde{k}^{-4}$, we find

$$\begin{aligned} A_2^P &= \Pi_1^P(k^2, 0)[(\epsilon_1 \cdot \epsilon_2)k^2 - (\epsilon_1 \cdot k)(\epsilon_2 \cdot k)], \qquad (5.3.9)\\ A_2^{NP} &= \Pi_1^{NP}(k^2, \tilde{k}^2)[(\epsilon_1 \cdot \epsilon_2)k^2 - (\epsilon_1 \cdot k)(\epsilon_2 \cdot k)] + \hat{\Pi}_2(k^2, \tilde{k}^2)[(\epsilon_1 \cdot \tilde{k})(\epsilon_2 \cdot \tilde{k})], \end{aligned}$$

where the standard gauge running is given by the formula

$$\Pi_1 = 4\alpha'^2 g_{D_p}^2 \int_0^\infty \mathrm{d}t\, Z(t) e^{-\frac{\tilde{k}^2}{4\alpha' t}\times a} \int_0^t \mathrm{d}x\; e^{-2\alpha' k^2 I_0} (\dot{I}_0)^2, \tag{5.3.10}$$

while the Lorentz-violating piece is given by

$$\hat{\Pi}_2 = 4\pi^2 g_{D_p}^2 \int_0^\infty \frac{dt}{t^2} Z(t) e^{-\frac{\tilde{k}^2}{4\alpha' t}} \int_0^t \mathrm{d}x\; e^{-2\alpha' k^2 I_0^{NP}}. \tag{5.3.11}$$

In eqn.(5.3.10) $a = 0$ or 1 for the planar or nonplanar case respectively. The term $Z(t)$ is the partition function for the model, which we shall take throughout to be

$$Z(t) = (8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} \eta(it)^{d-24}. \tag{5.3.12}$$

The parameter d is inserted to remove adjoint scalars from the theory: there are $p-1$ physical polarisations of the photon, and the remaining $25-p$ modes are scalars, so we can interpret the parameter d as removing d of these modes. This is performed either by considering string theory in $26-d$ dimensions [78,82], or taking the spacetime to be, for example, $\mathbb{R}_{p+1} \times \mathbb{R}_{25-p-d} \times \mathbb{R}_d/Z_N$ with the p-brane at a singularity such that the orbifold projection removes the scalars [73]. In this way, we can alternatively consider d as modelling the effect of compactified dimensions, and hence we shall refer to d "compact dimensions" throughout.

The forms (5.3.9)-(5.3.11) for the 1-loop amplitudes were already discussed in ref. [78] in the field theory limit. However it is important to note now that we have not taken any field theory limit and yet the $\tilde{k}$ dependence is already entirely contained within the $\tilde{k}^2$ term: the sole effect of noncommutativity is to truncate the Schwinger integration to $2\pi t > \frac{\tilde{k}^2}{\alpha'}$, even in the full string expression.

Thus there are two regimes that we will consider. The first is the regime where $\tilde{k}^2/\alpha' \gg 1$. In this case the Schwinger integral is truncated to the region $2\pi t \gg 1$ and the integral is well approximated by the $t \to \infty$ limit. The second regime is where $\tilde{k}^2/\alpha' \ll 1$. In this case much of the Schwinger integral is over the region where $2\pi t \ll 1$ and one expects the $t \to \infty$ limit to be a poor approximation. In this limit a good approximation to the integral requires a modular transformation of the ϑ and η functions to the closed string channel. It is natural to think of α' as playing the role of the UV cut-off to the field theory, $\alpha' \equiv \Lambda_{UV}^{-2}$, and then this regime corresponds precisely to

$$k \ll \frac{M_{NC}^2}{\Lambda_{UV}} \equiv \Lambda_{IR}, \tag{5.3.13}$$

i.e. the region in the deep IR where the field theory computation breaks down.

Indeed the integral will become sensitive to the global structure of the compactified dimensions since the $t \to 0$ UV end of it corresponds to closed string modes in the deep IR. Note that [83, 84] studied the connection between IR poles and closed string tachyons; we shall neglect these as we are interested in extracting results relevant to consistent theories. There may be other thresholds as well as the string scale where for example winding modes of the compact dimensions start to contribute in the integral. In order for there to be an effective field theory description below Λ_{IR} these effects should add contributions independent of p. In order to incorporate these effects one can divide the Schwinger integral into regions $t \in [0, 1]$ and $t \in [1, \infty]$ where the two approximations are valid.

5.3.1 Brief review of planar diagrams

The methods for obtaining the low energy behaviour of string diagrams in order to derive the effective four-dimensional field theories have been well covered elsewhere [82, 85]. Since the integrals do not contain any evidence of the non-commutativity, as can be seen from the Green's functions, the only difference from the $B = 0$

calculation in this case is a phase dependence on the ordering of the vertices. The reader is referred to the Appendix for details of the $t \to \infty$ limit for planar diagrams, which reviews some of the basic techniques that we will be using. We shall consider d dimensions transverse to the brane to be compactified with a radius close to or at the string scale, although we shall pay special attention to the case $d = 0$. The contributions to Π_1 and $\hat{\Pi}_2$ in this limit are given in equations (C.1.6) and (C.1.7).

The $t \to 0$ limit of the planar diagrams is the UV contribution, that is $t \in [0, 1]$ indicating momenta much higher than the string scale. It is well known as the planar contribution to the string threshold correction, however we present it here in order to emphasise the way that string theory is thought to render such contributions finite. Let us compute the $t \to 0$ contribution to the two point functions for a Dp-brane in 26 non-compact dimensions. We modular transform the expressions to give for the partition function

$$(8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} t^{12} e^{\frac{2\pi}{t}} (1 + 24\, e^{-\frac{2\pi}{t}} + \ldots), \tag{5.3.14}$$

where, since we are assuming no compact dimensions, there are no winding modes, and thus

$$\begin{aligned} \Pi^P_{1\,UV} \to \frac{g^2_{D_p}}{16\pi^2(8\pi^2\alpha')^{\frac{(p-3)}{2}}} \int_0^1 \mathrm{d}t\, t^{\frac{(21-p)}{2}} (2t)^{-2\alpha' k^2} \int_0^1 dy \\ \left[24|\cos\pi y|^2 |\sin\pi y|^{-2-2\alpha' k^2} + 8|\cos\pi y|^2 |\sin\pi y|^{-2\alpha' k^2} \right]. \end{aligned} \tag{5.3.15}$$

Here we cannot neglect the "pole" pieces, but perform the integral in terms of beta functions and analytically continue in the momentum, using

$$\int_0^1 \mathrm{d}y |\cos\pi y|^a |\sin\pi y|^b = \frac{2}{\pi} B(\frac{a+1}{2}, \frac{b+1}{2}) \tag{5.3.16}$$

to give the zero-momentum limit

$$\Pi^P_{1_{UV}} = \frac{5 g^2_{D_p}}{2\pi^2(23-p)(8\pi^2\alpha')^{\frac{(p-3)}{2}}}, \tag{5.3.17}$$

a threshold contribution to the gauge couplings which is finite when $p < 23$.

Now, of course this computation is a cheat because it assumes that transverse space was non-compact. In a compact space, sooner or later in the $t \to 0$ limit we need to sum over winding sectors in the measure of the integral. Once the winding sectors are included the effective p is $p \equiv 25$ and the integral diverges. However this divergence is resolved in a way that is at least qualitatively well understood: the natural way to write $A_{2_{UV}}$ is with the parameter $S = \frac{\alpha'^2}{T}$ which reveals the expression to be in the $\alpha' k^2 \ll 1$ limit simply an IR pole due to a massless closed string tadpole. Indeed in this limit and when $p = 25$ the contribution from level n is proportional to

$$\int_0^{\alpha'} \frac{dT}{T^2} e^{-\frac{4\pi^2\alpha'}{T}(n-1)} = \int_{\alpha'}^{\infty} dS\, e^{-\frac{4\pi^2}{\alpha'}(n-1)\,S}, \tag{5.3.18}$$

as appropriate for closed string states with $m^2 = \frac{4\pi^2}{\alpha'}(n-1)$. Such tadpoles are a signal that we are expanding around the wrong vacuum, and the solution is to give a VEV to the relevant fields in order to remove them. In this way the background is modified by the presence of the tadpoles and the nett effect is that systems with > 2 codimensions (i.e. $p < 23$) are insensitive to the moduli of the transverse dimensions, whereas those with 1 or 2 codimensions get threshold corrections that are respectively linearly or logarithmically dependent on the size of the transverse dimensions, but are still believed to be finite even when supersymmetry is broken by the construction. In principle in certain tachyon-free non-supersymmetric cases one can resum the tadpole contributions to the tree-level perturbation series to achieve a finite result. The precise details are rather subtle and beyond the scope of this work, and the reader is referred to refs. [86–88] for more details.

5.3.2 Non-planar diagrams in the $\tilde{k}^2/\alpha' \gg 1$ limit

We now turn to the non-planar diagrams. Once we turn on the B-field, the presence of the $e^{-\frac{\tilde{k}^2}{8\pi\alpha' t}}$ regulating factor will cause the celebrated UV/IR mixing. We may treat the UV and IR contribution at the same time in the two limits $\tilde{k}^2/\alpha' \ll 1$ and $\tilde{k}^2/\alpha' \gg 1$. Consider the second of these limits. The integrand is killed in the region where $t \ll \tilde{k}^2/\alpha'$ and hence we may always use the large t limit of the integrand.

We obtain

$$\begin{aligned}
\Pi_1 &= \frac{g_{D_p}^2}{(4\pi)^{\frac{p+1}{2}}}\int_{2\pi\alpha'}^{\infty} dT\, T^{-\frac{(p-1)}{2}}\int_0^1 dy\,\left[(24-d)(1-2y)^2-8\right]e^{-Tk^2(y-y^2)-\frac{\tilde{k}^2}{4T}} \\
&\to \frac{g_{D_p}^2}{(4\pi)^{\frac{p+1}{2}}}\int_0^1 \mathrm{d}y\,\left[(24-d)(1-2y)^2-8\right]\left[\frac{4k^2y(1-y)}{\tilde{k}^2}\right]^{(\frac{p-3}{4})} \\
&\quad \times K_{\frac{p-3}{2}}(\sqrt{y(1-y)k^2\tilde{k}^2}) \\
&\approx \begin{cases} \frac{d}{3}g_{D_p}^2(4\pi)^{-\frac{p+1}{2}}\ln k^2\tilde{k}^2, & p=3, \\ \frac{d}{3}g_{D_p}^2(4\pi)^{-\frac{p+1}{2}}2^{p-5}\Gamma(\frac{p-3}{2})|\tilde{k}|^{3-p}, & p>3, \end{cases}
\end{aligned} \tag{5.3.19}$$

where in the last step we assumed $|k||\tilde{k}| \ll 1$ or in other words momenta $|k| \ll M_{NC}$. In the case $p=3$ this gives the same logarithmic running to a free field theory in the IR observed in the field theory. When $p>3$ we find power law running in the IR as described in [89]. The Lorentz-violating term $\hat{\Pi}_2$ is given by

$$\begin{aligned}
\hat{\Pi}_2 &= \frac{g_{D_p}^2}{(4\pi)^{\frac{p+1}{2}}}\int_{2\pi\alpha'}^{\infty} dT\, T^{-\frac{(p+3)}{2}}\int_0^1 dy\,(24-d)e^{-Tk^2(y-y^2)-\frac{\tilde{k}^2}{4T}} \\
&\approx (24-d)\frac{g_{D_p}^2}{(4\pi)^{\frac{p+1}{2}}}2^{p-1}\Gamma(\frac{p+1}{2})|\tilde{k}|^{-(p+1)}
\end{aligned} \tag{5.3.20}$$

and shows a similar power law behaviour in the IR. For $p=3$ and $d=22$ we reproduce the result of [78]. This behaviour is entirely in line with what one would expect from the field theory.

5.3.3 Non-planar diagrams in the $\tilde{k}^2/\alpha' \ll 1$ limit

In this limit one expects to find behaviour differing from noncommutative field theory. We now have to split the integral into two halves, $t>1$ and $t<1$. The first IR part is treated similarly to the previous section, except in this case we simply set $\tilde{k}=0$ in the integrand when we consider $\alpha' \to 0$, and should thus obtain the same results as in the planar case; it is straightforward to show that for $p>3$

$$\begin{aligned}
\Pi_{1_{IR}}^{NP} &\approx \frac{d}{3}\frac{g_{D_p}^2}{(4\pi)^{\frac{p+1}{2}}}\frac{2}{(p-3)}(2\pi\alpha')^{\frac{3-p}{2}}, \\
\hat{\Pi}_{2_{IR}}^{NP} &\approx (24-d)\frac{g_{D_p}^2}{(4\pi)^{\frac{p+1}{2}}}\frac{2}{(p+1)}(2\pi\alpha')^{\frac{-(p+1)}{2}}.
\end{aligned} \tag{5.3.21}$$

The contributions are roughly constant, and equal to those of the $\tilde{k}^2/\alpha' \gg 1$ limit when $\tilde{k}^2 = 4\alpha'$.

The second, UV, contribution for $t < 1$ is the most interesting, as it is this contribution which in field theory gives IR poles. We now modular-transform the expressions, and expand in powers of $e^{-\frac{2\pi}{t}}$. For no compact dimensions, we have

$$\Pi_{1\ UV}^{NP} \rightarrow \frac{g_{D_p}^2}{16\pi^2(8\pi^2\alpha')^{\frac{(p-3)}{2}}} \int_0^1 \mathrm{d}t\, t^{\frac{(21-p)}{2}} (2t)^{-2\alpha' k^2} e^{-\frac{\tilde{k}^2}{8\pi\alpha' t}} e^{-\frac{\pi\alpha' k^2}{2t}} \int_0^1 dy\, \sin^2 2\pi y. \tag{5.3.22}$$

Note that for $\frac{\tilde{k}^2}{\alpha'} \rightarrow 0$ the integration is finite and the integral goes continuously to that of the commutative contribution, i.e. we have

$$\Pi_1^{NP}(\theta) = \Pi_1^{NP}(\theta = 0)\left(1 + \mathcal{O}(\frac{\tilde{k}^2}{\alpha'})\right), \tag{5.3.23}$$

as promised in the Introduction; in other words, at momenta $k \ll \Lambda_{IR}$ *the Wilsonian gauge couplings return to the values they would have had for a completely commutative theory with the same gauge group.* Note that this statement is expected to be true even when $p \geq 23$ and in compact spaces for the following reason. In the finite examples we have seen, the effect of string theory is clearly to allow the limit $\tilde{k} \rightarrow 0$ to be taken continuously, and to give the same physics as $\theta = 0$. If this is true of any consistent UV completion, then it seems reasonable to assume that what's good for the planar diagrams is good for the nonplanar ones. In other words, if the diagrams are formally divergent, continuity demands that the vacuum shifts which remove the UV divergences (i.e. closed string tadpoles) in the $B = 0$ theory should do so in the $B \neq 0$ theory as well, up to $\mathcal{O}(\tilde{k}^2/\alpha')$ corrections. Note that this is true even though the non-planar diagrams do not factorise onto disks in the closed string channel; IR singularities arise from divergences in the partition function which are regulated by the $e^{-\frac{\tilde{k}^2}{8\pi\alpha' t}} e^{-\frac{\pi\alpha' k^2}{2t}}$ term, and so when these divergences are cancelled, so are the IR poles.

This reasoning leads one to expect that the $\hat{\Pi}_2$ term is regulated, since it should tend to zero as $\theta \rightarrow 0$. Let us check this by computing the final contribution which is

$$\begin{aligned} \hat{\Pi}_{2_{UV}}^{NP} &\rightarrow \frac{24 g_{D_p}^2}{(8\pi^2\alpha')^{\frac{(p+1)}{2}}} \int_0^1 \frac{\mathrm{d}t}{t^4} t^{\frac{(25-p)}{2}} t^{-2\alpha' k^2} e^{-\frac{\tilde{k}^2}{8\pi\alpha' t}} e^{-\frac{\alpha' k^2 \pi}{2t}} \int_0^1 dy \\ &\approx \frac{24}{19-p} \frac{g_{D_p}^2}{(8\pi^2\alpha')^{\frac{(p+1)}{2}}}. \end{aligned} \tag{5.3.24}$$

5.4 Supersymmetric Models

To include the effects of worldsheet fermions we require the fermionic propagators [74]:

$$\langle\psi^\alpha(z_1)\psi^\beta(z_2)\rangle_\nu = G^{\alpha\beta}\frac{\theta_\nu(z_1-z_2)\theta_1'(0)}{\theta_\nu(0)\theta_1(z_1-z_2)}, \tag{5.4.1}$$

where the index ν specifies the spin structure, which must be summed over in the full amplitude. The above differs from the usual boundary fermion propagators purely by the replacement of the metric by the open string metric, but when we perform the rescaling of the external momenta and polarisations [78] it is transformed back to the standard propagators:

$$\begin{aligned}\langle\psi^\alpha(z_1)\psi^\beta(z_2)\rangle_\nu &\to \delta^{\alpha\beta}\frac{\theta_\nu(z_1-z_2)\theta_1'(0)}{\theta_\nu(0)\theta_1(z_1-z_2)} \\ &\equiv \delta^{\alpha\beta}G^\psi_\nu(z_1-z_2),\end{aligned} \tag{5.4.2}$$

which we shall use from now on.

We wish to calculate the one-loop amplitude for two spacetime bosons with an arbitrary amount of supersymmetry in the loop, which is defined by the compact dimensions - and thus only affects the amplitude via the partition function. The vertex operators are

$$V^0 = g_{Dp}\epsilon_\mu(i\dot{X}^\mu + 2\alpha' k\cdot\psi\psi^\mu)e^{ik\cdot X}\,, \tag{5.4.3}$$

and the resulting amplitude gives

$$\Pi_1 = 4g_{Dp}^2(\alpha')^2\int_0^\infty \mathrm{d}t\sum_\nu\, e^{2\alpha'\tilde{k}^2 J}Z_\nu(t)\int_0^t \mathrm{d}x e^{-2\alpha' k^2 I_0}\left[(\dot{I}_0)^2 - (G^\psi_\nu(z(x))^2\right] \tag{5.4.4}$$

and

$$\hat{\Pi}_2 = 4\pi^2 g_{Dp}^2\int_0^\infty \mathrm{d}t\sum_\nu\, Z_\nu(t)e^{-\frac{\tilde{k}^2}{4\alpha' t}}\int_0^t \mathrm{d}x e^{-2\alpha' k^2 I_0}, \tag{5.4.5}$$

where $Z_\nu(t)$ is the partition function for the theory, and

$$\begin{aligned} z^P &= ix, \\ z^{NP} &= ix - 1/2\,. \end{aligned} \tag{5.4.6}$$

Thus the spacetime fermionic component does not contribute to the Lorentz-violating term, since the kinematics for it are just the standard commutative gauge pieces.

The Lorentz-violating term thus derives from bosonic correlator exactly as in the bosonic string, the only difference being the partition function. Of course, if there is any supersymmetry then this term will vanish, as we expect, and the remaining Lorentz-preserving term can be calculated from the off-shell continuation of the fermionic piece. For $N \geq 1$ SUSY, Π_1 can be simplified using the identity

$$(G_\nu^\psi(z))^2 = \frac{\theta_\nu''(0)}{\theta_\nu(0)} - \partial^2 \log \theta_1(z) \tag{5.4.7}$$

to give

$$\Pi_1 = 4g_{Dp}^2(\alpha')^2 \int_0^\infty \mathrm{d}t \sum_\nu e^{2\alpha'\tilde{k}^2 J} Z_\nu(t) \frac{\theta_\nu''(0)}{\theta_\nu(0)} \int_0^t \mathrm{d}x e^{-2\alpha' k^2 I_0}. \tag{5.4.8}$$

Again this is essentially the usual expression for computing threshold corrections, but with an exponential factor inserted for non-planar diagrams.

To summarise the results of this and the previous two sections, as advertised in the Introduction, both bosonic and supersymmetric theories are found to tend continuously to the $B = 0$ theory as $k \to 0$. In particular the couplings freeze out below Λ_{IR} and the entire region above Λ_{IR} can now be consistently integrated out in the usual Wilsonian manner. The phenomenological footprint of the non-zero B field is then in the dispersion relation of massless particles, and in particular a birefringence of the trace-U(1) photon, which gets a polarisation dependent velocity shift of order

$$\Delta v \sim c \frac{M_{SUSY}^2 M_s^2}{M_{NC}^4}. \tag{5.4.9}$$

Whether the EM photon feels this effect is a model dependent question.

5.5 The two point function of the graviton

We now turn to the effect of the non-zero B field on gravity by focusing on the graviton two-point function. In particular, consider the corrections to the Newtonian force law of gravity due to the coupling of the graviton to gauge fields at one-loop. The momentum dependence of these corrections determines the running of Planck's constant, and our experience with gauge couplings suggests that this also may be subject to UV/IR mixing.

In the naive extension of noncommutative field theory of eq. (5.2.6), the one loop contributions divide into planar and non-planar exactly as they do for the trace U(1) photon. However in string theory the relevant diagram is an annulus with two graviton (closed string) vertices on the interior of the world sheet, and so the only way that planar could be distinguished from nonplanar would be either for there to be some kind of radial ordering effect in the vertices, or for there to be a limit in which the major contribution to the diagrams came from when the vertices were on the edges of the annulus. Neither of these possibilities is true and so, even before making any computation, it seems unlikely that there will be a simple field theory approximation involving Moyal products. The field theory limit has been the subject of a recent study in ref. [90] where it was indeed found to be a rather complicated issue. However for the present study we do not need to derive the effective action (and indeed we don't): we will instead examine the modification of the Newtonian force law between matter (open string) fields on the brane, by looking at the two point function determined at the string theory level.

By restricting our attention to the force law between matter fields, we are evading a significant technical difficulty, namely that in a sense we have two metrics, one for open strings and one for closed. We wish to examine the momentum dependence of the gravitational force between *open strings* confined to magnetised D-branes. In principle we ought to be doing this by factorising a four point open string amplitude on the graviton two point function. The relations for G, g and θ imply

$$g^{\mu\nu} = G^{\mu\nu} - \frac{(\theta G \theta)^{\mu\nu}}{4\pi^2(\alpha')^2}. \tag{5.5.1}$$

determining the coupling of the matter on the brane to gravity. We must choose a coordinate system where the components of the metric $g_{\mu\nu}$ are made small ($\alpha' F_{\mu\nu}$) for the dimensions in which magnetic field is turned on, so that the noncommutativity tensor $\theta^{\mu\nu}$ can be tuned to the desired values. Then the relevant momentum scale for the amplitude is given by the Mandelstam variables running through the loop, determined from the external momenta *as contracted with the open string metric.* Importantly the closed string metric is vastly different in the regimes of interest: for the exchange of a graviton with four-momentum q_μ between open strings, the

Mandelstam variables correspond to scales of order

$$q^2 \equiv q_\mu q_\nu G^{\mu\nu}; \tag{5.5.2}$$

but if we were interested in graviton exchange between external graviton states, it would be more appropriate to use

$$q_\mu q_\nu g^{\mu\nu} = q^2 - \frac{\tilde{q}^2}{4\pi^2(\alpha')^2}, \tag{5.5.3}$$

which, by definition, for $q \sim \Lambda_{IR}$, would be of order $\sim M_s^2$. In the former case, as we are only dealing with graviton propagators the difference is immaterial since we can always rescale the graviton states to absorb the difference, but the correct procedure for (or indeed physical meaning of) the latter is less clear. (For example we would probably want more information about the other contributions in such a process coming from B fields, and also more information about what the asymptotic states are – i.e. the effective field theory.) Thus in calculating the graviton correlator, we shall decompose the square of the momentum in terms of the open string quantities, and consider $\tilde{k}^2/\alpha' \gg \alpha' k^2$, while $k^2\tilde{k}^2 \ll 1$.

We shall restrict the discussion to exchanges between $D3$-brane states, for which (since there can be no orientifold planes) we need only consider the annulus for the induced gravity on the brane at one loop. A typical model for this scenario would be $D3/D7$-branes at a $C^6/\mathbb{Z}_N$ orbifold singularity [33, 91], with a magnetic flux on the 1 and 2 directions. However, we shall keep the discussion as general as possible. We proceed initially as in ref. [92, 93] to extract the correction to $\frac{M_{Pl}^2}{16\pi}$, denoted δ, by considering the following kinematic portion of the graviton two-point function:

$$\langle V_G(h^1, k) V_G(h^2, -k)\rangle \supset -\frac{\delta}{4} h^1_{\mu\nu} h^2_{\lambda\rho} g^{\mu\lambda} k^\nu k^\rho \equiv A_\delta \tag{5.5.4}$$

where the vertex operators are given by

$$\begin{aligned} &V_G(h,k) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.5.5)\\ &g_s^2 N_G \frac{2}{\alpha'} h_{\mu\nu} \big(\partial X^\mu(z) - \frac{i\alpha'}{2} : k\cdot\psi\psi^\mu(z) :\big)\big(\bar{\partial} X^\nu(\bar{z}) - \frac{i\alpha'}{2} : k\cdot\tilde{\psi}\tilde{\psi}^\nu(\bar{z}) :\big) e^{ik\cdot X(z,\bar{z})}. \end{aligned}$$

In the open string channel the Green's function is given by modular transforming the result of ref. [74][2]:

$$\mathcal{G}^{\mu\nu}(w_1,w_2) = -\frac{\alpha'}{2}I^C G^{\mu\nu} + J^C\frac{(\theta^{\mu\alpha}G_{\alpha\beta}\theta^{\beta\nu})}{8\pi^2\alpha'} - K^C\frac{\theta^{\mu\nu}}{2\pi}, \tag{5.5.6}$$

where

$$\begin{aligned}
I^C &= \ln\left|\frac{\theta_1(w_1-w_2,it)\theta_1(w_1+\bar{w}_2,it)}{4\pi^2\eta^6(it)}\right|^2 - \frac{4\pi}{t}|\Im(w_1-w_2)|^2,\\
J^C &= \ln\left|\frac{\theta_1(w_1-w_2,it)}{\theta_1(w_1+\bar{w}_2,it)}\right|^2 - \frac{4\pi}{t}\Big(|\Re(w_1+\bar{w}_2)|^2 - \Re(w_1) - \Re(w_2)\Big), \qquad (5.5.7)\\
K^C &= \ln\theta_1(w_1+\bar{w}_2,it) - \ln\theta_1(\bar{w}_1+w_2,it) - \frac{2\pi i}{t}\Im\left((w_1+\bar{w}_2+1/2)^2\right)\\
&\quad -2\pi i f(\Im(w_1-w_2))
\end{aligned}$$

and where $f(x) \equiv -[x/t]$, $[y]$ denotes the closest integer to y. Thus the self-contraction terms, with normal-ordering and the $w_1 \to w_2$ limit performed, are

$$\begin{aligned}
C^{\mu\nu}(w,\bar{w}) = &\; -\left(\frac{\alpha'}{2}G^{\mu\nu} + \frac{(\theta^{\mu\alpha}G_{\alpha\beta}\theta^{\beta\nu})}{8\pi^2\alpha'}\right)\ln\left|\frac{\theta_1(w+\bar{w},it)}{2\pi\eta^3(it)}\right|^2\\
&-\frac{(\theta^{\mu\alpha}G_{\alpha\beta}\theta^{\beta\nu})}{8\pi^2\alpha'}\frac{8\pi}{t}(2\Re^2(w) - \Re(w))\,.
\end{aligned} \tag{5.5.8}$$

The fermionic Green's functions are obtained from the torus functions using the doubling trick:

$$\psi^\mu(w) = \begin{cases} \psi^\mu(w), & \Re(w) > 0,\\ i(\frac{g+F}{g-F})^\mu_\nu\tilde{\psi}^\nu(-\bar{w}), & \Re(w) < 0. \end{cases} \tag{5.5.9}$$

We obtain

$$\begin{aligned}
\langle\psi^\alpha(z)\psi^\beta(w)\rangle_\nu &= g^{\alpha\beta}G^\psi_\nu(z-w),\\
\langle\tilde{\psi}^\alpha(\bar{z})\tilde{\psi}^\beta(\bar{w})\rangle_\nu &= g^{\alpha\beta}G^\psi_\nu(\bar{z}-\bar{w}),\\
\langle\psi^\alpha(z)\tilde{\psi}^\beta(\bar{w})\rangle_\nu &= -i\left(g^{\alpha\beta} + 2\frac{(\theta G\theta)^{\alpha\beta}}{4\pi^2(\alpha')^2} - 2\frac{\theta^{\alpha\beta}}{2\pi\alpha'}\right)G^\psi_\nu(z+\bar{w}).
\end{aligned} \tag{5.5.10}$$

[2]Note that this choice of propagator differs slightly from those given elsewhere [73, 78], but it was asserted in [74] that the additional terms are necessary to ensure periodicity and obedience to the equations of motion. They cause a discrepancy when the fields are taken to the boundary; (5.3.5, 5.3.6) are not obtained from (5.5.7) . However, the closed string propagators only differ by linear terms in J^C and K^C, plus the function f which plays no essential role in amplitudes (merely ensuring that the derivatives of the logs in the antisymmetric portion contain no discontinuities). The reader can check that (5.5.13) is unchanged by these, and since I^C is identical for both versions, so are all the other results in this section.

As for the gauge bosons, the physical behaviour naturally splits into long distance $\tilde{k}^2/\alpha' \ll 1$ and short distance $\tilde{k}^2/\alpha' \gg 1$ regimes. In the former, gravity will be dominated by the low energy modes, for which the usual corrections to Planck's constant apply. We can expand the amplitude as a power series in k^2 and $\tilde{k}^2$, and neglect the terms $O(k^2)$ relative to $O(\tilde{k}^2)$. In the short distance regime however such an expansion is no longer appropriate, but the amplitude still has terms with a prefactor of $\tilde{k}^2$ which we should consider dominating over those prefixed by k^2. In this way, we may consider the same correlators as being typical dominating terms in the amplitude for the non-zero B-field corrections to both limits; one such term is

$$A \supset \int_0^\infty \mathrm{d}t \int \mathrm{d}^2 z \int \mathrm{d}^2 w \; - g_s^2 \frac{\alpha'}{2} N_G^2 \langle \partial X^\mu(z) \bar{\partial} X^\lambda(\bar{w}) e^{ik\cdot X(z,\bar{z})} e^{-ik\cdot X(w,\bar{w})} \rangle \times \langle k \cdot \tilde{\psi}\tilde{\psi}^\nu(\bar{z}) k \cdot \psi\psi^\rho(w) \rangle \quad (5.5.11)$$

which has a leading contribution of the form

$$\int_0^\infty \mathrm{d}t \int \mathrm{d}^2 z \int \mathrm{d}^2 w \, g_s^2 N_G^2 \frac{\tilde{k}^2}{4\pi^2 \alpha'} Z(t) (G_\nu^\psi)^2 k_a k_b \partial_z \mathcal{G}^{a\mu} \bar{\partial}_w \mathcal{G}^{b\lambda} \langle e^{ik\cdot X(z,\bar{z})} e^{-ik\cdot X(w,\bar{w})} \rangle \equiv L_\delta + \dots. \quad (5.5.12)$$

Here L_δ is the component of this term which contributes to A_δ, and we have included in the partition function the Chan-Paton summation, which corresponds to summing over the Casimirs of the representations of the gauge group. We shall leave a complete analysis to future work, and consider the contribution from the corner of the moduli space where $t > 1$. Here we can take the derivatives of the Green's functions to be given by the leading order terms as $t \to \infty$ - as for the gauge theory case, this is equivalent to a field theory calculation, but it is more expedient to perform the calculation from string theory. We find that the behaviour is dominated by the correlator of the exponentials: this is given by

$$\begin{aligned}\langle e^{ik\cdot X(z_1)}e^{-ik\cdot X(z_2)}\rangle = & \left|\frac{\theta_1(z-w,it)}{2\pi\eta^3(it)}\right|^{-2\alpha' k^2-\frac{\tilde{k}^2}{4\pi^2\alpha'}}\left|\frac{\theta_1(z+\bar{w},it)}{2\pi\eta^3(it)}\right|^{-2\alpha' k^2+\frac{\tilde{k}^2}{4\pi^2\alpha'}} \\ & \left|\frac{\theta_1(z+\bar{z},it)}{2\pi\eta^3(it)}\right|^{\alpha' k^2+\frac{\tilde{k}^2}{8\pi^2\alpha'}}\left|\frac{\theta_1(w+\bar{w},it)}{2\pi\eta^3(it)}\right|^{\alpha' k^2+\frac{\tilde{k}^2}{8\pi^2\alpha'}} \\ & \exp\left[\frac{4\pi\alpha' k^2}{t}|\Im(z-w)|^2\right]\exp\left[\frac{-\tilde{k}^2}{\pi\alpha' t}|\Re(z-w)|^2\right]. \end{aligned} \tag{5.5.13}$$

Note that this correctly factorises onto the corresponding boundary amplitude. To take the field theory limit now, we write $T=\pi\alpha' t$, $y=\Im(z)/t$, use the translation invariance of the annulus to fix $\Im(w)=0$, and write $\Re(z)=x, \Re(w)=x'$, and insert the partition function and kinematic factors, with a sum over spin structures. Making use of the identity (5.4.7) and assuming $N\geq 1$ supersymmetry, so that after multiplying by the partition function all spin-structure-independent terms vanish, we obtain the prefactor

$$F(T)\equiv(8\pi T)^{-2}\sum_{\nu}Z_\nu(\frac{T}{\pi\alpha'})\frac{\theta_\nu''(0)}{\theta_\nu(0)}. \tag{5.5.14}$$

We shall assume $F(T)$ has the behaviour

$$\lim_{T\to\infty}T^2F(T)=\beta, \tag{5.5.15}$$

where β is a constant. If we now insert the factors from our "typical" contribution, we obtain

$$\begin{aligned}L_\delta^{FT} = g_s^2N_G^2\frac{\tilde{k}^2}{4\pi^2\alpha'}&\int_{\pi\alpha'}^{\infty}\mathrm{d}T\;TF(T)\int_0^1\mathrm{d}y(1-2y)^2e^{-4k^2Ty(1-y)} \\ &\int_0^{1/2}\mathrm{d}x\int_0^{1/2}\mathrm{d}x' e^{\frac{-\tilde{k}^2}{T}(x-x')^2}\left|\sin 2\pi x\sin 2\pi x'\right|^{\alpha' k^2+\frac{\tilde{k}^2}{8\pi^2\alpha'}}. \end{aligned}\tag{5.5.16}$$

As discussed, in contrast to a noncommutative field theory, there is no separation into planar and non-planar diagrams.

Since we are considering the regime $k^2\tilde{k}^2\ll 1$, we reorder the integration and use the leading behaviour of the Bessel function K_0 as in ref. [78] to give

$$\begin{aligned}L_\delta^{FT} &\approx g_s^2N_G^2\frac{\tilde{k}^2}{4\pi^2\alpha'}\frac{\beta}{6\pi}\log|k^2\tilde{k}^2|B^2\left(\frac{1}{2},\frac{1}{2}+\frac{\alpha' k^2}{2}+\frac{\tilde{k}^2}{16\pi^2\alpha'}\right) \\ &\approx g_s^2N_G^2\frac{\tilde{k}^2}{4\pi^2\alpha'}\frac{\beta}{6\pi}\frac{16\pi^3\alpha'}{\tilde{k}^2}\log|k^2\tilde{k}^2| \end{aligned}\tag{5.5.17}$$

and thus this contribution to the graviton renormalisation, after we include $N_G = (8\pi G_4)^{1/2}/2\pi$ (where G_4 is Newton's constant) is given by

$$\delta \supset -\frac{4g_s^2\beta G_4}{3\pi}\log|k^2\tilde{k}^2|. \tag{5.5.18}$$

Note that when we sum over all equivalent diagrams and thus remove the field theory singularity, the $\log\tilde{k}^2$ term still remains. However, this is of course not a singularity, as we have up to this point been considering $\tilde{k} \gg \alpha'$.

As $\tilde{k}$ decreases, the amplitude should smoothly revert to the correction for $\theta = 0$. To find the deviation from Newtonian behaviour at large distances we are interested in the variation of δ for small $\tilde{k}^2/\alpha'$, which as discussed above will be dominated by the same terms as in the large limit; for the term we have been considering we obtain

$$\begin{aligned} L_\delta^{FT} &= g_s^2 N_G^2 \frac{\beta\tilde{k}^2}{16\pi^2\alpha'}\int_{\pi\alpha'}^{\infty}\frac{\mathrm{d}T}{T}\int_0^1 \mathrm{d}y(1-2y)^2 e^{-4k^2Ty(1-y)} + O((\frac{\tilde{k}^2}{4\pi^2\alpha'})^2) \\ &= \frac{g_s^2\beta\tilde{k}^2 G_4}{8\pi^3\alpha'}2\int_0^1 \mathrm{d}y\; y(1-y)(1-\gamma_E-\log 4\pi-\log(\alpha' k^2 y(1-y))) \\ &\quad + O((\alpha' k^2)^2) \end{aligned} \tag{5.5.19}$$

from which we extract the contribution to the renormalisation:

$$\delta \supset -\frac{g_s^2\beta G_4}{24\pi^3}\frac{\tilde{k}^2}{\alpha'}\left(-\frac{5}{3}+\gamma_E+\log 4\pi+\log\alpha' k^2\right). \tag{5.5.20}$$

5.6 Phenomenology: modification of gravity at a *mm*

We now turn to phenomenological issues beginning briefly with the possibility of Lorentz violation in the photon. In the Introduction we mentioned birefringence of the trace U(1) photon which is constrained by astrophysical observations. Taking into account our analysis and the fact that the Lorentz violating operator Π_2 vanishes in a fully supersymmetric theory, the velocity shift is of order

$$\Delta v \sim c\frac{\lambda M_{SUSY}^2 M_s^2}{M_{NC}^4}. \tag{5.6.1}$$

Following ref. [94, 95] a relatively firm constraint comes from "time of flight" signals from pulsars;

$$\frac{\sqrt{\lambda}\, M_{SUSY} M_s}{M_{NC}^2} \sim \sqrt{\lambda}\, \frac{M_{SUSY}}{\Lambda_{IR}} < 2 \times 10^{-8}, \tag{5.6.2}$$

where λ is here a measure of the one-loop suppression in the gauge diagrams, and M_{SUSY} is a measure of the supersymmetry breaking. A natural question to ask is how low the IR cut-off can be; in other words, *is it likely that a regime that is well approximated by noncommutative gauge theory will ever be accessible?* Alas, the answer is no. Since λ is a loop suppression factor involving known gauge couplings it will be at least of order 10^{-3} assuming that the mixing between the physical photon and trace U(1) photon is of order unity. However supersymmetry is broken and transmitted, one should almost certainly take $M_{SUSY} > 1TeV$ giving

$$\Lambda_{IR} > 10^9 GeV. \tag{5.6.3}$$

This bound is comparable to those coming from atomic physics calculated in ref. [96];

$$M_{NC} > 10^{14} GeV. \tag{5.6.4}$$

Assuming that $M_s < M_{Pl}$, that bound translates into

$$\Lambda_{IR} > 2 \times 10^{10} GeV. \tag{5.6.5}$$

If the physical photon has significant mixing with the trace U(1) photon, it seems likely therefore that a non-zero B field would be felt as residual Lorentz violation rather than full blown noncommutative field theory. For more detailed discussion of these questions see ref. [72].

Consider instead the possibility that the physical photon does not mix with the trace U(1) photon. This could be the case if the trace U(1) photon forms part of a hidden sector, or if the trace U(1) is spontaneously broken by for example a Fayet-Iliopoulos term, if it is anomalous. In this case M_{NC} can be much lower and a significant effect can show up in gravitational interactions. Our general analysis shows that the graviton two point function in a theory with non-vanishing θ tends continuously to the commutative one with leading terms suppressed by factors of $\tilde{k}^2/\alpha'$. Neglecting the possible implications of a non-trivial tensor structure for the moment, the mildest effect one expects is a modification of the Newtonian force law which derives from it. The observable effects will make themselves felt as we probe the gravitational interaction at shorter distances. As we saw, there is something akin to a "nonplanar" one-loop contribution in the sense that $\tilde{G}(\mathbf{k})$ interpolates between the $\tilde{k}^2 \gg \alpha'$ regime and the $\tilde{k}^2 \ll \alpha'$ regime where it deviates from the purely commutative model as $\tilde{k}^2/\alpha'$. Neglecting tensor structure, we can therefore model the two point function as

$$\tilde{G}(\mathbf{k}) = \frac{1}{M_{Pl}^2 \mathbf{k}^2} \frac{1 + f(\frac{\tilde{\mathbf{k}}^2}{\alpha'})}{1+\lambda} \tag{5.6.6}$$

where $f(x) \to \lambda(1 + \mathcal{O}(x))$ for $x \ll 1$ and tends to the short range behaviour for $x \gg 1$. Here M_{Pl}^2 is the one loop Planck mass, which includes also tree level disk diagram contributions such as those considered in ref. [90]. For example if we assume that the one-loop contribution has power law behaviour $\sim |\tilde{k}|^{(3-p)}$ we can model the total tree and one-loop two point function as

$$\tilde{G}(\mathbf{k}) = \frac{1}{M_{Pl}^2 \mathbf{k}^2} \frac{1}{(1+\lambda)} \left(1 + \lambda \left(\frac{1}{1 + \frac{\tilde{\mathbf{k}}^2}{\alpha'}} \right)^{\frac{p-3}{2}} \right). \tag{5.6.7}$$

The coefficient λ encapsulates the one-loop open string contribution to Planck's constant in the commutative theory with $\theta = 0$, which can be significant and is model dependent. Indeed there are generic scenarios that lead to the extremes $\lambda \gg 1$ and $\lambda \ll 1$:

1. The ADD scenario [97,98]: the Standard Model is associated with a local brane configuration (for example in a "bottom-up" construction as per the previous section), with the 4D Einstein-Hilbert action deriving from the dimensionally reduced 10D action. In this case the one loop correction will be localised whereas the large tree-level M_{Pl}^2 is the result of a large volume. The one loop open string contribution will therefore be suppressed by a factor

$$\lambda \sim \frac{1}{V_{10-p}} \tag{5.6.8}$$

where V_{10-p} is the extra-dimensional volume in units of $\sqrt{\alpha'}$. 3-branes in the original ADD scenario with TeV scale gravity would therefore lead to a tiny

λ, but one could imagine the Standard Model localised on wrapped D7-branes for example, in which case intermediate values of λ are possible.

2. The DGP scenario [99]: gravity is localised to a 3-brane in infinite or large extra dimensions by one-loop diagrams with matter (brane) states in the loop. The novel feature is that gravity becomes higher dimensional at long distances, offering an explanation of the observed cosmological acceleration. In this case one expects $\lambda \gg 1$ in the region where gravity is 4 dimensional. In more detail, the full action consists of a bulk term and a one loop induced brane term;

$$M_{Pl}^2 \left(\int d^4x \sqrt{g_4} R^{(4)} + \rho_c^{D-4} \int d^D x \sqrt{g_D} R^{(D)} \right), \tag{5.6.9}$$

where $R^{(4)}$ is the curvature form the induced metric on the brane. Since ρ_c^{D-4} appears in the propagator with a factor k^2 it is natural that the cross-over length scale *above* which gravity appears D dimensional, generically given by [100]

$$R_c = \alpha'^{\frac{6-D}{4}} \rho_c^{\frac{D-4}{2}}. \tag{5.6.10}$$

This possibility has been analysed for (Type I) open string models in ref. [92, 93, 100], where in practice a number of different threshold effects are possible if the matter branes wrap some compact internal dimensions. The precise details of these other thresholds will not change our conclusions about the effect of UV/IR mixing.

To see the effect of the one-loop corrections on the potential between two point particles consider for example $\theta^{12} = \theta$. In this case

$$\tilde{\mathbf{k}}^2 = \theta^2(\mathbf{k}_1^2 + \mathbf{k}_2^2) \; = \; \theta^2 \mathbf{k}^2 \sin^2 \vartheta \, , \tag{5.6.11}$$

where ϑ is the angle to the 3 direction. The potential depends on the angle ϑ and is given by the retarded Green's function;

$$\begin{aligned} V(\mathbf{x}) &= \int dt \, G_R(t, \mathbf{x}) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{G}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} \end{aligned} \tag{5.6.12}$$

which leads to

$$V(r,\vartheta) = \frac{1}{8\pi M_{Pl}^2 r}\left(1 + \frac{1}{1+\lambda}\int_0^\infty \left(f\left(\frac{r_c^2}{r^2}y^2\right) - \lambda\right) e^{-y\cos\vartheta} J_0(y\sin\vartheta)\, dy\right) \tag{5.6.13}$$

where

$$r_c = \frac{\theta}{\sqrt{\alpha'}} = \frac{M_s}{M_{NC}^2}. \tag{5.6.14}$$

In the limit where $r\cos\vartheta \gg r_c$ we may expand f inside the integral. Using the identity

$$\int_0^\infty y^{m\geq 0} e^{-y\cos\vartheta} J_0(y\sin\vartheta)\, dy = (-1)^m m!\, P_m(\cos\vartheta) \tag{5.6.15}$$

we find that the leading deviation from Newtonian behaviour is a quadrupole moment that sets in at $r \sim r_c$: indeed if $f(x) = \lambda(1+\beta x + \ldots)$ we find

$$V(r,\vartheta) = \frac{1}{8\pi M_{Pl}^2 r}\left(1 + \frac{\lambda\beta\,(3\cos^2\vartheta - 1)}{(1+\lambda)}\frac{r_c^2}{r^2} + \mathcal{O}\left(\frac{r_c^4}{r^4}\right)\right). \tag{5.6.16}$$

The radius r_c is the distance above which Planck's constant tends to the $B=0$ one-loop value. This is a potential which can be compared directly with the experimental bounds presented in ref. [101]. Also note that there is a direction given by $\cos\vartheta = 0$ where the physics is identical to $\theta = 0$ physics.

At smaller distances the "nonplanar" contribution to Planck's constant diminishes. For $r \ll r_c$ we may use the identity

$$e^{-y\cos\vartheta} J_0(y\sin\vartheta) = \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{n!} P_n(\cos\vartheta) \tag{5.6.17}$$

and approximate

$$f\left(\frac{r_c^2}{r^2}y^2\right) = \lambda\left(\frac{1}{1+\frac{r_c^2}{r^2}y^2}\right)^{\frac{p-3}{2}} \tag{5.6.18}$$

to find the first few harmonics as

$$V(r,\vartheta) = \tag{5.6.19}$$
$$\frac{1}{8\pi M_{Pl}^2 r}\left(\frac{1}{1+\lambda} + \sum_{n=0}^{p-4} \frac{(-1)^n B(\frac{p-n-4)}{2}, \frac{n+1}{2})}{2n!} P_n(\cos\vartheta)\left(\frac{r}{r_c}\right)^{1+n} + \mathcal{O}\left(\frac{r}{r_c}\right)^{p-2}\right).$$

The leading term is the tree-level Planck's constant, and the sub-leading terms grow with radius, as they should, to build up the full one-loop Planck's constant at large distance.

The most notable general conclusion from this analysis is simply that the distance scale at which the modification of gravity takes place,

$$r_c = \frac{\theta}{\sqrt{\alpha'}} = \frac{M_s}{M_{NC}^2}, \tag{5.6.20}$$

can be much larger than the inherant distance scales in the model. For example if $M_s \sim M_{Pl}$ and $M_{NC} \sim 1TeV$ then[3] $r_c \sim 1mm$ (the same numerical coincidence as the large extra dimension scenarios with 2 extra dimensions).

5.7 Conclusions

Noncommutative field theory provides a theoretical framework to discuss effects of nonlocality and Lorentz symmetry violation. Proper understanding and better control of the UV/IR mixing has been a serious obstacle for the field theory. In this work, we have emphasised that the IR singularities are just a reflection of the fact that field theory is UV divergent. Consequently any attempt to resolve them without modifying the UV behaviour of the field theory is doomed, and they can only be consistently smoothed out in a UV finite theory. We have demonstrated this explicitly by considering noncommutative field theory as an approximation to open string theory with a background B-field. We showed that the noncommutative field theory description is valid only for the intermediate range of energy scale $\Lambda_{IR}^2 \equiv \alpha' M_{NC}^2 < k^2 < 1/\alpha'$ and explored what happens outside this range. The IR singularities are rendered harmless and in fact, long before they are reached, the singular IR physics of the noncommutative theory is replaced by regular physics that is dictated by the UV finiteness of strings. In many non-supersymmetric theories, tachyonic instabilities arise from the modified dispersion relation (5.1.4) [102], which our analysis implies are also resolved by embedding into an UV-complete theory, as discussed in the context of field theory in [72].

With the UV/IR mixing under control, one can now reliably study how noncommutative geometry modifies the IR physics. Below the noncommutative IR scale

[3]Note that the bound of equation (5.6.4) do not apply since we consider a VEV for a trace U(1) on a hidden brane, rather than the antisymmetric tensor or the visible electromagnetic field.

Λ_{IR}, normal Wilsonian behaviour is resumed and the low energy physics can be described in terms of ordinary local physics with residual Lorentz violating operators. Indeed the theory tends continuously to the commutative $B = 0$ field theory, with the Lorentz violating operators remaining as a footprint in the low energy phenomenology of the string scale physics. A second important example of how the low energy physics is modified arises in the gravitational sector. We studied how the noncommutative geometry may modify gravity by considering the graviton two point function. The departure from the ordinary Newtonian potential can be much more significant and happen at much lower energy scales than those suggested by any extra dimensions.

One aspect of the present study that requires further elaboration is the nature of the effective field theory in the gravity sector and the resulting cosmology. Because of the difficulty of extracting an effective field theory for the gravitational sector it is not clear how these features will turn out, or indeed if they lead to any strong observational constraints.

Chapter 6

Summary

This thesis comprises two rather different bodies of work, that both contribute to the current efforts of string phenomenology.

Chapter 3 examined in detail the calculation of one-loop amplitudes in intersecting brane worlds. We first used the techniques introduced in previous work [30] to calculate an amplitude which gives the running of Yukawa couplings, and found that it contained a divergence. The origin of the divergence was identified as being a closed string exchange, which indicated that it should cancel provided that the existing consistency conditions on such models were met. To show that this is indeed the case required examining the OPE of the boundary-changing operators and a simple assumption about the behaviour of the amplitudes without a periodic cycle between the boundaries of the worldsheet, and the Möbius strip diagrams.

The above calculation motivated a clarification and generalisation of the previous technology for calculating correlators of boundary-changing operators, extending it to general numbers of insertions and providing the phase matrix M^{ab} essential for such calculations. It also motivated an extension of the techniques to include hitherto uncalculable amplitudes, involving the string theory equivalent of loops without gauge fields. The phase matrix for these processes was also calculated. While the absolute evaluation of the amplitudes appears to be intractable, it is certainly possible to obtain useful information from the expressions given here; aside from confirming the assumption about the cancellation of divergences, it can also be used for examining the low energy theory, and indeed examining energies near the string scale, as illustrated by the analysis of section 3.4. That section examined the moderation of power-law running of the Yukawa coupling near the string scale, and argued that the behaviour can be approximated in the field theory by a string-inspired modification of the propagators in the loop.

Chapter 4 provided the necessary theoretical tools for the calculation of instanton contributions to the superpotential in toroidal intersecting brane worlds. In particular, the Pfaffian and determinant factor was calculated for the first time, including some new results relevant for gauge threshold corrections, and also some relevant correlators of spin fields. We then applied these tools to the issue of corrections to the Yukawa couplings, and found that they can contribute provided that the E2 instanton does not intersect any brane of the model. In this way, they can solve the "rank one Yukawa problem" from which these models suffer.

Chapter 5 investigated some of the effects of a non-zero antisymmetric tensor field in string theory models with D-Branes. The infrared singularity in noncommutative field theories was explcitly shown to be removed when embedded in string theory, provided that the theory is free of tadpoles. This was illustrated in bosonic string theory and the superstring. The effect of the antisymmetric tensor upon gravity was also examined, by computing a portion of the graviton propagator at one loop. This showed that there is an infrared-ultraviolet mixing in the running of Newton's constant, but not a separation into planar and non-planar diagrams as might have been naively expected. The violation of lorentz symmetry creates a directional dependence for the gravitational potential, which would be a clear experimental signal and in principle possible to observe at distances as large as a millimeter.

It is hoped that the ideas contained within this thesis, in addition to solving some interesting problems within the field of string phenomenology, present several avenues for further insights into string theory.

Appendix A

Material Regarding Intersecting Brane Worlds

A.1 Boundary-Changing Operators

The technically most significant part of any calculation involving chiral fields stretched between branes is the manipulation of the boundary-changing operators. They are primary fields on the worldsheet CFT which represent the bosonic vacuum state in the construction of vertex operators, so we require one for every stretched field. In factorisable setups such as we consider in this paper we require one for every compact dimension where there is a non-trivial intersection - so we will require three for each vertex operator in N=1 supersymmetric models.

In previous papers (e.g. [15, 16]) the boundary-changing operator for an angle θ has been denoted σ_θ, in analogy with closed-string twist-field calculations. It has conformal weight $h_\theta = \frac{1}{2}\theta(1-\theta)$, and we can thus write the OPE of two such fields as

$$\sigma_\nu(z_1)\sigma_\lambda(z_2) \sim \sum_k C^k_{\nu\lambda}\sigma_k(z_2)(z_2-z_1)^{h_k-h_\nu-h_\lambda} \tag{A.1.1}$$

Moreover, the "twists" are additive, so $k=\nu+\lambda$ for $\nu+\lambda<1$, or $k=\nu+\lambda-1$ for $\nu+\lambda>1$. However, they also carry on the worldsheet labels according to the boundaries that they connect; for a change from brane a to b we should write $\sigma_\theta^{(ab)}$. We then have non-zero OPEs only when the operators share a boundary, so that we modify the above to

$$\sigma_{\nu}^{(ab)}(z_1)\sigma_{\lambda}^{(cd)}(z_2) \sim \delta^{bc} C_{\nu\lambda}^{(abd)\nu+\lambda} \sigma_{\nu+\lambda}^{(ad)}(z_2)(z_2 - z_1)^{h_{\nu+\lambda}-h_{\nu}-h_{\lambda}} \tag{A.1.2}$$

We also find that new fields appear, representing strings stretched between parallel branes; the OPE coefficients and weights can all be found by analysing tree-level correlators. In order to normalise the fields and thus the OPEs we must compare the string diagrams to the low-energy field theory, specifically the Dirac-Born-Infeld action for the IIA theory:

$$S_6 = -T_6 \int \mathrm{d}^7\xi e^{-\Phi} tr\sqrt{-\det(G + B + 2\pi\alpha' F)} \tag{A.1.3}$$

where G and B are pull-backs of the metric and antisymmetric tensor to the brane world-volume, F is the gauge field strength, and e^{Φ} is the closed string coupling. If we consider gauge excitations only on the non-compact dimensions, the effective field theory has gauge kinetic function [15, 32]:

$$g_{D6}^{-2} = e^{-\Phi}\frac{1}{2\pi}\prod_{\kappa=1}^{3}\frac{L_a^{\kappa}}{2\pi\sqrt{\alpha'}} \tag{A.1.4}$$

where we have the volume of brane a equal to $L_a = \prod_{\kappa=1}^{3} L_a^{\kappa}$. If we now consider two, three and four-point gauge-boson amplitudes (for non-abelian gauge group) in the 4d effective theory, they must all have the same coupling; this is only possible if the gauge kinetic function is associated to the normalisation of the disc diagram, and not the fields, implying

$$(\alpha')^{-1} g_o^{-2}\langle 1\rangle_a = g_a^{-2} \tag{A.1.5}$$

where g_o is the open string coupling. Clearly the gauge bosons must be canonically normalised by a factor of g_o^{-1} to match the field theory, where the coupling is also absorbed into the fields. Note that the above can also be obtained by a boundary state analysis without recourse to the low energy supergravity [103, 104].

We now consider the two-point function for boundary-changing operators; we require

$$\langle \sigma_{\theta}^{ab}\sigma_{1-\theta}^{ba}\rangle = G_{C_{\theta}\bar{C}_{\theta}} = C_{\theta,1-\theta}^{(aba)0}\langle 1\rangle_a \tag{A.1.6}$$

where $G_{C_\theta \bar{C}_\theta}$ is the Kähler metric of the chiral multiplets [32]. This allows us to determine the OPE coefficients; however, for convenience we shall combine them (for now) with the string coupling since that is how they shall always appear:

$$(g_o)^2 \prod_{\kappa=1}^{3} C^{(aba)0}_{\theta^\kappa,1-\theta^\kappa} = (\alpha')^{-1} g_a^2 G_{C_{ab}\bar{C}_{ba}} \tag{A.1.7}$$

To obtain further correlators we must now consider four-point tree diagrams. First we consider $\mathcal{A}_{tree} \equiv g_o^4 \langle \sigma^{ab}_{\nu}(z_1) \sigma^{ba\|}_{1-\nu}(z_2) \sigma^{a\|c}_{1-\lambda}(z_3) \sigma^{ca}_{\lambda}(z_4) \rangle_{D_2}$, which was calculated for a particular limit in [15, 32], but for the general case in [16] (note that the product over dimensions is implied). The result was

$$\mathcal{A}_{tree} = C(\alpha')^{-1} g_o^2 \prod_{\kappa=1}^{3} z_{12}^{-2h_{\nu^\kappa}} z_{34}^{-2h_{\lambda^\kappa}} I^\kappa(x)^{-1/2} (1-x)^{-\nu^\kappa \lambda^\kappa}$$
$$\sum_{n_1,n_2} \exp\left[\frac{-\sin\pi\nu}{4\pi\alpha'} \frac{(v_{21}\tau - v_{32})^2 + (v_{21}\tau' + v_{32})^2}{\tau + \tau'} \right] \tag{A.1.8}$$

where $v_{21}(n_1) = L_0^{21,\kappa} + n_1 L_b^\kappa$ and $v_{32}(n_2) = L_0^{32,\kappa} + n_2 L_{a\|}^\kappa$ depend on the configuration; $L_0^{21,\kappa}$ is the distance between the intersections 1 and 2 etc; $x = (z_{12}z_{34}/z_{13}z_{24})$ is the only actually independent coordinate; and the various hypergeometric functions are given by:

$$\begin{aligned}
F_1^\kappa(1-x) &\equiv F(\nu^\kappa, \lambda^\kappa, \nu^\kappa + \lambda^\kappa, 1-x) \\
F_2^\kappa(1-x) &\equiv F(1-\nu^\kappa, 1-\lambda^\kappa, 2-\nu^\kappa-\lambda^\kappa, 1-x) \\
K_1^\kappa(x) &\equiv F(\nu^\kappa, \lambda^\kappa, 1, x) \\
K_2^\kappa(x) &\equiv F(1-\nu^\kappa, 1-\lambda^\kappa, 1, x) = (1-x)^{\nu^\kappa+\lambda^\kappa-1} K_1(x) \\
\tau^\kappa(x) &\equiv (1-x)^{1-\nu^\kappa-\lambda^\kappa} \frac{B(1-\nu, 1-\lambda^\kappa)}{B(\nu^\kappa, 1-\nu^\kappa)} \frac{F_2(1-x)}{K_1(x)} \\
&= \frac{B(1-\nu, 1-\lambda^\kappa)}{B(\nu^\kappa, 1-\nu^\kappa)} \frac{F_2(1-x)}{K_2(x)} \\
\tau^{\kappa\prime}(x) &\equiv \frac{B(\nu^\kappa, \lambda^\kappa)}{B(\nu^\kappa, 1-\nu^\kappa)} \frac{F_1(1-x)}{K_1(x)} \\
I^\kappa(x) &\equiv (1-x)^{1-\nu^\kappa-\lambda^\kappa} B(\nu^\kappa, 1-\nu^\kappa) K_1^\kappa(x) K_2^\kappa(x) (\tau^\kappa + \tau^{\kappa\prime})
\end{aligned} \tag{A.1.9}$$

The manifestly-$SL(2,C)$-invariant form of the above is obtained by premultiplying the amplitude by $(z_4)^{2h_\lambda}$ and taking $\{z_1, z_2, z_3, z_4\} \to \{0, x, 1, \infty\}$. To link the above expression with that in [15], we must put $d_2^\kappa = v_{32}^\kappa$, $d_1^\kappa = d_2^\kappa + \beta v_{21}$, where

$$\beta^\kappa = \tau^{\kappa\prime} - \tau^\kappa = \frac{\sin\pi(1-\nu^\kappa-\lambda^\kappa)}{\sin\pi\lambda^\kappa} \tag{A.1.10}$$

In the limit considered there, they set $d_2^\kappa = d_1^\kappa \tau/\tau'$. However, we continue with the above expression, and first normalise it by considering the limit $x \to 0$, i.e. $z_2 \to z_1$, $z_4 \to z_3$. This gives

$$\mathcal{A}_{tree}(x \to 0) \sim (\alpha')^{-1} g_o^2 \langle \sigma_0^{(aa^{\|})}(0) \sigma_0^{(a^{\|}a)}(1) \rangle_a \prod_{\kappa=1}^{3} C^{(aba^{\|})0}_{\nu^\kappa, 1-\nu^\kappa} C^{(a^{\|}ca)0}_{1-\lambda^\kappa, \lambda^\kappa} \tag{A.1.11}$$

where we now have *stretch* fields for strings stretched between parallel branes. In the effective field theory, these are highly massive for large separation, and thus non-supersymmetric; we normalise them for consistency with the case $a^{\|} = a$ to give $\langle 1 \rangle_a$, in which case we obtain

$$\mathcal{A}_{tree}(x \to 0) \sim (\alpha')^{-2} g_a^2 G_{C_{ab}\bar{C}_{ba}} G_{C_{ac}\bar{C}_{ca}} \tag{A.1.12}$$

Applying this to our expression, in this limit $\tau^{\kappa\prime} \to \tau \to \frac{-\sin \pi\nu^\kappa}{\pi} \ln x$, and so we the instanton sum over n_2 vanishes - requiring us to Poisson resum as in [15], giving

$$\mathcal{A}_{tree}(x \to 0) \sim C(\alpha')^{-1} g_o^2 \prod_{\kappa=1}^{3} z_{12}^{-2h_{\nu^\kappa}} z_{34}^{-2h_{\lambda^\kappa}} \frac{\sqrt{2\pi\alpha'}}{L_a^\kappa} x^{\frac{(y^\kappa)^2}{4\pi^2\alpha'}} \tag{A.1.13}$$

where y^κ is the perpendicular distance between branes a and $a^{\|}$ in sub-torus κ; the zero-mode of v_{32} is irrelevant. This gives us two pieces of information: first, we obtain the normalisation

$$C = \frac{e^\Phi}{\alpha' g_o^2} G_{C_{ab}\bar{C}_{ba}} G_{C_{ac}\bar{C}_{ca}} (2\pi)^{5/2} \tag{A.1.14}$$

(almost) in agreement with the similar case considered by [15]. Secondly, we obtain the conformal weight of the operators $\sigma^{aa^{\|}}$: $h^\kappa_{aa^{\|}} = \frac{(Y^\kappa)^2}{2\pi^2\alpha'}$.

Now we must determine the more general coefficients by taking the limit $z_3 \to z_2$, or equivalently $x \to 1$. In this case $\tau^\kappa \to 0$, $\tau^{\kappa\prime} \to \beta$, and we obtain

$$\mathcal{A}_{tree}(x \to 1) \sim C(\alpha')^{-1} g_o^2 \prod_{\kappa=1}^{3} (1-x)^{-\nu^\kappa\lambda^\kappa} Y^\kappa \sum_{n_1,n_2} \exp - \frac{A(n_1,n_2)}{2\pi\alpha'} \tag{A.1.15}$$

where

$$Y^\kappa = \begin{cases} \left(\frac{\Gamma(1-\nu^\kappa)\Gamma(1-\lambda^\kappa)\Gamma(\nu^\kappa+\lambda^\kappa)}{\Gamma(\lambda^\kappa)\Gamma(\nu^\kappa)\Gamma(1-\nu^\kappa-\lambda^\kappa)} \right)^{1/2} & \nu^\kappa + \lambda^\kappa < 1 \\ \left(\frac{\Gamma(\nu^\kappa)\Gamma(\lambda^\kappa)\Gamma(2-\nu^\kappa-\lambda^\kappa)}{\Gamma(1-\nu^\kappa)\Gamma(1-\lambda^\kappa)\Gamma(\nu^\kappa+\lambda^\kappa-1)} \right)^{1/2} & \nu^\kappa + \lambda^\kappa > 1 \end{cases} \tag{A.1.16}$$

and $A^\kappa(n_1^\kappa, n_2^\kappa)$ is the sum of the areas of the two possible Yukawa triangles formed by the intersection of the three branes wrapping in both directions, given by

$$\begin{aligned} A^\kappa(n_1^\kappa, n_2^\kappa) &= \frac{1}{2}\frac{\sin\pi\nu^\kappa \sin\pi\lambda^\kappa}{\sin\pi(1-\nu^\kappa-\lambda^\kappa)}\left((v_{32}^\kappa)^2 + (v_{21}^\kappa\beta^\kappa + v_{32}^\kappa)^2\right) \\ &= \frac{1}{2}\frac{\sin\pi\nu^\kappa \sin\pi\lambda^\kappa}{\sin\pi(1-\nu^\kappa-\lambda^\kappa)}\left((d_1^\kappa)^2(n_1^\kappa) + (d_2^\kappa)^2(n_2^\kappa)\right) \end{aligned} \tag{A.1.17}$$

(and a similar expression for $\lambda^\kappa + \nu^\kappa > 1$). With this expression, it becomes immediately clear how we can obtain the OPE coefficients:

$$\mathcal{A}_{tree}(x \to 1) \sim (\alpha')^{-1} g_o^{-2}\langle 1\rangle_c g_0^4 \prod_{\kappa=1}^{3} C^{(ba^\| c)1-\nu^\kappa-\lambda^\kappa}_{1-\nu^\kappa,1-\lambda^\kappa} C^{(cab)\nu^\kappa+\lambda^\kappa}_{\nu^\kappa,\lambda^\kappa} C^{(cbc)0}_{\lambda^\kappa,1-\lambda^\kappa}(1-x)^{-\nu^\kappa\lambda^\kappa} \tag{A.1.18}$$

and thus we infer

$$\begin{aligned} \sqrt{\alpha'} g_o e^{-\Phi/2}\prod_{\kappa=1}^{3} C^{(cab)\nu^\kappa+\lambda^\kappa}_{\nu^\kappa,\lambda^\kappa} &= \left(2\pi G^{C_{bc}\bar{C}_{cb}} G_{C_{ab}\bar{C}_{ba}} G_{C_{ac}\bar{C}_{ca}}\right)^{1/2} \\ &\qquad \prod_{\kappa=1}^{3}(\sqrt{2\pi}Y^\kappa)^{1/2}\sum_{m^\kappa}\exp -\frac{A^\kappa(m^\kappa)}{2\pi\alpha'} \end{aligned} \tag{A.1.19}$$

where now $A^\kappa(m^\kappa)$ is the area of the triangle bac, while the conjugate coefficient will have the area of $ba^\| c$ (zero if $a^\| = a$). This concludes the determination of all the relevant OPE coefficients in the theory. For use in the text, we define

$$Y^{(cab)} = g_o \prod_{\kappa=1}^{3} C^{(cab)}(G_{C_{bc}\bar{C}_{cb}})^{1/2} \tag{A.1.20}$$

and $Y^{ba^\| c}$ analogously; they represent the physical Yukawa couplings.

A.2 One-Loop Amplitudes With Gauge Bosons

A.2.1 Classical Part

The prescription of [57] applies to the evaluation of the classical part of boundary-changing operators, by the analogy with twist operators, after we have applied the doubling trick - the nett result being that we must halve the action that we obtain. Since it is the only consistent arrangement for two and three point diagrams, and the most interesting for four-point and above, we shall specialise to all operators lying on the imaginary axis, where the domain of the torus (doubled annulus) is $[-1/2, 1/2] \times [0, it]$; this provides simplifications in the calculation while providing the most interesting result. In the case of all operators on the other end of the string, we would simply define the doubling differently to obtain the same result, and since the amplitude only depends on differences between positions the quantum formulae given would be correct.

For L operators inserted, we must have $M = \sum_{i=1}^{L} \theta_i$ integer to have a non-zero amplitude; for vertices chosen to lie on the interior of a polygon we will have $M = L - 2$, which h shall be the case for the three and four point amplitudes we consider - in the case of a two-point amplitude we must have $M = 1$. Labelling the vertices in order, we denote the first $L - M$ vertices $\{z_\alpha\}$, and the remaining M vertices $\{z_\beta\}$, we then have a basis of $L - M$ functions for $\partial X(z)$ given by

$$\omega^\alpha = \theta_1(z - z_\alpha - Y) \prod_{j \in \{\alpha\} \neq \alpha}^{L-M} \theta_1(z - z_j) \prod_{k=1}^{L} \theta_1(z - z_k)^{\theta_k - 1} \tag{A.2.1}$$

where

$$Y = -\sum_{i=1}^{L-M} \theta_i \; z_i + \sum_{j=L-M+1}^{L} (1 - \theta_j) z_j \tag{A.2.2}$$

and the basis of M functions for $\partial \bar{X}(z)$

$$\omega^\beta = \theta_1(z - z_\beta + Y) \prod_{j \in \{\beta\} \neq \beta}^{L} \theta_1(z - z_j) \prod_{k=1}^{L} \theta_1(z - z_k)^{-\theta_k} \tag{A.2.3}$$

We are then required to choose a basis of closed loops on the surface. There are two cycles associated with the surface which we shall label A and B, and define as follows:

$$\begin{aligned} \oint_{\gamma_A} \mathrm{d}z &= \int_{it-1/2}^{-1/2} \mathrm{d}z \\ \oint_{\gamma_B} \mathrm{d}z &= \int_{-1/2}^{1/2} \mathrm{d}z \end{aligned} \tag{A.2.4}$$

The remaining integrals involve the boundary operators, and we define (not-closed) loops C_i by

$$\int_{C_i} \mathrm{d}z = \int_{z_i}^{z_{i+1}} \mathrm{d}z \tag{A.2.5}$$

Note that we have defined in total $L-1$ C-loops, and they are actually linearly dependent, since we can deform a contour around all of the operators to the boundary to give zero - we only require $L-2$. These could then be formed into (closed) Pochhammer loops by multiplication by a phase factor, but it actually turns out that this is not necessary. We form the above into a set $\{\gamma_a\} = \{\gamma_A, \gamma_B\} \cup \{C_i, i = 1, ..., L-2\}$, and define the $L \times L$ matrix W_a^i by

$$\begin{aligned} W_a^\alpha &\equiv \int_{\gamma_a} \mathrm{d}z \omega^\alpha(z) \\ W_a^\beta &\equiv \int_{\bar{\gamma}_a} \mathrm{d}\bar{z} \bar{\omega}^\beta(\bar{z}) \end{aligned} \tag{A.2.6}$$

The boundary operators induce branch cuts on the worldsheet, which we have a certain amount of freedom in arranging. We shall choose the prescription that the cuts run in a daisy-chain between the operators, with phases $\exp(i\alpha_i)$ when passing through the cut anticlockwise with respect to z_i defined as follows:

$$\begin{aligned} \alpha_1 &= 2\pi\theta_1 \\ \alpha_{L-1} &= -2\pi\theta_L \\ 2\pi\theta_i &= \alpha_i - \alpha_{i-1} \\ \alpha_i &= 2\pi \sum_{j=1}^{i} \theta_j \end{aligned} \tag{A.2.7}$$

noting that α_i is only defined modulo 2π. Each path γ_a is associated with a physical displacement v_a (which shall be determined later); i.e

$$\Delta_{\gamma_a} X_{cl} = v_a \tag{A.2.8}$$

and, since we can write ∂X and $\partial \bar{X}$ as linear combinations of ω^α and ω^β respectively, we obtain

$$S = \frac{1}{4\pi\alpha'} v_a \bar{v}_b \{ (W^{-1})^a_{i'} (\bar{W}^{-1})^b_{j'} (\omega^{i'}, \omega^{j'}) + (W^{-1})^a_{i''} (\bar{W}^{-1})^b_{j''} (\omega^{j''}, \omega^{i''}) \} \tag{A.2.9}$$

where the sum over primed indices is understood to be over $\{\alpha\}$, and over double-primed indices to be over $\{\beta\}$, and for reference we have used the complexification $X = \frac{1}{\sqrt{2}}(X_1 + iX_2)$. The inner product is defined to be

$$(\omega^{i'}, \omega^{j'}) = \int \mathrm{d}^2 z \omega^{i'}(z) \bar{\omega}^{j'}(\bar{z}) \equiv i W_a^{i'} \bar{W}_b^{j'} M^{ab} \tag{A.2.10}$$

and similarly

$$(\omega^{i''}, \omega^{j''}) = \frac{1}{4\pi\alpha'}\int d^2z\omega^{i''}(z)\bar{\omega}^{j''}(\bar{z}) \equiv i\bar{W}_a^{i''}W_b^{j''}\bar{M}^{ab} \tag{A.2.11}$$

and thus $\bar{M}^{ab} = -M^{ba}$. As in [57] we have $M^{AB} = -M^{BA} = 1$, but, after performing the canonical dissection on the torus for our arrangement of branch cuts and basis loops we determine:

$$\begin{aligned} M^{ml} &= \frac{2i}{\sin\left(\frac{\alpha_{L-1}}{2}\right)} e^{i\frac{\alpha_l-\alpha_m}{2}} \sin\left(\frac{\alpha_{L-1}-\alpha_l}{2}\right)\sin\frac{\alpha_m}{2} \quad m<l \\ M^{mm} &= \frac{2i}{\sin\left(\frac{\alpha_{L-1}}{2}\right)}\sin\left(\frac{\alpha_{L-1}-\alpha_m}{2}\right)\sin\frac{\alpha_m}{2} \end{aligned} \tag{A.2.12}$$

Inserted into the expression for the action, this gives $S = \frac{i}{4\pi\alpha'}v_a\bar{v}_b S^{ab}$, where

$$S^{ab} = (\bar{W}^{-1})^b_{j'}\bar{W}_d^{j'}M^{ad} + (W^{-1})^a_{i''}W_d^{i''}\bar{M}^{bd} \tag{A.2.13}$$

At this point we determine the displacements. Clearly, since the boundary $Re(z) = -1/2$ contains no boundary-changing operators, along γ_A the string end is fixed to one brane, which we shall label a. Thus, v_A represents the one-cycles of brane a, and is equal to $\frac{1}{\sqrt{2}}n_A L_a$, where L_a is the length of a and the factor of $1/\sqrt{2}$ is due to the complexification we chose. Now, the technique that we are using requires there to be a path around the boundary of the worldsheet where we do not cross any branch cuts: if the string has one end fixed on brane a along path γ_A, the other end (at $z=0$) must either reside on brane a or a brane parallel to it, which we shall denote $a^{\parallel}$. Path γ_B is related to the displacement between these branes. Consider the doubling used, for coordinates aligned along brane a:

$$\partial X(z) = \begin{cases} \partial X(z) & \Re(z) > 0 \\ -\bar{\partial}\bar{X}(-\bar{z}) & \Re(z) < 0 \end{cases} \tag{A.2.14}$$

and a similar relationship for $-\bar{\partial}X(-\bar{w})$ and $\partial\bar{X}$. Note that we have $\Delta_{\gamma_A}X = \Delta_{\gamma_a}\bar{X}$, in keeping with our identification of v_A. We have

$$\begin{aligned} \Delta_{\gamma_B} &= \int_{\gamma_B} dz\partial X + \int_{\gamma_B} d\bar{z}\bar{\partial}X(z) \\ &= \int_0^{1/2} dx\frac{d}{dx}X(x) - \frac{d}{dx}\bar{X}(x) \\ &= i\sqrt{2}(X_2(a) - X_2(a^{\parallel})) \\ &= i\sqrt{2}(\frac{n_B 4\pi^2 T_2}{L_a} + y) \end{aligned} \tag{A.2.15}$$

where y is the smallest distance between a and $a^{\|}$ and T_2 is the Kähler modulus of the torus, defined earlier. This resolves an ambiguity in [30]. The remaining paths have straightforward identifications, since they are essentially like the tree-level case; they are the displacements between physical vertices. We must identify each portion of the $Re(z) = 0$ boundary with a brane segment between intersections; so for the case

$$\langle \sigma_{\nu}^{(ab)}(z_1)\sigma_{1-\nu}^{(ba^{\|})}(z_2)\sigma_{1-\lambda}^{(a^{\|}c)}(z_3)\sigma_{\lambda}^{(ca)}(z_4)\rangle_{aa} \tag{A.2.16}$$

we have $v_1 = \frac{1}{\sqrt{2}}(n_1 L_b + L_{12})$ and $v_2 = \frac{1}{\sqrt{2}}(n_2 L_{a^{\|}} + L_{23})$. If we were to use Pochhammer loops for these, we would multiply these by Pochhammer factors - but these would be cancelled in the action by those associated with the loops on the worldsheet.

Note that although we are free to choose the arrangement of branch cuts on the worldsheet, we have no freedom in identifying the displacement vectors, and thus the daisy-chain prescription is the simplest, particularly since we are restricted in the permutations of the boundary-changing operators - we must ensure that each change of brane has the correct operator to mediate it.

A.2.2 Quantum Part

The quantum part of the correlator is given in terms of the variables defined in the previous section as

$$\begin{aligned}\langle \prod_{n=1}^{L} \sigma_{\theta_n}(z_n)\rangle = |W|^{-1/2}\theta_1(Y)^{L-2} \prod_{i<j}^{L-M} \theta_1(z_i - z_j) \prod_{L-M<i<j}^{L} \theta_1(z_i - z_j) \\ \prod_{i<j}^{L} \theta_1(z_i - z_j)^{-\theta_i\theta_j - (1-\theta_i)(1-\theta_j)}\end{aligned} \tag{A.2.17}$$

where $|W^{\kappa}|$ is the determinant of the matrix of integrals of cut differentials. Note that this expression does not depend upon the brane labels of the boundary-changing operators, as it is not sensitive to the particular boundary conditions.

The correlators for the fermionic component of the vertex operators do not receive worldsheet instanton corrections, so they are given by the relatively simple formula presented in [30]:

$$\langle \prod_{i=1}^{L} e^{iq_i H(z_i)} \rangle_\nu = \theta_\nu(\sum_{i=1}^{L} q_i z_i) \prod_{i<j} \theta_1(z_i - z_j)^{q_i q_j} \tag{A.2.18}$$

the above is a generalisation of the formulae for standard spin-operator correlators (see e.g. [37]) and is thus also valid for the $4d$ spin correlators.

A.2.3 4d Bosonic Correlators

Beginning with the Green's function for the annulus, obtained via the method of images on the torus:

$$\begin{aligned} G(z,w) = -\frac{\alpha'}{2} \ln|\theta_1(z-w)|^2 - \frac{\pi i \alpha'}{t}(\Im(z-w))^2 \\ - \frac{\alpha'}{2} \ln|\theta_1(z+\bar{w})|^2 - \frac{\pi i \alpha'}{t}(\Im(z+\bar{w}))^2 \end{aligned} \tag{A.2.19}$$

and specialising to all points at boundaries (i.e. $\bar{z} = -z$ or $1-z$), we obtain the amplitude on the annulus

$$\begin{aligned} A_X^4 \equiv \langle \prod_{i=1}^{M} e^{ik_i \cdot X(z_i)} \rangle &= C_X \prod_{i<j} \left[\theta_1(z_i - z_j) e^{\frac{-\pi}{t}(\Im(z_i - z_j))^2} \right]^{2\alpha' k_i \cdot k_j} \\ &= Z_X^4 \prod_{i<j} \left[\frac{\theta_1(z_i - z_j)}{\theta_1'(0)} e^{\frac{-\pi}{t}(\Im(z_i - z_j))^2} \right]^{2\alpha' k_i \cdot k_j} \\ &\equiv Z_X^4 \prod_{i<j} \chi^{2\alpha' k_i \cdot k_j} \end{aligned} \tag{A.2.20}$$

where

$$Z_X^4 = (8\pi^2 \alpha' t)^{-2} \eta(it)^{-2} \tag{A.2.21}$$

is the partition function for the non-compact bosons with the bc-ghost contribution included. We then use the green's function to obtain

$$\begin{aligned} \langle \partial X^{\mu_3}(z) \partial X^{\mu_4}(w) \prod_i e^{ik_i \cdot X(z_i)} \rangle = \langle \prod_{m=1}^{4} e^{ik_m \cdot X(z_m)} \rangle \\ \left\{ 2\eta^{\mu\nu} \partial_z \partial_w G(z,w) + 4 \sum_{i=1}^{4} \sum_{j=1}^{4} k_i^{\mu_3} k_j^{\mu_4} \partial_z G(z, z_i) \partial_w G(w, z_j) \right\} \end{aligned} \tag{A.2.22}$$

which is required for 4-point functions.

A.3 One-Loop Amplitudes Without Gauge Bosons

A.3.1 Classical Part

Cut Differentials

The cut differentials for diagrams on the doubled annulus where there exists no periodic cycle necessarily have modified boundary conditions. Where the diagram with no boundary changing operators is the partition function of strings stretched between branes a and b with angle $\pi\theta_{ab}$, the conditions for the one form $\partial X(w)$ are (see section 2.4.2):

$$\begin{aligned} \partial X(w+it) &= \partial X(w) \\ \partial X(w+1) &= e^{2\pi i\theta_{ab}}\partial X(w) \end{aligned} \tag{A.3.1}$$

while

$$\begin{aligned} \partial\bar{X}(w+it) &= \partial X(w) \\ \partial\bar{X}(w+1) &= e^{-2\pi i\theta_{ab}}\partial X(w) \end{aligned} \tag{A.3.2}$$

whereas the local monodromies remain the same as for the periodic case. Indeed, we can retain many of the elements of those differentials, including

$$\begin{aligned} \gamma_X(z) &= \prod_{i=1}^{L}\theta_1(z-z_i)^{\theta_i-1} \\ \gamma_{\bar{X}}(z) &= \prod_{i=1}^{L}\theta_1(z-z_i)^{-\theta_i} \end{aligned} \tag{A.3.3}$$

These satisfy the local monodromies; to construct a complete set of differentials satisfying the global monodromy conditions, we note the identities:

$$\begin{aligned} \theta\begin{bmatrix} 1/2+a \\ 1/2 \end{bmatrix}(z+m;\tau) &= \exp(2\pi i(1/2+a)m)\theta\begin{bmatrix} 1/2+a \\ 1/2 \end{bmatrix}(z;\tau) \qquad (A.3.4) \\ \theta\begin{bmatrix} 1/2+a \\ 1/2 \end{bmatrix}(z+m\tau;\tau) &= \exp(-\pi im-\pi im^2\tau-2\pi imz)\theta\begin{bmatrix} 1/2+a \\ 1/2 \end{bmatrix}(z;\tau) \end{aligned}$$

which show that we only need to modify one theta function from the periodic case; we have also

$$\theta\begin{bmatrix} c+a \\ b \end{bmatrix}(z;\tau) = \exp[2\pi ia(z+c)+a^2\pi i\tau]\theta\begin{bmatrix} c \\ b \end{bmatrix}(z+a\tau;\tau) \tag{A.3.5}$$

which is crucial for showing the equivalence between the approach that we are about to use, and the method of obtaining these amplitudes by factorising higher-order amplitudes calculated by the previous method. Denoting the theta-function

$$\theta_{\pm ab}(z,\tau) \equiv \theta \left[\begin{array}{c} 1/2 \pm \theta_{ab} \\ 1/2 \end{array} \right] (z;\tau) \tag{A.3.6}$$

we construct the set of $L-M$ differentials for $\partial X(z)$, similar to before

$$\omega^{\alpha}_{+ab}(z) = \gamma_X(z)\theta_{+ab}(z - z_\alpha - Y) \prod_{j\in\{\alpha\}\neq\alpha}^{L-M} \theta_1(z - z_j) \tag{A.3.7}$$

where Y is as defined before; and we have the set of M differentials for $\partial\bar{X}$

$$\omega^{\beta}_{-ab}(z) = \gamma_{\bar{X}}(z)\theta_{-ab}(z - z_\beta + Y) \prod_{j\in\{\beta\}\neq\beta}^{L} \theta_1(z - z_j) \tag{A.3.8}$$

We demonstrate that these are complete sets in the same way as in [57]: first, note that the function are independent, since $\omega^\alpha(z_{j\neq\alpha}) = 0$. Then suppose we had another differential $\omega'(z)$; we construct the doubly-periodic meromorphic function $\lambda(z)$

$$\lambda(z) = \frac{\omega'(z)}{\omega^1(z)} - \sum_{i=1}^{L-M} C_i \frac{\omega^i(z)}{\omega^1(z)} \tag{A.3.9}$$

At $z = z_1$ or $z \in \{z_\beta\}$, the above is not singular, while we can adjust the $L-M$ constants C_i to cancel the residues of the poles at $z = z_{i\neq 1}$ and $z_1 + Y - \theta_{ab} it$. Thus, since λ has no poles, and it is doubly periodic on the torus, it is a constant(the last point contains the only subtlety with respect to the earlier case; even though the differentials ω^α are not periodic on the torus, because they only acquire a phase, the differential λ is periodic). Hence, since C_1 just multiplies a constant, we can adjust it to set λ to zero. The same follows for ω^β.

Note that if we want to use the same theta-functions throughout, we could choose a basis obtained by factorising the functions used earlier. In this case, we obtain

$$\omega^{\alpha\prime}_{+ab}(z) = e^{\pi i(1-it)\theta_{ab}} e^{2\pi i\theta_{ab} z} \gamma_X(z)\theta_1(z - z_\alpha - Y + \theta_{ab} it) \prod_{j\in\{\alpha\}\neq\alpha}^{L-M} \theta_1(z - z_j) \tag{A.3.10}$$

and

$$\omega^{\beta\prime}_{-ab}(z) = e^{-\pi i(1-it)\theta_{ab}} e^{-2\pi i\theta_{ab} z} \gamma_{\bar{X}}(z)\theta_1(z - z_\beta + Y - \theta_{ab} it) \prod_{j\in\{\beta\}\neq\beta}^{L} \theta_1(z - z_j) \tag{A.3.11}$$

The relation between the bases is given simply by

$$\begin{aligned}\omega^{\alpha\prime}_{+ab}(z) &= e^{\pi t(\theta_{ab}+\theta^2_{ab})}e^{2\pi i\theta_{ab}(z_\alpha+Y)}\omega^{\alpha}_{+ab}(z)\\ \omega^{\beta\prime}_{-ab}(z) &= e^{-\pi t(\theta_{ab}+\theta^2_{ab})}e^{-2\pi i\theta_{ab}(z_\beta-Y)}\omega^{\beta}_{-ab}(z)\end{aligned}\tag{A.3.12}$$

The choice of basis is not important for the following section, and the amplitude will of course be independent of the basis choice.

Canonical Dissection

With our basis of cut differentials for the doubled annulus we require their Hermitian inner product to calculate the action. This is given by

$$(\omega^i,\omega^j) = \int_R \mathrm{d}^2 z\bar{\omega}^j\omega^i = i\oint_{\partial R}\bar{\omega}^j(\bar{z})\mathrm{d}\bar{z}\int_{z_0}^{z}\mathrm{d}z\omega^i \tag{A.3.13}$$

This is the canonical dissection of the surface, where the contour passes anticlockwise around the surface without crossing any branch cuts. The task is then to choose the most convenient arrangement of cuts, and express the above in terms of integrals of paths on the surface corresponding to physical displacements. Since we consider operators only on one boundary, we arrange them to be on the imaginary axis; since we always have one vertex fixed, we choose this to be z_L and place it at the origin. The integrals between vertices are then defined by

$$\int_{C_n} = \int_{z_{n+1}}^{z_n} \tag{A.3.14}$$

where $\Im(z_n) > \Im(z_{n+1})$. We also have the loops

$$\begin{aligned}\int_A &= \int_{-1+it}^{-1} = \int_{-1/2+it}^{-1/2}\\ \int_B &= \int_{-1}^{0}\end{aligned}\tag{A.3.15}$$

and the phases

$$\begin{aligned}\alpha_1 &= 2\pi\theta_1 - 2\pi\theta_{ab}\\ \alpha_{L-1} &= -2\pi\theta_L - 2\pi\theta_{ab}\\ 2\pi\theta_i &= \alpha_i - \alpha_{i-1}\\ \alpha_i &= 2\pi\sum_{j=1}^{i}\theta_j - 2\pi\theta_{ab}\end{aligned}\tag{A.3.16}$$

we also label the additional cycle γ_D:

$$\begin{aligned} \int_{\gamma_D} &= \int_{z_1}^{it} \\ \alpha_D &= -2\pi\theta_{ab} \end{aligned} \tag{A.3.17}$$

which we eliminate, along with C_1, via the equations

$$\begin{aligned} \gamma_D + C_1 + \sum_{n=2}^{L-1} C_n &= -\gamma_A \\ e^{i\alpha_D}\gamma_D + e^{i\alpha_1}C_1 + \sum_{n=2}^{L-1} e^{i\alpha_n}C_n &= -\gamma_A \end{aligned} \tag{A.3.18}$$

Hence we have chosen our set of paths to be $\{\gamma_A, \gamma_B\} \cup \{C_2..C_{L-1}\}$. With these definitions, we now perform the canonical dissection, and defining our matrices as before $W^{i'}_A = \int_A \omega^{i'}$ etc, the inner product is of the form

$$(\omega^{i'}, \omega^{j'}) = iW^{i'}_a \bar{W}^{j'}_b M^{ab} \tag{A.3.19}$$

and the results are

$$\begin{aligned} M^{AB} &= 1 \\ M^{AA} &= 2i\frac{\sin\frac{\alpha_1}{2}\sin\frac{\alpha_D}{2}}{\sin\frac{\alpha_D-\alpha_1}{2}} \\ M^{An} &= -2i\frac{e^{-i\alpha_n}}{\sin(\frac{\alpha_D-\alpha_1}{2})}\sin(\frac{\alpha_n-\alpha_1}{2})\sin\frac{\alpha_D}{2} \\ M^{mn} &= \frac{2i}{\sin(\frac{\alpha_D-\alpha_1}{2})}e^{i\frac{\alpha_m-\alpha_n}{2}}\sin(\frac{\alpha_m-\alpha_D}{2})\sin(\frac{\alpha_n-\alpha_1}{2}) \end{aligned} \tag{A.3.20}$$

where $m \geq n$ in the last line, $L-1 \geq n, m \geq 2$ and the elements reflected in the diagonal can be obtained from the above using $M^{dc} = -\bar{M}^{cd}$. We see that we can obtain appropriate expressions for the previous case simply by taking $\theta_{ab} = 0$, in which case the formulae are greatly simplified, with the AA and An-elements vanishing. The above expression then yields the classical action by equation (A.2.9).

A.3.2 Quantum Part

As for the classical part, the quantum part of the bosonic correlator for these diagrams may be determined in two ways; factorisation of a diagram with a larger number of operators inserted, or by calculating new green's functions and proceeding by the stress-tensor method. However, unlike for the classical part, it is straightforward to perform the factorisation.

To obtain the quantum amplitude for L boundary-changing operators with angles $\{\theta_i\}$ and overall periodicity θ_{ab}, we start with an amplitude with $L+1$ operators with angles $\{\theta_{ab}, \theta_i, \theta_L - \theta_{ab}\}$ which we assign to vertices at $\{z_0, z_i, 0\}$. The above choice of angles ensures that in both sets of equations M is the same, and we have

$$Y_{L+1} = Y_L - \theta_{ab} it \tag{A.3.21}$$

The full quantum correlator is

$$Z_{qu} = f(it)|W_{L+1}|^{-1/2}\theta_1(Y_{L+1})^{(L-1)/2} \prod_{0\le i<j}^{L-M} \theta_1(z_i - z_j)^{1/2} \prod_{L-M<i<j}^{L} \theta_1(z_i - z_j)^{1/2}$$
$$\prod_{0\le i<j}^{L} \theta_1(z_i - z_j)^{-\frac{1}{2}[1-\theta_i-\theta_j+2\theta_i\theta_j]} \tag{A.3.22}$$

where $f(it)$ is the normalisation. We then note that in the limit $z_0 \to it$, all except two of the integrals in the matrix W_{L+1} are finite. Indeed, only $\omega^0(z)$ develops a singularity, and only the integrals of it over the cycles γ_B and C_0 become infinite (note that we are using the set of curves $\{\gamma_A, \gamma_B, C_0..C_{L-1}\}$). The determinant becomes in the limit

$$|W_{L+1}| \to (W_{L+1})^0_0|W_L| - (W_{L+1})^0_B|W'_L| \tag{A.3.23}$$

where $|W'_L|$ is the determinant with the γ_B cycles and ω^0 integrals deleted: we then note that the rows are not linearly independent due to the identities (A.3.18), and it is therefore zero. We must finally evaluate $(W_{L+1})^0_0$.

$$(W_{L+1})^0_0 \sim -i(it - z_0)^{\theta_L - 1} B(1-\theta_L, \theta_L - \theta_{ab}) e^{\pi i(1-it)(\theta_{ab}-1)} \theta'_1(0)^{\theta_L - 2}$$
$$\theta_1(-Y_{L+1}) \prod_{i=1}^{L-M} \theta_1(it - z_i)^{\theta_i} \prod_{j=L-M+1}^{L-1} \theta_1(it - z_j)^{\theta_j - 1} \tag{A.3.24}$$

which, when we consider that the amplitude should factorise according to

$$\sigma^{(ca)}_{\theta_{ab}}(z_0)\sigma^{(ab)}_{\theta_L - \theta_{ab}}(0) \sim (it - z_0)^{-\theta_{ab}\theta_L + \theta_{ab}^2} C^{(bac)\theta_L}_{\theta_L - \theta_{ab}, \theta_{ab}} \sigma^{(bc)}_{\theta_L}(0) \tag{A.3.25}$$

we find that the quantum portion of the amplitude is

$$Z_{qu}^{(ab)} = g(it)|W_L'|^{-1/2}e^{2\pi iP}\theta_1(Y_L - \theta_{ab}it)^{(L-2)/2}\prod_{0<i<j}^{L-M}\theta_1(z_i - z_j)^{1/2}$$
$$\prod_{L-M<i<j}^{L}\theta_1(z_i - z_j)^{1/2}\prod_{0<i<j}^{L}\theta_1(z_i - z_j)^{-\frac{1}{2}[1-\theta_i-\theta_j+2\theta_i\theta_j]} \quad (A.3.26)$$

where now W_L contains integrals of the primed basis of cut differentials, and

$$P = Y + \frac{1}{2}\left(\sum_{i=1}^{L-M} z_i - \sum_{j=L-M+1}^{L} z_j\right) \quad (A.3.27)$$

Now we find that the "natural" basis for these functions is that which we defined in equations (A.3.7) and (A.3.8); in this basis, using the relations (A.3.12) we find the amplitude to be

$$Z_{qu}^{(ab)} = g(it)|W_L|^{-1/2}\theta_{-ab}(Y_L)^{(L-2)/2}\prod_{0<i<j}^{L-M}\theta_1(z_i - z_j)^{1/2}$$
$$\prod_{L-M<i<j}^{L}\theta_1(z_i - z_j)^{1/2}\prod_{0<i<j}^{L}\theta_1(z_i - z_j)^{-\frac{1}{2}[1-\theta_i-\theta_j+2\theta_i\theta_j]} \quad (A.3.28)$$

The above can also be obtained by repeating the analysis of [57]; most of the steps are the same, since the quantum amplitude does not depend upon the exact form of the greens' functions, only certain constraints upon them, which remain the same for these amplitudes.

A.3.3 Fermionic Correlators

The fermionic correlators for these amplitudes are easily obtained by simply factorising equation (A.2.18); we obtain

$$\langle\prod_{i=1}^{L} e^{i(\theta_i-1/2)H(z_i)}\rangle_{\nu,\theta_{ab}} = e^{2\pi i(\theta_{ab}-1/2)P}\theta_\nu(it(\theta_{ab}-1/2) + \sum_{i=1}^{L} q_i z_i)$$
$$\prod_{i<j}\theta_1(z_i - z_j)^{(\theta_i-1/2)(\theta_j-1/2)} \quad (A.3.29)$$

A.4 Theta Identities

Throughout we use the standard notation for the Jacobi Theta functions:

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau) = \sum_{n=-\infty}^{\infty} \exp\Big[\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)\Big] \tag{A.4.1}$$

and define $\theta_{\alpha\beta} \equiv \theta\begin{bmatrix} \alpha/2 \\ \beta/2 \end{bmatrix}$, so that we have the periodicity relations

$$\begin{aligned} \theta\begin{bmatrix} a \\ b \end{bmatrix}(z+m;\tau) &= \exp(2\pi iam)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau) \\ \theta\begin{bmatrix} a \\ b \end{bmatrix}(z+m\tau;\tau) &= \exp(-2\pi ibm)\exp(-\pi im^2\tau - 2\pi imz)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\tau) \end{aligned} \tag{A.4.2}$$

We also define, according to the usual conventions,

$$\begin{aligned} &\theta_1 \equiv \theta_{11} \quad \theta_2 \equiv \theta_{10} \\ &\theta_3 \equiv \theta_{00} \quad \theta_4 \equiv \theta_{01} \end{aligned} \tag{A.4.3}$$

Appendix B

Material Regarding Instanton Calculations in Intersecting Brane Worlds

B.1 4d Spin Field Correlators

In the evaluation of annulus contributions to the superpotential involving two fermionic fields and two supersymmetry modes, it is necessary to evaluate the correlator of four left-handed spin fields in four dimensions. In general, there may also be picture-changing operators in the amplitude, although there are not for the particular case in section 4.3. The calculation is performed using the techniques of [105]; the procedure is to construct a complete set of Lorentz structures and determine their coefficients by finding particular values for the spinor/Lorentz indices for which only one structure is non-zero, and evaluating the correlator in those cases. For the general case when there are two picture-changing operators inserted on the non-compact directions for four like-chirality spinors $\{u_1, u_2, u_3, u_4\}$, the amplitude is given by

$$\langle \Psi^{\mu}(z)\Psi^{\nu}(w)\prod_{i=1}^{4}(u_i)_{\alpha_i}\tilde{S}^{\alpha_i}(z_i)\rangle = -G(u_2u_4)(u_3C\Gamma^{\mu_3}\Gamma^{\mu_4}u_1)+J(u_1u_2)(u_3C\Gamma^{\mu_3}\Gamma^{\mu_4}u_4)$$

$$+ H(u_3u_2)(u_1C\Gamma^{\mu_3}\Gamma^{\mu_4}u_4) + 2B\eta^{\mu_3\mu_4}(u_1u_3)(u_2u_4) + 2C\eta^{\mu_3\mu_4}(u_1u_4)(u_2u_3) \quad \text{(B.1.1)}$$

where $(u_1 u_2) \equiv (u_1)_{\alpha_1} C^{\alpha_1\alpha_2}(u_2)_{\alpha_2}$, and the coefficient functions are given by

$$\begin{aligned}
G &= \langle \tilde{S}^2(z_1)\tilde{S}^2(z_2)\tilde{S}^2(z_3)\tilde{S}^1(z_4)\Psi^0(z)\Psi^{\bar{1}}(w)\rangle \\
J &= \langle \tilde{S}^1(z_1)\tilde{S}^2(z_2)\tilde{S}^2(z_3)\tilde{S}^2(z_4)\Psi^0(z)\Psi^{\bar{1}}(w)\rangle \\
H &= \langle \tilde{S}^2(z_1)\tilde{S}^2(z_2)\tilde{S}^1(z_3)\tilde{S}^2(z_4)\Psi^0(z)\Psi^{\bar{1}}(w)\rangle \\
B &= \langle \tilde{S}^2(z_1)\tilde{S}^1(z_2)\tilde{S}^1(z_3)\tilde{S}^2(z_4)\Psi^0(z)\Psi^{\bar{0}}(w)\rangle \\
C &= \langle \tilde{S}^1(z_1)\tilde{S}^2(z_2)\tilde{S}^1(z_3)\tilde{S}^2(z_4)\Psi^0(z)\Psi^{\bar{0}}(w)\rangle.
\end{aligned} \tag{B.1.2}$$

Note that we have written the functions in terms of gamma matrices rather than the standard Weyl-notation matrices $\sigma^{\mu\nu}$ since the amplitude with PCOs is summed over momenta - and it is then possible to cancel many terms via the on-shell conditions. We sacrifice obvious antisymmetry on the inserted operators, but it is straightforward to show that it is still antisymmetric on exchange of $\psi^{\mu_3}(z)$ and $\psi^{\mu_4}(w)$. To demonstrate that the above is a complete set, we require the standard Fierz identities, but we also require corresponding identities among the products of Jacobi Theta functions. In particular, we require

$$\begin{aligned}
&\theta_1(z_1-z_3)\theta_1(z_2-z_4)\theta_1(z-z_1)\theta_1(z-z_3)\theta_1(w-z_2)\theta_1(w-z_4) \\
&\qquad \theta_\nu(\frac{z_2+z_4-z_1-z_3}{2})\theta_\nu(\frac{z_1+z_3-z_2-z_4}{2}+z-w) \\
&\quad -\theta_1(z_1-z_2)\theta_1(z_1-z_3)\theta_1(z_2-z_3)\theta_1(z-z_4)\theta_1(w-z_4)\theta_1(z-w) \\
&\qquad \theta_\nu(\frac{z_4-z_1-z_3-z_2}{2}+z)\theta_\nu(\frac{z_3+z_2-z_4+z_1}{2}-w) \\
&\quad -\theta_1(z_1-z_4)\theta_1(z_2-z_3)\theta_1(z-z_2)\theta_1(z-z_3)\theta_1(w-z_1)\theta_1(w-z_4) \\
&\qquad \theta_\nu(\frac{z_1+z_4-z_2-z_3}{2})\theta_\nu(\frac{z_2+z_3-z_1-z_4}{2}+z-w) \\
&\quad -\theta_1(z_1-z_2)\theta_1(z_3-z_4)\theta_1(z-z_1)\theta_1(z-z_2)\theta_1(w-z_3)\theta_1(w-z_4) \\
&\qquad \theta_\nu(\frac{z_3+z_4-z_1-z_2}{2})\theta_\nu(\frac{z_1+z_2-z_3-z_4}{2}+z-w) \\
&\qquad\qquad = 0.
\end{aligned} \tag{B.1.3}$$

To prove this, it is easy to check that the periodicities of the terms are the same, and when one of the functions is zero, the remaining three sum to zero. Thus we can write any one of the functions as a constant multiple of the other three; since we can do this for any function the constant must be -1, so the identity holds in general.

The reader may then substitute $\theta_{\alpha_3}, \theta_{\alpha_4}$ for $(u_3)_{\alpha_3}, (u_4)_{\alpha_4}$. This results in many simplifications, because we need only keep structures involving $\theta_1\theta_2 = -\theta_2\theta_1$. However, to obtain the amplitude without PCO insertions, we can use the OPE of the Ψ fields in the above. Alternatively, we write the amplitude as

$$\langle\prod_{i=1}^{4}(u_i)_{\alpha_i}\tilde{S}^{\alpha_i}(z_i)\rangle = A_1(u_1u_3)(u_2u_4) + A_2(u_1u_4)(u_2u_3) \tag{B.1.4}$$

where

$$\begin{aligned} A_1 &= \langle\tilde{S}^1(z_1)\tilde{S}^2(z_2)\tilde{S}^2(z_3)\tilde{S}^1(z_4)\rangle \\ A_2 &= \langle\tilde{S}^1(z_1)\tilde{S}^2(z_2)\tilde{S}^1(z_3)\tilde{S}^2(z_4)\rangle. \end{aligned} \tag{B.1.5}$$

Here we have used

$$\begin{aligned} (u_1u_3)(u_2u_4) - (u_1u_4)(u_2u_3) &= (u_1u_2)(u_3u_4) \\ A_1 - A_2 &= \langle\tilde{S}^1(z_1)\tilde{S}^1(z_2)\tilde{S}^2(z_3)\tilde{S}^2(z_4)\rangle. \end{aligned} \tag{B.1.6}$$

Substitution of $\theta_{\alpha_3}, \theta_{\alpha_4}$ for $(u_3)_{\alpha_3}, (u_4)_{\alpha_4}$ and disregarding terms proportional to θ_1^2 and θ_2^2 results in the expression given in equation 4.3.2.

B.2 Spin-Structure Summation

It is possible to compute the spin-structure summation for the expression 4.3.2, since the non-compact spin structure is partially cancelled by the spin-dependent part of the superconformal ghost amplitude. The result is

$$\begin{aligned} &\langle V^0_{\phi_{ab}}V^{1/2}_{\psi_{bc}}V^{1/2}_{\psi_{ca}}V^{-1/2}_{\theta}V^{-1/2}_{\theta}\rangle_{a,E2} = \phi_{(ab)}\psi_{(bc)\alpha}C^{\alpha\beta}\psi_{(ca)\beta}\;\theta_1\theta_2\int \mathrm{d}t \\ &\prod_{i=1}^{3}\int_0^{it}\mathrm{d}z_i\; f(z_2-z_3,t)\lim_{x_1\to z_1,x_2\to z_2,x_3\to z_3}(x_1-z_1)(x_2-z_2)^{1/2}(x_3-z_3)^{1/2}\theta_1(z_2-z_3)^{1/4} \\ &\quad\langle\prod_{i=1}^{3}e^{ik_i\cdot X(z_i)}\rangle\sum_{\{y_1,y_2,y_3\}=P(x_1,x_2,x_3)}\prod_{\kappa=1}^{3}\sqrt{\frac{2}{\alpha'}}\langle\partial\bar{X}^\kappa(y_\kappa)\sigma_{\phi^\kappa_{ab}}(z_1)\sigma_{\phi^\kappa_{bc}}(z_2)\sigma_{\phi^\kappa_{ca}}(z_3)\rangle \\ &\qquad\langle e^{iH^\kappa(y_\kappa)}e^{i(\phi^\kappa_{(ab)}-1)H^\kappa(z_1)}e^{i(\phi^\kappa_{(bc)}-1/2)H^\kappa(z_2)}e^{i(\phi^\kappa_{(ca)}-1/2)H^\kappa(z_3)}\rangle_{(\nu\text{ indep.})} \\ &\qquad\qquad\theta_1((\phi^\kappa_{(ab)}-1)z_1+(\phi^\kappa_{(bc)}-1)z_2+(\phi^\kappa_{(ca)}-1)z_3+y_\kappa) \end{aligned} \tag{B.2.1}$$

where

$$\langle \prod_{i=1}^{N} e^{ia_i H(z_i)} \rangle_{(\nu \text{ indep})} \equiv \prod_{i<j} \theta_1(z_i - z_j)^{a_i a_j} \tag{B.2.2}$$

and

$$f(\delta, t) \equiv 2 \exp[\frac{\pi t}{2}] \prod_{j=1}^{2} \int_0^{it} \mathrm{d}w'_j \frac{\theta_1(w'_1 - w'_2 + \delta)\theta_4^2(w'_1 + w'_2)}{\theta_1(w'_1 + w'_2)\theta_2(w'_1)\theta_2(w'_2)\theta_2(w'_1 - \delta)\theta_2(w'_2 + \delta)} \tag{B.2.3}$$

encodes all of the dependence on the position of the V_θ insertions. It is an odd function of δ, and hence the amplitude does not have a pole at $z_2 = z_3$, as required.

Appendix C

Field Theory Limits of String Diagrams - a Review

First we divide the Schwinger integrals as described above so that

$$\Pi_1(k,-k) = \Pi_{1_{IR}} + \Pi_{1_{UV}}, \tag{C.1.1}$$

where UV and IR indicate $t \in [0,1]$ and $t \in [1,\infty]$ respectively. Considering the IR contribution of the planar diagram note that, if we reduce $\theta \longrightarrow 0$, then the field-theory limit should be the same for planar and non-planar diagrams; equivalently, they should be the same up to $O(\alpha')$ corrections. This is not immediately obvious from the Green's functions, but we must bear in mind that the planar diagrams have spurious poles on the worldsheet, and the string amplitude is strictly only defined after analytically continuing the momentum [82, 85]. The field theory limit is obtained by taking $t \gg 1$ and excising the regions around the poles - i.e. the region $|x-x'| < 1$ and $t - |x-x'| < 1$ - and then keeping terms of lowest order in w:

$$I_0^{P,NP} = -\frac{(x-x')^2}{t} + \pi|x-x'| \pm \Delta + O(e^{-2\pi t}), \tag{C.1.2}$$

where Δ is of order w, and the $-(+)$ preceding it applies to planar (non-planar) diagrams. We retain this term due to the presence of the tachyon, as in [78, 82]; it is given by

$$\Delta = e^{-2\pi x} + e^{2\pi(x-t)} \tag{C.1.3}$$

so that $\dot{\Delta}^2 = -4\pi^2 + O(w)$, but for the superstring we shall find that it is irrelevant. Inserting the above into (5.3.10) and extracting the contribution from the first level in the loop, we find

$$\Pi^P_{1_{IR}}(k^2) = -4\alpha'^2 g^2_{D_p} \int_1^\infty \mathrm{d}t\, (8\pi^2\alpha' t)^{-\frac{(p+1)}{2}} e^{2\pi t}(1 + (24-d)\, e^{-2\pi t} + \ldots)\times$$
$$\int_0^t \mathrm{d}x e^{-2\alpha' k^2 \pi (x - \frac{x^2}{t})} \left[\frac{-2\pi x}{t} + \pi + \dot{\Delta} + ...\right]^2$$
$$= \frac{-g^2_{D_p}}{(4\pi)^{\frac{(p+1)}{2}}} \int_{2\pi\alpha'}^\infty \mathrm{d}T\; T^{-\frac{(p-1)}{2}} \int_0^1 \mathrm{d}y\; e^{-Tk^2(y-y^2)} \left[(24-d)(1-2y)^2 - 8 + ...\right]. \tag{C.1.4}$$

This result looks just like the field theoretical Schwinger integral as it should (note the change to the parameters $T = 2\pi\alpha' t$ and $y = x/t$). We have not explicitly written the tachyonic contribution or contributions coming from states at higher excitation level: the tachyon because it is unphysical, and the higher states because their nonplanar counterparts in the IR ($p \to 0$) are all finite. For the moment we need only note that a contribution at level n yields a Schwinger integral of the form

$$\int_{2\pi\alpha'}^\infty dT\, T^{-\frac{(p-1)}{2}} \int_0^1 dy\, (1-2y)^2 e^{-T(k^2(y-y^2)+(n-1)\alpha'^{-1})}. \tag{C.1.5}$$

To obtain the field theory limit, we perform the integrals above and then take the $\alpha' \to 0$ limit; we can do this using the exponential integral. For example, when $p = 3$ we have the standard field theory behaviour, with

$$\Pi^P_{1_{IR}} = \frac{d}{3}\frac{g^2_{D_3}}{(4\pi)^2} \ln k^2 + O(1). \tag{C.1.6}$$

For $d = 22$ we obtain the beta function of ref. [78], but for the case $d = 0$, we find that the leading logarithm cancels, and we have the finite result

$$\Pi^P_{1_{IR}} = \frac{16}{3}\frac{g^2_{D_3}}{(4\pi)^2} + O(k^2). \tag{C.1.7}$$

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