A class of alternate strip-based domain decomposition methods for elliptic partial differential Equations

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A Class of Alternate Strip-Based Domain Decomposition Methods for Elliptic Partial Differential Equations

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A thesis presented for the degree of
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To my parents

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Abstract

The domain decomposition strategies proposed in this thesis are efficient preconditioning techniques with good parallelism properties for the discrete systems which arise from the finite element approximation of symmetric elliptic boundary value problems in two and three-dimensional Euclidean spaces.

For two-dimensional problems, two new domain decomposition preconditioners are introduced, such that the condition number of the preconditioned system is bounded independently of the size of the subdomains and the finite element mesh size. First, the alternate strip-based (ASB_2) preconditioner is based on the partitioning of the domain into a finite number of nonoverlapping strips without interior vertices. This preconditioner is obtained from direct solvers inside the strips and a direct fast Poisson solver on the edges between strips, and contains two stages. At each stage the strips change such that the edges between strips at one stage are perpendicular on the edges between strips at the other stage. Next, the alternate strip-based substructuring ($ASBS_2$) preconditioner is a Schur complement solver for the case of a decomposition with multiple nonoverlapping subdomains and interior vertices. The subdomains are assembled into nonoverlapping strips such that the vertices of the strips are on the boundary of the given domain, the edges between strips align with the edges of the subdomains and their union contains all of the interior vertices of the initial decomposition. This preconditioner is produced from direct fast Poisson solvers on the edges between strips and the edges between subdomains inside strips, and also contains two stages such that the edges between strips at one stage are perpendicular on the edges between strips at the other stage. The extension to three-dimensional problems is via solvers on slices of the domain.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, the Department of Mathematical Sciences, the University of Durham, England, between January 2001 and December 2004. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapter 1 contains an historical overview of domain decomposition and multigrid methods, and an outline of this thesis. Chapter 2 is a presentation of the symmetric elliptic boundary value model problem and a review of the essential principles of the domain decomposition and multigrid techniques. No claim of originality is made for this chapter. In Chapters 3 to 5, a new class of domain decomposition preconditioning techniques for the model problem in two and three-dimensional Euclidean space are described and analysed, and numerical examples for these techniques are presented. These chapters are all original work. The results of Chapter 4 have been presented at the 15th International Conference on Domain Decomposition Methods (Berlin, July 2003) and published in the subsequent proceedings [59]. In Chapter 6, conclusions and further remarks are addressed, and possible extensions of the new methods to time-dependent problems for parabolic operators are sketched.

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I have always regarded my time as a research student as a unique opportunity to explore new ideas, as well as to gain more insight into advanced mathematical concepts. The detailed and constructive comments on this thesis, of the external examiners Professor Ivan Graham, from the University of Bath, and Dr. Choi-Hong Lai, from the University of Greenwich, have been invaluable to me.

I am also conscious of a great debt of gratitude to the past, to my teachers back in Romania, those who gave me confidence in my personal resources and helped me to develop as a mathematician.

On a more personal note, very special thanks to my devoted family, my parents and sister, for their love and faith in me, for always sharing my hopes and dreams. My final thank you goes to my fiance, Simon E. Connell, for being a permanent source of strength, courage and optimism.

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Chapter 1

Introduction

Computational mathematics facilitates an approximation to the solution of a problem using limited computer resources. The discretisation of partial differential equations (PDE's) in many computational continuum mechanics problems, for example in fluid dynamics and elasticity, often leads to large linear systems of algebraic equations, sometimes several tens of thousands of unknowns for two-dimensional problems, and more than one million unknowns in three space dimensions. The direct solution of systems of this size is prohibitively expensive, both with respect to the amount of storage and to the computational work. Making a good choice of an iterative method for a particular problem is often difficult, since each method has its own advantages and liabilities. Suitable approaches are often based on physical insight in the underlying process or on insight into the structure of the mathematical model. Usually a method is selected according to its numerical convergence qualities and its ease of programming.

1.1 Significance of Domain Decomposition and Multigrid Methods

Classical iterative methods like the Jacobi, Gauss-Seidel and Successive Over Relaxation (SOR) have been used from the beginning of the numerical treatment of PDE's. Yet these methods share the disadvantage that the amount of work does not remain proportional to the number of unknowns, and the computer time needed to solve a problem grows more rapidly than the size of the problem (Varga (1962) [81], Young (1971) [89], Freund (1991) *et al.* [41], Nenvalina (1993) [62], Axelsson (1994) [1]).

In order to apply them efficiently, iterative methods need to be preconditioned. Domain decomposition[1] (DD) and multilevel methods are the basic techniques for parallelising PDE solvers and for constructing new parallel solvers, especially preconditioners.

The emergence of parallel computers and their potential for the numerical solution of difficult-to-solve problems, has led to a vast amount of research in DD methods, which provide a natural possibility to combine classical and well-tested single-processor algorithms with new parallel ones. In preconditioning, DD algorithms use a preconditioned conjugate gradient[2] (PCG) approach. The preconditioners are derived from exact or approximate solvers for large scale linear or nonlinear systems of equations arising from the discretisation of PDE's restricted to subdomains into which the given domain is subdivided (or from which it is originally assembled) to obtain fast solutions. Each subdomain can be associated with a set of nodes and a finite element subspace of a global finite element space. The DD techniques can also be used for describing complex geometries, for coupling physically different fields, and for coupling different discretisation techniques.

For most industrial and scientific problems, the most efficient discretisation is not explicit, it requires repeated solution of large systems of algebraic (usually non-linear and of mixed mathematical type) equations, and may require multigrid[3] (MG) algorithms to ensure rapid convergence of both short and long range solution components.

The MG approach combines two complementary ideas that lead to rapid convergence: the smoothing of the high frequency components of the error, and the coarse-grid correction of the low energy components. There are virtually unlimited choices of the interpolation and the coarse grid correction that may be used. Complying with the golden rule that "*the amount of computational work should be*

[1]The earliest known DD method was introduced by Hermann Amandus Schwarz (1870) [70] to prove existence and uniqueness of solutions to Laplace's equation on irregular domains. Sobolev (1936) [76] showed that the method converges for the equations of linear elasticity.

[2]The conjugate gradient (CG) method was developed by Hestenes and Stiefel (1952) [48], and first combined with a simple method for preconditioning by Reid (1971) [67].

[3]Probably the first working MG method was developed and analysed by Fedorenko (1964) [40] for the Laplace equation on the unit square. Bachvalov (1966) [2] considered the theoretical case of variable smooth coefficients. The beginning of a rapid development was marked by Brandt (1972) [13], (1977) [14], (1984) [15] who outlined the main principles and the practical utility of MG methods and Hackbusch (1980) [45], (1981) [46] who laid firm mathematical foundations and provided reliable methods.

proportional to the amount of real physical changes in the computed system" (Brandt (1984) [15], p. 1), MG methods offer the possibility of computational complexity and storage that is linearly proportional to the number of grid-points. For this reason MG methods are not usually considered as preconditioners for acceleration techniques, but rather as powerful iterative methods in themselves. However, when combined with DD, the amount of computational work needed to solve a problem to a particular accuracy may reduce considerably. DD's can also be regarded as MG methods employing potentially a more robust, localised smoother. Both DD and MG methods have a great potential for problem specific tuning and optimisation.

Recently a powerful abstract framework has been developed for the analysis of DD and MG algorithms. Texts on the theoretical foundations for MG techniques are the tutorial by Briggs (1987) [18] or Briggs *et al.* (2000) [19], the MG guide by Brandt (1984) [15], and the monographs by Hackbusch (1985) [47] and Wesseling (1992) [82]. Convergence and complexity issues are presented by Yserentant (1993) [90] and more extensively by Bramble (1993) [8]. An overview of the essential principles of DD is offered by Chan and Mathew (1994) [25], while a discussion of DD and MG algorithms, their implementation and analysis is presented by Smith *et al.* (1996) [75]. In Quarteroni and Valli (1999) [66], the fundamental mathematical concepts behind DD methods for a wide range of boundary value problems is treated. An introduction to the basic concepts of parallel computers, parallel programming, and the run-time behaviour of parallel algorithms to understand, develop, and implement parallel PDE solvers is Douglas *et al.* (2003) [30]. In Toselli and Widlund (2004) [80], some of the most successful and popular DD methods for finite and spectral element approximations of PDE's are presented. For a more extensive survey of recent advances in DD algorithmic techniques, implementation tools and applications, we refer to the proceedings of the annual DD meetings, which can be accessed via the web page of the Domain Decomposition Organisation (at `http://www.ddm.org`).

1.2 Scope of this Thesis

In DD, current research concentrates both on the improvement of existing algorithms as well as on the development of new ones. The goal is to design algorithms with a convergence rate and efficiency independent of the number of unknowns, coefficients, and geometry. In this thesis a new class of DD methods for symmetric elliptic boundary value problems in two and three-dimensional Euclidean spaces is proposed.

The remainder of this thesis is organised as follows. Chapter 2 is a brief review of the essential principles of the DD and MG methods in the case of second order symmetric elliptic boundary value problems. In particular, we introduce the basic mathematical concepts and develop the motivation behind the new DD approach.

For the two-dimensional case, in Chapters 3 and 4, two new domain decomposition preconditioners are introduced. These preconditioners are optimal with respect to the partitioning parameters, that is the condition number of the preconditioned system can be bounded independently of the size of the subdomains and the finite element mesh size.

Chapter 3 is dedicated to the formulation and analysis of the alternate strip-based (throughout this thesis, referred to as ASB_2) preconditioning technique. This technique is based on a partitioning of the given domain into a finite number of nonoverlapping strips without interior cross-points (i.e. all the vertices of the strips lie on the boundary of the domain). The strips may have high aspect ratio, that is they may be long and narrow. The new preconditioner is obtained from direct solvers inside the strips and a direct Poisson solver on the interfaces between strips. Perfect scalability, that is performance insensitive to the number of strips, is achieved in two stages. At each stage the strips change such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage. The applicability of this new DD method is restricted mainly to problems with constant coefficients.

When the coefficients of the given problem are varying, preconditioners with smaller subdomains better reflect the behaviour of the coefficients and give rise to more rapidly convergent algorithms. In Chapter 4, we present the alternate strip-based substructuring ($ASBS_2$) preconditioning technique. This technique applies to the case of a decomposition with multiple nonoverlapping subdomains and interior cross-points (i.e. there exist vertices of the subdomains which are situated in the interior of the domain). In general, cross-points are more difficult to handle, due to the fact that they represent strong coupling between subdomains. The main task of the $ASBS_2$ method is to determine the interface data between all subdomains, by solving iteratively the Schur complement problem obtained after the variables corresponding to the interior of the subdomains are block Gauss eliminated. In view of the ASB_2 solver (Chapter 3), for this Schur complement preconditioner, the separate treatment of the cross-points is avoided by assembling the subdomains into nonoverlapping strips such that: the ends of the strips are on the boundary of the given domain, the interfaces between strips align with the edges of the subdomains and their union contains all of the interior vertices of the initial decomposition. Then, the global interface between all subdomains is partitioned as a union of edges between strips and edges between subdomains that belong to the same strip (edges do not include their end-points). For the subproblems corresponding to the various edges, a direct fast Poisson solver is used. Scalability is again achieved in two stages. At each stage the strips change such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage.

The three-dimensional alternate strip-based substructuring ($ASBS_3$) preconditioning strategies, presented in Chapter 5, are direct extensions of the $ASBS_2$ techniques to three-dimensional problems. The $ASBS_3$ preconditioners are based on a decomposition of the given domain into a finite number of disjoint subdomains assembled into nonoverlapping slices such that: the edges of the slices lie on the boundary of the given domain, and the union of the faces between slices contains all of the interior vertices. For the subproblems corresponding to the faces between slices, a direct fast Poisson solver is used. Both scalability and efficiency are achieved in two stages. At each stage the slices change such that the faces between slices at one stage are orthogonal to the faces between slices at the other stage. The two stages of the ASB_2, $ASBS_2$, and $ASBS_3$ preconditioners allow the use of a two-grid V-cycle.

Each of the Chapters 3 to 5 contains three main parts. In the first part, the new algorithms are described and explained. Matrix notation is also used to help with the understanding of the implementation issues. A mathematical framework for the abstract analysis of the new DD techniques is developed in the second part. The third part contains numerical examples which confirm the theoretical estimates and illustrate the efficiency of these techniques. All computations were carried out in Matlab. Once it is understood why and how they work, these new DD methods can be extended to more general problems, defined on more complex geometries. In the final chapter, overall conclusions and final remarks are addressed. In addition, some possible extensions of these methods to time-dependent problems for parabolic operators are also sketched. This thesis ends with an appendix and references.

Chapter 2

Prerequisites and Notation

The aim of this chapter is to introduce the basic mathematical concepts and develop the motivation behind the new DD techniques proposed in this thesis. First, we formulate the elliptic PDE to be studied, then we focus on presenting the salient features od the DD and MG methods for the numerical solution of the given problem. For more comprehensive details and proofs, we recommend the relevant references in each section. The chapter ends with a summary.

In what follows, we shall denote continuous functions by using **boldface**, for example, $\mathbf{u}$, $\mathbf{f}$, and discrete functions (i.e. vectors) by using *italics*, such as u, f. Further notation is explained as it occurs.

2.1 Problem Formulation

In this section, we present the model problem to be studied, and its finite dimensional formulation in the framework of the piecewise polynomial finite elements. We focus on scalar, self-adjoint, second-order elliptic problems, including those with large variations in the coefficients. The classic results presented here can be found in various finite element monographs and course texts (see e.g. Lions and Magenes (1972) [52], Strang and Fix (1973) [77], Mitchell and Wait (1977) [60], Ciarlet (1978) [28], Yosida (1980) [88], Johnson (1987) [50], Quarteroni and Valli (1994) [65], Eriksson *et al.* (1996) [39], Renardy and Rogers (1996) [68], and Brenner and Scott (2002) [17]).

Differential Form. We consider the following symmetric elliptic boundary value problem:

$$\begin{cases} -\nabla \cdot (\alpha(x)\nabla \mathbf{u}(x)) = \mathbf{f}(x) \text{ in } \Omega \\ \mathbf{u}(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1.1}$$

where $\Omega \subset \mathbb{R}^D$ is a D-dimensional domain with Lipschitz boundary $\partial\Omega$. Without loss of generality we assume that Ω is a polygon, for $D = 2$, or a polyhedron, for $D = 3$, of unit diameter. Let $x = (x_1, \cdots, x_D)$ denote a generic element in Ω. The coefficient $\alpha(x) \geq \alpha_0$, for some positive constant α_0, will be taken as $\alpha(x) \equiv 1$ (for the Poisson equation) or piecewise constant in Ω; $\mathbf{f}(x)$ is a given datum.

Variational Form. First, we recall some function spaces, which are important for our subsequent analysis. $L^\infty(\Omega)$ is the Lebesgue space of real-valued, uniformly bounded functions on Ω. This is a Banach space for the ∞-norm:

$$\|\mathbf{u}\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |\mathbf{u}(x)|.$$

$L^2(\Omega)$ is the Lebesgue space of real-valued, square integrable functions on Ω. This is a Hilbert space for the scalar (inner) product and the associated 2-norm (Euclidean norm):

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}\mathbf{v}dx, \qquad \|\mathbf{u}\|^2_{L^2(\Omega)} = (\mathbf{u}, \mathbf{u}).$$

The following are Sobolev spaces:

$$H^1(\Omega) = \left\{ \mathbf{u} \in L^2(\Omega) | \ \frac{\partial \mathbf{u}}{\partial x_j} \in L^2(\Omega), \ j = 1, \cdots, D \right\},$$

with the associated seminorm and norm:

$$|\mathbf{u}|_{H^1(\Omega)} = \|\nabla \mathbf{u}\|_{L^2(\Omega)} = \left(\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{u} dx \right)^{1/2},$$

$$\|\mathbf{u}\|_{H^1(\Omega)} = \left(\|\mathbf{u}\|^2_{L^2(\Omega)} + |\mathbf{u}|^2_{H^1(\Omega)} \right)^{1/2},$$

and the subspace:

$$H^1_0(\Omega) = \left\{ \mathbf{u} \in H^1(\Omega) | \ \mathbf{u} = 0 \text{ on } \partial\Omega \right\}.$$

The above definitions imply:

$$\|\mathbf{u}\|_{L^2(\Omega)} \le C\|\mathbf{u}\|_{L^\infty(\Omega)} \qquad \text{and} \qquad \|\mathbf{u}\|_{L^2(\Omega)} \le C\|\mathbf{u}\|_{H^1(\Omega)}, \tag{2.1.2}$$

for some positive constants C. Thus the following ordering relations hold:

$$L^\infty(\Omega) \subset L^2(\Omega) \qquad \text{and} \qquad H^1(\Omega) \subset L^2(\Omega).$$

A function $\mathbf{u} \in H_0^1(\Omega)$ is said to be a *weak solution* of the differential equation (2.1.1), if

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}),\ \forall \mathbf{v} \in H_0^1(\Omega), \tag{2.1.3}$$

where

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \alpha \nabla\mathbf{u}\cdot\nabla\mathbf{v}dx \qquad \text{and} \qquad (\mathbf{f},\mathbf{v}) = \int_\Omega \mathbf{f}\mathbf{v}dx.$$

We assume that the bilinear form $a(\cdot,\cdot)$ is continuous (bounded), in the sense that there exists a positive constant c_0, such that:

$$|a(\mathbf{u},\mathbf{v})| \le c_0\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)},\ \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega). \tag{2.1.4}$$

We also require $a(\cdot,\cdot)$ to be coercive, i.e. there exists a positive constant c_1, such that:

$$a(\mathbf{u},\mathbf{u}) \ge c_1\|\mathbf{u}\|^2_{H^1(\Omega)},\ \forall \mathbf{u} \in H_0^1(\Omega). \tag{2.1.5}$$

Finally, we assume that there exists a positive constant c_2, such that:

$$|(\mathbf{f},\mathbf{v})| \le c_2\|\mathbf{v}\|_{H^1(\Omega)},\ \forall \mathbf{v} \in H_0^1(\Omega). \tag{2.1.6}$$

Note that $a(\cdot,\cdot)$ is an inner product in the space $H_0^1(\Omega)$. Thus we can define the corresponding energy norm $\|\mathbf{u}\|_a^2 = a(\mathbf{u},\mathbf{u})$, which through the coerciveness and continuity of the bilinear form $a(\cdot,\cdot)$ is equivalent to the norm in the Sobolev space $H^1(\Omega)$.

Theorem 2.1.1 (Lax-Milgram) When $a(\cdot,\cdot)$ is a continuous, coercive bilinear form on $H_0^1(\Omega)$, and (2.1.6) holds, there exists a unique $\mathbf{u} \in H_0^1(\Omega)$ such that (2.1.3) is satisfied. Furthermore, the following stability estimate holds: $\|\mathbf{u}\|_{H^1(\Omega)} \leq c_2/c_1$.

Proof: See e.g. Brenner and Scott (2002) [17], Section 2.7. □

Finite Element Approximation. Let $\Sigma^h = \{\sigma_h\}$ be a quasi uniform partitioning of the given domain Ω, into a finite number of polyhedra, such that (see e.g. Quarteroni and Valli (1994) [65], pp. 73-74; (1999) [66], pp. 41-43):

- $\{\sigma_h\}$ are nonoverlapping D-simplexes of size $h \in (0,1]$ (i.e. there exist positive constants C and c independent of h such that each simplex σ_h contains a ball of diameter ch and it is contained in a ball of diameter Ch). We define $S_h(\Omega)$ to be the piecewise linear finite element subspace of $H^1(\Omega)$ associated with Σ^h,

$$S_h(\Omega) = \{\mathbf{u} \in H^1(\Omega)|\ \mathbf{u}|_{\sigma_h} \in \mathcal{P}_1(\sigma_h), \forall \sigma_h \in \Sigma^h\},$$

where $\mathcal{P}_1(\sigma_h)$ is the set of linear polynomials (i.e. polynomials of degree less than or equal to 1 globally with respect to all space variables) defined in σ_h.

Similarly, let $S_h^0(\Omega)$ be the piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with Σ^h,

$$S_h^0(\Omega) = \{\mathbf{u} \in H_0^1(\Omega)|\ \mathbf{u}|_{\sigma_h} \in \mathcal{P}_1(\sigma_h), \forall \sigma_h \in \Sigma^h\},$$

or

- $\{\sigma_h\}$ are nonoverlapping D-cubes of size $h \in (0,1]$ (i.e. there exist positive constants C and c independent of h such that, for every element σ_h, each simplex formed by its vertices contains a ball of diameter ch and it is contained in a ball of diameter Ch). In this case, we define $S_h(\Omega)$ to be the piecewise bilinear (trilinear) finite element subspace of $H^1(\Omega)$ associated with Σ^h,

$$S_h(\Omega) = \{\mathbf{u} \in H^1(\Omega)|\ \mathbf{u}|_{\sigma_h} \in \mathcal{Q}_1(\sigma_h), \forall \sigma_h \in \Sigma^h\},$$

where $\mathcal{Q}_1$ is the space of bilinear (trilinear) polynomials (i.e. polynomials which are linear with respect to each variable).

Similarly, let $S_h^0(\Omega)$ be the piecewise bilinear (trilinear) finite element subspace of $H_0^1(\Omega)$ associated with Σ^h,

$$S_h^0(\Omega) = \{\mathbf{u} \in H_0^1(\Omega) |\ \mathbf{u}|_{\sigma_h} \in \mathcal{Q}_1(\sigma_h), \forall \sigma_h \in \Sigma^h\}.$$

The finite element (Galerkin) approximation for problem (2.1.1) is to find $\mathbf{u} \in S_h^0(\Omega)$ such that:

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}),\ \forall \mathbf{v} \in S_h^0(\Omega). \tag{2.1.7}$$

We observe that the converse inequalities to (2.1.2) are not true in general. However, if the function $\mathbf{u}$ belongs to the finite dimensional subspace $S_h(\Omega)$ of $H^1(\Omega)$, then the following inverse property holds.

Lemma 2.1.2 (the inverse property) For all $\mathbf{u} \in S_h(\Omega)$,

$$|\mathbf{u}|_{H^1(\Omega)} \leq C\frac{1}{h}||\mathbf{u}||_{L^2(\Omega)},$$

where the constant C depends on the domain Ω.

Proof: For proofs of this result in a more general case we refer to Ciarlet (1978) [28], Theorem 3.2.6. □

Algebraic Form of the Discretised Equation. To obtain the unknowns of the finite dimensional problem (2.1.7), given by the point values $\{u_i\}_{i=1}^m$ of $\mathbf{u}$ at the mesh points (m is the number of degrees of freedom for the grid), the test functions $\mathbf{v}$ are chosen equal to a set of piecewise linear (bilinear) basis functions, $\{\phi_i\}_{i=1}^m$, for $S_h^0(\Omega)$. Substitution of the discrete solution:

$$\mathbf{u}(x) = \sum_{i=1}^{m} u_i \phi_i(x)$$

into equation (2.1.7) generates the equivalent algebraic problem, in the form of the discrete linear system:

$$Au = f. \tag{2.1.8}$$

In (2.1.8), $u = \{u_i\}_{i=1}^m$ is the m vector of the point values, $A = \{a_{ij}\}_{i,j=1}^m$ is the $m \times m$ finite element stiffness matrix with entries:

$$a_{ij} = a(\phi_j, \phi_i),$$

and $f = \{f_i\}_{i=1}^m$ is the m-dimensional load vector with entries:

$$f_i = (\mathbf{f}, \phi_i).$$

The Conjugate Gradient Method. The stiffness matrix A in (2.1.8) is symmetric, i.e.

$$(Au, v) = (Av, u), \ \forall u, v \in \mathbb{R}^m,$$

and positive definite, i.e.

$$(Au, u) > 0, \ \forall u \in \mathbb{R}^m, \ u \neq 0,$$

where $(\cdot, \cdot)$ denotes the Euclidean scalar product. In particular all its eigenvalues are positive. Therefore the conjugate gradient (CG) method (see Luenberger (1973) [55], Golub *et al.* (1989) [43], Freund *et al.* (1992) [41], Saad (1996) [69]) can be applied. For the CG method, the decrease in the energy norm of the error after r steps can be bounded by:

$$2\left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1}\right)^r,$$

where

$$\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$$

is the condition number of the matrix A, with $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the maximum and minimum eigenvalue of the matrix A respectively. Therefore, for A with a low condition number or with clustered eigenvalues, the convergence of the CG method is very rapid. On the other hand, it can be proved that $\kappa(A) = \mathcal{O}(h^{-2})$ (see e.g. Johnson (1987) [50], Section 7.7; Quarteroni and Valli (1994) [65], Section 6.3.2; Smith *et al.* (1996) [75], Section 2.7), thus, for very small h, a direct application of the CG method will not be an efficient approach, since although sparse, the matrix A is no longer well conditioned. In the preconditioned conjugate gradient (PCG) method (Axelsson (1994) [1]) an easily invertible, symmetric positive definite (SPD) matrix B, which is spectrally close to A, is chosen, such that the relative condition number $\kappa(B^{-1}A)$ is much smaller than $\kappa(A)$. Let $b(\cdot,\cdot)$ be the bilinear form, also defined on $S_h^0(\Omega) \times S_h^0(\Omega)$, associated with the preconditioner B. Then the matrix $B^{-1}A$ is symmetric with respect to the inner products $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, i.e.

$$(AB^{-1}Au, v) = (AB^{-1}Av, u), \ \forall u, v \in \mathbb{R}^m,$$

and

$$(BB^{-1}Au, v) = (BB^{-1}Av, u), \ \forall u, v \in \mathbb{R}^m,$$

respectively.

Theorem 2.1.3 Let B be a SPD preconditioner for A, and let $b(\cdot,\cdot)$ be the inner product associated with B, such that:

$$cb(\mathbf{u},\mathbf{u}) \leq a(\mathbf{u},\mathbf{u}) \leq Cb(\mathbf{u},\mathbf{u}), \ \forall \mathbf{u} \in S_h^0(\Omega),$$

for some positive constants C and c. Then $c \leq \lambda_{\min}(B^{-1}A)$ and $\lambda_{\max}(B^{-1}A) \leq C$. Hence $\kappa(B^{-1}A) \leq C/c$.

Proof: See e.g. Dryja and Widlund (1994) [37], Section 2.3, or Brenner and Scott (2002) [17]), Section 7.1. □

The PCG method can be viewed as a CG method applied to the preconditioned system:

$$B^{-1}Au = B^{-1}f, \tag{2.1.9}$$

as follows:

- let u^0 be an initial iterate,

$$r^0 \leftarrow f - Au^0, \text{ the initial residual}$$
$$w^0 \leftarrow B^{-1}r^0, \text{ the initial preconditioned residual}$$
$$v^0 \leftarrow w^0, \text{ the initial search direction}$$

- for $l = 0, 1, \cdots$

  compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Av^l)}$

  update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

  update the residual: $r^{l+1} \leftarrow r^l + p_l A v^l$

  if $r^{l+1} \geq$ tolerance, then

  update the preconditioned residual: $w^{l+1} \leftarrow B^{-1} r^{l+1}$

  compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

  update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

  else end for.

Note that in the CG algorithm given above the matrices A and B^{-1} are not explicitly formed, but only applied to given vectors. In developing preconditioners, the goal is to find a B^{-1} that is inexpensive to apply in terms of floating point operations and interprocessor communication, and that provides fast convergence, hence requires a small number of iterations to achieve an accurate solution (see Golub and Van Loan (1989) [43], Section 10.3).

Parallel Subspace Correction. Many DD algorithms are interpreted and analysed within the framework of parallel subspace correction (PSC) method (Xu (1992) [86]), or the additive Schwarz method (Dryja and Widlund (1994) [37]) by constructing a partitioning of the solution into local subspaces and bounding the energy of each element in the partition. Let V_i, $i = 1, \cdot, N$ be a set of subspaces such that:

$$S_h^0(\Omega) = \sum_{i=1}^{N} V_i,$$

and $b_i(\cdot, \cdot)$ be an inner product defined on $V_i \times V_i$ such that:

$$a(\mathbf{u}, \mathbf{u}) \leq \omega b_i(\mathbf{u}, \mathbf{u}), \ \forall \mathbf{u} \in V_i,$$

for some positive constants ω. We define the operators $\mathbf{P}_i : S_h^0(\Omega) \rightarrow V_i$ such that:

$$b_i(\mathbf{P}_i \mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}), \ \forall \mathbf{v} \in V_i, \ \mathbf{P}_i \mathbf{u} \in V_i. \qquad (2.1.10)$$

Then the PSC operator $\mathbf{P} : S_h^0(\Omega) \to S_h^0(\Omega)$ is defined by:

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{P}_i, \tag{2.1.11}$$

and is analysed by the following result. We denote by P_i the equivalent matrix form of $\mathbf{P}_i$ $(i = 1, \cdots, N)$, and by P the equivalent matrix form of $\mathbf{P}$.

Theorem 2.1.4 Let

$$C = \max_{1 \le j \le N} \sum_{i=1}^{N} \delta_{ij}, \text{ where } \delta_{ij} = \begin{cases} 0, & \text{if } P_i P_j = 0 \\ 1, & \text{otherwise} \end{cases} \quad \text{is the Kronecker symbol.}$$

If for all $\mathbf{u} \in S_h^0(\Omega)$ there exists a representation (not necessarily unique)

$$\mathbf{u} = \sum_{i=1}^{N} \mathbf{u}_i, \qquad \mathbf{u}_i \in V_i, \qquad \text{such that} \qquad c \sum_{i=1}^{N} b_i(\mathbf{u}_i, \mathbf{u}_i) \le a(\mathbf{u}, \mathbf{u}),$$

for some positive constant c, then $\kappa(P) \le \omega C/c$.

Proof: See Dryja and Widlund (1994) [37], Section 2.3, Smith (1992) [74], Section 3, Chan and Mathew (1994) [25], Section 4.1, or Xu and Zou (1998) [87], Section 2.2. □

2.2 The Domain Decomposition Approach

DD methods are powerful preconditioned methods, where the preconditioners are designed from exact or approximate solvers for large scale linear or nonlinear systems of equations arising from the discretisation of PDE's restricted to subdomains into which the given domain is subdivided or from which it originally is assembled, to obtain fast solutions. In this section, we present a selective survey of several DD techniques, classified as either an overlapping or nonoverlapping subdomain procedure, which underlie the new DD methods introduced in this thesis. For extensive convergence and complexity issues we refer, for instance, to Smith *et al.* (1996) [75], Xu and Zou (1998) [87], Quarteroni and Valli (1999) [66], Brenner and Scott (2002) [17], Chapter 7, Toselli and Widlund (2004) [80], and the references therein.

2.2.1 Nonoverlapping Domain Decomposition

A DD without overlapping, of the domain Ω, consists of a number of mutually disjoint open subdomains Ω_i $(i = 1, \cdots, N)$ such that:

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i. \tag{2.2.1}$$

We assume that all subdomains Ω_i $(i = 1, \cdots, N)$ are of size H $(> h)$ in the sense that there exist positive constants C and c independent of H and h, such that Ω_i contains a ball of diameter cH and it is contained in a ball of diameter CH. We also assume that the mesh Σ^h is consistent with (2.2.1) in the sense that the boundary of every individual subdomain, $\partial\Omega_i$, can be written as a union of boundaries of elements in Σ^h (see Figure 2.2.1) and consider the elements (edges, vertices) of a subdomain to be direct projections of the corresponding elements in Ω.

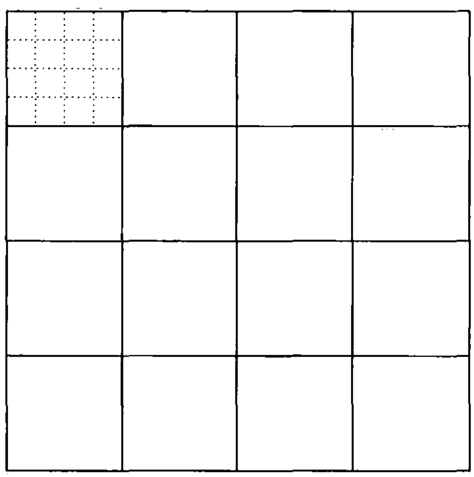

Figure 2.2.1: The initial partitioning of the domain $\Omega \subset \mathbb{R}^2$ into subdomains $\{\Omega_i\}_{i=1}^{N}$, with mesh refinement shown on one subdomain.

When the coefficient $\alpha(x)$ in (2.1.1) is piecewise constant, the subdomains are chosen such that $\alpha(x) \equiv \alpha_i$ in Ω_i. Corresponding to each subdomain Ω_i $(i = 1, \cdots, N)$, we denote:

$$a_i(\mathbf{u}, \mathbf{v}) = \int_{\Omega_i} \alpha_i \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx, \ \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega). \tag{2.2.2}$$

Thus,

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{N} a_i(\mathbf{u}, \mathbf{v}). \tag{2.2.3}$$

Let the global interface between all subdomains be:

$$\Gamma = \bigcup_{i=1}^{N} (\partial\Omega_i \cap \Omega).$$

Therefore:

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_N \cup \Gamma.$$

First, we define the local subspaces $S_h^0(\Omega_i) \subset S_h^0(\Omega)$ $(i = 1, \cdots, N)$, such that:

$$S_h^0(\Omega_i) = \{\mathbf{u} \in S_h^0(\Omega) | \ \mathbf{u}(x) = 0, \ \forall x \in \Omega \setminus \Omega_i\},$$

then, for each subdomain Ω_i, solve for $\mathbf{u}_i^I \in S_h^0(\Omega_i)$, the following local problem:

$$a(\mathbf{u}_i^I, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Omega_i).$$

We observe that the computation of $\mathbf{u}_i^I$ can be carried out independently and in parallel for all Ω_i. Let

$$\mathbf{u}^I \in V^I = S_h^0(\Omega_1) \bigoplus S_h^0(\Omega_2) \bigoplus \cdots \bigoplus S_h^0(\Omega_N)$$

be the solution of the problem:

$$a(\mathbf{u}^I, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in V^I.$$

We denote by $\mathbf{u}^E = \mathbf{u} - \mathbf{u}^I$ the part of the solution $\mathbf{u}$ which lies in the orthogonal complement of V^I in $S_h^0(\Omega)$:

$$V^E = \{\mathbf{u} \in S_h^0(\Omega) | \ a(\mathbf{u}, \mathbf{v}) = 0, \ \forall \mathbf{v} \in V^I\}.$$

Therefore, by the representation (2.2.3) of $a(\cdot, \cdot)$, on each subdomain Ω_i, the function $\mathbf{u}^E \in V^E$ satisfies:

$$a_i(\mathbf{u}^E, \phi) = \alpha_i(\nabla\mathbf{u}^E, \nabla\phi) = 0, \ \forall \phi \in S_h^0(\Omega_i).$$

The function $\mathbf{u}^E$ is called a piecewise discrete harmonic function. From the definition of V^E, we deduce:

$$a(\mathbf{u}^E, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}^I, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Omega),$$

or equivalently, when $\mathbf{v}^E \in V^E$ is similarly defined as $\mathbf{u}^E$,

$$a(\mathbf{u}^E, \mathbf{v}^E) = (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}^I, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Omega).$$

Note that:

$$a(\mathbf{u}, \mathbf{u}) = a(\mathbf{u}^I, \mathbf{u}^I) + a(\mathbf{u}^E, \mathbf{u}^E).$$

Since the value of $\mathbf{u}^E$ in Ω is uniquely determined by its value on Γ, it is convenient to consider only the restrictions on Γ of the functions in $S_h^0(\Omega)$. If we denote the finite element space of these restrictions by $S_h^0(\Gamma)$, then the relation between interface functions $\mathbf{u}$, $\mathbf{v} \in S_h^0(\Gamma)$ and their discrete harmonic extensions $\mathbf{u}^E$, $\mathbf{v}^E \in V^E$ respectively, can be established through the following bilinear form:

$$s(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}^E, \mathbf{v}^E), \ \forall \mathbf{u}, \mathbf{v} \in S_h^0(\Gamma). \tag{2.2.4}$$

When $a(\cdot,\cdot)$ and $s(\cdot,\cdot)$ are symmetric, positive definite, then the the bilinear form $s(\cdot,\cdot)$ has the minimisation property:

$$s(\mathbf{u}, \mathbf{u}) = \min_{\tilde{\mathbf{u}}|_\Gamma = \mathbf{u}} a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}), \ \tilde{\mathbf{u}} \in S_h^0(\Omega), \tag{2.2.5}$$

i.e. the discrete harmonic extension is the energy minimising extension (see e.g. Smith *et al.* (1996) [75], p. 145, or Brenner and Scott (2002) [17]), p. 194). The bilinear forms $a(\mathbf{u}^E, \mathbf{u}^E)$ and $s(\mathbf{u}, \mathbf{u})$ can be analysed using the seminorm $|\mathbf{u}|_{H^{1/2}(\Gamma)}$ defined by:

$$|\mathbf{u}|^2_{H^{1/2}(\Gamma)} = \min_{\tilde{\mathbf{u}}|_\Gamma = \mathbf{u}} |\tilde{\mathbf{u}}|^2_{H^1(\Omega)}, \ \tilde{\mathbf{u}} \in H^1(\Omega). \tag{2.2.6}$$

This is equivalent to the fractional order Sobolev seminorm:

$$|\mathbf{u}|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{(\mathbf{u}(\xi) - \mathbf{u}(\tau))^2}{|\xi - \tau|^2} d\xi d\tau, \tag{2.2.7}$$

where ξ and τ denote arc-length along Γ (see e.g. Xu and Zou (1998) [87], p. 871, or Brenner and Scott (2002) [17], p. 195). The associated space is:

$$H^{1/2}(\Gamma) = \{\mathbf{u} \in L^2(\Gamma) | \ |\mathbf{u}|^2_{H^{1/2}(\Gamma)} < \infty\}$$

is equipped with the norm:

$$||\mathbf{u}||^2_{H^{1/2}(\Gamma)} = ||\mathbf{u}||^2_{L^2(\Gamma)} + |\mathbf{u}|^2_{H^{1/2}(\Gamma)}. \tag{2.2.8}$$

Let Γ_i be an open curve in Γ, then the Hilbert space:

$$H^{1/2}_{oo}(\Gamma_i) = \{\mathbf{u} \in L^2(\Gamma_i) | \ \bar{\mathbf{u}} \in H^{1/2}(\Gamma)\},$$

where $\bar{\mathbf{u}}|_{\Gamma_i} = \mathbf{u}$, $\bar{\mathbf{u}}|_{\Gamma \backslash \Gamma_i} = 0$, is endowed with the norm defined by:

$$||\mathbf{u}||^2_{H^{1/2}_{oo}(\Gamma_i)} = \int_{\Gamma_i}\int_{\Gamma_i} \frac{(\mathbf{u}(\xi) - \mathbf{u}(\tau))^2}{|\xi - \tau|^D} d\xi d\tau + \int_{\Gamma_i} \frac{\mathbf{u}^2(\tau)}{|\tau - \gamma_1|} d\tau + \int_{\Gamma_i} \frac{\mathbf{u}^2(\tau)}{|\tau - \gamma_2|} d\tau, \tag{2.2.9}$$

where γ_1 and γ_2 are the end-points of Γ_i. For extended discussions of these Sobolev spaces and their properties, we refer to Lions and Magenes (1972) [52], and Grisvard (1985) [44].

The next results play important roles in the analysis of many DD algorithms.

Theorem 2.2.1 (trace theorem) For all functions $\mathbf{u} \in H^1(\Omega_i)$, there exists a continuous linear map $\gamma : H^1(\Omega_i) \to L^2(\Omega_i)$ such that $\gamma \mathbf{u} = \mathbf{u}|_{\partial\Omega_i}$. Furthermore,

$$||\gamma \mathbf{u}||_{H^{1/2}(\partial\Omega_i)} \leq C||\mathbf{u}||_{H^1(\Omega_i)},$$

for some positive constants C. Using this result, it can be proved that:

$$||\mathbf{u}||^2_{L^2(\partial\Omega_i)} \leq C||\mathbf{u}||_{L^2(\Omega_i)}||\mathbf{u}||_{H^1(\Omega_i)} \quad \text{and} \quad ||\mathbf{u}||_{L^2(\partial\Omega_i)} \leq C||\mathbf{u}||_{H^1(\Omega_i)},$$

for some positive constants C, where $||\mathbf{u}||_{L^2(\partial\Omega_i)}$ denotes the $L^2(\partial\Omega_i)$-norm of $\mathbf{u}|_{\partial\Omega_i}$.

Proof: See Lions and Magenes (1972) [52], v I, Sections 3 and 4, v II, Section 10, or Quarteroni and Valli (1994) [65], Section 1.3, or Brenner and Scott (2002) [17], Section 1.6. □

Lemma 2.2.2 (Poincaré - Friedrichs inequality) For all functions $\mathbf{u} \in H^1(\Omega_i)$,

$$\|\mathbf{u}\|^2_{L^2(\Omega_i)} \leq C\left(\|\mathbf{u}\|^2_{L^2(\partial\Omega_i)} + |\mathbf{u}|^2_{H^1(\Omega_i)}\right),$$

where C is a positive constant which depends on the domain Ω_i.

Proof: See e.g. Quarteroni and Valli (1994) [65], Section 1.3, or Brenner and Scott (2002) [17], Section 5.3. □

Theorem 2.2.3 If $\mathbf{u} \in S_h(\Omega)$ is discrete harmonic in Ω_i and $|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)}$ represents the $H^{1/2}(\partial\Omega_i)$-seminorm of $\mathbf{u}|_{\partial\Omega_i}$, then there exist positive constants C and c independent of the mesh parameter h and the number of subspaces N, such that:

$$c|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} \leq |\mathbf{u}|^2_{H^1(\Omega_i)} \leq C|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)}.$$

The left hand-side inequality holds for all functions $\mathbf{u} \in H^1(\Omega_i)$.

Proof: See e.g. Chan and Mathew (1994) [25], Section 4.3, and references therein, or Brenner and Scott (2002) [17]), p. 195). □

Let S be the stiffness matrix associated with the bilinear form $s(\cdot,\cdot)$ under the standard nodal basis functions in $S_h^0(\Gamma)$, then S is a Schur complement (SC) associated with the stiffness matrix A. If we write:

$$A = \begin{bmatrix} A_{II} & A_{IE} \\ A_{IE}^T & A_{EE} \end{bmatrix},$$

where A_{II} is the stiffness matrix associated with the nodes in $\Omega \setminus \Gamma$ and A_{EE} is the stiffness matrix associated with the nodes on Γ, then A can be expressed in factored form as:

$$A = \begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1}A_{IE} \\ 0 & I_{EE} \end{bmatrix},$$

where I_{II} and I_{EE} denote identity matrices, and $S = A_{EE} - A_{IE}^T A_{II}^{-1} A_{IE}$ is the SC matrix. Writing the original linear system (2.1.8) as:

$$\begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix} \times \begin{bmatrix} u^I \\ u^E \end{bmatrix} = \begin{bmatrix} f^I \\ f^E \end{bmatrix},$$

then eliminating u^I yields:

$$Su^E = f_S, \tag{2.2.10}$$

where $f_S = f^E - A_{IE}^T A_{II}^{-1} f^I$. We note that (2.2.4) can also be written as the following inner product:

$$(Su^E, v^E) = (u^E)^T S v^E,$$

and (2.2.5) can be expressed in matrix notation as:

$$(u^E)^T S u^E = \min_{u^I} \begin{bmatrix} (u^I)^T & (u^E)^T \end{bmatrix} A \begin{bmatrix} u^I \\ u^E \end{bmatrix}.$$

The condition number for S is much smaller than that for the matrix A, $\kappa(S) \leq \kappa(A)$ (see e.g. Smith *et al.* (1996) [75], Section 4.2). Therefore, we can iterate directly on S, that is apply the CG method to the system (2.2.10), then extend the result harmonically inside all the subdomains. When the SC is computed explicitly, the method is called direct substructuring. However, the explicit calculation of Schur complements is expensive and requires a large amount of memory since they are typically dense, though of much smaller dimension than the original stiffness matrix. Moreover, the condition number $\kappa(S)$ deteriorates with respect to the subdomain size H, the finite element mesh-size h, and the coefficients α_i:

$$\kappa(S) \leq C \frac{\max_i \alpha_i}{\min_i \alpha_i} \frac{H_{\text{max}}}{h H_{\text{min}}^2},$$

where H_{max} and H_{min} denote the maximum and the minimum diameters of the subdomains respectively (see e.g. LeTallec (1994) [78], Xu and Zou (1998) [87], or Brenner (1999) [16]). This may lead to a high number of iterations, and a suitable preconditioner may need to be considered. A large number of iterative substructuring methods have been proposed during the last decades (see Bramble *et al.* (1989) [11], Dryja *et al.* (1990) [36], Smith (1992) [74], Dryja *et al.* (1994) [35], Dryja *et al.* (1995) [38]). These preconditioners are referred to as interface solvers or interface preconditioners. Let M be a suitable preconditioner for the SC system. Then the preconditioner B for the whole matrix A is defined as:

$$B = \begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & 0 \\ 0 & M \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix} \tag{2.2.11}$$

and the preconditioned matrix is:

$$\begin{aligned} B^{-1}A = & \begin{bmatrix} I_{II} & -A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II}^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} \times \begin{bmatrix} I_{II} & 0 \\ -A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \\ & \times \begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix} \\ = & \begin{bmatrix} I_{II} & -A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix} \times \begin{bmatrix} I_{II} & 0 \\ 0 & M^{-1}S \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix}. \end{aligned}$$

Note that the matrix $B^{-1}A$ is equivalent to the block diagonal matrix:

$$\begin{bmatrix} I_{II} & 0 \\ 0 & M^{-1}S \end{bmatrix}.$$

Therefore the eigenvalues of $B^{-1}A$ are the eigenvalues of $M^{-1}S$ and the eigenvalue 1 of the identity matrix I.

In view of parallel implementation, for each subdomain Ω_i, let A_i denote the contribution to the stiffness matrix A obtained from the corresponding subdomain (see e.g. Quarteroni and Valli (1999) [66], Section 2.4):

$$A_i = \begin{bmatrix} A_{II}^{(i)} & A_{IE}^{(i)} \\ (A_{IE}^{(i)})^T & A_{EE}^{(i)} \end{bmatrix}. \tag{2.2.12}$$

The notation has the following meaning. $A_{II}^{(i)}$ is the principal submatrix associated with nodes in the interior of Ω_i, with entries:

$$(A_{II}^{(i)})_{jk} = a(\phi_k, \phi_j), \ \forall k, j = 1, \cdots, m_i,$$

the entries of $A_{IE}^{(i)}$ are:

$$(A_{IE}^{(i)})_{jr} = a(\psi_r, \phi_j), \ \forall j = 1, \cdots, m_i, \ \forall r = 1, \cdots, r_i$$

and $A_{EE}^{(i)}$ corresponds to the nodes on the interface $\Gamma \cap \partial\Omega_i$, and its entries are of the form:

$$(A_{EE}^{(i)})_{qr} = a(\psi_r, \psi_q), \ \forall r, q = 1, \cdots, r_i,$$

where ϕ_j $(j = 1, \cdots, m_i)$ are the finite element basis functions associated with the nodes in Ω_i, and ψ_r $(r = 1, \cdots, r_i)$ are those associated with the nodes on $\Gamma \cap \partial\Omega_i$.

Therefore, we can also write A in the split form:

$$A = \sum_{i=1}^{N} R_i^T A_i R_i, \tag{2.2.13}$$

where A_i is as introduced in (2.2.12), R_i is the restriction matrix from the full vector in Ω to the local vector in $\Omega_i \cup \Gamma_i$, and R_i^T denotes the prolongation by zero on the nodes external to $\Omega_i \cup \Gamma_i$ $(i = 1, \cdots, N)$.

We note that the interior nodes in each subdomain Ω_i are decoupled from the interior nodes in other subdomains, while for the interface nodes, more than one subdomain contribute. These features are useful for parallel processing, since the given problem can be decoupled into independent subproblems on subdomains and the communication needed will be only for the values on the interface between subdomains. We can write A_{II} as a block-diagonal matrix with block order i given by $A_{II}^{(i)}$ $(i = 1, \cdots, N)$:

$$A_{II} = blockdiag\left(A_{II}^{(i)}\right) = \begin{bmatrix} A_{II}^{(1)} & \cdots & 0 \\ \cdot & \cdots & \cdot \\ 0 & \cdots & A_{II}^{(N)} \end{bmatrix}$$

and

$$S = \sum_{i=1}^{N} \tilde{R}_i^T S_i \tilde{R}_i, \tag{2.2.14}$$

where $S_i = A_{EE}^{(i)} - (A_{IE}^{(i)})^T (A_{II}^{(i)})^{-1} A_{IE}^{(i)}$, $\tilde{R}_i$ is the restriction matrix from the vector on Γ to the vector on Γ_i, and $\tilde{R}_i^T$ is the matrix that extends by zero a nodal vector from Γ_i to Γ $(i = 1, \cdots, N)$. The Schur complements S_i can be formed independently and in parallel (see e.g. Quarteroni and Valli (1999) [66], Section 2.4).

When an iterative method is used to solve the linear system associated with $B^{-1}A$, at each iteration step a system with the coefficient matrix B needs to be solved. This requires first the inversion of $A_{II}^{(i)}$ for each $i = 1, \cdots, N$, i.e. the solution of N independent Dirichlet problems in the subdomains, which can be solved in parallel. Then backward substitution can be applied, starting from the block M to obtain the values along the interior boundary, followed by the block A_{II}, which again yields the solution of N independent Dirichlet problems in the subdomains.

The Null-Space Property. Very important for the iterative substructuring methods is the null-space property (see Bramble *et al.* (1986) [9], Mandel (1990) [56], Smith (1990) [73], Smith *et al.* (1996) [75]). This allows for global estimates to be derived from the local properties of the Schur complements and the interface preconditioners associated with the boundary of individual substructures. Let us write the SC operator in the form (2.2.14). We assume that the SC preconditioner M may also be written as:

$$M = \sum_{i=1}^{N} \tilde{R}_i^T M_i \tilde{R}_i,$$

such that for every subdomain Ω_i, the null-space for S_i and M_i are identical (note that for PDE's that are constant inside each subdomain, if a subdomain has no part of its boundary with given Dirichlet data, the corresponding null-space is equal to the constant functions, while to a subdomain with given Dirichlet data on part of its boundary corresponds the trivial null-space). If

$$c_i u^T \tilde{R}_i^T M_i \tilde{R}_i u \leq u^T \tilde{R}_i^T S_i \tilde{R}_i u \leq C_i u^T \tilde{R}_i^T M_i \tilde{R}_i u,$$

for some positive constants C_i and c_i, then summing over the subdomains implies:

$$(\min_i c_i) u^T M u \leq u^T S u \leq (\max_i C_i) u^T M u.$$

We refer to the quantity $\max_i C_i / \min_i c_i$ as the local bound. It follows that, when using M as a preconditioner for S, the convergence rate will not depend explicitly on the number of subdomains. When there are jumps between coefficients across the interface boundary between subdomains, such that:

$$a(\mathbf{u}, \mathbf{v}) = \sum_i \int_{\Omega_i} \alpha_i(x)(\nabla \mathbf{u}, \nabla \mathbf{v})dx,$$

and α_i is smooth inside the subdomains, we may define $\bar{M}_i = \bar{\alpha}_i M_i$, where M_i is a preconditioner associated with the Laplacian and $\bar{\alpha}_i$ is the average of $\alpha_i(x)$ on Ω_i. If there exist constants C_i and c_i independent of the jumps in $\alpha_i(x)$ such that:

$$c_i u^T \tilde{R}_i^T \bar{M}_i \tilde{R}_i u \leq u^T \tilde{R}_i^T S_i \tilde{R}_i u \leq C_i u^T \tilde{R}_i^T \bar{M}_i \tilde{R}_i u,$$

then the convergence rate is also independent of the jumps in $\alpha_i(x)$.

In the remaining part of this section we shall briefly review some of the most popular and successful DD methods and motivate the construction of the new DD approach to be introduced later in this thesis.

Decomposition without Interior Cross-Points. Let $\Omega = (0, l) \times (0, 1)$ be a rectangular domain, partitioned into $N \geq 2$ disjoint subdomains without interior cross-points (Figure 2.2.2). In this case, the SC matrix S can be written as a block-diagonal matrix, each block corresponding to the boundary $\partial\Omega_i \cap \Omega$. By dropping all the couplings between different edges, we obtain a block-diagonal matrix, each block corresponding to an interface Γ_i between two adjoint subdomains Ω_i and Ω_{i+1}, $(i = 1, \cdots, N-1)$:

$$\begin{bmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & S_{N-1} \end{bmatrix}.$$

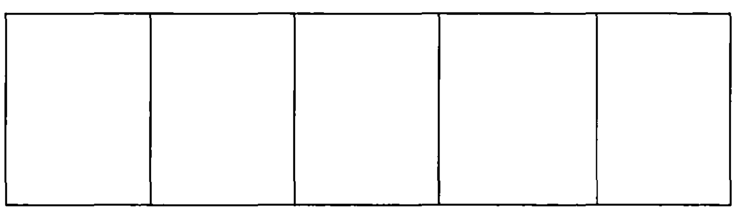

Figure 2.2.2: The partitioning of the domain $\Omega \subset \mathbb{R}^2$ into subdomains $\{\Omega_i\}_{i=1}^N$ without interior cross-points.

The J-Operator. We choose two adjacent subdomains $\Omega_1 = (0, l_1) \times (0, 1)$ and $\Omega_2 = (l_1, l_1 + l_2) \times (0, 1)$ with interface $\Gamma_1 = \{(x, y) : x = l_1, 0 < y < 1\}$ and assume that the grid corresponding to the union of the two subdomains is $(n_1 + 1 + n_2) \times n$, $l_k = (n_k + 1)h$ $(k = 1, 2)$ and $h = 1/(n + 1)$. For the Laplacian operator, S_1 can be expressed by an exact eigendecomposition (Bjørstad and Widlund (1986) [4], Chan (1987) [24]):

$$S_1 = F^{(n)} \Lambda^{(n)} F^{(n)},$$

where $F^{(n)}$ is the orthogonal sine transform matrix with entries:

$$F_{ij}^{(n)} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{ij\pi}{n+1}\right)$$

(hence $F^{(n)} = (F^{(n)})^T$) and $\Lambda^{(n)}$ is a diagonal matrix with elements given by:

$$\Lambda_i^{(n)} = \left(\frac{1 + \gamma_i^{n_1+1}}{1 - \gamma_i^{n_1+1}} + \frac{1 + \gamma_i^{n_2+1}}{1 - \gamma_i^{n_2+1}}\right) \sqrt{\psi_i + \frac{\psi_i^2}{4}},$$

where

$$\psi_i = 4 \sin^2\left(\frac{i\pi}{2(n+1)}\right) \qquad \text{and} \qquad \gamma_i = \left(1 + \frac{\psi_i}{2} - \sqrt{\psi_i + \frac{\psi_i^2}{4}}\right)^2.$$

Let $(\Psi^{(n)})^{1/2}$ denote the diagonal matrix with the diagonal entry of order i given by the positive square root of ψ_i, then S_1 can be preconditioned by the J-operator:

$$J_{2d} = F^{(n)} \left(\Psi^{(n)}\right)^{1/2} F^{(n)} = \frac{2}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}^{1/2}, \qquad (2.2.15)$$

which is a scaled version of the square root of the discrete one-dimensional Laplacian operator along the interface Γ_1, with zero Dirichlet boundary conditions on $\partial\Gamma_1$. This preconditioner was proposed by Dryja (1982) [31]. For computational issues regarding this preconditioner we also refer to the tutorial by Douglas *et al.* (2000) [30], Section 2.3, and the web resources for downloadable software announced in this tutorial.

The spectral technique used to approximate the solution along the edges can also be extended in higher dimensions. For the three-dimensional case, we assume that the interface Γ_1 between two adjacent subdomains is a rectangle with an $n_1 \times n_2$ mesh. The corresponding J-operator has the form:

$$J_{3d} = \left(F^{(n_2)} \bigotimes F^{(n_1)}\right) \left(\Psi^{(n_1)} \bigotimes I_{n_2} + I_{n_1} \bigotimes \Psi^{(n_2)}\right)^{1/2} \left(F^{(n_1)} \bigotimes F^{(n_2)}\right), \tag{2.2.16}$$

where $F^{(n_i)}$ and $\Psi^{(n_i)}$ are defined as above with n_i instead of n, I_{n_i} denotes the n_i-by-n_i identity matrix ($i = 1, 2$), while $\bigotimes$ represents the Kronecker (direct, tensor) product (see e.g. Smith *et al.* (1996) [75], p. 120). This operator guarantees a convergence rate that is independent of h, but depends on the aspect ratio of the subdomains.

When the mesh on Γ_1 is uniform, the preconditioner (2.2.15) can be solved in $\mathcal{O}(n \log(n))$ operations using the Fast Fourier (Sine) Transform (FFT). Moreover, when S_1 is associated with a quasi-uniform grid, (2.2.15) can still be used on the uniform grid to obtain an asymptotically well-conditioned preconditioner for S_1. For further details we refer to Dryja (1982) [31], (1984) [32], Smith *et al.* (1996) [75] pp. 119-120, Xu and Zou (1998) [87] p. 878, Quarteroni and Valli (1999) [66], pp. 77-78. For a survey of preconditioners for more general operators and discretisation, see Keyes and Gropp (1987) [51], Chan and Resasco (1987) [26] and (1988) [27].

Decomposition with Interior Cross-Points. We consider now the case of a partitioning (2.2.1) of the domain $\Omega \in \mathbb{R}^2$ into $N > 2$ disjoint subdomains with interior cross-points. When interior cross-points are present, in two dimensions, the global interface Γ can be partitioned as a union of edges and vertex-points (see Figure 2.2.3). The edges are the lines that separate two adjoint subdomains and do not include their end-points. The vertices are isolated points on the interface that are shared by more than two subdomains. We look for preconditioners for the SC system (2.2.10). These preconditioners should have good parallel properties on arbitrary elliptic operators and should be perfectly scalable, that is their performance should be insensitive to the number of subdomains. Thus global coupling between distant subdomains must also be provided.

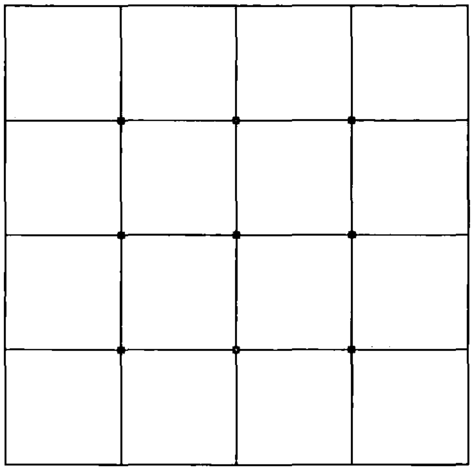

Figure 2.2.3: The global interface Γ as a union of edges and vertices in a partitioning of the domain $\Omega \subset \mathbb{R}^2$ into subdomains $\{\Omega_i\}_{i=1}^N$ with interior cross-points.

We reorder the unknowns u^E on the interface Γ, listing first those lying on each edge, then those at vertices. Thus,

$$u^E = (u_\epsilon, u_\chi)^T = (u_{\epsilon_1}, \cdots, u_{\epsilon_{n_\epsilon}}, u_\chi)^T.$$

In what follows we shall use deliberately a duplicity of notation and denote by ϵ_k the indices of the nodes lying on the edge ϵ_k $(k = 1, \cdots, n_\epsilon)$; similarly, denote by χ the indices of the vertex-points. With the above reordering we obtain the following block partitioning of S:

$$S = \begin{bmatrix} S_{\epsilon\epsilon} & S_{\epsilon\chi} \\ S_{\epsilon\chi}^T & S_{\chi\chi} \end{bmatrix}, \tag{2.2.17}$$

or equivalently:

$$S = \begin{bmatrix} S_{\epsilon_1\epsilon_1} & \cdots & S_{\epsilon_1\epsilon_{n_\epsilon}} & S_{\epsilon_1\chi} \\ S_{\epsilon_1\epsilon_2}^T & \cdots & S_{\epsilon_2 e_{n_\epsilon}} & S_{\epsilon_2\chi} \\ \cdots & \cdots & \cdots & \cdots \\ S_{\epsilon_1\chi}^T & \cdots & S_{\epsilon_{n_\epsilon}\chi}^T & S_{\chi\chi} \end{bmatrix}.$$

Note that $S_{\epsilon_i\epsilon_j} = 0$ whenever ϵ_i and ϵ_j are not part of the same subdomain, and that $S_{\chi\chi}$ is a diagonal matrix.

The survey below, of some of the preconditioners based on the partitioning (2.2.17), follows closely the presentations in Chan and Mathew (1994) [25], Smith *et al.* (1996) [75], Xu and Zou (1998) [87] and Quarteroni and Valli (1999) [66].

The Block-Jacobi Preconditioner. A simple block-Jacobi preconditioner M_J is obtained from S by dropping all the couplings between different edges and between edges and vertices. The result is a block diagonal preconditioner given by:

$$M_J = \begin{bmatrix} M_{\epsilon\epsilon} & 0 \\ 0 & S_{\mathcal{X}\mathcal{X}} \end{bmatrix},$$

where:

$$M_{\epsilon\epsilon} = \begin{bmatrix} M_{\epsilon_1\epsilon_1} & 0 & \cdots & 0 \\ 0 & M_{\epsilon_2\epsilon_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{n_\epsilon n_\epsilon} \end{bmatrix},$$

with $M_{\epsilon_k\epsilon_k}$ equal to either $S_{\epsilon_k\epsilon_k}$ or an interface preconditioner for the edge ϵ_k, like (2.2.15) introduced earlier.

If R_{ϵ_k} and $R_{\mathcal{X}}$ denote the pointwise restriction map from Γ onto the nodes on ϵ_k and the vertex-points respectively, then the block-Jacobi preconditioner M_J satisfies:

$$M_J^{-1} = \sum_{k=1}^{n_\epsilon} R_{\epsilon_k}^T M_{\epsilon_k\epsilon_k}^{-1} R_{\epsilon_k} + R_{\mathcal{X}}^T S_{\mathcal{X}\mathcal{X}}^{-1} R_{\mathcal{X}}. \qquad (2.2.18)$$

Since M_J does not involve global coupling between the subdomains, its spectral properties deteriorate as the number of subdomains increases. Then:

$$\kappa(M_J^{-1}S) \leq CH^{-2}\left(1 + \log\frac{H}{h}\right)^2,$$

where the positive constant C, independent of H and h, may depend on the coefficients of the given problem (see Bramble *et al.* (1986) [9], Dryja and Widlund (1994) [37]). The presence of the H^{-2} factor can be heuristically justified by the fact that the information is exchanged only between neighbouring subdomains, hence the number of steps required by the CG method to converge must be equal to the inverse of the diameter of Ω_i. On the other hand, the presence of the $\log(H/h)$ stems from the consideration that the global preconditioner is made up of local edge preconditioners $M_{\epsilon_k\epsilon_k}$ and of the vertex contributions $S_{\mathcal{X}\mathcal{X}}$. The latter yield pointwise values that should be controlled in terms of energy norm, which is possible at the expense of a logarithmic factor.

The Bramble-Pasciak-Schatz Preconditioner. In order to remove the H^{-2} term, Bramble *et al.* (1986) [9] inserted some mechanism of global coupling through a coarse grid problem based on a coarse mesh induced by the vertex-points. Let R_H^T and R_H denote the standard piecewise interpolation and restriction maps of coarse grid functions onto all the nodes of Γ, and $A_H = R_H A R_H^T$ be the associated stiffness matrix. Then the inverse of the modified preconditioner is defined as:

$$M_{BPS}^{-1} = \sum_{k=1}^{n_\epsilon} R_{\epsilon_k}^T M_{\epsilon_k\epsilon_k}^{-1} R_{\epsilon_k} + R_H^T A_H^{-1} R_H. \qquad (2.2.19)$$

This is also referred to as a direct sum preconditioner, since the sum of the dimensions of A_H and $M_{\epsilon_k\epsilon_k}$ equals the dimension of S.

The estimate of the relative condition number thus improves to:

$$\kappa(M_{BPS}^{-1} S) \le C \left(1 + \log \frac{H}{h}\right)^2,$$

where the positive constant C is independent of H and h, and also of the variation in the coefficients if they are constant in each subdomain Ω_i (see Bramble *et al.* (1986) [9], Widlund (1988) [83], Dryja *et al.* (1994) [35]).

In three dimensions, the interface Γ can be decomposed into faces φ, edges ϵ, and vertices χ. After reordering, the vector of interface unknowns can be expressed as:

$$u^E = (u_\varphi, u_\epsilon, u_\chi)^T.$$

Thus, the SC matrix may be written as:

$$S = \begin{bmatrix} S_{\varphi\varphi} & S_{\varphi\epsilon} & S_{\varphi\chi} \\ S_{\varphi\epsilon}^T & S_{\epsilon\epsilon} & S_{\epsilon\chi} \\ S_{\varphi\chi}^T & S_{\epsilon\chi}^T & S_{\chi\chi} \end{bmatrix}.$$

Droping the coupling between different faces, different edges, faces and edges, edges and vertices, faces and vertices yields:

$$M_{FEV} = \begin{bmatrix} M_{\varphi\varphi} & 0 & 0 \\ 0 & M_{\epsilon\epsilon} & 0 \\ 0 & 0 & A_H \end{bmatrix},$$

where $M_{\varphi\varphi}$ is block diagonal, such that each block is associated with one face and could be any face preconditioner, $M_{\epsilon\epsilon}$ is also block diagonal, where each block is associated with one edge and could be any edge preconditioner, A_H is a coarse grid operator obtained by using linear finite elements, with the subdomains themselves as mesh-elements. If R_{φ_i} denotes the restriction operator for each face, then:

$$M_{FEV}^{-1} = \sum_{i=1}^{n_\epsilon} R_{\varphi_i}^T M_{\varphi_i\varphi_i}^{-1} R_{\varphi_i} + \sum_{k=1}^{n_\epsilon} R_{\epsilon_k}^T M_{\epsilon_k\epsilon_k}^{-1} R_{\epsilon_k} + R_H^T A_H^{-1} R_H. \tag{2.2.20}$$

Then:

$$\kappa(M_{FEV}^{-1} S) \leq C \frac{H}{h} \left(1 + \log \frac{H}{h}\right)^2$$

(see Dryja *et al.* (1994) [35]). However, it is not possible to solve the coarse problem and the local problems in parallel while preserving both the null-space and the convergence properties.

The Vertex-Space Preconditioner. In general, whenever ϵ_i and ϵ_j are edges of the same subdomains, $S_{\epsilon_i\epsilon_j} \neq 0$. The preconditioners M_J and M_{BPS} both ignore this coupling, hence the logarithmic growth factor in the condition number. The aim is to remove the mild residual dependence on H/h and this is achieved by Smith (1990) [73], (1992) [74], where additional coupling between edges and vertex-points is introduced (see also Nepomnyaschikh (1986) [63]. The vertex-region ν_j is defined as the cross-shaped region centred at the vertex-point χ_j containing segments of length δH $(0 < \delta \leq 1)$ of all the edges that emanate from χ_j $(j = 1, \cdots, n_\chi)$ (see Figure 2.2.4).

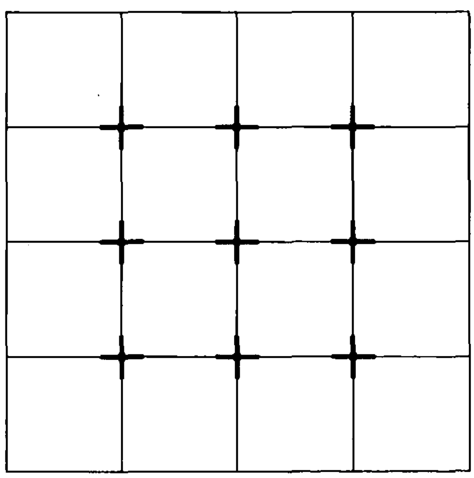

Figure 2.2.4: The vertex-regions as union of segments inside adjacent edges.

Let R_{ν_j} denote the restriction map that associates with full vectors the subvectors corresponding to the indices in ν_j, and S_{ν_j} denote the principal submatrix S corresponding to ν_j. Then:

$$S_{\nu_j} = R_{\nu_j} S R_{\nu_j}^T.$$

The vertex-space preconditioner is defined by:

$$M_{VS}^{-1} = M_{BPS}^{-1} + \sum_{j=1}^{n_X} R_{\nu_j}^T S_{\nu_j}^{-1} R_{\nu_j}. \tag{2.2.21}$$

Then:

$$\kappa(M_{VS}^{-1} S) \leq C \left(1 + \log \frac{H}{h}\right),$$

where C may depend on δ (see Smith (1990) [73], (1992) [74], also Dryja *et al.* (1994) [35]).

The Wire-Basket Preconditioner. For the three-dimensional case, in Smith (1990) [73] the vertex-space method is extended into a wire-basket based algorithm, by associating the vertex nodes and the edge nodes into one set called the wire-basket. The vector of interface unknowns becomes:

$$u^E = (u_\varphi, u_\omega)^T,$$

while the SC operator may be written as:

$$S = \begin{bmatrix} S_{\varphi\varphi} & S_{\varphi\omega} \\ S_{\varphi\omega}^T & S_{\omega\omega} \end{bmatrix}.$$

By dropping the couplings between various faces and faces and the wire-basket, the following preconditioner emerges:

$$M_{WB} = \begin{bmatrix} M_{\varphi\varphi} & 0 \\ 0 & M_{\omega\omega} \end{bmatrix}. \tag{2.2.22}$$

Then:

$$\kappa(M_{WB}^{-1} S) \leq C \left(1 + \log \frac{H}{h}\right)^2,$$

where C is also independent of coefficients (see Smith (1990) [73], also Dryja *et al.* (1994) [35], Dryja *et al.* (1996) [34]). This method is completely parallelizable.

The New Alternate Strip-Based Preconditioner. The convergence results corresponding to the substructuring methods via the M_J, M_{BPS}, and M_{VS} preconditioners indicate that interior cross-points are more difficult to handle. This is due to the fact that vertices represent strong coupling between interfaces. The BPS method is perhaps the first to treat vertices satisfactorily. In order to improve convergence, the VS method introduces additional local solvers associated with the points near each vertex-point. However, the local problems associated with the vertex-regions are usually expensive to solve, hence for these problems interface preconditioners may have to be considered to reduce computational complexity.

The new DD methods to be introduced later in this thesis may be viewed as direct extensions of the two-subdomains interface preconditioning technique via the J-operator, to the case of a decomposition with multiple nonoverlapping subdomains and interior cross-points. In two dimensions, the separate treatment of the vertex-points is avoided by assembling the original subdomains into nonoverlapping strips such that: the ends of the strips are on the boundary of the given domain, the interfaces between strips align with the edges of the subdomains and their union contains all of the interior vertices of the initial decomposition. Thus, the global interface between the subdomains can be partitioned as a union of edges between strips and edges between subdomains inside to the same strip (edges do not include their end-points). For the subproblems corresponding to the various edges, the J-operator is used. Since this interface preconditioner is sensitive to the aspect ratio of the subdomains, in order to achieve scalability, the new preconditioner is produced in two stages. At each stage the strips change such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage. This gives rise to an efficient method which is optimal with respect to the partitioning parameters (see Chapter 4). The extension to three-dimensional problems is via slice solvers (see Chapter 5).

2.2.2 Overlapping Domain Decomposition

Overlapping DD algorithms are based on a decomposition of the domain Ω into a number of overlapping subregions.

Decomposition with Two Overlapping Subdomains. Let $\tilde{\Omega}_1 \subset \Omega$ and $\tilde{\Omega}_2 \subset \Omega$ be two overlapping subdomains, which form a covering of Ω (see Figure 2.2.5), i.e. $\tilde{\Omega}_1 \bigcup \tilde{\Omega}_2 = \Omega$.

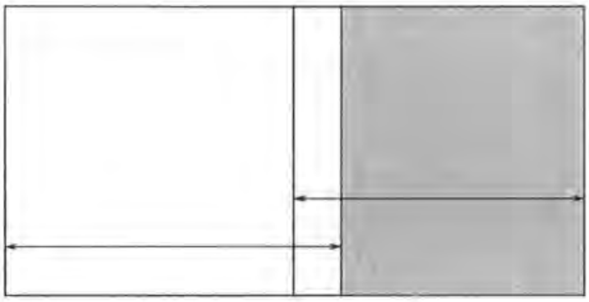

Figure 2.2.5: The covering of the domain $\Omega \subset \mathbb{R}^2$ with two overlapping subdomains.

In the Schwarz methods the computational domain is subdivided into overlapping subdomains and local Dirichlet problems are solved on each subdomain. The Schwarz alternating algorithm to solve (2.1.1) starts with a suitable initial guess $\mathbf{u}^0$ and generates a sequence of iterates $\mathbf{u}^1, \cdots, \mathbf{u}^k, \cdots$, as follows:

$$\begin{cases} -\nabla \cdot \left(\alpha \nabla \mathbf{u}_1^{k+1}\right) = \mathbf{f} \text{ in } \tilde{\Omega}_1 \\ \mathbf{u}_1^{k+1} = \mathbf{u}^k \text{ on } \Gamma_1 \\ \mathbf{u}_1^{k+1} = 0 \text{ on } \partial\tilde{\Omega}_1 \setminus \Gamma_1 \end{cases}$$

and

$$\begin{cases} -\nabla \cdot \left(\alpha \nabla \mathbf{u}_2^{k+1}\right) = \mathbf{f} \text{ in } \tilde{\Omega}_2 \\ \mathbf{u}_2^{k+1} = \mathbf{u}_1^{k+1} \text{ on } \Gamma_2 \\ \mathbf{u}_2^{k+1} = 0 \text{ on } \partial\tilde{\Omega}_2 \setminus \Gamma_2. \end{cases}$$

Then set:

$$\mathbf{u}^{k+1} = \begin{cases} \mathbf{u}_2^{k+1} \text{ in } \tilde{\Omega}_2 \\ \mathbf{u}_1^{k+1} \text{ in } \Omega \setminus \tilde{\Omega}_2. \end{cases}$$

Corresponding to each subregion $\tilde{\Omega}_i$, let $\tilde{I}_i$ denote the indices of the nodes in the interior of domains $\tilde{\Omega}_i$ and $card(\tilde{I}_i) = \tilde{n}_i$ denote the corresponding number of indices in each subdomain. Thus $\tilde{I}_1$ and $\tilde{I}_2$ form an overlapping set of indices for the unknown vector u and $\tilde{n}_1 + \tilde{n}_2 > n$, where n is the number of unknown in Ω. Let R_i be the restriction matrix whose action restricts a vector v of length n to a vector of size $\tilde{n}_i$ by choosing the entries with indices $\tilde{I}_i$ $(i = 1, 2)$. Its transpose R_i^T is an $n \times \tilde{n}_i$ matrix whose action extends by zeros a vector of nodal values in $\tilde{\Omega}_i$. Therefore the local subdomain matrices are:

$$A_i = R_i A R_i^T, \; i = 1, 2.$$

The Schwarz alternating algorithm can also be written as:

$$\begin{aligned} u^{k+1/2} &\leftarrow u^k + R_1 A_1^{-1} R_1^T (f - Au^k) \\ u^{k+1} &\leftarrow u^{k+1/2} + R_2 A_2^{-1} R_2^T (f - Au^{k+1/2}). \end{aligned} \tag{2.2.23}$$

By defining

$$B_i = R_i A_i^{-1} R_i^T, \; i = 1, 2,$$

(2.2.23) can be written as a single step process:

$$u^{k+1} \leftarrow u^k + (B_1 + B_2 - B_2 A B_1)(f - Au^k).$$

The multiplicative Schwarz preconditioner B_{ms} is given by:

$$B_{ms}^{-1} = B_1 + B_2 - B_2 A B_1,$$

and the convergence is governed by the multiplicative Schwarz iteration matrix:

$$(I - B_2 A)(I - B_1 A).$$

Although the matrices $B_i A$ are symmetric with respect to the $a(\cdot, \cdot)$ inner product (induced by the SPD matrix A), the multiplicative Schwarz iteration matrix is not so (see Chan and Mathew (1994) [25] pp. 66 and 92). A symmetrized version can be constructed by iterating one more half-step and obtain the iteration matrix:

$$(I - B_1 A)(I - B_2 A)(I - B_1 A),$$

therefore the CG method can be applied.

A more parallelizable version can be obtained by defining the additive Schwarz iteration as follows:

$$\begin{aligned} u^{k+1/2} &\leftarrow u^k + R_1 A_1^{-1} R_1^T (f - Au^k) \\ u^{k+1} &\leftarrow u^{k+1/2} + R_2 A_2^{-1} R_2^T (f - Au^k). \end{aligned} \tag{2.2.24}$$

Therefore the additive Schwarz preconditioner for A can be defined as:

$$B_{as}^{-1} = B_1 + B_2.$$

Hence $B^{-1}A$ is symmetric in the $a(\cdot,\cdot)$ scalar product, and can be used with a CG acceleration technique. The Schwarz alternating method may be used to solve classical boundary value problems for harmonic functions on domains that are the union of two subdomains by alternately solving the same boundary problem restricted to each subdomain. An extension to many subdomains is discussed in Lions (1988) [53].

Decomposition with Multiple Overlapping Subdomains. Consider the domain Ω as in Figure 2.2.6. Assume that the mesh diameter is $\mathcal{O}(h)$ and the subdomains $\tilde{\Omega}_i$ $(i = 1, \cdots, N)$ such that $\bigcup_{i=1}^{N} \tilde{\Omega}_i = \Omega$, are of diameter $\mathcal{O}(H)$ and overlap each other with a width of $\mathcal{O}(\delta)$. The number of nodes across Ω is $\mathcal{O}(1/h)$, the number of nodes across a subdomain is $\mathcal{O}(H/h)$, and the number of nodes across an overlap region is $\mathcal{O}(\delta/h)$.

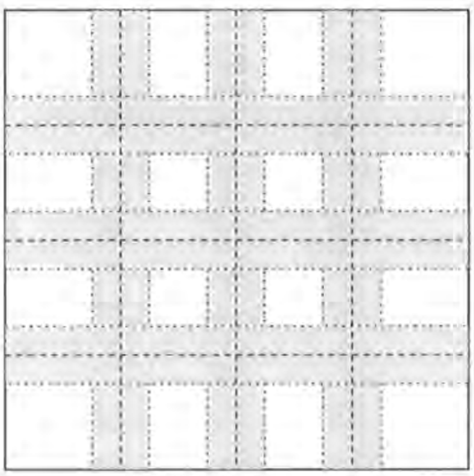

Figure 2.2.6: The covering of the domain $\Omega \subset \mathbb{R}^2$ with many overlapping subdomains $\{\tilde{\Omega}_i\}_{i=1}^N$.

The two subdomain additive Schwarz preconditioner can be extended to the multiple overlapping subdomains as follows:

$$B_{as,1}^{-1} = \sum_{i=1}^{N} R_i A_i^{-1} R_i^T,$$

where R_i and R_i^T are the restriction and interpolation maps respectively, corresponding to $\tilde{\Omega}_i$ and $A_i = R_i A R_i^T$ $(i = 1, \cdots, N)$. However, this algorithm is not scalable, since the condition number of the preconditioned system grows like the number of subdomains (Dryja and Widlund (1994) [37]):

$$\kappa(B_{as,1}^{-1} A) \leq C H^{-2}(1 + \delta^{-2}).$$

An improved additive Schwarz preconditioner is defined by:

$$B_{as,2}^{-1} = \sum_{i=1}^{N} R_i A_i^{-1} R_i^T + R_H^T A_H^{-1} R_H.$$

Then:

$$\kappa(B_{as,2}^{-1} A) \leq C(1 + \delta^{-2}),$$

where the positive constant C may depend on the variation of the coefficients (see Dryja and Widlund (1994) [37], Dryja *et al.* (1994) [35]). If the coefficients are constant or mildly varying within each coarse grid element, then:

$$\kappa(B_{as,2}^{-1} A) \leq C \left(1 + \log \frac{H}{h}\right),$$

where C may depend on the overlap parameter δ (see Dryja and Widlund (1994) [37]). In three dimensions this estimate deteriorates to:

$$\kappa(B_{as,3}^{-1} A) \leq C \frac{H}{h}$$

(see Dryja and Widlund (1994) [37]). On the other hand, the classical Dirichlet transmission conditions employed between subdomains lead to convergence rates which are not uniform with respect to frequency: high frequency components converge rapidly, whereas low frequency components converge only slowly (Gander (2000) [42]).

2.3 The Multigrid Technique

The MG methods provide optimal order algorithms for solving elliptic boundary value problems, in the sense that the amount of computation is determined only by the amount of real physical information. The error bounds of the approximate solution obtained from the full MG algorithm are comparable to the theoretical error bounds in the finite element method, while the amount of necessary computational work is proportional to the number of unknowns in the discretised equations (problems with N unknowns are solved with $\mathcal{O}(N)$ work and storage), for a large class of problems. The short presentation in this section is based the introductory tutorial on MG techniques by Briggs (1987) [18], Briggs *et al.* (2000) [19], and the monographs Hackbush (1985) [47] and Wesseling (1992) [82]. For theory regarding the more general case we recommend Scott and Zhang (1992) [72].

A MG method has two main features: smoothing on the current grid, and error correction on a coarser grid. The idea beyond the MG process is to damp all (locally) highly oscillating components of the error first, then approximate the remaining smooth part on the coarse grid. By alternately repeating the smoothing step and the coarse grid correction, an iterative method is obtained. From the beginning we introduce a sequence of grids obtained by doubling the step size from the smallest grid-size equal to h, to the largest possible grid-size equal to h_L: $h = h_0 < \cdots < h_l < \cdots < h_L$, where the index l is the number of the level and $h_l = 2^l h$ $(l = 0, \cdots, L)$ (Figure 2.3.1). This process of descending to a coarser grid is called coarsening, while the opposite process of ascending from the a coarser to a finer grid is called refinement.

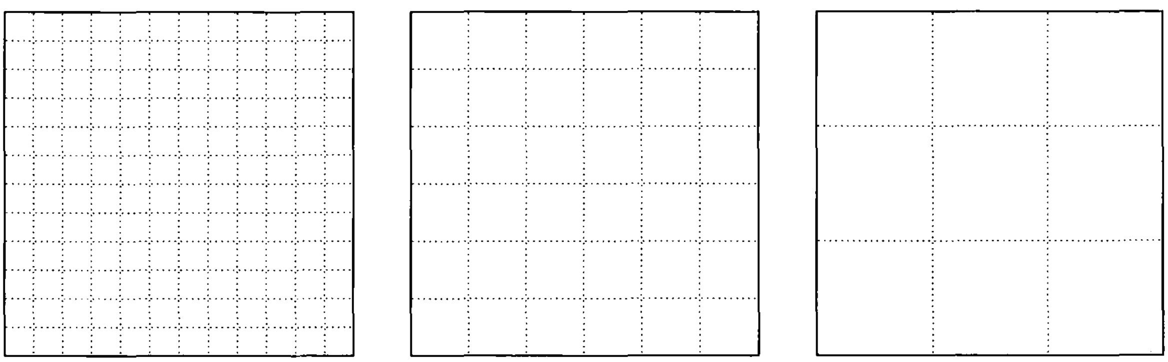

Figure 2.3.1: Sequence of nested grids associated with the domain $\Omega \subset \mathbb{R}^2$.

The Two-Level Case. We first assume only two grids, one fine and one coarse, of mesh-size h and $2h$ respectively. The two-grid (TG) method has the following steps:

Step (1) (Smoothing) first we consider the equation on the fine grid:

$$Au = f. \tag{2.3.1}$$

The goal of the TG iteration is the solution of (2.3.1). After a few iterations of a chosen iterative method (e.g. damped Jacobi, Gauss-Seidel, SOR, CG) the high frequencies become smooth. However, if low frequencies (smooth components) are also present, the convergence will become slow. Let $u^{1/2}$ be the approximate solution after this step has been completed. The TG technique is based on the observation that a complementary iteration needs to be constructed to reduce the smooth error. The error $u^{1/2} - u$ is to be approximated on a coarse grid. In order to do this, first, we calculate the initial defect on the fine grid:

$$r = Au^{1/2} - f.$$

Therefore:

$$A(u^{1/2} - u) = Au^{1/2} - f = r. \tag{2.3.2}$$

Then, translate the defect r into the coarse grid as:

$$r_c = Rr,$$

where $R : S_h^0(\Omega) \to S_{2h}^0(\Omega)$ is a restriction operator. Let A_c be the matrix of the original system (2.3.1) restricted to the coarse grid. Then the coarse grid approximation w of $u^{1/2} - u$ satisfies:

$$A_c w = r_c. \tag{2.3.3}$$

By the Galerkin approach, we can define $A_c = RAP$. Another approach is to rediscretise the given equation on the coarse grid and therefore obtain A_c in the same way A has been initially derived. The equation (2.3.3) can be solved exactly, or iteratively with an initial guess $w = 0$.

Step (2) (Coarse Grid Correction) The coarse grid approximation of the error is then translated into the fine grid, as Pw, by means of prolongation[1] operator, $P : S^0_{2h}(\Omega) \to S^0_h(\Omega)$. Finally the old value of u is updated as:

$$u^1 \leftarrow u^{1/2} - Pw.$$

This process can be expressed in a single coarse grid correction formula as:

$$u^1 \leftarrow u^{1/2} - PA_c^{-1}R(Au^{1/2} - f). \tag{2.3.4}$$

Having returned to the fine grid, we have completed a two-grid V-cycle and can check for convergence, by measuring the size of the residual $(I - APA_c^{-1}R)r$. The V-cycle is repeated until the error is below the required tolerance.

The Multilevel Algorithms. The notation in the description of the MG V-cycle is as follows: the index indicates the grid-size of the mesh-grid on which the system is described at each level, namely the index $2^l h$ indicates the discretisation on the grid of grid-size $h_l = 2^l h$ $(l = 0, \cdots, L)$; $P_{2^l h}$ denotes the interpolation from the $(l+1)$th grid level to the lth grid level, while $R_{2^l h}$ denotes the restriction from the $(l-1)$th grid level to the lth grid level.

Step (1_h) Pre-smooth $A_h u_h = f_h$ with some initial guess $\bar{u}_h$.

Compute $r_{2h} = R_{2h}(A_h u_h - f_h)$.

Step (1_{2h}) Pre-smooth $A_{2h} w_{2h} = r_{2h}$ with initial guess $w_{2h} = 0$.

Compute $r_{4h} = R_{4h}(A_{2h} w_{2h} - f_{2h})$.

Step (1_{4h}) Pre-smooth $A_{4h} w_{4h} = r_{4h}$ with initial guess $w_{4h} = 0$.

Compute $r_{8h} = R_{8h}(A_{4h} w_{4h} - f_{4h})$.

$\vdots$

Step (1_H) Pre-smooth $A_H w_{h_L} = r_{h_L}$

$\vdots$

Step (2_{4h}) Correct $\bar{w}_{4h} \leftarrow w_{4h} - P_{4h} w_{8h}$.

Post-smooth $A_{4h} w_{4h} = r_{4h}$ with initial guess $w_{4h} = \bar{w}_{4h}$.

[1] A popular prolongation is the piecewise linear interpolation (Hackbush (1985) [47], p. 59.)

Step (2_{2h}) Correct $\bar{w}_{2h} \leftarrow w_{2h} - P_{2h}w_{4h}$.

Post-smooth $A_{2h}w_{2h} = r_{2h}$ with initial guess $w_{2h} = \bar{w}_{2h}$.

Step (2_h) Correct $\bar{u}_h \leftarrow u_h - P_h w_{2h}$.

Post-smooth $A_h u_h = r_h$ with initial guess $u_h = \bar{u}_h$.

The difference from the TG V-cycle is that (2.3.3) is replaced by one multigrid iteration applied to the initial guess $w = 0$. If (2.3.3) is replaced by two multigrid iterations, then we obtain the W-cycle. A particular case of the V-cycle is the sawtooth cycle, derived by eliminating from the V-cycle process the pre-smoothing step.

Multilevel Schwarz Methods. Let the domain Ω be as represented by Figure 2.2.6 with the overlapping subdomains $\tilde{\Omega}_i$ ($i = 0, \cdots, N$), of diameter $\mathcal{O}(H)$ and overlap of width $\mathcal{O}(\delta)$. We assume that with the domain Ω, a sequence of grids with grid-sizes $h = h_0 < \cdots < h_l < \cdots < h_L$ are associated, denoted by $\Omega^{(l)}$. We also assume that the grids corresponding to each subdomain $\tilde{\Omega}_i$ ($i = 0, \cdots, N$) are inherited from the original grids on the domain Ω and denote by $\tilde{\Omega}_i^{(l)}$ the corresponding l-level subdomain of $\Omega^{(l)}$. We consider the interpolation maps $(R_i^{(l)})^T$ from the nodal values on the interior grid $\tilde{\Omega}_i^{(l)}$ to the finest grid and its corresponding restriction $R_i^{(l)}$ map on the interior nodes in $\tilde{\Omega}_i^{(l)}$ and denote by $A_i^{(l)} = R_i^{(l)} A (R_i^{(l)})^T$ the local stiffness matrix associated with the subregion $\tilde{\Omega}_i^{(l)}$. The additive two-level Schwarz process reads:

$$u^{1/2} \leftarrow (R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)} f$$

$$u^1 \leftarrow u^{1/2} + \sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)} f,$$

where $l = 0, \cdots, L-1$. The corresponding two-level additive Schwarz preconditioner is:

$$B_{tlas} = (R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)} + \sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)}.$$

Note that the local subproblems involving $(A_i^{(l)})^{-1}$ are much smaller than the original problem and can be solved by direct or iterative methods. If however the coarse problem involving $A^{(l+1)}$ is still large, the two-level additive overlapping preconditioner can be used again.

The multilevel additive Schwarz preconditioner is defined as follows:

$$B_{mlas}^{-1} = \sum_{l=1}^{L} \sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)}.$$

It has been shown (Zhang (1992) [91]) that:

$$\kappa(B_{mlas}^{-1} A) \leq C,$$

where the positive constant C is independent of h and the number of levels, but may depend on the variation in the coefficients.

The two-level multiplicative Schwarz process reads:

$$\begin{aligned} u^{1/2} &\leftarrow (R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)} f \\ u^1 &\leftarrow u^{1/2} + \sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)} (f - Au^{1/2}). \end{aligned}$$

The corresponding two-level multiplicative Schwarz preconditioner is:

$$\begin{aligned} B_{tlms} = {} & R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)} + \sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)} \\ & - \left(\sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)} \right) (R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)}. \end{aligned}$$

A symmetrized version can be obtained by iterating one more half-step as follows:

$$\begin{aligned} u^{1/3} &\leftarrow (R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)} f \\ u^{2/3} &\leftarrow u^{1/3} + \sum_{i=1}^{N} (R_i^{(l)})^T (A_i^{(l)})^{-1} R_i^{(l)} (f - Au^{1/3}) \\ u^1 &\leftarrow u^{2/3} + (R^{(l+1)})^T (A^{(l+1)})^{-1} R^{(l+1)} (f - Au^{2/3}). \end{aligned}$$

2.4 Summary

In this chapter, we have recalled some of the best known and most efficient DD and MG methods applied to second order, self-adjoint, coercive boundary value problems. We note that the convergence results for this methods remain valid if the exact solvers on every subdomain and/or on the coarse grid are replaced by spectrally equivalent inexact solvers and the meshes are assumed to be shape regular (for a definition of regularity of meshes see e.g. Ciarlet (1978) [28], Remark 3.1.3; Quarteroni and Valli (1994) [65], Section 3.1; (1999) [66], Section 2.1). In recent years, a unified framework for the analysis of both DD and MG methods has been developed via the PSC method or the additive Schwarz method. This generates the natural idea that new and more efficient algorithms may be devised, to draw upon the strengths of both DD and MG methodologies. The challenging point is how to do that and yet do not add to the already perceived complexity of DD algorithms.

Chapter 3

Alternate Strip-Based Domain Decomposition Algorithms for Symmetric Elliptic PDE's in 2D

3.1 Introduction

In this chapter, we propose a new class of DD preconditioners for the discrete linear system (2.1.8), in two dimensions. The new solvers are obtained from alternate decompositions of the domain $\Omega \subset \mathbb{R}^2$ into a finite number of nonoverlapping strips, and are perfectly scalable (i.e. their performance is insensitive to the partitioning parameters). Probably the earliest papers involving splitting the domain into subdomains without interior vertices are Buzbee *et al.* (1971) [20] and Buzbee and Dorr (1974) [21]. Later, preconditioners for two-dimensional elliptic boundary value problems together with analytic estimates of the convergence of the preconditioned iterative procedures were proposed by Bramble *et al.* (1986) [10] and Bjørstad and Widlund (1986) [4]. In Chan and Resasco (1987) [26], a fast direct Poisson solver on a rectangle divided into parallel strips or boxes is presented. These methods, however, are applicable only when the aspect ratio of each strip is not too high (i.e. when the strips are not too long and thin). In Mandel and Lett (1991) [57], DD preconditioners for p-version finite elements with high aspect ratio with better convergence properties are introduced. In Boglaev (1997) [5], (2000) [6] and Boglaev and Duoba (2004) [7], strip-based decompositions are used for solving singularly perturbed problems.

The alternate strip-based (ASB_2) preconditioner to be introduced in this chapter is based on exact solvers in the interior of the strips and the J-preconditioner (see Section 2.2) on the interfaces between strips (i.e. on the edges shared by two strips). Since this interface preconditioner is sensitive to the aspect ratio of the subdomains, in order to achieve scalability, the ASB_2 preconditioner is produced in two stages. At each stage the strips change such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage. The two stages allow the use of a two-grid V-cycle and guarantee a good rate of convergence of the preconditioned iterative procedures, which is optimal with respect to the partitioning parameters. Therefore, the new preconditioner is also applicable in the case of strips with high aspect ratio. This new DD approach extends in a straightforward manner to the three-dimensional case (see Chapter 5).

The rest of this chapter is organised as follows. In Section 3.2, we describe the strip-based (SB_2) and the alternate strip-based (ASB_2) preconditioning techniques. Section 3.3 is devoted to the theoretical investigation of these DD methods. In Section 3.4, numerical examples are presented to illustrate the behaviour of these methods. Conclusions and further remarks are addressed in Section 3.5.

3.2 Strip-Based Domain Decomposition

We consider the problem (2.1.1) with constant coefficients $\alpha(x) \equiv 1$, in the two-dimensional case. For clarity of presentation, we assume the domain $\Omega \subset \mathbb{R}^2$ to be the unit square $(0,1) \times (0,1)$. Let

$$\bar{\Omega} = \bigcup_{s=1}^{n_s} \bar{\Omega}^s \tag{3.2.1}$$

be a partitioning of this domain into strips Ω^s $(s = 1, \cdots, n_s)$, such that each Ω^s is an open rectangle in $\mathbb{R}^2$ having one dimension equal to 1 and all vertices situated on the boundary $\partial\Omega$ (see Figure 3.2.1 left or right). The interface between two strips, which we denote by Γ^j $(j = 1, \cdots, n_s - 1)$, is an open line in $\mathbb{R}$, of length 1. We assume shape regularity of the rectangular strips in the sense that there exists a maximum rectangle edge ratio $1/H$, such that for every strip Ω^s the width H^s

satisfies the double inequality $1 \leq 1/H^s \leq 1/H$. We say that the strip aspect ratio condition is satisfied with $1/H$. We emphasise that $1/H^s$ may be high, i.e. the strips may be long and narrow. We assume that the edges between strips align with a given finite element mesh Σ^h associated with Ω (see Figure 3.2.2 left or right).

Let $S_h^0(\Omega)$ be as described in Section 2.1 ($h < H$). For every strip Ω^s, we consider the restrictions on $\bar{\Omega}^s \cap \Omega$ of the functions in $S_h^0(\Omega)$, and denote the finite element space of these restrictions by $S_h^0(\bar{\Omega}^s)$. We define $S_h^0(\Omega^s)$ to be the subspace of $S_h^0(\bar{\Omega}^s)$ consisting of those functions which are zero on the boundary $\partial\Omega^s \cap \Omega$. We also consider the restrictions on $\partial\Omega^s \cap \Omega$ of the functions in $S_h^0(\Omega)$ and denote the finite element space of these restrictions by $S_h^0(\partial\Omega^s)$. For every edge between two strips $\Gamma^j \subset \partial\Omega^s$, we define $S_h^0(\Gamma^j)$ to be the subspace of $S_h^0(\partial\Omega^s)$ consisting of those functions which are zero on $(\partial\Omega^s \cap \Omega) \setminus \Gamma^j$.

Furthermore, let $\tilde{\Gamma}^j$ be the domain obtained from the union of the edge between two strips Γ^j with the neighbouring strips Ω^s. Note that the domains $\tilde{\Gamma}^j$ form an overlapping covering of Ω, such that every point in Ω is contained in at most two of these domains. With every such a domain, a subspace of $S_h^0(\Omega)$ is associated: we define $S_h^0(\tilde{\Gamma}^j)$ as the subspace of $S_h^0(\Omega)$ consisting of those functions with support in $\tilde{\Gamma}^j$. Then:

$$S_h^0(\Omega) = \sum_{j=1}^{n_s-1} S_h^0(\tilde{\Gamma}^j), \tag{3.2.2}$$

i.e. for all $\mathbf{u} \in S_h^0(\Omega)$, there exists a representation, which is not unique, of the form:

$$\mathbf{u} = \sum_{j=1}^{n_s-1} \mathbf{u}^j, \qquad \mathbf{u}^j \in S_h^0(\tilde{\Gamma}^j). \tag{3.2.3}$$

3.2.1 The Strip-Based (SB_2) Technique

We consider the linear system (2.1.8) and write the stiffness matrix A as:

$$A = \begin{bmatrix} A_{II} & A_{IE} \\ A_{IE}^T & A_{EE} \end{bmatrix},$$

where A_{II} is the stiffness matrix associated with the finite element nodes in $\bigcup_{s=1}^{n_s} \Omega^s$, and A_{EE}, the stiffness matrix associated with the finite element nodes on $\bigcup_{s=1}^{n_s} \partial\Omega^s \cap \Omega$.

Then:

$$A = \begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix},$$

where I_{II} and I_{EE} denote identity matrices, and $S = A_{EE} - A_{IE}^T A_{II}^{-1} A_{IE}$ is the SC matrix.

On the other hand, the matrix A can be split as:

$$A = \sum_{s=1}^{n_s} (R^s)^T A^s R^s, \tag{3.2.4}$$

where A^s is the finite element matrix associated with the given problem in the strip subregion Ω^s, with zero Dirichlet boundary conditions on $\partial\Omega^s \cap \partial\Omega$ and Neumann boundary conditions on $\partial\Omega^s \cap \Omega$, R^s is the restriction matrix from the full vector in Ω to local vectors in $\Omega^s \cup (\partial\Omega^s \cap \Omega)$, and $(R^s)^T$ is the corresponding prolongation by zero on the nodes external to $\Omega^s \cup (\partial\Omega^s \cap \Omega)$ (see also (2.2.13)). Therefore we can write A_{II} as a block-diagonal matrix with block of order s given by $A_{II}^{(s)}$:

$$A_{II} = blockdiag\left(A_{II}^{(s)}\right) = \begin{bmatrix} A_{II}^{(1)} & \cdots & 0 \\ \cdot & \cdots & \cdot \\ 0 & \cdots & A_{II}^{(n_s)} \end{bmatrix}.$$

Furthermore, by reordering the nodes, S can be expressed as a block-diagonal matrix, with each block corresponding to a boundary $\partial\Omega^s \cap \Omega$. First, in the SC matrix S, we drop all the couplings between different edges Γ^j, to obtain the block-diagonal matrix:

$$blockdiag\,(S_{\Gamma^j}),$$

each block S_{Γ^j} corresponding to an edge between two strips Γ^j. Then, we define the preconditioner M_{sb2} for S as follows. For every Γ^j, let $(-\partial^2/\partial\tau^2)_{\Gamma^j}$ be the one-dimensional Laplacian operator with domain of definition $H_0^1(\Gamma^j)$, and let δ_{Γ^j} denote the discrete operator defined on $S_h^0(\Gamma^j)$ by:

$$(\delta_{\Gamma^j}\mathbf{u}, \mathbf{v})_{\Gamma^j} = (\mathbf{u}', \mathbf{v}')_{\Gamma^j}, \ \forall \mathbf{v} \in S_h^0(\Gamma^j),$$

where the prime denotes differentiation with respect to the arc-length τ along Γ^j, and $(\cdot,\cdot)_{\Gamma^j}$ is the scalar product in $L^2(\Gamma^j)$. Note that δ_{Γ^j} represents a finite dimensional approximation of $(-\partial^2/\partial\tau^2)_{\Gamma^j}$. Since δ_{Γ^j} is symmetric and positive definite (SPD) in the inner product $(\cdot,\cdot)_{\Gamma^j}$, we can define the square root $\delta_{\Gamma^j}^{1/2}$ of δ_{Γ^j} (see Bramble *et al.* (1986) [9], pp. 108-109). We denote by J_{Γ^j} the matrix form of $\delta_{\Gamma^j}^{1/2}$, then set the approximation for S_{Γ^j} as J_{Γ^j} and the approximation for S as:

$$M_{sb2} = blockdiag\,(J_{\Gamma^j})\,.$$

We define the preconditioner B_{sb2} for the matrix A as:

$$B_{sb2} = \begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & 0 \\ 0 & M_{sb2} \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix}. \tag{3.2.5}$$

The preconditioned matrix is:

$$B_{sb2}^{-1} A = \begin{bmatrix} I_{II} & -A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix} \times \begin{bmatrix} I_{II} & 0 \\ 0 & M_{sb2}^{-1} S \end{bmatrix} \times \begin{bmatrix} I_{II} & A_{II}^{-1} A_{IE} \\ 0 & I_{EE} \end{bmatrix}.$$

A generic system $B_{sb2} w = r$ can now be written in terms of block matrices as:

$$\begin{bmatrix} I_{II} & 0 \\ A_{IE}^T A_{II}^{-1} & I_{EE} \end{bmatrix} \times \begin{bmatrix} A_{II} & A_{IE} \\ 0 & M_{sb2} \end{bmatrix} \times \begin{bmatrix} w^I \\ w^E \end{bmatrix} = \begin{bmatrix} r^I \\ r^E \end{bmatrix}, \tag{3.2.6}$$

where w^I represents the subvector of w associated with the finite element nodes in $\bigcup_{s=1}^{n_s} \Omega^s$, w^E represents the subvectors of w associated with the finite element nodes on $\bigcup_{s=1}^{n_s} \partial\Omega^s \cap \Omega$, and similarly for r^I and r^E. The solution $w = B_{sb2}^{-1} r$ can be derived as follows.

The SB$_2$ Procedure (algebraic form).

(I) compute $A_{II}^{-1} r^I$ and obtain the system equivalent to (3.2.6):

$$\begin{bmatrix} A_{II} & A_{IE} \\ 0 & M_{sb2} \end{bmatrix} \times \begin{bmatrix} w^I \\ w^E \end{bmatrix} = \begin{bmatrix} r^I \\ r^E - A_{IE}^T A_{II}^{-1} r^I \end{bmatrix}. \tag{3.2.7}$$

(II) using $A_{II}^{-1} r^I$ obtained in (I), solve for w^E the system (3.2.7).

(III) using w^E obtained in (II), solve for w^I the system (3.2.7), by backward substitution.

With the preconditioner B_{sb2} we can construct the following iterative method: start with u^0 as an initial approximation (without restricting the generality we can assume the starting approximation to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1} \leftarrow u^l + B_{sb2}^{-1}(f - Au^l).$$

This can be interpreted as a Richardson iterative procedure (see e.g. Smith *et al.* (1996) [75], Appendix, or Toselli and Widlund (2004) [80], C. 3).

Alternatively, since the new preconditioned matrix $B_{sb2}^{-1}A$ is symmetric and non-negative definite with respect to the $a(\cdot,\cdot)$ scalar product (induced by the SPD matrix A), the CG acceleration can be applied as follows (see also Chapter 2):

- let u^0 be an initial iterate,

$$r^0 \leftarrow f - Au^0, \text{ the initial residual}$$

$$w^0 \leftarrow B_{sb2}^{-1}r^0, \text{ the initial preconditioned residual}$$

$$v^0 \leftarrow w^0, \text{ the initial search direction}$$

- for $l = 0, 1, \cdots$

compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Av^l)}$

update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

update the residual: $r^{l+1} \leftarrow r^l + p_l A v^l$

if $r^{l+1} \geq$ tolerance, then

update the preconditioned residual: $w^{l+1} \leftarrow B_{sb2}^{-1}r^{l+1}$

compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

else end for.

The resulting SB_2 method has good parallelisation properties and a rate of convergence proportional to $1/\sqrt{H}$ (see Theorem 3.3.4 and Table 3.4.1).

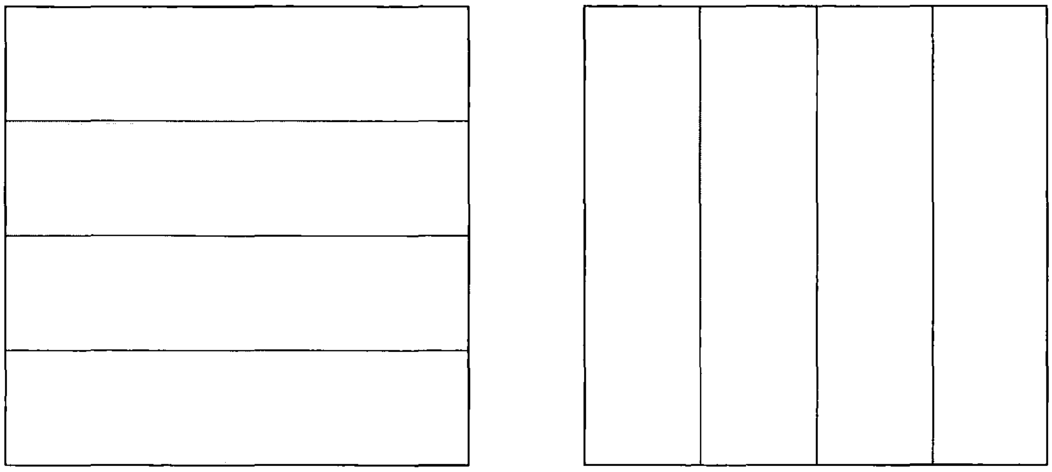

Figure 3.2.1: The horizontal (left) and vertical (right) partitioning into strips of the domain $\Omega \subset \mathbb{R}^2$.

Remark 3.2.1 We note that since the interior problems on each strip Ω^s are solved exactly, the variables corresponding to the interior of Ω^s can be eliminated from the iterative process, which then can be reduced to a boundary iteration on $\bigcup_{s=1}^{n_s} \partial\Omega^s \cap \Omega$. The resulting algorithm, though with a convergence rate still proportional to $1/\sqrt{H}$, is more efficient than the SB_2 algorithm presented above, as each iteration does not require the solution of interior strip problems (see Chan and Resasco (1987) [26]). However, our goal is to obtain a method which is optimal with respect to both partitioning parameters H and h. In the ASB_2 algorithms to be introduced below, the variables corresponding to the interior of the strips will play an essential role, and the SB_2 procedure as described earlier is preferable.

3.2.2 The Alternate Strip-Based (ASB_2) Technique

In order to be effective in a parallel environment, the conditioning of the preconditioned system should not deteriorate as the number of subregions (processors) increases. We continue our discussion with a two-stage extension of the SB_2 technique, which does not exhibit such growth in the condition number. We assume, for instance, that at the first stage the strips are horizontal (i.e. the edges between strips align in the horizontal direction $0x$), while at the second stage the strips are vertical (i.e. the edges between strips align in the vertical direction $0y$). Figure 3.2.1 shows the partitioning of the square Ω into disjoint, uniform strips at two different stages.

The Additive Alternate Strip-Based (ASB_{2a}) Algorithm. Let $B^{(1)}_{sb2}$ and $B^{(2)}_{sb2}$ denote the SB_2 preconditioner at the first and second stage respectively. It is easy to see that Ω is covered by the following overlapping subdomains: on one hand the strips and the edges between strips at the first stage, and on the other hand the strips and the edges between strips at the second stage. Therefore Schwarz algorithms can be built using the B_{sb2} preconditioner.

The (inexact) additive Schwarz method is: start with u^0 as an initial approximation (without restricting the generality we take this approximation to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1/2} \leftarrow u^l + (B^{(1)}_{sb2})^{-1}(f - Au^l)$$
$$u^{l+1} \leftarrow u^{l+1/2} + (B^{(2)}_{sbs2})^{-1}(f - Au^l).$$

This can also be written in one step as:

$$u^{l+1} \leftarrow u^l + \left((B^{(1)}_{sb2})^{-1} + (B^{(2)}_{sb2})^{-1}\right)(f - Au^l),$$

and interpreted as a Richardson iterative process with the two-stage SC preconditioner defined by:

$$B^{-1}_{asb2} = (B^{(1)}_{sb2})^{-1} + (B^{(2)}_{sb2})^{-1}.$$

The new preconditioned matrix $B^{-1}_{asb2}A$ can also be used with CG acceleration as follows (see also the SB_2 method above):

- let u^0 be an initial iterate,

$$r^0 \leftarrow f - Au^0, \text{ the initial residual}$$
$$w^0 \leftarrow B^{-1}_{asb2}r^0, \text{ the initial preconditioned residual}$$
$$v^0 \leftarrow w^0, \text{ the initial search direction}$$

- for $l = 0, 1, \cdots$

  compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Av^l)}$

  update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

  update the residual: $r^{l+1} \leftarrow r^l + p_l A v^l$

  if $r^{l+1} \geq$ tolerance, then

  update the preconditioned residual: $w^{l+1} \leftarrow B_{asb2}^{-1} r^{l+1}$

  compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

  update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

  else end for.

The following steps will compute $w^l = B_{asb2}^{-1} r^l$ $(l = 0, 1, \cdots)$:

$$
\begin{aligned}
w^{l+1/2} &\leftarrow (B_{sb2}^{(1)})^{-1} r^l \\
w^l &\leftarrow w^{l+1/2} + (B_{sb2}^{(2)})^{-1} r^l,
\end{aligned}
$$

or equivalently:

$$w^l \leftarrow \left((B_{sb2}^{(1)})^{-1} + (B_{sb2}^{(2)})^{-1}\right) r^l.$$

The resulting ASB$_{2a}$ method is optimal in the sense that the rate of convergence can be bounded independently of the partitioning parameters H and h (see Theorem 3.3.8 and Table 3.4.2).

Moreover, the ASB$_{2a}$ algorithm can be modified such that at the second stage the calculations are reduced to a coarser grid. The two-grid approach combines two ideas that lead to rapid convergence: the smoothing of the high frequency components of the error, and the coarse grid correction of the low energy components. Let $H^s = H$, for all $s = 1, \cdots, n_s$, and let $\Sigma^{2^p h} \subset \cdots \subset \Sigma^{2h} \subset \Sigma^h$ be a set of nested uniform square grids associated with the original domain Ω, such that $1 \leq p$ and $2^p h < H$. Figure 3.2.2 shows the partitioning of the square Ω into disjoint, uniform strips Ω^s at two different stages, with two levels of mesh-refinement. In order to describe the two-grid algorithm, we introduce the restriction R from grid Σ^h to grid $\Sigma^{2^p h}$ and the interpolation (prolongation, extension) $P = R^T$ from grid $\Sigma^{2^p h}$ to grid Σ^h. The prolongation and restriction operators can be defined blockwise, each block corresponding to a strip or to an edge between two strips. For A, the coarse grid reduced operator A_c is defined either by rediscretisation of the problem on the coarse grid $\Sigma^{2^p h}$, or by the relation $A_c = RAP$. Finally, let B^f_{sb2} denote the SB_2 preconditioner when the domain Ω is partitioned into horizontal strips and the grid is fine, and B^c_{sb2} denote the SB_2 preconditioner when the domain Ω is partitioned into vertical strips and the grid is coarse. Then, in the ASB_{2a} procedures above we can replace $(B^{(1)}_{sb2})^{-1}$ by $P(B^c_{sb2})^{-1}R$, and $(B^{(2)}_{sb2})^{-1}$ by $(B^f_{sb2})^{-1}$.

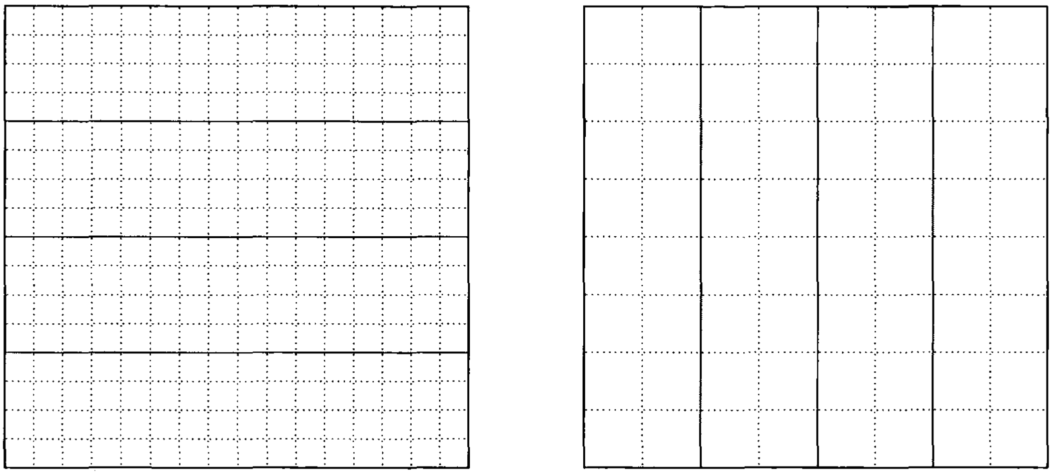

Figure 3.2.2: The horizontal (left) and vertical (right) partitioning into strips of the domain $\Omega \subset \mathbb{R}^2$, with two levels of mesh refinement.

The Two-Grid Alternate Strip-Based (ASB_{2ga}) Algorithm. The new additive two-grid method is: start with u^0 as an initial approximation (without restricting the generality we take this to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1/2} \leftarrow u^l + P(B^c_{sb2})^{-1}R(f - Au^l)$$
$$u^{l+1} \leftarrow u^{l+1/2} + (B^f_{sb2})^{-1}(f - Au^l).$$

This can also be written as:

$$u^{l+1} \leftarrow u^l + \left(P(B^c_{sb2})^{-1}R + (B^f_{sb2})^{-1}\right)(f - Au^l).$$

When this scheme is used to define a preconditioner for the CG method, the inverse of the new two-grid SC preconditioner is:

$$B^{-1}_{asb2g} = P(B^c_{sb2})^{-1}R + (B^f_{sb2})^{-1}.$$

The preconditioned SC matrix is $B^{-1}_{asb2g}A$. The resulting ASB$_{2ga}$ method is also optimal in the sense that the rate of convergence can be bounded independently of the partitioning parameters H and h (see Theorem 3.3.11 and Table 3.4.3).

Remark 3.2.2 We note two different possibilities that arise from the strip-splitting of the domain at the coarse level of the ASB$_{2ga}$ algorithm. First, if the same number of strips is maintained at both stages, then the size of the strip-subproblems at the coarse stage reduces by increasing the size of the coarse mesh. On the other hand, there is the possibility of reducing the number of strips while increasing the mesh size. The latter situation occurs, for instance, when at the coarse stage each strip is of width $2H$ while the mesh size is equal to H. Another case is that when the coarse strip is the whole domain and the size of the coarse mesh is equal to H.

3.3 Spectral Analysis for the SB$_2$ and ASB$_2$ Techniques

In this section, we analyse the behaviour of the strip-based preconditioners introduced in Section 3.2. In view of Theorem 2.1.3, we first collect some technical tools which will be used to prove our main results, then we state and prove the theorems concerning the spectral condition for the relevant operators in the PCG iterations described in Section 3.2. Throughout this section the notation introduced in Section 3.2 is maintained. Also, C and c denote generic positive constants which are independent of the partitioning parameters H and h. The actual values of these constants may not necessarily be the same in any two occurrences. Further notation is explained as it occurs.

We decompose functions $\mathbf{u} \in S_h^0(\Omega)$ as:

$$\mathbf{u} = \mathbf{u}^I + \mathbf{u}^E, \tag{3.3.1}$$

where

$$\mathbf{u}^I \in V^I = \bigoplus_{s=1}^{n_s} S_h^0(\Omega^s)$$

is the solution of the problem:

$$a(\mathbf{u}^I, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in V^I.$$

Note that this is equivalent to solving independently for each Ω^s the following local problem: find $\mathbf{u}_s^I \in S_h^0(\Omega^s)$, such that:

$$a(\mathbf{u}_s^I, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Omega^s).$$

We denote by $\mathbf{u}^E = \mathbf{u} - \mathbf{u}^I$ the part of the solution $\mathbf{u}$ which lies in the orthogonal complement of V^I in $S_h^0(\Omega)$:

$$V^E = \{\mathbf{u} \in S_h^0(\Omega) |\ a(\mathbf{u}, \mathbf{v}) = 0, \ \forall \mathbf{v} \in V^I\}.$$

The function $\mathbf{u}^E \in V^E$ is the piecewise discrete harmonic function in $S_h^0(\Omega)$, and the value of $\mathbf{u}^E$ in Ω is uniquely determined by its value on the global interface $\bigcup_{s=1}^{n_s} \partial\Omega^s \cap \Omega$ between all strips (see also Section 2.2). From the definition of V^E, we deduce:

$$a(\mathbf{u}^E, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}^I, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Omega),$$

or equivalently, when $\mathbf{v}^E \in V^E$ is similarly defined as $\mathbf{u}^E$,

$$a(\mathbf{u}^E, \mathbf{v}^E) = (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}^I, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Omega).$$

Note that:

$$a(\mathbf{u}, \mathbf{u}) = a(\mathbf{u}^I, \mathbf{u}^I) + a(\mathbf{u}^E, \mathbf{u}^E). \tag{3.3.2}$$

Next, we consider the bilinear form $\tilde{a}(\cdot, \cdot)$ on $S_h^0(\Omega) \times S_h^0(\Omega)$ defined by first setting:

$$\tilde{a}^s(\mathbf{u}, \mathbf{v}) = \int_{\Omega^s} \nabla\mathbf{u} \cdot \nabla\mathbf{v} dx, \tag{3.3.3}$$

for every $s = 1, \cdots, n_s$, then writing:

$$\tilde{a}(\mathbf{u}, \mathbf{v}) = \sum_{s=1}^{n_s} \tilde{a}^s(\mathbf{u}, \mathbf{v}). \tag{3.3.4}$$

It can be shown that the bilinear form $\tilde{a}(\cdot, \cdot)$ is equivalent to $a(\cdot, \cdot)$ (see e.g. Quarteroni and Valli (1999) [66], Section 2.4, also discussion in Section 2.2). Thus we can drop the *tilde* from this notation. The bilinear form $a(\mathbf{u}^E, \mathbf{u}^E)$ can now be analysed using the fractional order Sobolev seminorm on the boundaries of all strips:

$$\sum_{s=1}^{n_s} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega^s)},$$

where $|\mathbf{u}|_{H^{1/2}(\partial\Omega^s)}$ denotes the fractional seminorm of $\mathbf{u}|_{\partial\Omega^s} = \mathbf{u}^E|_{\partial\Omega^s}$ given by:

$$|\mathbf{u}|^2_{H^{1/2}(\partial\Omega^s)} = \int_{\partial\Omega^s}\int_{\partial\Omega^s} \frac{(\mathbf{u}(\xi) - \mathbf{u}(\tau))^2}{|\xi - \tau|^2} d\xi d\tau, \tag{3.3.5}$$

where ξ and τ denote arc-length along $\partial\Omega^s$ (see e.g. Xu and Zou (1998) [87], p. 866).

For every edge between two strips $\Gamma^j \subset \partial\Omega^s$, let

$$|\mathbf{u}|^2_{H^{1/2}(\Gamma^j)} = \int_{\Gamma^j}\int_{\Gamma^j} \frac{(\mathbf{u}(\xi) - \mathbf{u}(\tau))^2}{|\xi - \tau|^2} d\xi d\tau,$$

where ξ and τ denote arc-length along Γ^j. The associated space:

$$H^{1/2}(\Gamma^j) = \{\mathbf{u} \in L^2(\Gamma^j)|\ |\mathbf{u}|^2_{H^{1/2}(\Gamma^j)} < \infty\}$$

is equipped with the norm:

$$\|\mathbf{u}\|^2_{H^{1/2}(\Gamma^j)} = \|\mathbf{u}\|^2_{L^2(\Gamma^j)} + |\mathbf{u}|^2_{H^{1/2}(\Gamma^j)}$$

(see e.g. Xu and Zou (1998) [87], p. 868).

On the other hand, let $\mathbf{u}^j \in S_h^0(\Gamma^j)$, and let $\|\mathbf{u}^j\|_{H^{1/2}_{00}(\Gamma^j)}$ be the norm given by (2.2.9), or equivalently, by:

$$\|\mathbf{u}^j\|^2_{H^{1/2}_{00}(\Gamma^j)} = \int_{\Gamma^j}\int_{\Gamma^j} \frac{(\mathbf{u}^j(\xi) - \mathbf{u}^j(\tau))^2}{|\xi - \tau|^2} d\xi d\tau + \int_{\Gamma^j} \frac{(\mathbf{u}^j(\tau))^2}{dist(\tau, \partial\Gamma^j)} d\tau,$$

where $dist(\tau, \partial\Gamma^j)$ represents the distance of τ to the end-points of Γ^j. It can be shown that, when $\mathbf{u}^j$ is a smooth function defined on $\partial\Omega^s$, with support contained in the edge $\Gamma^j \subset \partial\Omega^s$,

$$c|\mathbf{u}^j|^2_{H^{1/2}(\partial\Omega^s)} \leq \|\mathbf{u}^j\|^2_{H^{1/2}_{00}(\Gamma^j)} \leq C|\mathbf{u}^j|^2_{H^{1/2}(\partial\Omega^s)} \tag{3.3.6}$$

(see e.g. Bramble *et al.* (1986) [9], p. 112, or Xu and Zou (1998) [87], p. 868).

Furthermore, if $\mathbf{u}^j \in S_h^0(\Gamma^j)$, then the following equivalence holds:

$$c\|\mathbf{u}^j\|^2_{H_{oo}^{1/2}(\Gamma^j)} \leq (\delta_{\Gamma_j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \leq C\|\mathbf{u}^j\|^2_{H_{oo}^{1/2}(\Gamma^j)} \tag{3.3.7}$$

(see e.g. Bramble *et al.* (1986) [9], p. 113).

Now we can define the bilinear form $b_{sb2}(\cdot,\cdot)$ associated with the preconditioner B_{sb2} as:

$$b_{sb2}(\mathbf{u},\mathbf{v}) = a(\mathbf{u}^I,\mathbf{v}^I) + \sum_j (\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{v}^j)_{\Gamma^j}, \tag{3.3.8}$$

where for every edge between two strips Γ^j, $\mathbf{u}^j$ is equal to $\mathbf{u}^E|_{\Gamma^j}$ on Γ^j, and zero on $\partial\Omega$ and on all the other strip boundaries, and $\mathbf{v}^j$ is similarly defined as $\mathbf{u}^j$.

The process of obtaining the solution $\mathbf{w} \in S_h^0(\Omega)$ of

$$b_{sb2}(\mathbf{w},\mathbf{v}) = (\mathbf{r},\mathbf{v}), \;\; \forall \mathbf{v} \in S_h^0(\Omega),$$

is equivalent to the following procedure.

The SB_2 Procedure (continuous form).

(I) for every strip $\Omega^s \subset \Omega$, solve for $\mathbf{w}_s^I \in S_h^0(\Omega^s)$ the following equation:

$$a(\mathbf{w}_s^I,\mathbf{v}) = (\mathbf{r},\mathbf{v}), \;\; \forall \mathbf{v} \in S_h^0(\Omega^s).$$

This can be done independently and in parallel for all Ω^s.

(II) for every edge between two strips Γ^j, solve for $\mathbf{w}^j \in S_h^0(\Gamma^j)$ the following equation:

$$\alpha_{\Gamma^j}(\delta_{\Gamma^j}^{1/2}\mathbf{w}^j, \mathbf{v}^j)_{\Gamma^j} = (\mathbf{r},\mathbf{v}) - a(\mathbf{w}_s^I,\mathbf{v}), \;\; \forall \mathbf{v} \in S_h^0(\tilde{\Gamma}^j),$$

where $\mathbf{v}^j = \mathbf{v}^E|_{\Gamma^j}$. This can be done independently and in parallel for all Γ^j.

(III) for every strip Ω^s, extend the values of $\mathbf{w}^j$, determined in (II), discrete harmonically into Ω^s. That is solve for $\mathbf{w}_s^E \in S_h^0(\bar{\Omega}^s)$ the following homogeneous equation:

$$a(\mathbf{w}_s^E,\mathbf{v}) = 0, \;\; \forall \mathbf{v} \in S_h^0(\Omega^s),$$

with $\mathbf{w}_s^E$ given by $\mathbf{w}^j$ from (II) on $\Gamma^j \subset \partial\Omega^s$. Then, set the solution $\mathbf{w}_s \in$

$S_h^0(\bar{\Omega}^s)$ as $\mathbf{w}_s = \mathbf{w}_s^I + \mathbf{w}_s^E$. Again, this can be done independently for all Ω^s.

Next, we define the bilinear form $b_{asb2}(\cdot,\cdot)$ associated with the preconditioner B_{asb2} as follows. Let $b_{sb2}^{(1)}(\cdot,\cdot)$ and $b_{sb2}^{(2)}(\cdot,\cdot)$ represent the bilinear form (3.3.8) associated with the preconditioners $B_{sb2}^{(1)}$ and $B_{sb2}^{(2)}$ respectively. We define:

$$b_{asb2}(\mathbf{u},\mathbf{v}) = b_{sb2}^{(1)}(\mathbf{u},\mathbf{v}) + b_{sb2}^{(2)}(\mathbf{u},\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v} \in S_h^0(\Omega). \tag{3.3.9}$$

Analogously, in order to define the bilinear form $b_{asb2g}(\cdot,\cdot)$ associated with the preconditioner B_{asb2g}, let $b_{sb2}^c(\cdot,\cdot)$ and $b_{sb2}^f(\cdot,\cdot)$ denote the bilinear form (3.3.8) associated with the preconditioners B_{sb2}^c and B_{sb2}^f respectively. We define:

$$b_{asb2g}(\mathbf{u},\mathbf{v}) = b_{sb2}^c(\mathbf{u},\mathbf{v}) + b_{sb2}^f(\mathbf{u},\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v} \in S_h^0(\Omega). \tag{3.3.10}$$

Lemma 3.3.1 Let $\Omega = (0,1)\times(0,1)$ be the unit square and let $\Omega^s = (0,1)\times(0,H)$ be a strip in the partitioning (3.2.1) of Ω. For all $\mathbf{u} \in H^1(\Omega^s)$, the following estimates hold:

(i) if $\mathbf{u}$ is equal to zero along one side of size H of Ω^s, then:

$$\|\mathbf{u}\|_{L^2(\Omega^s)}^2 \le C|\mathbf{u}|_{H^1(\Omega^s)}^2.$$

(ii) if $\mathbf{u}$ is equal to zero along one side of size 1 of Ω^s, then:

$$\|\mathbf{u}\|_{L^2(\Omega^s)}^2 \le CH^2|\mathbf{u}|_{H^1(\Omega^s)}^2.$$

(iii) if Γ^j is an edge of size 1 in $\partial\Omega^s$, and $\|\mathbf{u}\|_{L^2(\Gamma^j)}$ represents the $L^2(\Gamma^j)$-norm of $\mathbf{u}|_{\Gamma^j}$, then:

$$\|\mathbf{u}\|_{L^2(\Gamma^j)}^2 \le C\left(\frac{1}{H}\|\mathbf{u}\|_{L^2(\Omega^s)}^2 + H|\mathbf{u}|_{H^1(\Omega^s)}^2\right).$$

(iv) if $\mathbf{u} \in H^1(\Omega)$, then:

$$\|\mathbf{u}\|_{L^2(\Omega^s)}^2 \le CH^2\left(\frac{1}{H}\|\mathbf{u}\|_{L^2(\Omega)}^2 + |\mathbf{u}|_{H^1(\Omega)}^2\right).$$

In each of the estimates (i) – (iv), C denotes a generic positive constant which is independent of the function $\mathbf{u}$ and the partitioning parameter H.

Proof: These estimates can be obtained by direct integration and the Cauchy-Schwarz inequality. See Appendix for details. □

Lemma 3.3.2 Let Ω^s be a generic strip in the (3.2.1) partitioning of Ω and Γ^j denote a generic interface between two strips. If $\tilde{\mathbf{u}} \in S_h^0(\Omega)$ is discrete harmonic in Ω^s, $\mathbf{u} = \tilde{\mathbf{u}}|_{\partial\Omega^s}$, and $a^s(\cdot,\cdot)$ is defined by (3.3.3), then:

$$a^s(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \le C \sum_{\Gamma^j \subset \partial\Omega^s} (\delta_{\Gamma^j}^{1/2} \mathbf{u}, \mathbf{u})_{\Gamma^j}.$$

Proof: See Bramble *et al.* (1986) [9], Lemma 3.2 *(ii)*. □

Lemma 3.3.3 Let $\mathbf{u}$ be a continuous, piecewise quadratic function defined on the finite element mesh Σ^h of the domain Ω. If $I^h\mathbf{u}$ is its piecewise linear interpolant on the same mesh, then:

$$|I^h \mathbf{u}|_{H^1(\Omega^s)} \le C|\mathbf{u}|_{H^1(\Omega^s)},$$

where Ω^s is a generic strip in the (3.2.1) partitioning of Ω. The same type of bounds hold for the L^2, $H^{1/2}$, and $H_{oo}^{1/2}$ norms.

Proof: See Dryja and Widlund (1994) [37], Lemma 4. □

Theorem 3.3.4 For the SB_2 preconditioning technique, the relative condition number $\kappa(B_{sb2}^{-1}A)$ grows linearly as $1/H$, i.e.

$$\kappa(B_{sb2}^{-1}A) = \frac{\lambda_{\max}(B_{sb2}^{-1}A)}{\lambda_{\min}(B_{sb2}^{-1}A)} \le \frac{C}{H}.$$

Proof: Throughout this proof we maintain the notation adopted when defining (3.3.8). In order to show that the relative condition number satisfies $\kappa(B_{sb2}^{-1}A) \le C/H$, by Theorem 2.1.3, it suffices to show that:

$$cHb_{sb2}(\mathbf{u}, \mathbf{u}) \le a(\mathbf{u}, \mathbf{u}) \le Cb_{sb2}(\mathbf{u}, \mathbf{u}), \;\; \forall \mathbf{u} \in S_h^0(\Omega). \quad (3.3.11)$$

Note that, by the definition (3.3.8), of $b_{sb2}(\cdot,\cdot)$, and the representation (3.3.2), of $a(\cdot,\cdot)$, in order to prove (3.3.11), we only need to show that:

$$cH \sum_j (\delta_{\Gamma^j}^{1/2} \mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \le a(\mathbf{u}^E, \mathbf{u}^E) \le C \sum_j (\delta_{\Gamma^j}^{1/2} \mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j}. \quad (3.3.12)$$

Let Ω^s denote a generic strip in the (3.2.1) partitioning of Ω. The right hand-side inequality in (3.3.12) follows through the representation (3.3.4) of $a(\cdot,\cdot)$ and Lemma 3.3.2:

$$\begin{aligned} a(\mathbf{u}^E, \mathbf{u}^E) &= \sum_{s=1}^{n_s} a^s(\mathbf{u}^E, \mathbf{u}^E) \\ &\le C \sum_j (\delta_{\Gamma^j}^{1/2} \mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j}. \end{aligned}$$

Next, we show that the left hand-side inequality in (3.3.12) holds. Let $\mathbf{u} = \mathbf{u}^I + \mathbf{u}^E$ be the (3.3.1) decomposition of $\mathbf{u}$. For $\mathbf{u}^E$, we construct a representation of the form (3.2.3):

$$\mathbf{u}^E = \sum_j \tilde{\mathbf{u}}^j,$$

where $\tilde{\mathbf{u}}^j \in S_h^0(\tilde{\Gamma}^j)$, with $S_h^0(\tilde{\Gamma}^j)$ as in (3.2.2). We derive this representation as follows. Let η^j denote a continuous, piecewise linear function on the finite element nodes of $\partial\Omega^s$, that is zero at the ends of $\Gamma^j \subset \partial\Omega^s$ and everywhere else on $\partial\Omega^s \setminus \Gamma^j$, $0 \le \eta^j \le 1$, and its gradient is of order $\mathcal{O}(1)$. If I^h is the finite element interpolation operator onto the space $S_h^0(\partial\Omega^s)$ and $\mathbf{u}^s = \mathbf{u}^E|_{\partial\Omega^s}$, then we define:

$$\mathbf{u}^j = I^h(\eta^j \mathbf{u}^s).$$

Note that if $\{\eta^j\}$ form a partition of unity, then

$$\mathbf{u}^s = \sum_j \mathbf{u}^j.$$

Then we may choose $\tilde{\mathbf{u}}^j$ as the discrete harmonic extension of $\mathbf{u}^j$ in $\tilde{\Gamma}^j$, extended by zero to the rest of Ω. By Lemma 3.3.3 (for the $H_{oo}^{1/2}$ norm), when $\mathbf{v} = \eta^j \mathbf{u}^s$ (note that this is a continuous, piecewise quadratic function), in order to estimate $\|\mathbf{u}^j\|^2_{H_{oo}^{1/2}(\Gamma^j)}$, it suffices to estimate $\|\mathbf{v}\|^2_{H_{oo}^{1/2}(\Gamma^j)}$. We assume for simplicity that $\Gamma^j = (0,1)$. We divide the interval $[0,1]$ into two intervals of length $1/2$: $[0,1/2]$ and $[1/2,1]$ and take the tensor product $[0,1] \bigotimes [0,1]$. The double integral in the definition of $\|\mathbf{v}\|^2_{H_{oo}^{1/2}(\Gamma^j)}$ is then split into a sum of four double integrals. Due to the symmetry, we only need to consider one of them. We consider the diagonal term corresponding to the set

$[0, 1/2] \times [0, 1/2]$ and use the identity:

$$\begin{aligned}\mathbf{v}(\xi) - \mathbf{v}(\tau) &= 2\xi\mathbf{u}^s(\xi) - 2\tau\mathbf{u}^s(\tau)\\ &= (\xi+\tau)(\mathbf{u}^s(\xi) - \mathbf{u}^s(\tau)) + (\xi-\tau)(\mathbf{u}^s(\xi) + \mathbf{u}^s(\tau)).\end{aligned}$$

By the Cauchy-Schwarz inequality applied twice, we obtain:

$$\begin{aligned}\int_0^{1/2}\int_0^{1/2} \frac{|\mathbf{v}(\xi) - \mathbf{v}(\tau)|^2}{|\xi-\tau|^2} d\xi d\tau \leq& C\int_0^{1/2}\int_0^{1/2} \frac{(\xi+\tau)^2|\mathbf{u}^s(\xi) - \mathbf{u}^s(\tau)|^2}{|\xi-\tau|^2} d\xi d\tau\\ &+ C\int_0^{1/2}\int_0^{1/2} |\mathbf{u}^s(\xi) + \mathbf{u}^s(\tau)|^2 d\xi d\tau\\ \leq& C\int_0^{1/2}\int_0^{1/2} \frac{(\xi+\tau)^2|\mathbf{u}^s(\xi) - \mathbf{u}^s(\tau)|^2}{|\xi-\tau|^2} d\xi d\tau\\ &+ C\int_0^{1/2}\int_0^{1/2} |\mathbf{u}^s(\xi)|^2 d\xi d\tau\\ \leq& C\int_0^{1/2}\int_0^{1/2} \frac{|\mathbf{u}^s(\xi) - \mathbf{u}^s(\tau)|^2}{|\xi-\tau|^2} d\xi d\tau\\ &+ C\int_0^{1/2} (\mathbf{u}^s(\xi))^2 d\xi.\end{aligned}$$

Therefore:

$$\int_{\Gamma^j}\int_{\Gamma^j} \frac{|\mathbf{v}(\xi) - \mathbf{v}(\tau)|^2}{|\xi-\tau|^2} d\xi d\tau \leq C\|\mathbf{u}^s\|^2_{L^2(\Gamma^j)} + C|\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)}.$$

Now, we consider the single integral in the definition of $\|\mathbf{v}\|^2_{H^{1/2}_{oo}(\Gamma^j)}$:

$$\int_{\Gamma^j} \frac{(\mathbf{v}(\tau))^2}{dist(\tau, \partial\Gamma^j)} d\tau = \int_0^1 \frac{(\mathbf{v}(\tau))^2}{\tau} d\tau + \int_0^1 \frac{(\mathbf{v}(\tau))^2}{1-\tau} d\tau.$$

In this case, due to the symmetry, we only need to consider the following four single integrals:

$$\begin{aligned}\int_0^{1/2} \frac{(\mathbf{v}(\tau))^2}{\tau} d\tau &= \int_0^{1/2} \frac{4\tau^2(\mathbf{u}^s(\tau))^2}{\tau} d\tau = \int_0^{1/2} 4\tau(\mathbf{u}^s(\tau))^2 d\tau \leq C\int_0^{1/2} (\mathbf{u}^s(\tau))^2 d\tau,\\ \int_{1/2}^1 \frac{(\mathbf{v}(\tau))^2}{\tau} d\tau &= \int_{1/2}^1 \frac{4(1-\tau)^2(\mathbf{u}^s(\tau))^2}{\tau} d\tau \leq C\int_{1/2}^1 \frac{(\mathbf{u}^s(\tau))^2}{\tau} d\tau \leq C\int_{1/2}^1 (\mathbf{u}^s(\tau))^2 d\tau,\\ \int_0^{1/2} \frac{(\mathbf{v}(\tau))^2}{1-\tau} d\tau &= \int_0^{1/2} \frac{4\tau^2(\mathbf{u}^s(\tau))^2}{1-\tau} d\tau \leq C\int_0^{1/2} \frac{(\mathbf{u}^s(\tau))^2}{1-\tau} d\tau \leq C\int_0^{1/2} (\mathbf{u}^s(\tau))^2 d\tau,\\ \int_{1/2}^1 \frac{(\mathbf{v}(\tau))^2}{1-\tau} d\tau &= \int_{1/2}^1 \frac{4(1-\tau)^2(\mathbf{u}^s(\tau))^2}{1-\tau} d\tau \leq C\int_{1/2}^1 \frac{(\mathbf{u}^s(\tau))^2}{1-\tau} d\tau \leq C\int_{1/2}^1 (\mathbf{u}^s(\tau))^2 d\tau.\end{aligned}$$

Therefore:

$$\int_{\Gamma^j} \frac{(\mathbf{v}(\tau))^2}{dist(\tau,\partial\Gamma^j)} d\tau \le C\|\mathbf{u}^s\|^2_{L^2(\Gamma^j)}.$$

By the above evaluations, we obtain:

$$\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \le C\|\mathbf{u}^s\|^2_{L^2(\Gamma^j)} + C|\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)}.$$

From the above estimate, Lemma 3.3.1 (iii) and (i), and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{\Gamma^j\subset\partial\Omega^s} \|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} &\le C\sum_{\Gamma^j\subset\partial\Omega^s}\|\mathbf{u}^s\|^2_{L^2(\Gamma^j)} + C\sum_{\Gamma^j\subset\partial\Omega^s}|\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)}\\ &\le C\left(\frac{1}{H}\|\mathbf{u}^E\|^2_{L^2(\Omega^s)} + H|\mathbf{u}^E|^2_{H^1(\Omega^s)}\right) + C|\mathbf{u}^E|^2_{H^1(\Omega^s)}\\ &\le C\left(\frac{1}{H}+1\right)|\mathbf{u}^E|^2_{H^1(\Omega^s)}.\end{aligned}$$

Thus, through the equivalence (3.3.7) and the definition (3.3.3) of $a^s(\cdot,\cdot)$,

$$\sum_{\Gamma^j\subset\partial\Omega^s}(\delta^{1/2}_{\Gamma^j}\mathbf{u}^j,\mathbf{u}^j)_{\Gamma^j} \le C\left(\frac{1}{H}+1\right)a^s(\mathbf{u}^E,\mathbf{u}^E).$$

Since every Γ^j is shared by only two strips Ω^s, after summing over all $\Omega^s\subset\Omega$, through the representation (3.3.4) of $a(\cdot,\cdot)$, we obtain:

$$\sum_j(\delta^{1/2}_{\Gamma^j}\mathbf{u}^j,\mathbf{u}^j)_{\Gamma^j} \le C\left(\frac{1}{H}+1\right)a(\mathbf{u}^E,\mathbf{u}^E),$$

which is equivalent to the left hand-side inequality in (3.3.12).

Therefore, by Theorem 2.1.3 for the preconditioner B_{sb2}, $1/\lambda_{\min}(B^{-1}_{sb2}A)$ grows linearly as $1/H$. Since $1/\lambda_{\max}(B^{-1}_{sb2}A) \le C$, we conclude that:

$$\kappa(B^{-1}_{sb2}A) \le C/H. \;\Box$$

Remark 3.3.5 We observe that in the proof of Theorem 3.3.4, when Ω^s is a strip with only one edge in the interior of the domain Ω and the remaining edges on the boundary $\partial\Omega$, we can apply Lemma 3.3.1 (ii) instead of (i). Thus, for this strip:

$$\sum_{\Gamma^j\subset\partial\Omega^s}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \le C|\mathbf{u}^E|^2_{H^1(\Omega^s)}.$$

Lemma 3.3.6 Let $Q_{2^p h} : L^2(\Omega) \to S_{2^p H}(\Omega)$ be the L^2-projection associated with $S_{2^p h}(\Omega)$ ($h \leq 2^p h \leq H$, $p \in \mathbb{N}$). Then, for all $\mathbf{u} \in H^1(\Omega)$, the following estimates hold:

$$\|\mathbf{u} - Q_{2^p h}\mathbf{u}\|_{L^2(\Omega)} \leq C 2^p h |\mathbf{u}|_{H^1(\Omega)} \qquad \text{and} \qquad |Q_{2^p h}\mathbf{u}|_{H^1(\Omega)} \leq C|\mathbf{u}|_{H^1(\Omega)}.$$

Proof: See e.g. Bramble and Xu (1991) [12], Section 3. For related results we also refer to Xu (1989) [84] and (1991) [85]. □

Lemma 3.3.7 Let $Q_{2^p h} : L^2(\Omega) \to S_{2^p H}(\Omega)$ be the L^2-projection associated with $S_{2^p h}(\Omega)$ ($h \leq 2^p h < H$, $p \in \mathbb{N}$). Then, for all $\mathbf{u} \in S_h^0(\Omega)$,

$$b_{asb2}(Q_{2^p h}\mathbf{u}, Q_{2^p h}\mathbf{u}) \leq C a(\mathbf{u}, \mathbf{u}).$$

Proof: The proof of this result is based on the observation that the ASB$_{2a}$ preconditioner is obtained in two stages such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage. Throughout this proof we maintain the notation adopted when defining (3.3.8).

First, we show that for any $\mathbf{u}_o \in S_h^0(\Omega)$,

$$b_{asb2}(\mathbf{u}_o, \mathbf{u}_o) \leq C a(\mathbf{u}_o, \mathbf{u}_o). \tag{3.3.13}$$

Then, by replacing h by $2^p h$ and taking $\mathbf{u}_o = Q_{2^p h}\mathbf{u}$ in the above estimate, the lemma follows, through the definition of $a(\cdot,\cdot)$ and the second estimate in Lemma 3.3.6.

Let $\mathbf{u}_1 = \mathbf{u}_1^I + \mathbf{u}_1^E$ be the (3.3.1) decomposition of $\mathbf{u}_o$ at the first stage, and $\mathbf{u}_2 = \mathbf{u}_2^I + \mathbf{u}_2^E$ be the (3.3.1) decomposition of $\mathbf{u}_o$ at the second stage. By the definition (3.3.9) of $b_{asb2}(\cdot,\cdot)$, $\mathbf{u}_1^I$ and $\mathbf{u}_2^I$ are solved exactly. It remains to show that:

$$b_{sb2}^{(1)}(\mathbf{u}_1^E, \mathbf{u}_1^E) + b_{sb2}^{(2)}(\mathbf{u}_2^E, \mathbf{u}_2^E) \leq C a(\mathbf{u}_o, \mathbf{u}_o). \tag{3.3.14}$$

The proof of this estimate involves some auxiliary results from Chapter 4 (see also Remark 4.3.7). Let Γ be the union of the interfaces between strips at the first stage and the interfaces between strips at the second stage. Then Γ can be regarded as consisting of overlapping vertex-regions, such that each region is cross-shaped, is centered at a vertex-point, and contains parts of the interfaces between strips that are within a distance H from that vertex. Thus at most two such regions overlap and the overlap is uniform of order $\mathcal{O}(H)$. Let Γ^v denote a generic vertex-region as described above, restricted to the boundary $\partial\Omega_i^s$ of a generic subdomain Ω_i^s representing the intersection of a strip at the first stage with a strip at the second stage. Let Γ^{j_1} and Γ^{j_2} denote a generic edge between two strips at the first and at the second stage respectively, such that $\Gamma^v \subset \Gamma^{j_1} \cup \Gamma^{j_2}$. We introduce the following notation: $\tilde{\Gamma}^v = \tilde{\Gamma}^{j_1} \cap \tilde{\Gamma}^{j_2}$ and $S_h^0(\tilde{\Gamma}^v) = S_h^0(\tilde{\Gamma}^{j_1}) \cap S_h^0(\tilde{\Gamma}^{j_2})$, with $S_h^0(\tilde{\Gamma}^{j_1})$ and $S_h^0(\tilde{\Gamma}^{j_2})$ as in (3.2.2). If $\tilde{\mathbf{u}}_o$ represents the discrete harmonic extension in the sense described in Section 2.2.1 of the restriction $\mathbf{u}_o|_\Gamma$ into Ω, then we derive a representation:

$$\tilde{\mathbf{u}}_o = \sum_v \tilde{\mathbf{u}}^v,$$

where $\tilde{\mathbf{u}}^v \in S_h^0(\tilde{\Gamma}^v)$. We construct this representation as follows. Let η^v be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes of the boundary $\partial\tilde{\Gamma}^v$ and everywhere else on $\Omega \setminus \tilde{\Gamma}^v$, $0 \leq \eta^v \leq 1$, and its gradient is of order $\mathcal{O}(1/H)$. If I^h is the finite element interpolation operator onto the space $S_h^0(\Omega)$, then we define:

$$\bar{\mathbf{u}}^v = I^h(\eta^v \tilde{\mathbf{u}}_o).$$

Note that if $\{\eta^v\}$ form a partition of unity, then:

$$\tilde{\mathbf{u}}_o = \sum_v \bar{\mathbf{u}}^v.$$

We proceed to bound the energies of the parts of $\bar{\mathbf{u}}^v$ associated with the elements in Σ^h. If $\bar{\eta}^v$ is the average of η^v on a single mesh-element σ_h, then:

$$\|\eta^v - \bar{\eta}^v\|^2_{L^\infty(\sigma_h)} \leq C(h/H)^2.$$

By the Cauchy-Schwarz inequality, we can write:

$$|\bar{\mathbf{u}}^v|^2_{H^1(\sigma_h)} \leq 2|\bar{\eta}^v \tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} + 2|I^h(\bar{\eta}^v - \eta^v)\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)}. \tag{3.3.15}$$

Since $\|\bar{\eta}^v\|_{L^\infty(\sigma_h)} \leq 1$, the first term on the right hand-side of the inequality in

(3.3.15) can be bounded as:

$$2|\bar{\eta}^v \tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)}. \qquad (3.3.16)$$

For the second term on the right hand-side of the inequality in (3.3.15), from an the inverse estimate, the weighted norm on element σ_h of diameter h, and the bound on the gradient of η^v, we obtain:

$$2|I^h(\bar{\eta}^v - \eta^v)\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} \leq C\frac{1}{h^2}\|I^h(\bar{\eta}^v - \eta^v)\tilde{\mathbf{u}}_o\|^2_{L^2(\sigma_h)} \leq C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\sigma_h)}. \qquad (3.3.17)$$

Since each σ_h is associated with only four $\bar{\mathbf{u}}^v$, from (3.3.15), (3.3.16), and (3.3.17), we deduce:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} |\bar{\mathbf{u}}^v|^2_{H^1(\sigma_h)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\sigma_h)}.$$

After summing with respect to σ_h, we obtain:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} |\bar{\mathbf{u}}^v|^2_{H^1(\Omega_i^s)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^s)}.$$

We choose $\tilde{\mathbf{u}}^v$ to be the discrete harmonic extension of $\mathbf{u}^v = \bar{\mathbf{u}}^v|_{\Gamma^v}$ in $\tilde{\Gamma}^v$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5, of discrete harmonic functions, imply:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} |\tilde{\mathbf{u}}^v|^2_{H^1(\Omega_i^s)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^s)}.$$

Thus, by the left hand-side inequality in Theorem 2.2.3 and Lemma 4.3.1 (i),

$$\begin{aligned}\sum_{\Gamma^v \subset \partial\Omega_i^s} |\mathbf{u}^v|^2_{H^{1/2}(\partial\Omega_i^s)} &\leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^s)} \\ &\leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)}.\end{aligned}$$

From this estimate, by the definition (2.2.9), we deduce:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} \|\mathbf{u}^v\|^2_{H^{1/2}_{oo}(\Gamma^v)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)}.$$

Since each Γ^v is shared by only four subdomains Ω_i^s, after summing over all $\Omega_i^s \subset \Omega$, by the minimisation property of discrete harmonic functions, we obtain:

$$\sum_v \|\mathbf{u}^v\|^2_{H^{1/2}_{oo}(\Gamma^v)} \leq Ca(\tilde{\mathbf{u}}_o, \tilde{\mathbf{u}}_o) \leq Ca(\mathbf{u}_o, \mathbf{u}_o),$$

from which (3.3.14) follows. □

Theorem 3.3.8 For the ASB$_{2a}$ preconditioning technique, the relative condition number $\kappa(B^{-1}_{asb2}A)$ is bounded independently of the partitioning parameters H and h, i.e.

$$\kappa(B^{-1}_{asb2}A) = \frac{\lambda_{\max}(B^{-1}_{asb2}A)}{\lambda_{\min}(B^{-1}_{asb2}A)} \leq C.$$

Proof: This proof is based on the observation that the ASB$_{2a}$ preconditioner is of overlapping Schwarz type. In order to bound the condition number $\kappa(B^{-1}_{asb2}A)$, we need to find upper and lower bounds for the spectrum of $B^{-1}_{asb2}A$. To this end, we use Theorem 2.1.3 for the preconditioner B_{asb2}. Throughout this proof we maintain the notation adopted when defining (3.3.8) and (3.3.9).

First, we derive an upper bound for $\lambda_{\max}(B^{-1}_{asb2}A)$ as follows. Let $\mathbf{u} \in S^0_h(\Omega)$, and let $b^{(1)}_{sb2}(\cdot,\cdot)$ and $b^{(2)}_{sb2}(\cdot,\cdot)$ represent the bilinear form (3.3.8) associated with the preconditioners $B^{(1)}_{sb2}$ and $B^{(2)}_{sb2}$ respectively. By the Cauchy-Schwarz inequality and Theorem 3.3.4,

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) &\leq C\left(a(\mathbf{u},\mathbf{u}) + a(\mathbf{u},\mathbf{u})\right) \\ &\leq C\left(b^{(1)}_{sb2}(\mathbf{u},\mathbf{u}) + b^{(2)}_{sb2}(\mathbf{u},\mathbf{u})\right). \end{aligned}$$

From this estimate, the definition (3.3.9) of $b_{asb2}(\cdot,\cdot)$, and Theorem 2.1.3 for the preconditioner B_{asb2}, it follows that:

$$\lambda_{\max}(B^{-1}_{asb2}A) \leq C. \tag{3.3.18}$$

Next step of our proof is to determine a lower bound for $\lambda_{\min}(B^{-1}_{asb2}A)$. Let $\mathbf{u}_1 = \mathbf{u}^I_1 + \mathbf{u}^E_1$ be the (3.3.1) decomposition of $\mathbf{u}$ at the first stage, and $\mathbf{u}_2 = \mathbf{u}^I_2 + \mathbf{u}^E_2$ be the (3.3.1) decomposition of $\mathbf{u}$ at the second stage. By the definition (3.3.9) of $b_{asb2}(\cdot,\cdot)$, $\mathbf{u}^I_1$ and $\mathbf{u}^I_2$ are solved exactly. It remains to show that:

$$b^{(1)}_{sb2}(\mathbf{u}^E_1,\mathbf{u}^E_1) + b^{(2)}_{sb2}(\mathbf{u}^E_2,\mathbf{u}^E_2) \leq Ca(\mathbf{u},\mathbf{u}).$$

As in the proof of Lemma 3.3.7 above, we denote by Γ the union of the interfaces between strips at the first stage and the interfaces between strips at the second stage, and by Ω_i^s, a generic subdomain representing the intersection of a strip at the first stage with a strip at the second stage. Let $\tilde{\mathbf{u}}$ represent the discrete harmonic extension in the sense described in Section 2.2.1 of the restriction $\mathbf{u}|_\Gamma$ into Ω, and let $\tilde{\mathbf{u}}_o = Q_{H/2}\tilde{\mathbf{u}}$ be the L^2-projection of $\tilde{\mathbf{u}}$ onto $S^0_{H/2}(\Omega)$, and $\mathbf{u}_o = \tilde{\mathbf{u}}_o|_\Gamma$. Then, by the Cauchy-Schwarz inequality and Lemma 3.3.7,

$$\begin{aligned} b_{asb2}(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) &= b_{asb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o + \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o + \tilde{\mathbf{u}}_o) \\ &\leq C b_{asb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) + C b_{asb2}(\tilde{\mathbf{u}}_o, \tilde{\mathbf{u}}_o) \\ &\leq C b_{asb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) + C a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}). \end{aligned}$$

Since, by the minimisation property (2.2.5), of discrete harmonic functions,

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \leq C a(\mathbf{u}, \mathbf{u}),$$

it remains to show that:

$$b_{sb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq C a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}). \tag{3.3.19}$$

We demonstrate that:

$$b^{(1)}_{sb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq C a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \tag{3.3.20}$$

and

$$b^{(2)}_{sb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq C a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}). \tag{3.3.21}$$

Let $\mathbf{w} = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o$. At the first stage, for $\mathbf{w}$, we construct a representation of the form (3.2.3):

$$\mathbf{w} = \sum_j \tilde{\mathbf{u}}^j,$$

as follows. Let η^j be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes of the boundary $\partial\tilde{\Gamma}^j$ (hence at the ends of Γ^j as well) and everywhere else on $\Omega\backslash\tilde{\Gamma}^j$, $0 \leq \eta^j \leq 1$, and $\|\nabla\eta^j\|_{L^\infty(\tilde{\Gamma}^j)} \leq C/H$. If I^h is the finite element interpolation operator onto the space $S^0_h(\Omega)$, then we define:

$$\bar{\mathbf{u}}^j = I^h(\eta^j \mathbf{w}).$$

Note that if $\{\eta^j\}$ form a partition of unity, then

$$\mathbf{w} = \sum_j \bar{\mathbf{u}}^j.$$

We proceed to bound the energies of the parts of $\bar{\mathbf{u}}^j$ associated with the elements in Σ^h. Since the gradient of η^j is of order $\mathcal{O}(1/H)$, if $\bar{\eta}^j$ is the average of η^j on a single mesh-element σ_h, then:

$$\|\eta^j - \bar{\eta}^j\|^2_{L^\infty(\sigma_h)} \leq C(h/H)^2.$$

By the Cauchy-Schwarz inequality, we can write:

$$\begin{aligned} |\bar{\mathbf{u}}^j|^2_{H^1(\sigma_h)} &= |I^h(\eta^j \mathbf{w})|^2_{H^1(\sigma_h)} \\ &\leq 2|\bar{\eta}^j \mathbf{w}|^2_{H^1(\sigma_h)} + 2|I^h(\bar{\eta}^j - \eta^j)\mathbf{w}|^2_{H^1(\sigma_h)}. \end{aligned} \tag{3.3.22}$$

Since $\|\bar{\eta}^j\|_{L^\infty(\sigma_h)} \leq 1$, the first term on the right hand-side of the inequality in (3.3.22) can be bounded as:

$$2|\bar{\eta}^j \mathbf{w}|^2_{H^1(\sigma_h)} \leq C|\mathbf{w}|^2_{H^1(\sigma_h)}. \tag{3.3.23}$$

For the second term on the right hand-side of the inequality in (3.3.22), the inverse estimate in Lemma (2.1.2), the weighted norm on element σ_h of diameter h, and the bound on the gradient of η^j imply:

$$\begin{aligned} 2|I^h(\bar{\eta}^j - \eta^j)\mathbf{w}|^2_{H^1(\sigma_h)} &\leq C\frac{1}{h^2}\|I^h(\bar{\eta}^j - \eta^j)\mathbf{w}\|^2_{L^2(\sigma_h)} \\ &\leq C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\sigma_h)}. \end{aligned} \tag{3.3.24}$$

Since each σ_h is associated with only two $\bar{\mathbf{u}}^j$, from (3.3.22), (3.3.23) and (3.3.24), we deduce:

$$\sum_{\Gamma^j \subset \partial\Omega^s} |\bar{\mathbf{u}}^j|^2_{H^1(\sigma_h)} \leq C|\mathbf{w}|^2_{H^1(\sigma_h)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\sigma_h)}.$$

When we sum over all $\sigma_h \subset \Omega^s$, we obtain:

$$\sum_{\Gamma^j \subset \partial\Omega^s} |\bar{\mathbf{u}}^j|^2_{H^1(\Omega^s)} \leq C|\mathbf{w}|^2_{H^1(\Omega^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s)}.$$

We choose $\tilde{\mathbf{u}}^j$ as the discrete harmonic extension of $\mathbf{u}^j = \bar{\mathbf{u}}^j|_{\Gamma^j}$ into $\tilde{\Gamma}^j$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5), of discrete harmonic functions, imply:

$$\sum_{\Gamma^j \subset \partial\Omega^s} |\tilde{\mathbf{u}}^j|^2_{H^1(\Omega^s)} \leq C|\mathbf{w}|^2_{H^1(\Omega^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s)}.$$

From this estimate, through the equivalence (3.3.6) and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{\Gamma^j \subset \partial\Omega^s} \|\mathbf{u}^j\|^2_{H^{1/2}_{00}(\Gamma^j)} &\leq C \sum_{\Gamma^j \subset \partial\Omega^s} |\mathbf{u}^j|^2_{H^{1/2}(\partial\Omega^s)} \\ &\leq C|\mathbf{w}|^2_{H^1(\Omega^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s)}.\end{aligned}$$

Since every Γ^j is shared by only two strips Ω^s, after summing over all $\Omega^s \subset \Omega$, Lemma 3.3.6 implies:

$$\begin{aligned}\sum_j \|\mathbf{u}^j\|^2_{H^{1/2}_{00}(\Gamma^j)} &\leq C|\mathbf{w}|^2_{H^1(\Omega)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} \\ &\leq C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned} \tag{3.3.25}$$

Then, through the equivalence (3.3.7), the Cauchy-Schwarz inequality for the decomposition of Ω into nonoverlapping strips Ω^s, and the definition (3.3.3) of $a^s(\cdot,\cdot)$, we obtain:

$$\begin{aligned}\sum_j (\delta^{1/2}_{\Gamma^j}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} &\leq C \sum_{\Omega^s \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega^s)} \\ &\leq C \sum_{\Omega^s \subset \Omega} Ca^s(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}).\end{aligned}$$

Thus, by the representation (3.3.4) of $a(\cdot,\cdot)$,

$$\sum_j (\delta^{1/2}_{\Gamma^j}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \leq Ca(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}),$$

from which (3.3.20) follows. Analogously, at the second stage, we obtain (3.3.21).

Next, through the definition (3.3.9) of $b_{asb2}(\cdot,\cdot)$, (3.3.20) and (3.3.21) imply (3.3.19).

Therefore, by Theorem 2.1.3 for the preconditioner B_{asb2}, $\lambda_{\min}(B_{asb2}^{-1}A)$ is bounded independently of the partitioning parameters H and h. Since (3.3.18) also holds, we conclude that:

$$\kappa(B_{asb2}^{-1}A) \leq C. \ \square$$

Remark 3.3.9 We note that the arguments used to prove Theorem 3.3.8 cannot be applied to prove the growth of order $\mathcal{O}(1/H)$ for $\kappa(B_{sb2}^{-1}A)$ in Theorem 3.3.4. This is because, for Theorem 3.3.4, in the proof of Theorem 3.3.8 $\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o$ must be replaced by $\tilde{\mathbf{u}}$. Therefore, through the Poincaré - Friedrichs inequality, (3.3.25) becomes:

$$\begin{aligned}\sum_j \|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} &\leq C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}\|^2_{L^2(\Omega)} \\ &\leq C\left(1+\frac{1}{H^2}\right)|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned}$$

This leads to an upper bound of order $\mathcal{O}(1/H^2)$ for $\kappa(B_{sb2}^{-1}A)$.

Remark 3.3.10 We mention here that (3.3.19) also follows from Lemma 3.3.6 and (3.3.13), as:

$$b_{asb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq Ca(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq Ca(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}).$$

However, we chose to show that (3.3.20) and (3.3.21) also hold.

Theorem 3.3.11 For the ASB_{2ga} preconditioning technique, the relative condition number $\kappa(B_{asb2g}^{-1}A)$ is bounded independently of the partitioning parameters H and h, i.e.

$$\kappa(B_{asb2g}^{-1}A) = \frac{\lambda_{\max}(B_{asb2g}^{-1}A)}{\lambda_{\min}(B_{asb2g}^{-1}A)} \leq C.$$

Proof: We can prove this result in a similar manner as Theorem 3.3.8 above, by simply replacing the functions at first stage by those at the coarse level, and the functions at the second stage by those at the fine level. However, we present here a different approach for bounding the minimum eigenvalue, which is based on the observation that the ASB_{2ga} preconditioner is of a two-level type. This argument is also valid for Theorem 3.3.8, with the corresponding change of notation. Moreover, in view of Remark 3.3.9, this approach can be regarded as an extension of the argument used to demonstrate Theorem 3.3.4. Throughout this proof we maintain the notation adopted when defining (3.3.8) and (3.3.10).

First, we derive an upper bound for $\lambda_{\max}(B_{asb2g}^{-1}A)$ as follows. Let $\mathbf{u} \in S_h^0(\Omega)$, and let $b_{sb2}^c(\cdot,\cdot)$ and $b_{sb2}^f(\cdot,\cdot)$ represent the bilinear form (3.3.8) associated with the preconditioners B_{sb2}^c and B_{sb2}^f respectively. By the Cauchy-Schwarz inequality and Theorem 3.3.4 ,

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) &\leq C\,(a(\mathbf{u},\mathbf{u}) + a(\mathbf{u},\mathbf{u})) \\ &\leq C\left(b_{sb2}^c(\mathbf{u},\mathbf{u}) + b_{sb2}^f(\mathbf{u},\mathbf{u})\right). \end{aligned}$$

Thus, through the definition (3.3.10) of $b_{asb2g}(\cdot,\cdot)$, and Theorem 2.1.3 for the preconditioner B_{asb2g},

$$\lambda_{\max}(B_{asb2g}^{-1}A) \leq C. \tag{3.3.26}$$

Next, we find a lower bound for $\lambda_{\min}(B_{asb2g}^{-1}A)$. Let $\mathbf{u}_f = \mathbf{u}_f^I + \mathbf{u}_f^E$ be the (3.3.1) decomposition of $\mathbf{u}$ at the fine stage, and $\mathbf{u}_c = \mathbf{u}_c^I + \mathbf{u}_c^E$ be the (3.3.1) decomposition of $\mathbf{u}$ at the coarse stage. By the definition (3.3.10) of $b_{asb2g}(\cdot,\cdot)$, $\mathbf{u}_f^I$ and $\mathbf{u}_c^I$ are solved exactly. It remains to show that:

$$b_{sb2}^f(\mathbf{u}_f^E,\mathbf{u}_f^E) + b_{sb2}^c(\mathbf{u}_c^E,\mathbf{u}_c^E) \leq C a(\mathbf{u},\mathbf{u}).$$

As in the proofs of Lemma 3.3.7 and Theorem 3.3.8 above, let Γ be the union of the interfaces between strips at the first stage and the interfaces between strips at the second stage, and Ω_i^s denote a generic subdomain representing the intersection of a strip at the first stage with a strip at the second stage. Let $\tilde{\mathbf{u}}$ represent the discrete harmonic extension in the sense described in Section 2.2.1 of the restriction $\mathbf{u}|_\Gamma$ into Ω, and let $\tilde{\mathbf{u}}_o = Q_{H/2}\tilde{\mathbf{u}}$ be the L^2-projection of $\tilde{\mathbf{u}}$ onto $S_{H/2}^0(\Omega)$, and $\mathbf{u}_o = \tilde{\mathbf{u}}_o|_\Gamma$. Then, by the Cauchy-Schwarz inequality and Lemma 3.3.7,

$$\begin{aligned} b_{asb2g}(\tilde{\mathbf{u}},\tilde{\mathbf{u}}) &= b_{asb2g}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o + \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o + \tilde{\mathbf{u}}_o) \\ &\leq C b_{asb2g}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) + C b_{asb2g}(\tilde{\mathbf{u}}_o, \tilde{\mathbf{u}}_o) \\ &= C b_{asb2g}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) + C b_{asb2}(\tilde{\mathbf{u}}_o, \tilde{\mathbf{u}}_o) \\ &\leq C b_{asb2g}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) + C a(\tilde{\mathbf{u}},\tilde{\mathbf{u}}). \end{aligned}$$

Since, by the minimisation property (2.2.5), of discrete harmonic functions,

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \leq Ca(\mathbf{u}, \mathbf{u}),$$

it remains to show that:

$$b_{asb2g}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq Ca(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}). \tag{3.3.27}$$

We demonstrate that:

$$b^f_{sb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq Ca(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \tag{3.3.28}$$

and

$$b^c_{sb2}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o) \leq Ca(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}). \tag{3.3.29}$$

Let $\mathbf{w} = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o$. At the fine stage, we proceed as in the proof of Theorem 3.3.4, where we take $\mathbf{u}^s = \mathbf{w}|_{\partial\Omega^s}$. Then we obtain:

$$||\mathbf{u}^j||^2_{H^{1/2}_{oo}(\Gamma^j)} \leq C||\mathbf{u}^s||^2_{L^2(\Gamma^j)} + C|\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)}.$$

From the above estimate, Lemma 3.3.1 (*iii*), and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{\Gamma^j \subset \partial\Omega^s} ||\mathbf{u}^j||^2_{H^{1/2}_{oo}(\Gamma^j)} &\leq C \sum_{\Gamma^j \subset \partial\Omega^s} ||\mathbf{u}^s||^2_{L^2(\Gamma^j)} + C \sum_{\Gamma^j \subset \partial\Omega^s} |\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)} \\ &\leq C\frac{1}{H}||\mathbf{w}||^2_{L^2(\Omega^s)} + C|\mathbf{w}|^2_{H^1(\Omega^s)}.\end{aligned}$$

Since every Γ^j is shared by only two strips Ω^s, after summing over all $\Omega^s \subset \Omega$, Lemma 3.3.6 implies:

$$\begin{aligned}\sum_j ||\mathbf{u}^j||^2_{H^{1/2}_{oo}(\Gamma^j)} &\leq C\frac{1}{H}||\mathbf{w}||^2_{L^2(\Omega)} + C|\mathbf{w}|^2_{H^1(\Omega)} \\ &\leq C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned}$$

Then, through the equivalence (3.3.7), the Cauchy-Schwarz inequality for the decomposition of Ω into nonoverlapping strips Ω^s, and the definition (3.3.3) of $a^s(\cdot,\cdot)$, we obtain:

$$\sum_j (\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \le C \sum_{\Omega^s \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega^s)}$$
$$\le C \sum_{\Omega^s \subset \Omega} a^s(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}).$$

Thus, by the representation (3.3.4) of $a(\cdot,\cdot)$,

$$\sum_j (\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \le Ca(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}),$$

from which (3.3.28) follows. Analogously, at the coarse stage, we obtain (3.3.29). Next, through the definition (3.3.10) of $b_{asb2g}(\cdot,\cdot)$, (3.3.28) and (3.3.29) imply (3.3.27).

Therefore, by Theorem 2.1.3 for the preconditioner B_{asb2g}, $\lambda_{\min}(B_{asb2g}^{-1}A)$ is bounded independently of the partitioning parameters H and h. Since (3.3.26) also holds, we conclude that:

$$\kappa(B_{asb2g}^{-1}A) \le C. \ \square$$

3.4 Numerical Estimates

The purpose of this section is to illustrate the efficiency of the ASB_2 preconditioners when solving equations of the form (2.1.1) by the PCG method.

Example 3.4.1 We solve the Poisson equation:

$$\begin{cases} -\Delta\mathbf{u}(x) = \mathbf{f}(x) \text{ in } \Omega = (0,1)\times(0,1) \\ \mathbf{u}(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

In the computations, at each stage the unit square Ω is partitioned into $n_s = 1/H$ equal strips. The mesh size is h for the fine grid, and $H/2$ for the coarse grid. The iteration counts are for a reduction in error of 10^{-4}.

Discussion: Table 3.4.1 indicates that for the SB_2 preconditioning technique, the relative condition number grows linearly as the number of strips, $1/H$, and for a fixed number of strips it remains bounded independently of the mesh parameter h (see Theorem 3.3.4). Table 3.4.2 illustrates the theoretical results for the ASB_{2a} preconditioning technique, that is the relative condition number can be bounded

Table 3.4.1: Condition number and iteration counts for SB_2.

$1/H = n_s$	$1/h = 64$		128		256	
2	4.3606	10	4.3612	10	4.3613	10
4	6.0980	12	6.0991	12	6.0994	12
8	10.7012	16	10.7035	16	10.7040	16
16	20.6269	23	20.6314	23	20.6324	23

Table 3.4.2: Condition number and iteration counts for ASB_{2a}.

$1/H = n_s$	$1/h = 64$		128		256	
2	3.2545	8	3.4449	9	3.6298	9
4	3.1775	8	3.3734	9	3.5636	9
8	3.0088	8	3.2027	8	3.4000	9
16	2.9020	8	3.0188	8	3.2065	8

Table 3.4.3: Condition number and iteration counts for ASB_{2ga}.

$1/H = n_s$	$1/h = 64$		128		256	
2	4.1725	10	4.1788	10	4.1816	10
4	4.2897	10	4.3078	10	4.3151	10
8	4.3498	10	4.3900	10	4.4058	10
16	4.3755	10	4.4306	10	4.4601	10

independently of both partitioning parameters H and h (see Theorem 3.3.8). For the ASB_{2ga} preconditioning technique, in Table 3.4.3, although the relative condition number seems to increase slightly with $1/H$, the growth is asymptotic towards a value which has not yet been reached, thus it can also be bounded independently of the partitioning parameters H and h (see Theorem 3.3.11). The condition numbers in Table 3.4.2 are smaller than those in Table 3.4.3. However, the two-grid method has the advantage that the subproblems defined on the coarse grid are significantly smaller than those on the fine grid.

3.5 Summary

In this chapter, we have designed a perfectly scalable DD strategy for solving the symmetric elliptic boundary value problem (2.1.1) in two dimensions, when the domain is partitioned into long and narrow strips. The (two-stage) ASB_2 method presented here achieves scalability, and therefore optimal convergence properties, by alternating the (one-stage) SB_2 solver (obtained from direct solvers on the strip subproblems and the J-operator on the edges between strips) in the horizontal direction, with the SB_2 solver in the vertical direction. However, since the interior strip problems are solved exactly, to machine precision, this is an expensive computational procedure. Moreover, when the coefficients of the given problem are varying, preconditioners with smaller subdomains would better reflect the behaviour of the coefficients and give rise to more rapidly convergent algorithms. In the case of smooth coefficients, an alternative approach is to employ the overlapping Schwarz methods to solve the independent strip subproblems. When the coefficients are varying rapidly, the convergence estimates for well designed nonoverlapping DD algorithms are similar to those for smooth coefficients as long as the jumps align with subdomain boundaries.

Chapter 4

Alternate Strip-Based Substructuring Algorithms for Symmetric Elliptic PDE's in 2D

4.1 Introduction

The strip-based substructuring methods to be presented in this chapter are DD preconditioning techniques for the SC system (2.2.10) in the two-dimensional case, and may be viewed as simple generalisations of the two-subdomain iterative substructuring technique with the J-operator (see Section 2.2) to the case of a decomposition of $\Omega \subset \mathbb{R}^2$ into multiple nonoverlapping subdomains with interior vertices. In view of the strip based solvers introduced in Chapter 3, the separate treatment of the vertices is avoided by assembling the subdomains of the original decomposition into nonoverlapping strips such that: the ends of the strips are on the boundary of the given domain, the interfaces between strips (i.e. edges shared by two strips) align with the edges of the subdomains, and their union contains all of the interior vertices of the initial decomposition. Thus, the global interface between the subdomains can be partitioned as the union of edges between strips and edges between subdomains inside the same strip (edges do not include their end-points). For the subproblems corresponding to the various edges, the J-operator is used. Global coupling is again achieved in two stages. At each stage the strips change such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage. The two stages allow the use of a two-grid V-cycle and guarantee a good rate of convergence of the preconditioned iterative procedures, which is optimal with respect to the partitioning parameters. These new techniques have natural extensions for three-dimensional problems (see Chapter 5).

The rest of this chapter is organised as follows. In Section 4.2, we describe and give a brief account of the behaviour of the strip-based substructuring (SBS_2), and of the alternate strip-based substructuring ($ASBS_2$) preconditioning techniques. These are further analysed in Section 4.3. In Section 4.4, the performance of these DD methods is illustrated by numerical evaluations. This chapter is summarised in Section 4.5.

4.2 Strip-Based Substructuring

We consider the problem (2.1.1) in the two-dimensional case. For clarity of exposition, we assume $\Omega = (0,1) \times (0,1)$. Let (2.2.1) be the initial partitioning of Ω into subdomains such that each subdomain is an open square of uniform size $0 < H < 1$ (see Figure 2.2.1). In every subdomain, we consider the coefficient $\alpha(x)$ of (2.1.1) to be constant. We assemble the nonoverlapping subdomains in the initial partitioning of Ω into disjoint strips, Ω^s, such that: the vertices of each strip are on the boundary $\partial\Omega$, the interfaces between strips, Γ^j, align with the edges of the subdomains, and the union of these interfaces contains all of the vertices of the initial partitioning (see Figure 4.2.1 left or right). Thus the strips Ω^s form a partitioning of the form (3.2.1) of Ω. We denote by $\Omega_i^s \subset \Omega^s$ a generic subdomain inside the strip Ω^s, and by Γ_k a generic edge between two subdomains Ω_i^s inside the same strip Ω^s. The strips Ω^s are open rectangles in $\mathbb{R}^2$, and the edges Γ^j and Γ_k are open lines in $\mathbb{R}$ of length 1 and H respectively. If Γ is the global interface between all subdomains in the initial partitioning of Ω, then:

$$\Gamma = \left(\bigcup_k \Gamma_k\right) \bigcup \left(\bigcup_j \Gamma^j\right).$$

Let $S_h^0(\Omega)$ be as defined in Section 2.1 ($h < H$). As in Section 3.2, for every strip Ω^s, we consider the restrictions on $\bar{\Omega}^s \cap \Omega$ of the functions in $S_h^0(\Omega)$, and denote the finite element space of these restrictions by $S_h^0(\bar{\Omega}^s)$. We define $S_h^0(\Omega^s)$ to be the subspace of $S_h^0(\bar{\Omega}^s)$ consisting of those functions which are zero on the boundary $\partial\Omega^s \cap \Omega$. Next, for every subdomain $\Omega_i^s \subset \Omega^s$, we consider the restrictions on $\bar{\Omega}_i^s \cap \Omega$

of the functions in $S_h^0(\bar{\Omega}^s)$, and denote the finite element space of these restrictions by $S_h^0(\bar{\Omega}_i^s)$. We define $S_h^0(\Omega_i^s)$ to be the subspace of $S_h^0(\bar{\Omega}_i^s)$ consisting of those functions which are zero on the boundary $\partial\Omega_i^s \cap \Omega$. We also consider the restrictions on Γ of the functions in $S_h^0(\Omega)$ and denote the finite element space of these restrictions by $S_h^0(\Gamma)$. We define: $S_h^0(\partial\Omega^s)$, $S_h^0(\partial\Omega_i^s)$, $S_h^0(\partial\Omega_i^s \cap \Omega^s)$, $S_h^0(\Gamma^j)$, and $S_h^0(\Gamma_k)$ as the subspaces of $S_h^0(\Gamma)$ consisting of those functions which are zero on $\Gamma \setminus \partial\Omega^s$, $\Gamma \setminus \partial\Omega_i^s$, $\Gamma \setminus (\partial\Omega_i^s \cap \Omega^s)$, $\Gamma \setminus \Gamma^j$, and $\Gamma \setminus \Gamma_k$ respectively.

Furthermore, let $\tilde{\Gamma}_k$ be the domain obtained from the union of Γ_k with the neighbouring regions Ω_i^s inside the strip Ω^s, and $\tilde{\Gamma}^j$ be the domain obtained from the union of Γ^j with the neighbouring strips Ω^s. Note that these domains form an overlapping covering of Ω, such that every point in Ω is contained in at most four of these domains. We define $S_h^0(\tilde{\Gamma}_k)$ and $S_h^0(\tilde{\Gamma}^j)$ to be the subspaces of $S_h^0(\Omega)$ consisting of those functions with support in $\tilde{\Gamma}_k$ and $\tilde{\Gamma}^j$ respectively. Then:

$$S_h^0(\Omega) = \sum_k S_h^0(\tilde{\Gamma}_k) + \sum_j S_h^0(\tilde{\Gamma}^j), \tag{4.2.1}$$

i.e. for all $\mathbf{u} \in S_h^0(\Omega)$, there exists a representation, which is not unique, of the form:

$$\mathbf{u} = \sum_k \mathbf{u}_k + \sum_j \mathbf{u}^j, \qquad \mathbf{u}_k \in S_h^0(\tilde{\Gamma}_k),\ \mathbf{u}^j \in S_h^0(\tilde{\Gamma}^j). \tag{4.2.2}$$

4.2.1 The Strip-Based Substructuring (SBS$_2$) Technique

Let $x0y$ be a two-dimensional orthonormal system of coordinates. In $S_h^0(\Gamma)$, let $\{\psi^v\}$ be the set of finite element basis functions corresponding to the union of the vertex-points, and $\{\psi^x\}$ and $\{\psi^y\}$ be the set of finite element basis functions corresponding to the union of the edges that lie in the $0x$ and $0y$ direction respectively. The set of functions $\{\psi^x, \psi^v, \psi^y\}$ is a basis for $S_h^0(\Gamma)$ and so any function in $S_h^0(\Gamma)$ can be decomposed as a linear combination of this basis and represented by a vector of its coefficients. If we order these vectors as $[u^x\ u^v\ u^y]^T$ and consider the SC system (2.2.10), then the SC matrix S can be described in terms of block matrices as:

$$S = \begin{bmatrix} S_{xx} & S_{xv} & S_{xy} \\ S_{xv}^T & S_{vv} & S_{vy} \\ S_{xy}^T & S_{vy}^T & S_{yy} \end{bmatrix}.$$

We choose, for instance, the strips to be horizontal, i.e. Γ^j to align in the horizontal direction $0x$ and Γ_k to align in the vertical direction $0y$ (see Figure 4.2.1 left). Then S can be expressed in factored form as:

$$S = \begin{bmatrix} I_{xx} & 0 & S_{xy}S_{yy}^{-1} \\ 0 & I_{vv} & S_{vy}S_{yy}^{-1} \\ 0 & 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} & 0 \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} & 0 \\ 0 & 0 & S_{yy} \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{vv} & 0 \\ S_{yy}^{-1}S_{xy}^T & S_{yy}^{-1}S_{vy}^T & I_{yy} \end{bmatrix},$$

where I_{xx}, I_{vv} and I_{yy} denote identity matrices; S_{yy} corresponds to the union of edges Γ_k, and

$$\begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} \end{bmatrix} = \begin{bmatrix} S_{xx} & S_{xv} \\ S_{xv}^T & S_{vv} \end{bmatrix} - \begin{bmatrix} S_{xy} \\ S_{vy} \end{bmatrix} \times S_{yy}^{-1} \times \begin{bmatrix} S_{xy}^T & S_{vy}^T \end{bmatrix}$$

corresponds to the union of interfaces Γ^j (between strips).

We observe that, by reordering the finite element nodes, the matrix S_{yy} can be expressed as a block-diagonal matrix with each block corresponding to a boundary $\partial\Omega_i^s \cap \Omega^s$.

Remark 4.2.1 It can be shown that the matrix:

$$\begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} \end{bmatrix}$$

represents the SC matrix corresponding to the decomposition (3.2.1) of Ω into the nonoverlapping strips Ω^s and is equal to the matrix S in Section 3.2. We refer to the Appendix for detailed calculations. Therefore, by reordering the nodes, this matrix can be expressed as a block-diagonal matrix with each block corresponding to a boundary $\partial\Omega^s \cap \Omega$.

In order to construct the strip-based preconditioner for the SC system (2.2.10), we proceed as follows. First, we drop all the couplings between different edges Γ_k (inside strips), to obtain the block-diagonal matrix:

$$\mathit{blockdiag}\left(S_{\Gamma_k}\right),$$

each block S_{Γ_k} corresponding to an edge Γ_k. Then, for every Γ_k, let $(-\partial^2/\partial\xi^2)_{\Gamma_k}$ be the one-dimensional Laplacian operator with domain of definition $H_0^1(\Gamma_k)$, and let

δ_{Γ_k} denote the discrete operator defined on $S_h^0(\Gamma_k)$ by:

$$(\delta_{\Gamma_k}\mathbf{u}, \mathbf{v})_{\Gamma_k} = (\mathbf{u}', \mathbf{v}')_{\Gamma_k}, \ \forall \mathbf{v} \in S_h^0(\Gamma_k),$$

where the prime denotes differentiation with respect to the arc-length ξ along Γ_k, and $(\cdot,\cdot)_{\Gamma_k}$ is the scalar product in $L^2(\Gamma_k)$. Note that δ_{Γ_k} represents a finite dimensional approximation of $(-\partial^2/\partial\xi^2)_{\Gamma_k}$. Since δ_{Γ_k} is symmetric and positive definite (SPD) in the inner product $(\cdot,\cdot)_{\Gamma_k}$, we can define the square root $\delta_{\Gamma_k}^{1/2}$ of δ_{Γ_j} (see Bramble *et al.* (1986) [9], pp. 108-109). We denote by J_{Γ_k} the matrix form of $\delta_{\Gamma_k}^{1/2}$, then set the approximation for S_{Γ_k} as:

$$M_{\Gamma_k} = \alpha_{\Gamma_k} J_{\Gamma_k},$$

where α_{Γ_k} is a scaling factor equal to the average value of the coefficients inside the subdomains sharing the edge Γ_k. We set:

$$M_{yy} = \mathit{blockdiag}\,(M_{\Gamma_k}) \text{ as the approximation for } S_{yy}.$$

Analogously (see also Section 3.3), we drop all the couplings between different edges Γ^j (between strips), to obtain the block-diagonal matrix:

$$\mathit{blockdiag}\left(\tilde{S}_{\Gamma^j}\right),$$

each block $\tilde{S}_{\Gamma^j}$ corresponding to an edge Γ^j. Then, for every Γ^j, let $(-\partial^2/\partial\tau^2)_{\Gamma^j}$ be the one-dimensional Laplacian operator with domain of definition $H_0^1(\Gamma^j)$, and let δ_{Γ^j} denote the discrete operator defined on $S_h^0(\Gamma^j)$ by:

$$(\delta_{\Gamma^j}\mathbf{u}, \mathbf{v})_{\Gamma^j} = (\mathbf{u}', \mathbf{v}')_{\Gamma^j}, \ \forall \mathbf{v} \in S_h^0(\Gamma^j),$$

where the prime denotes differentiation with respect to the arc-length τ along Γ^j, and $(\cdot,\cdot)_{\Gamma^j}$ is the scalar product in $L^2(\Gamma^j)$. We denote by $\delta_{\Gamma^j}^{1/2}$ the square root of δ_{Γ^j}, and by J_{Γ^j} the matrix form of $\delta_{\Gamma^j}^{1/2}$, then set the approximation for S_{Γ^j} as:

$$M_{\Gamma^j} = \alpha_{\Gamma^j} J_{\Gamma^j},$$

where α_{Γ^j} is a scaling factor equal to the mean value of the coefficients inside the subdomains adjacent to Γ^j. We set:

$$\begin{bmatrix} M_{xx} & M_{xv} \\ M_{xv}^T & M_{vv} \end{bmatrix} = blockdiag\,(M_{\Gamma^j}) \text{ as the approximation for } \begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} \end{bmatrix}.$$

We define the preconditioner M_{sbs2} as:

$$M_{sbs2} = \begin{bmatrix} I_{xx} & 0 & S_{xy}M_{yy}^{-1} \\ 0 & I_{vv} & S_{vy}M_{yy}^{-1} \\ 0 & 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} M_{xx} & M_{xv} & 0 \\ M_{xv}^T & M_{vv} & 0 \\ 0 & 0 & M_{yy} \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{vv} & 0 \\ M_{yy}^{-1}S_{xy}^T & M_{yy}^{-1}S_{vy}^T & I_{yy} \end{bmatrix},$$

A generic system $M_{sbs2}w = r$ can now be written in terms of block matrices as:

$$\begin{bmatrix} I_{xx} & 0 & S_{xy}M_{yy}^{-1} \\ 0 & I_{vv} & S_{vy}M_{yy}^{-1} \\ 0 & 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} M_{xx} & M_{xv} & 0 \\ M_{xv}^T & M_{vv} & 0 \\ S_{xy}^T & S_{vy}^T & M_{yy} \end{bmatrix} \times \begin{bmatrix} w^x \\ w^v \\ w^y \end{bmatrix} = \begin{bmatrix} r^x \\ r^v \\ r^y \end{bmatrix}. \tag{4.2.3}$$

The solution $w = M_{sbs2}^{-1}r$ can be derived as follows.

The SBS$_2$ Procedure (algebraic form).

(I) compute the solution $M_{yy}^{-1}r^y$ and obtain the system equivalent to (4.2.3):

$$\begin{bmatrix} M_{xx} & M_{xv} & 0 \\ M_{xv}^T & M_{vv} & 0 \\ S_{xy}^T & S_{vy}^T & M_{yy} \end{bmatrix} \times \begin{bmatrix} w^x \\ w^v \\ w^y \end{bmatrix} = \begin{bmatrix} r^x - S_{xy}M_{yy}^{-1}r^y \\ r^v - S_{vy}M_{yy}^{-1}r^y \\ r^y \end{bmatrix}. \tag{4.2.4}$$

(II) using $M_{yy}^{-1}r^y$ obtained in (I), solve for w^x and w^v the system (4.2.4).

(III) using w^x and w^v obtained in (II), solve for w^y the system (4.2.4), by backward substitution.

With the preconditioner M_{sbs2} we can construct the following iterative method: start with u^0 as an initial approximation (without restricting the generality we can assume the starting approximation to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1} \leftarrow u^l + M_{sbs2}^{-1}(f_S - Su^l).$$

This can be interpreted as a Richardson iterative procedure (see e.g. Smith *et al.* (1996) [75], Appendix).

Alternatively, since the new preconditioned matrix $M_{sbs2}^{-1}S$ is symmetric and non-negative definite with respect to the $s(\cdot,\cdot)$ scalar product (induced by the SPD SC matrix S), the CG acceleration can be applied as follows (see also Chapter 2):

- let u^0 be an initial iterate,

$$r^0 \leftarrow f_S - Su^0, \text{ the initial residual}$$
$$w^0 \leftarrow M_{sbs2}^{-1}r^0, \text{ the initial preconditioned residual}$$
$$v^0 \leftarrow w^0, \text{ the initial search direction}$$

- for $l = 0, 1, \cdots$

compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Sv^l)}$

update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

update the residual: $r^{l+1} \leftarrow r^l + p_l S v^l$

if $r^{l+1} \geq$ tolerance, then

update the preconditioned residual: $w^{l+1} \leftarrow M_{sbs2}^{-1} r^{l+1}$

compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

else end for.

The resulting SBS_2 method has good parallelisation properties and a rate of convergence proportional to $1/\sqrt{H}$, when the strip aspect ratio is $r^s = 1/H$ (see Theorem 4.3.4 and Table 4.4.1).

Remark 4.2.2 We note that if the problems on each edge Γ_k (inside strips) are solved exactly, then the variables corresponding to these edges can be eliminated from the iterative process, which then reduces to an iteration on the edges between strips. The resulting algorithm coincides with that mentioned in Remark 3.2.1.

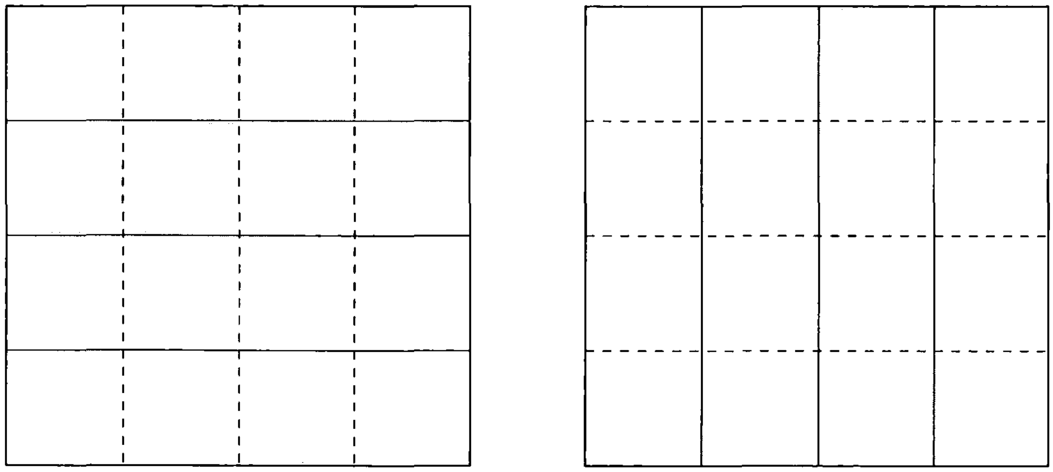

Figure 4.2.1: The horizontal (left) and vertical (right) association into strips of the subdomains of $\Omega \subset \mathbb{R}^2$.

4.2.2 The Alternate Strip-Based Substructuring ($ASBS_2$) Technique

We proceed with our discussion to the two-stage extension of the technique introduced in the previous section. At each stage the direction of the strips changes. We assume for instance that at the first stage the strips are horizontal, that is the edges between strip align in the horizontal direction $0x$, while at the second stage the strips are vertical, that is the edges between strips align in the vertical direction $0y$. Figure 4.2.1 shows the partitioning of the unit square Ω into disjoint, uniform strips Ω^s, at two different stages.

The Additive Alternate Strip-Based Substructuring ($ASBS_{2a}$) Algorithm. Let $M^{(1)}_{sbs2}$ and $M^{(2)}_{sbs2}$ denote the SBS_2 preconditioner at the first and second stage respectively. It is easy to see that the global interface Γ is covered by the union of the following two overlapping subdomains: on one hand the edges between strips and the edges between subdomains inside strips at the first stage, and on the other hand the edges between strips and the edges between subdomains inside strips at the second stage. Therefore Schwarz algorithms can be derived using the M_{sbs2} preconditioner. The (inexact) additive Schwarz method is: start with u^0 as an initial approximation (without restricting the generality we take this approximation to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1/2} \leftarrow u^l + (M^{(1)}_{sbs2})^{-1}(f_S - Su^l)$$
$$u^{l+1} \leftarrow u^{l+1/2} + (M^{(2)}_{sbs2})^{-1}(f_S - Su^l).$$

This can also be written in one step as:

$$u^{l+1} \leftarrow u^l + \left((M^{(1)}_{sbs2})^{-1} + (M^{(2)}_{sbs2})^{-1}\right)(f_S - Su^l),$$

and interpreted as a Richardson iterative process with the two-stage SC preconditioner defined by:

$$M^{-1}_{asbs2} = (M^{(1)}_{sbs2})^{-1} + (M^{(2)}_{sbs2})^{-1}.$$

The new preconditioned SC matrix $M^{-1}_{asbs2}S$ can also be used with CG acceleration:

- let u^0 be an initial iterate,

$r^0 \leftarrow f_S - Su^0$, the initial residual

$w^0 \leftarrow M^{-1}_{asbs2}r^0$, the initial preconditioned residual

$v^0 \leftarrow w^0$, the initial search direction

- for $l = 0, 1, \cdots$

compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Sv^l)}$

update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

update the residual: $r^{l+1} \leftarrow r^l + p_l S v^l$

if $r^{l+1} \geq$ tolerance, then

update the preconditioned residual: $w^{l+1} \leftarrow M^{-1}_{asbs2}r^{l+1}$

compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

else end for.

The following steps will compute $w^l = M_{asbs2}^{-1} r^l$ $(l = 0, 1, \cdots)$:

$$w^{l+1/2} \leftarrow (M_{sbs2}^{(1)})^{-1} r^l$$
$$w^l \leftarrow w^{l+1/2} + (M_{sbs2}^{(2)})^{-1} r^l,$$

or equivalently,

$$w^l \leftarrow \left((M_{sbs2}^{(1)})^{-1} + (M_{sbs2}^{(2)})^{-1}\right) r^l.$$

The resulting ASBS_{2a} method is optimal in the sense that the rate of convergence can be bounded independently of the partitioning parameters H and h (see Theorem 4.3.8 and Table 4.4.2).

However, reloading the problem at the second stage, when the direction of the strips changes, can be expensive. We therefore consider the possibility of reducing the calculations to a coarser grid at one of the stages, for instance when the edges between strips align in the vertical direction, Oy (see Figure 4.2.2). This will result in a two-grid process. First, we need to establish some further notation. Let $\Sigma^{2^p h} \subset \cdots \subset \Sigma^{2h} \subset \Sigma^h$ be a set of nested uniform square grids associated with the original domain Ω, such that $1 \leq p \in \mathbb{N}$ and $2^p h < H$. The coarse grid reduced operator for S, S_c, can be defined either by rediscretisation of the problem on the $\Sigma^{2^p h}$ grid, or by the relations $S_c = RSP$, where R is the restriction from grid Σ^h to grid $\Sigma^{2^p h}$ and $P = R^T$ is the prolongation from grid $\Sigma^{2^p h}$ to grid Σ^h. Finally, let M_{sbs2}^f and M_{sbs2}^c be the SBS_2 preconditioning matrix at the fine stage and the coarse stage respectively. Now, in the ASBS_{2a} procedures above we can simply replace $(M_{sbs2}^{(1)})^{-1}$ by $P(M_{sbs2}^c)^{-1}R$, and $(M_{sbs2}^{(2)})^{-1}$ by $(M_{sbs2}^f)^{-1}$.

The Two-Grid Alternate Strip-Based Substructuring (ASBS_{2ga}) Algorithm. The new additive two-grid method is: start with u^0 as an initial approximation (without restricting the generality we take this to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1/2} \leftarrow u^l + P(M_{sbs2}^c)^{-1}R(f_S - Su^l)$$
$$u^{l+1} \leftarrow u^{l+1/2} + (M_{sbs2}^f)^{-1}(f_S - Su^l).$$

This can also be written as:

$$u^{l+1} \leftarrow u^l + \left(P(M_{sbs2}^c)^{-1}R + (M_{sbs2}^f)^{-1}\right)(f_S - Su^l).$$

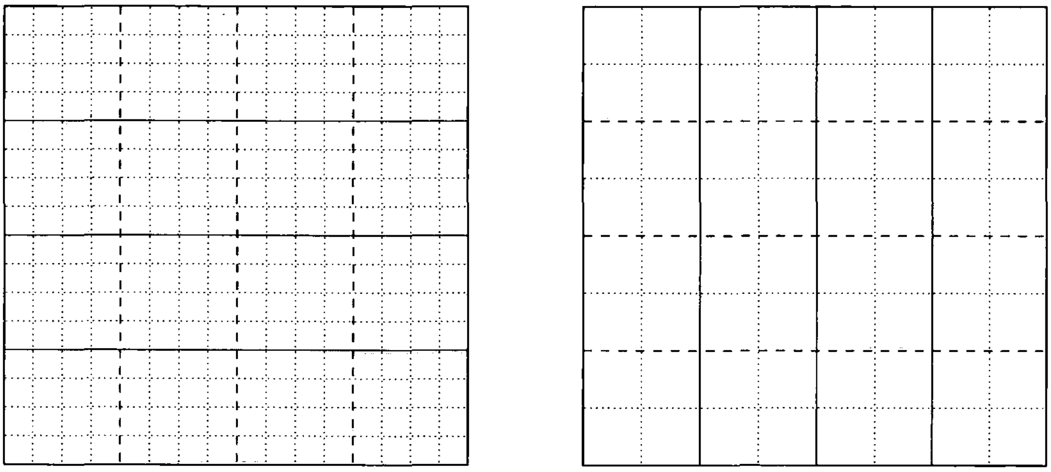

Figure 4.2.2: The horizontal (left) and vertical (right) association into strips of the subdomains of $\Omega \subset \mathbb{R}^2$, with two levels of mesh refinement.

When this scheme is used to define a preconditioner for the CG method, the inverse of the new two-grid SC preconditioner is:

$$M_{asbs2g}^{-1} = P(M_{sbs2}^{c})^{-1}R + (M_{sbs2}^{f})^{-1}.$$

The preconditioned SC matrix is $M_{asbs2g}^{-1}S$. The resulting $ASBS_{2ga}$ method is also optimal in the sense that the rate of convergence can be bounded independently of the partitioning parameters H and h (see Theorem 4.3.11 and Table 4.4.3).

4.3 Spectral Analysis for the SBS_2 and $ASBS_2$ Techniques

In this section, we concentrate on the abstract framework for the new strip-based substructuring algorithms described in Section 4.2. Our approach is via Theorem 2.1.3 applied to the SC matrix S and the new strip-based substructuring preconditioners. First, we present some technical tools which will be used to prove our spectral results, then we state and prove the theorems concerning the condition number for the relevant operators in the PCG iterations described in Section 4.2. Throughout this section the notation introduced in Section 4.2 is maintained. Also, C and c denote generic positive constants which are independent of the partitioning parameters H and h. The actual values of these constants may not necessarily be the same in any two occurrences. Further notation is explained as it occurs.

We decompose functions $\mathbf{u} \in S_h^0(\Gamma)$ as:

$$\mathbf{u} = \mathbf{u}^e + \mathbf{u}^s, \tag{4.3.1}$$

where

$$\mathbf{u}^e \in V^e = \sum_{s,i} S_h^0(\partial\Omega_i^s \cap \Omega^s)$$

is the solution of the following problem:

$$s(\mathbf{u}^e, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}), \ \forall \mathbf{v} \in V^e.$$

Note that this is equivalent to solving independently for each $\partial\Omega_i^s \cap \Omega^s$ the following local problem: find $\mathbf{u}_i^e \in S_h^0(\partial\Omega_i^s \cap \Omega^s)$, such that:

$$s(\mathbf{u}_i^e, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\partial\Omega_i^s \cap \Omega^s).$$

We denote by $\mathbf{u}^s = \mathbf{u} - \mathbf{u}^e$ the part of the solution $\mathbf{u}$ which lies in the orthogonal complement of V^e in $S_h^0(\Gamma)$:

$$V^s = \{\mathbf{u} \in S_h^0(\Gamma) | \ s(\mathbf{u}, \mathbf{v}) = 0, \ \forall \mathbf{v} \in V^e\}.$$

Thus, the value of the function $\mathbf{u}^s \in V^s$ on Γ is uniquely determined by its value on $\bigcup_j \Gamma^j$. From the definition of V^s, we deduce:

$$s(\mathbf{u}^s, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}) - s(\mathbf{u}^e, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Gamma),$$

or equivalently, when $\mathbf{v}^s$ is similarly defined as $\mathbf{u}^s$,

$$s(\mathbf{u}^s, \mathbf{v}^s) = (\mathbf{f}_S, \mathbf{v}) - s(\mathbf{u}^e, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Gamma).$$

Note that:

$$s(\mathbf{u}, \mathbf{u}) = s(\mathbf{u}^e, \mathbf{u}^e) + s(\mathbf{u}^s, \mathbf{u}^s). \tag{4.3.2}$$

Next, we consider the bilinear form $\tilde{s}(\cdot, \cdot)$ on $S_h^0(\Gamma) \times S_h^0(\Gamma)$ defined as follows. Let Ω_i be a generic subdomain in a partitioning (2.2.1) of Ω. First, we set:

$$\tilde{s}_i(\mathbf{u}, \mathbf{v}) = a_i(\mathbf{u}^E, \mathbf{v}^E), \ \forall \mathbf{u}, \mathbf{v} \in S_h^0(\Gamma), \tag{4.3.3}$$

where $a_i(\cdot,\cdot)$ is given by (2.2.2), and $\mathbf{u}^E$, $\mathbf{v}^E$ are the discrete harmonic extensions into Ω_i of $\mathbf{u}$, $\mathbf{v}$ respectively. Then, we define:

$$\tilde{s}(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{N} \tilde{s}_i(\mathbf{u},\mathbf{v}). \tag{4.3.4}$$

It can be shown that the bilinear form $\tilde{s}(\cdot,\cdot)$ is equivalent to $s(\cdot,\cdot)$, and we can drop the *tilde* from this notation (see e.g. Quarteroni and Valli (1999) [66], also discussion in Section 2.2). The bilinear form $s_i(\mathbf{u},\mathbf{u})$ can be analysed using the fractional order Sobolev seminorm $|\mathbf{u}|_{H^{1/2}(\partial\Omega_i)}$ given by:

$$|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} = \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{(\mathbf{u}(\xi)-\mathbf{u}(\tau))^2}{|\xi-\tau|^2} d\xi d\tau,$$

where ξ and τ denote arc-length along $\partial\Omega_i$ (see e.g. Xu and Zou (1998) [87], p. 866).

For every edge $\Gamma_k \subset \partial\Omega_i$, let

$$|\mathbf{u}|^2_{H^{1/2}(\Gamma_k)} = \int_{\Gamma_k}\int_{\Gamma_k} \frac{(\mathbf{u}(\xi)-\mathbf{u}(\tau))^2}{|\xi-\tau|^2} d\xi d\tau,$$

where ξ and τ denote arc-length along Γ_k. The associated space:

$$H^{1/2}(\Gamma_k) = \{\mathbf{u} \in L^2(\Gamma_k) | \ |\mathbf{u}|^2_{H^{1/2}(\Gamma_k)} < \infty\}$$

is equipped with the weighted norm:

$$||\mathbf{u}_k||^2_{H^{1/2}(\Gamma_k)} = \frac{1}{H}||\mathbf{u}_k||^2_{L^2(\Gamma_k)} + |\mathbf{u}_k|^2_{H^{1/2}(\Gamma_k)}$$

(see e.g. Xu and Zou (1998) [87], p. 868).

On the other hand, let $\mathbf{u}_k \in S_h^0(\Gamma_k)$, and $||\mathbf{u}_k||_{H^{1/2}_{oo}(\Gamma_k)}$ be the norm given by (2.2.9), or equivalently, by:

$$||\mathbf{u}_k||^2_{H^{1/2}_{oo}(\Gamma_k)} = \int_{\Gamma_k}\int_{\Gamma_k} \frac{(\mathbf{u}_k(\tau)-\mathbf{u}_k(\xi))^2}{|\tau-\xi|^2} d\tau d\xi + \int_{\Gamma_k} \frac{\mathbf{u}_k^2(\xi)}{dist(\xi,\partial\Gamma_k)} d\xi,$$

where τ and ξ denote arc-length along Γ_k, and $dist(\xi,\partial\Gamma_k)$ represents the distance of ξ to the end-points of Γ_k. It can be shown that, when $\mathbf{u}_k$ is a smooth function defined on $\partial\Omega_i^s$, with support contained in the edge $\Gamma_k \subset \partial\Omega_i^s$,

$$c|\mathbf{u}_k|^2_{H^{1/2}(\partial\Omega_i)} \le \|\mathbf{u}_k\|^2_{H^{1/2}_{oo}(\Gamma_k)} \le C|\mathbf{u}_k|^2_{H^{1/2}(\partial\Omega_i)} \tag{4.3.5}$$

(see e.g. Bramble *et al.* (1986) [9], p. 112, or Xu and Zou (1998) [87], p. 868).

Furthermore, the following equivalence holds:

$$c\|\mathbf{u}_k\|^2_{H^{1/2}_{oo}(\Gamma_k)} \le (\delta^{1/2}_{\Gamma_k}\mathbf{u}_k, \mathbf{u}_k)_{\Gamma_k} \le C\|\mathbf{u}_k\|^2_{H^{1/2}_{oo}(\Gamma_k)} \tag{4.3.6}$$

(see e.g. Bramble *et al.* (1986) [9], p. 113).

For an analogous discussion regarding the interfaces between strips Γ^j, we refer to Section 3.3.

The bilinear form $m_{sbs2}(\cdot,\cdot)$ associated with the preconditioner M_{sbs2} is defined by:

$$m_{sbs2}(\mathbf{u},\mathbf{v}) = \sum_k \alpha_{\Gamma_k}(\delta^{1/2}_{\Gamma_k}\mathbf{u}^e_k, \mathbf{v}^e_k)_{\Gamma_k} + \sum_j \alpha_{\Gamma^j}(\delta^{1/2}_{\Gamma^j}\mathbf{u}^j, \mathbf{v}^j)_{\Gamma^j}, \tag{4.3.7}$$

where for every edge Γ_k, $\mathbf{u}^e_k$ is equal to $\mathbf{u}^e|_{\Gamma_k}$ on Γ_k, and zero everywhere else on Γ and on $\partial\Omega$, and we recall that α_{Γ_k} is a scaling factor equal to the average value of the coefficients inside the subdomains sharing the common edge Γ_k; for every edge between two strips Γ^j, $\mathbf{u}^j$ is equal to $\mathbf{u}^s|_{\Gamma^j}$ on Γ^j, and zero everywhere else on Γ and on $\partial\Omega$, and α_{Γ^j} is a scaling factor equal to the mean value of the coefficients inside the subdomains adjacent to Γ^j; and $\mathbf{v}^e_k$ and $\mathbf{v}^j$ are similarly defined as $\mathbf{u}^e_k$ and $\mathbf{u}^j$ respectively.

The process of obtaining the solution $\mathbf{w} \in S^0_h(\Gamma)$ of

$$m_{sbs2}(\mathbf{w},\mathbf{v}) = (\mathbf{r},\mathbf{v}), \;\; \forall \mathbf{v} \in S^0_h(\Gamma)$$

is equivalent to the following procedure.

The SBS_2 Procedure (continuous form).

(I) for every edge $\Gamma_k \subset \Gamma$ (inside strips), solve for $\mathbf{w}^e_k \in S^0_h(\Gamma_k)$ the following equation:

$$\alpha_{\Gamma_k}(\delta^{1/2}_{\Gamma_k}\mathbf{w}^e_k, \mathbf{v})_{\Gamma_k} = (\mathbf{r},\mathbf{v}), \;\; \forall \mathbf{v} \in S^0_h(\Gamma_k).$$

This can be done independently and in parallel for all Γ_k.

(II) for every edge between two strips $\Gamma^j \subset \Gamma$, solve for $\mathbf{w}^j \in S_h^0(\Gamma^j)$ the following equation:

$$\alpha_{\Gamma^j}(\delta_{\Gamma^j}^{1/2}\mathbf{w}^j, \mathbf{v}^j)_{\Gamma^j} = (\mathbf{r}, \mathbf{v}) - \alpha_{\Gamma_k}(\delta_{\Gamma_k}^{1/2}\mathbf{w}_k^e, \mathbf{v})_{\Gamma_k}, \ \forall \mathbf{v} \in S_h^0(\Gamma),$$

where $\mathbf{v}^j = \mathbf{v}^s|_{\Gamma^j}$. This can be done independently and in parallel for all Γ^j.

(III) for every strip Ω^s, extend the values of $\mathbf{w}^j$, determined in (II), discrete harmonically onto all $\Gamma_k \subset \Omega^s$. That is solve for $\mathbf{w}^s \in \sum_{\Omega_i^s \subset \Omega^s} S_h^0(\partial\Omega_i^s)$ the homogeneous equation:

$$\sum_{\Gamma_k \subset \Omega^s} \alpha_{\Gamma_k}(\delta_{\Gamma_k}^{1/2}\mathbf{w}^s, \mathbf{v})_{\Gamma_k} = 0, \ \forall \mathbf{v} \in \sum_{\Omega_i^s \subset \Omega^s} S_h^0(\partial\Omega_i^s \cap \Omega^s),$$

with $\mathbf{w}^s$ given by $\mathbf{w}^j$ from (II) on $\Gamma^j \subset \partial\Omega^s$. Then, for each $\Gamma_k \subset \Omega^s$, set $\mathbf{w}_k = \mathbf{w}_k^e + \mathbf{w}^s|_{\Gamma_k}$. This can be done independently and in parallel for all Ω^s.

Next, we define the bilinear form $m_{asbs2}(\cdot,\cdot)$ associated with the preconditioner M_{asbs2} as follows. If $m_{sbs2}^{(1)}(\cdot,\cdot)$ and $m_{sbs2}^{(2)}(\cdot,\cdot)$ represent the bilinear form (4.3.7) associated with the preconditioners $M_{sbs2}^{(1)}$ and $M_{sbs2}^{(2)}$ respectively, we define:

$$m_{asbs2}(\mathbf{u}, \mathbf{v}) = m_{sbs2}^{(1)}(\mathbf{u}, \mathbf{v}) + m_{sbs2}^{(2)}(\mathbf{u}, \mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in S_h^0(\Gamma). \tag{4.3.8}$$

Similarly, in order to define the bilinear form $m_{asbs2g}(\cdot,\cdot)$ associated with the preconditioner M_{asbs2g}, let $m_{sbs2}^c(\cdot,\cdot)$ and $m_{sbs2}^f(\cdot,\cdot)$ denote the bilinear form (4.3.7) associated with the preconditioners M_{sbs2}^c and M_{sbs2}^f respectively, we define:

$$m_{asbs2g}(\mathbf{u}, \mathbf{v}) = m_{sbs2}^c(\mathbf{u}, \mathbf{v}) + m_{sbs2}^f(\mathbf{u}, \mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in S_h^0(\Gamma). \tag{4.3.9}$$

Lemma 4.3.1 Let $\Omega = (0,1) \times (0,1)$ and let $\Omega_i = (iH, (i+1)H) \times (0, H)$ be a subdomain in the partitioning (2.2.1) of Ω. For all $\mathbf{u} \in H^1(\Omega_i^s)$, the following estimates hold:

(i) if $\mathbf{u}$ is equal to zero along one side of Ω_i, then:

$$\|\mathbf{u}\|_{L^2(\Omega_i)}^2 \le CH^2|\mathbf{u}|_{H^1(\Omega_i)}^2.$$

(ii) if Γ_k is an edge in $\partial\Omega_i$, and $\|\mathbf{u}\|^2_{L^2(\Gamma_k)}$ represents the $L^2(\Gamma_k)$-norm of $\mathbf{u}|_{\Gamma_k}$, then:

$$\|\mathbf{u}\|^2_{L^2(\Gamma_k)} \leq C\left(\frac{1}{H}\|\mathbf{u}\|^2_{L^2(\Omega_i)} + H|\mathbf{u}|^2_{H^1(\Omega_i)}\right).$$

(iii) if $\Omega^s = (0,1)\times(0,H)$ is a strip such that $\Omega_i \subset \Omega^s$ and $\mathbf{u} \in H^1(\Omega^s)$, then:

$$\|\mathbf{u}\|^2_{L^2(\Omega_i)} \leq CH^2\left(\frac{1}{H}\|\mathbf{u}\|^2_{L^2(\Omega^s)} + |\mathbf{u}|^2_{H^1(\Omega^s)}\right).$$

In each of the estimates (i) – (iii), C denotes a generic positive constant which is independent of the function $\mathbf{u}$ and the partitioning parameter H.

Proof: These estimates can be derived by direct integration and by applying the Cauchy-Schwarz inequality (see also Lemma 3.3.1). □

Lemma 4.3.2 Let Ω_i be a subdomain in the (2.2.1) partitioning of Ω, and Γ_k denote a generic edge in $\partial\Omega_i$. If $\tilde{\mathbf{u}} \in S^0_h(\Omega)$ vanishes at the vertices of Ω_i, and $\mathbf{u} = \tilde{\mathbf{u}}|_{\partial\Omega_i}$, then:

$$s_i(\mathbf{u},\mathbf{u}) \leq C \sum_{\Gamma_k \subset \partial\Omega_i} (\delta^{1/2}_{\Gamma_k}\mathbf{u},\mathbf{u})_{\Gamma_k},$$

where $s_i(\cdot,\cdot)$ is defined by (4.3.3).

Proof: See Bramble *et al.* (1986) [9], Lemma 3.2 (ii). □

Lemma 4.3.3 Let $\mathbf{u}$ be a continuous, piecewise quadratic function defined on the finite element mesh Σ^h of the domain Ω. If $I^h\mathbf{u}$ is its piecewise linear interpolant on the same mesh, then:

$$|I^h\mathbf{u}|_{H^1(\Omega_i)} \leq C|\mathbf{u}|_{H^1(\Omega_i)},$$

where Ω_i is a generic subdomain in a (2.2.1) partitioning of Ω. The same type of bounds hold for the L^2, $H^{1/2}$, and $H^{1/2}_{oo}$ norms.

Proof: See Dryja and Widlund (1994) [37], Lemma 4. □

Theorem 4.3.4 For the SBS_2 preconditioning technique, the relative condition number $\kappa(M^{-1}_{sbs2}S)$ grows linearly as $1/H$, i.e.

$$\kappa(M^{-1}_{sbs2}S) = \frac{\lambda_{\max}(M^{-1}_{sbs2}S)}{\lambda_{\min}(M^{-1}_{sbs2}S)} \leq \frac{C}{H}.$$

Proof: Let $m_{sbs2}(\cdot,\cdot)$ be the bilinear form defined by (4.3.7). In order to show that the relative condition number satisfies $\kappa(M_{sbs2}^{-1}S) \le C/H$, through Theorem 2.1.3 for the matrix S and the preconditioner M_{sbs2}, it suffices to show that:

$$cHm_{sbs2}(\mathbf{u},\mathbf{u}) \le s(\mathbf{u},\mathbf{u}) \le Cm_{sbs2}(\mathbf{u},\mathbf{u}),\ \forall \mathbf{u} \in S_h^0(\Gamma). \tag{4.3.10}$$

Throughout this proof we maintain the notation adopted when defining (4.3.7).

First, in order to derive an upper bound for $\lambda_{\max}(M_{sbs2}^{-1}S)$, we show the right hand-side inequality in (4.3.10). By the Cauchy-Schwarz inequality,

$$s(\sum_k \mathbf{u}_k^e, \sum_k \mathbf{u}_k^e) \le C \sum_k s(\mathbf{u}_k^e, \mathbf{u}_k^e)$$

and

$$s(\sum_j \mathbf{u}^j, \sum_j \mathbf{u}^j) \le C \sum_j s(\mathbf{u}^j, \mathbf{u}^j).$$

Let $\tilde{\mathbf{u}}^j$ be the discrete harmonic extension of $\mathbf{u}^j$ in $\tilde{\Gamma}^j$, extended by zero to the rest of Ω. Therefore:

$$\sum_j s(\mathbf{u}^j, \mathbf{u}^j) = \sum_j a(\tilde{\mathbf{u}}^j, \tilde{\mathbf{u}}^j).$$

Then, by Lemma 3.3.2 and Lemma 4.3.2:

$$\sum_k s(\mathbf{u}_k^e, \mathbf{u}_k^e) \le C \sum_k \alpha_{\Gamma_k} (\delta_{\Gamma_k}^{1/2} \mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k}$$

and

$$\sum_j s(\mathbf{u}^j, \mathbf{u}^j) \le C \sum_j \alpha_{\Gamma^j} (\delta_{\Gamma^j}^{1/2} \mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j}$$

respectively.

Thus, by the decomposition (4.3.2) of $s(\cdot,\cdot)$, the above estimations, and the definition (4.3.7) of $m_{sbs2}(\cdot,\cdot)$, we deduce:

$$\begin{aligned} s(\mathbf{u},\mathbf{u}) =& s(\sum_k \mathbf{u}_k^e, \sum_k \mathbf{u}_k^e) + s(\sum_j \mathbf{u}^j, \sum_j \mathbf{u}^j) \\ \le& C \sum_k \alpha_{\Gamma_k} (\delta_{\Gamma_k}^{1/2} \mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} + C \sum_j \alpha_{\Gamma^j} (\delta_{\Gamma^j}^{1/2} \mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j}, \end{aligned}$$

from which the right hand-side inequality in (4.3.10) follows. Therefore, by Theorem 2.1.3, $\lambda_{\max}(M_{sbs2}^{-1}S) \le C$.

Next, in order to derive a lower bound for $\lambda_{\min}(M_{sbs2}^{-1}S)$, we show the left hand-side inequality in (4.3.10). The argument here is analogous to that used to prove Theorem 3.3.4. Let Ω^s denote a generic strip in Ω, and $\Omega_i^s \subset \Omega^s$, a generic subdomain in Ω^s. If $\mathbf{u} = \mathbf{u}^e + \mathbf{u}^s$ is the (4.3.1) decomposition of $\mathbf{u}$, we denote by $\tilde{\mathbf{u}}^e$ and $\tilde{\mathbf{u}}^s$ the discrete harmonic extensions of $\mathbf{u}^e$ and $\mathbf{u}^s$ respectively in Ω.

We show that:

$$\sum_k \alpha_{\Gamma_k}(\delta_{\Gamma_k}^{1/2}\mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} \leq Cs(\mathbf{u}^e, \mathbf{u}^e) \tag{4.3.11}$$

and

$$\sum_j \alpha_{\Gamma^j}(\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \leq C\left(1 + \frac{1}{H}\right) s(\mathbf{u}^s, \mathbf{u}^s). \tag{4.3.12}$$

Let η^j be defined as in the proof of Theorem 3.3.4, and η_k be a continuous, piecewise linear function on the finite element nodes of $\partial\Omega_i^s$ that is zero on the finite element nodes at the ends of $\Gamma_k \subset \partial\Omega_i^s$ and everywhere else on $\Gamma \setminus \Gamma_k$, grows linearly to 1 on the finite element nodes of Γ_k such that its gradient is of order $\mathcal{O}(1/H)$, and it is identically 1 on the remaining finite element nodes of Γ_k. If $\mathbf{u} = \mathbf{u}^e + \mathbf{u}^s$ is the (4.3.1) decomposition of $\mathbf{u}$, and I^h is the finite element interpolation operator onto the space $S_h^0(\Gamma)$, then we define:

$$\mathbf{u}_k^e = I^h(\eta_k \mathbf{u}^e) \qquad \text{and} \qquad \mathbf{u}^j = I^h(\eta^j \mathbf{u}^s).$$

Note that if $\{\eta_k\}$ and $\{\eta^j\}$ form partitions of unity, then:

$$\mathbf{u}^e = \sum_k \mathbf{u}_k^e \qquad \text{and} \qquad \mathbf{u}^s = \sum_j \mathbf{u}^j.$$

From Lemma 4.3.3 (for the $H_{oo}^{1/2}$ norm), we deduce that when $\mathbf{v} = \eta_k \mathbf{u}^e$ (note that this is a continuous, piecewise quadratic function), in order to estimate $||\mathbf{u}_k||^2_{H_{oo}^{1/2}(\Gamma_k)}$, it suffices to estimate $||\mathbf{v}||^2_{H_{oo}^{1/2}(\Gamma_k)}$. Let $\Gamma_k = (0, H)$ and Ω_i^s be a subdomain such that $\Gamma_k \subset \partial\Omega_i^s$. Then, we divide the interval $[0, H]$ in two parts $[0, H/2]$ and $[H/2, H]$, and take the tensor product $[0, H] \bigotimes [0, H]$. The double integral in the definition of $||\mathbf{v}||^2_{H_{oo}^{1/2}(\Gamma_k)}$ is then split into a sum of four double integrals. Due to the symmetry, we only need to consider one of them.

We consider the diagonal term corresponding to the set $[0, H/2] \times [0, H/2]$ and use the identity:

$$\begin{aligned}\mathbf{v}(\xi) - \mathbf{v}(\tau) &= \frac{2\xi \mathbf{u}^e(\xi) - 2\tau \mathbf{u}^e(\tau)}{H} \\ &= \frac{(\xi + \tau)(\mathbf{u}^e(\xi) - \mathbf{u}^e(\tau))}{H} + \frac{(\xi - \tau)(\mathbf{u}^e(\xi) + \mathbf{u}^e(\tau))}{H}.\end{aligned}$$

As for the diagonal term corresponding to the set $[0, 1/2] \times [0, 1/2]$ in the proof of Theorem 3.3.11, by the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\int_0^{H/2}\int_0^{H/2} \frac{|\mathbf{v}(\xi) - \mathbf{v}(\tau)|^2}{|\xi - \tau|^2} d\xi d\tau \leq & C \int_0^{H/2}\int_0^{H/2} \frac{(\xi + \tau)^2 |\mathbf{u}^e(\xi) - \mathbf{u}^e(\tau)|^2}{H^2 |\xi - \tau|^2} d\xi d\tau \\ & + C \int_0^{H/2}\int_0^{H/2} \frac{|\mathbf{u}^e(\xi) + \mathbf{u}^e(\tau)|^2}{H^2} d\xi d\tau \\ \leq & C \int_0^{H/2}\int_0^{H/2} \frac{(\xi + \tau)^2 |\mathbf{u}^e(\xi) - \mathbf{u}^e(\tau)|^2}{H^2 |\xi - \tau|^2} d\xi d\tau \\ & + C \int_0^{H/2}\int_0^{H/2} \frac{|\mathbf{u}^e(\xi)|^2}{H^2} d\xi d\tau \\ & + C \int_0^{H/2}\int_0^{H/2} \frac{|\mathbf{u}^e(\tau)|^2}{H^2} d\xi d\tau \\ \leq & C \int_0^{H/2}\int_0^{H/2} \frac{|\mathbf{u}^e(\xi) - \mathbf{u}^e(\tau)|^2}{|\xi - \tau|^2} d\xi d\tau \\ & + C \int_0^{H/2} \frac{|\mathbf{u}^e(\xi)|^2}{H} d\xi.\end{aligned}$$

Therefore:

$$\int_0^H \int_0^H \frac{|\mathbf{v}(\xi) - \mathbf{v}(\tau)|^2}{|\xi - \tau|^2} d\xi d\tau \leq C \frac{1}{H} \|\mathbf{u}^e\|^2_{L^2(\Gamma_k)} + C |\mathbf{u}^e|^2_{H^{1/2}(\Gamma_k)}.$$

In order to estimate the single integral in the definition of $\|\mathbf{v}\|^2_{H^{1/2}_{00}(\Gamma_k)}$, we write the closed set $\bar{\Gamma}_k$ as the interval $[0, H]$. Through the definition of η_k, we deduce:

$$\begin{aligned}
\int_{\Gamma_k} \frac{(\mathbf{v}(\xi))^2}{dist(\xi, \partial\Gamma_k)} d\xi &= \int_0^H \frac{(\mathbf{v}(\xi))^2}{\xi} d\xi + \int_0^H \frac{(\mathbf{v}(\xi))^2}{H-\xi} d\xi \\
&\le C \int_0^{H/2} \frac{\xi^2(\mathbf{u}^e(\xi))^2}{H^2\xi} d\xi + C \int_{H/2}^H \frac{(H-\xi)^2(\mathbf{u}^e(\xi))^2}{H^2\xi} d\xi \\
&+ C \int_0^{H/2} \frac{\xi^2(\mathbf{u}^e(\xi))^2}{H^2(H-\xi)} d\xi + C \int_{H/2}^H \frac{(H-\xi)^2(\mathbf{u}^e(\xi))^2}{H^2(H-\xi)} d\xi \\
&= C \int_0^{H/2} \frac{\xi(\mathbf{u}^e(\xi))^2}{H^2} d\xi + C \int_{H/2}^H \frac{(H-\xi)^2(\mathbf{u}^e(\xi))^2}{H^2\xi} d\xi \\
&+ C \int_0^{H/2} \frac{\xi^2(\mathbf{u}^e(\xi))^2}{H^2(H-\xi)} d\xi + C \int_{H/2}^H \frac{(H-\xi)(\mathbf{u}^e(\xi))^2}{H^2} d\xi \\
&\le C \int_0^H \frac{(\mathbf{u}^e(\xi))^2}{H} d\xi.
\end{aligned}$$

Therefore:

$$\int_{\Gamma_k} \frac{(\mathbf{v}(\xi))^2}{dist(\xi, \partial\Gamma_k)} d\xi \le C\frac{1}{H}\|\mathbf{u}^e\|^2_{L^2(\Gamma_k)}.$$

By the above evaluations, we obtain:

$$\|\mathbf{u}_k^e\|^2_{H_{00}^{1/2}(\Gamma_k)} \le C\frac{1}{H}\|\mathbf{u}^e\|^2_{L^2(\Gamma_k)} + C|\mathbf{u}^e|^2_{H^{1/2}(\Gamma_k)}.$$

From this estimate, Lemma 4.3.1 (*ii*) and (*i*), and Theorem 2.2.3, we deduce:

$$\begin{aligned}
\sum_{\Gamma_k \subset \partial\Omega_i^s} \alpha_{\Gamma_k} \|\mathbf{u}_k^e\|^2_{H_{00}^{1/2}(\Gamma_k)} &\le C \sum_{\Gamma_k \subset \partial\Omega_i^s} \frac{1}{H}\|\mathbf{u}^e\|^2_{L^2(\Gamma_k)} + C \sum_{\Gamma_k \subset \partial\Omega_i^s} |\mathbf{u}^e|^2_{H^{1/2}(\Gamma_k} \\
&\le C\frac{1}{H^2}\|\tilde{\mathbf{u}}^e\|^2_{L^2(\Omega_i^s)} + C|\tilde{\mathbf{u}}^e|^2_{H^1(\Omega_i^s)} \\
&\le C|\tilde{\mathbf{u}}^e|^2_{H^1(\Omega_i^s)} \\
&\le C|\mathbf{u}^e|^2_{H^{1/2}(\partial\Omega_i^s)}.
\end{aligned}$$

Thus, through the equivalence (4.3.6),

$$\sum_{\Gamma_k \subset \partial\Omega_i^s} \alpha_{\Gamma_k} (\delta_{\Gamma_k}^{1/2}\mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} \le Cs_i(\mathbf{u}^e, \mathbf{u}^e).$$

Since each Γ_k is shared by only two subdomains Ω_i^s, after summing over all $\Omega_i^s \subset \Omega$, through the representation (4.3.4) of $s(\cdot,\cdot)$, we obtain (4.3.11).

On the other hand, arguments similar to those in the proof of Theorem 3.3.4

yield:

$$\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \le C\|\mathbf{u}^s\|^2_{L^2(\Gamma^j)} + C|\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)}.$$

From this estimate, Lemma 3.3.1 (*iii*) and (*i*), and Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{\Gamma^j\subset\partial\Omega^s} \alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} &\le C\sum_{\Gamma^j\subset\partial\Omega^s}\|\mathbf{u}^s\|^2_{L^2(\Gamma^j)} + C\sum_{\Gamma^j\subset\partial\Omega^s}|\mathbf{u}^s|^2_{H^{1/2}(\Gamma^j)}\\ &\le C\frac{1}{H}\|\tilde{\mathbf{u}}^s\|^2_{L^2(\Omega^s)} + CH|\tilde{\mathbf{u}}^s|^2_{H^1(\Omega^s)} + C|\tilde{\mathbf{u}}^s|^2_{H^1(\Omega^s)}\\ &\le C\frac{1}{H}\|\tilde{\mathbf{u}}^s\|^2_{L^2(\Omega^s)} + C|\tilde{\mathbf{u}}^s|^2_{H^1(\Omega^s)}\\ &\le C\left(1+\frac{1}{H}\right)|\tilde{\mathbf{u}}^s|^2_{H^1(\Omega^s)}.\end{aligned}$$

Then, we apply the Cauchy-Schwarz inequality for the decomposition of the strips Ω^s into nonoverlapping subdomains Ω^s_i, and the right hand-side inequality in Theorem 2.2.3, to obtain:

$$\begin{aligned}\sum_{\Gamma^j\subset\partial\Omega^s} \alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} &\le C\left(1+\frac{1}{H}\right)\sum_{\Omega^s_i\subset\Omega^s}|\tilde{\mathbf{u}}^s|^2_{H^1(\Omega^s_i)}\\ &\le C\left(1+\frac{1}{H}\right)\sum_{\Omega^s_i\subset\Omega^s}|\mathbf{u}^s|^2_{H^{1/2}(\partial\Omega^s_i)}.\end{aligned}$$

Thus, through the equivalence (3.3.7),

$$\sum_{\Gamma^j\subset\partial\Omega^s} \alpha_{\Gamma^j}(\delta^{1/2}_{\Gamma^j}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \le C\left(1+\frac{1}{H}\right)\sum_{\Omega^s_i\subset\Omega^s} s_i(\mathbf{u}^s, \mathbf{u}^s).$$

Since each edge Γ^j is shared by only two strips Ω^s, after summing over all $\Omega^s \subset \Omega$, by the representation (4.3.4) of $s(\cdot,\cdot)$, we obtain (4.3.12).

From the estimates (4.3.11) and (4.3.12), the definition (4.3.7) of $m_{sbs2}(\cdot,\cdot)$, and the decomposition (4.3.2) of $s(\cdot,\cdot)$, we deduce:

$$m_{sbs2}(\mathbf{u},\mathbf{u}) \le C\left(1+\frac{1}{H}\right) s(\mathbf{u},\mathbf{u}),$$

which is equivalent to left hand-side inequality in (4.3.10).

Therefore, by Theorem 2.1.3 for the matrix S and the preconditioner M_{sbs2}, $1/\lambda_{\min}(M^{-1}_{sbs2}S)$ grows linearly as $1/H$. Since $1/\lambda_{\max}(M^{-1}_{sbs2}S) \le C$, we conclude that:

$$\kappa(M^{-1}_{sbs2}S) \le C/H. \ \square$$

Remark 4.3.5 We observe that in the proof of Theorem 4.3.4, when Ω^s is a strip with only one edge in the interior of the domain Ω and the remaining edges on the boundary $\partial\Omega$, we can apply Lemma 3.3.1 (*ii*) instead of (*i*). Thus, for this strip,

$$\sum_{\Gamma^j \subset \partial\Omega^s} \alpha_{\Gamma^j} \|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \leq C|\tilde{\mathbf{u}}^s|^2_{H^1(\Omega^s)}.$$

See also Remark 3.3.5.

Lemma 4.3.6 Let $\mathbf{u} \in S^0_h(\Gamma)$ and $\tilde{\mathbf{u}}$ be its discrete harmonic extension in Ω. If $Q_{2^p h} : L^2(\Omega) \to S^0_{2^p h}(\Omega)$ is the L^2-projection associated with $S^0_{2^p h}(\Omega)$ ($h \leq 2^p h < H$, $p \in \mathbb{N}$), we denote $\tilde{\mathbf{u}}_o = Q_{2^p h}\tilde{\mathbf{u}}$ and $\mathbf{u}_o = \tilde{\mathbf{u}}_{o|\Gamma}$. Then:

$$m_{asbs2}(\mathbf{u}_o, \mathbf{u}_o) \leq Cs(\mathbf{u}, \mathbf{u}).$$

Proof: The proof of this result is based on the observation that the $ASBS_{2a}$ preconditioning is obtained in two stages such that the interfaces between strips at one stage are perpendicular on the interfaces between strips at the other stage. Throughout this proof we maintain the notation adopted when defining (4.3.7).

First, we show that for any $\mathbf{u}_o \in S^0_h(\Gamma)$,

$$m_{asbs2}(\mathbf{u}_o, \mathbf{u}_o) \leq Cs(\mathbf{u}_o, \mathbf{u}_o). \tag{4.3.13}$$

Then, by replacing h by $2^p h$ and taking $\mathbf{u}_o = Q_{2^p h}\tilde{\mathbf{u}}_{o|\Gamma}$ in the above estimate, the lemma follows, through the definition of $s(\cdot,\cdot)$, Theorem 2.2.3, and the second estimate in Lemma 3.3.6.

Let $\mathbf{u}_1 = \mathbf{u}_1^e + \mathbf{u}_1^s$ be the (4.3.1) decomposition of $\mathbf{u}_o$ at the first stage, and $\mathbf{u}_2 = \mathbf{u}_2^e + \mathbf{u}_2^s$ be the (4.3.1) decomposition of $\mathbf{u}_o$ at the second stage. Note that the global interface Γ can be viewed as being covered by the union of the following two overlapping subdomains: on one hand the edges between strips and the edges between subdomains inside strips at the first stage, and on the other hand the edges between strips and the edges between subdomains inside strips at the second stage. Γ can also be viewed as being covered by the union of the interfaces between strips at the first stage, and the interfaces between strips at the second stage.

Furthermore, Γ can be regarded as consisting of overlapping vertex-regions, such that each region is cross-shaped, is centered at a vertex-point, and contains parts of the interfaces between strips that are within a distance H from that vertex. Thus at most two such regions overlap and the overlap is uniform of order $\mathcal{O}(H)$. Let Γ^v denote a generic vertex-region as described above, restricted to the boundary $\partial\Omega_i^s$ of a generic subdomain Ω_i^s. Then the restriction $\mathbf{u}^v = \mathbf{u}_{o|\Gamma^v}$, of $\mathbf{u}_o$ to Γ^v, can be analysed using the $H_{oo}^{1/2}(\Gamma^v)$ norm according to definition (2.2.9), as follows.

Let Γ^{j_1} and Γ^{j_2} denote a generic edge between two strips at the first and at the second stage respectively, such that $\Gamma^v \subset \Gamma^{j_1} \cup \Gamma^{j_2}$. We introduce the following notation: $\tilde{\Gamma}^v = \tilde{\Gamma}^{j_1} \cap \tilde{\Gamma}^{j_2}$ and $S_h^0(\tilde{\Gamma}^v) = S_h^0(\tilde{\Gamma}^{j_1}) \cap S_h^0(\tilde{\Gamma}^{j_2})$, with $S_h^0(\tilde{\Gamma}^{j_1})$ and $S_h^0(\tilde{\Gamma}^{j_2})$ as in (4.2.1). We denote by $\tilde{\mathbf{u}}_o$ the discrete harmonic extension of $\mathbf{u}_o$ in Ω and derive a representation:

$$\tilde{\mathbf{u}}_o = \sum_v \tilde{\mathbf{u}}^v,$$

where $\tilde{\mathbf{u}}^v \in S_h^0(\tilde{\Gamma}^v)$. We construct this representation as follows. Let η^v be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes of the boundary $\partial\tilde{\Gamma}^v$ and everywhere else on $\Omega \setminus \tilde{\Gamma}^v$, $0 \leq \eta^v \leq 1$, and its gradient is of order $\mathcal{O}(1/H)$. If I^h is the finite element interpolation operator onto the space $S_h^0(\Omega)$, then we define:

$$\bar{\mathbf{u}}^v = I^h(\eta^v \tilde{\mathbf{u}}_o).$$

Note that if $\{\eta^v\}$ form a partition of unity, then:

$$\tilde{\mathbf{u}}_o = \sum_v \bar{\mathbf{u}}^v.$$

We proceed to bound the energies of the parts of $\bar{\mathbf{u}}^v$ associated with the elements in Σ^h. If $\bar{\eta}^v$ is the average of η^v on a single mesh-element σ_h, then:

$$\|\eta^v - \bar{\eta}^v\|^2_{L^\infty(\sigma_h)} \leq C(h/H)^2.$$

By the Cauchy-Schwarz inequality, we can write:

$$\begin{aligned} |\bar{\mathbf{u}}^v|^2_{H^1(\sigma_h)} &= |I^h(\eta^v \tilde{\mathbf{u}}_o)|^2_{H^1(\sigma_h)} \\ &\leq 2|\bar{\eta}^v \tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} + 2|I^h(\bar{\eta}^v - \eta^v)\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)}. \end{aligned} \tag{4.3.14}$$

Since $\|\bar{\eta}^v\|_{L^\infty(\sigma_h)} \leq 1$, the first term on the right hand-side of the inequality in

(4.3.14) can be bounded as:

$$2|\bar{\eta}^v \tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)}. \tag{4.3.15}$$

For the second term on the right hand-side of the inequality in (4.3.14), from the inverse estimate in Lemma (2.1.2), the weighted norm on element σ_h of diameter h, and the bound on the gradient of η^v, we obtain:

$$\begin{aligned} 2|I^h(\bar{\eta}^v - \eta^v)\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} &\leq C\frac{1}{h^2}\|I^h(\bar{\eta}^v - \eta^v)\tilde{\mathbf{u}}_o\|^2_{L^2(\sigma_h)} \\ &\leq C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\sigma_h)}. \end{aligned} \tag{4.3.16}$$

Since each σ_h is associated with only four $\bar{\mathbf{u}}^v$, from (4.3.14), (4.3.15), and (4.3.16), we deduce:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} |\bar{\mathbf{u}}^v|^2_{H^1(\sigma_h)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\sigma_h)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\sigma_h)}.$$

After summing with respect to σ_h, we obtain:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} |\bar{\mathbf{u}}^v|^2_{H^1(\Omega_i^s)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^s)}.$$

We choose $\tilde{\mathbf{u}}^v$ to be the discrete harmonic extension of $\mathbf{u}^v = \bar{\mathbf{u}}^v|_{\Gamma^v}$ in $\tilde{\Gamma}^v$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5), of discrete harmonic functions, imply:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} |\tilde{\mathbf{u}}^v|^2_{H^1(\Omega_i^s)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^s)}. \tag{4.3.17}$$

Thus, by the left hand-side inequality in Theorem 2.2.3 and Lemma 4.3.1 (i),

$$\begin{aligned} \sum_{\Gamma^v \subset \partial\Omega_i^s} |\mathbf{u}^v|^2_{H^{1/2}(\partial\Omega_i^s)} &\leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^s)} \\ &\leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)}. \end{aligned}$$

From this estimate, by the definition (2.2.9) and the right hand-side inequality in

Theorem 2.2.3, we deduce:

$$\sum_{\Gamma^v \subset \partial\Omega_i^s} \|\mathbf{u}^v\|^2_{H_{oo}^{1/2}(\Gamma^v)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^s)}$$

$$\leq C|\mathbf{u}_o|^2_{H^{1/2}(\partial\Omega_i^s)}$$

$$\leq Cs_i(\mathbf{u}_o, \mathbf{u}_o).$$

Since each Γ^v is shared by only four subdomains Ω_i^s, after summing over all $\Omega_i^s \subset \Omega$, through the representation (4.3.4) of $s(\cdot,\cdot)$, we obtain:

$$\sum_v \|\mathbf{u}^v\|^2_{H_{oo}^{1/2}(\Gamma^v)} \leq Cs(\mathbf{u}_o, \mathbf{u}_o),$$

which implies (4.3.13). □

Remark 4.3.7 We note that, since Γ can be viewed as being covered by the union of the interfaces between strips at the first stage and the interfaces between strips at the second stage, the argument used in the proof of Lemma 4.3.6 can also be applied to prove Lemma 3.3.7. In order to achieve that, at the first stage, we take $\mathbf{u}_1^s$ of Lemma 4.3.6 as the restriction of $\mathbf{u}_1^E$ of Lemma 3.3.7 to the interfaces between strips, and $\mathbf{u}_1^e = 0$. Analogously, at the second stage, we take $\mathbf{u}_2^s$ of Lemma 4.3.6 as the restriction of $\mathbf{u}_2^E$ of Lemma 3.3.7 to the interfaces between strips, and $\mathbf{u}_2^e = 0$.

Theorem 4.3.8 For the ASBS$_{2a}$ preconditioning technique, the relative condition number $\kappa(M_{asbs2}^{-1}S)$ is bounded independently of the partitioning parameters H and h, i.e.

$$\kappa(M_{asbs2}^{-1}S) = \frac{\lambda_{\max}(M_{asbs2}^{-1}S)}{\lambda_{\min}(M_{asbs2}^{-1}S)} \leq C.$$

Proof: This proof is based on the observation that the ASBS$_{2a}$ preconditioner is of overlapping Schwarz type, and it is similar to that of Theorem 3.3.8. However, since the ASBS$_{2a}$ preconditioner applies to the SC system, here we use discrete harmonic extensions of the functions defined on Γ. Throughout the proof we maintain the notation adopted when defining (4.3.7) and (4.3.8).

In order to bound the condition number $\kappa(M_{asbs2}^{-1}S)$, we need upper and lower bounds for the spectrum of $M_{asbs2}^{-1}S$. In order to do this, we use Theorem 2.1.3 for the matrix S and the preconditioner M_{asbs2}. First, we find an upper bound for $\lambda_{\max}(M_{asbs2}^{-1}S)$. Let $\mathbf{u} \in S_h^0(\Gamma)$, and let $m_{sbs2}^{(1)}(\cdot,\cdot)$ and $m_{sbs2}^{(2)}(\cdot,\cdot)$ denote the bilinear form (4.3.7) associated with the preconditioners $M^{(1)}_{sbs2}$ and $M^{(2)}_{sbs2}$ respectively. By the Cauchy-Schwarz inequality and Theorem 4.3.4,

$$\begin{aligned} s(\mathbf{u},\mathbf{u}) &\le C(s(\mathbf{u},\mathbf{u}) + s(\mathbf{u},\mathbf{u})) \\ &\le C\left(m^{(1)}_{sbs2}(\mathbf{u},\mathbf{u}) + m^{(2)}_{sbs2}(\mathbf{u},\mathbf{u})\right). \end{aligned}$$

From this estimate, the definition (4.3.8) of $m_{asbs2}(\cdot,\cdot)$, and Theorem 2.1.3, it follows that:

$$\lambda_{\max}(M^{-1}_{asbs2}S) \le C. \tag{4.3.18}$$

Next step of our proof is to determine a lower bound for $\lambda_{\min}(B^{-1}_{asbs2}A)$. Let $\tilde{\mathbf{u}}$ be the discrete harmonic extension of $\mathbf{u}$ in Ω, and let $\tilde{\mathbf{u}}_o = Q_{H/2}\tilde{\mathbf{u}}$ be the L^2-projection of $\tilde{\mathbf{u}}$ onto $S^0_{H/2}(\Omega)$, and $\mathbf{u}_o = \tilde{\mathbf{u}}_o|_\Gamma$. Then, by the Cauchy-Schwarz inequality and Lemma 4.3.6,

$$\begin{aligned} m_{asbs2}(\mathbf{u},\mathbf{u}) &= m_{asbs2}(\mathbf{u}-\mathbf{u}_o+\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o+\mathbf{u}_o) \\ &\le Cm_{asbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + Cm_{asbs2}(\mathbf{u}_o,\mathbf{u}_o) \\ &\le Cm_{asbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + Cs(\mathbf{u},\mathbf{u}). \end{aligned}$$

It remains to show that

$$m_{asbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \le Cs(\mathbf{u},\mathbf{u}). \tag{4.3.19}$$

We demonstrate that:

$$m^{(1)}_{sbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \le Cs(\mathbf{u},\mathbf{u}) \tag{4.3.20}$$

and

$$m^{(2)}_{sbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \le Cs(\mathbf{u},\mathbf{u}). \tag{4.3.21}$$

Let $\mathbf{w} = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o$. Then $\mathbf{w}|_\Gamma = \mathbf{u} - \mathbf{u}_o$. At the first stage, for $\mathbf{w}$, we construct a representation of the form (4.2.2):

$$\mathbf{w} = \sum_k \tilde{\mathbf{u}}^e_k + \sum_j \tilde{\mathbf{u}}^j,$$

as follows. Let η^j be defined as in the proof of Theorem 3.3.8, and η_k be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes of the boundary $\partial\tilde{\Gamma}_k$ (hence at the ends of Γ_k as well) and everywhere else on $\Omega \setminus \tilde{\Gamma}_k$, grows linearly to 1 on the finite element nodes of Γ_k such that $\|\nabla \eta_k\|_{L^\infty(\tilde{\Gamma}_k)} \leq C/H$, and it is identically 1 on the remaining finite element nodes of Γ_k. If I^h is the finite element interpolation operator onto the space $S_h^0(\Omega)$, then we define:

$$\bar{\mathbf{u}}_k^e = I^h(\eta_k \mathbf{w}) \qquad \text{and} \qquad \bar{\mathbf{u}}^j = I^h(\eta^j \mathbf{w}).$$

Note that if $\{\eta_k\}$ and $\{\eta^j\}$ form partitions of unity, then:

$$\mathbf{w} = \sum_k \bar{\mathbf{u}}_k^e \qquad \text{and} \qquad \mathbf{w} = \sum_j \bar{\mathbf{u}}^j.$$

We show that:

$$\sum_k \alpha_{\Gamma_k} (\delta_{\Gamma_k}^{1/2} \mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} \leq Cs(\mathbf{u}, \mathbf{u}) \tag{4.3.22}$$

and

$$\sum_j \alpha_{\Gamma^j} (\delta_{\Gamma^j}^{1/2} \mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \leq Cs(\mathbf{u}, \mathbf{u}). \tag{4.3.23}$$

We proceed to bound the energies of the parts of $\bar{\mathbf{u}}_k^e$ associated with the elements in Σ^h. If $\bar{\eta}_k$ is the average of η_k on a single mesh-element σ_h, then:

$$\|\eta_k - \bar{\eta}_k\|_{L^\infty(\sigma_h)}^2 \leq C(h/H)^2.$$

By the Cauchy-Schwarz inequality, we can write:

$$\begin{aligned} |\bar{\mathbf{u}}_k^e|_{H^1(\sigma_h)}^2 &= |I^h(\eta_k \mathbf{w})|_{H^1(\sigma_h)}^2 \\ &\leq 2|\bar{\eta}_k \mathbf{w}|_{H^1(\sigma_h)}^2 + 2|I^h(\bar{\eta}_k - \eta_k)\mathbf{w}|_{H^1(\sigma_h)}^2. \end{aligned} \tag{4.3.24}$$

Since $\|\bar{\eta}_k\|_{L^\infty(\sigma_h)} \leq 1$, the first term on the right hand-side of the inequality in (4.3.24) can be bounded as:

$$2|\bar{\eta}_k \mathbf{w}|_{H^1(\sigma_h)}^2 \leq C|\mathbf{w}|_{H^1(\sigma_h)}^2. \tag{4.3.25}$$

For the second term on the right hand-side of the inequality in (4.3.24), from the inverse estimate in Lemma (2.1.2), the weighted norm on element σ_h of diameter h, and the bound on the gradient of η_k, we obtain:

$$\begin{aligned} 2|I^h(\bar{\eta}_k - \eta_k)\mathbf{w}|^2_{H^1(\sigma_h)} &\le C\frac{1}{h^2}\|I^h(\bar{\eta}_k - \eta_k)\mathbf{w}\|^2_{L^2(\sigma_h)} \\ &\le C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\sigma_h)}. \end{aligned} \tag{4.3.26}$$

Since each σ_h is associated with only two $\bar{\mathbf{u}}_k$, from (4.3.24), (4.3.25), and (4.3.26), we deduce:

$$\sum_{\Gamma_k \subset \partial\Omega_i^s} |\bar{\mathbf{u}}_k^e|^2_{H^1(\sigma_h)} \le C|\mathbf{w}|^2_{H^1(\sigma_h)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\sigma_h)}.$$

After summing with respect to σ_h, we obtain:

$$\sum_{\Gamma_k \subset \partial\Omega_i^s} |\bar{\mathbf{u}}_k^e|^2_{H^1(\Omega_i^s)} \le C|\mathbf{w}|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega_i^s)}.$$

We choose $\tilde{\mathbf{u}}_k^e$ to be the discrete harmonic extension of $\mathbf{u}_k^e = \bar{\mathbf{u}}_k^e|_{\Gamma_k}$ in $\tilde{\Gamma}_k$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5), of discrete harmonic functions, imply:

$$\sum_{\Gamma_k \subset \partial\Omega_i^s} |\tilde{\mathbf{u}}_k^e|^2_{H^1(\Omega_i^s)} \le C|\mathbf{w}|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega_i^s)}.$$

From this estimate, through the equivalence (4.3.5) and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned} \sum_{\Gamma_k \subset \partial\Omega_i^s} \alpha_{\Gamma_k}\|\mathbf{u}_k^e\|^2_{H^{1/2}_{oo}(\Gamma_k)} &\le C \sum_{\Gamma_k \subset \partial\Omega_i^s} |\mathbf{u}_k^e|^2_{H^{1/2}((\partial\Omega_i^s)} \\ &\le C|\mathbf{w}|^2_{H^1(\Omega_i^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega_i^s)}. \end{aligned}$$

Since each Γ_k is shared by only two subdomains Ω_i^s, after summing over all $\Omega_i^s \subset \Omega$, Lemma 3.3.6 implies:

$$\begin{aligned} \sum_k \alpha_{\Gamma_k}\|\mathbf{u}_k^e\|^2_{H^{1/2}_{oo}(\Gamma_k)} &\le C|\mathbf{w}|^2_{H^1(\Omega)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} \\ &\le C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}. \end{aligned} \tag{4.3.27}$$

Then, by the equivalence (4.3.6), the Cauchy-Schwarz inequality for the decomposition of Ω into nonoverlapping subdomains Ω_i^s, and the right hand-side inequality in Theorem 2.2.3,

$$\begin{aligned}\sum_k \alpha_{\Gamma_k}(\delta_{\Gamma_k}^{1/2}\mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} &\leq C \sum_{\Omega_i^s \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega_i^s)} \\ &\leq C \sum_{\Omega_i^s \subset \Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i^s)} \\ &\leq C \sum_{\Omega_i^s \subset \Omega} s_i(\mathbf{u}, \mathbf{u}).\end{aligned}$$

Thus, by the representation (4.3.4) of $s(\cdot,\cdot)$, we obtain (4.3.22).

Similar arguments yield:

$$\sum_{\Gamma^j \subset \partial\Omega^s} |\bar{\mathbf{u}}^j|^2_{H^1(\Omega^s)} \leq C|\mathbf{w}|^2_{H^1(\Omega^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s)}.$$

We choose $\tilde{\mathbf{u}}^j$ as the discrete harmonic extension of $\mathbf{u}^j = \bar{\mathbf{u}}^j|_{\Gamma^j}$ into $\tilde{\Gamma}^j$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5), of discrete discrete harmonic functions, imply:

$$\sum_{\Gamma^j \subset \partial\Omega^s} |\tilde{\mathbf{u}}^j|^2_{H^1(\Omega^s)} \leq C|\mathbf{w}|^2_{H^1(\Omega^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s)}.$$

From this estimate, through the equivalence (3.3.6) and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{\Gamma^j \subset \partial\Omega^s} \alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} &\leq C \sum_{\Gamma^j \subset \partial\Omega^s} |\mathbf{u}^j|^2_{H^{1/2}(\partial\Omega^s)} \\ &\leq C|\mathbf{w}|^2_{H^1(\Omega^s)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s)}.\end{aligned}$$

Since each edge Γ^j is shared by only two strips Ω^s, after summing over all $\Omega^s \subset \Omega$, Lemma 3.3.6 implies:

$$\begin{aligned}\sum_j \alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} &\leq C|\mathbf{w}|^2_{H^1(\Omega)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} \\ &\leq C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned} \tag{4.3.28}$$

Then, through the equivalence (3.3.7), the Cauchy-Schwarz inequality for the decomposition of Ω into nonoverlapping subdomains Ω_i^s, and the right hand-side inequality in Theorem 2.2.3,

$$\begin{aligned}\sum_j \alpha_{\Gamma^j}(\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} &\leq C \sum_{\Omega_i^s \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega_i^s)} \\ &\leq C \sum_{\Omega_i^s \subset \Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i^s)} \\ &\leq C \sum_{\Omega_i^s \subset \Omega} s_i(\mathbf{u}, \mathbf{u}).\end{aligned}$$

Thus, by the representation (4.3.4) of $s(\cdot,\cdot)$, we obtain (4.3.23). The estimates (4.3.22) and (4.3.23), and the definition (4.3.7) of $m_{sbs2}(\cdot,\cdot)$ imply (4.3.20). Analogously, at the second stage, we obtain (4.3.21). Then, through the definition (4.3.8) of $m_{asbs2}(\cdot,\cdot)$, (4.3.20) and (4.3.21) imply (4.3.19).

Therefore, by Theorem 2.1.3 for the matrix S and the preconditioner M_{asbs2}, $\lambda_{\min}(M_{asbs2}^{-1}S)$ is bounded independently of the partitioning parameters H and h. Since (4.3.18) also holds, we conclude that:

$$\kappa(M_{asbs2}^{-1}S) \leq C. \ \square$$

Remark 4.3.9 We note that the argument used in the proof of Theorem 4.3.8 cannot be applied to prove the growth of order $\mathcal{O}(1/H)$ for the relative condition number $\kappa(M_{sbs2}^{-1}S)$ in Theorem 4.3.4. This is because, for Theorem 4.3.4, in the proof of Theorem 4.3.8 we must take $\mathbf{w} = \tilde{\mathbf{u}}$. Therefore, through the Poincaré - Friedrichs inequality, (4.3.27) becomes:

$$\begin{aligned}\sum_k \alpha_{\Gamma_k} \|\mathbf{u}_k^e\|^2_{H_{00}^{1/2}(\Gamma_k)} &\leq C|\mathbf{w}|^2_{H^1(\Omega)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} \\ &\leq C\left(1 + \frac{1}{H^2}\right)|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}\end{aligned}$$

and (4.3.28) becomes:

$$\begin{aligned}\sum_j \alpha_{\Gamma^j} \|\mathbf{u}^j\|^2_{H_{00}^{1/2}(\Gamma^j)} &\leq C|\mathbf{w}|^2_{H^1(\Omega)} + C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} \\ &\leq C\left(1 + \frac{1}{H^2}\right)|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned}$$

These estimates lead to an upper bound of order $\mathcal{O}(1/H^2)$ for $\kappa(M_{sbs2}^{-1}S)$.

Remark 4.3.10 We mention here that (4.3.19) also follows from (4.3.13), the left hand-side inequality in Theorem 2.2.3, Lemma 3.3.6, and the right hand-side inequality in Theorem 2.2.3, as:

$$\begin{aligned} m_{asbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) &\le Cs(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \\ &\le C|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_o|^2_{H^1(\Omega)} \\ &\le Cs(\mathbf{u},\mathbf{u}). \end{aligned}$$

However, we chose to show that (4.3.20) and (4.3.21) hold as well.

Theorem 4.3.11 For the $ASBS_{2ga}$ preconditioning technique, the relative condition number $\kappa(M^{-1}_{asbs2g}S)$ is bounded independently of the partitioning parameters H and h, i.e.

$$\kappa(M^{-1}_{asbs2g}S) = \frac{\lambda_{\max}(M^{-1}_{asbs2g}S)}{\lambda_{\min}(M^{-1}_{asbs2g}S)} \le C.$$

Proof: This result can be demonstrated in a similar manner as Theorem 4.3.8 above, by simply replacing the functions at the first stage by those at the coarse level, and the functions at the second stage by those at the fine level. However, we present here a new approach for bounding the minimum eigenvalue, which is based on the observation that the $ASBS_{2ga}$ preconditioner is of a two-level type. This argument is also valid for Theorem 4.3.8, with the corresponding change of notation. Moreover, in view of Remark 4.3.9, this approch can be regarded as an extension of the argument used to demonstrate Theorem 4.3.4. Throughout the proof we maintain the notation adopted when defining (4.3.7) and (4.3.9).

First, we derive an upper bound for $\lambda_{\max}(M^{-1}_{asbs2g}S)$ as follows. Let $\mathbf{u} \in S^0_h(\Gamma)$, and let $m^c_{sbs2}(\cdot,\cdot)$ and $m^f_{sbs2}(\cdot,\cdot)$ represent the bilinear form (4.3.7) associated with the preconditioners M^c_{sbs2} and M^f_{sbs2} respectively. By the Cauchy-Schwarz inequality and Theorem 4.3.4,

$$\begin{aligned} s(\mathbf{u},\mathbf{u}) &\le C\left(s(\mathbf{u},\mathbf{u}) + s(\mathbf{u},\mathbf{u})\right) \\ &\le C\left(m^c_{sbs2}(\mathbf{u},\mathbf{u}) + m^f_{sbs2}(\mathbf{u},\mathbf{u})\right). \end{aligned}$$

Therefore, by the definition (4.3.9) of $m_{asbs2g}(\cdot,\cdot)$ and Theorem 2.1.3 for the matrix

S and the preconditioner M_{asbs2g},

$$\lambda_{\max}(M_{asbs2}^{-1}S) \leq C. \tag{4.3.29}$$

Next, we determine a lower bound for $\lambda_{\min}(M_{asbs2g}^{-1}S)$. Let $\tilde{\mathbf{u}}$ be the discrete harmonic extension of $\mathbf{u}$ in Ω, and let $\tilde{\mathbf{u}}_o = Q_{2^p h}\tilde{\mathbf{u}}$ be the L^2-projection of $\tilde{\mathbf{u}}$ onto $S^0_{2^p h}(\Omega)$, and $\mathbf{u}_o = \tilde{\mathbf{u}}_o|_\Gamma$. Then, by the Cauchy-Schwarz inequality and Lemma 4.3.6,

$$\begin{aligned}
m_{asbs2g}(\mathbf{u},\mathbf{u}) &= m_{asbs2g}(\mathbf{u}-\mathbf{u}_o+\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o+\mathbf{u}_o) \\
&\leq Cm_{asbs2g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + Cm_{asbs2g}(\mathbf{u}_o,\mathbf{u}_o) \\
&= Cm_{asbs2g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + Cm_{asbs2}(\mathbf{u}_o,\mathbf{u}_o) \\
&\leq Cm_{asbs2g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + Cs(\mathbf{u},\mathbf{u}).
\end{aligned}$$

It remains to show that

$$m_{asbs2g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \leq Cs(\mathbf{u},\mathbf{u}). \tag{4.3.30}$$

We demonstrate that:

$$m^f_{sbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \leq Cs(\mathbf{u},\mathbf{u}) \tag{4.3.31}$$

and

$$m^c_{sbs2}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \leq Cs(\mathbf{u},\mathbf{u}). \tag{4.3.32}$$

Let $\mathbf{w} = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o$. Then $\mathbf{w}|_\Gamma = \mathbf{u}-\mathbf{u}_o$. At the fine stage, in the proof of Theorem 4.3.4, we replace $\mathbf{u}$ by $\mathbf{u}_f$. Then we obtain:

$$\|\mathbf{u}^e_k\|^2_{H^{1/2}_{oo}(\Gamma_k)} \leq C\frac{1}{H}\|\mathbf{u}^e_f\|^2_{L^2(\Gamma_k)} + C|\mathbf{u}^e_f|^2_{H^{1/2}(\Gamma_k)}.$$

From the above estimate, Lemma 4.3.1 (*ii*), and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}
\sum_{\Gamma_k\subset\partial\Omega^s_i} \alpha_{\Gamma_k}\|\mathbf{u}^e_k\|^2_{H^{1/2}_{oo}(\Gamma_k)} &\leq C\sum_{\Gamma_k\subset\partial\Omega^s_i}\frac{1}{H}\|\mathbf{u}^e_f\|^2_{L^2(\Gamma_k)} + C\sum_{\Gamma_k\subset\partial\Omega^s}|\mathbf{u}^e_f|^2_{H^{1/2}(\Gamma_k)} \\
&\leq C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^s_i)} + C|\mathbf{w}|^2_{H^1(\Omega^s_i)}.
\end{aligned}$$

Since each Γ_k is shared by only two subdomains Ω^s_i, after summing over all $\Omega^s_i \subset \Omega$, Lemma 3.3.6 implies:

$$\sum_k \alpha_{\Gamma_k} \|\mathbf{u}_k^e\|^2_{H^{1/2}_{oo}(\Gamma_k)} \le C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} + C|\mathbf{w}|^2_{H^1(\Omega)}$$
$$\le C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.$$

Then, through the equivalence (4.3.6), the Cauchy-Schwarz inequality for the decomposition of Ω into nonoverlapping subdomains Ω_i^s, and the right hand-side inequality in Theorem 2.2.3, we obtain:

$$\sum_k \alpha_{\Gamma_k}(\delta^{1/2}_{\Gamma_k}\mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} \le C\sum_{\Omega_i^s\subset\Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega_i^s)}$$
$$\le C\sum_{\Omega_i^s\subset\Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i^s)}$$
$$\le C\sum_{\Omega_i^s\subset\Omega} s_i(\mathbf{u},\mathbf{u}).$$

Thus, by the representation (4.3.4) of $s(\cdot,\cdot)$,

$$\sum_k \alpha_{\Gamma_k}(\delta^{1/2}_{\Gamma_k}\mathbf{u}_k^e, \mathbf{u}_k^e)_{\Gamma_k} \le Cs(\mathbf{u},\mathbf{u}). \tag{4.3.33}$$

On the other hand, replacing $\mathbf{u}$ by $\mathbf{u}_f$ in the proof of Theorem 4.3.4, also yields:

$$\alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \le C\|\mathbf{u}_f^s\|^2_{L^2(\Gamma^j)} + C|\mathbf{u}_f^s|^2_{H^{1/2}(\Gamma^j)}.$$

From the above estimate and Lemma 3.3.1 *(iii)* and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\sum_{\Gamma^j\subset\partial\Omega^s} \alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \le C\sum_{\Gamma^j\subset\partial\Omega^s}\|\mathbf{u}_f^s\|^2_{L^2(\Gamma^j)} + C\sum_{\Gamma^j\subset\partial\Omega^s}|\mathbf{u}_f^s|^2_{H^{1/2}(\Gamma^j)}$$
$$\le C\frac{1}{H}\|\mathbf{w}\|^2_{L^2(\Omega^s)} + C|\mathbf{w}|^2_{H^1(\Omega^s)}.$$

Since each Γ^j is shared by only two strips Ω^s, after summing over all $\Omega^s\subset\Omega$, Lemma 3.3.6 implies:

$$\sum_j \alpha_{\Gamma^j}\|\mathbf{u}^j\|^2_{H^{1/2}_{oo}(\Gamma^j)} \le C\frac{1}{H}\|\mathbf{w}\|^2_{L^2(\Omega)} + C|\mathbf{w}|^2_{H^1(\Omega)}$$
$$\le C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.$$

Then, through the equivalence (3.3.7), the Cauchy-Schwarz inequality for the decomposition of Ω into nonoverlapping subdomains Ω_i^s, and the right hand-side inequality in Theorem 2.2.3, we obtain:

$$\begin{aligned}\sum_j \alpha_{\Gamma^j}(\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} &\leq C \sum_{\Omega_i^s \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega_i^s)} \\ &\leq C \sum_{\Omega_i^s \subset \Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i^s)} \\ &\leq C \sum_{\Omega_i^s \subset \Omega} s_i(\mathbf{u}, \mathbf{u}).\end{aligned}$$

Thus, by the representation (4.3.4) of $s(\cdot,\cdot)$,

$$\sum_j \alpha_{\Gamma^j}(\delta_{\Gamma^j}^{1/2}\mathbf{u}^j, \mathbf{u}^j)_{\Gamma^j} \leq Cs(\mathbf{u}, \mathbf{u}). \tag{4.3.34}$$

The estimates (4.3.33) and (4.3.34), and the definition (4.3.7) of $m_{sbs2}(\cdot,\cdot)$ imply (4.3.31). Analogously, at the coarse stage, we obtain (4.3.32). Then, through the definition (4.3.9) of $m_{asbs2g}(\cdot,\cdot)$, (4.3.31) and (4.3.32) imply (4.3.30).

Therefore, by Theorem 2.1.3 for the matrix S and the preconditioner M_{asbs2g}, $\lambda_{\min}(M_{asbs2}^{-1}S)$ is bounded independently of the partitioning parameters H and h. Since (4.3.29) also holds, we conclude that:

$$\kappa(M_{asbs2g}^{-1}S) \leq C. \ \square$$

4.4 Numerical Estimates

The purpose of this section is to illustrate the efficiency of the $ASBS_2$ preconditioners when solving the SC system (2.2.10) by the PCG method. The domain Ω is the unit square partitioned into $N = 1/H^2$ equal squares, and the coefficients α are either equal to 1 (Example 4.4.1), or are chosen random constants inside each square (Example 4.4.2). The mesh size is h for the fine grid, and $H/2$ for the coarse grid. The iteration counts are for a reduction in error of 10^{-4}.

Example 4.4.1 Consider the Poisson equation:

$$\begin{cases} -\Delta\mathbf{u}(x) = \mathbf{f}(x) \text{ in } \Omega = (0,1)\times(0,1) \\ \mathbf{u}(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

Table 4.4.1: Condition number and iteration counts for SBS_2 in the case of constant coefficients (Example 4.4.1).

$1/H^2 = N$	$1/h = 64$		128		256	
4	1.3350	4	1.3353	4	1.3354	4
16	1.8669	6	1.8674	6	1.8675	6
64	3.2761	8	3.2771	8	3.2774	8
256	6.3147	12	6.3168	12	6.3174	12

Table 4.4.2: Condition number and iteration counts for $ASBS_{2a}$ in the case of constant coefficients (Example 4.4.1).

$1/H^2 = N$	$1/h = 64$		128		256	
4	1.2549	4	1.2563	4	1.2572	4
16	1.3287	4	1.3269	4	1.3262	4
64	1.4327	5	1.4126	5	1.4036	5
256	1.5902	5	1.5056	5	1.4723	5

Table 4.4.3: Condition number and iteration counts for $ASBS_{2ga}$ in the case of constant coefficients (Example 4.4.1).

$1/H^2 = N$	$1/h = 64$		128		256	
4	2.3109	7	2.3112	7	2.3113	7
16	2.6319	7	2.6270	7	2.6257	7
64	2.8284	8	2.8085	8	2.8032	8
256	3.0722	8	2.9726	8	2.9464	8

Example 4.4.2 Consider the model problem:

$$\begin{cases} -\nabla \cdot \alpha(x)\nabla \mathbf{u}(x) = \mathbf{f}(x) \text{ in } \Omega = (0,1)\times(0,1) \\ \mathbf{u}(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

In this case, Ω is partitioned into four by four uniform square subdomains, as represented in Figure 4.2.1. The subdomains are Ω_i, $i = 1, \cdots, 16$, and their corresponding coefficients are indicated in Table 4.4.4. Inside each strip, the coefficients on the edge between two subdomains Ω_i and Ω_j, with coefficients α_i and α_j respectively, are taken to be equal to the average values $(\alpha_i + \alpha_j)/2$, for all i, j. For every edge between two strips, the coefficient is chosen as the average value of the coefficients corresponding to the adjacent subdomains.

Table 4.4.4: The specified discontinuous coefficients for Example 4.4.2.

$\alpha_1 = 10$	$\alpha_2 = 10^{-4}$	$\alpha_3 = 10^2$	$\alpha_4 = 10^{-2}$
$\alpha_5 = 10^4$	$\alpha_6 = 1$	$\alpha_7 = 10^{-3}$	$\alpha_8 = 10^2$
$\alpha_9 = 10^2$	$\alpha_{10} = 10^{-2}$	$\alpha_{11} = 15$	$\alpha_{12} = 10^{-4}$
$\alpha_{13} = 20^{-3}$	$\alpha_{14} = 10^2$	$\alpha_{15} = 10^4$	$\alpha_{16} = 5$

Table 4.4.5: Condition number and iteration counts in the case of discontinuous coefficients (Example 4.4.2).

$1/H^2 = N = 16$	$1/h = 64$		128		256	
SBS_2	1.8669	6	1.8674	6	1.8675	6
$ASBS_{2ga}$	2.6321	7	2.6275	7	2.6262	7

Discussion: Table 4.4.1 shows that for the SBS_2 preconditioning technique, the relative condition number grows linearly as $1/H$ and remains bounded independently of the mesh parameter h (see Theorem 4.3.4). In Tables 4.4.2 and 4.4.3, for the $ASBS_{2a}$ and $ASBS_{2ga}$ preconditioning techniques respectively, although the relative condition number seems to increase slightly with the number of subdomains, the growth is asymptotic towards a value which has not yet been reached, hence this condition number can be bounded independently of the partitioning parameters H and h (see Theorems 4.3.8 and 4.3.11). The values in Table 4.4.2 appear to be smaller than those in Table 4.4.3. However, the two-grid method has the advantage that the subproblems defined on the coarse grid are significantly smaller, than those defined on the fine grid. Note that the results in Table 4.4.5, for the case of discontinuous coefficients, differ negligibly from those given for the Laplace operator. Similar results were obtained in tests with other randomly chosen coefficients.

4.5 Summary

In this chapter, we have introduced a new class of strip-based iterative substructuring techniques for the SC system (2.2.10) in two dimensions. The new solvers can be regarded as extensions of the strip-based solvers presented in Chapter 3 to the case when each strip is a union of nonoverlapping subdomains. The global interface between all subdomains is the union of edges between strips and edges between subdomains inside the same strip. The interior problems on each subdomain being solved exactly, the variables corresponding to the interior of the subdomains are eliminated from the iterative process, which becomes a boundary iteration on the global interface between subdomains. The main task of the (two-stage) $ASBS_2$ process is to determine the interface data between all subdomains, by solving iteratively the SC problem. This method achieves scalability, and therefore optimal convergence properties, by alternating the (one-stage) SBS_2 solver (based on the J-operator on the edges between strips and on the edges between subdomains inside strips) in the horizontal direction, with the SBS_2 solver in the vertical direction.

We note that the convergence behaviour of the $ASBS_2$ preconditioners is comparable to that of the VS preconditioner (2.2.21) with overlap of order $\mathcal{O}(H)$. However, in (2.2.21) S_{ν_j} is dense and expensive to compute, hence for the VS method the design of appropriate approximations for the local problems associated with the vertex-regions still has to be considered to reduce computational complexity. In this context, the VS method can be regarded as a Schwarz method (Nepomnyaschikh (1986) [63]), while our new preconditioners can be viewed as inexact Schwarz solvers. Moreover, when the two-grid technique is applied, we solve one-dimensional problems on edges (between strips and between subdomains inside strips) at the coarse stage, and alternate the direction of the strips at the fine stage. The possibility of reducing the size of the coarse solver from two to only one dimension seems to offer an advantage, especially if H is small.

Chapter 5

Alternate Slice-Based Substructuring Algorithms for Symmetric Elliptic PDE's in 3D

5.1 Introduction

In this chapter, we derive and analyse some possible extensions to three-dimensional problems of the preconditioning techniques introduced in Chapter 4. The slice-based methods to be presented in this chapter are DD preconditioning techniques for the SC system (2.2.10) in the case of a decomposition of the domain $\Omega \subset \mathbb{R}^3$, into multiple disjoint subdomains with interior cross-points. In this case, the global interface $\Gamma \subset \Omega$ between subdomains contains faces, edges, and vertices of these subdomains. In order to avoid the separate treatment of the interior edges and the vertices (i.e. of the wire-basket, see Section 2.2), first the subdomains are assembled into disjoint bars, then the bars are assembled into disjoint slices. The vertices of each bar are on the boundary $\partial\Omega$, the interfaces between bars (i.e. faces shared by two bars) are strips which overlap with the faces of the subdomains, the interfaces between strips (i.e. edges shared by at least two strips) align with the edges of the subdomains, and the union of the interfaces between strips contains all of the interior vertices of the initial decomposition of the domain. The edges of each slice are on $\partial\Omega$, and the union of all the interfaces between slices (i.e. faces shared by two slices) contains all of the interfaces between strips. Therefore, the global interface Γ can be partitioned as a union of interfaces between slices, interfaces between bars, and interfaces between subdomains inside bars. For the subproblems corresponding to the various faces, the J-operator is used (see Section 2.2). As in the two-dimensional case, scalability is achieved in two stages. At each stage the slices change such that the interfaces between slices at one stage are orthogonal to the interfaces between slices at the second stage. The two stages allow the use of a two-grid V-cycle and guarantee a good rate of convergence of the preconditioned iterative procedures, which is optimal with respect to the partitioning parameters.

This chapter is organised as follows. In Section 5.2, we describe the slice-based substructuring (SBS_3) and the alternate slice-based substructuring ($ASBS_3$) preconditioning techniques. These are further analysed in Section 5.3. Numerical examples to illustrate the performance of these DD methods are given in Section 5.4. Section 5.5 summarises this chapter.

5.2 Slice-Based Substructuring

We consider the problem (2.1.1) in the three-dimensional case. For clarity of presentation, let the domain $\Omega \subset \mathbb{R}^3$ be the unit cube $(0,1) \times (0,1) \times (0,1)$. Let (2.2.1) be the initial partitioning of Ω into subdomains, such that each subdomain is an open cube of uniform size $0 < H < 1$ (Figure 5.2.1). In every subdomain, we consider the coefficient $\alpha(x)$ of (2.1.1) to be constant.

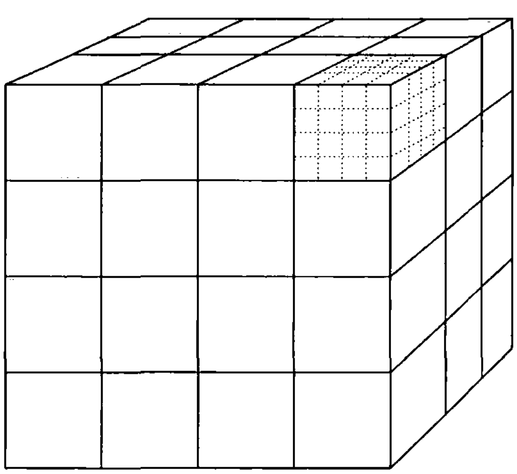

Figure 5.2.1: The initial partitioning of the domain $\Omega \subset \mathbb{R}^3$, with mesh refinement shown on one subdomain.

The boundary of each subdomain is partitioned into faces, edges, and vertices, such that the faces are open squares in $\mathbb{R}^2$, the edges are open lines in $\mathbb{R}$, defined to be the intersection of the boundaries of at least two faces, and the vertices are point sets which are the end-points of the edges. First, we assemble the subdomains in the initial partitioning of Ω into disjoint bars, Ω^b $(b = 1, \cdots, n_b)$, such that: the vertices of each bar are on the boundary $\partial\Omega$, the interfaces between bars are open (rectangular) strips which overlap with the faces of the subdomains, the interfaces between strips are lines which overlap with the edges of the subdomains, and the union of the interfaces between strips contains all of the interior vertices of the initial partitioning of Ω (see Figure 5.2.2-left). Thus the bars Ω^b form a partitioning of Ω:

$$\bar{\Omega} = \bigcup_{b=1}^{n_b} \bar{\Omega}^b. \tag{5.2.1}$$

Next, we assemble the bars Ω^b into disjoint slices, Ω^S $(S = 1, \cdots, n_S)$, such that: each face between two slices is an open square with its boundary on $\partial\Omega$, and the union of all the faces between slices contains all of the interfaces between strips (see Figure 5.2.2-right). Thus the slices Ω^S also form a partitioning of Ω.

$$\bar{\Omega} = \bigcup_{S=1}^{n_S} \bar{\Omega}^S. \tag{5.2.2}$$

Let Ω_b^S denote a generic bar in Ω^S, and Ω_i^b denote a generic subdomain (of uniform size H) in the bar Ω_b^S. We denote by F_l^S a generic face between two slices, by F_j^b a generic face between two bars inside the same slice, and by F_k^i a generic face between two subdomains in the same bar. Let Γ be the global interface between all subdomains in the initial partitioning of Ω. Then the union of the faces F_l^S, F_j^b, and F_k^i form a partitioning of Γ.

Let $S_h^0(\Omega)$ be as described in Section 2.1 $(h < H)$. For every slice Ω^S, we consider the restrictions on $\bar{\Omega}^S \cap \Omega$ of the functions in $S_h^0(\Omega)$ and denote the finite element space of these restrictions by $S_h^0(\bar{\Omega}_b^S)$. We define $S_h^0(\Omega^S)$ to be the subspace of $S_h^0(\bar{\Omega}^S)$ consisting of those functions which are zero on the boundary $\partial\Omega^S \cap \Omega$. Next, for every bar $\Omega_b^S \subset \Omega^S$, we consider the restrictions on $\bar{\Omega}_b^S \cap \Omega$ of the functions in $S_h^0(\bar{\Omega}^S)$ and denote the finite element space of these restrictions by $S_h^0(\bar{\Omega}_b^S)$. We define $S_h^0(\Omega_b^S)$ to be the subspace of $S_h^0(\bar{\Omega}_b^S)$ consisting of those functions which are zero on the boundary $\partial\Omega_b^S \cap \Omega$. Finally, for every subdomain $\Omega_i^b \subset \Omega_b^S$, we consider the restrictions on $\bar{\Omega}_i^b \cap \Omega$ of the functions in $S_h^0(\bar{\Omega}_b^S)$ and denote the finite element space of these restrictions by $S_h^0(\bar{\Omega}_i^b)$. We define $S_h^0(\Omega_i^b)$ to be the subspace of $S_h^0(\bar{\Omega}_i^b)$

consisting of those functions which are zero on the boundary $\partial\Omega_i^b \cap \Omega$.

We also consider the restrictions on Γ of the functions in $S_h^0(\Omega)$ and denote the finite element space of these restrictions by $S_h^0(\Gamma)$. We define $S_h^0(\partial\Omega^S)$, $S_h^0(\partial\Omega_b^S)$, $S_h^0(\partial\Omega_i^b)$, $S_h^0(\partial\Omega_i^b \cap \Omega_b^S)$, $S_h^0(\partial\Omega_i^b \cap \Omega^S)$, $S_h^0(\partial\Omega_b^S \cap \Omega^S)$, $S_h^0(F_l^S)$, $S_h^0(F_j^b)$, and $S_h^0(F_k^i)$ to be the subspaces of $S_h^0(\Gamma)$ consisting of those functions which are zero on $\Gamma \setminus \partial\Omega^S$, $\Gamma \setminus \partial\Omega_b^S$, $\Gamma \setminus \partial\Omega_i^b$, $\Gamma \setminus (\partial\Omega_i^b \cap \Omega_b^S)$, $\Gamma \setminus (\partial\Omega_i^b \cap \Omega^S)$, $\Gamma \setminus (\partial\Omega_b^S \cap \Omega^S)$, $\Gamma \setminus F_l^S$, $\Gamma \setminus F_j^b$, and $\Gamma \setminus F_k^i$ respectively.

Furthermore, let $\tilde{F}_k^i$ be the domain obtained by the union of F_k^i with the neighbouring regions Ω_i^b inside the bar Ω_b^S, $\tilde{F}_j^b$ be the domain obtained by the union of F_j^b with the neighbouring bars Ω_b^S inside the strips Ω^S, and $\tilde{F}_l^S$ be the domain obtained by the union of F_l^S with the neighbouring slices Ω^S. Note that these domains form an overlapping covering of Ω, such that every point in Ω is contained in at most six of these domains. We define $S_h^0(\tilde{F}_k^i)$, $S_h^0(\tilde{F}_j^b)$, and $S_h^0(\tilde{F}_l^S)$ to be the subspaces of $S_h^0(\Omega)$ consisting of those functions with support in $\tilde{F}_k^i$, $\tilde{F}_k^b$, and $\tilde{F}_l^S$ respectively. Then:

$$S_h^0(\Omega) = \sum_{i,k} S_h^0(\tilde{F}_k^i) + \sum_{b,j} S_h^0(\tilde{F}_j^b) + \sum_{S,l} S_h^0(\tilde{F}_l^S), \tag{5.2.3}$$

i.e. for all $\mathbf{u} \in S_h^0(\Omega)$, there exists a representation, which is not unique, of the form:

$$\mathbf{u} = \sum_{i,k} \mathbf{u}_k^i + \sum_{b,j} \mathbf{u}_j^b + \sum_{S,l} \mathbf{u}_l^S, \tag{5.2.4}$$

$\mathbf{u}_k^i \in S_h^0(\tilde{F}_k^i)$, $\mathbf{u}_j^b \in S_h^0(\tilde{F}_j^b)$, $\mathbf{u}_l^S \in S_h^0(\tilde{F}_l^S)$.

5.2.1 The Slice-Based Substructuring Technique

The slice-based substructuring (SBS_3) preconditioner is a natural extension to the three-dimensional case of the SBS_2 preconditioner introduced in Chapter 4. Let $0xyz$ be a three-dimensional orthonormal system of coordinates. In $S_h^0(\Gamma)$, let $\{\psi^1\}$, $\{\psi^2\}$, and $\{\psi^3\}$ be the sets of finite element basis functions corresponding to the union of faces in the planes $x0y$, $y0z$ and $z0x$ respectively; $\{\psi^v\}$ be the set of finite element basis functions corresponding to the union of the vertex-points; and $\{\psi^x\}$, $\{\psi^y\}$, and $\{\psi^z\}$, the sets of finite element basis functions corresponding to the union of the edges that lie in the $0x$, $0y$, and $0z$ direction respectively. The set of functions $\{\psi^1, \psi^2, \psi^3, \psi^v, \psi^x, \psi^y, \psi^z\}$ is a basis for $S_h^0(\Gamma)$, thus any function in $S_h^0(\Gamma)$ can be decomposed as a linear combination of this basis and represented by a vector of its coefficients. If we order these vectors as $[u^1\ u^2\ u^3\ u^v\ u^x\ u^y\ u^z]^T$ and consider the SC system (2.2.10), then the SC matrix S can be described in terms of block matrices as:

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{1v} & S_{1x} & S_{1y} & S_{1z} \\ S_{12}^T & S_{22} & S_{23} & S_{2v} & S_{2x} & S_{2y} & S_{2z} \\ S_{13}^T & S_{23}^T & S_{33} & S_{3v} & S_{3x} & S_{3y} & S_{3z} \\ S_{1v}^T & S_{2v}^T & S_{3v}^T & S_{3v} & S_{vx} & S_{vy} & S_{vz} \\ S_{1x}^T & S_{2x}^T & S_{3x}^T & S_{vx}^T & S_{xx} & S_{xy} & S_{xz} \\ S_{1y}^T & S_{2y}^T & S_{3y}^T & S_{vy}^T & S_{xy}^T & S_{yy} & S_{yz} \\ S_{1z}^T & S_{2z}^T & S_{3z}^T & S_{vz}^T & S_{xz}^T & S_{yz}^T & S_{zz} \end{bmatrix}.$$

We take, for instance, the faces F_k^i between subdomains Ω_i^b in the $y0z$ plane, the faces F_j^b between bars in the $x0z$ plane, such that each bar Ω^b is of size $1 \times H \times H$ (see Figure 5.2.2-left), and the faces F_l^S between slices in the $x0y$ plane, such that each slice Ω^S is of size $1 \times 1 \times H$ (see Figure 5.2.2-right).

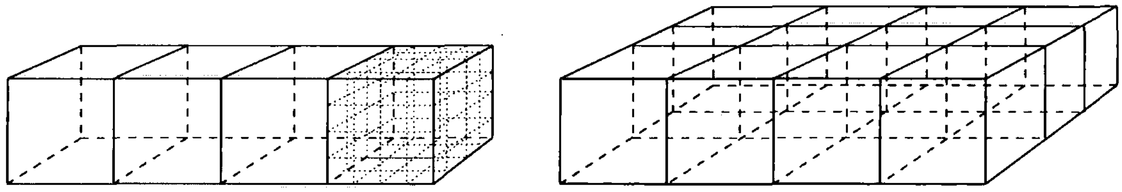

Figure 5.2.2: A bar (with mesh refinement shown on one subdomain) along the Ox axis (left) and the corresponding slice (right) of the domain $\Omega \subset \mathbb{R}^3$.

Then, we denote by $\Psi_i = [\psi_2]$, $\Psi_b = [\psi_3\ \psi_z]$, and $\Psi_S = [\psi_1\ \psi_v\ \psi_x\ \psi_y]$ the basis functions assembled into vectors corresponding to the interfaces between subdomains inside bars, the interfaces between bars, and the interfaces between slices respectively. Note that Ψ_i, Ψ_b, and Ψ_S also form a basis for $S_h^0(\Gamma)$ and so any function in $S_h^0(\Gamma)$ can be decomposed as a linear combination of this basis and represented by a vector of its coefficients. If we order these vectors as $[u^i\ u^b\ u^S]^T$, and consider the SC system (2.2.10), then S can be described in terms of block matrices as:

$$S = \begin{bmatrix} S_{ii} & S_{ib} & S_{iS} \\ S_{ib}^T & S_{bb} & S_{bS} \\ S_{iS}^T & S_{bS}^T & S_{SS} \end{bmatrix}.$$

We observe that, by reordering the nodes, S_{ii} can be expressed as block-diagonal matrix with each block corresponding to the boundary of a domain Ω_i^s, in the interior of a bar Ω_b^S.

Now, S can be expressed in factored form as:

$$S = \begin{bmatrix} I_{ii} & 0 & 0 \\ S_{ib}^T S_{ii}^{-1} & I_{bb} & 0 \\ S_{iS}^T S_{ii}^{-1} & 0 & I_{SS} \end{bmatrix} \times \begin{bmatrix} S_{ii} & 0 & 0 \\ 0 & \bar{S}_{bb} & \bar{S}_{bS} \\ 0 & \bar{S}_{bS}^T & \bar{S}_{SS} \end{bmatrix} \times \begin{bmatrix} I_{ii} & S_{ib} S_{ii}^{-1} & S_{iS} S_{ii}^{-1} \\ 0 & I_{bb} & 0 \\ 0 & 0 & I_{SS} \end{bmatrix},$$

where I_{ii}, I_{bb} and I_{SS} denote identity matrices, and

$$\begin{aligned} \begin{bmatrix} \bar{S}_{bb} & \bar{S}_{bS} \\ \bar{S}_{bS}^T & \bar{S}_{SS} \end{bmatrix} &= \begin{bmatrix} S_{bb} & S_{bS} \\ S_{bS}^T & S_{SS} \end{bmatrix} - \begin{bmatrix} S_{ib}^T \\ S_{iS}^T \end{bmatrix} \times S_{ii}^{-1} \times \begin{bmatrix} S_{ib} & S_{iS} \end{bmatrix} \\ &= \begin{bmatrix} S_{bb} - S_{ib}^T S_{ii}^{-1} S_{ib} & S_{bS} - S_{ib}^T S_{ii}^{-1} S_{iS} \\ S_{bS}^T - S_{iS}^T S_{ii}^{-1} S_{ib} & S_{SS} - S_{iS}^T S_{ii}^{-1} S_{iS} \end{bmatrix} \end{aligned}$$

corresponds to the union of faces F_j^b (between bars), and faces F_l^S (between slices).

Remark 5.2.1 It can be shown that the matrix:

$$\begin{bmatrix} \bar{S}_{bb} & \bar{S}_{bS} \\ \bar{S}_{bS}^T & \bar{S}_{SS} \end{bmatrix}$$

is equal to the SC matrix corresponding to the decomposition (5.2.1) of Ω into the disjoint bars Ω_b^S. The calculations are similar as for Remark 4.2.1. Therefore, by reordering the nodes, this matrix can be expressed as a block-diagonal matrix with each block corresponding to the boundary of a bar Ω_b^S, in the interior of a slice Ω^S.

On the other hand, the block diagonal matrix:

$$\begin{bmatrix} S_{ii} & 0 & 0 \\ 0 & \bar{S}_{bb} & \bar{S}_{bS} \\ 0 & \bar{S}_{bS}^T & \bar{S}_{SS} \end{bmatrix}$$

can be decomposed as:

$$\begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & \bar{S}_{bS}^T \bar{S}_{bb}^{-1} & I_{SS} \end{bmatrix} \times \begin{bmatrix} S_{ii} & 0 & 0 \\ 0 & \bar{S}_{bb} & 0 \\ 0 & 0 & \tilde{S}_{SS} \end{bmatrix} \times \begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & \bar{S}_{bS} \bar{S}_{bb}^{-1} \\ 0 & 0 & I_{SS} \end{bmatrix},$$

where

$$\begin{aligned}\tilde{S}_{SS} &= \bar{S}_{SS} - \bar{S}_{bS}^T \bar{S}_{bb}^{-1} \bar{S}_{bS} \\ &= S_{SS} - \begin{bmatrix} S_{iS}^T & S_{bS}^T \end{bmatrix} \times \begin{bmatrix} S_{ii} & S_{ib} \\ S_{ib}^T & S_{bb} \end{bmatrix}^{-1} \times \begin{bmatrix} S_{ib} \\ S_{bS} \end{bmatrix}\end{aligned}$$

corresponds to the union of faces F_l^S (between slices).

Remark 5.2.2 It can be shown that the matrix $\tilde{S}_{SS}$ is equal to the SC matrix corresponding to the decomposition (5.2.2) of Ω into the disjoint slices Ω^S. See also Remark 5.2.1. Therefore, by reordering the nodes, $\tilde{S}_{SS}$ can be expressed as block-diagonal matrix with each block corresponding to the boundary of a slice, in the interior of the domain Ω.

From the above factorisations, we deduce that S can be expressed equivalently as:

$$S = \begin{bmatrix} I_{ii} & 0 & 0 \\ S_{ib}^T S_{ii}^{-1} & I_{bb} & 0 \\ S_{iS}^T S_{ii}^{-1} & 0 & I_{SS} \end{bmatrix} \times \begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & \bar{S}_{bS}^T \bar{S}_{bb}^{-1} & I_{SS} \end{bmatrix} \times \begin{bmatrix} S_{ii} & S_{ib} S_{ii}^{-1} & S_{iS} S_{ii}^{-1} \\ 0 & \bar{S}_{bb} & \bar{S}_{bS} \bar{S}_{bb}^{-1} \\ 0 & 0 & \tilde{S}_{SS} \end{bmatrix}.$$

In order to construct the slice-based preconditioner for the SC system (2.2.10), we proceed as follows. First, in the SC matrix S, we drop all the couplings between different faces F_k^i (inside bars), to obtain the block-diagonal matrix:

$$\mathit{blockdiag}\left(S_{F_k^i}\right),$$

each block $S_{F_k^i}$ corresponding to a face F_k^i.

Similarly, we drop all the couplings between different faces F_j^b (between bars), to obtain the block-diagonal matrix:

$$\mathit{blockdiag}\left(\bar{S}_{F_j^b}\right),$$

each block $\bar{S}_{F_j^b}$ corresponding to a face F_j^b.

Finally, we drop all the couplings between different faces F_l^S (between slices), to obtain the block-diagonal matrix:

$$blockdiag\left(\bar{S}_{F_l^S}\right),$$

each block $\bar{S}_{F_l^S}$ corresponding to a face F_l^S.

Next, for every face $F \in \{F_k^i, F_j^b, F_l^S\}$, if $\mathbf{u} \in S_h^0(F)$, let $(-\Delta_F)$ be the two-dimensional Laplacian operator with domain of definition $H_0^1(F)$, and let δ_F denote the discrete operator defined on $S_h^0(F)$ by:

$$(\delta_F \mathbf{u}, \mathbf{v})_F = (\nabla \mathbf{u}, \nabla \mathbf{v})_F, \ \forall \mathbf{v} \in S_h^0(F),$$

where ∇ denotes the two-dimensional gradient on F, and $(\cdot,\cdot)_F$ is the scalar product in $L^2(F)$. Note that δ_F represents a finite dimensional approximation of $(-\Delta_F)$. Since δ_F is symmetric and positive definite (SPD) in the inner product $(\cdot,\cdot)_F$, we can define the square root $\delta_F^{1/2}$ of δ_F (see Bramble *et al.* (1989) [11], pp. 9-10). We denote by J_F the matrix form of $\delta_F^{1/2}$, then we choose:

$$M_{F_k^i} = \alpha_{F_k^i} J_{F_k^i} \quad \text{to be the approximation for } S_{F_k^i}$$
$$\bar{M}_{F_j^b} = \alpha_{F_j^b} J_{F_j^b} \quad \text{to be the approximation for } \bar{S}_{F_j^b}$$
$$\tilde{M}_{F_l^S} = \alpha_{F_l^S} J_{F_l^S} \quad \text{to be the approximation for } \tilde{S}_{F_l^S}.$$

Then, we set:

$$M_{ii} = blockdiag\left(M_{F_k^i}\right) \quad \text{to be the approximation for } S_{ii}$$
$$\bar{M}_{bb} = blockdiag\left(\bar{M}_{F_j^b}\right) \quad \text{to be the approximation for } \bar{S}_{bb}$$
$$\tilde{M}_{SS} = blockdiag\left(\tilde{M}_{F_l^S}\right) \quad \text{to be the approximation for } \tilde{S}_{SS}.$$

We define the preconditioner M_{sbs3} as:

$$M_{sbs3} = \begin{bmatrix} I_{ii} & 0 & 0 \\ S_{ib}^T M_{ii}^{-1} & I_{bb} & 0 \\ S_{iS}^T M_{ii}^{-1} & 0 & I_{SS} \end{bmatrix} \times \begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & \bar{S}_{bS}^T \bar{M}_{bb}^{-1} & I_{SS} \end{bmatrix} \times \begin{bmatrix} M_{ii} & S_{ib} M_{ii}^{-1} & S_{iS} M_{ii}^{-1} \\ 0 & \bar{M}_{bb} & \bar{S}_{bS} \bar{M}_{bb}^{-1} \\ 0 & 0 & \tilde{M}_{SS} \end{bmatrix}.$$

A generic system $M_{sbs3}w = r$ can now be written in terms of block matrices as:

$$\begin{bmatrix} I_{ii} & 0 & 0 \\ S_{ib}^T M_{ii}^{-1} & I_{bb} & 0 \\ S_{iS}^T M_{ii}^{-1} & 0 & I_{SS} \end{bmatrix} \times \begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & \bar{S}_{bS}^T \bar{M}_{bb}^{-1} & I_{SS} \end{bmatrix} \times \begin{bmatrix} M_{ii} & S_{ib}M_{ii}^{-1} & S_{iS}M_{ii}^{-1} \\ 0 & \bar{M}_{bb} & \bar{S}_{bS}\bar{M}_{bb}^{-1} \\ 0 & 0 & \tilde{M}_{SS} \end{bmatrix} \times \begin{bmatrix} w^i \\ w^b \\ w^S \end{bmatrix} = \begin{bmatrix} r^i \\ r^b \\ r^S \end{bmatrix}. \tag{5.2.5}$$

The solution $w = M_{sbs3}^{-1} r$ can be derived as follows.

The SBS$_3$ Procedure (algebraic form).

(I) compute the solution $M_{ii}^{-1} r^i$ and obtain the system equivalent to (5.2.5):

$$\begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & \bar{S}_{bS}^T \bar{M}_{bb}^{-1} & I_{SS} \end{bmatrix} \times \begin{bmatrix} M_{ii} & S_{ib}M_{ii}^{-1} & S_{iS}M_{ii}^{-1} \\ 0 & \bar{M}_{bb} & \bar{S}_{bS}\bar{M}_{bb}^{-1} \\ 0 & 0 & \tilde{M}_{SS} \end{bmatrix} \times \begin{bmatrix} w^i \\ w^b \\ w^S \end{bmatrix} = \begin{bmatrix} r^i \\ \bar{r}^b \\ \bar{r}^S \end{bmatrix}, \tag{5.2.6}$$

where $\bar{r}^b = r^b - S_{ib}^T M_{ii}^{-1} r^i$ and $\bar{r}^S = r^S - S_{iS}^T M_{ii}^{-1} r^i$.

(II) using $M_{ii}^{-1} r^i$ obtained in (I), compute the solution $\bar{M}_{bb}^{-1} \bar{r}^b$ and obtain the system equivalent to (5.2.6):

$$\begin{bmatrix} M_{ii} & S_{ib}M_{ii}^{-1} & S_{iS}M_{ii}^{-1} \\ 0 & \bar{M}_{bb} & \bar{S}_{bS}\bar{M}_{bb}^{-1} \\ 0 & 0 & \tilde{M}_{SS} \end{bmatrix} \times \begin{bmatrix} w^i \\ w^b \\ w^S \end{bmatrix} = \begin{bmatrix} I_{ii} & 0 & 0 \\ 0 & I_{bb} & 0 \\ 0 & -\bar{S}_{bS}^T \bar{M}_{bb}^{-1} & I_{SS} \end{bmatrix} \times \begin{bmatrix} r^i \\ \bar{r}^b \\ \bar{r}^S \end{bmatrix}, \tag{5.2.7}$$

where $\tilde{r}^S = \bar{r}^S - \bar{S}_{bS}^T \bar{M}_{bb}^{-1} \bar{r}^b = r^S - S_{iS}^T M_{ii}^{-1} r^i - \bar{S}_{bS}^T \bar{M}_{bb}^{-1} \bar{r}^b$.

(III) using $M_{ii}^{-1} r^i$ obtained in (I) and $\bar{M}_{bb}^{-1} \bar{r}^b$ obtained in (II), solve for w^S the system (5.2.7).

(IV) using w^S obtained in (III), solve for w^b the system (5.2.7), by backward substitution.

(V) using w^S obtained in (III) and w^b obtained in (IV), solve for w^b the system (5.2.6), by backward substitution.

As in the two-dimensional case (see Section 4.2), using the preconditioner M_{sbs3} we can construct the following iterative method: start with u^0 as an initial approximation (without restricting the generality we can assume the starting approximation to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1} \leftarrow u^l + M_{sbs3}^{-1}(f_S - Su^l).$$

This can be interpreted as a Richardson iterative procedure (see e.g. Smith *et al.* (1996) [75], Appendix).

Alternatively, since the new preconditioned matrix $M_{sbs3}^{-1}S$ is symmetric and non-negative definite with respect to the $s(\cdot, \cdot)$ scalar product, the CG acceleration can be applied as follows:

- let u^0 be an initial iterate,

$$r^0 \leftarrow f_S - Su^0, \text{ the initial residual}$$
$$w^0 \leftarrow M_{sbs3}^{-1}r^0, \text{ the initial preconditioned residual}$$
$$v^0 \leftarrow w^0, \text{ the initial search direction}$$

- for $l = 0, 1, \cdots$

compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Sv^l)}$

update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

update the residual: $r^{l+1} \leftarrow r^l + p_l S v^l$

if $r^{l+1} \geq$ tolerance, then

update the preconditioned residual: $w^{l+1} \leftarrow M_{sbs3}^{-1} r^{l+1}$

compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

else end for.

The resulting SBS_3 method has good parallelisation properties and a rate of convergence proportional to $1/\sqrt{H}$ (see Theorem 5.3.4 and Table 5.4.1).

Remark 5.2.3 We note that if the problems on the faces F_k^i (inside bars) are solved exactly, then the variables corresponding to these interfaces can be eliminated from the iterative process, which then reduces to an iteration on the boundaries of the bars. Moreover, if the problems on the faces F_j^b (between bars) are also solved exactly, then the variables corresponding to these interfaces can also be eliminated from the iterative process, which then reduces to an iteration on the faces F_l^S (between slices).

5.2.2 The Alternate Slice-Based Substructuring (ASBS$_3$) Technique

In order to obtain scalability with respect to H, we construct a two-stage preconditioner as follows. At each stage the slices change (see Figures 5.2.3) such that the interfaces between slices at one stage are orthogonal to the interfaces between slices at the second stage.

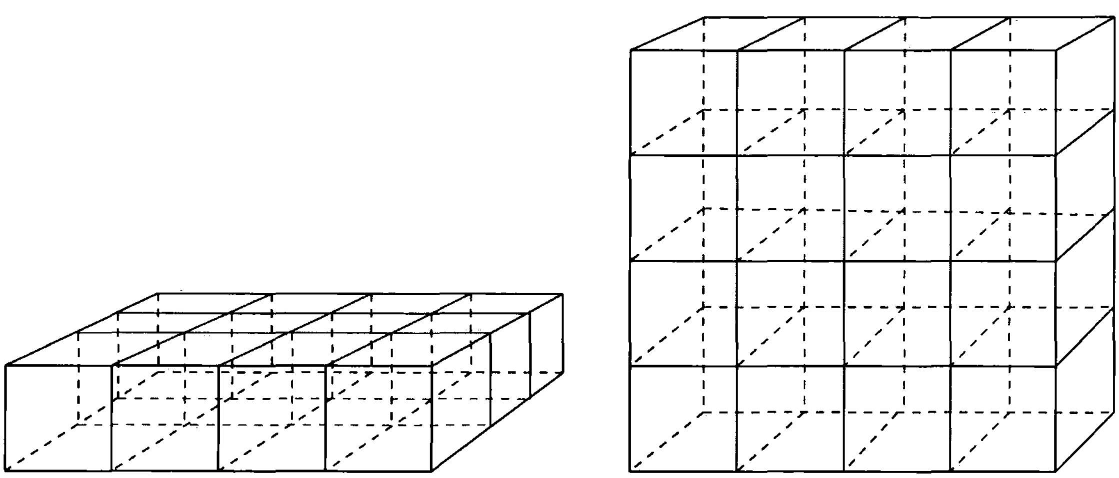

Figure 5.2.3: Slices of the domain $\Omega \subset \mathbb{R}^3$ at two different stages.

Remark 5.2.4 We note that in a two-stage process, the bars may change as well, such that the faces between bars at one stage are orthogonal to the faces between bars at the other stage. Figures 5.2.2-left and 5.2.4-left show bars and slices at two different stages. For clarity of presentation, throughout this section we shall restrict our attention to the case when, like the subdomains in the initial partitioning of Ω, the bars, once formed, do not change when the slices change. However, we remark that our subsequent analysis (see Section 5.3) is also valid in the case when the bars change with the slices.

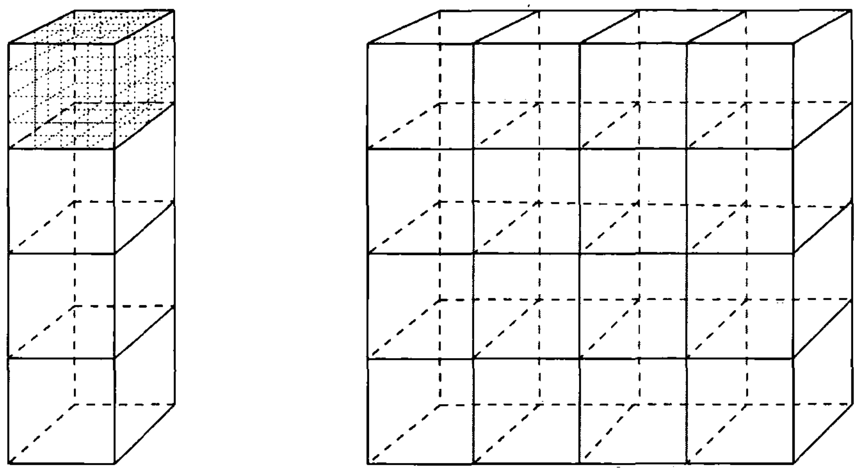

Figure 5.2.4: A bar (with mesh refinement shown on one subdomain) along the Oz axis (left) and the corresponding slice (right) of the domain $\Omega \subset \mathbb{R}^3$.

The Additive Alternate Slice-Based Substructuring ($\mathbf{ASBS}_{3a}$) Algorithm. Let $M^{(1)}_{sbs3}$ and $M^{(2)}_{sbs3}$ denote the SBS_3 preconditioner at the first and second stage respectively. It is easy to see that the global interface Γ is covered by the union of the following two overlapping subdomains: on one hand the interfaces between slices, between bars inside slices, and between subdomains inside bars at the first stage, and on the other hand the interfaces between slices, between bars inside slices, and the between subdomains inside bars at the second stage. Therefore Schwarz algorithms can be derived using the M_{sbs3} preconditioner. The (inexact) additive Schwarz method is: start with u^0 as an initial approximation (without restricting the generality we take this approximation to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1/2} \leftarrow u^l + (M^{(1)}_{sbs3})^{-1}(f_S - Su^l)$$
$$u^{l+1} \leftarrow u^{l+1/2} + (M^{(2)}_{sbs3})^{-1}(f_S - Su^l).$$

This can also be written in one step as:

$$u^{l+1} \leftarrow u^l + \left((M^{(1)}_{sbs3})^{-1} + (M^{(2)}_{sbs3})^{-1}\right)(f_S - Su^l),$$

and interpreted as a Richardson iterative process with the two-stage SC preconditioner defined by:

$$M^{-1}_{asbs3} = (M^{(1)}_{sbs3})^{-1} + (M^{(2)}_{sbs3})^{-1}.$$

The new preconditioned SC matrix $M_{asbs3}^{-1}S$ can also be used with CG acceleration:

- let u^0 be an initial iterate,

$$r^0 \leftarrow f_S - Su^0, \text{ the initial residual}$$
$$w^0 \leftarrow M_{asbs3}^{-1} r^0, \text{ the initial preconditioned residual}$$
$$v^0 \leftarrow w^0, \text{ the initial search direction}$$

- for $l = 0, 1, \cdots$

compute the direction coefficient: $p_l \leftarrow -\dfrac{(w^l, r^l)}{(v^l, Sv^l)}$

update the iterate: $u^{l+1} \leftarrow u^l - p_l v^l$

update the residual: $r^{l+1} \leftarrow r^l + p_l S v^l$

if $r^{l+1} \geq$ tolerance, then

update the preconditioned residual: $w^{l+1} \leftarrow M_{asbs3}^{-1} r^{l+1}$

compute the orthogonalisation coefficient: $q_l \leftarrow \dfrac{(w^{l+1}, r^{l+1})}{(w^l, r^l)}$

update the search direction: $v^{l+1} \leftarrow w^{l+1} + q_l v^l$

else end for.

The following steps will compute $w^l = M_{asbs3}^{-1} r^l$ $(l = 0, 1, \cdots)$:

$$w^{l+1/2} \leftarrow (M_{sbs3}^{(1)})^{-1} r^l$$
$$w^l \leftarrow w^{l+1/2} + (M_{sbs3}^{(2)})^{-1} r^l,$$

or equivalently,

$$w^l \leftarrow \left((M_{sbs3}^{(1)})^{-1} + (M_{sbs3}^{(2)})^{-1}\right) r^l.$$

The resulting ASBS_{3a} method is optimal in the sense that the rate of convergence can be bounded independently of the partitioning parameters H and h (see Theorem 5.3.8 and Table 5.4.2).

However, reloading the problem at the second stage, when the slices change, can be expensive. We therefore consider the possibility of reducing the calculations to a coarser grid at one of the stages (see Figures 5.2.5). Let $\Sigma^{2^p h} \subset \cdots \subset \Sigma^{2h} \subset \Sigma^h$ be a set of nested uniform square grids associated with the original domain Ω, such that $1 \leq p \in \mathbb{N}$ and $2^p h < H$. As in the two-dimensional case, the coarse grid reduced operator for S, S_c, can be defined either by rediscretisation of the problem on the $\Sigma^{2^p h}$ grid, or by the relations $S_c = RSP$, where R is the restriction from grid Σ^h to grid $\Sigma^{2^p h}$ and $P = R^T$ is the prolongation from grid $\Sigma^{2^p h}$ to grid Σ^h.

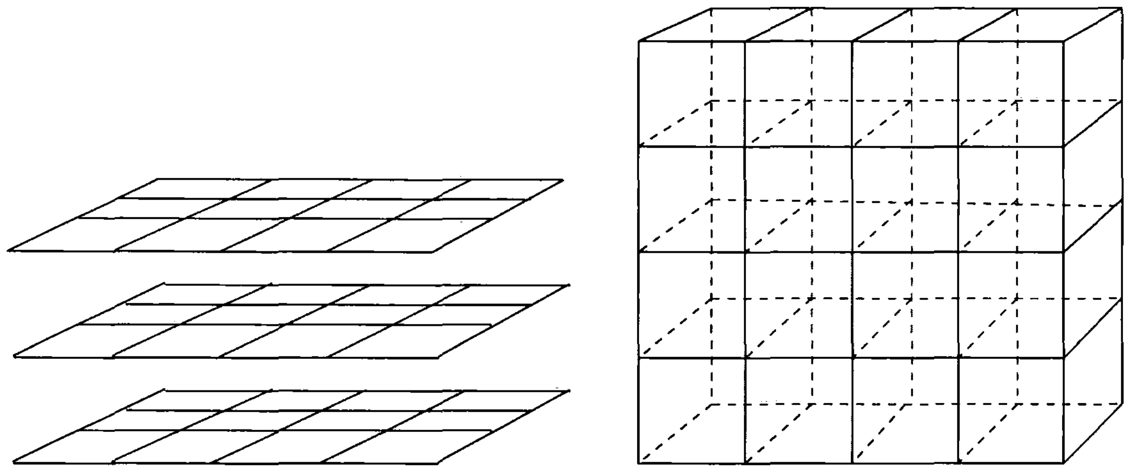

Figure 5.2.5: Slices of the domain $\Omega \subset \mathbb{R}^3$ at the coarse (left) and the fine (right) stage respectively.

The Two-Grid Alternate Slice-Based Substructuring ($\mathbf{ASBS}_{3ga}$) Algorithm. Let M^c_{sbs3} and M^f_{sbs3} be the SBS_3 preconditioner at the coarse stage and the fine stage respectively. The new additive two-grid method is: start with u^0 as an initial approximation (without restricting the generality we take this to be zero) and generate a sequence of iterates $u^1, \cdots, u^l, \cdots$, as follows:

$$u^{l+1/2} \leftarrow u^l + P(M^c_{sbs3})^{-1}R(f_S - Su^l)$$
$$u^{l+1} \leftarrow u^{l+1/2} + (M^f_{sbs3})^{-1}(f_S - Su^l).$$

This can also be written as:

$$u^{l+1} \leftarrow u^l + \left(P(M^c_{sbs3})^{-1}R + (M^f_{sbs3})^{-1}\right)(f_S - Su^l).$$

When this scheme is used to define a preconditioner for the CG method, the inverse of the new two-grid SC preconditioner is:

$$M^{-1}_{asbs3g} = P(M^c_{sbs3})^{-1}R + (M^f_{sbs3})^{-1}.$$

The preconditioned SC matrix is $M^{-1}_{asbs3g}S$. The resulting ASBS_{3ga} method is also optimal in the sense that the rate of convergence can be bounded independently of the partitioning parameters H and h (see Theorem 5.3.9).

5.3 Spectral Analysis for the SBS$_3$ and ASBS$_3$ Techniques

This purpose of this section is to present an abstract framework for the slice-based substructuring algorithms described in Section 5.2. First, we present some technical tools which will be used to prove our main results, then we state and prove the theorems concerning the condition number for the relevant operators in the PCG iterations described in Section 5.2. Throughout this section the notation introduced in Section 5.2 is maintained. Also, C and c denote generic positive constants which are independent of the partitioning parameters H and h. The actual values of these constants may not necessarily be the same in any two occurrences. Further notation is explained as it occurs.

We decompose functions $\mathbf{u} \in S_h^0(\Gamma)$ as:

$$\mathbf{u} = \mathbf{u}^{\varphi} + \mathbf{u}^{\varphi_b} + \mathbf{u}^{\varphi_S}, \tag{5.3.1}$$

where $\mathbf{u}^{\varphi}$, $\mathbf{u}^{\varphi_b}$, and $\mathbf{u}^{\varphi_S}$ are defined as follows. First,

$$\mathbf{u}^{\varphi} \in V^{\varphi} = \sum_{S,b,i} S_h^0(\partial\Omega_i^b \cap \Omega_b^S)$$

is the solution of the problem:

$$s(\mathbf{u}^{\varphi}, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}), \;\; \forall \mathbf{v} \in V^{\varphi}.$$

Note that this is equivalent to solving independently for each $\partial\Omega_i^b \cap \Omega_b^S$ the following local problem: find $\mathbf{u}_i^{\varphi} \in S_h^0(\partial\Omega_i^b \cap \Omega_b^S)$, such that:

$$s(\mathbf{u}^{\varphi}, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}), \;\; \forall \mathbf{v} \in S_h^0(\partial\Omega_i^b \cap \Omega_b^S).$$

Next, $\mathbf{u}^{\varphi_b}$ is the part of $\mathbf{u}$ which lies in the orthogonal complement of V^{φ} in $\sum_{S,b,i} S_h^0(\partial\Omega_i^b \cap \Omega^S)$:

$$V^{\varphi_b} = \{\mathbf{u} \in S_h^0(\partial\Omega_i^b \cap \Omega^S) |\; s(\mathbf{u}, \mathbf{v}) = 0, \;\forall \mathbf{v} \in V^{\varphi}\}.$$

Thus, the value of the function $\mathbf{u}^{\varphi_b} \in V^{\varphi_b}$ in $\Gamma \setminus \partial\Omega^S$ is uniquely determined by its value in $\bigcup_{b,j} F_j^b$. From the definition of V^{φ_b}, we deduce:

$$s(\mathbf{u}^{\varphi_b}, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}) - s(\mathbf{u}^{\varphi}, \mathbf{v}), \ \forall \mathbf{v} \in V^{\varphi} \bigoplus V^{\varphi_b}.$$

Note that:

$$s(\mathbf{u}^{\varphi_b}, \mathbf{v}) = 0, \ \forall \mathbf{v} \in V^{\varphi}.$$

We deduce that if $\mathbf{v}^{\varphi_b} \in V^{\varphi_b}$ is similarly defined as $\mathbf{u}^{\varphi_b}$, then:

$$s(\mathbf{u}^{\varphi_b}, \mathbf{v}^{\varphi_b}) = (\mathbf{f}_S, \mathbf{v}) - s(\mathbf{u}^{\varphi}, \mathbf{v}), \ \forall \mathbf{v} \in V^{\varphi} \bigoplus V^{\varphi_b}.$$

Finally, $\mathbf{u}^{\varphi_S} = \mathbf{u} - \mathbf{u}^{\varphi} - \mathbf{u}^{\varphi_b}$ is the part of $\mathbf{u}$ which lies in the orthogonal complement of $V^{\varphi} \bigoplus V^{\varphi_b}$ in $S_h^0(\Gamma)$:

$$V^{\varphi_S} = \{\mathbf{u} \in S_h^0(\Gamma) : s(\mathbf{u}, \mathbf{v}) = 0, \ \forall \mathbf{v} \in V^{\varphi} \bigoplus V^{\varphi_b}\}.$$

Thus, the value of the function $\mathbf{u}^{\varphi_S} \in V^{\varphi_S}$ in Γ is uniquely determined by its value in $\bigcup_{S,l} F_l^S$. From the definition of V^{φ_S}, we deduce:

$$s(\mathbf{u}^{\varphi_S}, \mathbf{v}) = (\mathbf{f}_S, \mathbf{v}) - s(\mathbf{u}^{\varphi}, \mathbf{v}) - s(\mathbf{u}^{\varphi_b}, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Gamma),$$

or equivalently, if $\mathbf{v}^{\varphi_S} \in V^{\varphi_S}$ is similarly defined as $\mathbf{u}^{\varphi_S}$, then:

$$s(\mathbf{u}^{\varphi_S}, \mathbf{v}^{\varphi_S}) = (\mathbf{f}_S, \mathbf{v}) - s(\mathbf{u}^{\varphi}, \mathbf{v}) - s(\mathbf{u}^{\varphi_b}, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(\Gamma).$$

Note that:

$$s(\mathbf{u}, \mathbf{u}) = s(\mathbf{u}^{\varphi}, \mathbf{u}^{\varphi}) + s(\mathbf{u}^{\varphi_b}, \mathbf{u}^{\varphi_b}) + s(\mathbf{u}^{\varphi_S}, \mathbf{u}^{\varphi_S}). \tag{5.3.2}$$

As in the two-dimensional case, we consider the bilinear form $\tilde{s}(\cdot,\cdot)$ on $S_h^0(\Gamma) \times S_h^0(\Gamma)$ defined as follows. Let Ω_i be a generic subdomain in a (2.2.1), (5.2.1), or (5.2.2) partitioning of Ω. First, we set:

$$\tilde{s}_i(\mathbf{u}, \mathbf{v}) = a_i(\mathbf{u}^E, \mathbf{v}^E), \ \forall \mathbf{u}, \mathbf{v} \in S_h^0(\Gamma), \tag{5.3.3}$$

where $a_i(\cdot,\cdot)$ is given by (2.2.2), and $\mathbf{u}^E$, $\mathbf{v}^E$ are the discrete harmonic extensions into Ω_i of $\mathbf{u}$, $\mathbf{v}$ respectively. Then, we define:

$$\tilde{s}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{N} \tilde{s}_i(\mathbf{u}, \mathbf{v}). \tag{5.3.4}$$

It can be shown that the bilinear form $\tilde{s}(\cdot, \cdot)$ is equivalent to $s(\cdot, \cdot)$, and we can drop the *tilde* from this notation (see e.g. Quarteroni and Valli (1999) [66], Section 2.4, also discussion in Section 2.2). The bilinear form $s_i(\mathbf{u}, \mathbf{u})$ can be analysed using the fractional order Sobolev seminorm $|\mathbf{u}|_{H^{1/2}(\partial\Omega_i)}$ given by:

$$|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} = \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{(\mathbf{u}(\xi) - \mathbf{u}(\tau))^2}{|\xi - \tau|^3} d\xi d\tau, \tag{5.3.5}$$

where ξ and τ denote areas on $\partial\Omega_i$ (see e.g. Xu and Zou (1998) [87], p. 866).

For every face $F \subset \partial\Omega_i$, let

$$|\mathbf{u}|^2_{H^{1/2}(F)} = \int_F \int_F \frac{(\mathbf{u}(\xi) - \mathbf{u}(\tau))^2}{|\xi - \tau|^3} d\xi d\tau,$$

where ξ and τ denote areas on $\partial\Omega_i$ (see e.g. Xu and Zou (1998) [87], p. 868).

When F is an face F_k^i (inside bars), the associated space:

$$H^{1/2}(F_k^i) = \{\mathbf{u} \in L^2(F_k^i) |\ |\mathbf{u}|^2_{H^{1/2}(F_k^i)} < \infty\}$$

is equipped with the weighted norm:

$$||\mathbf{u}||^2_{H^{1/2}(F_k^i)} = \frac{1}{H} ||\mathbf{u}||^2_{L^2(F_k^i)} + |\mathbf{u}|^2_{H^{1/2}(F_k^i)}. \tag{5.3.6}$$

Next, when F is a face F_j^b (between bars), the associated space:

$$H^{1/2}(F_j^b) = \{\mathbf{u} \in L^2(F_j^b) |\ |\mathbf{u}|^2_{H^{1/2}(F_j^b)} < \infty\}$$

is equipped with the weighted norm:

$$||\mathbf{u}||^2_{H^{1/2}(F_j^b)} = \frac{1}{H} ||\mathbf{u}||^2_{L^2(F_j^b)} + |\mathbf{u}|^2_{H^{1/2}(F_j^b)}. \tag{5.3.7}$$

Finally, when F is a face F_l^S (between slices), the associated space:

$$H^{1/2}(F_l^S) = \{\mathbf{u} \in L^2(F_l^S) |\ |\mathbf{u}|^2_{H^{1/2}(F_l^S)} < \infty\}$$

is equipped with the norm:

$$\|\mathbf{u}\|^2_{H^{1/2}(F_l^S)} = \|\mathbf{u}\|^2_{L^2(F_l^S)} + |\mathbf{u}|^2_{H^{1/2}(F_l^S)}. \tag{5.3.8}$$

On the other hand, let $\mathbf{u} \in S_h^0(F)$, and $\|\mathbf{u}\|_{H_{oo}^{1/2}(F)}$ be the norm defined by:

$$\|\mathbf{u}\|^2_{H_{oo}^{1/2}(F)} = \int_F \int_F \frac{(\mathbf{u}(\tau) - \mathbf{u}(\xi))^2}{|\tau - \xi|^3} d\tau d\xi + \int_F \frac{\mathbf{u}^2(\tau)}{dist(\tau, \partial F)} d\tau,$$

where τ and ξ denote areas on F, and $dist(\tau, \partial F)$ represents the distance from τ to the boundary ∂F of F. Conform Nečas (1967) [64], Chapter 2, Lemma 5.3 (see also Dryja (1988) [33] p. 47), in the case of a rectangle $F = (0, H_1) \times (0, H_2)$, in the definition (5.3.9),

$$\int_F \int_F \frac{(\mathbf{u}(\tau) - \mathbf{u}(\xi))^2}{|\tau - \xi|^3} d\tau d\xi$$

can be replaced equivalently by:

$$\int_0^{H_1} \int_0^{H_1} \frac{\|\mathbf{u}(\xi_1, \cdot) - \mathbf{u}(\tau_1, \cdot)\|^2_{L^2(0,H_2)}}{|\xi_1 - \tau_1|^2} d\xi_1 d\tau_1 + \int_0^{H_2} \int_0^{H_2} \frac{\|\mathbf{u}(\cdot, \xi_2) - \mathbf{u}(\cdot, \tau_2)\|^2_{L^2(0,H_1)}}{|\xi_2 - \tau_2|^2} d\xi_2 d\tau_2,$$

and

$$\int_F \frac{(\mathbf{u}(\tau))^2}{dist(\tau, \partial F)} d\tau$$

can be replaced equivalently by:

$$\int_0^{H_1} \frac{\|\mathbf{u}(\xi_1, \cdot)\|^2_{L^2(0,H_2)}}{\xi_1} d\xi_1 + \int_0^{H_1} \frac{\|\mathbf{u}(\xi_1, \cdot)\|^2_{L^2(0,H_2)}}{H_1 - \xi_1} d\xi_1 + \int_0^{H_2} \frac{\|\mathbf{u}(\cdot, \xi_2)\|^2_{L^2(0,H_1)}}{\xi_2} d\xi_2 + \int_0^{H_2} \frac{\|\mathbf{u}(\cdot, \xi_2)\|^2_{L^2(0,H_1)}}{H_2 - \xi_2} d\xi_2.$$

It can be shown that, when $\mathbf{u}$ is a smooth function defined on $\partial\Omega_i$, with support contained in the face F,

$$c|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} \leq \|\mathbf{u}\|^2_{H_{oo}^{1/2}(F)} \leq C|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} \tag{5.3.9}$$

(see e.g. Bramble *et al.* (1989) [11], p. 9, or Xu and Zou (1998) [87], p. 868).

Furthermore, the following equivalence holds:

$$c\|\mathbf{u}\|^2_{H^{1/2}_{oo}(F)} \leq (\delta_F^{1/2}\mathbf{u}, \mathbf{u})_F \leq C\|\mathbf{u}\|^2_{H^{1/2}_{oo}(F)} \tag{5.3.10}$$

(see e.g. Bramble *et al.* (1989) [11], pp. 9-10).

The bilinear form $m_{sbs3}(\cdot, \cdot)$ associated with the preconditioner M_{sbs3} is defined by:

$$\begin{aligned} m_{sbs3}(\mathbf{u}, \mathbf{v}) = &\sum_{i,k} \alpha_{F_k^i}(\delta_{F_k^i}^{1/2}\mathbf{u}_k^i, \mathbf{v}_k^i)_{F_k^i} + \sum_{b,j} \alpha_{F_j^b}(\delta_{F_j^b}^{1/2}\mathbf{u}_j^b, \mathbf{v}_j^b)_{F_j^b} \\ &+ \sum_{S,l} \alpha_{F_l^S}(\delta_{F_l^S}^{1/2}\mathbf{u}_l^S, \mathbf{v}_l^S)_{F_l^S}, \end{aligned} \tag{5.3.11}$$

where for every face F_k^i, $\mathbf{u}_k^i$ is equal to $\mathbf{u}^{\varphi}|_{F_k^i}$ on F_k^i, and zero everywhere else on Γ and on $\partial\Omega$, and we recall that $\alpha_{F_k^i}$ is a scaling factor equal to the average value of the coefficients inside the subdomains sharing the common face F_k^i; for every face F_j^b (between bars), $\mathbf{u}_j^b$ is equal to $\mathbf{u}^{\varphi_b}|_{F_j^b}$ on F_j^b, and zero everywhere else on Γ and on $\partial\Omega$, and $\alpha_{F_j^b}$ is a scaling factor equal to the average value of the coefficients inside the subdomains adjacent to F_j^b; for every face F_l^S (between slices), $\mathbf{u}_l^S$ is equal to $\mathbf{u}^{\varphi_S}|_{F_l^S}$ on F_l^S, and zero everywhere else on Γ and on $\partial\Omega$, and $\alpha_{F_l^S}$ is a scaling factor equal to the average value of the coefficients inside the subdomains adjacent to F_l^S, and $\mathbf{v}_k^i$, $\mathbf{v}_j^b$, and $\mathbf{v}_l^S$ are similarly defined as $\mathbf{u}_k^i$, $\mathbf{u}_j^b$, and $\mathbf{u}_l^S$ respectively.

The process of obtaining the solution $\mathbf{w} \in S_h^0(\Gamma)$ of

$$m_{sbs3}(\mathbf{w}, \mathbf{v}) = (\mathbf{r}, \mathbf{v}), \;\; \forall \mathbf{v} \in S_h^0(\Gamma)$$

is equivalent to the following procedure.

The SBS$_3$ Procedure.

(I) for every face F_k^i (inside bars), solve for $\mathbf{w}_{ik}^{\varphi} \in S_h^0(F_k^i)$ the following equation:

$$\alpha_{F_k^i}(\delta_{F_k^i}^{1/2}\mathbf{w}_{ik}^{\varphi}, \mathbf{v})_{F_k^i} = (\mathbf{r}, \mathbf{v}), \;\; \forall \mathbf{v} \in S_h^0(F_k^i).$$

This can be done independently and in parallel for all F_k^i.

(II) for every face F_j^b (between bars), solve for $\mathbf{w}_{bj}^{\varphi_b} \in S_h^0(F_j^b)$ the following equation:

$$\alpha_{F_j^b}(\delta_{F_j^b}^{1/2}\mathbf{w}_{bj}^{\varphi_b}, \mathbf{v})_{F_j^b} = (\mathbf{r}, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(F_j^b).$$

This can be done independently and in parallel for all F_j^b.

(III) for every face $F_l^S \subset \Gamma$ (between slices), solve for $\mathbf{w}_{Sl} \in S_h^0(F_l^S)$ the following equation:

$$\alpha_{F_l^S}(\delta_{F_l^S}^{1/2}\mathbf{w}_{Sl}, \mathbf{v})_{F_l^S} = (\mathbf{r}, \mathbf{v}), \ \forall \mathbf{v} \in S_h^0(F_l^S).$$

This can be done independently and in parallel for all F_l^S.

(IV) for every slice Ω^S, extend the values of $\mathbf{w}_{Sl}$, determined in (III), discrete harmonically onto all faces $F_j^b \subset \Omega^S$ (between bars). That is solve for $\mathbf{w}_b^{\varphi_S} \in \sum_{\Omega_b^S \subset \Omega^S} S_h^0(\partial \Omega_b^S)$ the following homogeneous equation:

$$\sum_{F_j^b \subset \partial \Omega_b^S} \alpha_{F_j^b}(\delta_{F_j^b}^{1/2}\mathbf{w}_b^{\varphi_S}, \mathbf{v})_{F_j^b} = 0, \ \forall \mathbf{v} \in \sum_{\Omega_b^S \subset \Omega^S} S_h^0(\partial \Omega_b^S \cap \Omega^S)$$

with $\mathbf{w}_b^{\varphi_S}$ given by $\mathbf{w}_{Sl}$ from (III) on $F_l^S \subset \partial \Omega^S$. Then, set $\mathbf{w}_b = \mathbf{w}_{bj}^{\varphi_b} + \mathbf{w}_{b|F_j^b}^{\varphi_S}$. This can be done independently and in parallel for all Ω^S.

(V) for every slice Ω^S, extend the values of $\mathbf{w}_{Sl}$, determined in (III), and the values of $\mathbf{w}_b^{\varphi_S}$, determined in (IV), discrete harmonically onto all faces $F_k^i \subset \Omega^S$. That is solve for $\mathbf{w}_i^S \in \sum_{\Omega_i^b \subset \Omega^S} S_h^0(\partial \Omega_i^b)$ the following homogeneous equation:

$$\sum_{F_k^i \subset \partial \Omega_i^b} \alpha_{F_k^i}(\delta_{F_k^i}^{1/2}\mathbf{w}_i^S, \mathbf{v})_{F_k^i} = 0, \ \forall \mathbf{v} \in \sum_{\Omega_i^b \subset \Omega^S} S_h^0(\partial \Omega_i^b \cap \Omega^S),$$

with $\mathbf{w}_i^S$ given by $\mathbf{w}_{Sl}$ from (III) on $F_l^S \subset \partial \Omega^S$, and by $\mathbf{w}_b^{\varphi_S}$ from (IV) on $F_j^b \subset \Omega^S$. Then, set $\mathbf{w}_{ik} = \mathbf{w}_{ik}^{\varphi} + \mathbf{w}_{i|\Gamma_k}^S$. Again, this can be done independently and in parallel for all Ω^S.

Next, we define the bilinear form $m_{asbs3}(\cdot,\cdot)$ associated with the preconditioner M_{asbs3} as follows. If $m_{sbs3}^{(1)}(\cdot,\cdot)$ and $m_{sbs3}^{(2)}(\cdot,\cdot)$ represent the bilinear form (5.3.11) associated with the preconditioners $M_{sbs3}^{(1)}$ and $M_{sbs3}^{(2)}$ respectively, we define:

$$m_{asbs3}(\mathbf{u}, \mathbf{v}) = m_{sbs3}^{(1)}(\mathbf{u}, \mathbf{v}) + m_{sbs3}^{(2)}(\mathbf{u}, \mathbf{v}), \qquad \forall \mathbf{u}, \mathbf{v} \in S_h^0(\Gamma). \qquad (5.3.12)$$

Similarly, in order to define the bilinear form $m_{asbs3g}(\cdot,\cdot)$ associated with the preconditioner M_{asbs3g}, let $m^c_{sbs3}(\cdot,\cdot)$ and $m^f_{sbs3}(\cdot,\cdot)$ denote the bilinear form (5.3.11) associated with the preconditioners M^c_{sbs3} and M^f_{sbs3} respectively, then we define:

$$m_{asbs3g}(\mathbf{u},\mathbf{v}) = m^c_{sbs3}(\mathbf{u},\mathbf{v}) + m^f_{sbs3}(\mathbf{u},\mathbf{v}), \qquad \forall \mathbf{u},\mathbf{v} \in S^0_h(\Gamma). \tag{5.3.13}$$

Lemma 5.3.1 Let $\Omega = (0,1)\times(0,1)\times(0,1)$ and let $\Omega^S = (0,1)\times(0,1)\times(0,H)$ be a slice in Ω. For $\mathbf{u} \in H^1(\Omega^S)$, the following estimates hold:

(i) if $\mathbf{u}$ is equal to zero on one face in $\partial\Omega^S \cap \partial\Omega$, then:

$$||\mathbf{u}||^2_{L^2(\Omega^S)} \le C|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

(ii) if $\mathbf{u}$ is equal to zero on one face in $\partial\Omega^S \cap \Omega$, then:

$$||\mathbf{u}||^2_{L^2(\Omega^S)} \le CH^2|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

(iii) if F is a square of size 1 in $\partial\Omega^S$, and $||\mathbf{u}||^2_{L^2(F)}$ represents the $L^2(F)$-norm of $\mathbf{u}|_F$, then:

$$||\mathbf{u}||^2_{L^2(F)} \le C\left(\frac{1}{H}||\mathbf{u}||^2_{L^2(\Omega^S)} + H|\mathbf{u}|^2_{H^1(\Omega^S)}\right).$$

(iv) if $\mathbf{u} \in H^1(\Omega)$, then:

$$||\mathbf{u}||^2_{L^2(\Omega^S)} \le CH^2\left(\frac{1}{H}||\mathbf{u}||^2_{L^2(\Omega)} + |\mathbf{u}|^2_{H^1(\Omega)}\right).$$

In each of the estimates $(i)-(iv)$, C denotes a generic positive constant which is independent of the function $\mathbf{u}$ and the partitioning parameter H.

Furthermore, let $\Omega^S_b = (0,1)\times(jH,(j+1)H)\times(0,H)$ be a bar in Ω^S, and $\Omega^b_i = (iH,(i+1)H)\times(jH,(j+1)H)\times(0,H)$ be a cube of size H in Ω^S_b. Then, estimates analogous to $(i)-(iv)$ hold when Ω^S and Ω are replaced by Ω^S_b and Ω^S respectively, and also when Ω^S and Ω are replaced by Ω^b_i and Ω^S_b respectively.

Proof: These estimates can be obtained by direct integration and the Cauchy-Schwarz inequality. See Appendix for details. □

Lemma 5.3.2 Let Ω_i be a subdomain in the (2.2.1) partitioning of Ω, and F denote a generic face in $\partial\Omega_i$. If $\tilde{\mathbf{u}} \in S_h^0(\Omega)$ vanishes on all edges and vertices in $\partial\Omega_i$, and $\mathbf{u} = \tilde{\mathbf{u}}|_{\partial\Omega_i}$, then:

$$s_i(\mathbf{u}, \mathbf{u}) \leq C \sum_{F \subset \partial\Omega_i} (\delta_F^{1/2}\mathbf{u}, \mathbf{u})_F,$$

where $s_i(\cdot, \cdot)$ is defined by (5.3.3).

Proof: This is the three-dimensional version of Lemma 4.3.2. See Bramble *et al.* (1989) [11], pp. 10-11. □

Lemma 5.3.3 Let $\mathbf{u}$ be a continuous, piecewise quadratic function defined on the finite element mesh Σ^h of the domain Ω. In three dimension, if $I^h\mathbf{u}$ is its piecewise linear interpolant on the same mesh, then:

$$|I^h\mathbf{u}|_{H^1(\Omega_i)} \leq C|\mathbf{u}|_{H^1(\Omega_i)},$$

where Ω_i is a generic subdomain in a (2.2.1), (5.2.1), or (5.2.2) partitioning of Ω. The same type of bounds hold for the L^2, $H^{1/2}$, and $H_{oo}^{1/2}$ norms.

Proof: This is the three-dimensional version of Lemmas 3.3.3 and 4.3.3. See Dryja and Widlund (1994) [37], Lemma 4. □

Theorem 5.3.4 For the SBS_3 preconditioning technique, the relative condition number $\kappa(M_{sbs3}^{-1}S)$ grows linearly as $1/H$, i.e.

$$\kappa(M_{sbs3}^{-1}S) = \frac{\lambda_{\max}(M_{sbs3}^{-1}S)}{\lambda_{\min}(M_{sbs3}^{-1}S)} \leq \frac{C}{H}.$$

Proof: Let $m_{sbs3}(\cdot, \cdot)$ be the bilinear form defined by (5.3.11). In order to show that the relative condition number satisfies $\kappa(M_{sbs3}^{-1}S) \leq C/H$, through Theorem 2.1.3 for the matrix S and the preconditioner M_{sbs3}, it suffices to show that:

$$cHm_{sbs3}(\mathbf{u}, \mathbf{u}) \leq s(\mathbf{u}, \mathbf{u}) \leq Cm_{sbs3}(\mathbf{u}, \mathbf{u}), \ \forall \mathbf{u} \in S_h^0(\Gamma). \tag{5.3.14}$$

Throughout this proof we maintain the notation adopted when defining (5.3.11).

First, in order to derive an upper bound for $\lambda_{\max}(M_{sbs3}^{-1}S)$, we show the right hand-side inequality in (5.3.14). By the Cauchy-Schwarz inequality,

$$s(\sum_{i,k} \mathbf{u}_k^i, \sum_{i,k} \mathbf{u}_k^i) \leq C \sum_{i,k} s(\mathbf{u}_k^i, \mathbf{u}_k^i)$$
$$s(\sum_{b,j} \mathbf{u}_j^b, \sum_{b,j} \mathbf{u}_j^b) \leq C \sum_{b,j} s(\mathbf{u}_j^b, \mathbf{u}_j^b)$$
$$s(\sum_{S,l} \mathbf{u}_l^S, \sum_{S,l} \mathbf{u}_l^S) \leq C \sum_{S,l} s(\mathbf{u}_l^S, \mathbf{u}_l^S).$$

On the other hand, by Lemma 5.3.2:

$$\sum_{i,k} s(\mathbf{u}_k^i, \mathbf{u}_k^i) \leq C \sum_{i,k} \alpha_{F_k^i} (\delta_{F_k^i}^{1/2} \mathbf{u}_k^i, \mathbf{u}_k^i)_{F_k^i}$$
$$\sum_{b,j} s(\mathbf{u}_j^b, \mathbf{u}_j^b) \leq C \sum_{b,j} \alpha_{F_j^b} (\delta_{F_j^b}^{1/2} \mathbf{u}_j^b, \mathbf{u}_j^b)_{F_j^b}$$
$$\sum_{S,l} s(\mathbf{u}_l^S, \mathbf{u}_l^S) \leq C \sum_{S,l} \alpha_{F_l^S} (\delta_{F_l^S}^{1/2} \mathbf{u}_l^S, \mathbf{u}_l^S)_{F_l^S}.$$

Thus, by the decomposition (5.3.2) of $s(\cdot,\cdot)$, the above estimations, and the definition (5.3.11) of $m_{sbs3}(\cdot,\cdot)$, we deduce:

$$s(\mathbf{u},\mathbf{u}) = s(\sum_{i,k} \mathbf{u}_k^i, \sum_{i,k} \mathbf{u}_k^i) + s(\sum_{b,j} \mathbf{u}_j^b, \sum_{b,j} \mathbf{u}_j^b) + s(\sum_{S,l} \mathbf{u}_l^S, \sum_{S,l} \mathbf{u}_l^S)$$
$$\leq C \sum_{i,k} \alpha_{F_k^i} (\delta_{F_k^i}^{1/2} \mathbf{u}_k^i, \mathbf{u}_k^i)_{F_k^i} + C \sum_{b,j} \alpha_{F_j^b} (\delta_{F_j^b}^{1/2} \mathbf{u}_j^b, \mathbf{u}_j^b)_{F_j^b} + C \sum_{S,l} \alpha_{F_l^S} (\delta_{F_l^S}^{1/2} \mathbf{u}_l^S, \mathbf{u}_l^S)_{F_l^S},$$

from which the right hand-side inequality in (5.3.14) follows. Therefore, by Theorem 2.1.3, $\lambda_{\max}(M_{sbs3}^{-1} S) \leq C$.

Next, in order to derive a lower bound for $\lambda_{\min}(M_{sbs3}^{-1} S)$, we show the left hand-side inequality in (5.3.14). The argument here is an extension to the three-dimensional case of that used to prove Theorem 4.3.4.

We show that:

$$\sum_{i,k} \alpha_{F_k^i} (\delta_{F_k^i}^{1/2} \mathbf{u}_k^i, \mathbf{u}_k^i)_{F_k^i} \leq C s(\mathbf{u}^{\varphi}, \mathbf{u}^{\varphi}), \tag{5.3.15}$$

$$\sum_{b,j} \alpha_{F_j^b} (\delta_{F_j^b}^{1/2} \mathbf{u}_j^b, \mathbf{u}_j^b)_{F_j^b} \leq C s(\mathbf{u}^{\varphi_b}, \mathbf{u}^{\varphi_b}), \tag{5.3.16}$$

and

$$\sum_{S,l} \alpha_{F_l^S} (\delta_{F_l^S}^{1/2} \mathbf{u}_l^S, \mathbf{u}_l^S)_{F_l^S} \leq \left(1 + \frac{1}{H}\right) s(\mathbf{u}^{\varphi_S}, \mathbf{u}^{\varphi_S}). \tag{5.3.17}$$

If $\mathbf{u} = \mathbf{u}^{\varphi} + \mathbf{u}^{\varphi_b} + \mathbf{u}^{\varphi_S}$ is the (5.3.1) decomposition of $\mathbf{u}$, we denote by $\tilde{\mathbf{u}}^{\varphi}$, $\tilde{\mathbf{u}}^{\varphi_b}$, and $\tilde{\mathbf{u}}^{\varphi_S}$ the discrete harmonic extensions of $\mathbf{u}^{\varphi}$, $\mathbf{u}^{\varphi_b}$ and $\mathbf{u}^{\varphi_S}$ in Ω respectively. Let η_k^i be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes on the boundary ∂F_k^i of F_k^i and everywhere else on $\Gamma \setminus F_k^i$, grows linearly to 1 such that $\|\nabla \eta_k^i\|_{L^\infty} \leq C/H$, and it is identically 1 on the remainder finite element nodes of F_k^i. Next, let η_j^b be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes on the boundary ∂F_j^b of F_j^b and everywhere else on $\Gamma \setminus F_j^b$, $0 \leq \eta_j^b \leq 1$, and $\|\nabla \eta_j^b\|_{L^\infty} \leq C/H$. Finally, let η_l^S be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes on the boundary ∂F_l^S of F_l^S and everywhere else on $\Gamma \setminus F_l^S$, $0 \leq \eta_l^S \leq 1$, and $\|\nabla \eta_l^S\|_{L^\infty} \leq C$.

If I^h is the finite element interpolation operator onto the space $S_h^0(\Gamma)$, then we define:

$$\mathbf{u}_k^i = I^h(\eta_k^i \mathbf{u}^{\varphi}), \qquad \mathbf{u}_j^b = I^h(\eta_j^b \mathbf{u}^{\varphi_b}), \qquad \text{and} \qquad \mathbf{u}_l^S = I^h(\eta_l^S \mathbf{u}^{\varphi_S}).$$

Note that if $\{\eta_k^i\}$, $\{\eta_j^b\}$, and $\{\eta_l^S\}$ form partitions of unity, then:

$$\mathbf{u}^{\varphi} = \sum_{i,k} \mathbf{u}_k^i, \qquad \mathbf{u}^{\varphi_b} = \sum_{b,j} \mathbf{u}_j^b, \qquad \text{and} \qquad \mathbf{u}^{\varphi_S} = \sum_{S,l} \mathbf{u}_l^S.$$

From Lemma 5.3.3 (for the $H_{oo}^{1/2}$ norm), we deduce that when $\mathbf{v} = \eta_k^i \mathbf{u}^{\varphi}$, in order to estimate $\|\mathbf{u}_k^i\|_{H_{oo}^{1/2}(F_k^i)}$, it is sufficient to estimate $\|\mathbf{v}\|_{H_{oo}^{1/2}(F_k^i)}$.

We consider a generic face $F_k^i = (0, H) \times (0, H)$. We divide the interval $[0, H]$ in two parts $[0, H/2]$ and $[H/2, H]$, and take the tensor product $[0, H] \bigotimes [0, H]$. The double integral in the definition of $\|\mathbf{v}\|^2_{H^{1/2}(F_k^i)}$ is then split into a sum of four double integrals. Due to the symmetry, we only need to consider one of them. By the definition of η_k^i and the Cauchy-Schwarz inequality, we obtain:

$$
\begin{aligned}
&\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{v}(\xi_1,\cdot)-\mathbf{v}(\tau_1,\cdot)\|^2_{L^2(0,H)}}{|\xi_1-\tau_1|^2}d\xi_1 d\tau_1 \\
&+\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{v}(\cdot,\xi_2)-\mathbf{v}(\cdot,\tau_2)\|^2_{L^2(0,H)}}{|\xi_2-\tau_2|^2}d\xi_2 d\tau_2 \\
&\le C\int_0^{H/2}\int_0^{H/2} \frac{(\xi_1+\tau_1)^2\|\mathbf{u}^\varphi(\xi_1,\cdot)-\mathbf{u}^\varphi(\tau_1,\cdot)\|^2_{L^2(0,H)}}{H^2|\xi_1-\tau_1|^2}d\xi_1 d\tau_1 \\
&+C\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{u}^\varphi(\xi_1,\cdot)+\mathbf{u}^\varphi(\tau_1,\cdot)\|^2_{L^2(0,H)}}{H^2}d\xi_1 d\tau_1 \\
&+C\int_0^{H/2}\int_0^{H/2} \frac{(\xi_2+\tau_2)^2\|\mathbf{u}^\varphi(\cdot,\xi_2)-\mathbf{u}^\varphi(\cdot,\tau_2)\|^2_{L^2(0,H)}}{H^2|\xi_2-\tau_2|^2}d\xi_2 d\tau_2 \\
&+C\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{u}^\varphi(\cdot,\xi_2)+\mathbf{u}^\varphi(\cdot,\tau_2)\|^2_{L^2(0,H)}}{H^2}d\xi_2 d\tau_2 \\
&\le C\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{u}^\varphi(\xi_1,\cdot)-\mathbf{u}^\varphi(\tau_1,\cdot)\|^2_{L^2(0,H)}}{|\xi_1-\tau_1|^2}d\xi_1 d\tau_1 \\
&+C\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2}d\xi_1 d\tau_1 \\
&+C\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{u}^\varphi(\cdot,\xi_2)-\mathbf{u}^\varphi(\cdot,\tau_2)\|^2_{L^2(0,H)}}{|\xi_2-\tau_2|^2}d\xi_2 d\tau_2 \\
&+C\int_0^{H/2}\int_0^{H/2} \frac{\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2}d\xi_2 d\tau_2.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&\int_0^{H}\int_0^{H} \frac{\|\mathbf{v}(\xi_1,\cdot)-\mathbf{v}(\tau_1,\cdot)\|^2_{L^2(0,H)}}{|\xi_1-\tau_1|^2}d\xi_1 d\tau_1 \\
&+\int_0^{H}\int_0^{H} \frac{\|\mathbf{v}(\cdot,\xi_2)-\mathbf{v}(\cdot,\tau_2)\|^2_{L^2(0,H)}}{|\xi_2-\tau_2|^2}d\xi_2 d\tau_2 \\
&\le C\frac{1}{H}\|\mathbf{u}^\varphi\|^2_{L^2(F_k^i)} + C|\mathbf{u}^\varphi|^2_{H^{1/2}(F_k^i)}.
\end{aligned}
$$

Next, we consider the single integrals in the definition of $\|\mathbf{v}\|^2_{H^{1/2}(F_k^i)}$. Through the definition of η_k^i, we deduce:

$$\begin{aligned}
&\int_0^{H/2} \frac{\|\mathbf{v}(\xi_1,\cdot)\|^2_{L^2(0,H)}}{\xi_1} d\xi_1 + \int_0^{H/2} \frac{\|\mathbf{v}(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H-\xi_1} d\xi_1 \\
&+ \int_0^{H/2} \frac{\|\mathbf{v}(\cdot,\xi_2)\|^2_{L^2(0,H)}}{\xi_2} d\xi_2 + \int_0^{H/2} \frac{\|\mathbf{v}(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H-\xi_2} d\xi_2 \\
&\le C \int_0^{H/2} \frac{\xi_1^2\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2\xi_1} d\xi_1 + C \int_{H/2}^{H} \frac{(H-\xi_1)^2\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2\xi_1} d\xi_1 \\
&+ C \int_0^{H} \frac{\xi_1^2\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2(H-\xi_1)} d\xi_1 + C \int_{H/2}^{H} \frac{(H-\xi_1)^2\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2(H-\xi_1)} d\xi_1 \\
&+ C \int_0^{H/2} \frac{\xi_2^2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2\xi_2} d\xi_2 + C \int_{H/2}^{H} \frac{(H-\xi_2)^2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2(\xi_2} d\xi_2 \\
&+ C \int_0^{H/2} \frac{\xi_2^2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2(H-\xi_2)} d\xi_2 + C \int_{H/2}^{H} \frac{(H-\xi_2)^2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2(H-\xi_2)} d\xi_2 \\
&= C \int_0^{H/2} \frac{\xi_1\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2} d\xi_1 + C \int_{H/2}^{H} \frac{(H-\xi_1)^2\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2\xi_1} d\xi_1 \\
&+ C \int_0^{H} \frac{\xi_1^2\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2(H-\xi_1)} d\xi_1 + C \int_{H/2}^{H} \frac{(H-\xi_1)\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H^2} d\xi_1 \\
&+ C \int_0^{H/2} \frac{\xi_2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2} d\xi_2 + C \int_{H/2}^{H} \frac{(H-\xi_2)^2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2(\xi_2} d\xi_2 \\
&+ C \int_0^{H/2} \frac{\xi_2^2\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2(H-\xi_2)}, d\xi_2 + C \int_{H/2}^{H} \frac{(H-\xi_2)\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H^2} d\xi_2 \\
&\le C \int_0^{H} \frac{\|\mathbf{u}^\varphi(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H} d\xi_1 + C \int_0^{H} \frac{\|\mathbf{u}^\varphi(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H} d\xi_2 .
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\int_0^{H} \frac{\|\mathbf{v}(\xi_1,\cdot)\|^2_{L^2(0,H)}}{\xi_1} d\xi_1 + \int_0^{H} \frac{\|\mathbf{v}(\xi_1,\cdot)\|^2_{L^2(0,H)}}{H-\xi_1} d\xi_1 \\
&+ \int_0^{H} \frac{\|\mathbf{v}(\cdot,\xi_2)\|^2_{L^2(0,H)}}{\xi_2} d\xi_2 + \int_0^{H} \frac{\|\mathbf{v}(\cdot,\xi_2)\|^2_{L^2(0,H)}}{H-\xi_2} d\xi_2 \\
&\le C\frac{1}{H}\|\mathbf{u}^\varphi\|^2_{L^2(F_k^i)} .
\end{aligned}$$

By the above evaluations, we obtain:

$$\|\mathbf{u}_k^i\|^2_{H_{00}^{1/2}(F_k^i)} \le C\frac{1}{H}\|\mathbf{u}_k^i\|^2_{L^2(F_k^i)} + C|\mathbf{u}_k^i|^2_{H^{1/2}(F_k^i)} .$$

From this estimate, Lemma 5.3.1 (iii) and (ii) for subdomains Ω_i^b inside bars, and

Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{F_k^i\subset\partial\Omega_i^b}\alpha_{F_k^i}\|\mathbf{u}_k^i\|^2_{H_{00}^{1/2}(F_k^i)} &\le C\sum_{F_k^i\subset\partial\Omega_i^b}\frac{1}{H}\|\mathbf{u}_k^i\|^2_{L^2(F_k^i)}+C\sum_{F_k^i\subset\partial\Omega_i^b}|\mathbf{u}_k^i|^2_{H^{1/2}(F_k^i)}\\ &\le C\frac{1}{H^2}\|\tilde{\mathbf{u}}^\varphi\|^2_{L^2(\Omega_i^b)}+C|\tilde{\mathbf{u}}^\varphi|^2_{H^1(\Omega_i^b)}\\ &\le C|\tilde{\mathbf{u}}^\varphi|^2_{H^1(\Omega_i^b)}\\ &\le C|\mathbf{u}^\varphi|^2_{H^{1/2}(\partial\Omega_i^b)}.\end{aligned}$$

Thus, through the equivalence (5.3.10) for the face F_k^i,

$$\sum_{F_k^i\subset\partial\Omega_i^b}\alpha_{F_k^i}(\delta_{F_k^i}^{1/2}\mathbf{u}_k^i,\mathbf{u}_k^i)_{F_k^i}\le Cs_i(\mathbf{u}^\varphi,\mathbf{u}^\varphi).$$

Since each F_k^i is shared by only two subdomains Ω_i^b, after summing over all $\Omega_i^s\subset\Omega$, by the representation (5.3.4) of $s(\cdot,\cdot)$, we obtain (5.3.15).

Next, similar evaluations are also possible on the faces $F_j^b=(0,1)\times(0,H)$ (between bars). Note that by the definition of η_j^b on F_j^b, we first need to divide this face into $(0,H]\times(0,H),\cdots,[1-H,1)\times(0,H)$. Then the double integral evaluated on $F_j^b\bigotimes F_j^b$ is split into the sum of $1/H^2$ double integrals. However, by symmetry, only one case needs to be considered, which is similar to the double integral for F_k^i. The single integral is also similar to that for F_k^i. Then we obtain:

$$\|\mathbf{u}_j^b\|^2_{H_{00}^{1/2}(F_j^b)}\le C\frac{1}{H}\|\mathbf{u}_j^b\|^2_{L^2(F_j^b)}+|\mathbf{u}_j^b|^2_{H^{1/2}(F_j^b)}.$$

From this estimate, Lemma 5.3.1 (*iii*) and (*ii*) for bars Ω_b^S inside slices, and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{F_j^b\subset\partial\Omega_b^S}\alpha_{F_j^b}\|\mathbf{u}_j^b\|^2_{H_{00}^{1/2}(F_j^b)} &\le \sum_{F_j^b\subset\partial\Omega_b^S}\frac{1}{H}\|\mathbf{u}_j^b\|^2_{L^2(F_j^b)}+\sum_{F_j^b\subset\partial\Omega_b^S}|\mathbf{u}_j^b|^2_{H^{1/2}(F_j^b)}\\ &\le C\frac{1}{H^2}\|\tilde{\mathbf{u}}^{\varphi_b}\|^2_{L^2(\Omega_b^S)}+C|\tilde{\mathbf{u}}^\varphi|^2_{H^1(\Omega_b^S)}\\ &\le C|\tilde{\mathbf{u}}^{\varphi_b}|^2_{H^1(\Omega_b^S)}.\end{aligned}$$

Then, we apply the Cauchy-Schwarz inequality for the decomposition of the bars Ω_b^S into disjoint subdomains Ω_i^b, and the right hand-side inequality in Theorem 2.2.3, to obtain:

$$\begin{aligned}\sum_{F_j^b \subset \partial\Omega_b^S} \alpha_{F_j^b} \|\mathbf{u}_j^b\|^2_{H_{00}^{1/2}(F_j^b)} &\le C|\tilde{\mathbf{u}}^{\varphi_b}|^2_{H^1(\Omega_b^S)} \\ &\le C \sum_{\Omega_i^b \subset \Omega_b^S} |\tilde{\mathbf{u}}^{\varphi_b}|^2_{H^1(\Omega_i^b)} \\ &\le C \sum_{\Omega_i^b \subset \Omega_b^S} |\mathbf{u}^{\varphi_b}|^2_{H^{1/2}(\partial\Omega_i^b)}.\end{aligned}$$

Thus, through the equivalence (5.3.10) for the face F_j^b,

$$\sum_{F_j^b \subset \partial\Omega_b^S} \alpha_{F_j^b} (\delta_{F_j^b}^{1/2} \mathbf{u}_j^b, \mathbf{u}_j^b)_{F_j^b} \le \sum_{\Omega_i^b \subset \Omega_b^S} s_i(\mathbf{u}^{\varphi_b}, \mathbf{u}^{\varphi_b}).$$

Since each face F_j^b is shared by only two bars Ω_b^S, after summing over all $\Omega_b^S \subset \Omega$, by the representation (5.3.4) of $s(\cdot,\cdot)$, we obtain (5.3.16).

Finally, analogous calculations can be carried out on the faces $F_l^S = (0,1)\times(0,1)$ (between slices). Note that by the definition of η_l^S on F_l^S, we first need to divide each interval $(0,1)$ as $(0,1/2]$ and $[1/2,1)$, then express F_l^S as the tensor product $(0,1)\bigotimes(0,1)$. Replacing H by 1 in the calculations carried out for F_k^i yields:

$$\|\mathbf{u}_l^S\|^2_{H_{00}^{1/2}(F_l^S)} \le C\|\mathbf{u}_l^S\|^2_{L^2(F_l^S)} + C|\mathbf{u}_l^S|^2_{H^{1/2}(F_l^S)}.$$

From this estimate, Lemma 5.3.1 (*iii*) for slices Ω^S, and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{F_l^S \subset \partial\Omega^S} \alpha_{F_l^S} \|\mathbf{u}_l^S\|^2_{H_{00}^{1/2}(F_l^S)} &\le \sum_{F_l^S \subset \partial\Omega^S} \|\mathbf{u}_l^S\|^2_{L^2(F_l^S)} + \sum_{F_l^S \subset \partial\Omega^S} |\mathbf{u}_l^S|^2_{H^{1/2}(F_l^S)} \\ &\le C\frac{1}{H}\|\tilde{\mathbf{u}}^{\varphi_S}\|^2_{L^2(\Omega^S)} + CH|\tilde{\mathbf{u}}^{\varphi_S}|^2_{H^1(\Omega^S)} + C|\tilde{\mathbf{u}}^{\varphi}|^2_{H^1(\Omega^S)} \\ &\le C\frac{1}{H}\|\tilde{\mathbf{u}}^{\varphi_S}\|^2_{L^2(\Omega^S)} + C|\tilde{\mathbf{u}}^{\varphi_S}|^2_{H^1(\Omega^S)} \\ &\le C\left(1+\frac{1}{H}\right)|\tilde{\mathbf{u}}^{\varphi_S}|^2_{H^1(\Omega^S)}.\end{aligned}$$

Then, we apply the Cauchy-Schwarz inequality for the decomposition of the slices Ω^S into disjoint bars Ω_b^S, and also for the decomposition of the into disjoint subdomains

Ω_i^b, and the right hand-side inequality in Theorem 2.2.3, to obtain:

$$\begin{aligned}\sum_{F_l^S \subset \partial\Omega^S} \alpha_{F_l^S} \|\mathbf{u}_l^S\|^2_{H_{00}^{1/2}(F_l^S)} &\le C\left(1+\frac{1}{H}\right)|\tilde{\mathbf{u}}^{\varphi_S}|^2_{H^1(\Omega^S)} \\ &\le C\left(1+\frac{1}{H}\right)\sum_{\Omega_b^S \subset \Omega^S} |\tilde{\mathbf{u}}^{\varphi_S}|^2_{H^1(\Omega_b^S)} \\ &\le C\left(1+\frac{1}{H}\right)\sum_{\Omega_b^S \subset \Omega^S}\sum_{\Omega_i^b \subset \Omega_b^S} |\tilde{\mathbf{u}}^{\varphi_S}|^2_{H^1(\Omega_i^b)} \\ &\le C\left(1+\frac{1}{H}\right)\sum_{\Omega_b^S \subset \Omega^S}\sum_{\Omega_i^b \subset \Omega_b^S} |\mathbf{u}^{\varphi_S}|^2_{H^{1/2}(\partial\Omega_i^b)}.\end{aligned}$$

Thus, through the equivalence (5.3.10) for the face F_l^S,

$$\sum_{F_l^S \subset \partial\Omega^S} \alpha_{F_l^S} (\delta_{F_l^S}^{1/2} \mathbf{u}_l^S, \mathbf{u}_l^S)_{F_l^S} \le C\left(1+\frac{1}{H}\right)\sum_{\Omega_b^S \subset \Omega^S}\sum_{\Omega_i^b \subset \Omega_b^S} s_i(\mathbf{u}^{\varphi_S}, \mathbf{u}^{\varphi_S}).$$

Since each face F_l^S is shared by only two slices Ω^S, after summing over all $\Omega^S \subset \Omega$, through the representation (5.3.4) of $s(\cdot,\cdot)$, we obtain (5.3.17).

From the estimates (5.3.15), (5.3.16), and (5.3.17), the definition (5.3.11) of $m_{sbs3}(\cdot,\cdot)$, and the decomposition (5.3.2) of $s(\cdot,\cdot)$, we deduce:

$$m_{sbs3}(\mathbf{u},\mathbf{u}) \le C\left(1+\frac{1}{H}\right) s(\mathbf{u},\mathbf{u}),$$

which is equivalent to the left hand-side inequality in (5.3.14).

Therefore, by Theorem 2.1.3 for the matrix S and the preconditioner M_{sbs3}, $1/\lambda_{\min}(M_{sbs3}^{-1}S)$ grows linearly as $1/H$. Since $1/\lambda_{\max}(M_{sbs2}^{-1}S) \le C$, we conclude that:

$$\kappa(M_{sbs3}^{-1}S) \le C/H. \ \square$$

Remark 5.3.5 We observe that in the proof of Theorem 5.3.4, when Ω^S is a slice with only one face in the interior of the domain Ω and the remaining faces on the boundary $\partial\Omega$, we can apply Lemma 5.3.1 *(ii)* instead of *(i)*. Thus, for this slice,

$$\sum_{F_l^S \subset \partial\Omega^S} \alpha_{F_l^S} \|\mathbf{u}_l^S\|^2_{H_{00}^{1/2}(F_l^S)} \le C \sum_{\Omega_b^S \subset \Omega^S}\sum_{\Omega_i^b \subset \Omega_b^S} |\mathbf{u}^{\varphi_S}|^2_{H^{1/2}(\partial\Omega_i^b)}.$$

Lemma 5.3.6 In three dimensions, let $Q_{2^ph} : L^2(\Omega) \to S_{2^pH}(\Omega)$ be the L^2-projection associated with $S_{2^ph}(\Omega)$ $(h \le 2^p h < H,\, p \in \mathbb{N})$. Then, for all $\mathbf{u} \in H_0^1(\Omega)$, the following estimates hold:

$$\|\mathbf{u} - Q_{2^ph}\mathbf{u}\|_{L^2(\Omega)} \le C2^p h |\mathbf{u}|_{H^1(\Omega)} \quad \text{and} \quad |Q_{2^ph}\mathbf{u}|_{H^1(\Omega)} \le C|\mathbf{u}|_{H^1(\Omega)}.$$

Proof: This is the three-dimensional version of Lemma 3.3.6. For a proof see e.g. Bramble and Xu (1991) [12], Section 3. □

Lemma 5.3.7 In three dimensions, let $\mathbf{u} \in S_h^0(\Gamma)$ and $\tilde{\mathbf{u}}$ be its discrete harmonic extension in Ω. If $Q_{2^ph} : L^2(\Omega) \to S_{2^ph}^0(\Omega)$ is the L^2-projection associated with $S_{2^ph}^0(\Omega)$ $(h \le 2^p h < H,\, p \in \mathbb{N})$, we denote $\tilde{\mathbf{u}}_o = Q_{2^ph}\tilde{\mathbf{u}}$ and $\mathbf{u}_o = \tilde{\mathbf{u}}_o|_\Gamma$. Then:

$$m_{asbs3}(\mathbf{u}_o, \mathbf{u}_o) \le Cs(\mathbf{u}, \mathbf{u}).$$

Proof: This is the three-dimensional version of Lemma 4.3.3. The proof is based on the observation that the $ASBS_{3a}$ preconditioner is obtained in two stages such that the interfaces between slices at one stage are perpendicular on the interfaces between slices at the other stage. Throughout this proof we maintain the notation adopted when defining (5.3.11). First, we show that for any $\mathbf{u}_o \in S_h^0(\Gamma)$,

$$m_{asbs3}(\mathbf{u}_o, \mathbf{u}_o) \le Cs(\mathbf{u}_o, \mathbf{u}_o). \tag{5.3.18}$$

Then, by replacing h by $2^p h$, and taking $\mathbf{u}_o = Q_{2^ph}\tilde{\mathbf{u}}_{o|\Gamma}$ in the above estimate, the lemma follows, through the definition of $s(\cdot,\cdot)$, Theorem 2.2.3, and the second estimate in Lemma 5.3.6.

Let $\mathbf{u}_1 = \mathbf{u}_1^{\varphi} + \mathbf{u}_1^{\varphi_b} + \mathbf{u}_1^{\varphi s}$ be the (5.3.1) decomposition of $\mathbf{u}_o$ at the first stage, and $\mathbf{u}_2 = \mathbf{u}_2^{\varphi} + \mathbf{u}_2^{\varphi_b} + \mathbf{u}_2^{\varphi s}$ be the (5.3.1) decomposition of $\mathbf{u}_o$ at the second stage. Note that the global interface Γ can be viewed as being covered by the union of the following two overlapping subdomains: on one hand the faces between slices, between bars inside slices, and between subdomains inside bars at the first stage, and on the other hand the faces between slices, between bars inside slices, and between subdomains inside bars at the second stage. Γ can also be viewed as consisting of the union of the faces between slices and between subdomains inside bars, at both stages.

Furthermore, the union of the interfaces between slices at both stages can be regarded as consisting of overlapping wire-basket regions, such that each region contains an edge of a bar and parts of the interfaces between slices that are within a distance H from that edge. Thus at most two such regions overlap and the width of the overlap is uniform of order $\mathcal{O}(H)$. Let Γ^v denote a generic wire-basket region as described above, restricted to the boundary $\partial\Omega_b^S$ of a generic bar Ω_b^S. Then the restriction $\mathbf{u}^v = \mathbf{u}_{o|\Gamma^v}$, of $\mathbf{u}_o$ to Γ^v, can be analysed using the $H_{oo}^{1/2}(\Gamma^v)$ norm according to definition (2.2.9).

Let F_{l1}^S and F_{l2}^S denote a generic face between two slices at the first and at the second stage respectively, such that $\Gamma^v \subset F_{l1}^S \cup F_{l2}^S$. We introduce the following notation: $\tilde{\Gamma}^v = \tilde{F}_{l1}^S \cap \tilde{F}_{l2}^S$ and $S_h^0(\tilde{\Gamma}^v) = S_h^0(\tilde{F}_{l1}^S) \cap S_h^0(\tilde{F}_{l2}^S)$, with $S_h^0(\tilde{F}_{l1}^S)$ and $S_h^0(\tilde{F}_{l2}^S)$ as in (5.2.3). We denote by $\tilde{\mathbf{u}}_o$, the harmonic extension of $\mathbf{u}_o$ in Ω, and derive a representation:

$$\tilde{\mathbf{u}}_o = \sum_v \tilde{\mathbf{u}}^v + \sum_k \tilde{\mathbf{u}}_k^i,$$

where $\tilde{\mathbf{u}}^v \in S_h^0(\tilde{\Gamma}^v)$ and $\tilde{\mathbf{u}}_k^i \in S_h^0(\tilde{F}_k^i)$, with $S_h^0(\tilde{F}_k^i)$ also as in (5.2.3). We construct this representation as follows. Let η^v be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes of the boundary $\partial\tilde{\Gamma}^v$ and everywhere else on $\Omega \setminus \tilde{\Gamma}^v$, $0 \le \eta^v \le 1$, and its gradient is of order $\mathcal{O}(1/H)$. If I^h is the finite element interpolation operator onto the space $S_h^0(\Omega)$, then we define:

$$\bar{\mathbf{u}}^v = I^h(\eta^v \tilde{\mathbf{u}}_o).$$

Analogously, let η_k^i be a continuous, piecewise linear function on the finite element nodes of Ω that is zero on the finite element nodes of the boundary $\partial\tilde{F}_k^i$ and everywhere else on $\Omega \setminus \tilde{F}_k^i$, $0 \le \eta_k^i \le 1$, and its gradient is of order $\mathcal{O}(1/H)$. Then we define:

$$\bar{\mathbf{u}}_k^i = I^h(\eta_k^i \tilde{\mathbf{u}}_o).$$

Note that if $\{\eta_k^i\}$ and $\{\eta^v\}$ form partitions of unity, then:

$$\tilde{\mathbf{u}}_o = \sum_v \bar{\mathbf{u}}^v + \sum_{i,k} \bar{\mathbf{u}}_k^i.$$

Like in the two-dimensional case (see the proof of Lemma 4.3.3), we obtain:

$$\sum_{\Gamma^v \subset \partial\Omega_b^S} |\bar{\mathbf{u}}^v|^2_{H^1(\Omega_b^S)} \le C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_b^S)} + C\frac{1}{H^2}||\tilde{\mathbf{u}}_o||^2_{L^2(\Omega_b^S)}.$$

We choose $\tilde{\mathbf{u}}^v$ to be the discrete harmonic extension of $\mathbf{u}^v = \bar{\mathbf{u}}^v|_{\Gamma^v}$ in $\tilde{\Gamma}^v$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5), of discrete harmonic functions, imply:

$$\sum_{\Gamma^v \subset \partial\Omega_b^S} |\tilde{\mathbf{u}}^v|^2_{H^1(\Omega_b^S)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_b^S)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_b^S)}.$$

Thus, by the left hand-side inequality in Theorem 2.2.3 and Lemma 5.3.1 (*ii*) for $\partial\Omega_b^S \cap \Omega^S$,

$$\sum_{\Gamma^v \subset \partial\Omega_b^S} |\mathbf{u}^v|^2_{H^{1/2}(\partial\Omega_b^S)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_b^S)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_b^S)}$$
$$\leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_b^S)}.$$

From this estimate, by the definition (2.2.9) and the right hand-side inequality in Theorem 2.2.3, we deduce:

$$\sum_{\Gamma^v \subset \partial\Omega_b^S} \|\mathbf{u}^v\|^2_{H^{1/2}_{oo}(\Gamma^v)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_b^S)}$$
$$\leq C|\mathbf{u}_o|^2_{H^{1/2}(\partial\Omega_b^S)}.$$

Since each Γ^v is shared by only four bars Ω_b^S, after summing over all $\Omega_b^S \subset \Omega$, we obtain:

$$\sum_v \|\mathbf{u}^v\|^2_{H^{1/2}_{oo}(\Gamma^v)} \leq C|\mathbf{u}_o|^2_{H^{1/2}(\Gamma)}. \tag{5.3.19}$$

Similarly, we obtain:

$$\sum_{F_k^i \subset \partial\Omega_i^b} |\bar{\mathbf{u}}_k^i|^2_{H^1(\Omega_i^b)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^b)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^b)}.$$

We choose $\tilde{\mathbf{u}}_k^i$ to be the discrete harmonic extension of $\mathbf{u}_k^i = \bar{\mathbf{u}}_k^i|_{F_k^i}$ in $\tilde{F}_k^i$, extended by zero to the rest of Ω. Then, the last estimate and the minimisation property (2.2.5), of discrete harmonic functions, imply:

$$\sum_{F_k^i \subset \partial\Omega_i^b} |\tilde{\mathbf{u}}_k^i|^2_{H^1(\Omega_i^b)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^b)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^b)}.$$

Thus, by the left hand-side inequality in Theorem 2.2.3 and Lemma 5.3.1 (*ii*) for

$\partial\Omega_i^b \cap \Omega_b^S$,

$$\sum_{F_k^i \subset \partial\Omega_i^b} |\mathbf{u}_k^i|^2_{H^{1/2}(\partial\Omega_i^b)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^b)} + C\frac{1}{H^2}\|\tilde{\mathbf{u}}_o\|^2_{L^2(\Omega_i^b)}$$
$$\leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^b)}.$$

From this estimate, by the definition (2.2.9) and the right hand-side inequality in Theorem 2.2.3, we deduce:

$$\sum_{F_k^i \subset \partial\Omega_i^b} \|\mathbf{u}_k^i\|^2_{H_{oo}^{1/2}(F_k^i)} \leq C|\tilde{\mathbf{u}}_o|^2_{H^1(\Omega_i^b)}$$
$$\leq C|\mathbf{u}_o|^2_{H^{1/2}(\partial\Omega_i^b)}.$$

Since each F_k^i is shared by only two subdomains $\Omega_i^b \subset \Omega_b^S$, after summing over all $\Omega_i^b \subset \Omega$, we obtain:

$$\sum_{i,k} \|\mathbf{u}_k^i\|^2_{H_{oo}^{1/2}(F_k^i)} \leq C|\mathbf{u}_o|^2_{H^{1/2}(\Gamma)}. \tag{5.3.20}$$

Finally, (5.3.19) and (5.3.20) imply (5.3.18). □

Theorem 5.3.8 For the ASBS$_{3a}$ preconditioning technique, the relative condition number $\kappa(M_{asbs3}^{-1}S)$ is bounded independently of the partitioning parameters H and h, i.e.

$$\kappa(M_{asbs3}^{-1}S) = \frac{\lambda_{\max}(M_{asbs3}^{-1}S)}{\lambda_{\min}(M_{asbs3}^{-1}S)} \leq C.$$

Proof: We observe that the ASBS$_{3a}$ preconditioner is of overlapping Schwarz type. Therefore it is possible to show this result by arguments extended from those used to prove Theorem 4.3.8. However, like in the two-dimensional case, this result can be demonstrated in a similar manner as shown in the proof of Theorem 5.3.9 below, by simply replacing the functions at the coarse level by those at the first stage, and the functions at the fine level by those at the second stage. □

Theorem 5.3.9 For the ASBS3_{ga} preconditioning technique, the relative condition number $\kappa(M_{asbs3g}^{-1}S)$ is bounded independently of the partitioning parameters H and h, i.e.

$$\kappa(M_{asbs3g}^{-1}S) = \frac{\lambda_{\max}(M_{asbs3g}^{-1}S)}{\lambda_{\min}(M_{asbs3g}^{-1}S)} \leq C.$$

Proof: In order to bound the condition number $\kappa(M_{asbs3g}^{-1}S)$, we need upper and lower bounds for the spectrum of $M_{asbs3g}^{-1}S$. To this end, we use Theorem 2.1.3 for the matrix S and the preconditioner M_{asbs3}. Throughout this proof we maintain the notation adopted when defining (5.3.11) and (5.3.13).

First, we find an upper bound for $\lambda_{\max}(M_{asbs3}^{-1}S)$. Let $\mathbf{u} \in S_h^0(\Gamma)$, and let $m_{sbs3}^c(\cdot,\cdot)$ and $m_{sbs3}^f(\cdot,\cdot)$ denote the bilinear form (5.3.11) associated with the preconditioners M_{sbs3}^c and M_{sbs3}^f respectively. By the Cauchy-Schwarz inequality and Theorem 5.3.4,

$$\begin{aligned} s(\mathbf{u},\mathbf{u}) &\leq C\,(s(\mathbf{u},\mathbf{u}) + s(\mathbf{u},\mathbf{u})) \\ &\leq C\left(m_{sbs3}^c(\mathbf{u},\mathbf{u}) + m_{sbs3}^f(\mathbf{u},\mathbf{u})\right). \end{aligned}$$

From this estimate, the definition (5.3.12) of $m_{asbs3}(\cdot,\cdot)$, and Theorem 2.1.3, it follows that:

$$\lambda_{\max}(M_{asbs3g}^{-1}S) \leq C. \tag{5.3.21}$$

Next, we develope a lower bound for $\lambda_{\min}(M_{asbs3g}^{-1}S)$. Let $\tilde{\mathbf{u}}$ be the discrete harmonic extension of $\mathbf{u}$ in Ω, and let $\tilde{\mathbf{u}}_o = Q_{2^ph}\tilde{\mathbf{u}}$ be the L^2-projection of $\tilde{\mathbf{u}}$ onto $S_{2^ph}^0(\Omega)$, and $\mathbf{u}_o = \tilde{\mathbf{u}}_o|_\Gamma$. Then, by the Cauchy-Schwarz inequality and Lemma 5.3.7,

$$\begin{aligned} m_{asbs3g}(\mathbf{u},\mathbf{u}) &= m_{asbs3g}(\mathbf{u}-\mathbf{u}_o+\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o+\mathbf{u}_o) \\ &\leq C m_{asbs3g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + C m_{asbs3g}(\mathbf{u}_o,\mathbf{u}_o) \\ &= C m_{asbs3g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + C m_{asbs3}(\mathbf{u}_o,\mathbf{u}_o) \\ &\leq C m_{asbs3g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) + C s(\mathbf{u},\mathbf{u}). \end{aligned}$$

It remains to show that

$$m_{asbs3g}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \leq C s(\mathbf{u},\mathbf{u}). \tag{5.3.22}$$

We demonstrate that:

$$m^f_{sbs3}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \le Cs(\mathbf{u},\mathbf{u}) \tag{5.3.23}$$

and

$$m^c_{sbs3}(\mathbf{u}-\mathbf{u}_o, \mathbf{u}-\mathbf{u}_o) \le Cs(\mathbf{u},\mathbf{u}). \tag{5.3.24}$$

Let $\mathbf{w} = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_o$. Then $\mathbf{w}|_\Gamma = \mathbf{u} - \mathbf{u}_o$. At the fine stage, in the proof of Theorem 5.3.4, we replace $\mathbf{u}$ by $\mathbf{u}_f$. Then we obtain:

$$\|\mathbf{u}^i_k\|^2_{H^{1/2}_{oo}(F^i_k)} \le \frac{1}{H}\|\mathbf{u}^\varphi_f\|^2_{L^2(F^i_k)} + |\mathbf{u}^\varphi_f|^2_{H^{1/2}(F^i_k)}.$$

From this estimate, Lemma 5.3.1 (*iii*) for subdomains Ω^b_i inside bars, and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{F^i_k \subset \partial\Omega^b_i} \alpha_{F^i_k}\|\mathbf{u}^i_k\|^2_{H^{1/2}_{oo}(F^i_k)} &\le \sum_{F^i_k \subset \partial\Omega^b_i} \frac{1}{H}\|\mathbf{u}^\varphi_f\|^2_{L^2(F^i_k)} + \sum_{F^i_k \subset \partial\Omega^b_i} |\mathbf{u}^\varphi_f|^2_{H^{1/2}(F^i_k)} \\ &\le C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega^b_i)} + C|\mathbf{w}|^2_{H^1(\Omega^b_i)}.\end{aligned}$$

Since each F^i_k is shared by only two subdomains Ω^b_i, after summing over all $\Omega^b_i \subset \Omega$, Lemma 5.3.6 implies:

$$\begin{aligned}\sum_{i,k} \alpha_{F^i_k}\|\mathbf{u}^i_k\|^2_{H^{1/2}_{oo}(F^i_k)} &\le C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} + C|\mathbf{w}|^2_{H^1(\Omega)} \\ &\le C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned}$$

Then, through the equivalence (5.3.10) for the face F^i_k, the Cauchy-Schwarz inequality for the decomposition of Ω into disjoint subdomains Ω^b_i, and the right hand-side inequality in Theorem 2.2.3, we obtain:

$$\begin{aligned}\sum_{i,k} \alpha_{F^i_k}(\delta^{1/2}_{F^i_k}\mathbf{u}^i_k, \mathbf{u}^i_k)_{F^i_k} &\le C\sum_{\Omega^b_i \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega^b_i)} \\ &\le C\sum_{\Omega^b_i \subset \Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega^b_i)} \\ &\le C\sum_{\Omega^b_i \subset \Omega} s_i(\mathbf{u},\mathbf{u}).\end{aligned}$$

Thus, by the representation (5.3.4) of $s(\cdot,\cdot)$,

$$\sum_{i,k} \alpha_{F_k^i} (\delta_{F_k^i}^{1/2} \mathbf{u}_k^i, \mathbf{u}_k^i)_{F_k^i} \le C s(\mathbf{u},\mathbf{u}). \tag{5.3.25}$$

Next, replacing $\mathbf{u}$ by $\mathbf{u}_f$ in the proof of Theorem 5.3.4, also yields:

$$\|\mathbf{u}_j^b\|^2_{H_{00}^{1/2}(F_j^b)} \le C\frac{1}{H}\|\mathbf{u}_f^{\varphi_b}\|^2_{L^2(F_j^b)} + C|\mathbf{u}^{\varphi_b}|^2_{H^{1/2}(F_j^b)}.$$

From this estimate, Lemma 5.3.1 (*iii*) for bars Ω_b^S inside slices, and the left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{F_j^b \subset \partial\Omega_b^S} \alpha_{F_j^b}\|\mathbf{u}_j^b\|^2_{H_{00}^{1/2}(F_j^b)} &\le \sum_{F_j^b \subset \partial\Omega_b^S} \frac{1}{H}\|\mathbf{u}_f^{\varphi_b}\|^2_{L^2(F_j^b)} + \sum_{F_j^b \subset \partial\Omega_b^S} |\mathbf{u}_f^{\varphi_b}|^2_{H^{1/2}(F_j^b)} \\ &\le C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega_b^S)} + C|\mathbf{w}|^2_{H^1(\Omega_b^S)}.\end{aligned}$$

Since each F_j^b is shared by only two bars Ω_b^S, after summing over all $\Omega_b^S \subset \Omega$, Lemma 5.3.6 implies:

$$\begin{aligned}\sum_{b,j} \alpha_{F_j^b}\|\mathbf{u}_j^b\|^2_{H_{00}^{1/2}(F_j^b)} &\le C\frac{1}{H^2}\|\mathbf{w}\|^2_{L^2(\Omega)} + C|\mathbf{w}|^2_{H^1(\Omega)} \\ &\le |\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned}$$

Then, through the equivalence (5.3.10) for the face F_j^b, the Cauchy-Schwarz inequality for the decomposition of Ω into disjoint subdomains Ω_i^b, and the right hand-side inequality in Theorem 2.2.3, we obtain:

$$\begin{aligned}\sum_{b,j} \alpha_{F_j^b} (\delta_{F_j^b}^{1/2} \mathbf{u}_j^b, \mathbf{u}_j^b)_{F_j^b} &\le C \sum_{\Omega_i^b \subset \Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega_i^b)} \\ &\le C \sum_{\Omega_i^b \subset \Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i^b)} \\ &\le C \sum_{\Omega_i^b \subset \Omega} s_i(\mathbf{u},\mathbf{u}).\end{aligned}$$

Thus, by the representation (5.3.4) of $s(\cdot,\cdot)$,

$$\sum_{b,j} \alpha_{F_j^b} (\delta_{F_j^b}^{1/2} \mathbf{u}_j^b, \mathbf{u}_j^b)_{F_j^b} \le C s(\mathbf{u},\mathbf{u}). \tag{5.3.26}$$

Finally, by replacing $\mathbf{u}$ by $\mathbf{u}_f$ in the proof of Theorem 5.3.4, also implies:

$$\|\mathbf{u}_l^S\|^2_{H^{1/2}_{oo}(F_l^S)} \leq C\|\mathbf{u}_f^{\varphi_S}\|^2_{L^2(F_l^S)} + C|\mathbf{u}_f^{\varphi_S}|^2_{H^{1/2}(F_l^S)}.$$

From this estimate, Lemma 5.3.1 (*iii*) for slices Ω^S, left hand-side inequality in Theorem 2.2.3, we deduce:

$$\begin{aligned}\sum_{F_l^S\subset\partial\Omega^S} \alpha_{F_l^S}\|\mathbf{u}_l^S\|^2_{H^{1/2}_{oo}(F_l^S)} &\leq \sum_{F_l^S\subset\partial\Omega^S} \|\mathbf{u}_f^{\varphi_S}\|^2_{L^2(F_l^S)} + \sum_{F_l^S\subset\partial\Omega^S} |\mathbf{u}_f^{\varphi_S}|^2_{H^{1/2}(F_l^S)}\\ &\leq C\frac{1}{H}\|\mathbf{w}\|^2_{L^2(\Omega^S)} + C|\mathbf{w}|^2_{H^1(\Omega^S)}.\end{aligned}$$

Since each F_l^S is shared by only two slices Ω^S, after summing over all $\Omega^S \subset \Omega$, Lemma 5.3.6 implies:

$$\begin{aligned}\sum_{S,l} \|\mathbf{u}_l^S\|^2_{H^{1/2}_{oo}(F_l^S)} &\leq C\frac{1}{H}\|\mathbf{w}\|^2_{L^2(\Omega)} + C|\mathbf{w}|^2_{H^1(\Omega)}\\ &\leq C|\tilde{\mathbf{u}}|^2_{H^1(\Omega)}.\end{aligned}$$

Then, through the equivalence (5.3.10) for the face F_l^S, the Cauchy-Schwarz inequality for the decomposition of Ω into disjoint subdomains Ω_i^b, and the right hand-side inequality in Theorem 2.2.3, we obtain:

$$\begin{aligned}\sum_{S,l} \alpha_{F_l^S}(\delta^{1/2}_{F_l^S}\mathbf{u}_l^S, \mathbf{u}_l^S)_{F_l^S} &\leq C\sum_{\Omega_i^b\subset\Omega} |\tilde{\mathbf{u}}|^2_{H^1(\Omega_i^b)}\\ &\leq C\sum_{\Omega_i^b\subset\Omega} |\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i^b)}\\ &\leq C\sum_{\Omega_i^b\subset\Omega} s_i(\mathbf{u},\mathbf{u}).\end{aligned}$$

Thus, by the representation (5.3.4) of $s(\cdot,\cdot)$,

$$\sum_{S,l} \alpha_{F_l^S}(\delta^{1/2}_{F_l^S}\mathbf{u}_l^S, \mathbf{u}_l^S)_{F_l^S} \leq Cs(\mathbf{u},\mathbf{u}). \tag{5.3.27}$$

The estimates (5.3.25), (5.3.26), and (5.3.27) imply (5.3.23). Analogously, at the coarse stage, we obtain (5.3.24). Then, through the definition (5.3.13) of $m_{asbs3g}(\cdot,\cdot)$, (5.3.23) and (5.3.24) imply (5.3.22).

Therefore, by Theorem 2.1.3 for the matrix S and the preconditioner M_{asbs3g}, $\lambda_{\min}(M_{asbs3g}^{-1}S)$ is bounded independently of the partitioning parameters H and h. Since (5.3.21) also holds, we conclude that:

$$\kappa(M_{asbs3g}^{-1}S) \leq C. \ \Box$$

5.4 Numerical Estimates

The aim of this section is to illustrate the efficiency of the ASBS$_3$ preconditioners when solving equations of the form (2.1.1) by the PCG method.

Example 5.4.1 We solve the Poisson equation:

$$\begin{cases} -\Delta \mathbf{u}(x) = \mathbf{f}(x) \text{ in } \Omega = (0,1) \times (0,1) \\ \quad \mathbf{u}(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

In the computations, at each stage the unit cube Ω is partitioned into $N = 1/H^3$ equal cubes. The mesh size is h for the fine grid, and $H/2$ for the coarse grid. The iteration counts are for a reduction in error of 10^{-4}.

Table 5.4.1: Condition number and iteration counts for SBS$_3$.

$1/H^3 = N$	$1/h = 16$		32		64	
2^3	3.2369	8	3.5697	9	3.6692	9
4^3	3.6275	9	3.7039	9	3.7250	9
8^3	5.7642	12	5.9023	12	5.9266	12
16^3	10.1532	16	10.9779	16	11.0591	16

Table 5.4.2: Condition number and iteration counts for ASBS$_{3a}$.

$1/H^3 = N$	$1/h = 16$		32		64	
2^3	3.0767	8	3.3672	9	3.4036	9
4^3	3.1926	8	3.5881	9	3.6512	9
8^3	3.3836	9	3.7624	9	3.9844	9
16^3	3.8914	9	4.0684	10	4.1180	10

Discussion: Table 5.4.1 shows that for the SBS_3 preconditioning technique, the relative condition number grows linearly as $1/H$ and remains bounded independently of the mesh parameter h (see Theorem 5.3.4). In Table 5.4.2, for the $ASBS_{3a}$ preconditioning technique, the relative condition number is bounded independently of the partitioning parameters H and h (see Theorem 5.3.8).

5.5 Summary

In this chapter, we have introduced a new class of slice-based iterative substructuring techniques for the SC system (2.2.10) in three dimensions. For the new solvers, the separate treatment of the wire-basket points (see Section 2.2) is avoided by partitioning the global interface between all subdomains as a union of faces between slices, faces between bars inside slices, and faces between subdomains inside bars. The (two-stage) $ASBS_3$ techniques presented here achieve scalability, and therefore optimal convergence properties, by alternating the (one-stage) SBS_3 solver (based on the J-operator on the various faces) in two orthogonal directions. When the two-grid technique is applied, we solve two-dimensional problems on interfaces (between slices, between bars inside slices, and between subdomains inside bars) at the coarse stage, and alternate the direction of the strips at the fine stage. The possibility of reducing the size of the coarse solver from three to only two dimensions seems to offer an advantage, especially when H is small.

Chapter 6

Conclusions and Further Remarks

6.1 Overview

In this thesis, we have considered the solution of discrete linear systems of equations, which result from the finite element approximation of second order symmetric elliptic PDE's on bounded domains, via a new class of DD methods. The alternate strip-based techniques described and analysed in this thesis are:

- optimal with respect to the partitioning parameters,
- suited for parallel computing architectures.

After the claimed convergence behaviour of these methods has been verified, the next step is to consider some more practical matters. Since the early implementations of DD methods on parallel computers, programming techniques and computer architectures have significantly evolved and developing efficient programs for these architectures needs in general some expertise in parallel programming. In a parallel setting, it is natural to match the number of subproblems to the number of processors available. In addition, the amount of storage requirements for each processor should be roughly similar. The new two-stage methods proposed in this thesis are all additive techniques. Corresponding multiplicative versions can also be derived. The convergence rate for additive methods is slower than for the multiplicative ones, and in general may require twice as many iterations as the multiplicative versions. On the other hand, additive algorithms tend to be easier to load balance, since the number of parallel tasks is significantly larger than in the multiplicative case.

An important issue in DD is choosing the subdomains. Very often the choice of the subdomains is dictated by geometric considerations, e.g. subdomains with regular geometry are preferable. From a purely computational complexity viewpoint, a small H provides a better, though more expensive, coarse grid approximation, and requires solving more subdomain problems of smaller size. However, for problems in two dimensions, when the $ASBS_2$ technique is applied, only one-dimensional problems need to be solved at both stages. Similarly, for problems in three dimensions, if the $ASBS_3$ technique is applied, then only two-dimensional problems need solving at both stages.

The results in Chapters 3 to 5 of this thesis remain valid if the solvers for the subproblems are replaced by spectrally equivalent solvers. In view of extending the (two-stage) ASB_2, $ASBS_2$, and $ASBS_3$ methods from the case of a quasi-uniform partition of Ω to the case when the elements of the coarse grid are shape regular, or to finite elements with polynomials of higher degree, we note that throughout our analysis the L^2-projection operator can be replaced by other local averaging operators (see Clément (1975) [29], Ciarlet (1978) [28], Section 3.2.3, Scott and Zhang (1990) [71], Xu and Zou (1998) [87], Section 4.2.1, or Brenner and Scott (2002) [17], Section 4.8).

6.2 Time Dependent Problems

Extending the alternate strip-based algorithms to parabolic problems is an area for future research. We now consider briefly one possible extension based on the observation that once a parabolic problem has been discretised in time, it can be viewed as a sequence of elliptic problems.

Differential Form We consider the following model parabolic problem:

$$\begin{cases} -\nabla \cdot \alpha \nabla \mathbf{u}(x,t) + \dfrac{\partial \mathbf{u}(x,t)}{\partial t} = \mathbf{f}(x,t) \text{ on } \Omega \times (0,T] \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \text{ on } \Omega \\ \mathbf{u}(x,t) = 0 \text{ on } \partial\Omega \times (0,T], \end{cases} \tag{6.2.1}$$

where the domain $\Omega \subset \mathbb{R}^D$ is a polygon, for $D = 2$, or a polyhedron, for $D = 3$; $x = (x_1, \cdots, x_D)$ denotes a generic element in Ω and $\mathbf{f}$ and $\mathbf{u}_0$ are given data. The coefficient $\alpha(x) \geq \alpha_0$, for some positive constant α_0, is $\alpha(x) \equiv 1$ (for the heat equation) or piecewise constant in Ω.

Variational Form Let

$$L^2\left(0,T;H_0^1(\Omega)\right) =$$
$$= \left\{\mathbf{u}:(0,T)\rightarrow H_0^1(\Omega)|\ \mathbf{u}\text{ - measurable in }(0,T),\ \int_0^T \|\mathbf{u}(t)\|^2_{H^1(\Omega)}dt<\infty\right\}$$

and

$$C^0\left([0,T];L^2(\Omega)\right) = \left\{\mathbf{u}:[0,T]\rightarrow L^2(\Omega)|\ \mathbf{u}\text{ - continuous in }[0,T]\right\}.$$

The weak form of problem (6.2.1) is: find $\mathbf{u} \in L^2\left(0,T;H_0^1(\Omega)\right)\bigcap C^0\left([0,T];L^2(\Omega)\right)$ such that:

$$\begin{cases} -(\nabla\cdot\alpha\nabla\mathbf{u}(t),\mathbf{v}) + \dfrac{d}{dt}(\mathbf{u}(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v}),\ \forall\mathbf{v}\in L^2\left(0,T;H_0^1(\Omega)\right) \\ \mathbf{u}(0)=\mathbf{u}_0, \end{cases} \tag{6.2.2}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

Finite Element Approximation (Thomée (1984) [79], Morton and Mayers (1994) [61]) The time discretisation for (6.2.1) is the discontinuous Galerkin method (see e.g. Johnson (1987) [50], Section 8.4.3). This is based on a uniform partition $0=t_0<t_1\cdots<t_{N_t}=T$ of the time interval $[0,T]$, with time-step $\Delta t = t_{p+1}-t_p$, $p=0,\cdots,N_t-1$, and a finite element formulation in time with piecewise polynomials of degree r. As in the time-independent case, at each given time t_p, the space discretisation is based on a uniform mesh of the domain Ω. Let h denote the space-step at each time t_p, $S_h(\Omega)$ be the space of continuous piecewise linear functions associated with the mesh Σ^h, and $S_h^0(\Omega)\subset S_h(\Omega)$, the subset of functions that vanish on $\partial\Omega$. Let:

$$V_p^r = \left\{\mathbf{u}\mid\mathbf{u}(x,t)=\sum_{j=0}^r t^j\phi_j(x),\ \phi_j\in S_h^0(\Omega),\ (x,t)\in\Omega\times(t_{p-1},t_p)\right\}$$

and

$$V_{\Delta t}^r = \left\{\mathbf{u}\mid\mathbf{u}|_{\Omega\times(t_{p-1},t_p)}\in V_p^r,\ \forall p=1,\cdots,N_t\right\}.$$

The full space-time discretisation for problem (6.2.1) is: find $\mathbf{u} \in V^r_{\Delta t}$ such that, for $p = 1, \cdots, N_t$,

$$\int_{t_{p-1}}^{t_p} ((-\nabla \cdot \alpha \nabla \mathbf{u}, \mathbf{v}) + \frac{d}{dt}(\mathbf{u}, \mathbf{v}))dt = (\mathbf{u}^-_{p-1} - \mathbf{u}^+_{p-1}, \mathbf{v}^+_{p-1}) + \int_{t_{p-1}}^{t_p} (\mathbf{f}, \mathbf{v})dt, \ \forall \mathbf{v} \in V^r_p, \tag{6.2.3}$$

where $\mathbf{u}^-_{p-1} = \lim_{s \to 0_-} \mathbf{u}(t_{p-1} + s)$ and $\mathbf{u}^+_{p-1} = \lim_{s \to 0^+} \mathbf{u}(t_{p-1} + s)$.

Algebraic Form of the Discretised Equation Let $\{\phi_j\}_{j=1}^m$ denote the nodal basis for $S^0_h(\Omega)$, where m is the number of degrees of freedom for the grid. If we take:

$$\mathbf{u}^p(x) = \sum_{k=1}^m u^p_k \phi_k(x), \ \forall p = 0, \cdots, N_t - 1,$$

and $\mathbf{v} = \phi_k$ $(k = 1, \cdots, m)$ in the case $r = 0$, then (6.2.3) generates by numerical integration (trapezoidal rule) the following linear system:

$$(\Delta t A + M)u^p = M u^{p-1} + \frac{\Delta t}{2}\left(f^{p-1} + f^p\right), \tag{6.2.4}$$

where A and M denote the stiffness and the mass matrix respectively, and u^p is the m vector of the point values $\{u^p_k\}_{k=1}^m$. We note that this is a simple modification of the backward Euler scheme (see e.g. Johnson (1987) [50], Section 8.4.2) where the term involving the data f involves an average over the time interval $[t_{p-1}, t_p]$ rather than the value f^p of f at t_p.

Domain Decomposition for Parabolic Equations When $L = \Delta t A + M$ in (6.2.4), the condition number $\kappa(L)$ is of order $\mathcal{O}(1 + \Delta t/h^2)$ (Quarteroni and Valli (1994) [65], Section 6.3.2). If $\Delta t = \mathcal{O}(h)$ or $\Delta t = \mathcal{O}(h^2)$, then $\kappa(L) < \kappa(A)$. In this case a coarse solver might not be needed for the global communication of information (see Bank and Dupont (1981) [3]). For developments in Schwarz preconditioners for parabolic problems we refer to Lions (1988) [53], Cai (1991) [22] and (1994) [23], Meurant (1991) [58], Israeli *et al.* (1993) [49], and Lube *et al.* [54].

Let Ω be partitioned into N nonoverlapping subdomains Ω_i, and Γ denote the global interface between all subdomains. If we write:

$$L = \begin{bmatrix} L_{II} & L_{IE} \\ L^T_{IE} & L_{EE} \end{bmatrix},$$

where L_{II} is the submatrix of L associated with the nodes in $\Omega \setminus \Gamma$, and L_{EE} is the submatrix of L associated with the nodes in Γ, then we can use the following LDU factorisation of L:

$$L = \begin{bmatrix} I & 0 \\ L_{IE}^T L_{II}^{-1} & I \end{bmatrix} \times \begin{bmatrix} L_{II} & 0 \\ 0 & S \end{bmatrix} \times \begin{bmatrix} I & L_{II}^{-1} L_{IE} \\ 0 & I \end{bmatrix},$$

where S is the SC matrix defined as $S = L_{EE} - L_{IE}^T L_{II}^{-1} L_{IE}$. When $\Omega \subset \mathbb{R}^2$, using the same notation as in Chapter 4, the preconditioners M_1 and M_2, at the first and second stage of the alternate strip-based substructuring process respectively, can be described in terms of block matrices as follows:

$$M_1 = \begin{bmatrix} I_{xx} & 0 & S_{xy} M_{yy}^{-1} \\ 0 & I_{vv} & S_{vy} M_{yy}^{-1} \\ 0 & 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} M_{xx}^1 & M_{xv}^1 & 0 \\ (M_{xv}^1)^T & M_{vv}^1 & 0 \\ 0 & 0 & M_{yy}^1 \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{vv} & 0 \\ M_{yy}^{-1} S_{xy}^T & M_{yy}^{-1} S_{vy}^T & I_{yy} \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} I_{xx} & 0 & 0 \\ S_{xv}^T M_{xx}^{-1} & I_{vv} & 0 \\ S_{xy}^T M_{xx}^{-1} & 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} M_{xx}^2 & 0 & 0 \\ 0 & M_{vv}^2 & M_{vy}^2 \\ 0 & (M_{vy}^2)^T & M_{yy}^2 \end{bmatrix} \times \begin{bmatrix} I_{xx} & M_{xx}^{-1} S_{xv} & M_{xx}^{-1} S_{xy} \\ 0 & I_{vv} & 0 \\ 0 & 0 & I_{yy} \end{bmatrix}.$$

In two and three space dimensions, if $\Delta t \leq CH$, then by using the one-stage strip-based preconditioner we would generally expect the condition number of the preconditioned SC system to be bounded independently of the space-time discretisation parameters H, h and Δt. For larger Δt, the same estimate would hold provided that a two-stage alternate strip-based preconditioner is employed.

Appendix A

Auxiliary Results

In this appendix we prove some of the results underlying the theorems in this thesis.

Proof of Lemma 3.3.1: Let $\Omega = (0,1) \times (0,1)$, $\Omega^s = (0,1) \times (0,H)$ be a strip-subregion of Ω, and $\mathbf{u} \in H^1(\Omega^s)$.

(i) If $\mathbf{u}$ is equal to zero along one side of size H of Ω^s, $\{0\} \times (0,H)$ say, then:

$$\mathbf{u}(x,y) = \int_0^x \frac{\partial \mathbf{u}(\tau,y)}{\partial \tau} d\tau.$$

By the Cauchy-Schwarz inequality,

$$\mathbf{u}^2(x,y) = \left(\int_0^x \frac{\partial \mathbf{u}(\tau,y)}{\partial \tau} d\tau \right)^2 \leq x \int_0^x \left(\frac{\partial \mathbf{u}(\tau,y)}{\partial \tau} \right)^2 d\tau \leq \int_0^1 \left(\frac{\partial \mathbf{u}(x,y)}{\partial x} \right)^2 dx.$$

To complete the proof of inequality (i), we integrate first with respect to y:

$$\int_0^H \mathbf{u}^2(x,y) dy \leq |\mathbf{u}|^2_{H^1(\Omega^s)},$$

then with respect to x:

$$\|\mathbf{u}\|^2_{L^2(\Omega^s)} = \int_0^1 \int_0^H \mathbf{u}^2(x,y) dy dx \leq |\mathbf{u}|^2_{H^1(\Omega^s)}.$$

(ii) If $\mathbf{u}$ is equal to zero along one side of size 1 of Ω^s, $[0,1] \times \{0\}$ say, then:

$$\mathbf{u}(x,y) = \int_0^y \frac{\partial \mathbf{u}(x,\tau)}{\partial \tau} d\tau.$$

By the Cauchy-Schwarz inequality,

$$\mathbf{u}^2(x,y) \le \left(\int_0^y \frac{\partial \mathbf{u}(x,\tau)}{\partial \tau} d\tau\right)^2 \le y \int_0^y \left(\frac{\partial \mathbf{u}(x,\tau)}{\partial \tau}\right)^2 d\tau \le H \int_0^H \left(\frac{\partial \mathbf{u}(x,y)}{\partial y}\right)^2 dy.$$

To complete the proof of inequality *(ii)* we integrate first with respect to x:

$$\int_0^1 \mathbf{u}^2(x,y)dx \le H|\mathbf{u}|^2_{H^1(\Omega^s)},$$

then with respect to y:

$$\|\mathbf{u}\|^2_{L^2(\Omega^s)} = \int_0^H \int_0^1 \mathbf{u}^2(x,y)dxdy \le H^2|\mathbf{u}|^2_{H^1(\Omega^s)}.$$

(iii) If Γ^j is a side of size 1 of Ω^s, $[0,1]\times\{0\}$ say, then:

$$\mathbf{u}(x,0) = \mathbf{u}(x,y) - \int_0^y \frac{\partial \mathbf{u}(x,\tau)}{\partial \tau} d\tau.$$

By the Cauchy inequality,

$$\mathbf{u}^2(x,0) \le 2\mathbf{u}^2(x,y) + 2\left(\int_0^y \frac{\partial \mathbf{u}(x,\tau)}{\partial \tau} d\tau\right)^2.$$

Next the Schwarz inequality and the bound on y imply:

$$\mathbf{u}^2(x,0) \le 2\mathbf{u}^2(x,y) + 2H\int_0^H \left(\frac{\partial \mathbf{u}(x,y)}{\partial y}\right)^2 dy.$$

Integrating with respect to x yields:

$$\int_0^1 \mathbf{u}^2(x,0)dx \le 2\int_0^1 \mathbf{u}^2(x,y)dx + 2H\int_0^1\int_0^H \left(\frac{\partial \mathbf{u}(x,y)}{\partial y}\right)^2 dydx.$$

Integrating again with respect to y implies:

$$H\|\mathbf{u}\|^2_{L^2(\Gamma^j)} \le 2\|\mathbf{u}\|^2_{L^2(\Omega^s)} + 2H^2|\mathbf{u}|^2_{H^1(\Omega^s)}.$$

Hence:

$$\|\mathbf{u}\|^2_{L^2(\Gamma^j)} \le \frac{2}{H}\|\mathbf{u}\|^2_{L^2(\Omega^s)} + 2H|\mathbf{u}|^2_{H^1(\Omega^s)}.$$

(iv) For $\mathbf{u} \in H^1(\Omega)$, we can write:

$$\mathbf{u}(x,y) = \mathbf{u}(iH,y) + \int_{iH}^{x} \frac{\partial \mathbf{u}(\tau,y)}{\partial \tau} d\tau.$$

For all $(x,y) \in \Omega_i$, by the Cauchy-Schwarz inequality, we deduce:

$$\begin{aligned}
\mathbf{u}^2(x,y) &\leq 2\mathbf{u}^2(iH,y) + 2\left(\int_{iH}^{x} \frac{\partial \mathbf{u}(\tau,y)}{\partial \tau} d\tau\right)^2 \\
&\leq 2\mathbf{u}^2(iH,y) + 2(x-iH)\int_{iH}^{x} \left(\frac{\partial \mathbf{u}(\tau,y)}{\partial \tau}\right)^2 d\tau \\
&\leq 2\mathbf{u}^2(iH,y) + 2H\int_{iH}^{(i+1)H} \left(\frac{\partial \mathbf{u}(x,y)}{\partial x}\right)^2 dx.
\end{aligned}$$

Thus:

$$\int_0^H \mathbf{u}^2(x,y)dy \leq 2\int_0^H \mathbf{u}^2(iH,y)dy + 2H|\mathbf{u}|^2_{H^1(\Omega_i)}$$

and

$$\begin{aligned}
\|\mathbf{u}\|^2_{L^2(\Omega_i)} &= \int_{iH}^{(i+1)H}\int_0^H \mathbf{u}^2(x,y)dydx \\
&\leq 2H\int_0^H \mathbf{u}^2(iH,y)dy + 2H^2|\mathbf{u}|^2_{H^1(\Omega^s)}.
\end{aligned}$$

On the other hand, it is easy to show that:

$$\int_0^H \mathbf{u}^2(iH,y)dy \leq 2\int_0^H \mathbf{u}^2(x,y)dy + 2H|\mathbf{u}|^2_{H^1(\Omega^s)}.$$

Therefore:

$$\int_0^H \mathbf{u}^2(iH,y)dy \leq 2\|\mathbf{u}\|^2_{L^2(\Omega^s)} + 2H|\mathbf{u}|^2_{H^1(\Omega^s)}.$$

Now we can write:

$$\|\mathbf{u}\|^2_{L^2(\Omega_i)} \leq 4H\|\mathbf{u}\|^2_{L^2(\Omega^s)} + 6H^2|\mathbf{u}|^2_{H^1(\Omega^s)}. \quad \square$$

Proof of Remark 4.2.1: As in Section 4.2, let (2.2.1) be the decomposition of the domain Ω into nonoverlapping subdomains with interior cross-points. We consider the SC system (2.2.10), and write the SC matrix S as:

$$S = \begin{bmatrix} S_{xx} & S_{xv} & S_{xy} \\ S_{xv}^T & S_{vv} & S_{vy} \\ S_{xy}^T & S_{vy}^T & S_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} A_{xx} & A_{xv} & A_{xy} \\ A_{xv}^T & A_{vv} & A_{vy} \\ A_{xy}^T & A_{vy}^T & I_{yy} \end{bmatrix} - \begin{bmatrix} A_{Ix}^T \\ A_{Iv}^T \\ A_{Iy}^T \end{bmatrix} \times A_{II}^{-1} \times \begin{bmatrix} A_{Ix} & A_{Iv} & A_{Iy} \end{bmatrix}.$$

Hence:

$$\begin{bmatrix} S_{xx} & S_{xv} & S_{xy} \\ S_{xv}^T & S_{vv} & S_{vy} \\ S_{xy}^T & S_{vy}^T & S_{yy} \end{bmatrix} = \begin{bmatrix} A_{xx} - A_{Ix}^T A_{II}^{-1} A_{Ix} & A_{xv} - A_{Ix}^T A_{II}^{-1} A_{Iv} & A_{xy} - A_{Ix}^T A_{II}^{-1} A_{Iy} \\ A_{xv}^T - A_{Iv}^T A_{II}^{-1} A_{Ix} & A_{vv} - A_{Iv}^T A_{II}^{-1} A_{Iv} & A_{vy} - A_{Iv}^T A_{II}^{-1} A_{Iy} \\ A_{xy}^T - A_{Iy}^T A_{II}^{-1} A_{Ix} & A_{vy}^T - A_{Iy}^T A_{II}^{-1} A_{Iv} & A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy} \end{bmatrix}. \tag{A1}$$

We also write S in the factored form:

$$S = \begin{bmatrix} I_{xx} & 0 & S_{xy}S_{yy}^{-1} \\ 0 & I_{vv} & S_{vy}S_{yy}^{-1} \\ 0 & 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} & 0 \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} & 0 \\ 0 & 0 & S_{yy} \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{vv} & 0 \\ S_{yy}^{-1}S_{xy}^T & S_{yy}^{-1}S_{vy}^T & I_{yy} \end{bmatrix},$$

where

$$\begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} \end{bmatrix} = \begin{bmatrix} S_{xx} & S_{xv} \\ S_{xv}^T & S_{vv} \end{bmatrix} - \begin{bmatrix} S_{xy} \\ S_{vy} \end{bmatrix} \times S_{yy}^{-1} \times \begin{bmatrix} S_{xy}^T & S_{vy}^T \end{bmatrix}$$

corresponds to the interfaces between disjoint strips into which the subdomains in (2.2.1) are assembled. Hence:

$$\begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} \end{bmatrix} = \begin{bmatrix} S_{xx} - S_{xy}S_{yy}^{-1}S_{xy}^T & S_{xv} - S_{xy}S_{yy}^{-1}S_{vy}^T \\ S_{xv}^T - S_{vy}S_{yy}^{-1}S_{xy}^T & S_{vv} - S_{vy}S_{yy}^{-1}S_{vy}^T \end{bmatrix}. \tag{A2}$$

When we replace the matrices on the right hand-side in (A2) by their equivalent forms from the right hand-side in (A1), we obtain:

$$
\begin{aligned}
\tilde{S}_{xx} =& A_{xx} - A_{Ix}^T A_{II}^{-1} A_{Ix} \\
&+ (A_{xy} - A_{Ix}^T A_{II}^{-1} A_{Iy})(A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy})^{-1}(A_{xy}^T - A_{Iy}^T A_{II}^{-1} A_{Ix}) \\
\tilde{S}_{xv} =& A_{xv} - A_{Ix}^T A_{II}^{-1} A_{Iv} \\
&+ (A_{xy} - A_{Ix}^T A_{II}^{-1} A_{Iy})(A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy})^{-1}(A_{vy}^T - A_{Iy}^T A_{II}^{-1} A_{Iv}) \\
\tilde{S}_{xv}^T =& A_{xv}^T - A_{Iv}^T A_{II}^{-1} A_{Ix} \\
&+ (A_{vy} - A_{Iv}^T A_{II}^{-1} A_{Iy})(A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy})^{-1}(A_{xy}^T - A_{Iy}^T A_{II}^{-1} A_{Ix}) \\
\tilde{S}_{vv} =& A_{vv} - A_{Iv}^T A_{II}^{-1} A_{Iv} \\
&+ (A_{vy} - A_{Iv}^T A_{II}^{-1} A_{Iy})(A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy})^{-1}(A_{vy}^T - A_{Iy}^T A_{II}^{-1} A_{Iv}).
\end{aligned}
\tag{A3}
$$

On the other hand, we assemble the subdomains in (2.2.1) into nonoverlapping strips, such that the strip interfaces align in the $0x$ direction. Then, we can write the SC matrix for the decomposition of Ω into strips as:

$$
S_s = \begin{bmatrix} A_{xx} & A_{xv} \\ A_{xv}^T & A_{vv} \end{bmatrix} - \begin{bmatrix} A_{Ix}^T & A_{xy} \\ A_{Iv}^T & A_{vy} \end{bmatrix} \times \begin{bmatrix} A_{II} & A_{Iy} \\ A_{Iy}^T & A_{yy} \end{bmatrix}^{-1} \times \begin{bmatrix} A_{Ix} & A_{Iv} \\ A_{xy}^T & A_{vy}^T \end{bmatrix}.
$$

Note that:

$$
\begin{aligned}
\begin{bmatrix} A_{II} & A_{Iy} \\ A_{Iy}^T & A_{yy} \end{bmatrix}^{-1} =& \begin{bmatrix} I_{II} & -A_{II}^{-1} A_{Iy} \\ 0 & I_{yy} \end{bmatrix} \times \begin{bmatrix} A_{II}^{-1} & 0 \\ 0 & (A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy})^{-1} \end{bmatrix} \\
& \times \begin{bmatrix} I_{II} & 0 \\ -A_{Iy}^T A_{II}^{-1} & I_{yy} \end{bmatrix}.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
S_s =& \begin{bmatrix} A_{xx} & A_{xv} \\ A_{xv}^T & A_{vv} \end{bmatrix} \\
&- \begin{bmatrix} A_{Ix}^T & A_{xy} - A_{Ix}^T A_{II}^{-1} A_{Iy} \\ A_{Iv}^T & A_{vy} - A_{Iv}^T A_{II}^{-1} A_{Iy} \end{bmatrix} \times \begin{bmatrix} A_{II}^{-1} & 0 \\ 0 & (A_{yy} - A_{Iy}^T A_{II}^{-1} A_{Iy})^{-1} \end{bmatrix} \\
&\times \begin{bmatrix} A_{Ix} & A_{Iv} \\ A_{xy}^T - A_{Iy}^T A_{II}^{-1} A_{Ix} & A_{vy}^T - A_{Iy}^T A_{II}^{-1} A_{Iv} \end{bmatrix}.
\end{aligned}
\tag{A4}
$$

After completing the calculations in (A4), we obtain:

$$S_s = \begin{bmatrix} \tilde{S}_{xx} & \tilde{S}_{xv} \\ \tilde{S}_{xv}^T & \tilde{S}_{vv} \end{bmatrix}, \text{ with } \tilde{S}_{xx},\ \tilde{S}_{xv},\ \tilde{S}_{xv}^T, \text{ and } \tilde{S}_{vv} \text{ as in (A3).} \ \square$$

Proof of Lemma 5.3.1: (i$_1$) We prove that if $\mathbf{u} \in H^1(\Omega_b^S)$ is equal to zero on one square face of Ω_b^S, $\{0\} \times (jH, (j+1)H) \times 0, H)$ say, then:

$$\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} \leq C|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

For $\mathbf{u} \in H^1(\Omega_b^S)$ equal to zero on $\{0\} \times (jH, (j+1)H) \times (0, H)$, we write:

$$\mathbf{u}(x,y,z) = \int_0^x \frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} d\tau.$$

Thus, for all $(x,y,z) \in \Omega_b^S$ by the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbf{u}^2(x,y,z) &\leq \left(\int_0^x \frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} d\tau \right)^2 \\ &\leq x \int_0^x \left(\frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} \right)^2 d\tau \\ &\leq \int_0^1 \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial x} \right)^2 dx. \end{aligned}$$

Now (i_1) follows by integrating first with respect to y and z:

$$\int_0^H \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydz \leq |\mathbf{u}|^2_{H^1(\Omega_b^S)},$$

then with respect to x:

$$\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} = \int_0^1 \int_0^H \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydzdx \leq |\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

(i$_2$) We show that if $\mathbf{u} \in H^1(\Omega^S)$ is equal to zero on one strip like face of Ω^S, $\{0\} \times (0,1) \times (0,H)$ say, then for all $(x,y,z) \in \Omega^S$:

$$\|\mathbf{u}\|^2_{L^2(\Omega^S)} \leq C|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

Let $\mathbf{u} \in H^1(\Omega^S)$ be zero on $\{0\} \times (0,1) \times (0,H)$, then:

$$\mathbf{u}(x,y,z) = \int_0^x \frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} d\tau.$$

By the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} \mathbf{u}^2(x,y,z) &\leq \left(\int_0^x \frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} d\tau \right)^2 \\ &\leq x \int_0^x \left(\frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} \right)^2 d\tau \\ &\leq \int_0^1 \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial x} \right)^2 dx. \end{aligned}$$

And (i_2) follows by integrating first with respect to y and z:

$$\int_0^H \int_0^1 \mathbf{u}^2(x,y,z) dy dz \leq |\mathbf{u}|^2_{H^1(\Omega^S)},$$

then with respect to x:

$$\|\mathbf{u}\|^2_{L^2(\Omega^S)} = \int_0^1 \int_0^H \int_0^1 \mathbf{u}^2(x,y,z) dy dz dx \leq |\mathbf{u}|^2_{H^1(\Omega^S)}.$$

(ii$_1$) We prove that if $\mathbf{u} \in H^1(\Omega_b^S)$ is equal to zero on one face of Ω_i^b, then:

$$\|\mathbf{u}\|^2_{L^2(\Omega_i^b)} \leq CH^2 |\mathbf{u}|^2_{H^1(\Omega_i^b)}.$$

Let $\mathbf{u}$ be equal to zero on $(iH,(i+1)H) \times (jh,(j+1)H) \times \{0\}$. For all $(x,y,z) \in \Omega_i^b$, we can write:

$$\mathbf{u}(x,y,z) = \int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau.$$

Thus:

$$\begin{aligned} \mathbf{u}^2(x,y,z) &\leq \left(\int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau \right)^2 \\ &\leq z \int_0^z \left(\frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} \right)^2 d\tau \\ &\leq H \int_0^H \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial z} \right)^2 dz. \end{aligned}$$

Now (ii_1) follows by integrating first with respect to x and y:

$$\int_{iH}^{(i+1)H} \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydx \le H|\mathbf{u}|^2_{H^1(\Omega_i^b)},$$

then with respect to z:

$$\|\mathbf{u}\|^2_{L^2(\Omega_i^b)} = \int_0^H \int_{iH}^{(i+1)H} \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydxdz \le H^2|\mathbf{u}|^2_{H^1(\Omega_i^b)}.$$

(ii$_2$) We show that if $\mathbf{u} \in H^1(\Omega_b^S)$ is equal to zero on one strip like face, $(0,1) \times (jh,(j+1)H) \times \{0\}$ say, then:

$$\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} \le CH^2|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

Let $\mathbf{u} \in H^1(\Omega_b^S)$ be equal to zero on $(0,1) \times (jh,(j+1)H) \times \{0\}$, then for all $(x,y,z) \in \Omega_b^S$:

$$\mathbf{u}(x,y,z) = \int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau.$$

Thus:

$$\begin{aligned}
\mathbf{u}^2(x,y,z) &\le \left(\int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau\right)^2 \\
&\le z \int_0^z \left(\frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau}\right)^2 d\tau \\
&\le H \int_0^H \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial z}\right)^2 dz.
\end{aligned}$$

Now (ii_2) follows by integrating first with respect to x and y:

$$\int_0^1 \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydx \le H|\mathbf{u}|^2_{H^1(\Omega_b^S)},$$

then with respect to z:

$$\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} = \int_0^H \int_0^1 \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydxdz \le H^2|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

(iii$_1$) We prove that if $F = (0,1) \times (jH,(j+1)H) \times \{0\}$ is a face on $\partial\Omega_b^S$, then:

$$\|\mathbf{u}\|^2_{L^2(F)} \le C\left(\frac{1}{H}\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} + H|\mathbf{u}|^2_{H^1(\Omega_b^S)}\right).$$

Let

$$\mathbf{u}(x,y,0) = \mathbf{u}(x,y,z) - \int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau.$$

By the Cauchy inequality,

$$\mathbf{u}^2(x,y,0) \le 2\mathbf{u}^2(x,y,z) + 2\left(\int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau\right)^2.$$

Next the Schwarz inequality and the bound on z imply:

$$\mathbf{u}^2(x,y,0) \le 2\mathbf{u}^2(x,y,z) + 2H \int_0^H \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial z}\right)^2 dy.$$

Integrating with respect to x and y yields:

$$\int_0^1 \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,0) dy dx \le 2 \int_0^1 \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z) dy dx + 2H \int_0^1 \int_{jH}^{(j+1)H} \int_0^H \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial z}\right)^2 dz dy dx.$$

Integrating again with respect to z implies:

$$H\|\mathbf{u}\|^2_{L^2(F)} \le 2\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} + 2H^2|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

Hence:

$$\|\mathbf{u}\|^2_{L^2(F)} \le \frac{2}{H}\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} + 2H|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

(iii$_2$) We show that if $F = (0,1) \times (0,1) \times \{0\}$ is a face on $\partial\Omega^S$, then:

$$\|\mathbf{u}\|^2_{L^2(F)} \le C\left(\frac{1}{H}\|\mathbf{u}\|^2_{L^2(\Omega^S)} + H|\mathbf{u}|^2_{H^1(\Omega^S)}\right).$$

Let

$$\mathbf{u}(x,y,0) = \mathbf{u}(x,y,z) - \int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau.$$

By the Cauchy inequality,

$$\mathbf{u}^2(x,y,0) \le 2\mathbf{u}^2(x,y,z) + 2\left(\int_0^z \frac{\partial \mathbf{u}(x,y,\tau)}{\partial \tau} d\tau\right)^2.$$

Next the Schwarz inequality and the bound on z imply:

$$\mathbf{u}^2(x,y,0) \leq 2\mathbf{u}^2(x,y,z) + 2H \int_0^H \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial z} \right)^2 dy.$$

Integrating with respect to x and y yields:

$$\int_0^1 \int_0^1 \mathbf{u}^2(x,y,0) dydx \leq 2 \int_0^1 \int_0^1 \mathbf{u}^2(x,y,z) dydx + 2H \int_0^1 \int_0^1 \int_0^H \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial z} \right)^2 dzdydx.$$

Integrating again with respect to z implies:

$$H\|\mathbf{u}\|^2_{L^2(F)} \leq 2\|\mathbf{u}\|^2_{L^2(\Omega^S)} + 2H^2|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

Hence:

$$\|\mathbf{u}\|^2_{L^2(F)} \leq \frac{2}{H}\|\mathbf{u}\|^2_{L^2(\Omega^S)} + 2H|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

(iv$_1$) We prove that for every $\mathbf{u} \in H^1(\Omega_b^S)$ the following estimate holds:

$$\|\mathbf{u}\|^2_{L^2(\Omega_i^b)} \leq CH^2 \left(\frac{1}{H}\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} + |\mathbf{u}|^2_{H^1(\Omega_b^S)} \right).$$

First we write:

$$\mathbf{u}(x,y,z) = \mathbf{u}(iH,y,z) + \int_{iH}^x \frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} d\tau.$$

For all $(x,y,z) \in \Omega_i^b$, by Cauchy-Schwarz inequality, we deduce:

$$\begin{aligned}
\mathbf{u}^2(x,y,z) &\leq 2\mathbf{u}^2(iH,y,z) + 2\left(\int_{iH}^x \frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} d\tau \right)^2 \\
&\leq 2\mathbf{u}^2(iH,y,z) + 2(x-iH) \int_{iH}^x \left(\frac{\partial \mathbf{u}(\tau,y,z)}{\partial \tau} \right)^2 d\tau \\
&\leq 2\mathbf{u}^2(iH,y,z) + 2H \int_{iH}^{(i+1)H} \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial x} \right)^2 dx.
\end{aligned}$$

Thus:

$$\int_0^H \int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z) dydz \leq 2 \int_0^H \int_{jH}^{(j+1)H} \mathbf{u}^2(iH,y,z) dydz + 2H|\mathbf{u}|^2_{H^1(\Omega_i^b)}$$

and

$$\|\mathbf{u}\|^2_{L^2(\Omega_i^b)} = \int_{iH}^{(i+1)H}\int_0^H\int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydzdx$$
$$\leq 2H\int_0^H\int_{jH}^{(j+1)H} \mathbf{u}^2(iH,y,z)dydz + 2H^2|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

On the other hand:

$$\int_0^H\int_{jH}^{(j+1)H} \mathbf{u}^2(iH,y,z)dydz \leq 2\int_0^H\int_{jH}^{(j+1)H} \mathbf{u}^2(x,y,z)dydz + 2H|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

Therefore:

$$\int_0^H\int_{jH}^{(j+1)H} \mathbf{u}^2(iH,y,z)dydz \leq 2\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} + 2H|\mathbf{u}|^2_{H^1(\Omega_b^S)}.$$

We deduce:

$$\|\mathbf{u}\|^2_{L^2(\Omega_i^b)} \leq 4H\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} + 6H^2|\mathbf{u}|^2_{H^1(\Omega_b^S)},$$

from which (iv_1) follows.

(iv$_2$) We show that for every $\mathbf{u} \in H^1(\Omega^S)$ the following estimate holds:

$$\|\mathbf{u}\|^2_{L^2(\Omega_b^S)} \leq CH^2\left(\frac{1}{H}\|\mathbf{u}\|^2_{L^2(\Omega^S)} + |\mathbf{u}|^2_{H^1(\Omega^S)}\right).$$

We write:

$$\mathbf{u}(x,y,z) = \mathbf{u}(x,jH,z) + \int_{jH}^{y} \frac{\partial \mathbf{u}(x,\tau,z)}{\partial \tau}d\tau.$$

For all $(x,y,z) \in \Omega^b$, by Cauchy-Schwarz inequality, we obtain:

$$\mathbf{u}^2(x,y,z) \leq 2\mathbf{u}^2(x,jH,z) + 2\left(\int_{jH}^{y} \frac{\partial \mathbf{u}(x,\tau,z)}{\partial \tau}d\tau\right)^2$$
$$\leq 2\mathbf{u}^2(x,jH,z) + 2\int_{jH}^{y} \left(\frac{\partial \mathbf{u}(x,\tau,z)}{\partial \tau}\right)^2 d\tau$$
$$\leq 2\mathbf{u}^2(x,jH,z) + 2H\int_{jH}^{(j+1)H} \left(\frac{\partial \mathbf{u}(x,y,z)}{\partial y}\right)^2 dy.$$

Therefore:

$$\int_0^1\int_0^H \mathbf{u}^2(x,y,z)dzdx \leq 2\int_0^1\int_0^H \mathbf{u}^2(x,jH,z)dzdx + 2H|\mathbf{u}|^2_{H^1(\Omega_b^S)}$$

and

$$\|\mathbf{u}\|^2_{L^2(\Omega^S_b)} = \int_{jH}^{(j+1)H} \int_0^1 \int_0^H \mathbf{u}^2(x,y,z)dzdxdy$$
$$\leq 2H \int_0^1 \int_0^H \mathbf{u}^2(x,jH,z)dzdx + 2H^2|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

Moreover,

$$\int_0^1 \int_0^H \mathbf{u}^2(x,jH,z)dzdx \leq 2\int_0^1 \int_0^H \mathbf{u}^2(x,y,z)dzdx + 2H|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

Thus:

$$\int_0^1 \int_0^H \mathbf{u}^2(x,jH,z)dzdx \leq 2\|\mathbf{u}\|^2_{L^2(\Omega^s)} + 2H|\mathbf{u}|^2_{H^1(\Omega^S)}.$$

Now (iv_2) follows from the inequality:

$$\|\mathbf{u}\|^2_{L^2(\Omega^S_b)} \leq 4H\|\mathbf{u}\|^2_{L^2(\Omega^S)} + 6H^2|\mathbf{u}|^2_{H^1(\Omega^S)}. \ \square$$

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