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Chains of solitons

Derek Harland

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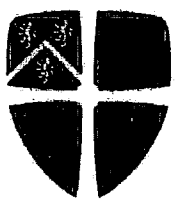
A Thesis presented for the degree of
Doctor of Philosophy



Centre for Particle Theory
Department of Mathematical Sciences
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UK

July 2008

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Dedicated to

Alan and Linda Harland

Chains of solitons

Derek Harland

Submitted for the degree of Doctor of Philosophy
July 2008

Abstract

We construct and analyse chains of solitons in various field theories. Particular emphasis is placed on the constituent structure, which appears to be be a generic feature of chains.

In Yang-Mills theory, we construct axially symmetric chains of instantons (calorons) with instanton charge 2, making essential use of the Nahm transform. We show that there are two distinct families of caloron, which can be distinguished using representation theory. We also construct calorons on hyperbolic space with instanton charge 1 and monopole charge 0. This generalises earlier work of Garland and Murray, in the same way that non-integer-mass hyperbolic monopoles generalise the integer-mass hyperbolic monopoles of Atiyah.

We study chains of skyrmions with charge 1 in both the Skyrme and planar Skyrme models, using various approximate analytic ansätze. In the Skyrme model chains are argued to exist and to have an energy per baryon number lower than the charge 2 skyrmion. In the planar Skyrme model, we show that the stability of chains depends on the choice of potential function.

We study chains and kinks in the $\mathbb{CP}^n$ sigma models analytically, in particular, we show that chains are kinks in a sigma model whose target is a homogeneous space for a loop group. This is the sigma model analog of the statement that a caloron is a monopole whose gauge group is a loop group.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapter 2 contains mostly background material which is included to place later work in context. The material in chapters 3-7 is my own work. The contents of chapters 3 and 4 appeared in my publications [Har07] and [Har08] respectively. The contents of chapters 5 and 6 represent my own contribution to the joint publications [HW08b] and [HW08a] with my supervisor, Prof. R. S. Ward.

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I am fortunate to have received financial support from the Science and Technology Facilities Council (formerly the Particle Physics and Astronomy Research Council), which has made my study possible.

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Chapter 1

Introduction

This thesis concerns static topological solitons in classical field theories. A static field theory consists of a set of fields, typically functions on a manifold or sections of fibre bundles, and an energy functional, which is expressed as an integral of an energy density and depends only on the fields. Fields which are stationary points of the energy functional solve the static field equations. A topological soliton is a field configuration which is a stationary point of the energy functional, and which is topologically non-trivial in some way. One can also consider dynamics of solitons and quantization of solitons, but both of these are beyond the scope of this thesis.

Typically, the topological properties of a soliton are described by a single integer, such as the degree of a map, which is called the charge. The first step towards understanding the structure of solitons is to look at their energy densities. For example, in most theories whose base manifold is Euclidean space, the energy density of a charge one soliton is concentrated in a small region of space. To a good approximation, the soliton looks and behaves just like a point particle. Similarly, the energy density of a charge two soliton is normally concentrated in two disjoint regions of space. A charge two soliton looks and behaves like a pair of particles, unless the two regions of high energy density are close enough to each other that they overlap. In that case the particle approximation breaks down, and some of the deeper structure of the solitons becomes apparent.

There are many different field theories which admit topological solitons, and each type of soliton has its own set of applications. An excellent introduction to the

subject is to be found in the book of Manton and Sutcliffe [MS04]. In this thesis we consider instantons and monopoles, which have applications in particle physics, skyrmions, which have applications in nuclear physics, and sigma models, which have applications in condensed matter physics. Our main area of interest will be in the mathematical structure of the solitons. In particular, we are fascinated that different types of soliton in different theories can exhibit apparently very similar behaviour.

The overarching theme of our work has been to study chains. A chain can be described in different ways. Firstly, if solitons a regarded as point particles in $\mathbb{R}^n$, then a chain consists of an infinitely long line of spatially separated solitons. Equivalently, a chain can be described as a topological soliton in a theory whose base manifold is the cartesian product $\mathbb{R}^{n-1} \times S^1$.

The classical example is a caloron, which is an instanton on $\mathbb{R}^3 \times S^1$. Just as instantons mediate tunnelling effects in quantum field theories at zero temperature, calorons mediate tunnelling effects in field theories at large temperatures. The period β of the chain (or the circumference of S^1) is inversely proportional to the temperature. Again, we can investigate the structure of calorons by looking at their energy densities - these depend qualitatively on the ratio β/λ, where λ represents the size of the instantons in the chain. If β/λ is large, the caloron looks like a chain of well-separated instantons. However, if β/λ is small the energy density is approximately independent of the periodic direction. One finds that the energy density is concentrated in two (or more) column-like regions, which are called "constituents". To us, it was very surprising that a charge one object splits into more than one constituent.

Closer analysis of calorons reveals that the constituents are monopoles. These were discovered independently by Kraan and van Baal [KvB98], and Lee and Lu [LL98], around ten years ago. Since then, chains of monopoles have been shown to exhibit a similar constituent structure [War05]. One of our main motivations has been to investigate to what extent constituents are a generic feature of chains of solitons. Perhaps the most important problem is to explain why the number of constituents is what it is: is there a simple rule which predicts the number of

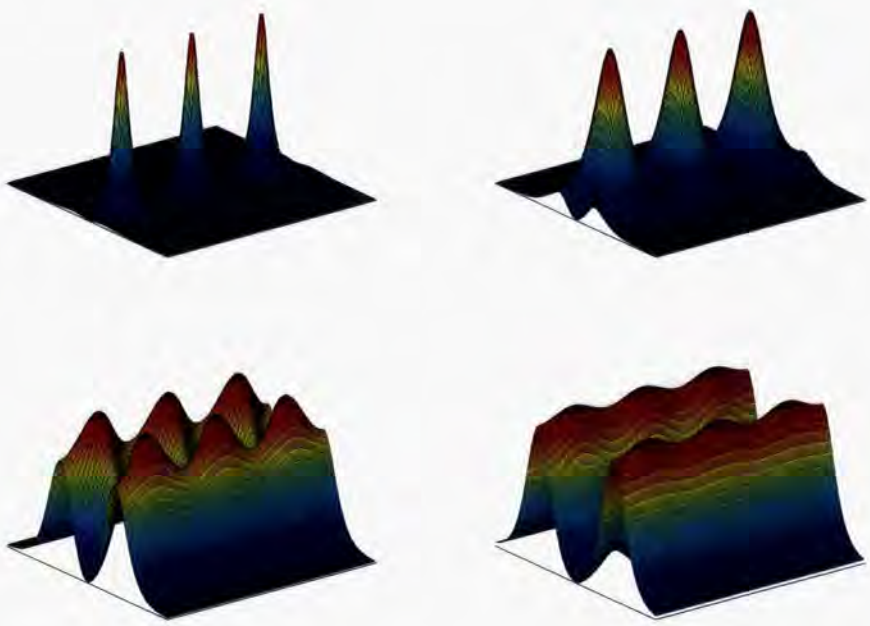

Figure 1.1: Energy densities of chains in the $\mathbb{CP}^1$ sigma model.

constituents in a chain, in any topological soliton model?

The second interesting feature of chains is their topology, which tends to be more complicated than that of isolated solitons. Calorons actually come equipped with not one but at least two topological charges, as we shall see in the next chapter. In later chapters we consider various models, each of whose field content is a map between manifolds. We will show that chains in these models possess a topological charge which is not equivalent to any classical topological invariant that we are aware of.

The discovery of consituent monopoles in calorons was greeted with excitement in the high energy physics community, since it was hoped that the monopoles would provide a mechanism to explain confinement in QCD. However, the problem of confinement is still not settled: other candidate models have been proposed which involve other types of topological object, most notably centre vortices. The problem of explaining confinement seems to be one of identifying the relevant degrees of freedom in the Yang-Mills background. This is tricky, because topological objects of one type can merge to form topological objects of another type; in lattice studies

it is not easy to distinguish between one type and another.

We shall now give a more detailed description of the work contained in this thesis. The three chapters following the introduction will be devoted to calorons. In chapter 2 we shall introduce calorons and their classification, and review standard methods of constructing them. In chapter 3 we show how some of these methods, combined with representation theory, can be applied to construct and classify calorons with high charge and axial symmetry. In chapter 4 we investigate a change of metric: we construct calorons on hyperbolic space. This is quite a natural step to take, in view of the close kinship between calorons and monopoles, and in view of the rich theory of monopoles on hyperbolic space.

In the final three chapters we consider chains of solitons in other models. In chapter 5 we consider the planar (or baby) Skyrme model, in chapter 6 we consider the Skyrme model, and in chapter 7 we consider the $\mathbb{CP}^n$ sigma-models. We make some closing remarks in chapter 8. We shall use chapter 7 to argue that constituents in topological soliton models can be understood using loop groups. All topological soliton models which appear in this thesis are based in some way on Lie groups: for monopoles, instantons (and calorons) the Lie group is the gauge group, while for the other models the target space is either a Lie group or a quotient space of a Lie group. The general pattern we observe is the following:

> *In a model associated with a gauge group G, the number of fundamental constituents in a chain is equal to the number of simple roots of the loop group $\mathbb{T}\tilde{\times}LG$.*

By "fundamental constituent", we mean a constituent cannot be separated further into smaller constituents. An additional qualification is that the boundary conditions of the chain must be sufficiently general in order to see the full number of constituents. For example, the hyperbolic calorons in chapter 3 exhibit one fundamental constituent rather than two, because their boundary condition is highly specialised. One could say that the second constituent is present, but can't be seen because it is massless.

A relationship between calorons and loop groups was observed by Hitchin (see [GM88]), which provides an explanation for the above statement for that case. In

chapter 7, we show that an analogous relationship also holds for chains in $\mathbb{CP}^n$ sigma models. Our hope is that this relationship can be extended to other topological soliton models, but this remains to be done. An alternative explanation for the number of constituents of a caloron was proposed using string theory [LY97], and this could in principle be extended to chains in other models (see for example [Ton02]). However, certain models, such as the Skyrme model, are harder to relate to string theory – this is why we have chosen to follow the loop group perspective.

It is worth considering here how a more precise meaning can be given to the statement "a chain splits into constituents". One definition could be the existence of limits of infinite separation, where all but one constituent can be separated to infinity. We shall show that limits of this type exist for examples of calorons in chapter 3. More generally, if the field equations admit a large moduli space of solutions, one could hope to find such limits. However, some models, such as the Skyrme and planar Skyrme models, are not integrable. For these models, our approach is more heuristic: we show that for small periods, chains are well-described by an ansatz which represents a superposition of constituents in some sense. An alternative approach could be to consider vibrational modes: one could investigate whether a chain has a low energy vibrational mode which corresponds to a splitting into constituents. This latter approach is beyond the scope of this thesis.

We have assumed that our reader is familiar with differential and Riemannian geometry. The books [GS87], [Mor01] contain most of the material we shall need. We also make use of standard material from topology, notably homotopy groups, cohomology groups, and Chern numbers. The last chapter also makes use of standard results from the theory of Lie groups, as can be found in [BtD85], and loop groups, as can be found in [PS86]. In addition to [MS04], background on various types of topological soliton may be found in [WW90, Ati79, Zak89, AH88].

Chapter 2

Yang-Mills fields

In this chapter we will review some well-known methods of constructing instantons, monopoles, and calorons. Most of the material in this section is not original, but it is included here because these methods will be utilised and generalised in later chapters.

2.1 Anti-self-dual gauge fields

The starting point for this discussion is the definition of a Yang-Mills action on a four-manifold and a Yang-Mills-Higgs energy on a three-manifold. Let M be a manifold of dimension four with Riemannian metric g_M and volume form ω_M. Let P be a rank n unitary vector bundle over M, and let $D = d + A$ be a unitary connection on E, where d is the exterior derivative and A is the 1-form part of D, also known as the gauge field. Let $F = [D, D] = dA + A \wedge A$ be the curvature of D, which may also be called the field strength tensor of A. The *Yang-Mills action* of D (or A) is

$$S := \int_M \|F\|^2 \omega_M. \tag{2.1}$$

Here $\|\cdot\|^2$ denotes the natural norm of F, induced from the metric g_M and the inner product $(X, Y) = -Tr(XY)$ of the Lie algebra $su(n)$.

Now let N be a three-manifold with Riemannian metric g_N and volume form ω_N. Let Q be a rank n unitary vector bundle over N with connection $D = d + A$, and let Φ denote a section of the adjoint bundle associated with N. Denoting the

curvature of A by F, the *Yang-Mills-Higgs energy* of the pair (A, Φ) is

$$E := \int_N \|F\|^2 + \|D\Phi\|^2 \omega_N. \tag{2.2}$$

There is a simple relation between S and E just defined. Suppose that (A, Φ) are a connection and adjoint bundle section on a vector bundle Q over a three-manifold N. Let $M = \mathbb{R} \times N$ and let $g_M = dt^2 + g_N$ be the Riemannian metric on M, where t is a coordinate on $\mathbb{R}$ and g_N is a metric on N. A bundle P over M is obtained by pulling back Q via the natural projection $M \to N$. Let $\tilde{A} = A + \Phi dt$ define a connection on P and let $\tilde{F}$ be its curvature; then $\|\tilde{F}\|^2 = \|F\|^2 + \|D\Phi\|^2$. It follows that $\tilde{A}$ has action

$$S = \int_{\mathbb{R}} E \, dt,$$

where E is the energy of (A, Φ).

The earliest motivation to study the Yang-Mills action and the Yang-Mills-Higgs energy came from particle physics. The standard model Lagrangian includes Yang-Mills-Higgs terms on a Lorentzian 4-manifold. E represents the static energy of a Lorentzian theory, so minima of E might represent observable physical objects. On the other hand, S is a Wick rotation of a Lorentzian action, and minima of S are believed to mediate tunnelling effects in quantum theories. So we see that minima of E and S are interesting to physicists. It has turned out that finding minima E or S is also a fascinating mathematical problem.

All successful attempts to minimize the Yang-Mills action rest on the identity:

$$\|F\|^2 \omega_M = Tr(F \wedge F) + \frac{1}{2}\|F + \star_M F\|^2 \omega_M.$$

Here $\star_M : \Lambda^2 M \to \Lambda^2 M$ is the Hodge star, defined by

$$u \wedge \star_M v = g_M(u, v)\, \omega_M.$$

It follows from the identity that

$$S \geq \int_M Tr(F \wedge F), \tag{2.3}$$

with equality if and only if

$$\star_M F = -F. \tag{2.4}$$

Equation (2.4) is called the *anti-self-dual equation*, and its solutions A are called *anti-self-dual gauge fields*. A similar argument shows that self-dual gauge fields, solving $F = \star_M F$, are also minima for the action. However, we shall restrict attention to anti-self-dual gauge fields, since on the manifolds we are interested in (such as $\mathbb{R}^4$), one can be obtained from the other simply by reversing the orientation.

The lower bound (2.3) is useful because the right hand side depends only on the topology of the situation and the boundary conditions imposed on A. For example, when M is compact, the lower bound is a multiple of the second Chern number of the vector bundle over which A is defined, so is in fact independent of the connection A.

A similar lower bound can be found for the Yang-Mills-Higgs energy. The identity

$$(\|F\|^2 + \|D\Phi\|^2)\omega_N = \|F + \star_N D\Phi\|^2\omega_N + 2Tr(F \wedge D\Phi)$$

implies that

$$E \geq 2\int_N Tr(F \wedge D\Phi) \tag{2.5}$$

with equality if and only if

$$F = -\star_N D\Phi. \tag{2.6}$$

Equation (2.6) is called the Bogomolny equation; it is easy to see that the Bogomolny equation for (A, Φ) is the same as the anti-self-dual equation for $\tilde{A} = A + \Phi dt$ defined as above, if we give M the orientation defined by the volume form $\omega_M = -dt \wedge \omega_N$.

Before moving on to specific examples, we point out here that the anti-self-dual equation is conformally invariant. In other words, replacing the metric g_M by $\lambda^2 g_M$ for some function $\lambda : M \to \mathbb{R}$ does not change the Hodge star. Similarly, the Yang-Mills action is conformally invariant. This observation will be important in chapter 4.

2.2 Examples

We have seen that a Yang-Mills action or a Yang-Mills-Higgs energy density can be defined on any four- or three-dimensional Riemannian manifold. However, it is not known whether the anti-self-dual equations or Bogomolny equations have solutions

on a general manifold. Most published work restricts attention to a particular manifold or set of manifolds; the Yang-Mills action has been most extensively studied on $\mathbb{R}^4$ and $\mathbb{R}^3 \times S^1$, while the Yang-Mills-Higgs energy has been most extensively studied on $\mathbb{R}^3$. These are the cases which are most relevant physically. In this section we shall review the topological charges and boundary conditions which determine the lower bounds (2.3), (2.5) in these cases. The remaining sections in this chapter will review methods of solving the anti-self-dual equation and Bogomolny equation in these cases.

2.2.1 Instantons on $\mathbb{R}^4$

An instanton on $\mathbb{R}^4$ is an anti-self-dual connection with finite action. Instantons have an integer topological charge; the standard way to show this is by comparing with instantons on S^4.

$\mathbb{R}^4$ is conformally equivalent to the sphere S^4 with a point removed. It follows directly that an instanton on S^4 defines an instanton on $\mathbb{R}^4$. By a theorem of Uhlenbeck, the converse is true. Vector bundles over S^4 are characterised by their second Chern number,

$$c_2 = \frac{1}{8\pi^2} \int_{S^4} Tr(F \wedge F),$$

which is in fact an integer. The second Chern number is called the charge of the instanton. The lower bound (2.3) may be written, $S \geq 8\pi^2 c_2$.

It will be useful to write everything down in tensor notation: we have

$$\begin{aligned} A &= A_\mu(x)\, dx^\mu \\ F &= \frac{1}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \end{aligned}$$

where $\mu, \nu = 0 \ldots 3$ and A_μ, $F_{\mu\nu}$ are anti-hermitian 2×2 matrix-valued functions of

$x \in \mathbb{R}^4$. Then

$$\begin{aligned} \omega &= dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ \|F\|^2 &= -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu}) \\ \star F &= \frac{1}{4} F^{\mu\nu} \epsilon_{\mu\nu\alpha\beta} dx^\alpha \wedge dx^\beta \\ Tr(F \wedge F) &= \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} Tr(F^{\alpha\beta} F^{\gamma\delta})\, \omega. \end{aligned}$$

2.2.2 Monopoles on $\mathbb{R}^3$

An $SU(n)$ monopole on $\mathbb{R}^3$ is a pair (A, Φ) solving the Bogomolny equation (2.6) with finite energy. Additional boundary conditions are normally imposed, which we summarise here, see [Mur84] for further details. The simplest type of monopole is the $SU(2)$ monopole, and monopoles for other gauge groups satisfying the condition of "maximal symmetry breaking" are just superpositions of embedded $SU(2)$ monopoles associated with the roots of the gauge group. We will describe this in detail here to emphasise some of the similarities between calorons and monopoles.

The topology of a monopole is described by n integers k_j which sum to zero, and the boundary conditions by n real numbers μ_j which also sum to zero. These must satisfy $\mu_j \geq \mu_{j+1}$ and $\sum_{i=1}^{j} k_j \geq 0$ for $j = 1, \ldots, n-1$. If one has $\mu_j > \mu_{j+1}$ for all j the monopole is said to have maximal symmetry breaking. Monopoles with maximal symmetry breaking tend to have nicer mathematical properties than those without.

Let $P = \mathbb{C}^n \times B^3$ denote the trivial bundle over the ball B^3. Let P_∞ denote the restriction of P to the boundary ∂B^3; given a connection 1-form a on P_∞, one can make P_∞ a holomorphic bundle. Then by Grothendieck's theorem [OSS80] there is a canonical decomposition $P_\infty = \bigoplus_{j=1}^{n} L^j$ as a sum of line bundles, whose Chern numbers $k_j = c_1(L^j)$ are uniquely determined up to permutation. One also has $a = \sum_{j=1}^{n} a^j$, where a^j is a connection 1-form on L^j; we assume that the curvatures $f^j = da^j$ are constant. Let ϕ be a section of the adjoint bundle associated with P_∞, such that $\phi s_j = i\mu_j s_j$ for all sections s_j of L^j (in other words, so that the L^j are eigenbundles for ϕ).

A monopole is required to satisfy the following:

- The gauge field A and Higgs field Φ admit extensions to the whole of the bundle P (where we have identifed $\mathbb{R}^3$ with the interior of B^3).
- Over ∂B^3, $\Phi = \phi$ and $A = a$.

One can evaluate the lower bound (2.5) in terms of μ_j and k_j: one finds that

$$2\int_{\mathbb{R}^3} Tr(F \wedge D\Phi) = 4\pi \sum_{j=1}^{n} k_j \mu_j.$$

In the simplest case $n = 2$, we have $k_2 = -k_1$ and $\mu_2 = -\mu_1$, since the k_j's and μ_j's sum to zero. We call $m := k_1$ the charge of the monopole and $\nu := \mu_1 - \mu_2$ the mass of the monopole. Its energy is $4\pi m\nu$.

Returning to the case of $SU(n)$, we define "constituent charges" $m_j = \sum_{i=1}^{j} k_i$ and "constituent masses" $\nu_j = \mu_j - \mu_{j+1}$ for $j = 1, \ldots, n-1$. An i-th fundamental monopole is a monopole with $m_j = 0$ and $\nu_j = 0$ except when $j = i$. The mass of this fundamental monopole is ν_i and its charge is m_i. The $n-1$ fundamental monopoles are in fact trivial embeddings of $SU(2)$ monopoles associated with the $n-1$ simple roots of $SU(n)$. A general $SU(n)$ monopole is roughly a superposition of fundamental (or constituent) monopoles with masses ν_j and charges m_j. More precisely, the moduli space of $SU(n)$ monopoles has limits in which the fundamental monopoles become infinitely separated. Notice that in terms of fundamental monopoles, the lower bound (2.5) on the energy is

$$2\int_{\mathbb{R}^3} Tr(F \wedge D\Phi) = 4\pi \sum_{j=1}^{n-1} m_j \nu_j.$$

Notice that the condition of maximal symmetry breaking implies that none of the constituent masses ν_j are zero.

2.2.3 Calorons on $\mathbb{R}^3 \times S^1$

A caloron is an anti-self-dual gauge field on $\mathbb{R}^3 \times S^1$ with finite action. Three standard references describing the topology of and boundary conditions for calorons are [GPY78], [GM88], and [Nye01]; an alternative perspective can be found in [EJ08]. Here we review the boundary conditions which are needed to evaluate the lower

bound on the action (2.3). Our treatment is based on [Nye01], but we have tried to simplify some of the proofs. More detailed analytic boundary conditions, which are required by the Nahm transform, can be found in [Nye01].

An $SU(n)$ caloron has $n+1$ topological charges $k_j \in \mathbb{Z}$, $j = 0\ldots,n$. k_0 is called the instanton charge and the remainder are called monopole charges. In addition, the boundary conditions imposed on a caloron are described by $n+1$ real numbers μ_j, $j = 0,\ldots,n$. μ_0 describes the periodic boundary condition: we set $\mu_0 = 2\pi/\beta$, where β is the period of the caloron (the circumference of S^1). The numbers $(k_0,\ldots,k_n,\mu_0,\ldots,\mu_n)$ are collectively called the boundary data of the caloron.

As in the case of monopoles, let P denote the trivial bundle over B^3. Again P_∞ has a decomposition into line bundles L^j, with compatible connections a^j and adjoint-valued section ϕ, described by k_j and μ_j for $j = 1\ldots n$. Let Q be an $SU(n)$ bundle over $B^3 \times S^1$, let Q_∞ denote its restriction to $\partial B^3 \times S^1$, and let $p : \partial B^3 \times S^1 \to \partial B^3$ be the obvious projection. Following [Nye01], the bundle Q is called *framed* if there exists a bundle isomorphism $Q_\infty \cong p^*P_\infty$.

Besides being anti-self-dual, a caloron A is required to satisfy the following boundary conditions:

- A must admit an extension to a framed bundle $Q \to B^3 \times S^1$ (here we have identified $\mathbb{R}^3$ with the interior of B^3)
- The restriction of A to $\partial B^3 \times S^1$ must take the form $A = (\phi \circ p)dx^0 + p^*\left(\sum_{j=1}^n a^j\right)$ (here $x^0 \in \mathbb{R}/\beta\mathbb{Z}$ is a coordinate on S^1).

The integers $k_j = c_1(L^j)$ are called the monopole charges of the caloron. We define the instanton charge k_0 as follows (this definition is our own, but is equivalent to those in [Nye01] and [GPY78]). Let $q : B^3 \times S^1 \to S^4$ be the map

$$q : (x, y) \mapsto (x, \sqrt{1 - \|x\|^2}y),$$

where $x \in \mathbb{R}^3$ satisfies $\|x\| \leq 1$ and represents a point in B^3, $y \in \mathbb{R}^2$ has $\|y\| = 1$ and represents a point in S^1, and S^4 is represented by unit vectors in $\mathbb{R}^5$. The map q is surjective, and maps the boundary $\partial B^3 \times S^1$ to a 2-sphere in S^4. Restricted to

$\partial B^3 \times S^1$, q agrees with the projection p composed with an inclusion from S^2 to S^4. Any framed bundle is in fact the pull-back via q of a bundle over S^4. k_0 is defined to be the second Chern number of the bundle over S^4, and is called the instanton charge of the caloron.

The following constraints are imposed on the boundary data (see [Nye01] and [GM88]).

$$\begin{aligned}
\sum_{j=1}^{n} k_j &= 0 \\
\sum_{j=1}^{n} \mu_j &= 0 \\
\sum_{j=0}^{n} k_j &\geq 0 \text{ for } j = 0, \ldots, n \\
\mu_j &\geq \mu_{j+1} \text{ for } j = 1, \ldots, n \\
\mu_1 - \mu_n &\leq \mu_0.
\end{aligned}$$

For any connection A satisfying the boundary conditions described above, there is a well-known formula

$$\frac{1}{8\pi^2} \int_{\mathbb{R}^3 \times S^1} Tr(F \wedge F) = \left(k_0 + \frac{1}{\mu_0} \sum_{j=1}^{n} \mu_j k_j \right). \tag{2.7}$$

We prove this formula as follows (see [Nye01] or [GPY78] for alternative proofs). First we consider the case where $\mu_j = 0$ for $j \geq 1$. Then A is the pull-back via q of a connection $\tilde{A}$ over S^4 [SS92]. We have

$$\begin{aligned}
\frac{1}{8\pi^2} \int_{B^3 \times S^1} Tr(F \wedge F) &= \frac{1}{8\pi^2} \int_{S^4} Tr(\tilde{F} \wedge \tilde{F}) \\
&= k_0.
\end{aligned}$$

Let A and A' be two connections and let $\alpha = A - A'$, the Chern-Simons formula [GS87] states that

$$Tr(F \wedge F) - Tr(F' \wedge F') = dTr\left(2\alpha \wedge F' + \alpha \wedge d\alpha + 2\alpha \wedge A' \wedge \alpha + \frac{2}{3} \alpha \wedge \alpha \wedge \alpha \right).$$

In the particular case where A satisfies the caloron boundary conditions with μ_j not all zero and $\alpha = A_0 dx^0$ is the component pointing in the circle direction, we deduce that

$$\frac{1}{8\pi^2} \int_{B^3 \times S^1} Tr(F \wedge F) = \frac{1}{8\pi^2} \int_{B^3 \times S^1} Tr(F' \wedge F') + \frac{1}{4\pi^2} \int_{\partial B^3 \times S^1} Tr(F' \wedge \alpha).$$

Since A' is a connection with the same topological charges as A but with $\mu_j = 0$ for $j = 1, \ldots, n$, the first term on the right is k_0. To evaluate the second term on the right, we use the prescribed form of A:

$$\begin{aligned}
\frac{1}{4\pi^2}\int_{\partial B^3 \times S^1} Tr(F' \wedge \alpha) &= \frac{1}{4\pi^2}\int_{\partial B^3 \times S^1} Tr(F'\phi) \wedge dx^0 \\
&= \frac{1}{2\pi\mu_0}\int_{\partial B^3} \sum_{j=1}^{n} if^j\mu_j \\
&= \frac{1}{\mu_0}\sum_{j=1}^{n} \mu_j c_1(L^j) \\
&= \frac{1}{\mu_0}\sum_{j=1}^{n} k_j\mu_j.
\end{aligned}$$

This concludes our discussion of the topology of calorons; now we describe the constituent monopoles of the caloron. Let

$$\begin{aligned}
m_j &= \sum_{i=0}^{j} k_j \text{ for } j = 1, \ldots, n \\
\nu_j &= \mu_j - \mu_{j+1} \text{ for } j = 1, \ldots, n-1 \\
\nu_n &= \mu_0 - \mu_1 + \mu_n.
\end{aligned}$$

These are respectively the charges and masses of the constituent monopoles of the caloron; if all of the masses ν_j are non-zero the caloron is said to have maximal symmetry breaking. One can think of an $SU(n)$ caloron with maximal symmetry breaking as being made of n fundamental constituent $SU(2)$ monopoles, each living on $\mathbb{R}^3$, just as an $SU(n)$ monopole with maximal symmetry breaking is made of $n-1$ fundamental constituent $SU(2)$ monopoles. The mathematical way to understand this lies in Garland and Murray's observation [GM88] that a caloron is the same thing as a monopole whose gauge group is a loop group. The loop group has n simple roots (see figure 2.1). The general pattern for monopoles is that the number of fundamental constituents is determined by the number of simple roots of the loop group, so one is lead to expect a caloron with maximal symmetry breaking to have n fundamental constituent monopoles. Alternatively, the constituent monopoles can be thought of as the T-dual description of the brane configuration in string theory which represents the caloron [LY97]. Note that if a caloron does not have maximal symmetry breaking, the constituent monopoles need not appear.

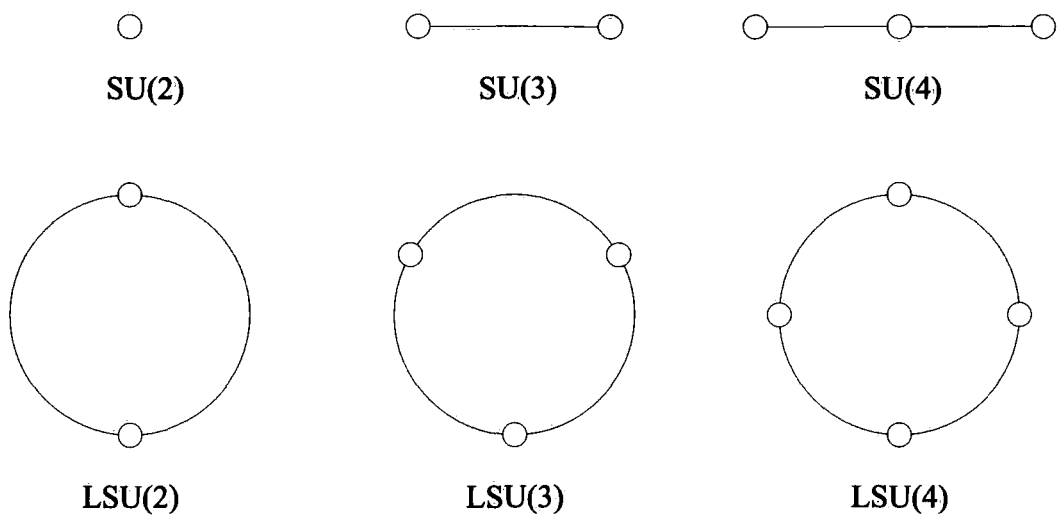

Figure 2.1: The Dynkin diagrams of various Lie groups and their loop groups

The constituent monopoles are defined up to cyclic permutation: the caloron with charges $(m_1, \ldots, m_n)$ and masses $(\nu_1, \ldots, \nu_n)$ is essentially the same as the caloron with charges $(m_2, \ldots, m_n, m_1)$ and masses $(\nu_2, \ldots, \nu_n, \nu_1)$. The map which permutes the constituent monopoles is called the rotation map. The rotation map is a large gauge transformation, that is, a gauge transformation which is not periodic, but which maps periodic configurations to periodic configurations. Such a gauge transformation will have a singularity, which accounts for the change in the second Chern number k_0. In the loop group picture, the rotation map can be thought of as a gauge transformation which permutes the roots of the loop group. Notice that, in terms of the constituent monopoles, the formula (2.7) takes the pleasing form:

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^3\times S^1} Tr(F\wedge F) = \frac{1}{\mu_0}\sum_{j=1}^{n} m_j\nu_j.$$

It is perhaps surprising that the lower bound on the action of a caloron is not simply a multiple of an integer, but can take any real value. One might ask whether this phenomenon is genuine: do calorons exist for which the integral (2.7) is not an integer? The answer to this question is yes; monopoles satisfy the boundary conditions of calorons, and can have arbitrary action.

Let (A, Φ) be an $SU(2)$ monopole with charge m, mass ν, and energy $E = 4\pi m\nu$ (we give $\mathbb{R}^3$ the volume form $-dx^1 \wedge dx^2 \wedge dx^3$ for consistency). Then, as described in the previous section, the monopole can be lifted to anti-self-dual gauge field over

$\mathbb{R}^4$ which is independent of one coordinate, and hence is periodic with any period β. Hence we obtain an $SU(2)$ caloron with instanton charge 0 and monopole charges $k_1 = m, k_2 = -m$, for which

$$\begin{aligned}\frac{1}{8\pi^2}\int_{\mathbb{R}^3\times S^1} Tr(\tilde{F}\wedge\tilde{F}) &= \frac{1}{4\pi\mu_0}\int_{\mathbb{R}^3} 2Tr(F\wedge D\Phi)\\ &= \frac{m\nu}{\mu_0}\end{aligned}$$

By varying the period, any value of the action can be attained from a single monopole.

Actually, we have been slightly careless here; if $\nu > \mu_0$ then the caloron boundary conditions as as described above are not satisfied. The solution is to make a large and singular gauge transformation so that the boundary conditions are satisfied. One obtains

$$(k_0, k_1, k_2, \mu_0, \mu_1, \mu_2) = \left(\left\lfloor\frac{\nu}{\mu_0}\right\rfloor, m, -m, \mu_0, \frac{\mu}{2} - \frac{\mu_0}{2}\left\lfloor\frac{\nu}{\mu_0}\right\rfloor, -\frac{\mu}{2} + \frac{\mu_0}{2}\left\lfloor\frac{\nu}{\mu_0}\right\rfloor\right).$$

If $\nu \leq \mu_0$ then the caloron has constituent monopole charges $(m_1, m_2) = (m, 0)$ and masses $(\nu_1, \nu_2) = (\nu, \mu_0 - \nu)$, which seems consistent with the fact that we started with a single monopole with charge m and mass ν. However, if $\nu > \mu_0$ both constituent monopoles have non-zero charge.

In what follows, we will be interested only in $SU(2)$ calorons, so for simplicity we summarise the boundary data for this case. We call k_1 the monopole charge of the caloron and set $k_2 = -k_1$. We also have $\mu_2 = -\mu_1$ and $0 \leq \mu_1 \leq \mu_0/2$. The constituent monopole charges and masses are

$$(m_1, m_2) = (k_0 + k_1, k_0) \tag{2.8}$$

$$(\nu_1, \nu_2) = (2\mu_1, \mu_0 - 2\mu_1). \tag{2.9}$$

We call a caloron with these boundary data an (m_1, m_2)-caloron. Calorons with $\mu_1 = 0$ or $\mu_1 = \mu_0/2$ are said to have trivial holonomy, otherwise we say the caloron has non-trivial holonomy or maximal symmetry breaking.

Notice that the monopole charge k_1 can usually be identified by looking at the asymptotic behaviour of A_0. Suppose that on near a coordinate patch on ∂B^3 the

gauge has been chosen so that

$$A_0 = i\mu_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{r}M + O(r^{-2})$$

and $A_r = O(r^{-3})$. Hence $F_{0r} = -\partial_r A_0 + O(r^{-3}) = Mr^{-2} + O(r^{-3})$. In the same gauge, the component of the curvature F pointing along ∂B^3 is

$$F' = \frac{-ik_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \omega.$$

where $\omega = \star(dx^0 \wedge dr)/r^2$ is the volume form on S^2. From the anti-self-dual equation, we deduce that

$$M = -\frac{ik_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2.3 Constructions of anti-self-dual gauge fields

The charge 1 instanton was constructed by Belavin, Polyakov, Schwarz, and Tyupkin [BPST75]. Following this discovery, a search began for instantons of higher charge. Among the many constructions for instantons which emerged, we shall focus on three: Witten's ansatz [Wit77], the Corrigan-Fairlie-'t Hooft (CF'tH) ansatz [CF77], and the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [AHDM78, CWS78].

Witten's ansatz is the easiest to implement and understand, but is also the least effective. Witten imposed an $SO(3)$ symmetry on his instantons, but there are many instantons which do not fall in to this category. We will return to Witten's ansatz in chapter 4, when we consider instantons on hyperbolic space. The CF'tH ansatz is still relatively simple to use, and more powerful than Witten's: it was shown by Manton [Man78] that all instantons written in Witten's ansatz can also be written in the CF'tH ansatz. We shall discuss the CF'tH ansatz further in the next section.

By far the most powerful ansatz is the ADHM construction. The ADHM construction is a non-linear transform which maps instantons to equivalence classes of matrices (called ADHM data) solving an equation (called the ADHM equation). This transform is a bijection, which means that every instanton can (in principle) be obtained using the ADHM construction. In general the ADHM equations are

much easier to solve than the anti-self-dual equations, which is what makes this construction useful. It can be tricky to obtain analytic formulae for gauge fields from ADHM data, but it is usually possible to obtain the gauge field numerically if required.

It is now realised that the ADHM transform is just one example of a wider set of transforms known as Nahm transforms [Jar04]. In general, Nahm transforms map anti-self-dual gauge fields on one manifold to anti-self-dual gauge fields on another "dual" manifold. Nahm transforms are normally invertible, and the inverse Nahm transform has the same form as the Nahm transform, as was first noted by Corrigan and Goddard [CG84].

The ADHM transform fits this pattern, since the ADHM equations can be interpreted as anti-self-dual equations for a gauge field defined over a manifold consisting of a single point. Another important example is the original Nahm transform: this relates monopoles to a set of matrix-valued functions on a line (called Nahm data) solving a differential equation (called the Nahm equation). As we saw above, the Bogomolny equation is the dimensional reduction of the anti-self-dual equation to three dimensions; similarly, the Nahm equation is the dimensional reduction of the anti-self-dual equation to one dimension.

In recent times, many other Nahm transforms have been discovered and analysed (see [Jar04] for a review). A large number of these apply to manifolds that can be written as a quotient of $\mathbb{R}^4$, but Nahm transforms have also been discovered for classes of curved manifolds. It is still an open question as to how general an object the Nahm transform is.

2.4 The ansatz of Corrigan, Fairlie and 't Hooft

Now we return to the CF'tH ansatz. This ansatz obtains anti-self-dual gauge fields from solutions to the Laplace equation on $\mathbb{R}^4$, with point singularities. A geometrical derivation of the CF'tH ansatz was provided by Atiyah, Hitchin and Singer [AHS78]. The advantage of this derivation is that it easily generalises manifolds other than $\mathbb{R}^4$. The derivation will be summarised here, following closely the treatment in [Lan05].

To begin, we recall some basic facts about Clifford algebras (see [Har90] or [JM89] for more details). Given a real vector space V with a symmetric blinear form g, one can form an algebra $Cl(V)$, called the Clifford algebra, which is generated by multiplying elements of V using a product which satisfies the identity $u \cdot v + v \cdot u = -2g(u, v)$.

We form a bracket $[\phi, \psi] = \phi \cdot \psi - \psi \cdot \phi$ on $Cl(V)$. The sub-algebra $spin(V) \subset Cl(V)$ is generated by adding together elements of the form $[u, v]$ for $u, v \in V$. It is closed under the Lie bracket, and acts on $V \subset Cl(V)$ according to $\phi : v \mapsto [\phi, v]$. $spin(V)$ is the Lie algebra of the Lie group $Spin(V)$, which can also be formed as a subset of $CL(V)$, and which is a double-cover for the special orthogonal group $SO(V)$. As such, there is an isomorphism between $spin(V)$ and the Lie algebra $so(V)$ of skew-adjoint linear maps from V to itself. The isomorphism is

$$a_{\mu\nu} v^\mu \theta_\nu \mapsto -\frac{1}{4} a_{\mu\nu} v^\mu \cdot v^\nu, \tag{2.10}$$

where v^μ denotes an orthonormal basis for V, θ_ν is the dual basis for V^* and $a_{\mu\nu}$ is anti-symmetric.

There is also a natural vector space isomorphism between $so(V)$ and $\Lambda^2 V$; this is obtained by raising indices using the metric. In the orthonormal basis, the isomorphism is

$$a_{\mu\nu} v^\mu \theta_\nu \mapsto a_{\mu\nu} v^\mu \wedge v^\nu.$$

It follows that $spin(V)$ is isomorphic to $\Lambda^2 V$ as a vector space. In fact, the algebras $Cl(V)$ and ΛV are isomorphic as vector spaces. The action of the Hodge star on ΛV is realised naturally as multiplication by the volume element in $Cl(V)$.

In the case where the dimension of V is four and g is positive definite, the space $\Lambda^2 V$ splits into self-dual and anti-self-dual components under the action of the Hodge star, $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. Likewise, the Lie algebra $spin(4) := spin(V)$ splits into two components. Each component is closed under the Lie bracket and is isomorphic to $su(2)$, we label them as $su(2)_\pm$. On the Lie group level this splitting is the well-known isomorphism $Spin(4) \cong SU(2) \times SU(2)$.

Let $e_\mu, \mu = 0,1,2,3$ be a basis for the quaternions $\mathbb{H}$, satisfying the identities,

$$\begin{aligned} e_0^2 &= e_0 \\ e_0 e_i &= e_i e_0 = e_i \\ e_i e_j &= -\delta_{ij} e_0 + \epsilon_{ijk} e_k \end{aligned} \tag{2.11}$$

for $i,j = 1,2,3$. Let γ be the mapping from V into $\mathbb{H}$, $\gamma(v^\mu) = e_\mu$. Then the splitting of $spin(4)$ is implemented by the mappings:

$$\gamma^+ : u \cdot w \mapsto -\gamma(u)\gamma(w)^\dagger \tag{2.12}$$

$$\gamma^- : u \cdot w \mapsto -\gamma(u)^\dagger\gamma(w) \tag{2.13}$$

(here we have identified $su(2)$ with the imaginary quaternions by setting $e_j = -i\sigma^j$).

Now let (M,g) be a four-dimensional Riemannian manifold. The 1-form part A of a connection on T^*M is (locally) a section of a bundle $so(4) \otimes \Lambda$. By the above discussion, it can equally regarded as a section of $\Lambda^2 \otimes \Lambda^1$ or $spin(4) \otimes \Lambda^1$. We may write $A = A_+ + A_-$, where $A_\pm \in \Lambda^2_\pm \otimes \Lambda^1 \cong su(2)_\pm \otimes \Lambda^1$. The curvature F of A splits as $F = F_+ + F_-$, where $F_\pm \in \Lambda^2_\pm \otimes \Lambda^2 \cong su(2)_\pm \otimes \Lambda^2$ are the curvatures of $A_\pm$. In the case where A is the Levi-Civita connection and $\bar{F}$ is the Riemannian curvature, there is a well-known splitting [AHS78, Lan05]

$$\begin{aligned} F_+ &= \left(W^+ - \frac{s}{12}\right) + R_0^* \\ W^+ - \frac{s}{12} &\in \Lambda^2_+ \otimes \Lambda^2_+ \\ R_0^* &\in \Lambda^2_+ \otimes \Lambda^2_- \\ F_- &= \left(W^- - \frac{s}{12}\right) + R_0 \\ W^- - \frac{s}{12} &\in \Lambda^2_- \otimes \Lambda^2_- \\ R_0 &\in \Lambda^2_- \otimes \Lambda^2_+. \end{aligned}$$

Here $W^\pm$ are the self-dual and anti-self dual Weyl tensors, s is the scalar curvature, and R_0 is the traceless Ricci tensor. In particular, the curvature F_+ of A_+ is anti-self-dual if and only if its self-dual part vanishes:

$$W^+ - \frac{s}{12} = 0.$$

Now suppose that (M, g) is a Riemannian spin manifold with $W^+ = 0$ (such a manifold is called anti-self-dual). Let $g' = \rho^2 g$ be a rescaling of g, where ρ is a positive real function on M. Let A be the Levi-Civita connection for g' and define A^+ as above. Since the Weyl tensor is conformally invariant, A^+ will be an anti-self-dual $SU(2)$ connection if and only if the scalar curvature s of A is zero.

We consider now the case the case $M = \mathbb{R}^4$, with the usual Euclidean metric g_E. If $g' = \rho^2 g_E$, then the Levi-Civita connection A of g' has Christoffel symbols:

$$\Gamma^{\nu}_{\alpha\mu} = \partial_\alpha \ln\rho\, \delta_{\mu\nu} + \partial_\mu \ln\rho\, \delta_{\alpha\nu} - \partial_\nu \ln\rho\, \delta_{\alpha\mu}.$$

We change to a basis orthonormal with respect to g': we let $\theta_\alpha = \rho^{-1}(\partial/\partial x^\alpha)$ and $v^\alpha = \rho dx^\alpha$. With respect to this basis, we write $A = \Gamma'^{\nu}_{\alpha\mu}\theta_\nu v^\mu dx^\alpha$, with

$$\Gamma'^{\nu}_{\alpha\mu} = \partial_\mu \ln\rho\, \delta_{\alpha\nu} - \partial_\nu \ln\rho\, \delta_{\alpha\mu}.$$

Note that this expression is antisymmetric in μ and ν, as an $so(4)$ connection should be.

Using the mappings (2.10) and (2.12), we find

$$\begin{aligned} A^+_0 &= -\partial_j \ln\rho \left(\frac{\sigma^j}{2i}\right) \\ A^+_j &= (+\partial_0 \ln\rho\, \delta_{jl} + \epsilon_{jlk}\partial_k \ln\rho) \left(\frac{\sigma^l}{2i}\right). \end{aligned}$$

More concisely, we have

$$A_\mu = \frac{1}{2}\bar{\eta}_{\mu\nu}\partial_\nu \ln\rho,$$

where $\bar{\eta}_{\mu\nu} = (e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu) = -\gamma^+([v^\mu, v^\nu])$ is the self-dual tensor introduced by 't Hooft.

This gauge field will be anti-self-dual exactly when the scalar curvature s of g' vanishes. We find the scalar curvature is

$$s = -6\rho^{-3}\Box_E \rho,$$

where $\Box_E = (\partial/\partial x^\alpha)^2$ is the usual Euclidean Laplace operator. Therefore the gauge field A^+ is anti-self-dual if and only if

$$\Box_E \rho = 0.$$

2.5 Calorons in the CF'tH ansatz

't Hooft considered solutions to the Laplace equation of the form

$$\rho = 1 + \sum_{j=1}^{N} \frac{\lambda_j^2}{|x - a_j|^2}$$

for some $\lambda_j \in \mathbb{R}$ and distinct $a_j \in \mathbb{R}^4$. The resulting gauge fields were instantons with charge N. The points a_j have the interpretation of instanton locations and the numbers λ_j of instanton sizes. A larger class of solutions, introduced by Jackiw, Nohl, and Rebbi in [JNR77], are

$$\rho = \sum_{j=1}^{N+1} \frac{\lambda_j^2}{|x - a_j|^2} \tag{2.14}$$

which also give rise to charge N instantons.

Harrington and Shepard [HS78] considered a generalisation of 't Hooft's ansatz, where the a_j are arranged in an infinite periodic chain. If one takes

$$a_j = (2\pi j/\mu_0, 0, 0, 0),\ \lambda_j = \lambda/\sqrt{\mu_0}$$

then ρ takes the form,

$$\rho = 1 + \frac{\lambda^2}{\mu_0} \sum_{j=-\infty}^{\infty} \frac{1}{|x - a_j|^2} = 1 + \frac{\lambda^2}{2r} \frac{\sinh(\mu_0 r)}{\cosh(\mu_0 r) - \cos(\mu_0 x^0)}. \tag{2.15}$$

The resulting gauge field is a caloron with constituent monopole charges $(1, 1)$ and masses $(0, \mu_0)$. More generally, consider the function

$$\rho = 1 + \sum_{j=1}^{N} \frac{\lambda_j^2}{2r_j} \frac{\sinh(\mu_0 r_j)}{\cosh(\mu_0 r_j) - \cos(\mu_0 (x^0 - t_j))} \tag{2.16}$$

where $\lambda_j > 0$, $r_j = |\mathbf{x} - \mathbf{a}_j|$ for $\mathbf{a}_j \in \mathbb{R}^3$, $t_j \in S^1$ and the points $(t_j, \mathbf{a}_j)$ are distinct. This generates a caloron with constituent monopole charges (N, N) and masses $(0, \mu_0)$, and will be called a Harrington-Shepard (N, N)-caloron. A Harrington-Shepard caloron is normally thought of as consisting of N $(1, 1)$-calorons, having locations $(t_j, \mathbf{a_j})$ and scales λ_j. These can be seen as distinct lumps of action density when the scales are small compared with the period, $2\pi/\mu_0$. When the period becomes infinitely large, one recovers the instantons of Corrigan, Fairlie and 't Hooft.

It is interesting to consider the opposite limit of Harrington-Shepard calorons, that is, the limit where the scales are large compared with the period. Letting all of the $\lambda_j \to \infty$ in equation (2.16) is equivalent to removing the constant term, since the gauge field is unaffected by constant rescalings of ρ. We are left with

$$\rho = \sum_{j=1}^{N} \frac{\lambda_j^2}{2r_j} \frac{\sinh(\mu_0 r_j)}{\cosh(\mu_0 r_j) - \cos(\mu_0(x^0 - t_j))}. \tag{2.17}$$

This function generates genuine calorons, which will be called JNR calorons, owing to the resemblance between (2.17) and the Jackiw-Nohl-Rebbi ansatz for instantons. The x^0-component of a JNR caloron gauge field at infinity has the form

$$A_0 = \frac{x^j}{r}\frac{\sigma^j}{2i}\left(-\frac{1}{r} + O(r^{-2})\right)$$

for large r. Therefore these calorons have $k_1 = -1$ and $\mu_1 = 0$.

One can evaluate k_0 for these calorons by evaluating the left hand side of (2.7). This is done using a method due to Jackiw et al [JNR77]. Since the gauge field is anti-self-dual, the inequality (2.3) is saturated. In the CFt'H ansatz the action density is given by $\mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta}) = \Box\Box \ln \rho$ [JNR77] (except at the points where ρ is singular). Therefore we have

$$\int_{\mathbb{R}^4} Tr(F \wedge F) = -\frac{1}{2}\int_{\mathbb{R}^3 \times S^1} \Box\Box \ln \rho \, d^4x, \tag{2.18}$$

where the right hand side must be evaluated while ignoring singularities. To remove the singularities, we replace ρ by

$$\rho' = \sum_{j=1}^{N} \frac{\lambda_j^2}{2r_j} \sinh(\mu_0 r_j) \prod_{i \neq j} (\cosh(\mu_0 r_i) - \cos(\mu_0(x^0 - t_i))).$$

This does not change the value of the integral, since $\Box\Box \ln(\cosh(\mu_0 r_i) - \cos(\mu_0(x^0 - t_i))) = 0$ away from the singularities of ρ.

When r is large, ρ' has the approximate form

$$\rho' \approx \left(2^{-N-1}\sum_{j=1}^{N}\lambda_j^2\right)\frac{\exp(N\mu_0 r)}{r}.$$

Since $\Box \ln(r\rho'/\exp(N\mu_0 r)) = O(r^{-2})$ we have,

$$\begin{aligned} \Box \ln \rho' &= \left(\partial_r^2 + \frac{2}{r}\partial_r\right)(N\mu_0 r - \ln r) + O(r^{-2}) \\ &= \frac{2N\mu_0}{r} + O(r^{-2}). \end{aligned}$$

This large r expansion is used to evaluate (2.18). Since $\Box\Box \ln \rho'$ has no singularities, we may apply Stokes' theorem, obtaining

$$\begin{aligned}\int_{\mathbb{R}^4} Tr(F \wedge F) &= -\frac{1}{2} \lim_{R\to\infty} \int_{S^2\times S^1} \partial_r \Box \ln \rho'|_{r=R} R^2 d\Omega_2 dx^0 \\ &= -\frac{1}{2} \lim_{R\to\infty} \int_{S^2\times S^1} \left(-\frac{2N\mu_0}{R^2} + O(R^{-3})\right) R^2 d\Omega_2 dx^0 \\ &= \frac{1}{2} \lim_{R\to\infty} \left(2N\mu_0(4\pi)(2\pi/\mu_0) + O(R^{-1})\right) \\ &= 8\pi^2 N.\end{aligned}$$

It follows from (2.7) that $k_0 = N$. So we see that the JNR caloron (2.17) is an $(N-1, N)$-caloron, with constituent monopole masses $(0, \mu_0)$.

Things are particularly simple when $N = 1$: the JNR $(0,1)$-caloron generated by

$$\rho = \frac{1}{2r} \frac{\sinh(\mu_0 r)}{\cosh(\mu_0 r) - \cos(\mu_0 x^0)}$$

has only one constituent monopole, so we expect it to be related to a 1-monopole by a large gauge transformation. This result was in fact obtained long ago by Rossi [Ros79]; Rossi's gauge transformation was an explicit example of a rotation map.

There is a final limit which we will consider. In equation (2.17), one can send one of the locations $\mathbf{a}_j$ to infinity. Consider the limit $|\mathbf{a}_N| \to \infty$, $\lambda_N \to \infty$ such that $\lambda_N^2/(2|\mathbf{a}_N|) \to 1$. Then we are left with a function,

$$\rho = 1 + \sum_{j=1}^{N-1} \frac{\lambda_j^2}{2r_j} \frac{\sinh(\mu_0 r_j)}{\cosh(\mu_0 r_j) - \cos(\mu_0(x^0 - t_j))}.$$

The resulting caloron is of course a Harrington-Shepard $(N-1, N-1)$-caloron. So, to summarise this discussion, we are able to obtain JNR calorons and Harrington-Shepard calorons as limits of each other, and these limiting processes always reduce the number of constituent monopoles.

2.6 The ADHM construction for instantons

The most comprehensive construction for instantons on $\mathbb{R}^4$ is the ADHM construction. The ADHM method constructs instantons from solutions to a set of algebraic

equations, called the ADHM equations. The construction is complete, in the sense that every instanton can be obtained from a solution to the ADHM equations. The ADHM equations are in general easier to solve than the anti-self-dual equations. The process of constructing an instanton from ADHM data can be tricky, but can be performed numerically if not analytically.

The ADHM data for a charge N $SU(2)$-instanton consists an $N \times N$ matrix of quaternions M and an N-row vector Λ of quaternions. These are combined into a single $N+1 \times N$ matrix Δ,

$$\Delta = \begin{pmatrix} \Lambda \\ M \end{pmatrix}.$$

The $N \times N$ matrix $\Delta^\dagger \Delta$ is required to be real and invertible.

Given a set of ADHM data, an instanton is constructed as follows. Let $u(x^\alpha)$ be an $N+1$-column vector of quaternions, and let $x = x^\alpha e_\alpha$ for any point $x^\alpha \in \mathbb{R}^4$ (where e_α is the basis (2.11) for the quaternions). The vector u is required to solve the equation,

$$\left(\Delta + \begin{pmatrix} 0 \\ I_N \end{pmatrix} x \right)^\dagger u(x) = 0,$$

and the normalisation condition,

$$u(x)^\dagger u(x) = e_0.$$

Then the gauge field

$$A_\alpha(x) = u^\dagger \partial_\alpha u$$

solves the anti-self-dual equation.

2.7 The Nahm transform for monopoles

The Nahm transform for monopoles is the analogue of the ADHM transform for instantons. Solutions of the Bogomolny equation are obtained from solutions of a first order ODE, which is in general easier to solve than the Bogomolny equation. The construction is complete.

The Nahm data for a charge k $SU(2)$-monopole consists of a set of four $k \times k$ hermitian matrix-valued functions $T^\alpha(s)$, defined for $s \in (-\mu, \mu)$. These must satisfy the Nahm equation,

$$\frac{d}{ds}T^j - i[T^0, T^j] - \frac{i}{2}\epsilon_{jkl}[T^k, T^l] = 0 \tag{2.19}$$

for $j = 1, 2, 3$, and the reality condition,

$$T^\alpha(-s) = T^\alpha(s)^t, \tag{2.20}$$

where ^{t} denotes matrix transpose. The boundary condition imposed is

$$T^j(s) = \frac{R^j}{s+\mu} + O((s+\mu)^{-2}) \tag{2.21}$$

where R^j are an irreducible $k \times k$ representation of the Lie algebra $su(2)$.

A monopole is constructed from its Nahm data as follows. Set $T = T^\alpha e_\alpha$ and $x = x^j e_j$ for $x^j \in \mathbb{R}^3$. Let $U(s, x^j)$ be a k-column vector of complex quaternions, that is, an element of $\mathbb{C}^k \otimes \mathbb{H}$. Suppose that U solves the Dirac equation

$$\frac{d}{ds}U(s, x^j) = i(T(s) + x)U(s, x^j) \tag{2.22}$$

and the normalisation condition

$$\int_{-\mu}^{\mu} U^\dagger U ds = e_0. \tag{2.23}$$

Then the pair (A, Φ) defined by

$$A_j = \int_{-\mu}^{\mu} U^\dagger \frac{\partial U}{\partial x^j} ds \tag{2.24}$$

$$\Phi = \int_{-\mu}^{\mu} isU^\dagger U ds \tag{2.25}$$

are a monopole.

A Nahm data gauge transformation is a function $g : [-\mu, \mu] \to U(2)$ which acts on a set of Nahm data in the following way:

$$\begin{aligned} T^j(s) &\mapsto g(s)T^j(s)g^{-1}(s) \\ T^0(s) &\mapsto g(s)T^0(s)g^{-1}(s) - i\frac{dg}{ds}(s)g^{-1}(s). \end{aligned}$$

These gauge transformations preserve conditions (2.19) and (2.21). The reality condition (2.20) is preserved provided $g(-s) = \bar{g}(s)$ ($\bar{}$denotes complex conjugation). It is easy to see that two monopoles arising from gauge equivalent Nahm data are identical. In fact that the Nahm transform is a bijection between the sets of gauge-equivalence classes of Nahm data and gauge-equivalence classes of monopoles.

Note that gauge transformations can be used to set $T^0 = 0$, which is the more usual form of the Nahm data.

2.8 The Nahm transform for calorons

The Nahm transform for calorons is similar to the Nahm transform for monopoles. Again, the construction is complete, and provides a useful tool in the sense that the Nahm equations are relatively easy to solve, and the construction can be implemented numerically. The proof of completeness was initiated by Nye [Nye01] and completed by Hurtubise and Charbonneau [CH]. Here we will describe the Nahm transform for $SU(2)$ calorons

2.8.1 Nahm data

The Nahm transform is formulated in terms of the constituent monopoles of the caloron. We begin with a circle $\mathbb{R}/\mu_0\mathbb{Z}$, which is divided into two intervals I_1, I_2 of length ν_1 and ν_2. We let s be a coordinate on the circle and take $I_1 = (-\mu_1, \mu_1)$ and $I_2 = (\mu_1, \mu_0 - \mu_1)$. There are two parts to the Nahm data, associated with the intervals I_p and their endpoints $\pm\mu_1$.

On each interval I_p we have a set of four Hermitian $m_p \times m_p$ Nahm matrices, $T_p^\alpha(s)$, where $\alpha = 0, 1, 2, 3$. These must satisfy the Nahm equation (2.19) and reality condition (2.20).

The second part of the Nahm data depends on the difference $m_1 - m_2$. We will only give the details for the two cases which will be relevant to our purposes. The first case is when $m_1 - m_2 = 0$; we will write $(m_1, m_2) = (N, N)$. Then the second part of the Nahm data consists of an N row vector W of quaternions and a purely imaginary unit quaternion τ (by "unit quaternion", we mean $\tau^\dagger\tau = 1$). The Nahm

data are required to satisfy the matching conditions,

$$T_2^j(\mu) - T_1^j(\mu) = i\Re(e_j W^\dagger P_1 W) \tag{2.26}$$

$$T_1^j(-\mu) - T_2^j(\mu_0 - \mu) = i\Re(e_j W^\dagger P_2 W) \tag{2.27}$$

for $j = 1, 2, 3$, with $P_1 := (e_0 + i\tau)/2$ and $P_2 := (e_0 - i\tau)/2$ elements of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$. Here $\Re$ refers to the real quaternionic part. In general † will act on complex-quaternionic matrices by transposing and taking quaternion and complex conjugates; so in the preceding formulae, $W^\dagger$ just denotes the quaternionic-conjugate transpose of W.

The second case we will consider is $m_1 - m_2 = 1$; we will write $(m_1, m_2) = (N, N-1)$. In this case the second part of the Nahm data is an $N \times (N-1)$ complex matrix X satisfying $X^\dagger X = I_{N-1}$. The Nahm data must satisfy the matching conditions

$$X^\dagger T_1^j(\mu) X = T_2^j(\mu) \tag{2.28}$$

$$X^t T_1^j(-\mu) \bar{X} = T_2^j(\mu_0 - \mu) \tag{2.29}$$

for $j = 1, 2, 3$, where $^-$ denotes complex conjugation.

2.8.2 Nahm transform

Now we will explain how to obtain a caloron from its Nahm data. Let $U_p(s, x^\alpha)$, $p = 1, 2$, be two m_p-component column-vectors of complex quaternions, defined for $s \in I_p$ and $(x^\alpha) \in \mathbb{R}^3 \times S^1$. Let $x = x^\alpha e_\alpha$ and let $T_p = T_p^\alpha e_\alpha$. Then (T_p, U_p, x) must satisfy the Dirac equation (2.22).

In the case $(m_1, m_2) = (N, N)$ we must also define a single quaternion $V(x^\alpha)$, and U_p and V must satisfy the matching condition,

$$U_2(\mu, x^\alpha) - U_1(\mu, x^\alpha) = iW^\dagger P_1 V(x^\alpha) \tag{2.30}$$

$$U_1(-\mu, x^\alpha) - U_2(\mu_0 - \mu, x^\alpha) = iW^\dagger P_2 V(x^\alpha). \tag{2.31}$$

U_p and V must also satisfy the normalisation condition,

$$\sum_{p=1}^{2} \int_{I_p} U_p^\dagger U_p ds + V^\dagger V = e_0. \tag{2.32}$$

Then the caloron gauge field is determined by

$$A_\alpha = \sum_{p=1}^{2} \int_{I_p} U_p^\dagger \frac{\partial U_p}{\partial x^\alpha} ds + V^\dagger \frac{\partial V}{\partial x^\alpha}. \tag{2.33}$$

In the case where $(m_1, m_2) = (N, N-1)$, there is no need to define V. The matching conditions (2.30), (2.31) are replaced by,

$$X^\dagger U_1(\mu, x^\alpha) = U_2(\mu, x^\alpha) \tag{2.34}$$

$$X^t U_1(-\mu, x^\alpha) = U_2(\mu_0 - \mu, x^\alpha). \tag{2.35}$$

The normalisation condition and the equation for the caloron gauge field are the same as in the (N, N) case, except that terms involving V are omitted.

As we have seen with monopoles, Nahm data are defined up to gauge equivalence. The action of gauge transformations is slightly complicated by the boundary conditions, so we give the details only for the cases we shall study later.

2.8.3 Gauge transformations: $(2, 2)$ case

There are two kinds of gauge transformation present for $(2, 2)$ Nahm data. The first is a $U(2)$ gauge transformation, which is defined by a $U(2)$-valued function $g(s)$. This satisfies a reality condition, $g(-s) = \bar{g}(s)$. g acts in the following way:

$$\begin{aligned}
T_p^j(s) &\mapsto g(s) T_p^j(s) g^{-1}(s) \\
T_p^0(s) &\mapsto g(s) T_p^0(s) g^{-1}(s) - i \frac{dg}{ds}(s) g^{-1}(s) \\
W &\mapsto P_1 W g^{-1}(\mu) + P_2 W g^{-1}(-\mu) \\
U_p(s) &\mapsto g(s) U_p(s) \\
V &\mapsto V.
\end{aligned}$$

The second kind of gauge transformation is a quaternion gauge transformation, defined by a unit quaternion h. The action of h on the Nahm data is

$$\begin{aligned}
V &\mapsto hV \\
W &\mapsto hW \\
\tau &\mapsto h\tau h^\dagger,
\end{aligned}$$

leaving T_p^α and U_p fixed.

It is straightforward to check that these gauge transformations map sets of Nahm data to sets of Nahm data, and leave the caloron gauge field unaffected. Given a set of Nahm data, it is always possible to make a $U(2)$ gauge transformation and a translation in x^0 so that

$$T_p^0(s) = \frac{\xi}{\mu_0}(\sigma^3 \cos\theta + \sigma^1 \sin\theta),$$

with constants $\xi \in [0, \pi]$ and $\theta \in [0, 2\pi)$, and where σ^j are the Pauli sigma matrices. By gauge rotation, one can then fix $\theta = 0$.

2.8.4 Gauge transformations: $(2, 1)$ case

We will divide the $(2, 1)$ Nahm data gauge transformations into two types. There are $U(2)$ gauge transformations $g_1(s)$ defined for $s \in I_1$, and $U(1)$ gauge transformations $g_2(s)$ defined for $s \in I_2$. These must satisfy the reality condition $g_p(-s) = \bar{g}_p(s)$. Their action is as follows:

$$\begin{aligned}
T_p^j(s) &\mapsto g_p(s) T_p^j(s) g_p^{-1}(s) \\
T_p^0(s) &\mapsto g_p(s) T_p^0(s) g_p^{-1}(s) - i\frac{dg_p}{ds}(s) g_p^{-1}(s) \\
X &\mapsto g_1(\mu) X g_2(\mu)^{-1} \\
U_p(s) &\mapsto g_p(s) U_p(s).
\end{aligned}$$

Using gauge transformations and x^0 translations, it is always possible to make $X = (\sin\beta/2, -\cos\beta/2)^t$ and fix $T_2^0 = 0$ and $T_1^0 = (\xi/\nu_1)\sigma^3$ for some constants $\xi \in [0, \pi]$ and $\beta \in [0, 2\pi)$.

Chapter 3

Euclidean calorons with axial symmetry

There is a natural action of $SU(2)$ on gauge fields over $\mathbb{R}^4$, where the group acts by rotation on $\mathbb{R}^3$ and by the adjoint representation on the Lie algebra. This symmetry was the basis of Witten's construction of multi-instantons [Wit77], and many monopoles have been found invariant under the action of $SU(2)$ and its subgroups [HMM96, HS96b, HS96a, Sut97]. Therefore it seems an important task to look for calorons invariant under the action of $SU(2)$ and its subgroups. In this section, we will show how to obtain Nahm data for calorons invariant under the action of a subgroup G of $SU(2)$, and we will give explicit Nahm data for $(2,2)$- and $(2,1)$-calorons which are invariant under the action of $U(1)$ ($(1,1)$-calorons have already been completely classified [KvB98, LL98]). At the end of the section, we shall investigate how these calorons are related to known instantons and monopoles.

The results of this section were published in [Har07]. This work is closely related to that of van Baal and co-workers [BvB02, BNvB03, BNvB04, N05] and Ward [War04b], who also construct high charge calorons using the Nahm transform. In fact, most of the calorons we construct here have been studied before, but our use of representation theory is novel. The representation theory allows us to classify families of $U(1)$-symmetric calorons, and to prove that the calorons constructed here are the only ones with the stated charges and symmetries. $U(1)$-symmetric calorons have also been studied by Chakrabarti [Cha82, Cha87], using a quite different ap-

proach.

3.1 Action of $SU(2)$ on calorons and Nahm data

The action of $SU(2)$ on calorons is as follows. Let R be an element of $SU(2)$, and let R_2 denote its image under the fundamental representation. Let R_3 denote the image of R in the irreducible 3-dimensional representation of $SU(2)$, so that the entries of R_3 are real. If $\mathbf{x} = (x^1, x^2, x^3)$, we can write $(R_3\mathbf{x})^j = R_3^{jk}x^k$. We have $R_2\sigma^j R_2^{-1} = R_3^{kj}\sigma^k$. Then the action of R on a caloron gauge field is

$$\begin{aligned} A_j(x^0, \mathbf{x}) &\mapsto R_3^{jk} R_2 A_k(x^0, R_3^{-1}\mathbf{x}) R_2^{-1} \\ A_0(x^0, \mathbf{x}) &\mapsto R_2 A_0(x^0, R_3^{-1}\mathbf{x}) R_2^{-1}. \end{aligned}$$

A caloron is said to be G-symmetric for some subgroup $G \subseteq SU(2)$ if it is invariant under the action of R for all $R \in G$.

If a caloron is G-symmetric, we expect its Nahm data to be invariant under some action of G. Consider the action,

$$T_p^j \mapsto R_3^{jk} R_N T_p^k R_N^{-1} \tag{3.1}$$

$$T_p^0 \mapsto R_N T_p^0 R_N^{-1} \tag{3.2}$$

$$W^\dagger \mapsto R_N(R_2 W^\dagger R_{2'}^{-1}) \tag{3.3}$$

$$\tau \mapsto R_{2'}\tau R_{2'}^{-1}, \tag{3.4}$$

where $R \mapsto R_N$ is a real N-dimensional orthogonal representation of G and $R \mapsto R_{2'}$ is a 2-dimensional unitary representation of G. In the above expressions, R_2 and $R_{2'}$ are acting on quaternions, which is made possible using the standard representation of the quaternions, $e_0 = I_2$ and $e_j = -i\sigma^j$. If a set of Nahm data is invariant under this action for some choice of representations R_N, $R_{2'}$, and for all $R \in G$, the corresponding caloron will be G-symmetric.

Similar constraints can be derived for the Nahm data of an $(N, N-1)$ caloron. In the case $N = 2$, a caloron will be G-symmetric if its Nahm data is invariant under

the action

$$T_1^j \mapsto R_3^{jk} R_N T_1^k R_N^{-1} \tag{3.5}$$

$$T_1^0 \mapsto R_N T_1^0 R_N^{-1} \tag{3.6}$$

$$T_2^j \mapsto R_3^{jk} T_2^k. \tag{3.7}$$

for all $R \in G$, where R_N is the image of R under some real 2-dimensional orthogonal representation of G.

So G-symmetric calorons can be found by constructing Nahm data satisfying the above symmetry conditions. To find this Nahm data, we first must choose the representations R_N and $R_{2'}$ so that the symmetry conditions have non-trivial solutions. In terms of representation theory, we must choose representations R_N and $R_{2'}$ of $SU(2)$ so that the induced representations of G (acting on the Nahm data) have trivial subrepresentations. For each such choice of R_N and $R_{2'}$, we attempt to solve the Nahm equations and matching conditions with G-invariant Nahm data.

3.2 $U(1)$-symmetric $(2,2)$-calorons

We will look for $(2,2)$-calorons invariant under the action of the $U(1)$ subgroup of $SU(2)$ generated by σ^2. Let P_0 denote the trivial representation of $U(1)$, and let Q_k denote the real 2-dimensional representation,

$$Q_k : \exp(i\theta\sigma^2) \mapsto \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$$

for $k \in \mathbb{Z}$. Note that $Q_0 = 2P_0$ and Q_{-k} is equivalent to Q_k. We will need to consider tensor products of such representations; we have

$$Q_k \otimes Q_l = Q_{k+l} \oplus Q_{k-l}.$$

Let us consider the representations in (3.1)-(3.4). We have $(R \mapsto R_2) = Q_1$ and $(R \mapsto R_3) = Q_2 \oplus P_0$, where P_0 acts on x^2 and Q_2 acts on the subspace spanned by x^1 and x^3. The representation R_N is 2-dimensional and real, so it must be of the form $R_N = Q_k(R)$ for some $k \in \mathbb{Z}$. We can make quaternion gauge transformations so that $R_{2'} = Q_l(R)$ for some $l \in \mathbb{Z}$.

Then the representation acting on the 2×2 Hermitian matrix T_p^0 in equation (3.2) is

$$Q_{2k} \oplus 2P_0,$$

where Q_{2k} acts on the subspace spanned by σ^1 and σ^3, and $2P_0$ acts on the subspace spanned by σ^2 and the identity matrix $\mathbf{1}_2$. The representation acting on T_p^j in equation (3.1) is a tensor product of this representation with the representation $(R \mapsto R_3)$ acting on the index j. So T_p^j is acted on by

$$(Q_2 \oplus P_0) \otimes (Q_{2k} \oplus 2P_0) = Q_{2+2k} \oplus Q_{2-2k} \oplus 2Q_2 \oplus Q_{2k} \oplus 2P_0.$$

In equation (3.3) there is a representation acting quaternions, $q \mapsto Q_1(R)qQ_l(R)^{-1}$. One can show that this representation is equal to $Q_{1+l} \oplus Q_{1-l}$, where Q_{1+l} acts on the subspace spanned by e_1 and e_3 and Q_{1-l} acts on the subspace spanned by e_0 and e_2. So the representation acting on W in (3.3) is

$$Q_k \otimes (Q_{1+l} \oplus Q_{1-l}) = Q_{k+l+1} \oplus Q_{k-l-1} \oplus Q_{k+l-1} \oplus Q_{k-l+1}.$$

We see that the trivial subrepresentations of these three representations are largest when (k,l) are equal to $(0,\pm 1)$, $(\pm 1, 0)$, or $(\pm 1, \pm 2)$. As we shall see below, the Nahm equations can be solved in two of these three cases.

3.2.1 R_N trivial

First we consider the case $(k,l) = (0,\pm 1)$, so that the representation $R \mapsto R_N$ is trivial. The invariant Nahm data must take the following form:

$$\begin{aligned}
T_p^0 &= (\xi/\mu_0)\sigma^3 \\
T_p^j &= 0 \\
T_1^2(s) &= \exp(iT_1^0 s) Y_1 \exp(-iT_1^0 s) \\
T_2^2(s) &= \exp(iT_2^0(s - \mu_0/2)) Y_2 \exp(-iT_2^0(s - \mu_0/2)) \\
W &= \begin{cases} \lambda(\cos(\beta/2)e_0, \sin(\beta/2)\exp(\alpha e_2)) & l = 1 \\ \lambda e_1(\cos(\beta/2)e_0, \sin(\beta/2)\exp(\alpha e_2)) & l = -1 \end{cases} \\
\tau &= e_2
\end{aligned}$$

for $p = 1, 2$ and $j = 1, 3$. Here $\lambda > 0$, $\beta \in [0, 2\pi)$ and $\alpha \in [0, 2\pi)$ are constants, and we have used gauge transformations and coordinate translations to fix the form of T_p^0 in terms of the constant $\xi \in [0, \pi]$. Y_1 and Y_2 are Hermitian matrices, which are constant as a consequence of the Nahm equation (2.19). We will write

$$Y_p = Y_p^0 \mathbf{1}_2 + Y_p^j \sigma^j,$$

and the reality condition (2.20) implies that $Y_p^2 = 0$.

This Nahm data now solves the Nahm equation and the reality condition, so it only remains to consider the matching conditions (2.26), (2.27). After some rearrangement these take the form,

$$\begin{aligned} Y_2^0 - Y_1^0 &= \pm\lambda^2/4 \\ Y_2^3 - Y_1^3 &= \pm(\lambda^2/4)\cos\beta \\ Y_2^1 \cos(\xi\nu_2/\mu_0) - Y_1^1 \cos(\xi\nu_1/\mu_0) &= \pm(\lambda^2/4)\sin\beta\cos\alpha \\ Y_2^1 \sin(\xi\nu_2/\mu_0) + Y_1^1 \sin(\xi\nu_1/\mu_0) &= (\lambda^2/4)\sin\beta\sin\alpha, \end{aligned} \tag{3.8}$$

where $\pm$ corresponds to the choice $l = \pm 1$. In the case where $\xi \neq 0$, the solution of the matching conditions is,

$$\begin{aligned} Y_1^0 &= 0 \\ Y_2^0 &= \pm\frac{\lambda^2}{4} \\ Y_1^3 &= \chi \\ Y_2^3 &= \chi \pm \frac{\lambda^2}{4}\cos\beta \\ Y_1^1 &= \frac{\lambda^2}{4}\frac{\sin\beta\sin(\alpha \mp \xi\nu_2/\mu_0)}{\sin\xi}. \\ Y_2^1 &= \frac{\lambda^2}{4}\frac{\sin\beta\sin(\alpha \pm \xi\nu_1/\mu_0)}{\sin\xi} \end{aligned} \tag{3.9}$$

Here χ is a real number and we have fixed $Y_1^0 = 0$ by making an x^2-translation. We see that this family of calorons depends on five parameters: λ, χ, α, β and ξ. In the case where $\xi = 0$, the solution of the matching conditions can be parametrised

in the following way:

$$\begin{aligned} Y_1^0 &= 0 \\ Y_2^0 &= \pm\frac{\lambda^2}{4} \\ Y_1^3 &= \chi \\ Y_2^3 &= \chi \pm \frac{\lambda^2}{4}\cos\beta \\ Y_1^1 &= 0 \\ Y_2^1 &= \pm\frac{\lambda^2}{4}\sin\beta, \end{aligned} \tag{3.10}$$

with $\alpha = 0$. Here the gauge has been chosen so that $Y_1^1 = 0$. This corresponds to a three-parameter sub-family of the five-parameter family.

The expressions (3.9), (3.10) remain valid when one of the monopoles is massless, that is, when $\nu_2 = 0$. In particular, when $\nu_2 = 0$ and $\alpha = 0$, the caloron obtained via the Nahm transform is a Harrington-Shepard $(2,2)$-caloron (2.16), with $\lambda_1 = \lambda\cos(\beta/2)$, $\lambda_2 = \lambda\sin(\beta/2)$, $t_1 = -t_2 = -\xi/\mu_0$, and $\mathbf{a_1} = -\mathbf{a_2} = (0, -\chi, 0)$.

The Nahm data (3.9), (3.10) were first studied in section 3.2 of [BvB02] , where a numerical Nahm transform was also implemented. There, the $(2,2)$-calorons were analysed in terms of their four constituent monopoles. The consituent monopoles appeared as lumps of action density, localised in $\mathbb{R}^3$ and smeared out over S^1. These lumps were located on the x^2-axis, at the points $x^2 = y_p^{\pm}$, where y_1^-, y_1^+ are the eigenvalues of Y_1 and y_2^-, y_2^+ are the eigenvalues of Y_2.

Interestingly, these eigenvalues satisfy the inequalities,

$$y_1^- \leq y_2^- \leq y_1^+ \leq y_2^+. \tag{3.11}$$

A proof of this is as follows. Let Y_p' denote the traceless part of Y_p, and let $\|A\| := \sqrt{-\det(A)}$ for any traceless hermitian matrix A. Then the eigenvalues of Y_p are

$$y_p^{\pm} = \frac{1}{2}\mathrm{Tr}(Y_p) \pm \|Y_p'\|,$$

and (3.11) is equivalent to

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}(Y_2 - Y_1) &\geq \|Y_2'\| - \|Y_1'\| \\ \frac{1}{2}\mathrm{Tr}(Y_2 - Y_1) &\geq \|Y_1'\| - \|Y_2'\| \\ \frac{1}{2}\mathrm{Tr}(Y_2 - Y_1) &\leq \|Y_1'\| + \|Y_2'\|. \end{aligned} \tag{3.12}$$

It follows from the matching conditions (3.8) that $(1/2)\mathrm{Tr}(Y_2 - Y_1) = \|Z_2 - Z_1\|$, where

$$\begin{aligned} Z_1 &= \exp(\xi\nu_1\sigma^3/2\mu_0)Y_1'\exp(-\xi\nu_1\sigma^3/2\mu_0) \\ Z_2 &= \exp(-\xi\nu_2\sigma^3/2\mu_0)Y_2'\exp(\xi\nu_2\sigma^3/2\mu_0). \end{aligned}$$

It is also true that $\|Z_p\| = \|Y_p'\|$, and this can be substituted to the right hand sides of the inequalities (3.8). Thus (3.12) can be shown to follow from the triangle inequality for $\|\cdot\|$.

3.2.2 R_N non-trivial

When $k = \pm 1$ the representation R_N is not trivial. We will only consider $k = 1$ here ($k = -1$ is similar up to minus signs); the invariant Nahm matrices must take the form

$$\begin{aligned} T_p^0(s) &= h_p(s)\sigma^2 + b_p(s)\mathbf{1}_2 \\ T_p^1(s) &= f_p(s)\exp(i\theta_p(s)\sigma^2)\sigma^1 \\ T_p^2(s) &= g_p(s)\sigma^2 + a_p(s)\mathbf{1}_2 \\ T_p^3(s) &= f_p(s)\exp(i\theta_p(s)\sigma^2)\sigma^3 \end{aligned}$$

for $p = 1, 2$, where $f_p, g_p, h_p, \theta_p, a_p, b_p$ are real functions of $s \in I_p$. By gauge transformations, we can make $T_p^0 = 0$. As we shall see, the Nahm equations and matching conditions can be solved when $l = \pm 2$, but the cannot be solved when $l = 0$. When $l = \pm 2$ the invariant forms for W and τ are

$$\begin{aligned} W &= \begin{cases} \lambda\exp(\alpha e_2)(e_0, e_2) & l = 2 \\ \lambda\exp(\alpha e_2)(e_1, e_3) & l = -2 \end{cases} \\ \tau &= e_2 \end{aligned}$$

where $\lambda > 0$ and $\alpha \in [0, 2\pi)$. One can make $\alpha = 0$ by gauge transformation.

The Nahm equations for this invariant Nahm data are

$$\begin{aligned} f_p' + 2f_p g_p &= 0 \\ f_p \theta_p' &= 0 \\ g_p' + 2f_p^2 &= 0 \\ a_p' &= 0, \end{aligned}$$

and the matching conditions are

$$\begin{aligned} f_2(\nu_1/2)\cos(\theta_2(\nu_1/2)) - f_1(\nu_1/2)\cos(\theta_1(\nu_1/2)) &= 0 \\ f_2(\nu_1/2)\sin(\theta_2(\nu_1/2)) - f_1(\nu_1/2)\sin(\theta_1(\nu_1/2)) &= 0 \\ g_2(\nu_1/2) - g_1(\nu_1/2) &= \lambda^2/2 \\ a_2(\nu_1/2) - a_1(\nu_1/2) &= \pm\lambda^2/2, \end{aligned}$$

where $\pm$ corresponds to the choice $l = \pm 2$. The Nahm equations have a trivial solution where $f_p = 0$, and g_p is constant. Since g_p is odd, we must have $g_1 = 0$, $g_2 = 0$. It then follows from the matching conditions that $\lambda = 0$, so there are no calorons in this case.

When f_p is non-zero, θ_p and a_p must both be constant. The most general solution to the Nahm equations for even f_p and odd g_p are

$$\begin{aligned} f_1(s) &= (D_1/2)\sec(D_1 s) \\ f_2(s) &= (D_2/2)\sec(D_2(s - \mu_0/2)) \\ g_1(s) &= -(D_1/2)\tan(D_1 s) \\ g_2(s) &= -(D_2/2)\tan(D_2(s - \mu_0/2)) \end{aligned}$$

for real constants D_1 and D_2, which can be assumed to be positive without loss of generality. In order that these functions remain finite, we must have $D_1 < \pi/\nu_1$ and $D_2 < \pi/\nu_2$. The matching conditions are easily seen to reduce to,

$$\theta_2 = \theta_1 \tag{3.13}$$

$$D_2 \sec(D_2\nu_2/2) = D_1 \sec(D_1\nu_1/2) \tag{3.14}$$

$$a_2 - a_1 = \pm\lambda^2/2 \tag{3.15}$$

$$D_1 \tan(D_1\nu_1/2) + D_2 \tan(D_2\nu_2/2) = \lambda^2. \tag{3.16}$$

Solving the matching conditions amounts to solving (3.14). This equation admits a one-parameter family of solutions, since the functions $D_p \mapsto D_p \sec(D_p\nu_p/2)$ are bijections from the intervals $(0, \pi/\nu_1)$ to $(0, \infty)$, hence invertible. However, an explicit parametrisation of the solutions is not known. Given a solution (D_1, D_2) of (3.14), the constants λ, a_1, a_2 are determined by (3.16) and (3.15). Notice that a_1 and a_2 are only determined up to addition of a constant, and that this constant can be fixed by x^2-translations. Notice also that θ_1 (and hence θ_2) can be made zero by gauge rotation. Therefore, this family of calorons depends on one parameter, if the position is fixed.

These expressions remain valid in the case where one monopole is massless ($\nu_2 = 0$); such calorons were considered in [War04b].

In the case where $\nu_1 = \nu_2$, the family of calorons considered here forms a subset of the "rectangular" configurations of Bruckman et al. Specifically, in section 6.3 of [BNvB03] an exact analytic solution of the Nahm equations with $\nu_1 = \nu_2$ is given in terms Jacobi elliptic functions. Our calorons correspond to the case where the elliptic parameter is zero. Bruckman et al. did not consider the case where $\nu_1 \neq \nu_2$. In reference [BNvB04] the action densities of these calorons were constructed, making use of a numerical implementation of the Nahm transform. The action density is concentrated in two rings in $\mathbb{R}^3$, smeared out over S^1. The rings are centred on the x^2-axis and lie in planes perpendicular to the x^2-axis. The location of each ring is $x^2 = -a_p$. The rings are identified with the two constituent 2-monopoles.

So far we have only considered Nahm data for which $l = \pm 2$. We also need to consider the case $l = 0$; it will turn out that the Nahm equations cannot be solved in this case. W takes the form

$$W = \lambda q(1, -e_2)$$

where $\lambda > 0$ and q is a quaternion of unit length ($q^\dagger q = 1$). Since R_2' is trivial, there is no restriction on τ. The solution of the Nahm equations proceeds in much the same way as before, but the matching condition (3.16) is replaced by

$$-D_2 \tan(D_2/\nu_2) - D_1 \tan(D_1\nu_1/2) = \lambda^2.$$

This can not be solved for nonzero D_1 and D_2, because the left hand side is less

than or equal to zero. Therefore no calorons are obtained in this case.

3.3 $U(1)$-symmetric $(2,1)$-calorons

Now we will look for $U(1)$-symmetric $(2,1)$-calorons. We start by considering the representations acting on the Nahm data in (3.5)-(3.7). We set $R_N = Q_k(R)$; then representation theory calculations tell us that we only need to consider invariant Nahm data in the cases $k = 0, \pm 1$.

When $k = 0$ (and R_N is trivial), the invariant Nahm data must have $T_p^1 = 0$ and $T_p^3 = 0$ for $p = 1, 2$. We choose a gauge so that $T_2^0 = 0$ and

$$\begin{aligned} T_1^0 &= (\xi/\nu_1)\sigma^3 \\ X &= \begin{pmatrix} \sin(\beta/2) \\ -\cos(\beta/2) \end{pmatrix} \end{aligned}$$

for $\xi \in [0, \pi]$ and $\beta \in [0, 2\pi)$. We write T_1^2 in the form,

$$T_1^2(s) = \exp(iT_1^0 s) Y \exp(-iT_1^0 s),$$

where Y is a Hermitian 2×2 matrix. The Nahm equations imply that Y and T_2^2 are constant. We write $Y = Y^0 \mathbf{1}_2 + Y^j \sigma^j$, and the reality condition (2.20) implies that $Y^2 = 0$. The matching conditions (2.28), (2.29) for this Nahm data are,

$$T_2^2 = Y^0 - Y^3 \cos\beta - Y^1 \sin\beta \cos\xi.$$

We choose to fix $Y^0 = 0$ by making an x^2-translation. Then we are left with a four-parameter family of calorons, parametrised by ξ, β, Y^1 and Y^3. Note that when $\xi = 0$ the gauge is no longer fixed, so $\xi = 0$ corresponds to a two-parameter subfamily. This Nahm data has not been considered before in the literature. We expect that there is some overlap between this family of calorons and those studied by Chakrabarti in [Cha87].

These expressions remain valid in the case where the 1-monopole becomes massless, $\nu_2 = 0$. In particular, when $\nu_2 = 0$ and $Y^1 = 0$, the calorons obtained from this Nahm data are JNR $(2,1)$-calorons (2.17), with $t_1 = -t_2 = -\xi/\mu_0$, $\mathbf{a}_1 = -\mathbf{a}_2 = (0, -Y^3, 0)$, $\lambda_1 = \cos(\beta/2)$ and $\lambda_2 = \sin(\beta/2)$. On the other hand,

when the 2-monopole becomes massless, the caloron is just a 1-monopole. This is because, when $\mu = 0$, a caloron with $k = 1$ can be gauge rotated to a caloron with $k = -1$.

By analogy with $(2,2)$-calorons, we expect the constituent monopoles of these $(2,1)$-calorons to be located on the x^2-axis with x^2-coordinates given by the eigenvalues of the matrices T_1^2 and T_2^2. If we denote the eigenvalues of T_1^2 by $y_1^- \leq y_1^+$, and write $y_2 = T_2^2$, then one can show that

$$y_1^- \leq y_2 \leq y_1^+.$$

This should be compared with the constraint (3.11) on the monopole locations for the family of $(2,2)$ calorons with R_N trivial.

We should also consider invariant Nahm data with $k = \pm 1$. However, no new calorons are obtained this way; the only solutions to the Nahm equations correspond to a subset of the Nahm data found for $k = 0$.

3.4 Large scale limits

The large scale limit of a $(1,1)$-caloron is a 1-monopole [Ros79, LL98]. For higher charges, the large scale limit of a caloron may be a monopole [Cha87], or it may be a caloron with constituent monopoles of unequal charge (as was the case for the Harrington-Shepard calorons). In this section we will take large scale limits of the $U(1)$-symmetric calorons constructed above.

As was observed in [LL98] for charge 1, taking infinite scale limits is the same as sending monopole locations to infinity. There are a limited number of ways of doing this for $(2,2)$-calorons. Either we send a 2-monopole to infinity, leaving behind a $(2,0)$-caloron (that is, a 2-monopole), or we send a 1-monopole to infinity, leaving a $(2,1)$-caloron. Since for the $U(1)$ symmetric 2-monopoles, the representation R_N on the Nahm data is non-trivial, one would expect the large scale limit of a $(2,2)$-caloron with non-trivial R_N to be a 2-monopole. One would anticipate that $(2,2)$-calorons with R_N trivial have a $(2,1)$-caloron as their large scale limit, for similar reasons. We will see that these predictions turn out to be correct; we will also show that

the large scale limit of a symmetric $(2,1)$-caloron is a $(1,1)$-caloron. We note that similar limits have been considered previously for monopoles [CW03].

3.4.1 $(2,2)$-calorons with R_N non-trivial

First we consider the family of $(2,2)$-calorons with R_N non-trivial, described in section 3.2.2. The large scale limit will be $\lambda \to \infty$. From equation (3.15) we see that $a_2 - a_1 \to \pm\infty$ in this limit. Recalling the interpretation of a_1 and a_2 as 2-monopole locations, this means that the separation of the monopoles tends to infinity. We will fix the first monopole at $a_1 = 0$ so that $a_2 \to \pm\infty$. This procedure is illustrated in figure 3.1, where we have represented each constituent 2-monopole by a ring.

From equation (3.16), $\lambda \to \infty$ implies that $D_1 \to \pi/\nu_1$ or $D_2 \to \pi/\nu_2$. But from equation (3.14), $D_1 \to \pi/\nu_1$ if and only if $D_2 \to \pi/\nu_2$. We conclude that, in the limit where $\lambda \to \infty$, the Nahm matrices on the interval I_2 diverge, while the Nahm matrices on the interval $I_1 = (-\mu, \mu)$ must converge to

$$\begin{aligned} T_1^2 &\to -(\pi/\mu)\tan(\pi s/2\mu)\sigma^2 \\ T_1^j &\to (\pi/\mu)\sec(\pi s/2\mu)\sigma^j \text{ for } j = 1,2. \end{aligned}$$

This is just the Nahm data for a 2-monopole with mass $\nu_1 = 2\mu$.

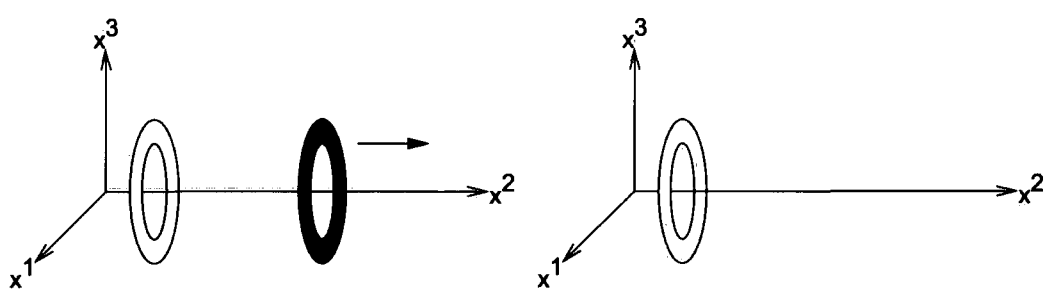

Figure 3.1: The large scale limit of a $(2,2)$-caloron with R_N non-trivial.

3.4.2 $(2,2)$-calorons with R_N trivial

Now we consider the family of $(2,2)$-calorons with R_N trivial, described in section 3.2.1. Recall that the eigenvalues $y_1^{\pm}$ and $y_2^{\pm}$ of Y_1 and Y_2 were interpreted as the

four constituent monopole locations. Our large scale limit will send only one of these four eigenvalues to infinity. We have illustrated this procedure in figure 3.2: the four constituent monopoles are represented by balls on the x^2-axis; the white balls have locations $x^2 = y_1^{\pm}$ and the black balls have locations $x^2 = y_2^{\pm}$.

We consider the case $l = 1$. In terms of the parametrisation given in (3.9), the eigenvalues are

$$\begin{aligned} y_1^{\pm} &= \pm\sqrt{\chi^2 + \frac{\lambda^4}{16}\frac{\sin^2(\beta)\sin^2(\alpha - \xi\nu_2/\mu_0)}{\sin^2\xi}} \\ y_2^{\pm} &= \frac{\lambda^2}{4} \pm \sqrt{\chi^2 + \chi\frac{\lambda^2}{2}\cos\beta + \frac{\lambda^4}{16}\left(\cos^2\beta + \frac{\sin^2(\beta)\sin^2(\alpha + \xi\nu_1/\mu_0)}{\sin^2\xi}\right)}. \end{aligned}$$

$y_1^{\pm}$ will stay finite in the limit where $\lambda \to \infty$ and $\alpha \to \xi\nu_2/\mu_0$, such that

$$\eta := \lim_{\lambda\to\infty,\, \alpha\to\xi\nu_2/\mu_0} \frac{\lambda^2}{4}\frac{\sin(\beta)\sin(\alpha - \xi\nu_2/\mu_0)}{\sin\xi}$$

is finite. In this limit, we have

$$\begin{aligned} Y_1 &\to \chi\sigma^3 + \eta\sigma^1 \\ y_2^+ &\to \infty \\ y_2^- &\to -\eta\cos\xi\sin\beta - \chi\cos\beta \\ v^- &\to \begin{pmatrix} \sin(\beta/2) \\ -\cos(\beta/2) \end{pmatrix}, \end{aligned}$$

where v^- is the eigenvector of Y_2 with eigenvalue y_2^-.

In order to get sensible Nahm data in this limit, we need to choose a gauge where T_2^2 is constant and diagonal. We make the gauge transformation $g(s) = g_2 g_1(s)$, where

$$g_1(s) = \begin{cases} \exp\left(i\frac{\xi}{\mu_0}\frac{\nu_2}{\nu_1}s\sigma^3\right) & s \in I_1 \\ \exp\left(-i\frac{\xi}{\mu_0}\left(s - \frac{\mu_0}{2}\right)\sigma^3\right) & s \in I_2 \end{cases}$$

makes $T_2^0 = 0$ and g_2 is the matrix which diagonalises Y_2. Now we take the large scale limit, discarding the part of Y_2 which becomes infinite, and undo the gauge rotation g_2. We are left with Nahm data,

$$\begin{aligned} T_1^0(s) &= (\xi/\nu_1)\sigma^3 \\ T_1^2(s) &= \exp(iT_1^0 s)(\chi\sigma^3 + \eta\sigma^1)\exp(-iT_1^0 s) \\ T_2^2(s) &= -\eta\cos\xi\sin\beta - \chi\cos\beta. \end{aligned}$$

This should be compared with the $(2,1)$ Nahm data found in section 3.3. We see that this Nahm data is exactly what we had before, with $Y^1 = \eta$ and $Y^3 = \chi$ and $X = \lim v^- = (\sin(\beta/2), -\cos(\beta/2))^t$.

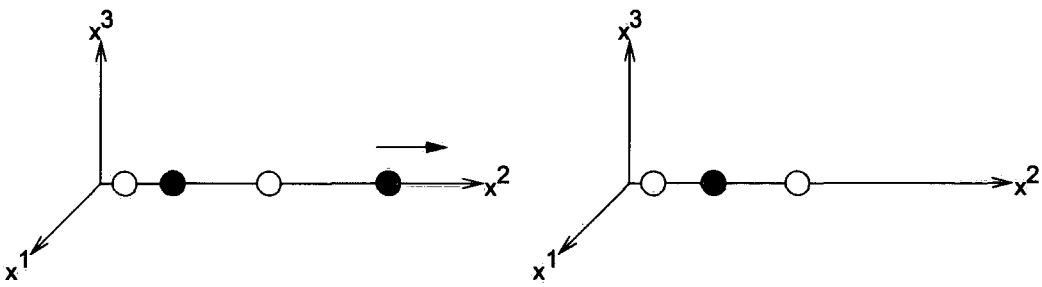

Figure 3.2: The large scale limit of a $(2,2)$-caloron with R_N trivial.

3.4.3 $(2,1)$-calorons

Finally, we show that the large the large scale limits of the $(2,1)$-calorons described in section 3.3 are $(1,1)$-calorons. We begin with the eigenvalues of T_1^2 and T_2^2:

$$\begin{aligned} y_1^{\pm} &= Y^0 \pm \sqrt{(Y^1)^2 + (Y^3)^2} \\ y_2 &= Y^0 - Y^3 \cos\beta - Y^1 \sin\beta \cos\xi. \end{aligned}$$

Recall that these are interpreted as constituent monopole locations. We can take a limit where $Y^0 \to \infty$, $Y^3 \to \infty$, $Y^1 \to 0$ and $\beta \to \pi$, such that $y_1^+ \to \infty$ and y_2 and y_1^- stay finite. This is the large scale limit of the $(2,1)$-caloron. To obtain sensible Nahm data in this limit, one should choose a gauge so that T_1^2 is constant and diagonal (which can always be done). By taking the large scale limit and discarding the part of this matrix which tends to infinity, we are left with 1×1 Nahm matrices $T_1^2 = \lim y_1^-$, $T_2^2 = \lim y_2$. These are Nahm matrices for a $(1,1)$-caloron, and the Nahm data is completed by taking $W = \sqrt{2}\lim(y_2 - y_1^-)$.

3.5 Large period limits

In this section we will consider large period limits of calorons. One can show that the $\mu_0 \to 0$ limit of caloron Nahm data is ADHM data for an instanton, so the large

period limit of a caloron is an instanton. We will study the ADHM data for the large period limit of the $(2,2)$-calorons constructed earlier; we will see that two distinct families of $U(1)$-symmetric 2-instanton are obtained. Previous, large period limits of calorons had been considered only in the case where one constituent monopole is massless [War04b].

First we consider Nahm data for an (N,N)-caloron. Let $\nu_1', \nu_2' \in [0,1]$ such that $\nu_1' + \nu_2' = 1$. Suppose that $\epsilon > 0$ and that for each $\mu_0 \in (0,\epsilon)$, $(T_1, T_2, W)(\mu_0)$ are a set of Nahm data which solve the Nahm equation and matching conditions with $\nu_p = \mu_0 \nu_p'$. We will assume that for each $\mu_0 \in (0,\epsilon)$, T_1 and T_2 have Taylor expansions which converge to T_1, T_2 on the intervals I_1, I_2.

By considering the Nahm equation (2.19) at $s = 0, \mu_0/2$, and substituting the Taylor expansions for T_p into the matching conditions (2.26), (2.27), we arrive at the following four equations, true for all $\mu_0 \in (0,\epsilon)$:

$$\begin{aligned}
\Im(T_1'(0)) &= -i\Im(T_1(0)T_1(0)^\dagger) \\
\Im(T_2'(\mu_0/2)) &= -i\Im(T_2(\mu_0/2)T_2(\mu_0/2)^\dagger) \\
2\Im(T_2(\mu_0/2) - T_1(0)) + O(\mu_0^2) &= -i\Im(W^\dagger \tau W) \\
-\Im(\nu_1' T_1'(0) + \nu_2' T_2'(\mu_0/2)) + O(\mu_0) &= -i\Im(W^\dagger W)/\mu_0.
\end{aligned}$$

In addition, we suppose that the gauge has been chosen so that $\Re(T_1)$ and $\Re(T_2)$ are constant on the intervals I_p and equal..

It follows from the above that if the matrix

$$\Delta := \lim_{\mu_0 \to 0} \begin{pmatrix} W/\sqrt{\mu_0} \\ T_1(0)^\dagger \end{pmatrix}$$

exists, then it satisfies the ADHM equation, $\Im(\Delta^\dagger \Delta) = 0$.

In a similar way, if (U_1, U_2, V) are solutions of (2.22)-(2.31) for the Nahm data $(T_1, T_2, W)(\mu_0)$ which are well-approximated by their Taylor expansions, then

$$u(x) := \lim_{\mu_0 \to 0} \begin{pmatrix} V(x) \\ U_1(0,x)/\sqrt{\mu_0} \end{pmatrix}$$

solves $(\Delta + x)^\dagger u = 0$ (if the limit exists). Also (2.32) implies that $u^\dagger u = e_0$ and (2.33) implies that $\lim_{\mu_0 \to 0} A_\alpha = u^\dagger \partial_\alpha u$. Therefore, the whole of the ADHM construction is recovered in a large period limit of the (N,N) Nahm construction.

Now we consider the large period limit of the $(2,2)$-calorons with R_N trivial (section 3.2.1). We let $\xi' := \xi/\mu_0$ and $\lambda' := \lambda/\sqrt{\mu_0}$ stay fixed as $\mu_0 \to 0$. Then the ADHM matrix is

$$\Delta = \begin{pmatrix} \lambda' \cos(\beta/2) e_0 & \lambda' \sin(\beta/2) \exp(\alpha e_2) \\ \xi' e_0 - \chi e_2 & -\lambda'^2/(4\xi') \sin\beta \sin\alpha e_2 \\ -\lambda'^2/(4\xi') \sin\beta \sin\alpha e_2 & -\xi' e_0 + \chi e_2 \end{pmatrix}$$

when $\xi' \neq 0$, and

$$\Delta = \begin{pmatrix} \lambda' \cos(\beta/2) e_0 & \lambda' \sin(\beta/2) e_0 \\ -\chi e_2 & 0 \\ 0 & \chi e_2 \end{pmatrix}$$

when $\xi' = 0$.

To take the large scale limit of the $(2,2)$-calorons with R_N non-trivial (section 3.2.2), we let $\lambda' = \lambda/\sqrt{\mu_0}$ stay fixed as $\mu_0 \to 0$. In this limit, the matching conditions (3.14), (3.16) are solved by $D_1 = D_2 = \sqrt{2}\lambda'$. The ADHM matrix is

$$\Delta = \frac{\lambda'}{\sqrt{2}} \begin{pmatrix} \sqrt{2} e_0 & \sqrt{2} e_2 \\ -e_3 & -e_1 \\ -e_1 & e_3 \end{pmatrix}.$$

There is a very simple geometrical description of the 2-instanton moduli space, due to Hartshorne [Har78, AM93]. Associated to each 2-instanton is a circle and an ellipse in $\mathbb{R}^4$ satisfying the Poncelet condition, which states that there exists a one-parameter family of triangles whose vertices lie on the circle and whose edges are tangent to the ellipse. There are two ways in which the circle and ellipse can be invariant under $U(1)$ rotations: either they lie in the axis of rotation (the plane $x^1 = x^3 = 0$); or they lie in a plane fixed by the rotation (x^2 =constant and x^0 =constant), such that the ellipse is a circle, and the centres of both circles sit on the axis of rotation.

It is possible to relate Hartshorne's description of instantons to the JNR ansatz (2.14) [AM93], and it is not hard to obtain ADHM matrices for instantons in the JNR ansatz [WW90]. Hence it is possible to compare Hartshorne's description of $U(1)$-symmetric 2-instantons with the ADHM matrices obtained above. It turns out that

the instanton limits of the calorons with R_N trivial correspond to the Hartshorne circle and ellipse lying in the axis of rotation, while the instanton limits of the calorons with R_N non-trivial correspond to the pair of concentric circles centred on the axis of rotation.

Finally, one can take the large period limits of $(N, N-1)$-calorons. The large period limit of an $(N, N-1)$-caloron is an $(N-1)$-instanton; in particular, the large period limits of the $U(1)$-symmetric $(2,1)$-calorons are 1-instantons.

3.6 Summary and open problems

For the sake of clarity, we summarise the results of this chapter here. We looked for calorons with monopole charges $(2,2)$ and $(2,1)$ which were invariant under a particular action of the group $U(1)$. We used some simple representation theory techniques to classify all possible actions of $U(1)$ on the Nahm data of such calorons. We were able to solve the Nahm equations and matching conditions explicitly in all these cases. We found one family of $(2,1)$-calorons and two distinct families of $(2,2)$-calorons, which could be distinguished by the action of $U(1)$ on their Nahm data. Finally, we considered limits: one family of $(2,2)$-calorons had the family of $(2,1)$-calorons as a limit, while the other had a family of 2-monopoles as a limit. We also obtained two distinct families of 2-instantons as large period limits.

The techniques we illustrated here could easily be applied to calorons with different symmetries, for example polyhedral groups. This would be an extension of [War04b], where polyhedral Nahm data was constructed for calorons with trivial holonomy. One outstanding question is whether these calorons have any interesting "hyperbolic" interpration. Instantons invariant under an action of $U(1)$ may be interpreted as monopoles on hyperbolic space (see the next chapter). Similarly, $U(1)$-symmetric calorons may be interpreted as monopoles on the quotient of hyperbolic space by a particular action of $\mathbb{Z}$ – it remains to be seen whether monopoles on this space have any interesting properties or applications.

Chapter 4

Hyperbolic calorons

In this chapter we shall consider anti-self-dual gauge fields on the space $\mathbb{H}^3 \times \mathbb{R}$. A finite-action anti-self-dual gauge field on this space will be called a hyperbolic instanton, while a finite action gauge field on the space $\mathbb{H}^3 \times S^1$ will be called a hyperbolic caloron.

The main reason to be interested in these spaces is that there is a conformal equivalence,

$$\mathbb{H}^3 \times S^1 \simeq \mathbb{R}^4 \backslash \mathbb{R}^2.$$

Since the anti-self-dual equations are conformally invariant, methods used to construct anti-self-dual gauge fields on $\mathbb{R}^4$ can be carried over to $\mathbb{H}^3 \times \mathbb{R}$, at least locally. This equivalence was first exploited by Atiyah, who initiated the study of monopoles on hyperbolic space [Ati87], as we shall now describe.

Let $SO(2)$ act on $\mathbb{R}^4$ with fixed subspace $\mathbb{R}^2$; then the action induced on $\mathbb{H}^3 \times S^1$ is just translation in the S^1 direction. Suppose that we have an instanton on $\mathbb{R}^4$ which is invariant under the action of $SO(2)$. Then this instanton is equivalently described as a translation-invariant gauge field on $\mathbb{H}^3 \times S^1$, which (by comments in an earlier chapter) is the same thing as a finite-energy solution to the Bogomolny equations on $\mathbb{H}^3$. Thus an axially-symmetric instanton defines a monopole on hyperbolic space. However, the converse is not true: there exist hyperbolic monopoles which do not map back to any instanton on $\mathbb{R}^3$ [Nas86].

To see why, one needs to look at the asymptotic value of the Higgs field of the hyperbolic monopole. Like Euclidean monopoles, hyperbolic monopoles are

required to satisfy $\|\Phi\| = \mu$ at infinity, for $\mu > 0$ a constant. Under the conformal equivalence, the boundary 2-sphere of $\mathbb{H}^3$ is identified with the subset $\mathbb{R}^2 \subset \mathbb{R}^4$ fixed by rotations. When one maps a hyperbolic monopole back to $\mathbb{R}^4$, one needs to check that no singularities appear on $\mathbb{R}^2$. It turns out that there will be a holonomy singularity unless

$$\mu = n/S,\ n \in \mathbb{Z},$$

where S is the radius of curvature of hyperbolic space.

So, $SO(2)$-symmetric instantons define monopoles on hyperbolic space, but not all hyperbolic monopoles can be obtained this way. An obvious jump to make from here is to consider instantons which are not necessarily $SO(2)$-symmetric. These will define finite-action anti-self-dual gauge fields on $\mathbb{H}^3 \times S^1$, in other words, hyperbolic calorons. This trick was exploited by Garland and Murray to study hyperbolic calorons and their twistor data [GM89]. However, as in the monopole case, there is a caveat: not all hyperbolic calorons can be obtained in this way. Specifically, the hyperbolic calorons obtained always satisfy the condition:

$$\beta = S\pi,$$

where β is the period of the caloron (the circumference of S^1) and S is the radius of curvature of $\mathbb{H}^3$.

In this chapter, we will construct hyperbolic calorons for which β and S take arbitrary values. Our method will be essentially to adapt standard ansätze on $\mathbb{R}^4$ to the hyperbolic case. We will find hyperbolic versions of both the Witten and the CF'tH ansätze. We will construct hyperbolic calorons and instantons explicitly in the Witten ansatz, and consider their relations with each other, with hyperbolic monopoles, and with Euclidean calorons. The work appearing in this chapter was published in [Har08]

4.1 Coordinates and metrics

The metric on $\mathbb{R}^4$ is

$$ds_E^2 = dt^2 + dr^2 + r^2 d\Omega^2,$$

where $t \in \mathbb{R}$, $r \geq 0$ is a radial coordinate on $\mathbb{R}^3$ and $d\Omega^2$ is the metric on S^2. Let y^1, y^2, y^3 be coordinates on the hyperbolic ball, satisfying $R := \sqrt{\sum_j (y^j)^2} < S$ for some $S > 0$. The metric on $\mathbb{H}^3 \times \mathbb{R}$ is

$$\begin{aligned} ds_H^2 &= d\tau^2 + \delta_{ij} dy^i dy^j \\ &= d\tau^2 + \Lambda^2 (dR^2 + R^2 d\Omega^2) \end{aligned}$$

where $\tau \in \mathbb{R}$, and $\Lambda := (1 - (R/S)^2)$.

Let $\mu = (S/2)\text{arctanh}(R/S)$. Then the hyperbolic metric is

$$ds_H^2 = d\tau^2 + d\mu^2 + \left(\frac{S}{2}\right)^2 \sinh^2\left(\frac{2\mu}{S}\right) d\Omega^2. \tag{4.1}$$

Let $z = \mu + i\tau$ and let $Z = r + it$. A pair (z, p) with $p \in S^2$ determines a point in $\mathbb{H}^3 \times \mathbb{R}$, similarly (Z, p) determines a point on $\mathbb{R}^4$. We define a map $M : \mathbb{H}^3 \times \mathbb{R} \to \mathbb{R}^4$ via

$$M : (z, p) \mapsto (Z, p) = (\tanh(z/S), p).$$

Then it is easy to check that this map is conformal: $M^* ds_E^2 = \xi^{-2} ds_H^2$, where

$$\xi := \left|\frac{\partial z}{\partial Z}\right| = \frac{S}{2}\left(\cosh\left(\frac{2\mu}{S}\right) + \cos\left(\frac{2\tau}{S}\right)\right).$$

The image of M is $(\mathbb{R}^4 \backslash S^2) \cup \{\infty\}$, where S^2 is the sphere $r = 1, t = 0$ and ∞ denotes the point at infinity. Note that the map M is periodic in τ with period $S\pi$; hence there is a conformal equivalence

$$(\mathbb{R}^4 \backslash S^2) \cup \{\infty\} \simeq \mathbb{H}^3 \times S^1$$

when $\beta = S\pi$.

Note that some papers on hyperbolic monopoles use the equivalence $\mathbb{R}^4 \backslash \mathbb{R}^2 \simeq \mathbb{H}^3 \times S^1$ rather than that described here. The two are obviously related, since $\mathbb{R}^2$ can be mapped conformally to S^2 in $\mathbb{R}^4$. We work with S^2 because it is fixed by an obvious $SO(3)$ symmetry.

Since $\mathbb{H}^3$ is topologically equivalent to $\mathbb{R}^3$, the topological classification of hyperbolic calorons and instantons is as in the Euclidean case.

4.2 Witten's ansatz

4.2.1 Dimensional Reduction

We make the following $SO(3)$-symmetric ansatz for a gauge field on $\mathbb{H}^3 \times \mathbb{R}$:

$$A = -\frac{1}{2}(Qa + \phi_1 dQ + (\phi_2 + 1)QdQ), \tag{4.2}$$

where

$$\begin{aligned} Q &= \hat{y}^j(-i\sigma^j) \\ a &= a_\mu d\mu + a_\tau d\tau. \end{aligned}$$

Here ϕ_1, ϕ_2, a_τ and a_μ are real functions of μ and τ only and $\hat{y}^j := y^j/R$.

Notice that $Q^2 = -1$, which implies that $QdQ = -dQQ$. Using this identity, one can show that the field strength tensor of A is

$$\begin{aligned} F \;=\; & -\frac{1}{2}Qda - \frac{1}{4}\left(1 - |\phi|^2\right) dQ \wedge dQ \\ & -\frac{1}{2}(Re(D\phi) + QIm(D\phi)) \wedge dQ. \end{aligned} \tag{4.3}$$

where $\phi = \phi_1 + i\phi_2$ and $D = d - ia$.

The action of this gauge field is

$$S = 4\pi \int \left(\frac{1}{2}\|da\|_h^2 + \frac{1}{2}\left(1 - |\phi|^2\right)^2 + \|D\phi\|_h^2\right) V_h. \tag{4.4}$$

where

$$\begin{aligned} ds_h^2 &= J^{-2}(d\mu^2 + d\tau^2) \\ V_h &= J^{-2} d\tau \wedge d\mu \\ J(\mu) &:= \frac{S}{2}\sinh\frac{2\mu}{S}. \end{aligned}$$

The factor of 4π arises because we have integrated over S^2. We have not specified the domain of integration for τ and μ, because this will change depending on whether we consider hyperbolic calorons or hyperbolic instantons. In proving (4.4), we have used the identities,

$$\begin{aligned} \|dQ\|_H^2 &= 2J^{-2} \\ \|QdQ\|_H^2 &= 2J^{-2} \\ \|dQ \wedge QdQ/2\|_H^2 &= J^{-4}. \end{aligned}$$

The action given in (4.4) is identical to the action for a 2-dimensional $U(1)$ Higgs model (or vortex model) with Higgs field ϕ and gauge field a on the manifold with metric g_h. (For instantons, the manifold on which the Higgs model lives is the universal cover of the hyperbolic plane with a point removed).

4.2.2 Gauge transformations

We consider the gauge freedom allowed by the ansatz (4.2). The most general $SO(3)$-symmetric gauge transformation is

$$g = \exp\left(-\frac{\lambda Q}{2}\right), \tag{4.5}$$

where λ is a real function of μ and τ. Consider the transformation $g(A) = g^{-1}Ag + g^{-1}dg$ of A. Using the identities,

$$\begin{aligned}
\exp\left(\frac{\lambda Q}{2}\right) dQ \exp\left(-\frac{\lambda Q}{2}\right) &= (\cos\lambda + Q\sin\lambda)dQ, \\
\exp\left(\frac{\lambda Q}{2}\right) d\left(\exp\left(-\frac{\lambda Q}{2}\right)\right) &= \frac{1}{2}(-Qd\lambda - \sin\lambda dQ + (\cos\lambda - 1)QdQ),
\end{aligned}$$

we find

$$g(A) = -\frac{1}{2}(Q(a + d\lambda) + (\phi_1\cos\lambda - \phi_2\sin\lambda)dQ + (\phi_2\cos\lambda + \phi_1\sin\lambda + 1)QdQ).$$

So, in the vortex interpretation, the $SU(2)$ gauge field $g(A)$ has Higgs field $e^{i\lambda}\phi$ and $U(1)$ gauge field $a + d\lambda$. Therefore the symmetric gauge transformation (4.5) is equivalent to a gauge transformation $e^{i\lambda}$ in the vortex model.

4.2.3 Anti-self-dual equations

The Hodge star can be shown to act in the following way:

$$\begin{aligned}
\star_H (d\tau \wedge dQ) &= (\star_h d\tau) \wedge QdQ \\
\star_H (d\mu \wedge dQ) &= (\star_h d\mu) \wedge QdQ \\
\star_H (Qd\tau \wedge d\mu) &= (\star_h (d\tau \wedge d\mu)) \wedge \left(\frac{1}{2} dQ \wedge dQ\right).
\end{aligned}$$

Therefore the anti-self-dual equations are

$$D\phi = -\star_h iD\phi \tag{4.6}$$

$$1 - |\phi|^2 = -\star_h da. \tag{4.7}$$

Alternatively, equations (4.6) and (4.7) can be derived directly from the vortex model action (4.4), as we shall now show. The vortex model has a topological charge,

$$k = \frac{1}{2\pi}\int da \tag{4.8}$$

which is an integer. The topological charge k forms a lower bound for the vortex action (4.4), by means of a standard argument [JT80].

We define

$$K = |\phi|^2 a + \frac{i}{2}\left(\bar{\phi}d\phi - \phi\overline{d\phi}\right).$$

The identities,

$$\begin{aligned} dK &= |\phi|^2 da - iD\phi \wedge \overline{D\phi} \\ (\star_h da + 1 - |\phi|^2)^2 V_h &= \|da\|_h^2 V_h + (1-|\phi|^2)^2 V_h + 2(1-|\phi|^2)da \\ \|D\phi + \star_h iD\phi\|_h^2 V_h &= 2\|D\phi\|_h^2 V_h + 2iD\phi \wedge \overline{D\phi} \end{aligned}$$

imply that

$$\begin{aligned} S &= 4\pi \int \frac{1}{2}\left((\star_h da + 1 - |\phi|^2)^2 + 2\|D\phi + \star_h iD\phi\|_h^2\right) V_h \\ &\quad +4\pi\int dK - 4\pi\int da. \end{aligned}$$

The first integral is positive, and vanishes when (4.6), (4.7) are satisfied. The second integral can be evaluated using Stokes' theorem, and vanishes if K decays sufficiently fast. Therefore, we have $S \geq -8\pi^2 k$, with equality if and only if (4.6), (4.7) hold.

Since (4.6), (4.7) are equivalent to the anti-self-dual equations, the above argument shows that

$$k = -c_2.$$

Probably this formula could also be proved directly, but we have not done so.

4.2.4 Solution of anti-self-dual equations

Our method of solving the anti-self-dual equations (4.6), (4.7) is similar to the method used by Witten in the Euclidean case. Notice that equation (4.6) is equivalent to

$$D_{\bar{z}}\phi = 0, \tag{4.9}$$

where $D_{\bar{z}} := 1/2(D_\mu + iD_\tau)$. Let us choose a gauge so that $\partial_\mu a_\mu + \partial_\tau a_\tau = 0$. This allows us locally to write

$$\begin{aligned} a_\mu &= -\partial_\tau \psi \\ a_\tau &= \partial_\mu \psi \end{aligned}$$

for some real function ψ. This gauge fixing condition still allows for gauge transformations $e^{i\lambda}$ for which $\triangle_z \lambda = 0$.

Equation (4.9) is then equivalent to

$$\partial_{\bar{z}}(e^{\psi}\phi) = 0.$$

So $e^{\psi}\phi$ must be a meromorphic function, which we denote f. Equation (4.7) now looks like

$$\triangle_z \psi = J^{-2} - J^{-2}e^{-2\psi}|f|^2 \tag{4.10}$$

We let

$$\rho = -\psi + \ln|f| - \ln J,$$

then the identities $\triangle_z \ln J = -J^{-2}$ and $\triangle \ln|f| = 0$ imply that (4.10) is equivalent to

$$\triangle_z \rho = e^{2\rho}.$$

This is known as Liouville's equation, and the general solution is

$$\rho = -\ln\left(\frac{1-|g|^2}{2}\right) + \ln|\partial_z g|,$$

where g is some meromorphic function.

In order to avoid singularities in the gauge field, f and $\partial_z g$ must have the same zeros and poles in the right half-plane $\mu > 0$. So we may define a function $h = \frac{f}{\partial_z g}$ which is analytic and non-zero in the right half-plane. Thus, the general solution to the equations of motion and the gauge-fixing condition is

$$\psi = \ln\left(1-|g|^2\right) - \ln 2J + \ln|h| \tag{4.11}$$

$$\phi = e^{-\psi} h \partial_z g \tag{4.12}$$

$$a_\mu = -\partial_\tau \psi \tag{4.13}$$

$$a_\tau = \partial_\mu \psi. \tag{4.14}$$

In these formulae, h is interpreted as specifying a choice of gauge, since it may be removed by the gauge transformation $e^{i\lambda}$, where $\lambda = (\ln \bar{h} - \ln h)/2$.

Therefore, given a meromorphic function g on the right half-plane, we may construct a symmetric anti-self-dual gauge field using equations (4.11)-(4.14). In order that ψ remains finite, we must impose the constraints

$$|g|^2 \leq 1 \text{ when } \mu \geq 0 \tag{4.15}$$

$$|g|^2 = 1 \text{ when } \mu = 0. \tag{4.16}$$

4.3 The Corrigan-Fairlie-'tHooft ansatz

Now we shall consider the hyperbolic analog of the CF'tH ansatz. Its derivation is essentially an application of the methods of section 2.4.

Let $g' = \rho^2 ds_H^2$, where ds_H^2 is the metric (4.1) on $\mathbb{H}^3 \times \mathbb{R}$. Orthonormal bases for the tangent and cotangent spaces with respect to g' are

$$\begin{aligned}
\theta_0 &= \frac{1}{\rho}\frac{\partial}{\partial \tau} \\
\theta_i &= \frac{1}{\rho\Lambda}\frac{\partial}{\partial y^i} \\
v^0 &= \rho d\tau \\
v^i &= \Lambda \rho dy^i.
\end{aligned}$$

Let A denote the Levi-Civita connection of g', which we write with respect to the orthonormal basis: $A = \Gamma'^{\delta}_{\alpha\gamma}\theta_\delta v^\gamma dy^\alpha$. The non-zero coefficents of A are

$$\begin{aligned}
\Gamma'^{k}_{ij} &= \left(\partial_j \ln\rho + \frac{2\Lambda}{S^2}y^j\right)\delta_{ik} - \left(\partial_k \ln\rho + \frac{2\Lambda}{S^2}y^k\right)\delta_{ij} \\
\Gamma'^{0}_{ij} &= -\Lambda\partial_0 \ln\rho\,\delta_{ij} \\
\Gamma'^{k}_{i0} &= \Lambda\partial_0 \ln\rho\,\delta_{ik} \\
\Gamma'^{k}_{00} &= -\frac{1}{\Lambda}\partial_k \ln\rho \\
\Gamma'^{0}_{0j} &= \frac{1}{\Lambda}\partial_j \ln\rho.
\end{aligned}$$

where we have used the convention $i, j, k \in \{1, 2, 3\}$.

Invoking the Lie algebra isomorphisms and splittings (2.10)-(2.13) , we find

$$A_0^+ = -\frac{1}{\Lambda}\partial_a \ln\rho\left(\frac{\sigma^a}{2i}\right) \tag{4.17}$$

$$A_j^+ = \left(+\Lambda\partial_0 \ln\rho\delta_{ja} + \epsilon_{jak}\left(\partial_k \ln\rho + \frac{2\Lambda}{S^2}y^k\right)\right)\left(\frac{\sigma^a}{2i}\right). \tag{4.18}$$

The scalar curvature s of the metric g' is

$$s = -6\rho^{-3}\left(\Box_H\rho + \frac{4}{S^2}\rho\right),$$

where $\Box_H$ is the Laplace-Beltrami operator for the metric ds_H^2:

$$\Box_H = \frac{\partial^2}{\partial\tau^2} + \frac{1}{\Lambda^2}\left(\frac{\partial}{\partial y^i}\right)^2 + \frac{2}{\Lambda S^2}y^i\frac{\partial}{\partial y^i}$$

Therefore A^+ will be anti-self-dual if and only if

$$\Box_H\rho = -\frac{4}{S^2}\rho. \tag{4.19}$$

4.3.1 Relation with Euclidean version

Recall that the hyperbolic metric is itself a conformal rescaling of the Euclidean metric: $ds_H^2 = \xi^2 ds_E^2$. This allows us to relate the above two constructions, and provides a check of what we have done so far. Let g' be a rescaling of the hyperbolic metric, $g' = \rho^2 ds_H^2$. Then we also have $g' = (\rho\xi)^2 ds_E^2$. As before the hyperbolic gauge field of equations (4.17) and (4.18) will be anti-self-dual if and only if g' has zero scalar curvature. From the Euclidean version, we know that g' has zero scalar curvature if and only if

$$\Box_E(\xi\rho) = 0.$$

We will show that this condition is equivalent to the previous condition, (4.19).

The following identity holds because ds_H^2 is a conformal rescaling of ds_E^2:

$$\xi^3\Box_H\rho = \xi\Box_E\rho + 2\partial_\mu\rho\partial_\mu\xi.$$

We also have, for any functions f, g,

$$\Box_E(fg) = f\Box_E g + g\Box_E g + 2\partial_\mu g\partial_\mu g.$$

Therefore

$$\Box_E(\xi\rho) = \xi^3\Box_H\rho + \rho\Box_E\xi.$$

We now compute $\Box_E\xi$. Note that for functions f depending only on r and t, we have

$$\Box_E f = \frac{1}{r}\triangle_Z(rf),$$

where $\triangle_Z = \partial_r^2 + \partial_t^2$. If we also let $\triangle_z = \partial_\mu^2 + \partial_\tau^2$, then

$$\triangle_Z = \left|\frac{\partial z}{\partial Z}\right|^2 \triangle_z.$$

So using the relations,

$$\begin{aligned}\left|\frac{\partial z}{\partial Z}\right|^2 &= \xi^2 \\ r &= \frac{S\sinh(2\mu/S)}{2\xi},\end{aligned}$$

we calculate $\Box_E\xi = 4\xi^3/S^2$. Therefore g' is flat and A^+ is anti-self-dual if and only if

$$\Box_H\rho = -\frac{4}{S^2}\rho,$$

which is exactly the same relation as we had before (4.19).

4.3.2 Relation with Witten's ansatz

We now have two methods for constructing anti-self-dual gauge fields on g_H; the Witten ansatz and the CF'tH ansatz. We show here that the two methods are related: all solutions constructed in the Witten ansatz can also be expressed in the CF'tH ansatz. This argument closely follows that of Manton [Man78].

Assuming that ρ depends only on μ and τ, the solutions of the anti-self-dual CF'tH ansatz can be written as:

$$\begin{aligned}A_0^+ &= -\hat{y}^a\partial_\mu \ln\rho\left(\frac{\sigma^a}{2i}\right) \\ A_j^+ &= \left(\Lambda\partial_\tau \ln\rho\delta_{ja} + \epsilon_{jak}\hat{y}^k\left(\partial_R \ln\rho + \frac{2\Lambda R}{S^2}\right)\right)\left(\frac{\sigma^a}{2i}\right).\end{aligned}$$

Meanwhile, the anti-self-dual $SO(3)$-symmetric gauge field is,

$$\begin{aligned}A_0 &= -\hat{y}^a\partial_\mu\psi\frac{\sigma^a}{2i} \\ A_j &= \left(\Lambda\partial_\tau\psi\hat{y}^j\hat{y}^a - \frac{\phi_2+1}{R}\epsilon_{jak}\hat{y}^k - \frac{\phi_1}{R}(\delta_{ja} - \hat{y}^j\hat{y}^a)\right)\left(\frac{\sigma^a}{2i}\right).\end{aligned}$$

These two solutions will be identical if

$$\psi = \ln \rho \tag{4.20}$$

$$\phi_1 = -\Lambda R \partial_\tau \ln \rho \tag{4.21}$$

$$\phi_2 = -R\partial_R \ln \rho - \left(\frac{2\Lambda R^2}{S^2} + 1\right). \tag{4.22}$$

If we assume that (4.20) defines ρ, equations (4.21) and (4.22) may be rewritten in (μ, τ) coordinates:

$$\begin{aligned} \phi_1 &= -J\partial_\tau \psi \\ \phi_2 &= -J\partial_\mu \psi - \partial_\mu J. \end{aligned}$$

These are equivalent to

$$e^\psi \phi = -2i\partial_z \left(J e^\psi\right).$$

Recall that $2Je^\psi = (1 - |g|^2)\,|h|$ and $e^\psi \phi = h\partial_z g$, so this equation is equivalent to

$$h\partial_z g = -i\partial_z \left((1 - |g|^2)\,|h|\right) \tag{4.23}$$

If we choose $h = -i(g-1)^{-2}$, we find

$$\begin{aligned} h\partial_z g &= i\frac{\partial}{\partial z}\left(\frac{1}{g-1}\right) \\ (1 - |g|^2)\,|h| &= -1 - \frac{1}{g-1} - \frac{1}{\bar{g}-1}. \end{aligned}$$

Therefore equation (4.23) is solved by this choice of h, and we note that

$$\rho = \frac{1}{2J}\frac{1 - |g|^2}{|g-1|^2} \tag{4.24}$$

$$= \frac{1}{2J}\left(-1 - \frac{1}{g-1} - \frac{1}{\bar{g}-1}\right). \tag{4.25}$$

4.4 Hyperbolic instantons

The goal of this section is to construct hyperbolic instantons. We focus our attention on instantons with $SO(3)$ symmetry, and use the hyperbolic Witten ansatz (4.11-4.16). In this context, a symmetric gauge field is completely determined by an analytic function g satisfying appropriate conditions. To demonstrate that such a

gauge field is indeed an instanton, it is necessary to prove that its action is finite. It is this task which will occupy most of the discussion of this chapter.

We consider the function,

$$g_1(z) = \exp\left(\frac{-2z}{S}\right)\frac{\lambda - 2z/S}{\lambda + 2z/S}, \tag{4.26}$$

where $\lambda > 0$ is a real parameter. One can readily verify that this function satisfies the conditions (4.15) and (4.16). Therefore to prove that the gauge field generated is an instanton, we need only verify that its action is finite. This is the first result of this subsection. This being done, we calulate the charge of the instanton.

We first recall the form of the action in Witten's ansatz, (4.4). In view of the anti-self-dual equations (4.7), (4.9), this may be written to

$$S = 4\pi\int_{\mu>0}\left(\frac{1}{2}\left(1-|\phi|^2\right)^2 J^{-2} + |D_z\phi|^2\right) d\tau\wedge d\mu. \tag{4.27}$$

The following proposition summarises the asymptotic behaviour of the terms in this action.

Proposition 4.4.1 Let the functions ϕ, a_μ, and a_τ be generated by the function $g_1(z)$ as in equations (4.11)-(4.14). Then the function $(1-|\phi|^2)/J(\mu)$ has a finite limit as $\mu \to 0$. In the limit $|z| \to \infty$, we have (for $\mu \geq 0$)

$$|\phi| = O(|z|^0) \tag{4.28}$$

$$\frac{1-|\phi|^2}{J(\mu)} = O(|z|^{-2}) \tag{4.29}$$

$$\left|\frac{D_z\phi}{\phi}\right| = O(|z|^{-2}). \tag{4.30}$$

Finally, when $\mu = 0$, $D_z\phi/\phi = 0$.

Proof: We choose the gauge $h = 1$ and calculate ϕ from g according to $\phi = e^{-\psi}\partial_z g$. ψ is calculated from equation (4.11):

$$\exp(-\psi_1) = \frac{S}{2}\exp\left(\frac{2\mu}{S}\right)\frac{|2z/S+\lambda|^2}{|2z/S|^2+\lambda^2+2\lambda+2\lambda p(\mu)},$$

where we have introduced $p(\mu) := (2\mu/S)\coth(2\mu/S) - 1$ for convenience. Meanwhile, $\partial_z g_1$ is given by

$$\partial_z g_1 = \frac{2}{S}\exp\left(-\frac{2z}{S}\right)\frac{(2z/S)^2-(\lambda^2+2\lambda)}{(2z/S+\lambda)^2}.$$

In order to proceed, we need the following lemma.

Lemma 4.4.2 The function $p(\mu)$ satisfies the following:

(a) $p(\mu) \to 0$ as $\mu \to 0$.

(b) $p(\mu) \geq -1 \ \forall \mu \geq 0$.

(c) $p(\mu)/\sinh(2\mu/S)$ is bounded.

(d) $p(\mu) = O(|z|^1)$ as $|z| \to \infty$

(e) $p'(\mu) \to 0$ as $\mu \to 0$, and $p'(\mu)$ is bounded

Proof:

(a) is easily computed. To see (b), simply note that $\coth(2\mu/S) \geq 1$ for $\mu > 0$, and hence $p(\mu) \geq (2\mu/S) - 1$. The function in (c) is finite in the limit $\mu \to 0$ by (a), and it is easily shown to converge as $\mu \to \infty$. Therefore the function is bounded.

To show (d), consider the function

$$p(\mu) - \frac{2\mu}{S} = \frac{2\mu}{S}\left(\coth\frac{2\mu}{S} - 1\right) - 1.$$

This function can be shown to have finite limits as μ tends to zero and infinity. Therefore the function is bounded. We have

$$\frac{2\mu}{S} - F \leq p(\mu) \leq \frac{2\mu}{S} + F$$

for some constant F, and (d) follows. (e) is a matter of computation. □

From the expressions preceding lemma, we have

$$|\phi| = \frac{|(2z/S)^2 - (\lambda^2 + 2\lambda)^2|}{|2z/S|^2 + \lambda^2 + 2\lambda + 2\lambda p(\mu)}.$$

The numerator of this expression is $O(|z|^2)$. Using part (b) of the lemma,

$$0 \leq \frac{1}{|2z/S|^2 + \lambda^2 + 2\lambda + 2\lambda p(\mu)} \leq \frac{1}{|2z/S|^2 + \lambda^2},$$

from which we see that

$$\left(|2z/S|^2 + \lambda^2 + 2\lambda + 2\lambda p(\mu)\right)^{-2} = O\left(|z|^{-2}\right).$$

Equation (4.28) follows.

From the expression for $|\phi|$ we compute

$$\frac{1-|\phi|^2}{J(\mu)} = \frac{2}{S}\frac{4(\lambda^2+2\lambda)\,(2\mu/S)^2 + 4\lambda p(\mu)\left(|2z/S|^2+\lambda^2+2\lambda\right) + 4\lambda^2 p(\mu)^2}{\sinh(2\mu/S)\left(|2z/S|^2+(\lambda^2+2\lambda)+2\lambda p(\mu)\right)^2}.$$

We note that $(2\mu/S)/(\sinh(2\mu/S))$ and $p(\mu)/\sinh(2\mu/S)$ are bounded (the second is part (c) of the lemma). This means in particular that $(1-|\phi|^2)/J(\mu)$ is bounded as $\mu \longrightarrow 0$, as claimed. Using the boundedness of these two functions and part (d) of the lemma, we have:

$$\begin{aligned}
\frac{(2\mu/S)^2}{\sinh(2\mu/S)} &= O(|z|) \\
\frac{p(\mu)}{\sinh(2\mu/S)}\left(\left|\frac{2z}{S}\right|^2+\lambda^2+2\lambda\right) &= O(|z|^2) \\
\frac{p(\mu)^2}{\sinh(2\mu/S)} &= O(|z|).
\end{aligned}$$

Combining these with earlier results proves equation (4.29).

From the expression for a, (4.13 and 4.14), we derive the identity $D_z = e^{\psi}\partial_z e^{-\psi}$. Applying this to $\phi = e^{-\psi}\partial_z g$, one arrives at the formula,

$$D_z\phi/\phi = \partial_z(\ln\partial_z g - 2\psi).$$

From the expressions for ψ_1 and $\partial_z g_1$ derived above, we have

$$\begin{aligned}
-\partial_z(2\psi) &= \frac{2}{S} + \frac{4/S}{2z/S+\lambda} - \frac{8\bar{z}/S^2+2\lambda p'(\mu)}{|2z/S|^2+\lambda^2+2\lambda+2\lambda p(\mu)} \\
\partial_z(\ln\partial_z g_1) &= -\frac{2}{S} - \frac{4/S}{2z/S+\lambda} + \frac{8z/S^2}{(2z/S)^2-(\lambda^2+2\lambda)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\partial_z(\ln\partial_z g_1 - 2\psi_1) = {} & \frac{(16\mu/S^2)(\lambda^2+2\lambda)+(16z/S^2)\lambda p(\mu)}{(|2z/S|^2+\lambda^2+2\lambda+2\lambda p(\mu))((2z/S)^2-(\lambda^2+2\lambda))} \\
& - \frac{2\lambda p'(\mu)}{|2z/S|^2+\lambda^2+2\lambda+2\lambda p(\mu)}.
\end{aligned}$$

One may easily check that the first of the terms on the right of this equation is $O(|z|^{-2})$, using parts (b) and (d) of lemma 1. The second term on the right hand side is $O(|z|^{-2})$, as can be deduced part (e) of the lemma. This proves equation (4.30). □

An immediate corollary of proposition 4.4.1 is the following:

Corollary 4.4.3 The anti-self-dual gauge field generated by the function $g_1(z)$ has finite action.

We may also use proposition 4.4.1 to find the topological charge of the instanton. This is the content of the next proposition.

Proposition 4.4.4 The topological charge k of the vortex generated by the function $g_1(z)$ converges absolutely. It is equal to -1. Therefore the instanton charge $c_2 = 1$.

Proof: Using the anti-self-dual equation (4.7), the vortex charge (4.8) is equal to

$$k = -\frac{1}{2\pi}\int \frac{1-|\phi|^2}{J(\mu)^2} d\tau \wedge d\mu.$$

The absolute convergence of this integral follows from equation (4.29). Notice that, since $|\phi|^2 < 1$, this formula tells us that $k < 0$.

The value of the vortex charge may be evaluated in a manner similar to the Euclidean case (see chapter II of [JT80]). We begin by writing ϕ in the form

$$\phi = |\phi|\, e^{i\theta},$$

for some real function $\theta(z)$. Then the covariant derivative of ϕ is

$$D\phi = d\,|\phi|\, e^{i\theta} + |\phi|\, e^{i\theta}(id\theta - ia).$$

Therefore we have

$$\left(\frac{D\phi}{\phi}\right) - \overline{\left(\frac{D\phi}{\phi}\right)} = 2i(d\theta - a).$$

This equation is integrated over the boundary of a large semicircle Γ of radius M to obtain

$$\int_\Gamma d\theta - \int_\Gamma a = \frac{1}{2i}\int_\Gamma \left(\frac{D\phi}{\phi}\right) - \overline{\left(\frac{D\phi}{\phi}\right)}.$$

The integral on the right of this equation converges to zero as M tends to ∞, using equation (4.30) of proposition 4.4.1. The second integral on the left converges to 2π times the vortex charge, using Stokes' theorem. The first integral on the left is 2π times the winding number of the Higgs field around Γ. This is equal to the number of zeros of the Higgs field, since the the Higgs field is the product of a non-zero real function and an analytic function. One can easily check that the Higgs field has only one zero, so we conclude that the vortex charge is $k = -1$. □

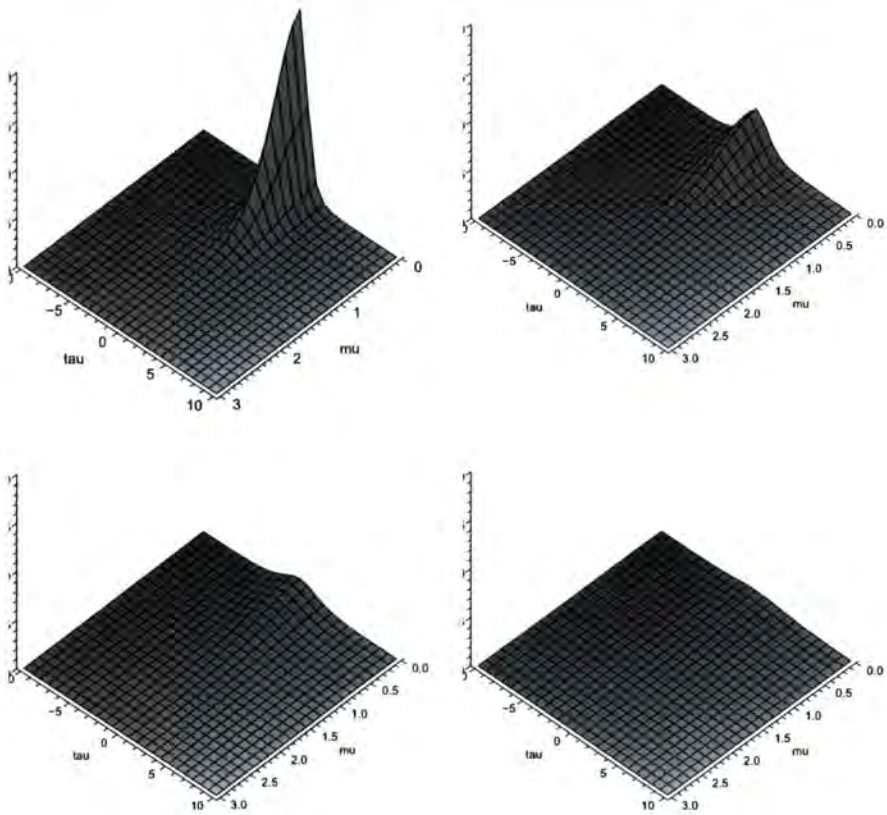

Figure 4.1: Graphs of field strength $\|F\|^2$ as a function of μ and τ for hyperbolic instantons with λ=1, 2, 3, 5.

The location of the vortex is taken to be the zero of the Higgs field. It is given by

$$(\mu_0, \tau_0) = \left(\frac{S}{2}\sqrt{\lambda^2 + 2\lambda}, 0\right).$$

We have observed that the location of the Higgs field zero is a scale parameter for the instanton; as μ_0 increases, the instanton action density becomes more spread out.

We may extend the family of solutions by considering translations along the time axis. This is equivalent to allowing λ to be complex with positive real part. In this case the imaginary part τ_0 of the vortex location specifies the location of the instanton on the time axis.

We propose that following generalisation of the function $g_1(z)$:

$$g_N(z) = \exp\left(-\frac{2z}{S}\right)\prod_{i=1}^{N}\frac{\lambda_i - 2z/S}{\bar{\lambda}_i + 2z/S} \tag{4.31}$$

where λ_i are complex numbers. The Higgs field generated by this function has N zeros, which suggest that the instanton charge should be $c_2 = N$. It is tempting to interpret the locations of these zeros as specifying the locations and scales of the constituent instantons, but we have no verification of this. We will not prove anything about higher charge instantons here, but we note that they motivate the construction of hyperbolic calorons, as we shall see in the next section.

4.5 Hyperbolic calorons with charge 1

In this section we present a three-parameter family of non-integral hyperbolic calorons with topological charge 1. As in the previous section, we consider calorons with $SO(3)$ symmetry. We will demonstrate directly that the action of these calorons is finite and that their charge is 1.

We begin this section with a direct construction of integral hyperbolic calorons from Euclidean instantons. At the end of the section, we will show how these integral calorons are included in the family of non-integral calorons.

4.5.1 Calorons from Euclidean instantons

Recall that a Euclidean instanton can be pulled back to an integral hyperbolic caloron, via the conformal equivalence $\mathbb{R}^4\backslash S^2 \simeq \mathbb{H}^3 \times S^1$ (with $\beta = S\pi$). Said more directly, a Euclidean instanton can be reinterpreted as an integral hyperbolic caloron simply by changing from Euclidean to hyperbolic coordinates. This procedure is particularly simple if we restrict attention to $SO(3)$-symmetric instantons and use Witten's ansatz; here we will spell out the details.

Witten's ansatz for a $SO(3)$-symmetric gauge field over $\mathbb{R}^4$:

$$A = -\frac{1}{2}(Qa + \phi_1 dQ + (\phi_2 + 1)QdQ).$$

Here $Q = \hat{x}^j(-i\sigma^j)$ (note that $\hat{x}^j = \hat{y}^j$) and $a = a_r dr + a_t dt$, and a_r, a_t, ϕ_1 and ϕ_2 are real functions of r and t. We write $\phi = \phi_1 + i\phi_2$ and $a_Z = (a_r - ia_t)/2$. Witten's solution of the anti-self-dual equation is

$$\begin{aligned} \psi &= \ln(1 - |g_E|^2) + \ln|h_E| - \ln(2r) \\ \phi &= \exp(-\psi) h_E \partial_Z g_E \\ a_Z &= -i\partial_Z \psi \end{aligned}$$

where g_E and h_E are analytic functions of Z (h_E represents a choice of gauge). If we change coordinates from (r, t) to (μ, τ) and choose the gauge $h_E = 1$, we obtain

$$\begin{aligned} \psi &= \ln(1 - |g_E|^2) - \ln(2r) \\ \phi &= \exp(\psi) \frac{\partial z}{\partial Z} \partial_z g_E \\ a_z &= -i\partial_z \psi. \end{aligned}$$

But, using the identity $J/r = |\partial z/\partial Z|$, we see that this is exactly matches our earlier solution (4.11)-(4.14), with $g(z) = g_E(Z(z))$ and $h = \partial z/\partial Z$.

For example, the charge 1 Euclidean instanton located at the origin has

$$g_E(Z) = \left(\frac{\alpha - Z}{\alpha + Z}\right)^2$$

where α is a real parameter. The corresponding hyperbolic caloron will have

$$g(z) = g_C(z) := \left(\frac{\alpha - \tanh(z/S)}{\alpha + \tanh(z/S)}\right)^2. \tag{4.32}$$

4.5.2 Non-integral charge 1 calorons

Recall the the function $g_N(z)$ which conjecturally generates charge N hyperbolic instantons (4.31). By thinking of a caloron as a sequence of equally-spaced instantons, we arrive at the following form for the g-function of a hyperbolic caloron with period β:

$$g_\infty(z) := \exp\left(-\frac{2z}{S}\right) \lim_{N \to \infty} \prod_{k=-N}^{N} \frac{\lambda + 2k\beta i/S - 2z/S}{\lambda - 2k\beta i/S + 2z/S}. \tag{4.33}$$

In this subsection we prove that the family of periodic anti-self-dual gauge fields generated by $g_\infty(z)$ are hyperbolic calorons. We do this first by showing that the

infinite product in $g_\infty(z)$ converges, and then by showing that the action is finite. We also evaluate the charges of these calorons. The following proposition establishes the convergence of the infinite product.

Proposition 4.5.1 The infinite product (4.33) converges uniformly on any compact set after singular terms have been removed. Moreover, the product has the following closed form:

$$g_\infty(z) = \exp\left(-\frac{2z}{S}\right)\frac{\sinh((z - S\lambda/2)\pi/\beta)}{\sinh((z + S\lambda/2)\pi/\beta)} \tag{4.34}$$

Proof:

First we demonstrate the uniform convergence of the product. The product may be re-arranged by multiplying the k-th and $-k$-th terms:

$$\prod_{k=-N}^{N}\frac{\lambda + 2k\beta i/S - 2z/S}{\lambda - 2k\beta i/S + 2z/S} = \frac{\lambda - 2z/S}{\lambda + 2z/S}\prod_{k=1}^{N}\left(1 - \frac{8\lambda z/S}{(\lambda + 2z/S)^2 + (2k\beta/S)^2}\right).$$

The product on the right-hand side of this expression converges uniformly as $N \to \infty$ on any compact set, after removal of singular terms, if and only if the infinite sum

$$\sum_{k=1}^{\infty}\left|\frac{8\lambda z/S}{(\lambda + 2z/S)^2 + (2k\beta/S)^2}\right|$$

converges uniformly on any compact set, after removal of singular terms.

The uniform convergence of the infinite sum is demonstrated as follows: fix a compact set and let R be such that $|z| < R$ for all z in the compact set. Then $|(\lambda + 2z/S)^2 + (2k\beta/S)^2| \geq (2k\beta/S)^2 - (\lambda + 2R/S)^2$. So we have

$$\sum_{k=M}^{\infty}\left|\frac{8\lambda z/S}{(\lambda + 2z/S)^2 + (2k\beta/S)^2}\right| \leq \sum_{k=M}^{\infty}\frac{8\lambda R/S}{(2k\beta/S)^2 - (\lambda + 2R/S)^2},$$

where M is chosen to make the terms on the right positive. The series on the right converges, so the series on the left converges uniformly (by the Weierstrass M-test). Hence the infinite product converges uniformly on any compact set, after singular terms have been removed.

To prove equation (4.34), we use the familiar product representation of sinh:

$$\sinh z = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{k^2\pi^2}\right).$$

Applying this to the right hand side of equation (4.34) yields

$$\frac{\sinh((z-S\lambda/2)\pi/\beta)}{\sinh((z+S\lambda/2)\pi/\beta)} = \frac{z-S\lambda/2}{z+S\lambda/2}\prod_{k=1}^{\infty}\frac{(\beta k)^2+(z-S\lambda/2)^2}{(\beta k)^2+(z+S\lambda/2)^2}.$$

After rearrangement, this is equal to the product in equation (4.33), and we have shown (4.34). □

The function $g_\infty(z)$ satisfies the basic requirements for generating an anti-self-dual gauge field (equations (4.15) and (4.16)); to conclude that the gauge field is a caloron we only need to show that its action is finite. The method will be similar to that used to show the instanton action is finite. We use the simplified vortex form of the action (4.27), which is valid for anti-self-dual gauge fields (but note that now τ is integrated between 0 and β).

We note here that the function $g_\infty(z)$ and the gauge field generated by it do not necessarily have period β, because of the factor $\exp(-2z/S)$. To make the gauge field strictly periodic, we make an aperiodic gauge transformation $h_\infty(z) = \exp(2z/S)$.

Proposition 4.5.2 Let the Higgs field ϕ and the $U(1)$ gauge field a_μ, a_τ be generated by the function $g_\infty(z)$ with gauge transformation $h_\infty(z)$. Then in the limit $\mu \to 0$, we have the following convergences:

$$\frac{1-|\phi|^2}{J(\mu)} \to 0 \tag{4.35}$$

$$\frac{D_z\phi}{\phi} \to 0 \tag{4.36}$$

In the limit $\mu \to \infty$, we have the following convergences (uniformly in τ):

$$|\phi| \to \exp\left(-\frac{S\pi\lambda}{\beta}\right) \tag{4.37}$$

$$\frac{1-|\phi|^2}{J(\mu)} = O\left(\exp\left(-\frac{2\mu}{S}\right)\right) \tag{4.38}$$

$$\left|\frac{D_z\phi}{\phi}\right| = O\left(\exp\left(-\frac{2\mu}{S}\right)\right). \tag{4.39}$$

Proof:

We compute:

$$h_\infty\partial_z g_\infty = \frac{\cosh(S\pi\lambda/\beta)+(S\pi/\beta)\sinh(S\pi\lambda/\beta)-\cosh(2\pi z/\beta)}{S\sinh^2((z+S\lambda/2)\pi/\beta)} \tag{4.40}$$

$$\exp(-\psi_\infty) = \frac{S\,|\sinh((z+S\lambda/2)\pi/\beta)|^2}{\cosh((2\mu+S\lambda)\pi/\beta)-\cos(2\pi\tau/\beta)+q(\mu)}, \tag{4.41}$$

where $q(\mu) := (\coth(2\mu/S)-1)\sinh(2\pi\mu/\beta)\sinh(S\pi\lambda/\beta)$ was introduced to simplify the exposition. The following lemma states all the properties of $q(\mu)$ we shall need, and is easily proved.

Lemma 4.5.3 The function $q(\mu)$ satisfies the following:

(a) $q(\mu) \to (S\pi/\beta)\sinh(S\pi\lambda/\beta)$ as μ tends to 0, and the convergence is $O(\mu^2)$.

(b) $q(\mu)\exp(-2\pi\mu/\beta) \to 0$ as μ tends to infinity.

(c) $q'(\mu) \to -(2\pi/\beta)\sinh(S\pi\lambda/\beta)$ as μ tends to 0.

(d) $q'(\mu)\exp(-2\pi\mu/\beta) \to 0$ as μ tends to infinity.

Continuing our calculation, we find that

$$|\phi|^2 = \frac{|\cosh(S\pi\lambda/\beta) + (S\pi/\beta)\sinh(S\pi\lambda/\beta) - \cosh(2\pi z/\beta)|^2}{(\cosh((2\mu + S\lambda)\pi/\beta) - \cos(2\pi\tau/\beta) + q(\mu))^2}.$$

In this last expression, the numerator and denominator both converge to the same function,

$$(\cosh(S\pi\lambda/\beta) + (S\pi/\beta)\sinh(S\pi\lambda/\beta) - \cos(2\pi\tau/\beta))^2,$$

as μ tends to 0. In both cases the convergence is faster that μ^2. Therefore $1 - |\phi|^2$ converges to 0 as μ tends to 0, and this convergence is $O(\mu^2)$. It follows that $(1 - |\phi|^2)/J(\mu)$ converges to 0 as μ tends to 0. This establishes (4.35).

From the above expression for $|\phi|^2$ one can check that $|\phi|^2$ tends to a constant, $\exp(-2S\pi\lambda/\beta)$, as μ tends to infinity. Therefore $(1 - |\phi|^2)/J(\mu)$ converges to zero exponentially fast as μ tends to infinity. This establishes (4.37) and (4.38).

We recall from the proof of proposition 4.4.1 that $D_z\phi/\phi = \partial_z(-2\psi + \ln(h\partial_z g))$. From the above expressions for e^ψ and $h_\infty \partial_z g_\infty$ above, we have

$$\begin{aligned}
\partial_z \ln(h_\infty \partial_z g_\infty) &= -\frac{S}{2} - \frac{2\pi}{\beta}\coth\left((z + S\lambda/2)\pi/\beta\right) \\
&\quad + \frac{(2\pi/\beta)\sinh(2\pi z/\beta)}{\cosh(2\pi z/\beta) - \cosh(S\pi\lambda/\beta) - (S\pi/\beta)\sinh(S\pi\lambda/\beta)} \\
-2\partial_z\psi &= \frac{S}{2} + \frac{2\pi}{\beta}\coth\left((z + S\lambda/2)\pi/\beta\right) \\
&\quad - \frac{(2\pi/\beta)(\sinh((2\mu + S\lambda)\pi/\beta) - i\sin(2\pi\tau/\beta)) + q'(\mu)}{\cosh((2\mu + S\lambda)\pi/\beta) - \cos(2\pi\tau/\beta) + q(\mu)}.
\end{aligned}$$

One can see by inspection that some terms cancel when these two expressions are added. We need to check show that their sum converges to 0 as μ tends to 0 and infinity, and that the convergence at infinity is exponentially fast. This can be done by hand, using lemma 4.5.3. We omit these calculations, as the expressions involved are complicated. □

It follows immediately from proposition 4.5.2 that the caloron action is finite:

Corollary 4.5.4 The action of the periodic anti-self-dual gauge field generated by $g_\infty(z)$ is absolutely convergent.

We may also use results from proposition 4.5.2 to determine the charge of the caloron.

Proposition 4.5.5 The topological charge k of the periodic vortex generated by $g_\infty(z)$ is absolutely convergent. It is equal to -1. We have $A_0 \to 0$ as $\mu \to 0$, and the monopole charge of the caloron is 0. Therefore, the constituent monopoles of this caloron have charges $(1,1)$ and masses $(0, 2\pi/\beta)$.

Proof:
The proof of this is essentially the same as in the instanton case. We use equation (4.38) of proposition 4.5.2 to establish the absolute convergence of the topological charge. As in the proof of proposition 4.4.4, we have

$$\frac{D\phi}{\phi} - \overline{\left(\frac{D\phi}{\phi}\right)} = 2i(d\theta + a).$$

We integrate over the boundary of the region $0 \leq \mu \leq M$ for some constant M. This consists of two curves located at $\mu = 0$ and $\mu = M$, which we denote Γ_0 and Γ_M. We recall that, since the gauge field is periodic, τ is integrated only between 0 and β.

The integral of the left hand side over Γ_0 is zero, since the integrand is zero (4.36). As M tends to infinity, the integral of the left hand side over Γ_M tends to zero, using equation (4.39). The integral of a over the whole boundary is the same as the integral of da over the region it encloses, and this converges to $2\pi k$ as M tends to infinity.

The integral of $d\theta$ over the whole boundary is the same as 2π times the winding number of the Higgs field over the whole boundary. Since the Higgs field is the product of a real non-zero function $e^{\psi}/|h_\infty|$ and an analytic function $h_\infty \partial_z g_\infty$, this is the same as the number of zeros of the Higgs field in the region it encloses. We see that the Higgs field has one zero on the interior, so we conclude that this integral is 2π when $M \to \infty$. Therefore the vortex charge k is equal to -1.

Let $\mu_1 = \lim_{\mu\to\infty} |A_0|$ and let $\mu_0 = 2\pi/\beta$. It is easily deduced from the formula given earlier for ψ_∞ that

$$\lim_{\mu\to\infty} \partial_\mu \psi = 0,$$

therefore, using (4.14), $\mu_1 = 0$.

The monopole charge k_1 of the caloron can be found by looking at the curvature tensor (4.3). The volume form on S^2 is $\omega = -QdQ \wedge dQ/2$. Hence the component of F on the sphere at infinity is

$$\lim_{\mu\to\infty} -\frac{1}{2}(1 - |\phi|^2)Q\omega = 0.$$

It follows that the Chern numbers $\pm k_1$ are zero.

Recall from (2.7) that

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^3\times S^1} Tr(F \wedge F) = k_0 + \frac{2\mu_1 k_1}{\mu_0}.$$

The left hand side is equal to minus the vortex charge k. Since $k = -1$, the left hand side is 1. Since $\mu_1 = 0$ the instanton charge $k_0 = 1$. The monopole charges and masses are read off from the formulae (2.8), (2.9). □

4.5.3 Relation between integral and non-integral hyperbolic calorons

A reasonable requirement for the family of non-integral hyperbolic calorons is that it includes the family of integral hyperbolic calorons. In this subsection we verify that this is the case.

Let A_C denote the integral hyperbolic caloron generated by the function $g_C(z)$ (4.32). A_C depends on two parameters, α and S. We will consider α in the range

$(0,1)$, since this corresponds to the components of the caloron being located at times $nS\pi$ (in preference to $(n+\frac{1}{2})S\pi$, which happens when $\alpha > 1$). The period of A_C is $S\pi$. Let A_∞ be the hyperbolic caloron generated by the function $g_\infty(z)$ (4.34) with gauge transformation $h_\infty(z)$. A_∞ depends on three parameters: S, β, and λ. We will find that A_∞ and A_C are gauge equivalent when

$$\begin{aligned} \beta &= S\pi, \\ \alpha^2 &= \tanh\frac{\lambda}{2}. \end{aligned}$$

This result is derived by using the fact that the zeros of the Higgs field are gauge invariant. The zeros of the Higgs field of the gauge field A_C are located at

$$\begin{aligned} \tau &= nS\pi, \quad n \in \mathbb{Z} \\ \mu &= S\,\mathrm{arctanh}\,\alpha. \end{aligned}$$

For the gauge field A_∞, the zeros are

$$\begin{aligned} \tau &= n\beta, \quad n \in \mathbb{Z} \\ \cosh\frac{2\pi\mu}{\beta} &= \cosh\frac{S\pi\lambda}{\beta} + \frac{S\pi}{\beta}\sinh\frac{S\pi\lambda}{\beta}. \end{aligned}$$

By requiring that the zeros of the Higgs field agree for A_C and A_∞, we arrive at the two equations relating the parameters β, S, α and λ which were given above.

Once the zeros of the Higgs fields agree for both calorons, they must be gauge equivalent. This is because both calorons are in fact Euclidean instantons, and we know that symmetric Euclidean instantons are completely determined by the zeros of their Higgs field [Wit77]. For explicitness, though, we give the precise gauge transformation taking A_C to A_∞ (when their parameters are appropriately related)

Recall that gauge transfomations of solutions were expressed by an analytic function h, with $\psi \mapsto \psi + \ln|h|$ and $\partial_z g \mapsto h\partial_z g$. So the gauge transformation taking A_C to A_∞ is $h = h_\infty \partial_z g_\infty / \partial_z g_C$. Explicitly,

$$h = -\frac{1+\alpha^2}{2\alpha}\frac{(\alpha\cosh(z/S)+\sinh(z/S))^4}{(\alpha^2\cosh(z/S)+\sinh(z/S))^2}$$

We check that the gauge transformation h also has the correct relation with ψ. For the solution A_∞, ψ is given by

$$\exp(-\psi_\infty) = \frac{S}{2}\frac{(1+\alpha^4)\cosh(2\mu/S) + 2\alpha^2\sinh(2\mu/S) + (\alpha^4-1)\cos(2\tau/S)}{(1+\alpha^2)((\alpha^2+1)\cosh(2\mu/S) + (\alpha^2-1)\cos(2\tau/S))}.$$

For A_C, ψ is given by

$$\exp(-\psi_C) = \frac{2S\left((\alpha^2+1)\cosh(2\mu/S) + (\alpha^2-1)\cos(2\tau/S) + 2\alpha\sinh(2\mu/S)\right)^2}{(\alpha^2+1)\cosh(2\mu/S) + (\alpha^2-1)\cos(2\tau/S)}.$$

Finally, $|h|$ is given by

$$|h| = \frac{1+\alpha^2}{4}\frac{\left((\alpha^2+1)\cosh(2\mu/S) + (\alpha^2-1)\cos(2\tau/S) + 2\alpha\sinh(2\mu/S)\right)^2}{(\alpha^4+1)\cosh(2\mu/S) + (\alpha^4-1)\cos(2\tau/S) + 2\alpha^2\sinh(2\mu/S)}.$$

From these formulae, it is clear that

$$\exp(-\psi_\infty) = \frac{\exp(-\psi_C)}{|h|},$$

which is the correct relation.

4.6 Taking limits of parameters

The family of calorons we have constructed depends on three parameters. They are: S, proportional to the radius of curvature of the hyperbolic space; β, the period of the caloron; and λ, which is related to the size of the caloron. In this section we will consider solutions obtained in various limits of these parameters. We will show how to take the flat space limit of a hyperbolic caloron to obtain a Euclidean caloron. We also show that the infinite period limit of a hyperbolic caloron is a hyperbolic instanton, and the infinite scale limit is a hyperbolic monopole. Thus the family of hyperbolic calorons constructed is directly analogous to the family of charge 1 Euclidean calorons which interpolates between instantons and monopoles.

4.6.1 Flat space limit

The limit as S tends to infinity of the hyperbolic metric ds_H^2 is the ordinary Euclidean metric. So one would hope that it is possible to find Euclidean calorons as some limit as S tends to infinity of the hyperbolic caloron A_∞. This turns out to be possible. The trick is to make sure the caloron retains finite size as S increases, or equivalently in the vortex picture, to fix the location of the vortex while S increases.

Recall that the locations of the vortices are given by:

$$\begin{aligned} \tau &= n\beta, \quad n \in \mathbb{Z} \\ \cosh\frac{2\pi\mu}{\beta} &= \cosh\frac{S\pi\lambda}{\beta} + \frac{S\pi}{\beta}\sinh\frac{S\pi\lambda}{\beta}. \end{aligned}$$

These will remain finite as S tends to infinity if β and λS^2 remain fixed. In this limit the second equation becomes

$$\cosh\frac{2\pi\mu}{\beta} = 1 + \left(\frac{\pi}{\beta}\right)^2 \lambda S^2.$$

We now consider the Higgs field and the $U(1)$ gauge field in this limit. To keep these expressions well-behaved, we make an additional gauge transformation, $h(z) = S/2$. We find

$$\begin{aligned}
\frac{S}{2} h_\infty \partial_z g_\infty &\to -1 + \frac{\pi^2 \lambda S^2}{2\beta^2} \mathrm{cosech}^2(\pi z/\beta) \\
\frac{S}{2}\exp(\psi_\infty) &\to 1 + \frac{\pi\lambda S^2}{2\beta\mu} \frac{\sinh(2\pi\mu/\beta)}{\cosh(2\pi\mu/\beta) - \cos(2\pi\tau/\beta)}.
\end{aligned}$$

We want to compare this limit with the caloron of Harrington and Shepherd [HS78]. Recall from (2.15) that Harrington and Shepherd's caloron was given in the CF'tH ansatz, with

$$\rho = 1 + \frac{\pi\nu^2}{\beta\mu} \frac{\sinh(2\pi\mu/\beta)}{\cosh(2\pi\mu/\beta) - \cos(2\pi\tau/\beta)}.$$

It is simple to convert this to vortex form. Using methods from Manton's paper [Man78], we have:

$$\begin{aligned}
\psi &= \ln\rho \\
\phi &= e^{-\psi} 2i\partial_z(\mu\rho).
\end{aligned}$$

For the Harrington and Shepherd caloron,

$$\begin{aligned}
e^\psi &= 1 + \frac{\pi\nu^2}{\beta\mu} \frac{\sinh(2\pi\mu/\beta)}{\cosh(2\pi\mu/\beta) - \cos(2\pi\tau/\beta)} \\
2i\partial_z(\mu\rho) &= i - i\frac{\pi^2\nu^2}{\beta^2}\mathrm{cosech}^2\frac{\pi z}{\beta}.
\end{aligned}$$

So, by directly comparing ϕ and ψ, one can see that the flat space limit of the hyperbolic caloron is (a gauge rotation of) the Harrington-Shepherd caloron, with

$$\nu = S\sqrt{\frac{\lambda}{2}}.$$

4.6.2 Instanton limit

In the Euclidean case, an instanton could be obtained as the limit of a caloron as its period tends to infinity. We will show that this is true in the hyperbolic case too. We show that the limit as β tends to infinity of the hyperbolic caloron A_∞ is a hyperbolic instanton, provided we keep S and λ fixed. In this limit, the location of the vortex at $\tau = 0$ tends to

$$\begin{aligned} \tau &= 0 \\ \mu &= \frac{S}{2}\sqrt{\lambda^2 + 2\lambda} \end{aligned}$$

By comparing vortex locations, this solution should be gauge equivalent to the charge one instanton generated by the function

$$g_1(z) = \exp\left(-\frac{2z}{S}\right)\frac{\lambda - (2z/S)}{\lambda + (2z/S)}.$$

We now verify this. Recall that, for the hyperbolic instanton,

$$\begin{aligned} \partial_z g_1 &= \frac{2}{S}\exp\left(-\frac{2z}{S}\right)\frac{(2z/S)^2 - (\lambda^2 + 2\lambda)}{(2z/S + \lambda)^2} \\ \exp(-\psi_1) &= \frac{S}{2}\exp\left(\frac{2\mu}{S}\right)\frac{(2\mu/S + \lambda)^2 + (2\tau/S)^2}{(2\mu/S)^2 + (2\tau/S)^2 + \lambda^2 + 2\lambda(2\mu/S)\coth(2\mu/S)}. \end{aligned}$$

If we consider the hyperbolic caloron without the gauge transformation h_∞, we find $\partial_z g_\infty \to -\partial_z g_1$ and $\psi_\infty \to \psi_1$ pointwise as β tends to infinity. So the instanton limit holds as expected, (note that there is a gauge rotation $h = -1$).

4.6.3 Monopole limit

The monopole limit of a Euclidean caloron was obtained by fixing the period of the caloron and allowing its scale to approach infinity. So we hope to obtain a hyperbolic monopole from a hyperbolic caloron by fixing β and S, and letting λ approach infinity.

To do so, we first need to express the hyperbolic monopole of Nash [Nas86] in the notation of this article. Nash's solution is equivalent to:

$$\begin{aligned} g_M(z) &= \exp\left(-\frac{2Bz}{S}\right) \\ h_M(z) &= \frac{i}{\partial_z g_M}. \end{aligned}$$

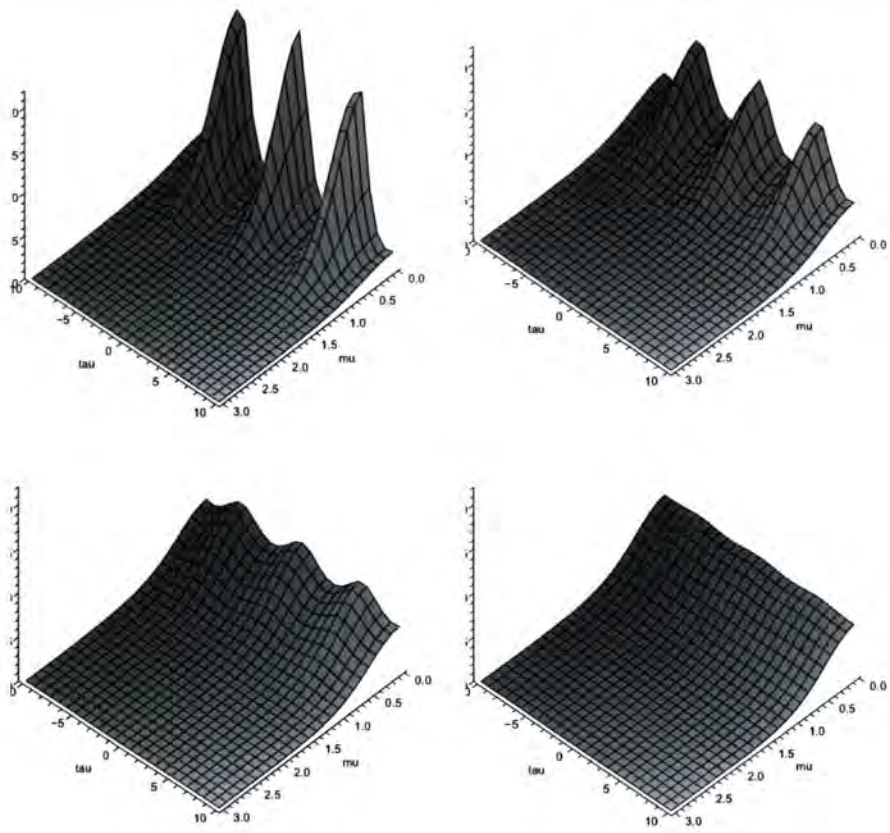

Figure 4.2: Graphs of $\|F\|^2$ as a function of μ and τ for hyperbolic calorons with $S = 2$, $\beta = 2\pi$ and $\lambda = 1.2, 2, 5$ and 10.

One may derive from these

$$\begin{aligned} \exp(-\psi_M) &= \frac{B\sinh(2\mu/S)}{\sinh(2B\mu/S)} \\ \phi &= ie^{-\psi}. \end{aligned}$$

The limit as λ tends to infinity of the caloron A_∞ is

$$\begin{aligned} h_\infty \partial_z g_\infty &\to \frac{2}{S}\left(1+\frac{S\pi}{\beta}\right)\exp\left(-\frac{2\pi z}{\beta}\right) \\ \exp(-\psi_\infty) &\to \frac{S}{2}\exp\left(\frac{2\pi\mu}{\beta}\right)\frac{\sinh(2\mu/S)}{\sinh((1+S\pi/\beta)(2\mu/S))}. \end{aligned}$$

If we gauge transform by $h = (iS/2)(1+S\pi/\beta)^{-1}\exp(2\pi z/\beta)$, this becomes identical to Nash's monopole, with

$$B = 1 + \frac{S\pi}{\beta}.$$

So by varying β and S, all symmetric charge 1 hyperbolic monopoles can be obtained as limits of symmetric charge 1 calorons.

4.7 Summary and open problems

We have demonstrated the existence of hyperbolic calorons by explicitly constructing them. Although our calorons are related to monopoles as one might expect, they do not exhibit the full constituent structure described in chapter 2, because they do not have maximal symmetry breaking. It remains to be seen whether hyperbolic calorons can be constructed satisfying more general boundary conditions. Following the example of Euclidean calorons, more general hyperbolic calorons might be constructed using a Nahm transform – but such a Nahm transform is currently not known.

In view of the emerging theory of Nahm transforms [Jar04], it would be interesting to investigate the possibility that a Nahm transform exists for hyperbolic calorons. Currently, a Nahm transform is known for hyperbolic monopoles, but only in the integral case [BA90]. Closely related is the problem of finding a Nahm transform for singular instantons. Instantons can in principle exist with mild singularities on a two-sphere – the types of singularity were classified in [SS92, Rad94], and more or less correspond to the types of boundary behaviour displayed by calorons. Only a few examples of such instantons exist [FHP81, FHP82]; in our view it seems quite possible that more could be shown to exist using some kind of Nahm transform.

We mention briefly a loop group interpretation for hyperbolic calorons. By analogy with the Euclidean case, we expect that a hyperbolic caloron can be identified as a hyperbolic monopole whose gauge group is a loop group. This monopole could be mapped conformally to an axially symmetric instanton whose gauge group is a loop group, subject to some integer-mass condition. So hyperbolic calorons could be used to study loop group instantons. Loop group instantons would be interesting to study, both for their novelty, and because they might tell us something about higher dimensional gauge theories.

A final open question concerns the boundary behaviour of hyperbolic calorons.

It was recently proved that hyperbolic monopoles are completely determined by the connection they induce on the sphere at infinity [Nor04], an example of a "holographic principle". It is natural to ask whether the same is true of calorons: alas, it appears not to be for our calorons. The connection induced on the sphere at infinity by our calorons is trivial, so cannot determine the caloron. If one applies a rotation map (a large gauge transformation), the caloron looks like a monopole at infinity, but even then the connection at infinity only appears to detect the location of the caloron, and not its scale. The failure of the holographic principle may be related to the degeneracy of the boundary condition: perhaps more general hyperbolic calorons are determined by their boundary values?

Chapter 5

Chains of planar skyrmions

In this chapter we shall consider chains in the planar (or baby) Skyrme model. The planar Skyrme model is essentially a generalisation of the $\mathbb{CP}^1$ sigma model (which will be studied in chapter 7): two terms are added to the sigma model Lagrangian which break integrability and conformal invariance. The planar Skyrme model was invented as a two-dimensional toy model for the three-dimensional Skyrme model. In a later chapter we will study chains in the Skyrme model, so this chapter represents a useful first step.

Our approach in this chapter is essentially analytic. We will consider a range of ansätze which are expected to make a good approximation to true minima of the planar Skyrme energy. However, since this model is not integrable, it will ultimately be necessary to compare our results with numerical simulations. Such simulations have been carried out by my supervisor, Prof. R. S. Ward, and appeared in our joint publication [HW08b].

An important feature of the planar Skyrme model is that it includes a potential function, which may be chosen arbitrarily. Different choices of potential function can result in different behaviour of the model, so it is important to consider a range of potentials. Ideally, one would like to make statements which depend only on a few qualitative features of the potential, but to do so is a hard problem which we do not address here.

We begin this chapter by introducing chains in the planar Skyrme model. We shall give a detailed account of the topology, since this is not standard. We shall

exhibit two lower bounds on the energies of chains: the first is the standard Bogomolny bound which depends only on the topology; the second depends on the period, and in particular diverges for small periods. After considering non-standard potentials for which the field equations have explicit solutions, we consider in the last two sections properties of chains for two standard choices of potential function.

5.1 The planar Skyrme model

The field content of the planar Skyrme model on a Riemannian manifold M of dimension two is a continuous function ϕ from M to S^2, called a planar Skyrme field. In components, we write $\phi = (\phi_1, \phi_2, \phi_3)$ with $\phi_1^2 + \phi_2^2 + \phi_3^2 = 1$. Let ν be the volume form on M, and let ω be the volume form on S^2. The planar Skyrme energy functional is

$$\begin{aligned} E[\phi] &= E_2[\phi] + \alpha_4 E_4[\phi] + \alpha_0 E_0[\phi] \\ E_2[\phi] &= \int_M \frac{1}{2}\|d\phi\|^2 \nu \\ E_4[\phi] &= \int_M \frac{1}{2}(\star_M(\phi^*\omega))^2 \nu \\ E_0[\phi] &= \int_M V(\phi)\nu. \end{aligned}$$

Here V is a real function on S^2 called the potential function, and the real α_0, α_4 are parameters of the model. Normally E_2 is called the sigma model term, E_4 is called the Skyrme term and E_0 is called the potential term. The planar Skyrme model was invented as a two-dimensional analog of the three-dimensional Skyrme model, and each term in the energy functional has a three-dimensional counterpart. Notice that, unlike in three dimensions, the potential term is indispensible here: without it, Derrick's theorem rules out the existence of static solitons on $\mathbb{R}^2$.

The choice of V is arbitrary, but the two most commonly studied choices are:

- $V(\phi) = (1 - \phi_3)$ (the old baby Skyrme model, see [PSZ95])
- $V(\phi) = \frac{1}{2}(1 - \phi_3^2)$ (the new baby Skyrme model, see [KPZ98, Wei99]).

Notice that the energy densities $\mathcal{E}_2, \mathcal{E}_4$ are invariant under $O(3)$ rotations of S^2, called isorotations. The choice of potential V breaks this symmetry; in the old

baby Skyrme model the isorotation group is broken to $O(2)$, while in the new baby Skyrme model it is broken to $O(2) \times \mathbb{Z}_2$ (where $\mathbb{Z}_2$ describes reflections which swap the poles $\phi_3 = \pm 1$).

In the case where $H^2(M) = \mathbb{Z}$ the map ϕ has a degree, denoted B and called the charge of ϕ. By the degree theorem,

$$B \int_{S^2} \omega' = \int_M \phi^* \omega'$$

for any volume form ω' on S^2. The energy is bounded from below by a multiple of the charge, as we now show. Let $R = R_1 + iR_2$ be the complex function on M defined by stereographic projection:

$$R = \frac{\phi_1 + i\phi_2}{1 + \phi_3} \tag{5.1}$$

$$\vec{\phi} = \left(\frac{2R_1}{|R|^2 + 1}, \frac{2R_2}{|R|^2 + 1}, \frac{1 - |R|^2}{|R|^2 + 1} \right). \tag{5.2}$$

Let z be a complex coordinate on M and such that the metric and volume form on M are $g = h/2(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ and $\nu = (ih/2)dz \wedge d\bar{z}$. We have identities

$$\frac{1}{2}\|d\phi\|^2 = \frac{4}{h} \frac{|\partial_z R|^2 + |\partial_{\bar{z}} R|^2}{(|R|^2 + 1)^2}$$

$$\star_M \phi^* \omega = \frac{4}{h} \frac{|\partial_z R|^2 - |\partial_{\bar{z}} R|^2}{(|R|^2 + 1)^2}$$

By integrating and applying the degree theorem, we obtain

$$E_2 \geq 4\pi B \tag{5.3}$$

with equality if and only if

$$\partial_{\bar{z}} R = 0. \tag{5.4}$$

On the other hand, we have an identity

$$\frac{\alpha_4}{2}(\star_M(\phi^*\omega))^2 + \alpha_0 V(\phi) = \frac{1}{2}\left(\sqrt{\alpha_4} \star_M (\phi^*\omega) - \sqrt{2\alpha_0 V}\right)^2 + \star_M \phi^* \left(\sqrt{2\alpha_0\alpha_4 V}\omega\right).$$

Integrating and applying the degree theorem, we obtain

$$\alpha_4 E_4 + \alpha_0 E_0 \geq B \int_{S^2} \sqrt{2\alpha_0\alpha_4 V}\omega. \tag{5.5}$$

This bound is saturated provided

$$\star_M(\phi^*\omega) = \sqrt{2\alpha_0 V/\alpha_4} \tag{5.6}$$

In summary,

$$E \geq B\left(4\pi + \int_{S^2} \sqrt{2\alpha_0\alpha_4 V}\omega\right) \tag{5.7}$$

In the case where $M = \mathbb{R}^2$, we have $E_i = \int_{\mathbb{R}^2} \mathcal{E}_i d^2x$ and $\mathcal{B} = \int_{\mathbb{R}^2} \mathcal{B} d^2x$, where

$$\begin{aligned}
\mathcal{B} &= \frac{1}{4\pi}\phi \cdot \partial_1\phi \times \partial_2\phi && (5.8)\\
\mathcal{E}_2 &= \frac{1}{2}(\partial_i\phi \cdot \partial_i\phi) && (5.9)\\
\mathcal{E}_4 &= \frac{1}{2}(\partial_1\phi \times \partial_2\phi)^2 && (5.10)\\
&= 8\pi^2\mathcal{B}^2 &&\\
\mathcal{E}_0 &= V(\phi). && (5.11)
\end{aligned}$$

For convenience, we also set

$$\mathcal{E} = \mathcal{E}_2 + \alpha_4\mathcal{E}_4 + \alpha_0\mathcal{E}_0. \tag{5.12}$$

In the case $M = \mathbb{R}^2$, the coefficients α_0, α_4 can be changed by rescaling the coordinates x^1, x^2. Therefore it is convenient to use this degeneracy to fix $\alpha_0 = \alpha_4 = \mu$ for a real parameter μ. The bound (5.7) applies to the planar skyrmions on $\mathbb{R}^2$, since finite-energy fields on $\mathbb{R}^2$ must tend to a constant at infinity, hence can be extended to maps from S^2 to S^2.

A finite-energy map $\phi : \mathbb{R}^2 \to S^2$ which minimizes the energy functional will be called a planar skyrmion. Only one charge 1 planar skyrmion is known in each of the old and new baby Skyrme models. They may be expressed in polar coordinates (r, θ) in the form of the hedgehog ansatz:

$$\phi = (\sin f(r)\cos(\theta - \chi),\ \sin f(r)\sin(\theta - \chi),\ \cos f(r)). \tag{5.13}$$

Here $\chi \in [0, 2\pi)$ is an angle and $f(r)$ is a function satisfying $f(0) = \pi$ and $f(r) \to 0$ as $r \to \infty$, which must be chosen to minimize E.

Notice that the hedgehog ansatz is symmetric: a spatial $SO(2)$ rotation can be compensated by a isorotation of the 2-sphere. So the orientation parameter χ might appear to be unimportant. However, we shall see later that when multiple planar skyrmions are placed next to each other, their relative orientations determine the forces between them, so it makes sense to leave χ in equation (5.13) for now.

We point out here that the charge one skyrmion generally fails to saturate the Bogomolny bounds (5.3), (5.5). The reason is that saturation of each bound requires solution of a first order differential equation, and the two differential equations (5.4), (5.6) are not in general compatible. It is interesting to ask the question: by how much does the skyrmion fail to saturate the Bogomolny bound? The answer takes the form of a graph, such as the solid curve in figure 5.1. There we have plotted the fractional excess of the energy E of a charge 1 skyrmion over its Bogomolny bound $E_{min} = 4\pi(1 + 4\mu/3)$, as a function of $\log \mu$. We worked with the old baby Skyrme potential. It is remarkable that the curve has such a symmetrical shape, and we also note that the excess becomes negligible for very large and very small values of μ. The $\mu = 0$ limit is the $\mathbb{CP}^1$ sigma model, while the $\mu = \infty$ theory was studied in [GP97]. In both limits, the Bogomolny bound is saturated although in the latter case, one must sacrifice smoothness of functions. (We note that the authors of [GP97] erroneously claim not to have saturated the Bogomolny bound: their mistake was to use a Bogomolny bound which is weaker than the usual one.)

In order to understand the shape of the curve better, we make a simple analogy. We replace fields ϕ by real numbers x and the fractional excess energy $(E[\phi] - E_{min})/E_{min}$ by the function

$$f : x \mapsto \frac{\alpha}{1+\gamma\mu}(x+1)^2 + \frac{\beta\mu}{1+\gamma\mu}(x-1)^2,$$

where $\mu, \alpha, \beta, \gamma$ are positive real parameters. This function is obviously bounded below by 0, but the bound is not saturated, since the "Bogomolny equations", $x = 1$ and $x = -1$, are not compatible. The true minimum of f is

$$\frac{4\alpha\beta\mu}{(1+\gamma\mu)(\alpha+\beta\mu)}.$$

The dashed curve in figure 5.1 is the graph of this function, with the parameters set to $\alpha = 0.042, \beta = 0.29, \gamma = 0.44$. We observe that this curve is a surprisingly good match for the graph of $(E - E_{min})/E_{min}$.

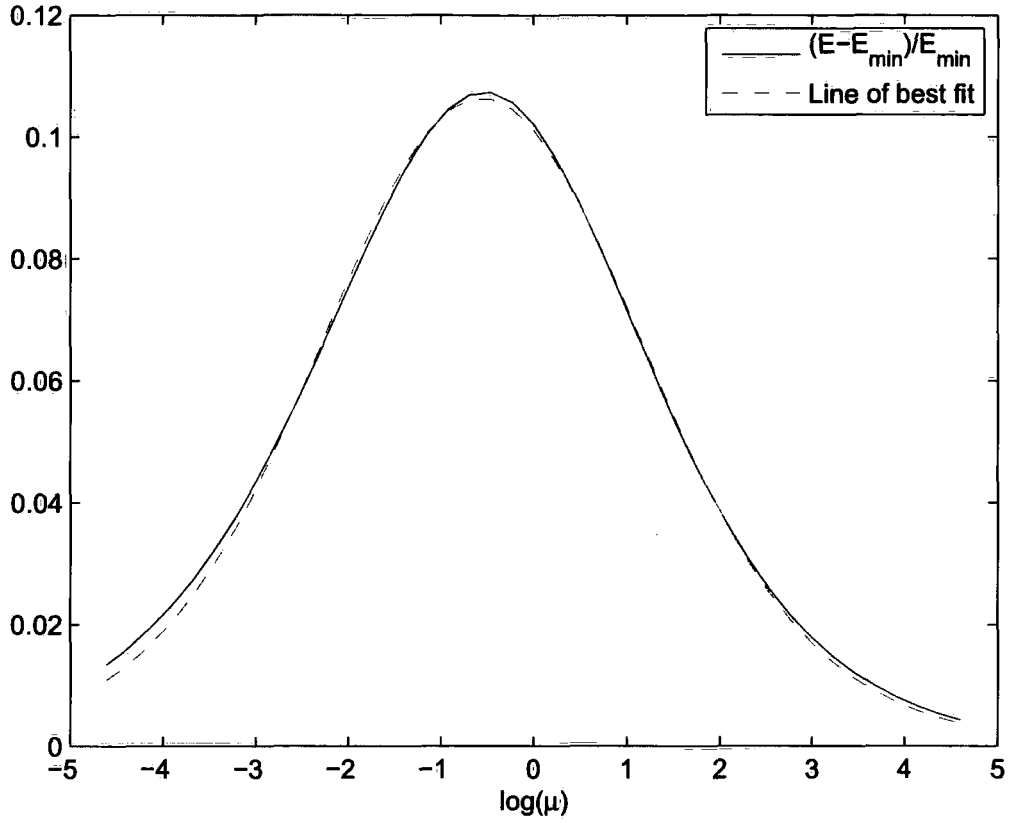

Figure 5.1: The quantity $(E - E_{min})/E_{min}$ as a function of $\log \mu$ for a 1-skyrmion with the old baby Skyrme potential.

5.2 Chains and their topology

We will be interested in periodic chains of planar skyrmions with a relative orientation between neighbouring planar skyrmions. These will be described by planar Skyrme fields ϕ periodic up to an isorotation. Let the period be $\beta > 0$, and let the isorotation be $\sigma \in SO(2)$. Then the planar Skyrme field is required to satisfy:

$$\phi(x^1, x^2 + \beta) = \sigma\phi(x^1, x^2). \tag{5.14}$$

In addition, we impose the boundary condition:

$$\phi(x^1, x^2) \to (0, 0, 1) \text{ as } x^1 \to \pm\infty. \tag{5.15}$$

In contrast, a domain wall in the planar Skyrme model is required to satisfy

$$\phi(x^1, x^2) \to (0, 0, \pm 1) \text{ as } x^1 \to \pm\infty. \tag{5.16}$$

Two special cases are as follows: if σ is the identity we call ϕ periodic, while if σ is not equal to the identity but σ^2 is we call the chain anti-periodic. Clearly, the only anti-periodic boundary condition is $\sigma = -Id$.

The energy and charge per unit length of a chain are computed by

$$\begin{aligned} E[\phi] &= \int_0^\beta \int_{-\infty}^{\infty} \mathcal{E}[\phi]\, dx^1\, dx^2 \\ B[\phi] &= \int_0^\beta \int_{-\infty}^{\infty} \mathcal{B}[\phi]\, dx^1\, dx^2. \end{aligned}$$

An important question to answer is whether the quantity B is guaranteed to be an integer. In the case where σ is the identity this is relatively simple to see: the boundary conditions ensure that ϕ can be extended to a map from T^2 to S^2. Since the second cohomology groups of these manifolds are both $\mathbb{Z}$, ϕ has a degree defined in the usual way, and the degree is computed by the integral B. In the general case this argument fails, but one still might expect B to be an integer, since ϕ is supposed to represent a chain of particle-like solitons.

It turns out that B is guaranteed to be an integer for all isorotations σ. This is a special case of the following theorem:

Theorem 5.2.1 Let Σ be an n-dimensional compact manifold without boundary, with volume form ω, and with $H^{n-1}(\Sigma) = 0$ and $H^n(\Sigma) = \mathbb{Z}$. Let M be an $(n-1)$-dimensional manifold such that $H^n(M \times S^1) = \mathbb{Z}$. Suppose that $SO(2)$ acts on Σ and that ω is $SO(2)$-invariant. Fix an element $\sigma \in SO(2)$ and let $\phi : M \times \mathbb{R} \to \Sigma$ be a map satisfying

$$\phi(x, y + \beta) = \sigma\phi(x, y)\ \forall x \in M, y \in \mathbb{R}.$$

Then ϕ has an integer degree, computed by the integral,

$$deg(\phi) = \frac{1}{Vol(\Sigma)} \int_0^\beta \int_M \phi^*\omega.$$

Furthermore, $deg(\phi)$ is independent of the choice of $SO(2)$-invariant volume form ω.

Proof: The idea of the proof is simple: we deform ϕ to a strictly periodic map using the $SO(2)$ action, and show that the integral is unchanged by this deformation.

First, we introduce some notation: we write the $SO(2)$ action as $R_s : \Sigma \to \Sigma$, with $s \in \mathbb{R}/\mathbb{Z} \cong SO(2)$. Let $X \in T\Sigma$ be the associated vector field. Let $t_0 \in \mathbb{R}/\mathbb{Z}$ be such that $R_{t_0} = \sigma^{-1}$, and let $t(x,y) = t_0 y/\beta$ be a function on $M \times \mathbb{R}$. We define a deformation,

$$\tilde{\phi}(x,y) = R_{t(x,y)} \circ \phi(x,y),\ x \in M, y \in \mathbb{R}.$$

Then $\tilde{\phi}$ is a strictly periodic map, hence has a degree computed by

$$deg(\tilde{\phi}) = \frac{1}{Vol(\Sigma)} \int_0^\beta \int_M \tilde{\phi}^* \omega.$$

Now we show that

$$\int_0^\beta \int_M \tilde{\phi}^* \omega = \int_0^\beta \int_M \phi^* \omega.$$

For any form $\theta \in \Lambda^* \Sigma$, one can show that

$$\tilde{\phi}^* \theta = \phi^* R_t^* \theta + \phi^* (i_X R_t^* \theta) \wedge dt.$$

Here i_X denotes the inner derivative of a form. In the particular case $\theta = \omega$, one has $R_t^* \omega = \omega$ (because the volume form is $SO(2)$-invariant). Hence

$$\tilde{\phi}^* \omega = \phi^* \omega + \phi^* (i_X \omega) \wedge dt.$$

By Cartan's formula, we have

$$L_X \omega = i_X d\omega + d i_X \omega.$$

where L_X denotes the Lie derivative. We see immediately that $d\omega = 0$, because $\omega \in \Lambda^n \Sigma$. On the other hand, $L_X \omega$ must vanish since ω is $SO(2)$-invariant. It follows that $i_X \omega$ is closed. Since $H^{n-1}(\Sigma) = 0$, $i_X \omega$ is exact, in other words, there exists a $\mu \in \Lambda^{n-2} \Sigma$ such that $i_X \omega = d\mu$. Therefore,

$$\begin{aligned} \tilde{\phi}^* \omega &= \phi^* \omega + \phi^* (d\mu) \wedge dt \\ &= \phi^* \omega + d(\phi^* \mu \wedge dt). \end{aligned}$$

Integrating and applying Stoke's theorem, we obtain

$$\int_0^\beta \int_M \tilde{\phi}^* \omega = \int_0^\beta \int_M \phi^* \omega + \left[\int_M \phi^* \mu \wedge dt \right]_0^\beta.$$

The boundary term vanishes since t is constant on the domain of integration, so we have the desired result.

That $deg(\phi)$ is independent of the choice of volume form follows from the corresponding property of the classical degree. If ω' is any other $SO(2)$-invariant volume form, then

$$\begin{aligned}\int_0^\beta \int_M \phi^*\omega' &= \int_0^\beta \int_M \tilde{\phi}^*\omega' \\ &= \int_0^\beta \int_M \tilde{\phi}^*\omega \\ &= \int_0^\beta \int_M \phi^*\omega.\end{aligned}$$

□

An useful corrollary of this theorem is that the lower bounds (5.3), (5.5) apply to chains, but the latter applies only in the case where the potential function $V(\phi)$ is $SO(2)$-symmetric. We stress that it is important that $B = deg(\phi)$ is independent of the choice of volume form, since the bounds (5.3), (5.5) are obtained using two different volume forms. It would be interesting to know if the degree $deg(\phi)$ can be defined without making a choice of volume form – this has not been done, as far as we are aware.

5.3 A period-dependent lower bound on the energy

It is interesting to consider chains in the limit where β is very small. For comparitive purposes, we first consider the planar Skyrme model on a manifold M with area form ν and with finite area. By the Cauchy-Schwarz inequality, one obtains

$$\int_M \nu \int_M (\star\phi^*\omega)^2\nu \geq \left(\int_M (\star\phi^*\omega)\nu\right)^2,$$

so there is a lower bound on the energy:

$$E \geq \alpha_4 E_4 \geq \frac{\alpha_4}{2}\frac{B^2\, Area(S^2)^2}{Area(M)}.$$

When M has large area, this lower bound is less than the standard Bogomolny lower bound. But as the area of M tends to zero, this lower bound diverges. Heuristically, one can say that squashing the Skyrme field into a small area has the effect of increasing the Skyrme term. We note that an analogous result holds in the full Skyrme model [Man87]. See [War04a, dIW01] for studies of planar skyrmions on compact manifolds.

Of course, the above argument says nothing about chains, because the area of $M = \mathbb{R} \times S^1$ is infinite. But one still might expect the energy of a chain to be large for small periods. One thing is clear: any lower bound which diverges for small periods must involve more than just the Skyrme term. This is because the Skyrme term is invariant under area-preserving maps. Given a chain $\phi(x, y)$ with period β, one can define a chain $\tilde{\phi}(x, y) = \phi(x/\lambda, y\lambda)$ which has period β/λ but the same Skyrme energy. So there cannot be a lower bound on the Skyrme energy that diverges as the period becomes small.

Now we shall prove the existence of a lower bound on the energy of a planar skyrmion chain which does diverge for small periods. Our method will be to consider both the Skyrme term and the sigma model term. We will first consider the case where both the domain and target of the Skyrme field are diffeomorphic to compact cylinders, and find a lower bound on the energy. Then we will show that provided a Skyrme field has non-zero winding number, there will exist regions in $\mathbb{R} \times S^1$ such that, when the field is restricted to these regions, the previously derived lower bound applies.

Lemma 5.3.1 Let $M = [0, W] \times S^1$ with Euclidean metric, let β be the circumference of S^1, and let Σ be a Riemannian 2-manifold diffeormorphic to M. Let $\phi : M \to \Sigma$ such that $\phi(\partial M) = \partial\Sigma$, and with degree 1. Suppose there exists a $\lambda > 0$ such that for all $x \in [0, W]$, the curve $\phi(x, \cdot)$ in Σ has length greater than or equal to λ. Then the energy of ϕ satisfies

$$E[\phi] \geq \frac{\sqrt{\mu}\,\lambda\, Area(\Sigma)}{\beta}.$$

Proof: Firstly, by the preceding argument, we have

$$E_4 \geq \frac{Area(\Sigma)^2}{2W\beta}.$$

Secondly, by the Cauchy-Schwarz inequality we have

$$\int_{[0,W]\times S^1} dx\,dy \int_{[0,W]\times S^1} |\partial_y\phi|^2 dx\,dy \geq \left(\int_{[0,W]\times S^1} |\partial_y\phi| dx\,dy\right)^2.$$

The right hand side is the square of $\int_0^W L[\phi(x,\cdot)]dx$, where

$$L[\gamma] := \int_{S^1} |\partial_t\gamma| dt$$

is the integral which computes the length of a closed curve γ. So by assumption, the right hand side is greater than or equal to $(W\lambda)^2$. Therefore

$$\begin{aligned} E_2 &\geq \frac{1}{2}\int_{[0,W]\times S^1} |\partial_y\phi|^2 dx\,dy \\ &\geq \frac{W\lambda^2}{2\beta}. \end{aligned}$$

Combining the above two inequalities, we have

$$\begin{aligned} E_2 + \mu E_4 &\geq \frac{W\lambda^2}{2\beta} + \frac{\mu\, Area(\Sigma)^2}{2W\beta} \\ &\geq \frac{\sqrt{\mu}\,\lambda\, Area(\Sigma)}{\beta}. \end{aligned}$$

□

In order to relate this lemma to spheres, we make the following definition:

Definition 5.3.2 Let $[a,b]$ be any interval and let $\psi : [a,b]\times S^1 \to S^2$ such that the curves $\psi(a,\cdot), \psi(b,\cdot)$ have length $\lambda \in (0, 2\pi)$. Let $\epsilon > 0$ and let $\tilde{\psi} : [a-\epsilon, b+\epsilon]\times S^1 \to S^2$ be an extension of ψ such that

$$\begin{aligned} \tilde{\psi}(x,y) &= \psi(x,y) \text{ for } x \in [a,b] \\ L[\tilde{\psi}(x,\cdot)] &\leq \lambda \text{ for } x \in [a-\epsilon, a] \cup [b, b+\epsilon] \\ \tilde{\psi}(a-\epsilon, y) &= p\ \forall y \in S^1 \\ \tilde{\psi}(b+\epsilon, y) &= q\ \forall y \in S^1, \end{aligned}$$

where p, q are two arbitrary points in S^2. Clearly $\tilde{\psi}$ is a map from S^2 to S^2, and hence has a winding number $w \in \pi_2(S^2) \cong \mathbb{Z}$. We define $W[\psi] := w$.

Proposition 5.3.3 $W[\psi]$ is independent of the choice of extension. W is additive in the following sense: let $a < b < c < d$ and let $\psi : [a,d]\times S^1 \to S^2$ be such that $L[\psi(x,\cdot)] = \lambda$ for $x = a, b, c, d$ and $L[\psi(x,\cdot)] \leq \lambda$ for $x \in [b,c]$. Then

$$W[\psi] = W[\psi|_{[a,b]\times S^1}] + W[\psi|_{[c,d]\times S^1}].$$

Proof: To see this, suppose that $\tilde{\psi}_1, \tilde{\psi}_2$ are two extensions of ψ. We need to show that they are homotopic. Consider the restrictions of $\tilde{\psi}_1, \tilde{\psi}_2$ to the interval $[a-\epsilon, a]$. These are two maps from D^2 to S^2 which agree on the boundary ∂D^2. We can glue them along their boundary to obtain a map $\chi : S^2 \to S^2$; if χ is contractible then the two restricted maps are homotopic.

We can show that this map is contractible by making use of the condition, $L[\tilde{\psi}_i(x,\cdot)] \leq \lambda < 2\pi \ \forall x \in [a-\epsilon, a]$. It follows from this condition that each curve $\tilde{\psi}_i(x,\cdot)$ lies within a hemisphere in S^2; hence between any two points y_1, y_2 in the curve, there is a unique shortest geodesic connecting them, which varies continuously with y_1 and y_2. Fix a base point $y_0 \in S^1$, and for all $y \in S^1$ let $\gamma^i_{(x,y)} : [0,1] \to S^2$ be the geodesic connecting $\tilde{\psi}_i(x,y)$ to $\tilde{\psi}_i(x,y_0)$. For definiteness, assume that $\gamma^i_{(x,y)}$ is parametrised proportional to its arclength. Then $\gamma^i_{(x,\cdot)}$ is a homotopy contracting the curve $\tilde{\psi}_i(x,\cdot)$ to a point. But we have defined $\gamma^i_{(x,y)}$ in such a way that they are continuous functions of x, y, t; in particular, when they are glued together, they define a homotopy contracting χ to a curve. Since a curve can be contracted to a point, the map χ is contractible.

A similar construction applies to the interval $[b, b+\epsilon]$. It follows that $W[\psi]$ is well-defined.

To show that W is additive, one needs to show that the restriction $\phi|_{[b,c]\times S^1}$ can be "pinched off" at a point; the result then follows from the corresponding one in standard homotopy theory. The pinching off can be achieved using again the fact that circles of length less than λ can be contracted along geodesics; we omit the details. □

Now we are ready to prove:

Theorem 5.3.4 Let ϕ be a sufficiently regular periodic planar Skyrme chain, or a domain wall, with period β and charge $B > 0$. Then

$$E[\phi] \geq \frac{4\pi^2\sqrt{\mu}B}{\beta}.$$

Proof: Let $\lambda \in (0, 2\pi)$ and let $X := \{x \in \mathbb{R} | L[\phi(x,\cdot)] \geq \lambda\}$. We assume that X is the union of a finite number of disjoint intervals, $X = \bigcup_{i=1}^n [a_i, b_i]$. This is what is meant by "sufficiently regular". Let $\phi_i = \phi|_{[a_i,b_i]\times S^1}$ be the restriction of ϕ to a

cylinder, and let $w_i = W[\phi_i]$ be its winding number. Then by the above,

$$deg(\phi) = \sum_{i=1}^{n} w_i.$$

The area enclosed by a curve of length λ in S^2 is less than $2\pi(1-\sqrt{1-(\lambda/2\pi)^2})$. It follows that

$$\left|Area\left[\phi([a_i,b_i]\times S^1)\right]\right| \geq 4\pi\left(|w_i| - 1 + \sqrt{1-\frac{\lambda^2}{4\pi^2}}\right).$$

Let $I = \{i|w_i > 0\}$. Then I has less than B elements and $\sum_{i\in I} w_i \geq B$. Then

$$\begin{aligned}
E[\phi] &\geq \sum_{i\in I} E\left[\phi|_{[a_i,b_i]\times S^1}\right] \\
&\geq \frac{\sqrt{\mu}\lambda}{\beta}\sum_{i\in I} Area\left[\phi([a_i,b_i]\times S^1)\right] \\
&\geq \frac{4\pi\sqrt{\mu}\lambda}{\beta}\sum_{i\in I}\left(w_i - 1 + \sqrt{1-\frac{\lambda^2}{4\pi^2}}\right) \\
&\geq \frac{4\pi\sqrt{\mu}\lambda}{\beta}\left(B - \left(1-\sqrt{1-\frac{\lambda^2}{4\pi^2}}\right)|I|\right) \\
&\geq \frac{4\pi B\sqrt{\mu}\lambda}{\beta}\sqrt{1-\frac{\lambda^2}{4\pi^2}}.
\end{aligned}$$

Notice that we used lemma 5.3.1 in going from the first line to the second line. We are left with a family of lower bounds parametrised by λ. The strongest lower bound is obtained when $\lambda^2 = 2\pi^2$. □

Of course, theorem 5.3.4 applies only to strictly periodic chains - a similar proof could be construced for chains with $\sigma \neq Id$, but we do not pursue this here. Instead, we are able to obtain a divergent lower bound in the anti-periodic case by using the fact that an anti-periodic chain with charge B and period β is also a periodic chain with charge $2B$ and period 2β. The result is:

Theorem 5.3.5 Let ϕ be a sufficiently regular anti-periodic planar Skyrme chain with period β and charge $B > 0$. Then

$$E[\phi] \geq \frac{2\pi^2\sqrt{\mu}B}{\beta}.$$

5.4 Baby skyrmions from sigma model lumps

In this section we will construct potentials V such that both Bogomolny bounds (5.3), (5.5) are saturated by planar skyrmion chains. The procedure [PZ95, War04a] is standard: we first consider the $\mu = 0$ limit of the planar Skyrme model. This limit is the $\mathbb{CP}^1$ sigma model, and exact solutions are known which saturate the Bogmolny bound (see chapter 7). For any given exact solution, it is straightforward to choose a potential so that the Bogomolny bound is still saturated for $\mu \neq 0$.

Let $z = x^1 + ix^2$, and let R be the complex function of z obtained by stereographic projection of ϕ (5.1). Then the Bogomolny equations (5.4), (5.6) are

$$\partial_{\bar{z}} R = 0 \tag{5.17}$$

$$V(R) = \frac{8|\partial_z R|^4}{(|R|^2+1)^4}. \tag{5.18}$$

In the sigma model limit $\mu = 0$, we only need to solve (5.17) in order to saturate the Bogomolny bound (5.3). Given a solution of (5.17), that is, a meromorphic function R, it is easy to find a potential function V so that (5.18) is also satisfied. If V is so chosen, then both Bogomolny bounds (5.3), (5.5) may be saturated for non-zero μ. In what follows we follow this procedure for the cases where R is a chain with $\sigma = \pm 1$.

5.4.1 Strictly periodic chains

The set of strictly periodic sigma model fields satisfying (5.17) were studied in [Sni94]. They are given by the rational function,

$$R(z) = \frac{c + d\exp(\mu_0 z)}{a + b\exp(\mu_0 z)},$$

where $\mu_0 = 2\pi/\beta$ and a, b, c, d are complex constants. The function R depends on six real parameters, of which two correspond to translations, three correspond to isorotations of S^2 and one is a non-trivial "size" parameter. Notice that it is impossible for the R to satisfy the boundary conditions (5.15). So a strictly periodic charge 1 skyrmion chain cannot saturate the energy bound (5.3).

Amongst all the fields R, we restrict attention to those satisfying the domain wall boundary condition,

$$\lim_{x^1 \to \pm\infty} \phi = (0, 0, \pm 1).$$

Then, up to a translation, R must take the form

$$R = \exp(-\mu_0 z).$$

The potential defined by (5.18) in this case is

$$V = \frac{1}{2}\mu_0^4(1 - \phi_3^2)^2. \tag{5.19}$$

We can compute the energy of this chain; the energy is determined by the lower bound (5.7), by construction. We compute

$$\int_{S^2} (1 - \phi_3^2)\omega = \frac{8\pi}{3}.$$

The components of the energy are

$$E_4 = E_0 = \frac{64\pi^4}{3\beta^2}$$

and $E_2 = 4\pi$. We have checked directly that $E_2 + \mu E_4$ exceeds the lower bound proved in theorem 5.3.4.

5.4.2 Anti-periodic chains

An anti-periodic sigma-model field satisfying the boundary condition of a planar skyrmion chain (5.15) is given by

$$R(z) = \frac{c}{a\exp(-\mu_0 z/2) + b\exp(\mu_0 z/2)}$$

with $\mu_0 = 2\pi/\beta$ as before and $a, b, c \in \mathbb{C}$ (see [Bru08]). By making translations and rescaling the constants, we can make $a = b = 1/2$. By making an isorotation, we can make c real. Then the potential defined by this field and (5.18) is

$$V = \frac{1}{2}\left(\frac{\mu_0}{2c}\right)^4 (1 - \phi_3)^2 \left[\left((c^2+1)\phi_3 + (c^2-1)\right)^2 + 4c^2\phi_2^2\right]. \tag{5.20}$$

This potential has three vacua, at the points

$$\vec{\phi} = (0, 0, 1), \left(\frac{\pm 2c}{1+c^2}, 0, \frac{1-c^2}{1+c^2}\right).$$

In the limit where $c \to \infty$, the potential (5.19) is obtained. In the limit where $c \to 0$, $\mu_0 \to 0$ such that c/μ_0 remains constant, the potential [PZ95] whose minima are isolated planar skyrmions is obtained.

Again, we can check that this configuration does not violate the lower bound proved in theorem 5.3.5. We have not been able to evaluate E_4 analytically, but we have found numerically that

$$\begin{aligned} \int_{S^2} \sqrt{2V}\omega &\geq \lim_{c\to\infty} \int_{S^2} \sqrt{2V}\omega \\ &= \left(\frac{\pi}{\beta}\right)^2 \int_{S^2} (1-\phi_3^2)\,\omega. \end{aligned}$$

Therefore $E_2 = 4\pi$ and $E_4 \geq 16\pi^4/(3\beta^2)$. We have checked that $E_2 + \mu E_4 \geq 2\pi^2\sqrt{\mu}/\beta$ for all β.

5.5 Chains with the old baby Skyrme potential

In this section, we consider chains in the old baby Skyrme model with $\sigma = \pm 1$. We will review two analytic ansätze which indicate some properties of chains.

5.5.1 The dipole approximation

An analytic ansatz [PSZ95] is known which describes well-separated planar skyrmions to a good approximation. The ansatz is as follows. Let $R^{(1)}, R^{(2)}$ be two planar skyrmions (5.13) with orientations $\chi^{(1)}, \chi^{(2)}$, expressed in stereographic coordinates (5.1). Suppose that the two planar skyrmions are given a seperation D, and let $R = R^{(1)} + R^{(2)}$ be their superposition. In the limit where $D \to \infty$, the energy of R is twice the energy E_1 of a planar skyrmion. It is possible to calculate the leading contribution to the difference,

$$I_2 := \bar{E}[R^{(1)} + R^{(2)}] - 2E_1,$$

in the limit where $D \to \infty$. Let K_n denote a modified Bessel function of order n. One finds

$$I_2 \approx \frac{p^2\mu^2}{\pi} \cos(\chi^{(2)} - \chi^{(1)}) K_0(\sqrt{\mu}D).$$

for large D, where p is a constant related to the decay of the profile function $f(r)$ by

$$f(r) \approx \frac{p\mu}{2\pi} K_1(\sqrt{\mu} r).$$

p is a function of μ, and must be determined numerically: for example, when $\mu^2 = 0.1$, one finds $p = 24.16$. This approximation is called the dipole approximation, because the asymptotic formula for I_2 matches the interaction energy of a pair of orthogonal scalar dipoles. Notice that the sign of I_2 depends on the relative orientation $\chi^{(2)} - \chi^{(1)}$; it follows that a pair of aligned planar skyrmions repel, while a pair of planar skyrmions with relative orientation π attract.

The same approximation can be applied to an infinite chain of planar skyrmions. For each $j \in \mathbb{Z}$, let $R^{(j)}$ be a planar skyrmion positioned at $(x^1, x^2) = (0, \beta j)$, with orientation $\chi = 0$. Then the fields

$$R^{\pm} := \sum_{j=-\infty}^{\infty} (\pm 1)^j R^{(j)}$$

are planar skyrmion chains with $\sigma = \pm 1$ respectively. Let $I_{\pm} = E[R^{\pm}] - E_1$ be the interaction energy per unit length of the chain. One can show analytically that

$$I_{\pm} \approx \sum_{j=1}^{\infty} (\pm 1)^j \frac{p^2 \mu^2}{\pi} K_0(\sqrt{\mu} j \beta)$$

for large β (notice that only the first term in the sum is significant in this limit). We find that (for large β) I_+ is a positive decreasing function of β and I_- is a negative increasing function of β. So, a chain of well separated planar skyrmions will tend extend for $\sigma = 1$ or contract for $\sigma = -1$.

5.5.2 The string ansatz

In [GP96] an ansatz was proposed for a string-like configuration in the old baby Skyrme model. The string satisfies the boundary conditions of a chain with $\sigma = -1$. The ansatz is

$$\phi = \begin{cases} (\sin f(x^1) \cos \nu x^2, \sin f(x^1) \sin \nu x^2, \cos f(x^1)) & x^1 \geq 0 \\ (\sin f(-x^1) \cos \nu x^2, -\sin f(-x^1) \sin \nu x^2, \cos f(-x^1)) & x^1 \leq 0 \end{cases}$$

where $f : [0,\infty) \to \mathbb{R}$ satisfies $f(0) = \pi$ and $f(t) \to 0$ as $t \to \infty$, and $\nu = \pi/\beta$. The energy density of this field is independent of x^2, but the field itself is not translation-invariant, because translations in x^2 cannot be compensated by isorotations. Therefore there is no reason to expect this ansatz to describe critical points of the energy functional.

The charge per unit length is one and the energy per unit length is

$$E' = \int_0^\infty \frac{\nu}{2} f'(t)^2 + \frac{1}{2\nu}\sin^2 f(t) + \frac{\mu}{2\nu}(f'(t)\sin f(t))^2 + \mu\nu(1-\cos f(t))\,dt. \quad (5.21)$$

This energy satisfies a Bogomolny-type inequality:

$$\begin{aligned} E' &= \int_0^\infty \frac{1}{2}\left(\sqrt{\nu + \frac{\mu}{\nu}\sin^2 f(t)}f'(t) + \sqrt{\frac{1}{\nu}\sin^2 f(t) + 2\mu\nu(1-\cos f(t))}\right)^2 \\ &\quad -\sqrt{\nu + \frac{\mu}{\nu}\sin^2 f(t)}\sqrt{\frac{1}{\nu}\sin^2 f(t) + 2\mu\nu(1-\cos f(t))}f'(t)\,dt \\ &\geq \int_0^\pi \sqrt{\nu + \frac{\mu}{\nu}\sin^2 f}\sqrt{\frac{1}{\nu}\sin^2 f + 2\mu\nu(1-\cos f)}df, \end{aligned}$$

with equality when

$$f'(t) = -\sqrt{\frac{\sin^2 f(t) + 2\mu\nu^2(1-\cos f(t))}{\nu^2 + \mu\sin^2 f(t)}}.$$

We can use the Bogomolny bound to determine the energy of a string, for any period and any values of μ and $\beta = \pi/\nu$. The Bogomolny bound is an elliptic integral.

5.5.3 Summary

In figure 5.2, we have plotted graphs of energy as a function of period for both ansätze in the case $\mu = 1$. The energy of a chain of hedgehogs was determined by numerically evaluating the energy of the analytic configuration. The minimum energy $2.45(4\pi)$ occurs in the hedgehog ansatz at a period of around 0.9π, and this is in reasonable agreement with the numerically determined minimum [HW08b]. Notice that the string ansatz is preferred for small periods.

5.6 Chains with new baby Skyrme potential

In this section, we consider chains in the new baby Skyrme model with $\sigma = \pm 1$. The ansatz of well-separated planar skyrmions, described above in the context of the old

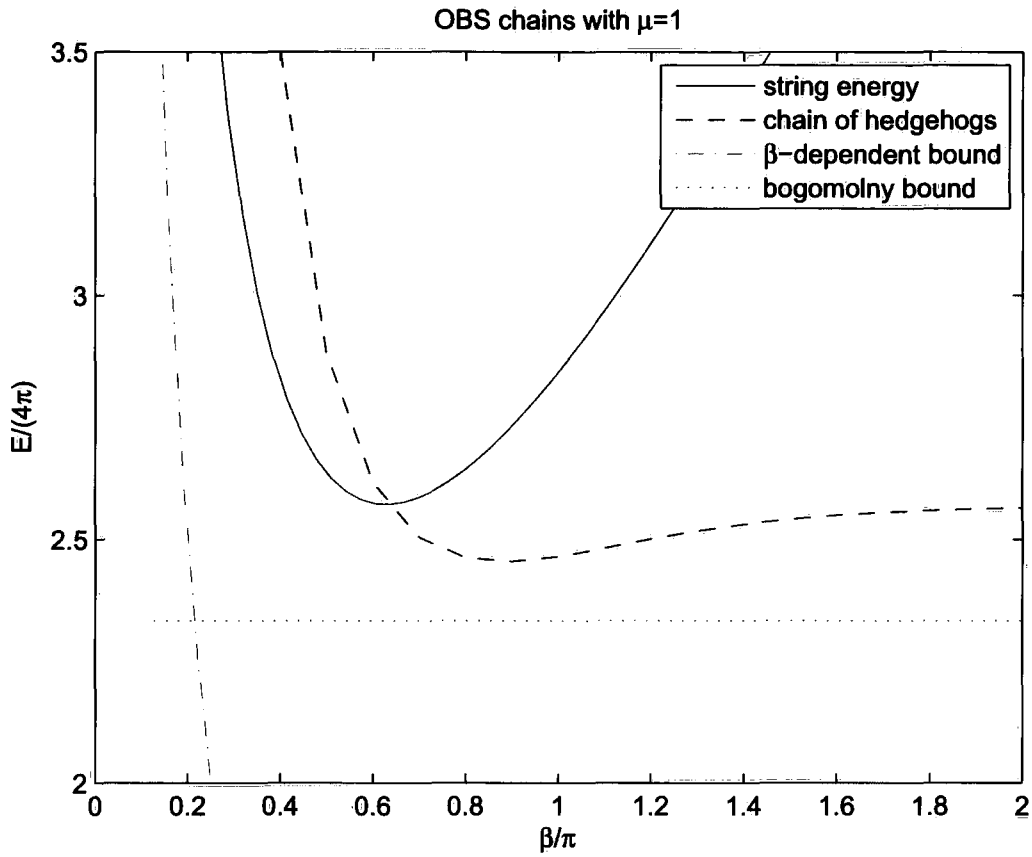

Figure 5.2: Energies of various ansätze for old baby Skyrme chains as functions of period.

baby Skyrme model, is also applicable to the new baby Skyrme model. The same result is obtained for the large β limit of the interaction energy $I_{\pm}$, except that a different value for the numerically determined constant p should be inserted. Also, the lower bound derived for small periods applies to the new baby Skyrme model. We present one additional analytic configuration below.

5.6.1 A domain wall

Consider the field defined by,

$$\vec{\phi} = (\sin f(x^1)\cos(\nu x^2 + \chi),\ \sin f(x^1)\sin(\nu x^2 + \chi),\ \cos f(x^1)), \tag{5.22}$$

in which ν is a real constant, χ is an angle and $f(x^1)$ is a function satisfying the boundary conditions $f(x^1) \to 0$ as $x^1 \to -\infty$ and $f(x^1) \to \pi$ as $x^1 \to \infty$. This

field defines a domain wall which separates the two vacua $\phi_3 = \pm 1$ of the new baby Skyrme potential. The old baby Skyrme model does not admit solutions of this type, because it has only one vacuum. Walls similar to this have been seen before in two different contexts: in [KPZ98] walls with $\nu = 0$ were studied dynamically, while in [Wei99] it was demonstrated that new baby skyrmions of any charge consist of walls similar to (5.22), but wrapped around a circle.

The charge per unit length of the domain wall is

$$\int_{-\infty}^{\infty} \mathcal{B} dx^1 = \frac{\nu}{2\pi},$$

and the energy per unit length is

$$\int_{-\infty}^{\infty} \mathcal{E} dx^1 = \int_{-\infty}^{\infty} \frac{1}{2}(f')^2 + \frac{1}{2}(\nu^2 + \mu)\sin^2 f + \frac{\mu\nu^2}{2}(f')^2 \sin^2 f \, dx^1.$$

Notice that when $\nu = 0$ or $\mu = 0$ the energy per unit length is the energy functional of the sine-Gordon model. The energy per unit length satisfies a Bogmolnyi inequality:

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{E} dx^1 &= \int_{-\infty}^{\infty} \frac{1}{2}\left(\sqrt{1+\mu\nu^2\sin^2 f}\, f' - \sqrt{\mu+\nu^2}\sin f\right)^2 dx^1 \\ &\quad + \sqrt{1+\mu\nu^2\sin^2 f}\sqrt{\mu+\nu^2}\sin f f' \, dx^1 \\ &\geq \int_0^{\pi} \sqrt{1+\mu\nu^2\sin^2 f}\sqrt{\mu+\nu^2}\sin f df, \end{aligned}$$

with equality when the Bogomolny equation,

$$f' = \frac{\sqrt{\mu+\nu^2}\sin f}{\sqrt{1+\mu\nu^2\sin^2 f}},$$

is solved. The Bogomolny equation can be solved analytically, to obtain x^1 as a function of f:

$$\begin{aligned} x^1 &= a^1 + \frac{1}{\sqrt{\mu+\nu^2}}\left(\sqrt{\mu}\nu \arctan\sqrt{\mu}\nu z + \frac{1}{2}\ln\frac{1+z}{1-z}\right) \\ z &= \frac{-\cos f}{\sqrt{1+\mu\nu^2\sin^2 f}}, \end{aligned} \tag{5.23}$$

for some constant a^1 which specifies the position of the wall. The energy per unit length of this field is determined by evaluating the Bogomolny bound analytically, and is equal to

$$\sqrt{\mu+\nu^2}\left(\left(\sqrt{\mu}\nu + \frac{1}{\sqrt{\mu}\nu}\right)\arcsin\sqrt{\frac{\mu\nu^2}{1+\mu\nu^2}} + 1\right).$$

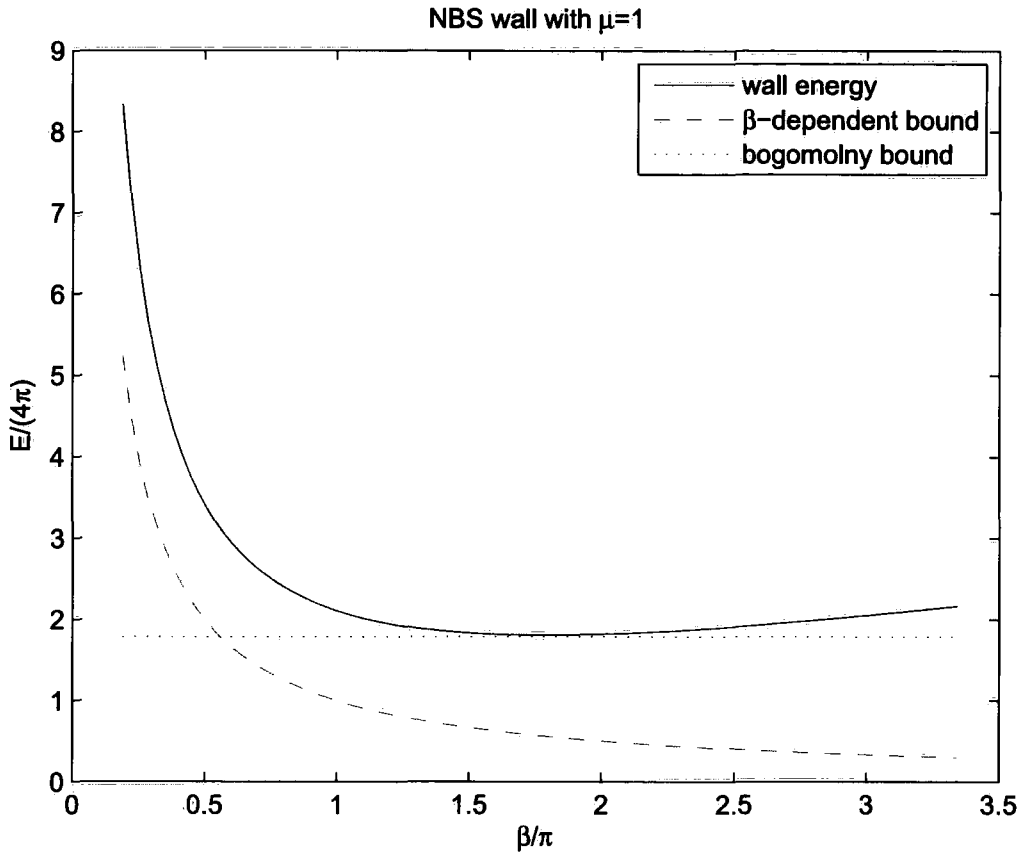

Figure 5.3: Energy of the new baby Skyrme domain wall as a function of period.

For fixed μ, the energy per unit length of this field is minimized when $\nu = 0$. However, the energy per unit charge,

$$\frac{2\pi}{\nu}\sqrt{\mu+\nu^2}\left(\left(\sqrt{\mu}\nu+\frac{1}{\sqrt{\mu}\nu}\right)\arcsin\sqrt{\frac{\mu\nu^2}{1+\mu\nu^2}}+1\right),$$

is minimized with respect to variation in ν when

$$\left(\sqrt{\mu}\nu-\frac{1}{\sqrt{\mu}\nu}-\frac{2\sqrt{\mu}}{\nu^3}\right)\arcsin\sqrt{\frac{\mu\nu^2}{1+\mu\nu^2}}+1=0.$$

We have plotted the energy per unit charge as a function of period with fixed $\mu = 1$ in figure 5.3, along with the two lower bounds on the energy. Notice that at its minimum, the energy is very close to the Bogomolny bound, so walls appear to be fundamental objects in the new baby Skyrme model.

5.6.2 A pair of domain walls

Now we will describe a field which resembles a pair of domain walls, and which satisfies the boundary conditions (5.14), (5.15) for a chain with $\sigma = -1$. Let $\phi^{(1)}$ be a domain wall (5.22) with $\nu^{(1)} = \pi/\beta$ and with profile function $f^{(1)}$ given by (5.23) and with location $a^1 = -D/2$ for some real number $D > 0$. Let $\phi^{(2)}$ be a second domain wall given by (5.22) with $\nu^{(2)} = -\pi/\beta$ and with profile function $f^{(2)}(x^1) = f^{(1)}(-x^1)$. Let $R(1)$, $R^{(2)}$ be the stereographic projections of these, as in (5.1). Finally define a superposition R by

$$\frac{1}{R} = \frac{1}{R^{(1)}} + \frac{1}{R^{(2)}}.$$

The field R satisfies the boundary conditions for a chain with period β and $\sigma = -1$, and the charge per unit period is 1. This field resembles a pair of parallel domain walls, separated by a distance D, with $\phi^3 \approx -1$ between them and $\phi_3 \approx 1$ on either side.

We will study the interaction energy

$$I_w := E[R] - E[R^{(1)}] - E[R^{(2)}]$$

of the superposition, which is a function of β and μ. This will give indications about the stability of new baby skyrmion chains with $\sigma = -1$.

$I_w \to 0$ as $\beta \to \infty$, and we have calculated the leading contribution to I_w in this limit. The calculation is similar to that performed for two planar skyrmions in [PSZ95], but is more complicated in practise for the following reason. In the case of two planar skyrmions, the leading contribution is found using an elegant argument based on the Euler-Lagrange equations. However, in the case of two walls the analogous term vanishes, so we must work to higher order.

We find that

$$I_w \approx \frac{16\pi \exp(4\sqrt{\mu}\nu \arctan\sqrt{\mu}\nu)}{\nu(1+\mu\nu^2)^2}(2\nu^4 + 2\mu\nu^2 - 1)D\exp(-2D\sqrt{\mu+\nu^2})$$

in the limit $D \to \infty$, where we have written $\nu = \pi/\beta$. It follows that a pair of well-separated domain walls will attract when

$$\beta^4 - 2\mu\pi^2\beta^2 - 2\pi^4 > 0$$

and repel otherwise. For large β, a chain is stable with respect to separation to two walls, but for some critical value β_c this is not the case. The above results suggest that

$$\beta_c \leq \pi\sqrt{\mu + \sqrt{\mu^2 + 2}}.$$

5.6.3 Summary

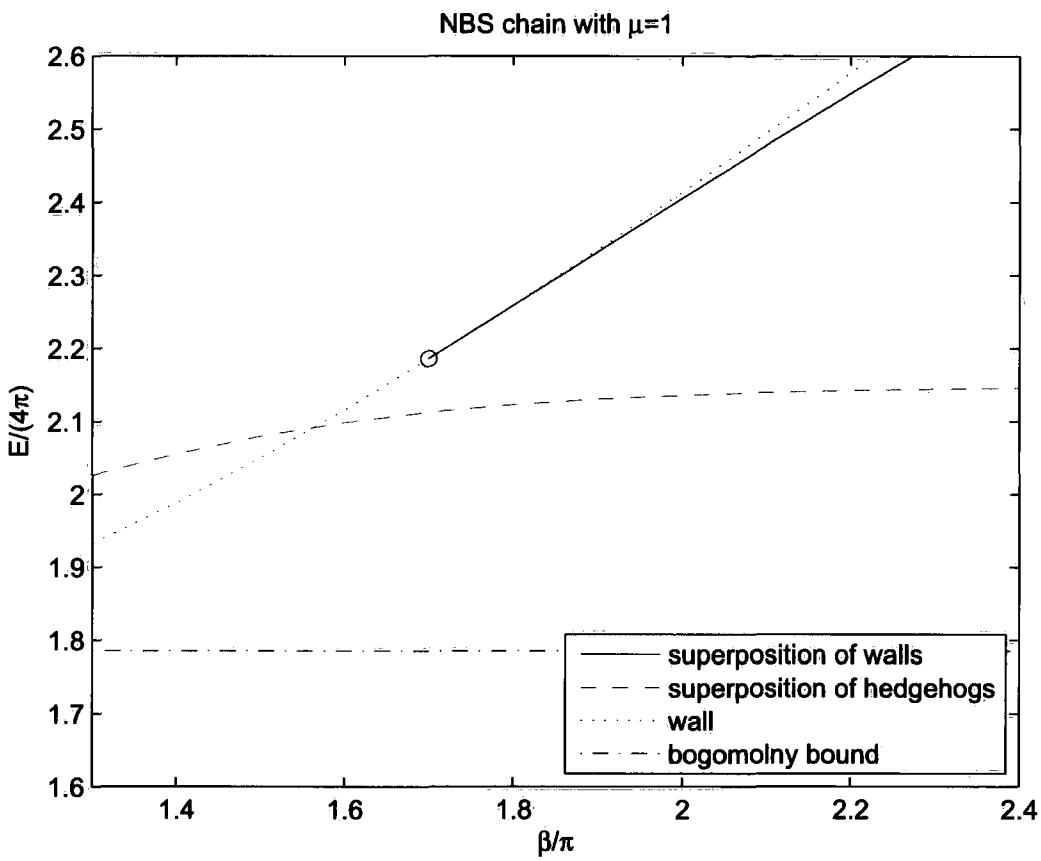

Figure 5.4: Energies of various ansätze for new baby Skyrme chains as functions of period.

In figure 5.4 we have plotted the energies of various ansätze as functions of the period, with $\mu = 1$. The energy of two isolated domain walls was determined analytically. The energy of a superposition of two walls was determined numerically, and was minimized with respect to variations in the separation D. For $\beta \leq 1.7\pi$ no minimum is found, and this agrees with the analytic formula $\beta_c = \sqrt{1+\sqrt{3}}\pi \approx 1.65\pi$. The energy of a chain of hedgehogs was determined numerically. We have

plotted the Bogomolny bound, but left out the period-dependent bound since it is very small for the periods shown in the graph.

Let $\beta_0 \approx 1.56\pi$ be the period at which the chain of hedgehogs and the domain wall have equal energies. We expect no stable chains to exist for $\beta \leq \beta_0$, and we expect stable chains to exist for $\beta \geq \beta_c$. Somewhere between β_0 and β_c there should be a critical period at which chains become unstable. Numerical simulations of the model indicate that the critical period is around 1.49π [HW08b], which is actually slightly less than β_0, but still close.

5.7 Summary and open questions

In this chapter, we have studied chains in the planar Skyrme model using a variety of approaches. Two theoretical results were obtained: we proved that chains have an integer topological charge, and we found a lower bound on their energy which tends to infinity as the period tends to zero. The former result is quite general and will be used again in later chapters.

We also studied chains in detail for three particular choices of potential. The first potential was chosen so that anti-periodic chains could be obtained analytically. The second two potentials were the well-studied old baby Skyrme and new baby Skyrme potentials, which have respectively one and two vacua. With one vacuum anti-periodic chains were found to exist for all values of the period, but with two vacua the chain developed an instability at small periods which was associated with a splitting into two infinitely separated domain walls. Thus, like calorons, planar Skyrme chains exhibit a constituent structure for small enough periods, but unlike calorons, the constituent structure has an associated dynamical instability.

An obvious next step is to consider chains in the three-dimensional Skyrme model: this is done in the next chapter. In chapter 7, we will analyse chains in the $\mathbb{CP}^1$ sigma-model, which is the $\mu = 0$ limit of the planar Skyrme model.

Chapter 6

Chains of skyrmions

In this chapter we shall study chains in the Skyrme model. The Skyrme model is a three-dimensional field theory with topological soliton solutions, which is believed to provide a good model of nuclear physics (the topological solitons are identified with protons and neutrons).

Topological solitons are well-studied in the Skyrme model up to relatively high charge. Extended configurations have also been considered: the Skyrme model is known to posess a "Skyrme lattice" solution (of dimension two) [BS98] and a "Skyrme crystal" (of dimension three) [KS88, KS89, CJJ+89]. "Skyrme chains" (of dimension one) were considered long ago [AWM+85], but have since been neglected.

The field equations of the Skyrme model are non-linear and no exact analytic solutions are known, so the only direct means of studying the Skyrme model is by numerical simulation. Since the theory is three-dimensional, numerical methods are not fast. However, a number alternative methods are known. These methods typically take the form of a simple ansatz, which is empirically observed to approximate the true (numerically determined) behaviour of the model. The three most prominent ansätze of this type are: Skyrme's product ansatz (see for example [Sch94]), the Atiyah-Manton ansatz [AM93], and the rational map ansatz [HMS98]. We have been able to adapt the first two to study chains, but the rational map ansatz appears unsuited to this task. Full numerical simulations were carried out by my supervisor, Prof. R. S. Ward, and appeared in our article [HW08a].

6.1 The Skyrme model and chains

The field content of the Skyrme model is a function $U : \mathbb{R}^3 \to SU(2)$. We define $L_i = U^{-1}\partial U/\partial x^i$. Then the energy density of the Skyrme model is

$$\mathcal{E} := -\frac{1}{2}\mathrm{Tr}(L_i L_i) = \frac{1}{16}\mathrm{Tr}([L_i, L_j][L_i, L_j]) \tag{6.1}$$

and the energy functional is

$$E[U] := \int_{\mathbb{R}^3} \mathcal{E} dx^1 dx^2 dx^3. \tag{6.2}$$

We will write

$$U(\vec{x}) = \sigma(\vec{x})Id + i\pi^j(\vec{x})\sigma^j$$

where σ, π^j are real functions of $\vec{x} \in \mathbb{R}^3$ satisfying $\sigma^2 + \Sigma(\pi^j)^2 = 1$, $Id \in SU(2)$ denotes the identity element, and σ^j are the Pauli matrices.

Skyrme fields with finite energy must tend to a constant at spatial infinity, and this constant is taken to be the identity element. So topologically, a finite energy Skyrme field is a map from $S^3 \to SU(2)$, and such a map has a degree $B \in \mathbb{Z}$. The degree can be conveniently computed by the integral,

$$B = \int_{\mathbb{R}^3} \mathcal{B}\, dx^1\, dx^2\, dx^3 \tag{6.3}$$

$$\mathcal{B} = \frac{1}{24\pi^2}\epsilon_{ijk}\mathrm{Tr}(L_i L_j L_k). \tag{6.4}$$

The degree of a Skyrme field forms a lower bound on its energy:

$$E \geq 12\pi^2 B.$$

Finite energy fields U which locally minimize E will be called skyrmions.

The Skyrme energy density (6.1) is invariant under the action of the group,

$$E_3 \times O(4)_{iso}.$$

Here E_3 is the group of isometries of $\mathbb{R}^3$ (including reflections) and $O(4)_{iso}$ is the group of isometries of the target $SU(2) \cong S^3$, called isorotations. If we represent $U = \sigma + i\pi^j\sigma^j$ by the column vector $U^c = (\sigma, \pi^2, \pi^2, \pi^3)^t$, then $O(4)_{iso}$ acts by multiplication from the left. The action of $g \in E_3$ is $g : U(\mathbf{x}) \mapsto U(g^{-1}\mathbf{x})$.

The boundary condition $U \to Id$ as $r \to \infty$ breaks the symmetry group to

$$E_3 \times O(3)_{iso}.$$

The charge density (6.4) is negated by orientation-reversing transformations of $\mathbb{R}^3$ or S^3, so the symmetry group of the whole model is

$$G = ker(\Pi)$$

where $\Pi : E_3 \times O(3)_{iso} \to \mathbb{Z}_2$ is the homomorphism, $\Pi(g,h) = det(g)det(h)$.

A Skyrme chain with period $\beta > 0$ and relative orientation $\sigma \in SO(3)_{iso}$ is a Skyrme field $U : \mathbb{R}^3 \to SU(2)$ satisfying

$$U(x,y,z+\beta) = \sigma \cdot U(x,y,z),$$

which minimises the energy functional

$$E[U] = \int_0^\beta \int_{\mathbb{R}^2} \mathcal{E}\, dx\, dy\, dz$$

and has finite energy. Skyrme chains are required to satisfy the boundary condition, $U \to Id$ as $x^2 + y^2 \to \infty$. By theorem 5.2.1, Skyrme chains have a degree $B \in \mathbb{Z}$ computed by the integral,

$$B[U] = \int_0^\beta \int_{\mathbb{R}^2} \mathcal{B}\, dx\, dy\, dz.$$

6.2 Symmetry of chains

Clearly, the only Skyrme field invariant under the whole group G is the constant field $U(\mathbf{x}) = Id$. Symmetries of other fields are conveniently described by twisted subgroups:

Definition 6.2.1 A *twisted subgroup* (H, ρ) of G consists of a subgroup $H \subset E_3$ and a homomorphism $\rho : H \to O(3)_{iso}$, such that $\Pi(h, \rho(h)) = 1$ for all $h \in H$. The twisted subgroup which describes all symmetries of a field is called the *isotropy group* of the field.

A simple example is provided by the charge 1 skyrmion: this can be written down in the "hedgehog" form:

$$U_h(\mathbf{x}) = (Id, m) \cdot \exp\left(if(r)\hat{x}^i\sigma^i\right) \tag{6.5}$$

$$= \exp\left(if(r)m_{ij}\hat{x}^j\sigma^i\right) \tag{6.6}$$

Here the function $f : [0, \infty) \to \mathbb{R}$ satisfies $f(0) \equiv \pi$ and $f(r) \to 0$ as $r \to \infty$, and is chosen to minimize the energy of U_h. $m \in SO(3)_{iso}$ specifies the orientation of the skyrmion and Id denotes the identity in E_3. The isotropy group of this skyrmion is (H, ρ), where $H = O(3) \subset E_3$ is the subgroup of rotations and reflections which fix the origin, and the homomorphism is $\rho : h \mapsto mh^{-1}m^{-1}$.

Isotropy groups are normally used to describe symmetry-breaking bifurcations; in our case they are merely a convenient notation for describing relative orientations of pairs of skyrmions. Let U_1, U_2 be two skyrmions with orientations m_1, m_2 and locations $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$, with $\mathbf{a}_1 \neq \mathbf{a}_2$. We denote their isotropy groups by $(O(3)_i, \rho_i)$ where $O(3)_i \subset E_3$ is the group which fixes $\mathbf{a}_i$.

The relative orientation is the isorotation which gives U_1 the same orientation as U_2, that is, $m_2 m_1^{-1} \in O(3)_{iso}$. We can also define a relative spatial rotation $R \in O(3)_1$, which is the spatial rotation that gives U_1 the orientation of U_2: we have $R = \rho_1^{-1}(m_2 m_1^{-1})$. A superposition of the pair (U_1, U_2) is a charge 2 Skyrme field which looks like U_1 in a neighbourhood of $\mathbf{a}_1$ and like U_2 in a neighbourhood of $\mathbf{a}_2$. It is reasonable to expect the isotropy group (H, ρ) of the superposition to satisfy the following:

(i) All elements $g \in H$ fix the set $\{\mathbf{a}_1, \mathbf{a}_2\} \subset \mathbb{R}^3$.

(ii) If $g \in H$ fixes each point $\mathbf{a}_i$ then $\rho(g) = \rho_1(g) = \rho_2(g)$.

We define the maximal isotropy group for a superposition to be the largest twisted subgroup (H, ρ) satisfying the above.

The simplest superposition of the fields U_1, U_2 is the product ansatz, $U = U_1 U_2$. This ansatz has all the symmetries we might expect, except that the reflection P which swaps $\mathbf{a}_1, \mathbf{a}_2$ is not a symmetry (essentially because $U_1 U_2 \neq U_2 U_1$). An ansatz which is better behaved with respect to symmetries is the relativised product

ansatz [NR88]. The isotropy group of the relativised product ansatz is the maximal isotropy group.

Both these ansätze can be used to predict the force between well-separated skyrmions. One finds that [Sch94]:

- When $R = Id$, the skyrmions repel. This case is called *aligned.*
- The strongest repulsive force occurs when R is a rotation through π about the line l joining $\mathbf{a}_1$ and $\mathbf{a}_2$. This case is called the *repulsive channel.*
- The strongest attractive force occurs when R is a rotation through π about an axis perpendicular to l. This case is called the *attractive channel.*

Proposition 6.2.2 The maximal isotropy groups of an aligned pair and a pair in the maximally repulsive channel are both $\langle O(2)_l, P\rangle$. Here $O(2)_l$ is the group of rotations and reflections which fix l, P is the reflection which swaps ∂_1 and ∂_2, and $\langle\cdots\rangle$ denotes the group generated by $\cdots$. The maximal isotropy group of a pair in the attractive channel is $\langle D_2, P\rangle$, where $D_2 \subset O(2)_l$ is the subgroup generated by reflections in two perpendicular planes whose intersection is l.

Proof: Let the skyrmions be U_1 and U_2, let their isotropy groups be $O(3)_1$ and $O(3)_2$, and observe that $O(2)_l = O(3)_1 \cap O(3)_2$. We have $U_2 = (TR, 1) \cdot U_1$, where T is the translation which maps $\mathbf{a}_1$ to $\mathbf{a}_2$. It follows that

$$\rho_2(g) = \rho_1(R^{-1}T^{-1}gTR)\ \forall g \in O(3)_2.$$

It follows from property (i) that either $H = \langle K, T\rangle$ or $H = K$, for some subgroup K of $O(2)_l$. It follows from property (ii) that every $g \in K$ satisfies $\rho_1(g) = \rho_2(g)$. Equivalently,

$$\rho_1(g) = \rho_1(R^{-1}T^{-1}gTR)\ \forall g \in K.$$

Since ρ_1 is injective and T commutes with $g \in O(2)_l$, this condition is equivalent to $gR = Rg$. So K can be no larger than the group $\{g \in O(2)_l | gR = Rg\}$. In the aligned and maximally repulsive cases, the maximal K is the whole of $O(2)_l$. In the maximally attractive case, K is a D_2 subgroup.

If $P \in H$, we must be able to find $\rho(P)$ such that

$$(P, \rho(P)) \cdot U_1 = U_2 \text{ and } (P, \rho(P)) \cdot U_2 = U_1.$$

Both conditions are satisfied by $\rho(P) := \rho_1(PTR)$ for the cases we are considering, so $P \in H$. It is easy to check that ρ is a homomorphism, as it should be. □

All of this is relevant to chains because if a chain posesses a continuous symmetry, it is much easier to analyse numerically. On the other hand, we are most interested to study chains whose energy is lower than that of an isolated skyrmion.

We think of a charge 1 chain as consisting of charge 1 skyrmions U_n located at points $\mathbf{a}_n = (0, 0, n\beta)$ and with orientations $\sigma^n m$, for $n \in \mathbb{Z}$. Each skyrmion has isotropy group $(O(3)_n, \rho_n)$, where $\rho_n(g) = \sigma^n m g^{-1} m^{-1} \sigma^{-n}$. The isotropy group (H, ρ) of the chain is required to satisfy:

(i) All elements $g \in H$ fix the set $\{\mathbf{a}_n | n \in \mathbb{Z}\} \subset \mathbb{R}^3$.

(ii) If $g \in H$ fixes a point $\mathbf{a}_n$ then $\rho(g) = \rho_n(g)$.

(iii) The translation $T : (x, y, z) \mapsto (x, y, z+\beta)$ is an element of H, and $\rho(T) = \sigma^{-1}$

The maximal isotropy group of a chain is the largest twisted subgroup satisfying the above. Notice that we do not consider the possibility that the chain has any symmetries which fix the z-axis but not the points $\mathbf{a}_n$, because by theorem 5.2.1 a chain with such a symmetry would have charge per period greater than 1.

We define R to be the rotation which gives U_0 the orientation of U_1, ie $R = \rho_0^{-1}(\sigma^{-1})$. Motivated by results for the case of two skyrmions, we make the following definitions

- A chain with $R = Id$ is called *aligned.*
- A chain for which R is a rotation through π about the z-axis is called *maximally repulsive.*
- A chain for which R is a rotation through π about an axis perpendicular to the z-axis is called *maximally attractive.*

Proposition 6.2.3 Let P here denote the reflection in the plane $z = 0$. The maximal isotropy groups of aligned and maximally repulsive chains are both $\langle O(2)_z, P, T\rangle$, where $O(2)_z$ fixes the z-axis. The maximal isotropy group of a maximally attractive chain is $\langle D_2, P, T\rangle$.

Proof: Properties (i) and (ii) imply that either $H = \langle K, P, T\rangle$ or $H = \langle K, T\rangle$, for some subgroup K of $O(2)_z$. K is determined just as in the case of a pair of skyrmions, and we set $\rho(g) = \rho_0(g)$ for $g \in K$. If $P \in H$ we must be able to find $\rho(P)$ that

$$(P, \rho(P))U_n = U_{-n} \; \forall n \in \mathbb{Z}.$$

The case $n = 0$ is satisfied only if $\rho(P) = \rho_0(P)$. It is easy to check that this choice of $\rho(P)$ also works for $n \neq 0$, for the cases we are interested in.

Finally, we must have $\rho(T) = \sigma^{-1} = \rho_0(R)$. It is easily checked that ρ is a homomorphism. □

The product ansatz for two skyrmions tells us that the forces between neighbouring skyrmions in maximally repulsive and attractive chains are repulsive and attractive respectively. Therefore, we expect that only the maximally attractive chain will have energy less than that of an isolated skyrmion. Disappointingly, it appears that a chain can have a continuous symmetry only if its energy exceeds that of an isolated skyrmion.

6.3 Skyrme chains from calorons

The Atiyah-Manton ansatz is method of obtaining Skyrme fields on $\mathbb{R}^3$ from $SU(2)$ gauge fields over $\mathbb{R}^4$. If the gauge field is a charge N instanton, then the Skyrme field will also have charge N. It turns out that, if the instanton is chosen so as to minimize the energy of the resulting Skyrme field, then the Skyrme field obtained is a surprisingly good approximation for the true charge N skyrmion.

The method of the Atiyah-Manton ansatz is to evaluate holonomies of the gauge field along a family of parallel lines. For example, we could choose the family of

lines parallel to the x^0-axis; then the Skyrme field is obtained via

$$U(x^1, x^2, x^3) = P\exp\left(\int_{-\infty}^{\infty} A_0(x)dx^0\right).$$

If we want to obtain a Skyrme chain, then we should start with a gauge field satisfying

$$A_\mu(x^0 + \beta, x^1, x^2, x^3) = \sigma \cdot A_\mu(x^0, x^1, x^2, x^3) \tag{6.7}$$

and integrate along a family of parallel lines perpendicular to the x^0-axis.

Calorons provide examples of gauge fields satisfying (6.7). Suppose we have a caloron with monopole charge $k_1 = 0$. Recall that asymptotically,

$$A_0 \to \mu_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

This is true not just in local gauges, but across the whole 2-sphere at infinity, since A_0 doesn't wind at infinity. Now we make a gauge transformation,

$$\begin{aligned} A'_\nu &= gA_\nu g^{-1} - g^{-1}\partial_\nu g \\ g &= \exp(i\mu_1 x^0 \sigma^3). \end{aligned}$$

In the new gauge, $A'_0 \to 0$ at infinity and the gauge field is no longer strictly periodic, but satisfies

$$A'_\nu(x^0 + \beta, x^1, x^2, x^3) = hA_\nu(x^0, x^1, x^2, x^3)h^{-1}$$

where $h = \exp(i\mu_1\beta\sigma^3)$. This gauge choice is known as the algebraic gauge.

So charge B Skyrme chains can be obtained by integrating holonomies of calorons with $k_0 = B$, $k_1 = 0$ in the algebraic gauge. Analytic expressions are known for all calorons $k_0 = 1$ and $k_1 = 0$ [KvB98]. So it seems feasible to study Skyrme chains using this family. Since we are ultimately interested in maximally attractive chains, we restrict attention to the calorons which yield $\sigma^2 = Id$ but $\sigma \neq Id$; these are the calorons with $\mu_1 = \mu_0/4 = \pi/2\beta$. Aligned chains could be obtained from calorons with $\mu_1 = 0$, such as the Harrington-Shepard calorons [HS78]. Notice that our procedure differs from [EK89, NZ89], where caloron holonomies are evaluated in the periodic direction in order to obtain skyrmions rather than Skyrme chains.

The calorons we are interested in are symmetric under rotations about an axis in $\mathbb{R}^3$. If we choose to evaluate holonomies along lines parallel to the axis, the Skyrme

chains will also have an $SO(2)$ symmetry: these will be maximally repulsive chains. If, on the other hand, we evaluate holonomies along lines perpendicular to the axis of rotation, we obtain maximally attractive chains.

We have implemented the Atiyah-Manton ansatz for maximally attractive chains numerically, and evaluated the energies of chains. The family of calorons we used is parametrised by a scale parameter ρ and a period β. The caloron with scale ρ and period β is in fact a rescaling of the caloron with scale ρ/β and period 1, and since the components of the Skyrme energy behave simply under rescalings it was sufficient only to consider calorons with fixed $\beta = 1$ and a range of values of ρ.

The holonomies were evaluated using the Runge-Kutta method. We evaluated energies in a finite box, and extrapolated both in the box size and the lattice spacing to obtain energies accurate to within 0.1%. We also calculated B to check the accuracy of our method. Our results are summarised in the following table:

ρ	E_2	E_4	$B-1$
0.2	10.56	509.2	2.0×10^{-4}
0.3	16.61	314.3	1.6×10^{-4}
0.4	23.20	213.7	1.1×10^{-4}
0.5	29.68	160.4	7.7×10^{-5}
0.6	35.40	135.4	6.3×10^{-5}
0.7	40.21	124.6	9.3×10^{-5}
0.8	44.25	119.6	1.9×10^{-4}

Table 6.1: Energies for caloron-generated Skyrme chains at $\beta = 1$

In figure 6.1, we have plotted a the minimum energy obtained at each period within the Atiyah-Manton ansatz. This graph was obtained by interpolating the data in table 6.1 to obtain E_2 and E_4 as polynomial functions of ρ, and minimizing the energy $E = E_2\beta + E_4/\beta$ with respect to variation in ρ.

6.4 The vortex ansatz

In this section we describe an alternative ansatz for chains, which is based on the idea that a chain can split into constituents. We define a Skyrme vortex to be a field,

$$U_v = \exp\left(\frac{\theta - \nu z}{2} i\sigma^3\right) \exp(f(r) i\sigma^1) \exp\left(\frac{\theta + \nu z}{2} i\sigma^3\right)$$

where (r, θ) are polar coordinates in $\mathbb{R}^2$ and z is the coordinate in the periodic direction. The profile function f should satisfy the boundary conditions, $f(0) = \pi/2$, $f(r) \to 0$ as $r \to \infty$. The boundary condition at $r = 0$ guarantees that U_v is well-defined there. U_v is periodic in z with period $2\pi/\nu$. Since U_v is independent of z at infinity, it can be compactified to a map from $S^3 \to SU(2)$, hence has an integer degree, or charge per unit period.

The energy density and charge density for U_v are

$$\begin{aligned} \mathcal{E} &= (f')^2 + \frac{\cos^2 f}{r^2} + \nu^2 \sin^2 f + (f')^2 \frac{\cos^2 f}{r^2} + (f')^2 \nu^2 \sin^2 f + \frac{\nu^2 \sin^2 f \cos^2 f}{r^2} \\ \mathcal{B} &= -\frac{1}{2\pi^2} \frac{\nu}{r} \sin f \cos f f'. \end{aligned}$$

We observe that U_v has infinite energy per unit period, which is a consequence of the fact that U_v is not constant on the torus at infinity. The charge per unit period is 1, which has been verified by integrating $\mathcal{B}$. The profile function f is chosen to solve the Euler-Lagrange equation for $E = (4\pi^2/\nu) \int_0^\infty \mathcal{E} r \, dr$.

The Skyrme vortex has been seen before. Kopeliovich and Stern [KS87] considered axially symmetric Skyrme fields: these resemble finite lengths of Skyrme vortex joined into a circle. The vortex loops are stable for $B = 2$ and unstable for $B > 2$. Here we will use the Skyrme vortex to construct an ansatz for a Skyrme chain.

Notice that the field $U(x, y, z) = U_v(x, -y, -z)$ also has charge 1, but winds in the opposite direction to U_v at infinity. We define a superposition of two vortices using the product ansatz:

$$U = U_1 U_2$$

with $U_1(x, y, z) = U_v(x - a, y, z)$ and $U_2(x, y, z) = U_v(x + a, -y, -z)$ This superposition satisfies the boundary conditions of a chain with period $\beta = \pi/\nu$ and $\sigma = diag(-1, -1, 1) \in SO(3)_{iso}$. The field represents a pair of vortices separated by

a distance $2a$, where $a > 0$. We obtain a similar, but more symmetric, field by using the relativised product ansatz [NR88]

$$U = (U_1U_2 + U_2U_1)/\sqrt{(det(U_1U_2 + U_2U_1)}$$

instead of the product ansatz. The relativised superposition has the same symmetries as a maximally attracting chain, so we expect this ansatz to describe a maximally attracting chain.

When the separation $2a$ of the vortices is large, they attract each other, as the following heuristic argument shows. Notice that, far from the centre of the vortex, the field is approximated by

$$U \approx \exp(i\theta\sigma^3).$$

Let $R > 0$ be sufficiently large that this approximation is valid for $r > R$, and let a be larger than R. We can evaluate the energy of the superposition by splitting $\mathbb{R}^2$ into three regions: the two discs of radius R centred on the vortex locations, and the exterior. The energy within each disc tends to a constant as a tends to infinity. The energy in the exterior diverges as $a \to \infty$; a calculation shows that the leading contribution at large a is

$$4\pi\beta \ln a.$$

So far we have not justified our choice of superposition procedure: it is important to ask whether there is another way to superpose two vortices to obtain a lower energy. Again, we have a heuristic argument why our superposition is the right thing to do, at least at large separations. Consider the following problem. Let F denote the exterior of the two discs D_1, D_2 of radius R and with centres $(x, y) = (\pm a, 0)$, as before. Let $\psi : F \to U(1)$ such that $\psi|_{\partial D_1}$ has winding number 1 and $\psi|_{\partial D_2}$ has winding number -1. We want to know what is the minimum value of the energy,

$$e = \beta \int_F \|\phi^{-1}d\phi\|^2 d^2x.$$

The ansatz we used before corresponds to taking $\psi = \exp(i(\theta_1 - \theta_2))$, where $\theta_1(x, y)$ is the angle between $(x - a, y)$ and the x-axis, and $\theta_2(x, y)$ is the angle between $(x + a, y)$ and the x-axis. If the energy of this field is close to the true minimum, then we know our ansatz is a good one. Notice that the Skyrme term has disappeared

from our energy functional; this is because the Skyrme term evaluates to zero for any $U(1)$ field.

The easiest way to find the minimum energy is to stereographically project from $\mathbb{R}^2$ to S^2; the energy is conformally invariant so we are allowed to do this. The stereographic projection can be chosen so that the two circles ∂D^1, ∂D^2 are described by $\theta = \alpha$ and $\theta = \pi - \alpha$ in spherical coordinates $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, where $\sin(\alpha) = R/a$. The energy functional is now written as

$$e = \beta \int_0^{2\pi} \int_\alpha^{\pi-\alpha} \left((\psi^{-1}\partial_\theta \psi)^2 + \sin^{-2}\theta (\psi^{-1}\partial_\phi \psi)^2 \right) \sin\theta \, d\theta \, d\phi.$$

A Bogomolny argument shows that this energy is minimized by $\psi(\theta, \phi) = \exp(i\phi)$; the minimum energy is

$$4\pi\beta \ln \cot(\alpha/2).$$

For large a, this agrees with our superposition to leading order.

β	E	a	$B - 1$
1.0	157.9	1.13	5.4×10^{-4}
1.1	153.1	1.15	3.4×10^{-4}
1.2	149.6	1.16	3.1×10^{-4}
1.3	147.1	1.18	2.1×10^{-4}
1.4	145.2	1.20	2.0×10^{-4}
1.5	144.0	1.21	1.5×10^{-4}
1.6	143.2	1.22	1.5×10^{-4}
1.7	142.8	1.23	1.2×10^{-4}
1.8	142.6	1.24	1.1×10^{-4}
1.9	142.8	1.24	9×10^{-5}
2.0	143.1	1.25	9×10^{-5}

Table 6.2: Energies of superposed vortices, minimized with respect to variation in a.

We have evaluated the energy of the superposition of two vortices for a range of values of β and a. The energies were evaluated in a finite box, and we extrapolated

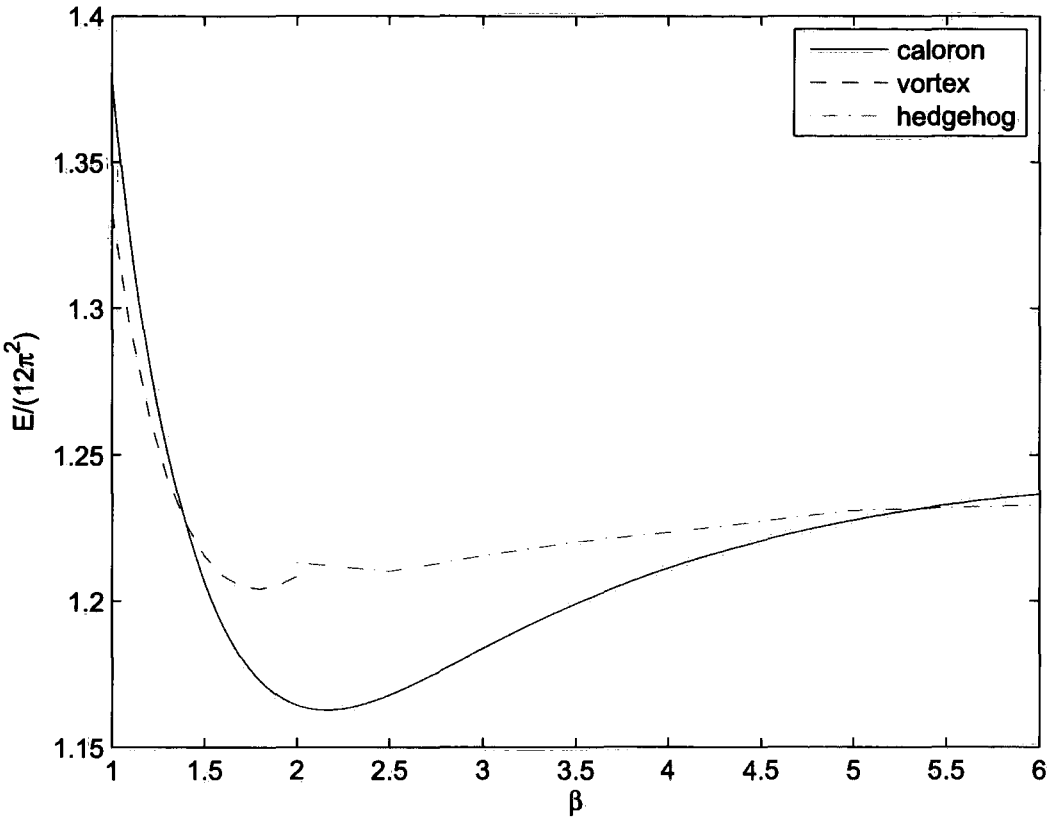

Figure 6.1: Energies of various ansätze for Skyrme chains as functions of period.

in the box size and the lattice spacing to obtain results accurate to within 0.1%. We tried using both the product ansatz and the relativised product ansatz, and found that the energies obtained agreed. We also evaluated B as a check on our methods. Table 6.2 shows the minimum energy of the superposition, together with the value of a for which this energy is attained.

6.5 Summary and open problems

In figure 6.1 we have plotted the energies of the ansätze described above, along with the numerically determined energy of a chain superposed of hedgehog-ansatz skyrmions. The minimum energy, $E = 1.16(12\pi^2)$, is obtained using the Atiyah-Manton ansatz, and occurs at a period $\beta = 2.15$. Numerical simulations of chains have been performed by my supervisor, Prof. Ward, and appeared in [HW08a].

These indicate that the Skyrme chain attains minimum energy $E = 1.143(12\pi^2)$ at period $\beta = 1.98$; the Atiyah-Manton ansatz provides a fairly accurate description of chains, at least for β close to 2.

The Atiyah-Manton ansatz appears to give reasonable energies for the whole range of periods, although we have not compared with numerical simulations for all values of β. For small periods the vortex ansatz has a lower energy than the Atiyah-Manton ansatz, while for large periods the superposed hedgehogs have a lower energy than the Atiyah-Manton ansatz. It is worth pointing out that the Atiyah-Manton ansatz takes much longer to implement numerically than the others, since one needs to solve a differential equation for each point on the grid taken to represent space.

The value ratio E/B for the chain lies between those of the charge 2 and charge 3 skyrmions. The value of $E/(12\pi^2 B)$ for a Skyrme crystal [KS88, KS89, CJJ+89] is 1.04, and for the Skyrme lattice [BS98] it is 1.06. The value 1.143 obtained numerically for a chain is much greater than either of these. It is quite likely that higher charge Skyrme chains exist, and would have lower energy than the charge one chain studied here, see [HW08a] for some numerical results.

We will mention briefly some things which we have not been able to do. Firstly, we have not found an analog to the rational map ansatz which works for chains. We tried an ansatz which obtains anti-periodic chains using anti-periodic meromorphic maps on the torus (such as the Jacobi elliptic functions), but the energies obtained exceeded those obtained in the vortex ansatz. We have also been unable to find a lower bound on the energy which diverges for small periods, analagous to theorems 5.3.4 or 5.3.5. Possibly the difficulty lies in the fact that $\mathbb{R}^2 \times S^1$ has two non-compact directions rather than one; it might be easier to find a lower bound for a Skyrme lattice.

One potential application of Skyrme chains is to constructing skyrmions. One can easily envisage a charge N skyrmion which consists of N periods of Skyrme chain wrapped around a circle; such a configuration would have a D_N symmetry. Although these D_N skyrmions would certainly have greater energy than the known polyhedral skyrmions, they may still be local minima of the energy functional.

The fact the vortex ansatz seems to well for small periods indicates that Skyrme chains, like other soliton chains, exhibit a constituent structure. However, the constituents are only be visible if the chain is squeezed beyond its preferred period.

Chapter 7

Chains in sigma models

In this chapter we will consider chains in the $\mathbb{CP}^n$ sigma models. These models are probably the easiest to work with amongst all models with topological solitons. For this reason they provide a good place to explore new ideas. There is a close resemblance between "lump" solitons in sigma models and instantons: both theories have conformally invariant energy functionals, and both sets of solitons have a holomorphic structure.

Chains were studied in sigma models a while ago in connection with finite temperature quantum field theory [MW89, Sni94], and their dynamics were explored in [Rom05]. More recently, it was observed that with different boundary conditions, a sigma model chain can exhibit constituents, just as a caloron does [Bru08]. The constituents of a caloron can be understood using loop groups; here we will show that the same is true of sigma model chains.

That a caloron exhibits constituents follows from two observations: firstly, that a caloron is a monopole whose gauge group is a loop group [GM88], and secondly that a monopole is a superposition of "fundamental" $SU(2)$ monopoles. We shall show that both of these statements have analogs in the $\mathbb{CP}^n$ sigma models. We will try to keep our discussion fairly simple, in order to make the ideas clear. We will not attempt to develop a general theory for sigma models and loop groups; instead we will present a few illustrative examples.

7.1 The $\mathbb{CP}^n$ sigma models

The field content of a sigma model is a map $\phi : M \mapsto N$ between two Riemannian manifolds. The energy functional is

$$E = \frac{1}{2}\int_M \|d\phi\|^2 \omega_M.$$

In the mathematical literature, a function ϕ which minimizes E is called harmonic. In the case where M has dimension two and N is Kähler, there is a lower bound

$$E \geq \int_M \phi^*(\omega_N)$$

which is saturated if and only if ϕ is holomorphic [Gue85]. In general, the lower bound will have a topological interpretation.

We will be interested in the case where M is $\mathbb{C}$ and N is n-dimensional complex projective space $\mathbb{CP}^n$. $\mathbb{CP}^n$ is one member of the family of symmetric spaces for $SU(n+1)$, all of which are Kähler manifolds. To construct these symmetric spaces, one first fixes a maximal torus T in $SU(n+1)$. All symmetric spaces for $SU(n+1)$ may be obtained as orbits of elements b of the Lie algebra $\mathfrak{t}$ of T under the adjoint action of G. The symmetric spaces can also be described as quotients $G/C(b)$, where $C(b) \subset G$ is the centraliser of b. In particular, if b is chosen so that $C(b)$ is isomorphic to $U(1)$, the symmetric space will be $\mathbb{CP}^n$. One can also represent symmetric spaces as quotients of complex groups $GL(n+1, \mathbb{C})$. Finally, $\mathbb{CP}^n$ can be represented geometrically as the set of lines in $\mathbb{C}^{n+1}$.

This last description allows us to represent points of $\mathbb{CP}^n$ by non-zero $n+1$-column vectors, defined up to multiplication by complex numbers. Thus the charge 1 $\mathbb{CP}^1$ sigma model lump may be written redundantly as

$$\phi : u \mapsto \begin{bmatrix} \lambda \\ u - a \end{bmatrix} \tag{7.1}$$

where $u = x + iy$ is a coordinate on $\mathbb{C}$ and $\lambda \in \mathbb{C}^*$, $a \in \mathbb{C}$ are parameters.

7.2 Multi-kinks

Multi-kinks are the sigma model analog of monopoles. Here we define a multi-kink to be a map $\phi : \mathbb{R}^2 \to \mathbb{CP}^n$ which obeys the constraint,

$$\frac{\partial \phi}{\partial y} = -X \cdot \phi$$

for $X \in su(n+1)$, and which minimizes the energy functional

$$E = \frac{1}{2} \int_{\mathbb{R}} \|d\phi\|^2 dx. \tag{7.2}$$

Multi-kinks wind in a spatial direction (the y-direction), so are similar Q-kinks [AT92b, AT92a], which wind in a temporal direction.

Given a multi-kink ϕ, we may set $\tilde{\phi} = \exp(yX)\phi$, so that $\tilde{\phi}$ is independent of y. In terms of $\tilde{\phi}$, the energy functional is

$$E = \frac{1}{2} \int_{\mathbb{R}} \|\partial_x \tilde{\phi}\|^2 + V(\tilde{\phi}) dx, \tag{7.3}$$

where $V : \mathbb{CP}^n \to \mathbb{R}$ is the function defined by take the norm squared of the vector field induced on $\mathbb{CP}^n$ by X. This second version of the energy functional, and its multi-kink minima, were studied in [GTT01].

We may choose coordinates so that $X = i\,diag(\mu_1, \ldots \mu_{n+1})$ for real numbers μ_i satisfying $\mu_i \geq \mu_{i+1}$. We assume further that these inequalities are strict; this is analogous to the condition of maximal symmetry breaking for monopoles. In order that the energy be finite, ϕ must tend to a fixed point of X, called a vacuum, as $x \to \pm\infty$. It is easy to see that there are $n+1$ vacua in $\mathbb{CP}^n$, written

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, v_{n+1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

There is a lower bound on the energy,

$$E \geq \int_{\mathbb{R}} \phi^* i_k(\omega).$$

Here ω is the standard Kähler form on $\mathbb{CP}^n$ and k is the vector field induced by X. Since $H^1(\mathbb{CP}^n) = 0$, the 1-form $i_k(\omega)$ is exact. In fact we can say more: there exists

a function ψ on $\mathbb{CP}^n$ such that k is the Hamiltonian vector field with Hamiltonian ψ, that is, $i_k\omega = d\psi$. We will show below that ψ can be constructed explicitly, and satisfies $\psi(v_i) = -2\mu_i$, hence the lower bound on the energy of a function ϕ satisfying $\phi(-\infty) = v_i$, $\phi(\infty) = v_j$ is $E \geq 2(\mu_i - \mu_j)$. The lower bound is saturated when ϕ is holomorphic, or equivalently, when $\tilde{\phi}$ solves

$$\frac{\partial}{\partial x}\tilde{\phi} = iX \cdot \tilde{\phi}. \tag{7.4}$$

To construct the function ψ, we work in the adjoint orbit model of $\mathbb{CP}^n$. Let $b = -2i/(n+1)\, diag(n, -1, \ldots, -1) \in su(n+1)$ be a basepoint, which we identify with v_1. Points $p \in \mathbb{CP}^1$ are represented by $p = gbg^{-1}$ for $g \in SU(2)$. The function ψ is defined by

$$\psi(p) = -Tr(pX).$$

The canonical choice of symplectic form (due to Kostant and Kirillov) is

$$\omega_p(\xi, \eta) = -Tr(p[\xi, \eta])$$

for $\xi, \eta \in su(n+1)$. The identity $i_k\omega = d\psi$ is equivalent to

$$\omega_p(X, Y) = L_Y\psi \; \forall Y \in su(n+1)$$

where L denotes the Lie derivative. We have $L_Y\psi = -Tr([Y, p], X)$ and $\omega_p(X, Y) = -Tr(p[X, Y])$, so the identity follows from the invariance of the trace:

$$Tr([Y, p]X) + Tr(p[Y, X]) = 0.$$

The simplest case of the above is when $n = 1$. Since X must be traceless, $\mu_1 > 0$ and $\mu_2 = -\mu_1$. The general solution of the Bogomolny equation is

$$\phi : u \mapsto \begin{bmatrix} a_1 \exp(-\mu_1 u) \\ a_2 \exp(-\mu_2 u) \end{bmatrix} \tag{7.5}$$

where $u = x + iy$ and $a_1, a_2 \in \mathbb{C}^*$ are defined up to rescaling. This solution satisfies the boundary conditions, $\phi(-\infty) = v_1$, $\phi(\infty) = v_2$. We define the mass of the kink ν to be half its energy: $\nu := \mu_1 - \mu_2$.

Actually, this $\mathbb{CP}^1$ kink is the familiar sine-Gordon kink. Consider a kink of the form,

$$\tilde{\phi} : x \mapsto \begin{bmatrix} \cos(f(x)) \\ e^{i\alpha}\sin(f(x)) \end{bmatrix},$$

for f a real function and $\alpha \in \mathbb{R}$. The energy (7.3) of such a kink is

$$E = \frac{1}{2}\int_{\mathbb{R}} (\partial_x f)^2 + \nu^2 \sin^2(f) dx,$$

which is the energy functional of the sine-Gordon model. The kink (7.5) has this form, and is identified with the sine-Gordon kink. We define the location of the kink to be the point $x = x_0$ where $f(x) = \pi/2$, that is, $x_0 = \nu^{-1}\ln|a_1/a_2|$. The second modulus of the wall corresponds to a $U(1)$ phase.

The next simplest case is $n = 2$. The general solution of the Bogomolny equation is

$$\phi : u \mapsto \begin{bmatrix} a_1 \exp(-\mu_1 u) \\ a_2 \exp(-\mu_2 u) \\ a_3 \exp(-\mu_3 u) \end{bmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{C}$ are defined up to rescaling, and at least two are non-zero. More concisely, we can say that $[a_1, a_2, a_3]^t \in \mathbb{CP}^2 \setminus V$, where $V = \{v_1, v_2, v_3\}$ is the set of vacua. When $a_3 = 0$, the solution satisfies the boundary condition $\phi(-\infty) = v_1$, $\phi(\infty) = v_2$, and the solution is an embedding of a $\mathbb{CP}^1$ kink with mass $\nu_1 := \mu_1 - \mu_2$. Similarly, when $a_1 = 0$ the solution is an embedding of the $\mathbb{CP}^1$ kink and satisfying the boundary conditions $\phi(-\infty) = v_2$, $\phi(\infty) = v_3$ and with mass $\nu_2 := \mu_2 - \mu_3$. In all other cases, the boundary condition satisfied is $\phi(-\infty) = v_1$ and $\phi(\infty) = v_3$, and the mass of the solution is $\nu_1 + \nu_2$.

The cases $a_3 = 0$ and $a_1 = 0$ can be considered fundamental solutions, and they can be obtained as limits of the general $a_1, a_3 \neq 0$ solution. The four real moduli of the general solution can be accounted for by thinking of it as the superposition the two fundamental kinks, each posessing two moduli. For this reason, the general solution will be referred to as a multi-kink.

It is easy to see how to generalise the multi-kink, and its fundamental kink constituents, the case of target space $\mathbb{CP}^n$. The general picture is that a $\mathbb{CP}^n$ multi-

kink is a superposition of n fundamental kinks of masses $\nu_i := \mu_i - \mu_{i+1}$, and posesses $2n$ moduli.

The structure of multi-kinks can be visualised using the Lie algebra model of $\mathbb{CP}^n$. Recall that, in this model, $\mathbb{CP}^n$ is the orbit of a point in the Lie algebra $su(n+1)$. This point represents the vacuum v_1, and lies on one of the walls of the fundamental Weyl chamber, which is a subset of the Lie algebra $\mathfrak{t}$ of the maximal torus T. We have illustrated this for the $\mathbb{CP}^2$ case in figure 7.1: the Lie algebra $\mathfrak{t}$ is two-dimensional, and is represented by the plane. The shaded region represents the fundamental Weyl chamber, and the point v_1 is indicated by a cross. The Weyl group W is generated by reflections in the lines H_1, H_2, H_0. The other vacua v_2, v_3 are the images of v_1 under the action of the Weyl group. The multi-kink, which interpolates between v_1 and v_3, is indicated by a dashed arrow. The fundamental kinks are indicated by the solid arrows.

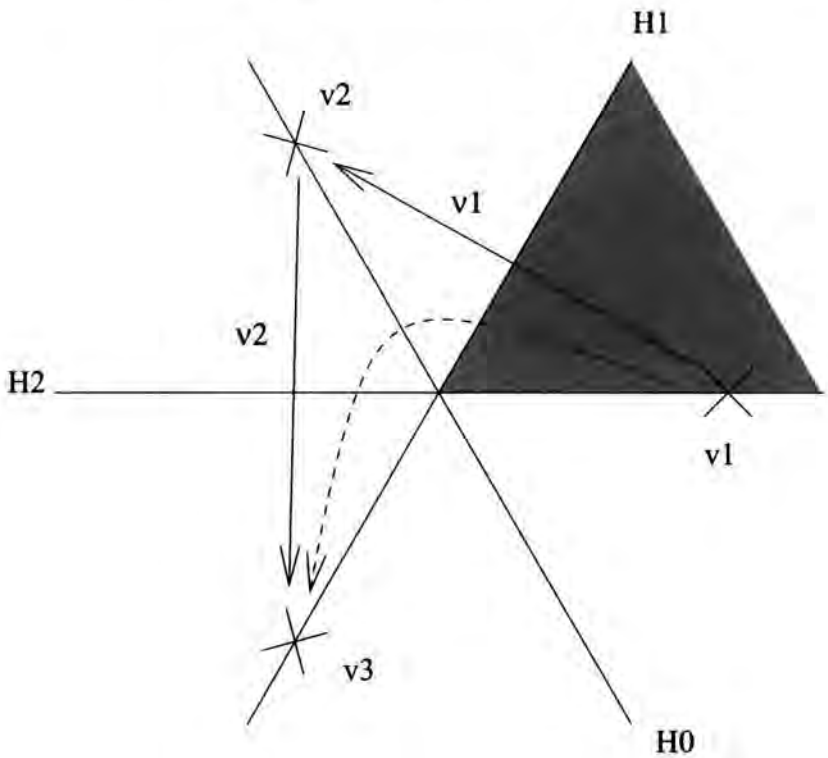

Figure 7.1: The $\mathbb{CP}^2$ multi-kink realised as an adjoint orbit in the Lie algebra of $SU(3)$.

Figure 7.1 suggests a connection between kinks and roots, as we now describe. Recall that the roots α of a Lie group are homomorphisms α from the Lie algebra $\mathfrak{t}$

of T to $\mathbb{R}$ which together describe the action of $\mathfrak{t}$ on the Lie algebra of G. The Weyl group $W := N(T)/T$ acts on $\mathfrak{t}$. Reflections in the hyperplanes $H_\alpha := ker(\alpha)$ are elements of W. W is generated by reflections in the walls of the fundamental Weyl chamber; the corresponding roots are called simple roots.

We observe that any pair of vacua $v_i, v_j \in \mathbb{CP}^n$ determine a unique reflection in the Weyl group which swaps them. Hence a kink interpolating between v_i and v_j may be associated with a root α. We have noticed (but not proven) that the mass of such a kink is equal to $\alpha(X) \in \mathbb{R}$. We have also noticed that a fundamental kink is associated with a reflection in a wall of the Weyl chamber, which is in turn associated with a simple root of the Lie group.

So it appears that the number of fundamental kinks is equal the number of simple roots in the Lie group (that is, the rank of the Lie group): $SU(n+1)$ has n simple roots, so a $\mathbb{CP}^n$ multi-kink has in general n fundamental kinks. Of course, this statement is just an observation; we have neither proved it, nor investigated it for other symmetric spaces or Lie groups.

7.3 Chains

A chain is a map $\phi : \mathbb{R}^2 \to \mathbb{CP}^n$ satisfying the periodicity condition

$$\phi(x, y + \beta) = g \cdot \phi(x, y)$$

which minimizes the energy functional,

$$E = \frac{1}{2} \int_0^\beta \int_{\mathbb{R}} \|d\phi\|^2 dx\, dy.$$

Here $\beta > 0$ is the period and $g \in SU(n+1)$. In order that the energy be finite, ϕ must tend to fixed points of g as $x \to \pm\infty$.

We assume that g is written in the form $g = \exp(-X\beta)$, with $X = idiag(\mu_1, \ldots \mu_{n+1})$. We assume that $\mu_i > \mu_{i+1}$ and $\mu_1 = \mu_{n+1} < \mu_0$, where $\mu_0 := 2\pi/\beta$. We do not consider at present the possibility that some of the μ_i are equal. Then the fixed points of g are the vacua $v_1, \ldots v_{n+1}$ described in the preceding section.

As usual, there is a Bogomolny bound on the energy and solutions of the Bogomolny equation are holomorphic. If ϕ tends to the same vacuum as $x \to \pm\infty$ then

the lower bound on the energy is a multiple of an integer, by theorem 5.2.1. In the general case, $\phi(-\infty) = v_i$ and $\phi(\infty) = v_j$, we have found that the lower bound is

$$\int_{\mathbb{R}\times S^1} \phi^*(\omega) = 4\pi \left(k_0 + \frac{\mu_i - \mu_j}{\mu_0} \right)$$

for an integer k_0.

This identity is proved in a similar manner to theorem 5.2.1. Let $\phi'(x,y) = \exp(yX)\phi(x,y)$ be a strictly periodic map, and let k_0 be its degree. We have an identity,

$$\phi'^*(\omega) = \phi^*(\omega) + \phi^*(i_k\omega) \wedge dy.$$

We recall from earlier the identity $i_k\omega = d\psi$. Integrating, we obtain

$$\begin{aligned} \int_{\mathbb{R}\times S^1} \phi^*(\omega) &= \int_{\mathbb{R}\times S^1} \phi'^*(\omega) - \beta \int_{\mathbb{R}} \phi^*(d\psi) \\ &= 4\pi k_0 - \beta \int_{v_i}^{v_j} d\psi \\ &= 4\pi k_0 + \frac{4\pi}{\mu_0}(\mu_i - \mu_j) \end{aligned}$$

as required.

The solution considered by Bruckmann [Bru08] was

$$\phi : u \mapsto \begin{bmatrix} a_1 \exp((\mu_0 - \mu_1)u) + a_0 \exp(-\mu_1 u) \\ b_0 \exp(-\mu_2 u) \end{bmatrix} \tag{7.6}$$

where a_0, a_{-1}, b_0 are non-zero complex parameters, defined up to an overall scaling. Note that $\mu_2 = -\mu_1$ because X is traceless. This map satisfies the boundary conditions $\phi(-\infty) = v_1$, $\phi(\infty) = v_1$, and its degree is 1.

Bruckmann observed that, for certain values of the parameters a_1, a_0, b_0 the chain resembles a superposition of two fundamental kinks, with masses $\nu_1 = \mu_1 - \mu_2$ and $\nu_2 = \mu_0 - \nu_1$. To see this, we fix some of the parameters: by making translations and phase rotations, we can choose $a_1 = -a_0 = 1$ and $b \in \mathbb{R}$, $b > 0$. The chain (7.6) becomes

$$\phi : u \mapsto \begin{bmatrix} b^{-1}(\exp(\nu_2 u) - \exp(\nu_1 u)) \\ 1 \end{bmatrix}.$$

When b is large, the field resembles two kinks of the form (7.5), with masses ν_1, ν_2 and locations $x_1 = -\nu_1^{-1} \ln b$, $x_2 = \nu_2^{-1} \ln b$. When b is small, the field resembles a

lump (7.1) located at $u = 0$, with scale parameter $\lambda = b/\mu_0$. Energy densities for various values of b are illustrated in figure 1.1.

It is easy to write down higher charge chains, including chains whose charge is fractional. It is also a simple matter to write down chains in different sigma models. For example, the analog of the Bruckmann chain in the $\mathbb{CP}^2$ sigma model is

$$\phi : u \mapsto \begin{bmatrix} a_1 \exp((\mu_0 - \mu_1)u) + a_0 \exp(-\mu_1 u) \\ b_0 \exp(-\mu_2 u) \\ c_0 \exp(-\mu_3 u) \end{bmatrix}. \tag{7.7}$$

This $\mathbb{CP}^2$ chain has three constituents, with masses $\nu_1 = \mu_1 - \mu_2$, $\nu_2 = \mu_2 - \mu_3$, $\nu_3 = \mu_0 - \nu_1 - \nu_2$. More generally, the charge 1 $\mathbb{CP}^n$ chain has $n + 1$ constituents. We would like to find a more systematic way to understand the constituents. We will argue below that the natural way to understand them is with loop groups.

7.4 Loop groups

A loop group LG is a group whose elements are maps from the circle S^1 to a Lie group G. The product of two loops is obtained by pointwise multiplication, using the product in G. We consider in particular the group of smooth maps, although there are other possibilities. The standard reference on loop groups is the book by Pressley and Segal [PS86]. Although loop groups are infinite-dimensional, their root structure is similar to that of finite-dimensional Lie groups, which is part of the reason why they are interesting to study.

Here we shall describe some symmetric spaces for loop groups. The starting point is to introduce the group $\mathbb{T}$ of rigid rotations of S^1, which acts on LG in an obvious way. We take a semi-direct product $\mathbb{T}\tilde{\times}LG$. Let T be a maximal torus in G, let G also denote the subgroup of LG of constant loops, and let T denote the subgroup of constant loops taking values in the torus. Then $\mathbb{T} \times T$ is a maximal torus in $\mathbb{T}\tilde{\times}LG$. Symmetric spaces of loop groups are obtained in an analogous way to symmetric spaces of finite-dimensional Lie groups: we choose a subgroups Z of the maximal torus $\mathbb{T} \times T$, and consider the quotient $\mathbb{T}\tilde{\times}LG/C(Z)$.

We are interested in the case $G = SU(n+1)$. We list three infinite-dimensional symmetric spaces for $SU(n+1)$ below.

- Suppose that $Z = \mathbb{T} \times T$. Then $C(Z) = \mathbb{T} \times T$, we denote the homogeneous space $\mathbb{T}\tilde{\times}LSU(n+1)/\mathbb{T} \times T = LSU(n+1)/T$ by $\mathcal{F}^{n+1}$.

- Suppose that $Z = \mathbb{T}$. Then $C(Z) = \mathbb{T}\times SU(n+1)$, we denote the homogeneous space $\mathbb{T}\tilde{\times}LSU(n+1)/\mathbb{T} \times SU(n+1) = LSU(n+1)/SU(n+1)$ by $\Omega SU(n+1)$.

- Suppose that Z is a $U(1)$ subgroup of T. Let Y denote the centraliser of Z in $SU(n+1)$. Then $C(Z) = \mathbb{T}\tilde{\times}LY$, we denote the homogeneous space $\mathbb{T}\tilde{\times}LSU(n+1)/\mathbb{T}\tilde{\times}LY = LSU(n+1)/LY$ by $L\mathbb{CP}^n$, it is identified with the space of loops in $\mathbb{CP}^n$.

The spaces $\Omega SU(n+1)$ and $\mathcal{F}^{n+1}$ are well studied; they are called respectively the fundamental homogeneous space and the periodic flag manifold. However, we shall be primarily interested in the space $L\mathbb{CP}^n$, which appears to be much less well-studied.

We will represent points of $L\mathbb{CP}^n$ by column vectors of functions $z^i : S^1 \to \mathbb{C}$:

$$\begin{bmatrix} z^1(\theta) \\ \vdots \\ z^{n+1}(\theta) \end{bmatrix}.$$

Here $\theta \in \mathbb{R}/2\pi\mathbb{Z} \cong S^1$. The functions z^i must not be all zero for any value of θ, and they are defined up to overall multiplication by functions from S^1 to $\mathbb{C}^*$.

The space $L\mathbb{CP}^n$ inherits a natural metric from $\mathbb{CP}^n$. Let $z = z(\theta)$ denote a point in $L\mathbb{CP}^n$; if two tangent vectors $\xi, \eta \in T_z L\mathbb{CP}^n$ can be represented by a functions $v, w : S^1 \to T_{z(\theta)}\mathbb{CP}^n$ then we define

$$g_z(\xi, \eta) = \frac{1}{2\pi}\int_{S^1} g_{z(\theta)}(v(\theta), w(\theta))d\theta.$$

The normalisation factor is chosen so that if z, ξ, η are all constant loops then their norm in $L\mathbb{CP}^n$ equals their norm in $\mathbb{CP}^n$. Notice that $L\mathbb{CP}^n$ also inherits a complex structure from $\mathbb{CP}^n$, so it is quite possible that $L\mathbb{CP}^n$ is a Kähler manifold.

Like the finite-dimensional symmetric spaces, infinite-dimensional symmetric spaces can be modelled as adjoint orbits in a Lie algebra. We consider the example of $L\mathbb{CP}^1$. The Lie algebra of the maximal torus $\mathbb{T} \times T$ is spanned by the vectors ∂_θ and t, where ∂_θ is a basis vector for the Lie algebra of $\mathbb{T}$ and t is a basis vector for the Lie algebra of $T \cong U(1)$. The stabiliser of the point t is the group $\mathbb{T}\tilde{\times}LT$; it follows that the orbit of t under the adjoint action of $\mathbb{T}\tilde{\times}LSU(2)$ is $L\mathbb{CP}^1$.

As in the finite-dimensional case, the fixed points of $\mathbb{T} \times T$ in the $L\mathbb{CP}^1$ are the images of the base point under the action of the Weyl group. The Weyl group for $\mathbb{T}\tilde{\times}LSU(2)$ is generated by the following transformations:

$$\begin{aligned} a\partial_\theta + bt &\mapsto a\partial_\theta - bt \\ a\partial_\theta + bt &\mapsto a\partial_\theta + (2b - a)t \end{aligned}$$

when t is suitably normalised, and where the coefficients $a, b \in \mathbb{R}$. It follows that $L\mathbb{CP}^1$ has two vacua, $v_1 = t$ and $v_2 = -t$. Written as vectors, these are

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The group $\mathbb{T}\tilde{\times}LSU(2)$ has two simple roots, which are identified with the two generators of the Weyl group introduced above. More generally, $\mathbb{T}\tilde{\times}LSU(n)$ has n simple roots (see figure 2.1)

7.5 A chain as a loop group multi-kink

We are now ready to apply loop groups to chains. We shall show that a $\mathbb{CP}^n$ chain is the same thing as a multi-kink in a sigma model whose target is $L\mathbb{CP}^n$. Let ϕ be a chain satisfying $\phi(x, y + \beta) = \exp(-\beta X)\phi(x, y)$. Define a map $\tilde{\phi} : \mathbb{R} \to L\mathbb{CP}^n$ by

$$\tilde{\phi}(x) = \exp(X\theta/\mu_0)\phi(x, \theta/\mu_0).$$

The Bogomolny equation for ϕ,

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\phi = 0,$$

is equivalent to

$$\frac{\partial}{\partial x}\tilde{\phi} = i\left(X - \mu_0\frac{\partial}{\partial\theta}\right)\tilde{\phi}.$$

This is formally identical to the Bogomolny equation for a multi-domain wall (7.4), since $X - \mu_0\partial_\theta$ is an element of the Lie algebra of $SU(n+1)\tilde{\times}\mathbb{T}$. Similarly, the energy functional for ϕ is equal to

$$E = \frac{\beta}{2}\int_{\mathbb{R}} \|\partial_x\tilde{\phi}\|^2 + V(\tilde{\phi})dx,$$

where the potential function $V : L\mathbb{CP}^n \to \mathbb{R}$ is equal to the norm squared of the vector field induced on $L\mathbb{CP}^n$ by $X - \mu_0\partial_\theta$. Here we are explicitly using the metric defined on $L\mathbb{CP}^n$.

As a concrete example, the Bruckmann chain (7.6) is mapped to

$$\tilde{\phi}(x) = \begin{bmatrix} a_1\exp((\mu_0-\mu_1)x)e^{i\theta} + a_0\exp(-\mu_1 x) \\ b_0\exp(-\mu_2 x) \end{bmatrix}.$$

We have illustrated this object in figure 7.2. The horizontal axis represents the Lie algebra of T and the vertical axis represents the Lie algebra of $\mathbb{T}$. The orbit of $v_1 = t$ is $L\mathbb{CP}^1$. The dashed arrow represents the chain, while the two solid arrow indicate the fundamental (consituent) kinks. The kink pointing from v_1 to v_2 has mass ν_1, the kink pointing from v_2 to v_1 has mass ν_2. These two types of kink are identified with the two simple roots of $\mathbb{T}\tilde{\times}LSU(2)$.

It is easy to see how figure 7.2 generalises to higher charge. For example, a dashed arrow which passes from v_1 to v_2 to v_1 to v_2 represents a chain with two constituents of mass ν_1 and one constituent of mass ν_2. The energy of such a chain is $(4\pi/\mu_0)(2\nu_1+\nu_2)$. It is also easy to guess what will happen for different target spaces. For example, the $\mathbb{CP}^2$ chain (7.7) is represented by an arrow in the figure 7.1 passing from a vacuum v_1 to v_2 to v_3 and then back to v_1.

Notice that there is a definite ordering to the fundamental kinks of a chain. For example, a kink lying to the right of kink of mass ν_1 always has mass ν_2. This is because, in figure 7.2, the arrow pointing from v_2 to v_1 is always associated with a kink of mass ν_2. The ordering prescription looks quite reminiscent of the ordering of constituent monopoles in axially-symmetric calorons, described by equation (3.11) and figure 3.2. It might be possible to relate axially-symmetric calorons and loop

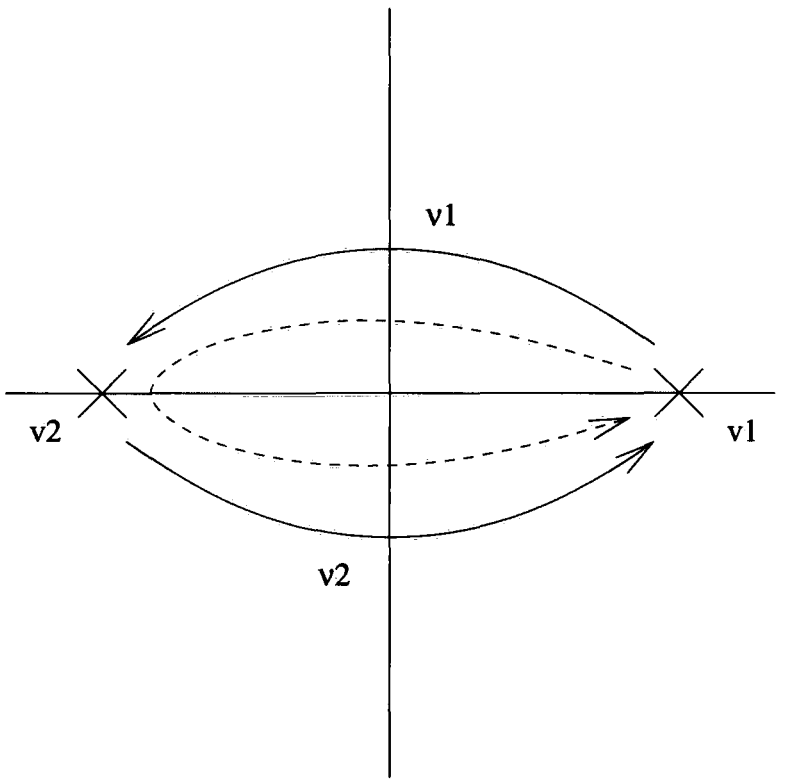

Figure 7.2: The charge 1 $\mathbb{CP}^1$ chain realised as an adjoint orbit in the Lie algebra of the loop groop

group multi-kinks using results of Atiyah [Ati84] or Charbonneau and Hurtubise [CH08].

7.6 Summary and open problems

In this chapter we have explored a connection between chains and multi-kinks in sigma models. We showed that the a $\mathbb{CP}^n$-chain is the same thing as an $L\mathbb{CP}^n$-multi-kink, and argued that this explains the appearance of fundamental kinks in a chain. We pointed out some connections between the structure of multi-kinks and the root structure of Lie groups. It would be interesting if these connections could be derived in a more fundamental way: could one deduce the properties of a multi-kink moduli space using instead machinery from group theory? And could such analysis be applied to more general symmetric spaces?

We point out here that sigma models have been studied in connection with loop

groups before. Atiyah has demonstrated the existence of a bijection between moduli spaces of $SU(n)$ Yang-Mills instantons and moduli spaces of lumps in sigma models with target $\Omega SU(n)$ [Ati84, Gue85]. Charbonneau and Hurtubise have explored a similar correspondence for calorons [CH08]. It remains to be seen whether the $L\mathbb{CP}^n$ sigma model, or its chain or multi-kink solitons, can be related to Yang-Mills theory in any way.

A final question concerns sphalerons, which are saddle points of the energy functional. Sphalerons have been found in certain sigma models, and have a nice mathematical structure [Zak89]; one wonders whether kink- or chain-type sphalerons also exist.

Chapter 8

Conclusion

We began this thesis by describing some simple properties of chains of solitons, and asking some basic questions about them. In the following three chapters we investigated in detail various aspects of calorons, which were historically the first type of chain to receive any attention. In the final three chapters we investigated chains in three more models, namely the planar Skyrme model, the Skyrme model, and the $\mathbb{CP}^n$ sigma models. We have seen that chains, and their constituent structure, are fairly generic. The only case where no constituents were found was for calorons on hyperbolic space – but this was to be expected, since our boundary conditions were degenerate.

At the end of each chapter we have mentioned some open problems and possible directions for generalisation. In view of the title of this thesis, we will make a few remarks here about the general status of research in chains. Monopole chains have also been studied and their Nahm transform and spectral curves analysed (see [War05] and references therein). It remains to be seen whether monopole chains inherit any other mathematical structure from monopoles, such as rational maps. A cursory glance at the contents page of [MS04] reveals that the only topological soliton model where chains have not been studied is the vortex model – but note the string-theoretic arguments at the end of [Ton02]. Lumps in the $\mathbb{CP}^n$ sigma model are known to describe semi-local vortices effectively, so a chain of semi-local vortices would presumably bear some resemblance to the the $\mathbb{CP}^n$ chains described in chapter 7.

Finally, as was emphasised in the introduction, it seems that that the constituent structures of chains in different models are in some way related to the root structure of loop groups. Loop groups have previously been related directly to calorons, and we have shown that they are also related to sigma model chains (and by extension, planar skyrmion chains). It remains to be seen whether monopole, skyrmion, or vortex chains can be described using loop groups.

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