Generalisations of the MHV lagrangian theory to D-dimensions and to super-space

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A Thesis presented for the degree of
Doctor of Philosophy

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December, 2009

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Submitted for the degree of Doctor of Philosophy

February, 2010

Abstract

It has been known that the CSW rules correctly reproduce all tree-level scattering amplitudes for perturbative non-abelian gauge theory but fail to explain some of the loop order results. In this thesis we generalise the lagrangian derivation of the rules and account for the missing amplitudes in dimensional regularisation scheme. We analyse generically when the equivalence between MHV rules lagrangian and Yang-Mills lagrangian theory is violated and hence the CSW rules do not apply. We find a type of generalised measure-preserving transformations which when applied to the Yang-Mills lagrangian also produce vertices that have same the helicity structure as the CSW rules. Among these transformations we find in 4-dimensions the canonical transformation generates the MHV vertices that are described by the Parke-Taylor formula. Finally we generalise the canonical transformation on supersymmetric theories. In light-cone gauge the physical components of the $\mathcal{N}=1$ SYM lagrangian are closed under a subgroup of the SUSY transformations. We find the $\mathcal{N}=1$ super Yang-Mills lagrangian can be rewritten in terms of chiral and anti-chiral superfields. In both $\mathcal{N}=1$ and $\mathcal{N}=4$ theories we perform a fermionic integral transformation on superfields analogous to Fourier transform which takes functions from coordinate space into momentum space. The on-shell SUSY generators we derive from the integral transformation agree with the prescription commonly used in the supersymmetry BCFW recursion formula. We apply the canonical transformation on both supersymmetric theories and compute the generic n-point MHV super-vertex. The $\mathcal{N}=4$ MHV super-vertices are shown to agree with Nair's formula which was

originally derived from WZW model.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and is all my own work unless referenced to the contrary in the text.

Acknowledgements

First and foremost, I would like to thank Paul Mansfield, my supervisor, for his patient guidance and encouragement throughout this project. Part of the work presented in this thesis was done in collaboration with James Ettle, Jonathan Fudger, Tim Morris and Zhiguang Xiao. I am thankful for the valuable discussions and insights they provided. I would also like to thank Chong-Sun Chu, Paolo Benincasa and Patrick Dorey for their help and encouragement. I express my gratitute to the Ogden Trust for financial support.

The PhD studies at Durham University has been a truely rewarding experience. Many of the joyful hours I spent in the department office, during tea time and in the meetings were attributed to my fellow graduate students, especially to Joao Pires, Nehir Ikizlerli, Jassem Al-Alawi, Dimitrios Giataganas, Marjan Safi-Samghabadi, Julian Le Witt and Tieqiang Li. Finally, I would like to thank my family for their unconditional support.

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Chapter 1

Introduction

In the Standard Model the fundamental forces of nature (except gravity) are described by gauge fields. The theory of electroweak interaction is given by the lagrangian that respects $SU(2) \times U(1)$ gauge symmetry and for the strong interaction by the lagrangian with $SU(3)$ symmetry [1]. However the standard perturbative calculation using Feynman rules is known to be challenging for non-abelian gauge theories. This is because the number of Feynman diagrams contributing to a scattering amplitude increases rapidly with the number of external legs. Even at tree-level, a 10-gluon amplitude requires the calculation of more than ten million diagrams, making the analytic computation practically impossible [2, 3].

In [4] Parke and Taylor conjectured a summarising formula for the squares of general n-point tree-level amplitudes with special helicity configurations. Later it was proved by Berends and Giele [5] that scattering amplitudes containing only positive helicity gluons and those containing all positive except one negative helicity gluons are vanishing.

$$A(+ + \cdots +) = 0, \tag{1.1}$$

$$A(+ \cdots - \cdots +) = 0, \tag{1.2}$$

The first non-vanishing scattering amplitudes, also called the maximally helicity violating (MHV) amplitudes, contain two negative helicity gluons, and were found to be given by the remarkably simple formula

$$A(+\cdots-\cdots-\cdots+) = g^{n-2}\,\frac{\langle i\,j\rangle^4}{\langle 12\rangle\,\langle 23\rangle\cdots\langle n1\rangle}. \tag{1.3}$$

A corresponding formula was found by Nair [6] which summarises all amplitudes that are related to the MHV amplitude by the supersymmetry Ward identity in the $\mathcal{N}=4$ supersymmetric Yang-Mills theory.

Inspired by the correspondence between twistor string theory and the gauge theory, Cachazo, Svrčeck and Witten (CSW) discovered a new set of rules which dramatically simplifies gluon amplitude calculations [7, 8]. A tree-level amplitude with generic helicity content in the CSW rules is constructed by gluing together off-shell continued MHV amplitudes with scalar propagators [9]. Later the rules were successfully generalised to one-loop level [11] and to include quarks and superpartners [12, 13].

In [14, 15] an on-shell recursion formula was found by Britto, Chachazo, Feng and Witten (BCFW) by analysing singularities of the amplitude when external leg momenta are shifted by a complex value. Using Cauchy's theorem it was shown that a generic n-point amplitude can be derived from scattering amplitudes of fewer particles whose momenta are shifted by an amount determined by the positions of poles. By shifting negative helicity legs Risager [16] provided a direct proof to the CSW rules using BCFW recursion. The method of BCFW recursion was shown to be a powerful tool for tree-level calculations [17] and was extended to theories containing massive particles and fermions [18, 19]. Combining with generalised unitarity the BCFW recursion relation was also extended to loop-level calculations [20]. Recently the recursion method was modified to be incorporated into $\mathcal{N}=4$ SYM theory [21]. All tree-level super-amplitudes were obtained by Drummond and Henn in [23] by shifting Nair's MHV super-amplitude formula.

Alternatively, it has been shown by Mansfield [24] and independently by Gorsky and Rosly [25] that the CSW rules can be directly derived by canonically transforming the Yang-Mills lagrangian in the light-cone gauge where the transverse components $\mathcal{A}$ and $\bar{\mathcal{A}}$ of the gauge field in light-cone coordinates were assumed to be functions of new field variables $\mathcal{B}$ and $\bar{\mathcal{B}}$.

$$\mathcal{A}_1 = \mathcal{B}_1 + \Upsilon_{123}\mathcal{B}_2\mathcal{B}_3 + \cdots \tag{1.4}$$

$$\bar{\mathcal{A}}_1 = \bar{\mathcal{B}}_1 + \Xi^2_{123}\bar{\mathcal{B}}_2\mathcal{B}_3 + \Xi^3_{123}\mathcal{B}_2\bar{\mathcal{B}}_3 + \cdots \tag{1.5}$$

The coefficients of the expansion (1.4), (1.5) were carefully chosen so that the self-dual part of the Yang-Mills lagrangian was transformed into a free field lagrangian in the new field variables.

$$\mathcal{L}^{-+}[\mathcal{A}] + \mathcal{L}^{-++}[\mathcal{A}] = \mathcal{L}^{-+}[\mathcal{B}] \tag{1.6}$$

The vertices in the new lagrangian were shown to have the helicity feature prescribed by CSW rules. However at one-loop level, amplitudes constructed from the $(-++)$ vertex which originally appeared in the LCYM lagrangian were found to be inexplicable by CSW rules. In [26] Brandhuber, Spence, Travaglini and Zoubos adopted the light-cone friendly regulator of Thorn [27] and the contribution to the "missing" all-plus 4-point amplitude was found to be provided by the counterterm. In Chapter 3 we take another approach and regularise the theory in dimensional regularisation scheme. We show in D-dimensions the translation kernels Υ and Ξ^k in (1.4) and (1.5) have non-trivial contributions to scattering amplitudes, which results in a violation to the equivalence theorem. In section (3.3) we analyse generically when such a violation happens.

In [24] it was argued that since in 4-dimensions the MHV amplitudes only receive contributions from vertices in the new lagrangian at tree-level the difference between MHV vertices and the Parke-Taylor formula can only consist of terms containing squares of external leg momenta, which vanish on-shell. Such differences can be argued to be non-existing from holomorphy. An explicit verification up to 5-points was given by Ettle and Morris in [29]. We show in chapter 4 that the holomorphic property originates from a condition implicitly taken in the canonical transformation. In section (4.1) we give a direct proof showing that a generic n-point MHV vertex in the new lagrangian has the same form as the Parke-Taylor formula.

The canonical transformation of Mansfield was generalised to QCD and SQCD by Ettle, Morris and Xiao by taking the transformation separately on quark, gluon and gluino fields [30, 31]. The corresponding MHV lagrangians generate the CSW rules for the physical field components. In chapter 5 we use the chiral superfields of Brink, Lindgren and Nilsson [32] to reconstruct the $\mathcal{N} = 1$ SYM lagrangian in light-cone gauge. We show that the first two terms in the superfield lagrangian can be arranged into the same form as the self-dual part of the pure Yang-Mills lagrangian, from which we derive a simple canonical transformation formula for chiral superfields. The general n-point vertices in the new lagrangian are shown to take a simple general form

$$V_{N=1}^{MHV}(1^+, 2^+ \ldots i^-, j^-, \ldots n^+) = \frac{\langle i\, j\rangle^3}{\langle 12\rangle\,\langle 23\rangle \cdots \langle n1\rangle} \sum_{i,j=1}^{n} \langle i\, j\rangle\, \eta_i \eta_j . \qquad (1.7)$$

At the end of the chapter we directly derive the super-amplitude formula of Nair [6] from a lagrangian point of view. By applying the canonical transformation to the $\mathcal{N} = 4$ SYM theory we show that the resulting MHV lagrangian naturally give rise to the supersymmetry generalisation of the CSW rules.

A summary of the background and notation we use in this thesis is given in chapter 2.

Chapter 2

Preliminaries

From the modern point of view, the existence of Quantum Field Theory is a direct consequence of the combination of the principles of quantum mechanics and the symmetries of nature [34]. According to quantum principles the probability amplitude of an event is given by the inner products of physical states vectors, which are rays in the Hilbert space labeled by symmetry generators. The dynamics of the physical states is given by the symmetry invariant action formed by contracting particle fields, and accordingly, the particle fields are required to be representations of the symmetry.

In addition to spacetime symmetry, the theories in the Standard Model are assumed to be invariant under gauge transformations. Quarks and leptons fields carry extra indices that label colours or hypercharges and can be arranged into fundamental representations of $SU(2)$ or $SU(3)$ group. Generically a gauge transformation can be coordinate-dependent

$$\psi \to e^{i\alpha(x)}\psi, \tag{2.1}$$

where $\alpha(x) = \alpha^a(x)T^a$ and T^a is a generator of the gauge group, normalised so that $tr\left(T^aT^b\right) = \delta^{ab}$. The kinetic term $\partial_\mu\psi$ in the lagrangian requires evaluation of the fermion fields at different points and does not have a simple transformation property. Therefore the assumption that the lagrangian be invariant under local gauge transformation (2.1) naturally brings about the introduction of a connection, which is itself a non-abelian gauge field $\mathcal{A}^a_\mu(x)$ in the adjoint representation. In the absence of fermions, the action of the system is given by the integral of the Yang-Mills lagrangian

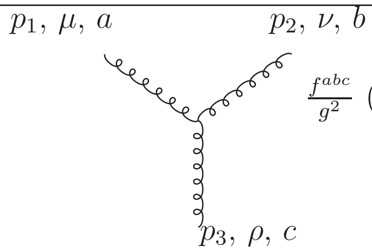
$$S = \frac{-4}{g^2} \int d^4x \, tr \left(F^{\mu\nu} F_{\mu\nu} \right), \tag{2.2}$$

with the field strength given by $F_{\mu\nu} = [D_\mu, D_\nu]$, and the covariant derivative is defined as $D_\mu = \partial_\mu + \mathcal{A}_\mu$. When calculating particle scattering amplitudes in this thesis the analysis we use does not depend on properties exclusive to any specific gauge group. In the following we shall assume that the lagrangian is invariant under a generic $SU(N_C)$ and the particles generated by the gauge field are commonly referred to as gluons. We compute the Green function perturbatively using functional integral, in which the gauge degree of freedom is fixed by the De Witt-Faddev-Popov method with the help of the introduction of the ghost fields. The particle scattering amplitudes observed in particle colliders are then calculated from Green functions using the LSZ reduction

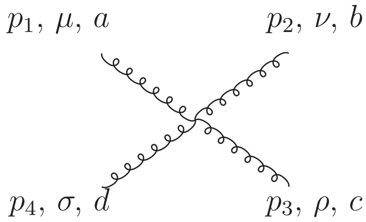
$$A(p_1, \sigma_1, a_1; \, p_2, \sigma_2, a_2 \cdots) = \lim_{p_i^2 \to 0} \prod_i \frac{p_i^2}{\sqrt{Z}} \left\langle \epsilon^{\sigma_1} \cdot \mathcal{A}^{a_1}(p_1) \, \epsilon^{\sigma_2} \cdot \mathcal{A}^{a_2}(p_2) \cdots \right\rangle, \tag{2.3}$$

where p_i, σ_i, a_i label the momenta, helicity, and the colour of the gluons respectively, and the factors ϵ^{σ_i}, Z appearing on the right hand side of the equation (2.3) are the polarisation and the field-strength renormalisation.

The Yang-Mills lagrangian (2.2) contains a 3-point and a 4-point self-interacting term:



$$\frac{f^{abc}}{g^2}\left(g^{\mu\nu}(p_2-p_1)^\rho+g^{\nu\rho}(p_3-p_2)^\mu+g^{\rho\mu}(p_1-p_3)^\nu\right)$$



$$\begin{aligned}\frac{1}{g^2}\Big(&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\&+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\\&\quad+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})\Big)\end{aligned}$$

Figure 2.1:

As noted in the introduction, a direct analytic calculation of the gluon scattering amplitude using perturbative method was found to be inefficient because the number of Feynman diagrams contributing to an amplitude increases rapidly with the number of external legs. (Asymptotically greater than $n!$ for an n-gluon scattering process. [35]) In this chapter we summarise the techniques that simplify the calculation. More thorough reviews can be found in [36, 37].

2.1 Colour decomposition and the colour ordered Feynman rules

In a series of papers [38, 39], an algebraic procedure was developed to manage the colour indices ubiquitous in the Feynman rules. From (Fig.2.1) we see that the colour dependence shows up in both the 3-point and the 4-point vertices through structure constants, which can be rewritten as a subtraction of two traces.

$$i\,f^{abc}=tr\left(T^aT^bT^c\right)-tr\left(T^bT^aT^c\right) \tag{2.4}$$

In a perturbative calculation the colour indices in these traces are linked by the Kronecker deltas δ^{ab} in the propagator. After using the Fierz rearrangement identity

$$(T^a)_i^{\ j}(T^a)_k^{\ l} = \delta_i^{\ l}\delta_k^{\ j} - \frac{1}{N_C}\delta_i^{\ j}\delta_k^{\ l} \tag{2.5}$$

to combine the traces with repeated dummy indices the Green function is given by a sum over products of traces containing only colour indices of the external legs multiplied by colour-ordered sub-amplitudes that depend on purely kinematic factors. The factorisation of the colour dependence into trace factors allows us to compute the trace part and the kinematic part separately. The traces associating with each sub-amplitude can be easily calculated from a diagrammatic approach and the kinematic sub-amplitude can be read off from the diagram using Feynman rules with colour dependence stripped off.

For tree-level amplitudes, the colour dependence is especially simple. The scattering amplitude factorises into single trace terms multiplied by the colour-ordered amplitude.

$$\begin{aligned}
&A^{tree}(p_1, \sigma_1, a_1; p_2, \sigma_2, a_2 \cdots) \\
&= (2\pi)^4\delta(\sum_{i=1}^{n} p_i)g^{n-2} \\
&\quad \times \sum_{j\in S_n/Z_n} tr(T^{a_{j_1}}T^{a_{j_2}}\cdots t^{a_{j_n}})A^{tree}(p_{j_1}, \sigma_{j_1}, a_{j_1}; p_{j_2}, \sigma_{j_2}, a_{j_2}\cdots)
\end{aligned} \tag{2.6}$$

where the summation runs through all possible permutations of the external legs that cannot be arranged into each other by a cyclic permutation.

For the purpose of calculation, it is also useful to compute the trace factors in an extended $U(N_C) = SU(N_C) \times U(1)$ theory, where the Fierz rearrange identity (2.5) does not have the $1/N_C$ term because of the inclusion of an auxiliary $U(1)$ gauge "photon" field $\mathcal{A}_\mu^0$. The zero colour identity "generator" $T^0 = I/\sqrt{N_C}$ associated with the new photon field commute with all other generators in the $SU(N_C)$, $i\, f^{0ab} = tr(T^aT^b) - tr(T^bT^a) = 0$, therefore introducing a $U(1)$ gauge photon into internal lines does not alter the value of gluon scattering amplitudes. By choosing one of the external legs to have zero colour we arrive at an identity relating colour-ordered amplitudes with different permutations. For example, if we assume the particle carrying momentum p_1 to be a photon, from the above argument the scattering amplitude vanishes. Collecting terms with the same colour dependent trace factor $tr(T^{a_2}T^{a_3}\cdots T^{a_n})$, we obtain the photon decomposition equation [39]

$$\begin{aligned}A(1^{\sigma_1},2^{\sigma_2},3^{\sigma_3},\cdots n^{\sigma_n})+A(2^{\sigma_2},1^{\sigma_1},3^{\sigma_3},\cdots n^{\sigma_n})\\ +A(2^{\sigma_2},3^{\sigma_3},1^{\sigma_1},\cdots n^{\sigma_n})+\cdots=0\end{aligned} \tag{2.7}$$

2.2 Spinor, spinor brackets, and the Parke-Taylor formula

Perhaps to one's surprise, the colour-ordered sub-amplitudes introduced in the previous section often take remarkably simple forms when expressed in terms of spinors. In this section we introduce the spinor and light-cone coordinate notations used throughout this thesis.

To start with, we adopted the shorthand notation ($\check{p}$, $\hat{p}$, p, $\bar{p}$) to describe covariant vectors in 4-dimensions in light-cone coordinates, which is related to Minkowski coordinates by

$$\check{p}=(p_0-p_3)\,,\;\hat{p}=(p_0+p_3)\,,\;p=(p_1-ip_2)\,,\;\bar{p}=(p_1+ip_2)\,. \tag{2.8}$$

In the light-cone coordinates the metric is off-diagonal. The inner product of two vectors is given by

$$p\cdot q=\left(\check{p}\hat{q}+\hat{p}\check{q}-p\bar{q}-\bar{p}q\right)/2. \tag{2.9}$$

To keep the notation simple, we directly use the number n with the appropriate decoration ($\check{n}$, $\hat{n}$, $\tilde{n}$, $\bar{n}$) to denote the momentum $p_{n\,\mu}$ of the n^{th} external leg of a scattering amplitude in light-cone coordinates.

A 4-vector can be written in the form of a bispinor

$$P_{\alpha\dot{\alpha}}=p^{\mu}\sigma_{\mu\alpha\dot{\alpha}}=\begin{pmatrix}\check{p} & -p\\ -\bar{p} & \hat{p}\end{pmatrix} \tag{2.10}$$

by contracting with $\sigma_\mu = (I_2, \overrightarrow{\sigma})$, where I_2 is the 2×2 identity matrix and σ stands for Pauli matrices, and we use a capital letter to distinguish the bispinor and the 4-vector participating the contraction. If p_μ is lightlike, $\check{p} = p\bar{p}/\hat{p}$ and the bispinor factorises as $p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$, where

$$\lambda_\alpha = \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}, \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} -\bar{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}. \tag{2.11}$$

Note that for complex value momenta the spinor $\bar{\lambda}_{\dot{\alpha}}$ is not necessarily related to λ_α by complex conjugation. Spinors $\lambda_{i\alpha}$, $\lambda_{j\alpha}$ associated with different massless particles carrying momenta $p_{i\,\mu}$ and $p_{j\,\mu}$ can be contracted to given an anti-symmetric Lorentz invariant angle bracket

$$\langle i\, j\rangle = \epsilon^{\alpha\beta} \lambda_{i\alpha} \lambda_{j\beta} = \frac{(i\, j)}{\sqrt{\hat{i}\,\hat{j}}}. \tag{2.12}$$

Similarly, we define a square bracket to be the spinor product of $\bar{\lambda}_{i\dot{\alpha}}$, $\bar{\lambda}_{j\dot{\alpha}}$

$$[i\, j] = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{i\dot{\alpha}} \bar{\lambda}_{j\dot{\beta}} = \frac{\{i, j\}}{\sqrt{\hat{i}\,\hat{j}}}, \tag{2.13}$$

The round bracket and the curly bracket in (2.12) and (2.13) are

$$(i\, j) = \hat{i}\,\tilde{j} - \hat{j}\,\tilde{i}, \; \{i\, j\} = \hat{i}\,\bar{j} - \hat{j}\,\bar{i} \tag{2.14}$$

We shall refer to spinor λ_α, the angle bracket and the round bracket as holomorphic, while the spinor $\bar{\lambda}_{\dot{\alpha}}$ and brackets associated with it are referred as anti-holomorphic because of the bar-component dependence. Using the definition for angle and square brackets the inner product of two null vectors can be expressed as

$$p \cdot q = \frac{1}{2} \langle p\, q\rangle\, [p\, q] \tag{2.15}$$

An advantage of introducing the spinor notation is that the polarisations of the gauge field have simple expressions in terms of spinors. In the textbook description the polarisation vectors are usually defined with the help of a light-like reference momentum η_μ. However, by assuming the polarisation as a function of 4-momentum and the reference momentum one always encounters a branch cut in the definition [8]. A convenient solution to the problem is to define the polarisation as a function of spinors. In [40] a formula was given by Xu, Zhang and Chang

$$\epsilon^{\pm}_{\mu}(\eta,\, p) = -\frac{\overline{u_{\mp}(\eta)}\,\gamma_\mu u_{\mp}(p)}{\overline{u_{\mp}(\eta)}\,u_{\pm}(p)}, \tag{2.16}$$

where $u_{\pm}$ and $v_{\pm}$ are the Dirac 4-spinors in the Weyl representation, $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$, and $\epsilon^{\pm}_{\mu}$ denotes the polarisations of positive and negative helicity gluons. Since both the gluon momentum and the reference momentum are null vectors, we can express the Dirac spinors in terms of the two-component spinors in (2.11)

$$u_+(p) = v_-(p) = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix},\; u_-(p) = v_+(p) = \begin{pmatrix} 0 \\ \epsilon^{\dot\alpha\dot\beta}\bar{\lambda}_{\dot\beta} \end{pmatrix} \tag{2.17}$$

Applying the algebraic identities $g^{\mu\nu}\,\sigma_{\mu\,\alpha\dot\alpha}\sigma_{\nu\,\beta\dot\beta} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$ and $\bar{\sigma}^{\dot\alpha\alpha}_{\mu} = \epsilon^{\beta\alpha}\sigma_{\mu\,\beta\dot\beta}\epsilon^{\dot\beta\dot\alpha}$ to the definition (2.16), we arrive at a compact notation for the polarisations in bispinor form $\epsilon^{\pm}_{\alpha\dot\alpha} = \epsilon^{\pm\,\mu}\sigma_{\mu\,\alpha\dot\alpha}$,

$$\epsilon^{+}_{\alpha\dot\alpha} = \frac{\eta_\alpha\bar{\lambda}_{\dot\alpha}}{\langle\eta\,\lambda\rangle},\; \epsilon^{-}_{\alpha\dot\alpha} = \frac{\lambda_\alpha\bar{\eta}_{\dot\alpha}}{[\lambda\,\eta]} \tag{2.18}$$

The polarisation vectors defined above can be easily checked to satisfy the following properties

$$\epsilon^{\pm}(\eta,\, p)\cdot\eta = \epsilon^{\pm}(\eta,\, p)\cdot p = 0, \tag{2.19}$$

$$\epsilon^{\pm}(\eta,\, p)\cdot\epsilon^{\pm}(q,\, p) = 0, \tag{2.20}$$

$$\epsilon^{\pm}(\eta,\, p)\cdot\epsilon^{\pm}(\eta,\, q) = 0, \tag{2.21}$$

$$\epsilon^{\pm}(\eta,\, p)\cdot\epsilon^{\mp}(p,\, q) = 0, \tag{2.22}$$

and $\epsilon^{\pm}$ are normalised so that

$$\epsilon^{+}(\eta,\, p)\cdot\epsilon^{-}(\eta,\, p) = 1. \tag{2.23}$$

2.2.1 MHV amplitudes

An argument was provided by Dixon in [36] using the identities (2.19) to (2.23) for polarisation vectors to show that the colour-ordered amplitudes for arbitrary number of external legs having all positive helicities and all positive except one negative helicities vanish at tree-level. In amplitudes that contain only positive helicity gluons, we can choose all of the reference momenta to be the same. Because an n-point gluon scattering amplitude can have at most n-2 vertices, there are at most n-2 momenta to be contracted with polarisation vectors, leaving at least one pair of $\epsilon^+ \cdot \epsilon^+$, which is zero from (2.21). For amplitudes with one negative helicity gluon, we choose the reference momenta for positive gluons to be the momentum carried by the negative helicity gluon. From (2.22) we have $\epsilon^- \cdot \epsilon^+ = 0$ for all possible pairs of polarisation vectors[1]. Since scattering amplitudes do not depend on reference momenta, the argument holds for other choices as well. The next amplitude is non-vanishing because even when all reference momenta of positive gluons are taken to be one of the negative gluon momenta, the polarisation of the other negative helicity gluon can have non-zero inner products with the polarisation of the positive helicity gluons. In [5] Berends and Giele proved from a recursion method that the generic n-point two-negative helicity amplitude at tree-level can be written as

$$A(1^+, 2^+, \cdots i^-, \cdots j^- \cdots n^+) = g^{n-2} \frac{\langle i\, j \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n\, 1 \rangle}. \tag{2.24}$$

The formula was first conjectured by Parke and Taylor and verified up to 6-points. Because the amplitude (2.24) has maximum number of positive helicity gluons, it is commonly referred to as Maximally Helicity Violating amplitude, or MHV for short.

Because taking complex conjugate of a polarisation vector reverses its helicity

[1] There is, however, one exception because the reference momentum of the polarisation $\epsilon^{\pm}_{\mu}(\eta, p)$ is not allowed to be parallel to p. In a three-particle scattering event the only combination of three null vectors that satisfy momentum conservation has all three vectors parallel to each other and therefore the reference momenta of leg 2 or leg 3 cannot be chosen as p_1. For real momenta we can still show that the amplitude $A(1^-, 2^+, 3^+)$ vanishes. We present the argument in section 2.4.1.

$(\epsilon^{\pm}_{\mu})^* = \epsilon^{\mp}_{\mu}$, the above argument can similarly be used to show that the all-negative and the all-negative-except-one-positive amplitudes are zero, and the two-positive helicity amplitudes are given by the Parke-Taylor formula (2.24) with all the angle brackets replaced by square brackets.

As an example we calculate the $A(1^-, 2^+, 3^-, 4^+)$ MHV amplitude. Choosing reference momenta $\bar{\eta}_1 = \bar{\eta}_3 = \bar{\lambda}_4$ and $\eta_2 = \eta_4 = \lambda_3$, from (2.19) to (2.23) we see that the only non-vanishing inner product between polarisation vectors is $\epsilon^-_1 \cdot \epsilon^+_2$, therefore the choice we made eliminates all except one of the Feynman graphs (Fig.2.2).

$$\begin{aligned}
&\left[\epsilon^+_4 \cdot \epsilon^-_1 \ (p_1 - p_4)^\mu + \epsilon^{-\mu}_1 \epsilon^+_4 \cdot (p_2 + p_3 - p_1) + \epsilon^{+\mu}_4 \epsilon^-_1 \cdot (p_4 - p_2 - p_3)\right] \\
&\times \left[\epsilon^+_2 \cdot \epsilon^-_4 \ (p_3 - p_2)_\mu + \epsilon^-_{3\,\mu} \epsilon^+_2 \cdot (p_2 + p_3 - p_1) + \epsilon^+_{2\,\mu} \epsilon^-_3 \cdot (p_4 - p_2 - p_3)\right] \\
&\times 1/(p_1 + p_4)^2
\end{aligned} \tag{2.25}$$

To compare the amplitude with Parke-Taylor formula we use (2.18) and (2.15) to express the polarisation vectors and the inner products in (2.25) as products of spinors. Because all of the external leg momenta are light-like, by contracting the total momentum in bispinor form $\sum_i \lambda_i \bar{\lambda}_i$ with an arbitrary pair of holomorphic and anti-holomorphic spinors we obtain an identity for products of brackets

$$\sum_i \langle j\, i\rangle\, [i\, k] = 0, \tag{2.26}$$

which allows us to convert (2.25) into

$$A(1^-, 2^+, 3^-, 4^+) = -g^2 \frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}. \tag{2.27}$$

2.3 CSW rules

Inspired by the twistor string theory, Cachazo, Svrček and Witten developed a set of rules which allows gluon scattering amplitudes of generic helicity contents to be more easily calculated [7]. In the CSW prescription the Parke-Taylor formula (2.24)

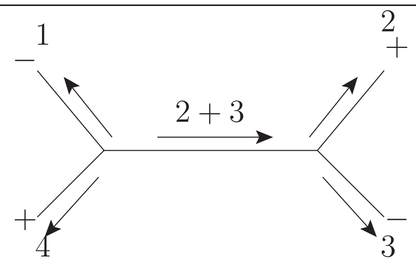


Figure 2.2:

which was originally used to describe MHV amplitudes is taken as the formula for the new vertices. The vertices are linked by scalar propagators $1/p^2$ to construct amplitudes of generic helicity structures. When an MHV vertex is connected to an internal line the off-shell continuation of the spinor associating to the line is defined by $\lambda_\alpha = P_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}/\,[\lambda\,\eta]$, where $\bar{\eta}$ is an arbitrary anti-holomorphic spinor. In [7] the CSW rules were verified to reproduce the known scattering amplitudes up to 7-points. At one-loop level however, amplitudes consisting of only positive helicity gluons cannot be built from MHV vertices, and yet such amplitudes were found to be non-vanishing [43–47].

For example the amplitude $A(1^+,2^+,3^+,4^+)$ was computed by Bern, Chalmers, Dixon Kosower and Mahlon in [43,44] to have non-vanishing quadruple cut, which is defined by replacing the four propagators in box integral by delta functions. The box diagram is therefore cut into four subamplitudes each contains one gluon scattered by two massive scalars:

$$g^4\,\frac{\overline{u_-(\eta_1)}\,\not{q}\,u_-(p_1)}{\langle\eta_1 1\rangle}\;\frac{\overline{u_-(\eta_2)}\,(\not{q}-\not{p}_2)\,u_-(p_2)}{\langle\eta_2 2\rangle}$$
$$\times\frac{\overline{u_-(\eta_3)}\,(\not{q}-\not{p}_2-\not{p}_3)\,u_-(p_3)}{\langle\eta_3 3\rangle}\;\frac{\overline{u_-(\eta_4)}\,(\not{q}+\not{p}_1)\,u_-(p_4)}{\langle\eta_4 4\rangle}, \quad (2.28)$$

and the cut conditions can be rearranged as

$$q^2-\mu^2=0,\; q\cdot p_2=0,\; q\cdot p_3=p_2\cdot p_3,\; q\cdot p_1=0 \quad (2.29)$$

Choosing reference spinors η_1,η_2 to be p_2,p_1 allows us to combine the numerators of the first two terms into a trace $tr(\not{p}_1(\not{q}-\not{p}_2)\not{p}_2\not{q})$. From (2.29) the trace yields $(p_1\cdot p_2)q^2=\langle 12\rangle[12]\mu^2$. Applying a similar method on the last two terms gives

$$A(1^+,2^+,3^+,4^+)\big|_{\text{quadruple cut}} = g^2\,\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\,\mu^4\,J, \quad (2.30)$$

where J is the jacobian obtained from integrating over four delta functions. In [48] it has been shown by Bern, Dixon, Dunbar and Kosower (BDDK) that a generic one-loop graph can be spanned by scalar box, triangle and bubble integrals upto a rational kinematic term, while the expanion coefficients can be quickly found by matching the discontinuities of the original LCYM box graph contribution and the discontinuities of the basis. The above analysis on quadruple cut determines the discontinuity that happens when the singularities from the surfaces $q^2 = 0$, $(q-p_2)^2 = 0$, $(q-p_2-p_3)^2 = 0$ and $(q+p_1)^2 = 0$ all collide with each other, which only occurs in box integral among the basis, so we have:

$$A(1^+, 2^+, 3^+, 4^+) = g^2 \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle} K_4 + \text{triangle} + \text{bubble} + \text{rational term}, \tag{2.31}$$

where K_4 is the dimension-regularised box integral with the loop momenta defined in (Fig.2.3).

$$K_4 = \frac{1}{(4\pi)^{4-2\epsilon}} \int d^{4-2\epsilon}q \, \frac{\mu^4}{q^2\,(q-p_1)^2(q-p_1-p_2)^2(q+p_4)^2} \tag{2.32}$$

Equation (2.31) shows that the all-plus amplitude is clearly nonzero at one-loop level. In the next chapter we shall provide a lagrangian point of view to explain the origin of this amplitude.

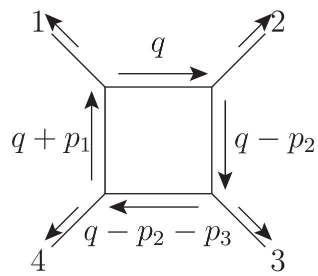


Figure 2.3:

2.4 BCFW recursion relation and a proof to the formula for general n-point MHV amplitudes

In [14] an on-shell recursion relation was found by Britto, Cachazo and Feng which allows us to quickly generate scattering amplitudes by reusing known amplitudes with fewer legs. The recursion relation was proved with the help of Witten using Cauchy's residue theorem [15]. Instead of calculating amplitudes directly using Feynman rules, we can consider the amplitude as a function of complex value momenta obtained by shifting the chosen external lines by a null vector η

$$p_i \to p_i + z\,\eta \tag{2.33}$$

where z is a complex variable and the internal lines are also shifted in accordance with momentum conservation. An inspection of the basic elements used in the Feynman rules shows that the singularities of the amplitude can either come from the denominator of polarisation vectors or from a propagator. By choosing the reference momenta of the shifted legs to be η we can eliminate all of the z dependence from the terms $\langle \eta\, \lambda \rangle$ and $[\eta\, \lambda]$ in the polarisation vector (2.18). Since scattering amplitudes do not depend on the reference momenta chosen, this suggests that singularities of the first type must cancel for arbitrary choices of reference momenta. Consider integrating $\frac{1}{z}\, A(z)$ over a contour at infinity. If $A(z) \to 0$ at infinity, nothing contributes to the integral. From Cauchy's Theorem, the sum over residues is zero, and the unshifted physical scattering amplitude $A(0)$ can be derived from adding up all other residues on the complex plane.

$$0 = \frac{1}{2\pi i} \oint_{\mathcal{C}\,\text{at}\,\infty} dz\, \frac{1}{z} A(z) = A(0) + \sum_{\text{rest of the poles}\, z_i} \frac{1}{z_i}\, Res_{zi} A(z_i) \tag{2.34}$$

Because η is light-like, $(p_i + z\,\eta)^2 = p_i^2 + 2\,z\,(p_i \cdot \eta)$, the singularities from the propagators contain only simple poles. At pole $z_i = -p_i^2/2(p_i \cdot \eta)$ the factor $1/z$ combines with the overall coefficient $2(p_i \cdot \eta)$ extracted from the simple pole term to restore the unshifted propagator $-1/p_i^2$.

The residue at z_i has a clear physical meaning [15]: The propagator responsible for producing the pole divides tree-level graphs into two parts, both of which retain the original kinematic dependence prescribed by Feynman rules but with legs shifted. If we allow complex value momenta the left and right part of all the divided graphs combine as before to yield amplitudes, where the dividing legs are guaranteed to be null by singularity conditions. Writing the left and right part of the subamplitudes as $A_L(z_i)$ and $A_R(z_i)$, the above identity can be written as

$$A(0) = \sum_{poles\, z_i \neq 0} A_L(z_i) \frac{1}{p_i^2} A_R(z_i), \tag{2.35}$$

Alternatively the above identity can be understood from an algebraic point of view: Imagine all of the differences from residues in $A(z)$ were forcibly extracted and collected as remainders, which would diverge as $z \to \infty$ due to the difference factor $(z - z_i)$ in the numerator. For amplitudes that vanish for large z such terms can only be canceled by each other, leaving the formula as (2.35). In the following we choose shifting directions η to derive the frmula for n-point MHV amplitude and to prove CSW rules.

2.4.1 Proving the Parke-Taylor formula

The BCFW recursion just introduced can be used to prove that the n-point MHV amplitude is described by the Parke-Taylor formula (2.24). First we need to derive the formula for the 3-point MHV amplitudes $A(1^-, 2^-, 3^+)$. In section 2.2.1 we showed that for specially chosen reference momenta all amplitudes containing all negative helicity or only one positive helicity gluons are vanishing. However as noted in footnote 1 for the 3-point amplitude such choices are forbidden by momentum conservation and on-shell conditions. Because gluons are massless, $p_3^2 = (p_1 + p_2)^2 = 0$, from (2.15) we have

$$\langle 12 \rangle \, [12] = 0. \tag{2.36}$$

Similarly, for the other two gluons,

$$\langle 23 \rangle \, [23] = 0, \tag{2.37}$$

$$\langle 31 \rangle \, [31] = 0. \tag{2.38}$$

For real value momenta, equations (2.36) to (2.38) give $|\langle 12\rangle|^2 = |\langle 23\rangle|^2 = |\langle 31\rangle|^2 = 0$ and all three spinors are parallel to each other. Since the reference momentum of the polarisation vector $\epsilon^-_\mu(\eta_1,\, p_1)$ is not allowed to be parallel to p_1 we cannot choose $\eta_1 = p_3$ to make $\epsilon^-(\eta_1,\, p_1)\cdot\epsilon^+(\eta_3,\, p_3)$ vanish. However the 3-point amplitude is straightforward to calculate,

$$A(1^-, 2^-, 3^+) = g\,\frac{\hat{3}}{\hat{1}\,\hat{2}}\,(12)\,. \tag{2.39}$$

When external leg momenta are real, $(12) = 0$ so the amplitude vanishes. The same argument shows that the 3-point $\overline{\text{MHV}}$ amplitude is also zero.

For complex momenta it is possible to have $[13] = 0$ so that we are again not allowed to choose reference momentum to let $\eta_1 = p_3$. Because angle brackets are no longer related to square brackets by complex conjugation, generically (12) can be non-zero and the 3-point MHV amplitude can be put into the form

$$A(1^-, 2^-, 3^+) = g\,\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle} \tag{2.40}$$

which agrees with the Parke-Taylor formula. Following the same analysis the 3-point $\overline{\text{MHV}}$ amplitude $A(1^+, 2^+, 3^-)$ is given by a similar formula with angle brackets in (2.40) replaced by square brackets

$$A(1^+, 2^+, 3^-) = g\,\frac{[12]^4}{[12]\,[23]\,[31]}. \tag{2.41}$$

To apply the BCFW recursion on an n-point MHV amplitude we label its external lines cyclically starting from one of the negative helicity gluons. The scattering amplitude is considered as a function of spinors associated with external lines: The momentum flowing through an internal line is defined by its bispinor (2.10), which is in turn related the spinors of external lines through momentum conservation; the scalar product of two vectors p, q (not necessarily null) appearing in the vertices is given by the contraction of their bispinors, $p\cdot q = \frac{1}{2}\,\epsilon^{\alpha\beta}P_{\alpha\dot{\alpha}}Q_{\beta\dot{\beta}}\,\epsilon^{\dot{\alpha}\dot{\beta}}$, and the polarisation vectors are defined by spinors of external line momenta and reference momenta through equation (2.18). We shift the adjacent legs p_1 and p_n of the MHV amplitude $A(1^-, 2^+, \cdots i^-, \cdots n^+)$ by

$$P'_{1\,\alpha\dot{\alpha}}(z) = \lambda_{1\,\alpha}\bar{\lambda}_{1\,\dot{\alpha}} - z\,\lambda_{1\,\alpha}\bar{\lambda}_{n\,\dot{\alpha}} \tag{2.42}$$

$$P'_{n\,\alpha\dot{\alpha}}(z) = \lambda_{n\,\alpha}\bar{\lambda}_{n\,\dot{\alpha}} + z\,\lambda_{1\,\alpha}\bar{\lambda}_{n\,\dot{\alpha}} \tag{2.43}$$

The above conditions leave λ_1 and $\bar{\lambda}_n$ unchanged, while $\bar{\lambda}_1'(z) = \bar{\lambda}_1 - z\,\bar{\lambda}_n$, and $\lambda_n'(n) = \lambda_n + z\,\lambda_1$. From the recursion formula (2.35) the amplitude receives contributions from (Fig.2.4) and (Fig.2.5). Reading off from graph (Fig.2.4) gives

$$\frac{\langle 1,\, i\rangle^4}{\langle 12\rangle\,\langle n-2,\, q'\rangle\,\langle q',\, 1\rangle}\;\frac{1}{\langle n-1,\, n\rangle\,[n-1,\, n]}\;\frac{[n-1,\, n]^4}{[n-1,\, n]\,[n,\, q']\,[q',\, n-1]} \tag{2.44}$$

where we used the condition $Q^2 = (P_n + P_{n-1})^2 = \langle n-1,\, n\rangle\,[n-1,\, n]$ to substitute the unshifted propagator. After replacing the remaining q' dependent terms by

$$\left\langle n-2,\, q'\right\rangle\left[q',\, n\right] = \langle n-2,\, n-1\rangle\,[n-1,\, n] \tag{2.45}$$

$$\left\langle 1,\, q'\right\rangle\left[q',\, n-1\right] = \langle 1,\, n\rangle\,[n,\, n-1] \tag{2.46}$$

equation (2.44) has the desired form (2.24).

The other graph can be shown to be vanishing: At pole the internal line in (Fig.2.5) carries momentum

$$Q'_{\alpha\dot{\alpha}}(z) = \lambda_{1\,\alpha}\bar{\lambda}_{1\,\dot{\alpha}} - z\,\lambda_{1\,\alpha}\bar{\lambda}_{n\,\dot{\alpha}} + \lambda_{2\,\alpha}\bar{\lambda}_{2\,\dot{\alpha}} = \lambda'_{q\,\alpha}\bar{\lambda}'_{q\,\dot{\alpha}}, \tag{2.47}$$

where $z = [12]/[n2]$. Contracting with $\lambda_{1\,\alpha}$ we obtain

$$\left\langle 1\, q'\right\rangle\bar{\lambda}'_{q\,\dot{\alpha}} = \langle 12\rangle\,\bar{\lambda}_{2\,\dot{\alpha}} \tag{2.48}$$

If λ_2 is not parallel to λ_1 then $\bar{\lambda}_2 = \frac{\langle 1\, q'\rangle}{\langle 12\rangle}\bar{\lambda}'_q$. Otherwise $\lambda_2 = c\,\lambda_1$ for some constant c and we have $\lambda'_q = \lambda_1$ and $\bar{\lambda}'_q = \bar{\lambda}_1 - \frac{[12]}{[n2]}\,\bar{\lambda}_n + c\,\bar{\lambda}_2$. In both cases $\left[q'\, 2\right] = 0$ and the 3-point $\overline{\text{MHV}}$ amplitude on the left hand side is zero. Since the 3-point MHV amplitude (2.40) agrees with the Parke-Taylor formula by induction this is generalised to n-points.

Note that in (Fig.2.4) we implicitly assumed that the other negative helicity leg does not happen to be the leg p_{n-1} on the right hand side of the graph. For amplitudes beyond 4-points we can choose to shift the other negative helicity so that after turning the graph upside down and relabeling the external lines we return to

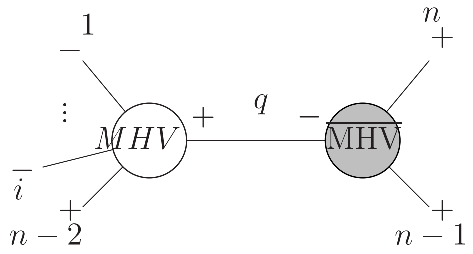


Figure 2.4:

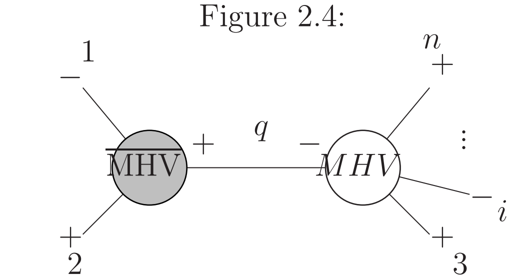


Figure 2.5:

the situation described by (Fig.2.4). The only amplitude left is $A(1^-, 2^+, 3^-, 4^+)$, which we computed directly in section (2.2.1).

2.4.2 Proving the CSW rules

A proof to the CSW rules was given by Risager in [16] using BCFW recursion. Starting with the next-to-MHV amplitude (NMHV) we shift the anti-holomorphic spinors of the three negative helicity legs p_i, p_j and p_k in the direction of an arbitrary spinor $\bar{\eta}$.

$$P'_{i\,\alpha\dot{\alpha}}(z) = \lambda_{i\,\alpha}\left(\bar{\lambda}_{i\,\dot{\alpha}} + z\,\langle j\,k\rangle\,\bar{\eta}_{\dot{\alpha}}\right) \tag{2.49}$$

$$P'_{j\,\alpha\dot{\alpha}}(z) = \lambda_{j\,\alpha}\left(\bar{\lambda}_{j\,\dot{\alpha}} + z\,\langle k\,i\rangle\,\bar{\eta}_{\dot{\alpha}}\right) \tag{2.50}$$

$$P'_{k\,\alpha\dot{\alpha}}(z) = \lambda_{k\,\alpha}\left(\bar{\lambda}_{k\,\dot{\alpha}} + z\,\langle i\,j\rangle\,\bar{\eta}_{\dot{\alpha}}\right) \tag{2.51}$$

Additional brackets $\langle i\,j\rangle$ are included in the shiftings so that the total momentum shifted in (2.49) is zero from Jacobi identity. The singularities in the propagators split the NMHV amplitude into two MHV amplitudes (Fig.2.6, 2.7).

The argument applies to the N^pMHV amplitudes which contain $2+p$ negative helicity gluons. This can be done either by recursively shifting any three of the

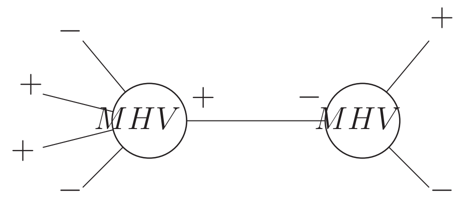


Figure 2.6:

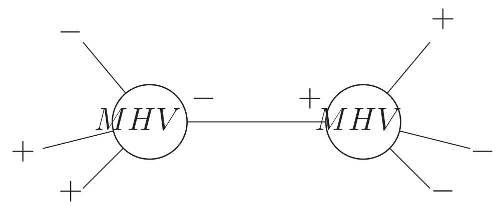


Figure 2.7:

negative helicity gluons to split the amplitude into two fewer leg amplitudes or shifting all of the negative helicity gluons by

$$P^{'}_{i\,\alpha\dot{\alpha}}(z) = \lambda_{i\,\alpha}\left(\bar{\lambda}_{i\,\dot{\alpha}} + z\,r_i\,\bar{\eta}_{\dot{\alpha}}\right) \tag{2.52}$$

with the coefficient r_i chosen so that $\sum_{i=1}^{p}\lambda_i r_i = 0$. From section 2.4.1 we saw the MHV amplitudes are described by the holomorphic Parke-Taylor formula, and therefore remain unchanged after the shifting (2.49) or (2.52). Applying BCFW recursion yields a product of $p+1$ MHV amplitudes linked by one unshifted propagator obtained from the first splitting together with $p-1$ shifted propagators $p^{'}_j$.

$$\sum_{poles\,z_i}\frac{1}{p_i^2}\prod_{j\neq i}\frac{1}{p^{'\,2}_j} \tag{2.53}$$

Consider a fictitious theory in which the building blocks in the Feynman rules contain scalar propagator and vertices that have the helicity structure prescribed by the CSW rules, but instead of contributing kinematic factors, in this fictitious theory the vertex factors are simply identities. Applying the same shifting (2.49) or (2.52) in such a theory on the same N^pMHV amplitude from BCFW recursion we obtain an identity $\prod_i \frac{1}{p_i^2} = \sum_i \frac{1}{p_i^2}\prod_j \frac{1}{p^{'\,2}_j}$, which allows us to replace (2.53) by the product of unshifted propagators. The recursion formula then agrees with the CSW prescription.

2.5 The MHV-rules lagrangian for pure Yang-Mills theory

In [24] and [25] a direct derivation of the CSW rules from the Yang-Mills lagrangian was developed by Mansfield, Gorsky and Rosly. When expressed as a bispinor, a generic off-shell 4-momentum can always be written as a sum of two bispinors of light-like momenta.

$$P_{\alpha\dot{\alpha}} = p^{\mu}\sigma_{\mu\,\alpha\dot{\alpha}} = \eta_{\alpha}\bar{\eta}_{\dot{\alpha}} + \lambda_{\alpha}\bar{\lambda}_{\dot{\alpha}} \tag{2.54}$$

We choose λ and $\bar{\lambda}$ to be the same holomorphic and anti-holomorphic spinors used in section 2.4.1 to derive n-point MHV amplitudes.

$$\eta_{\alpha} = \begin{pmatrix} \sqrt{\check{p} - p\bar{p}/\hat{p}} \\ 0 \end{pmatrix}, \; \bar{\eta}_{\dot{\alpha}} = \begin{pmatrix} \sqrt{\check{p} - p\bar{p}/\hat{p}} \\ 0 \end{pmatrix} \tag{2.55}$$

$$\lambda_{\alpha} = \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}, \; \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} -\bar{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix} \tag{2.56}$$

For massless particles η and $\bar{\eta}$ vanish on-shell and $P_{\alpha\dot{\alpha}}$ is separable. The polarisation vectors of the gauge field are defined as in (2.18)

$$\epsilon^{+}_{\alpha\dot{\alpha}} = \frac{\eta_{\alpha}\bar{\lambda}_{\dot{\alpha}}}{\langle\eta\,\lambda\rangle}, \; \epsilon^{-}_{\alpha\dot{\alpha}} = \frac{\lambda_{\alpha}\bar{\eta}_{\dot{\alpha}}}{[\lambda\,\eta]}, \tag{2.57}$$

where instead of allowing the reference spinors to be chosen later on we fix them as the spinors η and $\bar{\eta}$ in (2.55). Note that the same square roots $\sqrt{\check{p} - p\bar{p}/\hat{p}}$ factorise from both the numerator and the denominator so that we can use the limit of (2.57) to define the polarisation vectors for on-shell momenta. Equivalently we can extract the square root $\sqrt{\check{p} - p\bar{p}/\hat{p}}$ in advance and define the reference spinors η and $\bar{\eta}$ in (2.57) both to be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The polarisation vectors satisfy the condition

$$p \cdot \epsilon^{\pm} = \frac{1}{2}\,(\eta\bar{\eta} + \lambda\bar{\lambda})_{\alpha\dot{\alpha}}\,\epsilon^{\pm\alpha\dot{\alpha}} = 0 \tag{2.58}$$

even when off-shell.

2.5.1 Light-cone gauge Yang-Mills Theory

The reference momenta chosen in (2.57) allows us not only to identify the components needed in the Green function in order to calculate the scattering amplitude but also to associate helicities to gluons exchanged in the internal lines of a Feynman diagram. By comparing (2.57) and the bispinor expression of an arbitrary light-like vector (2.10), we see the polarisation vectors have the following components in light-cone coordinates

$$\check{\epsilon}^+ = -\frac{\bar{p}}{\hat{p}},\ \epsilon^+ = -1,\ \check{\epsilon}^- = \frac{p}{\hat{p}},\ \bar{\epsilon}^- = 1, \tag{2.59}$$

while the rest of the components are zero. Recalling that the inner product in light-cone coordinates is given by an off-diagonal metric, $p{\cdot}q = (\check{p}\hat{q} + \hat{p}\check{q} - p\bar{q} - \bar{p}q)/2$, the positive and negative helicity fields are

$$\epsilon^+ \cdot \mathcal{A} = -\frac{\bar{p}}{2\hat{p}}\hat{\mathcal{A}} + \frac{1}{2}\bar{\mathcal{A}},\ \epsilon^- \cdot \mathcal{A} = \frac{p}{2\hat{p}}\hat{\mathcal{A}} - \frac{1}{2}\mathcal{A}. \tag{2.60}$$

In light-cone gauge the $\hat{\mathcal{A}}$ component of the gauge field is taken to be zero, so a positive helicity gluon can only be created by the $\mathcal{A}$ field and annihilated by the $\bar{\mathcal{A}}$ field, and the reverse applies to negative helicity gluons. After integrating over the non-dynamical $\check{\mathcal{A}}$ component, we arrive at the light-cone Yang-Mills action containing only the physical components of the gauge field [2]

$$S = \frac{-8}{g^2}\int d\tau \mathcal{L} \tag{2.62}$$

where the lagrangian can be divided into a free field part, two 3-point interaction terms, and one 4-point interaction term.

---

[2]Alternatively, we can use the four possible combinations of spinor constructed from η, $\bar{\eta}$, λ and $\bar{\lambda}$ to span the off-shell momentum space gauge field,

$$\mathcal{A}_{\alpha\dot{\alpha}} = \eta_\alpha\bar{\eta}_{\dot{\alpha}}\mathcal{A}^{(\eta)} + \frac{\lambda_\alpha\bar{\eta}_{\dot{\alpha}}}{[\lambda\,\eta]}\mathcal{A}^{(+)} + \frac{\eta_\alpha\bar{\lambda}_{\dot{\alpha}}}{\langle\eta\,\lambda\rangle}\mathcal{A}^{(-)} + \lambda_\alpha\bar{\lambda}_{\dot{\alpha}}\mathcal{A}^{(\lambda)} \tag{2.61}$$

and use the identity $g_{\mu\nu} = \frac{1}{2}\,\epsilon^{\alpha\beta}\sigma_{\mu\,\alpha\dot{\alpha}}\sigma_{\nu\,\beta\dot{\beta}}\,\epsilon^{\dot{\alpha}\dot{\beta}}$ extensively to rewrite the Yang-Mills theory entirely in terms of spinor notation. The lagrangian we obtain after fixing gauge condition $\mathcal{A}^{(\lambda)} = 0$ and integrating out $\mathcal{A}^{(\eta)}$ is the same as (2.64) to (2.67) with $\mathcal{A}$ and $\bar{\mathcal{A}}$ replaced by $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$.

$$\mathcal{L} = \mathcal{L}^{-+} + \mathcal{L}^{-++} + \mathcal{L}^{++-} + \mathcal{L}^{--++}, \tag{2.63}$$

and

$$\mathcal{L}^{-+}[\mathcal{A}] = \int_{\tau} d^3\mathbf{x}\, \bar{\mathcal{A}} \left(\check{\partial}\hat{\partial} - \partial\bar{\partial}\right) \mathcal{A}, \tag{2.64}$$

$$\mathcal{L}^{-++}[\mathcal{A}] = \int_{\tau} d^3\mathbf{x}\, \hat{\partial}\bar{\mathcal{A}} \left[\mathcal{A},\, \bar{\partial}\hat{\partial}^{-1}\mathcal{A}\right], \tag{2.65}$$

$$\mathcal{L}^{+--}[\mathcal{A}] = \int_{\tau} d^3\mathbf{x}\, \hat{\partial}\mathcal{A} \left[\bar{\mathcal{A}},\, \partial\hat{\partial}^{-1}\bar{\mathcal{A}}\right], \tag{2.66}$$

$$\mathcal{L}^{--++}[\mathcal{A}] = \int_{\tau} d^3\mathbf{x}\, \left[\bar{\mathcal{A}},\, \hat{\partial}\mathcal{A}\right] \hat{\partial}^{-2} \left[\mathcal{A},\, \hat{\partial}\bar{\mathcal{A}}\right], \tag{2.67}$$

and the lagrangian is quantised on surface of constant light-cone "time" $\tau = (x^0 - x^3)$. Note that from the gauge condition the ghost effective action is $\int d^4x\, d^4y\, \omega^{*a}(x)\, \hat{\partial}\delta(x - y)\, \omega^b(y)$ so the ghost fields decouple. In a Green function calculation ghost loops factorise to cancel those from the vacuum bubbles and therefore can be neglected.

2.6 Canonically transforming the LCYM lagrangian

The Feynman graphs built from (2.64) to (2.67) clearly have well-defined helicity assignments associated with internal lines. The propagator connects negative helicity ends with positive helicity ends of the vertices as prescribed by the CSW rules, and the 3-point MHV vertex V^{--+} originates from the $\mathcal{L}^{--+}$ interaction term (2.66) can be put into the same form as that of the Parke-Taylor formula using momentum conservation, with spinors appearing in the formula generically associated with off-shell momenta defined from light-cone coordinate components (2.56).

$$\begin{aligned} V^{--+}(123) &= \frac{-8}{g^2}\,\frac{\hat{3}}{\hat{1}\hat{2}}\,(12) \\ &= \frac{-8}{g^2}\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \end{aligned} \tag{2.68}$$

It is easily seen that the 3-point MHV amplitude $A(1^-, 2^-, 3^+)$ receives contribution only from this vertex. As noted in section (2.4.1) the amplitude vanishes for real value momenta but not for complex momenta. The other 3-point vertex factor conjugate to the MHV vertex however is not included in the CSW construction.

$$
\begin{aligned}
V^{++-}(123) &= \frac{-8}{g^2}\frac{\hat{3}}{\hat{1}\hat{2}}\{12\} \\
&= \frac{-8}{g^2}\frac{[12]^4}{[12]\,[23]\,[31]}
\end{aligned} \tag{2.69}
$$

In addition the light-cone Yang-Mills lagrangian contains two 4-point vertices that have the same helicity contents as prescribed by CSW rules but do not agree with the Parke-Taylor formula and the interaction terms stop at 4-points.

$$
V^{--++}(1234) = \frac{8}{g^2}\frac{\hat{1}\hat{3}+\hat{2}\hat{4}}{(\hat{1}+\hat{4})^2}, \tag{2.70}
$$

$$
V^{-+-+}(1234) = \frac{-8}{g^2}\left(\frac{\hat{1}\hat{4}+\hat{2}\hat{3}}{(\hat{1}+\hat{2})^2}+\frac{\hat{1}\hat{2}+\hat{3}\hat{4}}{(\hat{1}+\hat{4})^2}\right) \tag{2.71}
$$

To produce a lagrangian that generates the CSW rules in the standard perturbative calculation it was assumed in [24,25] that the field variables $\mathcal{A}(\tau,\mathbf{p})$ and $\bar{\mathcal{A}}(\tau,\mathbf{p})$ at light-cone time τ are functionals of the new field variables $\mathcal{B}(\tau,\mathbf{p})$, $\bar{\mathcal{B}}(\tau,\mathbf{p})$ defined in 3-momentum space through expansion formulae

$$
\begin{aligned}
\mathcal{A}(p_1) = \mathcal{B}(p_1) &+ \int \Upsilon(123)\mathcal{B}(p_2)\mathcal{B}(p_3) + \cdots \\
&+ \int \Upsilon(123\cdots n)\mathcal{B}(p_2)\cdots\mathcal{B}(p_n) + \cdots ,
\end{aligned} \tag{2.72}
$$

$$
\begin{aligned}
\bar{\mathcal{A}}(p) = \bar{\mathcal{B}}(p) &+ \int \big(\Xi^2(123)\bar{\mathcal{B}}(p_2)\mathcal{B}(p_3) + \Xi^3(123)\mathcal{B}(p_2)\bar{\mathcal{B}}(p_3)\big) \\
&+ \cdots \int \sum_{k=2}^{n} \Xi^k(123\cdots n)\mathcal{B}(p_2)\cdots\bar{\mathcal{B}}(p_k)\cdots\mathcal{B}(p_n) + \cdots .
\end{aligned} \tag{2.73}
$$

The transformation in (2.72), (2.73) is assumed to be canonical, in the sense that the variation of the canonical conjugate variable $\hat{\partial}\bar{\mathcal{A}}$ in the functional integral is taken as the inverse of the variation of field variable $\mathcal{A}$,

$$
\hat{\partial}\bar{\mathcal{A}}^a(\tau,\mathbf{y}) = \int d^3\mathbf{x}\,\frac{\delta\mathcal{B}^b(\tau,\mathbf{x})}{\delta\mathcal{A}^a(\tau,\mathbf{y})}\hat{\partial}\bar{\mathcal{B}}^b(\tau,\mathbf{x}), \tag{2.74}
$$

and the unwanted $\overline{\text{MHV}}$ 3-point vertex (2.69) is absorbed into the free field lagrangian through the transformation

$$\mathcal{L}^{-+}[\mathcal{A}] + \mathcal{L}^{-++}[\mathcal{A}] = \mathcal{L}^{-+}[\mathcal{B}]\,. \tag{2.75}$$

It can be easily verified that the Jacobian determinant of a canonical transformation (2.74) is unity, therefore we can apply the functional integral method directly on the new lagrangian and treat the $\mathcal{B}$ and $\bar{\mathcal{B}}$ fields as the new integration variables. When combined together equations (2.74) and (2.75) provide us with the condition to determine the translation kernels Υ and Ξ^k

$$(\frac{\partial\bar{\partial}}{\hat{\partial}}\mathcal{A} + [\mathcal{A}, \frac{\bar{\partial}}{\hat{\partial}}\mathcal{A}])^a(\tau, \mathbf{x}) = \int d^3\mathbf{y}\, \frac{\partial\bar{\partial}}{\hat{\partial}}\mathcal{B}^b(\tau, \mathbf{y})\, \frac{\delta\mathcal{A}^a(\tau, \mathbf{x})}{\delta\mathcal{B}^b(\tau, \mathbf{y})} \tag{2.76}$$

Note that the transformation is designed so that to the lowest power an $\mathcal{A}$ or $\bar{\mathcal{A}}$ field is simply transformed into a $\mathcal{B}$ or $\bar{\mathcal{B}}$ field, and as one goes to higher powers terms the number of $\mathcal{B}$ fields increases but not the number of $\bar{\mathcal{B}}$ fields. Translating the remaining interaction terms $\mathcal{L}^{--+}$ and $\mathcal{L}^{--++}$ in the light-cone Yang-Mills lagrangian into new variables produces an infinite number of terms, each of them has two $\bar{\mathcal{B}}$s.

$$\mathcal{L} = \mathcal{L}^{-+}[\mathcal{B}] + \mathcal{L}^{--+}[\mathcal{B}] + \mathcal{L}^{--++}[\mathcal{B}] + \mathcal{L}^{--+++}[\mathcal{B}] + \cdots \tag{2.77}$$

Apparently the Feynman rules derived from the new lagrangian have the same helicity structure as the CSW rules. From the equivalence theorem, in the standard scattering amplitude calculation using LSZ reduction the higher power terms in the expansions (2.72) and (2.73) are generically suppressed by the p^2 factors in the on-shell limit, so $\mathcal{B}$ and $\bar{\mathcal{B}}$ can be interpreted as the helicity field components that generate positive and negative helicity gluons. However a few nontrivial exceptions exist when one generalises the canonical transformation method to obtain a D-dimensional MHV lagrangian theory. The higher power terms are found to be responsible to the all-plus amplitude which cannot be constructed from CSW rules alone. In chapter 3 we discuss all possible situations in the dimensional regularisation scheme for which the equivalence theorem can be violated.

In [29] Ettle and Morris found in 4-dimensions a generic n-th power term translation kernel can be summarised by the simple formula

$$\Upsilon\left(1\cdots n\right)=\frac{\hat{1}\hat{2}\cdots\widehat{n-1}}{(23)(34)\cdots(n-1,n)}, \tag{2.78}$$

and the coefficients for the $\bar{\mathcal{A}}$ expansion were found to be proportional to Υs

$$\Xi^{k}(1\cdots n)=-\frac{\hat{k}}{\hat{1}}\Upsilon(1\cdots n)=-\frac{\hat{k}\hat{2}\cdots\widehat{n-1}}{(23)(34)\cdots(n-1,n)} \tag{2.79}$$

where the definition for round brackets (2.14) were used to simplify the notation. Both formulae were proved in [29] by induction.

2.7 The MHV SQCD lagrangian

The canonical transformation discussed in the previous section were generalised by Ettle, Morris and Xiao to derive the MHV lagrangian theories for QCD and SQCD [30, 31]. In the standard approach to SQCD the lagrangian is constructed from two chiral superfields Φ_1, Φ_2 and the supersymmetric field-strength W^{α}, which contain a large quantity of field components. In the chiral superfields we have (ϕ_1, ψ_1, F_1), (ϕ_2, ψ_2, F_2), where ψ, ϕ represent the quark and squark fields. In additon, in the supersymmetric field-strength we have (Λ_α, $F_{\mu\nu}$, D), where Λ_α, $F_{\mu\nu}$ stand for the gaugino field and the gauge field field-strength. The rest of the fields F_1, F_2 and D are auxiliary fields.

$$\begin{aligned}\mathcal{L}_{SQCD}=&\int_{\tau}d^3x\,\left[\Phi_1^{\dagger}e^{2V^{*}}\Phi_1+\Phi_2^{\dagger}e^{2V}\Phi_2\right]\Big|_{\theta^4}\\&+\frac{4}{g^2}\left[W^{\alpha}W_{\alpha}|_{\theta^2}+h.c.\right]+m\left(\Phi_1\Phi_2|_{\theta^2}+h.c.\right)\end{aligned} \tag{2.80}$$

As in the pure Yang-Mills theory, the SQCD lagrangian is defined at light-cone time $\tau=(x^0-x^3)$. The following notations were assigned to quark, squark and gaugino fields in order to distinguish each component.

$$\psi_1^{\alpha}=(\bar{\beta}^{+},\bar{\alpha}^{+}),\;\bar{\psi}_1^{\dot{\alpha}}=(\beta^{-},\alpha^{-}),\;\psi_{2\alpha}=(\alpha^{+},\beta^{+}),\;\bar{\psi}_{2\dot{\alpha}}=(\bar{\alpha}^{-},\bar{\beta}^{-}), \tag{2.81}$$

$$\phi_1=\bar{\phi}^{+},\qquad \phi_1^{*}=\bar{\phi}_1^{-},\qquad \phi_2=\phi^{+},\qquad \phi_2^{*}=\phi^{-}, \tag{2.82}$$

$$\Lambda_{\alpha}=(\Lambda,T),\quad \bar{\Lambda}_{\dot{\alpha}}=(\bar{T},-\bar{\Lambda}) \tag{2.83}$$

After choosing the gauge fixing condition $\hat{A} = 0$, the non-dynamical components T, $\bar{T}$, $\beta^{\pm}$, $\bar{\beta}^{\pm}$ and the auxiliary fields were integrated out. The rest of the components were then canonically transformed on a pair by pair basis to produce the MHV SQCD lagrangian.

$$\left\{\mathcal{A}, \hat{\partial}\bar{\mathcal{A}}\right\} \to \left\{\mathcal{B}, \hat{\partial}\bar{\mathcal{B}}\right\}, \left\{\Lambda, \bar{\Lambda}\right\} \to \left\{\Pi, \bar{\Pi}\right\}, \tag{2.84}$$

$$\left\{\alpha^{\pm}, \bar{\alpha}^{\mp}\right\} \to \left\{\xi^{\pm}, \bar{\xi}^{\mp}\right\},$$

$$\left\{\phi^{-}, \hat{\partial}\phi^{+}\right\} \to \left\{\varphi^{-}, \hat{\partial}\bar{\varphi}^{+}\right\}, \left\{\bar{\phi}^{-}, \hat{\partial}\bar{\phi}^{+}\right\} \to \left\{\bar{\varphi}^{-}, \hat{\partial}\bar{\varphi}^{+}\right\} \tag{2.85}$$

In chapter 5 we focus on the gauge and gaugino field sector and introduce a different approach, starting by rewriting the $\mathcal{N} = 1$ super Yang-Mills lagrangian in the light-cone gauge as a functional of chiral superfield instead of super field-strength, the transformations in (2.84) can be combined into the canonical transformation of superfields. The corresponding $\mathcal{N} = 1$ generalisation of the MHV lagrangian can be used to calculate super-amplitudes which contain all physical scattering amplitudes related to the MHV amplitude (2.24) by SUSY Ward identity.

Chapter 3

Equivalence theorem evasion

In chapter 2 we saw that applying a canonical transformation on the LCYM lagrangian successfully produced an MHV lagrangian theory in which the vertices have the same helicity structure prescribed by the CSW rules. However at loop-level, non-trivial scattering amplitudes were found that cannot be explained by the CSW construction. For example in the LCYM theory the 4-point all-plus box diagram is constructed from connecting four $\overline{\text{MHV}}$ vertices. With the $\overline{\text{MHV}}$ interaction term absorbed into the free lagrangian during the transformation, it is not possible to rebuild another all-plus amplitude at one-loop using MHV vertices only. In [26] Brandhuber, Spence, Travaglini and Zoubos adopt the 4-dimensional light-cone friendly regularisation scheme given by Chakrabarti, Qiu and Thorn [27,61,62] and showed that the "missing" amplitude $A(1^+, 2^+, 3^+, 4^+)$ can be accounted for by counterterms.

In this chapter we take another approach and use the dimension regulator to regularise loop integrals. In the standard LSZ reduction scheme a generic scattering amplitude is given by the on-shell limit of the Green function multiplied by the appropriate polarisation vectors and momentum squares.

$$A(\cdots i^+, \cdots j^-, \cdots) = \lim_{p_i^2 \to 0} \prod_i \frac{p_i^2}{\sqrt{Z}} \left\langle \cdots \frac{1}{2}\bar{\mathcal{A}}(p_i) \cdots \frac{-1}{2}\mathcal{A}(p_j) \cdots \right\rangle . \qquad (3.1)$$

Using the expansion formulae (2.72), (2.73) to express $\mathcal{A}$ and $\bar{\mathcal{A}}$ into the new variables, the Green function is expanded into an infinite series

$$\begin{aligned}&\Big\langle \cdots \bar{\mathcal{A}}(p_i) \cdots \mathcal{A}(p_j) \cdots \Big\rangle \\ &= \sum_{m,n} \sum_{k=2}^{m} \int \; \Xi^k_{i\,23\cdots m}\, \Upsilon_{j\,2'3'\cdots n'} \\ &\qquad\qquad \times \Big\langle \cdots \big(\mathcal{B}(p_2) \cdots \bar{\mathcal{B}}(p_k) \cdots \mathcal{B}(p_m)\big) \cdots (\mathcal{B}(p_{2'}) \cdots \mathcal{B}(p_{n'})) \cdots \Big\rangle\end{aligned} \quad (3.2)$$

When a helicity field splits into a number of new fields, the momenta carried by the new fields are related to the momentum of the original helicity field by the law of conservation of momentum. After Wick contracting the new fields a number of propagators are created, but generically none of them has the same value as the propagator that would be produced if we performed the contraction on the original $\mathcal{A}$ and $\bar{\mathcal{A}}$ fields. If no other factor can cancel LSZ factors p_i^2 the higher order terms in the $\mathcal{A}$ and $\bar{\mathcal{A}}$ field expansions are suppressed in the on-shell limit, we then have an equivalent theory which allows us to calculate the same scattering amplitude using Green functions of the new variables. In D-dimensions, however, we find in certain special cases the translation kernels Υ and Ξ^k do generate an effective propagator to cancel the suppressing p_i^2. The contributions from these cases explain the non-vanishing amplitudes that appeared to be "missing" from the CSW viewpoint.

In order to regularise loop integrals, in section (3.1) we generalise the light-cone gauge Yang-Mills lagrangian to D-dimensions and perform the canonical transformation to obtain the corresponding MHV lagrangian. We find in the dimensional regularisation scheme the 4-point all-plus amplitude $A(1^+, 2^+, 3^+, 4^+)$ is explained by tadpole graphs constructed from self-contracting translation kernels (section (3.2.1)). In sections (3.2) and (3.3) we introduce a graphical notation which allows us to summarise the mathematical structure of the kernels and we discuss all of the possible situations where the higher order expansions in (3.2) can contribute to a scattering amplitude. Finally in section (3.4) we discuss some special cases where the singularities of the kernels need to be treated carefully to keep the loop integral well-defined.

The work was published in [28] and [59].

3.1 Dimension regularising the MHV lagrangian

The Yang-Mills lagrangian defined as the trace of the contracted fieldstrengths $tr(F_{\mu\nu}F^{\mu\nu})$ can be directly generalised to D-dimensions by allowing the indices to run through all D components. As in section (2.5) we express the lagrangian in light-cone coordinates before performing the canonical transformation. In D-dimensions the components of a covariant vector $p_\mu = (p_0,\, p_1, \cdots p_{D-1})$ in light-cone coordinates are defined as

$$\check{p} = (p_0 - p_{D-1})\,,\; \hat{p} = (p_0 + p_{D-1})\,, \tag{3.3}$$

$$p_I = (p_{2I-1} - ip_{2I})\,,\; \bar{p}_I = (p_{2I-1} + ip_{2I})\,. \tag{3.4}$$

In addition to the hat and check components, in D-dimensions we have $D/2-1$ pairs of holomorphic and anti-holomorphic transverse components p_I and $\bar{p}_I$, where the indices I range from 1 to $D/2-1$. The definitions for the round bracket and the curly bracket (2.14) generalise to

$$(p,\, q)_I = \hat{p}q_I - \hat{q}p_I,\; \{p,\, q\}_I = \hat{p}\bar{q}_I - \hat{q}\bar{p}_I \tag{3.5}$$

From (3.3) and (3.4) it is straightforward to see that the metric in D-dimensional light-cone coordinate system is off-diagonal. The inner product of two vectors is given by the formula

$$p \cdot q = (\check{p}\hat{q} + \hat{p}\check{q} - p_I\bar{q}_I - \bar{p}_I q_I)\,/2 \tag{3.6}$$

As in the 4-dimensional theory, we choose the gauge condition as $\hat{\mathcal{A}} = 0$ and integrate over the $\check{\mathcal{A}}$ component. The lagrangian can be divided into parts according to their helicity features.

$$S = \frac{-8}{g^2}\int d\tau \mathcal{L},\; \mathcal{L} = \mathcal{L}^{-+} + \mathcal{L}^{-++} + \mathcal{L}^{++-} + \mathcal{L}^{--++} \tag{3.7}$$

where

$$\mathcal{L}^{-+}[\mathcal{A}] = tr\int_\tau d^{D-1}\mathbf{x}\,\bar{\mathcal{A}}_I\left(\check{\partial}\hat{\partial} - \partial_J\bar{\partial}_J\right)\mathcal{A}_I, \tag{3.8}$$

$$\mathcal{L}^{++-}[\mathcal{A}] = tr\int_\tau d^{D-1}\mathbf{x}\,\bar{\partial}_I\mathcal{A}_J\left[\hat{\partial}\bar{\mathcal{A}}_J,\, \hat{\partial}^{-1}\mathcal{A}_I\right] + \bar{\partial}_I\bar{\mathcal{A}}_J\left[\hat{\partial}\mathcal{A}_J,\, \hat{\partial}^{-1}\mathcal{A}_I\right], \tag{3.9}$$

$$\mathcal{L}^{--+}[\mathcal{A}] = tr\int_\tau d^{D-1}\mathbf{x}\,\partial_I\mathcal{A}_J\left[\hat\partial\bar{\mathcal{A}}_J,\,\hat\partial^{-1}\bar{\mathcal{A}}_I\right] + \partial_I\bar{\mathcal{A}}_J\left[\hat\partial\mathcal{A}_J,\,\hat\partial^{-1}\bar{\mathcal{A}}_I\right], \tag{3.10}$$

$$\begin{aligned}\mathcal{L}^{--++}[\mathcal{A}] = \frac{1}{4}tr\int_\tau d^{D-1}\mathbf{x}\ &\left[\hat\partial\mathcal{A}_I,\,\bar{\mathcal{A}}_I\right]\hat\partial^{-2}\left[\hat\partial\mathcal{A}_J,\,\bar{\mathcal{A}}_J\right]\\ &-\left[\hat\partial\bar{\mathcal{A}}_I,\,\mathcal{A}_I\right]\hat\partial^{-2}\left[\hat\partial\bar{\mathcal{A}}_J,\,\mathcal{A}_J\right]\\ &+2\left[\hat\partial\mathcal{A}_I,\,\bar{\mathcal{A}}_I\right]\hat\partial^{-2}\left[\hat\partial\bar{\mathcal{A}}_J,\,\mathcal{A}_J\right]\\ &-\left[\mathcal{A}_I,\,\mathcal{A}_J\right]\left[\bar{\mathcal{A}}_I,\,\bar{\mathcal{A}}_J\right]-\left[\mathcal{A}_I,\,\bar{\mathcal{A}}_J\right]\left[\bar{\mathcal{A}}_I,\,\mathcal{A}_J\right],\end{aligned} \tag{3.11}$$

and we quantise the theory on the constant light-cone time surface $\tau = (x^0 - x^{D-1})$. Generically the correlation function of a given number of fields $\left\langle\cdots\mathcal{A}_I\cdots\bar{\mathcal{A}}_J\cdots\right\rangle$ can be computed from the functional integral in which we integrate the $D/2-1$ pairs of transverse components of the gauge field. To obtain the physical scattering amplitude we recall that in the LSZ reduction the amplitude is calculated from the correlation function of the values of gauge fields which are measured in the direction defined by polarisation vectors.

$$\epsilon^+\cdot\mathcal{A} = \frac{1}{2}\left(\mathcal{A}_1 + i\mathcal{A}_2\right),\ \epsilon^-\cdot\mathcal{A} = -\frac{1}{2}\left(\mathcal{A}_1 - i\mathcal{A}_2\right). \tag{3.12}$$

In light-cone coordinates (3.4) these two factors correspond to the transverse components $\mathcal{A}_I$ and $\bar{\mathcal{A}}_I$ when the index I equals one. So following the standard LSZ reduction scheme the dimensionally regularised scattering amplitude is obtained from the correlator $\left\langle\cdots\mathcal{A}_1\cdots\bar{\mathcal{A}}_1\cdots\right\rangle$ by multiplying by the appropriate LSZ factors, identifying the dimension D as $4-2\epsilon$, and then take the on-shell limit $p_i^2\to 0$.

From the equations (3.8) to (3.11) we read off the vertices in light-cone gauge as

$$V^{--+}(1^I,2^J,3^K) = \frac{(31)_J\,\delta_{KI}}{\hat 2} + \frac{(23)_I\,\delta_{JK}}{\hat 1} \tag{3.13}$$

$$V^{++-}(1^I,2^J,3^K) = \frac{\{31\}_J\,\delta_{KI}}{\hat 2} + \frac{\{23\}_I\,\delta_{JK}}{\hat 1} \tag{3.14}$$

As in 4-dimensions the lagrangian contains a 3-point MHV vertex, a 3-point $\overline{\text{MHV}}$ vertex, and two 4-point vertices. We apply the canonical transformation to rewrite the self-dual part as the free lagrangian of the new field variables. The transverse components are assumed to be functionals of $\mathcal{B}_I$, $\bar{\mathcal{B}}_I$, and the powers of the anti-holomorphic component fields are fixed in the expansions to ensure that in terms of the new fields the vertices will have exactly two negative helicity legs.

$$\mathcal{A}_I(p_1) = \mathcal{B}_I(p_1) + \int \Upsilon(1^I, 2^J, 3^K)\, \mathcal{B}_J(p_2)\, \mathcal{B}_K(p_3) + \cdots \tag{3.15}$$

$$\begin{aligned}\bar{\mathcal{A}}_I(p_1) = \bar{\mathcal{B}}_I(p_1) &+ \int \Xi^2(1^I, 2^J, 3^K)\, \bar{\mathcal{B}}_J(p_2)\, \mathcal{B}_K(p_3) \\ &+ \int \Xi^3(1^I, 2^J, 3^K)\, \mathcal{B}_J(p_2)\, \bar{\mathcal{B}}_K(p_3) + \cdots \end{aligned}\tag{3.16}$$

The translation kernels are iteratively solved from the condition

$$\mathcal{L}^{-+}[\mathcal{A}] + \mathcal{L}^{++-}[\mathcal{A}] = \mathcal{L}^{-+}[\mathcal{B}]\,. \tag{3.17}$$

Substituting $\mathcal{A}_I$ with (3.15) and collecting terms from both sides of the equation we find

$$\Upsilon(1^I, 2^J, 3^K) = \frac{1}{\hat{1}} \frac{V^{++-}(2^J, 3^K, 1^I)}{\frac{p_1^2}{\hat{1}} + \frac{p_2^2}{\hat{2}} + \frac{p_3^2}{\hat{3}}}. \tag{3.18}$$

Substituting (3.18) into the transformation condition again, we have

$$\begin{aligned}&\Upsilon(1^I, 2^J, 3^K, 4^L) \\ &= \frac{1}{\hat{1}} \frac{1}{\sum_{i=1}^4 p_i^2/\hat{p}_i} \left(\frac{1}{\hat{1}+\hat{2}} \frac{V^{++-}(2^J, 3+4^A, 1^I)\, V^{++-}(3^K, 4^L, 1+2^A)}{\frac{(p_1+p_2)^2}{\hat{1}+\hat{2}} + \frac{p_3^2}{\hat{3}} + \frac{p_4^2}{\hat{4}}} \right. \\ &\qquad \left. \frac{1}{\hat{1}+\hat{4}} \frac{V^{++-}(2+3^A, 4^L, 1^I)\, V^{++-}(2^J, 3^K, 1+4^A)}{\frac{(p_1+p_4)^2}{\hat{1}+\hat{4}} + \frac{p_2^2}{\hat{2}} + \frac{p_3^2}{\hat{3}}} \right). \end{aligned}\tag{3.19}$$

Note that in the above equations (3.18), (3.19) both Υs depend on the $\overline{\text{MHV}}$ vertex factor absorbed during the transformation.

3.2 Graphical conventions for the canonically transformed lagrangian theory

As noted at the beginning of this chapter the helicity field components can split into a number of $\mathcal{B}$ and $\bar{\mathcal{B}}$ fields through the canonical transformation so generically we need to consider a series of Green functions that contain all possible terms in the expansion (3.15), (3.16). In order to keep track of the contributions made by translation kernels we introduce graphical notation.

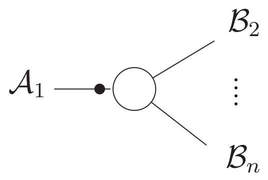
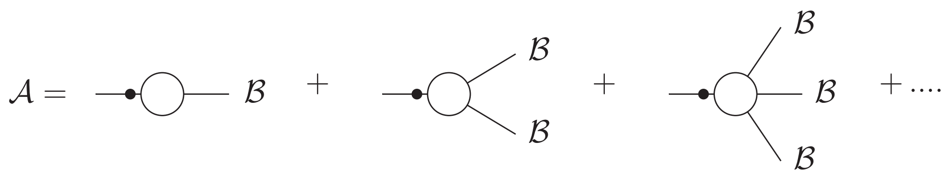


Figure 3.1: Graphical representation of an $(n-1)$-th order Υ kernel

When an $(n-1)$-th order term in the $\mathcal{A}$ field expansion is present in the Green function we draw a blank circle to represent the translation kernel $\Upsilon_{12\cdots n}$ and n lines stretching out from the circle to denote the fields associated to the kernel, $(n-1)$ of the lines represent the $(n-1)$ -tuple $\mathcal{B}$ fields created during the transformation, and one of the line represents the original $\mathcal{A}$ field being replaced by the transformation (Fig.3.2). We attach a small dot to indicate the negative helicity direction of the lines. In (Fig.3.2) the dot also distinguishes the line associated to the $\mathcal{A}$ field from those associated to the $\mathcal{B}$ fields. For simplicity in the following expressions we shall suppress the indices I that was introduced to denote the $(D/2-1)$ transverse field components in D-dimensions. Using the graphical convention just introduced the expansion formula (3.15) can be represented as



Similarly, we use the small dots to decorate the negative helicity ends of the lines that are attached to the original LCYM vertices. The MHV and the $\overline{\text{MHV}}$ vertices are represented by the graphs

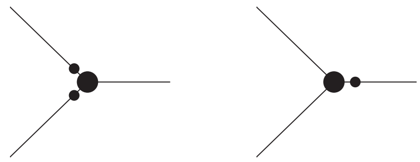


Figure 3.2: The MHV vertex and the $\overline{\text{MHV}}$ vertex

In section (3.1) we used the transformation condition (3.17) to solve for the translation kernel algebraically as a combination of the $\overline{\text{MHV}}$ vertex and the factor $1/(\sum_i p_i^2/\hat{p}_i)$. The same condition can also be expressed in graphical notation, which allows us to visualise the iterative pattern underlying the structure of the kernels and to relate kernels with the vertex factors in the original LCYM theory. Stripping off a factor $\hat{p}\bar{\mathcal{B}}(-p)$ from both sides, equation (3.17) becomes

$$\frac{p\bar{p}}{\hat{p}}\mathcal{A}(p) - \int d^{D-1}q \frac{1}{\hat{p}} V^{++-}(q, p-q, -p)\, \mathcal{A}(q)\mathcal{A}(p-q) = \int d^{D-1}q \frac{q\bar{q}}{\hat{q}} \mathcal{B}\frac{\delta \mathcal{A}(p)}{\delta \mathcal{B}(q)} \tag{3.20}$$

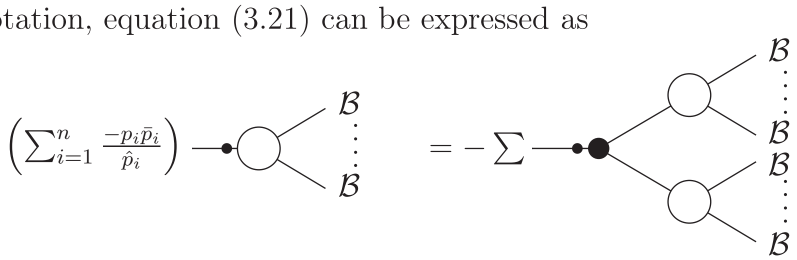
Using (3.15) to substitute the $\mathcal{A}$ field in the first term and collecting terms of the same powers in the new fields, we arrive at an identity which relates $\Upsilon_{12\cdots n}$ to translation kernels of lower order.

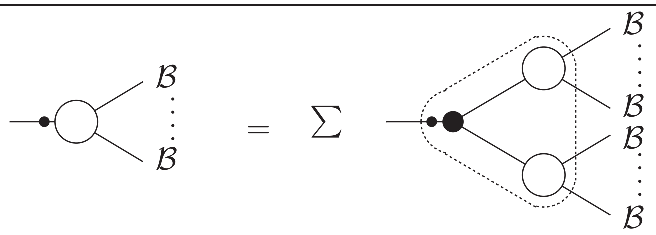
$$\left(\sum_{i=1}^{n} \frac{-p_i\bar{p}_i}{\hat{p}_i}\right) \Upsilon_{12\cdots n}\mathcal{B}(p_2)\ldots\mathcal{B}(p_n)$$

$$= \begin{array}{c}\text{terms that have } (n-1) - \text{th} \\ \text{power of } \mathcal{B} \text{ in}\end{array} \left[\int d^{D-1}q \frac{V^{++-}(q, -p_0 - q, p_0)}{\hat{p}} \mathcal{A}(q)\mathcal{A}(-p_0 - q)\right], \tag{3.21}$$

where when deriving the above equation we reversed the direction of p_1 to conform the convention for labeling particle fields by outgoing momenta.

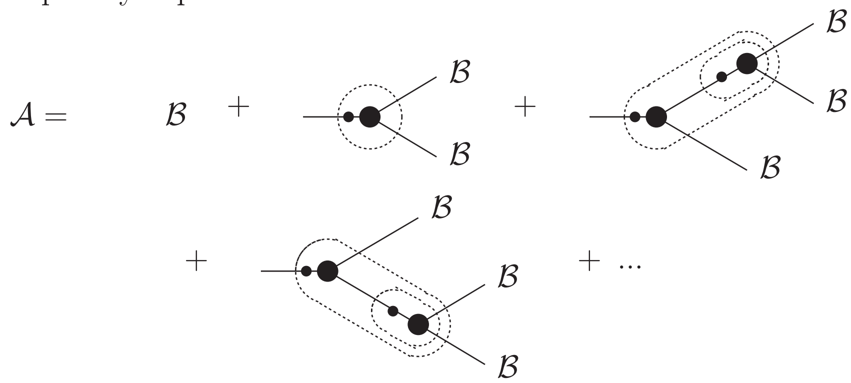
In graphical notation, equation (3.21) can be expressed as





Dividing $\left(\sum_{i=1}^{n}\frac{-p_i\bar{p}_i}{\hat{p}_i}\right)$ from both sides, we obtain the graphical identity

On the right-hand-side of the equation we introduce a dashed line bubble surrounding the graph to represent the factor $1/\left(\sum_{i=0}^{n}\frac{-p_i\bar{p}_i}{\hat{p}_i}\right)$, which receives contributions from all of the lines crossing through the bubble. The summation on the right hand side of the identity runs over combinations of subgraphs that add up to produce the same number of $\mathcal{B}$ fields as the translation kernel $\Upsilon_{12\cdots n}$. All graphical representation of the kernels can be solved by repeatedly substituting the lower order kernels into the right hand side of the identity. The expansion formula of $\mathcal{A}$ field can be more explicitly expressed as



It is clear that the contribution from each term can be easily read off from graphs. The structure underlying the kernels resembles the Feynman diagrams built from $\overline{\text{MHV}}$ vertices only. However note that the straight lines connecting vertices in the representations above do not contribute as propagators and are merely used to indicate the algebraic structure of the momenta flowing through the lines.

In the same spirit we can introduce a graphical notation that allows us to derive the $\overline{\text{MHV}}$ vertex dependence embedded in the $\bar{\mathcal{A}}$ field expansion formula. In order to distinguish the expansion coefficients Ξ^k from Υ we use a gray circle to denote the kernels $\Xi^k_{12\cdots n}$ in the $\bar{\mathcal{A}}$ transformation. The $(n-1)$-th order term in the expansion (3.16) is represented by the graph

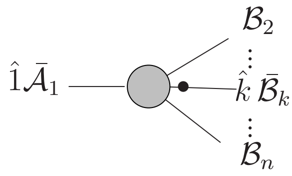
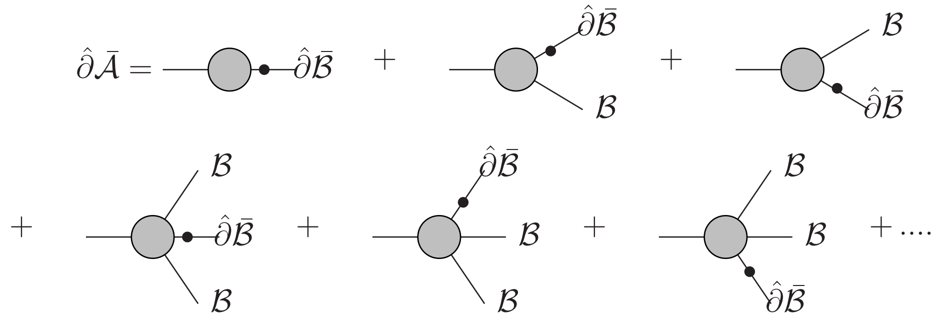


Figure 3.3: Graphical representation of an $(n-1)$-th order Ξ^k kernel

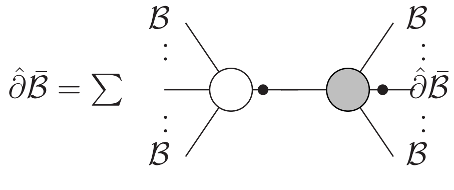
and a small dot is attached to the line representing the $\bar{\mathcal{B}}$ field being created during the transformation. In the graphical notation the negative helicity field $\bar{\mathcal{A}}$ is related to the new variables by the identity



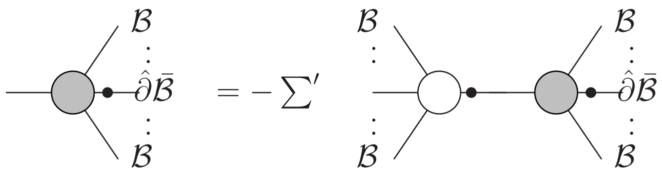
As in the case of the positive helicity field, kernels in the above identity can be expressed more explicitly in terms of $\overline{\text{MHV}}$ vertex factors. We recall that the kernel Ξ^k is related to Υ by the canonical transformation condition.

$$\int d^{D-1}y \frac{\delta\mathcal{A}(y)}{\delta\mathcal{B}(x)}\hat{\partial}\bar{\mathcal{A}}(y) = \hat{\partial}\bar{\mathcal{B}}(x) \tag{3.22}$$

Combining with the graphical conventions above, identity (3.22) yields

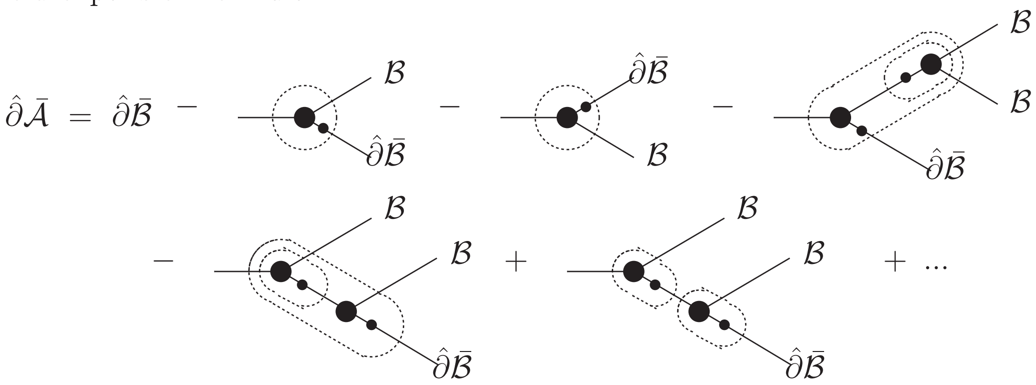


where on the right hand side we sum over all possible terms in the $\bar{\mathcal{A}}$ and $\mathcal{A}$ field expansion formulae. The Υ kernels appearing in the graph above came from the variation factor $\delta\mathcal{A}/\delta\mathcal{B}$ in equation (3.22). Note that mathematically the variation $\delta\mathcal{A}/\delta\mathcal{B}$ is obtained from stripping off one of the $\mathcal{B}$ fields in the $\mathcal{A}$ field expansion formula, so in the graphical expression of the identity (3.22) one of the lines branching from the blank circle is not attached with a $\mathcal{B}$ field. By matching terms that have the same number of $\mathcal{B}$ fields we arrive at an identity which allows us to express the kernel Ξ^k as a linear combination of Υ and Ξ^s of lower order. Alternatively we can start with the kernel graph (Fig.3.2) with the $\bar{\mathcal{A}}$ and the $\bar{\mathcal{B}}$ lines held fixed, pointing to the opposite directions, and draw all of the graphs that can be obtained by moving $\mathcal{B}$ field lines to the left hand side. Summing over all of the graphs and identifying the summation as zero yields the same identity as (3.22).



where in the above expression we used $\sum'$ to indicate that in the sum we include graphs with at least one $\mathcal{B}$ field leg moved to the left.

Substituting lower order kernel graphs repeatedly we can explicitly express the $\overline{\text{MHV}}$ vertex dependence and the bubble structure in all of the kernel factors Ξ^k in the $\bar{\mathcal{A}}$ field expansion formula.



Finally, we can apply the same procedure to obtain the graphical representations for MHV vertices. We use the 3-point and the 4-point LCYM vertices as the basic structure and replace the $\mathcal{A}$ and $\bar{\mathcal{A}}$ fields attached at the legs of the vertices by trees of $\mathcal{B}$ and $\bar{\mathcal{B}}$ that appear in the $\mathcal{A}$ and $\bar{\mathcal{A}}$ expansions. For example, the $(--++)$ 4-point MHV vertex is represented by the sum of the following four graphs.

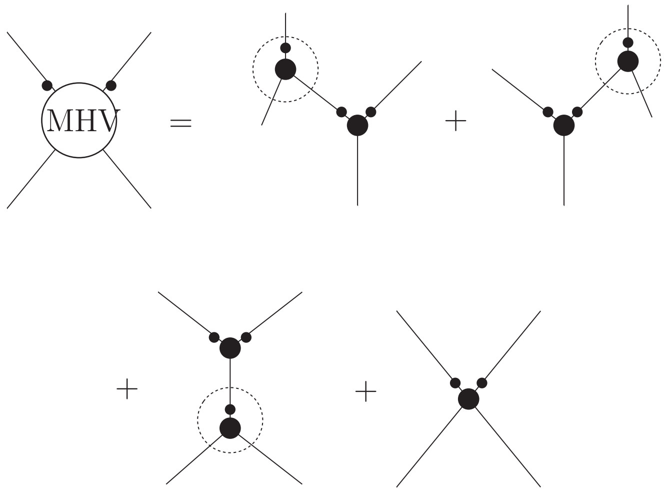


Figure 3.4: Graphs contributing to the 4-point MHV $(--++)$ vertex

3.2.1 The "missing" one-loop level $(++++)$ amplitude

In this section we show that the "missing" all-plus 4-point amplitude $A(1^+, 2^+, 3^+, 4^+)$ is explained by the contribution from the translation kernel. In LCYM theory the standard Feynman rules allow us to connect four pieces of $\overline{\mathrm{MHV}}$ vertices to form box diagrams, which are responsible for the non-vanishing all-plus amplitude at one-loop.

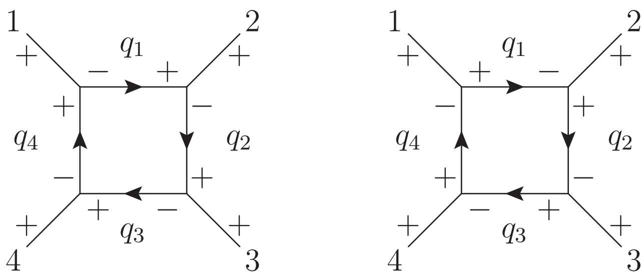


Figure 3.5: LCYM box graphs

From (Fig.3.5) we have the loop integral

$$A(1^+, 2^+, 3^+, 4^+) = \lim_{p_i^2 \to 0} \int \frac{d^D q}{(2\pi)^D} \left. \frac{X^{IJKL}}{q_1^2\, q_2^2\, q_3^2\, q_4^2} \right|_{I,J,K,L=1} , \tag{3.23}$$

where $q_1 = q$, $q_2 = q - p_2$, $q_3 = q - p_2 - p_3$, $q_4 = q + p_1$ are the momenta carried by the propagators in the loop and we used a symbol X^{IJKL} as an abbreviation of the vertex factors in the graph.

$$
\begin{aligned}
X^{IJKL} = {} & V^{++-}(-q_4^D, 1^I, q_1^A)\, V^{++-}(-q_1^A, 2^J, q_2^B) \\
& \times V^{++-}(-q_2^B, 3^K, q_3^C)\, V^{++-}(-q_3^C, 4^L, q_4^D)
\end{aligned} \tag{3.24}
$$

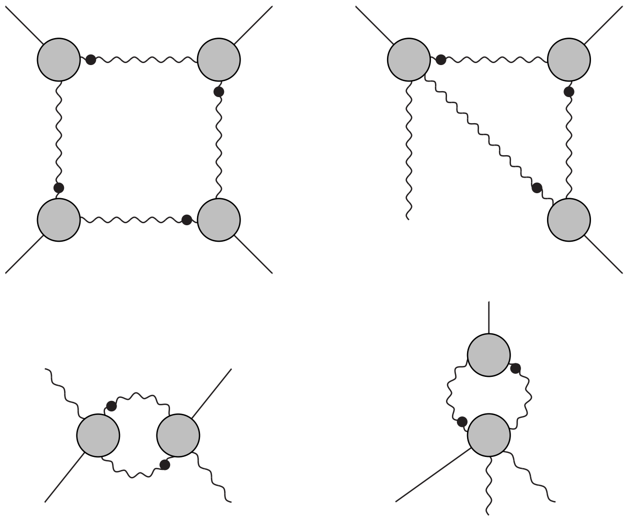


Figure 3.6: Graphs contributing to the $(++++)$ amplitude

For a physical scattering event which takes place in 4-dimensions we choose the values of the indices I, J, K, L to be 1 and the momenta associated with the external lines are taken to their on-shell limits at the end of the calculation. From the graphical representations given in the last section we saw that after the canonical transformation the original $\overline{\text{MHV}}$ interaction term is no longer included in the MHV lagrangian as a vertex, instead the same factor is implicitly contained in both types of the translation kernels Υ and Ξ^k, which can appear in an amplitude either through the helicity fields $\mathcal{A}$, $\bar{\mathcal{A}}$ being evaluated in the Green function or through an MHV vertex which has helicity fields being replaced by trees of new field variables during the transformation. In this case the contribution can only come from translating helicity fields because none of the possible contraction between MHV vertices or self-contraction among the legs of a MHV vertex can result in an all-plus amplitude at one-loop level. We find the types of graphs in (Fig.3.6) and (Fig.3.7) created from translation kernels have the required helicity structure. Note that in order to distinguish from the line structure within the kernels we use wavy lines to represent propagators.

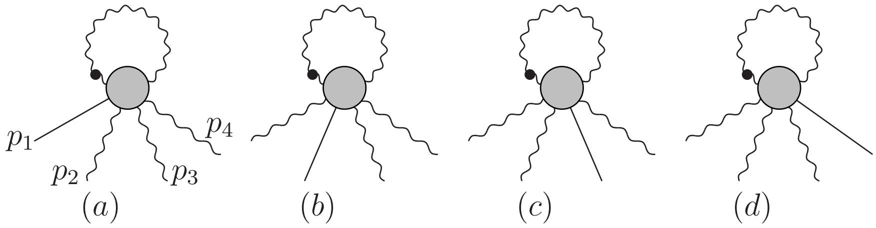


Figure 3.7: Tadpole graphs

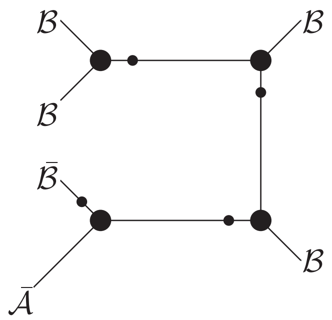


Figure 3.8: Vertex structure embedded in the kernel $\Xi^2_{1234556}$

From the iterative formula introduced in section (3.2) we see that the translation kernels are represented by tree graphs with $\overline{\text{MHV}}$ vertices implicitly embedded. A 6-point kernel Ξ^2_{123456} contains a series of graphs having tree structure of the form (Fig.3.8), each associated with different bubbles attached to its vertices. Contracting $\bar{\mathcal{B}}$ and $\mathcal{B}$ on the left side of the graphs to create 4-point tadpoles we see that the graph (Fig.3.7(a)) has the same $\overline{\text{MHV}}$ vertex factors as LCYM box diagrams. Collecting terms with all possible bubble structures we see that tadpole (Fig.3.7(a)) yields

$$\int \frac{d^D q}{(2\pi)^D} \frac{X^{IJKL}\, C_{(a)}}{q_1^2\, q_2^2\, q_3^2\, q_4^2}, \tag{3.25}$$

where X^{IJKL} contains the vertex factors given in (3.24). In the on-shell limit $p_1^2, p_2^2, p_3^2, p_4^2 \to 0$ we have

$$C_{(a)} = \frac{\frac{q_1^2}{\hat{q}_1} \frac{q_2^2}{\hat{q}_2} \frac{q_3^2}{\hat{q}_3}}{\left(\frac{q_1^2}{\hat{q}_1} - \frac{q_4^2}{\hat{q}_4}\right)\left(\frac{q_2^2}{\hat{q}_2} - \frac{q_4^2}{\hat{q}_4}\right)\left(\frac{q_3^2}{\hat{q}_3} - \frac{q_4^2}{\hat{q}_4}\right)}. \tag{3.26}$$

The contributions from other three tadpole graphs are obtained by cyclically permuting the indices in (3.26). Summing over all factors yields

$$C_{(a)} + C_{(b)} + C_{(c)} + C_{(d)} = 1 \tag{3.27}$$

so we find the tadpole graphs restore the same loop integral that was originally given by the box diagram in the LCYM theory and therefore reproduce the correct value for the all-plus 4-point amplitude. The other one-loop graphs in (Fig.3.6) all contain factors of the form $1/(\sum_i q_i^2/\hat{q}_i)$ in the integrand, which cannot be directly computed using the standard integration techniques. In fact the specific values of these integrals depends on the way we define the singularities in the translation kernel, though the sum of these graphs can be shown to produce vanishing result. In section (3.4) we shall show that with suitable arrangements with singularities we can neglect the contributions from translation kernels at one-loop level except for tadpole graphs.

3.3 Equivalence theorem evasion in general

Although introducing the numerous translation kernels may seem to have made the amplitude calculation more complicated, we find that these factors are negligible except in a few special cases. When an $\mathcal{A}$ or $\bar{\mathcal{A}}$ field in the Green function is expanded in terms of the new field variables the momentum it carries is redistributed into a number of $\mathcal{B}$ and $\bar{\mathcal{B}}$ fields. Generically the momenta flowing through the $\mathcal{B}$ field propagators deviate from the original $\mathcal{A}$ field momentum, leaving the LSZ factors $p_i^2 + i\epsilon$ uncanceled. If the Green function has a finite on-shell limit then the contribution to the scattering amplitude is suppressed by the LSZ factors. So the exceptions can only come from singularities of the Green function, where it is possible to produce an effective propagator $1/p_i^2$ asymptotically.

Tree-Level

(a) All-Plus-Except-One-Minus Amplitudes

The simplest case consists of graphs formed by kernels and no vertex. This happens when one of the $\mathcal{A}$ or $\bar{\mathcal{A}}$ in the Green function $\langle \bar{\mathcal{A}}_1 \mathcal{A}_2 \cdots \mathcal{A}_n \rangle$ is translated to higher order term and carries a kernel Υ or Ξ^k while the other $(n-1)$ fields are simply translated into single $\mathcal{B}$ or $\bar{\mathcal{B}}$. Contracting $\mathcal{B}$ and $\bar{\mathcal{B}}$ in pairs leads to tree-level graphs of the type

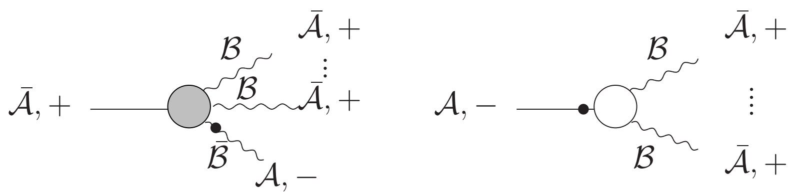


Figure 3.9: All-plus-except-one-minus amplitudes built from kernels

where all of the lines stretching from the kernel are linked to physical states which carry light-like momenta. In 4-dimensions all of the all-plus-except-one-minus amplitudes are known to be vanishing. However in the MHV lagrangian theory these amplitudes are vanishing not because they can be suppressed by LSZ factors, but because the $\overline{\text{MHV}}$ vertices contained in the kernel are zero. The simplest example is the 3-point $\overline{\text{MHV}}$ amplitude $A(1^+, 2^+, 3^-)$. Note that although no vertex can be used to connect $\langle \bar{\mathcal{B}}\bar{\mathcal{B}}\mathcal{B} \rangle$ at tree-level the amplitude receives contributions from (Fig3.10(a), (b) and (c)).

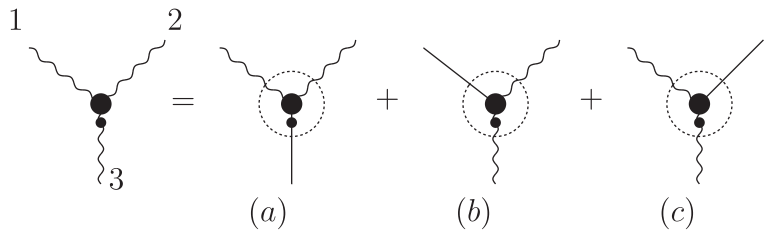


Figure 3.10: The sum of graph (a), (b) and (c) restores the LCYM $\overline{\text{MHV}}$ vertex

When combined together the three kernel graphs restore the original off-shell value $\overline{\text{MHV}}$ vertex.

$$
\begin{aligned}
&V^{++-}(p_1,p_2,p_3)\\
&= \frac{\frac{p_3^2+i\epsilon}{\hat{p}_3}}{\frac{p_1^2+i\epsilon}{\hat{p}_1}+\frac{p_2^2+i\epsilon}{\hat{p}_2}+\frac{p_3^2+i\epsilon}{\hat{p}_3}} V^{++-}(p_1,p_2,p_3)\\
&\qquad + \frac{\frac{p_1^2+i\epsilon}{\hat{p}_1}}{\frac{p_1^2+i\epsilon}{\hat{p}_1}+\frac{p_2^2+i\epsilon}{\hat{p}_2}+\frac{p_3^2+i\epsilon}{\hat{p}_3}} V^{++-}(p_1,p_2,p_3)\\
&\qquad\qquad + \frac{\frac{p_2^2+i\epsilon}{\hat{p}_2}}{\frac{p_1^2+i\epsilon}{\hat{p}_1}+\frac{p_2^2+i\epsilon}{\hat{p}_2}+\frac{p_3^2+i\epsilon}{\hat{p}_3}} V^{++-}(p_1,p_2,p_3)
\end{aligned} \tag{3.28}
$$

The vertex $V^{++-}(p_1,p_2,p_3)$ contains the curly bracket $\{12\}$. The curly bracket equals zero because for real values of momenta $p_3^2 = |\{12\}|/2\,\hat{1}\hat{2}$, but not for complex momenta. Therefore from equation (3.28) we see the MHV lagrangian reproduces the correct value for the $A(1^+,2^+,3^-)$ amplitude even for complex momenta. In section (3.2) we saw that the translation kernel can have an overall factor $1/(\sum_i p_i^2/\hat{p}_i)$ represented by the dashed line bubble. Because in an all-plus-except-one-minus amplitude all of the lines carry null momenta, such factor is singular in the on-shell limit.

We note that the value of a specific graph depends on the order we choose to send momenta on-shell despite there is no such a dependency in the standard Feynman rules derived from the Yang-Mills lagrangian. For example in equation (3.28) if the momenta are taken on-shell according to the order p_1, p_2 and then p_3, graphs (b) and (c) will be suppressed. After sending p_1^2 and p_2^2 to zero in the denominator of the kernel in graph (a) the factor $p_3^2/\hat{p}_3$ is left to cancel the LSZ factor, so in this order graph (a) is responsible for the amplitude $A(1^+,2^+,3^-)$. When we change the order to be p_2, p_3, p_1, the same mechanism picks graph (b). Generically the value of an amplitude is unchanged as long as we keep the same order for all of the graphs contributing to the amplitude.

In the graphical identity (Fig3.10) graphs (a), (b) and (c) are everywhere identical except each graph has a different propagator emerging from the bubble being replaced by a straight line. From the graphical convention defined in section (3.2) a straight line is used to indicate that the momentum flowing from one side to the other of the line remains the same value and the straight line itself does not contribute any numerical value to the graph. So effectively the straight line can also be regarded as a propagator multiplied by an additional p_i^2. Combining with the $1/\hat{p}_i$ factor carried by the $\overline{\text{MHV}}$ vertex embedded in the kernel, each graph contribute a factor of $p_i^2/\hat{p}_i$ in addition to the common factor. Therefore summing over graphs which have different propagators crossing through the bubble replaced by straight lines produces a new graph where the bubble factor $1/(\sum_i p_i^2/\hat{p}_i)$ is removed. By applying the graphical identity (Fig.3.10) repeatedly we can remove all of the bubbles and identify the graphs constructed from MHV lagrangian to those from the LCYM lagrangian. As an example in appendix (A) we restore a LCYM graph contributing to the amplitude $A(1^-, 2^+, 3^+, 4^+)$ from ten MHV lagrangian graphs which have more complicated kernel structure.

(b) Special Momentum Configurations

The other exception at tree-level happens only for some specific values of momenta. A graph consisting of translation kernel and vertices like (Fig.3.11) normally does not contribute because even when on-shell no factor in the graph can generate the effective propagator $1/p_1^2$. However when $p_2 + p_3$ and $p_5 + p_6$ happen to be light-like the factor $1/\sum(p_i^2 + i\epsilon)/\hat{p}_i$ in the kernel becomes $1/(p_1^2/\hat{p}_1)$ if we send p_4 on-shell before p_1. As in the previous case all of the lines linked to the kernel are null so the graph does not contribute, but not because of the suppressing LSZ factor. For this particular combination of momenta another graph which contains a translation kernel from $\bar{\mathcal{A}}(p_4)$ combines with the contribution from (Fig.3.11), and yields the same analytic form as for non-null $p_2 + p_3$ and $p_5 + p_6$. For practical calculations we can ignore these graphs and only consider the graphs that generically contribute. Despite the scattering amplitudes are not given by a direct application of LSZ reduction on Green functions of $\mathcal{B}$ and $\bar{\mathcal{B}}$ fields, we find at tree-level all amplitudes can be accounted for by the CSW rules.

One-Loop

(a) Tadpole graphs

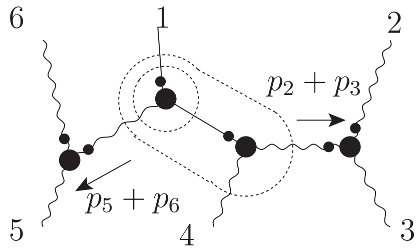


Figure 3.11: Tree-level graph constructed by kernel that can become non-vanishing for special combinations of leg momenta

At one-loop level there are more non-trivial graphs that can be built from translation kernels. By contracting a $\bar{\mathcal{B}}$ and a $\mathcal{B}$ in the same $\bar{\mathcal{A}}$ field expansion a kernel can be used to form a tadpole diagram. Because in a self-contraction the momenta flowing in and out of the loop must be equal and opposite the corresponding contributions to factor $1/\sum(p_i^2 + i\epsilon)/\hat{p}_i$ cancel each other, leaving only the momenta flowing into external lines to be summed over. Taking on-shell limit of this type of graphs is analogous to those of the all-plus-except-one-minus amplitudes, for which the order we send external line momenta on-shell determines whether a particular graph vanishes or not, but does not affect the sum of all graphs to contribute to the same amplitude. However unlike the all-plus-except-one-minus amplitudes, the sum of tadpole graphs do not vanish because the lines contracted to form loops are not null and therefore the $\overline{\text{MHV}}$ vertices contained in the kernel generically are not zero. As we saw in section (3.2.1) the tadpole graphs are responsible for the non-vanishing of the 4-point all-plus amplitude $A(1^+, 2^+, 3^+, 4^+)$.

(b) Dressing propagators

Another kind of cancellation of the LSZ factor happens when two of the legs from a translation kernel rejoin each other at a 3-point vertex. We divide this kind of graphs into two sub-types, the graphs in which the translation kernels are not attached to other external lines (Fig.3.12(a)), and otherwise (Fig.3.13). In both graphs we use a crossed circle to denote structure in the graph that is irrelevant to our current discussion.

In graph (Fig.3.12(a)) the legs of the kernel Υ_{123} are contracted with the legs of an MHV vertex. On the other side of the vertex a propagator is required by

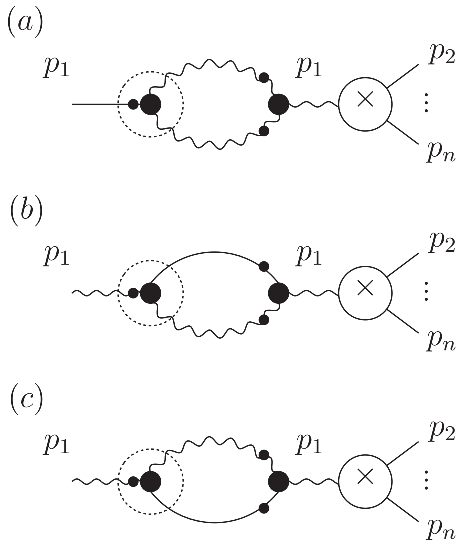


Figure 3.12: Single leg dressing propagator combines with MHV vertices as self-energy bubble

conservation of momentum to have the value of $1/p_1^2$. With the LSZ factor canceled the graph survives in the on-shell limit. From the identity (3.28) we see that combining (Fig.3.12(a)) with (Fig.3.12(b)) and (Fig.3.12(c)) eliminates the bubble factor $1/\sum(p_i^2 + i\epsilon)/\hat{p}_i$ and the sum contributes as the self-energy graph at one-loop which we absorb into the field renormalisation.

The other sub-type of the graphs contains kernels attached to external lines and cannot be absorbed into renormalisation factors. In (Fig.3.13) the propagator connecting with the MHV vertex contributes as $1/(p_1 + p_2)^2$. If the momentum $(p_1 + p_2)$ is also null, p_1 and p_2 become parallel to each other in the on-shell limit, and the inner bubble $1/\left(p_1^2/\hat{p}_1 + p_2^2/\hat{p}_2 + (p_1 + p_2)^2/(\hat{p}_1 + \hat{p}_2)\right)$ becomes singular.

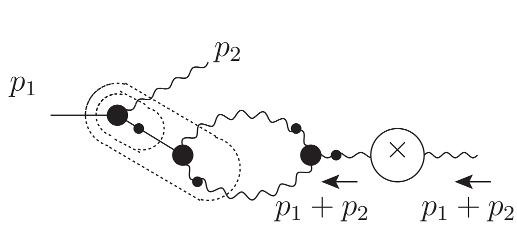


Figure 3.13: Multiple leg dressing propagator

Nevertheless this type of graphs does not contribute to the physical scattering amplitude because the left $\overline{\text{MHV}}$ vertex $V^{++-}(p_1, p_2, -p_1-p_2)$ vanishes for on-shell real value momenta. So the graphs containing dressing propagators can be neglect altogether for practical calculations.

(c) Kernels contracted with other structures to form a loop

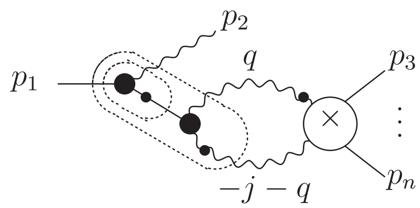


Figure 3.14: A loop contribution given by direclty contracting the kernel

In the most general case a translation kernel can contract with an arbitrary subgraph to form a loop. The contribution is given by loop integral of the form

$$(p_1^2+i\epsilon)\int d^Dq \frac{1}{\frac{p_1^2+i\epsilon}{\hat{p}_1}+\frac{p_2^2+i\epsilon}{\hat{p}_2}+\frac{q^2+i\epsilon}{\hat{q}}+\frac{(q+p_1+p_2)^2+i\epsilon}{-\hat{q}-\hat{p}_1-\hat{p}_2}} \frac{1}{q^2+i\epsilon} \frac{1}{(q+p_1+p_2)^2+i\epsilon} f(q) \tag{3.29}$$

where the first factor in the integrand is given by the outer bubble of the translation kernel and we use $f(q)$ to denote the factors represented by the subgraph on the right together with the rest of the factors contained in the kernel. An LSZ factor p_2^2 is canceled by the propagator $1/(p_2^2+i\epsilon)$ so we assume it is safe to send p_2 on-shell, and we rewrite the integral as

$$\int d^Dq \frac{1}{\frac{-p_1\bar{p}_1+i\epsilon}{\hat{p}_1}+\frac{-q\bar{q}+i\epsilon}{\hat{q}}-\frac{-(q+p_1+p_2)(\bar{q}+\bar{p}_1+\bar{p}_2)+i\epsilon}{\hat{q}+\hat{p}_1+\hat{p}_2}} \frac{1}{q^2+i\epsilon} \frac{1}{(q+p_1+p_2)^2+i\epsilon} f(q). \tag{3.30}$$

As noted at the beginning of this section, if a graph constructed from kernel is to produce a finite contribution then the Green function, which contains the loop integral (3.30) in this case, has to be divergent in the on-shell limit. This is possible if we have singularities colliding with one another from different sides of the integration contour. When we integrate over components of the loop momentum q^μ the position of the singularities of the propagator and of the kernel can be solved as functions of leg momenta p_1 and p_2. As p_1 approaches its on-shell limit these singularities move in the complex plane and can walk through the integration surface, producing a divergent integral. This type of source for the divergent behaviour is avoided if we are allowed to deform the contour. However the option is not available if there are other singularities pinching from the other side of the contour. A thorough discussion was given in [63] to show that the pinching has the effect of producing an effective pole in the space of parameters in the integrand.

Generically the translation kernel introduces a number of factors of the form $1/\sum_i(p_i^2/\hat{p}_i)$ to the integrand, which can not be spanned by Lorentz invariant products using the standard Passarino-Veltman technique, nor can it be combined with propagators to produce an easily calculated integrand with the help of introducing Feynman parameters. However the integral (3.30) can be simplified by first noting that the kernel is independent of check-component momenta, which allows us to integrate over the $\check{q}$ by closing the contour and pick up residues from propagators. The $\check{q}$ component singularities of the propagators can be in the upper half or the lower half plane, depending on the signs of $\hat{q}$ and $\hat{q}+\hat{p}_1+\hat{p}_2$. The sum of residues from either half of the complex plane gives

$$\begin{aligned}&(\theta(\hat{q})\theta(-\widehat{q}-\hat{p}_1-\hat{p}_2)-\theta(-\hat{q})\theta(\widehat{q}+\hat{p}_1+\hat{p}_2))\\ &\quad\times\frac{1}{\frac{-q\bar{q}+i\epsilon}{\hat{q}}-\check{p}_1-\check{p}_2-\frac{-(q+p_1+p_2)\overline{q+p_1+p_2}+i\epsilon}{\widehat{q}+\hat{p}_1+\hat{p}_2}}\frac{1}{\hat{q}(\hat{q}+\hat{p}_1+\hat{p}_2)}.\end{aligned} \tag{3.31}$$

We see in the on-shell limit $\check{p}_1=p_1\bar{p}_1/\hat{p}_1$, $\check{p}_2=p_2\bar{p}_2/\hat{p}_2$ the second factor in the above equation has the same form as the bubble factor $1/\sum_i(p_i^2/\hat{p}_i)$ in the integral (3.30). A propagator is forcibly extracted if we expand the product of the residue (3.31) and the kernel into a linear combination of these two terms.

$$\begin{aligned}&\frac{1}{p_1^2+i\epsilon}\int\prod_{i=1}^{D/2-1}dq_{x(i)}dq_{y(i)}d\hat{q}\,(\theta(\hat{q})\theta(-\widehat{q}-\hat{p}_1-\hat{p}_2)-\theta(-\hat{q})\theta(\widehat{q}+\hat{p}_1+\hat{p}_2))\\ &\times\Bigg[1/\Bigg(-\check{p}_1-\check{p}_2+\frac{-q\bar{q}+i\epsilon}{\hat{q}}-\frac{-(q+p_1+p_2)\overline{q+p_1+p_2}+i\epsilon}{\widehat{q}+\hat{p}_1+\hat{p}_2}\Bigg)\end{aligned}$$

$$-1/\left(\frac{-p_1\bar{p}_1+i\epsilon}{\hat{p}_1}+\frac{-p_2\bar{p}_2+i\epsilon}{\hat{p}_2}+\frac{-q\bar{q}+i\epsilon}{\hat{q}}-\frac{-(q+p_1+p_2)\overline{q+p_1+p_2}+i\epsilon}{\widehat{q}+\hat{p}_1+\hat{p}_2}\right)\Bigg]$$
$$\times\frac{\hat{p}f(q)}{\hat{q}(\hat{q}+\hat{p}_1+\hat{p}_2)} \tag{3.32}$$

With the LSZ factor canceled by the extracted $1/p_1^2+i\epsilon$, graph (Fig.3.14) will survive in the on-shell limit as long as the integral (3.32) is non-vanishing. However if the two terms in the square bracket have imaginary parts of the same signs then as we perform the subsequent integration the singularities from both terms are in the same half-plane. In the on-shell limit these singularities will coincide each other without crossing through the contour and picking additional contributions in the process. As a result the two terms in the square bracket cancel each other.

In the standard loop integral calculation the positions of the poles of propagators are determined by the $i\epsilon$ prescription. The infinitesimal imaginary part allows singularities to deviate slightly from the real axis for real values of momenta and keeps the integrand well-defined. In the first term in the square bracket we see that after we integrate over the $\hat{q}$ component, the two imaginary terms are fixed by step functions to have the same sign. Both $1/\hat{q}$ and $-1/(\hat{q}+\hat{p}_1+\hat{p}_2)$ are positive or negative when multiplied by $\theta(\hat{q})\theta(-\widehat{q}-\hat{p}_1-\hat{p}_2)$ or $\theta(-\hat{q})\theta(\widehat{q}+\hat{p}_1+\hat{p}_2)$ respectively. Determining the signs of the imaginary terms in the kernel bubble $1/\sum_i(p_i^2/\hat{p}_i)$ however requires more careful analysis. In the next section we show that the singularities of the kernels can be arranged so that its imaginary part remains the same sign with the residue term (3.31). For this specific arrangement the two terms in the square bracket of (3.32) cancel for arbitrary external line momenta. So we can neglect the contributions from graphs of the type (Fig.3.14).

3.4 Loop integrals that contain translation kernels

From the graphical representation introduced in section (3.2) we saw that the translation kernels generically contain a number of dashed line bubbles, which contribute factors of the form $1/\sum_i p_i^2/\hat{p}_i$, making the integral non-compatible to the standard techniques. In addition to practical reasons we find the singularities of the kernel need to be carefully defined in a few symmetrical graphs. For example the momenta

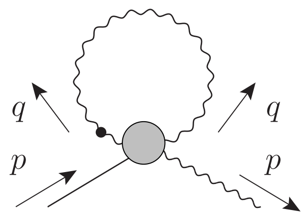


Figure 3.15: A divergent symmetrical loop graph constructed from the translation kernel

flowing in the legs of the kernel in self-energy graph (Fig.3.15) are required by conservation of momentum to be equal and opposite in pairs. In the denominator of the kernel a naive $i\epsilon$ prescription would give

$$\sum_i \frac{p_i^2 + i\epsilon}{\hat{p}_i} = \frac{p^2 + i\epsilon}{\hat{p}} + \frac{q^2 + i\epsilon}{\hat{q}} + \frac{q^2 + i\epsilon}{-\hat{q}} + \frac{p^2 + i\epsilon}{-\hat{p}} = 0 \tag{3.33}$$

The mirror symmetry of the graph causes a cancellation even for real values of momenta. Although all dashed line bubble factors are eventually canceled through the identity (3.28) when we combine graphs to give the physical amplitude, it is clear that we need to give a prescription to singularities of the kernels in the complex plane so that for arbitrary combinations of momenta $\{p_1, p_2, \cdots p_n\}$ the kernels $\Upsilon_{12\cdots n}$ and $\Xi^k_{12\cdots n}$ remain well-defined.

In equation (3.32) we showed that after integrating out the $\check{q}$ variable the sum of residues picked up from propagators is given by the formula

$$\frac{1}{-\check{p}_1 - \check{p}_2 + \frac{-q\bar{q}+i\epsilon}{\hat{q}} - \frac{-(q+p_1+p_2)\overline{q+p_1+p_2}+i\epsilon}{\hat{q}+\hat{p}_1+\hat{p}_2}}, \tag{3.34}$$

which has the same on-shell limit as the bubble factor in the kernel

$$\frac{1}{\frac{-p_1\bar{p}_1}{\hat{p}_1} + \frac{-p_2\bar{p}_2}{\hat{p}_2} + \frac{-q\bar{q}}{\hat{q}} - \frac{-(q+p_1+p_2)\overline{q+p_1+p_2}}{\hat{q}+\hat{p}_1+\hat{p}_2} + (\text{infinitesimal imaginary part})} \tag{3.35}$$

In the subsequent $\hat{q}$ component integral we need to cope with terms of the form

$$\begin{aligned} &\frac{-p\bar{p}}{\hat{p}} + \frac{-q\bar{q}}{\hat{q}} - \frac{-(q+j)(\bar{q}+\bar{j})}{\hat{q}+\hat{j}} \pm i\xi \\ &= \frac{-p\bar{p}}{\hat{p}} \frac{1}{\hat{q}} \frac{1}{\hat{q}+\hat{j}} \left[\hat{q} - \left(\alpha_{(+)} \mp i\epsilon_{(+)}(\xi)\right)\right] \left[\hat{q} - \left(\alpha_{(-)} \pm i\epsilon_{(-)}(\xi)\right)\right], \end{aligned} \tag{3.36}$$

where we introduced the symbol $j = p_1 + p_2$, $p = p_1$ to simplify the above expression and ξ denotes a positive infinitesimal term in propagator or in the kernel. The two roots $(\alpha_{(+)} \mp i\epsilon_{(+)})$ and $(\alpha_{(-)} \pm i\epsilon_{(-)})$ are given by

$$\alpha_{(\pm)} = \frac{-B \pm \sqrt{D}}{2A},\ \epsilon_{(\pm)} = \frac{\xi}{2A^2}\left(\pm(B+A) + \sqrt{D} + \frac{AC}{\sqrt{D}}\right), \tag{3.37}$$

where

$$\begin{aligned}
A &= \frac{p\bar{p}}{\hat{p}}\hat{j},\ B = -\frac{p\bar{p}}{\hat{p}}\hat{j} - q\bar{q} + (q+j)(\bar{q}+\bar{j}),\\
C &= \frac{p\bar{p}}{\hat{p}}\hat{j} - q\bar{q} - (q+j)(\bar{q}+\bar{j}),\\
D &= (p\bar{p})^2 + (q\bar{q})^2 + (q+j)^2(\bar{q}+\bar{j})^2\\
&\quad -2p\bar{p}q\bar{q} - 2q\bar{q}(q+j)(\bar{q}+\bar{j}) - 2p\bar{p}(q+j)(\bar{q}+\bar{j}).
\end{aligned} \tag{3.38}$$

Note that for real value momenta A, B, C and D are real.

When $D < 0$ the two roots are complex. So the singularities from the propagator (3.34) are at a finite distance from the real line, which is the integration contour in the $\hat{q}$ integral. If before the process of sending p_1 and p_2 on-shell we choose to start with off-shell continued parameters that are close enough to their on-shell values, the singularities of the kernel (3.35) will also approach the singularities of (3.34) from near by positions. Therefore for both sign options for the imaginary part of the translation kernel, the singularities of the kernel move to their destinations in the on-shell process without crossing through or deforming the contour. In the integral (3.32) the two terms in the square bracket cancel when $D < 0$.

If on the other hand $D > 0$ then $\alpha_{(\pm)}$ are real and the singularities of (3.34) and (3.35) are infinitesimally close to the real line. From triangular inequality it is straightforward to show that both $\epsilon_{(\pm)}$ are positive, which indicates that $(\alpha_{(\pm)} + i\epsilon_{(\pm)})$ are in the upper half plane and $(\alpha_{(\pm)} - i\epsilon_{(\pm)})$ are in the lower half plane. In equation (3.32) the sign of the imaginary part in the propagator term is determined by the sign of the factor $\frac{1}{\hat{q}} - \frac{1}{\hat{q}+\hat{j}}$. When the step function $\theta(-\hat{q})\theta(\hat{q}+\hat{j})$ in (3.32) is distributed to the propagator term its imaginary part is restricted to be negative, and the propagator term can be expanded as

$$\frac{\hat{q}(\hat{q}+\hat{j})}{\left(\frac{-p\bar{p}}{\hat{p}}\right)} \frac{1}{\hat{q}-\left(\alpha_{(+)}+i\epsilon_{(+)}(\xi)\right)} \frac{1}{\hat{q}-\left(\alpha_{(-)}-i\epsilon_{(-)}(\xi)\right)}\Bigg|_{\xi=\epsilon}, \tag{3.39}$$

where we have one singularity $\alpha_{(-)} - i\epsilon_{(-)}$ in the lower half plane and the other $\alpha_{(+)} + i\epsilon_{(+)}$ in the upper half plane. So if we assume that the translation kernel (3.35) also has a negative imaginary part $-i\xi_0$, its singularities are given by the same formulae as those of the propagator with ϵ replaced by ξ_0.

$$\frac{1}{\frac{-p_1\bar{p}_1}{\hat{p}_1}+\frac{-p_2\bar{p}_2}{\hat{p}_2}+\frac{-q\bar{q}}{\hat{q}}-\frac{-(q+p_1+p_2)\overline{q+p_1+p_2}}{\hat{q}+\hat{p}_1+\hat{p}_2}-i\xi_0}$$

$$= \frac{\hat{q}(\hat{q}+\hat{j})}{\left(\frac{-p\bar{p}}{\hat{p}}\right)} \frac{1}{\hat{q}-\left(\alpha_{(+)}+i\epsilon_{(+)}(\xi)\right)} \frac{1}{\hat{q}-\left(\alpha_{(-)}-i\epsilon_{(-)}(\xi)\right)}\Bigg|_{\xi=\xi_0}. \tag{3.40}$$

Apparently the singularities of the kernel are distributed in the complex plane in the same way as the singularities of the propagator term, so the two terms in the square bracket cancel. However in the integrand of (3.32) we have another possibility where the propagator term is multiplied by the step functions $\theta(\hat{q})\theta(-\hat{q}-\hat{j})$, which demands the imaginary part $\left(\frac{1}{\hat{q}}-\frac{1}{\hat{q}+\hat{j}}\right)i\epsilon$ of the denominator to be positive. In this term the pole near $\alpha_{(-)}$ is in the upper half plane and the other one near $\alpha_{(+)}$ is in the lower half plane.

$$\frac{\hat{q}(\hat{q}+\hat{j})}{\left(\frac{-p\bar{p}}{\hat{p}}\right)} \frac{1}{\hat{q}-\left(\alpha_{(+)}-i\epsilon_{(+)}(\xi)\right)} \frac{1}{\hat{q}-\left(\alpha_{(-)}+i\epsilon_{(-)}(\xi)\right)}\Bigg|_{\xi=\epsilon} \tag{3.41}$$

Therefore if we assume that the translation kernel (3.35) has a negative imaginary part $-i\xi_0$, then the singularities of the kernel will be sitting at the opposite side of the contour after we apply the on-shell condition. The two terms in the square bracket can be simplified as

$$\theta(\hat{q})\theta(-\hat{q}-\hat{j})\,\frac{\hat{q}(\hat{q}+\hat{j})}{\left(\frac{-p\bar{p}}{\hat{p}}\right)}\,\frac{1}{\alpha_{(+)}-\alpha_{(-)}}\left[\delta(\hat{q}-\alpha_{(-)})-\delta(\hat{q}-\alpha_{(+)})\right]$$

$$= -2\pi i\theta(\hat{q})\theta(-\hat{q}-\hat{j})\,\delta\Big(\frac{-p\bar{p}}{\hat{p}}+\frac{-q\bar{q}}{\hat{q}}-\frac{-(q+j)(\bar{q}+\bar{j})}{\hat{q}+\hat{j}}\Big), \tag{3.42}$$

where in the above equation we used the identity $1/(x-i\epsilon)-1/(x+i\epsilon) = 2\pi i\delta(x)$ to replace subtraction between terms by delta functions. Restoring the $\check{q}$ integral by inserting a delta function which forces $\check{q}$ to take the value at the singularity we picked, equation (3.32) can be rewritten as a double cut integral

$$\int d^D q\theta(\hat{q})\theta(-\hat{q}-\hat{j})\,\delta(q^2)\,\delta((q+j)^2)\,f(q). \tag{3.43}$$

Note that the step functions restrict that the loop momenta $q_1 = q$ and $q_2 = -q-j$ be on-shell and flow forward in light-cone time. For example if in the loop integral (3.32) $-p_1$, $-p_2$ correspond to incoming particle momenta and p_3 to p_n correspond to outgoing momenta, the cut integral is equivalent to the scattering amplitude $< p_3, \cdots p_n| - p_1, -p_2 >$ with two particle intermediate states inserted.

$$< p_3, \cdots p_n|q_1, q_2 > \frac{d^{D-1}q_1}{\hat{q}_1}\frac{d^{D-1}q_2}{\hat{q}_2} < q_1, q_2| - p_1, -p_2 > \tag{3.44}$$

In graphical representation the translation kernel that contains negative imaginary part contributes as the cut graph (Fig.3.16(a)).

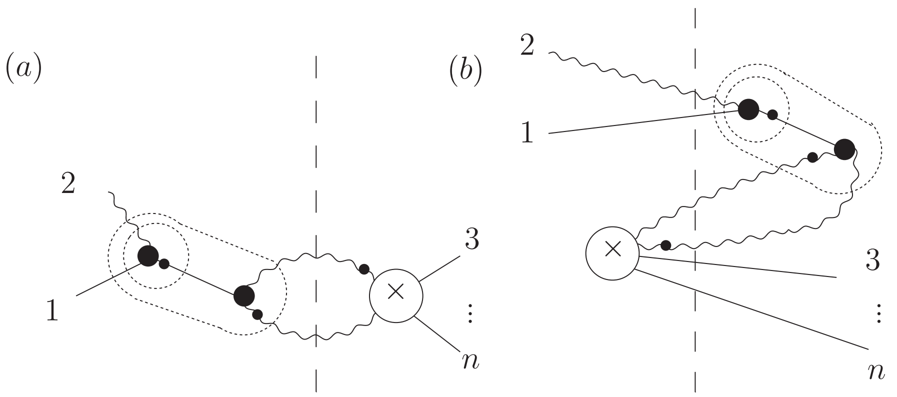


Figure 3.16: Kernel loop graphs contribute as cut diagrams

Following the same analysis as shown above, it is straightforward to show that the positive imaginary part option for the translation kernel favours the other term in the integrand which contains the step functions $\theta(-\hat{q})\theta(\hat{q}+\hat{j})$. For this choice the kernel restricts q_1 and q_2 to flow backward in (light-cone) time and the contribution is represented as in graph (Fig.3.16(b)). So in order to eliminate complicated loop integrals constructed from translation kernels and to keep symmetric graphs such as

(Fig.3.15) well-defined at the same time, we need to pick an $i\epsilon$ prescription for the kernel that switches sign according to the leg momenta and treats the legs of the kernel in an unsymmetrical way to avoid cancellation. Note that once a prescription has been decided the same translation kernel can also be used to construct other graphs in which the legs of the kernel are linked to different objects. An option that satisfies all of the requirements is to set up a priority system to determine the sign of the imaginary part. Starting with the naive $i\epsilon$ prescription with an infinitesimal imaginary part associated with each term

$$\begin{aligned} &\frac{-p\bar{p}+i\epsilon_p}{\hat{p}}+\frac{-q\bar{q}+i\epsilon_q}{\hat{q}}-\frac{-(q+j)(\bar{q}+\bar{j})+i\epsilon_{q+j}}{\hat{q}+\hat{j}} \\ \rightarrow\ &\frac{-p\bar{p}}{\hat{p}}+\frac{-q\bar{q}}{\hat{q}}-\frac{-(q+j)(\bar{q}+\bar{j})}{\hat{q}+\hat{j}}+i\xi, \end{aligned} \tag{3.45}$$

we redefine the overall sign of the imaginary part according to a number of rules:[1]

- If all of the momenta are 4-dimensional, we let the imaginary part be represented by the $i\epsilon$ term associated with the $\mathcal{A}$ field line, $i\xi = i\epsilon/\hat{p}$.

---

[1]

The set of rules presented here can be summarised as assigning a hierarchy of ϵs. Generically the dashed line bubble factor can be written as

$$\frac{p_1^2+i\epsilon_A}{\hat{p}_1}+\frac{p_2^2+i\epsilon_B}{\hat{p}_2}+\cdots\frac{p_n^2+i\epsilon_B}{\hat{p}_n} \tag{3.46}$$

In the above expression we associate ϵ_A to the line from which the $\mathcal{A}$ field is translated and ϵ_B to the legs that represent the $\mathcal{B}$ fields to avoid symmetry, and we have

$$\epsilon_A = \epsilon_2 + \epsilon_1\theta(\mu^2),\ \epsilon_B = \epsilon_3 + \epsilon_1\theta(\mu^2) \tag{3.47}$$

where $\theta(\mu^2)$ is the step function of the momentum square in the extended $D-4$ dimensions, $\mu^2 = \sum_{I=2}^{D/2-1} q_I\bar{q}_I$. The sign of the imaginary part of the translation kernel is determined by the dominant term in when we impose the priority condition $\epsilon_1 \gg \epsilon_2 \gg \epsilon_3$. Note that we fix the sign for every combination of momenta $\{p_1, p_2, \cdots p_n\}$ according to the hierarchy instead of keeping the sum $\frac{i\epsilon_A}{\hat{p}_1}+\frac{i\epsilon_B}{\hat{p}_2}+\cdots\frac{i\epsilon_B}{\hat{p}_n}$ as the imaginary part. In the later case the sign can still become undetermined because the proportionality between different $\hat{p}_i$ change dramatically when we integrate over loop momentum.

- If a subset or all of the legs extend to D-dimensions and their hat-components are of the same sign, then the imaginary part is determined by the $i\epsilon$ part of these terms, $i\xi = i\epsilon/\hat{q}$ or $i\xi = i\epsilon/(-\hat{q} - \hat{j})$.

- If a subset or all of the legs extend to D-dimensions and the signs of their hat-components disagree with each other, we assign an $i\epsilon$ to every one of them on an equal weighting. If the $i\epsilon$ terms are not completely canceled, the sum determines the imaginary part. $i\xi = \left(\frac{1}{\hat{q}} - \frac{1}{\hat{q}+\hat{j}}\right) i\epsilon$. If on the other hand they do cancel each other then we use the $i\epsilon$ term from the $\mathcal{A}$ field line. $i\xi = i\epsilon/\hat{p}$.

Note that we use D-dimension extension as a way to distinguish legs contracted in the loop and the legs contracted to form external lines. When the $i\epsilon$ prescription is modified by the above rules the imaginary part of the kernel in equation (3.32) adjusts its sign to agree with that of the propagator term both when $-\hat{q}$ and $\hat{q} + \hat{j}$ are positive or negative. So all loop integral of the form (3.32) can be neglected and the translation kernels only contribute to the scattering amplitude as tadpole graphs for practical calculations.

Applying the modified $i\epsilon$ prescription we see that in the symmetrical graph (Fig.3.15) the imaginary part is determined by the $\mathcal{A}$ field line and does not vanish. Equation (3.33) is modified as

$$\sum_i \frac{p_i^2 + i\epsilon}{\hat{p}_i} = \frac{i\epsilon}{\hat{p}} \tag{3.48}$$

So graph (Fig.3.15) vanishes because the translation kernel fails to generate a propagator $1/p^2$ to cancel the suppressing factor from LSZ reduction.

3.4.1 Correction terms originating from the modification on the $i\epsilon$ prescription

Although the $i\epsilon$ prescription just described allows us to get rid of loop integrals constructed from the kernel, the modified definition of the kernel also generate new vertices in the transformed lagrangian. Because the problem caused by graph (Fig.3.15) originates from symmetry we assigned different ϵs to the legs associated with $\mathcal{A}$ fields and $\mathcal{B}$ fields. A naive approach to realise this modification would be to adjust the canonical transformation condition so that in the $\mathcal{B}$ field free lagrangian the ϵ is different from that of the LCYM lagrangian.

$$\mathcal{L}_A^{-+}[\mathcal{A}] + \mathcal{L}^{++-}[\mathcal{A}] = \mathcal{L}_B^{-+}[\mathcal{B}], \tag{3.49}$$

where

$$\mathcal{L}_A^{-+}[\mathcal{A}] = \bar{\mathcal{A}}(p^2 + i\epsilon_A)\mathcal{A}(p),\ \mathcal{L}_B^{-+}[\mathcal{B}] = \bar{\mathcal{B}}(p^2 + i\epsilon_B)\mathcal{B}(p) \tag{3.50}$$

However comparing terms with the same number of B fields from both sides of the equation (3.49) results in contradictions. In particular, the lowest order term on the left hand side of the equation is $\frac{p^2+i\epsilon_A}{\hat{p}}B(p)$ while on the right hand side we have $\frac{p^2+i\epsilon_B}{\hat{p}}\mathcal{B}(p)$. So instead we modify the ϵs starting from the $\mathcal{B}^2$ term.

$$\mathcal{L}_A^{-+}[\mathcal{A}] + \mathcal{L}^{++-}[\mathcal{A}] = \mathcal{L}_A^{-+}[\mathcal{B}] + \mathcal{L}_\epsilon[\mathcal{B}] \tag{3.51}$$

where an infinite number of infinitesimal correction terms are generated to keep the transformation equation (3.51) self-consistent.

$$\begin{aligned}\mathcal{L}_\epsilon[\mathcal{B}] &= \int \left(\bar{\mathcal{B}} - \bar{\mathcal{A}}\right) i(\epsilon_A - \epsilon_B)\mathcal{B} \\ &= \left(\sum_{n=2}^{\infty}\int\prod_{i=1}^{n} d^{D-1}k_i\,\frac{\hat{k}}{\hat{p}}\Xi^k_{12\cdots n}\,\mathcal{B}_2\ldots\bar{\mathcal{B}}_k\ldots\mathcal{B}_n\right) i\left(\epsilon_A - \epsilon_B\right)\mathcal{B}(p)\end{aligned} \tag{3.52}$$

Following the graphical convention introduced in section (3.2) we use a gray circle to represent the translation kernel Ξ^k contained in the new effective vertex terms. We denote the infinitesimal factor $(\epsilon_A - \epsilon_B)$ by a small double circle. These new vertices generically can be neglected except in a few extremely divergent graphs which we discuss in the next section.

The translation kernels are determined from the transformation condition (3.51). In momentum space we have

$$\frac{p^2 + i\epsilon_A}{\hat{p}}A(p) + i\int d^{D-1}q[\frac{\bar{q}}{\hat{q}}A(q), A(p-q)]$$

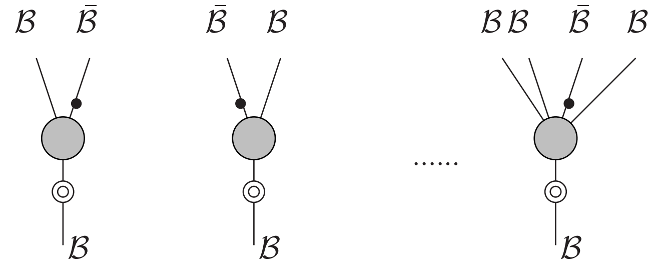


Figure 3.17: Infinitesimal vertices

$$
\begin{aligned}
&= \int d^{D-1}q \frac{q^2+i\epsilon_A}{\hat{q}} B(q) \frac{\delta A(p)}{\delta B(q)} \\
&\quad + \sum_{n=2}^{\infty} \int \prod_{i=2}^{n} d^{D-1}q_i \left(\sum_{j=2}^{n} \frac{i\,(\epsilon_A - \epsilon_B)}{\hat{q}_j} \right) \Upsilon_{12\ldots n} B(q_2)\ldots B(q_n) \\
&= \int d^{D-1}q \frac{q^2+i\epsilon_A}{\hat{q}} B(q) \frac{\delta A(p)}{\delta B(q)} + \frac{i\,(\epsilon_A-\epsilon_B)}{\hat{q}} B(q) \frac{\delta A(p)}{\delta B(q)} - \frac{i\,(\epsilon_A-\epsilon_B)}{\hat{p}} B(p). \quad (3.53)
\end{aligned}
$$

Matching both side of the equation we arrive at

$$
\Upsilon_{123} = \frac{\frac{1}{\hat{p}_1} V^{++-}(p_2,\, p_3,\, p_1)}{\frac{p_1^2+i\epsilon_A}{\hat{p}_1} + \frac{p_2^2+i\epsilon_B}{\hat{p}_2} + \frac{p_3^2+i\epsilon_B}{\hat{p}_3}}. \quad (3.54)
$$

As described in footnote (1) in the modified definition the imaginary part of the kernel is determined by a set of rules that picks the dominant term to replace the sum of $i\epsilon$ contributed from the legs. In the same spirit we use the factor $(\epsilon_A - \epsilon_B)$ in equation (3.52) to denote the infinitesimal correction required to balance the transformation equation (3.51). The factor $(\epsilon_A - \epsilon_B)$ is regarded as a function of the same leg momenta $\{p_1,\, p_2, \cdots p_n\}$ that appear in the $\Xi^k_{12\cdots n}$ to which the factor $(\epsilon_A - \epsilon_B)$ is multiplied to and its value can only be $+\epsilon$ or $-\epsilon$ which we determine by matching kernels $\Xi^k_{12\cdots n}$ and $\Upsilon_{12\cdots n}$ of the same set of leg momenta $\{p_1,\, p_2, \cdots p_n\}$ from both sides of the equation (3.51).

3.4.2 Self-energy graphs

In the LCYM theory the self-energy graphs contributing to the field renormalization at one-loop level can contain $\overline{\text{MHV}}$ vertices. These graphs appear missing when we canonically transform the lagrangian to absorb the $\overline{\text{MHV}}$ vertex. In section (3.3) we showed that in the MHV lagrangian theory the $\overline{\text{MHV}}$ vertex factor can be provided by the translation kernels in the form of tadpole graphs. The vertex factors in the

$\langle \bar{\mathcal{A}}\bar{\mathcal{A}} \rangle$ self-energy bubble in the original LCYM theory however are not explained by this type of graph. As discussed in section (3.4) the symmetrical tadpole graph (Fig.3.15) does not produce an effective propagator to cancel the LSZ factor so the graph is non-contributing. We find the $\overline{\text{MHV}}$ vertices are resupplied from the new vertices carrying infinitesimal correction ϵs. In this section we restore all self-energy bubble diagrams in LCYM at one-loop from the viewpoint of the MHV lagrangian theory.

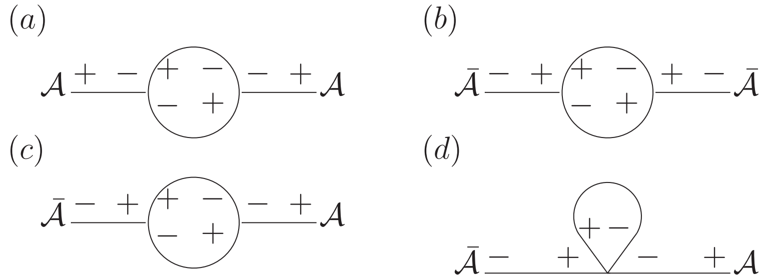


Figure 3.18: LCYM self-energy graphs

The $\langle \mathcal{A}\mathcal{A} \rangle$ self-energy graph (Fig.3.18(a)) is not altered in the MHV lagrangian theory because it only contains 3-point MHV vertices. Using the standard Passarino-Veltman reduction it is straightforward to show that the loop integral of a $\langle \mathcal{A}_\mu \mathcal{A}_\nu \rangle$ graph is proportional to the flat spacetime metric $\eta_{\mu\nu}$. Since the metric is off-diagonal in light-cone coordinates graph (Fig3.18(a)) yields zero.

The $\langle \bar{\mathcal{A}}\bar{\mathcal{A}} \rangle$ graph on the other hand has a more complicated structure. Both of the $\overline{\text{MHV}}$ factors in the original LCYM graph are supplied by the infinitesimal vertex. When self-contracted the kernel factor attached to the new vertex generates a singular factor $1/i\,(\epsilon_A - \epsilon_B)$ and cancels the infinitesimal factor carried by that vertex. Expanding the translation kernels represented by gray circles into graphs that contain $\overline{\text{MHV}}$ vertices and dashed line bubbles we find graphs (Fig.3.19(a), (b)) are replaced by the sum of (Fig.3.19 (c), (d), (e), (f)). Their contributions $Y_{(c)}$, $Y_{(d)}$, $Y_{(e)}$, $Y_{(f)}$ can be read off as

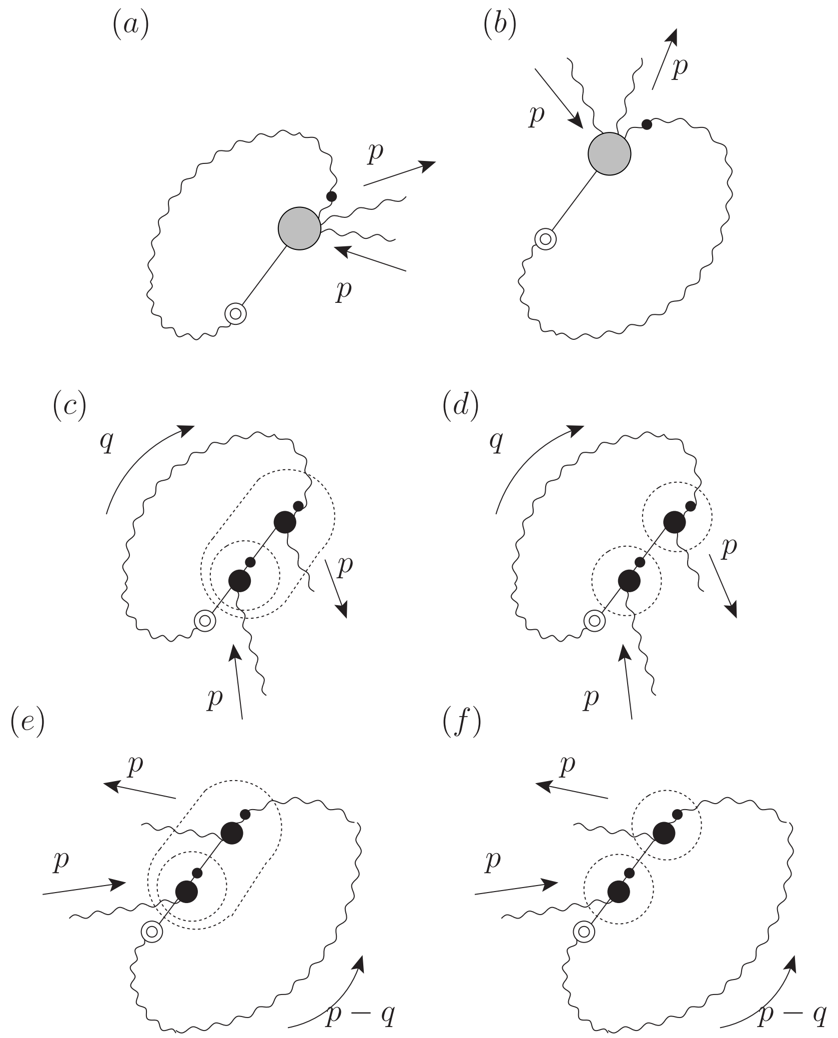


Figure 3.19: $\langle \bar{\mathcal{A}}\bar{\mathcal{A}} \rangle$ self-energy graphs

$$\begin{aligned}
Y_{(c)} = &-\int dq_I d\bar{q}_I d\hat{q} \frac{1}{\frac{p^2+i\epsilon_B}{-\hat{p}} + \frac{q^2+i\epsilon_B}{\hat{q}} + \frac{(p-q)^2+i\epsilon_A}{\hat{p}-\hat{q}}} \\
&\times \left(\theta(-\hat{q}) - \theta(\hat{q})\right) \frac{\pi i}{\hat{q}\left(\hat{p}-\hat{q}\right)} \\
&\times V^{++-}\left(p, q-p, -q\right) V^{++-}\left(-p, q, p-q\right),
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
Y_{(d)} = &\int dq_I d\bar{q}_I d\hat{q} \frac{\frac{1}{\hat{q}}}{\frac{1}{\hat{p}-\hat{q}} + \frac{1}{\hat{q}}} \frac{1}{\frac{q^2+i\epsilon_A}{\hat{q}} + \frac{p^2+i\epsilon_B}{-\hat{p}} + \frac{(p-q)^2+i\epsilon_B}{\hat{p}-\hat{q}}} \\
&\times \left(\theta(-\hat{q})\theta(\hat{p}-\hat{q}) - \theta(\hat{q})\theta(\hat{q}-\hat{p})\right) \frac{\pi i}{\hat{q}\left(\hat{p}-\hat{q}\right)} V^{++-}\left(p, q-p, -q\right) \\
&\times V^{++-}\left(-p, q, p-q\right),
\end{aligned} \tag{3.56}$$

$$Y_{(e)} = \int dq_I d\bar{q}_I d\hat{q} \frac{1}{\frac{p^2+i\epsilon_B}{-\hat{p}} + \frac{q^2+i\epsilon_B}{\hat{q}} + \frac{(p-q)^2+i\epsilon_A}{\hat{p}-\hat{q}}}$$
$$\times \left(\theta(-\hat{q}+\hat{p}) - \theta(\hat{q}-\hat{p})\right) \frac{\pi i}{\hat{q}\,(\hat{p}-\hat{q})}$$
$$\times V^{++-}\,(p, q-p, -q)\; V^{++-}\,(-p, q, p-q)\,, \tag{3.57}$$

$$Y_{(f)} = -\int dq_I d\bar{q}_I d\hat{q} \frac{\frac{1}{\hat{p}-\hat{q}}}{\frac{1}{\hat{p}-\hat{q}} + \frac{1}{\hat{q}}} \frac{1}{\frac{q^2+i\epsilon_A}{\hat{q}} + \frac{p^2+i\epsilon_B}{-\hat{p}} + \frac{(p-q)^2+i\epsilon_B}{\hat{p}-\hat{q}}}$$
$$\times \left(\theta(-\hat{q})\theta(\hat{p}-\hat{q}) - \theta(\hat{q})\theta(\hat{q}-\hat{p})\right) \frac{\pi i}{\hat{q}\,(\hat{p}-\hat{q})}$$
$$\times V^{++-}\,(p, q-p, -q)\; V^{++-}\,(-p, q, p-q)\,. \tag{3.58}$$

In the above expressions we integrated over the $\check{q}$ component loop momenta to make them more easily combined. The sum of these four graphs is given by

$$Y_{(c)} + Y_{(d)} + Y_{(e)} + Y_{(f)} = \int dq_I d\bar{q}_I d\hat{q} \frac{1}{\frac{q^2+i\epsilon_A}{\hat{q}} + \frac{p^2+i\epsilon_B}{-\hat{p}} + \frac{(p-q)^2+i\epsilon_B}{\hat{p}-\hat{q}}}$$
$$\times \left(\theta(-\hat{q})\theta(\hat{q}-\hat{p}) - \theta(\hat{q})\theta(-\hat{q}+\hat{p})\right) \frac{\pi i}{\hat{q}\,(\hat{p}-\hat{q})}$$
$$\times V^{++-}\,(p, q-p, -q)\; V^{++-}\,(-p, q, p-q)\,, \tag{3.59}$$

which is the same as the LCYM self-energy bubble with the $\check{q}$ component integrated over. Note that the loop integral (3.59) vanishes because the metric $\eta_{\mu\nu}$ does not have non-trivial diagonal elements. So in practice the new vertices can be neglected. However the reason for vanishing came from the structure of the $\overline{\text{MHV}}$ vertices instead of the infinitesimal nature of the new vertices.

The only non-trivial self-energy graphs are contained in $\langle \bar{\mathcal{A}}\mathcal{A}\rangle$, where $\overline{\text{MHV}}$ vertex is implicitly provided by the 4-point MHV vertex. Recall that the MHV vertices in the canonically transformed lagrangian are obtained from replacing the $\mathcal{A}$ and $\bar{\mathcal{A}}$ fields attached to the original LCYM vertices by terms in the expansion formulae (3.15) and (3.16). In graphical notation the contributions are represented by

In this chapter we saw that despite in the MHV lagrangian theory the scattering amplitudes are given by Green functions of the new field variables attached with

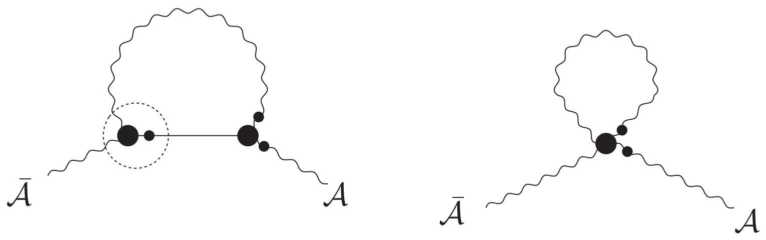


Figure 3.20: Contributions to the $\langle \bar{\mathcal{A}} \mathcal{A} \rangle$ at one-loop are given by factors in the MHV vertex

translation kernels, which violate the equivalent theorem, in most situations these kernels are suppressed in the on-shell limit. In the special cases where the kernel generates a singular factor to counteract the suppression mostly the vertex factors embedded in the kernel vanish for real value 4-dimensional null momenta. The only exception for which all these mechanisms fail to protect the equivalence theorem are the tadpole graphs. However as shown in section (3.4) the D-dimensional analytic continuation of the kernel has a complicated formula and is not compatible to the standard loop integration techniques. In the next chapter we shall present other integration measure preserving transformations which produce vertices with the same helicity structure as the canonically transformed lagrangian.

Chapter 4

The measure-preserving transformations that lead to MHV lagrangians

The canonical transformation introduced in chapters 2 and 3 re-expresses the helicity fields $\mathcal{A}$ and $\bar{\mathcal{A}}$ as series expansions in the new field variables. In the expansion formulae (2.72), (2.73) the powers of $\mathcal{B}$ fields increases while the powers of $\bar{\mathcal{B}}$ are held fixed, so that in terms of the new variables the 3-point and the 4-point interactions in the LCYM lagrangian are rewritten as an infinite number of new interaction terms, all of them contain two negative helicity fields.

$$\mathcal{L}^{--+}[\mathcal{A}] + \mathcal{L}^{--++}[\mathcal{A}] = \mathcal{L}^{--+}[\mathcal{B}] + \mathcal{L}^{--++}[\mathcal{B}] + \cdots + \mathcal{L}^{--+\cdots+}[\mathcal{B}] + \cdots \tag{4.1}$$

The Feynman rules derived from the new lagrangian allow us to construct scattering amplitudes from vertices and propagators that have the same helicity feature as prescribed by the CSW rules, with only a few exceptions coming from the translation kernels. In chapter 2 we show that at tree-level for generic values of leg momenta the contributions from translation kernels are suppressed by the LSZ factors p_i^2. So in the on-shell limit an MHV amplitude only receives the contribution from the vertex in the new lagrangian. In [24] it was argued that the new n-point vertex can only differ from the Parke-Taylor formula by squares of leg momenta. Because the translation kernels (2.78), (2.79), 3-point, 4-point vertices in the original LCYM lagrangian and the Parke-Taylor formula are all holomorphic, such factors must be absent. This was verified by explicit calculations of the MHV vertices up to 5-points by Ettle and Morris in [29].

We find the holomorphy of the translation kernels Υ and Ξ^k originates from a choice implicitly taken in the transformation condition. In (2.74) the canonical conjugate momentum $\hat{\partial}\bar{\mathcal{A}}^a(\tau, \mathbf{x})$ was assumed to transform inversely to that of the field variable $\mathcal{A}^a(\tau, \mathbf{x})$. The Jacobian determinant of the change of variables is field independent and does not introduce new factors to change the Feynman rules. However we find the inverse transformation condition can be relaxed to $f(\partial)\bar{\mathcal{A}}^a(x)$ with $f(\partial)$ an arbitrary function of differential operator. The transformation

$$f(\partial)\bar{\mathcal{A}}^a(\mathbf{y}) = \int d^D\mathbf{x} \frac{\delta \mathcal{B}^b(\mathbf{x})}{\delta \mathcal{A}^a(\mathbf{y})} f(\partial)\bar{\mathcal{B}}^b(\mathbf{x}) \tag{4.2}$$

performed in the full D-dimensions also leaves the integration measure invariant. Because the expansion formulae (4.3), (4.4) for the generalised measure-preserving transformation have the same helicity structure as the expansions for the canonical transformation, the vertices in the new lagrangian all have the same helicity assignments as the effective vertices of the CSW rules.

$$\mathcal{A}_1 = \mathcal{B}_1 + \Upsilon'_{123}\mathcal{B}_2\mathcal{B}_3 + \Upsilon'_{1234}\mathcal{B}_2\mathcal{B}_3\mathcal{B}_4 + \cdots \tag{4.3}$$

$$f(\partial_1)\bar{\mathcal{A}}_1 = f(\partial_1)\bar{\mathcal{B}}_1 + \Xi'^2_{123}\left(f(\partial_2)\bar{\mathcal{B}}_2\right)\mathcal{B}_3 + \Xi'^3_{123}\mathcal{B}_3\left(f(\partial_3)\bar{\mathcal{B}}_3\right) + \cdots \tag{4.4}$$

Nevertheless for a generic choice of $f(\partial)$ the translation kernels are not holomorphic and the MHV vertices are not described by the Parke-Taylor formula.

In chapter 3 we saw at loop-level the integral contribution from a symmetric diagram (Fig.3.15) that contains translation kernels requires careful definition to its singular behaviour. Generically the factor $1/\left(\sum p_i^2/\hat{p}_i\right)$ included in the kernels also makes the loop integral difficult to be carried out because standard methods such as Passarino-Veltman reduction do not apply. For practical calculations both of these problems can be avoided by a suitable choice of $f(\partial)$ in the transformation (4.2).

We divide this chapter into two parts. In section (4.0.3) to (4.0.4) we prove that the integration measure is invariant under the transformation (4.2) for arbitrary $f(\partial)$

and derive the modified kernels. In section (4.1) we calculate directly to show that in 4-dimensions, when the operator $f(\partial)$ is chosen to be $\hat{\partial}$, the generic n-point MHV vertex generated by the canonical transformation $\left\{\mathcal{A}, \hat{\partial}\bar{\mathcal{A}}\right\} \to \left\{\mathcal{B}, \hat{\partial}\bar{\mathcal{B}}\right\}$ matches with the Parke-Taylor formula.

This work was published in [58].

4.0.3 Proof of the measure invariance under the transformation $\{\mathcal{A}, f(\partial)\bar{\mathcal{A}}\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$

Before computing the Jacobian we divide the transformation $\{\mathcal{A}, f(\partial)\bar{\mathcal{A}}\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$ into intermediate steps. The same transformation can be achieved if we first define the mapping

$$\mathcal{A}'^a(x) = \mathcal{A}^a(x),\ \bar{\mathcal{A}}'^a(x) = f(\partial)\bar{\mathcal{A}}^a(x) \tag{4.5}$$

for every field variable labeled by the spacetime coordinates in D-dimensions and colour index, and then take $\{\mathcal{A}', \bar{\mathcal{A}}'\} \to \{\mathcal{B}', \bar{\mathcal{B}}'\}$ under the condition

$$\frac{\delta\bar{\mathcal{A}}'^a(x)}{\delta\bar{\mathcal{B}}'^b(y)} = \frac{\delta\mathcal{B}'^b(y)}{\delta\mathcal{A}'^a(x)}, \tag{4.6}$$

and followed by the mapping $\{\mathcal{B}', \bar{\mathcal{B}}'\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$

$$\mathcal{B}^b(y) = \mathcal{B}'^b(y),\ \bar{\mathcal{B}}^b(y) = f(\partial)\bar{\mathcal{B}}'^b(y) \tag{4.7}$$

In analogy with the canonical transformation used in [24,29] we assume that the field variable $\mathcal{A}'$ is a functional of $\mathcal{B}'$ in the form of a series expansion, and $\bar{\mathcal{A}}'$ is a functional of $\mathcal{B}'$ and $\bar{\mathcal{B}}'$ with the power of the bar-component field in every term fixed to be one. However in both of the expansion formulae (4.3), (4.4) the functional integrals run through the full D-dimensions instead of a constant light-cone time surface.

The Jacobian for $\mathcal{D}\mathcal{A}\,\mathcal{D}\bar{\mathcal{A}} \to \mathcal{D}\mathcal{A}'\,\mathcal{D}\bar{\mathcal{A}}'$ can be readily shown to be independent of field variables.

$$det \begin{pmatrix} \frac{\delta\mathcal{A}^a(x)}{\delta\mathcal{A}'^b(y)} & 0 \\ 0 & \frac{\delta\bar{\mathcal{A}}^a(x)}{\delta\bar{\mathcal{A}}'^b(y)} \end{pmatrix} = det \begin{pmatrix} \delta^a_b\,\delta(x-y) & 0 \\ 0 & \delta^a_b\, f(\partial)^{-1}\delta(x-y) \end{pmatrix} = const., \quad (4.8)$$

provided the inverse of the operator $f(\partial)$ exists. Similarly, for the transformation $\mathcal{DB}'\,\mathcal{D}\bar{\mathcal{B}}' \to \mathcal{DB}\,\mathcal{D}\bar{\mathcal{B}}$ the Jacobian is

$$det \begin{pmatrix} \frac{\delta\mathcal{B}'^a(x)}{\delta\mathcal{B}^b(y)} & 0 \\ 0 & \frac{\delta\bar{\mathcal{B}}'^a(x)}{\delta\bar{\mathcal{B}}^b(y)} \end{pmatrix} = det \begin{pmatrix} \delta^a_b\,\delta(x-y) & 0 \\ 0 & \delta^a_b\, f(\partial)\delta(x-y) \end{pmatrix} = const. \quad (4.9)$$

Multiplying the two determinants above yields the determinant of a unit matrix. From the inverse relation (4.6) and the assumption that $\mathcal{A}'$ is independent of $\mathcal{B}'$, we have

$$det \begin{pmatrix} \frac{\delta\mathcal{A}'^a(x)}{\delta\mathcal{B}^b(y)} & \frac{\delta\bar{\mathcal{A}}'^a(x)}{\delta\mathcal{B}'^b(y)} \\ 0 & \frac{\delta\bar{\mathcal{A}}'^a(x)}{\delta\bar{\mathcal{B}}'^b(y)} \end{pmatrix} = det(I) \quad (4.10)$$

So the combination of three successive transformations $\{\mathcal{A}, f(\partial)\bar{\mathcal{A}}\} \to \{\mathcal{A}', \bar{\mathcal{A}}'\}$, $\{\mathcal{A}', \bar{\mathcal{A}}'\} \to \{\mathcal{B}', \bar{\mathcal{B}}'\}$ and $\{\mathcal{B}', \bar{\mathcal{B}}'\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$ produces a unit Jacobian determinant.

Using the three transformation relations we find the $f(\partial)\bar{\mathcal{A}}$ can be expressed as

$$f(\partial)\bar{\mathcal{A}}^a(x) = \bar{\mathcal{A}}'^a(x) = \int d^4y\, \frac{\delta\bar{\mathcal{A}}'^a(x)}{\delta\bar{\mathcal{B}}'^b(y)}\bar{\mathcal{B}}'^b(y) = \int d^4y\, \frac{\delta\mathcal{B}^b(y)}{\delta\mathcal{A}^a(x)}\, f(\partial)\bar{\mathcal{B}}^b(y), \quad (4.11)$$

which is the same as equation (4.2), therefore we have proved that the transformation $\{\mathcal{A}, f(\partial)\bar{\mathcal{A}}\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$ leaves the integration measure invariant for an arbitrary field independent differential operator $f(\partial)$. Note that when substituting $\bar{\mathcal{A}}'^a(x)$ in the above equation (4.11) we used $\delta\bar{\mathcal{A}}'^a(x) = \int d^4y\, \frac{\delta\bar{\mathcal{A}}'^a(x)}{\delta\bar{\mathcal{B}}'^b(y)}\delta\bar{\mathcal{B}}'^b(y)$ together with the assumption that $\bar{\mathcal{A}}'^a(x)$ is linear in $\bar{\mathcal{B}}'^b(y)$, which allows us to interchange the variations $\delta\bar{\mathcal{A}}'^a(x)$, $\delta\bar{\mathcal{B}}'^b(y)$ with $\bar{\mathcal{A}}'^a(x)$ and $\bar{\mathcal{B}}'^b(y)$.

4.0.4 Translation kernels and the CSW rules generated by the measure-preserving transformation

The generalised transformation $\{\mathcal{A}, f(\partial)\bar{\mathcal{A}}\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$ can be applied on the LCYM theory as introduced in section (2.5). We impose the condition that the transformation rearranges the self-dual part of the original lagrangian into a free field theory

$$\mathcal{L}^{-+}[\mathcal{A}] + \mathcal{L}^{-++}[\mathcal{A}] = \mathcal{L}^{-+}[\mathcal{B}]\,. \tag{4.12}$$

Stripping off a factor of $f(\partial)\bar{\mathcal{B}}$ from both sides of (4.12) and inverting the variation $\delta\mathcal{B}^b(y)/\delta\mathcal{A}^a(x)$, we arrive at the equation

$$f^{-1}(\partial)\partial^2\mathcal{A} + \int d^Dx\, f^{-1}(\partial)V^{-++}\mathcal{A}_2\mathcal{A}_3 = \int d^Dy\, \frac{\delta\mathcal{A}(x)}{\delta\mathcal{B}(y)} f^{-1}(\partial)\partial^2\mathcal{B}, \tag{4.13}$$

from which we can solve the generic n-th order kernel $\Upsilon_{12\cdots n}$ iteratively. The translation kernel $\Xi^k_{12\cdots n}$ are then determined by the inverse transformation relation (4.6). It is straightforward to show that in momentum space the new transformation the formulae for $\Upsilon_{12\cdots n}$ and $\Xi^k_{12\cdots n}$ are given by the original formulae with the $\hat{p}_i$ dependence replaced by $f(p_i)$. For example, the lowest order translation kernel Υ_{123} is

$$\Upsilon_{123} = \frac{\frac{1}{f(p_3)}V^{-++}(p_1,\, p_2,\, p_3)}{\frac{p_1^2}{f(p_1)} - \frac{p_2^2}{f(p_2)} - \frac{p_3^2}{f(p_3)}}. \tag{4.14}$$

In section (3.3) we saw that when combined together the translation kernels Υ_{123}, Ξ^2_{123} and Ξ^3_{123} derived from the canonical transformation reproduce the absorbed LCYM $\overline{\text{MHV}}$ vertex (Fig.3.10 (a) (b) (c)). Since the cancellation of the bubble factors $1/(\sum p_i^2/\hat{p}_i)$ did not depend on specific properties of $\hat{p}$ we see that combining the generalised kernels with all $\hat{p}$ replaced by $f(p)$ yields the same result. The same argument also applies to more complicated identities between kernel and LCYM graphs such as the ones shown in appendix A. Similarly, we can recombine graphs from the lagrangian which underwent generalised measure-preserving transformation to form LCYM graphs since the canonically transformed lagrangian is known to reproduce the LCYM amplitudes as long as kernels are included in the formalism.

As noted at the beginning of this chapter we can simplify loop calculations by choosing a suitable function $f(p)$. One of the options is to have $f(p) = p^2$ so that in the denominator $\sum_i (\pm p_i^2/f(p_i))$ becomes a numerical factor. We see for this option, the vertex attached with a small dot in the graphical notation simply represents the

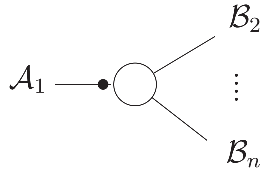
V^{-++} $\overline{\text{MHV}}$ vertex multiplied by the propagator associated with the dotted line, and the dash line bubble represents a numerical factor. So the n-th order translation kernel $\Upsilon_{12\cdots n}$ is proportional to the sum of contributions from graphs that are built from propagators and $\overline{\text{MHV}}$ vertices. Every such graph is proportional to a Feynman graph which contributes to the corresponding all-plus-except-one-minus amplitude. In 4-dimensions, we can apply BCFW recursion on the iterative formula which defines the translation kernel and repeat the argument used in section (2.4.1) to show that these graphs vanish on-shell for real values of momenta. In D-dimensions however, the $1/f(p_i)$ attached to the $\overline{\text{MHV}}$ vertex cancels the LSZ factor p_i^2 and generically the kernels are nonzero in the on-shell limit. This is not unexpected because in D-dimensions the all-plus-except-one-minus scattering amplitudes are known to be nonzero even at tree-level [57] and after the canonical transformation these amplitudes can only be constructed from translation kernels. For $f(p) = p^2$ the kernels are singular only when the p_i^2 associated to the dotted line is zero. This type of singularity is fixed by the standard $i\epsilon$ prescription. The kernels are also well defined in symmetrical graphs like ((Fig.3.15) so we do not need to distinguish the $i\epsilon$ in the $\mathcal{A}$ field and the $\mathcal{B}$ field theories or to introduce new vertices to explain the contributions to these graphs.

Another option is to choose $f(p) = 1$. In this option the denominator of a kernel becomes $(p_1^2 + i\epsilon) - \sum_{i\neq 1}(p_i^2 + i\epsilon)$ and remains nonzero for real value momenta if we adopt the usual $i\epsilon$ prescription. Although the denominator does not cancel the suppressing LSZ factor p_i^2 directly for generic external line momenta, as discussed in chapter 3, at tree-level when we apply the on-shell conditions to remove the p_i^2 in the denominators of a kernel one by one, there is always a last p_i^2 term left to cancel the LSZ factor, leaving one of the graphs non-vanishing. So the D-dimensional all-plus-except-one-minus amplitudes are generically nonzero. When the legs of a kernel are contracted in a loop, the denominator of the kernel $(p_1^2 + i\epsilon) - \sum_{i\neq 1}(p_i^2 + i\epsilon)$ can be combined with propagators using the standard Feynman parameter technique, and the loop integral can be straightforwardly computed.

4.1 MHV vertices in 4 dimensions and the Parke-Taylor formula

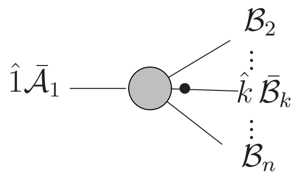
In the previous sections we saw that a generic change of variables which takes the form as the expansions (4.3), (4.4) and satisfies the inverse variation relation (4.2) preserves the integration measure of the functional integral. The vertices in the new lagrangian have the same helicity structure as the effective vertices used in CSW rules. Nevertheless an arbitrary choice of the operator $f(\partial)$ does not produce holomorphic translation kernels and the vertices generically can not be summarised by a simple formula.

For the canonical transformation in 4-dimensions $\left\{\mathcal{A}, \hat{\partial}\bar{\mathcal{A}}\right\} \to \left\{\mathcal{B}, \hat{\partial}\bar{\mathcal{B}}\right\}$, Ettle and Morris have shown from a recursive method that the n-th order kernel can be summarised by a simple formula [29].



$$\mathcal{A}_1 \to \frac{\hat{1}\,\hat{3}\hat{4}\cdots\widehat{n-1}}{(23)\cdots(n-1,n)}\,\mathcal{B}_2\mathcal{B}_3\cdots\mathcal{B}_n \tag{4.15}$$

Similarly, the $\bar{\mathcal{A}}$ expansion was shown to have the form



$$\hat{1}\bar{\mathcal{A}}_1 \to \frac{\hat{k}\,\hat{3}\hat{4}\cdots\widehat{n-1}}{(23)\cdots(n-1,n)}\,\mathcal{B}_2\mathcal{B}_3\cdots\left(\hat{k}\bar{\mathcal{B}}_k\right)\cdots\mathcal{B}_n \tag{4.16}$$

In the following sections we shall directly show that when the helicity fields $\mathcal{A}$, $\bar{\mathcal{A}}$ attached to the 3-point and the 4-point vertex terms in the LCYM lagrangian are transformed according to the above formulae (4.15), (4.16) the new vertices have the same form as the Parke-Taylor formula. In order to keep the notation simple, first we shall prove that when the negative helicity gluons are adjacent to each other both the MHV vertices described above and the Parke-Taylor formula $\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$ can be spanned by terms of the form

$$\begin{aligned}
&\frac{1}{(23)\,(34)\cdots(k-1,k)} \times \frac{1}{(k+1,k+2)\cdots(m-1,m)} \\
&\times \frac{1}{(m+1,m+2)\cdots(l-1,l)} \times \frac{1}{(l+1,l+2)\cdots(n,1)}
\end{aligned} \tag{4.17}$$

together with terms of the form

$$\frac{(12)}{(23)\,(34)\cdots(k-1,k)} \times \frac{1}{(k+1,k+2)\cdots(l-1,l)} \times \frac{1}{(l+1,l+2)\cdots(n,1)}, \tag{4.18}$$

then we shall check the coefficients of the expansion agree with each other. At the end of this chapter we extend the method to non-adjacent vertices. The denominators of (4.17) and (4.18) contain round brackets of adjacent legs arranged by the following rules: To obtain the products in (4.17) or (4.18) we consider chopping the cyclically labeled legs into three or four sets respectively. For example, for the 5-point case the legs can be arranged as $\{1\}\{2,3\}\{4,5\}$. We then write down the sequential products of brackets using labels in each set. ((23) and (45) in this example.) If there happens to be only one number in the set we simply drop off the corresponding leg.

The vertices and the Parke-Taylor formula are regarded as functions of tilde component variables $\tilde{p}$ contained in the round brackets while expansion coefficients depend only on hat components $\hat{p}$. For example, the 5-point Parke-Taylor formula can be rewritten as

$$\begin{aligned}
\frac{(12)^4}{(12)\,(23)\,(34)\,(45)\,(51)} &= A\,\frac{(12)}{(23)\,(34)} + B\,\frac{(12)}{(23)\,(45)} + C\,\frac{(12)}{(23)\,(51)} \\
&+D\,\frac{(12)}{(34)\,(45)} + E\,\frac{(12)}{(34)\,(51)} + F\,\frac{(12)}{(45)\,(51)} \\
&+G\,\frac{1}{(23)} + H\,\frac{1}{(34)} + I\,\frac{1}{(45)} + J\,\frac{1}{(51)}
\end{aligned} \tag{4.19}$$

The coefficients can be easily determined by the methods frequently used in partial fraction. To calculate A we let (23) and (34) be zero. These conditions allow us to solve $\tilde{3}$ and $\tilde{4}$ in terms of $\tilde{2}$.

$$\tilde{3} = \frac{\hat{3}}{\hat{2}}\tilde{2},\ \tilde{4} = \frac{\hat{4}}{\hat{2}}\tilde{2} \tag{4.20}$$

Brackets formed by momenta 3 and 4 with other legs $p\ q$ can therefore be replaced by brackets of 2 with $p\ q$.

$$(3,\ p) = \frac{\hat{3}}{\hat{2}}(2,\ p),\ (4,\ q) = \frac{\hat{4}}{\hat{2}}(2,\ q) \tag{4.21}$$

Together with momentum conservation the remaining brackets (45) and (51) can be expressed in terms of (12). Matching both sides of the equation gives us coefficient A. For terms like G that do not have (12) in the numerator we apply the conditions that (12) and (23) are zero. The other coefficients are then determined through the same procedure.

$$\begin{aligned} A &= \frac{\hat{2}\hat{3}\hat{5}}{\hat{1}\left(\hat{2}+\hat{3}+\hat{4}\right)},\ B = \frac{\hat{2}\left(\hat{1}+\hat{2}+\hat{3}\right)^2}{\hat{1}\left(\hat{2}+\hat{3}\right)},\ C = \frac{\hat{1}\hat{2}\hat{4}}{\left(\hat{1}+\hat{5}\right)\left(\hat{2}+\hat{3}\right)}, \\ D &= \frac{\hat{4}\left(\hat{3}+\hat{4}+\hat{5}\right)^2}{\hat{1}\hat{2}},\ E = \frac{-\hat{1}\hat{4}\left(\hat{1}+\hat{2}+\hat{5}\right)^2}{\hat{2}\hat{3}\left(\hat{1}+\hat{5}\right)},\ F = \frac{\hat{1}\hat{3}\hat{5}}{\hat{2}\left(\hat{1}+\hat{4}+\hat{5}\right)} \end{aligned} \tag{4.22}$$

Note that the Parke-Taylor formula contains a (12) dependence in the numerator so the expansion coefficients $G,\ H,\ I,\ J$ are all zero.

$$G = H = I = J = 0 \tag{4.23}$$

At the end of the argument we shall show this is generally also true for the n-point MHV vertex, but for convenience for the moment we will keep them in the expansion.

4.1.1 Partial fraction expansion

To justify the expansion we need to show (4.17) and (4.18) are sufficient for describing MHV vertices and Parke-Taylor formula. An n-point vertex in the MHV lagrangian theory consists of terms split from the 3-point and 4-point LCYM vertices. Translating $\mathcal{A}$ and $\bar{\mathcal{A}}$ fields associated with each leg into $\mathcal{B}$ and $\bar{\mathcal{B}}$ produces a series of bracket product. The contributions from the 4-point vertex (Fig.4.1) are naturally of the form (4.17). For vertices originate from the 3-point vertex (Fig.4.2), using the bilinear property the round bracket $(1 + \cdots (l+1)\,,\, 2 + \cdots k)$ in the numerator can be expanded into brackets of single leg momenta (p,q) with p and q running through 1 to $l+1$ and 2 to k respectively. Each term (p,q) can then be rewritten as a linear combination of brackets of adjacent momenta by noticing that

$$\begin{aligned}\frac{(p,q)}{\hat{p}\hat{q}} &= \frac{\tilde{q}}{\hat{q}} - \frac{\tilde{p}}{\hat{p}} = \frac{\tilde{q}}{\hat{q}} - \frac{\widetilde{p-1}}{\widehat{p-1}} + \frac{\widetilde{p-1}}{\widehat{p-1}} - \frac{\tilde{p}}{\hat{p}} \\ &= \frac{(p-1,q)}{\widehat{p-1}\hat{q}} + \frac{(p,p-1)}{\hat{p}\widehat{p-1}} \end{aligned} \tag{4.24}$$

Applying (4.24) repeatedly p and q can be moved toward 1 and 2, resulting in a term of the form (4.18) while brackets of adjacent momenta produced in the process cancel brackets in the denominator, resulting terms of the form (4.17).

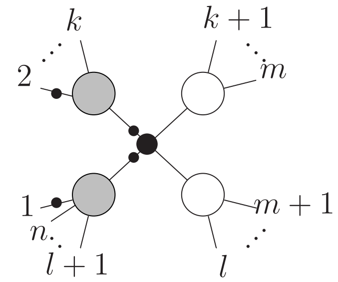


Figure 4.1: Translated 4-point vertex

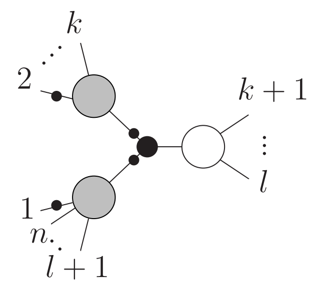


Figure 4.2: Translated 3-point vertex

To prove that the Parke-Taylor formula can also be spanned by (4.17) and (4.18) we need to express three of the four brackets (12) in the numerator of the Parke-Taylor formula as a linear combination of products $(ab)\,(cd)\,(ef)$ which contains three different brackets of adjacent legs, so that after cancellations the product $(ab)\,(cd)\,(ef)$ divide the cyclic product $(12)\,(23)\cdots(n1)$ in the denominator into three sequential parts, leaving one bracket (12) in the numerator.

Because (12) is itself an adjacent bracket, one (12) automatically cancels with the denominator. The other two (12) brackets can be replaced by other different brackets by first noticing that the momentum labels are defined cyclically. So from the definition of the round bracket we have the identity

$$\sum_{k=1}^{n} \frac{(k,k+1)}{\hat{k}\widehat{k+1}} = \sum_{k=1}^{n} \left(\frac{\widetilde{k+1}}{\widehat{k+1}} - \frac{\tilde{k}}{\hat{k}} \right) = 0 \tag{4.25}$$

In the case of a 5-point vertex, the identity reads

$$\frac{(12)}{\hat{1}\hat{2}} + \frac{(23)}{\hat{2}\hat{3}} + \frac{(34)}{\hat{3}\hat{4}} + \frac{(45)}{\hat{4}\hat{5}} + \frac{(51)}{\hat{5}\hat{1}} = 0 \tag{4.26}$$

Another identity which allows us to substitute the round bracket (12) comes from momentum conservation. $(12) + (32) + (42) + \cdots (n2) = 0$ Using equation (4.24) again to convert all brackets into adjacent ones, we obtain

$$(12) = \left(1 + \frac{\hat{4}}{\hat{3}} + \frac{\hat{5}}{\hat{3}}\right)(23) + \left(\frac{\hat{2}}{\hat{3}} + \frac{\hat{2}\hat{5}}{\hat{3}\hat{4}}\right)(34) + \frac{\hat{2}}{\hat{4}}(45) \tag{4.27}$$

The above two identities are both linear equations of the form

$$(12) = a_1 (23) + a_2 (34) + a_3 (45) + a_4 (51)\,, \tag{4.28}$$

$$(12) = b_1 (23) + b_2 (34) + b_3 (45) + b_4 (51)\,. \tag{4.29}$$

where the coefficients depends only on hat component variables. Multiplying equation (4.28) by (12), we obtain

$$(12)^2 = a_1 (12)(23) + a_2 (12)(34) + a_3 (12)(45) + a_4 (12)(51)\,. \tag{4.30}$$

The brackets (12) on the right hand side of the equation (4.30) can be further replaced by linear combinations of the brackets other than the one they are multiplied to. For example, for the term (12) (23) we used the identities (4.28) and (4.29) as simultaneous equations and solve (12) and (23) in terms of (34), (45), (51). Repeating the same procedure for other terms, we have

$$(12)^2 = \frac{(a_1 b_2 - a_2 b_1)^2}{(a_1 - b_1)(a_2 - b_2)}(23)(34) + \frac{(a_2 b_3 - a_3 b_2)^2}{(a_2 - b_2)(a_3 - b_3)}(34)(45)$$

$$+\frac{(a_3b_4-a_4b_3)^2}{(a_3-b_3)(a_4-b_4)}(45)(51)+\frac{(a_4b_1-a_1b_4)^2}{(a_4-b_4)(a_1-b_1)}(51)(23)$$
$$+\frac{(a_1b_3-a_3b_1)^2}{(a_1-b_1)(a_3-b_3)}(23)(45)+\frac{(a_2b_4-a_4b_2)^2}{(a_2-b_2)(a_4-b_4)}(34)(51) \qquad (4.31)$$

where the coefficients are

$$a_1=-\frac{\hat{1}}{\hat{3}},\ a_2=-\frac{\hat{1}\hat{2}}{\hat{3}\hat{4}},\ a_3=-\frac{\hat{1}\hat{2}}{\hat{4}\hat{5}},\ a_4=-\frac{\hat{2}}{\hat{5}} \qquad (4.32)$$

$$b_1=1+\frac{\hat{4}}{\hat{3}}+\frac{\hat{5}}{\hat{3}},\ b_2=\frac{\hat{2}}{\hat{3}}+\frac{\hat{2}\hat{5}}{\hat{3}\hat{4}},\ b_3=\frac{\hat{2}}{\hat{4}},\ b_4=0 \qquad (4.33)$$

Similarly, for the n-point Parke-Taylor formula for adjacent negative gluons we use the cyclic identity (4.25) and momentum conservation to re-express $(12)^3$ as three different adjacent brackets.

4.1.2 Matching expansion coefficients

Since the MHV vertices and the Parke-Taylor formula are spanned by functions of round brackets with coefficients depending on hat components only, as in the 5-point case shown at the beginning of this section we are free to adjust all of the tilde component variables on both sides of the expansion equation to solve for the coefficients. First let us check coefficients of (4.18). For the n-point MHV vertex, the contributions to (4.18) come solely from terms translated from the 3-point LCYM vertex (Fig.4.2). Following the graphical conventions introduced in [28] we read off the contribution as

$$\frac{\hat{2}}{\hat{2}+\cdots\hat{k}}\,\frac{\hat{2}\,\hat{3}\cdots\widehat{k-1}}{(23)\cdots(k-1,k)}$$
$$\times\frac{\hat{1}}{\widehat{l+1}+\cdots\hat{1}}\,\frac{\widehat{k+2}\cdots\widehat{l-1}}{(l+1,l+2)\cdots(n1)}\,\frac{\hat{1}\,\widehat{l+2}\cdots\hat{n}}{(k+1,k+2)\cdots(l-1,l)}$$
$$\times\frac{\widehat{k+1}+\cdots\hat{l}}{\left(\widehat{l+1}+\cdots\hat{1}\right)\left(\hat{2}+\cdots\hat{k}\right)}((l+1)+\cdots 1,\,2+\cdots k) \qquad (4.34)$$

Using conditions

$$\tilde{k}=\frac{\hat{k}}{\widehat{k-1}}\widetilde{k-1},\cdots,\tilde{3}=\frac{\hat{3}}{\hat{2}}\tilde{2} \qquad (4.35)$$

$$\widetilde{l+1} = \frac{\widehat{l+1}}{\widehat{l+2}}\widetilde{l+2}, \cdots, \tilde{n} = \frac{\hat{n}}{\hat{1}}\tilde{1} \tag{4.36}$$

$$\widetilde{k+1} = \frac{\widehat{k+1}}{\widehat{k+2}}\widetilde{k+2}, \cdots, \widetilde{l-1} = \frac{\widehat{l-1}}{\hat{l}}\tilde{l} \tag{4.37}$$

and conservation of momentum

$$\tilde{l} = -\tilde{1} - \tilde{2} - \cdots - \widetilde{l-1} - \widetilde{l+1} \cdots - \tilde{n} \tag{4.38}$$

the numerator simplifies to

$$((l+1) + \cdots 1,\, 2 + \cdots k) = \frac{\left(\widehat{l+1} + \cdots \hat{1}\right)\left(\hat{2} + \cdots \hat{k}\right)}{\hat{1}\hat{2}}(12) \tag{4.39}$$

Similarly, for the expanded Parke-Taylor formula we have

$$(k, k+1) = \frac{\hat{k}}{\hat{2}}\frac{\widehat{k+1}}{\hat{l}}\frac{\left(\widehat{l+1} + \cdots \hat{1}\right)}{\hat{1}}\frac{\hat{l}}{\left(\widehat{k+1} + \cdots \hat{l}\right)}(12) \tag{4.40}$$

$$(l, l+1) = \frac{\widehat{l+1}}{\hat{1}}\frac{\left(\hat{k} + \cdots \hat{2}\right)}{\hat{2}}\frac{\hat{l}}{\left(\widehat{k+1} + \cdots \hat{l}\right)}(12) \tag{4.41}$$

Collecting terms, both (4.34) and the Parke-Taylor formula give the same coefficient for (4.18).

$$\frac{\hat{1}\hat{2}\cdots\hat{n}}{\hat{k}\widehat{k+1}\,\hat{l}\,\widehat{l+1}}\frac{\left(\widehat{k+1} + \cdots \hat{l}\right)^2}{\left(\hat{k} + \cdots \hat{2}\right)\left(\widehat{l+1} + \cdots \hat{1}\right)} \tag{4.42}$$

As for the coefficient of terms (4.17), we receive contributions from graphs translated from the 4-point vertex (Fig.4.1), for which the translation kernels from the legs yield factors of the form (4.17), along with contributions from graphs using the 3-point vertex as backbone (Fig.4.3 (a) to (c)), in which case the bracket in the numerator cancels another bracket coming from the kernel and splits the denominator into two sets of continuous bracket products.

For simplicity we extract the common factors from each graph:

$$\begin{aligned}&\frac{\hat{2}\,\hat{3}\cdots\widehat{k-1}}{(23)\cdots(k-1,k)} \times \frac{\widehat{k+2}\cdots\widehat{m-1}}{(k,k+1)\cdots(m-1,m)}\\ &\times\frac{\widehat{m+2}\cdots\widehat{l-1}}{(m,m+1)\cdots(l-1,l)} \times \frac{\hat{1}\,\widehat{l+2}\cdots\hat{n}}{(l,l+1)\cdots(n,1)}\end{aligned} \tag{4.43}$$

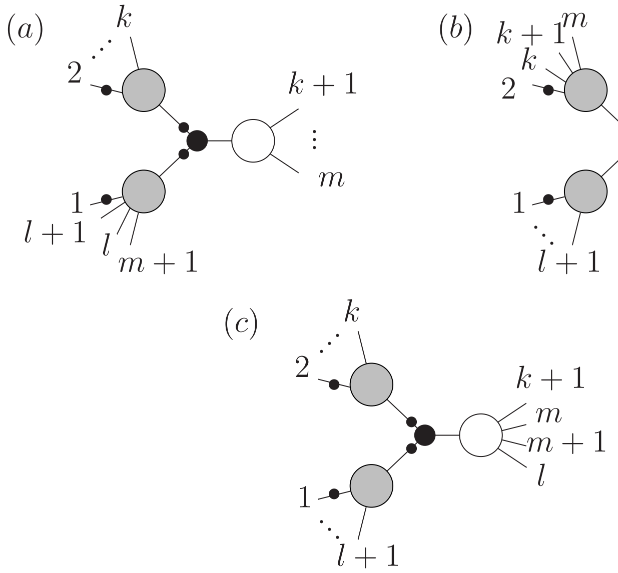


Figure 4.3: Contributions from the 3-point vertex

The remaining factors are then simplified by partial fractions. For graph (a), this is

$$\frac{\hat{l}\,\widehat{l+1}}{(l,l+1)}\,\frac{((m+1)+\cdots 1,\,2+\cdots k)}{\left(\widehat{m+1}+\cdots\hat{1}\right)\left(\hat{2}+\cdots\hat{k}\right)}\,\frac{\hat{1}\hat{2}\left(\widehat{k+1}+\cdots\hat{m}\right)}{\left(\widehat{m+1}+\cdots\hat{1}\right)\left(\hat{2}+\cdots\hat{k}\right)}=\frac{\hat{1}\hat{2}}{(a+d)^2}\,\frac{c^2d}{b}, \tag{4.44}$$

where a, b, c and d are the momenta carried by internal lines:

$$a=\widehat{l+1}+\cdots+\hat{n}+\hat{1} \tag{4.45}$$

$$b=\hat{2}+\hat{3}+\cdots+\hat{k} \tag{4.46}$$

$$c=\widehat{k+1}+\cdots+\hat{m} \tag{4.47}$$

$$d=\widehat{m+1}+\cdots+\hat{l} \tag{4.48}$$

Similarly for graph (b) we have

$$\frac{\hat{k}\,\widehat{k+1}}{(k,k+1)}\,\frac{((l+1)+\cdots 1,\,2+\cdots m)}{\left(\widehat{l+1}+\cdots\hat{1}\right)\left(\hat{2}+\cdots\hat{m}\right)}\,\frac{\hat{1}\hat{2}\left(\widehat{m+1}+\cdots\hat{l}\right)}{\left(\widehat{l+1}+\cdots\hat{1}\right)\left(\hat{2}+\cdots\hat{m}\right)}=\frac{\hat{1}\hat{2}}{(a+d)^2}\,\frac{cd^2}{a} \tag{4.49}$$

After simplification graph (c) is proportional to (12), therefore vanishes

$$((l+1)+\cdots 1,\, 2+\cdots k) = \frac{\widehat{l+1}+\cdots \hat{1}}{\hat{1}}\,\frac{\hat{2}+\cdots \hat{k}}{\hat{2}}\,(12) = 0 \tag{4.50}$$

Putting (4.44) and (4.49) together cancels the contribution from (Fig. 1)

$$-\frac{\hat{1}\hat{2}}{(a+d)^2}\,\frac{cd}{ab}\,(ac+bd) \tag{4.51}$$

we thus verified that all of the expansion coefficients for terms of the form (4.17) are zeros, as claimed at the beginning of this section.

4.1.3 MHV vertices with non-adjacent negative helicity gluons

For the cases where negative gluons are not adjacent to each other, we span the MHV vertices and the Parke-Taylor formula by

$$\begin{aligned}&\frac{1}{(h+1,h+2)\cdots(k-1,k)}\times\frac{1}{(k+1,k+2)\cdots(m-1,m)}\\&\times\frac{1}{(m+1,m+2)\cdots(l-1,l)}\times\frac{1}{(l+1,l+2)\cdots(h-1,h)}\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}&\frac{(i\,j)}{(l+1,l+2)\cdots(k-1,k)}\times\frac{1}{(k+1,k+2)\cdots(m-1,m)}\\&\times\frac{1}{(m+1,m+2)\cdots(l-1,l)}.\end{aligned} \tag{4.53}$$

From the formulae for translation kernels and the LCYM vertices we see the MHV vertices produced by the canonical transformation are readily given by linear combinations of terms of the form (4.52), (4.53). To justify that the Parke-Taylor formula with non-adjacent negative helicity gluons can also be spanned by these two types of products, again like in the adjacent case we need to show that we can express three of the round brackets $(i\,j)$ in the numerator as a linear combination of three different bracket products of adjacent legs $(ab)\,(cd)\,(ef)$. These three brackets then split the cyclic product in the denominator into three sequential products, creating a term described by (4.53). However since i and j are not adjacent we can not use

$(i\,j)$ to cancel the bracket in the denominator directly, instead we use the identity (4.24) to replace $(i\,j)$ by brackets of adjacent legs. For example, in the case of a 5-point Parke-Taylor formula, if $(i\,j) = (13)$, we have

$$\frac{(13)}{\hat{1}\hat{3}} = \frac{(12)}{\hat{1}\hat{2}} + \frac{(23)}{\hat{2}\hat{3}}. \tag{4.54}$$

Note that because the labels of momenta are assigned cyclically, using the identity (4.25) we can express $(i\,j)$ as brackets of legs which come from the opposite side of those in (4.54).

$$\frac{(13)}{\hat{1}\hat{3}} = \frac{(12)}{\hat{1}\hat{2}} + \frac{(23)}{\hat{2}\hat{3}} = -\frac{(34)}{\hat{3}\hat{4}} - \frac{(45)}{\hat{4}\hat{5}} - \frac{(51)}{\hat{5}\hat{1}} \tag{4.55}$$

The above equation gives us one more condition than in the case when the negative helicity gluons are adjacent. Combining (4.54), (4.55) and momentum conservation

$$(13) + (23) + (43) + (53) = 0 \tag{4.56}$$

we have three identities which allow us to express (13) in terms of brackets of adjacent legs.

$$(13) = a_1\,(12) + a_2\,(23) + \cdots a_5\,(51)\,, \tag{4.57}$$

$$(13) = b_1\,(12) + b_2\,(23) + \cdots b_5\,(51)\,, \tag{4.58}$$

$$(13) = c_1\,(12) + c_2\,(23) + \cdots c_5\,(51)\,. \tag{4.59}$$

Multiplying both sides of the equation (4.57) by a bracket (13), and substituting the brackets (13) on the right hand side by the solution to the simultaneous equations (4.57) and (4.58), we obtain an identity relating $(13)^2$ to a linear combination of products of two different brackets $(ab)\,(cd)$. Multiplying this identity by (13) again and substituting the brackets (13) on the right hand side by the solution to the three simultaneous equations (4.57), (4.58), (4.59) yields the identity relating $(13)^3$ to a linear combination of $(ab)\,(cd)\,(ef)$, which allows us to express the Parke-Taylor formula for non-adjacent negative helicity gluons as terms of the form (4.53). As in the adjacent case we find the expansion coefficients of the n-point MHV vertex and the Parke-Taylor formula agree with each other.

Chapter 5

Generating MHV super-vertices for the $\mathcal{N} = 1$ and the $\mathcal{N} = 4$ SYM theories

Recently, the BCFW recursion method introduced in section (2.4) has been generalised to tree-level amplitude computations in $\mathcal{N} = 4$ supersymmetry Yang-Mills theory [21]. Instead of shifting individual scattering amplitudes labeled by particle species and momenta, in the supersymmetry generalisation of the BCFW recursion one considers super-amplitudes whose initial and final states are described by momentum space super-wavefunctions. The super-wavefunction contains a superposition of all the single particle states in the $\mathcal{N} = 4$ supermultiplet, each of them being tagged by bookkeeping Grassmann variables η_A,

$$\Phi(p,\eta) = G^+(p) + \eta_A \Gamma^A(p) + \frac{1}{2}\eta_A\eta_B S^{AB} + \frac{1}{3!}\eta_A\eta_B\eta_C\epsilon^{ABCD}\bar{\Gamma}_D(p)$$
$$+\frac{1}{4!}\eta_A\eta_B\eta_C\eta_D\epsilon^{ABCD}G^-(p). \tag{5.1}$$

The conventional scattering amplitude is obtained from differentiating the super-amplitude with respect to the appropriate Grassmann variables which are determined by the species of the particles participating the scattering event. For example differentiating Nair's formula for the $\mathcal{N} = 4$ MHV super-amplitude

$$A^{MHV}_{N=4}(1,2,\cdots n) = \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\prod_{A=1}^{4}(\sum_{i,j=1}^{n}\langle i\,j\rangle\,\eta_{Ai}\eta_{Aj}) \tag{5.2}$$

with respect to η_{Ai} and η_{Aj}, from $A=1$ to 4, yields the familiar Parke-Taylor formula for gluon scattering, where the i-th and j-th legs are associated with negative helicity gluons. The supersymmetry BCFW recursion formula can be derived from applying Cauchy's theorem to the super-amplitude with leg momenta shifted according to (2.42) and (2.43) as in the pure Yang-Mills theory, together with the Grassmann variables associated with the selected leg shifted as $\eta_{A\,1}\to\eta_{A\,1}+z\eta_{A\,n}$. In [8] Witten introduced an on-shell representation for $\mathcal{N}=4$ SUSY generators

$$Q_{\alpha\,A}=\lambda_\alpha\eta_A,\ \bar{Q}^A_{\dot\alpha}=\bar\lambda_{\dot\alpha}\frac{\partial}{\partial\eta_A}, \tag{5.3}$$

and showed that Nair's MHV super-amplitude formula is superconformal invariant. However the connection between the standard of SUSY generators and the representation (5.3) is missing. The existence of a supersymmetry BCFW recursion relation that allows us to combine super-amplitudes labeled by η into super-amplitudes with more scattering particles also suggests that a corresponding set of supersymmetric CSW rules can be built.

In this chapter we generalise the canonical transformation introduced in section (2.5) to supersymmetric theories and derive the corresponding MHV lagrangians that directly yield super-amplitudes such as (5.2) from Feynman rules. In sections (5.1) to (5.4.2) we first test the method on the $\mathcal{N}=1$ SYM action. We show that in the light-cone gauge the standard supersymmetry Yang-Mills lagrangian built from vector superfields can be rewritten as a functional of the chiral and anti-chiral superfields given by Brink, Lindgren and Nilsson [32]. Then we perform a Grassmannian analogue to the Fourier transformation which replaces the standard super-space "coordinate" variables θ, $\bar\theta$ in the superfields by Grassmannian "momentum" η (section 5.2). We compute the generic n-point MHV super-amplitude for $\mathcal{N}=1$ SYM theory by applying BCFW recursion technique in section (5.3.1).

In section (5.4) we canonically transform $\mathcal{N}=1$ chiral superfields to produce an MHV lagrangian. The formula for translation kernel of the superfield automatically summarises the kernels of gluon and gluino fields, $\left\{\mathcal{A},\hat{\partial}\bar{\mathcal{A}}\right\}$ and $\{\Lambda,\bar{\Lambda}\}$, which are regarded as components of the superfield. We develop a set of off-shell manifestly supersymmetric CSW rules from functional integral over chiral superfield variables. In section (5.4.1) we use the method introduced in chapter 4 to prove that the $\mathcal{N}=1$ MHV super-vertices are given by the simple formula

$$V^{MHV}_{N=1}(1^+,2^+\dots i^-,j^-,\dots n^+) = \frac{\langle i\,j\rangle^3}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}(\sum_{i,j=1}^{n}\langle i\,j\rangle\,\eta_i\eta_j). \tag{5.4}$$

Finally, in section (5.5) we extend the method described above to $\mathcal{N}=4$ super Yang-Mills lagrangian in light-cone gauge and show that the MHV super-vertices have the same form as Nair's formula (5.2), in which the Grassmann variables η_A are interpreted as the super-space momenta.

This work was published in [49].

5.1 Chiral construction of the $\mathcal{N}=1$ SYM lagrangian

In the textbook approach, the supersymmetric non-abelian gauge theory is constructed from the fieldstrength of a vector superfield. In Wess-Zumino gauge the vector superfield contains gluon, gluino and an auxiliary field $\mathcal{A}_\mu$, ψ_α and D. The gauge invariant supersymmetric action is given by

$$\begin{aligned} S &= \frac{1}{4g^2}\int d^4x\,d^2\theta\,tr\left(W^\alpha W_\alpha + h.c.\right) \\ &= \frac{-16}{g^2}\int d^4x\,tr\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + \frac{1}{2}D^2\right). \end{aligned} \tag{5.5}$$

Under a SUSY transformation we have

$$\delta_\xi\mathcal{A}_\mu = \bar{\xi}\bar{\sigma}_\mu\psi + \bar{\psi}\bar{\sigma}_\mu\xi, \tag{5.6}$$

$$\delta_\xi\psi_\alpha = -\frac{i}{2}\sigma_\mu\bar{\sigma}_\nu\xi\,F^{\mu\nu} + \xi_\alpha D, \tag{5.7}$$

$$\delta_\xi D = -i(\bar{\xi}\bar{\sigma}^\mu D_\mu\psi - D_\mu\bar{\psi}\bar{\sigma}^\mu\xi), \tag{5.8}$$

and the SUSY transformation for a given parameter ξ_α is defined by $\delta_\xi\mathcal{A}_\mu = i(\xi Q + \bar{Q}\bar{\xi})\mathcal{A}_\mu$, where $Q_\alpha = \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix}$ and $\bar{Q}_{\dot{\alpha}} = \begin{pmatrix} \bar{Q}_{\dot{0}} \\ \bar{Q}_{\dot{1}} \end{pmatrix}$ are representations of the

SUSY generators. To arrive at an MHV lagrangian we can pick a convenient gauge, integrate over unphysical degrees of freedom, and canonically transform the helicity gluon and gluino fields separately. However the underlying supersymmetry implies that we can organise these physical fields into a conceptually simpler structure. In pursuit of this idea we may be tempted to apply the canonical transformation directly to the vector superfield, but the vector superfield depends on a large number of unphysical degrees of freedom and the lagrangian does not have an easily manipulated structure when it is written in terms of vector superfields. As a first step of the simplification, we extract the ξ_0 and the $\bar{\xi}_{\dot{0}}$ dependent terms from equations (5.6) to (5.8). These equations give the effect of the SUSY generators Q_1 and $\bar{Q}_{\dot{1}}$ operating on field components. We note that on the right hand side the quadratic terms all depends on the commutator of $\hat{A}$ with other fields. In light-cone gauge the physical fields are closed under this subalgebra.

$$Q_1\mathcal{A} = i\Lambda,\ \bar{Q}_{\dot{1}}\mathcal{A} = 0,\ Q_1\bar{\mathcal{A}} = 0,\ \bar{Q}_{\dot{1}}\bar{\mathcal{A}} = -i\bar{\Lambda}, \tag{5.9}$$

$$Q_1\Lambda = 0,\ \bar{Q}_{\dot{1}}\Lambda = \hat{\partial}\mathcal{A},\ Q_1\bar{\Lambda} = -\hat{\partial}\bar{\mathcal{A}},\ \bar{Q}_{\dot{1}}\bar{\Lambda} = 0, \tag{5.10}$$

where we used $\psi_\alpha = \begin{pmatrix} \bar{T} \\ \bar{\Lambda} \end{pmatrix}$, $\bar{\psi}_{\dot{\alpha}} = \begin{pmatrix} T \\ \Lambda \end{pmatrix}$ to denote gluino fields. With the quadratic terms eliminated by gauge condition, the transformations (5.9) and (5.10) are the same as their asymptotic forms.

The closure of SUSY subalgebra allows us to define chiral superfields without the help of auxiliary fields [32].

$$\Phi(x,\theta) = \mathcal{A}(y) + i\theta\,\Lambda(y), \tag{5.11}$$

$$\bar{\Phi}(x,\theta) = \bar{\mathcal{A}}(\bar{y}) + i\bar{\theta}\,\bar{\Lambda}(\bar{y}), \tag{5.12}$$

where gluons and gluinos with positive or negative helicities are enclosed into the same chiral or anti-chiral superfields, and $y = (x^+,\ x^- + \frac{1}{2}i\theta\bar{\theta},\ x^z,\ x^{\bar{z}})$. We introduce a shorthand notation for representations of the SUSY covariant derivatives and generators $D_1 = d$, $\bar{D}_{\dot{1}} = \bar{d}$, $Q_1 = q$, and $\bar{Q}_{\dot{1}} = \bar{q}$, which stand for

$$d = \frac{\partial}{\partial\theta} + \frac{i}{2}\bar{\theta}\hat{\partial}\,,\ \bar{d} = -\frac{\partial}{\partial\bar{\theta}} - \frac{i}{2}\,\theta\hat{\partial}, \tag{5.13}$$

$$q=\frac{\partial}{\partial\theta}-\frac{i}{2}\bar{\theta}\hat{\partial}\,,\,\bar{q}=-\frac{\partial}{\partial\bar{\theta}}+\frac{i}{2}\,\theta\hat{\partial}. \tag{5.14}$$

The definitions (5.11) and (5.12) satisfy the chiral constraints $\bar{d}\,\Phi=d\,\bar{\Phi}=0$. Operating q, $\bar{q}$ on Φ and $\bar{\Phi}$ reproduces the same effects as the transformation defined in (5.9) and (5.10) so Φ and $\bar{\Phi}$ are legitimate chiral and anti-chiral superfields.[1]

Using the chiral superfields just defined, we can construct a SUSY invariant lagrangian as a D-term integral

$$S=\frac{-4i}{g^2}\int d^4x\,d\theta\,d\bar{\theta}\,tr\left(\bar{\Phi}\frac{\partial^2}{\hat{\partial}}\Phi+[\Phi,\frac{\bar{\partial}}{\hat{\partial}}\Phi]\bar{\Phi}+[\bar{\Phi},\frac{\partial}{\hat{\partial}}\bar{\Phi}]\Phi-i\left[\Phi,\bar{d}\,\bar{\Phi}\right]\frac{1}{\hat{\partial}^2}\left[\bar{\Phi},d\,\Phi\right]\right) \tag{5.16}$$

It is straightforward to verify that after integrating θ, $\bar{\theta}$ equation (5.16) is the same as the standard $\mathcal{N}=1$ SYM lagrangian (5.5) with $\hat{\mathcal{A}}$ eliminated by light-cone gauge condition and all of the unphysical fields integrated out. This is not unexpected because to the lowest order in θ and $\bar{\theta}$ the chiral superfield lagrangian is the same as the pure Yang-Mills lagrangian in light-cone gauge. The rest of the terms are automatically determined by supersymmetry. A similar method was used by Ananth, Brink, Lindgren, Nilsson and Ramond to derive the $\mathcal{N}=4$ light-cone SYM lagrangian dimensionally reduced from 10-dimensions [32, 50].

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As in the case of pure Yang-Mills theory discussed in chapter 2, we can also identify the physical part of the gluino field from the spinor point of view. A generic off-shell gluino spinor field ψ_α can be spanned by the holomorphic 2-spinors of its momentum (2.55), (2.56).

$$\psi_\alpha=\eta_\alpha\bar{\Lambda}^{(\eta)}+\lambda_\alpha\bar{\Lambda}^{(\lambda)}, \tag{5.15}$$

where $P_{\alpha\dot{\alpha}}=\eta_\alpha\bar{\eta}_{\dot{\alpha}}+\lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$, and similarly we span the $\bar{\psi}_{\dot{\alpha}}$ by the anti-holomorphic spinors. Substituting (5.15) and (2.61) into the the formulae for SUSY transformation (5.6) to (5.8) it is easy to see that $\mathcal{A}^{(+)}$, $\bar{\mathcal{A}}^{(-)}$, $\Lambda^{(\lambda)}$ and $\bar{\Lambda}^{(\lambda)}$ satisfy the same relations as the fields in light-cone coordinates (5.9) and (5.10). Furthermore, the representations of the SUSY generators and covariant derivatives can also be spanned by the same basis. The q, d operators defined in (5.13), (5.14) correspond to $\langle Q\eta\rangle\,/\,\langle\lambda\eta\rangle$, $\langle D\eta\rangle\,/\,\langle\lambda\eta\rangle$. The Grassmann scalars θ, $\bar{\theta}$ used in (5.13), (5.14) are related to the standard Grassmann 2-spinors θ_α, $\bar{\theta}_{\dot{\alpha}}$ by $\theta=\langle\theta\lambda\rangle$ and $\bar{\theta}=[\lambda\theta]$. Other terms in the expansion such as $\langle\theta\eta\rangle$ do not show up in the above identities. From this point of view it is natural that $\mathcal{N}=1$ SYM lagrangian can be rearranged into a simpler form.

5.2 Transforming light-cone gauge SYM into the new representation

We note the superfields defined in (5.11) and (5.12) have two Grassmannian degrees of freedom and are constrained by chiral conditions. This suggests that we can simplify the superfields by introducing a new super-space variable. In the way analogous to a classical field satisfying field equations can be transformed into momentum space, where the energy variable is expressed in terms of the 3-momentum, we perform a fermionic integral transformation to replace the "coordinate" dependence θ, $\bar{\theta}$ by a Grassmannian "momentum" η.[2]

$$\bar{\Phi}(p,\,\theta) = -\int d\eta\, e^{-\frac{1}{2}\theta\bar{\theta}\hat{p}}\delta(\bar{\theta}\hat{p}^{\frac{1}{2}} - \eta)\,\bar{\phi}(p,\,\eta), \tag{5.19}$$

$$\bar{d}\,\bar{\Phi}(p,\,\theta) = -\hat{p}^{\frac{1}{2}}\int d\eta\, e^{\frac{1}{2}\theta\bar{\theta}\hat{p}}\delta(1 - \theta\eta\hat{p}^{\frac{1}{2}})\,\bar{\phi}(p,\,\eta), \tag{5.20}$$

$$\Phi(p,\,\theta) = \int d\eta\, e^{\frac{1}{2}\theta\bar{\theta}\hat{p}}\delta(1 - \theta\eta\hat{p}^{\frac{1}{2}})\,\phi(p,\,\eta), \tag{5.21}$$

$$d\,\Phi(p,\,\theta) = -\hat{p}^{\frac{1}{2}}\int d\eta\, e^{-\frac{1}{2}\theta\bar{\theta}\hat{p}}\delta(\bar{\theta}\hat{p}^{\frac{1}{2}} - \eta)\,\phi(p,\,\eta) \tag{5.22}$$

where we have used the Taylor expansion to convert functions of $y = (x^+,\, x^- + \frac{1}{2}i\theta\bar{\theta},\, x^z,\, x^{\bar{z}})$ into $f(y) = e^{\frac{i}{2}\theta\bar{\theta}\hat{\partial}}f(x)$ before transforming into the momentum space.

---

[2]When "negative energy" continuation for $\hat{p}$ is needed, we use

$$\bar{\Phi}(p,\,\theta) = -\int d\eta\, e^{-\frac{1}{2}\theta\bar{\theta}\hat{p}}\delta(\bar{\theta}|\hat{p}|^{\frac{1}{2}} - \eta)\,\bar{\phi}(p,\,\eta), \tag{5.17}$$

$$\bar{d}\,\bar{\Phi}(p,\,\theta) = -|\hat{p}|^{\frac{1}{2}}\int d\eta\, e^{\frac{1}{2}\theta\bar{\theta}\hat{p}}\delta(1 - \theta\eta\hat{p}/\hat{p}^{\frac{1}{2}})\,\bar{\phi}(p,\,\eta) \tag{5.18}$$

in place of (5.19) and (5.20) to take care of the possible phase ambiguity produced by square roots. The modification is more natural had we defined the superfields in spinor language as in footnote 1, where the $\bar{\theta}$ is identified as $[\lambda\theta]$. In that case we simply define the delta function as $\delta([\lambda\theta] - \eta)$. Note that equations (5.17) and (5.18) only affects the overall phase for a gluino field so the inclusion of a square root does not modify the kinematic part of the vertex. In the following we implicitly assume that the negative energy continuation (5.17) and (5.18) are used throughout the derivation. However we neglect the square root in the results for simplicity. The minus signs can be quickly restored from matching the MHV vertices with the known amplitudes that contain gluino pairs.

The same θ dependence in the equations (5.19) and (5.22) guarantees that $d\,\bar{\Phi} = d\,d\,\Phi = 0$, and similarly for the equations (5.20) (5.21).

In the integral transformations defined above we bring functions of two Grassmann variables θ and $\bar{\theta}$ to functions of η only. However the superfields satisfy chiral constraints so in the subspace where chiral superfields are defined the degrees of freedom are not reduced under these transformations. Their inverse transformations are given by

$$\phi(p,\,\eta) = -\hat{p}^{-\frac{1}{2}} \int d\theta e^{-\frac{1}{2}\theta\bar{\theta}\hat{p}} \delta(1 + \theta\eta\hat{p}^{\frac{1}{2}})\,\Phi(p,\,\theta), \tag{5.23}$$

$$\bar{\phi}(p,\,\eta) = -\frac{1}{\hat{p}} \int d\theta e^{-\frac{1}{2}\theta\bar{\theta}\hat{p}} \delta(1 + \theta\eta\hat{p}^{\frac{1}{2}})\,\bar{d}\,\bar{\Phi}(p,\,\theta), \tag{5.24}$$

and the momentum space superfields are

$$\phi(p,\eta) = i\bar{\Lambda}(p)\hat{p}^{\frac{1}{2}} + \eta\mathcal{A}(p), \tag{5.25}$$

$$\bar{\phi}(p,\eta) = \bar{\mathcal{A}}(p) + \eta i\Lambda(p)\hat{p}^{-\frac{1}{2}}. \tag{5.26}$$

After the integration the SUSY generators are also transformed into the new representation just like the symmetry generators of the Poincare group can both be expressed in terms of spacetime coordinates or of momenta. Operating the representations of SUSY generators (5.14) on the integral transformations (5.19) to (5.22) and moving the operators to the right gives the simple representations for the light-cone SUSY generators as

$$q = \hat{p}^{\frac{1}{2}}\eta,\ \bar{q} = \hat{p}^{\frac{1}{2}}\frac{\partial}{\partial\eta} \tag{5.27}$$

It is easy to see that the above representations in the new Grassmannian momentum space have the same effect as applying the (5.9), (5.10) directly on ϕ and $\bar{\phi}$, so they meet with the criteria as superfields. However because the number of Grassmann variables is reduced by one during the integral transformation, the new superfields ϕ and $\bar{\phi}$ in the momentum representation no longer satisfy chiral constraints. To see how a generic SUSY transformation is represented in the Grassmannian momentum space we recall that the light-cone SUSY generators are defined as the 1 and $\dot{1}$ components of the 2-spinor generators, $Q_1 = q$ and $\bar{Q}_{\dot{1}} = \bar{q}$. The effects of the other components can be derived from the definitions, equations (5.6) to

(5.8), by filling in the field components which were integrated out by their classical solutions. It is straightforward, though tedious, to verify that after neglecting all of the commutators appearing in the expressions, in the on-shell limit we have

$$Q_\alpha = \lambda_\alpha \eta,\ \bar{Q}_{\dot\alpha} = \bar{\lambda}_{\dot\alpha}\frac{\partial}{\partial\eta}, \tag{5.28}$$

which are the $\mathcal{N}=1$ version of the on-shell SUSY generators introduced by Witten in [8].[3]

5.2.1 3-point MHV and $\overline{\text{MHV}}$ vertices

The light-cone gauge SYM lagrangian can be quickly rewritten in terms of the new representation by applying (5.19) to (5.22) to its field contents. In addition to a momentum conservation delta function, the free field part contains a Grassmannian momentum delta function, which demands that gluons and gluinos cannot be interchanged into each other in the absence of an interaction.

$$\begin{aligned} S_{free} = \int d^4x\, d\theta d\bar\theta\, \mathcal{L}^{-+} &= \int d^4p_1\, d^4p_2\, d\theta\, d\bar\theta\, \bar\Phi \frac{p_2^2}{\hat{p}_2}\delta^{(4)}(p_1+p_2)\Phi \\ = \int d^4p_1\, d^4p_2\, d\eta_1 d\eta_2 \bar\phi(p_1,\eta_1)\delta^4(p_1+p_2)\delta(\eta_1+\eta_2)\, p_2^2\, \phi(p_2,\eta_2) \end{aligned} \tag{5.30}$$

For the $(--+)$ interaction term we apply the same transformation again to have

$$\int d^4x\, d\theta d\bar\theta\, \mathcal{L}^{--+} = tr \int d^4x\, d\theta d\bar\theta [\bar\Phi, \frac{\partial}{\hat\partial}\bar\Phi]\Phi \tag{5.31}$$

$$= tr \int d^4p_1 \ldots d\eta_1 \ldots \frac{(12)}{\hat{1}\hat{2}}\, \hat{3}^{\frac{1}{2}} (\hat{3}^{\frac{1}{2}}\eta_1\eta_2 + \hat{1}^{\frac{1}{2}}\eta_2\eta_3 + \hat{2}^{\frac{1}{2}}\eta_3\eta_1)\, \bar\phi_1\bar\phi_2\phi_3 \tag{5.32}$$

[3] Again the formulae given in (5.28) have a simpler explanation in spinor language. The 2-spinor SUSY generator can be spanned as

$$Q_\alpha = \lambda_\alpha \frac{\langle Q\eta\rangle}{\langle\lambda\eta\rangle} + \eta_\alpha \frac{\langle\lambda Q\rangle}{\langle\lambda\eta\rangle} \tag{5.29}$$

Following the same derivation as for the generators in light-cone coordinates, the operator $\langle Q\eta\rangle / \langle\lambda\eta\rangle$ in the momentum space can be shown to be represented by the multiplication of the Grassmann variable η. Since the spinor η_α vanishes on-shell (2.55), we have $Q_\alpha = \lambda_\alpha\eta$. The anti-holomorphic part can be similarly derived.

Using momentum conservation condition we can combine all of the hat-components and the round bracket into spinor brackets. Note that the vertex factor resembles the super-amplitude formula given by Nair for $\mathcal{N}=4$ SYM theory.

$$tr\int d^4p_1\dots d\eta_1\dots\frac{\langle 12\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\left(\langle 12\rangle\,\eta_1\eta_2+\langle 23\rangle\,\eta_2\eta_3+\langle 31\rangle\,\eta_3\eta_1\right)\bar{\phi}_1\bar{\phi}_2\phi_3 \quad (5.33)$$

We then compute the $(++-)$ term to give

$$\int d^4x\, d\theta d\bar{\theta}\,\mathcal{L}^{++-}=tr\int d^4x\, d\theta d\bar{\theta}[\Phi,\frac{\bar{\partial}}{\hat{\partial}}\Phi]\bar{\Phi}$$

$$=tr\int d^4p_1\dots d\eta_1\dots\frac{\{12\}}{\hat{1}\hat{2}}\,\hat{3}^{\frac{1}{2}}(\hat{1}^{\frac{1}{2}}\eta_1+\hat{2}^{\frac{1}{2}}\eta_2+\hat{3}^{\frac{1}{2}}\eta_3)\,\phi_1\phi_2\bar{\phi}_3 \quad (5.34)$$

$$=tr\int d^4p_1\dots d\eta_1\dots\frac{[12]^3}{[12]\,[23]\,[31]}\left([23]\,\eta_1+[31]\,\eta_2+[12]\,\eta_3\right)\phi_1\phi_2\bar{\phi}_3 \quad (5.35)$$

In the defining equations for integral transformations (5.19) to (5.22) we chose to replace the $\bar{\theta}\hat{p}^{\frac{1}{2}}$ dependence in $\bar{\Phi}$ and $d\,\Phi$ by the newly introduced Grassmann variable η. As an alternative we can choose our definition to the Grassmannian Fourier transform as replacing the $\theta\hat{p}^{\frac{1}{2}}$ in Φ and $\bar{d}\,\bar{\Phi}$. The transformation formula for $\bar{\Phi}$ follows from

$$\bar{\Phi}=\frac{\{d,\bar{d}\}}{-i\hat{\partial}}\bar{\Phi}=\frac{d}{-i\hat{\partial}}\left(\bar{d}\,\bar{\Phi}\right), \quad (5.36)$$

and in the expressions for the 3-point MHV and $\overline{\text{MHV}}$ vertices (5.33) and (5.35) we will have the same spinor bracket factors but with the $([23]\,\eta_1+[31]\,\eta_2+[12]\,\eta_3)$ and $(\langle 12\rangle\,\eta_1\eta_2+\langle 23\rangle\,\eta_2\eta_3+\langle 31\rangle\,\eta_3\eta_1)$ swapped. This is because by exchanging the conventions for η we modified the labeling of particle species associated with the legs of the vertex. The same correspondence between the η assignment of the super-wavefunction (5.1) and the η dependence in the super-amplitudes for $\mathcal{N}=4$ SYM was found by Drummond, Henn, Korchemsky and Sokatchev in [23, 51, 52].

The remaining two 4-point vertices can be calculated following the same procedure.

$$V^2_{1234}=\frac{1}{(\hat{2}+\hat{3})^2}\Big(\left(\hat{1}\hat{3}+\hat{2}\hat{4}\right)\eta_1\eta_2-\hat{1}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\left(\hat{2}+\hat{3}\right)\eta_2\eta_3+\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\left(\hat{1}+\hat{4}\right)\eta_1\eta_3$$

$$+2\,\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\eta_3\eta_4 - \hat{2}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\hat{1}+\hat{4}\right)\eta_4\eta_1 + \hat{1}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\hat{2}+\hat{3}\right)\eta_2\eta_4\Big)\,, \qquad (5.37)$$

and

$$V^3_{1234} = -\left(\frac{\hat{3}\hat{4}}{\left(\hat{1}+\hat{4}\right)^2} + \frac{\hat{1}\hat{4}}{\left(\hat{3}+\hat{4}\right)^2}\right)\eta_1\eta_3 - \frac{\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}\hat{4}}{\left(\hat{3}+\hat{4}\right)^2}\eta_2\eta_3 - \frac{\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\hat{4}}{\left(\hat{1}+\hat{4}\right)^2}\eta_1\eta_2$$

$$+\hat{3}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\frac{1}{\left(\hat{1}+\hat{4}\right)} - \frac{\hat{1}}{\left(\hat{3}+\hat{4}\right)^2}\right)\eta_4\eta_1 + \hat{4}^{\frac{1}{2}}\hat{1}^{\frac{1}{2}}\left(\frac{1}{\left(\hat{3}+\hat{4}\right)} - \frac{\hat{3}}{\left(\hat{1}+\hat{4}\right)^2}\right)\eta_3\eta_4$$

$$-\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\frac{1}{\left(\hat{1}+\hat{4}\right)^2} + \frac{1}{\left(\hat{3}+\hat{4}\right)^2}\right)\eta_2\eta_4, \qquad (5.38)$$

where we used V^2_{1234} and V^3_{1234} to denote the vertices that have adjacent and next-to-adjacent negative helicity legs.

$$\int d^4x\, d\theta d\bar{\theta}\, \mathcal{L}^{--++} = \frac{4}{g^2}\int d^4x\, d\theta d\bar{\theta}\left[\Phi, \bar{d}\,\bar{\Phi}\right]\frac{1}{\left(i\hat{\partial}\right)^2}\left[\bar{\Phi}, d\,\Phi\right]$$

$$= \frac{4}{g^2}\,tr\int V^{--++}_{1234}\bar{\phi}_1\bar{\phi}_2\phi_3\phi_4 + V^{-+-+}_{1234}\bar{\phi}_1\phi_2\bar{\phi}_3\phi_4 \qquad (5.39)$$

5.3 Calculating super-amplitudes using functional methods

One of the advantages of rewriting light-cone $\mathcal{N} = 1$ SYM lagrangian in terms of chiral superfields is that it allows us to compute super-amplitudes from a set of manifestly supersymmetric Feynman rules. In a system where supersymmetry is effectively unbroken it is reasonable that particles in the same supermultiplet can be treated altogether as a single object. Nevertheless, in the standard functional integral approach to Green function calculations there is a fundamental distinction between a field and the field of its superpartner. The gluon fields are bosonic while the gluino fields are taken as fermionic, both fields are regarded as independent variables to be integrated over in the functional integral. A naive attempt to combine these two fields by a change of variables reduces the degrees of freedom and does not make sense mathematically. So instead of on integration variables we focus on the generating functional that generates Green functions.

$$Z[J] = \int \mathcal{D}\mathcal{A}\mathcal{D}\bar{\mathcal{A}}\mathcal{D}\Lambda\mathcal{D}\bar{\Lambda}\, e^{iS+i\int j^{(A)}\mathcal{A}+\bar{\mathcal{A}}j^{(\bar{A})}+j^{(\Lambda)}\Lambda+\bar{\Lambda}j^{(\bar{\Lambda})}}, \tag{5.40}$$

where S is the $\mathcal{N}=1$ SYM action (5.16) in light-cone gauge, and we introduce generating currents $j^{(A)}$, $j^{(\bar{A})}$, $j^{(\Lambda)}$ and $j^{(\bar{\Lambda})}$ for every physical fields. We note that the current term integral can be simplified by the introduction of super-currents

$$\int d^4p\, j^{(A)}\mathcal{A} + \bar{\mathcal{A}}j^{(\bar{A})} + j^{(\Lambda)}\Lambda + \bar{\Lambda}j^{(\bar{\Lambda})} = \int d^4p\, d\eta\, J\phi + \bar{\phi}\bar{J}, \tag{5.41}$$

where we defined[4]

$$J = j^{(A)} - i\hat{p}^{\frac{1}{2}}\eta j^{(\Lambda)},\ \bar{J} = \eta j^{(\bar{A})} - i\hat{p}^{\frac{1}{2}}j^{(\bar{\Lambda})}. \tag{5.42}$$

As in the standard calculation we extract the interaction terms as variation operators of the generating currents. From equations (5.32) to (5.38) we saw the interactions in the lagrangian can be written as functionals of the momentum space superfields. This means that using chain rules the variations with respect to the currents associated with physical fields can be combined as $-i\frac{\delta}{\delta J}$ and $i\frac{\delta}{\delta \bar{J}}$, and the vertices are simply given by equations (5.32) to (5.38), where the variation with respect to a function of both bosonic and fermionic variables is defined as [53]

$$\frac{\delta J(p^{'},\eta^{'})}{\delta J(p,\eta)} = \delta^4(p^{'}-p)\,\delta(\eta^{'}-\eta) \tag{5.43}$$

Therefore we have

$$Z[J] = e^{iS_{int}\left[\frac{\delta}{\delta J}\right]}Z_0[J] \tag{5.44}$$

The free generating functional is calculated by integrating over all field variables.

$$Z_0[J] = e^{i\int j^{(\bar{A})}\Delta_{(A)}j^{(A)}+j^{(\bar{\Lambda})}\Delta_{(\Lambda)}j^{(\Lambda)}} = e^{i\int \bar{J}\Delta J} \tag{5.45}$$

[4]The signs of the square root in the super-currents are chosen so that after integrating η they cancel the $\hat{p}$ dependence in ϕ and $\bar{\phi}$ completely.

where the light-cone gauge gluon and gluino propagators are given by $\Delta_{(A)}(p_1,\, p_2) = \frac{1}{p_2^2}\delta^4(p_1+p_2)$, $\Delta_{(\Lambda)}(p_1,\, p_2) = \frac{\hat{p}_2}{p_2^2}\delta^4(p_1+p_2)$. We find that the currents associated with gluons and gluinos can be again organised into the super-currents (5.42), and the propagators of two different particle species are replaced by

$$\Delta(p_1, \eta_1;\, p_2, \eta_2) = \frac{1}{p_2^2}\delta^4(p_1 + p_2)\,\delta(\eta_1 + \eta_2) \tag{5.46}$$

Note that despite the free generating functional (5.45) was derived without treating the chiral superfields as field variables, the propagator (5.46) takes the form as the inverse of the free superfield lagrangian (5.30), allowing us to reintroduce superfields ϕ and $\bar{\phi}$ as auxiliary field variables, where we generalised the functional integral to fields labeled by both bosonic and fermionic indices p and η in the same way as in [53, 54] so that the integration over ϕ and $\bar{\phi}$ has the same properties as over ordinary fields. The interaction part of the action extracted as a functional of variation operators can be applied back to $Z_0\,[J]$ to restore the lagrangian as a functional of ϕ and $\bar{\phi}$. It is easy to see that the lagrangian has the same propagator and vertices given in (5.30) to (5.38).

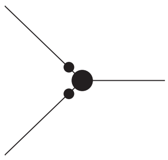
$$Z_0\,[J] = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\; e^{iS_{free}+\int\, J\phi+\bar{\phi}\bar{J}},\; Z\,[J] = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\; e^{iS+i\int\, \bar{J}\Delta J} \tag{5.47}$$

For the purpose of computing generating functionals and Green functions it makes no difference whether the functional integral was defined from the physical fields or the superfield.

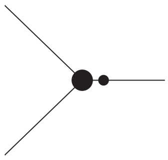
By using the standard Wick contraction and the LSZ reduction on (5.44) and (5.45) it is straightforward to derive a set of supersymmetric Feynman rules for $\mathcal{N} = 1$ SYM theory based on the super-momentum space lagrangian, and the method naturally leads to a combination of scattering amplitudes related to each other by supersymmetry transformation. We define the super-amplitude of a specific momentum and helicity configuration as

$$A(p_i, \sigma_i, \eta_i) = \lim_{p_i^2\to} \prod_i p_i^2 \left\langle \cdots \phi \cdots \bar{\phi} \cdots \right\rangle \tag{5.48}$$

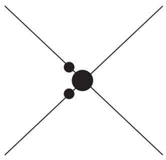
To convert the super-amplitude into the physical scattering amplitudes of gluons



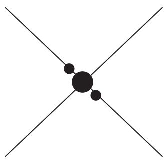
$$V^{--+}(123) = \frac{\langle 12\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\left(\langle 12\rangle\,\eta_1\eta_2 + \langle 23\rangle\,\eta_2\eta_3 + \langle 31\rangle\,\eta_3\eta_1\right)$$



$$V^{++-}(123) = \frac{[12]^3}{[12][23][31]}\left([23]\,\eta_1 + [31]\,\eta_2 + [12]\,\eta_3\right)$$



$$\begin{aligned} V^{--++}_{1234} &= \frac{1}{\left(\hat{2}+\hat{3}\right)^2}\Big(\left(\hat{1}\hat{3}+\hat{2}\hat{4}\right)\eta_1\eta_2 - \hat{1}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\left(\hat{2}+\hat{3}\right)\eta_2\eta_3 + \hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\left(\hat{1}+\hat{4}\right)\eta_1\eta_3 \\ &\quad +2\,\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\eta_3\eta_4 - \hat{2}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\hat{1}+\hat{4}\right)\eta_4\eta_1 + \hat{1}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\hat{2}+\hat{3}\right)\eta_2\eta_4\Big) \end{aligned}$$



$$\begin{aligned} V^{-+-+}_{1234} = &-\left(\frac{\hat{3}\hat{4}}{\left(\hat{1}+\hat{4}\right)^2}+\frac{\hat{1}\hat{4}}{\left(\hat{3}+\hat{4}\right)^2}\right)\eta_1\eta_3 - \frac{\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}\hat{4}}{\left(\hat{3}+\hat{4}\right)^2}\eta_2\eta_3 - \frac{\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\hat{4}}{\left(\hat{1}+\hat{4}\right)^2}\eta_1\eta_2 \\ &+\hat{3}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\frac{1}{\left(\hat{1}+\hat{4}\right)}-\frac{\hat{1}}{\left(\hat{3}+\hat{4}\right)^2}\right)\eta_4\eta_1 + \hat{4}^{\frac{1}{2}}\hat{1}^{\frac{1}{2}}\left(\frac{1}{\left(\hat{3}+\hat{4}\right)}-\frac{\hat{3}}{\left(\hat{1}+\hat{4}\right)^2}\right)\eta_3\eta_4 \\ &-\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}\hat{4}^{\frac{1}{2}}\left(\frac{1}{\left(\hat{1}+\hat{4}\right)^2}+\frac{1}{\left(\hat{3}+\hat{4}\right)^2}\right)\eta_2\eta_4 \end{aligned}$$

Figure 5.1: Super-vertices in the light-cone gauge $\mathcal{N}=1$ SYM lagrangian

and gluinos we extract terms with the Grassmann variables corresponding to the particle species participating the event:

$$\left\langle 1^+\cdots 2^+_\Lambda\cdots 3^-\cdots 4^-_\Lambda\right\rangle = \frac{\partial}{\partial\eta_4}\cdots\frac{\partial}{\partial\eta_1}\prod_i p_i^2\langle\mathcal{A}_1\cdots\frac{\Lambda_2}{\hat{p}_2^{\frac{1}{2}}}\cdots\bar{\mathcal{A}}_3\cdots\frac{\bar{\Lambda}_4}{\hat{p}_4^{\frac{1}{2}}}\rangle \qquad (5.49)$$

From the definitions of superfields (5.25), (5.26) we see a Grassmannian momentum η_i is present whenever there is a positive helicity gluon or a negative helicity gluino. The appropriate polarisations factors for the LSZ reduction formula are automatically included from the definition of a super-amplitude (5.48). Note that the superfields ϕ and $\bar{\phi}$ here are regarded as auxiliary fields introduced in the functional integral (5.47) which do not contain physical gluon or gluino fields as components. The expansion from the definition of a super-amplitude (5.48) into a series of physical scattering amplitudes (5.49) relies on current algebra. However since the chain rule of variations does not distinguish whether the current $j^{(A)} - i\hat{p}^{\frac{1}{2}}\eta j^{(\Lambda)}$ is multiplied by the combination $i\bar{\Lambda}(p)\hat{p}^{\frac{1}{2}} + \eta\mathcal{A}(p)$ or the newly introduced integration variable $\phi(p,\eta)$, the scattering amplitude calculated from integrating over gluon and gluino fields is the same as the amplitude calculated from integrating over $\phi(p,\eta)$.

5.3.1 Applying BCFW to calculate the $\mathcal{N} = 1$ MHV super-amplitudes

In [21, 22] the BCFW recursion method is generalised to $\mathcal{N} = 4$ SYM theory to compute super-amplitudes that have super-wavefunctions as initial and final states. We adapt the argument provided by Brandhuber, Heslop and Travaglini originally designed to apply on shifting two positive helicity legs [22] to shifting one positive, one negative leg in the $\mathcal{N} = 1$ theory and derive the formula for 4-point MHV super-amplitude. [5]

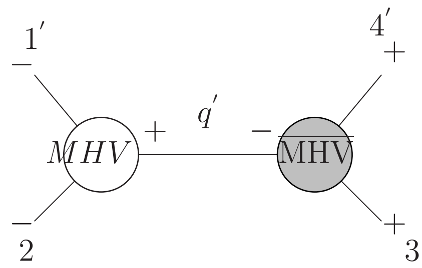


Figure 5.2: Shifting the super-amplitude $A(1^-, 2^-, 3^+, 4^+)$

Consider shifting the leg 1 and 4 of the super-amplitude $A(1^-, 2^-, 3^+, 4^+)$ (Fig.5.2). The momenta p_1 and p_4 are shifted in the same way as in the pure Yang-Mills theory (2.42), (2.43).

$$\begin{aligned} P'_{1\,\alpha\dot{\alpha}}(z) &= \lambda_{1\,\alpha}\bar{\lambda}_{1\,\dot{\alpha}} - z\,\lambda_{1\,\alpha}\bar{\lambda}_{4\,\dot{\alpha}}, \\ P'_{4\,\alpha\dot{\alpha}}(z) &= \lambda_{4\,\alpha}\bar{\lambda}_{4\,\dot{\alpha}} + z\,\lambda_{1\,\alpha}\bar{\lambda}_{4\,\dot{\alpha}} \end{aligned} \tag{5.50}$$

[5]The difference between the helicities of the shifted legs is because $\mathcal{N} = 4$ SYM theory is CPT self-conjugate and particles with opposite helicities are enclosed in the same chiral superfield. The relation between the helicity and the Grassmann variable assigned to the legs of a vertex factor will become clear when we derive the supersymmetric Feynman rules for $\mathcal{N} = 4$ SYM in section (5.5).

In addition to momenta we also shift the Grassmann variable associated with the negative helicity leg.

$$\eta_1^{'}(z) = \eta_1 - z\,\eta_4, \tag{5.51}$$

while all other momenta and Grassmann variables are unchanged. A super-amplitude defined in (5.48) generically contains a series of physical amplitudes, each of them multiplied by the corresponding Grassmann variables. The ratio between different physical amplitudes are fixed by the SUSY Ward identities. Because the on-shell SUSY generators Q_α, $\bar{Q}_{\dot\alpha}$ are momentum dependent, and therefore z dependent, generically so are the ratios. As $z \to \infty$ it is possible for the super-amplitude to diverge, causing the Cauchy's theorem fail to apply. However we note that the shifting given by equations (5.50), (5.51) have the effect of leaving the SUSY generators invariant

$$\bar{Q}_{\dot\alpha}^{'} = \sum_{i=1}^{4} \bar{\lambda}_{i\,\dot\alpha}^{'} \frac{\partial}{\partial \eta_i^{'}} = \bar{Q}_{\dot\alpha},\; Q_\alpha^{'} = \sum_{i=1}^{4} \lambda_\alpha^{'} \eta_i^{'} = Q_\alpha \tag{5.52}$$

so the ratios do not depend on the complex variable z. For the MHV super-amplitude $A(1^-, 2^-, 3^+, 4^+)$, we solve the ratios explicitly by repeatedly applying SUSY Ward identity with different SUSY transformation parameters, and the super-amplitude is proportional to

$$\langle 12\rangle\, \eta_1\eta_2 + \langle 23\rangle\, \eta_2\eta_3 + \langle 34\rangle\, \eta_3\eta_4 + \langle 41\rangle\, \eta_4\eta_1 + \langle 13\rangle\, \eta_1\eta_3 + \langle 24\rangle\, \eta_2\eta_4, \tag{5.53}$$

which is also invariant under the shifting (5.50) and (5.51).

In chapter 2 we saw in the case of pure gluon scattering, the amplitude shifted according to equation (5.50) vanishes as $z \to \infty$, which correspond to the coefficient of the $\eta_1\eta_2$ term. So the super-amplitude vanishes at infinity. From the above argument we also see that the generalisation to $\mathcal{N} = 1$ SYM does not introduce new singularities, therefore we have, from BCFW recursion,

$$A_4(0) = \int d\eta_q\, d\eta_{q'}\; A_L(z)\, \frac{\delta(\eta_q + \eta_{q'})}{q^2}\, A_R(z)\bigg|_{z=-\langle 34\rangle/\langle 31\rangle} \tag{5.54}$$

$$
\begin{aligned}
&= \frac{\langle 12\rangle^2}{\langle 2\,q'\rangle\,\langle q'\,1\rangle}\,\frac{1}{\langle 34\rangle\,[34]}\,\frac{[34]^2}{[4\,q']\,[q'\,3]} \\
&\times \left(\begin{array}{c} \langle 12\rangle\,[34]\,\eta_1'\eta_2 - \left\langle 2\,q'\right\rangle \left[4\,q'\right] \eta_2\eta_3 - \left\langle 2\,q'\right\rangle \left[q'\,3\right] \eta_2\eta_4 \\ + \left\langle q'\,1\right\rangle \left[4\,q'\right] \eta_1'\eta_3 + \left\langle q'\,1\right\rangle \left[q'\,3\right] \eta_1\eta_4 \end{array} \right),
\end{aligned}
\tag{5.55}
$$

where $q = p_3 + p_4$. Simplifying the above expression gives the 4-point super-amplitude

$$
A(1^-,2^-,3^+,4^+) = \frac{\langle 12\rangle^3}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}\left(\sum_{i,j=1}^{4}\langle i\,j\rangle\,\eta_i\eta_j\right). \tag{5.56}
$$

Since the argument for asymptotic behaviour and the algebraic derivation we just used do not depend on the number of legs, we can replace the amplitude $A_L(z)$ on the left hand side of the propagator by an $(n-1)$-point MHV super-amplitude. By induction an n-point MHV super-amplitude is given by the formula (5.4). The BCFW recursion also extends beyond MHV super-amplitudes because the argument for asymptotic behavior only rely on the fact that the SUSY generators are invariant under the shifting (5.50) and (5.51).

5.4 Super-space canonical transformation

So far we have derived a supersymmetry equivalence of the LCYM theory. A natural next step is to perform a canonical transformation on the field variables as in pure Yang-Mills theory [24] to absorb the unwanted $\overline{\text{MHV}}$ term so that in terms of the new variables the lagrangian automatically generates CSW rules for $\mathcal{N}=1$ SYM theory. In [31] Morris and Xiao applied the canonical transformation on a pair by pair basis. In the gauge field sector the gluon, gluino fields and their canonical conjugate momenta $\left\{\mathcal{A}, \hat{\partial}\bar{\mathcal{A}}\right\}$ and $\left\{\Lambda, \bar{\Lambda}\right\}$ were transformed into the corresponding new fields $\left\{\mathcal{B}, \hat{\partial}\bar{\mathcal{B}}\right\}$ and $\left\{\Pi, \bar{\Pi}\right\}$ according to the following expansions

$$
\mathcal{A}_1 = \mathcal{B}_1 + \Upsilon_{123}\mathcal{B}_2\mathcal{B}_3 + \cdots, \tag{5.57}
$$

$$
\begin{aligned}
\hat{\partial}\bar{\mathcal{A}}_1 &= \hat{\partial}\bar{\mathcal{B}}_1 + \Xi^2_{123}\hat{\partial}\bar{\mathcal{B}}_2\mathcal{B}_3 + \Xi^3_{123}\mathcal{B}_2\hat{\partial}\bar{\mathcal{B}}_3 + \cdots \\
&\quad + \Xi^2_{123}\bar{\Pi}_2\Pi_3 + \Xi^3_{123}\Pi_2\bar{\Pi}_3 + \Xi^2_{1234}\bar{\Pi}_2\Pi_3\mathcal{B}_4 + \Xi^2_{1234}\bar{\Pi}_2\mathcal{B}_3\Pi_4 + \cdots,
\end{aligned}
\tag{5.58}
$$

$$\Lambda_1 = \Pi_1 + \Upsilon_{123}\Pi_2\mathcal{B}_3 + \Upsilon_{123}\mathcal{B}_2\Pi_3 + \cdots, \tag{5.59}$$

$$\bar{\Lambda}_1 = \bar{\Pi}_1 + \Xi^2_{123}\bar{\Pi}_2\mathcal{B}_3 + \Xi^3_{123}\mathcal{B}_2\bar{\Pi}_3 + \cdots. \tag{5.60}$$

In order to keep the notation simple we neglected the overall momentum conservation delta functions in the above expressions. The coefficients Υ and Ξ^k in the expansion are the translation kernels originally defined for the Yang-Mills theory.

$$\begin{aligned}\Upsilon_{12\cdots n} &= \frac{\hat{1}\,\hat{3}\cdots\hat{n}}{(23)\,(34)\cdots(n-1,n)},\\ \Xi^k_{12\cdots n} &= \frac{\hat{k}\hat{3}\cdots\hat{n}}{(23)\,(34)\cdots(n-1,n)}.\end{aligned} \tag{5.61}$$

The transformation expansions (5.57) to (5.60) were verified to generate a unit Jacobian and have the effect of absorbing the $\mathcal{L}^{++-}_{A}$, $\mathcal{L}^{++-}_{\Lambda A}$ terms into the new lagrangian. In this section we take a different approach and apply the transformation on superfields directly. As noted in section (5.3) Green functions can be computed from functional integral over superfields labeled by super-space momenta p and η. Similarly a generating functional of currents in coordinate space originally derived from integrating over physical filed components $\mathcal{A}$, $\bar{\mathcal{A}}$, Λ, $\bar{\Lambda}$ can be reorganised as a functional of super-currents

$$\begin{aligned}Z_0[J] = &\int \mathcal{DA}\mathcal{D}\bar{\mathcal{A}}\mathcal{D}\bar{\Lambda}\mathcal{D}\Lambda\\ &exp\left\{iS\int d^4x\, d\theta d\bar{\theta}\,\mathcal{L}_{free} + i\int d^4x\, j^{(A)}\mathcal{A} + j^{(\Lambda)}\Lambda + \bar{\mathcal{A}}j^{(\bar{A})} + \bar{\Lambda}j^{(\bar{\Lambda})}\right\}\end{aligned} \tag{5.62}$$

$$= exp\left\{\int d^4x\, j^{(\bar{A})}\Delta^{(A)}j^{(A)} + j^{(\bar{\Lambda})}\Delta^{(\Lambda)}j^{(\Lambda)}\right\} = exp\left\{\int d^4x\, d\theta d\bar{\theta}\,\bar{J}\Delta J\right\} \tag{5.63}$$

where we defined the super-currents in coordinate space as

$$J(x,\theta) = \theta\bar{\theta}j^{(A)}(x) - i\bar{\theta}j^{(\Lambda)}, \tag{5.64}$$

$$\bar{J}(x,\theta) = \frac{1}{i\hat{\partial}}j^{(\bar{A})} - i\theta j^{(\Lambda)}, \tag{5.65}$$

and we have extracted the interaction part of the lagrangian as variation operators with respect to the supercurrents. As in the momentum space we introduce superfields as auxiliary field variables and restore the full action in chiral and anti-chiral superfields by operating the variation operators back onto the free generating functional.

$$Z\,[J] = \int \mathcal{D}\Phi(x,\theta)\mathcal{D}\bar{\Phi}(x,\theta)\, exp\left\{iS\,[\Phi] + i\int J\Phi + \bar{\Phi}\bar{J}\right\}, \qquad (5.66)$$

where $S\,[\Phi]$ is the $\mathcal{N} = 1$ SYM action in light-cone gauge introduced at the beginning of this chapter (5.16). Inspired by the canonical transformation originally applied on pure Yang-Mills to derive an MHV lagrangian [24] we make an analogous change of variables. The superfield $\Phi(\tau, \mathbf{x}, \theta)$ at light-cone time τ is assumed to be a functional of $\chi(\tau, \mathbf{y}, \xi)$ defined through power expansion. As in the pure Yang-Mills theory the expansion for the anti-chiral superfield $\bar{\Phi}(\tau, \mathbf{x}, \theta)$ is assumed to contain only one $\bar{\chi}(\tau, \mathbf{y}, \theta)$ in each term, while the power of $\chi(\tau, \mathbf{y}, \xi)$ increases term by term. This arrangement ensures that the new lagrangian will have exactly two anti-chiral superfields in every vertex as demanded by the CSW rules. In chapter 4 we proved that the integration measure is invariant under the transformation as long as the variation of the conjugate momentum $f(\partial)\bar{\mathcal{A}}$ with respect to $f(\partial)\bar{\mathcal{B}}$ is the inverse of $\delta\mathcal{A}/\delta\mathcal{B}$, where $f(\partial)$ is a generic function of differential operators. It is straightforward to generalise the proof to fields labeled by super-space coordinate variables and the function $f(\partial)$ is then generalised to an arbitrary function of derivatives and SUSY covariant derivatives. However, as pointed out in chapter 4 for an arbitrary choice of $f(\partial)$ the translation kernels are not holomorphic, and the MHV vertices generically differ from the Parke-Taylor formula by squares of external leg momenta. So we choose the operator for the transformation on $\mathcal{N} = 1$ SYM theory to be the covariant derivative $\bar{d}$, which is the natural extension of $\hat{\partial}$ into the supersymmetric theory. We assume the transformation is given by

$$\bar{d}\,\bar{\Phi}^a(x,\theta) = \int d^3y\, d\xi d\bar{\xi} \frac{\delta\chi^b(y,\xi)}{\delta\Phi^a(x,\theta)}\,\bar{d}\,\bar{\chi}^b(y,\xi), \qquad (5.67)$$

and after the transformation the unwanted super-vertex is absorbed into the new free field lagrangian $\mathcal{L}^{-+}\,[\Phi] + \mathcal{L}^{--+}[\Phi] = \mathcal{L}^{-+}\,[\chi]$.

We note that the nilpotency of the Grassmann variables $\theta\,\bar{\theta}$ allows us to replace the anti-chiral superfield by $\bar{\theta}\,\bar{d}\,\bar{\Phi}$,

$$tr\int d^4x\, d\theta d\bar{\theta}\,\bar{\Phi}\frac{\partial^2}{\hat{\partial}}\Phi + \bar{\Phi}[\Phi, \frac{\bar{\partial}}{\hat{\partial}}\Phi] = tr\int d^4x\, d\theta d\bar{\theta}\,\bar{\theta}\,\bar{d}\bar{\Phi}\left(\frac{\partial^2}{\hat{\partial}}\Phi + [\Phi, \frac{\bar{\partial}}{\hat{\partial}}\Phi]\right) \quad (5.68)$$

Using the condition (5.67) and stripping off an $\bar{\theta}\,\bar{d}\,\bar{\chi}$ from both sides of the equation, we have

$$\frac{\partial^2}{\hat{\partial}}\Phi(x,\theta) + [\Phi, \frac{\bar{\partial}}{\hat{\partial}}\Phi](x,\theta) = \int d^3y\, d\xi d\bar{\xi}\,\frac{\partial^2}{\hat{\partial}}\chi(y,\xi)\,\frac{\delta\Phi(x,\theta)}{\delta\chi(y,\xi)} \quad (5.69)$$

From (5.69) we determine the translation kernels in the expansions of Φ and $\bar{\Phi}$. Since the above condition is the same as the condition we used to solve for translation kernels in the pure Yang-Mills theory (2.76), we see that the kernels are simply given by the same formulae as in (5.61).

$$\Phi(\mathbf{x}_1,\theta) = \chi(\mathbf{x}_1,\theta) + \int \Upsilon_{123}\,\chi(\mathbf{x}_2,\theta)\,\chi(\mathbf{x}_3,\theta) + \cdots \quad (5.70)$$

$$\begin{aligned}\bar{d}\,\bar{\Phi}(\mathbf{x}_1,\theta) = \bar{d}\,\bar{\chi}(\mathbf{x}_1,\theta) &+ \int \Xi^2_{123}\,\left(\bar{d}\,\bar{\chi}(\mathbf{x}_2,\theta)\right)\,\chi(\mathbf{x}_3,\theta)\\ &+ \int \Xi^3_{123}\,\chi(\mathbf{x}_2,\theta)\,\left(\bar{d}\,\bar{\chi}(\mathbf{x}_3,\theta)\right)\cdots\end{aligned} \quad (5.71)$$

We Fourier transform the chiral superfields into momentum space and then apply the integrals defined in equations (5.19) to (5.22) to obtain the superfields in the new representation. The expansion formulae in momentum space are

$$\begin{aligned}\phi(p_1,\eta_1) &= \chi(p_1,\eta_1)\\ &+ \int \Upsilon_{123}\frac{-1}{\hat{1}^{\frac{1}{2}}}(-\eta_1\hat{1}^{\frac{1}{2}} + \eta_2\hat{2}^{\frac{1}{2}} + \eta_3\hat{3}^{\frac{1}{2}})\,\chi(p_2,\eta_2)\chi(p_3,\eta_3) + \cdots\\ &+ \int \Upsilon_{12\cdots n}\frac{-1}{\hat{1}^{\frac{1}{2}}}(-\eta_1\hat{1}^{\frac{1}{2}} + \eta_2\hat{2}^{\frac{1}{2}} + \cdots \eta_n\hat{n}^{\frac{1}{2}})\,\chi(p_2,\eta_2)\cdots\chi(p_n,\eta_n) + \cdots,\end{aligned} \quad (5.72)$$

and

$$\begin{aligned}\bar{\phi}(p_1,\eta_1) &= \bar{\chi}(p_1,\eta_1^{'})\\ &+ \int \Xi^2_{123}\frac{-\hat{2}^{\frac{1}{2}}}{\hat{1}}(-\eta_1\hat{1}^{\frac{1}{2}} + \eta_2\hat{2}^{\frac{1}{2}} + \eta_3\hat{3}^{\frac{1}{2}})\,\bar{\chi}(p_2,\eta_2)\chi(p_3,\eta_3) + \cdots\\ &+ \int \Xi^k_{12\cdots n}\frac{-\hat{k}^{\frac{1}{2}}}{\hat{1}}(-\eta_1\hat{1}^{\frac{1}{2}} + \eta_2\hat{2}^{\frac{1}{2}} + \cdots \eta_n\hat{n}^{\frac{1}{2}})\,\chi(p_2,\eta_2)\cdots\bar{\chi}(p_k,\eta_k)\cdots\chi(p_n,\eta_n) + \cdots.\end{aligned} \quad (5.73)$$

In order to avoid introducing too many symbols we slightly abuse the notation and use χ both for superfields before and after the integral transformations (5.19) to (5.22), in the same spirit as the same symbol is commonly used for wave functions before and after the Fourier transformation in the standard notation. The distinction between these two types of fields should be clear judging from the labels (x, θ) or (p, η) attached to the superfields. In equations (5.72) and (5.73) we neglected the overall momentum conservation delta function and the integrations are understood to be performed over momenta p_2 to p_n as well as superspace momenta η_2 to η_n.

The above expansion formulae can be conveniently summarised if we generalise the graphical notation introduced for pure Yang-Mills in chapter 3. When an n-th order term in (5.72) contribute to the calculation we use a blank circle follow by $(n+1)$ lines to represent the translation kernel, where one of the lines comes from the superfield ϕ being translated. For the $\bar{\phi}$ translation, we use a similar graph with the blank circle replaced by a gray circle.

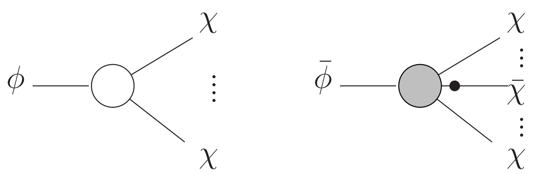


Figure 5.3: Graphical representations of superfield expansions

5.4.1 Generating MHV super-vertices

Following the same steps as for pure Yang-Mills theory the super-amplitude is generically transformed into a series, each of the term contains a number of translation kernels

$$\begin{aligned}
&\prod_l p_l^2 \left\langle \cdots \phi_i \cdots \bar{\phi}_j \cdots \right\rangle \\
&= \sum_{m,n} \prod_l p_l^2 \left\langle \cdots \left(\Upsilon_{i\, i_2 \cdots i_m} \chi_{i_2} \chi_{i_3} \cdots \chi_{i_m}\right) \cdots \left(\Xi^k_{j\, j_2 \cdots j_n} \chi_{j_2} \cdots \bar{\chi}_{j_k} \cdots \chi_{j_n}\right) \cdots \right\rangle
\end{aligned} \tag{5.74}$$

At tree-level these kernels are suppressed by LSZ factors, so an MHV super-amplitude simply equals the on-shell limit of the corresponding MHV super-vertex.

In section (5.3.1) we derived the formula for a generic n-point $\mathcal{N}=1$ MHV super-amplitude from supersymmetric BCFW recursion. Applying the same argument used in chapter 4, the formula for MHV super-vertices and super-amplitudes can only differ by squares of leg momenta. Since the expansion coefficients in (5.72) and (5.73) are holomorphic, such difference is absent, so an n-point MHV super-vertex is provided by the same formula as for the super-amplitude.

Alternatively, we can directly compute the n-point MHV super-vertex. An MHV super-vertex in the new lagrangian receives contributions from the original 3-point (5.32) and the 4-point vertices (5.37), (5.38), and the superfields attached to the legs of the vertices branch into trees of new field variables according to the expansion formulae (5.72) and (5.73). In the appendix B we prove that the formula for an n-point MHV super-vertex is the same as the super-amplitude (5.75) by matching their coefficients under partial fraction expansion.

$$V(1^+,2^+\dots i^-,j^-,\dots n^+)=\frac{\langle i\,j\rangle^3}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\left(\sum_{a,b=1}^{n}\langle a\,b\rangle\,\eta_a\eta_b\right). \tag{5.75}$$

From the generating functional we derive the CSW rules algebraically for $\mathcal{N}=1$ SYM theory with the vertices given by (5.75) and the propagator given by (5.46).

As noted in section (5.3), although we introduce ϕ and $\bar\phi$ as integration variables, from current algebra these variables can be interchanged by the corresponding combinations of gluon and gluino fields in a Green function calculation. Substituting (5.25) and (5.26) into the canonical transformation formulae of superfields (5.70) and (5.71) suggests that the MHV lagrangian can be as well derived by transforming physical field components separately. Writing the new field variables as

$$\chi(x,\theta)=\mathcal{B}(y)+i\theta\,\Pi(y), \tag{5.76}$$

$$\bar\chi(x,\theta)=\bar{\mathcal{B}}(\bar y)+i\bar\theta\,\bar\Pi(\bar y), \tag{5.77}$$

and integrating over Grassmann variables θ and $\bar\theta$, we find the same transformation relations as equations (5.57) to (5.60) originally given by Morris and Xiao in [31]. Operating the SUSY generators and covariant derivatives q, $\bar q$, d, $\bar d$ on ϕ, $\bar\phi$ and collecting terms we find the new fields χ, $\bar\chi$ satisfy the same chiral constraints, and the new components $\mathcal{B}$, $\bar{\mathcal{B}}$, Π, $\bar{\Pi}$ transform in the same way as positive and negative helicity gluon fields are transformed into gluino fields (5.9), (5.10). The chiral property of the new fields can also be seen by solving the inverse relation to the canonical transformations. By rewriting the expansion (5.70) as

$$\chi(\mathbf{x}_1,\theta) = \Phi(\mathbf{x}_1,\theta) - \int \Upsilon_{123}\,\chi(\mathbf{x}_2,\theta)\,\chi(\mathbf{x}_3,\theta) - \cdots \tag{5.78}$$

and substituting iteratively the new field is expressed as a functional of chiral superfield Φ. Because the translation kernels are independent of θ, $\bar{\theta}$, on the right hand side of the equation we have a series of chiral superfields. The series itself therefore also satisfy the chiral constraint.

5.4.2 SUSY Ward identity

In [8] Witten introduced an on-shell representation of the SUSY generators for $\mathcal{N} = 4$ SYM theory and verified that the MHV super-amplitude given by Nair [6] is superconformal invariant. We find both the on-shell SUSY generators and the super-amplitude for $\mathcal{N} = 4$ theory resemble the formulae we derived in (5.28) and (5.75). It is then straightforward to verify that the n-point $\mathcal{N} = 1$ MHV super-amplitude is SUSY invariant. The on-shell SUSY transformation operator $Q(\xi)$ is given by contracting generators with the parametric spinors ξ_α.

$$Q(\xi) = \sum_i^n \langle \xi i\rangle\,\eta_i + [i\xi]\,\frac{\partial}{\partial\eta_i} \tag{5.79}$$

The transformation operator consists of a multiplication part and a differentiation part. When operating on formula (5.75) we find the two parts are separately zero. Collecting terms having the same Grassmann numbers, the multiplication part vanishes because from Jacobi identity

$$\langle\xi 1\rangle\,\langle 23\rangle\,\eta_1\eta_2\eta_3 + \langle\xi 3\rangle\,\langle 12\rangle\,\eta_3\eta_1\eta_2 + \langle\xi 2\rangle\,\langle 31\rangle\,\eta_2\eta_3\eta_1 = 0 \tag{5.80}$$

for any three of the momenta carried by external lines. The differentiation part is proportional to

$$\sum_i [\xi i]\,\langle ij\rangle\,\eta_j = 0 \tag{5.81}$$

which vanishes from conservation of momentum. We note that supersymmetry is taken as a build-in property of the super-amplitude. Since the functional integral is invariant under SUSY transformation

$$\prod_i p_i^2 \left\langle Q(\xi)\phi_1\phi_2\bar{\phi}_3\cdots\bar{\phi}_n\right\rangle = 0. \tag{5.82}$$

Differentiating both sides of the identity (5.82) with respect to $\eta_1\ \eta_2\ \eta_j$ for example, gives the familiar SUSY Ward identity relating the amplitude that consists of a pair of gluino and the amplitude of all gluons.

$$\langle 21\rangle \left\langle 1^+_\Lambda, 2^+, 3^-, \cdots j^-_\Lambda \cdots n^-\right\rangle + \langle 2j\rangle \left\langle 1^+, 2^+, 3^-, \cdots n^-\right\rangle = 0 \tag{5.83}$$

5.5 MHV super-vertices of the $\mathcal{N}=4$ SYM lagrangian

In this section we extend the method discussed in this chapter to $\mathcal{N}=4$ SYM theory and derive the on-shell SUSY generators (5.3) from Grassmannian integral transformation. We find the super-vertices in the MHV lagrangian automatically assume the same form as Nair's formula for MHV super-amplitude (5.2).

The light-cone gauge action constructed as a functional of chiral superfields was given by Brink, Lindgren and Nilsson first in 10-dimensions and then reduced to the 4-dimensional Minkowski spacetime [32].

$$\begin{aligned} S = tr\int d^4x\, d^4\theta d^4\bar{\theta}\, \bar{\Phi}\frac{\partial^2}{\hat{\partial}^2}\Phi + \frac{2}{3}\left(\left[\Phi, \bar{\partial}\Phi\right]\frac{1}{\hat{\partial}}\bar{\Phi} + c.c.\right) \\ + \frac{1}{2}\left(\left[\Phi, \hat{\partial}\Phi\right]\frac{1}{\hat{\partial}^2}\left[\bar{\Phi}, \hat{\partial}\bar{\Phi}\right] - \frac{1}{2}\left[\Phi, \bar{\Phi}\right]\left[\Phi, \bar{\Phi}\right]\right) \end{aligned} \tag{5.84}$$

After integrating over eight Grassmann variables θ^A, $\bar{\theta}_A$ the action was found to agree with the standard $\mathcal{N}=4$ light-cone gauge action [55]. The chiral superfield is defined as

$$\begin{aligned} \Phi(x,\theta) = \frac{1}{\hat{\partial}}\mathcal{A}(y) + \frac{i}{\hat{\partial}}\theta^A\Lambda_A(y) + \frac{i}{2}\theta^A\theta^B\bar{C}_{AB}(y) \\ + \frac{i}{3!}\epsilon_{ABCD}\theta^A\theta^B\theta^C\bar{\Lambda}^D + \frac{1}{4!}\epsilon_{ABCD}\theta^A\theta^B\theta^C\theta^D\bar{\mathcal{A}}(y) \end{aligned} \tag{5.85}$$

where $y=(x^+,\, x^- + \frac{1}{2}i\theta^A\bar{\theta}_A,\, x^z,\, x^{\bar{z}})$. Unlike in the $\mathcal{N}=1$ theory, the superfield (5.85) contains both of the helicity fields of gluon and gluino and is CPT self-conjugate.

$$\bar{\Phi}(x,\theta)=\frac{1}{4!}\epsilon^{ABCD}d_A d_B d_C d_D\,\hat{\partial}^{-2}\Phi(x,\theta) \tag{5.86}$$

Using the above condition all of the Φ or $\bar{\Phi}$ dependence can be expressed in terms of the other. The superfields satisfy the constraints $d_A\,\bar{\Phi}=\bar{d}^A\,\Phi=0$ and transform as a representation under the SUSY generators. The $\mathcal{N}=1$ SUSY covariant derivatives and generators in light-cone coordinates are

$$\begin{aligned} d_A &= \frac{\partial}{\partial\theta^A}+\frac{i}{2}\bar{\theta}_A\hat{\partial}\,,\; \bar{d}^A=-\,\frac{\partial}{\partial\bar{\theta}_A}-\frac{i}{2}\,\theta^A\hat{\partial}, \\ q_A &= \frac{\partial}{\partial\theta^A}-\frac{i}{2}\bar{\theta}_A\hat{\partial}\,,\; \bar{q}^A=-\,\frac{\partial}{\partial\bar{\theta}_A}+\frac{i}{2}\,\theta^A\hat{\partial} \end{aligned} \tag{5.87}$$

We transform the anti-chiral superfield defined on the hypersurface of superspace which satisfy the chiral constraint in a way analogous to the transformation defined for the $\mathcal{N}=1$ anti-chiral superfield.

$$\bar{\Phi}_{N=4}(p,\,\theta)=\frac{1}{\hat{p}}\int d^4\eta\, e^{\frac{-1}{2}\theta^B\bar{\theta}_B\hat{p}}\delta^{(4)}(\bar{\theta}_A\hat{p}^{\frac{1}{2}}-\eta_A)\,\bar{\phi}_{N=4}(p,\,\eta) \tag{5.88}$$

We note that the integration $d^4\eta$, the delta functions and the exponentials together factorise into four copies of the same integral, each of them contains only Grassmann variables θ^A, $\bar{\theta}_A$ and η_A with the same index number for extended supersymmetry. As for the $\mathcal{N}=1$ SYM theory the integral transformation has the effect of removing the exponential factor originated from the y dependence in the definition (5.85) and replacing the $\bar{\theta}_A\hat{p}^{\frac{1}{2}}$ by η_A. After the transformation we have

$$\begin{aligned} \bar{\phi}_{N=4}(p,\,\eta) &= \bar{\mathcal{A}}(p)+\eta_A\frac{\bar{\Lambda}^A(p)}{\hat{p}^{\frac{1}{2}}}+\frac{i}{2}\eta_A\eta_B C^{AB}(p) \\ &+\frac{1}{3!}\eta_A\eta_B\eta_C\epsilon^{ABCD}\frac{\Lambda_D(p)}{\hat{p}^{\frac{1}{2}}}+\frac{1}{4!}\eta_A\eta_B\eta_C\eta_D\epsilon^{ABCD}\mathcal{A}(p), \end{aligned} \tag{5.89}$$

which takes a similar form to the super-wavefunction (5.1) taken as the end states of a super-amplitude in [17, 21, 22]. The light-cone gauge super-momentum space lagrangian can be readily computed by substituting all of the Φ that appear in the lagrangian by $\bar{d}^4\bar{\Phi}$ using the CPT self-conjugate condition (5.86) and transforming $\bar{\Phi}$ using the integral (5.88). Although the chiral and anti-chiral superfields are interchangeable in the $\mathcal{N} = 4$ SYM theory, we find the 3-point $\overline{\text{MHV}}$ interaction is given by the term that originally contains two Φ and one $\bar{\Phi}$ superfield as in the $\mathcal{N} = 1$ theory.

$$
\begin{aligned}
& tr \int d^4x \, d\theta d\bar{\theta} \left[\Phi, \bar{\partial}\Phi\right] \frac{1}{\hat{\partial}}\bar{\Phi} \\
&= tr \int d^4p_i d^4\eta_i \, \hat{1}\hat{2}\hat{3} \, \{32\} \prod_{A=1}^{4} \left(\frac{1}{\hat{2}^{\frac{1}{2}}\hat{3}^{\frac{1}{2}}}\eta_{1\,A} + \frac{1}{\hat{3}^{\frac{1}{2}}\hat{1}^{\frac{1}{2}}}\eta_{2\,A} + \frac{1}{\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}}\eta_{3\,A} \right) \\
&\qquad \times\bar{\phi}(p_1,\eta_1)\bar{\phi}(p_2,\eta_2)\bar{\phi}(p_3,\eta_3)
\end{aligned} \tag{5.90}
$$

Rotating cyclically we find the $\overline{\text{MHV}}$ super-vertex is

$$
V_{N=4}^{\overline{\text{MHV}}}(1,2,3) = \frac{1}{[12]\,[23]\,[31]} \prod_{A=1}^{4} \left([23]\,\eta_{1A} + [31]\,\eta_{2A} + [12]\,\eta_{3A}\right) \tag{5.91}
$$

which takes the same form as the formula for 3-point $\overline{\text{MHV}}$ super-amplitude in [23]. Similarly we derive the 3-point MHV super-vertex.

$$
V_{N=4}^{MHV}(1,2,3) = \frac{1}{\langle 12\rangle \langle 23\rangle \langle 31\rangle} \prod_{A=1}^{4} \left(\langle 12\rangle\,\eta_{1A}\eta_{2A} + \langle 23\rangle\,\eta_{2A}\eta_{3A} + \langle 31\rangle\,\eta_{3A}\eta_{1A}\right) \tag{5.92}
$$

The transformation (5.93) for chiral superfield in coordinate space was found by Feng and Huang [33] to have the effect of rearranging the self-dual part of the lagrangian into a free field lagrangian of the new variables.

$$
\hat{\partial}\Phi(x_1,\theta) = \hat{\partial}\chi(x_1,\theta) + \int \Upsilon_{123}\hat{\partial}\chi(x_2,\theta)\hat{\partial}\chi(x_3,\theta) + \cdots \tag{5.93}
$$

We again replace the chiral superfield dependence on both sides of the equation by $\bar{d}^4\bar{\Phi}$ and apply the integral transformation on anti-chiral superfields. The expansion formula in momentum space is given by

$$\begin{aligned}
&\bar{\phi}(p_1,\eta_1) = \bar{\chi}(p_1,\eta_1)\\
&+\int \Upsilon_{123}\frac{1}{\hat{1}^2}\prod_{A=1}^{4}(-\eta_{1\,A}\hat{1}^{\frac{1}{2}}+\eta_{2\,A}\hat{2}^{\frac{1}{2}}+\eta_{3\,A}\hat{3}^{\frac{1}{2}})\,\bar{\chi}(p_2,\eta_2)\bar{\chi}(p_3,\eta_3)+\cdots\\
&+\int \Upsilon_{12\cdots n}\frac{1}{\hat{1}^2}\prod_{A=1}^{4}(-\eta_{1\,A}\hat{1}^{\frac{1}{2}}+\eta_{2\,A}\hat{2}^{\frac{1}{2}}+\eta_{3\,A}\hat{3}^{\frac{1}{2}}+\cdots\eta_{n\,A}\hat{n}^{\frac{1}{2}})\\
&\qquad\qquad\times\bar{\chi}(p_2,\eta_2)\bar{\chi}(p_3,\eta_3)\cdots\bar{\chi}(p_n,\eta_n)\,+\cdots
\end{aligned}\tag{5.94}$$

Translating all of the chiral superfields into the new field variables we arrive at the MHV lagrangian for $\mathcal{N}=4$ SYM theory. By matching the coefficients in the partial fraction expansion as in appendix (B) it is straightforward, though tedious, to show that a generic n-point super-vertex in the new lagrangian agrees with Nair's formula (5.2)

$$V^{MHV}_{N=4}(1,2,\cdots n)=\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\prod_{A=1}^{4}\left(\sum_{i,j}\langle ij\rangle\,\eta_{iA}\eta_{jA}\right). \tag{5.95}$$

As in the pure Yang-Mills and the $\mathcal{N}=1$ SYM theories we can also argue from holomorphy that the n-point super-vertex must given by the above formula. An $\mathcal{N}=4$ MHV super-vertex can only differ from corresponding super-amplitude by squares of momenta that vanish on-shell. Because the 3-point and 4-point supersymmetry LCYM vertices and the translation kernel (5.94) are all holomorphic in 4-dimensions, the super-vertex must also be holomorphic.

After the integral transformation, we find the holomorphic and anti-holomorphic SUSY generators in light-cone coordinates are the same as the Q_1 and $\bar{Q}_{\dot{1}}$ components of the on-shell SUSY generators introduced by Witten in [8].

$$Q_{\alpha\,A}=\lambda_\alpha\eta_A,\;\bar{Q}^A_{\dot{\alpha}}=\bar{\lambda}_{\dot{\alpha}}\frac{\partial}{\partial\eta_A} \tag{5.96}$$

The other components can be verified to match (5.96) by substituting the field components which were integrated out by their classical values.

Despite the similarities between $\mathcal{N}=1$ and $\mathcal{N}=4$ theories allows us to generalise the methods discussed in this chapter directly we notice there is a major difference in the Feynman rules of these two theories. In $\mathcal{N}=4$ SYM theory since both positive and negative helicity fields of the gluon and gluino are included in the same superfield we can not associate helicities to the lines connecting to super-vertices. Because physical field components only show up in the lagrangian through superfields their corresponding generating currents can be combined into super-currents so that the generating functional $Z[J]$ is a functional of supercurrents. As in previous sections we introduce $\bar{\phi}$ as an auxiliary field, and we have

$$Z[J] = \int \mathcal{D}\bar{\phi}\, e^{iS + \int \bar{\phi}\, \bar{J}}. \tag{5.97}$$

With all of the ϕ replaced $\bar{\phi}$ the generating functional is similar to that of a scalar field theory for which we are allowed to connect any two legs of the vertices by a propagator. Generically a super-amplitude or a super-vertex is a series of Grassmann numbers with the coefficients given by physical amplitudes or vertices that have different helicity contents. The helicity of a leg is instead determined by the Grassmann variables assigned to that leg. It is easy to see that the propagator

$$\frac{1}{p_2^2}\, \delta^4(p_1 + p_2)\, \delta^4(\eta_{1\,A} + \eta_{2\,A}) \tag{5.98}$$

makes sure that the same particle species is interchanged when we connect super-vertices and the super-amplitude defined as in (5.48) from a direct generalisation of the LSZ reduction to superfields

$$\prod_i p_i^2\, \langle \bar{\phi}_1 \bar{\phi}_2 \cdots \bar{\phi}_n \rangle \tag{5.99}$$

reproduces the same super-amplitude defined from the transition amplitude between super-wavefunction end states.

Chapter 6

Summary and discussions

In the past few years a considerable amount of the progress has been made in gluon scattering amplitude calculations based on the observation made by Cachazo, Svrčeck and Witten [7,8] that the calculations can be drastically simplified when we treat the MHV amplitudes [4,5] as the new building blocks. The CSW rules however were found to be incomplete. None of the graphs given by the CSW prescription yields an all-plus helicity amplitude at one-loop level and yet amplitudes of this type were found to be non-vanishing. In chapter 3 we generalised the canonical transformation method which was originally used in [24] to rewrite the D-dimensional light-cone gauge Yang-Mills theory in terms of the transformed new field variables. The canonical transformation reorganises the self-dual part of the LCYM lagrangian as the new free lagrangian. We found the all-plus helicity amplitude that appeared "missing" in the CSW rules are explained by the translation kernels which were created during the canonical transformation. The translation kernels entered the scattering amplitude when the helicity fields $\mathcal{A}$ and $\bar{\mathcal{A}}$ in the Green function were replaced by terms in the expansion formulae (3.15), (3.16). The inclusion of these kernel factors as a patch to the Feynman rules that were derived from the MHV lagrangian restores the correct result for scattering amplitudes to all order (Appendix A). From the graphical notation developed in section (3.2) we saw that the $\overline{\text{MHV}}$ vertex absent in the CSW rules are implicitly carried by the kernels. We found the translation kernels generically do not contribute to the amplitude either because they are suppressed by LSZ factors or because when the on-shell condition is applied the kernels equal the all-plus-except-one-minus amplitude $(- + + \cdots +)$, which vanishes for real value momenta. We showed that for a suitable choice of sign convention for the $i\epsilon$ prescription in the kernel, the tadpole graphs are the only exceptions to the CSW rules for practical amplitude calculations.

In chapter 4 we showed that the canonical transformation belongs to a class $\{\mathcal{A}, f(\partial)\bar{\mathcal{A}}\} \to \{\mathcal{B}, f(\partial)\bar{\mathcal{B}}\}$ in which all transformations preserve the integration measure and have the effect of rearranging the lagrangian into the helicity structure as implied by the CSW rules. By adjusting the operator $f(\partial)$ we demonstrated that the kernels can be chosen to have simpler singular behaviour and no additional infinitesimal vertex terms are needed to account for symmetrical graphs. For $f(\partial) = 1$ or $f(\partial) = \partial^2$ the loop integral that contains translation kernels can be evaluated using the standard Passarino-Veltman technique. The MHV vertices produced by the generalised measure-preserving transformation are generically not the same as the Parke-Taylor formula. In [24] Mansfield argued that for the canonical transformation these two objects are the same because of holomorphy. The vertices were calculated by Ettle and Morris up to 5-points [29]. In section (4.1) we proved that for canonical transformation the formula applies to n-points in 4-dimensions.

Following the same spirit we generalised the canonical transformation on supersymmetric theories. In the $\mathcal{N} = 1$ SYM theory we found in light-cone coordinates the physical components of the gluon fields $\mathcal{A}$, $\bar{\mathcal{A}}$ together with the gluino fields Λ, $\bar{\Lambda}$ are closed under the SUSY subalgebra Q_1, $\bar{Q}_{\bar{1}}$. The field components possessing the same helicities can be packed into the superfields Φ and $\bar{\Phi}$ of Brink, Lindgren and Nilsson [32]. We found that when light-cone gauge condition is applied and all of the auxiliary components are integrated out the $\mathcal{N} = 1$ SYM lagrangian can be rewritten in terms of these superfields. The superfields Φ and $\bar{\Phi}$ chiral and anti-chiral constraint respectively, which eliminates one degree of freedom. In the way analogous to a Fourier transform converts a field variable from coordinate space to momentum space we introduce a Grassmannian integral transform which converts a superfield from the surface of super-space where chiral constraint is satisfied to a Grassmannian momentum space where the superfields are labeled by single variable η. The integral transformation provides a link between the MHV super-amplitude formula commonly used in the recent amplitude calculation using the BCFW recursion approach [21, 22] and the light-cone gauge super Yang-Mills lagrangian constructed from chiral superfields. We found when expressed in Grassmannian momentum space the 3-point MHV and $\overline{\text{MHV}}$ vertices in the $\mathcal{N} = 1$ SYM lagrangian take similar forms to the formulae given by Nair [6] for $\mathcal{N} = 4$ MHV super-amplitudes. We adapted the argument developed by Brandhuber, Heslop and Travaglini [22] to show that the BCFW recursion technique can be applied to $\mathcal{N} = 1$ SYM super-amplitudes. As an example we derived the formula for n-point MHV super-amplitudes. Algebraically, we found the generating functional of the $\mathcal{N} = 1$ SYM theory is equivalent to a functional integral over superfield variables which are reintroduced as auxiliary fields after the physical field components were integrated over. By performing the canonical transformation on superfields we arrived at an MHV lagrangian. The formula for the n-point MHV super-vertex was proved to agree with the super-amplitude. At the end of the chapter we extended the method to $\mathcal{N} = 4$ and calculated the MHV and $\overline{\text{MHV}}$ vertices.

The SUSY Feynman rules derived in chapter 5 can be immediate applied to compute tree level graphs with generic helicity configurations. It is appearant that the formalism generalises to loop-level provided a suitable regulator is defined such as the one used in the "light-cone friendly" 4-dimensional regulariation scheme [62]. Alternatively the chiral lagrangian may be analytically continued to extra-dimenions, which will enable the dimension regulator to be applied on loop level diagrams. In [55] a similar super-space formalism was used to argue that $\mathcal{N} = 4$ SYM theory is UV finite. It is possible that the same can be seen from the structure of the supersymmetric light-cone gauge action or the MHV action.

In both pure and supersymmetric MHV lagrangian theories the derivation of the MHV vertex formula depends heavily on holomorphy of the translation kernels and also of the LCYM 3-point MHV vertex and 4-point vertices. When extended to higher dimensions these factors will no lnger remain holomorphic due to the fact that the inner product becomes $p \cdot q = (pq)_I\{pq\}_I/\hat{p}\hat{q}$ and the contraction between transverse direction indices I prevent brackets from factorisation, therefore the anti-holomorphic dependence does not cancel completely. In this case one can still reorganise (super-)vertices or amplitudes by choosing more complicated basis and concentrate on analysing the poles related to holomorphic variables. However the amplitudes in higher dimensions may not have a simple formula in terms of the bracket notation. Nevertheless it is still possible to develope a BCFW recursion base on the computing these singularities.

Appendix A

Converting between LCYM graphs and the MHV lagrangian graphs

As shown in chapter 2 in the LCYM lagrangian theory the building blocks of a scattering amplitude are the 3-point $\overline{\text{MHV}}$, MHV vertices and two different 4-point vertices. When the canonical transformation is applied, in the new lagrangian these are replaced by the an infinite number of MHV vertices and translation kernels $\Upsilon_{12\cdots n}$ and $\Xi^k_{12\cdots n}$. Generically any LCYM graph can be derived from a sum of graphs in the MHV lagrangian theory using the graphical identity shown in (Fig.3.10). As an example, we demonstrate how to recover (Fig.A.1) from MHV graphs contributing to the same tree-level amplitude $A(1^-, 2^+, 3^+, 4^+)$.

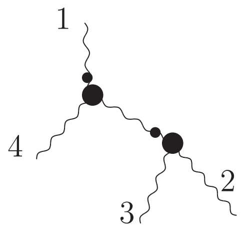


Figure A.1: A $(-+++)$ LCYM graph

In the MHV lagrangian theory only the translation kernels have the helicity configuration that fits into the amplitude $A(1^-, 2^+, 3^+, 4^+)$ at tree-level. One type of the contributions comes from translating one of the helicity fields in $\langle \mathcal{A}_1, \bar{\mathcal{A}}_2, \bar{\mathcal{A}}_3, \bar{\mathcal{A}}_4 \rangle$ and connect all of the legs stretching from the kernel directly with the rest of the fields. From the graphical expansions of $\mathcal{A}$ and $\bar{\mathcal{A}}$ fields we find the following six graphs have the same tree structure as (Fig.A.1).

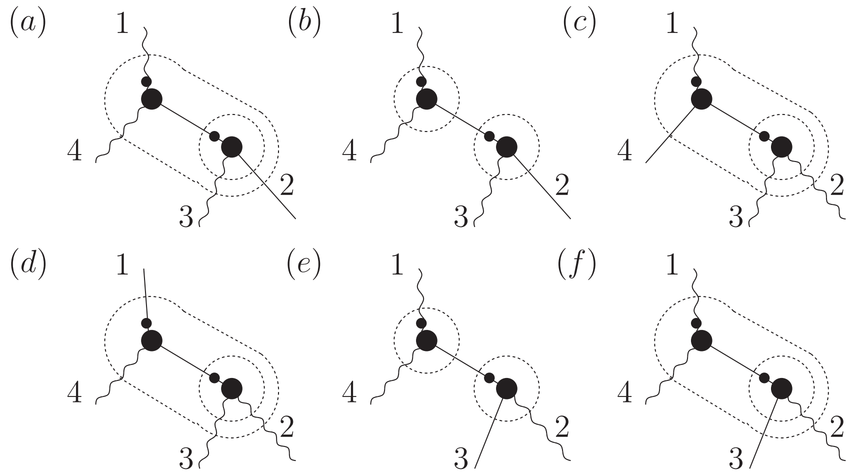


Figure A.2: All-plus-except-one-minus graphs constructed from kernels

We use wavy lines to indicate propagators. The dashed line bubble represents a factor of $1/(\sum_i p_i^2/\hat{p}_i)$ summing over the momenta crossing through the bubble. In addition to the graphs shown in (Fig.A.2), the contribution to the amplitude $A(1^-, 2^+, 3^+, 4^+)$ can also come from joining two translation kernels which originates from any two of the helicity fields in $\langle \mathcal{A}_1, \bar{\mathcal{A}}_2, \bar{\mathcal{A}}_3, \bar{\mathcal{A}}_4 \rangle$.

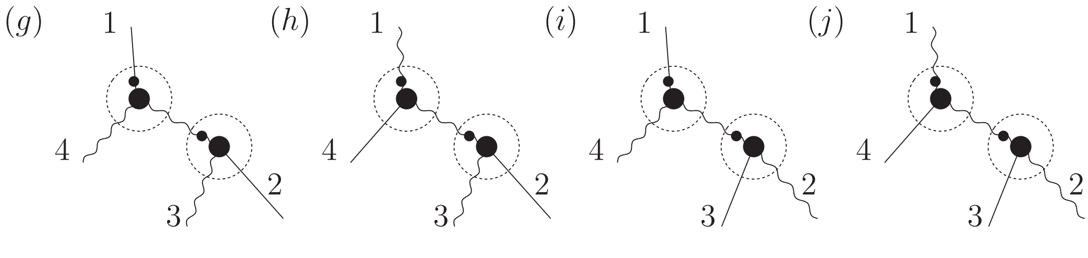
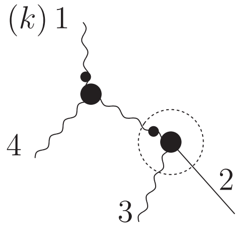
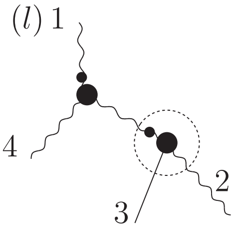


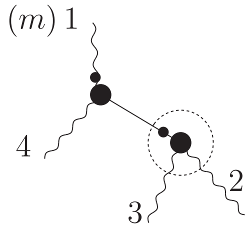
Figure A.3: Graphs constructed from joining two kernels

From the graphical convention introduced in chapter 3 a straight line is equivalent to the propagator multiplied by p^2, therefore summing over a series of graphs each has one of the propagators stretching out of the same bubble replaced by a straight line yields a graph with the bubble removed. We divide the graphs into three groups according to whether the leg 2 and 3 are propagators or straight lines. Summing over (b) (g) (h) and (e) (i) (j) removes the upper bubble.

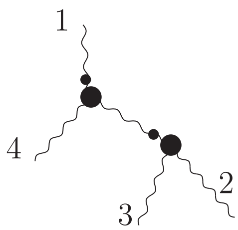
$(b) + (g) + (h) =$ 

$(e) + (i) + (j) =$ 

Adding the rest of the graphs together removes the outer dashed line bubble.

$(a) + (c) + (d) + (e) =$ 

Finally, collecting the sums yields the LCYM graph (Fig.A.1)

$(k) + (l) + (m) =$ 

By applying the same graphical identity repeatedly we can restore LCYM graphs with an arbitrary helicity configuration. Note that the identity we use applies to off-shell momenta as well, therefore the method extends to loop-level graphs.

Appendix B

The $\mathcal{N} = 1$ MHV super-vertices

In this appendix we prove that when the superfields ϕ and $\bar{\phi}$ are canonically transformed into new fields χ and $\bar{\chi}$ the original 3-point and 4-point vertices in the light-cone gauge SYM generate MHV super-vertices of the form (5.2). For simplicity we show this is true when the two negative helicity particles are adjacent. The method outlined here generalise to arbitrary configurations.

Labeling the negative helicity leg momenta as leg 1 and 2 the formula (5.2) reads

$$\begin{aligned} V(1^-, 2^-, 3^+ \dots n^+) &= \frac{\langle 1\,2\rangle^3}{\langle 12\rangle\,\langle 23\rangle\cdots\langle n1\rangle} \left(\sum_{a,b=1}^{n} \langle a\, b\rangle\, \eta_a \eta_b \right) \\ &= \frac{(12)^2}{(23)\cdots(n1)} \frac{\hat{3}\cdots\hat{n}}{\hat{1}^{\frac{1}{2}}\hat{2}^{\frac{1}{2}}} \left(\sum_{a,b=1}^{n} \frac{(a\,b)}{\hat{a}^{\frac{1}{2}}\hat{b}^{\frac{1}{2}}} \eta_a \eta_b \right) \end{aligned} \tag{B.0.1}$$

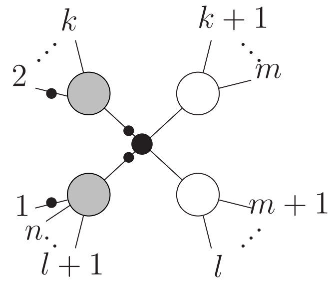


Figure B.1: Translated 4-point vertex

From (5.61) we saw the factors Υ and Ξ^k appearing in the translation formula (5.72) and (5.73) contain in the denominators a sequential product of round brackets. After the canonical transformation the 3-point and 4-point vertices constitute three

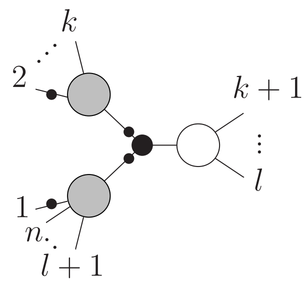


Figure B.2: Translated 3-point vertex

and four groups of products of round brackets (Fig B.2 and B.1). Therefore when regarded as functions of tilde component variables $\tilde{p}$ which are contained in the round brackets, both formula (B.0.1) and the translated 3-point and 4-point vertices can be spanned by terms of the form

$$\frac{1}{(23)\,(34)\cdots(k-1,k)} \times \frac{1}{(k+1,k+2)\cdots(m-1,m)}$$

$$\times \frac{1}{(m+1,m+2)\cdots(l-1,l)} \times \frac{1}{(l+1,l+2)\cdots(n,1)} \tag{B.0.2}$$

together with terms of the form

$$\frac{(12)}{(23)\,(34)\cdots(k-1,k)} \times \frac{1}{(k+1,k+2)\cdots(l-1,l)} \times \frac{1}{(l+1,l+2)\cdots(n,1)} \tag{B.0.3}$$

where the expansion coefficients depend on pairs of Grassmann numbers $\eta_i\eta_j$ with $i\ j$ running through all possible combinations of external legs and on hat component momenta $\hat{p}$.

To obtain the coefficients we use the method of partial fractions. For terms of the form (B.0.2) we adjust tilde components to set

$$(12) = 0 \tag{B.0.4}$$

$$(23) = \cdots = (k-1,k) = 0 \tag{B.0.5}$$

$$(k+1,k+2) = \cdots = (m-1,m) = 0 \tag{B.0.6}$$

$$(m+1,m+2) = \cdots = (l-1,l) = 0 \tag{B.0.7}$$

$$(l+1,l+2) = \cdots = (n,1) = 0 \tag{B.0.8}$$

Similarly, the coefficient of term (B.0.3) can be obtained by applying the conditions

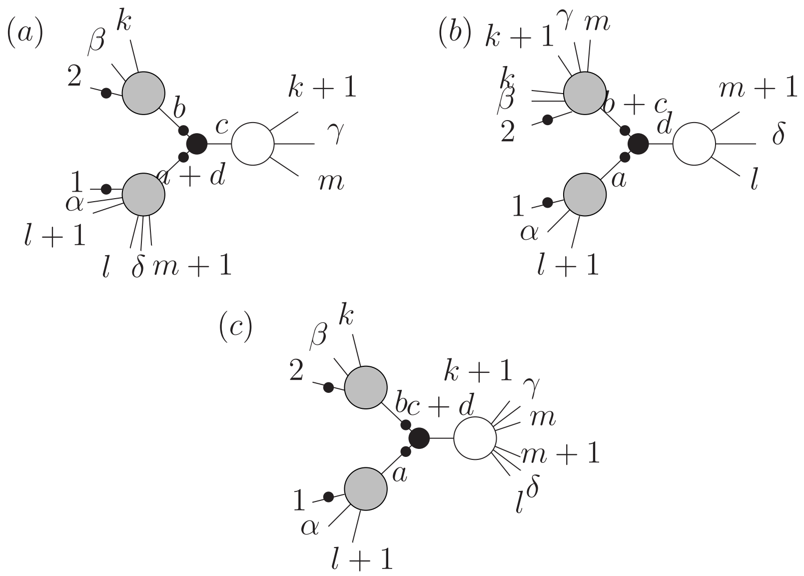
$$(23) = \cdots = (k-1, k) = 0 \tag{B.0.9}$$

$$(k+1, k+2) = \cdots = (l-1, l) = 0 \tag{B.0.10}$$

$$(l+1, l+2) = \cdots = (n, 1) = 0 \tag{B.0.11}$$

We shall prove that the translated 3-point and 4-point vertices combine to give the formula (B.0.1) by showing their expansion coefficients agree with each other.

First we check the coefficients for four sequential products of brackets (B.0.2). Applying conditions (B.0.8) on formula (B.0.1) gives zero because of the (12) dependence in the numerator. The 3-point vertex contribution to the coefficient of (B.0.2) can be from the following three cases



In each graph one of the sequential products of brackets splits into two. Summing over contributions from these three graphs gives

$$\sum_{\alpha,\beta,\gamma,\delta} \frac{1}{c^{\frac{1}{2}} d^{\frac{1}{2}} (b+c)^2} \Big((ac+bd)\, \hat{\alpha}^{\frac{1}{2}} \hat{\beta}^{\frac{1}{2}} \eta_\alpha \eta_\beta - a\,(b+c)\, \hat{\beta}^{\frac{1}{2}} \hat{\gamma}^{\frac{1}{2}} \eta_\beta \eta_\gamma - b\,(a+d)\, \hat{\delta}^{\frac{1}{2}} \hat{\alpha}^{\frac{1}{2}} \eta_\delta \eta_\alpha$$
$$+2\,ab \hat{\gamma}^{\frac{1}{2}} \hat{\delta}^{\frac{1}{2}} \eta_\gamma \eta_\delta + b\,(a+d)\, \hat{\alpha}^{\frac{1}{2}} \hat{\gamma}^{\frac{1}{2}} \eta_\alpha \eta_\gamma + a\,(b+c)\, \hat{\beta}^{\frac{1}{2}} \hat{\delta}^{\frac{1}{2}} \eta_\beta \eta_\delta \Big) \tag{B.0.12}$$

where we used η_α η_β η_γ η_δ to denote Grassmann variables associated with legs from each of the four branches emerging from the original 4-point vertex. The index

α is to be summed over from $(l+1)$ to 1, β from 2 to k, γ from $(k+1)$ to m, and finally δ from $(m+1)$ to l. For simplicity we denote the four internal lines of (Fig. 4-point) by a, b, c and d.

$$a = \widehat{l+1} + \cdots + \hat{n} + \hat{1} \tag{B.0.13}$$

$$b = \hat{2} + \hat{3} + \cdots + \hat{k} \tag{B.0.14}$$

$$c = \widehat{k+1} + \cdots + \hat{m} \tag{B.0.15}$$

$$d = \widehat{m+1} + \cdots + \hat{l} \tag{B.0.16}$$

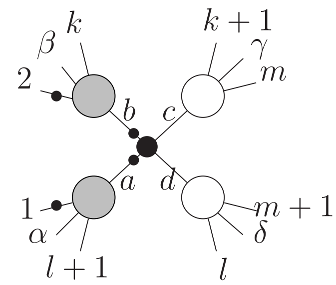


Figure B.3: Notation for the 4-point vertex expansion

The contribution from the original 4-point vertex can be readily derived by translating the superfields ϕ and $\bar{\phi}$ attached to (5.37), which cancels (B.0.12). The expansion coefficient of terms (B.0.2) vanishes, therefore agrees with the coefficients obtained by expanding the formula (B.0.1)

The coefficients of term (B.0.3) can be calculated using the same method. However we note that since in the pure YM case the LCYM 3-point vertex was verified to give the same expansion coefficients as the Parke-Taylor formula, which corresponds to the $\eta_1\eta_2$ term of the formula (B.0.1), the proof is complete as long as the ratio between the expansion coefficients of each $\eta_i\eta_j$ term from the original 3-point vertex is the same as ratio of coefficients of $\eta_i\eta_j$ from the formula (B.0.1). Using α, β and γ to denote external legs from each of the three branches of (Fig.B.4), we find the translated 3-point vertex contribute to the expansion coefficient of (B.0.3) as

$$\sum_{\alpha,\beta,\gamma} c\,\hat{\alpha}^{\frac{1}{2}}\hat{\beta}^{\frac{1}{2}}\eta_\alpha\eta_\beta + a\,\hat{\beta}^{\frac{1}{2}}\hat{\gamma}^{\frac{1}{2}}\eta_\beta\eta_\gamma + b\,\hat{\gamma}^{\frac{1}{2}}\hat{\alpha}^{\frac{1}{2}}\eta_\gamma\eta_\alpha \tag{B.0.17}$$

where a, b and c here stand for

$$a = \widehat{l+1} + \cdots + \hat{n} + \hat{1} \tag{B.0.18}$$

$$b = \hat{2} + \hat{3} + \cdots + \hat{k} \tag{B.0.19}$$

$$c = \widehat{k+1} + \cdots + \hat{l} \tag{B.0.20}$$

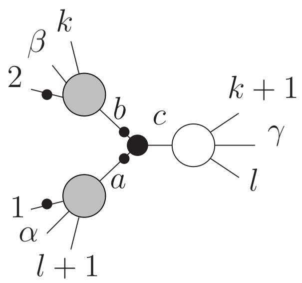


Figure B.4: Notation for the 3-point vertex expansion

The numerator of the formula (B.0.1) can accordingly be written as

$$\sum_{\alpha,\beta,\gamma} \langle\alpha\beta\rangle\, \eta_\alpha\eta_\beta + \langle\beta\gamma\rangle\, \eta_\beta\eta_\gamma + \langle\gamma\delta\rangle\, \eta_\gamma\eta_\alpha \tag{B.0.21}$$

Applying (B.0.11) this becomes

$$\frac{(12)}{\hat{1}\hat{2}}\,\frac{1}{c}\left(\sum_{\alpha,\beta,\gamma} c\,\hat{\alpha}^{\frac{1}{2}}\hat{\beta}^{\frac{1}{2}}\eta_\alpha\eta_\beta + a\,\hat{\beta}^{\frac{1}{2}}\hat{\gamma}^{\frac{1}{2}}\eta_\beta\eta_\gamma + b\,\hat{\gamma}^{\frac{1}{2}}\hat{\alpha}^{\frac{1}{2}}\eta_\gamma\eta_\alpha\right) \tag{B.0.22}$$

The ratio between the coefficients of the $\eta_\alpha\eta_\beta$ term, $\eta_\beta\eta_\gamma$ term and $\eta_\gamma\eta_\alpha$ term are the same as the ratio in (B.0.17).

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