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# STUDIES IN PARTITION THEORY

by

S. MANZUR HUSSAIN, M.A.

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY,

JULY, 1954.

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## DECLARATION.

The work in my thesis, except Section 4 which contains the unpublished results of Dr. Atkin and except where explicit reference is made, is entirely my own.


## ACKNOWLEDGMENTS.

I wish to express my gratitude to Dr. Atkin for his constant help and encouragement, and many useful suggestions during the writing of this thesis. I am also indebted to him for his kind permission to include in the thesis some of his unpublished results.

I also wish to thank Professor Burchnall and Dr. Atkin for their guidance in advanced studies in my first year of research.

## CONTENTS.

## INTRODUCTION.

1. Ramanujan (1) discovered, and later proved, three beautiful arithmetical properties of $p(n)$ where $p(n)$ denotes the number of unrestricted partitions of a positive integer $n$. These are

$$p(5n+4) \equiv 0 \pmod 5 \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{1.3}$$

Later on Dyson (2) discovered empirically a remarkable combinatorial method of dividing the partitions of $5n+4$ and $7n+5$ into 5 and 7 equal classes respectively. He defined the rank of a partition as the largest part minus the number of parts and found that the partitions of $5n+4$ fell into 5 equally numerous classes according to the least positive residue mod 5 of their ranks. The case of $7n+5$ was similar, but the corresponding conjecture for $11n+6$ was false. These conjectures together with many other similar results were proved by Atkin and Swinnerton-Dyer (3).

We shall adopt throughout the notation of Atkin and Swinnerton-Dyer and refer to their Paper as (ASD).

The main object of this thesis is to give the values of the $r_{\ell c}(d)$ when $q = 11$ in a form similar to that of (ASD) for $q = 5$ and $q = 7$. These values were first obtained empirically; not by a direct examination of power series (which, in view of the results of Theorem 1, would hardly have been possible), but as a by-product of the investigation of certain congruences.

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In fact, we observed that (i)* each $R_{lc}(d)/P^3(o)$ is congruent (mod 11) to certain rational functions of the P(a), (ii) there are linear congruent relations between the $R_{lc}(d)$ for different values of d, (iii) each $R_{lc}(d)$ $(d \neq 6)$ is expressible (mod 11) in terms P(o), P(a), and the $R_{lc}(6)$, and (iv) there are several congruent relations between the various $R_{lc}(6)$ and the P(a). Some of these congruent relations appeared to be identities when verified by power series and on this assumption we obtained conjectural values for all the $R_{lc}(d)$, which we prove in Theorem 1.

In addition the thesis also contains two minor results. One is the replacement of the last parts of the proofs of Theorems 4 and 5 (ASD) by the solution of a set of simultaneous equations which we explain in §3; the other, the evaluation of three determinants of a special type corresponding to $q = 5$, $q = 7$, and $q = 11$. The first two determinants are evaluated in § 2 in quite a straightforward manner, but the one corresponding to $q = 11$ (evaluated in § 5) requires a more elaborate treatment.

The general plan of the thesis is as follows:-
In Section 2 we obtain the values of the two determinants corresponding to $q = 5$ and $q = 7$. Section 3 contains the solutions of two sets of simultaneous equations. In Section 4 we obtain preliminary results for $q = 11$ and its corresponding

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* $R_{lc}(d)$ is defined in §6.

determinant is evaluated in Section 5. Sections 6 and 7 deal respectively with rank congruences and rank identities for $q = 11$.

We may note here that we have been obliged to use some letters with more than one significance. For example, $i$ occurs both as the current variable in $\prod(1 - x^i)$ in (5.1), (6.56) and in r, s, t, u, v, relations in (4.1), (4.4), (4.6), (4.8), (4.9), (4.10) and (4.11). In every case, however, it will be found that the duplicate usesare sufficiently distinct to cause no confusion.

## EVALUATION OF $\Delta_5$ AND $\Delta_7$.

2. In this section we obtain the values* of two determinants, $\Delta_5$ and $\Delta_7$, which arise out of the simultaneous equations (3.2) to (3.6) and (3.29) to 3.35) in §3. They correspond to q = 5 and q = 7 respectively. The determinant corresponding to q = 11 we evaluate in §5 by a different method. These determinants can be factorised by a method similar to that used for circulants.

Lemma 1. Suppose $\Delta_5$ is equal to

$$\begin{vmatrix} \frac{P(2)}{P(1)} & 0 & 0 & \frac{-x^5 P(1)}{P(2)} & -x^5 \\ -1 & \frac{P(2)}{P(1)} & 0 & 0 & \frac{-x^5 P(1)}{P(2)} \\ \frac{-P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} & 0 & 0 \\ 0 & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} & 0 \\ 0 & 0 & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} \end{vmatrix}.$$

Then
$$\Delta_5 = \frac{P^5(2)}{P^5(1)} - x^{10}\frac{P^5(1)}{P^5(2)} - 11x^5. \tag{2.1}$$

Denote $x^{-1}P(2)/P(1)$, $-1$, and $-xP(1)/P(2)$ by a, b, and c respectively, and let $A = a^2 + b^2 + c^2$, $B = ab + bc$, and $C = ac$. If now we multiply each row by $x^{-1}$ the determinant can be written as

-----------

* The values have in fact been obtained previously by Watson (4) from the forms in equations (2.2) and (2.8).

$$x^5 \begin{vmatrix} a & o & o & cx^3 & bx^4 \\ bx^{-1} & a & o & o & cx^3 \\ cx^{-2} & bx^{-1} & a & o & o \\ o & cx^{-2} & bx^{-1} & a & o \\ o & o & cx^{-2} & bx^{-1} & a \end{vmatrix}.$$

Multiplying the first, second, third, fourth and fifth rows by 1, $xw_\rho$, $x^2 w_\rho^2$, $x^3 w_\rho^3$, and $x^4 w_\rho^4$ respectively, and adding, it is found that $(a + bw_\rho + cw_\rho^2)$ is a common factor, where $w_\rho$ ($\rho = 1$ to 5) is a fifth root of unity. Thus $\Delta_5/x^5 = k \prod_{\rho=1}^{5} (a + bw_\rho + cw_\rho^2)$ where k is a constant, and on comparing the coefficients of $a^5$ we obtain $k = 1$. Hence

$$\begin{aligned} \Delta_5/x^5 &= \prod_{\rho=1}^{5} (a + bw_\rho + cw_\rho^2), \\ &= (a + b + c)(a + bw + cw^2) \\ &\quad (a + bw^2 + cw^4)(a + bw^3 + cw) \\ &\quad (a + bw^4 + cw^3), \end{aligned} \tag{2.2}$$

where $w$ ($\neq 1$) is a fifth root of unity.

Let now $\alpha = w + w^4$, and $\beta = w^2 + w^3$; we have

$$\alpha + \beta = -1, \tag{2.3}$$

$$\alpha\beta = -1, \tag{2.4}$$

$$\alpha^2 + \beta^2 = 3. \tag{2.5}$$

By combining the second and fifth and third and fourth factors in the right-hand side of (2.2) we obtain

$$\begin{aligned} \Delta_5/x^5 &= (a + b + c)(A + \alpha B + \beta C)(A + \beta B + \alpha C), \\ &= (a + b + c)(A^2 - B^2 - C^2 - AB - AC + 3\,BC), \\ &= (a + b + c)(a^4 + b^4 + c^4 - a^3 b - a^3 c + a^2 b^2 + a^2 c^2 + b^2 c^2 + \\ &\qquad + 2a^2 bc + 2\,ab\,c^2 - ab^3 - ac^3 - 3\,ab^2 c - b^3 c - \\ &\qquad - bc^3), \\ &= a^5 + b^5 + c^5 + 5\,a^2 bc^2 - 5ab^3 c, \end{aligned} \tag{2.6}$$

which is equivalent to Lemma 1.

Lemma 2. If $\Delta_7$ is equal to

$$\left|\begin{array}{ccccccc}
\frac{P(2)}{P(1)} & 0 & \frac{x^7 P(1)}{P(3)} & 0 & 0 & -x^7 & -\frac{x^7 P(3)}{P(2)} \\
-\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{x^7 P(1)}{P(3)} & 0 & 0 & -x^7 \\
-1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{x^7 P(1)}{P(3)} & 0 & 0 \\
0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{x^7 P(1)}{P(3)} & \\
0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{x^7 P(1)}{P(3)} \\
\frac{P(1)}{P(3)} & 0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 \\
0 & \frac{P(1)}{P(3)} & 0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)}
\end{array}\right|,$$

then
$$\Delta_7 = \frac{P^7(2)}{P^7(1)} - \frac{x^7 P^7(3)}{P^7(1)} + \frac{x^{35} P^7(1)}{P^7(3)} -$$
$$- 8x^{14} + 7\left[\frac{x^7 P^5(2)}{P^4(1)P(3)} - \frac{x^{28} P^5(1)}{P(2)P^4(3)} - \frac{x^7 P^5(3)}{P(1)P^4(2)}\right] +$$
$$+ 14\left[\frac{x^{21} P^3(1)}{P^2(2)P(3)} + \frac{x^{14} P^3(2)}{P(1)\cdot P^2(3)} - \frac{x^7 P^3(3)}{P^2(1)P(2)}\right]. \quad (2.7)$$

Denote $x^{-1}\cdot P(2)/P(1)$, $-P(3)/P(2)$, $-x$, and $x^4 P(1)/P(3)$ by a, b, c, and d respectively, and let $A = a^2 + b^2 + c^2 + d^2$, $B = ab + bc$, $C = ac + ad$, and $D = bd + cd$. If now we multiply each row by $x^{-1}$ the determinant can be written as

$$x^7\left|\begin{array}{ccccccc}
a & 0 & dx^2 & 0 & 0 & cx^5 & bx^6 \\
bx^{-1} & a & 0 & dx^2 & 0 & 0 & cx^5 \\
cx^{-2} & bx^{-1} & a & 0 & dx^2 & 0 & 0 \\
0 & cx^{-2} & bx^{-1} & a & 0 & dx^2 & 0 \\
0 & 0 & cx^{-2} & bx^{-1} & a & 0 & dx^2 \\
dx & 0 & 0 & cx^{-2} & bx^{-1} & a & 0 \\
0 & dx & 0 & 0 & cx^{-2} & bx^{-1} & a
\end{array}\right|.$$

Now by a similar process to that used in Lemma 1 we can show that

$$\begin{aligned}\Delta_7/x^7 &= \prod_{\rho=1}^{7} (a + bw_\rho + cw_\rho^2 + dw_\rho^5), \\ &= (a + b + c + d)(a + bw + cw^2 + dw^5) \\ &\quad (a + bw^2 + cw^4 + dw^3)(a + bw^3 + cw^6 + dw) \\ &\quad (a + bw^4 + cw + dw^6)(a + bw^5 + cw^3 + dw^4) \\ &\quad (a + bw^6 + cw^5 + dw^2). \qquad (2.8)\end{aligned}$$

where $w_\rho$ ( $\rho$ = 1 to 7) is a seventh root of unity and w is one of the imaginary roots.

Let now $\alpha = w + w^6$, $\beta = w^2 + w^5$, and $\gamma = w^3 + w^4$; we have

$$\Sigma\alpha = -1, \qquad (2.9)$$

$$\Sigma\alpha\beta = -2, \qquad (2.10)$$

$$\Sigma\alpha^2 = 5, \qquad (2.11)$$

$$\Sigma\alpha^3 = -4, \qquad (2.12)$$

$$\Sigma\alpha\beta^2 = -4, \qquad (2.13)$$

$$\Sigma\alpha\gamma^2 = 3, \qquad (2.14)$$

$$\alpha\beta\gamma = 1, \qquad (2.15)$$

where $\Sigma$ denotes the sum of three terms obtained by permuting the typical term in cyclic order.

By combining the second and seventh, third and sixth, and fourth and fifth factors in the right-hand side of (2.8) we obtain

$$
\begin{aligned}
\Delta_7/x^7 &= (a + b + c + d)(A + \alpha B + \beta C + \gamma D)(A + \beta B + \gamma C + \alpha D) \\
&\quad (A + \gamma B + \alpha C + \beta D), \\
&= (a + b + c + d)(A^3 + B^3 + C^3 + D^3 - AC^2 - A^2 B - A^2 D + 3\ ABC + \\
&\quad + 3\ ABD + 3\ ACD - 2\ AB^2 - 2\ AC^2 - 2\ AD^2 + 3B^2C + 3\ BD^2 + \\
&\quad + 3\ C^2 D - 4\ B^2 D - 4\ C^2 b - 4\ CD^2 - BCD), \\
&= a^7 + b^7 + c^7 + d^7 + 7abd(a^3 b + b^3 d + ad^3) - \\
&\quad - 7c^3 (a^3 b + b^3 d + ad^3) - 7c(a^5 d + bd^5 + ab^5) - \\
&\quad - 7\ a^2 b^2 cd^2 + 14\ abc^4 d + 14c^2 (a^3 d^2 + a^2 b^3 + b^2 d^3), \qquad (2.16)
\end{aligned}
$$

which is equivalent to Lemma 2.

We are indebted to Prof. Burchnall for suggesting an interesting matrix treatment of the determinants discussed above. We shall not only explain the cases when $q = 5$, and $q = 7$; but also the third case when $q = 11$.

Let $I_n$ be the unit matrix of the nth order, $E_n = \begin{bmatrix} 0 & 1_{n-1} \\ 0 & 0 \end{bmatrix}$, $F_n = \begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix}$, and $U_n^2 = F_n^2 + yE_n^{n-2}$; then, when $q = 5$, we have

$$
\begin{bmatrix}
a & 0 & 0 & by & cy \\
c & a & 0 & 0 & by \\
b & c & a & 0 & 0 \\
0 & b & c & a & 0 \\
0 & 0 & b & c & a
\end{bmatrix}
= aI_5 + byE_5^3 + cyE_5^4 + cF_5 + bF_5^2 = aI_5 + cU_5 + bU_5^2 . \qquad (2.17)
$$

Next, when $q = 7$, we have

$$\begin{bmatrix} a & 0 & by & 0 & 0 & cy & dy \\ d & a & 0 & by & 0 & 0 & cy \\ c & d & a & 0 & by & 0 & 0 \\ 0 & c & d & a & 0 & by & 0 \\ 0 & 0 & c & d & a & 0 & by \\ b & 0 & 0 & c & d & a & 0 \\ 0 & b & 0 & 0 & c & d & a \end{bmatrix} = aI_7 + byE_7^2 + cyE_7^5 + dyE_7^6 + dF_7 + cF_7^2 + bF_7^5 = aI_7 + dU_7 + cU_7^2 + bU_7^5 \tag{2.18}$$

Finally, when $q = 11$, then

$$\begin{bmatrix} a & 0 & 0 & 0 & by & 0 & cy & dy^2 & 0 & ey & fy \\ f & a & 0 & 0 & 0 & by & 0 & cy & dy^2 & 0 & ey \\ e & f & a & 0 & 0 & 0 & by & 0 & cy & dy^2 & 0 \\ 0 & e & f & a & 0 & 0 & 0 & by & 0 & cy & dy^2 \\ dy & 0 & e & f & a & 0 & 0 & 0 & by & 0 & cy \\ c & dy & 0 & e & f & a & 0 & 0 & 0 & by & 0 \\ 0 & c & dy & 0 & e & f & a & 0 & 0 & 0 & by \\ b & 0 & c & dy & 0 & e & f & a & 0 & 0 & 0 \\ 0 & b & 0 & c & dy & 0 & e & f & a & 0 & 0 \\ 0 & 0 & b & 0 & c & dy & 0 & e & f & a & 0 \\ 0 & 0 & 0 & b & 0 & c & dy & 0 & e & f & a \end{bmatrix}$$

$$= aI_{11} + byE_{11}^4 + cyE_{11}^6 + dy^2E_{11}^7 + eyE_{11}^4 + fyE_{11}^{10} + fF_{11} + eF_{11}^2 + dyF_{11}^4 + cF_{11}^5 + bF_{11}^7$$

$$= aI_{11} + fU_{11} + eU_{11}^2 + dyU_{11}^4 + cU_{11}^5 + bU_{11}^7 \tag{2.19}$$

If we replace $U_n$ by $x$ and $I_n$ by $I$ then we find that right-hand sides of (2.17) to (2.19) are equivalent to those right-hand sides of Lemma 6 (ASD) for $q = 5$, $7$ and $11$ respectively.

SOLUTION OF SIMULTANEOUS EQUATIONS.

3. We now proceed to describe an alternative approach to the proofs of Theorems 4 and 5 (ASD). The final stages of their proofs involve in effect a previous knowledge of the values of the $r_{\ell c}(d)$, obtained by empirical methods, and then showing that these are in fact the unique solutions of a set of simultaneous equations. In this section we replace the last parts of their proofs by solutions of the simultaneous equations using standard determinantal methods. In theory this method is of more general application, but in practice it is ineffective for $q > 7$ and in fact it could have hardly been successful in obtaining the results in § 7.

We start from the top of page 102 of their Paper and obtain all the results of their Theorem 4 but, for convenience, restrict ourselves to one set of results of the type $r_{23}(d)$ in their Theorem 5.

Theorem 4 (ASD) Part 1.

By Lemma 6, (2.13) and (6.10) (ASD) we have

$$S(o) + S(5) - S(2) - S(3) = P(o)\left[P(2)/P(1) - x - x^2 P(1)/P(2)\right] \times \sum_{d=0}^{4} r_{02}(d)\, x^d \,. \tag{3.1}$$

Equating the coefficients of $x^0$, $x$, $x^2$, $x^3$, and $x^4$ after substituting the values of S(o), S(2), S(3), and S(5) given in Table 1 (Page 71 ), we have

$$\frac{P(2)\ r_{o2}(o)}{P(1)} - \frac{yP(1) r_{o2}(3)}{P(2)} - \frac{y r_{o2}(4)}{1} = \frac{P(2)}{P(1)} + \frac{P(o)\ P^2(2)}{P^3(1)} - \frac{2yP(2)\ \xi(1,0)}{P(0)P(1)} + \frac{y^2 P(1)\ \xi(2,o)}{P(o)P(2)}\ , \qquad (3.2)$$

$$- r_{o2}(o) + \frac{P(2)\ r_{o2}(1)}{P(1)} - \frac{yP(1)\ r_{o2}(4)}{P(2)} = \frac{2y\ \xi(1,o)}{P(o)} + 1\ , \qquad (3.3)$$

$$\frac{-P(1) r_{o2}(o)}{P(2)} - r_{o2}(1) + \frac{P(2)\ r_{o2}(2)}{P(1)} = \frac{P(1)}{P(2)} - \frac{2P(o)}{P(1)} + \frac{2yP(1)\ \xi(1,o)}{P(o)P(2)}\ , \qquad (3.4)$$

$$\frac{-P(1)\ r_{o2}(1)}{P(2)} - r_{o2}(2) + \frac{P(2)\ r_{o2}(3)}{P(1)} = \frac{-P(o)}{P(2)} - \frac{yP(2)\xi(2,o)}{P(o)P(1)}\ , \qquad (3.5)$$

$$\frac{-P(1)\ r_{o2}(2)}{P(2)} - r_{o2}(3) + \frac{P(2)\ r_{o2}(4)}{P(1)} = \frac{y\ \xi(2,o)}{P(o)}\ . \qquad (3.6)$$

For convenience we write $r_{o2}(o) = \left[r'_{o2}(o) - 2y\ \xi(1,o)/P(o)\right]$ and $r_{o2}(3) = \left[r'_{o2}(3) - y\ \xi(2,o)/P(o)\right]$ in the above equations which can be rewritten as

$$\frac{P(2)\ r'_{o2}(o)}{P(1)} - \frac{yP(1) r'_{o2}(3)}{P(2)} - y r_{o2}(4) = \frac{-P(2)}{P(1)} + \frac{P(o)\ P^2(2)}{P^3(1)}, \qquad (3.7)$$

$$- r'_{o2}(o) + \frac{P(2)\ r_{o2}(1)}{P(1)} - \frac{yP(1)\ r_{o2}(4)}{P(2)} = 1, \qquad (3.8)$$

$$\frac{-P(1)\ r'_{o2}(o)}{P(2)} - r_{o2}(1) + \frac{P(2)\ r_{o2}(2)}{P(1)} = \frac{P(1)}{P(2)} - \frac{2P(o)}{P(1)}\ , \qquad (3.9)$$

$$\frac{-P(1)\ r_{o2}(1)}{P(2)} - r_{o2}(2) + \frac{P(2)\ r'_{o2}(3)}{P(1)} = \frac{-P(o)}{P(2)}\ , \qquad (3.10)$$

$$\frac{-P(1) r_{o2}(2)}{P(2)} - r'_{o2}(3) + \frac{P(2) r_{o2}(4)}{P(1)} = o. \qquad (3.11)$$

Adopting now the usual determinantal method of solving the simultaneous equations we see that $r_{o2}(2)$ is equal to

$$\begin{vmatrix} -\frac{P(2)}{P(1)}+\frac{P(o)P^2(2)}{P^3(1)} & \frac{P(2)}{P(1)} & o & \frac{-yP(1)}{P(2)} & -y \\ 1 & -1 & \frac{P(2)}{P(1)} & o & -\frac{yP(1)}{P(2)} \\ \frac{P(1)}{P(2)}-\frac{2P(o)}{P(1)} & -\frac{P(1)}{P(2)} & -1 & o & o \\ -\frac{P(o)}{P(2)} & o & -\frac{P(1)}{P(2)} & \frac{P(2)}{P(1)} & o \\ o & o & o & -1 & \frac{P(2)}{P(1)} \end{vmatrix} \Big/ \Delta_5 .$$

By adding the second column to the first column and taking out common

(i) $P(2)/P(1)$ from the first row,

(ii) $P(o)P(2)/P^2(1)$ from the first column, and

(iii) $P(2)/P(1)$ from the third column, we find that $r_{o2}(2)$ is equal to

$$\begin{vmatrix} 1 & 1 & o & \frac{-yP^2(1)}{P^2(2)} & -\frac{yP(1)}{P(2)} \\ o & -1 & 1 & o & -\frac{yP(1)}{P(2)} \\ -2 & -1 & -1 & o & o \\ -\frac{P^2(1)}{P^2(2)} & o & -\frac{P^2(1)}{P^2(2)} & \frac{P(2)}{P(1)} & o \\ o & o & o & -1 & \frac{P(2)}{P(1)} \end{vmatrix} \times \frac{P(o)P^3(2)}{P^4(1)\Delta_5} .$$

If we subtract the sums of the second and the third columns from the first column we clearly obtain* $r_{o2}(2) = o$.

Again we have $r_{o2}(4)$ is equal to

$$\begin{vmatrix} -\frac{P(2)}{P(1)}+\frac{P(o)P^2(2)}{P^3(1)} & \frac{P(2)}{P(1)} & o & o & -\frac{yP(1)}{P(2)} \\ 1 & -1 & \frac{P(2)}{P(1)} & o & o \\ \frac{P(1)}{P(2)}-\frac{2P(o)}{P(1)} & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} & o \\ -\frac{P(o)}{P(2)} & o & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} \\ o & o & o & -\frac{P(1)}{P(2)} & -1 \end{vmatrix} \times {}^1/_{\Delta_5}.$$

Adding the second column to the first column and repeating the steps (i) to (iii) we see that $r_{o2}(4)$ is equal to

$$\begin{vmatrix} 1 & 1 & o & o & -\frac{yP\ (1)}{P\ (2)} \\ o & -1 & 1 & o & o \\ -2 & -1 & -1 & \frac{P(2)}{P(1)} & o \\ -\frac{P^2(1)}{P^2(2)} & o & \frac{-P^2(1)}{P^2(2)} & -1 & \frac{P(2)}{P(1)} \\ o & o & o & -\frac{P(1)}{P(2)} & -1 \end{vmatrix} \times \frac{P(o)P^3(2)}{P^4(1)\Delta_5}.$$

If now we subtract the sums of the second and third columns from the first column we clearly find that

$$r_{o2}(4) = o. \tag{3.13}$$

---

* Of course in choosing to evaluate $r_{o2}(2)$ and $r_{o2}(4)$ we are effectively using Dyson's conjectures but not, however, the subsequent conjectures of Atkin and Swinnerton-Dyer.

Using (3.12) and (3.13) in the equations (3.2) to (3.6) we easily get the following results:

$$r_{02}(o) = P(o)P(2)/P^2(1) - 1 - 2y\,\Sigma(1,o)/P(o), \tag{3.14}$$

$$r_{02}(1) = P(o)/P(1), \tag{3.15}$$

$$r_{02}(3) = -y\,\Sigma(2,o)/P(o) \tag{3.16}$$

Part 2.

From Lemma 6, (2.13) and (6.10) (ASD) we have

$$S(1) + S(4) - S(2) - S(3) = P(o)\left[P(2)/P(1) - x - x^2 P(1)/P(2)\right] \times \sum_{d=0}^{4} r_{02}(d)\, x^d. \tag{3.17}$$

We substitute the values of S(1), S(2), S(3) and S(4) from Table 1 (Page 71 ), and equate the coefficients of $x^0$ to $x^4$ after replacing $r_{12}(o)$ and $r_{12}(3)$ by $\left[r'_{12}(o) + y\,\Sigma(1,o)/P(o)\right]$ and $\left[r'_{12}(3) - 2y\,\Sigma(2,o)/P(o)\right]$ respectively. We thus obtain

$$\frac{P(2)r'_{12}(o)}{P(1)} - \frac{yP(1)r'_{12}(3)}{P(2)} - yr_{12}(4) = \frac{yP(o)P^2(1)}{P^3(2)}, \tag{3.18}$$

$$-r'_{12}(o) + \frac{P(2)r_{12}(1)}{P(1)} - \frac{yP(1)r_{12}(4)}{P(2)} = o, \tag{3.19}$$

$$\frac{-P(1)r'_{12}(o)}{P(2)} - r_{12}(1) + \frac{P(2)r_{12}(2)}{P(1)} = \frac{P(o)}{P(1)}, \tag{3.20}$$

$$\frac{-P(1)r_{12}(1)}{P(2)} - r_{12}(2) + \frac{P(2)r'_{12}(3)}{P(1)} = \frac{-2P(o)}{P(2)}, \tag{3.21}$$

$$\frac{-P(1)r_{12}(2)}{P(2)} - r'_{12}(3) + \frac{P(2)r_{12}(4)}{P(1)} = o. \tag{3.22}$$

As before we have $r_{12}(1)$ is equal to

$$
- \left| \begin{array}{ccccc}
\dfrac{yP(o)P^2(1)}{P^3(2)} & \dfrac{P(2)}{P(1)} & o & \dfrac{-yP(1)}{P(2)} & -y \\[2ex]
o & -1 & o & o & -\dfrac{yP(1)}{P(2)} \\[2ex]
\dfrac{P(o)}{P(1)} & -\dfrac{P(1)}{P(2)} & \dfrac{P(2)}{P(1)} & o & o \\[2ex]
-\dfrac{2P(o)}{P(2)} & o & -1 & \dfrac{P(2)}{P(1)} & o \\[2ex]
o & o & \dfrac{-P(1)}{P(2)} & -1 & \dfrac{P(2)}{P(1)}
\end{array} \right| / \Delta_5 .
$$

Taking out common

(i) $P(2)/P(1)$ from the first row,

(ii) $P(1)/P(2)$ from the third row, and

(iii) $P(o)/P(2)$ from the first column, we obtain

$r_{12}(1)$ is equal to

$$
- \left| \begin{array}{ccccc}
\dfrac{yP^3(1)}{P^3(2)} & 1 & o & -\dfrac{yP^2(1)}{P^2(2)} & -\dfrac{yP(1)}{P(2)} \\[2ex]
o & -1 & o & o & -\dfrac{yP(1)}{P(2)} \\[2ex]
\dfrac{P^2(2)}{P^2(1)} & -1 & \dfrac{P^2(2)}{P^2(1)} & o & o \\[2ex]
-2 & o & -1 & \dfrac{P(2)}{P(1)} & o \\[2ex]
o & o & \dfrac{-P(1)}{P(2)} & -1 & \dfrac{P(2)}{P(1)}
\end{array} \right| \times \frac{P(o)}{P(2)\Delta_5} .
$$

If we multiply the fourth column by $-P(1)/P(2)$ and add to the third column then it is obvious that the new third column and the first column are identical. Hence

$$r_{12}(1) = o. \qquad (3.23)$$

Again we have $r_{12}(4)$ is equal to

$$\begin{vmatrix} \frac{yP(o)P^2(1)}{P^3(2)} & \frac{P(2)}{P(1)} & o & o & -\frac{yP(1)}{P(2)} \\ o & -1 & \frac{P(2)}{P(1)} & 0 & o \\ \frac{P(o)}{P(1)} & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} & o \\ -\frac{2P(o)}{P(2)} & o & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} \\ o & o & o & -\frac{P(1)}{P(2)} & -1 \end{vmatrix} \Big/ \Delta_5$$

Repeating the steps (i) to (iii) we see that $r_{12}(4)$ equals

$$\begin{vmatrix} \frac{yP^3(1)}{P^3(2)} & 1 & o & o & -\frac{yP^2(1)}{P^2(2)} \\ o & -1 & \frac{P(2)}{P(1)} & o & o \\ \frac{P^2(2)}{P^2(1)} & -1 & -\frac{P(2)}{P(1)} & \frac{P^2(2)}{P^2(1)} & o \\ -2 & o & -\frac{P(1)}{P(2)} & -1 & \frac{P(2)}{P(1)} \\ o & o & o & -\frac{P(1)}{P(2)} & -1 \end{vmatrix} \frac{P(o)}{P(2)\,\Delta_5}.$$

If we multiply the last column by $-P(1)/P(2)$ and add to the fourth column, then it is clear that the new fourth column and the first column are identical. Hence

$$r_{12}(4) = o. \tag{3.24}$$

By using (3.23) and 5.24) we can easily determine the remaining results given below:

$$r_{12}(o) = y\,\ell(1,o)/P(o) \tag{3.25}$$

$$r_{12}(2) = yP(o)/P(2) \tag{3.26}$$

$$r_{12}(3) = -P(o)P(1)/P^2(2) - 2y\,\ell(2,o)/P(o). \tag{3.27}$$

Theorem 5 (ASD) ($r_{23}(d)$ only).

We shall use the following* three equivalent relations without explicit reference:

$$\frac{P(3)}{P(2)} - \frac{P^2(2)}{P(1)P(3)} + \frac{yP^2(1)}{P^2(3)} = o ,$$

$$\frac{P^2(3)}{P^2(2)} - \frac{P(2)}{P(1)} + \frac{yP^2(1)}{P(2)P(3)} = o,$$

$$\frac{P^2(3)}{P(1)P(2)} - \frac{P^2(2)}{P^2(1)} + \frac{yP(1)}{P(3)} = o.$$

From Lemma 6, (2.13) and (6.10) (ASD) we have

$$S(2) + S(5) - S(3) - S(4) = P(o)\left[P(2)/P(1) - xP(3)/P(2) - x^2 + x^5P(1)/P(3)\right] \times \sum_{d=o}^{6} r_{23}(d)\ x^d . \tag{3.28}$$

For convenience we write $r_{23}(2) = \left[r'_{23}(2) - y\,\Sigma(3,o)/P(o)\right]$ and $r_{23}(6) = \left[r'_{23}(6) + 2y\,\Sigma(2,o)/P(o)\right]$ in the above equation. Now we substitute the values of S(2), S(3), S(4) and S(5) given in Table 2 (Page 12 ) and equate the coefficients of $x^o$ to $x^6$ obtaining the following set of simultaneous equations:

----------

* Cf. Lemma 4 (ASD).

$$\frac{P(2)r_{23}(0)}{P(1)} + \frac{yP(1)r'_{23}(2)}{P(3)} - yr_{23}(5) - \frac{yP(3)r'_{23}(6)}{P(2)} = \frac{-yP(0)P(1)}{P^2(2)}, \quad (3.29)$$

$$-\frac{P(3)r_{23}(0)}{P(2)} + \frac{P(2)r_{23}(1)}{P(1)} + \frac{yP(1)r_{23}(3)}{P(3)} - yr_{23}(\bar{6}) = 0, \quad (3.30)$$

$$-r_{23}(0) - \frac{P(3)r_{23}(1)}{P(2)} + \frac{P(2)r'_{23}(2)}{P(1)} + \frac{yP(1)r_{23}(4)}{P(3)} = \frac{-yP(0)P(1)}{P^2(3)}, \quad (3.31)$$

$$-r_{23}(1) - \frac{P(3)r'_{23}(2)}{P(2)} + \frac{P(2)r_{23}(3)}{P(1)} - \frac{yP(1)r_{23}(5)}{P(3)} = \frac{P(0)}{P(1)}, \quad (3.32)$$

$$-r'_{23}(2) - \frac{P(3)r_{23}(3)}{P(2)} + \frac{P(2)r_{23}(4)}{P(1)} + \frac{yP(1)r'_{23}(\bar{6})}{P(3)} = \frac{-2P(0)P(3)}{P^2(2)}, \quad (3.33)$$

$$\frac{P(1)r_{23}(0)}{P(3)} - r_{23}(3) - \frac{P(3)r_{23}(4)}{P(2)} + \frac{P(2)r_{23}(5)}{P(1)} = 0, \quad (3.34)$$

$$\frac{P(1)r_{23}(1)}{P(3)} - r_{23}(4) - \frac{P(3)r_{23}(5)}{P(2)} + \frac{P(2)r'_{23}(\bar{6})}{P(1)} = \frac{2P(0)}{P(3)}. \quad (3.35)$$

Now we have $r_{13}(0)$ is equal to

$$
\left|\begin{matrix}
-\frac{yP(0)P(1)}{P^2(2)} & 0 & \frac{yP(1)}{P(3)} & 0 & 0 & -y & -\frac{yP(3)}{P(2)} \\
0 & \frac{P(2)}{P(1)} & 0 & \frac{yP(1)}{P(3)} & 0 & 0 & -y \\
-\frac{P(0)P(1)}{P^2(3)} & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{yP(1)}{P(3)} & 0 & 0 \\
\frac{P(0)}{P(1)} & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{yP(1)}{P(3)} & 0 \\
-\frac{2P(0)P(3)}{P^2(2)} & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & \frac{yP(1)}{P(3)} \\
0 & 0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 \\
\frac{2P(0)}{P(3)} & \frac{P(1)}{P(3)} & 0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)}
\end{matrix}\right| \Big/ \Delta_7
$$

If we now (i) multiply the second column by $y$, and

(ii) multiply the last column by $P(2)/P(1)$ $\frac{P(1)}{P(3)}$ and add to the second and fourth columns respectively, and

(iii) take out common $-y$ and $P(0)/P(2)$ from the first row and first column respectively,

we obtain $r_{13}(0)$ is equal to

$$
- \begin{vmatrix}
\frac{P(1)}{P(2)} & \frac{P(3)}{P(1)} & -\frac{P(1)}{P(3)} & \frac{P(1)}{P(2)} & 0 & 1 \\
-\frac{yP(1)P(2)}{P^2(3)} & -y\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & y\frac{P(1)}{P(3)} & 0 \\
\frac{P(2)}{P(1)} & -y & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & y\frac{P(1)}{P(3)} \\
-2\frac{P(3)}{P(2)} & y\frac{P(2)}{P(3)} & -1 & -\frac{P(3)}{P(2)}+y\frac{P^2(1)}{P^2(3)} & \frac{P(2)}{P(1)} & 0 \\
0 & 0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} \\
\frac{2P(2)}{P(3)} & \frac{P^2(2)}{P^2(1)}+y\frac{P(1)}{P(3)} & 0 & \frac{P(2)}{P(3)} & -1 & -\frac{P(3)}{P(2)}
\end{vmatrix} \times \frac{yP(0)}{P(2)\Delta_7}.
$$

If we subtract the first column from the fourth column and multiply the fifth column by $P(2)/P(3)$ we find that the new fourth and the fifth columns are identical. Hence

$$r_{23}(0) = 0. \tag{3.36}$$

Again we have $r_{23}(1)$ equals

$$
- \begin{vmatrix}
-\frac{yP(0)P(1)}{P^2(2)} & \frac{P(2)}{P(1)} & y\frac{P(1)}{P(3)} & 0 & 0 & 0 & -\frac{yP(3)}{P(2)} \\
0 & -\frac{P(3)}{P(2)} & 0 & y\frac{P(1)}{P(3)} & 0 & 0 & -y \\
-\frac{yP(0)P(1)}{P^2(3)} & -1 & \frac{P(2)}{P(1)} & 0 & y\frac{P(1)}{P(3)} & 0 & 0 \\
\frac{P(0)}{P(1)} & 0 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & y\frac{P(1)}{P(3)} & 0 \\
-2\frac{P(0)P(3)}{P^2(2)} & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 & y\frac{P(1)}{P(3)} \\
0 & \frac{P(1)}{P(3)} & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & 0 \\
\frac{2P(0)}{P(3)} & 0 & 0 & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)}
\end{vmatrix} \Big/ \Delta_7
$$

If we

(i) multiply the second column by y,

(ii) multiply the last column by $-P(3)/P(2)$ and $P(1)/P(3)$ and add to the second and fourth columns respectively, and

(iii) take out common $P(o)/P(2)$ and y from the first column and first row respectively,

we obtain $r_{23}(1)$ is equal to

$$
-\begin{vmatrix}
-\frac{P(1)}{P(2)} & \frac{P(2)}{P(1)}+\frac{P^2(3)}{P^2(2)} & \frac{P(1)}{P(3)} & -\frac{P(1)}{P(2)} & o & -1 \\
-\frac{yP(1)P(2)}{P^2(3)} & -y & \frac{P(2)}{P(1)} & o & \frac{yP(1)}{P(3)} & o \\
\frac{P(2)}{P(1)} & o & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & o & y\frac{P(1)}{P(3)} \\
-\frac{2P(3)}{P(2)} & -y\frac{P(1)}{P(2)} & -1 & -\frac{P(3)}{P(2)}+y\frac{P^2(1)}{P^2(3)} & \frac{P(2)}{P(1)} & o \\
o & y\frac{P(1)}{P(3)} & o & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} \\
2\frac{P(2)}{P(3)} & -\frac{P(3)}{P(1)} & o & \frac{P(2)}{P(3)} & -1 & -\frac{P(3)}{P(2)}
\end{vmatrix} \times \frac{yP(o)}{P(2)\Delta_7}.
$$

Now subtracting the first column from the fourth column and multiplying the fifth column by $P(2)/P(3)$ we find that the new columns thus obtained are identical. Hence

$$r_{23}(1) = o. \tag{3.37}$$

Also we have $r'_{23}(2)$ is equal to

$$\left|\begin{matrix} -\frac{yP(o)P(1)}{P^2(2)} & \frac{P(2)}{P(1)} & o & o & o & -y & -\frac{yP(3)}{P(2)} \\ o & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & y\frac{P(1)}{P(3)} & o & o & -y \\ -\frac{yP(o)P(1)}{P^2(3)} & -1 & -\frac{P(3)}{P(2)} & o & y\frac{P(1)}{P(3)} & o & o \\ \frac{P(o)}{P(1)} & o & -1 & \frac{P(2)}{P(1)} & o & y\frac{P(1)}{P(3)} & o \\ -\frac{2P(o)P(3)}{P^2(2)} & o & o & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & o & \frac{yP(1)}{P(3)} \\ o & \frac{P(1)}{P(3)} & o & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} & o \\ \frac{2P(o)}{P(3)} & o & \frac{P(1)}{P(3)} & o & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} \end{matrix}\right| \Big/ \Delta_7 .$$

If we now

(i) multiply the second and third columns by y,

(ii) multiply the last column by $-P(3)/P(2)$, $P(2)/P(1)$, and $\frac{P(1)}{P(3)}$ and add to the second, third, and fourth columns respectively, and

(iii) take out common $-y$ and $P(o)/P(2)$ from the first row and first column respectively,

we obtain $r'_{23}(2)$ =

$$P(0)\begin{vmatrix} \frac{P(1)}{P(2)} & -\frac{P(2)}{P(1)}-\frac{P^2(3)}{P^2(2)} & \frac{P(3)}{P(1)} & \frac{P(1)}{P(2)} & 0 & 1 \\ -y\frac{P(1)P(2)}{P^2(3)} & -y & -y\frac{P(3)}{P(2)} & 0 & y\frac{P(1)}{P(3)} & 0 \\ \frac{P(2)}{P(1)} & 0 & -y & \frac{P(2)}{P(1)} & 0 & y\frac{P(1)}{P(3)} \\ -2\frac{P(3)}{P(2)} & -\frac{yP(1)}{P(2)} & \frac{yP(2)}{P(3)} & -\frac{P(3)}{P(2)}+\frac{yP^2(1)}{P^2(3)} & \frac{P(2)}{P(1)} & 0 \\ 0 & \frac{yP(1)}{P(3)} & 0 & -1 & -\frac{P(3)}{P(2)} & \frac{P(2)}{P(1)} \\ 2\frac{P(2)}{P(3)} & -\frac{P(3)}{P(1)} & y\frac{P(1)}{P(3)}+\frac{P^2(2)}{P^2(1)} & \frac{P(2)}{P(3)} & -1 & -\frac{P(3)}{P(2)} \end{vmatrix} \times {}^1/P(2)\Delta,$$

If we subtract the first column from the fourth column and multiply the fifth column by $P(2)/P(3)$ we find that the new columns thus obtained are identical. Hence $r'_{23}(2) = 0$. (3.38)

By using (3.36), (3.37) and (3.38) we can easily obtain the remaining results given below:

$$r_{23}(3) = P(0)/P(2) \tag{3.39}$$

$$r_{23}(4) = -P(0)/P(3) \tag{3.40}$$

$$r_{23}(5) = 0 \tag{3.41}$$

$$r'_{23}(6) = P(0)P(1)/P(2)P(3). \tag{3.42}$$

## PRELIMINARY RESULTS FOR q = 11.

4. It is not practicable to use the P(a) themselves directly in our work with q = 11; instead we introduce certain rational functions of the P(2) (r, s, t, u, v, $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\lambda$ and $\mu$ below) which we regard as fundamental, and develop the relations between the P(a) for q = 11 systematically in terms of these new functions.

We define

$$r = -y^2 P(1)/P(3)P(5),$$
$$s = -yP(2)/P(1)P(5),$$
$$t = P(4)/P(1)P(2),$$
$$u = yP(3)/P(2)P(4),$$
$$v = yP(5)/P(3)P(4), \qquad (4.1)$$

and $\alpha = r/s$, $\beta = s/t$, $\gamma = t/u$, $\delta = u/v$, $\epsilon = v/r$. (4.2)

Further we write

$$\lambda = \alpha\gamma + \beta\delta + \gamma\epsilon + \delta\alpha + \epsilon\beta,$$
$$\mu = \alpha^2\gamma + \beta^2\delta + \gamma^2\epsilon + \delta^2\alpha + \epsilon^2\beta. \qquad (4.3)$$

From (4.6) to (4.10) and Lemma 8 (ASD) we have

$$t - s = su/r = P^2(3)/P^2(1)P(4) = (2g(1)-g(2)+1)/P^2(o),$$
$$u - t = tv/s = -P^2(5)/P^2(2)P(3) = (g(4)-2g(2)-1)/P^2(o),$$
$$v - u = ur/t = -y^3 P^2(1)/P^2(4)P(5) = (2g(4) + g(3))/P^2(o),$$
$$r - v = vs/u = -yP^2(2)/P(1)\cdot P^2(3) = (2g(3) + g(5))/P^2(o),$$
$$s - r = rt/v = -yp^2(4)/P(2)\cdot P^2(5) = (2g(5) + g(1))/P^2(o), \qquad (4.4)$$

It is clear that any equation derived from (4.4) in r, s, t, u, and v remains valid if these quantities are permuted cyclically. Accordingly we shall usually exhibit one equation for each set of five symmetrical equations. The same remarks apply to $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$, and in connection with both sets of quantities we shall use the symbol $\Sigma$ to denote a sum of five terms obtained by permuting the typical term cyclically. Thus $\lambda = \Sigma\alpha\gamma$ and $\mu = \Sigma\alpha^2\gamma$.

The following equations (each typical of a set of five) are immediate deductions from (4.4):

$$r/s + t/u + u/r = o,$$

$$s/u + u/r + r/s + s/r = o,$$

$$rt/s - r - u = o,$$

$$rs/t - r + s + v = o. \tag{4.5}$$

and we have the single equations

$$\Sigma\, rs = o, \tag{4.6}$$

$$g(1) + g(3) + g(4) + g(5) - g(2) = o. \tag{4.7}$$

and the congruence

$$4r - 2s + t + 5u + 3v = 1/P^2(o) \pmod{11}. \tag{4.8}$$

We also give here, for convenience, some further relations; (4.9) are required in §6,
(4.10) in §7, and
(4.11) in both §6 and §7.

$$rs/v = -r - s - su/v,$$

$$rs/u = -s + rt/u,$$

$$rv/u = r - v - rt/u,$$

$$r^2/s = -u - rt/u,$$

$$r^2/t = -v - rt/u,$$

$$r^2/v = r - s + rt/u,$$

$$rsu/tv = s - su/v,$$

$$rtu/v^2 = -s - u + su/v,$$

$$rsu/t^2 = -s - u - uv/t, \tag{4.9}$$

$$r^2 s = -r^2 v - rsu - uvs + vrt,$$

$$r^2 t = -r^2 v - uvs + vrt,$$

$$r^2 u = -tur + vrt,$$

$$stu = -rsu + stv, \tag{4.10}$$

$$3g(2) + g(5) + 1 = P^2(o) \times (-r + s + t - 2u + v),$$

$$g(1) - g(2) - 2g(5) + 1 = P^2(o) \times (r - 2s + t),$$

$$-2g(1) + g(5) - g(3) - 2 = P^2(o) \times (-r + 2s - 2t + u),$$

$$g(1) + 2g(3) + g(4) + 1 = P^2(o) \times (r - s + t - u),$$

$$-g(3) - 2g(4) = P^2(o) \times (u - v). \tag{4.11}$$

It can easily be seen that of the relations between $r$, $s$, $t$, $u$ and $v$ in (4.4) only three are independent.

In terms of $\alpha$, $\beta$, $\gamma$, $\delta$, and $\epsilon$, the relations between $r$, $s$, $t$, $u$, $v$ in (4.4) and (4.5) become

$$\alpha + \delta + \beta\gamma = o$$

$$1/\alpha = 1 - \beta - \epsilon = 1 + \gamma\delta . \tag{4.12}$$

Adding the five equations typified by $\alpha + \delta + \beta\gamma = o$ we have

$$2\Sigma\alpha + \Sigma\alpha\beta = o.$$

Similarly from $1 = \alpha - \alpha\beta - \alpha\epsilon$, we obtain

$$\Sigma\alpha - 2\Sigma\alpha\beta = 5.$$

Hence $\Sigma\alpha = 1,$ (4.12.1)

and $\Sigma\alpha\beta = -2.$ (4.12.2)

Also $\Sigma\alpha\beta + \Sigma\alpha\gamma = \lambda - 2.$ (4.12.3)

Again $\beta^2\gamma = -\alpha\beta - \beta\delta ,$

so that $\Sigma\beta^2\gamma = -\Sigma\alpha\beta - \Sigma\beta\delta,$

whence $\Sigma\alpha^2\beta = 2 - \lambda .$ (4.12.4)

Similarly $\qquad \Sigma\alpha^2\epsilon \quad = \quad 2 - \lambda \quad .$ (4.12.5)

Also $\qquad \alpha \quad = \alpha^2 - \alpha^2\beta - \alpha^2\epsilon \quad ,$

so that $\qquad \Sigma\alpha \quad = \Sigma\alpha^2 - \Sigma\alpha^2\beta - \Sigma\alpha^2\epsilon \quad .$

By (4.12.1), 4.12.4), and (4.12.5), we have

$$\Sigma\alpha^2 \quad = \quad 5 - 2\lambda \, . \tag{4.12.6}$$

Further we have $\alpha^2 + \alpha\delta \quad + \alpha\beta\gamma \quad = \quad o,$

so that $\quad \Sigma\alpha^2 + \Sigma\alpha\gamma \quad + \Sigma\alpha\beta\gamma \quad = o.$

Hence $\qquad \Sigma\alpha\beta\gamma \qquad = \lambda \quad - 5.$ (4.12.7)

Also $\qquad 1 \quad = \quad \alpha \quad + \alpha\gamma\delta \, ,$

so that $\qquad 5 \quad = \quad \Sigma\alpha \quad + \Sigma\alpha\gamma\delta,$

whence $\quad \Sigma\alpha\gamma\delta \quad = \quad 4.$ (4.12.8)

Adding (4.12.7) and (4.12.8) we obtain

$$\Sigma\alpha\beta\gamma + \Sigma\alpha\gamma\delta \quad = \quad \lambda - 1. \tag{4.12.9}$$

Again $\qquad \beta \quad = \alpha\beta \quad + \alpha\beta\gamma\delta \, ,$

so that $\qquad \Sigma\alpha \quad = \Sigma\alpha\beta \quad + \Sigma\alpha\beta\gamma\delta,$

and hence $\quad \Sigma\alpha\beta\gamma\delta \quad = \quad 3.$ (4.12.10)

Finally $\quad \alpha\beta\gamma\delta\epsilon \quad = \quad 1.$ (4.12.11)

From (4.12.1), (4.12.3), (4.12.9), 4.12.10), and (4.12.11) we see that $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$ are the roots of the equation

$$z^5 - z^4 - 2z^3 + z^2 + 3z - 1 + \lambda(z^3 - z^2) = o. \tag{4.13}$$

There are four independent relations between $\alpha$, $\beta$, $\gamma$ $\delta$, and $\epsilon$; three derived from (4.4), and $\alpha\beta\gamma\delta\epsilon = 1$. We can express any three of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$, rationally in terms of the other two, which are themselves connected by a symmetrical rational equation typified by one of

$$\alpha\gamma(\alpha+\gamma) - (\alpha+\gamma) + 1 = 0,$$

$$(\alpha\beta)^2 - \alpha\beta(\alpha+\beta) + 2\alpha\beta - (\alpha+\beta) = 0. \tag{4.14}$$

We now prove that any cyclically symmetric function of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$, can be expressed rationally in terms of $\lambda$ and $\mu$. In fact we have

<u>Lemma 3. Any expression of the form $\sum \alpha^l \beta^m \gamma^n \delta^p \epsilon^q$, where l, m, n, p and q are positive, negative or zero integers, is equal to</u>

$$\underline{Q_1(\lambda) + \mu Q_2(\lambda),}$$

<u>where $Q_1(\lambda)$ and $Q_2(\lambda)$ are polynomials in $\lambda$ with integral coefficients</u>.

Since $\alpha\beta\gamma\delta\epsilon = 1$, we may clearly suppose without loss of generality that l, m, n, p and q, are greater than or equal to zero. Assume now that the Lemma is true for all values of l, m, n, p and q with $l+m+n+p+q \leq k$ where $k \geq 1$ and consider any $\sum \alpha^l \beta^m \gamma^n \delta^p \epsilon^q$ with $l+m+n+p+q = k+1$. If any two of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$, adjacent in the cyclic order, have non-zero indices, we can express $\sum \alpha^l \beta^m \gamma^n \delta^p \epsilon^q$ as a linear combination of similar expressions with $l+m+n+p+q \leq k$ by using $\beta\gamma = -\alpha-\delta$ and the other four similar equations; and so by the induction hypothesis it is equal to $Q_1(\lambda) + \mu Q_2(\lambda)$.

Next, $\sum \alpha^l$ can be expressed as $Q_1(\lambda)$, by ordinary equation theory applied to (4.13). Finally consider $\sum \alpha^l \gamma^n$ where $l \geq 2$, $n \geq 2$. Since $\lambda \sum \alpha^{l-1} \gamma^{n-1}$ is equal to plus a linear combination of cyclically symmetric sums in $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, <u>in which adjacent letters occur</u>, we have $\Sigma\alpha^{l}\gamma^{n} = Q_1'(\lambda) + \mu Q_2'(\lambda) + \lambda[Q_1''(\lambda) + \mu Q_2''(\lambda)]$,

$$= Q_1(\lambda) + \mu Q_2(\lambda).$$

Moreover $\Sigma\alpha^2\gamma^2 = \lambda^2 - 4\lambda + 2$, $\Sigma\alpha^2\gamma = \mu$, $\Sigma\alpha\gamma^2 = -\mu - 3$, $\Sigma\alpha\gamma = \lambda$. Thus if the Lemma is true for $l+m+n+p+q \leq k$, it is true for all $l$, $m$, $n$, $p$ and $q$ with $l+m+n+p+q = k+1$. But the Lemma is clearly true for $k = 1$, and hence it is true for all values of $k$ by the strong form of mathematical induction.

In particular $\mu^2$ can be expressed in terms of $\mu$ and powers of $\lambda$; in fact

$$\mu^2 + 3\mu + \lambda^3 - 8\lambda^2 + 11\lambda + 34 = 0; \qquad (4.15)$$

which we now prove. We have

$$(\Sigma\alpha^2\gamma^2)(\Sigma\alpha\gamma) = \lambda^3 - 4\lambda^2 + 2\lambda . \qquad (4.15.1)$$

But the left-hand side of (4.15.1) is equal to

$$\Sigma\alpha^3\gamma^3 + \Sigma\alpha^2\beta\gamma^2\delta + \Sigma\alpha^2\gamma^3\epsilon + \Sigma\alpha^3\gamma^2\delta + \Sigma\alpha^2\gamma^2\beta\epsilon$$

$$= \Sigma\alpha^3\gamma^3 + 2\lambda^2 - 4\lambda - 12; \text{ since}$$

$$\Sigma\alpha^2\beta\gamma^2\delta = \Sigma\frac{\alpha\gamma}{\epsilon} = \Sigma\alpha\gamma(1-\alpha-\delta) = \Sigma\alpha\gamma - \Sigma\alpha^2\gamma - \Sigma\alpha\gamma\delta$$

$$= \lambda - \mu - 4,$$

$$\Sigma\alpha^2\gamma^3\epsilon = \Sigma\alpha\gamma^3(-1+1/\gamma),$$

$$= \Sigma\alpha^2\delta - \Sigma\alpha^3\delta,$$

$$= \lambda^2 - 3\lambda - \mu - 5,$$

$$\Sigma\alpha^3\gamma^2\delta = \Sigma\alpha^3\gamma(-1+1/\alpha),$$

$$= \Sigma\alpha^2\gamma - \Sigma\alpha^3\gamma$$

$$= \lambda^2 - 3\lambda + \mu - 2, \text{ and}$$

$$\Sigma\alpha^2\beta\gamma^2\epsilon = \Sigma\alpha\gamma(1-\epsilon-\gamma),$$

$$= \Sigma\alpha\gamma - \Sigma\alpha\gamma\epsilon - \Sigma\alpha\gamma^2,$$

$$= \lambda + \mu - 1. \qquad (4.15.2)$$

Hence $\Sigma\alpha^3\gamma^3 = \lambda^3 - 6\lambda^2 + 6\lambda + 12.$

Similarly we obtain from $(\Sigma\alpha^2\gamma)(\Sigma\alpha\gamma^2) = -\mu(3+\mu)$ the relation

$$\Sigma\alpha^3\gamma^3 = -\mu(3+\mu) + 2\lambda^2 - 5\lambda - 22. \qquad (4.15.3)$$

Now equating the right-hand sides of (4.15.2 and 4.15.3) we obtain the required relation.

In our application of Lemma 3 we require to express some function F(y), known to be a linear combination of sums $\Sigma\alpha^l\beta^m\gamma^n\delta^p\epsilon^q$, in terms of $\lambda$ and $\mu$. Since the lowest powers of y in the expansions of $\lambda$ and $\mu$ are $y^{-2}$ and $y^{-3}$ respectively, we assume a form $C_p\lambda^p + C_{p-1}\lambda^{p-1} + \dots\dots\dots + C_0 + \mu(C'_{p-2}\lambda^{p-2} + C'_{p-3}\lambda^{p-3} + \dots + C'_0)$ if $y^{2p}$. F(y) is a Taylor series, and $C_p\lambda^p + C_{p-1}\lambda^{p-1} + \dots\dots\dots + C_0 + \mu(C'_{p-1}\lambda^{p-1} + C'_{p-2}\lambda^{p-2} + \dots + C'_0)$ if $y^{2p+1}$ F(y) is a Taylor series. The values of the constants $C_i$ and $C'_i$ are then obtained by equating coefficients in the expansions of F(y), $\lambda^j$ and $\mu\lambda^j$ in the powers of y.

We may note here that in future L.H.S. and R.H.S. will be used for left-hand and right-hand sides respectively.

## EVALUATION OF $\Delta_{11}$.

5. We now write

$$f(z) = \prod (1 - z^{i}).$$

$$\text{Thus } f(y) = P(o)P(1)P(2)P(3)P(4)P(5),$$

$$f(y^{11}) = P(o),$$

$$1/f(x) = \sum p(n)x^{n}. \tag{5.1}$$

We also write

$$a = -x^{-4}P(2)/P(1), \quad b = x^{-5}P(4)/P(2), \quad c = x^{2}P(3)/P(4),$$

$$d = -x^{-3}\frac{P(5)}{P(3)}, \text{ and } e = -x^{10}P(1)/P(5). \tag{5.2}$$

It is easily verified that

$$a^{11} = -\alpha^{-5}\beta^{2}\gamma\delta^{-2}, \quad b^{11} = -\beta^{-5}\gamma^{2}\delta\epsilon^{-2}, \quad c^{11} = -\gamma^{-5}\delta^{2}\epsilon\alpha^{-2},$$

$$d^{11} = -\delta^{-5}\epsilon^{2}\alpha\beta^{-2}, \text{ and } e^{11} = -\epsilon^{-5}\alpha^{2}\beta\gamma^{-2} \tag{5.3}$$

and we also have, by Lemma 6 (ASD) when q = 11

$$x^{-5}f(x)/f(y^{11}) = (a + b + c + d + e + 1). \tag{5.4}$$

In (5.4) we replace x by $xw_\rho$ where $w_\rho$ ( $\rho = 1$ to 11) is one of the eleventh roots of unity and multiply the resulting eleven equations, obtaining

$$y^{-5}f^{12}(y)/f^{12}(y^{11}) = \prod_{\rho=1}^{11}(aw_\rho^{-4} + bw_\rho^{-5} + cw_\rho^{2} + dw^{-3} + ew_\rho^{10} + 1), \tag{5.5}$$

since $\prod_{\rho=1}^{11}(1-x^{\sigma}w_\rho^{\sigma}) = (1-x^{11\sigma})$, when $\sigma \not\equiv o \pmod{11}$, and
$\prod_{\rho=1}^{11}(1-x^{\sigma}w_\rho^{\sigma}) = (1-x^{\sigma})^{11}$ when $\sigma \equiv o \pmod{11}$.

Now as $w_\rho$ runs through the eleventh roots of unity, so does $w_\rho^{4}$, so that the product on the right-hand side of (5.5) is also equal to $\prod_{\rho=1}^{11}(aw_\rho^{-5} + bw_\rho^{2} + cw_\rho^{-3} + dw_\rho^{10} + ew_\rho^{-4} + 1)$, and is thus unchanged if a, b, c, d and e are interchanged cyclically. The product is thus a linear combination of terms $\Sigma\, a^{l} b^{m} c^{n} d^{p} e^{q}$ , denoting as in Section 4 a sum of five terms obtained by permuting the typical term cyclically. Moreover, considering the left-hand side of (5.5), such terms as occur can only involve x in terms of $y = x^{11}$. Thus if $a^{l} \times b^{m} \times c^{n} \times d^{p} \times e^{q}$ occurs we must have

$$-4l - 5m + 2n - 3p + 10q \equiv 0 \pmod{11}. \qquad (5.6)$$

Hence

$$-(a^{l}\, b^{m}\, c^{n}\, d^{p}\, e^{q})^{11} = (\alpha\beta\gamma\delta\epsilon)^{l-5m+4n-4p+8q} \times \alpha^{-5l-2n+p+2q} \times \beta^{-5m-2p+q+2l} \times \gamma^{-5n-2q+l+2m} \times \delta^{-5p-2l+m+2n} \times \epsilon^{-5q-2m+n+2p},$$

where the indices of $\alpha$, $\beta$, $\gamma$, $\delta$, and $\epsilon$ on the right-hand side are multiples of 11 by (5.6). Thus every term occurring in the right-hand side of (5.5) is of the form $\alpha^{l'} \beta^{m'} \gamma^{n'} \delta^{p'} \epsilon^{q'}$ where $l'$, $m'$, $n'$, $p'$ and $q'$ are positive, negative integers or zero, and such terms occur in symmetrical sets of five terms each. Now it is easily seen that

$$y^{-5} f^{12}(y)/f^{12}(y^{11}) = (1 - 12y + 54y^{2} - 88y^{3} - 99y^{4} + 540y^{5} \ldots \ldots\ldots)/y^{5},$$

$$\lambda = -(1 + 2y + y^{2} + \ldots\ldots\ldots)/y^{2},$$

$$\lambda^{2} = (1 + 4y + 6y^{2} + 14y^{3} + 37y^{4} + \ldots\ldots)/y^{4},$$

$$\mu = -(1 + 3y + 7y^{2} + 14y^{3} + \ldots\ldots)/y^{4},$$

and

$$\lambda\mu = (1 + 5y + 14y^{2} + 36y^{3} + 75y^{4} + 134y^{5} + \ldots)/y^{5}.$$

Hence by Lemma 3, we have

$$y^{-5} f^{12}(y)/f^{12}(y^{11})^{*} = A + B\lambda + C\mu + D\lambda^{2} + E\lambda\mu, \qquad (5.7)$$

where A, B, C, D and E are constants.

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* This is, in fact, as will be shown in Lemma 4, the value of $\Delta_{11}/\Delta$

On comparing the coefficients of powers of y on each side of (5.7) we get

$$
\begin{aligned}
E &= 1,\\
5E + D &= -12,\\
14E + 5D - C &= 54,\\
36E + 6E - 3C - B &= -88,\\
75E + 14D - 7C - 2B &= -99,\\
134E + 37D - 14C - B + A &= 540,
\end{aligned}
$$

whence we have $A = -131$, $B = 346$, $C = -108$, $D = -17$, and $E = 1$. Hence the left-hand side of (5.7) is equal to

$$\lambda\mu - 17\lambda^2 + 346\lambda - 108\mu - 131. \tag{5.8}$$

We can prove by a similar method that

$$a'' + b'' + c'' + d'' + e'' = \lambda\mu - 6\lambda^2 + 82\lambda - 9\mu - 297. \tag{5.9}$$

We may assume that $\Sigma a'' = A\lambda\mu + B\lambda^2 + C\lambda + D\mu + E$; since $\Sigma a'' = (1 - y - y^2 - 55y^3 - 110y^4 - 341y^5 - \ldots\ldots)/y^5$.

On comparing the coefficients of powers of y in both sides of (5.9) we have

$$
\begin{aligned}
A &= 1,\\
5A + B &= -1,\\
14A + 4B - D &= -1,\\
36A + 6B - C - 3D &= -55,\\
65A + 14B - 2C - 7D &= -110,\\
134A + 37B - C - 14D + E &= -341,
\end{aligned}
$$

whence we obtain $A - 1$, $B = -6$, $C = 82$, $D = -9$, and $E = -297$.

We are now in a position to evaluate $\Delta_{11}$ referred to in Section 2.

<u>Lemma 4</u>. If $\Delta_{11}^{*}/y^5$ is equivalent to

$$\begin{vmatrix}
b & 0 & 0 & 0 & cx^4 & 0 & x^6 & ex^7 & 0 & dx^9 & ax^{10} \\
ax^{-1} & b & 0 & 0 & 0 & cx^4 & 0 & x^6 & ex^7 & 0 & dx^9 \\
dx^{-2} & ax^{-1} & b & 0 & 0 & 0 & cx^4 & 0 & x^6 & ex^7 & 0 \\
0 & dx^{-2} & ax^{-1} & b & 0 & 0 & 0 & cx^4 & 0 & x^6 & ex^7 \\
ex^{-4} & 0 & dx^{-2} & ax^{-1} & b & 0 & 0 & 0 & cx^4 & 0 & x^6 \\
x^{-5} & ex^{-4} & 0 & dx^{-2} & ax^{-1} & b & 0 & 0 & 0 & cx^4 & 0 \\
0 & x^{-5} & ex^{-4} & 0 & dx^{-2} & ax^{-1} & b & 0 & 0 & 0 & cx^4 \\
cx^{-7} & 0 & x^{-5} & ex^{-4} & 0 & dx^{-2} & ax^{-1} & b & 0 & 0 & 0 \\
0 & cx^{-7} & 0 & x^{-5} & ex^{-4} & 0 & dx^{-2} & ax^{-1} & b & 0 & 0 \\
0 & 0 & cx^{-7} & 0 & x^{-5} & ex^{-4} & 0 & dx^{-2} & ax^{-1} & b & 0 \\
0 & 0 & 0 & cx^{-7} & 0 & x^{-5} & ex^{-4} & 0 & dx^{-2} & ax^{-1} & b
\end{vmatrix}$$

Then $\Delta_{11}/y^5 = \lambda\mu - 17\lambda^2 + 346\lambda - 108\mu - 131.$ (5.10)

By a similar process to that used in Lemma 1 we can show that

$$\begin{aligned}
\Delta_{11}/y^5 &= \prod_{\rho=1}^{11}(aw_\rho + b + cw_\rho^7 + dw_\rho^2 + ew_\rho^{15} + w_\rho^5), \\
&= \prod_{\rho=1}^{11}(aw_\rho^{-4} + bw_\rho^{-5} + cw_\rho^2 + dw_\rho^{-3} + ew_\rho^{10} + 1). \quad (5.11)
\end{aligned}$$

which is equal to the right-hand side of (5.5) and hence equal to the value obtained for this in (5.8). This is Lemma 4.

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* $\Delta_{11}$ has been derived from the L.H.S. of the simultaneous equations (7.56) to (7.66) and written in the above form after each row has been multiplied by $x^{-5}$. The a, b, c, d and e occurring are those defined in (5.2).

## RANK CONGRUENCES FOR q = 11.

6. We now introduce some further notation. We write

$$R_{01}(0) = P(1)(r_{01}(0) + 1 + 3y^2\Sigma(2,0)/P(0)),$$
$$R_{12}(0) = P(0)(r_{12}(0) - y^2\Sigma(2,0)/P(0)),$$
$$R_{34}(4) = P(2)(r_{34}(4) + y^3\Sigma(4,0)/P(0)),$$
$$R_{45}(4) = P(2)(r_{45}(4) - 2y^3\Sigma(4,0)/P(0)),$$
$$R_{01}(7) = P(3)(r_{01}(7) - y^3\Sigma(5,0)/P(0)),$$
$$R_{12}(7) = P(3)(r_{12}(7) + 2y^3\Sigma(5,0)/P(0)),$$
$$R_{23}(7) = P(3)(r_{23}(7) - y^3\Sigma(5,0)/P(0)),$$
$$R_{23}(9) = P(4)(r_{23}(9) + y^2\Sigma(3,0)/P(0)),$$
$$R_{34}(9) = P(4)(r_{34}(9) - 2y^3\Sigma(3,0)/P(0)),$$
$$R_{45}(9) = P(4)(r_{45}(9) + y^3\Sigma(3,0)/P(0)),$$
$$R_{12}(10) = P(5)(r_{12}(10) - 1/y - \Sigma(1,0)/P(0)),$$
$$R_{23}(10) = P(5)(r_{23}(10) + 2/y + 2\Sigma(1,0)/P(0)),$$
$$R_{34}(10) = P(5)(r_{34}(10) - 1/y - \Sigma(1,0)/P(0)),$$

and for all other values of b and c,

$$R_{bc}(0) = P(1)r_{bc}(0),$$
$$R_{bc}(1) = P(2)P(3)r_{bc}(1)/P(5),$$
$$R_{bc}(2) = P(1)P(4)r_{bc}(2)/P(3),$$
$$R_{bc}(3) = P(1)P(3)r_{bc}(3)/P(2),$$
$$R_{bc}(4) = P(2)r_{bc}(4),$$
$$R_{bc}(5) = P(2)P(5)r_{bc}(5)/P(4),$$
$$R_{bc}(6) = yr_{bc}(6),$$
$$R_{bc}(7) = P(3)r_{bc}(7),$$
$$R_{bc}(8) = P(4)P(5)r_{bc}(8)/yP(1),$$
$$R_{bc}(9) = P(4)r_{bc}(9),$$
$$R_{bc}(10) = P(5)r_{bc}(10).$$

Investigating the existence of possible congruences between the $R_{lc}(d)$, we observed that

(1) each $R_{lc}(d)/P^3(o)$ is congruent (mod 11) to certain rational functions of r, s, t, u and v (these congruences are given in Lemma 5),

(2) there are linear congruent relations between the $R_{lc}(d)$ for different values of d (given in (6.57)),

(3) each $R_{lc}(d)$ ($\neq 6$) is congruent (mod 11) to certain functions of P(o), r, s, t, u, v, and the $R_{lc}(6)$, and

(4) there are several congruent relations between the various $R_{lc}(6)$ and r, s, t, u v.

<u>Verification by power series of these congruences gave conclusive empirical evidence that some of them (given in (6.56)) were in fact identical relations</u>. We next found empirically five independent relations between the $R_{lc}(6)$ and r, s, t, u, v, when we obtained the values of the $R_{lc}(6)$ and consequently those of the $R_{lc}(d)$. That these conjectural values are in fact correct we prove in Theorem 1.

<u>Lemma 5</u>.

$$R_{01}(o) \equiv P^3(o)(-2r-2s+5t+3v+4ut/s) \pmod{11} \quad (6.1)$$

$$R_{01}(1) \equiv P^3(o)(3r+2s+t+u-3v-4su/v) \pmod{11} \quad (6.2)$$

$$R_{01}(2) \equiv P^3(o)(-t+4u+rt/u) \pmod{11} \quad (6.3)$$

$$R_{01}(3) \equiv P^3(o)(4r+2s+3t+5u+3v+2ut/s) \pmod{11} \quad (6.4)$$

$$R_{01}(4) \equiv P^3(o)(-r+2s+5u-2v) \pmod{11} \quad (6.5)$$

$$R_{01}(5) \equiv P^3(o)(4r+2u+2v-5uv/t) \pmod{11} \quad (6.6)$$

$$R_{01}(6) \equiv P^3(o)(-rt+2su-tv-2ur-2vs) \pmod{11} \quad (6.7)$$

$$R_{01}(7) \equiv P^3(o)(-4r-4s+2t-3u-5v-2su/v) \pmod{11} \quad (6.8)$$

$$R_{01}(8) \equiv P^3(o)(-4r+3t-5u-5v+vt/r) \pmod{11} \quad (6.9)$$

$$R_{01}(9) \equiv P^3(o)(-2r+t+u-3v) \pmod{11} \quad (6.10)$$

$$R_{01}(10) \equiv P^3(o)(-r-t-2u+5v) \pmod{11} \quad (6.11)$$

$$R_{12}(o) \equiv P^3(o)(2u+3r+3v-5t-5tu/s) \pmod{11} \quad (6.12)$$

$$R_{12}(1) \equiv P^3(o)(4r-3s-u-3v-2su/v) \pmod{11} \quad (6.13)$$

$$R_{12}(2) \equiv P^3(o)(-2r+t+u-3v) \pmod{11} \quad (6.14)$$

$$R_{12}(3) \equiv P^3(o)(2s-3t-4u-2tu/s) \pmod{11} \quad (6.15)$$

$$R_{12}(4) \equiv P^3(o)(-3r+2s+t-5u-5v) \pmod{11} \quad (6.16)$$

$$R_{12}(5) \equiv P^3(o)(-4r-2s+5u-5v-5uv/t) \pmod{11} \quad (6.17)$$

$$R_{12}(6) \equiv P^3(o)(4rt-4su+tv+2ur) \pmod{11} \quad (6.18)$$

$$R_{12}(7) \equiv P^3(o)(4r-t+3u+3v+4su/v) \pmod{11} \quad (6.19)$$

$$R_{12}(8) \equiv P^3(o)(2s-t-6u-tv/r) \pmod{11} \quad (6.20)$$

$$R_{12}(9) \equiv P^3(o)(-4r+2s+4u) \pmod{11} \quad (6.21)$$

$$R_{12}(10) \equiv P^3(o)(4r+2t-4u+5v+tv/r) \pmod{11} \quad (6.22)$$

$$R_{23}(o) \equiv P^3(o)(3r-2s+3u-3v) \pmod{11} \quad (6.23)$$

$$R_{23}(1) \equiv P^3(o)(4r+4s+u+3v) \pmod{11} \quad (6.24)$$

$$R_{23}(2) \equiv P^3(o)(-2r+2s-u+3v-rt/u) \pmod{11} \quad (6.25)$$

$$R_{23}(3) \equiv P^3(o)(-4r+3t-3u+2tu/s) \pmod{11} \quad (6.26)$$

$$R_{23}(4) \equiv P^3(o)(4r+2s-t+5u-v) \pmod{11} \quad (6.27)$$

$$R_{23}(5) \equiv P^3(o)(t-3u+2v+2uv/t) \pmod{11} \quad (6.28)$$

$$R_{23}(6) \equiv P^3(o)(4rt+4ur+5vs) \pmod{11} \quad (6.29)$$

$$R_{23}(7) \equiv P^3(o)(-2s+t+2u+5v-2su/v) \pmod{11} \quad (6.30)$$

$$R_{23}(8) \equiv P^3(o)(-3r+2t-5u-3v) \pmod{11} \quad (6.31)$$

$$R_{23}(9) \equiv P^3(o)(3r+2s-4u+4rt/u) \pmod{11} \quad (6.32)$$

$$R_{23}(10) \equiv P^{3}(o)(4r-2s-2t-u-2v-2tv/r)\ (\mathrm{mod}\ 11) \qquad (6.33)$$
$$R_{34}(0) \equiv P^{3}(o)(5r+2s-4u+5v)\ (\mathrm{mod}\ 11) \qquad (6.34)$$
$$R_{34}(1) \equiv P^{3}(o)(3r+5s+5u+3v+2su/v)\ (\mathrm{mod}\ 11) \qquad (6.35)$$
$$R_{34}(2) \equiv P^{3}(o)(-2s+u+3v+5rt/u)\ (\mathrm{mod}\ 11) \qquad (6.36)$$
$$R_{34}(3) \equiv P^{3}(o)(-3r-2s+4t-2u-3v+4tu/s)\ (\mathrm{mod}\ 11) \qquad (6.37)$$
$$R_{34}(4) \equiv P^{3}(o)(-r-2s+t+3u-5v-3uv/t)\ (\mathrm{mod}\ 11) \qquad (6.38)$$
$$R_{34}(5) \equiv P^{3}(o)(4r+2s-t+5u-v)\ (\mathrm{mod}\ 11) \qquad (6.39)$$
$$R_{34}(6) \equiv P^{3}(o)(-rt+2su+tv-5ur)\ (\mathrm{mod}\ 11) \qquad (6.40)$$
$$R_{34}(7) \equiv P^{3}(o)(-4r+2s+4u)\ (\mathrm{mod}\ 11) \qquad (6.41)$$
$$R_{34}(8) \equiv P^{3}(o)(-r+t+5u)\ (\mathrm{mod}\ 11) \qquad (6.42)$$
$$R_{34}(9) \equiv P^{3}(o)(r-2s+t-u+3v+3rt/u)\ (\mathrm{mod}\ 11) \qquad (6.43)$$
$$R_{34}(10) \equiv P^{3}(o)(3r+2s+2t+u+3v+tv/r)\ (\mathrm{mod}\ 11) \qquad (6.44)$$
$$R_{45}(0) \equiv P^{3}(o)(-r+2s+5u-2v)\ (\mathrm{mod}\ 11) \qquad (6.45)$$
$$R_{45}(1) \equiv P^{3}(o)(4r+3s+5v+2su/v)\ (\mathrm{mod}\ 11) \qquad (6.46)$$
$$R_{45}(2) \equiv P^{3}(o)(-r+5u+5v-rt/u)\ (\mathrm{mod}\ 11) \qquad (6.47)$$
$$R_{45}(3) \equiv P^{3}(o)(3r-5t+2u+3v-5tu/s)\ (\mathrm{mod}\ 11) \qquad (6.48)$$
$$R_{45}(4) \equiv P^{3}(o)(-3r-4s-3v-5uv/t)\ (\mathrm{mod}\ 11) \qquad (6.49)$$
$$R_{45}(5) \equiv P^{3}(o)(-4r+t+u-v-5uv/t)\ (\mathrm{mod}\ 11) \qquad (6.50)$$
$$R_{45}(6) \equiv P^{3}(o)(rt+2su-tv-5vs)\ (\mathrm{mod}\ 11) \qquad (6.51)$$
$$R_{45}(7) \equiv P^{3}(o)(4r+4s+u+3v)\ (\mathrm{mod}\ 11) \qquad (6.52)$$
$$R_{45}(8) \equiv P^{3}(o)(-3r-2s+5u+3v-tv/r)\ (\mathrm{mod}\ 11) \qquad (6.53)$$
$$R_{45}(9) \equiv P^{3}(o)(-r-2s-5u-3v+4tr/u)\ (\mathrm{mod}\ 11) \qquad (6.54)$$
$$R_{45}(10) \equiv P^{3}(o)(-3r-u+2v)\ (\mathrm{mod}\ 11) \qquad (6.55)$$

We shall prove (6.1) to (6.11); the other congruent relations fall into four more sets which can be proved similarly. We shall use the relations (4.5), (4.9) and

(4.11) in the proof without explicit mention.

From Lemma 6, (2.13) and (6.10) (ASD), we obtain

$$\sum_{d=0}^{10} r_{01}(d)x^d = \frac{1}{\prod(1-x^k)} \sum \frac{(x^0 + x^{11n}) - (x^n + x^{10n})}{1-x^{11n}} \times x^{\frac{n}{2}(3n+1)},$$

$$= \frac{1}{\prod(1-x^k)} \left[ 2\, S(o) - S(11) - S(1) \right].$$

Now substituting the values of S(o), S(1), and S(11) from Table 3 and using the relation $1/\pi\,(1-x^k) = \sum_{b=0}^{10} \phi(b)\, x^b$ together with the congruences in Theorem 3 (ASD) we have

$$\sum_{d=0}^{10} r_{01}(d)x^d \equiv -\frac{3y^2\sum(2,o)}{P(o)} - 1 + \frac{y^3\sum(5,o)x^7}{P(o)} +$$

$$+ P^3(o)\Big[ \frac{1}{P(1)} + \frac{P(5)x}{P(2)P(3)} + \frac{2P(3)x^2}{P(1)P(4)} +$$

$$+ \cancel{\frac{2P(3)x}{P(1)P(4)x}} + \frac{3P(2)x^3}{P(1)P(3)} + \frac{5x^4}{P(2)} +$$

$$+ \frac{7P(4)x^5}{P(2)P(5)} + \frac{4x^7}{P(3)} + \frac{6yP(1)x^8}{P(4)P(5)} +$$

$$+ \frac{8x^9}{P(4)} + \frac{9x^{10}}{P(5)}\Big] \Big[ (-r+s+t-2u+v) +$$

$$+ \frac{3yx}{P(3)} - \frac{3P(3)x^2}{P(1)P(2)} + \frac{P(3)P(4)x^3}{P(1)P(2)P(5)} -$$

$$- \frac{3yP(1)P(5)x^4}{P(2)P(3)P(4)} + \frac{3P(3)P(4)x^7}{P^2(2)P(5)} + \frac{yP(2)x^7}{P(3)P(5)} -$$

$$- \frac{y^2P(1)P(2)x^8}{P(4)\,P^2(5)} - \frac{x^9}{P(2)}\Big] \pmod{11} \tag{6.56}$$

On comparing the coefficients of $x^d$ (d = o to 10) in both sides of (6.56) we have

$$R_{01}(o) \equiv P^3(o)\Big[(-r+s+t-2u+v) - \frac{2yP(3)}{P(2)P(4)} - \frac{-3y^3P(1)P^2(2)}{P(3)P(4)P^2(5)} + \frac{5y^2P(1)}{P(3)P(5)} + \frac{15yP(1)P(3)P(4)}{P^3(2)\;P(5)} - \frac{12y^2P^2(1)P(5)}{P(2)P^2(3)P(4)} + \frac{6y^2P(1)P(3)}{P(2)P^2(5)} - \frac{24yP(3)}{P(2)P(4)} + \frac{27y^2\cdot P(1)}{P(3)P(5)}\Big] \pmod{11},$$

$$\equiv P^3(o)\;\big[(-r+s+t-2u+v) - 2u - 3rs/t - 5r + 15\,rtu/sv - 12rv/s - 6ru/v - 24u - 27r\big] \pmod{11},$$

$$\equiv P^3(o)\;\big[-24r - 2s + 16t - 8v + 15tu/s\big] \pmod{11}$$

$$\equiv P^3(o)\times\big[(-2r - 2s + 5t + 3v + 4tu/s)\big] \pmod{11}$$

$$R_{01}(1) \equiv P^3(o)\Big[(-r+s+t-2u+v) - \frac{5y^3P(1)P(2)P(3)}{P(4)P^3(5)} + \frac{7y^2P(2)P(4)}{P^3(5)} + \frac{21yP^2(3)\;P^2(4)}{P^2(2)P^3(5)} - \frac{18y^3P^2(1)}{P^2(4)P(5)} - \frac{19yP^2(3)}{P(1)P^2(5)}\Big] \pmod{11},$$

$$\equiv P^3(o)\big[(-r+s+t-2u+v) - 5rsu/tv + 7rs/v - 21\,rtu/v^2 + 18\,ru/t + 19\,su/v\big] \pmod{11},$$

$$\equiv P^3(o)\times(3r + 2s + t + u - 3v - 4su/v) \pmod{11}.$$

$$R_{01}(2) \equiv P^{3}(o)\Big[\ 2(-r + s + t - 2u + v) - \frac{3P(4)}{P(1)P(2)} + \frac{3yP(1)P(4)P(5)}{P(2)P^{3}(3)} - \frac{5yP(1)P(4)}{P^{2}(2)P(3)} - \frac{7y^{3}P^{2}(1)P(4)}{P(3)P^{3}(5)} - \frac{24y^{2}P^{2}(1)P(5)}{P(2)P^{2}(3)P(4)} + \frac{9yP^{2}(4)}{P(2)P^{2}(5)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[\ 2(-r + s + t - 2u + v) - 3t + 3\,rtv/us - 5rt/s - 7r^{2}/v - 24\,rv/s - 9rt/v)\Big] \pmod{11}$$

$$\equiv P^{3}(o) \times (-t + 4u + rt/u) \pmod{11}$$

$$R_{01}(3) \equiv P^{3}(o)\Big[\ 3(-r + s + t - 2u + v) + \frac{P^{2}(3)P(4)}{P(1)P^{2}(2)P(5)} - \frac{3P(3)P(5)}{P^{3}(2)} + \frac{6yP(3)}{P(2)P(4)} - \frac{7yP(1)P(3)P(4)}{P^{3}(2)P(5)} - \frac{27\,y^{2}P^{2}(1)}{P^{2}(2)P(4)} + \frac{4y^{2}P(1)}{P(3)P(5)} + \frac{12\,yP(1)P(3)P(4)}{P^{3}(2)P(5)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[\ 3(-r + s + t - 2u + v) + tu/v + 3tu/s + 6u + 5rtu/sv - 27\,ru/s - 4r\ \Big] \pmod{11},$$

$$\equiv P^{3}(o) \times (4r + 2s + 3t + 5u + 3v + 2tu/s) \pmod{11}.$$

$$R_{01}(4) \equiv P^{3}(o)\Big[\, 5(-r + s + t - 2u + v) - \frac{3yP(5)}{P(3)P(4)} + \frac{P(4)}{P(1)P(2)} -$$

$$- \frac{6P^{2}(3)}{P^{2}(1)P(4)} + \frac{9yP^{2}(2)}{P(1)P^{2}(3)} + \frac{2y^{3}P(1)P^{2}(2)}{P(3)P(4)P^{2}(5)} +$$

$$+ \frac{18\,y^{2}P(1)P(3)}{P(2)P^{2}(5)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[\, 5(-r + s + t - 2u + v) - 3v + t - 6\,su/r -$$

$$- 9\,sv/u + 2\,rs/t - 18\,ru/v \pmod{11},$$

$$\equiv P^{3}(o) \times (-r + 2s + 5u - 2v) \pmod{11}.$$

$$R_{01}(5) \equiv P^{3}(o)\Big[\, 7(-r + s + t - 2u + v) - \frac{3yP(1)P^{3}(5)}{P(2)P^{2}(3)P^{2}(4)} +$$

$$+ \frac{2P^{2}(3)}{P^{2}(1)P(4)} - \frac{9P(2)P(5)}{P^{2}(1)\,P(4)} - \frac{6y^{4}P^{2}(1)P^{2}(2)}{P^{3}(4)P^{2}(5)} +$$

$$+ \frac{8\,y^{2}P(2)}{P(3)P(4)} + \frac{24yP(3)}{P(2)P(4)} \pmod{11},$$

$$\equiv P^{3}(o)\Big[\, 7(-r + s + t - 2u + v) + 3\,v^{2}/s + 2\,su/r -$$

$$- 9\,sv/r - 6\,rsu/t^{2} - 8sv/t + 24u\Big] \pmod{11},$$

$$\equiv P^{3}(o) \times (4r + 2u - 2v - 5uv/t) \pmod{11}.$$

$$R_{01}(6) \equiv P^{3}(o)\Big[ - \frac{6y^{2}P(5)}{P(2)P^{2}(4)} + \frac{3yP(4)}{P^{2}(1)P(5)} - \frac{15P(3)}{P(1)P^{2}(2)} +$$

$$+ \frac{21y^{2}P(4)}{P(2)P(3)P(5)} - \frac{6y^{3}P(1)}{P(2)P(4)P(5)} - \frac{8y^{4}P(1)P(2)}{P^{2}(4)P^{2}(5)} +$$

$$+ \frac{9y^{3}P(2)}{P(3)P^{2}(5)} + \frac{27y^{2}P(3)P(4)}{P^{2}(2)P^{2}(5)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[ -6uv - 3st - 15tu - 21rt + 6ur - 8rsu/t +$$

$$+ 9rs - 27rtu/v\Big] \pmod{11},$$

$$\equiv P^{3}(o) \times (-rt + 2stu - tv - 2ur - 2vs) \pmod{11}.$$

$$R_{01}(7) \equiv P^{3}(o)\Big[4(-r+s+t-2u+v) + \frac{3P^{2}(3)P(4)}{P(1)P^{2}(2)P(5)} +$$

$$+ \frac{yP(2)}{P(1)P(5)} - \frac{9yP(5)}{P(3)P(4)} + \frac{5P^{2}(3)P(4)}{P(1)P^{2}(2)P(5)} -$$

$$- \frac{21P^{2}(3)P(4)}{P(1)P^{2}(2)P(5)} - \frac{8yP(3)}{P(2)P(4)} -$$

$$- \frac{9yP^{3}(1)P(2)P(3)}{P(4)P^{3}(5)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[\,4(-r+s+t-2u+v) - 13tu/v - s - 9v$$

$$-8u - 9rst/tv\,\Big] \pmod{11},$$

$$\equiv P^{3}(o) \times (-4r - 4s + 2t - 3u - 5v - 2su/v) \pmod{11}.$$

$$R_{01}(8) \equiv P^{3}(o)\Big[\,6\,(-r+s+t-2u+v) - \frac{yP(2)}{P(1)P(5)} +$$

$$+\frac{13P(4)P(5)}{P(1)P^{2}(3)} + \frac{3P^{2}(4)P(5)}{yP(1)P^{3}(2)} - \frac{15P^{2}(5)}{P(3)P^{2}(2)} +$$

$$+\frac{7P(3)P^{3}(4)}{yP^{2}(1)P^{2}(2)P(5)} - \frac{9P(4)}{P(1)P(2)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[\,6(-r+s+t-2u+v) + s + 13tv/u - 3t^{2}/s +$$

$$+ 15\,tv/s + 7t^{2}/v - 9t\Big] \pmod{11},$$

$$\equiv P^{3}(o) \times (-4r + 3t - 5u - 5v + tv/r) \pmod{11}.$$

$$R_{01}(9) \equiv P^{3}(o)\Big[\quad 8\,(-r+s+t-2u+v) - \frac{13\,P(4)}{P(1)P(2)} + \frac{17y^{2}P(1)}{P(3)P(5)} +$$

$$+\frac{2yP(2)}{P(1)P(5)} + \frac{6\,P^{2}(3)P(4)}{P(1)P^{2}(2)P(5)} - \frac{21P(1)P(4)}{P(3)P^{2}(2)}\Big] \pmod{11},$$

$$\equiv P^{3}(o)\Big[\,8\,(-r+s+t-2u+v) - 13t - 17r - 2s +$$

$$+ 6tu/v - 21\,rt/s\,\Big] \pmod{11},$$

$$\equiv P^{3}(o) \times (-2r + t + u - 3v) \pmod{11}.$$

$$R_{01}(10) \equiv P^3(o)\Big[\, 9(-r+s+t-2u+v) - \frac{P^2(5)}{P^2(2)P(3)} - \frac{2y^2P(2)P(3)}{P^2(4)P(5)} + \frac{3yP(2)}{P(1)P^2(3)} + \frac{13P(4)}{P(1)P(2)} - \frac{18yP(3)}{P(2)P(4)} + \frac{24yP(5)}{P(3)P(4)}\Big] \pmod{11},$$

$$\equiv P^3(o)\Big[\, 9(-r+s+t-2u+v) + tv/s + 2\,su/t - 3sv/u + 13t - 18u + 24v \Big] \pmod{11},$$

$$\equiv P^3(o) * (-r-t-2u+5v) \pmod{11}.$$

From Lemma 5 we derived some further congruences which appeared to be identical relations, because in every one of them the first four terms of the power series on both sides were found to be identical. These are:

$$R_{35}(o) \equiv R_{23}(1) \equiv R_{13}(2) - R_{45}(2) - P(o) \equiv R_{02}(3) \equiv R_{12}(4) - P(o) \equiv R_{35}(5) - R_{12}(5) \equiv R_{45}(7) \equiv R_{02}(8) - R_{34}(8) - P(o) \equiv R_{23}(9) - R_{45}(9) \equiv R_{35}(10) - R_{02}(10) - P(o) \equiv R_{25}(6)/t,$$

$$R_{24}(o) \equiv R_{12}(1) + R_{45}(1) \equiv R_{02}(2) + R_{45}(2) \equiv R_{24}(3) - R_{45}(3) - P(o) \equiv R_{23}(4) + P(o) \equiv R_{34}(5) + P(o) \equiv R_{23}(7) - R_{01}(7) + P(o) \equiv R\ (8) - R\ (8) - R\ (8) + P(o) \equiv R_{02}(9) + R_{23}(9) + R_{45}(9) - P(o) \equiv R_{45}(10) \equiv R_{15}(6)/t \equiv -R_{02}(6)/s,$$

$$R_{45}(o) \equiv R_{34}(1) - R_{45}(1) \equiv R_{23}(2) - R_{45}(2) \equiv R_{12}(3) + R_{45}(3) - R_{34}(3) + P(o) \equiv R_{01}(4) \equiv R_{45}(5) - R_{01}(5) - P(o) \equiv R_{01}(7) - R_{23}(7) + R_{34}(7) - P(o) \equiv R_{02}(8) - R_{23}(8) \equiv P(o) - R_{01}(9) + R_{23}(9) - R_{45}(9) \equiv R_{34}(10) - R_{12}(10) \equiv R_{34}(6)/t,$$

$$R_{01}(o) - R_{45}(o) + 3R_{15}(o) + P(o) \equiv R_{24}(1) + R_{05}(1) + R_{23}(1) + P(o)$$
$$\equiv 2R_{12}(2) + 3R_{24}(2) + R_{45}(2) + P(o)$$
$$\equiv 2R_{15}(3) + R_{01}(3) + P(o)$$
$$\equiv R_{12}(4) + R_{34}(4) + R_{15}(4) - P(o)$$
$$\equiv R_{25}(5) + R_{03}(5) = R_{05}(7)$$
$$\equiv R_{01}(8) + R_{45}(8) - P(o)$$
$$\equiv R_{24}(9) + R_{23}(9) + P(o)$$
$$= R_{14}(10) = -R_{23}(6)/r,$$

$$R_{45}(4) \equiv R_{02}(5) - R_{45}(5) + P(o) \equiv R_{13}(6)/t,$$

$$R_{13}(2) \equiv R_{01}(9) + R_{34}(9) - R_{45}(9) - P(o) \equiv R_{34}(6)/u,$$

$$R_{15}(1) \equiv R_{13}(7) \equiv R_{24}(6)/v - P(o),$$

$$R_{12}(o) = R_{45}(3) \equiv -\frac{u}{t} R_{25}(6) - \frac{R_{12}(6)}{t} - \frac{r}{t}\left[\frac{R_{24}(6)}{v} - \frac{R_{12}(6)}{t}\right] +$$
$$+\frac{s}{t}\left[\frac{R_{45}(6)}{r} + \frac{R_{23}(6)}{v}\right] + P(o)(r + t - 2s - u)/t.$$

$$R_{12}(8) \equiv R_{25}(10) - P(o) \equiv -\frac{R_{34}(6)}{r} \qquad (6.57)$$

We may note that the first four sets of identities are relations between the $R_{lc}(d)$ for all values of d, whereas the remaining five sets are odd ones.

We may observe here that Dr. Atkin had previously obtained empirically and later proved some rank congruences; writing

$$\rho_1(d) = r_{23}(d) - 4r_{34}(d) + 5r_{45}(d),$$
$$\rho_2(d) = r_{12}(d) - 4r_{23}(d) + 6r_{34}(d) - 4r_{45}(d),$$

and $$\rho_3(d) = r_{01}(d) - 4r_{12}(d) + 5r_{23}(d) - 4r_{45}(d),$$

he obtained

$$
\begin{aligned}
\rho_1(0) &\equiv 0 \pmod{11},\\
\rho_2(1) - \rho_1(1) &\equiv 0 \pmod{11},\\
2\rho_3(2) - \rho_2(2) - 4\rho_1(2) &\equiv 0 \pmod{11},\\
2\rho_3(3) + 2\rho_2(3) + 2\rho_1(3) &\equiv 0 \pmod{11},\\
3\rho_3(4) + 2\rho_2(4) - \rho_1(4) &\equiv 0 \pmod{11},\\
\rho_3(5) + \rho_1(5) &\equiv 0 \pmod{11},\\
\rho_2(6) &\equiv 0 \pmod{11},\\
\rho_1(6) + 2\rho_3(6) &\equiv 0 \pmod{11},\\
4\rho_3(7) + \rho_2(7) - 6\rho_1(7) &\equiv 0 \pmod{11},\\
\rho_3(8) - \rho_2(8) - \rho_1(8) &\equiv 0 \pmod{11},\\
\rho_3(9) + \rho_2(9) - \rho_1(9) &\equiv 0 \pmod{11},\\
4\rho_3(10) - 4\rho_2(10) - \rho_1(10) &\equiv 0 \pmod{11}.
\end{aligned}
\tag{6.58}
$$

All of these are in fact equivalent to

$$\text{II} - \text{III} - 5\text{I} \equiv 0 \pmod{11},$$

where I, II, and III denote respectively the first, second, and third sets of relations in (6.57).

We next obtained the following five independent congruent relations between the $R_{\ell c}(6)$ and r, s, t, u, v:

$$
\begin{aligned}
-vR_{01}(6) + tR_{02}(6) - (v+t)R_{03}(6) + vR_{04}(6) + \\
+ vR_{05}(6) &\equiv 0 \pmod{11},\\
sR_{01}(6) - tR_{02}(6) - sR_{05}(6) &\equiv 0 \pmod{11},\\
-uR_{01}(6) - uR_{02}(6) + tR_{03}(6) + (2u-t)R_{05}(6) &\equiv 0 \pmod{11},\\
-vR_{01}(6) + (v-s)R_{02}(6) + sR_{03}(6) - uR_{04}(6) + uR_{05}(6) \\
&\equiv 0 \pmod{11},\\
(r-s)R_{01}(6) - (r+t)R_{02}(6) + tR_{03}(6) + sR_{05}(6) &\equiv -rtP(0) \pmod{11}.
\end{aligned}
\tag{6.59}
$$

These again appeared to be identities, and solving them as a simultaneous set we found

$$R_{01}(6) = D(rsu + stv - 4\,tur - 3uvs + 3vrt),$$
$$R_{02}(6) = D(-rsu - 3uvs + 2vrt),$$
$$R_{03}(6) = D(-rsu - 3tur - uvs + vrt),$$
$$R_{04}(6) = D(-stv - 2tur - uvs + 4vrt),$$
$$R_{05}(6) = D(rsu - 2tur - 3uvs + vrt), \qquad (6.60)$$

where $D = P(o)/(rt + su + tv + ur + vs)$.

Finally using the relations in (6.57) together with those in (6.60) we obtained a complete set of conjectural values of the $R_{\ell c}(d)$ which we prove immediately in Section 7.

RANK IDENTITIES FOR $q = 11$.

7. Theorem 1.

$R_{01}(0) = D^{*}(3u^2 + 5rt + su + tv - 3ur + vs)$, (7.1)

$R_{01}(1) = D(-2s^2 - 2rt + 3su + tv - 2ur - vs)$ (7.2)

$R_{01}(2) = D(-3r^2 - 5rt - tv + 3ur)$, (7.3)

$R_{01}(3) = D(-4u^2 + rt - su + tv - 6ur + vs)$ (7.4)

$R_{01}(4) = D(-3rt - su + ur + 3vs)$, (7.5)

$R_{01}(5) = D(2v^2 + rt - 3ur + vs)$, (7.6)

$R_{01}(6) = D(rsu + stv - 4tur - 3uvs + 3vrt)$, (7.7)

$R_{01}(7) = D(-s^2 - rt + 4su + 2tv - 5ur + 2vs)$, (7.8)

$R_{01}(8) = D(-t^2 - rt + 5tv - ur + 2vs)$, (7.9)

$R_{01}(9) = D(5rt + tv - 2ur - vs)$, (7.10)

$R_{01}(10) = D(-3rt - tv + 4ur - 2vs)$, (7.11)

$R_{12}(0) = D(-u^2 - 2rt - 3ur + vs)$, (7.12)

$R_{12}(1) = D(-s^2 + rt - 2su + 2ur - vs)$, (7.13)

$R_{12}(2) = D(5rt + tv - 2ur - vs)$, (7.14)

$R_{12}(3) = D(4u^2 - su - tv + 4ur)$, (7.15)

$R_{12}(4) = D(2rt - su + tv - ur + 2vs)$, (7.16)

$R_{12}(5) = D(2v^2 - rt + su + 2ur)$, (7.17)

$R_{12}(6) = D(-2rsu - stv + 4tur - vrt)$, (7.18)

$R_{12}(7) = D(2s^2 + rt - 4su - tv + 5ur + vs)$, (7.19)

$R_{12}(8) = D(t^2 - su - 3tv + ur)$, (7.20)

$R_{12}(9) = D(-rt - su + 3ur)$, (7.21)

$R_{12}(10) = D(-t^2 + rt + 4tv - 3ur - 2vs)$, (7.22)

$R_{23}(0) = D(-2rt + su + 5ur - vs)$, (7.23)

----------

* D is the same as defined in (6.60).

$$R_{23}(1) = D(rt - 2su - 2ur + vs), \tag{7.24}$$

$$R_{23}(2) = D(3r^2 - rt - su + 2ur + vs), \tag{7.25}$$

$$R_{23}(3) = D(-4u^2 - rt + tv - ur), \tag{7.26}$$

$$R_{23}(4) = D(rt - su - tv + ur - 4vs), \tag{7.27}$$

$$R_{23}(5) = D(-3v^2 + tv - ur), \tag{7.28}$$

$$R_{23}(6) = D(-3tur + 2uvs - vrt), \tag{7.29}$$

$$R_{23}(7) = D(-s^2 + 3su + tv - 4ur - 2vs), \tag{7.30}$$

$$R_{23}(8) = D(2rt + 2tv - ur - vs), \tag{7.31}$$

$$R_{23}(9) = D(-r^2 - su - 3ur), \tag{7.32}$$

$$R_{23}(10) = D(2t^2 + rt + su - 6tv + 2ur + 3vs), \tag{7.33}$$

$$R_{34}(o) = D(4rt - su - 3ur - 2vs), \tag{7.34}$$

$$R_{34}(1) = D(s^2 - 2rt + su + ur + vs), \tag{7.35}$$

$$R_{34}(2) = D(-4r^2 - 3rt + su - 2ur + vs), \tag{7.36}$$

$$R_{34}(3) = D(3u^2 + 2rt + su + ur - vs), \tag{7.37}$$

$$R_{34}(4) = D(-v^2 - 3rt + su + tv - ur + 3vs), \tag{7.38}$$

$$R_{34}(5) = D(rt - su - tv + ur - 4vs), \tag{7.39}$$

$$R_{34}(6) = D(rsu - stv + tur + 3\,vrt), \tag{7.40}$$

$$R_{34}(7) = D(-rt - su + 3ur), \tag{7.41}$$

$$R_{34}(8) = D(-3rt + tv + ur), \tag{7.42}$$

$$R_{34}(9) = D(2r^2 - rt + su + tv + 2ur + vs), \tag{7.43}$$

$$R_{34}(10) = D(-t^2 - 2rt - su + 4tv - 2ur + vs), \tag{7.44}$$

$$R_{45}(o) = D(-3rt - su + ur + 3vs), \tag{7.45}$$

$$R_{45}(1) = D(s^2 + rt + 2su - 2vs), \tag{7.46}$$

$$R_{45}(2) = D(3r^2 + 2rt + ur - 2vs), \tag{7.47}$$

$$R_{45}(3) = D(-u^2 - 2rt - 3ur + vs), \tag{7.48}$$

$$R_{45}(4) = D(2v^2 + 2rt + 2su + ur - 3vs), \tag{7.49}$$

$$R_{45}(5) = D(2v^2 - rt + tv - ur + 5vs), \tag{7.50}$$

$$R_{45}(6) = D(rsu + stv - 2uvs - 3vrt), \tag{7.51}$$

$$R_{45}(7) = D(rt - 2su - 2ur + vs), \tag{7.52}$$

$$R_{45}(8) = D(t^2 + 2rt + su - 2tv - ur + vs), \tag{7.53}$$

$$R_{45}(9) = D(-r^2 - rt + su - ur - vs), \tag{7.54}$$

$$R_{45}(10) = D(2rt + 2ur - 3vs). \tag{7.55}$$

We shall base our proof on those of Atkin and Swinnerton-Dyer for their Theorems 4 and 5, that is, we shall show that the values of the $R_{\ell c}(d)$ are, in fact, the unique solutions of five sets of simultaneous equations. We shall use the relations in (4.10) and (4.11) without explicit reference.

From (2.13), Lemma 6 (ASD) and our Table 3 we have the following set of simultaneous equations:

$$\frac{P(4)r_{01}(0)}{P(2)} + \frac{yP(3)r_{01}(4)}{P(4)} + yr_{01}(6) - \frac{y^2P(1)r_{01}(7)}{P(5)} - \frac{yP(5)r_{01}(9)}{P(3)} - \frac{yP(2)r_{01}(10)}{P(1)} = P(0)(pr + s + t - 2u + v) - \frac{3y^2P(4)\,\varepsilon(2,0)}{P(0)P(2)} - \frac{P(4)}{P(2)}, \tag{7.56}$$

$$-\frac{P(2)r_{01}(0)}{P(1)} + \frac{P(4)r_{01}(1)}{P(2)} + \frac{yP(3)r_{01}(5)}{P(4)} + yr_{01}(7) - \frac{y^2P(1)r_{01}(8)}{P(5)} - \frac{yP(5)r_{01}(10)}{P(3)} = \frac{3y^2P(2)\,\varepsilon(2,0)}{P(0)P(1)} + \frac{P(2)}{P(1)} - \frac{y^4\varepsilon(5,0)}{P(0)} + \frac{3yP(0)}{P(3)}, \tag{7.57}$$

$$
\begin{aligned}
&-\frac{P(5)r_{01}(o)}{P(3)} - \frac{P(2)r_{01}(1)}{P(1)} + \frac{P(4)r_{01}(2)}{P(2)} + \frac{yP(3)r_{01}(6)}{P(4)} + \\
&+ yr_{01}(8) - \frac{y^2 P(1)r_{01}(9)}{P(5)} = \frac{3y^2 P(5)\mathcal{E}(2,o)}{P(o)P(3)} + \frac{P(5)}{P(3)} - \\
&\qquad - \frac{3P(o)P(3)}{P(1)P(2)},
\end{aligned}
\tag{7.58}
$$

$$
\begin{aligned}
&-\frac{P(5)r_{01}(1)}{P(3)} - \frac{P(2)r_{01}(2)}{P(1)} + \frac{P(4)r_{01}(3)}{P(2)} + \frac{yP(3)r_{01}(7)}{P(4)} + \\
&+ yr_{01}(9) - \frac{y^2 P(1)r_{01}(10)}{P(5)} = \frac{y^4 P(3)\mathcal{E}(5,o)}{P(o)P(4)} + \frac{P(o)P(3)P(4)}{P(1)P(2)P(5)} .
\end{aligned}
\tag{7.59}
$$

$$
\begin{aligned}
&-\frac{yP(1)r_{01}(0)}{P(5)} - \frac{P(5)r_{01}(2)}{P(3)} - \frac{P(2)r_{01}(3)}{P(1)} + \frac{P(4)r_{01}(4)}{P(2)} + \\
&+ \frac{yP(3)r_{01}(8)}{P(4)} + yr_{01}(10) = \frac{3y^3 P(1)\mathcal{E}(2,o)}{P(o)P(5)} + \frac{yP(1)}{P(5)} - \\
&\qquad - \frac{3yP(o)P(1)P(5)}{P(2)P(3)P(4)}
\end{aligned}
\tag{7.60}
$$

$$
\begin{aligned}
&r_{01}(o) - \frac{yP(1)r_{01}(1)}{P(5)} - \frac{P(5)r_{01}(3)}{P(3)} - \frac{P(2)r_{01}(4)}{P(1)} + \\
&\qquad + \frac{P(4)r_{01}(5)}{P(2)} + \frac{yP(3)r_{01}(9)}{P(4)} = -\frac{3y^2\mathcal{E}(2,o)}{P(o)} + 1.
\end{aligned}
\tag{7.61}
$$

$$
\begin{aligned}
&r_{01}(1) - \frac{yP(1)r_{01}(2)}{P(5)} - \frac{P(5)r_{01}(4)}{P(3)} - \frac{P(2)r_{01}(5)}{P(1)} + \\
&\qquad + \frac{P(4)r_{01}(6)}{P(2)} + \frac{yP(3)r_{01}(10)}{P(4)} = 0.
\end{aligned}
\tag{7.62}
$$

$$\frac{P(3)r_{01}(0)}{P(4)} + r_{01}(2) - \frac{yP(1)r_{01}(3)}{P(5)} - \frac{P(5)r_{01}(5)}{P(3)} - \frac{P(2)r_{01}(6)}{P(1)} + \frac{P(4)r_{01}(7)}{P(2)} = - \frac{3y^2 P(3)\,\mathcal{E}(2,0)}{P(0)P(4)} - \frac{P(3)}{P(4)} + \frac{3P(0)P(3)P(4)}{P^2(2)P(5)} + \frac{yP(0)P(2)}{P(3)P(5)}, \tag{7.63}$$

$$\frac{P(3)r_{01}(1)}{P(4)} + r_{01}(3) - \frac{yP(1)r_{01}(4)}{P(5)} - \frac{P(5)r_{01}(6)}{P(3)} - \frac{P(2)r_{01}(7)}{P(1)} + \frac{P(4)r_{01}(8)}{P(2)} = - \frac{y^3 P(2)\,\mathcal{E}(5,0)}{P(0)P(1)} - \frac{y^2 P(0)P(1)P(2)}{P(4)P^2(5)}, \tag{7.64}$$

$$\frac{P(3)r_{01}(2)}{P(4)} + r_{01}(4) - \frac{yP(1)r_{01}(5)}{P(5)} - \frac{P(5)r_{01}(7)}{P(3)} - \frac{P(2)r_{01}(8)}{P(1)} + \frac{P(4)r_{01}(9)}{P(2)} = - \frac{y^3 P(5)\,\mathcal{E}(5,0)}{P(0)P(3)} - \frac{P(0)}{P(2)}, \tag{7.65}$$

$$\frac{P(3)r_{01}(3)}{P(4)} + r_{01}(5) - \frac{yP(1)r_{01}(6)}{P(5)} - \frac{P(5)r_{01}(8)}{P(3)} - \frac{P(2)r_{01}(9)}{P(1)} + \frac{P(4)r_{01}(10)}{P(2)} = 0. \tag{7.66}$$

We now multiply the equations (7.56) to (7.66) by 1, $yP(1)/P(5)$, $y/P(2)$, $P(3)/P(2)$, $yP(1)/P(3)$, $P(4)/P(2)$, $y/P(1)$ $y^2/P(5)$, $y^2/P(4)$, $P(4)/P(1)$, and $y^2/P(3)$ respectively and denote the transformed equations by Roman letters I to XI.

It is clearly sufficient for the truth of (7.1) to (7.11) to show that the equations I to XI are reduced to a set of identities on the substitution of the values of $R_{01}(0)$ to $R_{01}(10)$ given in Theorem 1.

Equation I is:

$$tR_{01}(o) + uR_{01}(4) + \overline{R}_{01}(6) + rR_{01}(7) - vR_{01}(9) + sR_{01}(10) -$$
$$-(-r + s + t - 2u + v)\,P(o) = 0.$$

$$\text{L.H.S.} = D(4rt^2 - 9urt - 3\,vrt - 4\,rst + su^2 - uvs + 9\,rsu -$$
$$- s^2u + 3vrt - stv + 2\,tuv - 2v^2t + 3\,u^2r - 4\,ur^2 +$$
$$+ uvr + 3vrs - 3\,s^2v + 2\,uvs + 3\,u^2t - rs^2),$$
$$= D\big[4(stv - vrt) - 9tur - 3vrt - 4(rsu - vrt) +$$
$$+ (tur - rsu) - uvs + 9\,rsu - (rsu - rs^2 - vrt) +$$
$$- 3vrt - stv + 2(tur - stv) - 2(uvs - stv) +$$
$$+ 3(tur - stv - u^2t) - 4(vrt - uvs) + (-tur + uvs) +$$
$$+ 3(vrt - uvs) - 3(rsu - uvs) + 2\,uvs + 3\,u^2t -$$
$$- rs^2\big],$$

Equation II is:

$$s\,[R_{01}(o) - R_{01}(5) + R_{01}(8)] + u\,[R_{01}(1) - R_{01}(5) + R_{01}(8)] +$$
$$+ r\,[R_{01}(1) - R_{01}(7) - R_{01}(8) + R_{01}(10)] + 3rP(o) = 0,$$

$$\text{L.H.S.} = D\big[r(-s^2 + t^2 + 2su - 4vt + 11ur - 4vs) +$$
$$+ s(3u^2 - 2v^2 - t^2 + 3rt - su + 6tv - ur + 2vs) +$$
$$+ u(-t^2 - 2v^2 - 2s^2 - 4rt + 3su + 6tv)\big],$$
$$= D\big[-rs^2 + (stv - vrt) + 2rsu - 4vrt + 11\,(vrt - tur) -$$
$$- 4(vrt - uvs) + 3(tur - rsu) - 2(uvs - v^2u - tur) -$$
$$- st^2 + 3(rsu - vrt) + (rsu - s^2r - vrt) + 6\,stv -$$
$$- rsu + 2(rsu - uvs) - (stv - t^2s - tur - rsu) -$$
$$-2v^2u - 2(rsu - vrt - s^2r) - 4tur + 3(tur - rsu) +$$
$$+ 6\,(tur - stv)\big],$$

Equation III is:

$$- v\left[R_{01}(o) + R_{01}(1) + R_{01}(8) - R_{01}(9)\right] + u\left[R_{01}(2) + R_{01}(8) - R_{01}(9)\right] + uR_{01}(6)/t + 3uP(o) = 0,$$

$$\text{L.H.S.} = D\left[u(-3r^2 - t^2 - 3v^2 - 11rt + 2su + 6tv + 3ur + 8sv) + v(t^2 - 3u^2 + 2s^2 + 3rt - 4su - 6tv + 4ur - 3vs)\right],$$

$$= D\left[-3(vrt - tur) - (stv - tur - rsu - t^2 s) - 3uv^2 - 11tur + 2(tur - rsu) + 6(tur - stv) + 3(tur - u^2 t - stv)) + 8\,uvs + (stv - t^2 s - rsu) - 3(tur - u^2 t - uvs - stv) + 2(rsu - uvs) + 3\,vrt - 4\,uvs - 6(uvs - stv) + 4(uvs - tur) - 3(uvs - tur - v^2 u)\right],$$

Equation IV is:

$$u\left[R_{01}(1) + R_{01}(7) + R_{01}(9) + R_{01}(10)\right] - t\left[R_{01}(1) + R_{01}(2) - R_{01}(3)\right] + s\left[R_{01}(2) + R_{01}(10)\right] - utP(o)/v = 0,$$

$$\text{L.H.S.} = D\left[S(-3r^2 - 7rt + su - tv + 8ur - vs) + t(-4u^2 + 2s^2 + 3r^2 + 7rt - 5su - 8ur + vs) + u(-3s^2 + 8su + 4\,tv - 4\,ur - vs)\right],$$

$$= D\left[-3(-r^2 v - rsu - uvs + vrt) - 7(-vrt + rsu) + (-s^2 r - vrt + rsu) - stv + 8rsu - (rsu - suv) - 4u^2 t + 2(-s^2 r - stv - vrt + rsu) + 3(vrt - r^2 v - uvs) + 7(stv - vrt) - 5(stv - rsu) - 8tur + stv - 3(-s^2 r - vrt + rsu) + 8(tur - rsu) + 4(tur - stv) - 4(-u^2 t - stv + tur) - uvs\right],$$

Equation V is:

$$r\left[R_{01}(o) + R_{01}(3) + R_{01}(4) - R_{01}(10)\right] - v\left[R_{01}(2) + R_{01}(3) + \right.$$
$$\left. + R_{01}(8)\right] + u\left[R_{01}(4) + R_{01}(8)\right] + 3vrP(o)/s = 0,$$

$$\text{L.H.S.} = D\left[r\,(-u^2 + 3rt - 4\,su - 15ur + 4vs) - \right.$$
$$-v\,(-3r^2 - 4u^2 - t^2 - 8rt - 4su + 2tv - 7ur)$$
$$\left. + u(-t^2 - 7\,rt - 4su + 2tv - 3ur + 2vs)\right] \;,$$
$$= D\left[-(-u^2 t - stv + tur) + 3(-r^2 v - uvs + vrt) - \right.$$
$$-4rsu - 15(-tur + vrt) + 4\,(vrt - uvs) + 3vr^2 +$$
$$+ 4(-u^2 t - uvs - stv + tur) + (stv - t^2 s - rsu) +$$
$$+ 8vrt + 4\,uvs - 2(uvs - stv) + 7(uvs - tur) -$$
$$- (stv - rsu - tur - t^2 s) - 7tur - 4(-rsu + tur) +$$
$$\left. + 2(tur - stv) - 3(tur - u^2 t - stv) + 2uvs)\right] \;,$$

Equation VI is:

$$t\left[R_{01}(o) - R_{01}(3) - R_{01}(4) + R_{01}(5)\right] - u\left[R_{01}(1) + R_{01}(5) - \right.$$
$$\left. - R_{01}(9)\right] - r\left[R_{01}(1) + R_{01}(3) + R_{01}(5)\right] = 0,$$

$$\text{L.H.S.} = D\left[-r(-2s^2 - 4u^2 + 2v^2 + 2su + 2tv - 11ur + vs) + \right.$$
$$+ t\,(7u^2 + 2v^2 + 8rt + 3su - ur - 2sv) -$$
$$\left. - u(-2s^2 + 2v^2 - 6rt + 3su - 3ur + vs)\right] \;,$$
$$= D\left[2rs^2 + 4(-u^2 t - stv + tur) - 2(-v^2 u - vrt - tur + \right.$$
$$+ uvs) - 2rsu - 2vrt + 11(vrt - tur) - (vrt - uvs) -$$
$$+ 7tu^2 + 2(uvs - stv) + 8(stv - vrt) + 3(stv - rsu) -$$
$$- tur - 2stv + 2(rsu - vrt - s^2 r) - 2uv^2 +$$
$$+ 6tur - 3(tur - rsu) + 3(tur - u^2 t - stv) -$$
$$\left. - uvs\right] \;,$$

Equation VII is:

$$u\left[R_{01}(1) + R_{01}(2) - R_{01}(4) - R_{01}(10)\right] + t\left[-R_{01}(1) +\right.$$

$$\left. + R_{01}(4) + R_{01}(5)\right] + tR_{01}(6)/s = 0,$$

$$\text{L.H.S.} = D\left[u(-2s^2 - 3r^2 - rt + 4su + tv - 4ur - 2vs) +\right.$$

$$\left. + t(2v^2 + 2s^2 - 4u^2 - 3rt - 4su - 3ur + 8vs)\right],$$

$$= D\left[-2(-s^2 r - vrt + rsu) - 3(vrt - tur) - tur +\right.$$

$$+ 4(tur - rsu) + tur - stv) - 4(tur - u^2 t - stv) -$$

$$- 2uvs + 2(uvs - stv) + 2(rsu - s r - stv - vrt) -$$

$$\left. - 4u^2 t - 3(stv - vrt) - 4(stv - rsu) - 3(tur + 8\, stv\right],$$

Equation VIII is:

$$-u\left[R_{01}(o) + R_{01}(2) - R_{01}(5) + R_{01}(7)\right] - r\left[R_{01}(3) -\right.$$

$$\left. - R_{01}(5) + R_{01}(7)\right] + R_{01}(6) + (3t - r + 3ut/s)P(o) = 0,$$

$$\text{L.H.S.} = D\left[r(4u^2 + s^2 + 2v^2 - 4su - 4tv + 7ur - 3vs) +\right.$$

$$+ 3t(rt + su + tv + ur + vs) + u(2v^2 - 3t^2 + 3r^2 +$$

$$\left. + \underset{+ (rsu + stv - 4tur - 3uvs + 3vrt)}{s^2 + 5rt - 5su + 10tv - 4ur - 2vs) +}\right],$$

$$= D\left[4(tur - u^2 t - stv) + rs^2 + 2(uvs - v^2 u - vrt -\right.$$

$$- tur) - 4rsu - 4vrt + 7(vrt - tur) - 3(vrt -$$

$$- uvs) + 3(stv - vrt) + 3(stv - rsu) + 3(stv -$$

$$- t^2 s - rsu) + 3\, tur + 3\, stv + 2uv^2 - 3(stv -$$

$$- t^2 s - tur - rsu) + 3(vrt - tur) +$$

$$+ (rsu - s^2 r - vrt) + 5tur - 5(tur - rsu) +$$

$$+ 10(tur - stv) - 4(tur - u^2 t - stv) - 2uvs +$$

$$\left. + rsu + stv - 4tur - 3uvs + 3\, vrt\right],$$

Equation IX is:

$$u\left[R_{01}(1) + R_{01}(4) - R_{01}(8)\right] + s\left[-R_{01}(3) + R_{01}(4) + \right.$$
$$\left. + R_{01}(7) - R_{01}(8)\right] - R_{01}(6) + (r + rs/v)P(o) = 0,$$

$$\text{L.H.S.} = D\left[r(rt + st + tv + ur + vs) + s(t^2 + 4u^2 - r^2 + \right.$$
$$+ 2su - 4tv + 4\,ur + 2vs) + u(-2s^2 + t^2 - 4rt +$$
$$\left. + 2su - 4tv)\ ) - (rsu + stv - 4tur - 3uvs + 3vrt)\right],$$
$$= D\left[(vrt - r^2v - uvs) + rsu + vrt + (vrt - tur) + \right.$$
$$+ (vrt - uvs) + st^2 + 4(tur - rsu) - (vrt -$$
$$- r^2v - rsu - uvs) + 2(rsu - s^2r - vrt) -$$
$$- 4stv + 4rsu + 2(rsu - uvs) - 2(rsu - s^2r -$$
$$- vrt) + (stv - t^2s - tur - rsu) - 4tur +$$
$$+ 2(tur - rsu) - 4(tur - stv) - (rsu + stv -$$
$$\left. - 4tur - 3uvs + 3vrt)\right],$$

Equation X is:

$$t\left[R_{01}(2) + R_{01}(4) + R_{01}(9) - R_{01}(7)\right] + s\left[-R_{01}(2) + \right.$$
$$\left. + R_{01}(5) + R_{01}(8)\right] - r\left[R_{01}(5) + R_{01}(7)\right] + tP(o) = 0,$$

$$\text{L.H.S.} = D\left[-r(-s^2 + 2v^2 + 4su + 2tv - 8ur + 3vs) + \right.$$
$$+ s(2v^2 - t^2 + 3r^2 + 5rt + 6tv - 7ur + 3vs) +$$
$$\left. + t(-3r^2 + s^2 - rt - 4su - tv + 8ur + vs)\right],$$
$$= D\left[rs^2 - 2(uvs - v^2u - vrt - tur) - 4rsu - 2vrt + \right.$$
$$+ 8(vrt - tur) - 3(vrt - uvs) + 2(uvs - v^2u -$$
$$- tur) - st^2 + 3(vrt - r^2v - rsu - uvs) +$$
$$+ 5(rsu - vrt) + 6stv - 7rsu + 3(rsu - uvs) -$$
$$-3(vrt - r^2v - uvs) + (rsu - s^2r - stv - vrt) -$$
$$- (stv - vrt) - 4(stv - rsu) - (stv - t^2s -$$
$$\left. - rsu) + 8\,tur + stv\right],$$

Equation XI is:

$$-s\left[R_{01}(3) + R_{01}(5) - R_{01}(9) + R_{01}(10)\right] + r\left[R_{01}(5) + \right.$$
$$\left. + R_{01}(8) + R_{01}(10)\right] + rR_{01}(6)/v = 0,$$

$$\begin{aligned}
\text{L.H.S.} &= D\left[r(s^2 - t^2 + 2v^2 - 5us + 3tv + 4ur + vs) + \right.\\
&\qquad \left. + s(4u^2 - 2v^2 + 6rt + su + tv + 3ur - vs)\right], \\
&= D\left[rs^2 - (stv - vrt) + 2(uvs - v^2u - vrt - tur) - \right.\\
&\qquad - 5rsu + 3vrt + 4(vrt - tur) + (vrt - uvs) + \\
&\qquad + 4(tur - rsu) - 2(uvs - v^2u - tur) + \\
&\qquad + 6(rsu - vrt) + (rsu - s^2r - vrt) + stv + 3rsu - \\
&\qquad \left. - (rsu - uvs)\right],
\end{aligned}$$

and the coefficients of each of the quantities $r^2v$, $s^2r$, $t^2s$, $u^2t$, $v^2u$, rsu, stv, tur, uvs and vrt in every one of these expressions, which we have from the left-hand sides of equations I to XI, are zero.

In this proof of (7.1) to (7.11) we have had in effect to show that certain homogeneous functions of degree 3 in r, s, t, u and v are zero. When these functions were expressed in terms of $r^2v$, $s^2r$, $t^2s$, $u^2t$, $v^2u$, rsu, stv, tur, uvs and vrt we found in every case that the coefficients of these quantities are separately zero. We shall, therefore, in future proceed only to the point of obtaining these homogeneous functions and leave the reader the task of verifying that they are zero.

Adopting now a similar method to that used for the results in (7.1) to (7.11) we have the following simultaneous equations for the next set of eleven results (7.12) to (7.22):

Equation 1 is:

$$tR_{12}(0) + uR_{12}(4) + \bar{R}_{12}(6) + rR_{12}(7) - vR_{12}(9) + sR_{12}(10) - (r + t - 2s)P(o) = 0,$$

$$\text{L.H.S.} = D\Big[r(2s^2 - 5su - 2tv + 4ur) + s(-t^2 + 3rt + 2su + 6\bar{t}v - ur) + t(-u^2 - 3rt - su - tv - 4ur) + u(2rt - su + tv - ur + 2vs) + v(rt + su - 3ur) + (-2rsu - stv + 4tur - vrt)\Big],$$

Equation II is:

$$s\big[R_{12}(o) - R_{12}(5) + R_{12}(8)\big] + u\big[R_{12}(1) - R_{12}(5) + R_{12}(8)\big] + r\big[R_{12}(1) - R_{12}(7) - R_{12}(8) + R_{12}(10)\big] - (r + u)P(o) = 0,$$

$$\text{L.H.S.} = D\Big[r(-3s^2 - 2t^2 + 2su + 7tv - 8ur - 5vs) + s(t^2 - u^2 - 2v^2 - rt - 2su - 3tv - 4ur + vs) + u(-s^2 + t^2 - 2v^2 + rt - 5su - 4tv - 2vs)\Big],$$

Equation III is:

$$-v\big[R_{12}(o) + \underline{R_{12}(1)} + R_{12}(8) - R_{12}(9)\big] + u\big[R_{12}(2) + R_{12}(8) - R_{12}(9)\big] + u\bar{R}_{12}(6)/t - uP(o) = 0,$$

$$\text{L.H.S.} = D\ u(t^2 + 6rt + su - 3tv - ur - 6vs) + v(s^2 - t^2 + u^2 + 2su + 3tv + 3ur)\Big],$$

Equation IV is:

$$u\big[R_{12}(1) + R_{12}(7) + R_{12}(9) + R_{12}(10)\big] - t\big[R_{12}(1) + R_{12}(2) - R_{12}(3)\big] + s\big[R_{12}(2) + R_{12}(10)\big] + (2t - 3u - 2s)P(o) = 0,$$

$$\text{L.H.S.} = D\Big[s(-t^2 + 4rt - 2su + 3tv - 7ur - 5vs) + t(4u^2 + s^2 - 4rt + 3su + 6ur + 4vs) + u(s^2 - t^2 - rt - 10su + 4ur - 5vs)\Big],$$

Equation V is:

$$r\left[R_{12}(o) + R_{12}(3) + R_{12}(4) - R_{12}(10)\right] - v\left[R_{12}(2) + R_{12}(3) + R_{12}(8)\right] + u\left[R_{12}(4) + R_{12}(8)\right] - vrP(o)/s = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & r\,(t^2 + 3u^2 - su - 3tv + 4ur + 6sv) \\ & + u(t^2 + 3rt - su - tv + ur + 3vs) \\ & - v(t^2 + 4u^2 + 6rt - su - 2tv + 4ur)\Big], \end{aligned}$$

Equation VI is:

$$t\left[R_{12}(o) - R_{12}(3) - R_{12}(4) + R_{12}(5)\right] - u\left[R_{12}(1) + R_{12}(5) - R_{12}(9)\right] - r\left[(R_{12}(1) + R_{12}(3) + R_{12}(5)\right] = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & -r(-s^2 + 4u^2 + 2v^2 - 2su - tv + 8ur - vs) + \\ & + t(-5u^2 + 2v^2 - 5rt + 3su - 4ur - vs) - \\ & - u(-s^2 + 2v^2 + rt + ur - vs)\Big], \end{aligned}$$

Equation VII is:

$$u\left[R_{12}(1) + R_{12}(2) - R_{12}(4) - R_{12}(10)\right] + t\left[-R_{12}(1) + R_{12}(4) + R_{12}(5)\right] + tR_{12}(6)/s - tP(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & t(s^2 + 4u^2 + 2v^2 + su - tv - ur + 2vs) + \\ & + u(-s^2 + t^2 + 3rt - su - 4tv + 4ur - 2vs)\Big], \end{aligned}$$

Equation VIII is:

$$-u\left[R_{12}(0) + R_{12}(2) - R_{12}(5) + R_{12}(7)\right] - r\left[R_{12}(3) - R_{12}(5) + R_{12}(7)\right] + R_{12}(6) + (2r - u + ru/s)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & -r(2s^2 + 4u^2 - 2v^2 - 8su - 4tv + 5ur - vs) \\ & + u(r^2 - 2s^2 + 2v^2 - 9rt + 4su - tv + 4ur - vs) \\ & + (-2rsu - stv + 4tur - vrt)\Big]. \end{aligned}$$

Equation IX is:

$$u\left[R_{12}(1) + R_{12}(4) - R_{12}(8)\right] + s\left[-R_{12}(3) + R_{12}(4) + \right.$$
$$\left. + R_{12}(7) - R_{12}(8)\right] - R_{12}(6) - 2(r + rs/v)P(o) = 0,$$
$$\text{L.H.S.} = D\left[-2r(rt + su + tv + ur + vs) + \right.$$
$$+ s(2r^2 - t^2 - 4u^2 - 5rt + su + 4tv - 3ur + 3vs) +$$
$$+ u(-s^2 - t^2 + 3rt - 2su + 4tv + vs)$$
$$\left. -(-2rsu - stv + 4tur - vrt)\right],$$

Equation X is:

$$t\left[(R_{12}(2) + R_{12}(4) - R_{12}(7) + R_{12}(9)\right] + s\left[-R_{12}(2) + \right.$$
$$\left. + R_{12}(5) + R_{12}(8)\right] - r\left[R_{12}(5) + R_{12}(7)\right] - 2tP(o) = 0,$$
$$\text{L.H.S.} = D\left[-r(2v^2 + 2s^2 - 3su - tv + 7ur + vs) + \right.$$
$$+ s(t^2 + 2v^2 - 6rt - 4tv + 5ur + vs) +$$
$$\left. + t(-2s^2 + 3rt + tv - 7ur - 2vs)\right].$$

Equation XI is:

$$-s\left[R_{12}(3) + R_{12}(5) - R_{12}(9) + R_{12}(10)\right] + r\left[R_{12}(5) + \right.$$
$$\left. + R_{12}(8) + R_{12}(10)\right] + rR_{12}(6)/v - stP(o)/u = 0,$$
$$\text{L.H.S.} = D\left[r(-2s^2 + 2v^2 + 2su + 3tv - 3ur - vs) \right.$$
$$+ s(t^2 - 4u^2 - 2v^2 - 2rt - 2su - 4tv - ur + vs) +$$
$$\left. + t(rt + su + tv + ur + vs)\right].$$

For the third set of results we have the following equations:-

Equation 1 is:

$$tR_{23}(0) + uR_{23}(4) + R_{23}(6) + rR_{23}(7) - vR_{23}(9) + sR_{23}(10) -$$
$$- (-r + 2s - 2t + u)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & r(-s^2 + rt + 4su + 2tv - 3ur - vs) + \\ & + s(2t^2 - rt - su - 8tv + vs) + \\ & + t(3su + 2tv + 7ur + vs) - \\ & - u(2sv + 2tv + 5vs) + v(r^2 + su + 3ur) + \\ & + (-3tur + 2uvs - vrt)\Big] . \end{aligned}$$

Equation II is:

$$s\left[R_{23}(0) - R_{23}(5) + R_{23}(8)\right] + u\left[R_{23}(1) - R_{23}(5) + R_{23}(8)\right] +$$
$$+ r\left[R_{23}(1) - R_{23}(7) - R_{23}(8) + R_{23}(10)\right] + 2uP(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & r(s^2 + 2t^2 - 4su - 9tv + 5ur + 7vs) + \\ & + s(3v^2 + su + tv + 5ur - 2vs) + \\ & + u(3v^2 + 5rt + 3tv + 2vs)\Big] . \end{aligned}$$

Equation III is:

$$-v\left[R_{23}(0) + R_{23}(1) + R_{23}(8) - R_{23}(9)\right] + u\left[R_{23}(2) + R_{23}(8) -\right.$$
$$\left.- R_{23}(9)\right] + uR_{23}(6)/t + (v - r - u)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\Big[ & -r(rt + su + tv + ur + vs) + \\ & + u(4r^2 + 2v^2 + rt - su + tv) + \\ & + v(-r^2 + su - tv - 4ur + 2vs)\Big] . \end{aligned}$$

Equation IV is:

$$u\left[R_{23}(1) + R_{23}(7) + R_{23}(9) + R_{23}(10)\right] - t\left[R_{23}(1) + R_{23}(2) - \right.$$
$$\left. - R_{23}(3)\right] + s\left[R_{23}(2) + R_{23}(10)\right] + (3u - t + s)P(o) = 0,$$

$$\text{L.H.S.} = D\left[ s(3r^2 + 2t^2 + rt + su - 5tv + 5ur + 5vs) + \right.$$
$$+ t(-3r^2 - 4u^2 - 2rt + 2su - 2ur - 3vs) +$$
$$\left. + u(-r^2 - s^2 + 2t^2 + 5rt + 4su - 2tv - 4ur + 5vs\right].$$

Equation V is:

$$r\left[R_{23}(0) + R_{23}(3) + R_{23}(4) - R_{23}(10)\right] - v\left[R_{23}(2) + R_{23}(3) + \right.$$
$$\left. + R_{23}(8)\right] + u\left[R_{23}(4) + R_{23}(8)\right] = 0,$$

$$\text{L.H.S.} = D\left[ r(-2t^2 - 4u^2 - 3rt - su + 6tv + 3ur - 8vs) - \right.$$
$$\left. - v(3r^2 - 4u^2 - su + 3tv) + u(3rt - su + tv - 5vs)\right].$$

Equation VI is:

$$t\left[R_{23}(0) - R_{23}(3) - R_{23}(4) + R_{23}(5)\right] - u\left[R_{23}(1) + R_{23}(5) - \right.$$
$$\left. - R_{23}(9)\right] - r\left[(R_{23}(1) + R_{23}(3) + R_{23}(5)\right] - tP(o) = 0.$$

$$\text{L.H.S.} = D\left[ r(4u^2 + 3v^2 + 2su - 2vt + 4ur - vs) + \right.$$
$$+ t(4u^2 - 3v^2 - 3rt + su + 3ur + 2vs) -$$
$$\left. - u(r^2 - 3v^2 + rt - su + tv + vs)\right].$$

Equation VII is:

$$u\left[R_{23}(1) + R_{23}(2) - R_{23}(4) - R_{23}(10)\right] + t\left[-R_{23}(1) + R_{23}(4) + \right.$$
$$\left. + R_{23}(5)\right] + tR_{23}(6)/s + 2tP(o) = 0,$$

$$\text{L.H.S.} = D\left[ t(-3u^2 - 3v^2 + 3rt + 3su + 2tv + 2ur - 5vs) + \right.$$
$$\left. + u(3r^2 - 2t^2 - 2rt - 3su + 7tv - 3ur + 3vs)\right].$$

Equation VIII is:

$$-u\left[R_{23}(o) + R_{23}(2) - R_{23}(5) + R_{23}(7)\right] - r\left[R_{23}(3) - R_{23}(5) + R_{23}(7)\right] + R_{23}(6) - rP(o) = 0,$$

$$\text{L.H.S.} = D\left[u(-3r^2 + s^2 - 3v^2 + 3rt - 3su - 4ur + 2vs) - r(-s^2 - 4u^2 + 3v^2 + 4su + 2tv - 3ur - vs) + (-3tur + 2uvs - vrt)\right].$$

Equation IX is:

$$u\left[R_{23}(1) + R_{23}(4) - R_{23}(8)\right] + s\left[-R_{23}(3) + R_{23}(4) + R_{23}(7) - R_{23}(8)\right] - R_{23}(6) + (s - us/v)\,P(o) = 0,$$

$$\text{L.H.S.} = D\left[-u(3su + 3tv + 2vs) + s(3u^2 + rt - 3tv + 3ur - 4vs) + (3tur - 2uvs + vrt)\right].$$

Equation X is:

$$t\left[R_{23}(2) + R_{23}(4) - R_{23}(7) + R_{23}(9)\right] + s\left[-R_{23}(2) + R_{23}(5) + R_{23}(8)\right] - r\left[R_{23}(5) + R_{23}(7)\right] + (2t - s + r)P(o) = 0,$$

$$\text{L.H.S.} = D\left[-r(-s^2 - 3v^2 - rt + 2su + tv - 6ur - 3vs) + s(-3r^2 - 3v^2 + 2rt + 2tv - 5ur - 3vs) + t(2r^2 + s^2 + 2rt - 4su + 6ur + vs)\right].$$

Equation XI is:

$$-s\left[R_{23}(3) + R_{23}(5) - R_{23}(9) + R_{23}(10)\right] + r\left[R_{23}(5) + R_{23}(8) + R_{23}(10)\right] + rR_{23}(6)/v + (2st/u + r)P(o) = 0,$$

$$\text{L.H.S.} = D\left[r(2t^2 - 3v^2 + rt - su - 4tv + 2ur + vs) - s(r^2 + 2t^2 - 4u^2 - 3v^2 - 2rt - 6tv + ur + vs) - 2t(rt + su + tv + ur + vs)\right].$$

For the fourth set of results (7.34) to (7.44) we have the following equations:

Equation 1 is:

$$tR_{34}(o) + uR_{34}(4) + R_{34}(6) + rR_{34}(7) - vR_{34}(9) + sR_{34}(10) -$$
$$-(r - s + t - u)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\big[ & t(3rt - 2su - tv - 4ur - 3vs) + \\ & + u(-v^2 - 2rt + 2su + 2tv + 4vs) + \\ & + (rsu - stv + tur + 3vrt) + \\ & + r(-2rt - 2su - tv + 2ur - vs) - \\ & - v(2r^2 - rt + su + tv + 2ur + vs) + \\ & + s(-t^2 - rt + 5tv - ur + 2vs)\big] . \end{aligned}$$

Equation II is:

$$s\left[R_{34}(o) - R_{34}(5) + R_{34}(8)\right] + u\left[R_{34}(1) - R_{34}(5) + R_{34}(8)\right] +$$
$$+ r\left[R_{34}(1) - R_{34}(7) - R_{34}(8) + R_{34}(10)\right] - uP(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\big[ & r(s^2 - t^2 + su + 3tv - 5ur + 2vs) + \\ & + s(2tv - 3ur + 2vs) + \\ & + u(s^2 - 7rt + su + tv + 4vs)\big] . \end{aligned}$$

Equation III is:

$$-v\left[R_{34}(o) + R_{34}(1) + R_{34}(8) - R_{34}(9)\right] + u\left[R_{34}(2) + R_{34}(8) -\right.$$
$$\left. - R_{34}(9)\right] + uR_{34}(6)/t - 2(v - r - u)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\big[ & 2r(rt + su + tv + ur + vs) + \\ & + u(-6r^2 - 6rt + su + 2tv + 5sv) - \\ & - v(-2r^2 + s^2 + 2rt + su + 2tv - ur)\big] . \end{aligned}$$

Equation IV is:

$$u\left[R_{34}(1) + R_{34}(7) + R_{34}(9) + R_{34}(10)\right] - t\left[R_{34}(1) + R_{34}(2) - \right.$$
$$\left. - R_{34}(3)\right] + s\left[R_{34}(2) + R_{34}(10)\right] - uP(o) = 0,$$

$$\text{L.H.S.} = D\Big[\, s(-4r^2 - t^2 - 5rt + 4tv - 4ur + 2vs) -$$
$$-t(-4r^2 + s^2 - 3u^2 - 7\,rt + su - 2ur + 3vs) +$$
$$+ u(2r^2 - t^2 + s^2 - 7rt - su + 4tv + 3ur + 2vs)\Big] .$$

Equation V is:

$$r\left[R_{34}(o) + R_{34}(3) + R_{34}(4) - R_{34}(10)\right] - v\left[R_{34}(2) + R_{34}(3) + \right.$$
$$\left. + R_{34}(8)\right] + u\left[R_{34}(4) + R_{34}(8)\right] - rP(o) = 0,$$

$$\text{L.H.S.} = D\Big[\, r(t^2 + 3u^2 - v^2 + 2rt + su - 4tv = 2ur - 2vs) -$$
$$-v(-4r^2 + 3u^2 - 4rt + 2su + tv) +$$
$$+u(-v^2 - 6rt + su + 2tv + 3vs)\Big] .$$

Equation VI is:

$$t\left[R_{34}(o) - R_{34}(3) - R_{34}(4) + R_{34}(5)\right] - u\left[R_{34}(1) + R_{34}(5) - \right.$$
$$\left. - R_{34}(9)\right] - r\left[R_{34}(1) + R_{34}(3) + R_{34}(5)\right] + (r + 2t)P(o) = 0,$$

$$\text{L.H.S.} = D\Big[-r(s^2 + 3u^2 - 2tv + 2ur - 5vs) +$$
$$+ t(-3u^2 + v^2 + 8rt - 2su - 6vs) -$$
$$- u(-2r^2 + s^2 - su - 2tv - 4vs)\Big] .$$

Equation VII is:

$$u\left[R_{34}(1) + R_{34}(2) - R_{34}(4) - R_{34}(10)\right] + t\left[-R_{34}(1) + R_{34}(4) + \right.$$
$$\left. + R_{34}(5)\right] + tR_{34}(6)/s + (v - u)\,P(o) = 0,$$

$$\text{L.H.S.} = D\Big[\, u(-4r^2 + s^2 + t^2 + v^2 - rt + su - 6tv + ur - 3vs) +$$
$$+ t(-s^2 + u^2 - v^2 - 3rt - su - tv + 4ur - 2vs) +$$
$$+ v(rt + su + tv + ur + vs)\Big].$$

Equation VIII is:

$$-u\left[R_{34}(o) + R_{34}(2) - R_{34}(5) + R_{34}(7)\right] - r\left[R_{34}(3) - R_{34}(5) + \right.$$
$$\left. + R_{34}(7)\right] + R_{34}(6) = 0,$$
$$\text{L.H.S.} = D\left[-r(3u^2 + su + tv + 3ur + 3vs) + \right.$$
$$+ u(4r^2 + rt\ )\ tv + 3ur - 3vs) +$$
$$\left. + (rsu - stv + tur) + 3vrt)\right].$$

Equation IX is:

$$u\left[R_{34}(1) + R_{34}(4) - R_{34}(8)\right] + s\left[- R_{34}(3) + R_{34}(4) + \right.$$
$$\left. + R_{34}(7) - R_{34}(8)\right] - R_{34}(6) + uP(o) + 0,$$
$$\text{L.H.S.} = D\left[u(s^2 - v^2 - rt + 3su + tv + 5vs) + \right.$$
$$+ s(-3u^2 - v^2 - 3rt - su + 4sv) -$$
$$\left. - (rsu - stv + tur + 3vrt)\right].$$

Equation X is:

$$t\left[R_{34}(2) + R_{34}(4) - R_{34}(7) + R_{34}(9)\right] + s\left[(-R_{34}(2) + \right.$$
$$\left. + R_{34}(5) + R_{34}(8)\right] - r\left[R_{34}(5) + R_{34}(7)\right] + 2(s - t - r)P(o) = 0,$$
$$\text{L.H.S.} = D\left[-r(2rt + tv + 6ur - 2vs) + \right.$$
$$+ s(4r^2 + 3rt + 2tv + 6ur - 3vs) +$$
$$\left. + t(-2r^2 - v^2 - 8rt + 2su - 6ur + 3vs)\right].$$

Equation XI is:

$$-s\left[R_{34}(3) + R_{34}(5) - R_{34}(9) + R_{34}(10)\right] + r\left[R_{34}(5) + R_{34}(8) + \right.$$
$$\left. + R_{34}(10)\right] + rR_{34}(6)/v - (s - t + r)P(o) = 0,$$
$$\text{L.H.S.} = D\left[r(s^2 - t^2 - 2rt - 2su + 4tv - 2ur - 4vs) - \right.$$
$$-s(-2r^2 - t^2 + 3u^2 + 3rt - su + 3tv - ur - 4vs) +$$
$$\left. + t(rt + su + tv + ur + vs)\right].$$

Finally, for the last set of results we have the following equations:-

Equation I is:

$$tR_{45}(o) + uR_{45}(4) + R_{45}(6) + rR_{45}(7) - vR_{45}(9) + sR_{45}(10) + $$
$$+ (v - u)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\big[\, & r(rt - 2su - 2ur + vs) + \\ & + s(2rt + 2ur - 3vs) + \\ & + t\,(-3rt - su + ur + 3vs) + \\ & + u(2v^2 + rt + su - tv - 4vs) + \\ & + v(r^2 + 2rt + tv + 2ur + 2vs) + \\ & + (rsu + stv - 2uvs - 3vrt)\big] . \end{aligned}$$

Equation II is:

$$s\left[R_{45}(o) - R_{45}(5) + R_{45}(8)\right] + u\left[R_{45}(1) - R_{45}(5) + R_{45}(8)\right] +$$
$$+ r\left[R_{45}(1) - R_{45}(7) - R_{45}(8) + R_{45}(10)\right] = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\big[\, & r(s^2 - t^2 + 3su + 2tv + 5ur - 7vs) + \\ & + s(t^2 - 2v^2 - 3tv + ur - vs) + \\ & + u(s^2 + t^2 - 2v^2 + 4rt + 3su - 3tv - 6vs)\big] . \end{aligned}$$

Equation III is:

$$-v\left[R_{45}(o) + R_{45}(1) + R_{45}(8) - R_{45}(9)\right] + u\left[R_{45}(2) + R_{45}(8) -\right.$$
$$\left. - R_{45}(9)\right] + uR_{45}(6)/t + (v - r - u)P(o) = 0,$$

$$\begin{aligned} \text{L.H.S.} = D\big[\, & -r(rt + su + tv + ur + vs) \\ & + u(4r^2 + t^2 - 2v^2 + 7rt - 2su - 3tv - 4vs) - \\ & -v(s^2 + t^2 + r^2 - 3tv + 2vs)\big] . \end{aligned}$$

Equation IV is:

$$u\left[R_{45}(1) + R_{45}(7) + R_{45}(9) + R_{45}(10)\right] - t\left[R_{45}(1) + R_{45}(2) - R_{45}(3)\right] + s\left[R_{45}(2) + R_{45}(10)\right] = 0,$$

$$\text{L.H.S.} = D\Big[s(3r^2 + 4rt + 3ur - 5vs) - t(3r^2 + s^2 + u^2 + 5rt + 2su + 4ur - 5vs) + u(-r^2 + s^2 + 3rt + su - ur - 5vs)\Big].$$

Equation V is:

$$r\left[R_{45}(o) + R_{45}(3) + R_{45}(4) - R_{45}(10)\right] - v\left[R_{45}(2) + R_{45}(3) + R_{45}(8)\right] + u\left[R_{45}(4) + R_{45}(8)\right] + 2rP(o) = 0,$$

$$\text{L.H.S.} = D\Big[r(-u^2 + 2v^2 - 3rt + 3su + 2tv - ur + 6vs) - v(3r^2 + t^2 - u^2 + 2rt + su - 2tv - 3ur) + u(t^2 + 2v^2 + 4rt + 3su - 2tv - 2sv)\Big].$$

Equation VI is:

$$t\left[R_{45}(o) - R_{45}(3) - R_{45}(4) + R_{45}(5)\right] - u\left[R_{45}(1) + R_{45}(5) - R_{45}(9)\right] - r\left[R_{45}(1) + R_{45}(3) + R_{45}(5)\right] - (2r + t)P(o) = 0,$$

$$\text{L.H.S.} = D\Big[-r(s^2 - u^2 + 2v^2 + 4su + 3tv - 2ur + 6vs) + t(u^2 - 5rt - 4su + ur + 9vs) - u(r^2 + s^2 + 2v^2 + rt + su + tv + 4vs)\Big].$$

Equation VII is:

$$u\left[R_{45}(1) + R_{45}(2) - R_{45}(4) - R_{45}(10)\right] + t\left[-R_{45}(1) + R_{45}(4) + R_{45}(5)\right] + tR_{45}(6)/s + 2(u - v - t)P(o) = 0,$$

$$\text{L.H.S.} = D\Big[t(-s^2 + 4v^2 + rt - 2su - 6ur + 4vs) + u(3r^2 + s^2 - 2v^2 + rt + 2su + 2tv + 4vs) - 2v(rt + su + tv + ur + vs)\Big].$$

Equation VIII is:

$$-u\left[R_{45}(o) + R_{45}(2) - R_{45}(5) + R_{45}(7)\right] - r\left[R_{45}(3) - \right.$$
$$\left. -R_{45}(5) + R_{45}(7)\right] + R_{45}(6) = 0,$$
$$\text{L.H.S.} = D\left[r(u^2 + 2v^2 + 2su + tv + 4ur + 3vs) - \right.$$
$$-u(3r^2 - 2v^2 + rt - 3su - tv + ur - 3vs) +$$
$$\left. + (rsu + stv - 2uvs - 3vrt)\right].$$

Equation IX is:

$$u\left[R_{45}(1) + R_{45}(4) - R_{45}(8)\right] + s\left[-R_{45}(3) + R_{45}(4) + \right.$$
$$\left. + R_{45}(7) - R_{45}(8)\right] - R_{45}(6) - 2UP(o) = 0,$$
$$\text{L.H.S.} = D\left[s(-t^2 + u^2 + 2v^2 + 3rt - su + 2tv + 3ur - 4vs) + \right.$$
$$+ u(s^2 - t^2 + 2v^2 - rt + su - 8vs) -$$
$$\left. - (rsu + stv - 2uvs - 3vrt)\right].$$

Equation X is:

$$t\left[R_{45}(2) + R_{45}(4) - R_{45}(7) + R_{45}(9)\right] + s\left[-R_{45}(2) + R_{45}(5) + \right.$$
$$\left. + R_{45}(8)\right] - r\left[R_{45}(5) + R_{45}(7)\right] - (s - t - r)P(o) = 0,$$
$$\text{L.H.S.} = D\left[-r(2v^2 - rt - 3su - 4ur + 5vs) + \right.$$
$$+ s(-3r^2 + t^2 + 2v^2 - 2rt + 2tv - 4ur + 7vs) +$$
$$\left. + t(2r^2 + 2v^2 + 3rt + 6su + tv + 4ur - 6vs)\right].$$

Equation XI:

$$-s\left[R_{45}(3) + R_{45}(5) - R_{45}(9) + R_{45}(10)\right] + r\left[R_{45}(5) + \right.$$
$$\left. + R_{45}(8) + R_{45}(10)\right] + rR_{45}(6)/v + rP(o) = 0,$$
$$\text{L.H.S.} = D\left[r(s^2 + t^2 + 2v^2 + rt + 2su - tv + ur + 4vs) - \right.$$
$$\left. - s(r^2 - u^2 + 2v^2 - su + tv - ur + 4vs)\right].$$

This completes the Proof of the Theorem.

Table 1. ($q = 5$).

This table gives us the values of $S(b)$ when $b = 0$ and $1$. The rule is to

(i) select the elements in the column headed by $S(b)$,

(ii) multiply each by the power of $x$ noted opposite to it in the first column, and

(iii) finally add all such products. For instance,

$$S(1) = -g(2)x^0 - \left[P(0)/P(2) + y\Pi(1-x^2)\zeta(2,0)/P(0)\right]x^3.$$

The same rule holds also in the cases when $q = 7$ and $q = 11$.

---

$S(2) = 0$, $S(3) = -S(1)$, $S(4) = -S(0)$, and $S(5) = S(0) + 1 - \Pi(1-x^2)$.

| | $S(0)$ | $S(1)$ |
|---|---|---|
| $x^0 \times$ | $g(1) - y\Pi(1-x^2)\zeta(1,0)/P(0)$ | $-g(2)$ |
| $+\ x \times$ | 0 | 0 |
| $+\ x^2 \times$ | $-P^2(0)/P(1)$ | 0 |
| $+\ x^3 \times$ | 0 | $-P^2(0)/P(2) - y\Pi(1-x^2)\zeta(2,0)/P(0)$ |
| $+\ x^4 \times$ | 0 | 0 |

Table 2. ($q = 7$).

$S(3) = 0$, $S(4) = -S(2)$, $S(5) = -S(1)$, $S(6) = -S(0)$, and $S(7) = S(0)+1-\Pi(1-x^2)$.

| | | S(0) | S(1) | S(2) |
|---|---|---|---|---|
| $x^0$ | X | $-1-g(1)+\Pi(1-x^2)\left[1+y\frac{\zeta(1,0)}{P(0)}\right]$ | $-g(3)$ | $g(2)$ |
| $+x$ | X | $\frac{P^2(0)P(2)}{P^2(1)}$ | 0 | 0 |
| $+x^2$ | X | 0 | $y\frac{P^2(0)P(1)}{P^2(3)}+y^2\Pi(1-x^2)\frac{\zeta(3,0)}{P(0)}$ | 0 |
| $+x^3$ | X | 0 | $-\frac{P^2(0)}{P(1)}$ | 0 |
| $+x^4$ | X | 0 | 0 | $-\frac{P^2(0)P(3)}{P^2(2)}$ |
| $+x^5$ | X | $-\frac{P^2(0)}{P(2)}$ | 0 | 0 |
| $+x^6$ | X | 0 | 0 | $\frac{P^2(0)}{P(3)}+y\Pi(1-x^2)\frac{\zeta(2,0)}{P(0)}$ |

Table 3.(q =11).

$S(5) = 0, S(6) = -S(4), S(7) = -S(3), S(8) = -S(2), S(9) = -S(1), S(10) = -S(0)$, and $S(11) = 1 + S(0) - \Pi(1-x^2)$

| | $S(0)$ | $S(1)$ | $S(2)$ | $S(3)$ | $S(4)$ |
|---|---|---|---|---|---|
| $x^0 \times$ | $g(2) - y^2\Pi(1-x^2)\Sigma(2,0)/P(0)$ | $-g(5)$ | $-1-g(1)$ | $g(3)$ | $-g(4)$ |
| $+x \times$ | $\frac{yP^2(0)}{P(3)}$ | 0 | $-\frac{P^2(0)P(3)P(5)}{P(1)P(2)P(4)}$ | 0 | 0 |
| $+x^2 \times$ | $-\frac{P^2(0)P(3)}{P(1)P(2)}$ | 0 | 0 | $\frac{yP^2(0)P(1)P(5)}{P(2)P^2(3)}$ | 0 |
| $+x^3 \times$ | 0 | $-\frac{P^2(0)P(3)P(4)}{P(1)P(2)P(5)}$ | $-\frac{yP^2(0)}{P(4)}$ | 0 | 0 |
| $+x^4 \times$ | $-\frac{yP^2(0)P(1)P(5)}{P(2)P(3)P(4)}$ | 0 | 0 | 0 | $yP^2(0)/P(5) + y^3\Pi(1-x^2)\Sigma(4,0)/P(0)$ |
| $+x^5 \times$ | 0 | 0 | 0 | $-\frac{P^2(0)}{P(1)}$ | $-y^3\frac{P^2(0)P(1)P(2)}{P(3)P(4)P(5)}$ |
| $+x^6 \times$ | 0 | 0 | $\frac{P^2(0)P(4)}{P(1)P(5)}$ | 0 | $-\frac{P^2(0)P(3)P(5)}{P(1)P^2(4)}$ |
| $+x^7 \times$ | $\frac{P^2(0)P(3)P(4)}{P^2(2)P(5)}$ | $-yP^2(0)P(2)/P(3)P(5) - y^3\Pi(1-x^2)\Sigma(5,0)/P(0)$ | 0 | 0 | 0 |
| $+x^8 \times$ | 0 | $\frac{y^2P^2(0)P(1)P(2)}{P(4)P^2(5)}$ | 0 | 0 | $\frac{P^2(0)P(5)}{P(2)P(4)}$ |
| $+x^9 \times$ | 0 | $\frac{P(0)}{P(2)}$ | 0 | $\frac{P^2(0)P(2)P(4)}{P(1)P(3)P(5)} + \frac{y^2\Pi(1-x^2)\Sigma(3,0)}{P(0)}$ | 0 |
| $+x^{10} \times$ | 0 | 0 | $-\Pi(1-x^2)\left[\frac{\Sigma(1,0)}{P(0)} + \frac{1}{y}\right] + \frac{P^2(0)P(2)P(4)}{yP^2(1)P(3)}$ | $-\frac{yP^2(0)P(1)}{P(3)P(4)}$ | 0 |

## NOTATION.

$\Delta_5$ and $\Delta_7$ are defined in § 2.

$\Delta_{11}$ is defined in § 5.

For $q = 11$,

$r = -y^2 P(1)/P(3)P(5)$,

$s = -yP(2)/P(1)P(5)$,

$t = P(4)/P(1)P(2)$,

$u = yP(3)/P(2)P(4)$,

$v = yP(5)/P(3)P(4)$,

$\alpha = r/s, \quad \beta = s/t, \quad \gamma = t/u, \quad \delta = u/v, \quad \epsilon = v/r$,

$\lambda = \Sigma \alpha\gamma \quad , \qquad \mu = \Sigma \alpha^2\gamma \quad .$

$a = -x^{-4} P(2)/P(1), \quad b = x^{-5} P(4)/P(2), \quad c^2 = x P(3)/P(4)$,

$d = -x^{-3} P(5)/P(3), \quad e = -x^{10} P(1)/P(5)$,

$D = P(o)/(rt + su + tv + ur + vs)$.

$F(\nu_1)$ is defined in § 4.

$R_{bc}(d)$ for different values of b, c, and d is defined in § 6.

There are other notations which last only for the length of the relevant proof. These are:

a, b, c, d, e and $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ in Lemmas 1 and 2;

A, B, C, D, E in § 4;

$\omega_\rho$ in § 2, § 4 and § 5;

and $\rho_1(d)$, $\rho_2(d)$, $\rho_3(d)$ in § 6.

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