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# Isospectral Deformations of Eguchi-Hanson Spaces: a Case Study in Noncompact Noncommutative Geometry

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A Thesis presented for the degree of
Doctor of Philosophy

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*Dedicated to*

My parents Bin-Hao Yang and Hui-Lin Gu.

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## Abstract

We study the isospectral deformations of the Eguchi-Hanson spaces along a torus isometric action in the noncompact noncommutative geometry. We concentrate on locality, smoothness and summability conditions of the nonunital spectral triples, and relate them to some geometric conditions to be noncommutative spin manifolds.

# Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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# Contents

# Chapter 1

# Introduction

A fundamental theorem given by Gelfand and Naimark [1] of $C^*$-algebras shows the equivalence between the category of commutative $C^*$-algebras with $*$-homomorphisms and the category of Hausdorff locally compact spaces with base-point preserving continuous maps. Under this correspondence, a locally compact Hausdorff topological space $X$ is mapped to the $C^*$-algebra $C_0(X)$ of continuous complex valued functions of $X$ vanishing at infinity. Conversely, a commutative $C^*$-algebra $A$ is mapped to the space of characters $M(A)$ consisting of algebra homomorphisms from $A$ to $\mathbb{C}$. $M(A)$ is endowed with a Hausdorff topology defined by the supremum norm. This correspondence motivates the idea to study topological spaces through the theory of $C^*$-algebra.

Around the 80's, Alain Connes started to generalise this idea to study Riemannian manifolds through algebraic data. By further allowing noncommutativity of the algebra, he laid the foundation of *noncommutative differential geometry.* Before a noncommutative geometry can be obtained, the key is to rewrite a Riemannian geometry algebraically. The main issue in doing this is to first find the algebraic descriptions of coordinate charts so that a differentiable manifold is obtained and secondly the description of a Riemannian metric. This fundamental problem of reconstruction Riemannian manifolds in noncommutative geometry was announced by Connes in 1996 [2], where compact and spin or spin$^c$ Riemannian manifolds are considered as a first attempt.

From a compact Riemanian spin manifold $M$ of metric $g$ and spinor bundle $\mathcal{S}$, one can extract algebraic information as follows: (1) the algebra $\mathcal{A} = C^\infty(M)$ of smooth complex-valued functions of $M$; (2) the space $\Gamma^\infty(M, \mathcal{S})$ of smooth sections of the spinor bundle $\mathcal{S}$, which is a finitely generated projective $\mathcal{A}$-module by the Serre-Swan theorem [3]; (3) the Dirac operator $D$ as a first order differential operator acting on the Hilbert space completion $\mathcal{H}$ of $\Gamma^\infty(M, \mathcal{S})$ under the usual $L^2$-norm. The algebraic data $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is represented as operators on $\mathcal{H}$, is called the *spectral triple* associated to $M$. From the geometrical properties of $M$, one may deduce the set of algebraic properties $\mathbb{X} = \{X_1, \ldots, X_i, \ldots\}$ of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

The reverse of this procedure can be considered as the reconstruction of Riemannian spin manifolds: *finding a finite set of geometric conditions from the set* $\mathbb{X}$ *of all the algebraic properties of any given commutative spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *so as to reconstruct a compact Riemannian spin manifold.* Here the commutativity of the spectral triple means the commutativity of the operator algebra $\mathcal{A}$.

The idea is that once such a set of geometric conditions is found for commutative spectral triples, they can be modified for noncommutativite spectral triples. The resulting set of geometric conditions (or axioms, if the independence of the conditions is shown) can be regarded as the definition of *noncommutative Riemannian manifolds.* To anticipate a bit more, characterisations of these noncommutative spectral triples can lead us to fundamental problems in describing the standard model coupled with gravity as explored in [4].

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ be a unital involutive commutative algebra represented on $\mathcal{H}$ and $\mathcal{D}$ be a self-adjoint operator on $\mathcal{H}$. The following are the geometric conditions of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ proposed by Connes [2] for a compact commutative Riemannian spin manifold.

1. *(Metric dimension.) The operator* $\mathcal{D}^{-1}$ *is an infinitesimal of order* $1/p$, *for a given positive integer* $p$.

   This means that $\mu_m(\mathcal{D}^{-1}) = \mathcal{O}(m^{-1/p})$ where $\mu_m(\mathcal{D}^{-1})$ is the decreasing sequence of eigenvalues of $|\mathcal{D}^{-1}|$.

2. *(First order.)* $[[\mathcal{D}, f], g] = 0$ *for any* $f, g \in \mathcal{A}$.

3. *(Regularity.) For any $a \in \mathcal{A}$ both $a$ and $[\mathcal{D}, a]$ belong to the domain of $\delta^m$ for any integer $m$.*

   $\delta\cdot$ is the derivation on the space of linear operators of $\mathcal{H}$ given by $[|\mathcal{D}|, \cdot]$.

4. *(Orientability.) For $p$ even, there exists a Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ and an operator $\chi$ such that $\pi_{\mathcal{D}}(c) = \chi$ satisfying*

$$\chi = \chi^*; \quad \chi^2 = 1, \quad \chi\mathcal{D} = -\mathcal{D}\chi.$$

   *For $p$ odd, $\pi_{\mathcal{D}}(c) = 1$ for some $p$-dimensional Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$.*

   The representation $\pi_{\mathcal{D}}$ of Hochschild cycle in $Z_p(\mathcal{A}, \mathcal{A})$ is defined by the formula

$$\pi_{\mathcal{D}}(a^0 \otimes a^1 \otimes \cdots \otimes a^p) := a^0[\mathcal{D}, a^1] \cdots [\mathcal{D}, a^n].$$

5. *(Finiteness and absolute continuity.) The space $\mathcal{H}_\infty := \cap_m Dom(\mathcal{D}^m)$ is a finite projective (left) $\mathcal{A}$-module. Moreover, the following equality defines a hermitian structure $(\cdot, \cdot)$ on the module by*

$$\langle a\xi, \eta \rangle = ⨍ a(\xi, \eta)|\mathcal{D}|^{-p}, \quad \forall a \in \mathcal{A}, \xi, \eta \in \mathcal{H}_\infty.$$

   $⨍$ is the Dixmier trace of measurable operators. The measurability of $a|\mathcal{D}|^{-p}$ for $a \in \mathcal{A}$ can be implied by the orientability condition and Connes' character theorem [5].

6. *(Poincaré duality.) The intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z}$ of $K$-groups of $\mathcal{A}$ is invertible.*

7. *(Reality.) There exists an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that $JaJ^{-1} = a^*$ for $a \in \mathcal{A}$ and $J^2 = \epsilon$, $J\mathcal{D} = \epsilon'\mathcal{D}J$, and $J\chi = \epsilon''\chi J$, where $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ are given by the following table from the value of $p$ modulo 8.*

How exactly these geometric conditions can be deduced from a compact Riemannian spin manifold is given as Theorem 11.1 in [6]. The converse is the reconstruction problem, which takes the following form [2] [6].

| p | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\epsilon$ | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| $\epsilon'$ | 1 | -1 | 1 | 1 | 1 | -1 | 1 | 1 |
| $\epsilon''$ | 1 | | -1 | | 1 | | -1 | |

**Theorem 1.0.1** *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ be a unital involutive algebra represented on $\mathcal{H}$ and $\mathcal{D}$ be a self-adjoint operator on $\mathcal{H}$.*

*(a) Let $\pi$ be any unitary representation of $\mathcal{A}$ and $\mathcal{D}$ on the Hilbert space $\mathcal{H}$ satisfying the above seven geometric conditions, then the spectrum of $\mathcal{A}$ defines a differentiable manifold $M$, such that $C^\infty(M) \cong \mathcal{A}$. There is a unique Riemannian metric $g$ of $M$ such that the geodesic distance between any two points $x, y \in M$ is given by*

$$d(x,y) = \sup\{|a(x) - a(y)| : a \in \mathcal{A}, \|[\mathcal{D}, \pi(a)]\| \leq 1\}. \tag{1.1}$$

*(b) The metric $g$ depends only on the unitary equivalence class $[\pi]$ of $\pi$. There is a finite collection of affine spaces of unitary equivalence classes $\{\mathcal{A}_\sigma\}$ in which each equivalence class gives rise to the metric $g$ as described in (a). The finite collection of affine spaces is parametrised by the spin structures $\sigma$ on $(M, g)$.*

*(c) The action functional $⨍ \cdot |\mathcal{D}|^{-(p-2)}$ is a positive quadratic form on each affine space $\mathcal{A}_\sigma$ with a unique minimum $\pi_\sigma$.*

*(d) The minimum $\pi_\sigma$ is the representation of $\mathcal{A}$ and $\mathcal{D}$ on $L^2(M, \mathcal{S}_\sigma)$ such that $\mathcal{A}$ is represented as multiplication of operators, while $\mathcal{D}$ is represented as the Dirac operator $D$ operating on the sections of spinor bundle $\mathcal{S}_\sigma$ with respect to the spin structure $\sigma$.*

*(e) The value of $⨍ |D|^{-(p-2)}$ on $\pi_\sigma$ is given by the Einstein-Hilbert action,*

$$-c_n \int R\sqrt{g} d^n x, \quad c_n = \frac{n-2}{12\,(4\pi)^{p/2}} \Gamma\left(\frac{p}{2} + 1\right)^{-1} 2^{[p/2]}.$$

The formulation and the proof of the theorem, as well as the phrasing of the geometric conditions themselves are considered extensively since then by [7] [6] [8] [9].

In [6] (Theorem 11.2), a simpler version of the above theorem is considered: the algebra $\mathcal{A}$ is assumed to be the space of smooth functions of some compact manifold to start with. This assumption is nontrivial and its proof is rather technically involved as the rest of the above references show.

Assuming the validity of the reconstruction theorem, Connes also provides the modification of the above geometric conditions in adapting to noncommutative spectral triples [2], where only the first order, Poincaré duality and orientability conditions require modification.

As a generalisation of the compact case, the noncompact noncommutative geometry is the study of nonunital spectral triples [10] [11] [12]. Geometric conditions for nonunital spectral triples are conjectured by [13] and [12] for the noncommutative case directly, which we will present in chapter 9. The credibility of these conditions to serve as the definition of noncommutative noncompact manifolds will be determined by a reconstruction theorem for noncompact commutative manifolds assuming the commutative version of the conjectured conditions. It is however still too soon to make any conclusion along this line. At the moment, we may look at nonunital examples, commutative or not, and see how the nonunital geometric conditions fit examples.

There are various nonunital examples [13] [14] [12] [15]. In this dissertation, we follow the construction of [14] to find nonunital spectral triples as isospectral deformations of Eguchi-Hanson (EH-) spaces [16], which are geodesically complete noncompact Riemannian spin manifolds.

The Eguchi-Hanson spaces are of interest in both Riemannian geometry and physics. Geometrically, they are the simplest asymptotic locally Euclidean (ALE) spaces, for which a complete classification is provided by Kronheimer through the method of hyper-Kähler quotients [17]. This construction realises the family of EH-spaces as a resolution of a singular conifold. In physics, where they first appeared, EH-spaces are known as gravitational instantons. Due to their hyper-Kähler structures, the ADHM construction [18] of Yang-Mills instantons, is generalised on the EH-spaces in an elegant way [19] [20]. The nonunital spectral triples from isospectral deformations of Eguchi-Hanson spaces may thus link various perspectives.

Isospectral deformation is a simple method to deform a commutative spectral triple. It traces back to the Moyal type of deformation from quantum mechanics. Rieffel's insight is to consider Lie group actions on function spaces and hence explain the Moyal product between functions by oscillatory integrals over the group actions [21]. Apart from the well-known Moyal planes and noncommutative tori [22], this scheme allows more general deformations. Connes and Landi in [23] deform spheres and more general compact spin manifold with isometry group containing a two-torus. Connes and Dubois-Violette in [14] observe that this works equally well for noncompact spin manifolds. We will obtain the isospectral deformation of EH-spaces in this way.

As in the appendix of [10], it is possible to realise such noncompact examples in the nonunital framework there. Closely assocatied to the geometric conditions of nonunital spectral triples, there are some analytical and homological properties: locality, smoothness and summability [10] [11]. Our aim is to concentrate on these properties of the deformed spectral triples of the EH-spaces and further see how the modified geometric conditions follows. We will however leave out an important condition from the point of view of the reconstruction problem, namely the Poincaré duality, for future work.

The organisation of the rest of the dissertation is as follows. Chapter two to five serve as preliminary material. They describe operator algebras, noncommutative integration, $C^*$-modules and spin geometry. The main results are contained in chapters six onwards. In chapter six, we describe Eguchi-Hanson spaces in spin geometry. In chapter seven, we consider algebras of functions over EH-spaces, the deformation quantization of algebras, and representations of algebras as operators on the Hilbert space of spinors. We also obtain the projective module description of the spinor bundle of the EH-space. In chapter eight, we define spectral triples of the deformed EH-spaces and study their summability properties. In chapter nine, we examine how the spectral triple fits into the modified nonunital geometric conditions. In chapter ten, we draw conclusions.

# Chapter 2

# Operator algebras

The first two sections of this chapter are on Fréchet spaces and some examples. The references are [24] [25]. The rest of the chapter is on $C^*$-algebras, where we refer to [26] [27] [28] [6] for references.

## 2.1 Fréchet spaces

A *topological vector space* is a vector space endowed with a topology in such a way that the scalar multiplication and addition of the vector space is continuous with respect to the underlying topology.

A *seminorm* on a vector space $V$ is a map $q : V \to [0, \infty)$ such that $q(x+y) \leq q(x) + q(y)$ and $q(\alpha x) = |\alpha| q(x)$ for $\alpha \in \mathbb{C}$, for all $x, y \in V$. A family of seminorms $\{q_m\}_{m \in M}$ is said to *separate points* if $q_m(x) = 0$ for all $m \in M$ implies $x = 0$.

A *locally convex space* is a vector space $X$ (over $\mathbb{C}$ say) with a family of seminorms $\{q_m\}_{m \in A}$ separating points. The natural topology on a locally convex space $(X, \{q_m\}_{m \in A})$ is the weakest topology in which all the seminorms $q_m$'s are continuous and the operation of addition is continuous. The condition of separating points implies further that the induced topology is Hausdorff.

Two families of seminorms, say $\{q_m\}_{m \in M}$ and $\{d_n\}_{n \in N}$, are said to be *equivalent* if they induce the same natural topology on a vector space $X$. There is the following fact.

**Proposition 2.1.1** *The families of seminorms $\{q_m\}_{m\in M}$ and $\{d_n\}_{n\in N}$ on the vector space $X$ are equivalent if and only if for each $m \in M$, there are $n_1, \dots, n_k \in N$ and $C > 0$ so that for all $x \in X$ $q_m(x) \leq C(d_{n_1}(x) + \cdots + d_{n_k}(x))$ and conversely for each $n \in N$, there are $m_1, \dots, m_j \in M$ and $D > 0$ so that for all $x \in X$, $d_n(x) \leq D(q_{m_1}(x) + \cdots + q_{m_j}(x))$.*

There is a particular class of locally convex spaces, whose topology can be generated by a metric, they are called *metrisable.* The fact is that a locally convex space $X$ is metrisable if and only if the topology on $X$ is generated by some *countable* family of seminorms $\{q_m\}_{m=1,2,\dots}$. In fact, such family defines a metric $d : X\times X \longrightarrow [0,\infty)$ by

$$d(x,y) := \sum_{m=1}^{\infty} \frac{q_m(x-y)}{2^m[1+q_m(x-y)]}, \quad \forall x, y \in X. \tag{2.1}$$

The natural topology induced by the metric $d$ is the same as the topology generated by the family of seminorms $\{q_m\}_{m=1,2,\dots}$.

Given a locally convex space $X$ with the natural topology defined by the family of seminorms $\{q_m\}_{m\in M}$ separating points, a net $\{x_\beta\}$ in $X$ is called *Cauchy* if and only if for any $\varepsilon > 0$ and each seminorm $q_m$, there is a $\beta_0$ so that $q_m(x_{\beta_1} - x_{\beta_2}) < \varepsilon$ when $\beta_1, \beta_2 > \beta_0$. A net $\{x_\beta\}$ *converges* to $x \in X$, denoted as $x_\beta \rightarrow x$, if and only if $q_m(x_\beta - x) \rightarrow 0$, for any $m \in A$. The locally convex space $X$ is called *complete* if every Cauchy net converges.

In the case of metrisable locally convex space $X$, a net $\{x_\beta\}$ is called *Cauchy with respect to the metric* if and only if for any $\varepsilon > 0$, there is a $\beta_0$ so that $d(x_{\beta_1}, x_{\beta_2}) < \varepsilon$, when $\beta_1, \beta_2 > \beta_0$. A net $\{x_\beta\}$ *converges* to $x \in X$, denoted as $x_\beta \rightarrow x$, if and only if $d(x_\beta, x) \rightarrow 0$. The metrisable locally convex space $X$ is called *complete* if every Cauchy net converges. The fact is that a metrisable locally convex space $X$ is complete as a metric space if and only if it is complete as a locally convex space.

**Definition 2.1.1** *A Fréchet space is a topological vector space which is locally convex, metrisable and complete.*

## 2.2 Examples

Let $X$ be a locally compact differentiable manifold of dimension $n$. We consider the following spaces of complex-valued functions on $X$. Let $C_c^\infty(X)$ be the space of smooth functions on $X$ of compact support. Let $C_0^\infty(X)$ be the space of smooth functions on $X$ vanishing at infinity. That is, $f \in C_0^\infty(X)$ if and only if for any $\varepsilon > 0$, there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ when $x \in X \backslash K$.

Let $C_b^\infty(X)$ be the space of smooth functions $f$ whose derivatives are bounded to all degrees. That is, for any local coordinate charts $\mathcal{U} := \{U_a, \phi_a : U_a \to \mathbb{R}^n\}_{a\in A}$ of $X$ with a partition of unity $\{h_a\}_{a\in A}$ subordinate to it so that $supp(h_a) \subset U_a$, then $f \in C_b^\infty(X)$ if

$$|h_a \partial^{|\alpha|}(f \circ \phi_a^{-1})(\phi_a(x))| < \infty, \quad \forall x \in U_a, a \in A,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ are any multi-indices of *length* $|\alpha| := \sum_{i=1}^n \alpha_n$ and $\partial^{|\alpha|} := \sum \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\partial_i$ for $i = 1, \ldots, n$ is the partial derivative with respect to the $i$-th coordinates in $U_a$. The definition of $C_b^\infty(X)$ is independent of the choice of $\mathcal{U}$.

We can topologise vector spaces $C_c^\infty(X)$, $C_0^\infty(X)$ and $C_b^\infty(X)$ by a family of countable seminorms, so that each of them becomes metrisable, locally convex topological vector spaces. Furthermore, we will show that both $C_0^\infty(X)$ and $C_b^\infty(X)$ are complete and hence Fréchet spaces.

With respect to a choice of local coordinate charts and a corresponding partition of unity, say $\mathcal{U} = \{U_a, \phi_a; h_a\}_{a\in A}$, we may define the following seminorms $q_m^{\mathcal{U}} : C_b^\infty(X) \longrightarrow [0, \infty)$ by

$$q_m^{\mathcal{U}}(f) := \sum_{a\in A} \sup_{|\alpha|\leq m} \left( \sup_{x\in U_a} |h_a(x)\partial^\alpha(f(x))| \right), \quad m = 0, 1, \ldots. \tag{2.2}$$

for any $f \in C_b^\infty(X)$.

From definitions, the family of the seminorms $\{q_m^{\mathcal{U}}\}_{m=0,1,2,\ldots}$ separating points. Similarly for their restriction on the subalgebras $C_c^\infty(X), C_0^\infty(X)$. Therefore

**Lemma 2.2.1** *Under the natural topology induced by the family of seminorms $\{q_m^{\mathcal{U}}\}$, each of the spaces $C_b^\infty(X)$, $C_0^\infty(X)$ and $C_c^\infty(X)$ is metrisable and locally convex topological space.*

**Lemma 2.2.2** *Two families of seminorms defined by (2.2) from different choices of local coordinate charts are equivalent families seminorms on the space $C_b^\infty(X)$. Similarly, this is also true for spaces $C_0^\infty(X)$ and $C_c^\infty(X)$.*

**Proof**: Suppose there are two local coordinate charts on $X$ with their corresponding partition of unity, $\mathcal{U} = \{U_a, \phi_a; h_a\}_{a\in A}$, and $\mathcal{W} = \{W_b, \psi_b; g_b\}_{b\in B}$. With respect to each of the open covering, we can define two families of seminorms $\{q_m^{\mathcal{U}}\}_{m=0,1,\dots}$ and $\{q_n^{\mathcal{W}}\}_{n=0,1,\dots}$. By Proposition 2.1.1, it suffices to show that for each $m = 0, 1, \dots,$ there are $n_1, \dots, n_l = 0, 1, \dots$ and a constant $C > 0$ such that for any $f \in C_b^\infty(X)$,

$$q_m^{\mathcal{U}}(f) \leq C(q_{n_1}^{\mathcal{W}}(f) + \cdots + q_{n_l}^{\mathcal{W}}(f)); \tag{2.3}$$

and conversely, for each $n = 0, 1, \dots,$ there are $m_1, \dots, m_k = 0, 1, \dots$ and a constant $D > 0$ such that for any $f \in C_b^\infty(X)$, $q_m^{\mathcal{W}}(f) \leq D(q_{m_1}^{\mathcal{V}}(f) + \cdots + q_{m_k}^{\mathcal{V}}(f))$. By the symmetry of the two families of the seminorms, it suffices to show (2.3). For $f \in C_b^\infty(X)$ and any fixed $m = 0, 1, 2, \dots,$

$$q_m^{\mathcal{U}}(f) = \sum_{a\in A} \sup_{|\alpha|\leq m} \sup_{x\in U_a} |h_a(x)\partial^\alpha(f(x))|. \tag{2.4}$$

Then for each coordinate chart $U_a$,

$$\begin{aligned} \sup_{x\in U_a} |h_a(x)^{(a)}\partial^\alpha(f(x))| &= \sup_{x\in\cup_{b\in B}(U_a\cap W_b)} |h_a(x)^{(a)}\partial^\alpha f(x)| \\ &\leq \sum_{b\in B} \sup_{x\in U_a\cap W_b} |h_a(x)^{(a)}\partial^\alpha f(x)|, \end{aligned} \tag{2.5}$$

where we write ${}^{(a)}\partial^\alpha$ instead of $\partial^\alpha$ to indicate that it is with respect to the coordinate chart $U_a$.

Since the function $f$ is bounded to all degrees, we may find a constant $C_{ab}^\alpha$ such that the transition satisfies

$$|h_a(x)^{(a)}\partial^\alpha f(x)| \leq C_{ab}^\alpha |h_a(x)^{(b)}\partial^\alpha f(x)| \leq C_{ab}^\alpha |{}^{(b)}\partial^\alpha f(x)|, \quad \forall x \in U_a \cap W_b. \tag{2.6}$$

Since $A$ and $B$ are finite set, we may define the positive constant $C^\alpha$ by $C^\alpha := |A| \max\{C_{ab}^\alpha; a \in A, b \in B\}$, where the factor $|A|$ is the number of open cover in $\mathcal{U}$.

Thus (2.4), (2.5) and (2.6) imply that

$$\begin{aligned}
q_m^{\mathcal{U}}(f) &\leq \sum_{a\in A}\sup_{|\alpha|\leq m}\left(\sum_{b\in B}\sup_{x\in U_a\cap W_b}\frac{C^\alpha}{|A|}|^{(b)}\partial^\alpha f(x)|\right)\\
&\leq \sum_{a\in A}\sup_{|\alpha|\leq m}\left(\sum_{b\in B}\sup_{x\in U_a\cap W_b}\frac{C^\alpha}{|A|}|g_b(x)^{(b)}\partial^\alpha f(x)|\right)\\
&\leq |A|\sup_{|\alpha|\leq m}\left(\sum_{b\in B}\sup_{x\in W_b}\frac{C^\alpha}{|A|}|g_b(x)^{(b)}\partial^\alpha f(x)|\right)\\
&\leq C_m\sum_{b\in B}\sup_{|\alpha|\leq m}\sup_{x\in W_b}|g_b(x)^{(b)}\partial^\alpha f(x)| = C_m q_m^{\mathcal{W}}(f),
\end{aligned}$$

where $C_m := \max\{C^\alpha : |\alpha| \leq m\}$ is a positive constant. In the second inequality, we redefine $C^\alpha$ to take care of the multiplication of $g_b$ if necessary. Therefore (2.3) is satisfied. The same procedure works for subalgebras $C_0^\infty(X)$ and $C_c^\infty(X)$. This completes the proof. □

The topology induced by the countable family of seminorms (2.2) is called the *topology of uniform convergence of all derivatives.* Under the respective topology of uniform convergence of all derivatives, the spaces $C_c^\infty(X)$, $C_0^\infty(X)$ and $C_b^\infty(X)$ are all metrisable, locally convex spaces. In the following we will further see that $C_0^\infty(X)$ and $C_b^\infty(X)$ are both complete and hence Fréchet spaces.

**Lemma 2.2.3** *For $X$ a locally compact differentiable manifold, the space $C_b^\infty(X)$ of bounded functions to all degrees is a Fréchet space with respect to the topology of uniform convergence of all derivatives.*

**Proof**: To show that $C_b^\infty(X)$ is complete with respect to the family of seminorms $\{q_m^{\mathcal{U}}\}_{m=0,1,\ldots}$, let $\{f_\beta\}$ be a Cauchy sequence in $C_b^\infty(X)$ with respect to each of the seminorms $\{q_m^{\mathcal{U}}\}_{m=0,1,\ldots}$ in (2.4). That is, for any $\varepsilon > 0$, there exists $\beta_0$ such that

$$q_m^{\mathcal{U}}(f_{\beta_1} - f_{\beta_2}) < \varepsilon, \quad \text{as } \beta_1, \beta_2 > \beta_0. \tag{2.7}$$

This implies that for any fixed $a \in A$, $\alpha$ and fixed point $x \in U_a$,

$$|h_a(x)\partial^\alpha(f_{\beta_1} - f_{\beta_2})(x)| \to 0, \quad \text{as } \beta_1, \beta_2 \to \infty. \tag{2.8}$$

In other words, for any fixed $a \in A$, $\alpha$ and $x \in U_a$, there is a Cauchy sequence in the complex plane $\mathbb{C}$,

$$\{h_a(x)\partial^\alpha(f_\beta)(x)\}_\beta. \tag{2.9}$$

Since $\mathbb{C}$ is complete, the Cauchy sequence (2.9) converges to a complex number, say $h_a(x)\,{}^{(a)}g_x^\alpha$. That is,

$$\partial^\alpha f_\beta(x) \longrightarrow {}^{(a)}g_x^\alpha, \qquad \text{as } \beta \to \infty. \tag{2.10}$$

Thus for any fixed $a \in A$ and $\alpha$, we can define a map ${}^{(a)}g^\alpha : U_a \longrightarrow \mathbb{C}$ by ${}^{(a)}g^\alpha(x) := {}^{(a)}g_x^\alpha$ as given in (2.10) for any point $x \in U_a$.

Observe that the map ${}^{(a)}g^\alpha$ is a continuous map since it is the uniform limit of continuous functions $\partial^\alpha f_\beta : U_a \longrightarrow \mathbb{C}$ as $\beta \to \infty$. A second observation is as follows. If $x \in U_a \cap U_{a'}$ for $a, a' \in A$, then the corresponding functions defined by

$${}^{(a)}g^\alpha|_{U_a} := \lim_{\beta\to\infty} {}^{(a)}\partial^\alpha f_\beta|_{U_a}; \quad {}^{(a')}g^\alpha|_{U_{a'}} := \lim_{\beta\to\infty} {}^{(a')}\partial^\alpha f_\beta|_{U_a'}$$

agree on $x \in U_a \cap U_{a'}$. For the simple reason that for any $\beta$, ${}^{(a)}\partial^\alpha f_\beta(x) = {}^{(a')}\partial^\alpha f_\beta(x)$, for $x \in U_a \cap U_{a'}$.

With the two observations, we can define a global continuous function $g^\alpha : X \longrightarrow \mathbb{C}$ given by $g^\alpha(x) := {}^{(a)}g^\alpha(x)$ for $x \in U_a$. The independence of choices of $U_a$ containing $x$ is stated in the second observation while the continuity of the function is stated by the first observation.

We make another observation, which is crucial in the later proof. For any $U_a$,

$$g^\alpha|_{U_a} = {}^{(a)}\partial^\alpha g^{\mathbf{0}}|_{U_a}. \tag{2.11}$$

In fact, for any $x \in U_a$,

$${}^{(a)}g^\alpha(x) \overset{(2.10)}{=} \lim_{\beta\to\infty} {}^{(a)}\partial^\alpha f_\beta(x) = {}^{(a)}\partial^\alpha \left(\lim_{\beta\to\infty} f_\beta(x)\right) = {}^{(a)}\partial^\alpha g^{\mathbf{0}}(x).$$

What is left is to show the following: (i) $q_m^{\mathcal{U}}(f_\beta - g^{\mathbf{0}}) \longrightarrow 0$, as $\beta \to \infty$, for any $m = 0, 1, \ldots$; (ii) $g^{\mathbf{0}} \in C_b^\infty(X)$.

To show $(i)$, we assume that $\beta > \beta_0$ where $\beta_0$ is defined in (2.7), so that $q_m^{\mathcal{U}}(f_\beta -$

$f_{\beta'}) < \varepsilon$ for $\beta, \beta' > \beta_0$. We have

$$\begin{aligned}
q_m^{\mathcal{U}}(f_\beta - g^0) &= \sum_{a\in A} \sup_{|\alpha|\le m} \sup_{x\in U_a} |h_a(x)\partial^\alpha f_\beta(x) - h_a(x)\partial^\alpha g^0(x)| \\
&\overset{(2.11)}{=} \sum_{a\in A} \sup_{|\alpha|\le m} \sup_{x\in U_a} |h_a(x)\partial^\alpha f_\beta(x) - h_a(x)^{(a)}g^\alpha(x)| \\
&\overset{(2.10)}{=} \sum_{a\in A} \sup_{|\alpha|\le m} \sup_{x\in U_a} |h_a(x)\partial^\alpha f_\beta(x) - h_a(x) \lim_{\beta'\to\infty} \left({}^{(a)}\partial^\alpha f_{\beta'}\right)_x| \\
&\le \sup_{\beta'\ge\beta_0} \sum_{a\in A} \sup_{|\alpha|\le m} \sup_{x\in U_a} |h_a(x)\partial^\alpha (f_\beta - f_{\beta'})_x| \\
&= \sup_{\beta'\ge\beta_0} q_m^{\mathcal{U}}(f_\beta - f_{\beta'}) < \varepsilon,
\end{aligned}$$

since both $\beta$ and $\beta'$ are greater than $\beta_0$. Hence (i) is shown.

To prove (ii). It suffices to show that the limit $g^0$ of the Cauchy sequence $\{f_\beta\}$ in $C_b^\infty(X)$ satisfies

$$\sum_{a\in A} \sup_{x\in U_a} |h_a(x)\partial^\alpha g^0(x)| < \infty, \quad \forall\alpha. \tag{2.12}$$

We have

$$\begin{aligned}
\sum_{a\in A} \sup_{x\in U_a} |h_a(x)\partial^\alpha g^0(x)| &= \sum_{a\in A} \sup_{x\in U_a} |h_a(x)^{(a)}g^\alpha(x)| \\
&= \sum_{a\in A} \sup_{x\in U_a} |h_a(x) \lim_{\beta\to\infty} \partial^\alpha f_\beta(x)| \\
&= \lim_{\beta\to\infty} \sum_{a\in A} \sup_{x\in U_a} |h_a(x)\partial^\alpha f_\beta(x)|.
\end{aligned}$$

Note that each $f_\beta$ satisfies that $\sum_{a\in A} \sup_{x\in U_a} |h_a(x)\partial^\alpha f_\beta(x)| < \infty$ by the assumption that $f_\beta \in C_b^\infty(X)$. Therefore, the limit at $\beta \to \infty$ is also finite and we have $\sum_{a\in A} \sup_{x\in U_a} |h_a(x)\partial^\alpha g^0(x)| < \infty$, which gives that $g^0 \in C_b^\infty(X)$. This completes the proof that $C_b^\infty(X)$ is Fréchet. □

**Lemma 2.2.4** *For $X$ a locally compact differentiable manifold, the space $C_0^\infty(X)$ of smooth functions vanishing at infinity is a Fréchet space with respect to the topology of uniform convergence of all derivatives.*

**Proof**: The first part of the proof is obtained by replacing the algebra $C_b^\infty(X)$ by $C_0^\infty(X)$ everywhere in the proof of Lemma 2.2.3 until the claim (ii).

The second part of the proof is as follows. We claim that (ii)' If the Cauchy sequence $\{f_\beta\}$ is in the algebra $C_0^\infty(X)$, then the limit $g^0$ is also in $C_0^\infty(X)$.

The smoothness of $g^0$ is can be shown easily since it is in $C_b^\infty(X)$. Thus, it suffices to show that for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $|g^0(x)| < \varepsilon$, for $x \in X\backslash K$.

We assume on the contrary that there exists $\eta > 0$ such that $|g^0(x)| \geq \eta$ when $x \in X\backslash K$ for any compact set $K \subset X$. Then since $\{f_\beta\}$ converges to $g^0$ pointwisely, for any fixed $x \in X$ and $\epsilon < \eta$ there exists $B$ such that $|g^0(x) - f_\beta(x)| < \epsilon$ when $\beta > B$. The pointwise inequality

$$|f_\beta(x)| \geq |g^0(x)| - |g^0(x) - f_\beta(x)| > |g^0(x)| - \epsilon,$$

together with the assumption imply that $|f_\beta(x)| \geq \eta - \epsilon$ for $x \in X\backslash K$ for any compact set $K \subset X$ when $\beta > B$. This contradicts to the fact that each $f_\beta$ is a function vanishing at infinity. Hence $g^0 \in C_0^\infty(X)$ and the space $C_0^\infty(X)$ is Fréchet. □

**Definition 2.2.5** *An algebra is a Fréchet algebra if it is a Fréchet space and furthermore each seminorm of the family of seminorms $\{q_m\}$ is submultiplicative, i.e. $q_m(fg) \leq q_m(f)q_m(g)$.*

Both $C_b^\infty(X)$ and $C_0^\infty(X)$ of the topology of uniform convergence of all derivatives are examples of Fréchet algebras.

## 2.3 $C^*$-algebras and Gelfand transform

A norm $\|\cdot\|$ on an algebra $A$ is *submultiplicative* if $\|ab\| \leq \|a\|\,\|b\|$ for $a, b \in A$. The pair $(A, \|\cdot\|)$ is called a *normed algebra.* A complete normed algebra is called a *Banach algebra.* An algebra $A$ is *unital* if it has a *unit* $1_A$ such that $a1_A = 1_Aa$ for $a \in A$.

A Banach algebra $A$ can be *unitized* as $A^+ := A \times \mathbb{C}$, where the multiplication $(a, \lambda)\,(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$ for $(a, \lambda), (b, \mu) \in A^+$ and the norm $\|(a, \lambda)\|_{A^+} := \sup\{\|ab + \lambda b\| : \|b\| \leq 1\}$. The unit of $A^+$ is $1_{A^+} := (0, 1)$. $A^+$ is a unital Banach algebra.

We write $A^\times$ as the set of all invertible elements in a unital Banach algebra $A$. The *spectrum* of an element $a$ in $A$ is $\sigma_A(a) := \{\lambda \in \mathbb{C} : a - \lambda 1_A \notin A^\times\}$. The *spectral* *radius* of $a$ in $A$ is the supremum $r_A(a) := \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$. The *spectral radius formula* says that $r_A(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. When the algebra $A$ is nonunital, we obtain the corresponding definitions in its unitization $A^+$.

An algebra $A$ is called a $*$*-algebra* or *involutive algebra* if it is endowed with an *involution* $* : a \mapsto a^*$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for $a, b \in A$. A *Banach* $*$*-algebra* is a Banach algebra $(A, \|\cdot\|)$ endowed with an involution $*$ and satisfying that $\|a^*\| = \|a\|$ for $a \in A$. An *(abstract)* $C^*$*-algebra* is a Banach $*$-algebra which satisfies the $C^*$*-identity*

$$\|a^*a\| = \|a\|^2, \quad \forall a \in A.$$

If $A$ is a $C^*$-algebra, then its unitization $A^+$ is also a $C^*$-algebra.

An algebra homomorphism of $*$-algebras $\phi : A \to B$ is a $*$*-homomorphism* if $\phi(a^*) = \phi(a)^*$ for $a \in A$. A homomorphism is a $*$-homomorphism if and only if it maps self-adjoint elements to self-adjoint elements. A bijective $*$-homomorphism is a $*$*-isomorphism*. A $*$-homomorphism between unital $*$-algebras is called *unital* $*$*-homomorphism* if it preserve the units.

A useful property of $*$-homomorphism of $C^*$-algebras is as follows

**Proposition 2.3.1** *If $\alpha : A \to B$ is a $*$-homomorphism between $C^*$-algebras $A$ and $B$, then $\|\alpha\| \leq 1$. In particular, if $\alpha$ is a $*$-isomorphism, then it is isometric.*

**Proof**: Replacing $A$ and $B$ by their unitizations $A^+$ and $B^+$ if necessary, we assume that both $A$ and $B$ are unital $C^*$-algebra. We need to show that

$$\|\alpha(a)\| \leq \|a\|, \quad \forall a \in A. \tag{2.13}$$

Taking the square of (2.13) and applying the $C^*$-identity, we obtain $\|\alpha(a^*a)\| \leq \|a^*a\|$. Since $a^*a$ is self-adjoint for any $a \in A$, it suffices to show (2.13) for any self-adjoint element $b \in A$. Such an element $b$ is in particular normal, i.e. $b^*b = bb^*$. Together with the $C^*$-identity, $b$ satisfies that $\|b^2\| = \|b\|^2$. Indeed,

$$\|b^2\|^2 = \|(b^2)^*b^2\| = \|(b^*b)^*(b^*b)\| = \|b^*b\|^2 = \|b\|^4.$$

Thus $\|b\| = \lim_{n\to\infty} \|b^n\|^{1/n} = r_A(b)$. Similarly, $\|\alpha(b)\| = r_B(\alpha(b))$, since $\alpha(b)$ is self-adjont and hence is a normal element in $B$. The simple observation that

$\sigma_B(\alpha(b)) \subset \sigma_A(b)$ implies that $r_B(\alpha(b)) \leq r_A(b)$ and hence $\|\alpha(b)\| \leq \|b\|$. This completes the proof. □

The dual of a commutative Banach algebra $A$ is the set $M(A)$ of non-zero, continuous algebra homomorphisms $\alpha : A \to \mathbb{C}$. The continuity is with respect to the respective norm topology on $A$ and $\mathbb{C}$. $M(A)$ is a locally compact Hausdorff space in the topology of pointwise convergence. We denote by $C_0(M(A))$ the algebra of continuous complex-valued functions on $M(A)$ vanishing at infinity. Equipped with the supremum norm $\|\cdot\|_\infty$, $C_0(M(A))$ is a Banach $*$-algebra. The *Gelfand transform* $A \to C_0(M(A))$ is given by $a \mapsto \hat{a}$ such that

$$\hat{a}(\alpha) = \alpha(a), \quad \forall \alpha \in M(A).$$

A simple result of Gelfand asserts that

**Lemma 2.3.1** *When $A$ is a unital Banach algebra, $\sigma_A(a) = \sigma_{C(M(A))}(\hat{a})$. In particular, $r_A(a) = r_{C(M(A))}(\hat{a})$.*

**Proof**: This is implied by that fact that $a \in A$ is invertible if and only if its Gelfand transform $\hat{a} \in C(M(A))$ is invertible. □

Let $A$ be a unital Banach $*$-algebra. Endowed with the involution defined by the complex conjugation, $C(M(A))$ is a $*$-algebra. An element $\hat{a}$ in $C(M(A))$ is self-adjoint if and only if it has real spectrum. By Lemma 2.3.1, the Gelfand transform of a $*$-algebra $A$ is a $*$-homomorphism if and only if all the self-adjoint elements in $A$ have real spectrum.

The following theorems of Gelfand and Naimark link commutative $C^*$-algebras to topological spaces.

**Theorem 2.3.2** *If $A$ is a commutative $C^*$-algebra, then the Gelfand transform is an isometric $*$-isomorphism from $A$ onto $C_0(M(A))$.*

**Proof**: If $A$ is nonunital, the Gelfand transform of $A$ is defined as that of $A^+$. Thus we may assume $A$ is unital. Every element $a$ of the commutative $C^*$-algebra $A$ is normal, and hence $\|a\| = r_A(a)$.

On the other hand, one can show that for any $\hat{a} \in C_0(M(A))$, $\|\hat{a}\|_\infty = r_{C(M(A))}(\hat{a})$. Lemma 2.3.1 further implies that $\|\hat{a}\|_\infty = r_A(a)$. Therefore $\|\hat{a}\|_\infty = \|a\|$ for $a \in A$

and hence the Gelfand transform is an isometry.

We use the fact that self-adjoint elements of a $C^*$-algebra have real spectra to conclude that the Gelfand transform of a $C^*$-algebra is a $*$-homomorphism. It is easy to check the injectivity of the Gelfand transform and the surjectivity follows from the Stone-Weierstrass theorem. □

At the level of categories, there is the following celebrated theorem.

**Theorem 2.3.3**

1. *The category of commutative, unital $C^*$-algebras and unital $*$-homomorphisms is equivalent to the opposite category of compact Hausdorff spaces and continuous maps.*

2. *The category of nonunital commutative $C^*$-algebras and $*$-homomorphisms is equivalent to the opposite category of locally compact Hausdorff spaces and base-point preserving continuous maps.*

**Proof**: We consider the unital case first. For any unital $*$-homomorphism $F : A \to B$ of commutative $C^*$-algebras $A$ and $B$, we induce $M(F) : M(B) \to M(A)$ by $M(F)(\beta)(a) = \beta(F(a)), \forall a \in A$ for $\beta \in M(B)$. Conversely, for any continuous map $\hat{F} : \hat{B} \to \hat{A}$ between compact Hausdorff topological spaces $\hat{B}$ and $\hat{A}$, we induce a $*$-homomorphism $C(\hat{F}) : C(\hat{A}) \to C(\hat{B})$ by $C(\hat{F})(\hat{a})(\beta) = \hat{a}(\hat{F}(\beta)), \forall \beta \in \hat{B}$ for $\hat{a} \in C(\hat{A})$. It is not hard to show that the functors thus defined yield the equivalence between the categories.

For the nonunital case, if $\hat{A}$ is a locally compact Hausdorff space and if $A := C_0(\hat{A})$ then $A^+ \cong C(\hat{A}^+)$ where $\hat{A}^+$ is the one-point compactification of $\hat{A}$. Similarly a locally compact Hausdorff space $\hat{B}$ defines $B := C_0(\hat{B})$ with the one-point compactification $\hat{B}^+$.

A $*$-homomorphism of the nonunital $C^*$-algebras $A \to B$ induce a unital $*$-homomorphism from $A^+ \to B^+$. By the proof for the unital case, it induces a base-point preserving continuous map from $\hat{B}^+$ to $\hat{A}^+$. If we define the category of locally compact Hausdorff spaces in such a way that the morphisms from $\hat{B}$ to $\hat{A}$ are the base-point preserving maps from $\hat{B}^+$ to $\hat{A}^+$, then it is equivalent to the category of nonunital $C^*$-algebras with $*$-homomorphisms. □

We end this section by giving the notion of positivity of elements in a $C^*$-algebra. A self-adjoint element $a$ of a $C^*$-algebra is *positive*, written as $a \geq 0$, if its spectrum $\sigma_A(a)$ is non-negative. An element $a$ in $A$ is positive if and only if $a = b^*b$ for some $b \in A$. Positivity induces a partial order $\leq$ on $A$ by $a \leq b$ if and only if $b - a \geq 0$ for $a, b \in A$. A useful property for a $C^*$-algebra $A$ is that

$$0 \leq b \leq a \Longrightarrow \|b\| \leq \|a\|, \quad \forall a, b \in A. \tag{2.14}$$

## 2.4 Representation of $C^*$-algebras

A *concrete $C^*$-algebra* $A$ is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra of the algebra of bounded linear operators $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The algebra of Hilbert space operators satisfies the $C^*$-identity automatically. This implies that any concrete $C^*$-algebra is an abstract $C^*$-algebra. The *GNS (Gelfand-Naimark-Segal) construction* on the other hand implies that each abstract $C^*$-algebra admits a representation on some Hilbert space, which further realises it as a concrete $C^*$-algebra.

A *$*$-representation* of an abstract $C^*$-algebra $A$ is a $*$-homomorphism $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$. The $*$-representation is called *faithful* if $\pi$ is injective. Two representations $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$ and $\pi' : A \longrightarrow \mathcal{B}(\mathcal{H}')$ are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $U\pi(a)U^* = \pi'(a)$ for all $a \in A$.

An important fact is that the spectrum of an element $a$ in a $C^*$-algebra $A$ is the same as the spectrum of $a$ as an operator in a faithful $*$-representation on some Hilbert space $\mathcal{H}$, that is

$$\sigma_A(a) = \sigma_{\mathcal{B}(\mathcal{H})}(a). \tag{2.15}$$

Recall that a a bounded self-adjoint operator $T$ on a Hilbert space $\mathcal{H}$ of inner product $\langle \cdot, \cdot \rangle$ is positive (written as $T \geq 0$) if and only if the spectrum of $T$ is non-negative; if and only if $T$ is of the form $S^*S$ for some bounded operators $S$; if and only if $\langle \xi, T\xi \rangle \geq 0$, for any $\xi \in \mathcal{H}$.

Now (2.15) in particular implies that $a$ is positive as an element in a $C^*$-algebra $A$ if and only if $\pi(a)$ is a positive operator on a Hilbert space through a faithful representation.

A *state* on a unital $C^*$-algebra $A$ is a positive linear functional $\phi : A \longrightarrow \mathbb{C}$ such that $\phi(1_A) = 1$. We will restrict to the case when $A$ is unital, while referring to [6] for the nonunital case.

**Theorem 2.4.1 (GNS construction)** *Let $\phi$ be a state on a unital $C^*$-algebra $A$. There is a representation on a Hilbert space $\mathcal{H}$, $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$ such that $\phi(a) = \langle \xi, \pi(a)\,\xi \rangle$ for all $a \in A$, and such that the subspace $\pi(A)\xi$ is dense in $\mathcal{H}$. The pair $(\pi, \xi)$ is unique up to unitary equivalence.*

**Proof**: The idea is to construct a Hilbert space from the vector space underlying $A$ and hence define the required representation.

The state $\phi$ induces a sesquilinear Hermitian form $\langle \cdot, \cdot \rangle : A \times A \longrightarrow \mathbb{C}$ by

$$(a, b) \mapsto \langle a, b \rangle := \phi(a^* b), \quad \forall (a, b) \in A \times A.$$

If $\langle \cdot, \cdot \rangle$ were further positive definite, then it would be an inner product of $A$. However, the set $N := \{a \in A : \langle a, a \rangle = 0\}$ is not the null set. Nonetheless, the Cauchy-Schwarz inequality $|\langle a, b \rangle|^2 \leq \langle a, a \rangle \langle b, b \rangle$, $\forall a, b \in A$, together with the linearity of the Hermitian form implies that $N$ is a vector subspace of $A$. Thus we obtain the quotient space $A/N$ on which the restriction of $\langle \cdot, \cdot \rangle$ is positive definite and hence an inner product. Specifically, the inner product $\langle \cdot, \cdot \rangle_{A/N} : A/N \times A/N \longrightarrow \mathbb{C}$ is defined by

$$\langle a + N, b + N \rangle_{A/N} := \langle a, b \rangle, \quad \text{for } a + N, b + N \in A/N.$$

We thus obtain a Hilbert space $\mathcal{H}$ by the completion of the vector space $A/N$ under the inner product $\langle \cdot, \cdot \rangle_{A/N}$.

To obtain a representation of $A$ on $\mathcal{H}$, we observe that $A/N$ is a left ideal in $A$ by using the inequality $\langle ab, ab \rangle \leq \|a\|^2 \langle b, b \rangle$, for $a, b \in A$. Thus, for any $a \in A$ we may define an operator $\pi(a)$ on $A/N$ by $\pi(a)(b + N) := ab + N$. $\pi(a)$ extends by continuity to a bounded operator on $\mathcal{H}$. In this way, we obtain a representation $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$.

Furthermore, we define the unit vector $\xi := 1_A + N \in \mathcal{H}$, then

$$\langle \xi, \pi(a)\xi \rangle_{A/N} = \langle 1_A + N, a + N \rangle_{A/N} = \langle 1_A, a \rangle = \phi(a), \quad \forall a \in A$$

as required. This completes the existence part of the proof.

To show the uniqueness, suppose that $\pi' : A \longrightarrow \mathcal{B}(\mathcal{H}')$ is another representation with the unit vector $\xi'$ such that $\langle \xi', \pi'(a)\xi' \rangle_{A/N'} = \phi(a)$. Then the map $\pi(a)\xi \mapsto \pi'(a)\xi'$ defines an isometry from the dense subspace $\pi(A)\xi \subset \mathcal{H}$ to $\mathcal{H}'$. Indeed,

$$\|\pi(a)\xi\|_{\mathcal{H}}^2 = \langle \xi, \pi(a^*a)\xi \rangle_{A/N} = \phi(a^*a) = \langle \xi', \pi'(a^*a)\xi' \rangle_{A/N'} = \|\pi'(a)\xi'\|_{\mathcal{H}'}^2, \quad \forall a \in A.$$

Since $\pi'(A)\xi'$ is dense in $\mathcal{H}'$, then the isometry extends to a unitary isomorphism $U : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $U\pi(a)U^* = \pi'(a)$ and $U\xi = \xi'$. That is $U$ defines the unitary equivalence between representations $(\pi, \xi)$ and $(\pi', \xi')$ and hence uniqueness is shown. □

The following *Gelfand-Naimark representation theorem* clarifies the equivalence between the definitions of an abstract $C^*$-algebra and a concrete $C^*$-algebra.

**Theorem 2.4.2** *Every abstract $C^*$-algebra $A$ is isometrically $*$-isomorphic to a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.*

**Proof**: If $A$ is a nonunital $C^*$-algebra, then its unitization $A^+$ is a $C^*$-algebra. Thus, we restrict to the case when $A$ is a unital $C^*$-algebra. For each $a \in A$, let $\phi_a$ be a state such that $\phi_a(a^*a) = \|a^*a\|$, whose existence can be shown by Hahn-Banach theorem. Let $\pi_a : A \longrightarrow \mathcal{B}(\mathcal{H}_a)$ be the representation from the GNS construction. We may define the representation $\pi : A \longrightarrow \mathcal{H}$ by direct sum representation $\pi := \oplus_{a\in A}\pi_a$ on the direct sum of Hilbert spaces $\mathcal{H} := \oplus_{a\in A}\mathcal{H}_a$.

Since for any $b \in A$ $\|\pi(b^*b)\| = \sup_{a\in A} \|\pi_a(b^*b)\|$. Then,

$$\|\pi(b)\|^2 = \|\pi(b^*b)\| = \sup_{a\in A} \|\pi_a(b^*b)\| \geq \|\pi_b(b^*b)\| = \phi_b(b^*b) = \|b^*b\| = \|b\|^2.$$

The other direction $\|\pi(b)\|^2 \leq \|b\|^2$ is by Proposition 2.3.1. Thus $\pi$ is an isometric $*$-homomorphism from $A$ to $\mathcal{B}(\mathcal{H})$. □

## 2.5 Holomorphic functional calculus and pre-$C^*$-algebras

Since commutative $C^*$-algebras are equivalent to locally compact Hausdorff topological spaces, noncommutative $C^*$-algebras are considered as *noncommutative topo-*

*logical spaces*. This is the foundation of noncommutative geometry. To study differentiability in noncommutative geometry, $C^*$-algebras are too large. Thus one may consider differentiable dense subalgebras instead. On the other hand, such subalgebras are required to preserve certain topological properties of the original $C^*$-algebras, for example K-theories. See Section 3.8 of [6]. The pre-$C^*$-algebras are such candidates. We will define the holomorphic functional calculus of Banach algebras [27] and then define pre-$C^*$-algebras and some of their properties.

Assume that $(A, \|\cdot\|)$ is a Banach space and that $U \subset \mathbb{C}$ is an open subset. A mapping $f : U \longrightarrow A$ is said to be *holomorphic* on $U$ when it is differentiable at each point $z_0$ of $U$. That is, the limit

$$\lim_{\Delta z \to 0} \frac{\|f(z_0 + \Delta z) - f(z_0)\|}{|\Delta z|}$$

exists in the norm topology of $A$. In this case, we denote the limit by $f'(z_0) \in A$. For $f$ holomorphic and $\rho : A \longrightarrow \mathbb{C}$ a bounded linear functional, the composition $\rho \circ f : U \longrightarrow \mathbb{C}$ is a holomorphic function in the usual sense.

One defines the line integral of $A$-valued functions over complex plane as follows: let $f : \mathbb{C} \longrightarrow A$ be a continuous function and $C : [a, b] \longrightarrow \mathbb{C}$ that maps $t$ to $z(t)$ be a smooth curve in the complex plane. That is, $z(t)$ is a differentiable complex-valued function on $[a, b]$. We may define the line integral

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

as the norm limit of Riemann sums of the form $\sum_{j=1}^n f(z(t_j'))\,[z(t_j) - z(t_{j-1})]$ where $a = t_0 < t_1 < \cdots < t_n = b$, $t_{j-1} \leq t_j' \leq t_j$, for $j = 0, \ldots, n$. The limit is taken as $\max\{|t_j - t_{j-1}| : j = 1, \ldots, n\}$ tends to 0.

Suppose that $f : U \longrightarrow A$ is holomorphic and $\rho : A \longrightarrow \mathbb{C}$ is a bounded linear functional, one can show that $\rho\left(\int_C f(z)dz\right) = \int_C \rho(f(z))dz$. This identity together with the usual Cauchy theorem implies that $\int_C f(z)dz = 0$, where $C$ is a contour in the complex plane. This can be seen as a generalised Cauchy theorem for $A$-valued holomorphic functions.

We are ready to introduce the holomorphic functional calculus.

**Lemma 2.5.1** *Let $A$ be a unital Banach algebra, then*

$$a^n = -\frac{1}{2\pi i}\int_C z^n(a - z1_A)^{-1}dz, \tag{2.16}$$

*where $n$ is a non-negative integer, $a \in A$ and $C$ is a smooth closed curve whose interior contains $\sigma_A(a)$.*

**Proof**: Using the fact that $z \mapsto (a - z1_A)^{-1}$ is holomorphic on the open subset $U := \mathbb{C}\backslash\sigma_A(a)$, the map $f : z \longrightarrow z^n(a - z1_A)^{-1}$ is also holomorphic on $U$. Let $C'$ be a large circle centered at the origin and of radius greater than $\|a\|(\geq r_A(a))$ so that $f$ is holomorphic on $C'$. By the generalised Cauchy theorem, the following integrals agree

$$-\frac{1}{2\pi i}\int_C z^n(a - z1_A)^{-1}dz = -\frac{1}{2\pi i}\int_{C'} z^n(a - z1_A)^{-1}dz.$$

Therefore, it suffices to show (2.16) with $C$ replaced by $C'$.

Since any $z \in C'$ satisfies that $|z| > \|a\|$, then the Neumann series of $(a - z1_A)^{-1}$ is defined. I.e., $(a - z1_A)^{-1} = -\sum_{k=0}^{\infty} a^k z^{-k-1}, z \in C'$. Thus

$$-\frac{1}{2\pi i}\int_{C'} z^n(a - z1_A)^{-1}dz = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{C'} z^n a^k z^{-k-1}dz = a^n,$$

since $\int_{C'} z^l dz = 0$ for any integer $l \neq -1$ and $\int_{C'} z^{-1}dz = 2\pi i$. This completes the proof. □

Lemma 2.16 implies immediately that

$$f(a) = -\frac{1}{2\pi i}\int_C f(z)(a - z1_A)^{-1}dz \tag{2.17}$$

for each $f$ in the polynomial ring of complex coefficients $\mathbb{C}[z]$ over a smooth curve $C$ whose interior contains $\sigma_A(a)$. The integral (2.17) is called the *Dunford integral.* The map $f \mapsto f(a)$ defines an algebra homomorphism from $\mathbb{C}[z] \rightarrow A(a)$, where $A(a)$ denotes the closed subalgebra of $A$ generated by $a$. In the case when $A$ is a Banach algebra, one can at best replace the algebra $\mathbb{C}[z]$ by the algebra of holomorphic functions over $\mathbb{C}\backslash\sigma_A(a)$. Specifically, let $H(a)$ be the set of holomorphic functions in some open set in $\mathbb{C}$ containing $\sigma_A(a)$. Applying definitions, one can prove that the mapping $f \mapsto f(a)$ of the Dunford integral defines an algebra homomorphism $H(a) \rightarrow A(a)$. This is the *holomorphic functional calculus.* When the

Banach algebra $A$ is nonunital, the holomorphic functional calculus is defined for its unitization $A^+$ and restricted on holomorphic functions vanishing at 0.

A subalgebra $B$ of a unital Banach algebra $A$ is *stable under the holomorphic functional calculus* of $A$ if (i) $B$ is complete under some locally convex topology finer than the topology of $A$; (ii) $f(b)$ defined by the holomorphic functional calculus of $A$ is an element of $B$ for all $b \in B$. When $A$ is nonunital, we require $f$ to vanish at 0. $B$ is stable under the holomorphic functional calculus of $A$ if and only if $B^\times = A^\times \cap B$.

**Definition 2.5.2** *A pre-$C^*$-algebra is a subalgebra of a $C^*$-algebra that is stable under the holomorphic functional calculus.*

Very often a pre-$C^*$-algebra is Fréchet. The importance of these conditions lies in the following property (Theorem 3.44 [6]):

**Theorem 2.5.3** *If $\mathcal{A}$ is Fréchet pre-$C^*$-algebra with $C^*$-completion $A$, the inclusion $i : \mathcal{A} \hookrightarrow A$ induces an isomorphism $K_0(i) : K_0(\mathcal{A}) \to K_0(A)$.*

To study K-theories of algebras, there is no loss in replacing a $C^*$-algebra by a dense subalgebra which is both a pre-$C^*$-algebra and Fréchet. Such subalgebra is a *smooth algebra* [10]:

**Definition 2.5.4** *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a $C^*$-algebra $A$ which is stable under the holomorphic functional calculus.*

**Example 2.5.1** Let $X$ be a locally compact Hausdorff topological space, Lemma 2.2.3 implies that $C_b^\infty(X)$ is Fréchet under the topology of uniform convergence of all derivatives. The zero-th seminorm in the family of seminorms is the supremum norm $\|\cdot\|_\infty$ which is a $C^*$-norm. The $C^*$-completion of $C_b^\infty(X)$ under $\|\cdot\|_\infty$ is the $C^*$-algebra $C_b(X)$ of continuous complex valued functions on $X$. $C_b^\infty(X)$ is a pre-$C^*$-algebra. Indeed, any $f \in C_b^\infty(X)$ is invertible in $C_b(X)$ if and only if it does not vanish on $X$, and then its inverse $1/f$ is also a smooth function in $C_b^\infty(X)$. This implies that $C_b^\infty(X)$ is closed under the holomorphic calculus of $C_b(X)$. Therefore $C_b^\infty(X)$ is a smooth algebra.

Using Lemma 2.2.4 and a similar argument to the above, the algebra $C_0^\infty(X)$ is also a smooth algebra whose $C^*$-completion is the algebra $C_0(X)$ of continuous functions on $X$ vanishing at infinity.

# Chapter 3

# Noncommutative integration

We give some background on noncommutative integration in noncommutative geometry. For details on this topic we refer to [5] [6] [29].

## 3.1 Compact operators

We consider the space of compact operators $\mathcal{K}(\mathcal{H})$ on a separable and infinite dimensional Hilbert space $\mathcal{H}$, whose orthonormal basis is countably infinite. Recall that an operator is *compact* if it is a norm limit of a family of finite-rank operators, whose ranges are finite dimensional.

Let $T$ be a positive compact operator on $\mathcal{H}$, then the spectrum $\sigma_{\mathcal{B}(\mathcal{H})}(T)$ of $T$ consists of countably many non-negative eigenvalues of finite multiplicity $\{s_0, s_1, \dots\}$. We may choose an orthonormal basis $\{u_k\}$ of $\mathcal{H}$ by assembling eigenvectors so that $T$ has the expansion

**Lemma 3.1.1**

$$T = \sum_{k \geq 0} s_k |u_k\rangle\langle u_k|, \tag{3.1}$$

*where the ketbra notation means* $|r\rangle\langle s| : v \mapsto r\langle s, v\rangle_{\mathcal{H}}, \forall r, s, v \in \mathcal{H}$.

**Proof**: To see the convergence of the series, we may rearrange the eigenvalues in a decreasing sequence $\{s_0, s_1, \dots\}$ such that $s_k \to 0$ as $k \to \infty$. Thus for any $\varepsilon > 0$, there exists an integer $N := N(\varepsilon)$ such that $s_0, \dots, s_{N-1}$ are the only eigenvalues greater than $\varepsilon$. In this way,

$$\left\| T - \sum_{k=0}^{N-1} s_k |u_k\rangle\langle u_k| \right\| = \left\| \sum_{k\geq N} s_k |u_k\rangle\langle u_k| \right\| < \varepsilon \sum_{k\geq N} \| |u_k\rangle\langle u_k| \| = \varepsilon.$$

Hence the series converges to $T$. □

If $T \in \mathcal{K}(\mathcal{H})$ is any compact operator, then $|T| := (T^*T)^{1/2}$ is a compact positive operator, admitting an expansion as (3.1) $|T| = \sum_{k\geq 0} s_k |u_k\rangle\langle u_k|$, where $u_k$'s are eigenvectors of $|T|$ and form an orthonormal basis of the closure of the graph $T(\mathcal{H})$.

The *polar decomposition* $T = U|T|$ is defined by $U(|T|\xi) := T\xi$ for $\xi \in \mathcal{H}$ and $U\eta := 0$ for $\eta \in Ker(T)$. This implies that $U$ is a partial isometry uniquely determined by $T$. Now let $v_k := Uu_k$, then

$$T = U|T| = \sum_{k\geq 0} s_k |Uu_k\rangle\langle u_k| = \sum_{k\geq 0} s_k |v_k\rangle\langle u_k|. \tag{3.2}$$

This is the *canonical expansion* of the compact operator $T$ and eigenvalues $s_k$'s of $|T|$ are called *singular values* of $T$. Singular values are fixed under unitary transformation. That is, $s_k(U_1 T U_2) = s_k(T)$ for any unitary operators $U_1, U_2$ in $\mathcal{B}(\mathcal{H})$.

## 3.2 Ideals of compact operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is an *infinitesimal* if for each $\epsilon > 0$, there exists a finite dimensional vector subspace $E$ of $\mathcal{H}$, such that $\|T|_{E^\perp}\| < \epsilon$. It turns out that the set of infinitesimals is the set of compact operators.

Several subclasses of infinitesimals may be determined by imposing suitable conditions on the singular values.

**Definition 3.2.1** *For $1 \leq p < \infty$, we may define the Schatten-p-class $\mathcal{L}^p(\mathcal{H})$ in $\mathcal{K}(\mathcal{H})$ by requiring that $T \in \mathcal{L}^p(\mathcal{H})$ if and only if $\sum_{k\geq 0} s_k^p < \infty$, where $\{s_k\}$ are singular values of $T$.*

For each $1 \leq p < \infty$, the Schatten-$p$-class $\mathcal{L}^p(\mathcal{H})$ is an ideal of $\mathcal{K}(\mathcal{H})$. The function $\|\cdot\|_p : \mathcal{L}^p(\mathcal{H}) \to \mathbb{R}$ given by $\|T\|_p := \left(\sum_{k\geq 0} s_k^p\right)^{1/p}$ defines a norm on $\mathcal{L}^p(\mathcal{H})$. One can further show that $\mathcal{L}^p(\mathcal{H})$ is a Banach space. It is also true that $\|ATB\|_p = \|A\|\,\|T\|_p\|B\|$ for any $T \in \mathcal{L}^p(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})$, which makes $\|\cdot\|_p$ a *symmetric norm* on $\mathcal{L}^p(\mathcal{H})$. Each Banach space $\mathcal{L}^p(\mathcal{H})$ is thus a *symmetrically normed ideal.* [6]

The first Schatten class $\mathcal{L}^1(\mathcal{H})$ is called the *trace class*, for the reason that the trace of $T$ exists if and only if $T \in \mathcal{L}^1(\mathcal{H})$. The second Schatten class $\mathcal{L}^2(\mathcal{H})$ is called the *Hilbert-Schmidt class.*

For any infinitesimal $T$, we may rearrange its singular values as a decreasing non-negative sequence $\{s_0, s_1, \dots\}$. We say that $T$ is an *infinitesimal of order* $\alpha$ if the sequence $\{s_n\}$ decays like $n^{-\alpha}$ as $n \to \infty$. By definition, elements in the trace class $\mathcal{L}^1(\mathcal{H})$ are infinitesimals of order 0. Such a class turns out to be too small, it is the class of infinitesimals of order 1 which is relavent in noncommutative integration. We may consider the partial sums of the sequence of singular values associated to $T$, $\sigma_n(T) := s_0 + \cdots + s_n$. Note that if $T$ is of order one, then $s_n \to \mathcal{O}(1/n)$ and the series $\sigma_n(T)$ grows logarithmically, i.e. $\sigma_n(T) \to \mathcal{O}(\log(n))$. Then the supremum $\sup_n \frac{\sigma_n(T)}{\log(n)}$ exists. Note that although the sequence $\frac{\sigma_n(T)}{\log(n)}$ is bounded, it need not to be convergent.

We can define a norm on $\mathcal{K}(\mathcal{H})$ by

$$||T||_{1,\infty} := \sup_{n \geq e} \frac{\sigma_n(T)}{log(n)}$$

whose domain is exactly the space of infinitesimals of order one, denoted as $\mathcal{L}^{(1,\infty)}(\mathcal{H})$. That is to say $T \in L^{(1,\infty)}(\mathcal{H})$ if and only if $||T||_{1,\infty} < \infty$. $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ can be shown as an ideal of $\mathcal{K}(\mathcal{H})$ and is called the *Dixmier trace ideal.*

For any $1 < p < \infty$, we consider infinitesimals of order $1/p$. The $(p, \infty)$-norm can be defined similarly as

$$||T||_{p,\infty} := \sup_{n \geq e} \frac{\sigma_n}{n^{(p-1)/p}}$$

for $T \in \mathcal{K}(\mathcal{H})$ and the *generalised Schatten-p-class* $\mathcal{L}^{(p,\infty)}(\mathcal{H})$ can be defined by requiring that $T \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ if and only if $||T||_{p,\infty} < \infty$. We have the useful property of generalised Schatten classes,

**Lemma 3.2.2** *If* $1 < p < \infty$ *and* $T \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$ *is positive, then* $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

The following lemma relates Schatten classes and generalised Schatten classes.

**Lemma 3.2.3** *For* $1 \leq p < s$, $\mathcal{L}^p(\mathcal{H}) \subset \mathcal{L}^{(p,\infty)}(\mathcal{H}) \subset \mathcal{L}^s(\mathcal{H})$.

## 3.3 Dixmier traces and measurability

To obtain a trace functional whose domain is the Dixmier trace ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$, we consider the partial sum $\sigma_n(T)$ of $T$ in the Dixmier trace ideal. Each $\sigma_n$ is a norm on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$. It can be characterised by

$$\sigma_n(T) = \inf\{\|R\|_1 + n\|S\| : R, S \in \mathcal{K}(\mathcal{H}), R + S = T\}.$$

One can then extend the sequence of norms $\{\sigma_n\}$ parametrised by $n \in \mathbb{N}$ of the Dixmier trace ideal to a family of norms $\{\sigma_\lambda\}$ parametrised by $\lambda \in [0, \infty)$ by defining

$$\sigma_\lambda(T) := \inf\{\|R\|_1 + \lambda\|S\| : R, S \in \mathcal{K}(\mathcal{H}), R + S = T\}.$$

An important property of these norms is that

$$\sigma_\lambda(A + B) \leq \sigma_\lambda(A) + \sigma_\lambda(B) \leq \sigma_{2\lambda}(A + B), \quad \lambda \in [0, \infty),$$

for any positive operators $A, B \in \mathcal{L}^{1,\infty}(\mathcal{H})$. This implies that for large $\lambda$, $\sigma_\lambda(A+B)$ almost equals $\sigma_{2\lambda}(A + B)$ and hence $\sigma_\lambda(A + B)$ almost equals $\sigma_\lambda(A) + \sigma_\lambda(B)$. I.e., $\sigma_\lambda/\log\lambda$ is almost additive and defines a trace functional on the cone of positive operators in $\mathcal{L}^{1,\infty}(\mathcal{H})$. Since this is not exactly the case, we need to obtain a trace functional by taking averaging of the norm $T \mapsto \sigma_\lambda(T)/\log\lambda$ on $\lambda \in [3, \infty)$ as follows. The *Cesáro mean* of the function $\lambda \mapsto \sigma_\lambda(T)/\log\lambda$ is defined by

$$\tau_\lambda(T) := \frac{1}{\log\lambda}\int_3^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u}, \quad \lambda \geq 3.$$

Connes and Moscovici [30] show the "asymptotic additivity" property of $\tau_\lambda$,

$$\tau_\lambda(A + B) = \tau_\lambda(A) + \tau_\lambda(B) + \mathcal{O}\left(\frac{\log\log\lambda}{\log\lambda}\right), \quad \lambda \to \infty.$$

Note that the function $\lambda \mapsto \tau_\lambda(A)$ is in the $C^*$ algebra $C_b([3, \infty])$ and also note that $(\log\log\lambda)/\log\lambda$ is in the $C^*$-subalgebra $C_0([3, \infty))$. One defines the quotient $C^*$-algebra by

$$B_\infty := C_b([3, \infty])/C_0([3, \infty))$$

and let $\tau(A) \in B_\infty$ be the equivalence class defined by $\lambda \mapsto \tau_\lambda(A)$ in $C_b([3, \infty))$. In this way $\tau$ is positive homogeneous and additive, i.e.,

$$\tau(cA) = c\tau(A), \quad c \geq 0$$

$$\tau(A+B) = \tau(A) + \tau(B), \quad \forall A, B \in \mathcal{L}^{1,\infty}(\mathcal{H}).$$

One can further show that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{L}^{1,\infty}(\mathcal{H})$.

To define a trace formula with domain $\mathcal{L}^{1,\infty}(\mathcal{H})$, we need to further compose the map $\tau : \mathcal{L}^{1,\infty}(\mathcal{H}) \to B_\infty$ with a state of the $C^*$-algebra $\omega : B_\infty \to \mathbb{C}$.

**Definition 3.3.1** *To each state $\omega$ of the commutative $C^*$-algebra $B_\infty$, there corresponds a Dixmier trace*

$$Tr_\omega(T) := \omega(\tau(T)),$$

*whose domain is $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.*

**Definition 3.3.2** *An operator $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ is measurable if the function $\lambda \to \tau_\lambda(T)$ converges as $\lambda \to \infty$. In and only in that case, the Dixmier traces $Tr_\omega(T) = \lim_{\lambda\to\infty} \tau_\lambda(T)$ is independent of $\omega$. We denote the Dixmier trace of a measurable operator $T$ by*

$$Tr^+(T) := Tr_\omega(T), \quad \forall\omega.$$

If the sequence $\{\frac{\sigma_n(T)}{\log n}\}_n$ converges, then any Dixmier trace takes the value of its limit. This implies that $T$ is measurable. Furthermore, when $T$ is positive the existence of the limit of the sequence is equivalent to the existence of the limit of

$$\lim_{s\to 1^+} (s-1)\zeta(s)$$

where $\zeta(s) := Trace(|T|^s)$ $s$ is a complex number such that $Re(s) > 1$ (see page 306 of [5]). This can be seen as a measurability criterion in the unital case and we will get to its nonunital version in Section 8.3.

# Chapter 4

# $C^*$-modules

Modules of algebras are basic objects in K-theory. We will first give the definitions of modules over unital rings in general and then give the notion of $C^*$-modules over unital $C^*$-algebras, and finally give the notion of finitely generated projective modules, which is an equivalent description of spaces of sections of vector bundles. We refer for the details to references [6] [31].

## 4.1 Modules over unital rings

We start with some terminology on modules over unital rings. Let $A$ be a unital ring with unit $1_A$, a *right $A$-module* is an additive abelian group $E$ together with a (scalar) multiplication $E \times A \to E$ which maps $(s, a) \mapsto s\,a$ satisfying

$$\begin{aligned}
&(s+t)\,a = s\,a + t\,a, \\
&s\,(a+b) = s\,a + s\,b, \\
&s\,(ab) = (s\,a)\,b, \\
&s\,1_A = s, \quad \forall a, b \in A, s, t \in E.
\end{aligned}$$

An *right $A$-linear map* $f : E \to F$ between two right $A$-modules $E$ and $F$ is a homomorphism of the additive groups that satisfies $f(s\,a) = f(s)\,a$ for any $a \in A$ and $s \in E$. We can similarly define left $A$-modules and left $A$-linear maps of them. An *$A$-isomorphism* between two right (left) $A$-modules is a right (left) $A$-homomorphism with an inverse right (left) $A$-homomorphism. An $A$-linear map from a right (left) $A$-module $E$ to itself is called an *endomorphism* of $E$.

Let $E$ and $F$ be both right (left) $A$-modules. The set of all right (left) $A$-linear maps from $E$ to $F$ is denoted as $Hom_A(E, F)$ and the set of all right (left) endomorphisms of $E$ is denoted as $End_A(E)$.

Let $A$ be a unital ring and $E$ be a right $A$-module. The set of elements $S := \{s_1, \dots, s_m\} \subset E$ is a *set of generators* of $E$ if any element $m \in E$ can is an $A$-linear combination of elements of $S$. That is, $m = s_1\, a_1 + \cdots s_m\, a_m$ for $a_1, \dots, a_m \in A$. We say that $E$ is *finitely generated* if it has a finite set of generators. A set of generators $S$ of $M$ is an *$A$-basis* if it is $A$-linearly independent. That is, $s_1\, a_1 + \cdots + s_m\, a_m = 0$ for $a_1, \dots, a_m$ in $A$ implies that $a_1 = \cdots = a_m = 0$ . A right $A$-module $E$ is *free* if it has an $A$-basis. A free module is finitely generated if and only if it has a finite $A$-basis. Similar definitions can be obtained for left $A$-modules.

**Example 4.1.1** Suppose that $A$ is a nontrivial unital ring and $m$ is a positive integer, then the $m$-fold direct sum $A^m := A \oplus \cdots \oplus A$ is the *standard free right $A$-module* with an $A$-basis $\{e_1, \dots, e_m\}$ where $e_j = (0, \dots, 0, 1_A, 0, \dots, 0)^t$ with the $j$-th component the unit of $A$ for $j = 1, \dots, m$. $(\cdot)^t$ means taking transpose of a row vector to a column vector. The right $A$-module action on any $s = \sum_i e_i a_i \in A^m$ is given by

$$(a_1, \dots, a_m)^t\, b = (a_1\, b, \dots, a_m\, b)^t, \quad \forall b \in A.$$

This is called the standard one because any finitely generated free right $A$-module $F$ is $A$-isomorphic to $A^m$ for some integer $m$ by matching basis.

**Example 4.1.2** We denote ${}^mA := A \oplus \cdots \oplus A$ as the free left $A$-module with $A$-basis $\{f_1, \dots, f_m\}$ where $f_j = (0, \dots, 0, 1_A, 0, \dots, 0)$ with the $j$-th component the unit of $A$ for $j = 1, \dots, m$. The left $A$-module action on any $s = \sum_i a_i f_i \in {}^mA$ is given by

$$b\,(a_1, \dots, a_m) = (b\, a_1, \dots, b\, a_m), \quad \forall b \in A.$$

Let $A$ and $B$ be two arbitrary unital rings. If $E$ is both a right $A$-module and a left $B$-module, such that $b\,(s\,a) = (b\,s)\,a$ for any $a \in A$, $b \in B$ and $s \in E$, then $E$ is called a $B$-$A$-bimodule. When $A = B$, $E$ is called an *$A$-bimodule*.

For $A$ a unital ring, we may define the *tensor product* $E \otimes_A F$ of a right $A$-module $E$ and a left $A$-module $F$ as the abelian group generated by simple tensors $s \otimes t$ with $s \in E$ and $t \in F$, subjecting to the relations

$$s_1 \otimes t + s_2 \otimes t = (s_1 + s_2) \otimes t, \quad \forall s_1, s_2 \in E,\ t \in F,$$

$$s \otimes t_1 + s \otimes t_2 = s \otimes (t_1 + t_2), \quad \forall s \in E,\ t_1, t_2 \in F,$$

$$s\,a \otimes t = s \otimes a\,t, \quad \forall a \in A,\ s \in E,\ t \in F.$$

In particular, if both $E$ and $F$ are $A$-bimodules, then $E \otimes_A F$ is also an $A$-bimodule.

## 4.2 $C^*$-modules

Let $A$ be a unital $C^*$-algebra, a *right pre-$C^*$-($A$-)module* is a right $A$-module $E$ equipped with an $A$-valued inner product $(\cdot,\cdot) : E \times E \longrightarrow A$ satisfying the following conditions

$$(s, t_1 + t_2) = (s, t_1) + (s, t_2), \quad \forall s, t_1, t_2 \in E,$$

$$(s, t\,a) = (s, t)\,a, \quad \forall s, t \in E,\ a \in A,$$

$$(r, s) = (s, r)^*, \quad \forall r, s \in E,$$

$$(r, r) \geq 0; \quad (r, r) = 0 \Leftrightarrow r = 0, \quad \forall r \in E.$$

The last condition is the property of positive definitness. Positivity refers to self-adjoint elements in the $C^*$-algebra. The self-adjointness of $(r, r)$ follows from the third condition.

The $C^*$-norm $||\cdot||$ of $A$ together with the $A$-valued inner product induce a norm $||\cdot||_E$ on the right pre-$C^*$-module $E$ as

$$||s||_E := ||(s, s)||^{1/2}, \quad \forall s \in E. \tag{4.1}$$

To see that (4.1) is a norm, we first show the general Cauchy-Schwarz inequality,

**Lemma 4.2.1** *If $E$ is a right pre-$C^*$-$A$-module, then $||(r, s)|| \leq ||r||_E\, ||s||_E$, for $r, s \in E$.*

**Proof**: By the fact that $a^*ca \leq \|c\|a^*a$ for any positive $c \in A$ and $a \in A$, we have $a^*(r,r)a \leq \|r\|_E^2 a^*a$ for any $r \in E$. Thus

$$\begin{aligned} 0 &\leq (r\,a - s, r\,a - s) \\ &= a^*(r,r)a + (s,s) - a^*(r,s) - (s,r)a \\ &\leq a^*\|r\|_E^2 a + \|s\|_E^2 - a^*(r,s) - (s,r)a. \end{aligned}$$

By taking $a = \|r\|_E^{-2}(r,s)$, and using the fact that $0 \leq a \leq b \Longrightarrow \|a\| \leq \|b\|$ for $a, b \in A$ by (2.14), we obtain the required inequality. □

The triangle inequality for $\|\cdot\|_E$ follows immediately. Since

$$\begin{aligned} \|r+s\|_E^2 &= \|(r,r) + (s,s) + (r,s) + (s,r)\| \\ &\leq \|(r,r)\| + \|(s,s)\| + \|(r,s)\| + \|(s,r)\| \\ &\leq \|r\|_E^2 + \|s\|_E^2 + \|r\|_E\|s\|_E + \|s\|_E\|r\|_E \\ &= (\|r\|_E + \|s\|_E)^2, \quad r, s \in E, \end{aligned}$$

then $\|r+s\|_E \leq \|r\|_E + \|s\|_E$ and $\|\cdot\|_E$ is a norm.

Furthermore, the *Banach module condition* holds, i.e. $\|r\,a\|_E \leq \|r\|_E\|a\|$, for $r \in E, a \in A$. One can complete $E$ in this norm and the resulting Banach space is called a *right $C^*$-$A$-module,* or simply a $C^*$-module if the underlying $C^*$-algebra $A$ is understood. One can similarly define a *left $C^*$-module.*

**Example 4.2.1** Any $C^*$-algebra $(A, \|\cdot\|)$ is a right $C^*$-module over itself. We may define the $A$-valued inner product as $(a,b) := a^*b$ for $a, b \in A$ and the induced norm is the same as the $C^*$-norm. Indeed, $\|a\|_A^2 = \|(a,a)\| = \|a^*a\| = \|a\|^2$, where the $C^*$-identity is applied, and hence $\|a\|_A = \|a\|$.

**Example 4.2.2** We consider the right $A$-module $A^m$ as in Example 4.1.1 and define a standard $A$-valued inner product $(\cdot,\cdot)_{A^m} : A^m \times A^m \to A$ by

$$(s,t)_{A^m} := s_1^*t_1 + \cdots s_m^*t_m,$$

where $s = (s_1, \ldots, s_m)^t, t = (t_1, \ldots, t_m)^t \in A^m$. And the induced norm is thus

$$\|s\|_{A^m} := \|(s,s)_{A^m}\|^{1/2} = \left\|\sum_{k=1}^m s_k^*s_k\right\|^{1/2}, \quad \forall s \in A^m.$$

The completion of $A^m$ by the above norm gives us a right $C^*$-module.

Actually,

**Lemma 4.2.2** *$A^m$ is complete with respect to the norm $\|\cdot\|_{A^m}$ and hence has a right $C^*$-module structure.*

**Proof**: We define the *coefficient map* $Q_k : A^m \to A$ by

$$Q_k(a_1, \ldots, a_m,)^t := a_k, \quad k = 1, \ldots, m.$$

First we see that $Q_k$ is norm decreasing. The norm on $A^m$ gives $\|(a_1, \ldots, a_m)^t\|_{A^m}^2 = \|\sum_{l=1}^m a_l^* a_l\|$ while the norm square of its image $a_k$ under $Q_k$ is $\|a_k\|^2 = \|a_k^* a_k\|$. Since

$$a_1^* a_1 + \cdots + a_{k-1}^* a_{k-1} + a_{k+1}^* a_{k+1} + \cdots + a_m^* a_m \geq 0$$

for any fixed $k$, then $\sum_{l=1}^m a_l^* a_l \geq a_k^* a_k$. As implied by (2.14), we can take norm and obtain $\|\sum_{l=1}^m a_l^* a_l\| \geq \|a_k^* a_k\|$, for each $k = 1, \ldots, m$. This shows that $Q_k$ is norm-decreasing.

In this way, any Cauchy sequence $\{s^{(\beta)} := (a_1^{(\beta)}, \ldots, a_m^{(\beta)})^t\}_\beta$ in $A^m$ defines $m$ Cauchy sequences $\{Q_k(s^{(\beta)})\}_\beta$ where $k = 1, \ldots, m$ in $A$. Since $A$ is complete the limit $a_k := \lim_{\beta \to \infty} Q_k(s^{(\beta)})$ is an element of $A$ for each $k$. Furthermore, $(a_1, \ldots, a_m)^t \in A^m$ is the limit of the Cauchy sequence $\{s^{(\beta)}\}$. Thus $A^m$ is complete. □

Before the next example of a $C^*$-module, we give the following lemmas from [6].

**Lemma 4.2.3** *If $A$ is a $C^*$-algebra, then so is the matrix algebra $M_m(A)$ for any positive integer $m$.*

**Proof**: If $A$ is a $C^*$-algebra, then it is isometrically $*$-isomorphic to a $C^*$-subalgebra of operators on a Hilbert space $\mathcal{H}$. In other words, there is a faithful $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$. We may induce a representation $\pi^{(m)} : M_m(A) \to \mathcal{B}(\mathbb{C}^m \otimes \mathcal{H})$ by

$$(\pi^{(m)}(a)\eta)_i := \sum_{j=1}^n \pi(a_{ij})\eta_j, \quad i = 1, \ldots, n,$$

where $a = (a_{ij}) \in M_m(A)$ and $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{C}^m \otimes \mathcal{H}$. This representation further realises $M_m(A)$ as a $C^*$-algebra. □

**Lemma 4.2.4** *An element $a = (a_{ij}) \in M_m(A)$ as a $C^*$-algebra in Lemma 4.2.3 is positive if and only if it is a sum of matrices of the form $(a_{ij} = a_i^* a_j)$ with $a_1, \ldots, a_m \in A$.*

**Proof**: For a matrix $a$ of the form $(a_{ij} = a_i^* a_j)$ with $a_1, \dots, a_n \in A$, we may define a matrix $b = (b_{ij}) \in M_m(A)$ such that $b_{1j} := a_j$ for $j = 1, \dots, m$ and $b_{ij} := 0$ for $i = 2, \dots, m$, then $a_i^* a_j$ is the $(i, j)$-th entry of the matrix $b^* b$, i.e., $a = b^* b$, which is positive in the $C^*$-algebra $M_m(A)$. Conversely, if $a$ is a positive element in $M_m(A)$, then there exists a matrix $c$ such that $a = c^* c$, i.e., $a_{ij} = \sum_{k=1}^m c_{ki}^* c_{kj}$. That is, $a = b^1 + \cdots + b^m$ where the matrix $b^{(k)} = (b_{ij}^{(k)})$ is defined by $b_{ij}^{(k)} := c_{ki}^* c_{kj}$ for $k = 1, \dots, m$. This completes the proof. □

Another useful fact is (Proposition 1.20 [6])

**Lemma 4.2.5** *An element $a = (a_{ij})$ in the $C^*$-algebra $M_m(A)$ is positive if and only if $\sum_{i,j=1}^m c_i^* a_{ij} c_j$ is positive in $A$ for all $c_1, \dots c_n \in A$.*

**Example 4.2.3** Consider the left $A$-module ${}^mA$ defined in Example 4.1.2 and notice that it is also a right $M_m(A)$-module. We define an $M_m(A)$-valued inner product $(\cdot, \cdot) : {}^mA \times {}^mA \to M_m(A)$ by $(s, t) := (s_i^* t_j) \in M_m(A)$, where $s = (s_1, \dots, s_m), t = (t_1, \dots, t_m) \in {}^mA$. To see that it is positive definite, it suffices to check that

$$\sum_{i,j=1}^{m} c_i^* s_i^* s_j c_j = \left(\sum_i s_i c_i\right)^* \left(\sum_j s_j c_j\right), \quad \forall c_1, \dots, c_m \in A,$$

is positive in $A$ by Lemma 4.2.5. Therefore ${}^mA$ is a right $C^*$-$M_m(A)$-module.

## 4.3 Finitely generated projective modules

Assuming that $A$ is a unital $C^*$-algebra, an $A$-module $E$ is called *projective* when it is a direct summand of some free $A$-module. $E$ is a finitely generated projective right (left) $A$-module if and only if $E$ is isomorphic as modules to a direct summand in the right (left) $A$-module $A^m$ (${}^mA$) for some integer $m$. Furthermore, every finitely generated projective $A$-module can be endowed with the structure of a $C^*$-module over $A$.

We consider the finitely generated projective right $A$-module $pA^m$ for some *projection* $p = (p_{ij}) \in M_m(A)$. That is, $p^2 = p = p^*$. A basis of $pA^m$ can be generated by the standard basis $\{e_1, \dots, e_m\}$ of $A^m$ in Example 4.1.1 as follows. Any element $\xi \in pA^m$ is written as $\xi = pt$ for some $t \in A^m$. In this basis, we may write an element in $A^m$ as $e_1\xi_1 + \cdots e_m\xi_m$ for $\xi_i \in A$ and $i$ ranges from 1 to $m$. Thus, $\xi = p(e_1\xi_1 + \cdots e_m\xi_m) = f_1t_1 + \cdots + f_mt_m$, where $f_i := pe_i = \sum_j p_{ji}e_j$, which is the $i$-th column $(p_{1i}, \cdots, p_{mi})^t$ of the matrix $p$. $\{f_1, \ldots, f_m\}$ is a basis of the projective module $pA^m$, under which any element $\xi = \sum f_i\xi_i \in pA^m$ has coordinates $(\xi_1, \ldots, \xi_m)$ satisfying $\xi_i = \sum_j p_{ij}\xi_j$.

The right $A$-action on $\xi \in pA^m$ is simply

$$(\xi_1, \ldots, \xi_m)^t\, b = (\xi_1 b, \ldots, \xi_m b)^t, \quad \forall b \in A.$$

We may define the standard $A$-valued inner product on the right projective module $pA^m$, $(\cdot, \cdot) : pA^m \times pA^m \to A$ by

$$(\xi, \eta) := \sum_i \xi_i^*\eta_i, \quad \xi, \eta \in pA^m, \tag{4.2}$$

where $\xi = (\xi_1, \ldots, \xi_m)^t$ with $\xi_i = \sum_j p_{ij}\xi_j$ and similarly $\eta = (\eta_1, \ldots, \eta_m)^t$ with $\eta_i = \sum_j p_{ij}\eta_j$.

To see that the inner product (4.2) is positive definite, we note that for any $\xi \in pA^m$,

$$(\xi, \xi) = \sum_{i,j=1}^m \xi_i^* p_{ij}\xi_j.$$

Implied by Lemma 4.2.5, we can conclude that $(\xi, \xi)$ is positive in $A$ if $p = (p_{ij})$ is positive in $M_m(A)$. Now since $p$ is a projection so that $p^2 = p = p^*$ and the $(i, j)$-the entry of $p$ is in the form of $p_{ij} = \sum_{k,l=1}^m p_{ik}p_{kj} = \sum_{k,l=1}^m p_{ki}^*p_{kj}$. In other words, $p_{ij} = \sum_{k=1}^m p_{ij}^{(k)}$, $\quad p_{ij}^{(k)} := p_{ki}^*p_{kj}$. Thus, $p = (p_{ij}) = \sum_{k=1}^m (p_{ij}^{(k)})$ is positive by Lemma 4.2.4. Indeed, since $p$ is linear combination of matrices $(p_{ij}^{(k)})$ whose $(i, j)$-th component is in the form of $p_{ki}^*p_{kj}$ for $p_{k1}, \ldots, p_{km} \in A$. Thus we conclude that $(\xi, \xi)_A \geq 0$ in $A$ for all $\xi \in pA^m$.

We end this section with an analogue of Lemma 4.2.2,

**Lemma 4.3.1** *The finitely generated projective right $A$-module $pA^m$ is complete and hence a $C^*$-module.*

**Proof**: Note that each element $s \in pA^m$ can be written as $\xi = \sum_{i=1}^m f_i\xi_i = \sum_{i,k=1}^m p_{ki}e_k\xi_i = \sum_{i,k=1}^m p_{ki}\xi_ie_k$. We define the coefficient map $Q_k : pA^m \to A$ by

$$Q_k(\xi) = Q_k\left(\sum_{i,k=1}^m p_{ki}\xi_ie_k\right) := \sum_i^m p_{ki}\xi_i, \quad k = 1, \ldots, m.$$

We now show that $Q_k$ is norm decreasing. The norm square of $Q_k(\xi)$ is

$$\|Q_k(\xi)\|^2 = \left\|\sum_{i=1}^{m} p_{ki}\xi_i\right\|^2 = \left\|\sum_{i,j=1}^{m} (p_{ik}\xi_i)^*(p_{kj}\xi_j)\right\|, \quad k = 1, \ldots, m.$$

The norm square of $\xi$ is given by

$$\begin{aligned} \|\xi\|^2_{pA^m} &= \left\|\sum_{i,j=1}^{m} \xi_i^* p_{ij}\xi_j\right\| = \left\|\sum_{i,j=1}^{m} \xi_i^* \left(\sum_{k=1}^{m} p_{ik}p_{kj}\right)\xi_j\right\| \\ &= \left\|\sum_{k,i,j=1}^{m} \xi_i^* p_{ki}^* p_{kj}\xi_j\right\| = \left\|\sum_{k,i,j=1}^{m} (p_{ki}\xi_i)^*(p_{kj}\xi_j)\right\| \end{aligned}$$

Lemma 4.2.4 implies that for each fixed $k$, the sum $\sum_{i,j=1}^{m}(p_{ki}\xi_i)^*(p_{kj}\xi_j)$ is positive in $A$, thus

$$(0 \leq) \sum_{i,j=1}^{m} (p_{ki}\xi_i)^*(p_{kj}\xi_j) \leq \sum_{i,j,k=1}^{m} (p_{ki}\xi_i)^*(p_{kj}\xi_j).$$

By (2.14), we can take norm and obtain

$$\left\|\sum_{i,j=1}^{m} (p_{ki}\xi_i)^*(p_{kj}\xi_j)\right\| \leq \left\|\sum_{i,j,k=1}^{m} (p_{ki}\xi_i)^*(p_{kj}\xi_j)\right\|, \quad k = 1, \ldots, m.$$

In other words, $\|Q_k(\xi)\| \leq \|\xi\|_{pA^m}$ for any $\xi \in pA^m$. This shows that $Q_k$ is norm decreasing.

Now any Cauchy sequence $\{\xi^{(\beta)} = \sum_{i,k=1}^{m} p_{ki}\xi_i^{(\beta)}e_k\}_\beta$ in $pA^m$ defines $m$ Cauchy sequences $\{\sum_{i=1}^{m} p_{ki}\xi_i^{(\beta)}\}_\beta$ where $k = 1, \ldots, m$. $A$ is complete so that the limit

$$\lim_{\beta\to\infty} \sum_{i=1}^{m} p_{ki}\xi_i^{(\beta)} = \sum_{i=1}^{m} p_{ki}\left(\lim_{\beta\to\infty} \xi_i^{(\beta)}\right) =: \sum_{i=1}^{m} p_{ki}\xi_i$$

satisfies that $\xi_i \in A$ for each $k$. The element

$$\left(\sum_{i=1}^{m} p_{1i}\xi_i, \ldots, \sum_{i=1}^{m} p_{mi}\xi_i\right)^t = p(\xi_1, \ldots, \xi_m)^t$$

in $pA^m$ is the limit of the Cauchy sequence $\{\xi^{(\beta)}\}_\beta$. Thus $pA^m$ is complete and hence is endowed with a $C^*$-module structure. □

The projective module discussed above is for a unital $C^*$-algebra $A$. In general for a unital $*$-algebra $\mathcal{A}$, we can define the *projective right $\mathcal{A}$-module* by $p\mathcal{A}^m$ for some $p \in M_m(\mathcal{A})$ satisfying $p^2 = p = p^*$ for some positive integer $m$. The left modules can also be defined directly. We will further discuss the notion of smooth module [10] in Chapter 7.

# Chapter 5

# Spin geometry

We will give a summary of the relevant facts in spin geometry necessary for formulating noncommutative geometry. We will give the definition of Clifford algebras, Clifford actions and spinor bundles over Riemannian spin manifolds. References on spin geometry are [32] [33] [34].

## 5.1 Clifford algebras

Let $V$ be a $n$-dimensional vector space over the commutative field $k$ of characteristic $\neq 2$ (e.g. $k = \mathbb{R}$ or $k = \mathbb{C}$) with a nondegenerate quadratic form $q$. We let $\mathcal{T}(V) := \sum_r \mathcal{T}^r(V)$, where $\mathcal{T}^r(V) := V \otimes \cdots \otimes V(r\text{copies of } V$, be the tensor algebra of $V$ and $\mathcal{I}_q$ be the ideal in the tensor algebra generated by all the elements of the form $v \otimes v + q(v)\mathbf{1}$ for $v \in V$. The *Clifford algebra* is defined as the quotient algebra

$$Cl(V, q) := \mathcal{T}(V)/\mathcal{I}_q.$$

The equivalence class $v_1 \otimes v_2 \cdots \otimes v_p + \mathcal{I}_q(V)$ is denoted as $v_1 \cdot v_2 \cdots v_p \in Cl(V, q)$. The induced multiplication on the Clifford algebra is denoted by $\cdot$, called the *Clifford multiplication.*

Equivalently, the Clifford algebra $(Cl(V, q), \cdot)$ is defined to be the algebra generated by elements of the vector space $V \subset Cl(V, q)$ and an identity element $\mathbf{1}$ subject to the relation $v \cdot v = -q(v)\mathbf{1}, \quad \forall v \in V$. This is equivalent to $v \cdot w + w \cdot v = -2q(v, w)\mathbf{1}, \quad \forall v, w \in V$, where the symmetric bilinear form $2q(v, w) := q(v + w) - q(v) - q(w)$ is the *polarization* of $q$.

There is a natural filtration of the tensor algebra $\mathcal{T}(V)$ as $\tilde{\mathcal{F}}^0(V) \subset \tilde{\mathcal{F}}^1(V) \subset \cdots \subset \tilde{\mathcal{F}}^r(V) \subset \cdots \subset \mathcal{T}(V)$ satisfying $\tilde{\mathcal{F}}^r(V) \otimes \tilde{\mathcal{F}}^s(V) \subset \tilde{\mathcal{F}}^{r+s}(V)$, where

$$\tilde{\mathcal{F}}^r(V) := \sum_{s \leq r} \mathcal{T}^s(V).$$

For each $r \in \mathbb{N}$, we define $\mathcal{F}^r(V) := \pi_q(\tilde{\mathcal{F}}^r(V))$ where $\pi_q$ denotes the restriction of the quotient map of $\mathcal{T}(V)$ to $Cl(V, q)$ by $\mathcal{I}_q(V)$. Then this gives a natural filtration of the Clifford algebra, $\mathcal{F}^0(V) \subset \mathcal{F}^1(V) \subset \cdots \subset \mathcal{F}^r(V) \subset \cdots \subset Cl(V, q)$ satisfying $\mathcal{F}^r(V) \cdot \mathcal{F}^s(V) \subset \mathcal{F}^{r+s}(V)$. Thus the Clifford multiplication desends to a map $\mathcal{G}^r(V) \times \mathcal{G}^s(V) \to \mathcal{G}^{r+s}$ where $\mathcal{G}^r(V) := \mathcal{F}^r(V)/\mathcal{F}^{r-1}(V)$ for $r \in \mathbb{N}$. $\mathcal{G}^*(V) := \oplus_{r \geq 0} \mathcal{G}^r(V)$ is the associated graded algebra. This algebra is isomorphic to the exterior algebra as a graded algebra.

Suppose that $(V, q)$ are as before. We may write $Cl(V, q)$ as $Cl(V)$ if $q$ can be deduced from the content. The following is the defining property of the Clifford algebra $Cl(V, q)$. If $\alpha : V \to A$ is a linear map into an associative $k$-algebra $(A, \cdot_A)$ with unit $1_A$ such that $\alpha(v) \cdot_A \alpha(v) = -q(v) 1_A, \quad \forall v \in V$, then $\alpha$ extends uniquely to a $k$-algebra homomorphism $\tilde{\alpha} : Cl(V) \to A$ such that $\alpha = \tilde{\alpha} \circ i$ where $i$ is the inclusion of $V$ into $Cl(V)$. Furthermore, $Cl(V)$ is the unique associative $k$-algebra with this property.

By the defining property of a Clifford algebra, the linear map $\alpha : V \longrightarrow V$ such that $\alpha(v) = -v$ can be uniquely extended to an automorphism $\widetilde{\alpha} : Cl(V, q) \longrightarrow Cl(V, q)$, called the *canonical automorphism of the Clifford algebra* satisfying $\widetilde{\alpha}^2 = \mathbf{1}$. The Clifford algebra is thus split into *even part* and *odd part* defined by the $+1$ and $-1$-eigenspaces of $\widetilde{\alpha}$ respectively

$$Cl(V) = Cl(V)^0 \oplus Cl(V)^1. \tag{5.1}$$

There are several important subgroups contained in the Clifford algebra $Cl(V, q)$. The *multiplicative group* $Cl^\times(V, q)$ consists of invertible elements in the Clifford algebra $Cl(V, q)$. The *Pin group* is the subgroup of $Cl^\times(V, q)$ generated by unit vectors in $V$. That is,

$$Pin(V, q) := \{u \in Cl^\times(V, q) : u = u_1 \cdots u_r \text{ with } u_j \in V, q(u_j) = \pm 1, j = 1, \ldots, r\}.$$

The *Spin group* is defined to be the even part of the Pin group. That is $Spin(V, q) = Pin(V, q) \cap Cl(V)^0$.

There is an important representation of $Cl^\times(V, q)$ on $V$. We define the *twisted adjoint representation* $\widetilde{Ad} : Cl^\times(V, q) \to Aut(Cl(V, q))$ by

$$\widetilde{Ad}_w(v) := \widetilde{Ad}(w)(v) := \tilde{\alpha}(w) \cdot v \cdot w^{-1}, \quad w \in Cl^\times(V, q), v \in Cl(V, q),$$

where $\tilde{\alpha}$ is the canonical automorphism of Clifford algebras defined above. In the case $w \in V$ such that $q(v) \neq 0$ and $v \in V \subset Cl(V)$, the map $\widetilde{Ad}_w(v) = w - \frac{2q(v,w)}{q(v)}$. This is the reflection $\rho_w(v)$ of $v$ with respect to the hyperplane $w^\perp$ of normal direction $w$ in $V$. Note that both $w$ and $-w$ give the same reflection. More generally, when $w$ is in $Pin(V, q)$ or $Spin(V, q)$ the geometric pictures are still available.

For instance, when the map $\widetilde{Ad}$ is restricted on the subgroup $Spin(V, q)$. If we further restrict the respesentation on the space $V$ considered as a subspace of $Cl(V, q)$, we obtain the surjective map $\widetilde{Ad} : Spin(V, q) \to SO(V)$ such that $\widetilde{Ad}(v_1 \cdots v_r) = \rho_{v_1} \circ \cdots \circ \rho_{v_r}$ in the form of composition of even number of reflections. When the field $k = \mathbb{R}$ the kernel of $\tilde{A}d$ is $\{1, -1\} \cong \mathbb{Z}_2$. Thus we have the following short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin(n) \xrightarrow{\xi_0} SO(n) \longrightarrow 1,$$

where $\xi_0 := \widetilde{Ad}$. This thus gives a double covering of $SO(n)$. When $n \geq 3$, $SO(n)$ is connected and $Spin(n)$ is simply connected so that

$$\xi_0 : Spin(n) \longrightarrow SO(n) \tag{5.2}$$

is the universal covering of $SO(n)$.

Consider the vector space $V = \mathbb{R}^n$ with its usual inner product as the symmetric bilinear form and let $\{e_j\}_{j=1\ldots n}$ be an *oriented* orthonormal basis of $V$. The Clifford algebra $Cl(n) := Cl(\mathbb{R}^n)$ is generated by $\{e_1, \ldots, e_n\}$ subject to the relation $e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}1$.

We define the *chirality operator* acting on the Clifford algebra to itself by the Clifford multiplication of

$$\chi := e_1 \cdots e_n \tag{5.3}$$

in $Cl(n)$. Note that this definition is independent of choices of the basis. $\chi$ satisfies $\chi \cdot \chi = (-1)^{\frac{n(n+1)}{2}}$.

In the complex case, we consider $\mathbb{C}^n$ and suppose the quadratic form $q_{\mathbb{C}} = q \otimes \mathbb{C}$ of $\mathbb{C}^n$ is obtained from the complexification of a quadratic form $q$ of $\mathbb{R}^n$. We denote its associated Clifford algebra as $\mathbb{C}l(n) := Cl(\mathbb{C}^n, q_{\mathbb{C}})$. This is simply the complexification of the real Clifford algebra $Cl(\mathbb{R}^n)$. That is, $\mathbb{C}l(n) \cong Cl(\mathbb{R}^n, q) \otimes_{\mathbb{R}} \mathbb{C}$.

In $\mathbb{C}l(n)$, we can also define a corresponding *complex chirality operator*

$$\chi_{\mathbb{C}} := i^{[\frac{n+1}{2}]} \chi, \tag{5.4}$$

where $\chi$ is defined in (5.3). For $n$ even, $\chi_{\mathbb{C}} = i^{n/2} e_1 \cdots e_n$. In particular when $n = 4$, $\chi_{\mathbb{C}} = -e_1 \cdots e_4$. Again $\chi_{\mathbb{C}}$ is independent of choices of the orthonormal oriented basis $e_j$ of $\mathbb{R}^n$. One can also show that $\chi_{\mathbb{C}}^2 = 1$. The corresponding $\pm 1$-eigenspace decomposition of $\chi_{\mathbb{C}}$ gives

$$\mathbb{C}l(n) = \mathbb{C}l(n)^+ \oplus \mathbb{C}l(n)^-$$

where $\mathbb{C}l(n)^{\pm} := (1 \pm \chi_{\mathbb{C}})\mathbb{C}l(n)$. This is known as the *self-dual (SD)* and *antiself dual (ASD) decomposition.* This can further be restricted to the even part of the Clifford algebra $\mathbb{C}l(n)^0$ as in (5.1) $\mathbb{C}l(n)^0 = \mathbb{C}l(n)^{0,+} \oplus \mathbb{C}l(n)^{0,-}$, where $\mathbb{C}l(n)^{0,\pm} := \{v \in \mathbb{C}l(n)^0 : \chi_{\mathbb{C}} \cdot v = \pm v\}$.

## 5.2 Clifford actions and Clifford modules

Let $V$ be a vector space over a field $k$ with a quadratic form $q$. Let $K \supset k$ be a field containing $k$. Then a *$K$-representation* of the Clifford algebra $Cl(V, q)$ is a $k$-algebra homomorphism

$$\rho : Cl(V, q) \longrightarrow Hom_K(W, W) \tag{5.5}$$

into the algebra of linear transformations of a finite dimensional vector space $W$ over $K$. The space $W$ is called a *$Cl(V, q)$-module* over $K$. We call the representation $\rho(v)(w) = v \cdot w$ for $v \in Cl(V, q)$ and $w \in W$ the *Clifford multiplication* by $v$.

A $K$-representation as (5.5) is said to be *reducible* if the vector space $W$ can be written as a direct sum over $K$. That is, $W = W_1 \oplus W_2$ such that $\rho(v)(W_j) \subset W_j$

for $j = 1,2$ and $v \in Cl(V,q)$. In that case we can write $\rho = \rho_1 \oplus \rho_2$ where $\rho_j(v) := \rho(v)|_{W_j}$ for $j = 1,2$. A representation is called *irreducible* if it is not reducible. A $K$-representation of $\rho$ of a Clifford algebra $Cl(V,q)$ can always be decomposed into direct sum of irreducible representations.

Two representations $\rho_j : Cl(V,q) \longrightarrow Hom_K(W_j, W_j)$ for $j = 1,2$ are said to be *equivalent* if there exists a $K$-linear isomorphism $F : W_1 \longrightarrow W_2$ such that $F \circ \rho_1(v) \circ F^{-1} = \rho_2(v)$ for all $v \in Cl(V,q)$.

Let $\rho : Cl(n) \to Hom_{\mathbb{R}}(W,W)$ be an irreducible real representation where $n = 4m$, and consider the splitting

$$W = W^+ \oplus W^-$$

where $W^\pm = (1 \pm \rho(\chi))W$, where $\omega$ is the chirality operator of $\mathbb{R}^n$. Then each of the subspaces $W^+$ and $W^-$ is invariant under the even subalgebra $Cl(n)^0$. The analogous statements are true in the complex case $\mathbb{C}l(n)$ with $n$ even.

Recall that the Clifford algebra has a $\mathbb{Z}_2$-graded algebra with respect to the canonical automorphism. We concentrate on actions of Clifford algebra on modules over $\mathbb{R}$ or $\mathbb{C}$ which are also $\mathbb{Z}_2$-graded. That is, we consider the module $E = E^0 \oplus E^1$, such that the Clifford action $Cl(V) \to End(E)$ is even with respect to the grading on the Clifford algebra:

$$Cl(V)^0(E^{0/1}) \subset E^{0/1}, \quad Cl(V)^1(E^{0/1}) \subset E^{1/0}.$$

Such a module is called the *Clifford module.*

One example of a Clifford module is the exterior algebra $\Lambda(V)$. We define the representation $c : V \to End(\Lambda(V))$ by

$$c(v)(u) := E_v(u) - I_v(u), \quad \forall u \in \Lambda(V),$$

where $E_v : \Lambda(V) \to \Lambda(V)$ is defined by $E_v(u) = v \wedge u, \quad \forall u \in \Lambda(V)$ and $I_v : \Lambda(V) \to \Lambda(V)$ is defined by requiring that $\langle v \wedge u, w \rangle = \langle u, I_v w \rangle$, $\forall u, w \in \Lambda(V)$. The inner product $\langle \cdot, \cdot \rangle$ on $\Lambda(V)$ is induced from $q$ of $V$. The action $c$ of $V$ can be uniquely extended to the action $\tilde{c}$ of the Clifford algebra $Cl(V)$ on $\Lambda(V)$. $\Lambda(V)$ is a Clifford module with respect to its $\mathbb{Z}_2$-grading given by the parity of its exterior degree $\Lambda(V) = \Lambda(V)^0 \oplus \Lambda(V)^1$.

By evaluating the action $\tilde{c}$ on the identity element of $\Lambda(V)$, we obtain the *symbol map* between the Clifford algebra and the exterior algebra, $\sigma : Cl(V) \longrightarrow \Lambda V$ as $\sigma(a) = \tilde{c}(a)(\mathbf{1})$ where $\mathbf{1}$ is the identity element of $\Lambda V$. The symbol map has an inverse. Let $e_i$ be an orthonormal basis of $V$, and denote $e_i$ as $c_i$ when considered in $Cl(V)$, then $\mathbf{c} : \Lambda V \longrightarrow Cl(V)$ defined by $\mathbf{c}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = c_{i_1} \cdots c_{i_k}$ is the inverse and we call $\mathbf{c}$ the *quantization map.*

## 5.3 Spinor representation

We assume that $V = \mathbb{R}^n$. The Clifford algebra $Cl(n)$ with the Lie bracket defined by $[v, w] := v \cdot w - w \cdot v$ for $w \in Cl(n)$ is a Lie algebra.

The subspace $\mathbf{c}(\Lambda^2 V)$ is a Lie subalgebra of $Cl(n)$. One can show that it is isomorphic to the Lie algebra $\mathfrak{so}(n)$ under the map $\tau : \mathbf{c}(\Lambda^2 V) \longrightarrow \mathfrak{so}(n)$ defined by

$$\tau(a)(v) = [a, v], \tag{5.6}$$

where $\tau(a)$ act on $v \in V \cong \mathbf{c}(\Lambda^1 V)$. This implies that a matrix $a \in \mathfrak{so}(V)$ corresponds to the Clifford element $\tau^{-1}(a) = 1/2 \sum_{i<j}(ae_i, e_j)c_i \cdot c_j$. If we identify $a \in \mathfrak{so}(V)$ with an element of $\Lambda^2 V$ by the isomorphism that maps $a$ to $\sum_{i<j}(ae_i, e_j)e_i \wedge e_j$, then $\mathbf{c}(a) = \sum_{i<j}(ae_i, e_j)c_i \cdot c_j$. The Lie group obtained as the image of the exponential map of the Lie subalgebra $\mathbf{c}(\Lambda^2 V)$ is exactly the Spin group $Spin(\mathbb{R}^n, q)$. We will denote it as $Spin(n)$.

The map $\tau$ in (5.6) between Lie algebras is then exponentiated to a map between Lie groups, $\tilde{\tau} : Spin(n) \longrightarrow SO(n)$, such that

$$\tilde{\tau}(g)(v) = \tilde{\tau}(\exp(a))(v) = \exp(\tau(a))(v) = gvg^{-1},$$

for any $g = \exp(a) \in Spin(n)$ and $v \in V$. $\tilde{\tau}$ is a double covering if $n > 1$. This is the *adjoint representation map* of $Spin(n)$, which we can compare it with the twisted adjoint representation defined previously.

The importance of the Spin group in a Clifford algebra is that any Clifford module restricts to a representation of the Spin group.

We will now construct the spinor representation of a Clifford algebra $Cl(V, q)$. Recall that the chirality operator $\chi_{\mathbb{C}}$ in (5.4 ) satisfies $\chi_{\mathbb{C}}^2 = 1$. Furthermore, for

$v \in \mathbb{C}l(n)$, $\chi_{\mathbb{C}} \cdot v = -v \cdot \chi_{\mathbb{C}}$ if $n$ is even and $\chi_{\mathbb{C}} \cdot v = v \cdot \chi_{\mathbb{C}}$ if $n$ is odd. It also decomposes $\mathbb{C}l(V)^0$ to SD and ASD parts.

Let $E$ be a Clifford module (possibly over the field $\mathbb{C}$) with Clifford action $c : \mathbb{C}l(V) \to End(E)$. We can define a $\mathbb{Z}_2$-grading on $E$ with respect to the chirality operator $\chi_{\mathbb{C}} \in Cl(V \otimes \mathbb{C}, q_{\mathbb{C}})$ by

$$E^{\pm} := \{v \in E : c(\gamma)v = \pm v\}.$$

We call the decomposition as *SD/ASD decomposition* of the Clifford module $E$.

**Remark**: In the case when $n = 4m$ for integer $m$, $\chi_{\mathbb{C}}$ actually belongs to the real Clifford algebra $Cl(V)$.

We quote the following results from Proposition 3.19 of [34].

**Proposition 5.3.1** *If $V$ is an oriented vector space with even dimension $n$, then there exists a unique $\mathbb{Z}_2$-graded irreducible Clifford module $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, called the spinor module, such that $Cl(V \otimes \mathbb{C}, q \otimes \mathbb{C}) \cong End(\mathbb{S})$. In particular, $dim_{\mathbb{C}}(\mathbb{S}) = 2^{n/2}$ and $dim_{\mathbb{C}}(\mathbb{S}^+) = dim_{\mathbb{C}}(\mathbb{S}^-) = 2^{n/2-1}$.*

An explicit example of such spinor module $\mathbb{S}$ is given by the exterior algebra of a polarization of $V \otimes \mathbb{C}$. Start with the even dimensional vector space $(V, q)$, and denote its complexification as $(V \otimes \mathbb{C}, q_{\mathbb{C}})$. A *polarization* of the complexified space $V \otimes \mathbb{C}$ is a subspace satisfying $q_{\mathbb{C}}(w, w) = 0$ for all $w \in P$ and $V \otimes \mathbb{C} = P \oplus \overline{P}$. The polarization is oriented when there is an oriented orthonormal basis $\{e_i\}$ of $V$ such that $P$ is spanned by the vectors $\{e_{2j-1} - ie_{2j} : 1 \le j \le n/2\}$. Under such an oriented polarization $P$, we may take $\mathbb{S}$ as the exterior algebra $\Lambda P$ of $P$. The explicit Clifford action of elements in $Cl(V \otimes \mathbb{C}, q_{\mathbb{C}})$ on $\mathbb{S}$ is as follows:

$$c(w) \cdot s = 2^{1/2} E_w(s), \quad \forall w \in P \subset P \oplus \overline{P} = V \otimes \mathbb{C} \cong Cl(V \otimes \mathbb{C}, q_{\mathbb{C}}),$$
$$c(\overline{w}) \cdot s = -2^{1/2} I_{\overline{w}}(s), \quad \forall \overline{w} \in \overline{P} \subset Cl(V \otimes \mathbb{C}, q_{\mathbb{C}}).$$

This action can be further shown to satisfy $Cl(V \otimes \mathbb{C}, q_{\mathbb{C}}) \cong End(\mathbb{S})$.

Note that for such oriented polarization $P$, the operator $c(\chi)$ on $\mathbb{S} = \Lambda P$ is equal to $(-1)^k$ on $\Lambda^k P$, so that

$$\mathbb{S}^+ = \Lambda P^0, \quad \mathbb{S}^- = \Lambda P^1.$$

This can be shown by rewriting $\chi_{\mathbb{C}} = 2^{-n/2}(w_1\overline{w}_1 - \overline{w}_1 w_1)\dots(w_{n/2}\overline{w}_{n/2} - \overline{w}_{n/2}w_{n/2})$, where $w_j := 2^{-1/2}(e_{2j-1} - ie_{2j})$.

Since $Spin(V) \subset Cl(V)^0 \subset Cl(V \otimes \mathbb{C}, q_{\mathbb{C}})$, the restriction of the representation $\rho : Spin(V) \to End(\mathbb{S})$ is called the *spinor representation*.

## 5.4 Clifford bundles and spinor bundles

We generalise the action of Clifford algebras on Clifford modules to actions of bundles of Clifford algebras, constructed from dual tangent bundles of Riemannian manifolds.

Consider a Riemannian manifold $M$ of metric $g$. The tangent space $T_x(M)$ at a point $x \in M$ is a Euclidean vector space, with the quadratic form defined by the pointwise evaluation of metric tensor $g_x$. We can thus construct a Clifford algebra $Cl(T_x(M), g_x)$ over each point $x$. The Clifford bundle can be defined from the tangent bundle by the procedure of associative bundle construction:

Given a principal $G$-bundle $\pi : P_G \to M$ over a space $M$ and $F$ be a vector space with the group of homomorphisms $Homeo(F)$ endowed with the compact-open topology, each continuous morphism $\rho : G \to Homeo(F)$ defines a fibre bundle over $M$ with fibre $F$ as follows. Consider the free left action of $G$ on the product $P_G \times F$ given by $\phi_g(p, f) = (pg^{-1}, \rho(g)f)$, for $g \in G$ and $(p, f) \in P_G \times F$. Define $P_G \times_\rho F$ to be the quotient space of this action. Hence the projection $P_G \times F \to P_G \xrightarrow{\pi} M$ descends to a mapping $\pi_\rho : P_G \times_\rho F \to M$ which is the fibre bundle over $M$ with fibre $F$. It is called the bundle associated to $P_G$ by $\rho$.

Let $M$ be an oriented Riemannian manifold and let $P_{SO(n)}(M)$ be the principal $SO(n)$-bundle of positively oriented orthonormal frames. Let $\rho_n : SO(n) \to SO(\mathbb{R}^n)$ be the fundamental representation of the space of $n \times n$ matrices over $\mathbb{R}$ over space of $n$-vectors over $\mathbb{R}$.

To obtain the Clifford bundle, we first note that there is a representation $cl(\rho_n) : SO(n) \to Aut(Cl(n))$ from the fact that each orthogonal transformation on $\mathbb{R}^n$ induces an orthogonal transformation of $Cl(n)$. The *Clifford bundle* of rank $n$ oriented vector bundle $E \longrightarrow M$ is the bundle

$$Cl(E) := P_{SO(n)}(E) \times_{cl(\rho_n)} Cl(n).$$

The fibrewise multiplication on $Cl(E)$ gives an algebra structure to the space of sections of $Cl(E)$.

Notions intrinsic to Clifford algebras carry over to Clifford bundles. For example there is a decomposition with respect to parity: $Cl(E) = Cl(E)^0 \oplus Cl(E)^1$. When $E = T^*(M)$ we denote the corresponding Clifford bundle as $Cl(M)$ and it is the *Clifford bundle* of $M$.

As we have seen from Proposition 5.3.1, for an even $n$ dimensional vector space $(V, q)$ there always exists an irreducible representation $\mathbb{S}$ of $Cl(V \otimes \mathbb{C})$. We consider globally the representation of the Clifford bundle $Cl(E \otimes \mathbb{C})$. One may ask whether there exists a spinor module $\mathcal{S}$ as a globally defined vector bundle over $M$. The existence of such bundle is equivalent to the existence of *spin structures* for the bundle $E$. And further the existence of a spin structure is determined by the vanishing of the second Stiefel-Whitney class of the oriented bundle $E$. [32]

Suppose $n \geq 3$, then a *spin structure* on $E$ is a principal $Spin(n)$-bundle $P_{Spin}(E)$ over $X$ together with a two sheeted covering

$$\xi : P_{Spin}(E) \longrightarrow P_{SO}(E)$$

such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin}(E)$ and all $g \in Spin(\mathbb{R}^n)$, where $\xi_0$ is the double covering given by the twisted adjoint representation as in (5.2). Fibrewise $P_{Spin}(E)_x \longrightarrow P_{SO}(E)_x$ for any $x \in X$ is nothing but the double covering map $\xi_0 : Spin(n) \longrightarrow SO(n)$.

If $E = TM$, the tangent bundle of $M$, has spin structure, then we call $M$ a *spin manifold*. We omit $n$ indicating the dimension of the structure groups in the principal bundles. For a principal bundle of spin structure $\xi$, there is an associated spinor bundle. The pointwise restriction of such spinor bundle is the spinor representation. We will consider the a specific construction of the spinor bundle over the Eguchi-Hanson space in Chapter 6.

# Chapter 6

# Spin geometry of Eguchi-Hanson spaces

In this chapter, we first describe the metric and the Levi-Civita connection of the Eguchi-Hanson space, and then introduce its spinor bundle, the spin connection and the Dirac operator. Finally, we write down the torus action through parallel propagators on the spinor bundle.

## 6.1 Metrics, connections and torus isometric actions

The Eguchi-Hanson (EH-) spaces, which are Riemannian manifold of dimension four, were originally constructed as gravitational instantons [16]. Generalized by Gibbons and Hawking, they fall into a new category of solutions of the Einstein's equation, known as the multicenter solutions [35]. In local coordinates, the metric is

$$ds^2 = \Delta^{-1} dr^2 + r^2 \left[ (\sigma_x^2 + \sigma_y^2) + \Delta\, \sigma_z^2 \right] , \tag{6.1}$$

where $\Delta := \Delta(r) := 1 - a^4/r^4$ and $\{\sigma_x, \sigma_y, \sigma_z\}$ is the standard Cartan basis for the 3-sphere,

$$\begin{aligned}
\sigma_x &= \frac{1}{2}(-\cos\psi\, d\theta - \sin\theta\, \sin\psi\, d\phi),\\
\sigma_y &= \frac{1}{2}(\sin\psi\, d\theta - \sin\theta\, \cos\psi\, d\phi),\\
\sigma_z &= \frac{1}{2}(-d\psi - \cos\theta\, d\phi),
\end{aligned}$$

with $r \geq a, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \psi < 2\pi.$

**Remark**: The convention that the period of $\psi$ is $2\pi$ rather than $4\pi$ as in the original construction is suggested in [35] to remove the singularity at $r = a$, so that the manifold becomes geodesically complete.

The EH-space is diffeomorphic to the tangent bundle of a 2-sphere $T(\mathbb{S}^2)$. Modulo a distortion of the metric, the base as a unit two sphere $\mathbb{S}^2$ is parametrised by parameters $\phi$ and $\theta$, with $\theta = 0$ as the south pole and $\theta = \pi$ as the north pole. The angle $\phi$ parametrises the circle defined by a constant $\theta$. Over each point, say $(\theta, \phi)$ on the 2-sphere, the tangent plane is parametrised by $(r, \psi)$. The number $r$ parametrises the radial direction with $r = a$ at the origin of the plane. Circles of constant radius $r$ are parametrised by $\psi$. The identification of $\psi = \psi + 2\pi$ is the identification the antipodal points on the circle of constant radius. Together with the metric, this implies that the space at large enough $r$ is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$, so that it is an ALE space.

The parameter $a$ in the metric (6.1) is a non-negative real number parametrising a family of EH-spaces. When $a = 0$, the metric degenerates to the conifold $\mathbb{R}^4/\mathbb{Z}_2$ and the rest of the family is a resolution of the conifold. This appears as the simplest case in Kronheimer's classification of ALE spaces [17]. We will only concentrate on the smooth case so that $a$ is assumed to be positive.

Choose the local coordinates $\{x_i\}$ with $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$, $x_4 = \psi$. We will write the coordinates $(r, \theta, \phi, \psi)$ and $(x_1, x_2, x_3, x_4)$ interchangeably throughout the content, because the former give a clear geometric picture while the latter are convenient in tensorial expressions. The corresponding basis on the tangent space $T_x(EH)$ of any point $x \in EH$ are $\left\{\partial_i := \frac{\partial}{\partial x_i}\right\}$, and the dual basis on the cotangent space $T_x^*(EH)$ are $\{dx^j\}$. The corresponding metric tensor $g_{ij}(x)\, dx^i \otimes dx^j$ can be written as entries of the matrix $G = (g_{ij})$ as

$$G(x) = \frac{1}{4}\begin{pmatrix} 4\,\Delta^{-1} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & \rho & r^2\,\Delta\,\cos\theta \\ 0 & 0 & r^2\,\Delta\,\cos\theta & r^2\,\Delta \end{pmatrix} \tag{6.2}$$

where $\rho := \rho(r,\theta) := \left(r^4 - a^4\cos^2\theta\right)/r^2$. We always assume Einstein's summation convention.

In the same coordinate chart, the Christoffel symbols of the Levi-Civita connection of (6.1) , defined by $\nabla_i\partial_j = \Gamma^k_{ij}\partial_k$, are explicitly,

$$\begin{aligned}
&\Gamma^1_{11} = -\frac{\Delta'}{\Delta}, \quad \Gamma^1_{22} = -\frac{r\,\Delta}{4}, \quad \Gamma^1_{33} = -\frac{\Delta\,\rho^+}{4\,r}, \\
&\Gamma^1_{34} = -\frac{r\,\Delta^+\Delta\,\cos\theta}{4}, \quad \Gamma^1_{44} = -\frac{r\,\Delta^+\Delta}{4}, \\
&\Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^2_{33} = -\frac{a^4\sin 2\theta}{2\,r^4}, \quad \Gamma^2_{34} = \frac{\Delta\,\sin\theta}{2}, \\
&\Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \frac{\cot\theta\,\Delta^+}{2}, \quad \Gamma^3_{24} = -\frac{\Delta}{2\,\sin\theta}, \\
&\Gamma^4_{13} = \frac{2\,a^4\,\cos\theta}{r(r^4 - a^4)}, \quad \Gamma^4_{14} = \frac{\Delta^+}{r\,\Delta}, \quad \Gamma^4_{23} = -\frac{\rho^+}{2\,r^2\,\sin\theta}, \quad \Gamma^4_{24} = \frac{\cot\theta\,\Delta}{2}.
\end{aligned} \tag{6.3}$$

where

$$\Delta^+ := \Delta^+(r) := 1 + \frac{a^4}{r^4}, \quad \Delta' := \frac{\partial\Delta}{\partial r}, \quad \rho^+ := \rho^+(r,\theta) := \frac{r^4 + a^4\,\cos^2\theta}{r^2}.$$

The identity $\Gamma^i_{jk} = \Gamma^i_{kj}$, implied by the torsion free property of the connection, generates another set of symbols and all the rest of the Christoffel symbols vanish.

The isometry group of the metric (6.1) is $U(1) \times SU(2)$. The Killing vector $\partial_\psi$ generates the group $U(1)$. Another Killing vector is $\partial_\phi$. Its action on the restriction of the space at $r = a$ is analogous to one of the three typical generators of the Lie algebra of the Lie group $SU(2)$ on a standard two-sphere. These are the two Killing vectors which define a torus action $\sigma$ on the Eguchi-Hanson space.

Supoose that $\{\hat{e}_3, \hat{e}_4\}$ is a basis of the Lie algebra $\mathfrak{U}(1) \times \mathfrak{U}(1)$ of the Lie group $U(1) \times U(1)$. Elements in $\mathfrak{U}(1) \times \mathfrak{U}(1)$ are written as

$$v = v_e\hat{e}_3 + v_4\hat{e}_4 = (v_3, v_4), \quad 0 \leq v_3, v_4 < 2\pi.$$

Elements in $U(1) \times U(1)$ are images of the exponential map of the Lie algebra $\exp : \mathfrak{U}(1) \times \mathfrak{U}(1) \to U(1) \times U(1)$ as

$$\exp(v_3\hat{e}_3 + v_4\hat{e}_4) = (e^{iv_3}, e^{iv_4}) \in U(1) \times U(1), \quad v_3\hat{e}_3 + v_4\hat{e}_4 \in \mathfrak{U}(1) \times \mathfrak{U}(1).$$

We define the Lie algebra isometric action
$sigma : \mathfrak{U}(1) \times \mathfrak{U}(1) \to Aut(EH)$ by matching the basis $\{\hat{e}_3, \hat{e}_4\}$ with the Killing vectors $\{\partial_\phi, \partial_\psi\}$ as follows. For any element $v = v_3\hat{e}_3 + v_4\hat{e}_4 \in \mathfrak{U}(1) \times \mathfrak{U}(1)$

$$\sigma_v := \sigma(v) : (r, \theta, \phi, \psi) \mapsto (r, \theta, \phi + v_3, \psi + v_4), \quad \forall (r, \theta, \phi, \psi) \in EH. \tag{6.4}$$

The corresponding Lie group isometric action $\tilde{\sigma} : U(1) \times U(1) \to Aut(EH)$ is defined by

$$\tilde{\sigma}(e^{iv_3}, e^{iv_4}) := \sigma_v, \quad 0 \leq v_3, v_4 < 2\pi$$

where $v = (v_3, v_4) \in \mathfrak{U}(1) \times \mathfrak{U}(1)$ is the pre-image of $(e^{iv_3}, e^{iv_4})$ under the exponential map. The isometric torus action will determine the isospectral deformation later.

## 6.2 The stereographic projection and orthonormal basis

We choose an orthonormal basis to trivialize the cotangent bundle of the EH-space and obtain the corresponding transition functions. Since the EH-space is locally the same as $T(\mathbb{S}^2)$, we may obtain another set of coordinates by taking the stereographic projection of the $\mathbb{S}^2$ part, while keeping the coordinates on the tangent space unchanged.

The EH-space (6.1) can be covered by two open neighbourhoods $U_N$ and $U_S$, where $U_N$ covers the whole space except at $\theta = \pi$ and $U_S$ covers the whole space except at $\theta = 0$. We may define the map $f_N : U_N \longrightarrow \mathbb{C} \times \mathbb{R}^2$ by taking a stereographic projection of the base two sphere to $\mathbb{C}$. I.e., $f_N(\phi, \theta, r, \psi) = (z, r, \psi)$. For the coordinate chart $U_S$, we similarly define the projection map $f_S : N_S \longrightarrow \mathbb{C} \times \mathbb{R}^2$, by $f_S(\phi, \theta, r, \psi) = (w, r, \psi)$, where

$$z := \cot\frac{\theta}{2}\, e^{-i\phi}, \quad w := \tan\frac{\theta}{2}\, e^{i\phi}.$$

For any point $x \in U_N \cap U_S$, the transition function from the coordinate charts $U_S$ to $U_N$ is

$$(w, r, \psi) = (\frac{1}{z}, r, \psi),$$

and the transition function from $U_N$ to $U_S$ is $(z, r, \psi) = (\frac{1}{w}, r, \psi)$.

The restriction of metric (6.1) to the $U_N$ chart with coordinates $(z, r, \psi)$ is,

$$ds^2 = \frac{r^2}{(1+z\bar{z})^2} dz d\bar{z} + \frac{r^2 \Delta}{4} \left[ d\psi + \frac{z\bar{z}-1}{z\bar{z}+1} \frac{i}{2} (\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}) \right]^2 .$$

To obtain a local orthonormal basis of $T^*(EH)_{U_N}$ we may simply define

$$l := \frac{r}{\sqrt{2}\,(1+z\bar{z})} dz, \quad m := \frac{\Delta^{-1/2}}{\sqrt{2}} dr + \frac{r\,\Delta^{1/2}}{4\sqrt{2}} \left[ \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \frac{1-z\bar{z}}{1+z\bar{z}} + 2i\, d\psi \right],$$

with their complex conjugates $\bar{l}$, $\overline{m}$ so that the metric tensor over $U_N$ is $ds^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \overline{m} + \overline{m} \otimes m$.

A real orthonormal frame $\{\vartheta^\alpha\}$ of $T^*(EH)_{U_N}$ is thus defined by

$$\vartheta^1 := \frac{1}{\sqrt{2}}(l + \bar{l}), \quad \vartheta^2 := -\frac{i}{\sqrt{2}}(l - \bar{l}), \quad \vartheta^3 := -\frac{i}{\sqrt{2}}(m - \overline{m}), \quad \vartheta^4 := \frac{1}{\sqrt{2}}(m + \overline{m}).$$

such that the metric on $U_N$ is diagonalized as $ds^2 = \delta_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$. The coordinate transformations $\vartheta^\alpha = h_i^\alpha dx^i$ are determined by the matrix $H = (h_i^\alpha)$,

$$H = \frac{1}{2} \begin{pmatrix} 0 & -r\,\cos\phi & -r\,\sin\theta\,\sin\phi & 0 \\ 0 & r\,\sin\phi & -r\,\sin\theta\,\cos\phi & 0 \\ 0 & 0 & r\,\Delta^{1/2}\cos\theta & r\,\Delta^{1/2} \\ \frac{2}{\Delta^{1/2}} & 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

whose inverse $H^{-1} = (\tilde{h}_\beta^j)$ from $dx^j = \tilde{h}_\beta^j \vartheta^\beta$ is

$$H^{-1} = 2 \begin{pmatrix} 0 & 0 & 0 & \frac{\Delta^{1/2}}{2} \\ -\frac{\cos\phi}{r} & \frac{\sin\phi}{r} & 0 & 0 \\ -\frac{\sin\phi}{r\,\sin\theta} & -\frac{\cos\phi}{r\,\sin\theta} & 0 & 0 \\ \frac{\cos\theta\,\sin\phi}{r\,\sin\theta} & \frac{\cos\theta\,\cos\phi}{r\,\sin\theta} & \frac{1}{r\,\Delta^{1/2}} & 0. \end{pmatrix} \tag{6.6}$$

The above construction on the $U_N$ chart can be carried out the same way on the $U_S$ coordinates. We denote orthonormal frames over $U_S$ by adding $'$s to $l, m, \vartheta^\alpha$, $x_j$ and etc.

Local frames $\{\vartheta^\alpha\}$ on $U_N$ define a local trivialization of the cotangent bundle, $F_N : T^*(EH)_{U_N} \to U_N \times \mathbb{R}^4$ by $F_N(x; a_1\vartheta^1 + \cdots + a_4\vartheta^4) := (x; a_1, \ldots, a_4)$, where $a_\alpha$'s are real-valued functions over $U_N$. In a similar way, the choice of local frames $\{\vartheta'^\alpha\}$ on $U_S$ defines a local trivialization of the cotangent bundle, $F_S : T^*(EH)_{U_S} \longrightarrow U_N \times \mathbb{R}^4$.

The transition functions $f_\alpha^\beta$'s such that $\vartheta'^\beta = f_\alpha^\beta \vartheta^\alpha$ are elements of the matrix $F_{SN} := F_N \circ F_S^{-1}$ as

$$F_{SN} = \begin{pmatrix} -\frac{\bar{z}^2+z^2}{2\,z\bar{z}} & -i\,\frac{\bar{z}^2-z^2}{2\,z\bar{z}} & 0 & 0 \\ i\,\frac{\bar{z}^2-z^2}{2\,z\bar{z}} & -\frac{\bar{z}^2+z^2}{2\,z\bar{z}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos 2\phi & \sin 2\phi & 0 & 0 \\ -\sin 2\phi & -\cos 2\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.7}$$

The inverse transition function is given by the inverse of the matrix $F_{SN}$, $F_{NS} := F_S \circ F_N^{-1} = F_{SN}^{-1}$. The cotangent bundle is thus

$$T^*(EH) = (U_N \times \mathbb{R}^4) \cup (U_S \times \mathbb{R}^4)/\sim, \tag{6.8}$$

where $(x; a_1, \ldots, a_4) \in U_N \times \mathbb{R}^4$ and $(x'; a'_1, \ldots, a'_4) \in U_S \times \mathbb{R}^4$ are defined to be equivalent if and only if $x = x'$ and $F_{NS}(a_1, \ldots, a_4)^t = (a'_1, \ldots, a'_4)^t$.

## 6.3 Spin structures and spinor bundles

Following a standard procedure from [32], we obtain the spinor bundle of the EH-space. In coordinate charts $\{U_N, U_S\}$, the frame bundle $P_{SO(4)}$ of the EH-space is the $SO(4)$-principal bundle with transition functions $F_{NS}$ in (6.7) and its inverse $F_{SN}$.

Recall that the covering map of groups,

$$\xi_0 : Spin(4) \longrightarrow SO(4), \tag{6.9}$$

is defined by the twisted adjoint representation $\widetilde{Ad}$ of $Spin(4)$ as

$$\xi_0(w)x = \widetilde{Ad}_w(x) = \tilde{\alpha}(w) \cdot x \cdot w^{-1}, \forall x \in \mathbb{R}^4 (\subset Cl(4)),$$

where $w = v_1 \cdots v_m \in Spin(4)$, $m$ is even and $v_i \in \mathbb{R}^4$ for $i = 1, \ldots, m$. Geometrically (as we discussed in Section 5.1), $\xi_0(w) = \rho(v_1) \circ \cdots \circ \rho(v_m)$, where $\rho(v_i)$ is the reflection of the space $\mathbb{R}^4$ with respect to the hyperplane with normal vector $v_i$.

Locally, the upper left block of the transition matrix (6.7) is a rotation in the plane spanned by $\{\vartheta^1, \vartheta^2\}$ through an angle $2\phi + \pi$. Such a rotation can be decomposed into two reflections say $\rho(v_2) \circ \rho(v_1)$, with

$$v_1 := \vartheta^1, \quad v_2 := -\sin\phi\,\vartheta^1 + \cos\phi\,\vartheta^2.$$

**Remark**: As a result of the double covering map, another choice is $\rho(-v_2) \circ \rho(v_1)$, which gives the same rotation as an element of $SO(2)$.

$v_2 \cdot v_1 \in Spin(4)$ is a lifting of $\rho(v_2) \circ \rho(v_1) \in SO(4)$ under the covering map (6.9). Thus, in the local coordinate chart $U_N$, $\widetilde{F_{SN}} := v_2 \cdot v_1$ in $Spin(4)$ defines a lifting of the action $F_{NS} \in SO(4)$ as in (6.7) under the double covering (6.9).

To obtain a global lifting of the frame bundle, we consistently define the transition matrix $\widetilde{F_{NS}}$ as a lifting in the group $Spin(4)$ over $x' \in U_S$ by $\widetilde{F_{NS}} = -v_2' \cdot v_1'$, where

$$v_1' := \vartheta'^1, \quad v_2' := \sin\phi\,\vartheta'^1 + \cos\phi\,\vartheta'^2.$$

The following confirms the consistency of the liftings on two coordinate charts.

**Lemma 6.3.1** *Transition functions $\{\widetilde{F_{NS}}, \widetilde{F_{SN}}\}$ satisfy the cocycle condition, $\widetilde{F_{NS}} \circ \widetilde{F_{SN}} = \widetilde{F_{SN}} \circ \widetilde{F_{NS}} = \mathbf{1}$.*

**Proof**: Applying the transformation from $\vartheta^\alpha$'s to $\vartheta'^\beta$'s by (6.7), we have $\vartheta'^1 \cdot \vartheta'^2 = \vartheta^1 \cdot \vartheta^2$. Thus,

$$\begin{aligned}
\widetilde{F_{NS}} \circ \widetilde{F_{SN}} &= -v_2' \cdot v_1' \cdot v_2 \cdot v_1 \\
&= -(\sin\phi\,\vartheta'^1 + \cos\phi\,\vartheta'^2) \cdot \vartheta'^1 \cdot (-\sin\phi\,\vartheta^1 + \cos\phi\,\vartheta^2) \cdot \vartheta^1 \\
&= \sin^2\phi - \sin\phi\,\cos\phi(\vartheta^1 \cdot \vartheta^2 + \vartheta^2 \cdot \vartheta^1) - \cos^2\phi\,\vartheta^2 \cdot \vartheta^1 \cdot \vartheta^2 \cdot \vartheta^1 = \mathbf{1},
\end{aligned}$$

by using identities $\vartheta^\alpha \cdot \vartheta^\alpha = -\mathbf{1}$ and $\vartheta^\alpha \cdot \vartheta^\beta + \vartheta^\beta \cdot \vartheta^\alpha = 0$ for $\alpha \neq \beta$, of elements of the orthonormal bases $\vartheta^\alpha$'s and those of $\vartheta'^\beta$'s. Similarly, $\widetilde{F_{SN}} \circ \widetilde{F_{NS}} = \mathbf{1}$. □

Therefore, the principal $Spin(4)$-bundle can be defined by

$$P_{Spin(4)} := (U_N \times Spin(4) \cup U_S \times Spin(4))/\sim . \tag{6.10}$$

where $(x, \tilde{g}) \in U_N \times Spin(4)$ and $(x', \tilde{g}') \in U_S \times Spin(4)$ are defined to be equivalent if and only if $x = x'$ and $\tilde{g}' = \tilde{F}_{NS}\, \tilde{g}$.

The double covering of bundles (6.10) over the EH-space defines a *spin structure* on it. We will always assume this choice of spin structure.

The spinor bundle can be defined as an associative bundle of typical fibre $\mathbb{C}^4$ of the principal $Spin(4)$-bundle (6.10), by specifying a representation of $Spin(4)$ on $GL_{\mathbb{C}}(4)$. We know that locally, for any $x \in EH$, there exists a unique irreducible representation space $\Lambda$ of complex dimension 4 of the Clifford algebra $Cl(T_x^*(EH))$ through the Clifford action $c : Cl(T_x^*(EH)) \longrightarrow End(\Lambda)$. We define the representation of $Spin(4)$ in $End(\Lambda)(\cong GL_{\mathbb{C}}(4))$ simply by the restriction of $c$ from the Clifford algebra, and obtain the *spinor bundle* $S$ of typical fibre $\Lambda$, with transition functions $\{c(\widetilde{F_{NS}}), c(\widetilde{F_{SN}})\}$ in the coordinate charts $\{U_N, U_S\}$.

With respect to the orthonormal basis, say $\{\vartheta^\alpha\}$ of $T^*(EH)_{U_N}$, there exists a unitary frame $\{f_\alpha\}$ of the representation space $\Lambda \cong \mathbb{C}^4$, such that the Clifford representations $\gamma^\alpha := c(\vartheta^\alpha(x))$ for $\alpha = 1, \ldots, 4$ can be represented as constant matrices,

$$\gamma^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{6.11}$$

The fact is that there exist a frame $\{f'_\beta\}$ on the coordinate chart $U_S$ so that the representation of $c(\vartheta'^\beta)$'s are also the constant matrices $\gamma^\beta$'s as above.

Under the chosen frames $\{f_\alpha\}$ and $\{f'_\beta\}$, we may represent the transition functions of the spinor bundle as follows. Define maps $P, Q : U_N \cap U_S \longrightarrow GL_{\mathbb{C}}(4)$ by

$$P := c(\widetilde{F_{SN}}) = -\sin\phi\gamma^1\gamma^1 + \cos\phi\gamma^2\gamma^1 = diag(-\frac{i\,\bar{z}}{|z|}, \frac{i\,z}{|z|}, \frac{i\,z}{|z|}, -\frac{i\,\bar{z}}{|z|}), \tag{6.12}$$

$$Q := c(\widetilde{F_{NS}}) = -\sin\phi\gamma^1\gamma^1 - \cos\phi\gamma^2\gamma^1 = diag(\frac{i\,\overline{w}}{|w|}, -\frac{i\,w}{|w|}, -\frac{i\,w}{|w|}, \frac{i\,\overline{w}}{|w|}). \qquad (6.13)$$

$diag(a, b, c, d)$ stands for the diagonal matrix with diagonal elements $a, b, c, d$.

The spinor bundle $\mathcal{S}$ is thus,

$$\mathcal{S} := (U_N \times \mathbb{C}^4 \cup U_S \times \mathbb{C}^4)/\sim, \qquad (6.14)$$

where $(x; s_1, \cdots, s_4) \in U_N \times \mathbb{C}^4$ and $(x'; s'_1, \cdots, s'_4) \in U_S \times \mathbb{C}^4$ are defined to be equivalent if and only if $x = x'$ and $(s'_1, \cdots, s'_4)^t = Q(s_1, \cdots, s_4)^t$. One can easily see that the cocycle condition of the transition functions $P \circ Q = Q \circ P = \mathbf{1}$ holds.

The *chirality operator* is defined by

$$\chi := c(\vartheta^1)\,c(\vartheta^2)\,c(\vartheta^3)\,c(\vartheta^4) = \gamma^1\,\gamma^2\,\gamma^3\,\gamma^4 = diag(-1, -1, 1, 1), \qquad (6.15)$$

such that $\chi^2 = \mathbf{1}$. The representation space $\Lambda = \Lambda^+ \oplus \Lambda^-$ is decomposed as $\pm 1$-eigenspaces of the operator $\chi$, with $dim_{\mathbb{C}}\,\Lambda^+ = dim_{\mathbb{C}}\,\Lambda^- = 2$. This fibrewise splitting extends to the global decomposition of spinor bundle as subbundles over the EH-space, $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, with each of the complex subbundles $\mathcal{S}^+$ and $\mathcal{S}^-$ of rank 2. Therefore, any element $s \in \mathcal{S}$ can be decomposed as $s = (s^+, s^-)^t$. The *charge conjugate operator* on the spinor bundle $J : \mathcal{S} \to \mathcal{S}$ is defined by

$$J\begin{pmatrix} s^+ \\ s^- \end{pmatrix} := \begin{pmatrix} -\overline{s}^- \\ \overline{s}^+ \end{pmatrix}. \qquad (6.16)$$

## 6.4 Spin connections and Dirac operators of spinor bundles

Following the general procedure in [6], we can induce the spin connection $\nabla^{\mathcal{S}}$ of the spinor bundle $\mathcal{S}$ from the Levi-Civita connection of the EH-space.

We will only work on the $U_N$ coordinate chart and the construction on $U_S$ is similar. In the orthonormal frame $\{\vartheta^\alpha\}$, the corresponding Levi-Civita connection on the dual tangent bundle, $T^*(EH)_{U_N}$, can be expressed as $\nabla^{T^*EH}\vartheta^\beta = -\widetilde{\Gamma}^\beta_{i\alpha}\,dx^i \otimes \vartheta^\alpha$. The metric compatibility of the Levi-Civita connection implies that $\widetilde{\Gamma}^\alpha_{i\beta} = -\widetilde{\Gamma}^\beta_{i\alpha}$.

We may represent $\widetilde{\Gamma}^\beta_{i\alpha}$'s in terms of the Christoffel symbols $\Gamma^k_{ij}$'s of $\nabla$ in the $dx^i$'s (6.3) by

$$\widetilde{\Gamma}^\beta_{i\alpha} = \tilde{h}^j_\alpha\left(h^\beta_k\,\Gamma^k_{ij} - \partial_i h^\beta_j\right), \qquad (6.17)$$

where $h_i^\alpha$'s and $\tilde{h}_\beta^j$'s are the matrix entries of $H$ in (6.5) and $H^{-1}$ in (6.6), respectively. Modulo the anti-symmetric condition between $\alpha$ and $\beta$ indices, all the nonvanishing Christoffel symbols are

$$\begin{aligned}
&\widetilde{\Gamma}^1_{23} = \frac{1}{2}\Delta^{1/2}\sin\phi, \quad \widetilde{\Gamma}^1_{24} = -\frac{1}{2}\Delta^{1/2}\cos\phi,\\
&\widetilde{\Gamma}^3_{22} = \frac{1}{2}\Delta^{1/2}\cos\phi, \quad \widetilde{\Gamma}^4_{22} = -\frac{1}{2}\Delta^{1/2}\sin\phi,\\
&\widetilde{\Gamma}^1_{33} = -\frac{1}{2}\Delta^{1/2}\sin\theta\,\cos\phi, \quad \widetilde{\Gamma}^1_{34} = -\frac{1}{2}\Delta^{1/2}\sin\theta\,\sin\phi,\\
&\widetilde{\Gamma}^1_{32} = -1-\frac{1}{2}\Delta^{+}\cos\theta, \quad \widetilde{\Gamma}^3_{32} = -\frac{1}{2}\Delta^{1/2}\sin\theta\,\sin\phi\\
&\widetilde{\Gamma}^4_{32} = \frac{1}{2}\Delta^{1/2}\sin\theta\,\cos\phi, \quad \widetilde{\Gamma}^4_{33} = -\frac{1}{2}\Delta^{+}\cos\theta\\
&\widetilde{\Gamma}^1_{42} = \frac{1}{2}\Delta, \quad \widetilde{\Gamma}^4_{43} = -\frac{1}{2}\Delta^{+}.
\end{aligned} \tag{6.18}$$

We define $\gamma_\alpha := \gamma^\alpha$, then the *spin connection* $\nabla^{\mathcal{S}} : \mathcal{S} \to \mathcal{S} \otimes \Omega^1(EH)$ is

$$\nabla^{\mathcal{S}} := d - \frac{1}{4}\widetilde{\Gamma}^\beta_{i\alpha}\, dx^i \otimes \gamma^\alpha\gamma_\beta. \tag{6.19}$$

The covariant derivative $\nabla^{\mathcal{S}}_i := \nabla^{\mathcal{S}}(\partial_i)$, for $i = 1, \ldots, 4$, equals $\nabla^{\mathcal{S}}_i = \partial_i - \omega_i$, where $\omega_i = \frac{1}{4}\widetilde{\Gamma}^\beta_{i\alpha}\gamma^\alpha\gamma_\beta$. The *Dirac operator* $\mathcal{D} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ can be defined by

$$\mathcal{D}(\psi) := -i\,\gamma^j\,\nabla^{\mathcal{S}}_j\psi, \quad \forall\psi \in \Gamma(\mathcal{S}), \tag{6.20}$$

where $\gamma^j := c(dx^j) = \tilde{h}^j_\beta\gamma^\beta$. We note that the compatibility of the spin connection with respect to the spin structure implies commutativity between the Dirac operator and the charge conjugate operator, i.e. $[\mathcal{D}, J] = 0$.

## 6.5 Torus actions on the spinor bundle

A torus action on the spinor bundle $\mathcal{S}$ can be induced from the torus isometric action on a general Riemannian manifold [23], [14]. In this section, we will represent such torus action (6.4) through parallel transporting spinors along geodesics.

Recall that the isometric action $\sigma$ is generated by the two Killing vectors $\partial_3 = \partial_\phi$ and $\partial_4 = \partial_\psi$. Let $c_k : \mathbb{R} \to EH$ be the geodesics obtained as integral curves of the Killing vector field $\partial_k$ for $k = 3, 4$.

The equation of parallel transport with respect to the spin connection along any curve $c(t)$ is $\nabla^{\mathcal{S}}_{c'(t)}\psi = 0$, where $c'(t) := dc(t)/dt$, for $\psi \in \Gamma(\mathcal{S})$. Substituting (6.19), we obtain

$$\frac{d\psi}{dt} - A(c(t))\,\psi = 0, \quad A(c(t)) := \frac{1}{4}\tilde{\Gamma}^{\beta}_{i\alpha}\, dx^i(c'(t)) \otimes \gamma^{\alpha}\gamma_{\beta}. \tag{6.21}$$

When the curve is $c_3(t)$, the corresponding matrix $A(c_3(t))$ is

$$A(c_3(t)) = \frac{1}{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i\,(1+\Delta^{+}\cos\theta) & -\Delta^{1/2}\sin\theta\, e^{i\phi} \\ 0 & 0 & \Delta^{1/2}\sin\theta\, e^{-i\phi} & i\,(1+\Delta^{+}\cos\theta), \end{pmatrix} \tag{6.22}$$

where $r, \theta$ and $\phi$ are understood as components of coordinates on the curve $c_3(t)$. When the curve is $c_4(t)$, the corresponding matrix $A(c_4(t))$ is

$$A(c_4(t)) = \frac{i}{2}\, diag(-\frac{a^4}{r^4}, \frac{a^4}{r^4}, -1, 1), \tag{6.23}$$

where $r$ is understood as one of the components of coordinates on the curve $c_4(t)$.

The corresponding *parallel propagator* is a map $P_{c(t)}(t_0, t_1) : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ defined by parallel transporting any section $\psi$ along the curve $c(t)$ with $t \in [t_0, t_1]$. In a chosen frame of the local trivialization of the vector bundle, it is a matrix-valued function evaluated at $c(t_1)$. If $\psi(c(t_0))$ is of coordinate $(\psi_1, \dots, \psi_4)^t$, then the vector at $c(t_1)$ of coordinate $(\psi'_1, \dots, \psi'_4)^t := P_{c(t)}(t_0, t_1)(\psi_1, \dots, \psi_4)^t$ is the image of $\psi(c(t_0))$ through the parallel transport along the curve $c(t), t_0 \le t \le t_1$. If the curve $c(t)$ passes two distinct local trivializations of the bundle, then we can track the transportation by using transition functions of the bundle.

The propagator can be represented by an iterated integration of the equation (6.21) [36]. For geodesics $c_k(t)$, $k = 3, 4$, the corresponding matrix is formally solved as

$$P_{c_k(t)}(t_0, t_1) = \mathcal{P}\exp\left(\int_{t_i}^{t_f} A_k(t)dt\right), \tag{6.24}$$

where $\mathcal{P}$ is the path-ordering operator.

We can write down the torus action of the spinor bundle specifically. We define the Lie algebra action $V : \mathfrak{u}(1) \times \mathfrak{u}(1) \to End(\mathcal{S})$ as follows. For any fixed $v =$

$(v_3, v_4)$ in the Lie algebra and any point $x = (r, \theta, \phi, \psi) \in EH$, we write

$$x_1 = \sigma_{(0,-v_3)}(x) = (r, \theta, \phi - v_3, \psi), \quad x_0 = \sigma_{(-v_3,-v_4)}(x) = (r, \theta, \phi - v_3, \psi - v_4).$$

Let $C_4 : [0, v_4] \to EH$ be the integral curve of $\partial_\psi$ starting at $x_0$ and ending at $x_1$. Let $C_3 : [0, v_3] \to EH$ be the integral curve of $\partial_\phi$ starting at $x_1$ and ending at $x$.

Let $P_4(x_1)$ be the matrix of parallel transport of vectors from $x_0$ to $x_1$ along $C_4$ and $P_3(x)$ be the matrix of parallel transport of vectors from $x_1$ to $x$ along $C_3$. Both $P_4(x_1)$ and $P_3(x)$ can be formally written down in the form of (6.24). Their composition gives a matrix $P_{34}(x)$ evaluated at $x$. We define the Lie algebra action $V_v := V(v)$ of $v = (v_3, v_4) \in \mathfrak{U}(1) \times \mathfrak{U}(1)$ by

$$V_v(\psi)(x) := P_{34}(x)\psi(\sigma_{-v}(x)), \quad \forall \psi \in \Gamma(\mathcal{S}), x \in EH. \tag{6.25}$$

Let $\mathcal{H}$ be the Hilbert space completion with respect to the $L^2$-inner product on the space of $L^2$-integrable sections of the spinor bundle $\mathcal{S}$. The action $V$ extends to $V : \mathfrak{U}(1) \times \mathfrak{U}(1) \to \mathcal{L}(\mathcal{H})$. Since the spin connection is compatible with the metric of the EH-space, the pointwise inner product of the images of any two sections under parallel transport along the geodesics $c_k(t)$ remains unchanged. This further implies that their $L^2$-integrations remain the same. That is

$$||V_v(\psi)||_{L^2} = ||\psi||_{L^2}, \quad \forall \psi \in \mathcal{H}.$$

Note that the adjoint operator $V_v^* = V_{-v}$ so that $||V_v^*(\psi)||_{L^2} = ||\psi||_{L^2}$. This implies that $V_v$ is a unitary operator on $\mathcal{H}$ for any Lie algebra element $v$.

**Remark**: In general, a double cover of the torus is required to define an action on the spinor bundle [14]. The reason is as follows. When a loop on the base space is crossing two trivializations of the spinor bundle, the parallel transport of spinors along the loop may flip between the two sheets (depending on the transition functions). As a result, after one loop the vector say $\psi(x)$ could be $\pm\psi(x)$. However, if instead of winding once on the loop, we wind twice, then the image will always be $\psi(x)$. In our case we do not need to consider a double cover of the torus action, since the trajectory of the torus action of any point in the $EH$-space remains in the same local trivialization of the spinor bundle.

# Chapter 7

# Deformation quantization and smooth modules

In this chapter, we first consider various algebras of differentiable functions over the Eguchi-Hanson spaces and their deformations as differential algebras. We secondly obtain representations of the deformed differentiable algebras as operators on the Hilbert space of spinors and in particular obtain $C^*$-norms by operator norm. By completion of the deformed Fréchet algebra under $C^*$-norms we obtain the deformation quantization of $C^*$-algebras. We finally find the projective module description of the spinor bundle and see that it is a smooth module [10].

## 7.1 Algebras of differentiable functions

The notions of smooth algebras [10] and examples given in Chapter 2 can apply to the case when $X$ is the Eguchi-Hanson space. For the $EH$-space, we may use the coordinate charts $\mathcal{U} = \{U_N, U_S\}$ defined in Section 6.2, with a partition of unity $\{h_N, h_S\}$ subordinated to them. The family of seminorms (2.2) can be written as

$$q_m^{\mathcal{U}}(f) = \sup_{|\alpha| \leq m} \sup_{x \in U_N} |h_N(x)\, \partial^\alpha f(x)| + \sup_{|\alpha'| \leq m} \sup_{x' \in U_S} |h_S(x')\, \partial^{\alpha'} f(x')|. \qquad (7.1)$$

Algebras $C_b^\infty(EH)$ and $C_0^\infty(EH)$ are both Fréchet in the topology of uniform convergence of all derivatives and furthermore they are both smooth algebras. The algebra $C_c^\infty(X)$ is not complete in the topology of uniform convergence of all derivatives. However, it is complete in the *topology of inductive limit* as the inductive limit of the topology obtained by restriction on a family of algebras $C_c^\infty(K_n)$, where $\{K_n\}_{n\in\mathbb{N}}$ is an increasing family of compact subsets in $EH$. The algebra $C_c^\infty(EH)$ is dense in the Fréchet algebra $C_0^\infty(EH)$.

Apart from algebras of functions which can be represented as operators, there are algebras of functions which may define projective modules as representation spaces. Decay conditions at infinity and integrability conditions of functions become important when considering noncompact spaces. We consider the following algebras of integrable functions.

The $(k,p)$*-th Sobolev norm* of a function $f$, say in $C_b^\infty(EH)$, is given as

$$\|f\|_{H_k^p} := \sum_{m=0}^{k} \left(\int_{EH} |\nabla^m f|^p dVol\right)^{1/p}, \tag{7.2}$$

where $k$ is a non-negative integer and $p$ is a positive integer. (We will not consider the case where $p$ is a real number). We define subspaces in $C_b^\infty(EH)$ which contain functions with finite Sobolev norm,

$$C_k^p(EH) := \{f \in C_b^\infty(EH) : \|f\|_{H_k^p} < \infty\}.$$

Let $H_k^p(EH)$ be the Banach space obtained by the completion of the algebra $C_k^p(EH)$ with respect to the Sobolev norm. In particular, $H_0^p(EH) \supset \cdots \supset H_k^p(EH) \supset H_{k+1}^p(EH) \supset \cdots$.

**Remark**: Notice that the algebra $C_c^\infty(EH)$ is contained in $H_k^p(EH)$ for any $k \in \mathbb{N}$. The completion of $C_c^\infty(EH)$ with respect to $\|\cdot\|_{H_k^p}$ gives us the Banach space, $H_{k,0}^p(EH)$ such that $H_{k,0}^p(EH) \subset H_k^p(EH)$. The equality does not hold in general. However, in the circumstances of a complete Riemannian manifold with Ricci curvature bounded up to degree $k-2$, and positive injective radius (which is satisfied by the $EH$-space), $H_{k,0}^p(EH) = H_k^p(EH)$ when $k \geq 2$ [37].

**Lemma 7.1.1** *For a fixed non-negative integer $p$, the intersection defined as*

$$C_p^\infty(EH) := \cap_k H_k^p(EH)$$

*is a Fréchet algebra in the topology defined by the family of norms* $\{\|\cdot\|_{H_k^p}\}_{k\in\mathbb{N}}$.

**Proof**: The topology is easily seen to be locally convex and metrisable. To show that it is complete, let $\{f_\beta\}$ be any Cauchy sequence in $C_p^\infty(EH)$, then there exists a limit $f_k^p$ of $\{f_\beta\}$ under the norm $\|\cdot\|_{H_k^p}$ in $H_k^p(EH)$ for each $k \in \mathbb{N}$. For any two indices $k_1, k_2$ such that $k_1 \leq k_2$, the norm $\|\cdot\|_{H_{k_2}^p}$ is stronger than the norm $\|\cdot\|_{H_{k_1}^p}$. The Cauchy sequence $\{f_\beta\}$ with the limit $f_{k_2}^p$ in the norm $\|\cdot\|_{H_{k_2}^p}$ is also a Cauchy sequence with the limit $f_{k_1}^p$ in the norm $\|\cdot\|_{H_{k_1}^p}$. Uniqueness of the limit implies that $f_{k_2}^p = f_{k_1}^p$. Since $k_1, k_2$ are arbitrary, the limits $f_k^p$ for any $k \in \mathbb{N}$ agree. We denote the limit as $f$ so that the Cauchy sequence converges to $f \in C_p^\infty(EH)$ with respect to any of the norms. Thus the topology is complete and $C_p^\infty(EH)$ is a Fréchet algebra. □

When $p = 2$, the Fréchet algebra $C_2^\infty(EH)$ belongs to the chain of continuous inclusions,

$$C_c^\infty(EH) \hookrightarrow C_2^\infty(EH) \hookrightarrow C_0^\infty(EH), \tag{7.3}$$

with respect to their aforementioned topologies.

## 7.2 Deformation quantization of differentiable algebras

Rieffel's deformation quantization of a differentiable Fréchet algebra in [21] (Chapter 1, 2) can be summarized as follows. Let $\mathcal{A}$ be a Fréchet algebra whose topology is defined by a family of seminorms $\{q_m\}$. We assume that there there is an isometric action $\alpha$ of the vector space $V := \mathbb{R}^d$ considered as a $d$-dimensional Lie algebra acting on $\mathcal{A}$. We also assume that the algebra is smooth with respect to the action $\alpha$, i.e. $\mathcal{A} = \mathcal{A}^\infty$ in the notation of the reference.

Under the choice of a basis $\{X_1, \ldots, X_d\}$ of the Lie algebra, the action $\alpha_{X_i}$ of $X_i$ defines a partial differentiation on $\mathcal{A}$. One can define a new family of seminorms from $q_m$ by taking into account the action of $\alpha$. For any $f \in \mathcal{A}$,

$$\|f\|_{j,k} := \sum_{m \leq j, |\mu| \leq k} q_m(\delta^\mu f), \tag{7.4}$$

where $\mu$ are the multi-indices $(\mu_1, \ldots, \mu_d)$ and $\delta^\mu = \alpha_{X_1}^{\mu_1} \ldots \alpha_{X_d}^{\mu_d}$. It is a fact that if each of the seminorm in the family of seminorms $\{q_m\}$ is submultiplicative then each of the seminorm in the family of seminorms $\{\|\cdot\|_{j,k}$ is submultiplicative. The deformation quantization of the algebra $\mathcal{A}$ can be carried out in three steps:

*Step 1.* Let $C_b(V \times V, \mathcal{A})$ be the space of bounded continuous functions from $V \times V$ to $\mathcal{A}$. One can induce the family of seminorms $\{\|\cdot\|^C_{j,k}\}$ on the space $C_b(V \times V, \mathcal{A})$ by

$$\|F\|^C_{j,k} := \sup_{w \in V \times V} \|F(w)\|_{j,k}, \tag{7.5}$$

for $F$ in $C_b(V \times V, \mathcal{A})$ and $\|\cdot\|_{j,k}$ on $\mathcal{A}$ as in (7.4).

Let $\tau$ be an action of $V \times V$ on the space $C_b(V \times V, \mathcal{A})$ defined by translation. That is, $\tau_{w_0}(F)(w) = F(w + w_0)$ for any $w_0, w \in V \times V$ and $F \in C_b(V \times V, \mathcal{A})$. The action $\tau$ is an isometry action with respect to the seminorms (7.5). We define $\mathcal{B}^{\mathcal{A}}(V \times V)$ to be the maximal subalgebra such that $\tau$ is strongly continuous and whose elements are all smooth with respect to the action of $\tau$.

In the same way as one induces from the family of seminorms $\{q_m\}$ and obtains the seminorms $\|\cdot\|_{j,k}$ of $\mathcal{A}$ in (7.4), one may induce the family of seminorms on $\mathcal{B}^{\mathcal{A}}(V \times V)$ from (7.5) by taking into account of the action of $\tau$. For any $F \in \mathcal{B}^{\mathcal{A}}(V \times V)$, let

$$\|F\|^{\mathcal{B}}_{j,k;l} := \sum_{(l,m)\le(j,k)} \sum_{|\nu|\le l} \|\delta^\nu F\|^C_{l,m}, \tag{7.6}$$

where $\nu$ are the multi-indices and $\delta^\nu$ denotes the partial differentiation operator associated to $\tau$ of $V \times V$.

*Step 2.* The following is the fundamental result of the deformation quantization of a differentiable algebra. See Proposition 1.6 in [21]. One can define an $\mathcal{A}$-valued oscillatory integral over $V \times V$ of $F \in \mathcal{B}^{\mathcal{A}}(V \times V)$ by

$$\int_{V \times V} F(u, v) e(u \cdot v)\, du dv, \tag{7.7}$$

where $e(t) := e^{it})$ for $t \in \mathbb{R}$ and $\cdot$ is the natural inner product on $V$.

The integral is shown to be convergent in the family of seminorms $\{\|\cdot\|_{j,k}\}$ on $\mathcal{A}$ so that it is $\mathcal{A}$-valued. Specifically, for large enough $l$, there exists a constant $C_l$ such that

$$\left\| \int_{V \times V} F(u, v) e(u \cdot v)\, du dv \right\|_{j,k} \le C_l \, \|F\|^{\mathcal{B}}_{j,k;l} < \infty,$$

where the seminorm $\|\cdot\|^{\mathcal{B}}_{j,k;l}$ is defined in (7.6).

*Step 3.* Any two functions $f, g \in \mathcal{A}$ define an element $F^{f,g} \in \mathcal{B}^{\mathcal{A}}(V \times V)$ by

$$F^{f,g}(u,v) := \alpha_{Ju}(f)\alpha_v(g) \in \mathcal{A}, \quad \forall (u,v) \in V \times V, \tag{7.8}$$

for any invertible matrix $J$ acting on $V$. The *deformed product* $f \times_J g$ is thus defined by the integral (7.7) of $F^{f,g}(u,v)$ as,

$$f \times_J g := \int_V \int_V \alpha_{Ju}(f)\alpha_v(g)e(u \cdot v)\, dudv. \tag{7.9}$$

The algebra $\mathcal{A}$ with its deformed product $\times_J$, together with its undeformed seminorms $\{\|\cdot\|_{j,k}\}$, defines the deformed Fréchet algebra $\mathcal{A}_J$. This is called the *deformation of the algebra $\mathcal{A}$ (in the direction of $J$)* as a differentiable Fréchet algebra.

In the following, we obtain deformation quantizations of various algebras of functions on EH-spaces. We induce the torus action $\alpha$ on the algebra $C_*^\infty(EH)$ where $C_*^\infty(EH)$ stands for $C_b^\infty(EH)$, $C_0^\infty(EH)$ or $C_2^\infty(EH)$ from the isometric action $\sigma$ (6.4) as follows. For any $v \in \mathfrak{U}(1) \times \mathfrak{U}(1)$ its action through $\alpha : \mathfrak{U}(1) \times \mathfrak{U}(1) \to Aut(C_*^\infty(EH))$ is defined by

$$\alpha_v(f)(x) := \alpha(v)(f)(x) = f(\sigma_{-v}(x)), \quad \forall, f \in C_*^\infty(EH) x \in EH.$$

In coordinates, if $v = (v_3, v_4)$ then $\alpha_v(f)(r,\theta,\phi,\psi) = f(r,\theta,\phi - v_3, \psi - v_4)$ where $(r,\theta,\phi,\psi) \in EH$.

Under the choice of the covering $\{U_N, U_S\}$, the orbit of any point $x \in EH$ lies in the same coordinate chart as $x$. We assume that the partition of unity $h_N$ and $h_S$ only depend on the coordinate $\theta$ so that they are invariant under the torus action $\alpha$.

One can easily show that the torus action $\alpha$ is isometric with respect to the family of seminorms (7.1). We also note that each of the Fréchet algebras $C_b^\infty(EH)$ and $C_0^\infty(EH)$ is already smooth with respect to the action $\alpha$. Thus, each of $C_b^\infty(EH)$ and $C_0^\infty(EH)$, with the isometric action $\alpha$, regarded as a periodic action of $V = \mathbb{R}^2$, appears exactly as the starting point as $(\mathcal{A}, \{q_m\})$ of Rieffel's deformation quantization. We can carry out step 1 to step 3 and obtain the product $\times_J$ on the respective algebras,

$$f \times_J g := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \alpha_{Ju}(f)\, \alpha_v(g)\, e(u \cdot v)\, dudv, \tag{7.10}$$

where the inner product $u \cdot v$ is the one on $\mathbb{R}^2$ and $J$ is a skew-symmetric linear operator on $\mathbb{R}^2$. In the following we assume $J := \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$, for some $\theta \in \mathbb{R}\backslash\{0\}$, and denote $\times_J$ as $\times_\theta$.

The algebra $C_b^\infty(EH)$ with its deformed product $\times_\theta$, together with its undeformed family of seminorms (7.1) defines the deformed Fréchet algebra $C_b^\infty(EH)_\theta$ as the *deformation quantization of* $C_b^\infty(EH)$. Similarly, $C_0^\infty(EH)_\theta$ is the deformation quantization of the algebra $C_0^\infty(EH)$.

For the Fréchet algebra $C_2^\infty(EH)$, the torus action $\alpha$ is isometric with respect to the family of norms $\{\|\cdot\|_{H_k^2}\}_{k\in\mathbb{N}}$, because it is isometric with respect to the Riemannian metric. We can similarly obtain the Fréchet algebra $C_2^\infty(EH)_\theta$ as deformation quantization of the algebra $C_2^\infty(EH)$.

**Remark**: For any of the algebras in our example, the family of seminorms $\|\cdot\|_{j,k}$ induced from $q_m$'s as in *Step 1* is equivalent to the original family of seminorms. Indeed, the torus action is defined by the normal differentiation with respect to coordinates.

There follows some immediate observations.

**Lemma 7.2.1** *The algebra $C_2^\infty(EH)_\theta$ is an ideal of the algebra $C_b^\infty(EH)_\theta$.*

**Proof**: Let $f \in C_2^\infty(EH)$ and $g \in C_b^\infty(EH)$. Considered as elements of the algebra $C_b^\infty(EH)$, they define $F^{f,g} \in \mathcal{B}^{C_b^\infty(EH)}(\mathbb{R}^2 \times \mathbb{R}^2)$ by (7.8). We claim that $F^{f,g}$ lies in $\mathcal{B}^{C_2^\infty(EH)}(\mathbb{R}^2 \times \mathbb{R}^2)$ so that its oscillatory integral, or product of $f \times_\theta g$ by definition, will be finite in the family of seminorms on $C_2^\infty(EH)$ and hence $C_2^\infty(EH)$-valued. In fact,

$$\begin{aligned} \int_{EH} |F^{f,g}(u,v)(x)|^2 dVol(x) &= \int_{EH} |f(Ju+x)g(v+x)|^2 dVol(x) \\ &\leq \sup_{x\in EH} |g(x)|^2 \int_{EH} |f(Ju+x)|^2 dVol(x) \\ &= \sup_{x\in EH} |g(x)|^2 \int_{EH} |f(x)|^2 dVol(x) < \infty. \end{aligned}$$

The last equality follows from the invariance of the volume form of the integration with respect to the torus isometric action. The finiteness is because $g$ is a bounded function and $f \in C_2^\infty(EH)$.

Higher orders can be shown as follows. For any non-negative integer $k$, we may expand $\nabla^k(f(Ju+x)g(v+x))$ by the Leibniz rule to a summation of terms in the form of $\nabla^l f(Ju+x)\nabla^m g(v+x)$ with $l+m=k$. By the assumption that $\nabla^k f$ is $L^2$-integrable for any $k$ and $\nabla^l g$ is bounded for any $l$, each term in the summation is $L^2$-integrable. Thus $\nabla^k(f(Ju+x)g(v+x))$ is $L^2$-integrable for any $k$ and $F^{f,g}(u,v) \in C_2^\infty(EH)$ for any $(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2$. As a result, the product $f \times_\theta g$ is $C_2^\infty(EH)$-valued and $C_2^\infty(EH)_\theta$ is an ideal. □

Restriction of the product (7.10) of the algebra $C_b^\infty(EH)_\theta$ to the algebra $C_c^\infty(EH)$ gives the deformed algebra $C_c^\infty(EH)_\theta$. We see that it is closed as an algebra as follows. For any $f, g \in C_c^\infty(EH)$, the integral (7.10) vanishes outside the compact set $Orb(supp(f)) \cap Orb(supp(g))$, where $Orb(U) := \{\alpha_{\mathbb{T}^2}(x) : x \in U \subset EH\}$. Therefore, $f \times_\theta g$ is of compact support and $C_c^\infty(EH)_\theta$ is thus closed. We assign the topology of inductive limit on $C_c^\infty(EH)_\theta$ from that of $C_c^\infty(EH)$. Using definitions, we have

**Lemma 7.2.2** *$C_c^\infty(EH)_\theta$ is an ideal of the algebras $C_0^\infty(EH)_\theta$ and $C_b^\infty(EH)_\theta$.*

**Proof**: For $f \in C_c^\infty(EH)_\theta$ and $g \in C_b^\infty(EH)_\theta$, the integral (7.10) vanishes outside the compact set $Orb(supp(f))$. Hence $f \times_\theta g$ is $C_c^\infty(EH)$-valued, so that $C_c^\infty(EH)_\theta$ is an ideal of the algebras $C_b^\infty(EH)_\theta$. The proof for the algebra $C_0^\infty(EH)_\theta$ is the same. □

The torus action $\alpha$ as a compact action defines a spectral decomposition of a function $f$ in the algebra $C_b^\infty(EH)$ or $C_0^\infty(EH)$, by

$$f = \sum_s f_s, \quad f_s(x) = e^{-is_3\phi}\, e^{-is_4\psi}\, h_s(r,\theta),$$

where $s = (s_3, s_4) \in \mathbb{Z}^2$, $f_s$ satisfies $\alpha_v f_s = e(s \cdot v) f_s, \forall v \in \mathfrak{U}(1) \times \mathfrak{U}(1)$, and the series converges in the topology of uniform convergence of all derivatives. Under the decomposition, the product of (7.10) takes a simple form (Chapter 2, [21]). Let $f = \sum_r f_r$ and $g = \sum_s g_s$, in their respective decompositions, be both in the algebra $C_b^\infty(EH)$ (or $C_0^\infty(EH)$), then

$$f \times_\theta g = \sum_{r,s} \sigma(r,s) f_r g_s. \tag{7.11}$$

where $\sigma(r,s) := e(-s \cdot Jr) = e(\theta(r_4 s_3 - r_3 s_4))$ and $r = (r_3, r_4), s = (s_3, s_4) \in \mathbb{Z}^2$. The expression (7.11) can also be restricted to the algebra $C_c^\infty(EH)_\theta$.

**Lemma 7.2.3** *$C_0^\infty(EH)_\theta$ is an ideal of $C_b^\infty(EH)_\theta$.*

**Proof**: For any $f \in C_0^\infty(EH)_\theta$ and $g \in C_b^\infty(EH)_\theta$, it suffices to show that $f \times_\theta g \in C_0^\infty(EH)_\theta$. For $g$ being zero, this is trivial. We thus assume that $g$ is nonzero. The convergence of the series (7.11) implies that for any $\varepsilon/2 > 0$, there exists an integer $N$ such that

$$|f \times_\theta g(x)| < \left| \sum_{|r|,|s| \le N} \sigma(r,s) f_r(x) g_s(x) \right| + \frac{\varepsilon}{2},$$

for any $x \in EH$, where $|r| := |r_3| + |r_4|$ and $|s| := |s_3| + |s_4|$.

Since $f_r \in C_0^\infty(EH)$, for each $|r| \le N$, there exists a compact set $K(f_r) \subset EH$ such that

$$|f_r(x)| < \frac{\varepsilon}{2C}, \quad \forall x \in EH \backslash K(f_r),$$

for any fixed constant $C$.

Therefore, for any $\varepsilon > 0$, we may choose $N$ and $K(f_r)$ as above and define the union of finitely many compact sets as $K := \cup_{|r|\le N} K(f_r)$, so that $x \in EH\backslash K$ implies that

$$|f \times_\theta g(x)| < \left| \sum_{|r|,|s| \le N} \sigma(r,s) f_r(x) g_s(x) \right| + \frac{\varepsilon}{2} < \frac{\varepsilon\, A_N}{2C} \sup_{x\in EH} |\sigma(r,s) g(x)| + \frac{\varepsilon}{2},$$

where $A_N$ is a finite non-negative integer counting numbers of indices $r$ and $s$ satisfying $|r|, |s| \le N$. If we fix the constant $C = \sup_{x\in EH} |\sigma(r,s)g(x)| A_N$, then the above inequalities give $|f \times_\theta g(x)| < \varepsilon$, whenever $x \in EH\backslash K$. Therefore, $f \times_\theta g$ is $C_0^\infty(EH)$-valued, and $C_0^\infty(EH)_\theta$ is an ideal of $C_b^\infty(EH)_\theta$. □

We will end this section by introducing local algebras.

**Definition 7.2.4** [10] *An algebra $\mathcal{A}_c$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}_c$, there exists $\phi \in \mathcal{A}_c$ such that for each $i$, $\phi\, a_i = a_i\, \phi = a_i$.*

*Let $\mathcal{A}$ be a Fréchet algebra such that $\mathcal{A}_c \subset \mathcal{A}$ is a dense ideal with local units, then $\mathcal{A}$ is called a local algebra.*

**Proposition 7.2.1** *The algebra $C_c^\infty(EH)_\theta$ has local units and the algebra $C_0^\infty(EH)_\theta$ is a local $*$-algebra.*

**Proof**: For any finite set of elements $\{f_\beta\}_{\beta=1}^n \subset C_c^\infty(EH)_\theta$, there exists a compact set $K$ large enough to contain the union of supports $\cup_\beta supp(f_\beta)$. Let $\phi$ be a function equal to 1 on $K$ and decaying only with respect to the $r$-variable to zero outside $K$. Thus defined $\phi$ satisfies $\phi = \phi_{(0,0)}$ in the spectral decomposition so that $\phi \times_\theta f_\beta = f_\beta \times_\theta \phi = f_\beta$ for all $\beta$. Thus, $(C_c^\infty(EH), \times_\theta)$ is an algebra with units.

The fact that $C_c^\infty(EH)$ is dense in $C_0^\infty(EH)$ with respect to the topology of uniform convergence of all derivatives implies that $C_c^\infty(EH)_\theta$ is dense in $C_0^\infty(EH)_\theta$, since the family of seminorms is not deformed. $C_c^\infty(EH)_\theta$ is an ideal in $C_0^\infty(EH)_\theta$ by Lemma 7.2.2.

The involution $*$ of $C_0^\infty(EH)_\theta$ is simply defined by the complex conjugation. Thus $C_0^\infty(EH)_\theta$ is a local $*$-algebra. □

Lemma 3 of [10] says that there exists *a local approximate unit* $\{\phi_n\}_{n\geq 1}$ for a local algebra $(\mathcal{A}_c \subset)\mathcal{A}$. In this example, we choose a family of compact sets $K_0 \subset K_1 \subset \ldots$ in the $EH$-space, increasing in the $r$-direction. For instance,

$$K_n := \{x \in EH : r \leq n\}, \quad \forall n \in \mathbb{N}.$$

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a family of functions with compact support $K_n \subset supp(\phi_n) \subset K_{n+1}$ such that $\phi_n$ is constant 1 on $K_n$ and decays only with respect to $r$ to zero on $K_{n+1}$. This gives a local approximate unit. It is not hard to see that each $\phi_i$ actually commutes with functions in the algebra $C_0^\infty(EH)_\theta$. Furthermore, the union of the algebras $\cup_{n\in\mathbb{N}}[C_0^\infty(EH)_\theta]_n$, where

$$[C_0^\infty(EH)_\theta]_n := \{f \in C_0^\infty(EH)_\theta : \phi_n \times_\theta f = f \times_\theta \phi_n = f\},$$

is the algebra $C_c^\infty(EH)_\theta$.

## 7.3 Algebras of operators, deformation quantization of $C^*$-algebras

Let $C_*^\infty(EH)_\theta$ stand for the algebras $C_c^\infty(EH)_\theta$, $C_0^\infty(EH)_\theta$ or $C_b^\infty(EH)_\theta$. Following the construction of [23] [14], we may obtain representations of these differentiable algebras on the Hilbert space $\mathcal{H}$ of spinors by the torus isometric action.

The operator representation of $C_*^\infty(EH)_\theta$ on the Hilbert space $\mathcal{H}$ is defined by

$$L_f^\theta := \sum_{r\in\mathbb{Z}^2} M_{f_r} V_r^\theta, \tag{7.12}$$

where $M_{f_r}$ is the normal multiplication by $f_r$ and $V_r^\theta$ is defined to be the unitary operator $V_v$ (6.25), evaluated at $v = (v_3, v_4) := (\theta r_4, -\theta r_3)$.

**Remark**: Geometrically, $V_r^\theta = V_{(\theta r_4, -\theta r_3)}$ is the action of parallel transporting any section by $-\theta r_3$ along the $\psi$ direction followed by a parallel transporting by $\theta r_4$ along the $\phi$ direction.

With the involution on $C_*^\infty(EH)_\theta$ defined by the complex conjugation of functions, we can use the property $(f^*)_r = (f_{-r})^*$ and $V_r^\theta h_s = h_s V_r^\theta \sigma(r, s)$ for any simple component $h_s$ from $\sum_s h_s$, to show that the representation (7.12) is a faithful $*$-representation of $C_*^\infty(EH)_\theta$.

We may define the $C^*$-norm of $C_*^\infty(EH)_\theta$ by the operator norm $\|\cdot\|_{op}$ of the representation on $\mathcal{H}$. The series of operators (7.12) converges uniformly in the operator norm. We denote the $C^*$-completion of the algebra $C_b^\infty(EH)_\theta$ by $C_b(EH)_\theta$. It is a deformation of $C_b(EH)$ as a $C^*$-algebra.

As a Banach algebra, $C_b(EH)_\theta$ satisfies holomorphic functional calculus. We see that the subalgebra $C_b^\infty(EH)_\theta$ is stable under the holomorphic functional calculus of $C_b(EH)_\theta$. Indeed, for any invertible $f \in C_b^\infty(EH)_\theta$ of inverse $f^{-1}$ as an element in $C_b(EH)_\theta$, we apply derivatives to $f \times_\theta f^{-1} = 1$. By the Leibniz rule and the boundedness of derivatives of $f$ to all degrees, we conclude that $f^{-1}$ is bounded of all derivatives and hence an element in $C_b^\infty(EH)_\theta$. As we will see that the $C^*$-norm is weaker than the family of seminorms which define the Fréchet topology on $C_b^\infty(EH)_\theta$, $C_b^\infty(EH)_\theta$ is a smooth algebra from Definition 2.5.4.

The $C^*$-completion $C_0(EH)_\theta$ of the algebra $C_0^\infty(EH)_\theta$ defines a deformation of $C_0(EH)$ as a $C^*$-algebra. $C_0^\infty(EH)_\theta$ is also stable under the holomorphic functional calculus of $C_0(EH)_\theta$ and hence a pre-$C^*$-algebra. Endowed with the Fréchet topology, $C_0^\infty(EH)_\theta$ is also a smooth algebra.

In the commutative case, one can show that $\|\cdot\|_{op}$ is bounded by the zero-th seminorm $q_0$ in the family of seminorms (2.2). Hence the $C^*$-norm is weaker than the family of seminorms (2.2). To see that the same holds in the deformed case, we note that in Rieffel's construction, the deformed Fréchet algebras can be represented on the space of Schwarz functions associated with a natural inner product (page 23 [21]) and completed to $C^*$-algebras. Furthermore, the correspondent $C^*$-norm is shown to be weaker than the family of seminorms defining the Fréchet topology (Proposition 4.10 [21]). We may induce a $*$-homomorphism from the $C^*$-algebra representing on $\mathcal{H}$ to the $C^*$-algebra representing on the space of Schwarz functions by the identity map of functions. Since any $*$-homomorphism between $C^*$-algebras is norm-decreasing by Proposition 2.3.1, we conclude that the $C^*$-norm on $C_*^\infty(EH)_\theta$ represented on $\mathcal{H}$ is also weaker than the family of seminorms (2.2) defining the topology of uniform convergence of all derivatives.

## 7.4 Nonunital Serre-Swan theorem

The link between vector bundles over compact space and projective modules is the Serre-Swan theorem [3]. It is generalised for vector bundles of finite type, of which there exists a finite number of open sets in the open cover of the base manifold such that the bundle is trivialized on each open set [38]. The smooth version of the result is as follows.

**Theorem 7.4.1** *The category of spaces of smooth sections of complex vector bundles of finite type over any differentiable manifold $X$ is equivalent to the category of finitely generated projective $C_b^\infty(X)$-modules.*

**Remark**: There exists an alternative version of the generalised Serre-Swan theorem [10] for vector bundles over noncompact manifolds, proved by using certain compactification of the base manifolds. Since the simplest one-point compactification of the Eguchi-Hanson space gives an orbifold due to the $\mathbb{Z}_2$-identification, it is not straightforward to apply the construction there.

In the following, we will use Theorem 7.4.1 to find the projective module associated to the spinor bundle $S$ of the EH-space as defined in Section 6.3. In the coordinate charts $U_N$ and $U_S$ of the EH-space, we may choose a partition of unity

$\{h_N, h_S\}$ by

$$h_N(x) := \cos^2\frac{\theta}{2}, \quad h_S(x) := \sin^2\frac{\theta}{2}, \quad x \in EH. \tag{7.13}$$

Recall that in the unitary basis $\{f_\alpha\}$ of $\mathcal{S}_{U_N}$ and $\{f'_\beta\}$ of $\mathcal{S}_{U_S}$, the transition functions $P^\alpha_\beta$'s and $Q^\alpha_\beta$'s, such that $f_\beta = P^\alpha_\beta f'_\alpha$ and $f'_\beta = Q^\alpha_\beta f_\alpha$, are matrix entries of $P$ in (6.12) and $Q$ in (6.13), respectively.

The idea is to extend the basis $\{f_\alpha\}$ on $U_N$ across the "north pole" $N$ and $\{f'_\alpha\}$ on $U_S$ across the "south pole" $S$ so that one can take the summation of both extended global sections to obtain a generating set of the space of smooth bounded sections of the spinor bundle $\Gamma^\infty_b(\mathcal{S})$.

To extend $\{f_\alpha\}$ across $N$, we may rescale it by the function $h_N$,

$$F_\alpha := \begin{cases} f_\alpha h_N & \text{on } U_N \\ 0 & \text{at } N \end{cases}, \tag{7.14}$$

so that $F_\alpha$'s now decay to zero smoothly at $N$. Similarly, we may rescale the basis $\{f'_\alpha\}$ by the function $h_S$ by defining

$$F'_\alpha := \begin{cases} f'_\alpha h_S & \text{on } U_S \\ 0 & \text{at } S \end{cases}. \tag{7.15}$$

Note that on the intersection $U_N \cap U_S$, the transition function satisfies $P^\alpha_\beta h_N \to 0$ whenever $h_N \to 0$, and similarly $Q^\alpha_\beta h_S \to 0$ whenever $h_S \to 0$.

**Lemma 7.4.2** *The set of global sections $\{F_\alpha, F'_\alpha\}$, where $\alpha = 1, \ldots, 4$, are the generating set of the space of bounded smooth sections of the spinor bundle $\Gamma^\infty_b(\mathcal{S})$.*

**Proof**: The restriction $\{F_\alpha|_{U_N}\}$ where $\alpha = 1, \ldots, 4$ is a basis for $\mathcal{S}_{U_N}$. Indeed, any section $\psi \in \Gamma^\infty_b(\mathcal{S})$ can be written as $\psi|_{U_N} = \psi^\alpha f_\alpha = a^\alpha f_\alpha h_N = a^\alpha F_\alpha|_{U_N}$, where $a^\alpha = \psi^\alpha / h_N$. Similarly, the restriction $\{F'_\alpha|_{U_S}\}$ gives a basis for $\mathcal{S}_{U_S}$, since any section $\psi$ can be written as $\psi|_{U_S} = \psi'^\alpha f'_\alpha = b^\alpha f'_\alpha h_S = b^\alpha F'_\alpha|_{U_S}$, where $b^\alpha = \psi'^\alpha / h_S$.

On the intersection,

$$F_\alpha|_{U_N \cap U_S} = h_N\, P^\beta_\alpha\, F'_\beta\, h_S^{-1}, \quad F'_\alpha|_{U_N \cap U_S} = h_S\, Q^\beta_\alpha\, F_\beta\, h_N^{-1}.$$

Let $\{k_N, k_S\}$ be a new partition of unity such that the $supp(k_N) \subset U_N$ and $supp(k_S) \subset U_S$. Furthermore, $k_N$ ($k_S$, respectively) is required to decay faster than $h_N$ around

$N$ ($h_S$ around $S$, respectively). We may choose for instance,[1]

$$k_N(x) := \cos^2(\frac{\pi}{2}\sin^2\frac{\theta}{2}), \quad k_S(x) := \sin^2(\frac{\pi}{2}\sin^2\frac{\theta}{2}), \quad x \in EH.$$

Therefore, $a^\alpha k_N \to 0$ on $U_N$, whenever $h_N \to 0$, and $b^\alpha k_S \to 0$ on $U_S$, whenever $h_S \to 0$. Thus, we can extend the coefficient functions $a^\alpha$'s and $b^\alpha$'s by zero,

$$A^\alpha := \begin{cases} a^\alpha k_N & \text{on } U_N \\ 0 & \text{at } N \end{cases}, \quad B^\alpha := \begin{cases} b^\alpha k_S & \text{on } U_S \\ 0 & \text{at } S \end{cases}.$$

so that $\psi = A^\alpha F_\alpha + B^\alpha F'_\alpha$. In fact,

$$A^\alpha F_\alpha + B^\alpha F'_\alpha = \begin{cases} \psi^\alpha k_N f_\alpha + \psi'^\alpha k_S f'_\alpha & \text{on } U_N \cap U_S \\ \psi'^\alpha k_S f'_\alpha & \text{at } N \\ \psi^\alpha k_N f_\alpha & \text{at } S \end{cases} = \begin{cases} \psi^\alpha f_\alpha & \text{on } U_N \\ \psi'^\alpha f'_\alpha & \text{on } U_S \end{cases} \tag{7.16}$$

which is the section $\psi$ in $\Gamma_b^\infty(\mathcal{S})$. Therefore, $\{F_\alpha, F'_\alpha\}$ with $\alpha = 1, \ldots, 4$ is a generating set of $\Gamma_b^\infty(\mathcal{S})$. □

By construction, we may obtain a projection in $M_8(C_b^\infty(EH))$ corresponding to the spinor bundle $\mathcal{S}$. Under the standard basis of the free $C_b^\infty(EH)$-module $C_b^\infty(EH)^8$, we define the matrix,

$$p := \begin{pmatrix} k_N \mathbf{1} & k_N P \\ k_S Q & k_S \mathbf{1} \end{pmatrix} \tag{7.17}$$

where $P$ and $Q$ are $4 \times 4$ complex matrices from (6.12) and (6.13) and $\mathbf{1}$ is the four by four identity matrix.

**Proposition 7.4.1** *$\Gamma_b^\infty(\mathcal{S})$ is a finitely generated projective right $C_b^\infty(EH)$-module,*

$$pC_b^\infty(EH)^8 \cong \Gamma_b^\infty(\mathcal{S}). \tag{7.18}$$

**Proof**: It is easy to check that $p^2 = p = p^*$. To show that (7.18) is an isomorphism, any section can be represented as an element in $pC_b^\infty(EH)^8$ by construction. Conversely, the matrix $p$ maps any element $(t_1, \ldots, t_4, t'_1, \ldots, t'_4)^t$ of $C_b^\infty(EH)^8$ to

$$((t_1 + P_1^\beta t'_\beta)\, k_N, (t_2 + P_2^\beta t'_\beta)\, k_N, (t_3 + P_3^\beta t'_\beta)\, k_N, (t_4 + P_4^\beta t'_\beta)\, k_N,$$

[1]These functions are kindly suggested by Derek Harland.

$$(t'_1 + Q^\beta_1 t_\beta)\, k_S, (t'_2 + Q^\beta_2 t_\beta)\, k_S, (t'_3 + Q^\beta_3 t_\beta)\, k_S, (t'_4 + Q^\beta_4 t_\beta)\, k_S)^t.$$

Let $A^\alpha = (t^\alpha + P^\beta_\alpha t'_\beta)k_N$ and $B^\alpha = (t'^\alpha + Q^\beta_\alpha t_\beta)k_S$, for $\alpha = 1, \cdots, 4$, then the image gives a section in $\Gamma^\infty_b(\mathcal{S})$ in the form of (7.16). Therefore, (7.18) is an isomorphism. □

Columns of the matrix $p = (p^\alpha_\beta)$ give a generating set of $\Gamma^\infty_b(\mathcal{S})$. We may define $p_k := (p^k_1, \cdots, p^k_8)^t$ for $k = 1, \ldots, 8$, then any element $\xi \in pC^\infty_b(EH)^8$ can be written as $\xi = \sum p_k \xi_k$ for functions $\xi_k \in C^\infty_b(EH)$.

## 7.5 Smooth modules of the spinor bundle

In addition to the description of a vector bundle as a finitely generated projective module, the integrability conditions of the sections become vital when the base manifold is noncompact. The notion of *smooth module* [10] is proposed to integrate the two aspects. We will give the relevant background from the reference.

Let $\mathcal{A}_0$ be an ideal in a smooth unital algebra $\mathcal{A}_b$. Suppose that $\mathcal{A}_0$ is further a local algebra containing a dense subalgebra of local units $\mathcal{A}_c$. Assuming the topology on $\mathcal{A}_0$ is the one making it local and the topology on $\mathcal{A}_b$ is the one making it smooth, if the inclusion $i : \mathcal{A}_0 \hookrightarrow \mathcal{A}_b$ is continuous, then $\mathcal{A}_0$ is a *local ideal*. It is further called *essential* if $\mathcal{A}_0\, b = \{0\}$ for some $b \in \mathcal{A}_b$ implies $b = 0$.

Let $\mathcal{A}_0$ be a closed essential local ideal in a smooth unital algebra $\mathcal{A}_b$ and $p \in M_m(\mathcal{A}_b)$ be a projection. By pulling back the projective modules $\mathcal{E}_b$ defined by $p\mathcal{A}^m_b$ through inclusion maps $i : \mathcal{A}_c \hookrightarrow \mathcal{A}_b$, one can define the $\mathcal{A}_b$-finite projective $\mathcal{A}_c$-module $\mathcal{E}_c$ by $p\mathcal{A}^m_c$. Similarly, one can define the $\mathcal{A}_b$-finite projective $\mathcal{A}_0$-module $\mathcal{E}_0$ by $p\mathcal{A}^m_0$.

By using the Hermitian form on the projective modules $(\xi, \eta) := \sum \xi^*_i \eta_i$ as the convention given in Section 4.3. One may obtain the topology on $\mathcal{E}_c$ induced from the topology of the inductive limit on $\mathcal{A}_c$, the Fréchet topology on $\mathcal{E}_0$ induced from the Fréchet topology on $\mathcal{A}_0$ and the Fréchet topology on $\mathcal{E}_b$ induced from the Fréchet topology on $\mathcal{A}_b$. Hence one has the following continuous inclusions of projective modules, $\mathcal{E}_c \hookrightarrow \mathcal{E}_0 \hookrightarrow \mathcal{E}_b$.

**Definition 7.5.1** *A smooth $\mathcal{A}_b$-module $\mathcal{E}_2$ is a Fréchet space with a continuous* *action of $\mathcal{A}_b$ such that*

$$\mathcal{E}_c \hookrightarrow \mathcal{E}_2 \hookrightarrow \mathcal{E}_0,$$

*as linear spaces, where the inclusions are all continuous.*

Returning to our example, we may choose $\mathcal{A}_c$ as $C_c^\infty(EH)_\theta$, $\mathcal{A}_0$ as $C_0^\infty(EH)_\theta$, $\mathcal{A}_2$ as $C_2^\infty(EH)_\theta$ and $\mathcal{A}_b$ as the unital smooth algebra $C_b^\infty(EH)_\theta$.

**Proposition 7.5.1** *Assuming that $C_c^\infty(EH)_\theta$ is the algebra of units, the algebras $C_c^\infty(EH)_\theta$, $C_2^\infty(EH)_\theta$ and $C_0^\infty(EH)_\theta$ are all essential local ideals of $C_b^\infty(EH)_\theta$ under the topology of uniform convergence of all derivatives.*

**Proof**: $C_0^\infty(EH)_\theta$ is an ideal of $C_b^\infty(EH)_\theta$ by Lemma 7.2.3. Since the topology on $C_0^\infty(EH)_\theta$ and $C_b^\infty(EH)_\theta$ are both the topology of uniform convergence of all derivatives, the inclusion $C_0^\infty(EH)_\theta \hookrightarrow C_b^\infty(EH)_\theta$ is continuous.

To show that the ideal $C_0^\infty(EH)_\theta$ is essential, we suppose that $f \in C_b^\infty(EH)_\theta$ satisfies $g \times_\theta f = 0$ for all $g \in C_0(EH)_\theta$. Taking $g = 1/r$, $g \times_\theta f = g \times f = 0$. This implies that $f = 0$, since $1/r$ is nowhere zero. Thus, $C_0^\infty(EH)_\theta$ is an essential ideal.

$C_2^\infty(EH)_\theta$ is an ideal of $C_b^\infty(EH)_\theta$ by Lemma 7.2.1. Similar to the proof for $C_0^\infty(EH)_\theta$, $C_2^\infty(EH)_\theta$ is further an essential ideal.

$C_c^\infty(EH)_\theta$ is an ideal of $C_b^\infty(EH)_\theta$ by Lemma 7.2.2. $C_c^\infty(EH)_\theta$ carrying the topology of inductive limit is a local essential ideal, as is implied by Corollary 7 of [10] directly. □

With the differential topologies the same as their commutative restriction, there is a chain of continuous inclusions,

$$C_c^\infty(EH)_\theta \hookrightarrow C_2^\infty(EH)_\theta \hookrightarrow C_0^\infty(EH)_\theta \hookrightarrow C_b^\infty(EH)_\theta. \tag{7.19}$$

One may define the following projective modules $pC_c^\infty(EH)_\theta^8$, $pC_0^\infty(EH)_\theta^8$ and $pC_b^\infty(EH)_\theta^8$ by the projection $p$ in the form of (7.17) while considered as an element in $M_8(C_b^\infty(EH)_\theta)$. It is not hard to see that $p^2 = p = p^*$ still holds in this deformed case.

The family of seminorms, say $\{Q_m\}$'s, on the projective modules is induced from the family of seminorms on the algebra, say $\{q_m\}$'s, by composing with the Hermitian form $(\cdot,\cdot)$ on the projective modules as $Q_m(\xi) := q_m((\xi,\xi))$ for any $\xi$ in the projective module. The topologies on the projective modules are defined by the induced family of seminorms. In this way, the chain of algebras (7.19) induces the chain of projective modules,

$$pC_c^\infty(EH)_\theta^8 \hookrightarrow pC_2^\infty(EH)_\theta^8 \hookrightarrow pC_0^\infty(EH)_\theta^8 \hookrightarrow pC_b^\infty(EH)_\theta^8.$$

Note that the action of $C_b^\infty(EH)_\theta$ on $pC_2^\infty(EH)_\theta^8$ is continuous. Indeed, if a sequence of elements $\{\xi_\beta\}$ in $pC_2^\infty(EH)_\theta^8$ satisfies that $Q_m(\xi_\beta) \to 0$ as $\beta \to \infty$, then for any $f \in C_b^\infty(EH)_\theta$,

$$Q_m(\xi_\beta f) = q_m((\xi_\beta f, \xi_\beta f)) = q_m(f^*(\xi_\beta, \xi_\beta)f) = q_m(f^*)Q_m(\xi_\beta)q_m(f) \to 0,$$

where $q_m$ stands for $||\cdot||_{H_m^2}$ defined in (7.2). Therefore, $pC_2^\infty(EH)_\theta^8$ is a smooth module.

# Chapter 8

# Nonunital spectral triples and summability

In this chapter, we define nonunital spectral triples and consider their summability. We also consider the regularity and measurability of the spectral triples of the isospectral deformations of EH-spaces.

Rennie (Theorem 12, [11]) provides a measurability criterion of operators from *local* nonunital spectral triples. Within the locality framework, a generalised Connes trace theorem over a commutative geodesically complete Riemannian manifold is also given (Proposition 15, [11]). The Dixmier trace of such measurable operator agrees with the Wodzicki residue of the operator [39].

Gayral and his coworkers [40] carry out a detailed study on summability of the nonunital spectral triples from isospectral deformations. Their results are also of a local kind.

## 8.1 Nonunital spectral triples and the local $(p,\infty)$-summability

**Definition 8.1.1** [10] *A nonunital spectral triple* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *is given by*

1. *A representation* $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ *of a local* $*$*-algebra* $\mathcal{A}$*, containing some algebra* $\mathcal{A}_c$ *of local units as a dense ideal, on the Hilbert space* $\mathcal{H}$*.* $\mathcal{A}$ *admits a*

*suitable unitization $\mathcal{A}_b$.*

2. *A self-adjoint (unbounded, densely defined) operator $\mathcal{D} : dom\mathcal{D} \longrightarrow \mathcal{H}$ such that $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}_b$ and $a\,(\mathcal{D}-\lambda)^{-1}$ is compact for $\lambda \notin \mathbb{R}$ and all $a \in \mathcal{A}$. This is the compact resolvent condition for nonunital triples.*

We omit $\pi$ if no ambiguity arises. The spectral triple is *even* if there exists an operator $\chi = \chi^*$ such that $\chi^2 = 1$, $[\chi, a] = 0$ for all $a \in \mathcal{A}$ and $\chi\mathcal{D} + \mathcal{D}\chi = 0$. Otherwise, it is *odd*.

To obtain the nonunital spectral triple of the isospectral deformation of the EH-space, let $\mathcal{A}$ be the local $*$-algebra $C_0^\infty(EH)_\theta$ (Proposition 7.2.1) which contains the algebra of local units $C_c^\infty(EH)_\theta$ as a dense ideal. The unitization $\mathcal{A}_b$ is chosen as $C_b^\infty(EH)_\theta$. The representation $\pi$ is defined by the representation $L^\theta_\bullet : C_b^\infty(EH)_\theta \to \mathcal{B}(\mathcal{H})$ from (7.12). The boundedness of $L^\theta_f$ where $f = \sum_r f_r$ can be seen as follows,

$$\|L^\theta_f\|_{op} = \left\|\sum_r M_{f_r} V^\theta_r\right\|_{op} \leq \sum_r \|M_{f_r} V^\theta_r\|_{op} \leq \sum_r \|M_{f_r}\|_{op} = \sum_r \|f_r\|_\infty < \infty,$$

where the summations are over $\mathbb{Z}^2$. The second inequality is implied by the fact that $V^\theta_r$ is unitary.

Let $\mathcal{D}$ be the extension of the Dirac operator of the spinor bundle to the Hilbert space $\mathcal{H}$. Since the Eguchi-Hanson space is geodesically complete, the extended operator is self-adjoint. We will see in the next subsection that the operator $[\mathcal{D}, L^\theta_f]$ is of degree 0 as a pseudodifferential operator and hence bounded.

The operator $\chi$ is chosen to be the chirality operator defined in (6.15), such that $\chi = \chi^*$ and $\chi^2 = 1$. Since $\chi$ can be realised as a fibrewise constant matrix operating on the spinor bundle, its commutativity with respect to any $L^\theta_f = \sum_r M_{f_r} V^\theta_r$ holds. The identity $\chi\mathcal{D} + \mathcal{D}\chi = 0$ can be deduced from spin geometry [32].

The data $(C_0^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ will be a nonunital spectral triple once the compact resolvent condition is shown. Before that, we consider the following proposition.

**Proposition 8.1.1** *For any $f \in C_c^\infty(EH)_\theta$,*

$$L^\theta_f(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{4,\infty}(\mathcal{H}), \quad \forall \lambda \notin \mathbb{R}. \tag{8.1}$$

**Proof**: The proof is a straightforward generalisation of Proposition 15 of [11] and references therein.

With respect to the local trivializations $\{U_N, U_S\}$ of the spinor bundle $\mathcal{S}$ coming from the stereographic projection as before, we may show the summability of the operator (8.1) by showing the summability of the restrictions of the operator on each trivialization. Indeed, for any $f \in C_c^\infty(EH)_\theta$, the operator $L_f^\theta = \sum_r M_{f_r} V_r^\theta$ is defined by summations of normal multiplications by $f_r$ following parallel transporting in the $\phi$ and $\psi$ directions, so that it is well-defined when restricted on either $U_N$ or $U_S$. We may choose some partition of unity so that each function $f$ can be decomposed as $f = f_N + f_S$ with $f_N \in C_c^\infty(U_N)$ and $f_S \in C_c^\infty(U_S)$. It suffices to show that

$$L_f^\theta(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(L^2(\mathcal{S}_{U_N})), \quad \forall f \in C_c^\infty(U_N), \tag{8.2}$$

and similarly for $U_S$.

For any fixed $f \in C_c^\infty(U_N)_\theta$, we can find a positive constant $R > a$ big enough, and a constant $\Theta > 0$ small enough such that the compact region defined by

$$W_{R,\Theta} := \{x \in U_N : r \leq R,\, \theta \geq \Theta\} \subset U_N,$$

contains the compact support of $f$. Notice that with the restricted metric from the EH-space, the region $W_{R,\Theta}$ is a compact manifold with a boundary $\partial W_{R,\Theta}$ defined by $r = R$ and $\theta = \Theta$. We will fix $R$ and $\Theta$ from now on, and write $W$ instead of $W_{R,\Theta}$ and denote the restriction of the spinor bundle $\mathcal{S}$ on $W_{R,\Theta}$ by $\mathcal{S}_W$. Because the integral curve starting through any point in $W$ along the $\phi$ or $\psi$ direction still lies within $W$, the action of $L_f^\theta$ can be restricted on sections of the subbundle $\mathcal{S}_W$.

To prove (8.2), it suffices to prove that

$$L_f^\theta(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(L^2(\mathcal{S}_W)).$$

Let $\widetilde{W} := W \cup_{\partial W} (-W)$ be the *invertible double* [41] of the compact manifold $W$ with boundary $\partial W$, and let the corresponding spinor bundle be $\widetilde{\mathcal{S}} \to \widetilde{W}$ and the corresponding Dirac operator be $\mathcal{D}_I$. Applying the Weyl's lemma [42] on $\widetilde{\mathcal{S}} \to \widetilde{W}$ as a vector bundle over a compact manifold without boundary, we obtain $(\mathcal{D}_I - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(L^2(\widetilde{\mathcal{S}}))$, for $\lambda \notin \mathbb{R}$. That is,

$$\|(\mathcal{D}_I - \lambda)^{-1}\|_{4,\infty}^{\widetilde{W}\to\widetilde{W}} < \infty, \quad \forall \lambda \notin \mathbb{R}, \tag{8.3}$$

where the norm is the $(4, \infty)$-Schatten norm and we indicate the domain and image of operators as superscript on the norms.

As to the action of $L^\theta_f$, we may extend the function $f \in C^\infty_c(W)$ to a function $\tilde{f} \in C^\infty_c(\widetilde{W})$ by zero. Correspondingly, we may extend the operator $L^\theta_f : L^2(W, \mathcal{S}) \to L^2(W, \mathcal{S})$ to

$$L^\theta_{\tilde{f}} : L^2(\widetilde{W}, \widetilde{\mathcal{S}}) \to L^2(\widetilde{W}, \widetilde{\mathcal{S}}).$$

Using the resolvent identity $[L^\theta_{\tilde{f}}, (\mathcal{D}_I - \lambda)^{-1}] = (\mathcal{D}_I - \lambda)^{-1}[\mathcal{D}_I, L^\theta_{\tilde{f}}](\mathcal{D}_I - \lambda)^{-1}$, we have

$$L^\theta_{\tilde{f}}\,(\mathcal{D}_I - \lambda)^{-1} = (\mathcal{D}_I - \lambda)^{-1}(\mathcal{D}_I\, L^\theta_{\tilde{f}} - L^\theta_{\tilde{f}}\,\mathcal{D}_I)(\mathcal{D}_I - \lambda)^{-1} + (\mathcal{D}_I - \lambda)^{-1}\, L^\theta_{\tilde{f}}. \tag{8.4}$$

By composing $L^\theta_{\tilde{f}}$ with the restriction of sections of $L^2(\widetilde{W}, \widetilde{\mathcal{S}})$ to $L^2(W, \mathcal{S})$, we obtain an operator in the same notation, $L^\theta_{\tilde{f}}$ mapping from $L^2(\widetilde{W}, \widetilde{\mathcal{S}})$ to $L^2(W, \mathcal{S})$. Let $\iota : W \hookrightarrow \widetilde{W}$ be the inclusion map, the composition of $\iota$ with the identity (8.4) then gives,

$$L^\theta_{\tilde{f}}\,(\mathcal{D}_I - \lambda)^{-1}\,\iota = (\mathcal{D} - \lambda)^{-1}(\mathcal{D}\, L^\theta_{\tilde{f}} - L^\theta_{\tilde{f}}\,\mathcal{D}_I)(\mathcal{D}_I - \lambda)^{-1}\,\iota + (\mathcal{D} - \lambda)^{-1}\, L^\theta_{\tilde{f}}\,\iota, \tag{8.5}$$

as operators maps from $L^2(W, \mathcal{S})$ to itself.

Applying (8.5), we obtain

$$\begin{aligned}
& \|L^\theta_f\,(\mathcal{D} - \lambda)^{-1}\|^{W \to W}_{4,\infty} \\
= \; & \|L^\theta_{\tilde{f}}(\mathcal{D}_I - \lambda)^{-1}\iota\|^{W \to W}_{4,\infty} \\
= \; & \|(\mathcal{D} - \lambda)^{-1}\, L^\theta_{\tilde{f}}\,\iota + (\mathcal{D} - \lambda)^{-1}(\mathcal{D}\, L^\theta_{\tilde{f}} - L^\theta_{\tilde{f}}\,\mathcal{D}_I)(\mathcal{D}_I - \lambda)^{-1}\,\iota\|^{W \to W}_{4,\infty} \\
\leq \; & \|(\mathcal{D} - \lambda)^{-1}\, L^\theta_{\tilde{f}}\,\iota\|^{W \to W}_{4,\infty} + \|(\mathcal{D} - \lambda)^{-1}(\mathcal{D}\, L^\theta_{\tilde{f}} - L^\theta_{\tilde{f}}\,\mathcal{D}_I)(\mathcal{D}_I - \lambda)^{-1}\,\iota\|^{W \to W}_{4,\infty}.
\end{aligned}$$

We consider the two terms in the last line separately. Since the inclusion $\iota$ is an isometry, the first term is bounded as

$$\begin{aligned}
\|(\mathcal{D} - \lambda)^{-1}\, L^\theta_{\tilde{f}}\,\iota\|^{W \to W}_{4,\infty} & \leq \|(\mathcal{D} - \lambda)^{-1}\, L^\theta_{\tilde{f}}\,\|^{\widetilde{W} \to W}_{4,\infty} \\
& \leq \|(\mathcal{D}_I - \lambda)^{-1}\,\|^{\widetilde{W} \to \widetilde{W}}_{4,\infty}\|L^\theta_{\tilde{f}}\|^{\widetilde{W} \to W}_{op} < \infty,
\end{aligned} \tag{8.6}$$

where $\|L^\theta_{\tilde{f}}\|^{\widetilde{W} \to W}_{op} < \infty$ is because $L^\theta_{\tilde{f}}$ is the trivial extension of the bounded operator $L^\theta_f$ from $L^2(W, \mathcal{S})$ to itself and the finiteness of $\|(\mathcal{D}_I - \lambda)^{-1}\,\|^{\widetilde{W} \to \widetilde{W}}_{4,\infty}$ is by (8.3). The second term is bounded as

$$\begin{aligned}
& \|(\mathcal{D}-\lambda)^{-1}(\mathcal{D}\,L^\theta_{\tilde f} - L^\theta_{\tilde f}\,\mathcal{D}_I)(\mathcal{D}_I-\lambda)^{-1}\,\iota\|_{4,\infty}^{W\to W} \\
\leq\ & \|(\mathcal{D}-\lambda)^{-1}(\mathcal{D}\,L^\theta_{\tilde f} - L^\theta_{\tilde f}\,\mathcal{D}_I)(\mathcal{D}_I-\lambda)^{-1}\|_{4,\infty}^{\widetilde{W}\to W} \\
\leq\ & \|(\mathcal{D}-\lambda)^{-1}\|_{op}^{W\to W}\,\|\mathcal{D}\,L^\theta_{\tilde f} - L^\theta_{\tilde f}\,\mathcal{D}_I\|_{op}^{\widetilde{W}\to W}\,\|(\mathcal{D}_I-\lambda)^{-1}\|_{4,\infty}^{\widetilde{W}\to\widetilde{W}} < \infty. \quad (8.7)
\end{aligned}$$

Indeed, the finiteness of $\|(\mathcal{D}-\lambda)^{-1}\|_{op}^{W\to W}$ is by the fact that $(D-\lambda)^{-1}$ is a bounded operator on $\mathcal{S}\to W$ as the restriction of the bounded operator on $L^2(\mathcal{S})$. For the finiteness of $\|\mathcal{D}\,L^\theta_{\tilde f} - L^\theta_{\tilde f}\,\mathcal{D}_I\|_{op}^{\widetilde{W}\to W}$, we have

$$\|\mathcal{D}\,L^\theta_{\tilde f} - L^\theta_{\tilde f}\,\mathcal{D}_I\|_{op}^{\widetilde{W}\to W} = \|[\mathcal{D}, L^\theta_f]\|_{op}^{W\to W} \leq \|[\mathcal{D}, L^\theta_f]\|_{op}^{EH\to EH} < \infty,$$

since $\tilde f$ extends $f$ by zero and the boundedness of $[\mathcal{D}, L^\theta_f]$ will be shown in the next section. The the finiteness of $\|(\mathcal{D}_I-\lambda)^{-1}\|_{4,\infty}^{\widetilde{W}\to\widetilde{W}}$ is again by (8.3).

Summation of the inequalities (8.6) and (8.7) implies that

$$\|L^\theta_f\,(\mathcal{D}-\lambda)^{-1}\|_{4,\infty}^{W\to W} < \infty. \quad (8.8)$$

The proof for the coordinate patch $U_S$ is the same. □

As pointed out by Rennie, Proposition 8.1.1 implies the compact resolvent condition.

**Lemma 8.1.2** *For any $f \in C_0^\infty(EH)_\theta$, $L^\theta_f\,(\mathcal{D}-\lambda)^{-1} \in \mathcal{K}(\mathcal{H})$ with $\lambda \notin \mathbb{R}$.*

**Proof**: Let $\{f_\beta\}$ is be a sequence of functions in $C_c^\infty(EH)_\theta$, which converges to the function $f \in C_0^\infty(EH)_\theta$ in the topology of uniform convergence, then $\{L^\theta_{f_\beta}\}$ converges to $L^\theta_f$ in the $C^*$-operator norm, for the norm-topology is weaker than the topology of uniform convergence. This further implies that the sequence of operators $\{L^\theta_{f_\beta}\,(\mathcal{D}-\lambda)^{-1}\}$ converges uniformly to $L^\theta_f\,(\mathcal{D}-\lambda)^{-1}$ in the operator norm. The $(4,\infty)$-summability of each $L^\theta_{f_\beta}\,(\mathcal{D}-\lambda)^{-1}$ by (8.8) implies that they are all compact operators. As the uniform limit of a sequence of compact operators, $L^\theta_f\,(\mathcal{D}-\lambda)^{-1}$ is also compact. □

In summary,

**Proposition 8.1.2** *The spectral data $(C_0^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ of the isospectral deformations of the Eguchi-Hanson spaces are even nonunital spectral triples.*

**Definition 8.1.3** [11] *A (nonunital) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called local, if there exists a local approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying*

$$\Omega_{\mathcal{D}}(\mathcal{A}_c) = \cup_n \Omega_{\mathcal{D}}(\mathcal{A})_n,$$

*where $\Omega_{\mathcal{D}}(\mathcal{A})$ is the algebra of operators on $\mathcal{H}$ generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ and $\Omega_{\mathcal{D}}(\mathcal{A})_n := \{\omega \in \Omega_{\mathcal{D}}(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}$.*

*For $p \geq 1$, the local spectral triple is called local $(p, \infty)$-summable if $a\,(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{p,\infty}(\mathcal{H})$, $\lambda \notin \mathbb{R}$, for any $a \in \mathcal{A}_c$.*

Local $(p, \infty)$-summability implies that (Proposition 10 [11])

$$T(1 + \mathcal{D}^2)^{-s} \in \mathcal{L}^{2p/s,\infty}(\mathcal{H}), \quad 1 \leq Re(2s) \leq p, \tag{8.9}$$

for any $T \in \mathcal{B}(\mathcal{H})$ such that $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. If $Re(2s) > p$, the operator is of trace class.

In considering the (local) summability of the spectral triples, we restrict ourselves on the spectral triple $(C_c^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$.

**Lemma 8.1.4** *The spectral triple $(C_c^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ is local $(4, \infty)$-summable.*

**Proof**: First we show that the spectral triple is local. We may choose the local approximate unit $\{\phi_n\}$ as defined in Section 7.2 so that each of $\phi_n$ remains commutative. As operators, they act only by normal multiplication $M_{\phi_n}$ on spinors.

Define $[C_c^\infty(EH)_\theta]_n$ to be the subalgebra of $C_c^\infty(EH)_\theta$ consisting of elements $L_f^\theta$ such that $L_f^\theta M_{\phi_n} = M_{\phi_n} L_f^\theta = L_f^\theta$, then $C_c^\infty(EH)_\theta = \cup_{n\in\mathbb{N}}[C_c^\infty(EH)_\theta]_n$. Thus

$$\Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta) = \Omega_{\mathcal{D}}(\cup_{n\in\mathbb{N}}[C_c^\infty(EH)_\theta]_n) = \cup_{n\in\mathbb{N}}\Omega_{\mathcal{D}}([C_c^\infty(EH)_\theta]_n).$$

We claim that this equals $\cup_{n\in\mathbb{N}}[\Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta]_n$, where

$$[\Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta]_n := \{\omega \in \Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta) : \omega M_{\phi_n} = M_{\phi_n}\omega = \omega\}.$$

By the fact that the orbit of the torus action of any point $x \in K_n$ remains in $K_n$, $M_{\phi_n} L_f^\theta = L_f^\theta M_{\phi_n}$ whenever $supp(f) \subset K_n$. That the Dirac operator preserves support implies

$$M_{\phi_n}[\mathcal{D}, L_f^\theta] = [\mathcal{D}, L_f^\theta]M_{\phi_n} = [\mathcal{D}, L_f^\theta].$$

This further gives that $\cup_{n\in\mathbb{N}}\Omega_{\mathcal{D}}([C_c^\infty(EH)_\theta]_n) \subset \cup_{n\in\mathbb{N}}[\Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta]_n$. The other direction is obvious. Therefore, $\Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta) = \cup_{n\in\mathbb{N}}[\Omega_{\mathcal{D}}(C_c^\infty(EH)_\theta]_n$, and the spectral triple is local.

The local $(4,\infty)$-summability of the spectral triple $(C_c^\infty(EH)_\theta, \mathcal{D}, \mathcal{H})$ is implied by Proposition 8.1.1. □

## 8.2 Regularity of spectral triples

For a given spectral triple $(\mathcal{A}, \mathcal{D}, \mathcal{H})$, we can define a derivation $\delta$ on the space $\mathcal{L}(\mathcal{H})$ of linear operators on the Hilbert space by

$$\delta(T) := [|\mathcal{D}|, T], \quad \forall T \in \mathcal{L}(\mathcal{H}).$$

A linear operator $T$ is in the domain of the derivation $dom\,\delta \subset \mathcal{L}(\mathcal{H})$ if $\psi \in dom(|\mathcal{D}|)$ implies $T(\psi) \in dom(|\mathcal{D}|)$ for $\psi \in \mathcal{H}$. For any positive integer $k$, $T$ is in the domain of the $k$-th derivation $dom\,\delta^k \subset \mathcal{L}(\mathcal{H})$ if $\delta^{k-1}(T) \in dom\,\delta$, where $\delta^{k-1}(T) = [|\mathcal{D}|, [|\mathcal{D}|, \ldots, [|\mathcal{D}|, T]\ldots]]$, with $(k-1)$ brackets.

The intersection of domains of $\delta$ with all possible degree $dom^\infty\delta := \cap_{k\in\mathbb{N}} dom\,\delta^k$ is the *smooth domain of the derivation* $\delta$. When $k=0$, $dom\,\delta^0$ is simply the space of bounded operator $\mathcal{B}(\mathcal{H})$. Therefore, an operator $T \in dom\delta^k$ if $\delta^k(T)$ is a bounded operator.

**Definition 8.2.1** *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is regular if $\Omega_{\mathcal{D}}(\mathcal{A}) \subset dom^\infty\delta$.*

The regularity condition is crucial in considering differential functional calculus associated to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. For $T \in \mathcal{A}$ we define the family of seminorms by

$$q_{ni}(T) := \|\delta^n(T)d^i(T)\|, \quad n \geq 0,\ i = 0, 1,$$

where $d(T) := [\mathcal{D}, T]$. The natural topology of $\mathcal{A}$ induced by this family of seminorms is called the *$\delta$-topology*. Let $\mathcal{A}_\delta$ be the completion of $\mathcal{A}$ under the $\delta$-topology. An important observation in [10] (Lemma 16) says that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a regular spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a regular spectral triple. Furthermore, the algebra $\mathcal{A}_\delta$ is a smooth algebra. This allows one to consider the spectral triple $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$

instead. The completeness of $\mathcal{A}_\delta$ allows the $C^\infty$ *functional calculus* to hold by Proposition 22 of [10]. We also remark that these are the techniques necessary in the reconstruction theorem to obtain coordinate charts and differentiable structures [8] [9].

Before considering the regularity of the spectral triple $(C_0(EH)_\theta, \mathcal{H}, \mathcal{D})$, we collect some related properties of operators $L_f^\theta$ and $\mathcal{D}$ as pseudodifferential operators. We refer to [43] for background on pseudodifferential operators on noncompact manifolds. The Dirac operator $\mathcal{D}$ on the spinor bundle $\mathcal{S}$ is a first order differential operator with a principal symbol $\sigma^{\mathcal{D}}(x,\xi) = c(\xi_j dx^j)$, where $\xi$ as a section in the cotangent bundle $T^*(EH)$ is of coordinates $(\xi_1, \ldots, \xi_4)$ with respect to the basis $\{dx^i\}$ defined in the beginning of Section 6.1 and $c$ is the Clifford action.

The operator $\mathcal{D}^2$ is a second-order differential operator with a principal symbol

$$\sigma^{\mathcal{D}^2}(x,\xi) = g(\xi,\xi)\,\mathbf{1}_4, \tag{8.10}$$

where $g$ is the induced metric tensor on the cotangent bundle from that on the tangent bundle (6.2).

**Lemma 8.2.2** *The principal symbol of the pseudodifferential operator $M_f$ is*

$$\sigma^{M_f}(x,\xi) = M_f(x) = diag_4(f(x)), \tag{8.11}$$

*where $diag_r(g)$ denotes the $r\times r$ diagonal matrix of $g$ on the diagonal. The principal symbol of the pseudodifferential operator $L_f^\theta$ is*

$$\sigma^{L_f^\theta}(x,\xi) = \sum_{r=(r_3,r_4)} M_{f_r}(x) P^\theta(x)\, e(\theta\,(r_3\xi_4 - r_4\xi_3)), \tag{8.12}$$

*where the matrix-valued function $P^\theta(x) = P_{34}(x)$ is defined by (6.25) with $(v_3, v_4) := (\theta r_4, -\theta r_3)$.*

**Proof**: Applying $M_f$ where $f = \sum_r f_r$ on the inverse Fourier transformation of a spinor $\psi$,

$$M_f\left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} e^{ix\cdot\xi}\hat{\psi}(\xi)d\xi\right) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} diag_4(f(x))e^{ix\cdot\xi}\hat{\psi}(\xi)d\xi,$$

we see that $M_f$ is an order zero classical pseudodifferential operator with principal symbol (8.11).

The pointwise evaluation of the operator $L_f^\theta$ is

$$L_f^\theta \psi(x) = \sum_r M_{f_r}(P_3 \circ P_4)(\psi(x + (0, 0, -\theta r_4, \theta r_3))), \tag{8.13}$$

where $C_4$ is the integral curve of the Killing field $\partial_\psi$ starting at $(x_1, x_2, x_3 - \theta r_4, x_4 + \theta r_3)$ and ending $(x_1, x_2, x_3 - \theta r_4, x_4)$, and $P_4$ is assumed to be the parallel propagator with respect to the spin connection along the $C_4$. It is evaluated at the point $(x_1, x_2, x_3 - \theta r_4, x_4)$ as a four by four matrix. Similarly, $C_3$ is the integral curve of the Killing field $\partial_\phi$ starting at $(x_1, x_2, x_3 - \theta r_4, x_4)$ and ending at $(x_1, x_2, x_3, x_4)$. $P_3$ is assumed to be the parallel propagator with respect to the spin connection along the $C_3$ as defined by (6.24). In (8.13), their composition is evaluated at the point $(x_1, x_2, x_3, x_4)$ as a four by four matrix.

Applying $L_f^\theta$ on the inverse Fourier transformation of $\psi$,

$$L_f^\theta \psi(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \sum_r M_{f_r} P^\theta(x) \exp(i((x + (0, 0, -\theta r_4, \theta r_3))) \cdot \xi) \hat{\psi}(\xi) d\xi,$$

one obtains the symbol of $L_f^\theta$. With respect to the $\xi$ variable, the complete symbol is bounded by a constant and hence is of degree 0 and it can be chosen to be its principal symbol, which takes the form of (8.12). □

**Proposition 8.2.1** *The spectral triple $(C_0^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ is regular.*

**Proof**: We write $L_f^\theta$ by $f$ for notational simplicity here. As indicated in the proof of Proposition 20 in [10], $f, [\mathcal{D}, f] \in dom^\infty \delta$ for any $f \in C_0^\infty(EH)_\theta$ if and only if $f, [\mathcal{D}, f] \in dom_{k,l \geq 0} L^k R^l$, where

$$L(f) := (1 + \mathcal{D}^2)^{-1/2} [\mathcal{D}^2, f], \quad R(f) := [\mathcal{D}^2, f](1 + \mathcal{D}^2)^{-1/2},$$

for the reason that $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$ is bounded. The rest of the proof is a direct generalisation of the standard method in the unital case, see for instance [6]. Denote $ad(\mathcal{D}^2)^m(\cdot) = [\mathcal{D}^2, \dots, [\mathcal{D}^2, \cdot] \dots]$, with $m$ brackets, so that

$$L^k(f) = (1 + \mathcal{D}^2)^{-k/2} ad(\mathcal{D}^2)^k(f), \quad R^l(f) = ad(\mathcal{D}^2)^l(f)(1 + \mathcal{D}^2)^{-l/2},$$

where $k, l \in \mathbb{N}$. Their composition is

$$L^k R^l(f) = (1 + \mathcal{D}^2)^{-k/2} ad(\mathcal{D}^2)^{k+l}(f)(1 + \mathcal{D}^2)^{-l/2}.$$

The operator $ad(\mathcal{D}^2)(f) = [\mathcal{D}^2, f]$ is of order at most 1, since the commutator of the principal symbols (8.10) and (8.12) vanishes. Similarly, the operator $ad(\mathcal{D}^2)^{k+l}(f)$ is of order at most $k + l$. This implies that the operator $L^k R^l(f)$ is of order at most zero and hence a bounded pseudodifferential operator on $\mathcal{H}$. This holds for any $k$ and $l$ in $\mathbb{N}$. Hence $f \in dom_{k,l \geq 0} L^k R^l$, for any $f \in C_0^\infty(EH_\theta)$.

Since $[\mathcal{D}, M_f]$ is a bounded operator of degree 0 and $V_r^\theta$ is of degree 0, seen from (8.12), $[\mathcal{D}, L_f^\theta]$ is also a bounded operator of degree 0. The above proof holds if $f$ is replaced by $[\mathcal{D}, L_f^\theta]$. Thus $[\mathcal{D}, L_f^\theta] \in dom_{k,l \geq 0} L^k R^l$, for any $f \in C_0^\infty(EH)_\theta$. Since $L^k R^l(T) \in dom L^0 R^0 = \mathcal{B}(\mathcal{H})$ for any $k, l$ where $T \in \mathcal{B}(\mathcal{H})$ is equivalent to $T \in dom_{k,l \geq 0} L^k R^l$ for any $k, l$, we obtain $\Omega_{\mathcal{D}}(C_0^\infty(EH)_\theta) \subset dom^\infty \delta$. Hence the spectral triple is regular. □

The regularity of the spectral triple $(C_0(EH)_\theta, \mathcal{H}, D)$ allows us to define a new regular spectral triple by replacing $C_0(EH)_\theta$ by its completion under the $\delta$-topology if required.

## 8.3 Measurability in the nonunital case

The following is the measurability criterion of operators from a local nonunital spectral triple [11] as a generalisation of the criterion given by Connes in the unital case [5] (page 306), which we mentioned in the end of Chapter 3.

**Theorem 8.3.1** *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a regular, local $(p, \infty)$-summable spectral triple with $p \geq 1$. Suppose that $T \in \mathcal{B}(\mathcal{H})$ such that $\psi T = T \psi = T$ for some $\psi \geq 0$ in $\mathcal{A}_c$. If the limit*

$$\lim_{s \to \frac{p}{2}^+} \left(s - \frac{p}{2}\right) Trace\left(T(1 + \mathcal{D}^2)^{-s}\right) \tag{8.14}$$

*exists, then the operator $T(1 + \mathcal{D}^2)^{-p/2}$ is measurable and its Dixmier trace equals the limit up to a factor of $2/p$,*

$$Tr^+\left(T(1 + \mathcal{D}^2)^{-p/2}\right) = \frac{2}{p} \lim_{s \to \frac{p}{2}^+} \left(s - \frac{p}{2}\right) Trace\left(T(1 + \mathcal{D}^2)^{-s}\right). \tag{8.15}$$

Implied by [40], the operators $L_f^\theta(1 + \mathcal{D}^2)^{-2}$, for $f \in C_c^\infty(EH)_\theta$ from the spectral triple $(C_c^\infty(EH)_\theta, \mathcal{D}, \mathcal{H})$ satisfy the measurability criterion (8.14) and hence the Dixmier trace can be uniquely defined. We will show this in the following.

We have seen that the operator $L_f^\theta(1+\mathcal{D}^2)^{-2}$ for $f \in C_c^\infty(EH)_\theta$ arising from the spectral triple $(C_c^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ is in the Dixmier trace ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$. We consider the measurability of such an operator. Note that since $f$ is of compact support, we can always find a function $\phi$ of value one on the support of $f$ and decaying to zero only with respect to the $r$ variable so that $L_\phi^\theta = M_\phi$ and hence $L_f^\theta M_\phi = M_\phi L_f^\theta = L_f^\theta$ holds. By taking $p = 4$, Theorem 8.3.1 implies that the measurability of $L_f^\theta(1+\mathcal{D}^2)^{-2}$ is the same as the existence of the limit

$$\lim_{s \to 2^+} (s-2)\, Trace\left(L_f^\theta(1+\mathcal{D}^2)^{-s}\right). \tag{8.16}$$

Before finding the limit, we want to consider Schwartz kernels of operators involved. The operator $M_f$ can be represented as an integral operator on $\mathcal{H}$ as

$$M_f(\psi)(x) = \int_{EH} K_{M_f}(x, x')\psi(x')dVol(x'), \quad \forall \psi \in \mathcal{H}, \tag{8.17}$$

where the Schwartz kernel $K_{M_f} : U \times U \to \mathbb{C}$ and $U$ is the local coordinate chart around the point $x$.

**Lemma 8.3.2** *The Schwartz kernel of $M_f$ is*

$$K_{M_f}(x, x') = \sum_{r=(r_3, r_4)} M_{f_r}(x)\delta_x^g(x'), \tag{8.18}$$

*where $\delta_x^g(x')$ is defined by requiring that*

$$\psi(x) = \int_{EH} \delta_x^g(x')\psi(x')dVol(x'), \quad \forall \psi \in \mathcal{H}. \tag{8.19}$$

**Proof**: Applying (8.19), we have

$$M_f\psi(x) = M_f(x)\int_{EH} \delta_x^g(x')\psi(x')dVol(x') = \int_{EH} M_f(x)\delta_x^g(x')\psi(x')dVol(x')$$

Comparing with (8.17), we obtain (8.18). □

Using the property (8.9), the local $(4,\infty)$-summability of the spectral triple $(C_c^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ implies that

**Lemma 8.3.3** *The operators $L_f^\theta(1+\mathcal{D}^2)^{-s}$ is of trace class for $s > 2$. Similarly, the operators $M_f(1+\mathcal{D}^2)^{-s}$ is of trace class for $s > 2$.*

As Corollary 3.10 of [40], we have the following

**Lemma 8.3.4** *$Trace(L^\theta_f(1+\mathcal{D}^2)^{-s}) = Trace(M_f(1+\mathcal{D}^2)^{-s})$ for $s > 2$.*

**Proof**: We write $h(\mathcal{D}) := (1+\mathcal{D}^2)^{-s}$ for convenience.

$$\begin{aligned} Trace(L^\theta_f h(\mathcal{D})) &= \sum_r Trace((M_{f_r}V^\theta_r)h(\mathcal{D})) \\ &= \sum_r Trace(V_z M_{f_r} V_{-z} V^\theta_r h(\mathcal{D})), \quad \forall z \in \mathfrak{U}(1)\times\mathfrak{U}(1) \\ &= \sum_r Trace(M_{\alpha_z(f_r)} V^\theta_r h(\mathcal{D})) \\ &= \sum_r e(r\cdot z) Trace(M_{f_r} V^\theta_r h(\mathcal{D})), \end{aligned}$$

using $\alpha_z(f_r) = e(r\cdot z) f_r$. The identity

$$\sum_r Trace(M_{f_r}V^\theta_r h(\mathcal{D})) = \sum_r e(r\cdot z) Trace(M_{f_r}V^\theta_r h(\mathcal{D})), \quad \forall z \in \mathfrak{U}(1)\times\mathfrak{U}(1)$$

implies that $r=0$ in the summations and the sum takes the value $Trace(M_{f_0}V^\theta_0 h(\mathcal{D}))$, which is nothing but $Trace(M_f h(\mathcal{D}))$. Therefore, we obtain $Trace(L^\theta_f h(\mathcal{D})) = Trace(M_f h(\mathcal{D}))$. □

**Proposition 8.3.1** *The limit*

$$\lim_{s\to 2^+} (s-2)Trace(L^\theta_f(1+\mathcal{D}^2)^{-s}), \quad \forall f \in C^\infty_c(EH)_\theta$$

*exists and hence the operator $L^\theta_f(1+\mathcal{D}^2)^{-s}$ is measurable, whose Dixmier trace is*

$$Tr^+(L^\theta_f(1+\mathcal{D}^2)^{-2}) = \frac{2}{(2\pi)^2}\int_{EH} f(x)dVol(x) \tag{8.20}$$

**Proof**: We adopt the proof used in [40] and references therein. By Lemma 8.3.3 and Lemma 8.3.4, $Trace(L^\theta_f(1+\mathcal{D}^2)^{-s})$ exists and equals $Trace(M^\theta_f(1+\mathcal{D}^2)^{-s})$ for $s > 2$. Thus it suffices to show the existence of the limit,

$$\lim_{s\to 2^+}(s-2)Trace(M_f(1+\mathcal{D}^2)^{-s}), \quad \forall f \in C^\infty_c(EH). \tag{8.21}$$

We will show this by representing the Trace of the operator $M_f(1+\mathcal{D}^2)^{-s}$ by its kernel, which can further be written as the composition of the kernel of $M_f$ and that of $(1+\mathcal{D}^2)^{-s}$. The former is simple, we estimate the later by heat kernel methods [40]. For $s > 2$,

$$
\begin{aligned}
& Trace(M_f\,(1+\mathcal{D}^2)^{-s}) \\
= \;& \int_{EH} K_{M_f\,(1+\mathcal{D}^2)^{-s}}(x,x)dVol(x) \\
= \;& \int_{EH}\int_{EH} K_{M_f}(x,x')\,K_{(1+\mathcal{D}^2)^{-s}}(x',x)dVol(x')dVol(x) \\
\overset{(8.18)}{=} \;& \int_{EH} tr_{M_4}\left(\sum_{r\in\mathbb{Z}^2} diag(f_r(x))_4\right) K_{(1+\mathcal{D}^2)^{-s}}(x,x)dVol(x) \\
= \;& 4\int_{EH} f(x)\,K_{(1+\mathcal{D}^2)^{-s}}(x,x)dVol(x)
\end{aligned}
$$

We compute the kernel $K_{(1+\mathcal{D}^2)^{-s}}(x,x)$ by Laplacian transformation,

$$
K_{(1+\mathcal{D}^2)^{-s}}(x,x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t}K_t(x,x)dt, \tag{8.22}
$$

where $K_t(x,x')$ is the kernel of the heat operator $e^{-t\mathcal{D}^2}$. Here $K_t(x,x')$ is a smooth strictly positive function on $EH\times EH$. For any constant $0<\epsilon<1$, we may rewrite the integration (8.22) as

$$
\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t}K_t(x,x)dt = \frac{1}{\Gamma(s)}\left[\int_0^\epsilon t^{s-1}e^{-t}K_t(x,x)dt + \int_\epsilon^\infty t^{s-1}e^{-t}K_t(x,x)dt\right].
$$

We consider the two integrations separately. For the first integration, we use the asymptotic approximation of the heat kernel on the diagonal [42] (Lemma 4.14) as follows,

$$
K_t(x,x) = 2^2(4\pi\,t)^{-2} + \mathcal{O}(t^{-1}), \quad \text{as } t\to 0
$$

so that

$$
\begin{aligned}
\lim_{\epsilon\to 0^+}\int_0^\epsilon t^{s-1}e^{-t}K_t(x,x)dt &= \lim_{\epsilon\to 0^+}\int_0^\epsilon t^{s-1}e^{-t}(2^2(4\pi\,t)^{-2})dt \\
&= (2\pi)^{-2}\lim_{\epsilon\to 0^+}\int_0^\epsilon t^{s-2-1}e^{-t}dt.
\end{aligned}
$$

For the second integration, it is dominated by the factor $e^{-t}$, so that we may replace $K_t(x,x)$ by $2^2(4\pi\, t)^{-2}$. Therefore,

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\lim_{\epsilon\to 0^+}\left[\int_0^{\epsilon} t^{s-1}e^{-t}K_t(x,x)dt+\int_{\epsilon}^{\infty} t^{s-1}e^{-t}K_t(x,x)dt\right]\\
=\;&\frac{1}{\Gamma(s)}(2\pi)^{-2}\int_0^{\infty} t^{s-2-1}e^{-t}dt\\
=\;&\frac{1}{\Gamma(s)}(2\pi)^{-2}\,\Gamma(s-2)
\end{aligned}$$

Substitute back to (8.21),

$$\begin{aligned}
&\lim_{s\to 2^+}(s-2)Trace(M_f\,(1+\mathcal{D}^2)^{-s})\\
=\;&4\int_{EH} f(x)dVol(g)\,(2\pi)^{-2}\lim_{s\to 2^+}(s-2)\Gamma(s-2)\frac{1}{\Gamma(s)}\\
=\;&4\int_{EH} f(x)dVol(g)\,(2\pi)^{-2}\lim_{s\to 2^+}\Gamma(s-2+1)\frac{1}{\Gamma(s)}\\
=\;&\frac{4}{(2\pi)^2}\int_{EH} f(x)dVol(g)<\infty.
\end{aligned}$$

since the pole of the Gamma function at $s=2$ cancels with the zero $(s-2)$. Thus the limit $\lim_{s\to 2^+}(s-2)Trace(L_f^{\theta}\,(1+\mathcal{D}^2)^{-s})$ also exists. By applying Theorem 8.3.1, both of the operators $M_f\,(1+\mathcal{D}^2)^{-2}$ and $L_f^{\theta}\,(1+\mathcal{D}^2)^{-2}$ are measurable. Theorem 8.3.1 also implies also that (8.20). □

We end this part by computing the Wodzicki residue of the operator $M_f(1+\mathcal{D}^2)^{-2}$ and comparing it with its Dixmier trace (8.20).

The principal symbol of the classical pseudodifferential operator $(1+\mathcal{D}^2)^{-2}$ is

$$\sigma^{(1+\mathcal{D}^2)^{-2}}(x,\xi)=(g(\xi,\xi)))^{-2}\,\mathbf{1}_4,\quad \forall x\in EH. \tag{8.23}$$

We obtain the principal symbol of $M_f(1+\mathcal{D}^2)^{-2}$ by those of $M_f$ and $(1+\mathcal{D}^2)^{-2}$ by symbol calculus,

$$\sigma^{M_f(1+\mathcal{D}^2)^{-2}}(x,\xi)=\sigma^{M_f}(x,\xi)\,\sigma^{(1+\mathcal{D}^2)}(x,\xi)=f(x)g(\xi,\xi)^{-n/2}\mathbf{1}_4.$$

The *Wodzicki density* of $M_f\,(1+\mathcal{D}^2)^{-2}$ at any point $x\in EH$ is given by the integration over the cosphere tangent space $S_x^*(M)$ at $x$

$$\begin{aligned}
wres_x\left(M_f\,(1+\mathcal{D}^2)^{-2}\right)&=\int_{S_x^*(M)} tr_{M_4(\mathbb{C})}\left(\sigma^{M_f(1+\mathcal{D}^2)^{-2}}(x,\xi)\right)d\xi\,dx\\
&=4f(x)\int_{S_x^*(M)} g(\xi,\xi)^{-2}d\xi\,dx\\
&=4f(x)\Omega_4\,dVol(x)
\end{aligned}$$

where $\Omega_4 = 2(2\pi)^2$.

The *Wodzicki residue* of the operator $M_f\,(1+\mathcal{D}^2)^{-n/2}$ is given by the integration of the Wodzicki density

$$Wres\left(M_f\,(1+\mathcal{D}^2)^{-2}\right) = 8(2\pi)^2 \int_{EH} f(x)\, dVol(x), \quad f \in C_c^\infty(EH) \qquad (8.24)$$

Recall that the Connes trace theorem for the unital case (Theorem 7.18 [6]) implies that for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$,

$$Tr^+\left(a\,(1+\mathcal{D}^2)^{-p/2}\right) = \frac{1}{p(2\pi)^p}\, Wres(a\,(1+\mathcal{D}^2)^{-p/2}), \qquad (8.25)$$

where $\mathcal{D}$ is the Dirac operator of some $p$-dimensional spin manifold, $a\,(1+\mathcal{D}^2)^{-p/2}$ is considered as a elliptic pseudodifferential operator on the complex spinor bundle.

Despite a full understanding of (8.25) in the noncommutative nonunital case, the above Wodzicki residue computation (8.24) of $M_f^\theta(1+\mathcal{D}^2)^{1/2}$ for $f \in C_c^\infty(EH)$ compared with (8.20) satisfies the formula (8.25) for $a = f$ and $p = 4$. This also serves as an example of Proposition 15 in [11] where a geodesically complete manifold is considered.

# Chapter 9

# Geometric conditions

In this chapter, we see how the spectral triples of the isospectral deformations of the EH-spaces fit into the proposed geometric conditions of noncompact noncommutative spin manifolds [13], [12].

For a nonunital spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as in Definition 8.1.1, the geometric conditions (except the Poincaré duality) are as follows,

1. *(Metric dimension.) There is a unique non-negative integer p, the metric dimension, for which* $a\,(1+\mathcal{D}^2)^{-1/2}$ *belongs to the generalised Schatten ideal* $\mathcal{L}^{p,\infty}(\mathcal{H})$ *for* $a \in \mathcal{A}$. *Moreover,* $Tr^+(a\,(1+\mathcal{D}^2)^{-p/2})$ *is defined and not identically zero. This p is even if and only if the spectral triple is even.*

2. *(Regularity.) Bounded operators a and* $[\mathcal{D}, a]$, *for* $a \in \mathcal{A}$, *lie in the smooth domain of the derivation* $\delta\cdot = [|\mathcal{D}|, \cdot]$ on $\mathcal{L}(\mathcal{H})$.

3. *(Finiteness.) The algebra* $\mathcal{A}$ *and its preferred unitization* $\mathcal{A}_b$ *are pre-*$C^*$*-algebras. There exists an ideal* $\mathcal{A}_2$ *of* $\mathcal{A}_b$ *with the same* $C^*$*-completion as* $\mathcal{A}$, *such that the subspace of smooth vectors in* $\mathcal{H}$

$$\mathcal{H}_\infty := \cap_{m\in\mathbb{N}} dom(\mathcal{D}^m)$$

*is an* $\mathcal{A}_b$ *finitely generated projective* $\mathcal{A}_2$*-module.*

*Furthermore, the noncommutative integration defines a Hermitian form on the projective module* $\mathcal{H}_\infty$ *through the identity*

$$\langle \xi, \eta\, a\rangle = Tr^+((\xi, \eta)\, a\,(1+\mathcal{D}^2)^{-p/2}), \quad \forall \xi, \eta \in \mathcal{H}_\infty,\ a \in \mathcal{A}_2, \tag{9.1}$$

*where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}_\infty$ restricted from that of $\mathcal{H}$.*

We are using the convention that $\mathcal{H}_\infty$ is a right $\mathcal{A}_2$-module instead of the original version discussed in the introduction.

4. *(Reality.) There is an antiunitary operator $J$ on $\mathcal{H}$, such that*

$$[a, Jb^*J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}_b.$$

*Thus the map $b \mapsto Jb^*J^{-1}$ is a commuting representation on $\mathcal{H}$ of the opposite algebra $\mathcal{A}_b^\circ$. Moreover, for the metric dimension $p = 4$,*

$$J^2 = -1, \quad J\mathcal{D} = \mathcal{D}J, \quad J\chi = \chi J.$$

*For other dimensions, we refer to the table in chapter one.*

5. *(First order.) The bounded operator $[\mathcal{D}, a]$ commutes with the opposite algebra representation: $[[\mathcal{D}, a], Jb^*J^{-1}] = 0$ for all $a, b \in \mathcal{A}_b$.*

6. *(Orientability.) There is a Hochschild $p$-cycle $c$ on $\mathcal{A}_b$, with values in $\mathcal{A}_b \otimes \mathcal{A}_b^\circ$. The $p$-cycle is a finite sum of terms like $(a \otimes b^\circ) \otimes a_1 \otimes \cdots \otimes a_p$ and its natural representation $\pi_\mathcal{D}(c)$ on $\mathcal{H}$ is defined by*

$$\pi_\mathcal{D}((a_0 \otimes b_0^\circ) \otimes a_1 \otimes \cdots \otimes a_p) := a_0 J b_0^* J^{-1} [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k].$$

*The volume form $\pi_\mathcal{D}(c)$ solves the equation $\pi_\mathcal{D}(c) = \chi$ in the even case and $\pi_\mathcal{D}(c) = 1$ in the odd case.*

## 9.1 Metric dimensions

One might show $p = 4$ for the triples $(C_0^\infty(EH)_\theta, \mathcal{H}, \mathcal{D})$ by considering the measurability of the operator $L_f^\theta(1 + \mathcal{D}^2)^{-2}$ for $f \in C_0^\infty(EH)_\theta$. However, the algebra $C_0^\infty(EH)_\theta$ is not integrable, which is necessary for the computation of the Wodzicki residue [39] of the operator $L_f^\theta(1 + \mathcal{D}^2)^{-2}$. Thus $L_f^\theta(1 + \mathcal{D}^2)^{-2}$ may not be measurable. Nonetheless, Proposition 8.3.1 implies that operators $L_f^\theta(1 + \mathcal{D}^2)^{-2}$ for $f \in C_c^\infty(EH)_\theta$ are measurable. The Dixmier trace is evaluated as

$$Tr^+(L_f^\theta(1 + \mathcal{D}^2)^{-2}) = \frac{2}{(2\pi)^2} \int f \, dVol,$$

which is finite and nonzero. We do not know whether this remains true for some general integrable algebras, for instance $C_2^\infty(EH)_\theta$, lying between $C_c^\infty(EH)_\theta$ and $C_0^\infty(EH)_\theta$.

## 9.2 Regularity

The regularity condition is implied by Proposition 8.2.1.

## 9.3 Finiteness

By the construction of the ideal $C_2^\infty(EH)$ in Section 7.1, we see that the $C_b^\infty(EH)$ projective $C_2^\infty(EH)$-module $pC_2^\infty(EH)^8$, with $p$ as in (7.17), is the smooth domain of the Dirac operator in $\mathcal{H}$. In the deformed case, we recall that $pC_2^\infty(EH)_\theta^8$ is a $C_b^\infty(EH)_\theta$ projective $C_2^\infty(EH)_\theta$-module.

By matching generators, we have the isomorphism between the finitely generated projective modules $pC_2^\infty(EH)_\theta^8 \cong pC_2^\infty(EH)^8$. Therefore,

$$\mathcal{H}_\infty \cong pC_2^\infty(EH)_\theta^8.$$

The Fréchet algebra $C_2^\infty(EH)_\theta$ is of the same $C^*$-completion $C_0(EH)_\theta$ as that of the algebra $C_0^\infty(EH)_\theta$.

For the second part of the finiteness condition, we may obtain the Hermitian form on the projective module $\mathcal{H}_\infty$ by the standard one (4.2) as follows. Let $\xi = (\xi_1, \dots, \xi_8)^t \in \mathcal{H}_\infty$ where the coordinates satisfying $\xi_i = \sum_j p_{ij} \times_\theta \xi_j$ and similarly for $\eta = (\eta_1, \dots, \eta_8)^t \in \mathcal{H}_\infty$. The Hermitian inner product on $\mathcal{H}_\infty$ is defined by

$$(\xi, \eta) := \sum_i \xi_i^* \times_\theta \eta_i, \quad \xi, \eta \in \mathcal{H}_\infty. \tag{9.2}$$

By restriction, we obtain a Hermitian inner product in the form of (9.2) of projective module $pC_c^\infty(EH)_\theta^8$. As an application of a general construction considering smooth projective modules in [10], we may define a $\mathbb{C}$-valued inner product on the projective module. Since the Hermitian form on the projective module $pC_c^\infty(EH)_\theta^8$ is $C_c^\infty(EH)_\theta$-valued, composing with the Dixmier trace, one may define an inner product on $pC_c^\infty(EH)_\theta^8$ by

$$\tau(\xi,\eta\, a) := Tr^+\left(L^\theta_{(\xi|\eta)\times_\theta a}(1+\mathcal{D}^2)^{-2}\right), \quad \forall \xi,\eta \in pC_c^\infty(EH)_\theta^8,\ a \in C_2^\infty(EH)_\theta. \tag{9.3}$$

Lemma 8.3.4 implies that

$$Tr^+\left(L^\theta_{(\xi,\eta)\times_\theta a}(1+\mathcal{D}^2)^{-2}\right) = Tr^+\left(M_{(\xi,\eta)\times_\theta a}(1+\mathcal{D}^2)^{-2}\right),$$

where the element $(\xi,\eta)\times_\theta a = \sum_i \xi_i^* \times_\theta \eta_i \times_\theta a \in C_c^\infty(EH)_\theta$ on the right hand side is considered as a function in $C_c^\infty(EH)$. The identity (8.20) implies that

$$Tr^+\left(M_{(\xi,\eta)\times_\theta a}(1+\mathcal{D}^2)^{-2}\right) = \frac{2}{(2\pi)^2}\int (\xi,\eta)\times_\theta a\, dVol,$$

where we recognise that the right hand side is in the form of the usual inner product on the Hilbert space $\mathcal{H}$. In particular for the commutative case, the completion of $pC_c^\infty(EH)^8$ under the inner product $\tau$ gives us exactly the usual Hilbert space $\mathcal{H} = L^2(EH,\mathcal{S})$.

Notice that the inner product (9.3) can be defined because of the measurability of the operator $L_f^\theta(1+\mathcal{D}^2)^{-2}$ for $f \in C_c^\infty(EH)_\theta$ as we have seen in Proposition 8.3.1. If the above construction holds when the module $pC_c^\infty(EH)_\theta^8$ is replaced by $pC_2^\infty(EH)_\theta^8$, then the finiteness condition will hold completely. The validity of the generalisation is decided by the measurability of the operator $L_f^\theta(1+\mathcal{D}^2)^{-2}$ for $f \in C_2^\infty(EH)_\theta$, which we do not know yet. This is the same problem that arises in considering the condition of metric dimensions.

## 9.4 Reality

The proof of the reality condition is based on the lecture notes [44]. With respect to the decomposition of spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ as in Section 6.3, we have the corresponding Hilbert space completions under the inner product coming from the $L^2$-norms, and their sum is the Hilbert space completion of $\mathcal{S}$, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Any element $\psi \in \mathcal{H}$ can thus be decomposed as $\psi = (\psi^+,\psi^-)^t$. The operator $J$ defined on the spinor bundle (6.16) can be extended to the Hilbert space as an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ by

$$J\begin{pmatrix}\psi^+\\ \psi^-\end{pmatrix} := \begin{pmatrix}-\overline{\psi}^-\\ \overline{\psi}^+\end{pmatrix},$$

satisfying $J^2 = -1$.

We define the representation of the opposite algebra $\mathcal{A}_b^\circ$ of $\mathcal{A}_b = C_b^\infty(EH)_\theta$ on $\mathcal{H}$, $R^\theta_\bullet : \mathcal{A}_b^\circ \to \mathcal{B}(\mathcal{H})$ by $R^\theta_h := J\, L^{\theta\,*}_h J^{-1}$. Specifically, for $h = \sum_s h_s$, the representation is

$$R^\theta_h = \sum_s J M_{h_s^*}\, V^\theta_{-s} J^{-1} = \sum_s M_{h_s}\, V^\theta_{-s}.$$

The commutativity of operators $L^\theta_f$ and $R^\theta_h$ where $f = \sum_r f_r$ is seen as follows,

$$\begin{aligned}
[L^\theta_f, R^\theta_h] &= \sum_{r,s} f_r\, V^\theta_r\, h_s\, V^\theta_{-s} - h_s\, V^\theta_{-s}\, f_r\, V^\theta_r \\
&= \sum_{r,s} f_r\, h_s \sigma(r,s)\, V^\theta_r\, V^\theta_{-s} - h_s\; f_r \sigma(-s,r) V^\theta_{-s}\, V^\theta_r \\
&= \sum_{r,s} [f_r, h_s] \sigma(r,s)\, V^\theta_{r-s} = 0, 
\end{aligned} \tag{9.4}$$

where identities $\sigma(r,s) = \sigma(-s,r)$ and $V^\theta_r V^\theta_{-s} = V^\theta_{-s} V^\theta_r = V^\theta_{r-s}$ are applied.

As in the commutative case, $\mathcal{D}\, J = J\, \mathcal{D}$ and $J\, \chi = \chi\, J$ where $\chi$ is the chirality operator (6.15).

## 9.5 First order

The proof of the first order condition is again from [44]. For any $f = \sum_r f_r$ and $h = \sum_s h_s$ in $C_b^\infty(EH)_\theta$, the first order property $[[\mathcal{D}, f_r], h_s] = 0$ in the commutative case implies that,

$$[[\mathcal{D}, L^\theta_f], R^\theta_h] = \sum_{r,s} [[\mathcal{D}, f_r]\, V^\theta_r, h_s\, V^\theta_{-s}] = \sum_{r,s} [[\mathcal{D}, f_r], h_s] \sigma(r,s)\, V^\theta_{r-s} = 0.$$

## 9.6 Orientability

In Riemannian geometry, the volume form determines the orientation of a manifold. Translated to the spectral triple language, the volume form is replaced by a Hochschild cycle $c$ which can be represented on $\mathcal{H}$ such that $\pi_D(c) = \chi$ in the even case. For background on Hochschild homology we refer to Loday [45] and for a discussion on the orientability condition we refer to [6].

We may obtain a Hochschild 4-cycle of the spectral triple from the classical volume form of the Eguchi-Hanson space. We will only give the construction on the coordinate chart $U_N$, that for the other chart $U_S$ is similar and the global construction can be obtained by a partition of unity. We will consider the commutative case first and then the deformed case.

Define a new set of coordinates by $u_1 = x_1$, $u_2 = x_2$, $u_3 = e^{ix_3}$, $u_4 = e^{ix_4}$, so that the transition of differential forms $dx^i = v^i_j du^j$ is given by the diagonal matrix $V = (v^i_j) := diag(1, 1, -\frac{i}{u_3}, -\frac{i}{u_4})$. Composing with the $\vartheta^\alpha = h^\alpha_i dx^i$ where $h^\alpha_i$ are components of the matrix $H$ in (6.5), the transition of differential forms $\vartheta^\alpha = k^\alpha_i du^i$ is given by the matrix $K = (k^\alpha_i) := HV$. In components,

$$k^\alpha_1 = h^\alpha_1, \quad k^\alpha_2 = h^\alpha_2, \quad k^\alpha_3 = h^\alpha_3 \frac{-i}{u_3}, \quad k^\alpha_4 = h^\alpha_4 \frac{-i}{u_4}, \quad \alpha = 1, \cdots, 4. \tag{9.5}$$

Similarly, the transition $du^j = \tilde{v}^j_i dx^i$ is given by the inverse matrix $V^{-1} = (\tilde{v}^j_i)$ of $V$. Composing with $dx^j = \tilde{h}^j_\beta \vartheta^\beta$ where $\tilde{h}^j_\beta$ are elements of the inverse matrix $H^{-1}$ in (6.6) , we obtain $du^i = \tilde{k}^i_\beta \vartheta^\beta$ with $\tilde{k}^i_\beta$ as the elements of the inverse matrix $K^{-1} = V^{-1}H^{-1}$. In components,

$$\tilde{k}^1_\beta = \tilde{h}^1_\beta, \quad \tilde{k}^2_\beta = \tilde{h}^2_\beta, \quad \tilde{k}^3_\beta = i\, u_3\, \tilde{h}^3_\beta, \quad \tilde{k}^4_\beta = i\, u_4\, \tilde{h}^4_\beta, \quad \beta = 1, \cdots, 4.$$

To avoid ambiguity, if the $u$-coordinates and $x$-coordinates appear in the same formula, we will distinguish them by adding $'$ to indices of the $u$-coordinates. By tensor transformations, we may obtain the Dirac operator satisfying $\mathcal{D}(s) = -i\gamma^{j'}\nabla^S_{j'} s$ in the coordinates $\{u'_i\}$'s from (6.20) in the coordinates $\{x_i\}$'s as,

$$\begin{aligned} \mathcal{D} \;=\; & -i\,\tilde{h}^{1'}_\eta\,\gamma^\eta \left(\partial_{1'} - \frac{1}{4}\widetilde{\Gamma}^\beta_{1\alpha}\gamma^\alpha\gamma_\beta\right) - i\,\tilde{h}^{2'}_\eta\,\gamma^\eta \left(\partial_{2'} - \frac{1}{4}\widetilde{\Gamma}^\beta_{2\alpha}\gamma^\alpha\gamma_\beta\right) \\ & + u_{3'}\,\tilde{h}^{3'}_\eta\,\gamma^\eta \left(\partial_{3'} + \frac{1}{4}\frac{i}{u_{3'}}\,\widetilde{\Gamma}^\beta_{3\alpha}\gamma^\alpha\gamma_\beta\right) + u_{4'}\,\tilde{h}^{4'}_\eta\,\gamma^\eta \left(\partial_{4'} + \frac{1}{4}\frac{i}{u_{4'}}\,\widetilde{\Gamma}^\beta_{4\alpha}\gamma^\alpha\gamma_\beta\right), \end{aligned}$$

where $\widetilde{\Gamma}^\beta_{i\alpha}$'s are from (6.18) and $\gamma_\alpha = \gamma^\alpha$'s are from (6.11).

The volume form of the Eguchi-Hanson space can be represented in the orthonormal basis on $U_N$ as

$$\begin{aligned} \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \wedge \vartheta^4 \;&=\; k^1_{i_1} du^{i_1} \wedge k^2_{i_2} du^{i_2} \wedge k^3_{i_3} du^{i_3} \wedge k^4_{i_4} du^{i_4} \\ &=\; k^4_{i_4}\, k^3_{i_3}\, k^2_{i_2}\, k^1_{i_1}\, du^{i_1} \wedge du^{i_2} \wedge du^{i_3} \wedge du^{i_4}. \end{aligned} \tag{9.6}$$

We may define a Hochschild 4-chain $c_0$ in $C_4(\mathcal{A}_b, \mathcal{A}_b \otimes \mathcal{A}^\circ_b)$, with $\mathcal{A}_b = C^\infty_b(EH)$ and

$\mathcal{A}_b^\circ$ as the opposite algebra of $\mathcal{A}_b$, by

$$\begin{aligned} c_0 &:= \frac{1}{4!}\sum_{\sigma\in S_4}(-1)^\sigma(k^{\sigma(4)}_{i_{\sigma(4)}}\otimes 1^\circ)(k^{\sigma(3)}_{i_{\sigma(3)}}\otimes 1^\circ)(k^{\sigma(2)}_{i_{\sigma(2)}}\otimes 1^\circ)(k^{\sigma(1)}_{i_{\sigma(1)}}\otimes 1^\circ) \\ &\qquad \otimes u^{i_{\sigma(1)}}\otimes u^{i_{\sigma(2)}}\otimes u^{i_{\sigma(3)}}\otimes u^{i_{\sigma(4)}}, \end{aligned} \tag{9.7}$$

where $\sigma$ is an element in the permutation group $S_4$ and $(-1)^\sigma$ indicates the sign of the permutation. Thus defined Hochschild 4-chain, which we will see below actually a Hochschild cycle, is antisymmetrised. It is however considered as an open problem whether antisymmetry of the cycle is required in the orientability condition concerning the reconstruction of manifolds as in [9]. On the $\mathcal{A}_b$-bimodule $\mathcal{A}_b\otimes\mathcal{A}_b^\circ$, $\mathcal{A}_b$ acts as $a'(a\otimes b^0)a'' := a'aa''\otimes b^\circ$, for $a\otimes b^\circ\in\mathcal{A}_b\otimes\mathcal{A}_b^\circ$ and $a', a''\in\mathcal{A}_b$.

**Lemma 9.6.1** *The Hochschild* 4*-chain (9.7) defines a Hochschild cycle. That is,* $b(c_0)=0$, *where $b$ is the boundary operator of a Hochschild chain.*

**Proof**: Recall that the Hochschild boundary operator $b$ acts on a simple $n$-chain $a=(a_0\otimes b_0^\circ)\otimes a_1\otimes\cdots\otimes a_n$ in $C_n(\mathcal{A}_b,\mathcal{A}_b\otimes\mathcal{A}_b^\circ)$ by

$$\begin{aligned} b(a) &= (a_0\otimes b_0^\circ)a_1\otimes a_2\otimes\cdots\otimes a_n \\ &\quad +\sum_{j=1}^{n-1}(-1)^j(a_0\otimes b_0^\circ)\otimes a_1\otimes\cdots\otimes a_j\,a_{j+1}\otimes\cdots\otimes a_n \\ &\quad +(-1)^n a_n(a_0\otimes b_0^\circ)\otimes a_1\otimes\cdots\otimes a_{n-1}. \end{aligned} \tag{9.8}$$

Elements of $b(c_0)$ are of three types.

The first type corresponds to the second line in (9.8),

$$\begin{aligned} &(-1)^\sigma(-1)^j(k^{\sigma(4)}_{i_{\sigma(4)}}\otimes 1^\circ)(k^{\sigma(3)}_{i_{\sigma(3)}}\otimes 1^\circ)(k^{\sigma(2)}_{i_{\sigma(2)}}\otimes 1^\circ)(k^{\sigma(1)}_{i_{\sigma(1)}}\otimes 1^\circ) \\ &\otimes u^{i_{\sigma(1)}}\otimes\cdots\otimes u^{i_{\sigma(j)}}\,u^{i_{\sigma(j+1)}}\otimes\cdots\otimes u^{i_{\sigma(4)}}. \end{aligned}$$

In the summation of all $\sigma\in S_4$, each such term can be cancelled by a term from another $\sigma'$ which obtain from the composition of $\sigma$ by a transition between $\sigma(j)$ and $\sigma(j+1)$, as

$$\begin{aligned} &(-1)^{|\sigma'|}(-1)^j(k^{\sigma'(4)}_{i_{\sigma'(4)}}\otimes 1^\circ)(k^{\sigma'(3)}_{i_{\sigma'(3)}}\otimes 1^\circ)(k^{\sigma'(2)}_{i_{\sigma'(2)}}\otimes 1^\circ)(k^{\sigma'(1)}_{i_{\sigma'(1)}}\otimes 1^\circ) \\ &\otimes u^{i_{\sigma(1)}}\otimes\cdots\otimes u^{i_{\sigma(j+1)}}\,u^{i_{\sigma(j)}}\otimes\cdots\otimes u^{i_{\sigma(4)}}. \end{aligned}$$

Indeed, since $(-1)^\sigma = -(-1)^{|\sigma'|}$ and the elements in the first term from the bimodule are commuting, the summation of such pairs is

$$(-1)^\sigma(-1)^j(k^{\sigma(4)}_{i_{\sigma(4)}} \otimes 1^\circ)(k^{\sigma(3)}_{i_{\sigma(3)}} \otimes 1^\circ)(k^{\sigma(2)}_{i_{\sigma(2)}} \otimes 1^\circ)(k^{\sigma(1)}_{i_{\sigma(1)}} \otimes 1^\circ)$$
$$\otimes u^{i_{\sigma(1)}} \otimes \cdots \otimes \left(u^{i_{\sigma(j)}}\, u^{i_{\sigma(j+1)}} - u^{i_{\sigma(j+1)}}\, u^{i_{\sigma(j)}}\right) \otimes \cdots \otimes u^{i_{\sigma(4)}} = 0.$$

It vanishes since $u_{i_{\sigma(j)}}\, u_{i_{\sigma(j+1)}} = u_{i_{\sigma(j+1)}}\, u_{i_{\sigma(j)}}$ as elements in $\mathcal{A}_b$.

The second type corresponds to the first line in (9.8). After the $\mathcal{A}_b$-bimodule action from the right, it is in the following form,

$$\left(\left(k^{\sigma(4)}_{i_{\sigma(4)}} k^{\sigma(3)}_{i_{\sigma(3)}} k^{\sigma(2)}_{i_{\sigma(2)}} k^{\sigma(1)}_{i_{\sigma(1)}} u^{i_{\sigma(1)}}\right) \otimes 1^\circ\right) \otimes u^{i_{\sigma(2)}} \otimes u^{i_{\sigma(3)}} \otimes u^{i_{\sigma(4)}}.$$

The third type of component corresponds to the third line in (9.8). After the $\mathcal{A}_b$-bimodule action from the left, it is in the following form,

$$\left(\left(u^{i_{\sigma'(4)}} k^{\sigma'(4)}_{i_{\sigma'(1)}} k^{\sigma'(3)}_{i_{\sigma'(3)}} k^{\sigma'(2)}_{i_{\sigma'(2)}} k^{\sigma'(1)}_{i_{\sigma'(1)}}\right) \otimes 1^\circ\right) \otimes u^{i_{\sigma'(1)}} \otimes u^{i_{\sigma'(2)}} \otimes u^{i_{\sigma'(3)}}.$$

By commutativity of $\mathcal{A}_b$, the summation of all $\sigma$ of the second type and the third type cancel exactly when the permutation $\sigma'$ differs from $\sigma$ by a transition between $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ to $(\sigma(4), \sigma(1), \sigma(2), \sigma(3))$. Indeed, such $\sigma$ and $\sigma'$ are of opposite sign. Therefore, all three types cancel in the summation of $\sigma \in S_4$, and $b(c_0) = 0$. This shows that $c_0$ is a Hochschild 4-cycle. □

We define the representation $\pi_{\mathcal{D}}$ of the Hochschild cycle $c_0$ on the Hilbert space by $\pi_{\mathcal{D}}(a_0 \otimes b_0^\circ \otimes a_1 \otimes \cdots \otimes a_4) := M_{a_0}M_{b_0}[\mathcal{D}, M_{a_1}][\mathcal{D}, M_{a_2}][\mathcal{D}, M_{a_3}][\mathcal{D}, M_{a_4}]$.

**Proposition 9.6.1** *The operator $\pi_{\mathcal{D}}(c^0) = \chi$.*

**Proof**:

$$\begin{aligned}
4!\,\pi_{\mathcal{D}}(c_0) &= \sum_{\sigma \in S_4} (-1)^\sigma M_{k^{\sigma(4)}_{i_{\sigma(4)}}} M_{k^{\sigma(3)}_{i_{\sigma(3)}}}) M_{k^{\sigma(2)}_{i_{\sigma(2)}}} M_{k^{\sigma(1)}_{i_{\sigma(1)}}} \\
&\qquad c(du^{i_{\sigma(1)}})\, c(du^{i_{\sigma(2)}})\, c(du^{i_{\sigma(3)}}) c(du^{i_{\sigma(4)}}) \\
&= \sum_{\sigma \in S_4} (-1)^\sigma M_{k^{\sigma(4)}_{i_{\sigma(4)}}} M_{k^{\sigma(3)}_{i_{\sigma(3)}}}) M_{k^{\sigma(2)}_{i_{\sigma(2)}}} M_{k^{\sigma(1)}_{i_{\sigma(1)}}} \\
&\qquad \tilde{k}^{i_{\sigma(1)}}_{\alpha_1} \gamma^{\alpha_1}\, \tilde{k}^{i_{\sigma(2)}}_{\alpha_2} \gamma^{\alpha_2}\, \tilde{k}^{i_{\sigma(3)}}_{\alpha_3} \gamma^{\alpha_3}\, \tilde{k}^{i_{\sigma(4)}}_{\alpha_4} \gamma^{\alpha_4} \\
&= \sum_{\sigma \in S_4} (-1)^\sigma \delta^{\sigma(1)}_{\alpha_1} \delta^{\sigma(2)}_{\alpha_2} \delta^{\sigma(3)}_{\alpha_3} \delta^{\sigma(4)}_{\alpha_4} \gamma^{\alpha_1} \gamma^{\alpha_2} \gamma^{\alpha_3} \gamma^{\alpha_4} \\
&= \sum_{\sigma \in S_4} (-1)^\sigma \gamma^{\sigma(1)} \gamma^{\sigma(2)} \gamma^{\sigma(3)} \gamma^{\sigma(4)} = 4!\, \gamma^1 \gamma^2 \gamma^3 \gamma^4.
\end{aligned}$$

Thus $\pi_{\mathcal{D}}(c_0) = \chi$. □

Now we consider the noncommutative case. Let $\mathcal{A}_{b,\theta}$ be $C_b^\infty(EH)_\theta$ and $\mathcal{A}_{b,\theta}^\circ$ be the opposite algebra. On the $\mathcal{A}_{b,\theta}$-bimodule $\mathcal{A}_{b,\theta} \otimes \mathcal{A}_{b,\theta}^\circ$, $\mathcal{A}_{b,\theta}$ acts as $a'(a \otimes b^0)a'' := (a' \times_\theta a \times_\theta a'') \otimes b^\circ$, for $a \otimes b^\circ \in \mathcal{A}_{b,\theta} \otimes \mathcal{A}_{b,\theta}^\circ$ and $a', a'' \in \mathcal{A}_{b,\theta}$.

The antisymmetric Hochschild 4-chain in $C_4(\mathcal{A}_{b,\theta}, \mathcal{A}_{b,\theta} \otimes \mathcal{A}_{b,\theta}^\circ)$ is defined by

$$c := \frac{1}{4!} \sum_{\sigma \in S_4} (-1)^\sigma K_{i_{\sigma(4)}}^{\sigma(4)} K_{i_{\sigma(3)}}^{\sigma(3)} K_{i_{\sigma(2)}}^{\sigma(2)} K_{i_{\sigma(1)}}^{\sigma(1)} \otimes u^{i_{\sigma(1)}} \otimes u^{i_{\sigma(2)}} \otimes u^{i_{\sigma(3)}} \otimes u^{i_{\sigma(4)}}, \tag{9.9}$$

where $K_i^j$ is the corresponding element of $k_i^j$ in the bimodule $\mathcal{A}_{b,\theta} \otimes \mathcal{A}_{b,\theta}^\circ$. They are chosen as,

$$K_1^4 := \Delta(u_1)^{-1/2} \otimes 1^\circ, \quad K_2^1 := -\left(\frac{u_1}{2} \otimes 1^\circ\right) \varkappa(u_3), \quad K_2^2 := \left(\frac{u_1}{2} \otimes 1^\circ\right) \varrho(u_3),$$

$$K_3^1 := \left(-\frac{u_1}{2} \sin u_2 \otimes 1^\circ\right) \varrho(u_3) \left(\frac{-i}{u_3} \otimes 1^\circ\right),$$

$$K_3^2 := \left(-\frac{u_1}{2} \sin u_2 \otimes 1^\circ\right) \varkappa(u_3) \left(\frac{-i}{u_3} \otimes 1^\circ\right),$$

$$K_3^3 := \left(\frac{u_1}{2} \Delta(u_1)^{1/2} \cos u_2 \otimes 1^\circ\right) \left(\frac{-i}{u_3} \otimes 1^\circ\right),$$

$$K_4^3 := \left(\frac{u_1}{2} \Delta(u_1)^{1/2} \otimes 1^\circ\right) \left(\frac{-i}{u_4} \otimes 1^\circ\right),$$

where

$$\Delta(u_1) := 1 - a^4/u_1^2, \quad \varkappa(u_3) := \frac{1}{2}\left((u_3^{1/2} \otimes (u_3^{1/2})^\circ + \bar{u}_3^{1/2} \otimes (\bar{u}_3^{1/2})^\circ)\right),$$

$$\varrho(u_3) := \frac{1}{2i}\left((u_3^{1/2} \otimes (u_3^{1/2})^\circ - \bar{u}_3^{1/2} \otimes (\bar{u}_3^{1/2})^\circ)\right).$$

**Remark**: The choices of $K_i^j$'s are based on the following observation. If $e^{ir\phi} \in \mathcal{A}_{b,\theta}$ is of spectral homogeneous degree $-r$, then $e^{i\frac{r}{2}\phi} \otimes (e^{i\frac{r}{2}\phi})^\circ$ as an element in the $\mathcal{A}_{b,\theta}$-bimodule is of the bimodule action satisfying

$$e^{is\psi}(e^{i\frac{r}{2}\phi} \otimes (e^{i\frac{r}{2}\phi})^\circ) = (e^{i\frac{r}{2}\phi} \otimes (e^{i\frac{r}{2}\phi})^\circ)e^{is\psi},$$

for any $e^{is\psi}$ of homogeneous degree $-s$ in the algebra $\mathcal{A}_{b,\theta}$. The same holds when $\phi$ and $\psi$ swap. In this way, all the $u_3$ appearing in the matrix $H$ of $K = HV$ can be "commutatised".

**Lemma 9.6.2** *The Hochschild* 4*-chain (9.9) defines a Hochschild cycle in* $Z_4(\mathcal{A}_{b,\theta}, \mathcal{A}_{b,\theta} \otimes \mathcal{A}^{\circ}_{b,\theta})$. *I.e.,* $b(c) = 0$, *where* $b$ *is the boundary operator of a Hochschild chain.*

**Proof**: As in the commutative case, elements of $b(c)$ are of three types. The first type is,

$$(-1)^{\sigma}(-1)^{j}(K^{\sigma(4)}_{i_{\sigma(4)}} \times_\theta K^{\sigma(3)}_{i_{\sigma(3)}} \times_\theta K^{\sigma(2)}_{i_{\sigma(2)}} \times_\theta K^{\sigma(1)}_{i_{\sigma(1)}} \otimes u^{i_{\sigma(1)}} \otimes \cdots \otimes u^{i_{\sigma(j)}} \times_\theta u^{i_{\sigma(j+1)}} \otimes \cdots \otimes u^{i_{\sigma(4)}}. \tag{9.10}$$

Firstly, from Remark 9.6, we may observe that the noncommutative part of any $K^j_i$ has only contributions from terms like $\frac{i}{u_i} \times_\theta \cdot$, for $i = 3, 4$. Secondly, any term containing the product $\frac{-i}{u_3} \times_\theta \frac{-i}{u_4}$ contains the product $u_4 \times_\theta u_3$ and their product is,

$$\frac{-i}{u_3} \times_\theta \frac{-i}{u_4} \times_\theta u_4 \times_\theta u_3 = e^{-i\theta}\,\frac{-i}{u_3}\frac{-i}{u_4} e^{i\theta}\,u_4 u_3 = -1.$$

This also holds when 3 and 4 swap. These observations imply that the noncommutativity factor coming from the first line of (9.10) always cancels with the noncommutativity factor coming from the second line. Therefore, it reduces to the commutative case. By the same matching of $\sigma$'s in the proof Lemma 9.6.1 for terms of the first type, summation of all the terms of first type is zero.

The second type is

$$\left(K^{\sigma(4)}_{i_{\sigma(4)}} \times_\theta K^{\sigma(3)}_{i_{\sigma(3)}} \times_\theta K^{\sigma(2)}_{i_{\sigma(2)}} \times_\theta K^{\sigma(1)}_{i_{\sigma(1)}}\right) u^{i_{\sigma(1)}} \otimes u^{i_{\sigma(2)}} \otimes u^{i_{\sigma(3)}} \otimes u^{i_{\sigma(4)}}.$$

Notice that $K^{\sigma(1)}_{i_{\sigma(1)}}$ commutes with $u_{i_{\sigma(1)}}$. The third type is

$$u_{i_{\sigma'(4)}}\left(K^{\sigma'(4)}_{i_{\sigma'(4)}} \times_\theta K^{\sigma'(3)}_{i_{\sigma'(3)}} \times_\theta K^{\sigma'(2)}_{i_{\sigma'(2)}} \times_\theta K^{\sigma'(1)}_{i_{\sigma'(1)}}\right) \otimes u^{i_{\sigma'(1)}} \otimes u^{i_{\sigma'(2)}} \otimes u^{i_{\sigma'(3)}}.$$

Notice that $u_{i_{\sigma'(4)}}$ commutes with $K^{\sigma'(4)}_{i_{\sigma'(4)}}$. As in the commutative case, we may pair $\sigma$ and $\sigma'$ which are related by $\sigma'(1) = \sigma(4), \sigma'(2) = \sigma(1), \sigma'(3) = \sigma(2), \sigma'(4) = \sigma(3)$ so that they are canceled through the summation of $\sigma$. Three cases altogether give us $b(c) = 0$, and hence the proof. □

We represent the Hochschild cycle $c$ on the Hilbert space $\mathcal{H}$ by

$$\pi_{\mathcal{D}}(a_0 \otimes b_0^{\circ} \otimes a_1 \otimes \cdots \otimes a_4) := L^\theta_{a_0} R^\theta_{b_0}[\mathcal{D}, L^\theta_{a_1}][\mathcal{D}, L^\theta_{a_2}][\mathcal{D}, L^\theta_{a_3}][\mathcal{D}, L^\theta_{a_4}],$$

for $a_0 \otimes b_0^{\circ} \otimes a_1 \otimes \cdots \otimes a_4 \in Z_4(\mathcal{A}_{b,\theta}, \mathcal{A}_{b,\theta} \otimes \mathcal{A}^{\circ}_{b,\theta})$. A straightforward fact follows,

**Lemma 9.6.3** $\pi_{\mathcal{D}}(\varkappa(u_3)) = M_{\cos\phi}$ *and* $\pi_{\mathcal{D}}(\varrho(u_3)) = M_{\sin\phi}$.

**Proposition 9.6.2** *The operator* $\pi_{\mathcal{D}}(c) = \chi$.

**Proof**: By using the commutativity between the Dirac operator and $V_r^\theta$, we can write down the formula for the commutators:

$$[\mathcal{D}, L^\theta_{u_i}] = c(du^i), \quad [\mathcal{D}, L^\theta_{u_3}] = c(du_3)V^\theta_{(-1,0)}, \quad [\mathcal{D}, L^\theta_{u_4}] = c(du_4)V^\theta_{(0,-1)}$$

where $i = 1, 2$. By Lemma 9.6.3, all the nonvanishing representation of coefficients in the bimodule of the Hochschild cycle $c$ are

$$\pi_{\mathcal{D}}(K_1^4) = M_{\Delta(u_1)^{-1/2}}, \quad \pi_{\mathcal{D}}(K_2^1) = -M_{\frac{u_1}{2}}M_{\cos\phi}, \quad \pi_{\mathcal{D}}(K_2^2) = M_{\frac{u_1}{2}}M_{\sin\phi}$$

$$\pi_{\mathcal{D}}(K_3^1) = -M_{\frac{u_1}{2}\sin u_2}M_{\sin\phi}L^\theta_{\frac{-i}{u_3}}, \quad \pi_{\mathcal{D}}(K_3^3) = M_{\frac{u_1}{2}\Delta(u_1)^{1/2}\cos u_2}L^\theta_{\frac{-i}{u_3}}$$

$$\pi_{\mathcal{D}}(K_3^4) = -M_{\frac{u_1}{2}\sin u_2}M_{\cos\phi}L^\theta_{\frac{-i}{u_3}}, \quad \pi_{\mathcal{D}}(K_4^3) = M_{\frac{u_1}{2}\Delta(u_1)^{1/2}}L^\theta_{\frac{-i}{u_4}}.$$

The representation $\pi_{\mathcal{D}}(c)$ is thus

$$\begin{aligned}\pi_{\mathcal{D}}(c) \;=\; & \frac{1}{4!}\sum_{\sigma\in S_4}(-1)^\sigma \pi_{\mathcal{D}}(K^{\sigma(4)}_{i_{\sigma(4)}})\,\pi_{\mathcal{D}}(K^{\sigma(3)}_{i_{\sigma(3)}})\,\pi_{\mathcal{D}}(K^{\sigma(2)}_{i_{\sigma(2)}})\,\pi_{\mathcal{D}}(K^{\sigma(1)}_{i_{\sigma(1)}}) \\ & c(du^{i_{\sigma(1)}})V^\theta_{i_{\sigma(1)}}\,c(du^{i_{\sigma(2)}})V^\theta_{i_{\sigma(2)}}\,c(du^{i_{\sigma(3)}})V^\theta_{i_{\sigma(3)}}\,c(du^{i_{\sigma(4)}})V^\theta_{i_{\sigma(4)}},\end{aligned}$$

where $V^\theta_{i_{\sigma(k)}} := V^\theta_{deg(u_{i_{\sigma(k)}})}$. For any fixed component in the summation we may compare the expression of $\pi_{\mathcal{D}}(K_i^j)$ and $[\mathcal{D}, L^\theta_{u_k}]$. The result is that whenever there is a noncommutative factor generated by some $\pi_{\mathcal{D}}(K_i^j)$ as $V^\theta_{deg(1/u_i)}$ there is a corresponding noncommutative factor generated by $[\mathcal{D}, L^\theta_{u_i}]$ as $V^\theta_{deg(u_i)}$. Furthermore, these paired noncommutative factors cancel consistently. Thus, each component in the summation is simply the same as that in the commutative case. Applying Proposition 9.6.1, the summation gives $\chi$ again and this completes the proof of the orientability condition, $\pi_{\mathcal{D}}(c) = \chi$. □

# Chapter 10

# Conclusion

We have obtained the nonunital spectral triples of the isospectral deformations of the Eguchi-Hanson spaces along torus isometric actions and studied analytical properties of the triple. We have also tested the proposed geometric conditions of a noncompact noncommutative geometry on this example.

There are possible generalisations in the following directions. Firstly, we may further consider the Poincaré duality of nonunital spectral triples [46]. Secondly, we may take the conical singularity limit of EH-spaces and consider the spectral triple of the conifold. Thirdly, we may realise the spectral triple as a complex noncommutative geometry defined by [47]. Finally, we may deform the EH-spaces, and possibly for more general ALE-spaces, by using the hyper-Kähler quotient structures.

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