The Modal Logic of the Calculus of Probability

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Abstract

A new class of models which does not require an additional operator on the Boolean algebra of formulas to capture some notion of necessity, possibility, and impossibility is introduced and investigated. These models permit to combine efficiently logic and probability via a bridge principle. In particular, a second intension to the mode of impossibility resolves, canonically, the problem of the existence of almost identical events, which are usually considered equivalent. In fact, these events reveal a critical aspect of uncertainty; their systematic study in relation to uncertainty modelling is the domain of formal agnoiology.

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Declaration

This thesis is based on research carried out at the University of Durham in candidacy for the degree of Doctor of Philosophy. No part of it has been submitted elsewhere for any other degree or qualification, and it is my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

1.1 Motivation

This study presents a tentative solution to the problem of combining efficiently logic and probability. The adverb 'efficiently' is to be taken seriously. For logic and probability need not merely be combined; they need to be combined in a way which is primarily satisfactory to the probabilist, that is to say, the mathematician. For if the logician, especially, the philosophical logician, is always keen to combine logic and probability, few mathematicians venture into the foundations of probability beyond those established by Kolmogorov, the general consensus being that a revamping of those foundations is, to paraphrase Tarski, neither necessary nor desirable.

Accordingly, any viable attempt to combine logic and probability must leave the axiomatic treatment of the calculus of probability formulated by Kolmogorov [16, 17] unscathed; in particular, any viable attempt to combine logic and probability must be carried out using the tools and methods of measure theory. Only if these requirements are met will a mathematician be interested in further investigations into the foundations of probability. Of course, if, in passing, a problem relating to those foundations is resolved, or the importance of a neglected chapter of the calculus of probability is revealed, such investigations will be viewed more favourably.

Investigations into the foundations of probability essentially aim at answering two questions: *What are events? What are probabilities?* Following Kolmogorov [17], events form a field of sets with respect to the set-theoretic operations of union, intersection, and complementation. A probability is a normed measure on a field of sets. In all empirical cases, probabilities can be considered finitely additive. For mathematical reasons, however, it is more convenient to assume that probabilities are countably additive. This can always be achieved via Kolmogorov's axiom of continuity so that, by the extension theorem, the study of events and probabilities can be reduced to that of countably additive, normed measures on σ-fields of sets.

This set-theoretic approach has become standard. A second model, closer to the empirical origins of the subject, is, however, possible. Hence, following Kolmogorov [16], alternatively, events form a Boolean algebra with respect to the lattice operations of join, meet, and complementation. A probability is a normed measure on a Boolean algebra of events. Probabilities are finitely additive and strictly positive in the sense that they vanish at the impossible event only. The countable additivity of probability measures, finally, is achieved via a metric extension so that the study of events and probabilities can be reduced to that of countably additive, strictly positive, normed measures on σ-complete Boolean algebras.

The question arises whether the Boolean model is more suitable than the set-theoretic one. In fact, since every Boolean algebra is, by the Stone representation theorem, isomorphic to a field of sets, the study of finitely additive probability measures on Boolean algebras can be reduced to that of finitely additive probability measures on fields of sets. And since every σ-complete Boolean algebra is, by the Loomis–Sikorski theorem, isomorphic to the quotient of a σ-field of sets by a σ-ideal, the study of countably additive probability measures on σ-complete Boolean algebras can the reduced to that of countably additive probability measures on σ-fields of sets by identifying the probability of every set with that of its equivalence class of measurable sets. Hence, the two models are equivalent, and a generalisation to σ-complete Boolean algebras is not essential.

This is, of course, abstraction made of probability logic, whose usual approach to directly ascribe probabilities to sentences reduces to ascribe probabilities to elements of a Boolean algebra [8, 27]. In particular, ordinary Boolean algebras constitute the algebraic semantics of classical sentential logic. It is not clear, however, whether events truly abide by the rules of that logic. For if, intuitively, an event is something which can occur or not, that is to say, something which is possible, then events rather abide by the rules of modal sentential logic, an argument supported by the fact that the meet of two possible events need not be possible in general.

Modal sentential logic has a long tradition in mathematics and philosophy. The current view, inherited from the work of Lewis [19] and Gödel [9] (the 'Lewis–Gödel paradigm'), consists in adding to the language of classical sentential logic an operator $\Box$ to capture the meaning of the predicate *is necessary* and an operator $\Diamond$ to capture the meaning of the predicate *is possible.* Various axiomatic systems based on Lewis's insight and, later, that of other logicians, are then postulated and investigated using different methods. It must be emphasised that this paradigm has been tremendously successful not only in philosophical logic but also in mathematics, with the advent, for example, of the class of Boolean algebras with operators for which purely mathematical results have been established.

Yet no serious attempts have been made by mathematicians to develop the calculus of probability on, say, McKinsey and Tarski's closure algebras [22], which constitute the algebraic semantics of Lewis's modal system S4, or Halmos's monadic Boolean algebras [11], which constitute the algebraic semantics of Lewis's modal system S5. This is, of course, not surprising. For ordinary Boolean algebras are more general and easier to handle than Boolean algebras with operators; moreover, it is unlikely that the specialist in, say, stochastic processes will ever abandon ordinary Boolean algebras for, say, closure algebras. This suggests, a bit provocatively and, given the inexperience of the author, a bit impertinently, that, if, indeed, events truly abide by the rules of modal sentential logic whilst forming an ordinary Boolean algebra, then, perhaps, a mistake has been made and that a reconsideration of the foundations of modal logic in light of the foundations of probability is in order.

The rest of the thesis consists in one section and two working chapters. The next section is an extended abstract of the thesis. Readable in less than one hour, this abstract presents, with additional context the author found helpful during his own investigations, the main ideas of the subsequent chapters. Chapter 2 and Chapter 3 are concerned with the foundations of modal logic and the foundations of probability, respectively. More precisely, Chapter 2 introduces a new class of models for modal sentential logic which does not require an additional operator on the Boolean algebra of formulas to capture some notion of necessity, possibility, and impossibility, whilst Chapter 3 shows how these models can be used to combine efficiently logic and probability via a bridge principle. In particular, the problem of the existence of almost identical events, which have no probabilistic interpretation, is resolved canonically via a second intension to the mode of impossibility; this in turn lays the foundations of a field of research called *formal agnoiology*.

Note that Chapter 2 and Chapter 3 both contain a preliminary section where some results in algebra, topology, and measure theory are stated with full proofs. All these results are corollaries of well-known theorems, and no claim of originality in their proofs, which all proceed from equally well-known and well-documented arguments, is made. An effort, however, has been made to present these proofs in a way which is accessible to a broad audience; an effort has also been made to attribute these theorems to the author(s) who discovered them, with the exception of Theorems 3.1.8 and 3.1.10, which, although attributed to Fremlin, are part of the general folklore. Sections 2.2, 2.3, and 3.2, on the other hand, are original.

1.2 Extended abstract

Investigations into systems of logic invariably start with the specification of the language to be considered. In this study, the language of interest is one of infinitary sentential logic. The adjective 'sentential' is clear: If $\mathcal{L}$ is the language under consideration, then all the formulas of $\mathcal{L}$, that is to say, all the well-formed expressions built up from the sentential variables and operations of $\mathcal{L}$, are sentences; in other words, the formulas of $\mathcal{L}$ contain no free variables. The adjective 'infinitary' is less transparent. A classification based on the quantity of sentential variables and the length of formulas is, however, possible [26]. Hence, if κ, λ are cardinals, then the language $\mathcal{L}_\lambda^\kappa$ has a supply of κ distinct sentential variables and permits the formation of disjunctions and conjunctions of all lengths $< \lambda$.

In this study, it is assumed that $\kappa \geq \omega$ and that $\lambda = \omega_1$ (cardinals are identified with the initial ordinal of their number class); in other words, the language $\mathcal{L}^{\kappa}_{\omega_1}$ of infinitary sentential logic discussed herein, namely, that of countable sentential calculi, has a supply of at least ω distinct sentential variables and permits the formation of disjunctions $\bigvee$ and conjunctions $\bigwedge$ of finite or denumerable lengths. Of course, the usual operations of negation $-$ and material implication $\rightarrow$ as well as parentheses (and), whose role is to disambiguate syntactic constructions, also belong to $\mathcal{L}^{\kappa}_{\omega_1}$.

Let $\mathcal{S}$ be the least set of formulas of $\mathcal{L}^{\kappa}_{\omega_1}$, or $\mathcal{L}$ for convenience, no other languages being investigated henceforward. Remarkably, some formulas of $\mathcal{L}$ are tautologies (they are true whatever the truth-value of their components), whilst, conceivably, others are, in some sense to be made precise, logically equivalent. For example, if F denotes a formula of $\mathcal{L}$, then $F \vee -F$ is a tautology, whilst no logicians will object if $F \vee F$ is reduced to F *simpliciter*.

In fact, consider a deductive system for countable sentential calculi, that is to say, some set of distinguished tautologies of $\mathcal{L}$ and rules of inference, and define an equivalence relation $\equiv$ on $\mathcal{S}$, any two formulas F_1, F_2 of $\mathcal{L}$ being logically equivalent if and only if they materially imply each other, that is to say,

$$F_1 \equiv F_2 \text{ if and only if } F_1 \rightarrow F_2 \text{ and } F_2 \rightarrow F_1.$$

Then the structure $\mathfrak{S} / \equiv\, = (\mathcal{S} / \equiv; -, \rightarrow, \bigvee, \bigwedge)$, where $\mathcal{S} / \equiv$ is the set of formulas of the language $\mathcal{L}$ modulo logical equivalence, $-$ is the lattice operation of complementation, $\rightarrow$ is the lattice operation of relative pseudo-complementation, $\bigvee$ is the lattice operation of finite or denumerable join, and $\bigwedge$ is the lattice operation of finite or denumerable meet, is a σ-complete Boolean algebra called the *Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi.*

Lindenbaum–Tarski algebras $\mathfrak{S} / \equiv$ have interesting properties. For example, an infinite $\mathfrak{S} / \equiv$ is atomic with cardinality exactly that of the continuum $\mathfrak{c} = 2^{\aleph_0}$ if and only if it is countably generated and atomless with cardinality $> \mathfrak{c}$ if and only if it is uncountably generated. More importantly, and concomitantly, the axioms (or axiom schemas) of the deductive system in question being tautologies, and tautologies only, $\mathfrak{S} / \equiv$ is free in the class of all similar Boolean algebras, every mapping of the set of sentential variables generating $\mathfrak{S} / \equiv$ into a Boolean algebra $\mathfrak{B}$ similar to $\mathfrak{S} / \equiv$ having at most one homomorphic extension of $\mathfrak{S} / \equiv$ into $\mathfrak{B}$.

This last property is fundamental in mathematical logic. For if a logician is capable of assigning a truth-value to each sentential variable, then, certainly, by freeness, he can evaluate the truth-value of every formula of $\mathcal{L}$ with no danger of encountering a contradiction. From the standpoint of the foundations of probability, however, the freeness of $\mathfrak{S} / \equiv$ is not desirable, every Boolean algebra carrying a strictly positive, countably additive probability measure being complete, whilst no infinite free complete Boolean algebra exists in Zermelo–Fraenkel set theory with the axiom of choice (ZFC) [7, 10].

By an *almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi*, in this study, is meant a Lindenbaum–Tarski algebra $\mathfrak{S} \,/\, \equiv$ which satisfies the countable chain condition, or is c.c.c. for short, and is weakly countably distributive. Clearly, by definition, every almost measurable Lindenbaum–Tarski algebra is a Lindenbaum–Tarski algebra, but not conversely. Almost measurable Lindenbaum–Tarski algebras are denoted by $\mathfrak{S}^* \,/\, \equiv$.

The countable chain condition and the property of weak countable distributivity are necessary for the existence of a strictly positive, countably additive probability measure on a σ-complete Boolean algebra [23]. Incidentally, the weak countable distributivity of $\mathfrak{S} \,/\, \equiv$ in the definition of an almost measurable Lindenbaum–Tarski algebra is redundant. For every free σ-complete Boolean algebra being isomorphic to the σ-field of Baire subsets of a Cantor space [25], every Lindenbaum–Tarski algebra $\mathfrak{S} \,/\, \equiv$ is countably distributive, hence weakly countably distributive. The countable chain condition and the property of weak countable distributivity, however, because of their necessary character, are considered here indissociable. Of course, every c.c.c. σ-complete Boolean algebra being complete, $\mathfrak{S}^* \,/\, \equiv$ is not free.

Almost measurable Lindenbaum–Tarski algebras are the primary object of contemplation in Chapter 2, where a new class of models for modal sentential logic is introduced and investigated. Note in passing, as the reader may suspect, that an almost measurable Lindenbaum–Tarski algebra may not be measurable, the countable chain condition and the property of weak countable distributivity being not sufficient for the existence of a strictly positive, countably additive probability measure on $\mathfrak{S}^* / \equiv$ [32]. This remark is revisited in detail later.

Consider now a set Z of maximal filters of $\mathfrak{S}^* / \equiv$, and let γ be a Boolean homomorphism of $\mathfrak{S}^* / \equiv$ into the power set $\mathcal{P}(Z)$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $\gamma([F]) = \{\mathbf{F} \in Z : [F] \in \mathbf{F}\}$. Then Z equipped with the topology for which the $\gamma([F])$'s form an open base is a zero-dimensional compact Hausdorff space called the *Stone space of* $\mathfrak{S}^* / \equiv$. In fact, since $\mathfrak{S}^* / \equiv$ is c.c.c., hence complete, Z is extremally disconnected; and since $\mathfrak{S}^* / \equiv$ is c.c.c. and weakly countably distributive, every meagre set in Z is nowhere dense. In other words, the Stone space Z of an almost measurable Lindenbaum–Tarski algebra $\mathfrak{S}^* / \equiv$ of the language $\mathcal{L}$ of countable sentential calculi can be described as an *extremally disconnected compact Hausdorff space, or Stonean space, in which every meagre set is nowhere dense.* Of course, every extremally disconnected compact Hausdorff space is zero-dimensional.

Let $\text{Clop}(Z) = \{\gamma([F]) : [F] \in \mathfrak{S}^* / \equiv\}$ be the field of open-closed subsets of Z (that $\text{Clop}(Z)$ coalesces with the set $\{\gamma([F]) : [F] \in \mathfrak{S}^* / \equiv\}$ of basic open sets is a direct consequence of the zero-dimensionality and compactness of Z). Then every almost measurable Lindenbaum–Tarski algebra $\mathfrak{S}^* / \equiv$ is, by the Stone representation theorem for Boolean algebras [29, 30], isomorphic to $\text{Clop}(Z)$, which is, therefore, an example of a field of sets which is also a complete Boolean algebra. Note, however, that Clop(Z) is not a complete field of sets. For if $\{\gamma([F_n]) : n \in \omega\}$ is an indexed family of open-closed sets, then $\bigcup_{n\in\omega} \gamma([F_n])$ is not closed.

In fact, let $\mathcal{B}a(Z)$ be the σ-field of Baire subsets of Z, that is to say, the least σ-field of sets containing Clop(Z). Then $\bigcup_{n\in\omega} \gamma([F_n])$ is an open Baire set which symmetrically differs from its closure $\mathrm{cl} \bigcup_{n\in\omega} \gamma([F_n])$, which is open-closed because Z is extremally disconnected, by its boundary $\mathrm{cl} \bigcup_{n\in\omega} \gamma([F_n]) \setminus \bigcup_{n\in\omega} \gamma([F_n])$, which is nowhere dense or, more generally, meagre. By the Baire category theorem for compact Hausdorff spaces [4], which guarantees that each equivalence class of Baire subsets of Z has at most one open-closed representative, $\mathfrak{S}^* / \equiv$ is isomorphic to the quotient algebra $\mathcal{B}a(Z) / \mathcal{M}$, where $\mathcal{M} \subseteq \mathcal{B}a(Z)$ is the σ-ideal of meagre sets.

The representation of almost measurable Lindenbaum–Tarski algebras as the quotient of a σ-field of Baire sets by a σ-ideal of meagre sets is a instance of the Loomis–Sikorski theorem [20, 28]. In the present context, however, the representation can be strengthened. For recall that every meagre set in Z is nowhere dense. Since every nowhere dense set is meagre, it follows immediately that every almost measurable Lindenbaum–Tarski algebra $\mathfrak{S}^* / \equiv$ is isomorphic to the quotient algebra $\mathcal{B}a(Z) / \mathcal{N}$, where $\mathcal{N} \subseteq \mathcal{B}a(Z)$ is the σ-ideal of nowhere dense sets; $\mathcal{B}a(Z) / \mathcal{N}$ is called the *algebra of Baire subsets of Z modulo nowhere dense sets* and plays an important role in the upcoming models for classical and modal sentential calculi.

By a *model for the classical (two-valued) sentential calculus*, in this study, is meant an ordered system $(\mathfrak{A}, Z, \mathcal{F}, \xi)$, where $\mathfrak{A} = \mathfrak{S}^* \,/\, \equiv$ is an almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z is its Stone space, $\mathcal{F} = \{\emptyset, Z\} \,/\, \mathcal{N}$ is the trivial algebra of Baire subsets of Z modulo nowhere dense sets, and ξ is a Boolean epimorphism of $\mathfrak{A}$ into $\mathcal{F}$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(I) $$\text{int cl } \xi([F])^* = \emptyset \text{ if and only if } \xi([F])^* = \emptyset,$$

the open-closed representative $\xi([F])^*$ of the image $\xi([F])$ of $[F]$ being nowhere dense in Z ('false') if and only if it is empty.

Schema (I) probably deserves an explanation. For if it is clear that the property of 'nowhere denseness' rather applies to sets than equivalence classes of sets, that is to say, sets of sets, the reader may wonder why the representative $\xi([F])^*$ of $\xi([F])$ is open-closed at all. The point, of course, is that $\xi([F])^*$ must be uniquely determined for Schema (I) to be necessary and sufficient. And, by definition, each equivalence class $\xi([F])$ of Baire subsets of Z has at most one open-closed representative because $\mathcal{F} \subseteq \mathcal{B}a(Z) \,/\, \mathcal{N}$.

Suppose now, for convenience, that $\xi(-[F])^*$ is nowhere dense in Z ('false') if and only if it is empty. Since ξ is a Boolean homomorphism, and the complement of a nowhere dense set is *non-boundary* [35] (i.e. open and dense); and since, moreover, a sentence is true if and only if its negation is false, it follows immediately that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(II) $$\text{cl int } \xi([F])^* = Z \text{ if and only if } \xi([F])^* = Z,$$

the open-closed representative $\xi([F])^*$ of the image $\xi([F])$ of $[F]$ being non-boundary in Z ('true') if and only if it is Z.

At this point, a reader familiar with model theory probably recognised in Schema (II) the epitome of a T-schema and, more generally, Tarski's semantic conception of truth [33]. Hence, Schema (II) is not formula of $\mathcal{L}$ but a formula of a stronger language which contains $\mathcal{L}$, namely, that $\mathcal{L}^E$ of mathematical English viewed as an informal version of the first-order language of Zermelo–Fraenkel set theory with the axiom of choice (ZFC). In particular, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $\xi([F])^*$ in the definiendum of Schema (II) is the name of $[F]$ to which a predicate applies, cl int $\cdot = Z$ is the predicate itself (here: *is true*), and $\xi([F])^* = Z$, the definiens, is the translation of $[F]$ into $\mathcal{L}^E$. In Tarski's phraseology, constructions of that kind are called *formally correct.*

Formal correctness is essential to resolve the Liar paradox and other similar semantic antinomies [33, pp. 157–158]. More importantly, however, is Tarski's Convention T [ibid. pp. 187–188], a definition of the predicate *is true*, formulated in the metalanguage $\mathcal{L}^E$ of mathematical English, being *materially adequate* if it implies all instances of a T-schema, that is to say, in the present context, all instances of Schema (II). Note in passing that the condition of material adequacy can only be satisfied using the recursive method; this follows, of course, from the fact that the language $\mathcal{L}$ of countable sentential calculi contains infinitely many sentences.

Let $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ be a model for the classical sentential calculus. The following schemas constitute, as a whole, a formally correct and materially adequate definition of the predicate *is true*:

cl int $\xi([F])^* = Z$ if and only if $\xi([F])^* = Z$;

cl int $\xi(-[F])^* = Z$ if and only if int cl $\xi([F])^* = \emptyset$;

cl int $\xi(\bigvee_{n\in\omega}[F_n])^* = Z$ if and only if cl int $\xi([F_n])^* = Z$ for some n;

cl int $\xi(\bigwedge_{n\in\omega}[F_n])^* = Z$ if and only if cl int $\xi([F_n])^* = Z$ for every n;

cl int $\xi([F_1] \to [F_2])^* = Z$ if and only if

$$\text{cl int } \xi(-[F_1])^* = Z \text{ or cl int } \xi([F_2])^* = Z,$$

the open-closed representative $\xi([F])^*$ of the image $\xi([F])$ of $[F]$ being non-boundary in Z if and only if it is Z; or $\xi(-[F])^*$ is non-boundary in Z if and only if $\xi([F])^*$ is nowhere dense in Z; or $\xi(\bigvee_{n\in\omega}[F_n])^*$ is non-boundary in Z if and only if at least one $\xi([F_0])^*, \xi([F_1])^*, \xi([F_2])^*, \ldots$ is non-boundary in Z; or $\xi(\bigwedge_{n\in\omega}[F_n])^*$ is non-boundary in Z if and only if all $\xi([F_0])^*, \xi([F_1])^*, \xi([F_2])^*, \ldots$ are non-boundary in Z; or $\xi([F_1] \to [F_2])^*$ is non-boundary in Z if and only if $\xi(-[F_1])^*$ or $\xi([F_2])^*$ is non-boundary in Z.

The material adequacy of the above definition of truth presents no particular difficulties. It should be noted, however, that although a countable union of non-boundary sets is always non-boundary, a countable intersection of non-boundary sets may not be non-boundary in general; in fact, it can even be codense (i.e. have empty interior), the set $\mathbb{R} \setminus \mathbb{Q}$ of irrationals in $\mathbb{R}$ equipped with the usual topology being a classic example. In other words, the topology of Z induced by the algebraic properties of $\mathfrak{A}$ is here essential. These intricacies are discussed in detail in Chapter 2.

By a *model for the modal sentential calculus*, in this study, is meant an ordered system $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$, where $\mathfrak{A} = \mathfrak{S}^* / \equiv$ is an almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z is its Stone space, $\mathcal{E} = \mathcal{B}a(Z) / \mathcal{N}$ is the algebra of Baire subsets of Z modulo nowhere dense sets, and ζ is the canonical Boolean isomorphism of $\mathfrak{A}$ into $\mathcal{E}$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

$$\text{(III)} \qquad \text{int cl } \zeta([F])^* = \emptyset \text{ if and only if } \zeta([F])^* = \emptyset,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being nowhere dense in Z ('impossible') if and only if it is empty.

That Schema (III) is, like Schema (I), a theorem of the metalanguage $\mathcal{L}^E$ of mathematical English if and only if $\zeta([F])^*$ is open-closed is clear. For suppose to dispel any remaining doubts that $\zeta([F])^*$ is closed. Then the sufficiency of Schema (III) certainly holds because the empty set is a closed, nowhere dense Baire subset of Z, whilst its necessity fails because the empty set is not the only subset of Z which satisfies these properties, the homeomorphic copy of the Cantor set in Z being a classic example.

Suppose now, for convenience, that $\zeta(-[F])^*$ is nowhere dense in Z ('impossible') if and only if it is empty. Since ζ is a Boolean homomorphism, and, once again, the complement of a nowhere dense set is non-boundary; and since, moreover, a sentence is necessary if and only if its negation is impossible, it follows immediately that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(IV) $$\text{cl int } \zeta([F])^* = Z \text{ if and only if } \zeta([F])^* = Z,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being non-boundary in Z ('necessary') if and only if it is Z.

In fact, recall that $\zeta([F])^*$ is nowhere dense in Z ('impossible') if and only if it is empty. Since ζ is, more precisely, a Boolean isomorphism, every equivalence class $[F]$ of formulas of $\mathcal{L}$ having, therefore, at least two possible truth-values, and a set is somewhere dense if and only if it is not nowhere dense; and since, moreover, a sentence is possibly true if and only if it is not impossible, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(V) $$\text{int cl } \zeta([F])^* \neq \emptyset \text{ if and only if } \zeta([F])^* \neq \emptyset,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being somewhere dense in Z ('possibly true') iff it is non-empty.

Schema (V) captures the meaning of the predicate *is possible* as usually understood by logicians and philosophers alike. It is convenient, however, to have a second definition of possibility. Hence, recall that $\zeta([F])^*$ is non-boundary in Z ('necessary') if and only if it is Z. Since ζ is a Boolean isomorphism, and a set is *somewhere codense* if and only if it is not non-boundary; and since, moreover, a sentence is possibly false if and only if it is not necessary, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(VI) $$\text{cl int } \zeta([F])^* \neq Z \text{ if and only if } \zeta([F])^* \neq Z,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being somewhere codense in Z ('possibly false') iff it is not Z.

Schemas (III)–(VI) are noteworthy. For, contrary to the Lewis–Gödel paradigm [9, 19] currently operating in full force, no additional symbol to the language $\mathcal{L}$ of countable sentential calculi, that is to say, no additional operator on the Lindenbaum–Tarski algebra $\mathfrak{A}$, was required to capture some notion of necessity, possibility, and impossibility. This follows, of course, from the fact that the aforementioned modes of truth, in a Tarskian fashion, were defined at the level of the metalanguage $\mathcal{L}^E$ of mathematical English instead.

In particular, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $\xi([F])^*$ is non-boundary in Z whenever $\zeta([F])^*$ is non-boundary in Z, but not conversely; $\zeta([F])^*$ is somewhere dense in Z whenever $\zeta([F])^*$ is non-boundary in Z, but not conversely; and $\zeta([F])^*$ is somewhere dense in Z whenever $\xi([F])^*$ is non-boundary in Z, but not conversely. In other words, necessity implies truth, but not conversely; necessity implies possibility, but not conversely; and truth implies possibility, but not conversely. These properties are common desiderata in modal logic. Interestingly, in the Lewis–Gödel paradigm [9, 19], the first two are axioms (or axiom schemas) formulated in the customised language of modal logic, whilst, in this study, they are theorems of the metalanguage $\mathcal{L}^E$ of mathematical English.

Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus.

The following schemas constitute, as a whole, a formally correct and, in the sense that they only imply some instances of Schema (IV), *weakly materially adequate* definition of the predicate *is necessary*:

cl int $\zeta([F])^* = Z$ if and only if $\zeta([F])^* = Z$;

cl int $\zeta(-[F])^* = Z$ if and only if int cl $\zeta([F])^* = \emptyset$;

cl int $\zeta(\bigvee_{n\in\omega}[F_n])^* = Z$ whenever cl int $\zeta([F_n])^* = Z$ for some n;

cl int $\zeta(\bigwedge_{n\in\omega}[F_n])^* = Z$ if and only if cl int $\zeta([F_n])^* = Z$ for every n;

cl int $\zeta([F_1] \to [F_2])^* = Z$ whenever

cl int $\zeta(-[F_1])^* = Z$ or cl int $\zeta([F_2])^* = Z$,

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being non-boundary in Z if and only if it is Z; or $\zeta(-[F])^*$ is non-boundary in Z if and only if $\zeta([F])^*$ is nowhere dense in Z; or $\zeta(\bigvee_{n\in\omega}[F_n])^*$ is non-boundary in Z whenever at least one $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ is non-boundary in Z; or $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ is non-boundary in Z if and only if all $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ are non-boundary in Z; or $\zeta([F_1] \to [F_2])^*$ is non-boundary in Z whenever $\zeta(-[F_1])^*$ or $\zeta([F_2])^*$ is non-boundary in Z.

In the same vein, the following schemas constitute, as a whole, a formally correct and, in the sense that they are only implied by some instances of Schema (V) (compare with necessity), *weakly materially adequate* definition of the predicate *is possible*:

int cl $\zeta([F])^* \neq \emptyset$ if and only if $\zeta([F])^* \neq \emptyset$;

int cl $\zeta(-[F])^* \neq \emptyset$ if and only if cl int $\zeta([F])^* \neq Z$;

int cl $\zeta(\bigvee_{n\in\omega}[F_n])^* \neq \emptyset$ if and only if int cl $\zeta([F_n])^* \neq \emptyset$ for some n;

int cl $\zeta([F_n])^* \neq \emptyset$ for every n whenever int cl $\zeta(\bigwedge_{n\in\omega}[F_n])^* \neq \emptyset$;

int cl $\zeta([F_1] \rightarrow [F_2])^* \neq \emptyset$ if and only if

$$\text{int cl } \zeta(-[F_1])^* \neq \emptyset \text{ or int cl } \zeta([F_2])^* \neq \emptyset,$$

the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ being somewhere dense in Z if and only if it is non-empty; or $\zeta(-[F])^*$ is somewhere dense in Z if and only if $\zeta([F])^*$ is somewhere codense in Z; or $\zeta(\bigvee_{n\in\omega}[F_n])^*$ is somewhere dense in Z if and only if at least one $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ is somewhere dense in Z; or all $\zeta([F_0])^*$, $\zeta([F_1])^*, \zeta([F_2])^*, \ldots$ are somewhere dense in Z whenever $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ is somewhere dense in Z; or $\zeta([F_1] \rightarrow [F_2])^*$ is somewhere dense in Z if and only if $\zeta(-[F_1])^*$ or $\zeta([F_2])^*$ is somewhere dense in Z.

The weak material adequacy of the above definitions of necessity and possibility conforms with the intuition. For if a conjunction of sentences is necessary if and only if all the sentences forming the conjunction are necessary, a disjunction of sentences may be necessary although none of the sentences forming the disjunction are necessary, a result naturally extending, *mutatis mutandis*, to material implication (note that the formula cl int $\zeta([F_1] \rightarrow [F_2])^* = Z$ does not capture Lewis's strict implication [19] but a weak form of material implication). And if a disjunction of sentences is possible if and only if at least one of the sentences forming the disjunction is possible, a result naturally extending, *mutatis mutandis*, to material implication, a conjunction of sentences may not be possible although all the sentences forming the conjunction are possible.

It is tempting, at this point, to conclude that the analysis is complete. This is not, however, quite yet the case. For recall that any two Baire subsets of Z belong to the same equivalence class (of Baire sets) if and only if their symmetric difference is nowhere dense. Then, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(VII) $\quad$ int cl $(\zeta([F])^* \bigtriangleup B) = \emptyset$ if and only if $B \in \zeta([F])$,

the symmetric difference of the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ with an arbitrary Baire set B being nowhere dense in Z ('impossible') if and only if B belongs to $\zeta([F])$.

At first glance, the reader may contend that Schema (VII) is inconsequential: $\zeta([F])^* \bigtriangleup B$ is nowhere dense in Z, that is to say, $\zeta([F])^*$ and B are impossible to discriminate from each other, if and only if $\zeta([F])^*$ and B belong to the same equivalence class of Baire sets, and the equivalence relation defined on $\mathcal{B}a(Z)$ naturally inducing a congruence relation on $\mathcal{E} = \mathcal{B}a(Z) \,/\, \mathcal{N}$, they are interchangeable, in Leibniz's phraseology, *salva veritate.*

But the present state of affairs is more subtle. For recall, as previously illustrated, that Schemas (III)–(VI) are theorems of the metalanguage $\mathcal{L}^E$ of mathematical English if and only if $\zeta([F])^*$ is open-closed. Then although $\zeta([F])^* \bigtriangleup B$ is nowhere dense in Z, that is to say, although $\zeta([F])^*$ and B are impossible to discriminate from each other, they are *not* interchangeable *salva veritate.*

Schema (VII) uncovers, therefore, a particularly acute problem: Unbeknown to the logician, Schemas (III)–(VI) can be false in the sense that they are not theorems of $\mathcal{L}^E$. Moreover, no remedy is available. For suppose that $\mathcal{L}^{E^+}$ is a meta-metalanguage capable of assessing the truth-value of each schema. Then this assessment must be indubitably true to be conclusive. A contradiction because $\mathcal{F}$ and, indeed, every isomorphic copy thereof are subalgebras of $\mathcal{E}$.

The impossibility to discriminate a true or false or necessary or possible or impossible sentence which is 'true' from a true or false or necessary or possible or impossible sentence which is 'false' is called *agnoia* (from Ancient Greek ἄγνοια, 'ignorance'). The systematic study of agnoia in relation to uncertainty modelling is the domain of *formal agnoiology* (the term 'agnoiology' was coined by Ferrier [5], who saw in agnoiology, the theory of ignorance, one of the key divisions of metaphysics alongside epistemology and ontology). Incidentally, agnoia introduces an asymmetry in the general definition of impossibility; this is formally acknowledged in Chapter 2.

Recall that a model for the modal sentential calculus, in this study, is an ordered system $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$, where $\mathfrak{A} = \mathfrak{S}^* / \equiv$ is an almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z is its Stone space, $\mathcal{E} = \mathcal{B}a(Z) / \mathcal{N}$ is the algebra of Baire subsets of Z modulo nowhere dense sets, and ζ is the canonical Boolean isomorphism of $\mathfrak{A}$ into $\mathcal{E}$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(III) $$\operatorname{int}\operatorname{cl}\ \zeta([F])^* = \emptyset \text{ if and only if } \zeta([F])^* = \emptyset,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being nowhere dense in Z ('impossible') if and only if it is empty.

The question arises as to what extent $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ is compatible with the axiomatic treatment of the calculus of probability formulated by Kolmogorov [16, 17]. Clearly, since $\mathfrak{A}$ and $\mathcal{E}$, as Boolean algebras, are isomorphic, this question immediately reduces to that of the existence of a strictly positive, countably additive probability measure $\mathbf{p}$ on $\mathfrak{A}$, which, it is recalled, satisfies the countable chain condition (c.c.c.) and is weakly countably distributive. Von Neumann [23] famously asked whether these conditions are sufficient for the existence of a strictly positive, countably additive measure on a σ-complete Boolean algebra; by a theorem of Talagrand [32], it is now acknowledged that this is not the case in general.

Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus. There exists, therefore, an almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ of the language $\mathcal{L}$ of countable sentential calculi which does not carry a strictly positive, countably additive probability measure $\mathbf{p}$ so that the question of the compatibility of $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ with the calculus of probability, which first morphed into that of the existence of a strictly positive, countably additive probability measure $\mathbf{p}$ on $\mathfrak{A}$, can yet be reformulated as that of the identification of a necessary and sufficient criterion for the existence of $\mathbf{p}$.

In particular, it is known that every complete Boolean algebra carrying a strictly positive, finitely additive probability measure carries a strictly positive, countably additive probability measure if and only if it is weakly countably distributive [15]. Hence, since every almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ is, as previously recalled, c.c.c. and weakly countably distributive, it certainly carries a strictly positive, countably additive probability measure $\mathbf{p}$ if and only if it carries a strictly positive, finitely additive probability measure $\mathbf{q}$. This result, however, although workable, is not entirely satisfactory, for it leaves open the question of the existence of a necessary and sufficient criterion for the existence of $\mathbf{q}$ on $\mathfrak{A}$.

In fact, technical details aside, for every subset $\mathcal{A} \subseteq \mathfrak{A}$ of equivalence classes of formulas of $\mathcal{L}$, define, combinatorially, a quantity $\#(\mathcal{A})$ called the *intersection number of* $\mathcal{A}$. Then, by a theorem of Kelley [15], $\mathfrak{A}$ carries a strictly positive, finitely additive probability measure $\mathbf{q}$ if and only if $\mathfrak{A} \setminus \{[\mathbf{0}]\}$ is the union of countably many sets of equivalence classes of formulas of $\mathcal{L}$, each of which has a positive intersection number. That is to say, an almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ of the language $\mathcal{L}$ of countable sentential calculi carries a strictly positive, countably additive probability measure $\mathbf{p}$ if and only if $\mathfrak{A} \setminus \{[\mathbf{0}]\}$ is the union of countably many sets of equivalence classes of formulas of $\mathcal{L}$, each of which has a positive intersection number.

An almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ which satisfies Kelley's criterion is called *measurable*. By definition, every measurable Lindenbaum–Tarski algebra is almost measurable, but not conversely; in particular, every measurable Lindenbaum–Tarski algebra is complete. Measurable Lindenbaum–Tarski algebras are denoted by $\mathfrak{A}^*$. An ordered system $(\mathfrak{A}^*, \mathbf{p})$, where $\mathfrak{A}^*$ is a measurable Lindenbaum–Tarski algebra, and $\mathbf{p}$ is a strictly positive, countably additive probability measure on $\mathfrak{A}^*$, is called a *Lindenbaum–Tarski probability algebra of the language $\mathcal{L}$ of countable sentential calculi.* Lindenbaum–Tarski probability algebras $(\mathfrak{A}^*, \mathbf{p})$ are the primary object of contemplation in Chapter 3, where a tentative solution to the problem of combining efficiently logic and probability is presented.

Consider now a set Z^* of maximal filters of $\mathfrak{A}^*$, and, using, for convenience, the same notation as for almost measurable Lindenbaum–Tarski algebras $\mathfrak{A}$, let γ be a Boolean homomorphism of $\mathfrak{A}^*$ into the power set $\mathcal{P}(Z^*)$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $\gamma([F]) = \{\mathbf{F} \in Z^* : [F] \in \mathbf{F}\}$. Then Z^* equipped with the topology for which the $\gamma([F])$'s form an open base is a zero-dimensional compact Hausdorff space called the *Stone space of* $\mathfrak{A}^*$. Of course, since $\mathfrak{A}^*$ is c.c.c. and weakly countably distributive, Z^* is extremally disconnected, and every meagre set in Z^* is nowhere dense. As previously noted, an extremally disconnected compact Hausdorff space is also called a Stonean space.

Clearly, at this point, a representation theorem for Lindenbaum–Tarski probability algebras $(\mathfrak{A}^*, \mathbf{p})$ would be useful. Let $(\mathcal{B}a(Z^*) \,/\, \mathcal{N}_{\mathbb{P}}, \mathbf{P})$, where $\mathcal{B}a(Z^*) \,/\, \mathcal{N}_{\mathbb{P}}$ is the algebra of Baire subsets of Z^* modulo $\mathbb{P}$-negligible sets, be the probability algebra of the probability space $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ naturally associated with $(\mathfrak{A}^*, \mathbf{p})$. Since every measurable Lindenbaum–Tarski algebra $\mathfrak{A}^*$ is, by the Loomis–Sikorski theorem [20, 28], isomorphic to the algebra $\mathcal{B}a(Z^*) \,/\, \mathcal{N}$ of Baire subsets of Z^* modulo nowhere dense sets, it suffices to construct a measure-preserving Boolean isomorphism of $\mathfrak{A}^*$ into $\mathcal{B}a(Z^*) \,/\, \mathcal{N}$ such that $\mathcal{N}_{\mathbb{P}} = \mathcal{N}$ to obtain the desired representation. The interaction between the Stone topology $\mathcal{T}_S^*$ on Z^*, that is to say, the topology generated by the sets of the form $\{\mathbf{F} \in Z^* : [F] \in \mathbf{F}\}$, and the probability measure $\mathbb{P}$ on $\mathcal{B}a(Z^*)$ in the construction of that Boolean isomorphism, however, is not clear. In particular, it is not clear in what sense $\mathcal{T}_S^*$ and $\mathbb{P}$ are compatible with each other.

In fact, let $(Z^*, \mathcal{T}_S^*)$ be the Stone space of $\mathfrak{A}^*$. Then the ordered system $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ resulting from combining $(Z^*, \mathcal{T}_S^*)$ and $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is a Radon probability space. Indeed, since Z^* is zero-dimensional, and $\mathcal{B}a(Z^*)$ is the least σ-field of sets containing the open-closed subsets of Z^*, every open set in Z^* is $\mathbb{P}$-measurable; and since every Baire set in Z^* is $\mathbb{P}$-negligible if and only if it is nowhere dense, and nowhere dense sets form a σ-ideal in Z^*, $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is Carathéodory complete; finally, $\mathbb{P}$ is inner regular with respect to the compact sets because every Baire subset of Z^* is approximable from within by compact sets.

Let $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ be the Radon probability space naturally associated with the Lindenbaum–Tarski probability algebra $(\mathfrak{A}^*, \mathbf{p})$. A Radon probability measure on $(Z^*, \mathcal{B}a(Z^*))$ is called *normal* if and only if every nowhere dense Baire set in Z^* is negligible [2]. A

Stonean space in which the union of the supports of all normal probability measures is dense is said to be *hyperstonean* (in French: *espace hyperstonien* [2]). It is easy to see that Z^* is hyperstonean. The Radon probability space $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ is called the *Stone probability space of* $(\mathfrak{A}^*, \mathbf{p})$. Of course, in the final analysis, every Lindenbaum–Tarski probability algebra $(\mathfrak{A}^*, \mathbf{p})$ of the language $\mathcal{L}$ of countable sentential calculi is isomorphic to the probability algebra $(\mathcal{B}a(Z^*) \;/\; \mathcal{N}, \mathbf{P})$ of its Stone probability space $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$.

By a *model for the calculus of probability*, in this study, is meant an ordered system $(\boldsymbol{\mathfrak{A}}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$, where $\boldsymbol{\mathfrak{A}} = (\mathfrak{A}^*, \mathbf{p})$ is a Lindenbaum–Tarski probability algebra of the language $\mathcal{L}$ of countable sentential calculi, $\boldsymbol{Z} = (Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ is its Stone probability space, $\boldsymbol{\mathcal{E}} = (\mathcal{B}a(Z^*) \;/\; \mathcal{N}, \mathbf{P})$ is the probability algebra of $\boldsymbol{Z}$, and ζ is the canonical Boolean isomorphism of $\boldsymbol{\mathfrak{A}}$ into $\boldsymbol{\mathcal{E}}$ such that, for every equivalence class $[F]$ of formulas of the language $\mathcal{L}$,

(VIII) $\quad \mathbb{P}(\zeta([F])^*) = 0$ if and only if $\operatorname{int} \operatorname{cl} \zeta([F])^* = \emptyset$,

the probability of the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ being 0 if and only if $\zeta([F])^*$ is nowhere dense in Z^* ('impossible').

That Schema (VIII) is, like Schemas (I) and (III), a theorem of the metalanguage $\mathcal{L}^E$ of mathematical English hardly needs an explanation and follows immediately from the representation of Lindenbaum–Tarski probability algebras, every Baire subset of the Stone space Z^* of $\mathfrak{A}^*$, hence every open-closed subset thereof, being $\mathbb{P}$-negligible if and only if it is nowhere dense as already noted.

Suppose now, for convenience, that $\mathbb{P}(\zeta(-[F])^*) = 0$ if and only if $\zeta(-[F])^*$ is nowhere dense in Z^* ('impossible'). Since ζ is a Boolean homomorphism, and the complement of a nowhere dense set is non-boundary; and since, moreover, $\mathbb{P}$ is a probability measure, that is to say, a measure whose total mass is 1, it follows immediately that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(IX) $$\mathbb{P}(\zeta([F])^*) = 1 \text{ if and only if } \mathrm{cl}\ \mathrm{int}\ \zeta([F])^* = Z^*,$$

the probability of the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ being 1 if and only if $\zeta([F])^*$ is non-boundary in Z^* ('necessary').

In fact, recall that $\mathbb{P}(\zeta([F])^*) = 0$ if and only if $\zeta([F])^*$ is nowhere dense in Z^* ('impossible'). Since ζ is, more precisely, a Boolean isomorphism, and a set is somewhere dense if and only if it is not nowhere dense; and since, moreover, $\mathbb{P}$ is a non-negative measure, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(X) $$\mathbb{P}(\zeta([F])^*) > 0 \text{ if and only if } \mathrm{int}\ \mathrm{cl}\ \zeta([F])^* \neq \emptyset,$$

the probability of the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ being > 0 iff $\zeta([F])^*$ is somewhere dense in Z^* ('possibly true').

Schema (X) associates every statement of probability whose value is > 0 with something which can occur. Echoing a previous argument for possibility, it is sometimes convenient to associate every statement of probability whose value is < 1 with something which can not occur. Hence, recall that $\mathbb{P}(\zeta([F])^*) = 1$ if and only if $\zeta([F])^*$ is non-boundary in Z ('necessary'). Since ζ is a Boolean isomorphism, and a set is somewhere codense if and only if it is not non-boundary; and since, moreover, $\mathbb{P}$ is a measure whose total mass is 1, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(XI) $\quad \mathbb{P}(\zeta([F])^*) < 1$ if and only if $\mathrm{cl}\ \mathrm{int}\ \zeta([F])^* \neq Z^*$,

the probability of the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ being < 1 iff $\zeta([F])^*$ is somewhere codense in Z^* ('possibly false').

Schemas (VIII)–(XI) define the probability of every $\zeta([F])^*$ in terms of its mode of truth. Since Schemas (III)–(VI) define the mode of truth of every $\zeta([F])^*$ in terms of sets, every statement of probability $\mathbb{P}(\zeta([F])^*)$, as a sentence of the metalanguage $\mathcal{L}^E$, can be decomposed into a statement of probability and a statement of modal logic by combining Schemas (VIII)–(XI) and (III)–(VI). For example, combining Schemas (VIII) and (III), it follows immediately that $\mathbb{P}(\zeta([F])^*) = 0$ if and only if $\mathrm{int}\ \mathrm{cl}\ \zeta([F])^* = \emptyset$ if and only if $\zeta([F])^* = \emptyset$. Of course, in the ordinary conduct of his affairs, a probabilist will simply say that $\mathbb{P}(\emptyset) = 0$.

Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. A *semantic bridge* is a finite sequence of logical equivalences, formulated in the metalanguage $\mathcal{L}^E$ of mathematical English, combining a statement of probability and a statement of modal logic. A *semantic decomposition* is a finite sequence of semantic bridges. An *event* is an element of the σ-field $\mathcal{B}a(Z^*)$ of Baire subsets of Z^*. In particular, for every equivalence class $[F]$ of formulas of the language $\mathcal{L}$, $\zeta([F])^*$ is called the *event corresponding to* $[F]$.

In fact, combining Schemas (VIII) and (III), Schemas (IX) and

(IV), and Schemas (X) and (V), and by the material adequacy of impossibility and possibility for countable joins, there exists a semantic decomposition for each of the axioms of probability

(P1) $\mathbb{P}(\zeta([F])^*) \geq 0$ for every $\zeta([F])^* \in \mathcal{B}a(Z^*)$;

(P2) $\mathbb{P}(Z^*) = 1$;

(P3) $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) = \sum_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*)$,

the probability of every event $\zeta([F])^*$ corresponding to an equivalence class $[F]$ of formulas of $\mathcal{L}$ being non-negative; the probability of the necessary event being 1; and the probability of every event $\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*$ corresponding to the join of an indexed family $\{[F_n] : n \in \mathbb{N}\}$ of pairwise disjoint equivalence classes of formulas of $\mathcal{L}$ being the sum of the probabilities of each event $\zeta([F_n])^*$ corresponding to each equivalence class of formulas (countable additivity)

In the same vein, combining Schemas (IX) and (IV) and Schemas (XI) and (VI), and by the material adequacy of necessity and possible falsity for countable meets (compare with the previous argument), there exists a semantic decomposition such that

(P4) $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = \prod_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*)$,

the probability of the event $\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*$ corresponding to the meet of an indexed family $\{[F_n] : n \in \mathbb{N}\}$ of equivalence classes of formulas of $\mathcal{L}$, the occurrence of any of which unaffecting that of any other, being the product of the probabilities of each event $\zeta([F_n])^*$ corresponding to each equivalence class of formulas.

Finally, combining Schemas (XI) and (VI) and, once again, by the material adequacy of possible falsity for countable meets, there certainly exists a semantic decomposition such that

(P5) $$\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) \neq \prod_{n=0}^{\infty}\mathbb{P}(\zeta([F_n])^*),$$

the probability of the event $\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*$ corresponding to the meet of an indexed family $\{[F_n] : n \in \mathbb{N}\}$ of equivalence classes of formulas of $\mathcal{L}$ differing, in general, from the product of the probabilities of each event $\zeta([F_n])^*$ corresponding to each equivalence class of formulas whenever the probability of $\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*$ is < 1.

Note in passing that the existence of a semantic decomposition for each of the axioms of probability (P1)–(P3) immediately implies the existence of a semantic decomposition for the *complement rule*, *monotonicity*, and the *addition rules*. Likewise, the existence of a semantic decomposition for (P4) immediately implies the existence of a semantic decomposition for *independence* and *mutual independence*. Finally, the existence of a semantic decomposition for (P5) immediately implies the existence of a semantic decomposition for *dependence* and, provided the ad hoc introduction of the notion of conditional probability, *conditional dependence* and the *product rule*. These properties are formally proved in Chapter 3; their only interest is to show how modal logic and probability are combined in practice.

Semantic decompositions bring to the fore the inner modal logic of the calculus of probability. One remaining aspect of the calculus of probability, however, needs attention. For recall that every Baire subset of Z^* is $\mathbb{P}$-negligible if and only if it is nowhere dense.

Since the symmetric difference of two Baire sets is Baire,

(XII-a) $\mathbb{P}(\zeta([F])^* \triangle B) = 0$ if and only if $\text{int cl } (\zeta([F])^* \triangle B) = \emptyset$,

the probability of the symmetric difference of the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ with an arbitrary Baire set B being 0 if and only if $\zeta([F])^* \triangle B$ is nowhere dense in Z^* ('impossible'), whence, combining Schema (XII-a) and Schema (VII),

(XII-b) $\mathbb{P}(\zeta([F])^* \triangle B) = 0$ if and only if $B \in \zeta([F])$,

the probability of the symmetric difference of the open-closed representative $\zeta([F])^*$ of the equivalence class $\zeta([F])$ with an arbitrary Baire set B being 0 if and only if B belongs to $\zeta([F])$.

Schema (XII-b) is the probabilistic version of Schema (VII). Likewise, at first glance, the reader may contend that it is inconsequential: $\zeta([F])^* \triangle B$ is $\mathbb{P}$-negligible, that is to say, $\zeta([F])^*$ and B are $\mathbb{P}$-impossible to discriminate from each other, if and only if $\zeta([F])^*$ and B belong to the same equivalence class of $\mathbb{P}$-measurable sets (in which case, $\mathbb{P}(\zeta([F])^*) = \mathbb{P}(B)$), and the equivalence relation defined on $\mathcal{B}a(Z^*)$ naturally inducing a congruence relation on $\mathcal{B}a(Z^*) \,/\, \mathcal{N}$, they are interchangeable *salva veritate.*

But the present state of affairs is, once again, more subtle. For recall that every statement of probability $\mathbb{P}(\zeta([F])^*)$, as a sentence of the metalanguage $\mathcal{L}^E$ of mathematical English, can be decomposed into a statement of probability and a statement of modal logic by combining Schemas (VIII)–(XI) and (III)–(VI). Then $\mathbb{P}(\zeta([F])^*)$ is a theorem of $\mathcal{L}^E$ if and only if $\zeta([F])^*$ is open-closed because Schemas (III)–(VI) are theorems of $\mathcal{L}^E$ if and only if $\zeta([F])^*$ is open-closed. Hence, although $\zeta([F])^* \bigtriangleup B$ is $\mathbb{P}$-negligible, that is to say, although $\zeta([F])^*$ and B are $\mathbb{P}$-impossible to discriminate from each other, they are not interchangeable *salva veritate.*

Schema (XII-b) uncovers, therefore, a probabilistic version of agnoia: Unbeknown to the probabilist, every statement of probability $\mathbb{P}(\zeta([F])^*)$, as a sentence of the metalanguage $\mathcal{L}^E$, can be false in the sense that its semantic decomposition is not a theorem of $\mathcal{L}^E$, in which case $\mathbb{P}(\zeta([F])^*)$, as a probability measure, is meaningless. For example, consider the following semantic decomposition

$$\mathbb{P}(\zeta([F])^*) = 0 \text{ if and only if } \operatorname{int}\operatorname{cl}\zeta([F])^* = \emptyset$$
$$\text{if and only if } \zeta([F])^* = \emptyset,$$

and suppose that $\zeta([F])^*$ is closed. If $\zeta([F])^*$ is closed and empty (i.e. the empty set), then it is nowhere dense in Z^*, hence $\mathbb{P}$-negligible; whilst if $\zeta([F])^*$ is closed and $\mathbb{P}$-negligible, then it is nowhere dense in Z^*, although not necessarily empty, the homeomorphic copy of the Cantor set in Z^* being a classic example.

The analysis, however, is still incomplete. For the question arises whether a finer measure of agnoia than $\mathbb{P}$ exists. In fact, let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. Then there exist no finer measures of agnoia than $\mathbb{P}$, that is to say, if P is a probability measure on the σ-field $\mathcal{B}a(Z^*)$ of Baire subsets of the Stone space Z^* of a measurable Lindenbaum–Tarski algebra $\mathfrak{A}^*$, then P is absolutely continuous with respect to $\mathbb{P}$. This result is to be considered the first theorem of formal agnoiology.

This concludes the author's investigations into the modal logic of the calculus of probability. The reader may wonder where do these investigations leave him and what would future work look like. It is immediately clear that the lack of a completeness theorem for the classical sentential calculus and the modal sentential calculus as conceived herein must be addressed in priority. The completeness of the classical sentential calculus presents no particular difficulties; that of the modal sentential calculus is a bit more intricate, but the author is confident that a proof can be found. More importantly, however, is the development of formal agnoiology for which everything remains to be done. Mathematically, the study of agnoia reduces to that of probability measures with nowhere dense support in the Stone space of almost measurable algebras (i.e. Stonean spaces in which meagre sets are nowhere dense). This may potentially pave the way to the discovery of interesting results in uncertainty modelling, especially in relation to decision theory and topological games.

Chapter 2

Foundations of modal logic

2.1 Preliminaries

Let $\kappa \geq \omega$ be an infinite cardinal. The language $\mathcal{L}$ of countable sentential calculi discussed in this study has a supply of κ distinct sentential variables and permits the formation of disjunctions and conjunctions of all lengths $< \omega_1$. In other words, by definition, in addition to the usual operations of negation $-$ and material implication $\rightarrow$ common to all sentential calculi, finite or denumerable disjunctions $\bigvee$ and finite or denumerable conjunctions $\bigwedge$ are well defined in $\mathcal{L}$. Auxiliary parentheses (and), whose role is to disambiguate syntactic constructions, are also used unreservedly.

Let $\mathcal{S}$ be the least set of well-formed expressions built up from the sentential variables and operations of $\mathcal{L}$. Elements of $\mathcal{S}$ are called *formulas of the language* $\mathcal{L}$ *of countable sentential calculi*, or simply *formulas of* $\mathcal{L}$, and will be denoted by F with a subscript if necessary. In particular, the following rules hold:

(i) Every sentential variable is a formula of $\mathcal{L}$;

(ii) If F is a formula of $\mathcal{L}$, then so is $-F$;

(iii) If F_1, F_2 are formulas of $\mathcal{L}$, then so is $F_1 \to F_2$;

(iv) If $F_0, F_1, F_2, \ldots$ are formulas of $\mathcal{L}$, then so is $\bigvee_{n\in\omega} F_n$;

(v) If $F_0, F_1, F_2, \ldots$ are formulas of $\mathcal{L}$, then so is $\bigwedge_{n\in\omega} F_n$.

2.1.1 Theorem (Lindenbaum [unpublished], Tarski [34]) *The structure* $\mathfrak{S} = (\mathcal{S}\,; -, \to, \bigvee, \bigwedge)$, *where* $\mathcal{S}$ *is the least set of formulas of the language* $\mathcal{L}$ *of countable sentential calculi,* $-$ *is the operation of negation,* $\to$ *is the operation of material implication,* $\bigvee$ *is the operation of finite or denumerable disjunction, and* $\bigwedge$ *is the operation of finite or denumerable conjunction, is a generalised abstract algebra.*

PROOF As an immediate consequence of the definition of $\mathcal{L}$. ∎

The generalised abstract algebra $\mathfrak{S} = (\mathcal{S}\,; -, \to, \bigvee, \bigwedge)$ described in Theorem 2.1.1 is called the *algebra of formulas of the language* $\mathcal{L}$ *of countable sentential calculi.* Of course, there is nothing special about $\mathfrak{S}$; it is just an abstract algebra in which finite or denumerable disjunctions $\bigvee$ and conjunctions $\bigwedge$ are well defined.

2.1.2 Definition (Karp [14], Rasiowa and Sikorski [24]) The deductive system for countable sentential calculi comprises the following axiom schemas (tautologies) and rules of inference:

A1. $F \to F$;

A2. $(F_1 \to F_2) \to ((F_2 \to F_3) \to (F_1 \to F_3))$;

A3. For every index $m = 0, 1, 2, \ldots,$ $F_m \to \bigvee_{n\in\omega} F_n$;

A4. $(\bigwedge_{n\in\omega} (F_n \to F)) \to (\bigvee_{n\in\omega} F_n \to F)$;

A5. For every index $m = 0, 1, 2, \ldots,$ $\bigwedge_{n\in\omega} F_n \to F_m$;

A6. $(\bigwedge_{n\in\omega} (F \to F_n)) \to (F \to \bigwedge_{n\in\omega} F_n)$;

A7. $(F_1 \to (F_2 \to F_3)) \to ((F_1 \wedge F_2) \to F_3)$;

A8. $((F_1 \wedge F_2) \to F_3) \to (F_1 \to (F_2 \to F_3))$;

A9. $(F_1 \wedge -F_1) \to F_2$;

A10. $F \vee -F$;

R1. From $F_1 \to F_2$ and F_1, infer F_2 (*modus ponens*);

R2. From $F_0, F_1, F_2, \ldots,$ infer $\bigwedge_{n\in\omega} F_n$ (conjunction rule).

2.1.3 Theorem (Lindenbaum [unpublished], Tarski [34]) *Let $\mathcal{S}$ be the least set of formulas of the language $\mathcal{L}$ of countable sentential calculi and $\equiv$ be an equivalence relation defined on $\mathcal{S}$, any two formulas F_1, F_2 of $\mathcal{L}$ being logically equivalent if and only if they materially imply each other, that is to say,*

(1) $F_1 \equiv F_2$ *if and only if* $F_1 \to F_2$ *and* $F_2 \to F_1$.

If all instances of Schemas A1–A10 *are theorems of $\mathcal{L}$, then the structure $\mathfrak{S} / \equiv\ = (\mathcal{S} / \equiv; -, \to, \bigvee, \bigwedge)$, where $\mathcal{S} / \equiv$ is the set of formulas of the language $\mathcal{L}$ of countable sentential calculi modulo logical equivalence, $-$ is the lattice operation of complementation, $\to$ is the lattice operation of relative pseudo-complementation, $\bigvee$ is the lattice operation of finite or denumerable join, and $\bigwedge$ is the lattice operation of finite or denumerable meet, is a σ-complete Boolean algebra.*

PROOF For clarity, the argument is split into five lemmas.

Lemma 1 *For every formula* F_1, F_2 *of* $\mathcal{L}$, *write*

(2) $$F_1 \leq F_2 \text{ if and only if } F_1 \to F_2.$$

If all instances of Schemas A1–A2 *are theorems of* $\mathcal{L}$, *then the relation* $\leq$, *to be read 'is contained in' or 'is a part of', is reflexive and transitive, that is to say,* $\leq$ *is a quasi-ordering on* $\mathcal{S}$.

PROOF Clearly, $F \leq F$ because $F \to F$ by Schema A1.

Suppose now that $F_1 \leq F_2$ and that $F_2 \leq F_3$, that is to say, that $F_1 \to F_2$ and that $F_2 \to F_3$ by (2). If $F_1 \to F_2$, then $(F_2 \to F_3) \to (F_1 \to F_3)$ by Schema A2 and *modus ponens*, which implies, once again, by *modus ponens*, that $F_1 \to F_3$ because $F_2 \to F_3$. Hence, $F_1 \leq F_3$ by (2), which completes the proof. ▮

Lemma 2 *For every formula* F_1, F_2 *of* $\mathcal{L}$, *write*

(3) $$[F_1] \leq [F_2] \text{ if and only if } F_1 \leq F_2.$$

If all instances of Schemas A1–A2 *are theorems of* $\mathcal{L}$, *then the relation* $\leq$ *on the quotient set* $\mathcal{S} \,/\, \equiv$ *is antisymmetric, every quasi-ordering on* $\mathcal{S}$ *naturally inducing a partial ordering on* $\mathcal{S} \,/\, \equiv$.

PROOF Clearly, $\leq$ is a quasi-ordering on $\mathcal{S} \,/\, \equiv$ by Lemma 1.

Suppose now that $[F_1] \leq [F_2]$ and that $[F_2] \leq [F_1]$. Then $F_1 \leq F_2$ and $F_2 \leq F_1$ by (3) if and only if $F_1 \to F_2$ and $F_2 \to F_1$ by (2). Hence, $F_1 \equiv F_2$ by (1), that is to say, by the method of identification of equivalent elements, $[F_1] = [F_2]$ as required. ▮

Lemma 3 *If all instances of Schemas* A1–A2 *and* A3–A6 *are theorems of $\mathcal{L}$, then the partial ordering $\leq$ on $\mathcal{S} / \equiv$ is a σ-complete lattice ordering, and $(\mathcal{S} / \equiv; \bigvee, \bigwedge)$ is a σ-complete lattice. Moreover, for every indexed family $\{F_n : n \in \omega\}$ of formulas of $\mathcal{L}$,*

$$(4) \qquad \textstyle\bigvee_{n\in\omega} [F_n] = [\bigvee_{n\in\omega} F_n] \ \text{ and } \ \bigwedge_{n\in\omega} [F_n] = [\bigwedge_{n\in\omega} F_n],$$

$\equiv$ being a congruence relation with respect to the operations of disjunction $\bigvee$ and conjunction $\bigwedge$ in the algebra $\mathfrak{S}$ of formulas of $\mathcal{L}$.

PROOF Let $\{[F_n] : n \in \omega\}$ be an arbitrary indexed family of equivalence classes of formulas of $\mathcal{L}$. Then $\bigvee_{n\in\omega} [F_n]$ is certainly an upper bound for $\{[F_n] : n \in \omega\}$ because, for every index $m = 0, 1, 2, \ldots$, $F_m \to \bigvee_{n\in\omega} F_n$ by Schema A3 if and only if $[F_m] \leq [\bigvee_{n\in\omega} F_n]$ by (2) and (3) if and only if $[F_m] \leq \bigvee_{n\in\omega} [F_n]$ by (4) (that the relation $\equiv$ is a congruence relation on $\mathcal{S}$, that is to say, that $\equiv$ satisfies the Substitution Property, is a routine verification).

In fact, let F be a formula of $\mathcal{L}$ such that $F_0 \to F$, $F_1 \to F$, $F_2 \to F, \ldots$. Then $\bigwedge_{n\in\omega} (F_n \to F)$ by the conjunction rule, whence $\bigvee_{n\in\omega} F_n \to F$ by Schema A4 and *modus ponens* if and only if $[\bigvee_{n\in\omega} F_n] \leq [F]$ if and only if $\bigvee_{n\in\omega} [F_n] \leq [F]$. This implies that $\bigvee_{n\in\omega} [F_n]$ is the least upper bound of $\{[F_n] : n \in \omega\}$ as claimed.

Of course, by Schemas A5–A6, a contradual argument shows that $\bigwedge_{n\in\omega} [F_n]$ is the greatest lower bound of $\{[F_n] : n \in \omega\}$. It follows that every indexed family of equivalence classes of formulas of $\mathcal{L}$ has a least upper bound and a greatest lower bound because $\{[F_n] : n \in \omega\}$ is arbitrary. By definition, $\leq$ is a σ-complete lattice ordering and $(\mathcal{S} / \equiv ; \bigvee, \bigwedge)$ a σ-complete lattice. ▮

Lemma 4 *If all instances of Schemas* A1–A6 *and* A7–A8 *are theorems of $\mathcal{L}$, then $(\mathcal{S} / \equiv ; \rightarrow, \bigvee, \bigwedge)$ is a relatively pseudo-complemented, σ-complete lattice. Moreover, for every formula F_1, F_2 of $\mathcal{L}$,*

(5) $$[F_1] \rightarrow [F_2] = [F_1 \rightarrow F_2],$$

$\equiv$ being a congruence relation with respect to the operation of material implication $\rightarrow$ in the algebra $\mathfrak{S}$ of formulas of $\mathcal{L}$.

PROOF Recall that the *pseudo-complement of* $[F_1]$ *relative to* $[F_2]$, denoted by $[F_1] \rightarrow [F_2]$, is the largest equivalence class $[F]$ of formulas of $\mathcal{L}$, if it exists, such that $[F_1] \wedge [F] \leq [F_2]$. In practice, it is shown that, for every equivalence class $[F_3]$ of formulas of $\mathcal{L}$,

$$[F_3] \leq [F_1] \rightarrow [F_2] \text{ if and only if } [F_1] \wedge [F_3] \leq [F_2].$$

Indeed, suppose that $[F_3] \leq [F_1] \rightarrow [F_2]$, that is to say, that $[F_3] \leq [F_1 \rightarrow F_2]$ by (5) if and only if $F_3 \rightarrow (F_1 \rightarrow F_2)$ by (3) and (2). Then, certainly, $(F_3 \wedge F_1) \rightarrow F_2$ by Schema A7 and *modus ponens* if and only if $[F_3 \wedge F_1] \leq [F_2]$ by (2) and (3) if and only if $[F_3] \wedge [F_1] \leq [F_2]$ by (4). Since the operation of meet $\wedge$ is commutative in every lattice, $[F_1] \wedge [F_3] \leq [F_2]$ as required.

Conversely, suppose that $[F_3] \wedge [F_1] \leq [F_2]$, that is to say, that $[F_3 \wedge F_1] \leq [F_2]$ if and only if $(F_3 \wedge F_1) \rightarrow F_2$. Then $F_3 \rightarrow (F_1 \rightarrow F_2)$ by Schema A8 and *modus ponens* if and only if $[F_3] \leq$

$[F_1 \to F_2]$ if and only if $[F_3] \leq [F_1] \to [F_2]$. It follows that the pseudo-complement of $[F_1]$ relative to $[F_2]$ exists; since $[F_1], [F_2]$ are arbitrary, $(\mathcal{S} \,/\, \equiv; \to, \bigvee, \bigwedge)$ is relatively pseudo-complemented. ▮

Lemma 5 *If all instances of Schemas* A1–A8 *and* A9–A10 *are theorems of* $\mathcal{L}$, *then* $(\mathcal{S} \,/\, \equiv; -, \to, \bigvee, \bigwedge) = \mathfrak{S} \,/\, \equiv$ *is a* σ*-complete Boolean algebra. Moreover, for every formula* F *of the language* $\mathcal{L}$,

$$-[F] = [-F], \tag{6}$$

$\equiv$ *being a congruence relation with respect to the operation of negation* $-$ *in the algebra* $\mathfrak{S}$ *of formulas of the language* $\mathcal{L}$.

PROOF Notice first that $(\mathcal{S} \,/\, \equiv; \to, \bigvee, \bigwedge)$, as a relatively pseudo-complemented lattice, contains the unit element $[\mathbf{1}]$. Indeed, by definition, and for every equivalence class $[F_1], [F_2]$ of formulas of $\mathcal{L}$, $[F_2] \leq [F_1] \to [F_1]$ if and only if $[F_1] \wedge [F_2] \leq [F_1]$. Hence, in particular, if $[F_1] \wedge [F_2] \leq [F_1]$, which holds in every lattice, then $[F_2] \leq [F_1] \to [F_1]$, whence $[F_1] \to [F_1] = [\mathbf{1}]$ because $[F_2]$ is arbitrary.

In fact, since $(F_1 \wedge -F_1) \to F_2$ by Schema A9 if and only if $[F_1 \wedge -F_1] \leq [F_2]$ by (2) and (3) if and only if $[F_1] \wedge -[F_1] \leq [F_2]$ by (4) and (6), $(\mathcal{S} \,/\, \equiv; \to, \bigvee, \bigwedge)$ also contains a zero element $[\mathbf{0}]$ and, consequently, is a pseudo-Boolean algebra with pseudo-complement defined as $-[F] = [F] \to [\mathbf{0}]$ for every equivalence class $[F]$ of formulas of $\mathcal{L}$. Of course, this new algebra $(\mathcal{S} \,/\, \equiv; -, \to, \bigvee, \bigwedge)$ inherits the structure of $(\mathcal{S} \,/\, \equiv; \to, \bigvee, \bigwedge)$ and is σ-complete.

Finally, suppose that a formula of $\mathcal{L}$ is identifiable with the unit element $\mathbf{1}$ in $\mathfrak{S} = (\mathcal{S}\,; -, \to, \bigvee, \bigwedge)$ if and only if it is a theorem of $\mathcal{L}$. Then $F \vee -F = \mathbf{1}$ because $F \vee -F$ by Schema A10. It follows immediately that $[F] \vee -[F] = [\mathbf{1}]$ in the pseudo-Boolean algebra $(\mathcal{S}\ /\ \equiv\,; -, \to, \bigvee, \bigwedge)$, which implies that the pseudo-complement $-[F]$ in $(\mathcal{S}\ /\ \equiv\,; -, \to, \bigvee, \bigwedge)$ is the complement of $[F]$. By definition, $(\mathcal{S}\ /\ \equiv\,; -, \to, \bigvee, \bigwedge) = \mathfrak{S}\ /\ \equiv$ is a σ-complete Boolean algebra. ▮

This completes the proof of Theorem 2.1.3. ▮

The σ-complete Boolean algebra $\mathfrak{S}\ /\ \equiv$ of formulas of $\mathcal{L}$ modulo logical equivalence of Theorem 2.1.3 is called the *Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi.*

2.1.4 Definition Let $\mathfrak{S}\ /\ \equiv$ be the Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi. Then $\mathfrak{S}\ /\ \equiv$ satisfies the *countable chain condition* (or *is c.c.c.*) if every disjoint family of equivalence classes of formulas of $\mathcal{L}$ is at most denumerable.

2.1.5 Definition Let $\mathfrak{S}\ /\ \equiv$ be the Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi. Then $\mathfrak{S}\ /\ \equiv$ is *weakly countably distributive* if, for every indexed family $\{[F_{m,n}] : m, n \in \omega \times \omega\}$ of equivalence classes of formulas of the language $\mathcal{L}$ such that $[F_{m,n}] \leq [F_{m,n+1}]$, the join $\bigvee_{n\in\omega} [F_{m,n}]$ as well as the meets $\bigwedge_{m\in\omega} [F_{m,\varphi(m)}]$ and $\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}]$ exist in $\mathfrak{S}\ /\ \equiv$, and

(7) $$\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}] = \bigvee_{\varphi\in\omega^\omega} \bigwedge_{m\in\omega} [F_{m,\varphi(m)}],$$

where ω^ω denotes the set of all mappings φ of ω into ω.

2.1.6 Theorem (von Neumann [23]) *The Lindenbaum–Tarski algebra $\mathfrak{S} \,/ \equiv$ of the language $\mathcal{L}$ of countable sentential calculi carries a strictly positive, countably additive probability measure* $\mathbf{p}$ *only if it satisfies the countable chain condition and is weakly countably distributive.*

PROOF Clearly, $\mathfrak{S} \,/ \equiv$ is c.c.c. For a disjoint family of equivalence classes of formulas of $\mathcal{L}$ cannot contain, for every integer $n \geq 1$, as many as n elements whose probability is greater than $1 \,/\, n$.

To see that $\mathfrak{S} \,/ \equiv$ must also be weakly countably distributive, let $\{[F_{m,n}] : m, n \in \omega \times \omega\}$ be an indexed family of equivalence classes of formulas of $\mathcal{L}$ such that $[F_{m,n}] \leq [F_{m,n+1}]$, and suppose that every relevant join and meet exists in $\mathfrak{S} \,/ \equiv$. Clearly, $\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}] \geq \bigvee_{\varphi\in\omega^\omega} \bigwedge_{m\in\omega} [F_{m,\varphi(m)}]$ holds, so it suffices to show that $\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}] \leq \bigvee_{\varphi\in\omega^\omega} \bigwedge_{m\in\omega} [F_{m,\varphi(m)}]$ whenever $\mathfrak{S} \,/ \equiv$ carries a strictly positive, countably additive probability measure $\mathbf{p}$ [12].

In fact, since $\mathbf{p}$ is countably additive, and, by definition, $[F_{m,n}] \leq [F_{m,n+1}]$, $\mathbf{p}(\bigvee_{n\in\omega} [F_{m,n}]) = \lim_{n\to\infty} \mathbf{p}([F_{m,n}])$. Hence, there exist a φ and an $\epsilon > 0$ such that $\mathbf{p}(\bigvee_{n\in\omega} [F_{m,n}]) - 2^{-m-1}\epsilon < \mathbf{p}([F_{m,\varphi(m)}])$ if and only if $\mathbf{p}(\bigvee_{n\in\omega} [F_{m,n}] \wedge -[F_{m,\varphi(m)}]) < 2^{-m-1}\epsilon$, whence

$$\text{(8)} \qquad \mathbf{p}(\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}] \wedge \bigvee_{m\in\omega} -[F_{m,\varphi(m)}]) < \epsilon.$$

Indeed, it follows immediately from the infinite distributive law

$$\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}] \wedge \bigvee_{m\in\omega} -[F_{m,\varphi(m)}]$$
$$= \bigvee_{m\in\omega} (\bigwedge_{m\in\omega} \bigvee_{n\in\omega} [F_{m,n}] \wedge -[F_{m,\varphi(m)}])$$

[18, Lemma 1.33(b), p. 22] that $\mathbf{p}(\bigwedge_{m\in\omega}\bigvee_{n\in\omega}[F_{m,n}] \wedge \bigvee_{m\in\omega} -[F_{m,\varphi(m)}])$ $= \mathbf{p}(\bigvee_{m\in\omega}(\bigwedge_{m\in\omega}\bigvee_{n\in\omega}[F_{m,n}] \wedge -[F_{m,\varphi(m)}])) = \sum_{m=0}^{\infty}\mathbf{p}(\bigwedge_{m\in\omega}\bigvee_{n\in\omega}[F_{m,n}]$ $\wedge -[F_{m,\varphi(m)}]) \leq \sum_{m=0}^{\infty}\mathbf{p}(\bigvee_{n\in\omega}[F_{m,n}] \wedge -[F_{m,\varphi(m)}]) < \sum_{m=0}^{\infty} 2^{-m-1}\epsilon = \epsilon$ because the series $\sum_{m=0}^{\infty} 2^{-m-1}$ converges to 1. This proves (8).

Suppose now that there exists a $[F_0] \in \mathfrak{S} / \equiv$ such that $[F_0]$ is an upper bound for all the $\bigwedge_{m\in\omega}[F_{m,\varphi(m)}]$'s (i.e. $[F_0] \geq \bigwedge_{m\in\omega}$ $[F_{m,\varphi(m)}]$ for every φ), and set $[F] = \bigwedge_{m\in\omega}\bigvee_{n\in\omega}[F_{m,n}]$. Then

$$\begin{aligned}
\mathbf{p}([F_0]) &\geq \mathbf{p}([F]) - \mathbf{p}([F]) + \mathbf{p}(\textstyle\bigwedge_{m\in\omega}[F_{m,\varphi(m)}]) \\
&\geq \mathbf{p}([F]) - \mathbf{p}([F] \wedge -\textstyle\bigwedge_{m\in\omega}[F_{m,\varphi(m)}]) \\
&\geq \mathbf{p}([F]) - \mathbf{p}([F] \wedge \textstyle\bigvee_{m\in\omega} -[F_{m,\varphi(m)}]) \\
&> \mathbf{p}([F]) - \epsilon
\end{aligned}$$

by (8), whence $\mathbf{p}([F_0]) \geq \mathbf{p}([F])$ because ϵ is arbitrary. It follows that $\mathbf{p}([F_0] \wedge [F]) = \mathbf{p}([F])$ because $\mathbf{p}([F_0] \wedge [F]) \geq \mathbf{p}([F] \wedge [F])$ $\geq \mathbf{p}([F_0] \wedge [F])$, which implies, since $\mathbf{p}$ is strictly positive, that $[F_0] \wedge [F] = [F]$ if and only if $[F] \leq [F_0]$. Hence, there exists no upper bound $[F_0]$ for the $\bigwedge_{m\in\omega}[F_{m,\varphi(m)}]$'s such that $[F_0] < [F]$, whence $\bigwedge_{m\in\omega}\bigvee_{n\in\omega}[F_{m,n}] \leq \bigvee_{\varphi\in\omega^\omega}\bigwedge_{m\in\omega}[F_{m,\varphi(m)}]$ as required. ∎

A Lindenbaum–Tarski algebra $\mathfrak{S} / \equiv$ of the language $\mathcal{L}$ of countable sentential calculi which is both c.c.c. and weakly countably distributive is called *almost measurable*. Every almost measurable Lindenbaum–Tarski algebra is a Lindenbaum–Tarski algebra in the sense of Theorem 2.1.3, but not conversely. Almost measurable Lindenbaum–Tarski algebras are denoted by $\mathfrak{S}^* / \equiv$. Of course, being c.c.c., every almost measurable Lindenbaum–Tarski algebra is complete.

2.1.7 Theorem (Stone [30]) *Let* $\mathfrak{S}^* / \equiv$ *be an almost measurable Lindenbaum–Tarski algebra of the language* $\mathcal{L}$ *of countable sentential calculi,* Z *be a set of maximal filters of* $\mathfrak{S}^* / \equiv$*, and* γ *be a Boolean homomorphism of* $\mathfrak{S}^* / \equiv$ *into the power set* $\mathcal{P}(Z)$ *such that, for every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,* $\gamma([F]) = \{\mathbf{F} \in Z : [F] \in \mathbf{F}\}$*. Then* Z *equipped with the topology for which the* $\gamma([F])$*'s form an open base is a zero-dimensional compact Hausdorff space.*

PROOF Of course, since γ is a Boolean homomorphism, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $\gamma([F]) = Z \setminus \gamma(-[F])$ is also closed in Z. It follows that every basic open set $\gamma([F])$ is open-closed, which in turn implies that Z is zero-dimensional.

To see that Z is compact, let $\mathbf{E}$ be a filter of closed subsets of Z (note that $\mathbf{E}$ has the finite intersection property); it is shown that $\bigcap \mathbf{E} \neq \emptyset$, that is to say, that $\mathbf{E}$ has non-empty intersection. Indeed, write $\overline{\mathbf{E}} = \{\gamma([F]) : \gamma([F]) \supseteq \mathrm{E}$ for some $\mathrm{E} \in \mathbf{E}\}$. Since the $\gamma([F])$'s form a closed base for the topology, $\bigcap \mathbf{E} = \bigcap \{\gamma([F]) : \gamma([F]) \in \overline{\mathbf{E}}\}$; and since $\overline{\mathbf{E}}$ is a proper filter in Z, it is contained in a maximal filter $\mathbf{F}$ by the Ultrafilter Lemma [18, Proposition 2.16, p. 33]. It follows that $\mathbf{F} \in \bigcap \mathbf{E}$ because $\mathbf{F} \in \gamma([F])$ for every $\gamma([F]) \in \overline{\mathbf{E}}$, whence $\mathbf{E}$ has non-empty intersection as required.

Finally, let $\mathbf{F}_1, \mathbf{F}_2$ be maximal filters in Z such that $\mathbf{F}_1 \neq \mathbf{F}_2$, and suppose, without loss of generality, that $[F] \in \mathbf{F}_1 \setminus \mathbf{F}_2$. Then, certainly, $-[F] \in \mathbf{F}_2$ because $[F] \notin \mathbf{F}_2$. Since $\mathbf{F}_1 \in \gamma([F])$ and $\mathbf{F}_2 \in \gamma(-[F])$; and since, moreover, $\gamma([F])$ and $\gamma(-[F])$ are disjoint open sets, Z is Hausdorff as claimed. The proof is complete. ▮

The zero-dimensional compact Hausdorff space Z of Theorem 2.1.7 is known as the *Stone space of* $\mathfrak{S}^* / \equiv$. Stone spaces Z are essential to a deeper understanding of the structure of almost measurable Lindenbaum–Tarski algebras. Theorems 2.1.9 and 2.1.11 below describe two important properties of Z; the next result is an instance of the Stone representation theorem for Boolean algebras.

2.1.8 Theorem (Stone [29, 30]) *Let Z be the Stone space of an almost measurable Lindenbaum–Tarski algebra $\mathfrak{S}^* / \equiv$ of the language $\mathcal{L}$ of countable sentential calculi. Then $\mathfrak{S}^* / \equiv$ is isomorphic, as a Boolean algebra, to the field* $\mathrm{Clop}(Z)$ *of open-closed subsets of Z.*

PROOF It is first verified that the open-closed subsets of Z are exactly the basic open sets of Z. Indeed, since Z is zero-dimensional, $\{\gamma([F]) : [F] \in \mathfrak{S}^* / \equiv\} \subseteq \mathrm{Clop}(Z)$. Conversely, let C be an open-closed subset of Z. Since C is open, and the $\gamma([F])$'s form a base, there exists a family $\mathcal{C} \subseteq \mathfrak{S}^* / \equiv$ of equivalence classes of formulas of $\mathcal{L}$ such that $C = \bigcup\{\gamma([F]) : [F] \in \mathcal{C}\}$; and since C is closed, hence compact, there exists a finite family $\mathcal{D} \subseteq \mathcal{C}$ of equivalence classes of formulas of $\mathcal{L}$ such that $C = \bigcup\{\gamma([F]) : [F] \in \mathcal{D}\}$. It follows that $C = \gamma(\bigvee\{[F] : [F] \in \mathcal{D}\})$ because γ is a (finite) Boolean homomorphism of $\mathfrak{S}^* / \equiv$ into $\mathcal{P}(Z)$, whence

$\mathrm{Clop}(Z) \subseteq \{\gamma([F]) : [F] \in \mathfrak{S}^* / \equiv\}$ because C is arbitrary.

To see that γ is a Boolean isomorphism into $\mathrm{Clop}(Z)$, let $[F_1]$, $[F_2]$ be equivalence classes of formulas of $\mathcal{L}$ such that $[F_1] \neq [F_2]$. It is shown that $\gamma([F_1]) \neq \gamma([F_2])$, that is to say, that γ is a bijective Boolean homomorphism. Indeed, since $[F_1] \wedge -[F_2] \neq [\mathbf{0}]$, let $\mathbf{E} = \{[F] \in \mathfrak{S}^* / \equiv\ : [F] \geq [F_1] \wedge -[F_2]\}$. Then $\mathbf{E}$ is a filter of $\mathfrak{S}^* / \equiv$ which, by the Ultrafilter Lemma, is contained in a maximal filter $\mathbf{F}$. Since $[F_1] \geq [F_1] \wedge -[F_2]$ and $-[F_2] \geq [F_1] \wedge -[F_2]$, it follows that $[F_1] \in \mathbf{F}$ and $-[F_2] \in \mathbf{F}$. This implies that $\mathbf{F} \in \gamma([F_1])$ and $\mathbf{F} \notin \gamma([F_2])$, whence $\gamma([F_1]) \neq \gamma([F_2])$ as required. ▮

2.1.9 Theorem (Stone [31]) *The closure of every open subset of the Stone space Z of an almost measurable Lindenbaum–Tarski algebra $\mathfrak{S}^* / \equiv$ of the language $\mathcal{L}$ of countable sentential calculi is open.*

PROOF Let $\mathrm{Clop}(Z)$ be the field of open-closed subsets of Z and $\{\gamma([F_n]) : n \in \omega\}$ be an indexed family of open-closed sets in that same space Z. Clearly, $\bigcup_{n\in\omega} \gamma([F_n])$ is open, with $\bigvee_{n\in\omega} \gamma([F_n])$ being the least upper bound of $\mathrm{Clop}(Z)$ because $\mathrm{Clop}(Z)$ is c.c.c. as an isomorphic copy of $\mathfrak{A}$ by Theorem 2.1.8. Let $U = \bigcup_{n\in\omega} \gamma([F_n])$ and $V = \bigvee_{n\in\omega} \gamma([F_n])$. It is shown that $\mathrm{cl}\ U = V$.

Indeed, since V is an upper bound for the family $\{\gamma([F_n]) : n \in \omega\}$, $U \subseteq V$, whence $\mathrm{cl}\ U \subseteq V$ because V is (open-)closed. Suppose now that $V \setminus \mathrm{cl}\ U$ is non-empty. Then it must contain an open-closed set W because Z is zero-dimensional. It follows that cl $U \subseteq W \subseteq V$, which contradicts the premiss that V is the least upper bound of U. Hence, cl $U = V$ as required. ∎

A topological space in which the closure of every open set is open is said to be *extremally disconnected*. An extremally disconnected compact Hausdorff space is also called a *Stonean space*. Of course, every Stonean space is a Stone space, but not conversely.

2.1.10 Definition In every topological space, a set is

(i) *nowhere dense* if the interior of its closure is empty;

(ii) *meagre* if it is a countable union of nowhere dense sets.

2.1.11 Theorem (Kelley [15]) *Every meagre subset M of the Stone space Z of an almost measurable Lindenbaum–Tarski algebra $\mathfrak{S}^* / \equiv$ of the language $\mathcal{L}$ of countable sentential calculi is nowhere dense.*

PROOF Let Clop(Z) be the field of open-closed subsets of Z. The point is to notice first, following Kelley's ingenious insight, that a set N in Z is nowhere dense if and only if the greatest lower bound of all open-closed subsets of Z containing N is empty, that is to say, since Clop(Z) is c.c.c., if and only if, for some indexed family $\{\gamma([F_n]) : n \in \omega\}$ of open-closed subsets of Z such that $\gamma([F_n]) \supseteq N$, $\bigwedge_{n\in\omega} \gamma([F_n]) = \text{int} \bigcap_{n\in\omega} \gamma([F_n]) = \emptyset$. Of course, the aforementioned family may be assumed to be decreasing.

Now, since Clop(Z) is also weakly countably distributive, there exists a decreasing indexed family $\{\gamma([F_{m,n}]) : m, n \in \omega \times \omega\}$ of open-closed subsets of Z such that, whenever $\bigwedge_{n\in\omega}\gamma([F_{m,n}])$ as well as $\bigvee_{m\in\omega}\gamma([F_{m,\varphi(m)}])$ and $\bigvee_{m\in\omega}\bigwedge_{n\in\omega}\gamma([F_{m,n}])$ exist in $\mathrm{Clop}(Z)$,

$$(9) \qquad \bigvee_{m\in\omega}\bigwedge_{n\in\omega}\gamma([F_{m,n}]) = \bigwedge_{\varphi\in\omega^\omega}\bigvee_{m\in\omega}\gamma([F_{m,\varphi(m)}]),$$

where ω^ω denotes the set of all mappings φ of ω into ω.

Let $M = \bigcup_{m\in\omega} N_m$ be a meagre subset of Z. Since each N_m is nowhere dense, there exists a decreasing indexed family $\{\gamma([F_{m,n}]) : m,n \in \omega\times\omega\}$ of open-closed subsets of Z such that $\gamma([F_{m,n}]) \supseteq N_m$ and $\bigwedge_{n\in\omega}\gamma([F_{m,n}]) = \emptyset$. It follows that $\bigvee_{m\in\omega}\gamma([F_{m,n}]) \supseteq \bigcup_{m\in\omega} N_m$ and $\bigvee_{m\in\omega}\bigwedge_{n\in\omega}\gamma([F_{m,n}]) = \emptyset$, whence $\bigwedge_{\varphi\in\omega^\omega}\bigvee_{m\in\omega}\gamma([F_{m,\varphi(m)}]) = \emptyset$ by (9). Since $\bigvee_{m\in\omega}\gamma([F_{m,\varphi(m)}]) \supseteq \bigcup_{m\in\omega} N_m$, with $\bigvee_{m\in\omega}\gamma([F_{m,\varphi(m)}])$ open-closed because Z is extremally disconnected, every meagre subset M of Z is nowhere dense as claimed. The proof is complete. ▮

Note that $\mathrm{Clop}(Z)$ in the proof of Theorems 2.1.9 and 2.1.11, although a complete Boolean algebra, is not a complete field of sets. For if $\{\gamma([F_n]) : n \in \omega\}$ is an indexed family of open-closed subsets of Z, then $\bigcup_{n\in\omega}\gamma([F_n])$ is certainly not open-closed; yet, in some sense, it can be considered open-closed, as the next result, which is an instance of the Loomis–Sikorski theorem, shows.

2.1.12 Definition A *Baire set* in Z is an element of the least σ-field of sets containing the open-closed subsets of Z.

2.1.13 Theorem (Loomis [20], Sikorski [28]) *Every almost measurable Lindenbaum–Tarski algebra* $\mathfrak{S}^* / \equiv$ *of the language* $\mathcal{L}$ *of count-*

able sentential calculi is isomorphic to the quotient algebra $\mathcal{B}a(Z)\ /\ \mathcal{N}$, where $\mathcal{B}a(Z)$ is the σ-field of Baire subsets of the Stone space Z of $\mathfrak{S}^*\ /\ \equiv$, and $\mathcal{N} \subseteq \mathcal{B}a(Z)$ is the σ-ideal of nowhere dense sets.

PROOF Let $\mathrm{Clop}(Z)$ be the field of open-closed subsets of Z. Since $\mathrm{Clop}(Z)$ is an isomorphic copy of $\mathfrak{S}^*\ /\ \equiv$, it suffices to show that $\mathcal{B}a(Z)\ /\ \mathcal{N}$ is isomorphic to $\mathrm{Clop}(Z)$. Let $\triangle(Z)$ be the set of all subsets S of Z whose symmetric difference $\triangle$ with an open-closed set is meagre. It is first shown that $\triangle(Z)$ is a σ-field of sets.

Indeed, $\emptyset \triangle \emptyset = \emptyset$ and $Z \setminus (S \triangle \gamma([F])) = (Z \setminus S) \triangle \gamma([F])$ for every S in $\triangle(Z)$, whence $\triangle(Z)$ contains the empty set and is closed under the formation of complements. Consider now two indexed families $\{S_n : n \in \omega\}$ and $\{\gamma([F_n]) : n \in \omega\}$ of arbitrary and open-closed subsets of Z, respectively, and suppose that $S_n \triangle \gamma([F_n])$ is meagre for every n. Since $\bigcup_{n\in\omega} S_n \triangle \bigcup_{n\in\omega} \gamma([F_n]) \subseteq \bigcup_{n\in\omega} (S_n \triangle \gamma([F_n]))$, and meagre sets form a σ-ideal, $\bigcup_{n\in\omega} S_n \triangle \bigcup_{n\in\omega} \gamma([F_n])$ is certainly meagre. Recalling that $\mathrm{cl} \bigcup_{n\in\omega} \gamma([F_n])$ is, by Theorem 2.1.9, open, hence open-closed, and that $\bigcup_{n\in\omega} \gamma([F_n]) \triangle \mathrm{cl} \bigcup_{n\in\omega} \gamma([F_n])$, as a boundary set, is nowhere dense, that is to say, meagre, $\triangle(Z)$ is closed under the formation of countable unions as required.

Note that $\triangle(Z)$ contains all the open-closed and, consequently, all the Baire subsets of Z, the former generating the latter by definition. In fact, to every Baire set B corresponds exactly one open-closed set $\gamma([F])$ such that $B \triangle \gamma([F])$ is meagre. For suppose that there exists another open-closed set $\gamma([F'])$ such that $B \triangle \gamma([F'])$ is meagre. Then so is $\gamma([F]) \triangle \gamma([F']) \subseteq (B \triangle \gamma([F])) \cup (B \triangle \gamma([F']))$, which contradicts the Baire category theorem for compact Hausdorff spaces [4, Theorem 3.9.3, p. 197], whereby the interior of meagre sets is empty, because $\gamma([F]) \triangle \gamma([F'])$ has no empty interior.

Hence, since $(Z \setminus B) \triangle \gamma([F])$ and, for every family $\{B_n : n \in \omega\}$ and $\{\gamma([F_n]) : n \in \omega\}$ of Baire and open-closed subsets of Z, $\bigcup_{n\in\omega} B_n \triangle \mathrm{cl} \bigcup_{n\in\omega} \gamma([F_n])$ are meagre; and since, moreover, every open-closed set is Baire, the mapping which sends every Baire set B to an open-closed set $\gamma([F])$ is a Boolean epimorphism whose kernel is exactly the σ-ideal $\mathcal{M}$ of meagre sets. By the Homomorphism Theorem [18, Proposition 5.23, p. 77], $\mathcal{B}a(Z) \,/\, \mathcal{M}$ is isomorphic to Clop(Z). A glance at Theorem 2.1.11 completes the proof. ∎

2.2 Semantic theory of truth

2.2.1 Definition A *model for the classical (two-valued) sentential calculus* is an ordered system $(\mathfrak{A}, Z, \mathcal{F}, \xi)$, where $\mathfrak{A} = \mathfrak{S}^* \,/\, \equiv$ is an almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z is its Stone space, $\mathcal{F} = \{\emptyset, Z\} \,/\, \mathcal{N} \subseteq \mathcal{B}a(Z) \,/\, \mathcal{N}$ is the trivial algebra of Baire subsets of Z modulo nowhere dense sets, and ξ is a Boolean epimorphism of $\mathfrak{A}$ into $\mathcal{F}$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(I) $\quad$ int cl $\xi([F])^* = \emptyset$ if and only if $\xi([F])^* = \emptyset$, the open-closed representative $\xi([F])^*$ of the image $\xi([F])$ of $[F]$ being nowhere dense in Z ('false') if and only if it is empty.

Schema (I) probably deserves an explanation. For if it is clear that the property of 'nowhere denseness' rather applies to sets than equivalence classes of sets, that is to say, sets of sets, the reader may wonder why the representative $\xi([F])^*$ of $\xi([F])$ is open-closed at all. The point, of course, is that $\xi([F])^*$ must be uniquely determined for Schema (I) to be necessary and sufficient. And, by definition, each equivalence class $\xi([F])$ of Baire subsets of Z has at most one open-closed representative because $\mathcal{F} \subseteq \mathcal{B}a(Z) \,/\, \mathcal{N}$.

2.2.2 Definition In every topological space, a set is

(i) *non-boundary* if the closure of its interior is the space;

(ii) *comeagre* if it is a countable intersection of non-boundary sets.

(Non-boundary sets were first investigated by Wallace [35].)

2.2.3 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ *be a model for the classical sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(II) $\quad$ $\mathrm{cl}\ \mathrm{int}\ \xi([F])^* = Z$ *if and only if* $\xi([F])^* = Z$,

the open-closed representative $\xi([F])^*$ *of the image* $\xi([F])$ *of* $[F]$ *being non-boundary in* Z *('true') if and only if it is* Z*.*

PROOF Suppose that $\mathrm{int}\ \mathrm{cl}\ \xi(-[F])^* = \emptyset$ if and only if $\xi(-[F])^* = \emptyset$, that is to say, that $\xi(-[F])^*$ is nowhere dense in Z ('false') if and only if it is empty. Since $\mathrm{int}\ \xi(-[F])^* = Z \setminus \mathrm{cl}\ Z \setminus$

$\xi(-[F])^*$, and ξ is a Boolean homomorphism, it follows that

$$\begin{aligned} & Z \setminus \text{int cl } \xi(-[F])^* = Z \text{ if and only if } Z \setminus \xi(-[F])^* = Z \\ \Longleftrightarrow \quad & \text{cl } Z \setminus \text{cl } \xi(-[F])^* = Z \text{ if and only if } Z \setminus \xi(-[F])^* = Z \\ \Longleftrightarrow \quad & \text{cl int } Z \setminus \xi(-[F])^* = Z \text{ if and only if } Z \setminus \xi(-[F])^* = Z \\ \Longleftrightarrow \quad & \text{cl int } \xi([F])^* = Z \text{ if and only if } \xi([F])^* = Z, \end{aligned}$$

whence if a sentence is true if and only if its negation is false (principle of bivalence common to all two-valued calculi), then $\xi([F])^*$ is non-boundary in Z ('true') if and only if it is the space Z as claimed, which completes the proof of the theorem. ▮

At this point, a reader familiar with model theory probably recognised in Schema (II) the epitome of a T-schema and, more generally, Tarski's semantic conception of truth [33]. Hence, Schema (II) is not formula of $\mathcal{L}$ but a formula of a stronger language which contains $\mathcal{L}$, namely, that $\mathcal{L}^E$ of mathematical English viewed as an informal version of the first-order language of Zermelo–Fraenkel set theory with the axiom of choice (ZFC). In particular, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $\xi([F])^*$ in the definiendum of Schema (II) is the name of $[F]$ to which a predicate applies, cl int $\cdot = Z$ is the predicate itself (here: *is true*), and $\xi([F])^* = Z$, the definiens, is the translation of $[F]$ into $\mathcal{L}^E$. In Tarski's phraseology, constructions of that kind are called *formally correct.*

2.2.4 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ *be a model for the classical sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

$$\text{cl int } \xi(-[F])^* = Z \text{ if and only if } \text{int cl } \xi([F])^* = \emptyset,$$

the open-closed representative $\xi(-[F])^*$ *of the image* $\xi(-[F])$ *of* $-[F]$ *being non-boundary in* Z *if and only if the open-closed representative* $\xi([F])^*$ *of the image* $\xi([F])$ *of* $[F]$ *is nowhere dense in* Z*.*

PROOF Clearly, $\text{cl int } \xi(-[F])^* = \text{cl int } Z \setminus \xi([F])^*$ because ξ is a Boolean homomorphism. And, certainly, $\text{cl int } Z \setminus \xi([F])^* = \text{cl } Z \setminus \text{cl } \xi([F])^* = Z \setminus \text{int cl } \xi([F])^*$ by the dual interdefinability of the operations of closure and interior of a set. Hence, $\text{cl int } \xi(-[F])^* = Z \setminus \text{int cl } \xi([F])^*$ from which it follows immediately that $\text{cl int } \xi(-[F])^* = Z$ if and only if $\text{int cl } \xi([F])^* = \emptyset$ as required. ∎

2.2.5 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ *be a model for the classical sentential calculus. For every equivalence class* $[F_0], [F_1], [F_2], \ldots$ *of formulas of the language* $\mathcal{L}$ *of countable sentential calculi,*

$$\text{cl int } \xi(\textstyle\bigvee_{n\in\omega} [F_n])^* = Z \text{ if and only if } \text{cl int } \xi([F_n])^* = Z \text{ for some } n \in \omega,$$

the open-closed representative $\xi(\bigvee_{n\in\omega} [F_n])^*$ *of the image* $\xi(\bigvee_{n\in\omega} [F_n])$ *of* $\bigvee_{n\in\omega} [F_n]$ *being non-boundary in* Z *if and only if at least one open-closed representative* $\xi([F_0])^*, \xi([F_1])^*, \xi([F_2])^*, \ldots$ *of at least one image* $\xi([F_0]), \xi([F_1]), \xi([F_2]), \ldots$ *of* $[F_0], [F_1], [F_2], \ldots$ *is non-boundary in* Z*.*

PROOF The sufficiency poses no difficulties. For suppose that $\text{cl int } \xi([F_n])^* = Z$ for some n. Then, certainly, $\text{cl int } \xi(\bigvee_{n\in\omega} [F_n])^* = Z$ because $\xi([F_n])^* \subseteq \bigcup_{n\in\omega} \xi([F_n])^*$ for every n if and only if cl int

$\xi([F_n])^* \subseteq$ cl int $\bigcup_{n\in\omega} \xi([F_n])^*$ if and only if cl int $\xi([F_n])^* \subseteq$ cl int $\xi(\bigvee_{n\in\omega} [F_n])^*$ because ζ is a Boolean homomorphism.

Conversely, suppose that cl int $\xi(\bigvee_{n\in\omega} [F_n])^* = Z$. It is shown that cl int $\xi([F_n])^* = Z$ for some n. For suppose, *reductio ad absurdum*, that this is not the case. Then int cl $\xi([F_n])^* = \emptyset$ for every n, whence int cl $\bigcup_{n\in\omega} \xi([F_n])^* =$ int cl $\xi(\bigvee_{n\in\omega} [F_n])^* = \emptyset$ by Theorem 2.1.11. A contradiction. Hence, cl int $\xi(\bigvee_{n\in\omega} [F_n])^* = Z$ if and only if cl int $\xi([F_n])^* = Z$ for some n as claimed. ∎

Note that the Stone topology generated by the countable chain condition and the weak countable distributivity of $\mathfrak{A}$ is essential to the proof of Theorem 2.2.5, a countable union of nowhere dense sets being not nowhere dense in general; in fact, it can even be (everywhere) dense, the set $\mathbb{Q}$ of rationals in $\mathbb{R}$ equipped with the usual topology being a classic example. Moreover, a set which is not non-boundary is not nowhere dense in general; in other words, that ξ is a Boolean epimorphism of $\mathfrak{A}$ into $\mathcal{F} = \{\emptyset, Z\} / \mathcal{N}$ is equally essential to the proof of Theorem 2.2.5.

2.2.6 Theorem *Every comeagre subset $\bar{M}$ of the Stone space Z of an almost measurable Lindenbaum–Tarski algebra $\mathfrak{A} = \mathfrak{S}^* / \equiv$ of the language $\mathcal{L}$ of countable sentential calculi is non-boundary.*

PROOF Let Clop(Z) be the field of open-closed subsets of Z. The point is to notice first that, contradually to Theorem 2.1.11, a set $\bar{N}$ in Z is non-boundary if and only if the least upper bound of all open-closed subsets of Z contained in $\bar{N}$ is Z, that is to say, since $\mathrm{Clop}(Z)$ is c.c.c, if and only if, for some indexed family $\{\gamma([F_n]) : n \in \omega\}$ of open-closed subsets of Z such that $\gamma([F_n]) \subseteq \bar{N}$, $\bigvee_{n\in\omega} \gamma([F_n]) = \mathrm{cl} \bigcup_{n\in\omega} \gamma([F_n]) = Z$. Of course, the aforementioned family may be assumed to be increasing.

Let $\bar{M} = \bigcap_{m\in\omega} \bar{N}_m$ be a comeagre subset of Z. Since each $\bar{N}_m$ is non-boundary, there certainly exists an increasing indexed family $\{\gamma([F_{m,n}]) : m, n \in \omega \times \omega\}$ of open-closed subsets of Z such that $\gamma([F_{m,n}]) \subseteq \bar{N}_m$ and $\bigvee_{n\in\omega} \gamma([F_{m,n}]) = Z$. It follows immediately that $\bigwedge_{m\in\omega} \gamma([F_{m,n}]) \subseteq \bigcap_{m\in\omega} \bar{N}_m$ and $\bigwedge_{m\in\omega} \bigvee_{n\in\omega} \gamma([F_{m,n}]) = Z$, whence $\bigvee_{\varphi\in\omega^\omega} \bigwedge_{m\in\omega} \gamma([F_{m,\varphi(m)}]) = Z$ because $\mathrm{Clop}(Z)$ is weakly countably distributive. Since $\bigwedge_{m\in\omega} \gamma([F_{m,\varphi(m)}]) \subseteq \bigcap_{m\in\omega} \bar{N}_m$, with $\bigwedge_{m\in\omega} \gamma([F_{m,\varphi(m)}])$ open-closed because Z is extremally disconnected, every comeagre subset $\bar{M}$ of Z is non-boundary as claimed. ∎

2.2.7 Theorem *Let $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ be a model for the classical sentential calculus. For every equivalence class $[F_0], [F_1], [F_2], \ldots$ of formulas of the language $\mathcal{L}$ of countable sentential calculi,*

$$\mathrm{cl\ int}\ \xi(\textstyle\bigwedge_{n\in\omega} [F_n])^* = Z \text{ if and only if }$$

$$\mathrm{cl\ int}\ \xi([F_n])^* = Z \text{ for every } n \in \omega,$$

the open-closed representative $\xi(\bigwedge_{n\in\omega} [F_n])^*$ of the image $\xi(\bigwedge_{n\in\omega} [F_n])$ of $\bigwedge_{n\in\omega} [F_n]$ being non-boundary in Z if and only if all open-closed representatives $\xi([F_0])^*, \xi([F_1])^*, \xi([F_2])^*, \ldots$ of all images $\xi([F_0]), \xi([F_1])^*$ $\xi([F_2]), \ldots$ of $[F_0], [F_1], [F_2], \ldots$ are non-boundary in Z.

PROOF The argument is similar to that of Theorem 2.2.5. For suppose that cl int $\xi(\bigwedge_{n\in\omega}[F_n])^* = Z$, that is to say, that cl int $\bigcap_{n\in\omega}\xi([F_n])^* = Z$. Then cl int $\xi([F_n])^* = Z$ for every n because $\bigcap_{n\in\omega}\xi([F_n])^* \subseteq \xi([F_n])^*$ for every n if and only if cl int $\bigcap_{n\in\omega}\xi([F_n])^* \subseteq$ cl int $\xi([F_n])^*$ if and only if cl int $\xi(\bigwedge_{n\in\omega}[F_n])^* \subseteq$ cl int $\xi([F_n])^*$. The converse follows immediately from Theorem 2.2.6, every comeagre subset of Z, that is to say, every countable intersection of non-boundary subsets of Z, being non-boundary. ▮

2.2.8 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ *be a model for the classical sentential calculus. For every equivalence class* $[F_1], [F_2]$ *of formulas of* $\mathcal{L}$*,*

$$\text{cl int } \xi([F_1] \to [F_2])^* = Z \text{ if and only if } \text{cl int } \xi(-[F_1])^* = Z \text{ or } \text{cl int } \xi([F_2])^* = Z,$$

the open-closed representative $\xi([F_1] \to [F_2])^*$ *of the image* $\xi([F_1] \to [F_2])$ *of* $[F_1] \to [F_2]$ *being non-boundary in* Z *iff at least one open-closed representative* $\xi(-[F_1])^*$ *or* $\xi([F_2])^*$ *of at least one image* $\xi(-[F_1])$ *of* $-[F_1]$ *or* $\xi([F_2])$ *of* $[F_2]$ *is non-boundary in* Z*.*

PROOF Recall that, in every Lindenbaum–Tarski algebra $\mathfrak{A}$, the relative pseudo-complement $[F_1] \to [F_2]$ is the largest equivalence class $[F]$ of formulas of $\mathcal{L}$ such that $[F_1] \wedge [F] = [F_2]$. It follows immediately that $[F_1] \wedge [F] = [F_2]$ if and only if $[F_1] \wedge [F] \wedge -[F_2] = [\mathbf{0}]$ if and only if $([F_1] \wedge -[F_2]) \wedge [F] = [\mathbf{0}]$ if and only if $[F] = -([F_1] \wedge -[F_2])$ if and only if $[F] = -[F_1] \vee [F_2]$ by De Morgan's laws. A glance at Theorem 2.2.5 completes the proof. ▮

2.2.9 Convention T (Tarski [33]) A formally correct definition of the predicate *is true*, formulated in the metalanguage $\mathcal{L}^E$ of mathematical English, is *materially adequate* if it implies all instances of a T-schema, that is to say, all instances of Schema (II).

2.2.10 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{F}, \xi)$ *be a model for the classical sentential calculus. The following schemas constitute a formally correct and materially adequate definition of the predicate* is true*:*

$\mathrm{cl\ int}\ \xi([F])^* = Z$ *if and only if* $\xi([F])^* = Z$;

$\mathrm{cl\ int}\ \xi(-[F])^* = Z$ *if and only if* $\mathrm{int\ cl}\ \xi([F])^* = \emptyset$;

$\mathrm{cl\ int}\ \xi(\bigvee_{n\in\omega}[F_n])^* = Z$ *if and only if* $\mathrm{cl\ int}\ \xi([F_n])^* = Z$ *for some* n;

$\mathrm{cl\ int}\ \xi(\bigwedge_{n\in\omega}[F_n])^* = Z$ *if and only if* $\mathrm{cl\ int}\ \xi([F_n])^* = Z$ *for every* n;

$\mathrm{cl\ int}\ \xi([F_1] \to [F_2])^* = Z$ *if and only if*

$$\mathrm{cl\ int}\ \xi(-[F_1])^* = Z \ \text{ or } \ \mathrm{cl\ int}\ \xi([F_2])^* = Z,$$

the open-closed representative $\xi([F])^*$ *of the image* $\xi([F])$ *of* $[F]$ *being non-boundary in* Z *if and only if it is* Z*; or* $\xi(-[F])^*$ *is non-boundary in* Z *if and only if* $\xi([F])^*$ *is nowhere dense in* Z*; or* $\xi(\bigvee_{n\in\omega}[F_n])^*$ *is non-boundary in* Z *if and only if at least one* $\xi([F_0])^*, \xi([F_1])^*, \xi([F_2])^*, \ldots$ *is non-boundary in* Z*; or* $\xi(\bigwedge_{n\in\omega}[F_n])^*$ *is non-boundary in* Z *if and only if all* $\xi([F_0])^*, \xi([F_1])^*, \xi([F_2])^*, \ldots$ *are non-boundary in* Z*; or* $\xi([F_1] \to [F_2])^*$ *is non-boundary in* Z *if and only if* $\xi(-[F_1])^*$ *or* $\xi([F_2])^*$ *is non-boundary in* Z*.*

PROOF By Theorems 2.2.3, 2.2.4, 2.2.5, 2.2.7, and 2.2.8.

More precisely, formal correctness follows immediately from Theorem 2.2.3, whilst material adequacy follows from Theorems 2.2.4, 2.2.5, 2.2.7, and 2.2.8. That the predicate *is non-boundary in* Z, finally, has the same extension as the predicate *is true* is an immediate consequence of the existence of a canonical Boolean isomorphism of $\mathcal{F}$ into the two-element Boolean algebra $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$. ∎

2.3 Modal extension

2.3.1 Definition A *model for the modal sentential calculus* is an ordered system $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$, where $\mathfrak{A} = \mathfrak{S}^* / \equiv$ is an almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z is its Stone space, $\mathcal{E} = \mathcal{B}a(Z) / \mathcal{N}$ is the algebra of Baire subsets of Z modulo nowhere dense sets, and ζ is the canonical Boolean isomorphism of $\mathfrak{A}$ into $\mathcal{E}$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

$$\text{(III)} \qquad \text{int cl } \zeta([F])^* = \emptyset \text{ if and only if } \zeta([F])^* = \emptyset,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being nowhere dense in Z ('impossible') if and only if it is empty.

That Schema (III) is, like Schema (I), a theorem of the metalanguage $\mathcal{L}^E$ of mathematical English if and only if $\zeta([F])^*$ is open-closed is clear. For suppose to dispel any remaining doubts that $\zeta([F])^*$ is closed. Then the sufficiency of Schema (III) certainly holds because the empty set is a closed, nowhere dense Baire subset of Z, whilst its necessity fails because the empty set is not the only subset of Z which satisfies these properties, the homeomorphic copy of the Cantor set in Z being a classic example.

2.3.2 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(IV) $\quad \text{cl int } \zeta([F])^* = Z$ *if and only if* $\zeta([F])^* = Z$,

the open-closed representative $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *being non-boundary in* Z *('necessary') if and only if it is* Z*.*

PROOF The argument is similar to that of Theorem 2.2.3. For suppose that int cl $\zeta(-[F])^* = \emptyset$ if and only if $\zeta(-[F])^* = \emptyset$, that is to say, that $\zeta(-[F])^*$ is nowhere dense in Z ('impossible') if and only if it is empty. Since int $\zeta(-[F])^* = Z \setminus \text{cl } Z \setminus \zeta(-[F])^*$, and ζ is a Boolean homomorphism, it follows immediately that

$$\begin{aligned} & Z \setminus \text{int cl } \zeta(-[F])^* = Z \text{ if and only if } Z \setminus \zeta(-[F])^* = Z \\ \iff & \text{cl } Z \setminus \text{cl } \zeta(-[F])^* = Z \text{ if and only if } Z \setminus \zeta(-[F])^* = Z \\ \iff & \text{cl int } Z \setminus \zeta(-[F])^* = Z \text{ if and only if } Z \setminus \zeta(-[F])^* = Z \\ \iff & \text{cl int } \zeta([F])^* = Z \text{ if and only if } \zeta([F])^* = Z, \end{aligned}$$

whence if a sentence is necessary if and only if its negation is impossible (principle of duality common to all classical modal calculi), then $\zeta([F])^*$ is non-boundary in Z ('necessary') if and only if it is the space Z, which completes the proof of the theorem. ∎

2.3.3 Definition In every topological space, a set is

(i) *somewhere dense* if and only if it is not nowhere dense;

(ii) *somewhere codense* if and only if it is not non-boundary.

2.3.4 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(V) $\quad$ int cl $\zeta([F])^* \neq \emptyset$ *if and only if* $\zeta([F])^* \neq \emptyset$,

the open-closed representative $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *being somewhere dense in* Z *('possibly true') iff it is non-empty.*

PROOF Recall that int cl $\zeta([F])^* = \emptyset$ if and only if $\zeta([F])^* = \emptyset$. Then int cl $\zeta([F])^* \neq \emptyset$ if and only if $\zeta([F])^* \neq \emptyset$. Hence, since a set is, by Definition 2.3.3(i), somewhere dense in Z if and only if it is not nowhere dense, and a sentence is possibly true if and only if it is not impossible, $\zeta([F])^*$ is somewhere dense in Z ('possibly true') if and only if it is non-empty. ∎

2.3.5 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(VI) $\quad$ cl int $\zeta([F])^* \neq Z$ *if and only if* $\zeta([F])^* \neq Z$,

the open-closed representative $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *being somewhere codense in* Z *('possibly false') iff it is not* Z*.*

PROOF Recall that cl int $\zeta([F])^* = Z$ if and only if $\zeta([F])^* = Z$. Then cl int $\zeta([F])^* \neq Z$ if and only if $\zeta([F])^* \neq Z$. Hence, since a set is, by Definition 2.3.3(ii), somewhere codense in Z if and only if it is not non-boundary, and a sentence is possibly false if and only if it is not necessary, $\zeta([F])^*$ is somewhere co-dense in Z ('possibly false') if and only if it is not Z. ∎

Schemas (III)–(VI) are noteworthy. For, contrary to the Lewis–Gödel paradigm [9, 19] currently operating in full force, no additional symbol to the language $\mathcal{L}$ of countable sentential calculi, that is to say, no additional operator on the Lindenbaum–Tarski algebra $\mathfrak{A}$, was required to capture some notion of necessity, possibility, and impossibility. This follows, of course, from the fact that the aforementioned modes of truth, in a Tarskian fashion, were defined at the level of the metalanguage $\mathcal{L}^E$ of mathematical English instead.

2.3.6 Theorem *Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus. For every equivalence class $[F]$ of formulas of $\mathcal{L}$,*

$$\text{cl int } \xi([F])^* = Z \quad \textit{whenever} \quad \text{cl int } \zeta([F])^* = Z,$$

the open-closed representative $\xi([F])^*$ of the image $\xi([F])$ of $[F]$ being non-boundary in Z whenever that $\zeta([F])^*$ of the image $\zeta([F])$ of the same $[F]$ is non-boundary in Z, but not conversely.

PROOF Recall that ζ is a Boolean isomorphism of $\mathfrak{A}$ into $\mathcal{E}$ and that ξ is a Boolean epimorphism of $\mathfrak{A}$ into $\mathcal{F}$; moreover, let θ be a Boolean epimorphism of $\mathcal{E}$ into $\mathcal{F} \subsetneq \mathcal{E}$. Then the composition $\theta \circ \zeta$ is a Boolean epimorphism of $\mathfrak{A}$ into $\mathcal{F}$ because every Boolean isomorphism is a Boolean epimorphism, and the composition of two Boolean epimorphisms is a Boolean epimorphism.

In fact, set $\theta \circ \zeta = \xi$ so that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, $(\theta \circ \zeta)([F])$ and $\xi([F])$ denote the same element in $\mathcal{F}$, and suppose that cl int $\zeta([F])^* = Z$, that is to say, by Theorem 2.3.2, that $\zeta([F])^* = Z$. Then $\xi([F])^* = (\theta \circ \zeta)([F])^* = \theta(\zeta([F])^*) = \theta(Z) = Z$, whence cl int $\xi([F])^* = Z$ by Theorem 2.2.3 as required. That the converse fails follows immediately from the fact that θ, as a Boolean epimorphism, is not invertible. ∎

2.3.7 Theorem *Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus. For every equivalence class $[F]$ of formulas of $\mathcal{L}$,*

$$\text{int cl } \zeta([F])^* \neq \emptyset \text{ whenever } \text{cl int } \zeta([F])^* = Z,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being somewhere dense in Z whenever that $\zeta([F])^*$ of the image $\zeta([F])$ of the same $[F]$ is non-boundary in Z, but not conversely.

PROOF Indeed, it follows immediately from the rules of the topological calculus that cl int $\zeta([F])^* = Z$ if and only if int cl int $\zeta([F])^* =$ int Z if and only if int cl $\zeta([F])^* = Z$ because $\zeta([F])^*$ is open(-closed) and int $Z = Z$. This in turn implies that int cl $\zeta([F])^* \neq \emptyset$ as required. That the converse fails is obvious. ∎

2.3.8 Theorem *Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus. For every equivalence class $[F]$ of formulas of $\mathcal{L}$,*

$$\text{int cl } \zeta([F])^* \neq \emptyset \text{ whenever } \text{cl int } \xi([F])^* = Z,$$

the open-closed representative $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *being somewhere dense in* Z *whenever that* $\xi([F])^*$ *of the image* $\xi([F])$ *of the same* $[F]$ *is non-boundary in* Z*, but not conversely.*

PROOF Recall that θ is a Boolean epimorphism of $\mathcal{E}$ into $\mathcal{F} \subsetneq \mathcal{E}$ so that the composition $\theta \circ \zeta$ is a Boolean epimorphism of $\mathfrak{A}$ into $\mathcal{F}$, and set $\theta \circ \zeta = \xi$ so that $(\theta \circ \zeta)([F])$ and $\xi([F])$ denote the same element in $\mathcal{F}$; moreover, suppose that $\mathcal{F}$ is non-degenerate and that cl int $\xi([F])^* = Z$, that is to say, by Theorem 2.2.3, that $\xi([F])^* = Z$. Then $Z = (\theta \circ \zeta)([F])^* = \theta(\zeta([F])^*)$, which immediately implies, despite the fact that θ is not invertible, that $\zeta([F])^* \neq \emptyset$ because $\theta(\zeta([F])^*) = \emptyset$ whenever $\zeta([F])^* = \emptyset$. Hence, int cl $\zeta([F])^* \neq \emptyset$ by Theorem 3.2.4 whenever cl int $\xi([F])^* = Z$ as claimed. That the converse fails is obvious. ∎

It follows immediately from Theorems 2.3.6, 2.3.7, and 2.3.8 that a sentence which is necessary is true, but not conversely; that a sentence which is necessary is possible (i.e. possibly true), but not conversely; and that a sentence which is true is possible, but not conversely. These results conform with the intuition and capture the fundamental properties of necessity and possibility.

Note, in particular, that Theorems 2.3.6 and 2.3.8 rely on the relatively strong assumption that $\theta \circ \zeta = \xi$, that is to say, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, that $\theta(\zeta([F]))$ and $\xi([F])$ denote the same element in $\mathcal{F} = \{\emptyset, Z\} / \mathcal{N}$. This assumption is, of course, not true in general but is essential here to guarantee that $\theta(\zeta([F])^*)$ and $\xi([F])^*$ are comparable.

2.3.9 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

$$\text{cl int } \zeta(-[F])^* = Z \textit{ if and only if } \text{int cl } \zeta([F])^* = \emptyset,$$

the open-closed representative $\zeta(-[F])^*$ *of the image* $\zeta(-[F])$ *of* $-[F]$ *being non-boundary in* Z *if and only if the open-closed representative* $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *is nowhere dense in* Z*.*

PROOF The argument is similar to that of Theorem 2.2.4. For cl int $\zeta(-[F])^*$ = cl int $Z \setminus \zeta([F])^*$ because ζ is a Boolean homomorphism. And, certainly, cl int $Z \setminus \zeta([F])^* = Z \setminus$ int cl $\zeta([F])^*$ by the dual interdefinability of the operations of closure and interior of a set from which the conclusion follows immediately. ▮

2.3.10 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F_0], [F_1], [F_2], \ldots$ *of formulas of the language* $\mathcal{L}$ *of countable sentential calculi,*

$$\text{cl int } \zeta(\textstyle\bigvee_{n\in\omega}[F_n])^* = Z \textit{ whenever}$$
$$\text{cl int } \zeta([F_n])^* = Z \textit{ for some } n \in \omega,$$

the open-closed representative $\zeta(\bigvee_{n\in\omega}[F_n])^*$ *of the image* $\zeta(\bigvee_{n\in\omega}[F_n])$ *of* $\bigvee_{n\in\omega}[F_n]$ *being non-boundary in* Z *whenever at least one open-closed representative* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *of at least one image* $\zeta([F_0])$*,* $\xi([F_1]), \zeta([F_2]), \ldots$ *of* $[F_0], [F_1], [F_2], \ldots$ *is non-boundary in* Z*.*

PROOF The sufficiency part of the argument is similar to that of Theorem 2.2.5. For suppose that cl int $\zeta([F_n])^* = Z$ for some n. Then cl int $\zeta(\bigvee_{n\in\omega}[F_n])^* = Z$ because $\zeta([F_n])^* \subseteq \bigcup_{n\in\omega}\zeta([F_n])^*$ for every n if and only if cl int $\zeta([F_n])^* \subseteq$ cl int $\bigcup_{n\in\omega}\zeta([F_n])^*$.

To see that the necessity fails in general, suppose that $\zeta([F])^*$ is neither non-boundary nor nowhere dense in Z, that is to say, that cl int $\zeta([F])^* \neq Z$ and int cl $\zeta([F])^* \neq \emptyset$. Then $\zeta(-[F])^*$ is, by Theorem 2.3.15 below, neither nowhere dense nor non-boundary in Z. Yet the set $\zeta([F])^* \cup \zeta(-[F])^* = \zeta([F] \vee -[F])^* = Z$ is clearly non-boundary. Hence, cl int $\zeta(\bigvee_{n\in\omega}[F_n])^* = Z$ does not necessarily imply that cl int $\zeta([F])^* = Z$ for some n as claimed. ∎

2.3.11 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F_0], [F_1], [F_2], \ldots$ *of formulas of the language* $\mathcal{L}$ *of countable sentential calculi,*

$$\text{cl int } \zeta(\textstyle\bigwedge_{n\in\omega}[F_n])^* = Z \textit{ if and only if}$$
$$\text{cl int } \zeta([F_n])^* = Z \textit{ for every } n \in \omega,$$

the open-closed representative $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ *of the image* $\zeta(\bigwedge_{n\in\omega}[F_n])$ *of* $\bigwedge_{n\in\omega}[F_n]$ *being non-boundary in* Z *if and only if all open-closed representatives* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *of all images* $\zeta([F_0]), \zeta([F_1])^*$ $\zeta([F_2]), \ldots$ *of* $[F_0], [F_1], [F_2], \ldots$ *are non-boundary in* Z.

PROOF The argument is similar to that of Theorem 2.2.7. For suppose that cl int $\zeta(\bigwedge_{n\in\omega}[F_n])^* = Z$, that is to say, that cl int

$\bigcap_{n\in\omega} \zeta([F_n])^* = Z$. Then cl int $\zeta([F_n])^* = Z$ for every n because $\bigcap_{n\in\omega} \zeta([F_n])^* \subseteq \zeta([F_n])^*$ for every n. The converse follows immediately from Theorem 2.2.6, every countable intersection of non-boundary subsets of Z being non-boundary. The proof is complete. ∎

2.3.12 Theorem *Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus. For every equivalence class $[F]$ of formulas of $\mathcal{L}$,*

$$\text{cl int } \zeta([F_1] \to [F_2])^* = Z \quad \textit{whenever}$$

$$\text{cl int } \zeta(-[F_1])^* = Z \quad \textit{or} \quad \text{cl int } \zeta([F_2])^* = Z,$$

the open-closed representative $\zeta([F_1] \to [F_2])^*$ of the image $\zeta([F_1] \to [F_2])$ of $[F_1] \to [F_2]$ being non-boundary in Z whenever at least one open-closed representative $\zeta(-[F_1])^*$ or $\zeta([F_2])^*$ of at least one image $\zeta(-[F_1])$ of $-[F_1]$ or $\zeta([F_2])$ of $[F_2]$ is non-boundary in Z.

PROOF The sufficiency part of the argument is similar to that of Theorem 2.2.8. For recall that the relative pseudo-complement $[F_1] \to [F_2]$ is the largest equivalence class $[F]$ of formulas of $\mathcal{L}$ such that $[F] = -[F_1] \vee [F_2]$. Then cl int $\zeta([F_1] \to [F_2])^* = Z$ whenever cl int $\zeta(-[F_1])^* = Z$ or cl int $\zeta([F_2])^* = Z$ by Theorem 2.3.10.

To see that the necessity fails in general, suppose that $\zeta([F])^*$, in a similar vein to Theorem 2.3.10, is neither non-boundary nor nowhere dense. Then $\zeta(-[F])^*$ is neither nowhere dense nor non-boundary. Yet $\zeta(-[F])^* \cup \zeta([F])^* = \zeta(-[F] \vee [F])^* = \zeta([F] \to [F])^*$ is non-boundary in Z because $[F] \to [F]$ is a tautology. ∎

2.3.13 Definition A formally correct definition of the predicate *is necessary*, formulated in the metalanguage $\mathcal{L}^E$ of mathematical English, is said to be *weakly materially adequate* if it implies some instances of Schemas (IV). (Compare with Definition 2.2.9.)

2.3.14 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. The following schemas constitute a formally correct and weakly materially adequate definition of the predicate* is necessary*:*

cl int $\zeta([F])^* = Z$ *if and only if* $\zeta([F])^* = Z$;

cl int $\zeta(-[F])^* = Z$ *if and only if* int cl $\zeta([F])^* = \emptyset$;

cl int $\zeta(\bigvee_{n\in\omega}[F_n])^* = Z$ *whenever* cl int $\zeta([F_n])^* = Z$ *for some* n;

cl int $\zeta(\bigwedge_{n\in\omega}[F_n])^* = Z$ *if and only if* cl int $\zeta([F_n])^* = Z$ *for every* n;

cl int $\zeta([F_1] \to [F_2])^* = Z$ *whenever*

cl int $\zeta(-[F_1])^* = Z$ *or* cl int $\zeta([F_2])^* = Z$,

the open-closed representative $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *being non-boundary in* Z *if and only if it is* Z*; or* $\zeta(-[F])^*$ *is non-boundary in* Z *if and only if* $\zeta([F])^*$ *is nowhere dense in* Z*; or* $\zeta(\bigvee_{n\in\omega}[F_n])^*$ *is non-boundary in* Z *whenever at least one* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *is non-boundary in* Z*; or* $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ *is non-boundary in* Z *if and only if all* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *are non-boundary in* Z*; or* $\zeta([F_1] \to [F_2])^*$ *is non-boundary in* Z *whenever* $\zeta(-[F_1])^*$ *or* $\zeta([F_2])^*$ *is non-boundary in* Z.

PROOF By Theorems 2.3.2, 2.3.9, 2.3.10, 2.3.11, and 2.3.12. ∎

The definition of necessity formulated in Theorem 2.3.14 conforms with the intuition. Hence, it follows from Theorem 2.3.9 that a sentence is necessary if and only if its negation is impossible (i.e. not possible) and from Theorem 2.3.11 that a conjunction of sentences is necessary if and only if all the sentences forming the conjunction are necessary. By Theorem 2.3.10, however, a disjunction of sentences may be necessary although none of the sentences forming the disjunction are necessary, a result naturally extending, *mutatis mutandis*, to material implication by Theorem 2.3.12.

2.3.15 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

$$\text{int cl } \zeta(-[F])^* \neq \emptyset \textit{ if and only if } \text{cl int } \zeta([F])^* \neq Z,$$

the open-closed representative $\zeta(-[F])^*$ *of the image* $\zeta(-[F])$ *of* $-[F]$ *being somewhere dense in* Z *iff the open-closed representative* $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *is somewhere codense in* Z*.*

PROOF The argument is similar to that of Theorem 2.2.4. For int cl $\zeta(-[F])^* = \text{int cl } Z \setminus \zeta([F])^*$ because ζ is a Boolean homomorphism. And, certainly, int cl $Z \setminus \zeta([F])^* = Z \setminus \text{cl int } \zeta([F])^*$ by the dual interdefinability of the operations of closure and interior of a set from which the conclusion follows immediately. ▮

2.3.16 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F_0], [F_1], [F_2], \ldots$ *of formulas* *of the language $\mathcal{L}$ of countable sentential calculi,*

$$\text{int cl } \zeta(\bigvee_{n\in\omega}[F_n])^* \neq \emptyset \text{ if and only if}$$
$$\text{int cl } \zeta([F_n])^* \neq \emptyset \text{ for some } n \in \omega,$$

the open-closed representative $\zeta(\bigvee_{n\in\omega}[F_n])^*$ *of the image* $\zeta(\bigvee_{n\in\omega}[F_n])$ *of* $\bigvee_{n\in\omega}[F_n]$ *being somewhere dense in* Z *iff at least one open-closed representative* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *of at least one image* $\zeta([F_0])$, $\xi([F_1]), \zeta([F_2]), \ldots$ *of* $[F_0], [F_1], [F_2], \ldots$ *is somewhere dense in* Z.

PROOF The argument is similar to that of Theorem 2.2.5. For suppose that int cl $\zeta([F_n])^* \neq \emptyset$ for some n. Then int cl $\zeta(\bigvee_{n\in\omega}[F_n])^*$ $\neq \emptyset$ because $\zeta([F_n])^* \subseteq \bigcup_{n\in\omega}\zeta([F_n])^*$ for every n if and only if int cl $\zeta([F_n])^* \subseteq$ int cl $\bigcup_{n\in\omega}\zeta([F_n])^*$ = int cl $\zeta(\bigvee_{n\in\omega}[F_n])^*$.

Conversely, suppose that int cl $\zeta(\bigvee_{n\in\omega}[F_n])^* \neq \emptyset$. It is shown that int cl $\zeta([F_n])^* \neq \emptyset$ for some n. For suppose, *reductio ad absurdum*, that this is not the case. Then int cl $\zeta([F_n])^* = \emptyset$ for every n, whence int cl $\bigcup_{n\in\omega}\zeta([F_n])^*$ = int cl $\zeta(\bigvee_{n\in\omega}[F_n])^* = \emptyset$ by Theorem 2.1.11. A contradiction. Hence, int cl $\zeta(\bigvee_{n\in\omega}[F_n])^* \neq \emptyset$ if and only if int cl $\xi([F_n])^* \neq \emptyset$ for some n as claimed. ∎

2.3.17 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F_0], [F_1], [F_2], \ldots$ *of formulas of the language* $\mathcal{L}$ *of countable sentential calculi,*

$$\text{int cl } \zeta([F_n])^* \neq \emptyset \text{ for every } n \in \omega$$
$$\text{whenever int cl } \zeta(\bigwedge_{n\in\omega}[F_n])^* \neq \emptyset,$$

all open-closed representatives $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *of all images* $\zeta([F_0]), \xi([F_1]), \zeta([F_2]), \ldots$ *of* $[F_0], [F_1], [F_2], \ldots$ *being somewhere dense in* Z *whenever the open-closed representative* $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ *of the image* $\zeta(\bigwedge_{n\in\omega}[F_n])$ *of* $\bigwedge_{n\in\omega}[F_n]$ *is somewhere dense in* Z*.*

PROOF The sufficiency part of the argument is similar to that of Theorem 2.2.7. For suppose that int cl $\zeta(\bigwedge_{n\in\omega}[F_n])^* \neq \emptyset$, that is to say, that int cl $\bigcap_{n\in\omega}\zeta([F_n])^* \neq \emptyset$. Then int cl $\zeta([F_n])^* \neq \emptyset$ for every n because $\bigcap_{n\in\omega}\zeta([F_n])^* \subseteq \zeta([F_n])^*$ for every n.

To see that the necessity fails in general, suppose that $\zeta([F_1])^*$, $\zeta([F_2])^*$ are somewhere dense in Z, that is to say, that int cl $\zeta([F_1])^* \neq \emptyset$ and int cl $\zeta([F_2])^* \neq \emptyset$, and suppose that $\zeta([F_1])^*$, $\zeta([F_2])^*$ are disjoint. Then $\zeta([F_1])^* \cap \zeta([F_2])^* = \zeta([F_1] \wedge [F_2])^* = \emptyset$. A contradiction. Hence, int cl $\zeta([F_n])^* \neq \emptyset$ for every n does not necessarily imply that int cl $\zeta(\bigwedge_{n\in\omega}[F_n])^* \neq \emptyset$ as claimed. ∎

2.3.18 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

$$\text{int cl } \zeta([F_1] \to [F_2])^* \neq \emptyset \text{ if and only if}$$
$$\text{int cl } \zeta(-[F_1])^* \neq \emptyset \text{ or int cl } \zeta([F_2])^* \neq \emptyset,$$

the open-closed representative $\zeta([F_1] \to [F_2])^*$ *of the image* $\zeta([F_1] \to [F_2])$ *of* $[F_1] \to [F_2]$ *being somewhere dense in* Z *iff at least one open-closed representative* $\zeta(-[F_1])^*$ *or* $\zeta([F_2])^*$ *of at least one image* $\zeta(-[F_1])$ *of* $-[F_1]$ *or* $\zeta([F_2])$ *of* $[F_2]$ *is somewhere dense in* Z*.*

PROOF The argument is similar to that of Theorem 2.2.8. For recall that the relative pseudo-complement $[F_1] \to [F_2]$ is defined as the largest equivalence class $[F]$ of formulas of $\mathcal{L}$ such that $[F] = -[F_1] \vee [F_2]$. A glance at Theorem 2.3.16 completes the proof. ▮

2.3.19 Definition A formally correct definition of the predicate *is possible*, formulated in the metalanguage $\mathcal{L}^E$ of mathematical English, is said to be *weakly materially adequate* if it is implied by some instances of Schema (V). (Compare with Definition 2.3.13.)

2.3.20 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. The following schemas constitute a formally correct and weakly materially adequate definition of the predicate* is possible*:*

int cl $\zeta([F])^* \neq \emptyset$ *if and only if* $\zeta([F])^* \neq \emptyset$;

int cl $\zeta(-[F])^* \neq \emptyset$ *if and only if* cl int $\zeta([F])^* \neq Z$;

int cl $\zeta(\bigvee_{n\in\omega}[F_n])^* \neq \emptyset$ *if and only if* int cl $\zeta([F_n])^* \neq \emptyset$ *for some* n;

int cl $\zeta([F_n])^* \neq \emptyset$ *for every* n *whenever* int cl $\zeta(\bigwedge_{n\in\omega}[F_n])^* \neq \emptyset$;

int cl $\zeta([F_1] \to [F_2])^* \neq \emptyset$ *if and only if*

$$\text{int cl } \zeta(-[F_1])^* \neq \emptyset \text{ or int cl } \zeta([F_2])^* \neq \emptyset,$$

the open-closed representative $\zeta([F])^*$ *of* $\zeta([F])$ *being somewhere dense in* Z *if and only if it is non-empty; or* $\zeta(-[F])^*$ *is somewhere dense in* Z *if and only if* $\zeta([F])^*$ *is somewhere codense in* Z*; or* $\zeta(\bigvee_{n\in\omega}[F_n])^*$ *is somewhere dense in* Z *if and only if at least one* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *is somewhere dense in* Z*; or all* $\zeta([F_0])^*$,

$\zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *are somewhere dense in* Z *whenever* $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ *is somewhere dense in* Z*; or* $\zeta([F_1] \to [F_2])^*$ *is somewhere dense in* Z *if and only if* $\zeta(-[F_1])^*$ *or* $\zeta([F_2])^*$ *is somewhere dense in* Z*.*

PROOF By Theorems 2.3.4, 2.3.15, 2.3.16, 2.3.17, and 2.3.18. ▮

The definition of possibility formulated in Theorem 2.3.20, like that of necessity earlier, conforms with the intuition. Hence, it follows from Theorem 2.3.15 that a sentence is possible (i.e. possibly true) if and only is its negation is not necessary (i.e. possibly false) and from Theorem 2.3.16 that a disjunction of sentences is possible if and only if at least one of the sentences forming the disjunction is possible, a result naturally extending, *mutatis mutandis*, to material implication by Theorem 2.3.18. By Theorem 2.3.17, however, a conjunction of sentences may not be possible although all the sentences forming the conjunction are possible.

2.3.21 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(VII) $\quad \text{int cl}\ (\zeta([F])^* \bigtriangleup B) = \emptyset$ *if and only if* $B \in \zeta([F])$*,*

the symmetric difference of the open-closed representative $\zeta([F])^*$ *of the image* $\zeta([F])$ *of* $[F]$ *with an arbitrary Baire set* B *being nowhere dense in* Z *('impossible') if and only if* B *belongs to* $\zeta([F])$*.*

PROOF Suppose that $B \in \zeta([F])$. Then $\zeta([F])^* \bigtriangleup B$ is certainly nowhere dense in Z, by definition. Conversely, suppose that $\zeta([F])^* \bigtriangleup$

B is nowhere dense in Z; it is shown that $B \in \zeta([F])$. For suppose that this is not the case. Then there exists an equivalence class $[F']$ of formulas of $\mathcal{L}$ such that $B \in \zeta([F'])$. It follows that $\zeta([F'])^* \triangle B$ is nowhere dense in Z as well as $\zeta([F])^* \triangle \zeta([F'])^*$ because $\zeta([F])^* \triangle \zeta([F'])^* \subseteq (\zeta([F])^* \triangle B) \cup (\zeta([F'])^* \triangle B)$, contradicting the Baire category theorem for compact Hausdorff spaces [4, Theorem 3.9.3, p. 197] whereby the interior of meagre sets, hence, here, that of nowhere dense sets, is empty because $\zeta([F])^* \triangle \zeta([F'])^*$ has no empty interior. Hence, $B \in \zeta([F])$ as required. ∎

At first glance, the reader may contend that Schema (VII) is inconsequential: $\zeta([F])^* \triangle B$ is nowhere dense in Z, that is to say, $\zeta([F])^*$ and B are impossible to discriminate from each other, if and only if $\zeta([F])^*$ and B belong to the same equivalence class of Baire sets, and the equivalence relation defined on $\mathcal{B}a(Z)$ naturally inducing a congruence relation on $\mathcal{E} = \mathcal{B}a(Z) \,/\, \mathcal{N}$, they are interchangeable, in Leibniz's phraseology, *salva veritate.*

But the present state of affairs is more subtle. For recall, as previously illustrated, that Schemas (III)–(VI) are theorems of the metalanguage $\mathcal{L}^E$ of mathematical English if and only if $\zeta([F])^*$ is open-closed. Then although $\zeta([F])^* \triangle B$ is nowhere dense in Z, that is to say, although $\zeta([F])^*$ and B are impossible to discriminate from each other, they are *not* interchangeable *salva veritate.*

Schema (VII) uncovers, therefore, a particularly acute problem:

Unbeknown to the logician, Schemas (III)–(VI) can be false in the sense that they are not theorems of $\mathcal{L}^E$. Moreover, no remedy is available. For suppose that $\mathcal{L}^{E^+}$ is a meta-metalanguage capable of assessing the truth-value of each schema. Then this assessment must be indubitably true to be conclusive. A contradiction because $\mathcal{F}$ and, indeed, every isomorphic copy thereof are subalgebras of $\mathcal{E}$.

2.3.22 Definition Let $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ be a model for the modal sentential calculus. The impossibility to discriminate a true or false or necessary or possible or impossible sentence which is 'true' from a true or false or necessary or possible or impossible sentence which is 'false' is called *agnoia*. The systematic study of agnoia in relation to uncertainty modelling is the domain of *formal agnoiology*.

2.3.23 Definition A formally correct definition of the predicate *is impossible*, formulated in the metalanguage $\mathcal{L}^E$, is said to be *weakly materially adequate* if it implies some instances of Schema (III) plus Schema (VII). (Compare with Definitions 2.3.13 and 2.3.19.)

2.3.24 Theorem *Let* $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ *be a model for the modal sentential calculus. The following schemas constitute of formally correct and weakly materially adequate definition of the predicate* is impossible*:*

int cl $\zeta([F])^* = \emptyset$ *if and only if* $\zeta([F])^* = \emptyset$;

int cl $\zeta(-[F])^* = \emptyset$ *if and only if* cl int $\zeta([F])^* = Z$;

int cl $\zeta(\bigvee_{n\in\omega}[F_n])^* = \emptyset$ *if and only if* int cl $\zeta([F_n])^* = \emptyset$ *for every* n;

$\text{int cl } \zeta(\bigwedge_{n\in\omega}[F_n])^* = \emptyset$ *whenever* $\text{int cl } \zeta([F_n])^* = \emptyset$ *for some* n;

$\text{int cl } \zeta([F_1] \to [F_2])^* = \emptyset$ *if and only if*

$$\text{int cl } \zeta(-[F_1])^* = \emptyset \text{ and } \text{int cl } \zeta([F_2])^* = \emptyset;$$

$\text{int cl } (\zeta([F])^* \triangle B) = \emptyset$ *if and only if* $B \in \zeta([F])$,

the open-closed representative $\zeta([F])^*$ *of* $\zeta([F])$ *being nowhere dense in* Z *if and only if it is empty; or* $\zeta(-[F])^*$ *is nowhere dense in* Z *if and only if* $\zeta([F])^*$ *is non-boundary in* Z*; or* $\zeta(\bigvee_{n\in\omega}[F_n])^*$ *is nowhere dense in* Z *if and only if all* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *are nowhere dense in* Z*; or* $\zeta(\bigwedge_{n\in\omega}[F_n])^*$ *is nowhere dense in* Z *whenever at least one* $\zeta([F_0])^*, \zeta([F_1])^*, \zeta([F_2])^*, \ldots$ *is nowhere dense in* Z*; or* $\zeta([F_1] \to [F_2])^*$ *is nowhere dense in* Z *if and only if* $\zeta(-[F_1])^*$ *and* $\zeta([F_2])^*$ *are nowhere dense in* Z*; or* $\zeta([F])^* \triangle B$ *is nowhere dense in* Z *if and only if* B *belongs to* $\zeta([F])$.

PROOF By Definition 2.3.1 and the contraduals of Theorems 2.3.9, 2.3.10, 2.3.11, and 2.3.12; Theorem 2.3.21 completes the proof. ∎

Chapter 3

Foundations of probability

3.1 Preliminaries

Recall that by a model for the modal sentential calculus, in this study, is meant an ordered system $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$, where $\mathfrak{A} = \mathfrak{S}^* / \equiv$ is an almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z is its Stone space, $\mathcal{E} = \mathcal{B}a(Z) / \mathcal{N}$ is the algebra of Baire subsets of Z modulo nowhere dense sets, and ζ is the canonical Boolean isomorphism of $\mathfrak{A}$ into $\mathcal{E}$ such that, for every equivalence class $[F]$ of formulas of $\mathcal{L}$,

(III) $$\text{int cl } \zeta([F])^* = \emptyset \text{ if and only if } \zeta([F])^* = \emptyset,$$

the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ being nowhere dense in Z ('impossible') if and only if it is empty.

The question arises as to what extent $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ is compatible with the axiomatic treatment of the calculus of probability formulated by Kolmogorov [16, 17]. Clearly, since $\mathfrak{A}$ and $\mathcal{E}$, as Boolean algebras, are isomorphic, this question immediately reduces to that of the existence of a strictly positive, countably additive probability measure $\mathbf{p}$ on $\mathfrak{A}$, which, it is recalled, satisfies the countable chain condition (c.c.c.) and is weakly countably distributive. Von Neumann [23] famously asked whether these conditions are sufficient for the existence of a strictly positive, countably additive measure on a σ-complete Boolean algebra; the next result shows, in the context of this study, that this is not the case in general.

3.1.1 Theorem (Talagrand [32]) *There exists an almost measurable (i.e. a c.c.c. and weakly countably distributive) Lindenbaum–Tarski algebra $\mathfrak{A}$ of the language $\mathcal{L}$ of countable sentential calculi which does not carry a strictly positive, countably additive probability measure.*

SKETCH OF PROOF The argument is based on two lemmas.

Let $\mathfrak{B}$ be a Boolean algebra. A *probability submeasure* on $\mathfrak{B}$ is a function $\nu : \mathfrak{B} \to [0, 1]$ such that (i) $\nu(\mathbf{0}) = 0$ and $\nu(\mathbf{1}) = 1$; (ii) If $a, b \in \mathfrak{B}$ and $a \leq b$, then $\nu(a) \leq \nu(b)$ (monotonicity); (iii) if $a, b \in \mathfrak{B}$, then $\nu(a \vee b) \leq \nu(a) + \nu(b)$ (subadditivity). In particular, if $\nu(a \vee b) = \nu(a) + \nu(b)$ for every pairwise disjoint elements $a, b \in \mathfrak{B}$ (finite additivity), then ν is called a *probability measure.*

A probability submeasure ν is (i) *strictly positive* if $\nu(a) > 0$ if and only if $a \neq \mathbf{0}$; (ii) *exhaustive* if $\lim_{n\to\infty} \nu(a_n) = 0$ for every indexed family $\{a_n : n \in \mathbb{N}\}$ of pairwise disjoint elements of

$\mathfrak{B}$; (iii) *uniformly exhaustive* if, for every $\epsilon > 0$, there is a n such that no indexed family $\{a_0, a_1, \ldots, a_n\}$ of pairwise disjoint elements of $\mathfrak{B}$ with $\nu(a_i) \geq \epsilon$, for every $i \leq n$, exists. Clearly, every uniformly exhaustive probability submeasure is exhaustive.

Still in the same vein, a probability submeasure ν_1 is (i) *continuous* if $\lim_{n\to\infty} \nu_1(a_n) = 0$ whenever $\{a_n : n \in \mathbb{N}\}$ is a decreasing family of elements of $\mathfrak{B}$ such that $\bigwedge_{n\in\mathbb{N}} a_n = \mathbf{0}$; (ii) *absolutely continuous with respect to a probability submeasure* ν_2 if $\nu(a_1) = 0$ whenever $\nu(a_2) = 0$. In particular, two probability submeasures ν_1, ν_2 are said to be *equivalent* if ν_1 is absolutely continuous with respect to ν_2, and ν_2 is absolutely continuous with respect to ν_1.

It is known that a probability submeasure is absolutely continuous with respect to a probability measure if and only if it is uniformly exhaustive [13]. Since every uniformly exhaustive probability submeasure is exhaustive, the question arises whether an exhaustive probability submeasure is absolutely continuous with respect to a probability measure. The next result shows that the answer is negative.

Lemma 1 *There exists a non-zero, exhaustive probability submeasure* ν *on the field* $\mathrm{Clop}(2^{\mathbb{N}})$ *of open-closed subsets of the Cantor space* $2^{\mathbb{N}}$ *which is not uniformly exhaustive, hence not absolutely continuous with respect to a probability measure. Moreover, no non-zero probability measure* μ *on* $\mathrm{Clop}(2^{\mathbb{N}})$ *is absolutely continuous with respect to* ν*.*

PROOF See Talagrand [32, Sections 2–6, pp. 985–1006]. ∎

An ordered system $(\mathfrak{B}, \bar{\nu})$, where $\mathfrak{B}$ is a σ-complete Boolean algebra, and $\bar{\nu}$ is a strictly positive, continuous probability submeasure on $\mathfrak{B}$, is called a *subprobability algebra*. In particular, if $\bar{\nu}$ is a probability measure, then $(\mathfrak{B}, \bar{\nu})$ is a *probability algebra*. Every probability algebra is a subprobability algebra; D. Maharam [21] asked whether, in general, the converse is true, that is to say, whether every submeasure algebra is a measure algebra. The next result shows, in probabilistic terms, that this is not the case.

Lemma 2 *There exists a subprobability algebra $(\mathfrak{B}, \bar{\nu})$ which is not a probability algebra. In fact, neither a strictly positive probability measure nor a non-zero, continuous probability measure exists on $\mathfrak{B}$.*

PROOF By Lemma 1, there exists an exhaustive probability submeasure ν on $\mathrm{Clop}(2^{\mathbb{N}})$ which is not equivalent to a probability measure μ, neither of them being absolutely continuous with respect to the other. It is first shown that ν extends to a strictly positive, continuous probability submeasure $\bar{\nu}$ on a σ-complete Boolean algebra.

For every open-closed subset A_1, A_2 of $2^{\mathbb{N}}$, write

$$\rho(A_1, A_2) = \nu(A_1 \triangle A_2).$$

It is easy to verify that ρ is a pseudometric and, since ν is strictly positive, that $(\mathrm{Clop}(2^{\mathbb{N}}), \rho)$ is a metric space. Let $(\mathfrak{B}, \bar{\rho})$ be its completion. Then $\mathfrak{B}$ is a Boolean algebra (the operations being defined by continuity), with the extension $\bar{\nu}$ of ν on $\mathfrak{B}$ being strictly positive because $\bar{\nu}(B_1 \triangle B_2) = 0$ if and only if $\bar{\rho}(B_1, B_2)$

$= 0$ if and only if $B_1 = B_2$ if and only if $B_1 \triangle B_2 = \mathbf{0}$.

In particular, $\bar{\nu}$ is exhaustive. For let $\{E_n : n \in \mathbb{N}^*\}$ be an indexed family of pairwise disjoint elements of $\mathfrak{B}$, and, for every $n \geq 1$, find an element A_n in $\mathrm{Clop}(2^{\mathbb{N}})$ such that $\bar{\nu}(A_n \triangle E_n) \leq 2^{-n}\epsilon$ for some $\epsilon > 0$; moreover, write $A'_n = A_n - (A_1 \vee A_2 \vee \cdots \vee A_{n-1})$. Since $E_n = E_n - (E_1 \vee E_2 \vee \cdots \vee E_{n-1})$,

$$\bar{\nu}(A'_n \triangle E_n) \leq \textstyle\sum_{m=1}^{n} \bar{\nu}(A_m \triangle E_m) \leq \sum_{m=1}^{n} 2^{-m}\epsilon < \epsilon,$$

whence $\lim\sup_{n\to\infty} \bar{\nu}(A'_n \triangle E_n) = 0$ because $\epsilon > 0$ is arbitrary. Since $\lim_{n\to\infty} \bar{\nu}(A'_n) = 0$ as an immediate consequence of $\{A'_n : n \in \mathbb{N}^*\}$ being a disjoint family of open-closed subsets of $2^{\mathbb{N}}$ (in which case $\bar{\nu}$ and ν coalesce), $\lim_{n\to\infty} \bar{\nu}(E_n) = 0$ as required.

In fact, since $\bar{\nu}$ is exhaustive, every indexed family $\{B_n : n \in \mathbb{N}^*\}$ of elements of $\mathfrak{B}$ is a Cauchy sequence (otherwise, there would exist indices $m(k) < n(k) \leq m(k+1) < n(k+1) \cdots$ and an $\epsilon > 0$ such that $\bar{\nu}(B_{n(k)} - B_{m(k)}) \geq \epsilon$, which contradicts exhaustivity). Hence, $\lim_{n\to\infty} \bar{\nu}(B_n) = 0$ whenever $\{B_n : n \in \mathbb{N}^*\}$ is a decreasing family of elements of $\mathfrak{B}$ such that $\bigwedge_{n\in\mathbb{N}^*} B_n = \mathbf{0}$, which immediately implies that $\bar{\nu}$ is continuous and that $\mathfrak{B}$ is σ-complete.

Note, in passing, that $\bar{\nu}$ is countably subadditive. This follows from the fact that, for every Cauchy sequence $\{B_n : n \in \mathbb{N}^*\}$ of elements of $\mathfrak{B}$, if $B'_m = \bigvee_{n\in\mathbb{N}^*} B_n - \bigvee_{1\leq n\leq m} B_n$, then

$$\bar{\nu}(\textstyle\bigvee_{n\in\mathbb{N}^*} B_n) = \bar{\nu}(B'_m \vee \bigvee_{1\leq n\leq m} B_n) \leq \bar{\nu}(B'_m) + \sum_{n=1}^{m} \bar{\nu}(B_n)$$

$$= \lim_{m\to\infty} (\bar{\nu}(B'_m) + \textstyle\sum_{n=1}^{m} \bar{\nu}(B_n)) = \sum_{n=1}^{\infty} \bar{\nu}(B_n)$$

because $\bar{\nu}$ is subadditive, continuous, and the indexed family $\{B'_m : m \in \mathbb{N}^*\}$ is a decreasing Cauchy sequence of elements of $\mathfrak{B}$ such that $\bigwedge_{m\in\mathbb{N}^*} B'_m = \mathbf{0}$. This proves the first part of Lemma 2.

Let μ be a probability measure on $\mathrm{Clop}(2^{\mathbb{N}})$. Since μ is not absolutely continuous with respect to ν, there is an $\epsilon > 0$ such that, for every $n \geq 1$, $\mu(A_n) > \epsilon$ and $\nu(A_n) \leq 2^{-n}$, where A_n is an open-closed subset of $2^{\mathbb{N}}$. Let $B_n = \bigvee_{m\geq n} A_m$ and $\bar{\nu}$ be the continuous extension of ν on $\mathfrak{B}$. Since $\bar{\nu}$ is countably subadditive, $\bar{\nu}(B_n) \leq \sum_{m=n}^{\infty} 2^{-m} \leq 2^{-n+1}$, whence if $B = \bigwedge_{n\in\mathbb{N}^*} B_n$, then $\bar{\nu}(B) = 0$, and $B = \mathbf{0}$ because $\bar{\nu}$ is continuous. Yet $\mu(B_n) > \epsilon$ for every n. Hence, $\mu(\bigwedge_{n\in\mathbb{N}^*} B_n) = \mu(B) > 0$, which implies that μ cannot be extended to a continuous probability measure on $\mathfrak{B}$ as claimed.

Likewise, since ν is not absolutely continuous with respect to μ, there exists an $\epsilon > 0$ such that, for every $n \geq 1$, $\nu(A_n) > \epsilon$ and $\mu(A_n) \leq 2^{-n}$. Let $B_n = \bigvee_{m\geq n} A_m$, $B = \bigwedge_{n\in\mathbb{N}^*} B_n$, and $\bar{\nu}$ be the continuous extension of ν. Then $\bar{\nu}(B_n) > \epsilon$ and $\bar{\nu}(B) > \epsilon$. In fact, let $B' \leq B$, $B' \neq \mathbf{0}$, and suppose that B' can be covered by a finite subset of $\{A_m : m \geq n\}$. Then $\mu(B') \leq \sum_{m=n}^{\infty} 2^{-m}$, which immediately implies that $\mu(B') = 0$ as n grows. It follows that μ is not strictly positive, which concludes the proof of the lemma. ▮

It follows from Lemma 2 that there exists a σ-complete Boolean algebra $\mathfrak{B}$ which does not carry a strictly positive, continuous probability measure. Since every σ-complete Boolean algebra is the Lindenbaum–Tarski algebra of some language of countable sentential calculi, and every continuous, finitely additive probability measure has a unique countably additive extension, it suffices to show that $\mathfrak{B}$ is c.c.c. and weakly countably distributive to prove Theorem 3.1.1.

Indeed, since the probability submeasure $\bar{\nu}$ is strictly positive and exhaustive, $\mathfrak{B}$ is c.c.c., for there can only exist finitely many $B \in \mathfrak{B}$ such that $\bar{\nu}(B) \geq 1 / n$ for some $n \geq 1$. That $\mathfrak{B}$, finally, is weakly countably distributive whenever it carries a strictly positive, countably subadditive probability submeasure $\bar{\nu}$ follows, *mutatis mutandis*, from the same argument as that in Theorem 2.1.6 for a strictly positive, countably additive probability measure. ▮

Theorem 3.1.1 shows that not every almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ carries a strictly positive, countably additive probability measure $\mathbf{p}$. It follows that the question of the compatibility of any model $(\mathfrak{A}, Z, \mathcal{E}, \zeta)$ for the modal sentential calculus with the calculus of probability, which first morphed into that of the existence of a strictly positive, countably additive probability measure $\mathbf{p}$ on $\mathfrak{A}$, can yet be reformulated as that of the identification of a necessary and sufficient criterion for the existence of $\mathbf{p}$.

3.1.2 Theorem (Kelley [15]) *An almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ of the language $\mathcal{L}$ of countable sentential calculi carries a strictly positive, countably additive probability measure $\mathbf{p}$ if and only if it carries a strictly positive, finitely additive probability measure $\mathbf{q}$.*

PROOF That every almost measurable Lindenbaum–Tarski algebra carrying a strictly positive, countably additive probability measure carries a strictly positive, finitely additive probability measure is clear. Conversely, let $\mathbf{q}$ be a strictly positive, finitely additive probability measure on $\mathfrak{A}$, and, for every equivalence class $[F]$ of formulas of $\mathcal{L}$, set $[F] = [F_1] \vee [F_2] \vee \cdots$, $[F_1] \wedge [F_2] \wedge \cdots = [\mathbf{0}]$. Then

$$\mathbf{p}([F]) = \inf\,(\mathbf{q}([F_1]) + \mathbf{q}([F_2]) + \cdots)$$

defines a countably additive probability measure, more precisely, the greatest countably additive probability measure $\mathbf{p} \leq \mathbf{q}$ on $\mathfrak{A}$. (This construction is due to Ryll-Nardzewski [15, Addendum, p. 1176].)

In fact, since $\mathbf{p}([\mathbf{0}]) = 0$, $[F] \neq [\mathbf{0}]$ whenever $\mathbf{p}([F]) > 0$, by definition. To verify the converse and, consequently, that $\mathbf{p}$ is strictly positive, suppose that $[F] \neq [\mathbf{0}]$, and, without loss of generality, set $\bigwedge_{m \in \mathbb{N}} \bigvee_{n \in \mathbb{N}} [F_{m,n}] = [\mathbf{1}]$ for some increasing family $\{[F_{m,n}] : m, n \in \mathbb{N} \times \mathbb{N}\}$ of equivalence classes of formulas of $\mathcal{L}$. Then, certainly, $\bigvee_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigwedge_{m \in \mathbb{N}} [F_{m,\varphi(m)}] = [\mathbf{1}]$ because $\mathfrak{A}$ is weakly countably distributive. There exists, therefore, at least one φ such that $\bigwedge_{m \in \mathbb{N}} [F_{m,\varphi(m)}] \neq [\mathbf{0}]$ if and only if $[F] \wedge \bigwedge_{m \in \mathbb{N}} [F_{m,\varphi(m)}] \neq [\mathbf{0}]$, whence $\mathbf{p}([F] \wedge \bigwedge_{m \in \mathbb{N}} [F_{m,\varphi(m)}]) \neq \mathbf{p}([\mathbf{0}])$ if and only if $\mathbf{p}([F]) \times \mathbf{p}(\bigwedge_{m \in \mathbb{N}} [F_{m,\varphi(m)}]) > 0$. This implies that $\mathbf{p}([F]) > 0$, which completes the proof. ∎

Theorem 3.1.2 offers a workable criterion for the existence of a strictly positive, countably additive probability measure $\mathbf{p}$ on an almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$. This result, however, is not entirely satisfactory, for it leaves open the question of the existence of a necessary and sufficient criterion for the existence of a strictly positive, finitely additive probability measure $\mathbf{q}$ on $\mathfrak{A}$.

3.1.3 Definition (Kelley [15]) Let $\mathcal{A} \subseteq \mathfrak{A}$ be a non-empty set of equivalence classes of formulas of $\mathcal{L}$. For each indexed family $[\mathbf{F}] = \{[F_1], [F_2], \ldots, [F_n]\} \subseteq \mathcal{A}$ of (not necessarily disjoint) equivalence classes thereof, let $n([\mathbf{F}])$ be the number n of elements in $[\mathbf{F}]$ and $m([\mathbf{F}])$ be the maximum number of elements of $[\mathbf{F}]$ having non-empty intersection. The *intersection number of* $\mathcal{A}$ is the quantity

$$\#(\mathcal{A}) = \inf \{m([\mathbf{F}]) \,/\, n([\mathbf{F}]) \;:\; [\mathbf{F}] \text{ is a finite family in } \mathcal{A}\}.$$

3.1.4 Theorem (Kelley [15]) *An almost measurable Lindenbaum–Tarski algebra* $\mathfrak{A}$ *of the language* $\mathcal{L}$ *of countable sentential calculi carries a strictly positive, finitely additive probability measure* $\mathbf{q}$ *if and only if* $\mathfrak{A} \setminus \{[\mathbf{0}]\}$ *is the union of countably many sets of equivalence classes of formulas of* $\mathcal{L}$*, each of which has a positive intersection number.*

PROOF Let Z be the Stone space of $\mathfrak{A}$, $\mathrm{Clop}(Z)$ be the field of open-closed subsets of Z, and γ be the Stone map (Boolean isomorphism) of $\mathfrak{A}$ into $\mathrm{Clop}(Z)$; moreover, set $\bar{\mathbf{q}}(\gamma([F])) = \mathbf{q}([F])$ so that γ is measure-preserving. The argument is based on two lemmas.

Lemma 1 *If* $\bar{\mathbf{q}}$ *is a finitely additive probability measure defined on the field* $\mathrm{Clop}(Z)$ *of open-closed subsets of the Stone space* Z *of* $\mathfrak{A}$*,* *and* $\mathcal{B}$ *is a non-empty subset of* Clop(Z)*, then* $\inf\{\bar{\mathbf{q}}(\gamma([F])) : \gamma([F]) \in \mathcal{B}\} \leq \#(\mathcal{B})$*, where* $\#(\mathcal{B})$ *is the intersection number of* $\mathcal{B}$*.*

PROOF Note at the outset that if $\mathcal{B} \subseteq \text{Clop}(Z)$ is a non-empty set of open-closed subsets of Z, then, for every indexed family $\gamma([\mathbf{F}]) = \{\gamma([F_1]), \gamma([F_2]), \ldots, \gamma([F_n])\} \subseteq \mathcal{B}$ of (not necessarily disjoint) open-closed subsets of Z, the maximum number of elements of $\gamma([\mathbf{F}])$ having non-empty intersection can be defined as the quantity

$$m(\gamma([\mathbf{F}])) = \sup\{\textstyle\sum_{i=1}^{n} \mathbf{1}_{\gamma([F_i])}(\mathbf{E}) : \mathbf{E} \text{ is a maximal filter in } Z\},$$

where $\mathbf{1}_{\gamma([F_i])}(\mathbf{E})$ denotes the indicator function of $\gamma([F_i])$.

For convenience, write $\bar{\mathbf{r}} = \inf\{\bar{\mathbf{q}}(\gamma([F])) : \gamma([F]) \in \mathcal{B}\}$. Then

$$\int \textstyle\sum_{i=1}^{n} \mathbf{1}_{\gamma([F_i])}(\mathbf{E})\, d\bar{\mathbf{q}} = \sum_{i=1}^{n} \bar{\mathbf{q}}(\gamma([F_i])) \geq n\bar{\mathbf{r}},$$

whence $\sum_{i=1}^{n} \mathbf{1}_{\gamma([F_i])}(\mathbf{E}) \geq n\bar{\mathbf{r}}$ for some $\mathbf{E}$ in Z. It follows that the maximum number $m(\gamma([\mathbf{F}]))$ of elements of $\gamma([\mathbf{F}])$ which intersect is at least $n\bar{\mathbf{r}}$ and, consequently, that $m(\gamma([\mathbf{F}])) \,/\, n(\gamma([\mathbf{F}])) \geq \bar{\mathbf{r}}$. By taking the infimum for all the $\gamma([\mathbf{F}])$'s, $\#(\mathcal{B}) \geq \bar{\mathbf{r}}$ as claimed. ▮

Lemma 2 *If* $\mathcal{B} \subseteq \text{Clop}(Z)$ *is a non-empty set of open-closed subsets of the Stone space* Z *of* $\mathfrak{A}$*, then there exists a finitely additive probability measure* $\bar{\mathbf{q}}$ *on* Clop(Z) *such that* $\inf\{\bar{\mathbf{q}}(\gamma([F])) : \gamma([F]) \in \mathcal{B}\} = \#(\mathcal{B})$*, where* $\#(\mathcal{B})$ *is the intersection number of* $\mathcal{B}$*.*

PROOF It has already been established that $\inf\{\bar{\mathbf{q}}(\gamma([F])) : \gamma([F]) \in \mathcal{B}\} \leq \#(\mathcal{B})$. To see the converse, let $C(Z)$ be the class of all real-valued, continuous functions defined on Z equipped with the usual supremum norm $\|\cdot\|$, $\mathrm{Ind}(\mathcal{B})$ be the class of all indicator functions of members of $\mathcal{B}$, and $\mathcal{H}$ be the least convex set of $\mathrm{Ind}(\mathcal{B})$. It is first shown that if $h \in \mathcal{H}$, then $\|h\| \geq \#(\mathcal{B})$.

For suppose, using the same notation as in the proof of Lemma 1, that $\left\| \sum_{i=1}^{q} t_i \, \mathbf{1}_{\gamma([F_i])}(\mathbf{E}) \right\| = \bar{\mathbf{r}}$, where $0 < t_i \leq 1$ and $\sum_{i=1}^{q} t_i = 1$. Then, for some $\epsilon > 0$, $|1 \,/\, n| \left\| \sum_{i=1}^{q} p_i \, \mathbf{1}_{\gamma([F_i])}(\mathbf{E}) \right\| < \bar{\mathbf{r}} + \epsilon$ whenever $p_1, p_2, \ldots, p_q$ are positive integers such that $\sum_{i=1}^{q} p_i = n$. Considering the family in $\mathcal{B}$ obtained by counting each $\gamma([F_i])$ p_i times, it follows that at least $n \times \#(\mathcal{B})$ of its elements intersect, whence $\#(\mathcal{B}) \leq |1 \,/\, n| \left\| \sum_{i=1}^{q} p_i \, \mathbf{1}_{\gamma([F_i])}(\mathbf{E}) \right\|$ and $\#(\mathcal{B}) \leq \bar{\mathbf{r}}$ as required.

Let $\mathbf{B}_{\#(\mathcal{B})}(0) = \{b : \|b\| < \#(\mathcal{B})\}$ be the open ball in $C(Z)$ of radius $\#(\mathcal{B})$ centred at 0. Since $\|h\| \geq \#(\mathcal{B})$ for every $h \in \mathcal{H}$, each member of the sum $\mathcal{H} + \mathbf{B}_{\#(\mathcal{B})}(0)$ is positive. Let $C^{+}(Z)$ be the class of positive, continuous functions on Z and $\mathcal{G} \subseteq \mathcal{H}$ be the convex cone $\{\alpha(h + b) + \beta c : \alpha, \beta \geq 0,\ h \in \mathcal{H},\ b \in \mathbf{B}_{\#(\mathcal{B})}(0),\ c \in C^{+}(Z)\}$. Then no member of $\mathcal{G}$ is negative; moreover, by the Hahn–Banach theorem [3, Theorem 10, p. 62], there certainly exists a linear functional ϕ such that $\phi(-1) < \phi(g)$ for every $g \in \mathcal{G}$.

For convenience, and without loss of generality, set $\phi(-1) = -1$, and suppose that $\phi(g) \geq 0$ (such a ϕ always exists because ϕ is closed under positive scalar multiplication). Then, for every $h \in \mathcal{H}$ and every $-\#(\mathcal{B}) + \epsilon \in \mathbf{B}_{\#(\mathcal{B})}(0)$, where $\epsilon > 0$, $h - \#(\mathcal{B}) + \epsilon \in \mathcal{G}$, whence $\phi(h - \#(\mathcal{B}) + \epsilon) \geq 0$ if and only if $\phi(h) - \#(\mathcal{B}) + \epsilon$

≥ 0 if and only if $\phi(h) \geq \#(\mathcal{B})$. Setting $\bar{\mathbf{q}}(\gamma([F])) = \phi(\mathbf{1}_{\gamma([F])}(\mathbf{E}))$ for every $\gamma([F]) \in \mathrm{Clop}(Z)$, $\bar{\mathbf{q}}$ is a finitely additive probability measure with $\inf\{\bar{\mathbf{q}}(\gamma([F])) : \gamma([F]) \in \mathcal{B}\} = \#(\mathcal{B})$ as claimed. ▮

Suppose now, in the final analysis, that $\mathrm{Clop}(Z) \setminus \{\emptyset\}$ is the union of countably many sets $\mathcal{B}_n$ of open-closed subsets of Z, each of which has a positive intersection number, that is to say, such that $\#(\mathcal{B}_n) > 0$ for every n. Then $\bar{\mathbf{q}} = \sum_{n=1}^{\infty} \bar{\mathbf{q}}_n$ such that inf $\{\bar{\mathbf{q}}_n(\gamma([F])) : \gamma([F]) \in \mathcal{B}_n\} = \#(\mathcal{B}_n)$ for every n is a strictly positive, countably additive, hence finitely additive, probability measure on $\mathrm{Clop}(Z)$ by Lemma 2. Conversely, if $\bar{\mathbf{q}}$ is a strictly positive, finitely additive probability measure on $\mathrm{Clop}(Z)$, then $\mathrm{Clop}(Z) \setminus \{\emptyset\}$ can be written as the union of countably many sets $\mathcal{B}_n$ of open-closed subsets of Z, each of which has a positive intersection number by Lemma 1. The proof of Theorem 3.1.4 is complete. ▮

3.1.5 Theorem (Kelley [15]) *An almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$ of the language $\mathcal{L}$ of countable sentential calculi carries a strictly positive, countably additive probability measure* $\mathbf{p}$ *if and only if $\mathfrak{A} \setminus \{[\mathbf{0}]\}$ is the union of countably many sets of equivalence classes of formulas of $\mathcal{L}$, each of which has a positive intersection number.*

PROOF Immediate by combining Theorems 3.1.2 and 3.1.4. ▮

An almost measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi which satisfies either Theorem 3.1.2

or 3.1.4 is called *measurable.* Every measurable Lindenbaum–Tarski algebra is almost measurable, but not conversely; in particular, every measurable Lindenbaum–Tarski algebra is c.c.c., hence complete. Measurable Lindenbaum–Tarski algebras are denoted by $\mathfrak{A}^*$.

3.1.6 Definition An ordered system $(\mathfrak{A}^*, \mathbf{p})$, where $\mathfrak{A}^*$ is a measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, and $\mathbf{p}$ is a strictly positive, countably additive probability measure defined on $\mathfrak{A}^*$, is called a *Lindenbaum–Tarski probability algebra of the language $\mathcal{L}$ of countable sentential calculi.*

3.1.7 Theorem (Stone [30]) *Let $\mathfrak{A}^*$ be a measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, Z^* be a set of maximal filters of $\mathfrak{A}^*$, and γ be a Boolean homomorphism of $\mathfrak{A}^*$ into the power set $\mathcal{P}(Z^*)$ such that, for every equivalence class $[F]$ of formulas of the language $\mathcal{L}$, $\gamma([F]) = \{\mathbf{F} \in Z^* : [F] \in \mathbf{F}\}$. Then the set Z^* equipped with the topology for which the $\gamma([F])$'s form an open base is a zero-dimensional compact Hausdorff space.*

PROOF The argument is identical to that of Theorem 2.1.7. Indeed, the zero-dimensionality of Z^* follows immediately from the fact that γ is a Boolean homomorphism. To see that Z^* is compact, consider a filter $\overline{\mathbf{F}}$ of closed subsets of Z^*, and notice that $\overline{\mathbf{F}}$ has non-empty intersection. Finally, Z^* is Hausdorff because any two maximal filters in Z^* are separated by disjoint open(-closed) sets. ∎

The zero-dimensional compact Hausdorff space Z^* of Theorem 3.1.7 is called the *Stone space of* $\mathfrak{A}^*$. Of course, in the same way that every measurable Lindenbaum–Tarski algebra $\mathfrak{A}^*$ of the language $\mathcal{L}$ of countable sentential calculi is almost measurable, every Stone space Z^* of $\mathfrak{A}^*$ is, by definition, an extremally disconnected compact Hausdorff space in which every meagre set is nowhere dense (since $\mathfrak{A}^*$ is c.c.c. and weakly countably distributive). As previously noted in the preliminaries of Chapter 2, an extremally disconnected compact Hausdorff space is also called a Stonean space.

3.1.8 Theorem (Fremlin [6]) *Every Lindenbaum–Tarski probability algebra* $(\mathfrak{A}^*, \mathbf{p})$ *of the language* $\mathcal{L}$ *of countable sentential calculi is isomorphic, as a probability algebra, to the probability algebra* $(\mathcal{B}a(Z^*) / \mathcal{N}, \mathbf{P})$ *of the probability space* $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$*, where* Z^* *is the Stone space of* $\mathfrak{A}^*$*,* $\mathcal{B}a(Z^*)$ *is the* σ*-field of Baire subsets of* Z^**, and* $\mathbb{P}$ *is a countably additive probability measure on* $\mathcal{B}a(Z^*)$*.*

PROOF Since every measurable Lindenbaum–Tarski algebra $\mathfrak{A}^*$ of the language $\mathcal{L}$ of countable sentential calculi is σ-complete, by Theorem 2.1.13, it is isomorphic to the algebra $\mathcal{B}a(Z^*) / \mathcal{N}$ of Baire subsets of Z^* modulo nowhere dense sets. Let ζ^{-1} be the canonical Boolean isomorphism of $\mathcal{B}a(Z^*) / \mathcal{N}$ into $\mathfrak{A}^*$, and, for every Baire set B, write $\eta(B) = \zeta^{-1}(\zeta([F]))$. Then η is a Boolean epimorphism of $\mathcal{B}a(Z^*)$ into $\mathfrak{A}^*$ whose kernel is the σ-ideal of nowhere dense sets, that is to say, $\ker \eta = \{B \in \mathcal{B}a(Z^*) : \eta(B) = [\mathbf{0}]\} = \mathcal{N}$.

Suppose now that $\mathbb{P}(B) = \mathbf{p}(\eta(B))$ for every $B \in \mathcal{B}a(Z^*)$. Then $\mathbb{P}(\emptyset) = \mathbf{p}(\eta(\emptyset)) = \mathbf{p}([\mathbf{0}]) = 0$, $\mathbb{P}(Z^*) = \mathbf{p}(\eta(Z^*)) = \mathbf{p}([\mathbf{1}]) = 1$, and, for every indexed family $\{B_n : n \in \mathbb{N}\}$ of Baire subsets of Z^*,

$$\mathbb{P}(\bigcup_{n\in\mathbb{N}} B_n) = \mathbf{p}(\eta(\bigcup_{n\in\mathbb{N}} B_n)) = \mathbf{p}(\bigvee_{n\in\mathbb{N}} \eta(B_n)) =$$

$$\mathbf{p}(\bigvee_{n\in\mathbb{N}} [F_n]) = \sum_{n=0}^{\infty} \mathbf{p}([F_n]) = \sum_{n=0}^{\infty} \mathbf{p}(\eta(B_n)) = \sum_{n=0}^{\infty} \mathbb{P}(B_n),$$

whence $\mathbb{P}$ is a countably additive probability measure on $\mathcal{B}a(Z^*)$, and the ordered system $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is a probability space.

In fact, since $\mathbb{P}(B) = 0$ if and only if $\mathbf{p}(\eta(B)) = 0$ if and only if $\eta(B) = [\mathbf{0}]$ if and only if $B \in \mathcal{N}$, a Baire set B in Z^* is $\mathbb{P}$-negligible if and only if it is nowhere dense. It follows that the probability algebra of $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ can be identified with the ordered system $(\mathcal{B}a(Z^*) \,/\, \mathcal{N}, \mathbf{P})$ by setting $\mathbf{P}(\zeta([F])) = \mathbb{P}(B) = \mathbf{p}(\eta(B)) = \mathbf{p}([F])$. This implies that ζ is a measure-preserving Boolean isomorphism and, consequently, that $(\mathfrak{A}^*, \mathbf{p})$ and $(\mathcal{B}a(Z^*) \,/\, \mathcal{N}, \mathbf{P})$, as probability algebras, are isomorphic. The proof is complete. ∎

Theorem 3.1.8 can be regarded as the preliminary version of a Stone-like representation theorem for Lindenbaum–Tarski probability algebras $(\mathfrak{A}^*, \mathbf{p})$. The interaction between the Stone topology $\mathcal{T}_S^*$ on Z^*, that is to say, the topology generated by the sets of the form $\{\mathbf{F} \in Z^* : [F] \in \mathbf{F}\}$, and the probability measure $\mathbb{P}$ on $\mathcal{B}a(Z^*)$ in the construction of the measure-preserving Boolean isomorphism ζ in Theorem 3.1.8, however, is not clear. In particular, it is not clear in what sense $\mathcal{T}_S^*$ and $\mathbb{P}$ are compatible with each other.

3.1.9 Definition Let Z^* be the Stone space of a measurable Lindenbaum–Tarski algebra $\mathfrak{A}^*$ of the language $\mathcal{L}$ of countable sentential calculi, $\mathcal{T}_S^*$ be the Stone topology on Z^*, $\mathcal{B}a(Z^*)$ be the σ-field of Baire subsets of Z^*, and $\mathbb{P}$ be a countably additive probability measure on $\mathcal{B}a(Z^*)$. Then the ordered system $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P}) = (Z^*, \mathcal{T}_S^*) \oplus (Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is a *Radon probability space* if

(i) $\mathcal{T}_S^* \subseteq \mathcal{B}a(Z^*)$ (every open set in $\mathcal{T}_S^*$ is $\mathbb{P}$-measurable);

(ii) $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is Carathéodory complete (every subset of every $\mathbb{P}$-negligible set is $\mathbb{P}$-measurable and has probability 0);

(iii) $\mathbb{P}(B) = \sup\{\mathbb{P}(K) : K \subseteq B, K \text{ compact}\}$ for every Baire set B ($\mathbb{P}$ is inner regular with respect to the compact sets).

3.1.10 Theorem (Fremlin [6]) *Let Z^* be the Stone space of a measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi, $\mathcal{T}_S^*$ be the Stone topology on Z^*, $\mathcal{B}a(Z^*)$ be the σ-field of Baire subsets of Z^*, and $\mathbb{P}$ be a countably additive probability measure on $\mathcal{B}a(Z^*)$. Then the ordered system $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ $= (Z^*, \mathcal{T}_S^*) \oplus (Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is a Radon probability space.*

PROOF Since every basic open set $\gamma([F]) = \{\mathbf{F} \in Z^* : [F] \in \mathbf{F}\}$ is open-closed, and a Baire set is an element of the least σ-field of sets containing the open-closed subsets of Z^*, it follows immediately that $\mathcal{T}_S^* \subseteq \mathcal{B}a(Z^*)$; and since every Baire subset of Z^* is, by Theorem 3.1.8, $\mathbb{P}$-negligible if and only if it is nowhere dense, and nowhere dense sets form a σ-ideal in Z^*, the probability space $(Z^*, \mathcal{B}a(Z^*), \mathbb{P})$ is certainly Carathéodory complete.

Finally, to see that $\mathbb{P}$ is inner regular with respect to the compact subsets of Z^*, write $\mathcal{D} = \{[F] \in \mathfrak{A}^* : \gamma([F]) \subseteq B\}$ for some $B \in \mathcal{B}a(Z^*)$. Then there exists an $[F] \in \mathcal{D}$ such that

$$\sup_{[F]\in\mathcal{D}} \mathbb{P}(\gamma([F])) \leq \sup\{\mathbb{P}(K) : K \subseteq B,\ K \text{ compact}\} \leq \mathbb{P}(B)$$

(recall that a compact set in a Hausdorff space is closed). Suppose now that $\sup_{[F]\in\mathcal{D}} \mathbb{P}(\gamma([F])) < \mathbb{P}(B)$, and write $[F'] = \bigvee \mathcal{D}$. Since $\mathcal{D}$ is an upward directed set, $\mathbb{P}(\gamma([F'])) = \sup_{[F]\in\mathcal{D}} \mathbb{P}(\gamma([F]))$, whence $\mathbb{P}(\gamma([F'])) < \mathbb{P}(B)$. There exists, therefore, an equivalence class $[F_0]$ of formulas of $\mathcal{L}$ such that $\gamma([F_0]) \subseteq B \setminus \gamma([F'])$, contradicting the premiss that $\mathbb{P}(\gamma([F']))$ is a least upper bound. It follows that $\mathbb{P}(B) = \sup\{\mathbb{P}(K) : K \subseteq B,\ K \text{ compact}\}$, that is to say, that $\mathbb{P}$ is inner regular with respect to the compact sets as required. ▮

3.1.11 Definition (Dixmier [2]) Let Z^* be the Stone space of a measurable Lindenbaum–Tarski algebra $\mathfrak{A}^*$. A Radon probability measure on $(Z^*, \mathcal{B}a(Z^*))$ is called *normal* if and only if every nowhere dense Baire set in Z^* is negligible. A Stonean space in which the union of the supports of all normal probability measures is dense is said to be *hyperstonean* (in French: *espace hyperstonien*).

3.1.12 Theorem (Dixmier [2]) *Let $\mathfrak{A}^*$ be a measurable Lindenbaum–Tarski algebra of the language $\mathcal{L}$ of countable sentential calculi and Z^* be its Stone space. Then Z^* is a hyperstonean space.*

PROOF It suffices to show that the union of the supports

$$\operatorname{supp}(\mathbb{P}) = Z^* \setminus \bigcup \{G \subseteq Z^*,\ G \text{ open} : \mathbb{P}(G) = 0\}$$

is dense in Z^*. Indeed, since every Baire set in Z^* is, by Theorem 3.1.8, $\mathbb{P}$-negligible if and only if it is nowhere dense, $\mathbb{P}$ is normal, and all $\mathbb{P}$-negligible open sets in Z^* are nowhere dense as well as their union by Theorem 2.1.11. And since the complement of a nowhere dense set is non-boundary, hence dense, all supports of all normal probability measures $\mathbb{P}$ are dense in Z^* as well as their union because every superset of a dense set is dense. ∎

Of course, by definition, every hyperstonean space is Stonean, but not conversely. In particular, let Z be the Stone space of an almost measurable Lindenbaum–Tarski algebra $\mathfrak{A}$. Then there exists a unique disjoint decomposition $Z = Z^* \sqcup Z^{**}$ such that

(i) Z^* is hyperstonean. Every meagre set is nowhere dense, and every normal probability measure has dense support;

(ii) Z^{**} is not hyperstonean. Meagre sets remain nowhere dense, but every probability measure has nowhere dense support (whence no probability measure is normal but that which is identically 0). For an example of a Stonean space Z^{**}, see Dixmier [2].

3.1.13 Definition Let $(\mathfrak{A}^*, \mathbf{p})$ be a Lindenbaum–Tarski probability algebra of the language $\mathcal{L}$ of countable sentential calculi. The Radon probability space $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$, where Z^* is the Stone space of $\mathfrak{A}^*$, is called the *Stone probability space of* $(\mathfrak{A}^*, \mathbf{p})$.

3.1.14 Theorem *Every Lindenbaum–Tarski probability algebra* $(\mathfrak{A}^*, \mathbf{p})$ *of the language* $\mathcal{L}$ *of countable sentential calculi is isomorphic, as a probability algebra, to the probability algebra* $(\mathcal{B}a(Z^*) \,/\, \mathcal{N}, \mathbf{P})$ *of its Stone probability space* $(Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$*, where* Z^* *is the Stone space of* $\mathfrak{A}^*$ *(which, it is recalled, is hyperstonean),* $\mathcal{T}_S^*$ *is the Stone topology on* Z^**,* $\mathcal{B}a(Z^*)$ *is the* σ*-field of Baire subsets of* Z^**, and* $\mathbb{P}$ *is a countably additive probability measure on* $\mathcal{B}a(Z^*)$*.*

PROOF By combining Theorems 3.1.8, 3.1.10, and 3.1.12. ▮

3.2 Modal probability spaces

3.2.1 Definition A *model for the calculus of probability* is an ordered system $(\boldsymbol{\mathfrak{A}}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$, where $\boldsymbol{\mathfrak{A}} = (\mathfrak{A}^*, \mathbf{p})$ is a Lindenbaum–Tarski probability algebra of the language $\mathcal{L}$ of countable sentential calculi, $\boldsymbol{Z} = (Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ is its Stone probability space, $\boldsymbol{\mathcal{E}} = (\mathcal{B}a(Z^*) \,/\, \mathcal{N}, \mathbf{P})$ is the probability algebra of $\boldsymbol{Z}$, and ζ is the canonical Boolean isomorphism of $\boldsymbol{\mathfrak{A}}$ into $\boldsymbol{\mathcal{E}}$ such that, for every equivalence class $[F]$ of formulas of the language $\mathcal{L}$,

(VIII) $\quad \mathbb{P}(\zeta([F])^*) = 0$ if and only if $\operatorname{int}\operatorname{cl}\, \zeta([F])^* = \emptyset$,

the probability of the open-closed representative $\zeta([F])^*$ of $\zeta([F])$ being 0 if and only if $\zeta([F])^*$ is nowhere dense in Z^* (‘impossible’).

That Schema (VIII) is, like Schemas (I) and (III), a theorem of the metalanguage $\mathcal{L}^E$ of mathematical English hardly needs an explanation and follows immediately from the representation of Lindenbaum–Tarski probability algebras, every Baire subset of the Stone space Z^* of $\mathfrak{A}^*$, hence every open-closed subset thereof, being $\mathbb{P}$-negligible if and only if it is nowhere dense as already noted.

3.2.2 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

$$\text{(IX)} \qquad \mathbb{P}(\zeta([F])^*) = 1 \text{ if and only if } \mathrm{cl}\ \mathrm{int}\ \zeta([F])^* = Z^*,$$

the probability of the open-closed representative $\zeta([F])^*$ *of* $\zeta([F])$ *being 1 if and only if* $\zeta([F])^*$ *is non-boundary in* Z^* *('necessary').*

PROOF For suppose that $\mathbb{P}(\zeta(-[F])^*) = 0$ if and only if int cl $\zeta(-[F])^* = \emptyset$, that is to say, that $\mathbb{P}(\zeta(-[F])^*) = 0$ if and only if $\zeta(-[F])^*$ is nowhere dense in Z^* ('impossible'). Since int $\zeta(-[F])^* = Z^* \setminus \mathrm{cl}\ Z^* \setminus \zeta(-[F])^*$, and ζ is a Boolean homomorphism; and since, moreover, $\mathbb{P}$ is, by definition, a probability measure, that is to say, a measure whose total mass is 1, it follows immediately that

$$\begin{aligned} & 1 - \mathbb{P}(\zeta(-[F])^*) = 1 \text{ if and only if } Z^* \setminus \mathrm{int}\ \mathrm{cl}\ \zeta(-[F])^* = Z^* \\ \iff\ & \mathbb{P}(Z^* \setminus \zeta(-[F])^*) = 1 \text{ if and only if } \mathrm{cl}\ Z^* \setminus \mathrm{cl}\ \zeta(-[F])^* = Z^* \\ \iff\ & \mathbb{P}(Z^* \setminus \zeta(-[F])^*) = 1 \text{ if and only if } \mathrm{cl}\ \mathrm{int}\ Z^* \setminus \zeta(-[F])^* = Z^* \\ \iff\ & \mathbb{P}(\zeta([F])^*) = 1 \text{ if and only if } \mathrm{cl}\ \mathrm{int}\ \zeta([F])^* = Z^*, \end{aligned}$$

whence if a sentence is necessary if and only if its negation is impossible (principle of duality common to all classical modal calculi as already noted), then $\zeta([F])^*$ has probability 1 if and only if it is non-boundary in Z^* ('necessary'). The proof is complete. ∎

3.2.3 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(X) $\mathbb{P}(\zeta([F])^*) > 0$ *if and only if* $\text{int cl } \zeta([F])^* \neq \emptyset$,

the probability of the open-closed representative $\zeta([F])^*$ *of* $\zeta([F])$ *being* > 0 *iff* $\zeta([F])^*$ *is somewhere dense in* Z^* *('possibly true').*

PROOF Recall that, by Definition 3.2.1, $\mathbb{P}(\zeta([F])^*) = 0$ if and only if $\text{int cl } \zeta([F])^* = \emptyset$. Then $\mathbb{P}(\zeta([F])^*) \neq 0$ if and only if int cl $\zeta([F])^* \neq \emptyset$, that is to say, since $\mathbb{P}$ is a non-negative measure, $\mathbb{P}(\zeta([F])^*) > 0$ if and only if int cl $\zeta([F])^* \neq \emptyset$ as claimed. ∎

3.2.4 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(XI) $\mathbb{P}(\zeta([F])^*) < 1$ *if and only if* $\text{cl int } \zeta([F])^* \neq Z^*$,

the probability of the open-closed representative $\zeta([F])^*$ *of* $\zeta([F])$ *being* < 1 *iff* $\zeta([F])^*$ *is somewhere codense in* Z^* *('possibly false').*

PROOF The argument is similar to that of Theorem 3.2.3. For recall that, by Theorem 3.2.2, $\mathbb{P}(\zeta([F])^*) = 1$ if and only if cl int $\zeta([F])^* = Z^*$. Then $\mathbb{P}(\zeta([F])^*) \neq 1$ if and only if cl int $\zeta([F])^* \neq Z^*$, that is to say, since $\mathbb{P}$ is a measure whose total mass of 1, $\mathbb{P}(\zeta([F])^*) < 1$ if and only if cl int $\zeta([F])^* \neq Z^*$ as claimed. ∎

Schemas (VIII)–(XI) define the probability of every $\zeta([F])^*$ in terms of its mode of truth. Since Schemas (III)–(VI) define the mode of truth of every $\zeta([F])^*$ in terms of sets, every statement of probability $\mathbb{P}(\zeta([F])^*)$, as a sentence of the metalanguage $\mathcal{L}^E$, can be decomposed into a statement of probability and a statement of modal logic by combining Schemas (VIII)–(XI) and (III)–(VI). For example, combining Schemas (VIII) and (III), it follows immediately that $\mathbb{P}(\zeta([F])^*) = 0$ if and only if $\text{int cl } \zeta([F])^* = \emptyset$ if and only if $\zeta([F])^* = \emptyset$. Of course, in the ordinary conduct of his affairs, a probabilist will simply say that $\mathbb{P}(\emptyset) = 0$.

3.2.5 Definition Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. A *semantic bridge* is a finite sequence of logical equivalences, formulated in the metalanguage $\mathcal{L}^E$ of mathematical English, combining a statement of probability and a statement of modal logic. A *semantic decomposition* is a finite sequence of semantic bridges.

3.2.6 Definition Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. An *event* is an element of the σ-field $\mathcal{B}a(Z^*)$ of Baire subsets of Z^*. In particular, for every equivalence class $[F]$ of formulas of the language $\mathcal{L}$, the open-closed representative $\zeta([F])^*$ of the image $\zeta([F])$ of $[F]$ is called the *event corresponding to* $[F]$.

Of course, as an immediate corollary of Theorem 2.1.13, to every equivalence class $[F]$ of formulas of $\mathcal{L}$ corresponds exactly one event $\zeta([F])^*$; and, conversely, to every event $\zeta([F])^*$ corresponds exactly one equivalence class $[F]$ of formulas of $\mathcal{L}$. Note also that, ζ being a measure-preserving Boolean homomorphism, $\zeta([F])^*$ and $[F]$ are equal in probability, that is to say, $\mathbb{P}(\zeta([F])^*) = \mathbf{p}([F])$. Indeed, since every open-closed set is Baire, and $\boldsymbol{\mathcal{E}}$ is the probability algebra of $\boldsymbol{Z}$, $\mathbb{P}(\zeta([F])^*) = \mathbf{P}(\zeta([F]))$; and since, moreover, $\boldsymbol{\mathcal{E}}$ and $\mathfrak{A}$, as probability algebras, are, by Theorem 3.1.14, isomorphic, $\mathbf{P}(\zeta([F])) = \mathbf{p}([F])$, whence the conclusion follows immediately.

3.2.7 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F])^*) \geq 0 \quad \textit{for every} \quad \zeta([F])^* \in \mathcal{B}a(Z^*),$$

the probability of every event $\zeta([F])^*$ *corresponding to an equivalence class* $[F]$ *of formulas of the language* $\mathcal{L}$ *being non-negative.*

PROOF As previously noted, $\mathbb{P}(\zeta([F])^*) = 0$ if and only if int cl $\zeta([F])^* = \emptyset$ if and only if $\zeta([F])^* = \emptyset$ by Schemas (VIII) and (III). Similarly, $\mathbb{P}(\zeta([F])^*) > 0$ if and only if int cl $\zeta([F])^* \neq \emptyset$ if and only if $\zeta([F])^* \neq \emptyset$ by Schemas (X) and (V). There exist, therefore, semantic bridges such that $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\zeta([F])^*) > 0$ if and only if $\zeta([F])^* \neq \emptyset$, whence there exists a semantic decomposition such that $\mathbb{P}(\zeta([F])^*) \geq 0$ for every $\zeta([F])^* \in \mathcal{B}a(Z^*)$. ▮

3.2.8 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(Z^*) = 1,$$

the probability of the necessary event (*or sure event*) *being* 1.

PROOF Indeed, it follows immediately from Schemas (IX) and (IV) that $\mathbb{P}(\zeta([F])^*) = 1$ if and only if cl int $\zeta([F])^* = Z^*$ if and only if $\zeta([F])^* = Z^*$, whence there exists a semantic bridge, hence a semantic decomposition, such that $\mathbb{P}(Z^*) = 1$ as claimed. ∎

3.2.9 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) = \sum_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*),$$

the probability of the event $\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*$ *corresponding to the join of an indexed family* $\{[F_n] : n \in \mathbb{N}\}$ *of pairwise disjoint equivalence classes of formulas of* $\boldsymbol{\mathcal{L}}$ *being the sum of the probabilities of each event* $\zeta([F_n])^*$ *corresponding to each equivalence class of formulas.*

PROOF For suppose first that $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) = 0$. It follows immediately from Schemas (VIII) and (III) plus Theorem 2.3.24 for countable joins that $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) = 0$ if and only if int cl $\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^* = \emptyset$ if and only if int cl $\zeta([F_n])^* = \emptyset$ for every n if and only if $\zeta([F_n])^* = \emptyset$ for every n if and only if int cl $\zeta([F_n])^* = \emptyset$ for every n if and only if $\mathbb{P}(\zeta([F_n])^*) = 0$ for every n if and only if $\sum_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*) = 0$ because $\mathbb{P}$ is a non-negative measure. Hence, there exist semantic bridges such that $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) = \sum_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*)$ whenever $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) = 0$.

Suppose now that $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) > 0$. Similarly, it follows from Schemas (X) and (V) plus Theorem 2.3.20 for countable joins that $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^*) > 0$ if and only if int cl $\zeta(\bigvee_{n\in\mathbb{N}}[F_n])^* \neq \emptyset$ if and only if int cl $\zeta([F_n])^* \neq \emptyset$ for some n if and only if $\zeta([F_n])^* \neq \emptyset$

for some n if and only if int cl $\zeta([F_n])^* \neq \emptyset$ for some n if and only if $\mathbb{P}(\zeta([F_n])^*) > 0$ for some n if and only if $\sum_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*) > 0$ because $\mathbb{P}$ is a non-negative measure. Hence, there exist semantic bridges such that $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}} [F_n])^*) = \sum_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*)$ whenever $\mathbb{P}(\zeta(\bigvee_{n\in\mathbb{N}} [F_n])^*) > 0$ because $\{[F_n] : n \in \mathbb{N}\}$ is an indexed family of pairwise disjoint equivalence classes of formulas of $\mathcal{L}$. A glance at the definition of a semantic decomposition completes the proof. ▮

Theorem 3.2.7, 3.2.8, and 3.2.9 establish the existence of a semantic decomposition for the axioms of probability [17]. Some level of pedantry in the proofs was necessary to show how logic and probability are combined. Note, in particular, that these theorems did not aim at proving that $\mathbb{P}$ is a non-negative, countably additive measure whose total mass is 1 (a definition which held *a priori*), but that the class of models for modal sentential logic described in Chapter 2 is consistent with the calculus of probability, that is to say, that it does not lead to a contradiction.

3.2.10 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta(-[F])^*) = 1 - \mathbb{P}(\zeta([F])^*),$$

the probability of the event $\zeta(-[F])^*$ *corresponding to the complement of an equivalence class* $[F]$ *of formulas of* $\mathcal{L}$ *being the difference to* 1 *of the probability of the event* $\zeta([F])^*$ *corresponding to* $[F]$.

PROOF Of course, since $\zeta([F] \vee -[F])^* = Z^*$, there exists a semantic bridge such that $\mathbb{P}(\zeta([F] \vee -[F])^*) = 1$ by Theorem 3.2.8; and since $[F]$ and $-[F]$ are disjoint, that is to say, since $[F] \wedge -[F] = [\mathbf{0}]$, there exist semantic bridges such that $\mathbb{P}(\zeta([F] \vee -[F])^*) = \mathbb{P}(\zeta([F])^*) + \mathbb{P}(\zeta(-[F])^*)$ by Theorem 3.2.9. Hence, there exists a semantic decomposition such that $1 = \mathbb{P}(\zeta([F] \vee -[F])^*) = \mathbb{P}(\zeta([F])^*) + \mathbb{P}(\zeta(-[F])^*)$, whence the conclusion follows immediately. ∎

3.2.11 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F_1])^*) \leq \mathbb{P}(\zeta([F_2])^*) \quad \textit{whenever} \quad \zeta([F_1])^* \subseteq \zeta([F_2])^*,$$

the probability of the event $\zeta([F_1])^*$ *corresponding to an equivalence class* $[F_1]$ *of formulas of* $\mathcal{L}$ *being at most that of the event* $\zeta([F_2])^*$ *corresponding to an equivalence class* $[F_2]$ *of formulas of the same language* $\mathcal{L}$ *whenever* $\zeta([F_1])^*$ *is a subset of* $\zeta([F_2])^*$.

PROOF The point is to notice first that $\zeta([F_1])^* \subseteq \zeta([F_2])^*$ if and only if $Z^* \setminus \zeta([F_1])^* \cup \zeta([F_2])^* = Z^*$ if and only if $\zeta(-[F_1] \vee [F_2])^* = Z^*$ if and only if $\zeta([F_1] \rightarrow [F_2])^* = Z^*$ by definition of $\rightarrow$ in $\mathfrak{A}^*$ if and only if $\mathrm{cl\ int}\ \zeta([F_1] \rightarrow [F_2])^* = Z^*$ by Schema (IV) if and only if $\zeta([F_1] \rightarrow [F_2])^* = Z^*$ if and only if $[F_1] \rightarrow [F_2] = [\mathbf{1}]$ because ζ is a Boolean isomorphism if and only if $[F_1] \leq [F_2]$.

In fact, write $[F_3] = [F_2] \wedge -[F_1]$. Then $[F_1] \wedge [F_3] = [\mathbf{0}]$ and $[F_1] \vee [F_3] = [F_2]$, whence there certainly exist semantic bridges such that $\mathbb{P}(\zeta([F_2])^*) = \mathbb{P}(\zeta([F_1] \vee [F_3])^*) = \mathbb{P}(\zeta([F_1])^*) + \mathbb{P}(\zeta([F_3])^*)$ by Theorem 3.2.9. Since there exist semantic bridges such that $\mathbb{P}(\zeta([F_3])^* \geq 0$ by Theorem 3.2.7, there exists a semantic decomposition such that $\mathbb{P}(\zeta([F_1])^*) \leq \mathbb{P}(\zeta([F_2])^*)$ whenever $\zeta([F_1])^* \subseteq \zeta([F_2])^*$. ∎

Theorems 3.2.10 and 3.2.11 establish the existence of a semantic decomposition for the *complement rule* and *monotonicity.* In particular, Theorem 3.2.11 can be reformulated as follows:

$$\mathbb{P}(\zeta([F_1])^*) \leq \mathbb{P}(\zeta([F_2])^*) \quad \text{whenever} \quad \mathrm{cl}\ \mathrm{int}\ \zeta([F_1] \to [F_2])^* = Z^*,$$

the probability of the event $\zeta([F_1])^*$ corresponding to an equivalence class $[F_1]$ of formulas of $\mathcal{L}$ being at most that of the event $\zeta([F_2])^*$ corresponding to an equivalence class $[F_2]$ of formulas of $\mathcal{L}$ whenever the event $\zeta([F_1] \to [F_2])^*$ corresponding to the pseudo-complement of $[F_1]$ relative to $[F_2]$ is non-boundary in Z^* ('necessary'). This result is easily derived from Theorem 3.2.11.

3.2.12 Theorem *Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F_1] \vee [F_2])^*) = \mathbb{P}(\zeta([F_1])^*) + \mathbb{P}(\zeta([F_2])^*) - \mathbb{P}(\zeta([F_1] \wedge [F_2])^*),$$

the probability of the event $\zeta([F_1] \vee [F_2])^*$ corresponding to the join of two arbitrary equivalence classes $[F_1], [F_2]$ of formulas of $\mathcal{L}$ being the sum of the probabilities of each event $\zeta([F_1])^*, \zeta([F_2])^*$ corresponding to each equivalence class of formulas minus the probability of the event $\zeta([F_1] \wedge [F_2])^*$ corresponding to the meet of $[F_1]$ and $[F_2]$.

PROOF For write $[F_1] \vee [F_2] = ([F_1] \wedge -([F_1] \wedge [F_2])) \vee ([F_1] \wedge [F_2]) \vee ([F_2] \wedge -([F_1] \wedge [F_2]))$. Since $([F_1] \wedge -([F_1] \wedge [F_2]))$, $([F_1] \wedge [F_2])$, and $([F_2] \wedge -([F_1] \wedge [F_2]))$ are disjoint, there exist semantic bridges such that $\mathbb{P}(\zeta([F_1] \vee [F_2])^*) = \mathbb{P}(\zeta([F_1] \wedge -([F_1] \wedge [F_2]))^*) + \mathbb{P}(\zeta([F_1] \wedge [F_2])^*) + \mathbb{P}(\zeta([F_2] \wedge -([F_1] \wedge [F_2]))^*)$ by Theorem 3.2.9; and since $[F_1] \wedge [F_2] \leq [F_1]$ and $[F_1] \wedge [F_2] \leq [F_2]$, there exist semantic bridges such that $\mathbb{P}(\zeta([F_1] \wedge -([F_1] \wedge [F_2]))^*) + \mathbb{P}(\zeta([F_1] \wedge [F_2])^*) + \mathbb{P}(\zeta([F_2] \wedge -([F_1] \wedge [F_2]))^*) = \mathbb{P}(\zeta([F_1])^*) - \mathbb{P}(\zeta([F_1] \wedge [F_2])^*) + \mathbb{P}(\zeta([F_1] \wedge [F_2])^*) + \mathbb{P}(\zeta([F_2])^*) - \mathbb{P}(\zeta([F_1] \wedge [F_2])^*)$ by the same Theorem 3.2.9. Hence, there exists a semantic decomposition such that $\mathbb{P}(\zeta([F_1] \vee [F_2])^*) = \mathbb{P}(\zeta([F_1])^*) + \mathbb{P}(\zeta([F_2])^*) - \mathbb{P}(\zeta([F_1] \wedge [F_2])^*)$ as claimed. ∎

3.2.13 Theorem *Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta(\textstyle\bigvee_{1\leq k\leq n}[F_k])^*) = \sum_i \mathbb{P}(\zeta([F_i])^*) - \sum_{i<j} \mathbb{P}(\zeta([F_i] \wedge [F_j])^*) + \sum_{i<j<k} \mathbb{P}(\zeta([F_i] \wedge [F_j] \wedge [F_k])^*) - \cdots + (-1)^{n+1}\mathbb{P}(\zeta([F_1] \wedge \cdots \wedge [F_n])^*),$$

the probability of the event $\zeta(\bigvee_{1\leq k\leq n}[F_k])$ corresponding to the join of a finite family $\{[F_1], \ldots, [F_n]\}$ of arbitrary equivalence classes of formulas of $\mathcal{L}$ being the sum of the probabilities of each event $\zeta([F_i])^*$ corresponding to each equivalence class of formulas minus the sum of the probabilities of the events $\zeta([F_i] \wedge [F_j])^*$ corresponding to the meet of $[F_i]$ and $[F_j]$ for every $i < j$, plus the sum of the probabilities of the events $\zeta([F_i] \wedge [F_j] \wedge [F_k])^*$ corresponding to the meet of $[F_i]$,

$[F_j], [F_k]$ *for every* $i < j < k$, *etc. plus or minus, depending on the cardinality of the family* $\{[F_1], \ldots, [F_n]\}$, *the probability of the event* $\zeta([F_1] \wedge \cdots \wedge [F_n])^*$ *corresponding to the meet of* $[F_1], \ldots, [F_n]$.

PROOF For there certainly exist semantic bridges for $n = 2$ by Theorem 3.2.12. Suppose now that there exist semantic bridges at rank n. Since $\bigvee_{1 \leq k \leq n+1} [F_k] = \bigvee_{1 \leq k \leq n} [F_k] \vee [F_{n+1}]$, it follows that

$$\mathbb{P}(\zeta(\bigvee_{1 \leq k \leq n+1} [F_k])^*) = \mathbb{P}(\zeta(\bigvee_{1 \leq k \leq n} [F_k])^*) + \mathbb{P}(\zeta([F_{n+1}])^*) - \mathbb{P}(\zeta(\bigvee_{1 \leq k \leq n} [F_k] \wedge [F_{n+1}])^*),$$

whence, by applying the induction hypothesis twice,

$$\mathbb{P}(\zeta(\bigvee_{1 \leq k \leq n+1} [F_k])^*) = \sum_i \mathbb{P}(\zeta([F_i])^*) - \sum_{i<j} \mathbb{P}(\zeta([F_i] \wedge [F_j])^*) + \sum_{i<j<k} \mathbb{P}(\zeta([F_i] \wedge [F_j] \wedge [F_k])^*) - \cdots + (-1)^{n+1} \mathbb{P}(\zeta([F_1] \wedge \cdots \wedge [F_n])^*) + \mathbb{P}(\zeta([F_{n+1}])^*) - \sum_i \mathbb{P}(\zeta([F_i] \wedge [F_{n+1}])^*) + \sum_{i<j} \mathbb{P}(\zeta([F_i] \wedge [F_j] \wedge [F_{n+1}])^*) - \cdots + (-1)^{(n+1)+1} \mathbb{P}(\zeta([F_1] \wedge \cdots \wedge [F_{n+1}])^*).$$

Hence, there exist, after arrangement and simplification, semantic bridges at rank $n + 1$ by Theorem 3.2.9, which implies that there exists a semantic decomposition for every n as required. ▮

3.2.14 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta(\bigvee_{1 \leq k \leq n} [F_k])^*) \leq \sum_{k=1}^{n} \mathbb{P}(\zeta([F_k])^*),$$

the probability of the event $\zeta(\bigvee_{1 \leq k \leq n} [F_k])^*$ *corresponding to the join of a finite family* $\{[F_1], \ldots, [F_n]\}$ *of arbitrary (i.e. not necessarily pairwise disjoint) equivalence classes of formulas of the language* $\mathcal{L}$ *be-*

ing at most the sum of the probabilities of each event $\zeta([F_1])^*, \ldots,$ $\zeta([F_n])^*$ *corresponding to each equivalence class of formulas.*

PROOF Indeed, as a variation on the same theme, there certainly exist semantic bridges for $n = 2$ by Theorem 3.2.12. Suppose now that there exist semantic bridges at rank n. It follows that

$$\mathbb{P}(\zeta(\bigvee_{1\leq k\leq n+1}[F_k])^*) = \mathbb{P}(\zeta(\bigvee_{1\leq k\leq n}[F_k])^*) + \mathbb{P}(\zeta([F_{n+1}])^*) - \mathbb{P}(\zeta(\bigvee_{1\leq k\leq n}[F_k] \wedge [F_{n+1}])^*) \leq \mathbb{P}(\zeta(\bigvee_{1\leq k\leq n}[F_k])^*) + \mathbb{P}(\zeta([F_{n+1}])^*) \leq \sum_{k=1}^{n} \mathbb{P}(\zeta([F_k])^*) + \mathbb{P}(\zeta([F_{n+1}])^*) = \sum_{k=1}^{n+1} \mathbb{P}(\zeta([F_k])^*).$$

Hence, in the same vein as Theorem 3.2.13, there exist semantic bridges at rank $n + 1$ by Theorem 3.2.9, which implies that there exists a semantic decomposition for every n as required. ∎

Theorem 3.2.12 is called the *addition rule*, whilst Theorem 3.2.13 is called *Poincaré's identity* or the *inclusion-exclusion formula*; Theorem 3.2.14, for its part, is known as *Boole's inequality.*

3.2.15 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = \prod_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*),$$

the probability of the event $\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*$ *corresponding to the meet of an indexed family* $\{[F_n] : n \in \mathbb{N}\}$ *of equivalence classes of formulas of the language* $\mathcal{L}$*, the occurrence of any of which unaffecting that of any other, being the product (multiplication) of the probabilities of each event* $\zeta([F_n])^*$ *corresponding to each equivalence class of formulas.*

PROOF For suppose first that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = 1$. It follows immediately from Schemas (IX) and (IV) plus Theorem 2.3.14 for countable meets that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = 1$ if and only if cl int $\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^* = Z^*$ if and only if cl int $\zeta([F_n])^* = Z^*$ for every n if and only if $\zeta([F_n])^* = Z^*$ for every n if and only if cl int $\zeta([F_n])^* = Z^*$ for every n if and only if $\mathbb{P}(\zeta([F_n])^*) = 1$ for every n if and only if $\prod_{n=0}^{\infty}\mathbb{P}(\zeta([F_n])^*) = 1$ because $\mathbb{P}$ has a total mass of 1. Hence, there exist semantic bridges such that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = \prod_{n=0}^{\infty}\mathbb{P}(\zeta([F_n])^*)$ whenever $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = 1$.

Suppose now that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) < 1$. Similarly, it follows from Schemas (XI) and (VI) plus the contradual of Theorem 2.3.20 for countable meets that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) < 1$ if and only if cl int $\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^* \neq Z^*$ if and only if cl int $\zeta([F_n])^* \neq Z^*$ for some n if and only if $\zeta([F_n])^* \neq Z^*$ for some n if and only if cl int $\zeta([F_n])^* \neq Z^*$ for some n if and only if $\mathbb{P}(\zeta([F_n])^*) < 1$ for some n if and only if $\prod_{n=0}^{\infty}\mathbb{P}(\zeta([F_n])^*) < 1$ because $\mathbb{P}$ has a total mass of 1. Hence, there exist semantic bridges such that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) = \prod_{n=0}^{\infty}\mathbb{P}(\zeta([F_n])^*)$ whenever $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}}[F_n])^*) < 1$ because $\{[F_n] : n \in \mathbb{N}\}$ is an indexed family of equivalence classes of formulas of $\mathcal{L}$, the occurrence of any of which unaffecting that of any other. A glance at Definition 3.2.5 completes the proof of the theorem. ▮

3.2.16 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) = \mathbb{P}(\zeta([F_1])^*) \times \mathbb{P}(\zeta([F_2])^*),$$

the probability of the event $\zeta([F_1] \wedge [F_2])^*$ *corresponding to the meet of two equivalence classes* $[F_1], [F_2]$ *of formulas of the language* $\mathcal{L}$*, the occurrence of any of which unaffecting that of the other, being the product (multiplication) of the probabilities of each event* $\zeta([F_1])^*$, $\zeta([F_2])^*$ *corresponding to each equivalence class of formulas.*

PROOF As an immediate consequence of Theorem 3.2.15. ▮

3.2.17 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that, for every family* $\{[F_1], \ldots, [F_n]\}$ *of equivalence classes of formulas of* $\mathcal{L}$*,*

$$\mathbb{P}(\zeta([F_{k_1}] \wedge \cdots \wedge [F_{k_j}])^*) = \mathbb{P}(\zeta([F_{k_1}])^*) \times \cdots \times \mathbb{P}(\zeta([F_{k_j}])^*),$$

the probability of the event $\zeta([F_{k_1}] \wedge \cdots \wedge [F_{k_j}])^*$ *corresponding to the meet, for every* $2 \leq j \leq n$ *and every* $1 \leq k_1 < \cdots < k_j \leq n$*, of each finite subfamily* $\{[F_{k_1}], \ldots, [F_{k_j}]\} \subseteq \{[F_1], \ldots, [F_n]\}$ *of equivalence classes of formulas of* $\mathcal{L}$*, the occurrence of any of which unaffecting that of any other, being the product (multiplication) of the probabilities of each event* $\zeta([F_{k_1}])^*, \ldots, \zeta([F_{k_j}])^*$ *corresponding to each equivalence class of formulas in each finite subfamily* $\{[F_{k_1}], \ldots, [F_{k_j}]\}$.

PROOF As an immediate consequence of Theorem 3.2.16. ▮

Theorem 3.2.16 captures the notion of *independence.* Theorem 3. 2.17, which generalises Theorem 3.2.16 to the case of any finite family $\{[F_1], \ldots, [F_n]\}$ of equivalence classes of formulas of $\mathcal{L}$, is sometimes called *mutual independence* to highlight that every finite subfamily $\{[F_{k_1}], \ldots, [F_{k_j}]\}$, $2 \leq j \leq n$, $1 \leq k_1 < \cdots < k_j \leq n$, must be independent. This amounts, in practice, to verify that independence holds for $\sum_{j=2}^{n} \binom{n}{j} = 2^n - n - 1$ finite subfamilies.

3.2.18 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*) \neq \prod_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*),$$

the probability of the event $\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*$ *corresponding to the meet of an indexed family* $\{[F_n] : n \in \mathbb{N}\}$ *of equivalence classes of formulas of* $\boldsymbol{\mathcal{L}}$ *differing, in general, from the product of the probabilities of each event* $\zeta([F_n])^*$ *corresponding to each equivalence class of formulas whenever the probability of* $\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*$ *is less than* 1.

PROOF That $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*)$ must be less that 1 is obvious.

Suppose now that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*) < 1$. Then, similarly to Theorem 3.2.15, $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*) < 1$ if and only if cl int $\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^* \neq Z^*$ if and only if cl int $\zeta([F_n])^* \neq Z^*$ for some n if and only if $\zeta([F_n])^* \neq Z^*$ for some n if and only if cl int $\zeta([F_n])^* \neq Z^*$ for some n if and only if $\mathbb{P}(\zeta([F_n])^*) < 1$ for some n if and only if $\prod_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*) < 1$, which is not a sufficient condition to conclude that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*) = \prod_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*)$ in general. Hence, there exists a semantic decomposition such that $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*) \neq \prod_{n=0}^{\infty} \mathbb{P}(\zeta([F_n])^*)$ whenever $\mathbb{P}(\zeta(\bigwedge_{n\in\mathbb{N}} [F_n])^*) < 1$ as claimed. ∎

3.2.19 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) \neq \mathbb{P}(\zeta([F_1])^*) \times \mathbb{P}(\zeta([F_2])^*),$$

the probability of the event $\zeta([F_1] \wedge [F_2])^*$ *corresponding to the meet of two equivalence classes* $[F_1], [F_2]$ *of formulas of the language* $\mathcal{L}$ *differing, in general, from the product of the probabilities of each event* $\zeta([F_1])^*, \zeta([F_2])^*$ *corresponding to each equivalence class of formulas whenever the probability of* $\zeta([F_1] \wedge [F_2])^*$ *is less than* 1.

PROOF As an immediate consequence of Theorem 3.2.18. ▮

3.2.20 Definition Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability and $\zeta([F_1])^*, \zeta([F_2])^*$ be two events corresponding to two equivalence classes $[F_1], [F_2]$ of formulas of $\mathcal{L}$ such that the probability of $\zeta([F_2])^*$ corresponding to $[F_2]$ is positive. Then

$$\mathbb{P}(\zeta([F_1])^* \mid \zeta([F_2])^*) = \frac{\mathbb{P}(\zeta([F_1] \wedge [F_2])^*)}{\mathbb{P}(\zeta([F_2])^*)}$$

is a probability called the *conditional probability of the event* $\zeta([F_1])^*$ *corresponding to* $[F_1]$ *given the event* $\zeta([F_2])^*$ *corresponding to* $[F_2]$.

3.2.21 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) = \mathbb{P}(\zeta([F_1])^* \mid \zeta([F_2])^*) \times \mathbb{P}(\zeta([F_2])^*),$$

the probability of the event $\zeta([F_1] \wedge [F_2])^*$ *corresponding to the meet of two not necessarily independent equivalence classes* $[F_1], [F_2]$ *of for-*

mulas of the language $\mathcal{L}$ being the conditional probability of the event $\zeta([F_1])^*$ corresponding to $[F_1]$ given the event $\zeta([F_2])^*$ corresponding to $[F_2]$, provided that the probability of $\zeta([F_2])^*$ is positive, times the probability of the event $\zeta([F_2])^*$ corresponding to $[F_2]$.

PROOF Suppose that $\mathbb{P}(\zeta([F_2])^*) > 0$. If $[F_1]$ and $[F_2]$ are independent equivalence classes of formulas, then $\mathbb{P}(\zeta([F_1])^* \mid \zeta([F_2])^*) = \mathbb{P}(\zeta([F_1])^*)$, whence $\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) = \mathbb{P}(\zeta([F_1])^*) \times \mathbb{P}(\zeta([F_2])^*)$, which implies that there exist semantic bridges such that $\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) = \mathbb{P}(\zeta([F_1])^* \mid \zeta([F_2])^*) \times \mathbb{P}(\zeta([F_2])^*)$ by Theorem 3.2.16.

On the other hand, if $[F_1]$ and $[F_2]$ are not independent, then $\mathbb{P}(\zeta([F_1])^* \mid \zeta([F_2])^*) \neq \mathbb{P}(\zeta([F_1])^*)$, whence there exist semantic bridges such that $\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) \neq \mathbb{P}(\zeta([F_1])^*) \times \mathbb{P}(\zeta([F_2])^*)$ by Theorem 3.2.19 provided that $\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) < 1$. Assuming that $[F_1] \neq [\mathbf{1}]$ so that this last condition is satisfied, there exists a semantic decomposition such that $\mathbb{P}(\zeta([F_1] \wedge [F_2])^*) = \mathbb{P}(\zeta([F_1])^* \mid \zeta([F_2])^*) \times \mathbb{P}(\zeta([F_2])^*)$ as a particular case. The proof is complete. ▮

3.2.22 Theorem *Let $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ be a model for the calculus of probability. Then there exists a semantic decomposition such that*

$$\mathbb{P}(\zeta([F_1] \wedge \cdots \wedge [F_n])^*) = \mathbb{P}(\zeta([F_1])^*) \times \mathbb{P}(\zeta([F_2])^* \mid \zeta([F_1])^*) \times \mathbb{P}(\zeta([F_3])^* \mid \zeta([F_1] \wedge [F_2])^*) \times \cdots \times \mathbb{P}(\zeta([F_n])^* \mid \zeta([F_1] \wedge \cdots \wedge [F_{n-1}])^*),$$

the probability of the event $\zeta([F_1] \wedge \cdots \wedge [F_n])^*$ corresponding to the meet of a finite family $\{[F_1], \ldots, [F_n]\}$ of not necessarily independent *equivalence classes of formulas of the language* $\mathcal{L}$ *being the probability of the event* $\zeta([F_1])^*$ *corresponding to* $[F_1]$ *times the probability of the event* $\zeta([F_2])^*$ *corresponding to* $[F_2]$ *given the event* $\zeta([F_1])^*$ *corresponding to* $[F_1]$*, times the probability of the event* $\zeta([F_3])^*$ *corresponding to* $[F_3]$ *given the event* $\zeta([F_1] \wedge [F_2])^*$ *corresponding to the meet of* $[F_1]$ *and* $[F_2]$*, etc. times the probability of the event* $\zeta([F_n])^*$ *corresponding to* $[F_n]$ *given the event* $\zeta([F_1] \wedge \cdots \wedge [F_{n-1}])^*$ *corresponding to the meet of* $[F_1], \ldots, [F_{n-1}]$*, provided that the probability of the event* $\zeta([F_1] \wedge \cdots \wedge [F_{n-1}])^*$ *in question is positive.*

PROOF There certainly exist semantic bridges for $n = 2$. Suppose now that there exist semantic bridges at rank n. Since $[F_1] \wedge \cdots \wedge [F_{n+1}] = ([F_1] \wedge \cdots \wedge [F_n]) \wedge [F_{n+1}]$, it follows that

$$\mathbb{P}(\zeta([F_1] \wedge \cdots \wedge [F_{n+1}])^*) = \mathbb{P}(\zeta(([F_1] \wedge \cdots \wedge [F_n]) \wedge [F_{n+1}])^*) =$$

$$\mathbb{P}(\zeta([F_1] \wedge \cdots \wedge [F_n])^*) \times \mathbb{P}(\zeta([F_{n+1}])^* \mid \zeta([F_1] \wedge \cdots \wedge [F_n])^*) = \mathbb{P}(\zeta([F_1])^*)$$

$$\times \mathbb{P}(\zeta([F_2])^* \mid \zeta([F_1])^*) \times \cdots \times \mathbb{P}(\zeta([F_n])^* \mid \zeta([F_1] \wedge \cdots \wedge [F_{n-1}])^*)$$

$$\times \mathbb{P}(\zeta([F_{n+1}])^* \mid \zeta([F_1] \wedge \cdots \wedge [F_n])^*),$$

provided that the probability of $\zeta([F_1] \wedge \cdots \wedge [F_n])^*$ is positive. Hence, there exist semantic bridges at rank $n + 1$, which implies that there exists a semantic decomposition for every n as required. ▮

Theorem 3.2.19 captures a general notion of *dependence*, whilst Theorem 3.2.21 makes it precise via the notion of conditional probability; Theorem 3.2.22, for its part, is known as the *product rule.*

Other properties can be derived (e.g. Bayes' theorem); since they arise as corollaries of corollaries, they are not reproduced here. Note that the notion of conditional probability has been introduced by definition. This follows from the fact that although the quotient of two probability measures is well defined in $\mathbb{R}$, that of two equivalence classes of formulas in $\mathfrak{A}^*$ (or two events in $\mathcal{B}a(Z^*)$) is not, conditional probabilities appearing, therefore, as a surconstruction of probability theory. Yet this does not impair the existence of, say, a semantic decomposition for the product rule because its existence does not depend on the definition of conditional probability.

3.2.23 Theorem *Let* $(\boldsymbol{\mathfrak{A}}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. For every equivalence class* $[F]$ *of formulas of* $\mathcal{L}$*,*

(XII-a) $\mathbb{P}(\zeta([F])^* \triangle B) = 0$ *if and only if* $\operatorname{int}\operatorname{cl}(\zeta([F])^* \triangle B) = \emptyset$,

the probability of the symmetric difference of the open-closed representative $\zeta([F])^*$ *of* $\zeta([F])$ *with an arbitrary Baire set* B *being* 0 *if and only if* $\zeta([F])^* \triangle B$ *is nowhere dense in* Z^* *('impossible').*

PROOF That the set $\zeta([F])^* \triangle B = (\zeta([F])^* \cup B) \setminus (\zeta([F])^* \cap B) = (\zeta([F])^* \setminus B) \cup (B \setminus \zeta([F])^*)$ is a Baire subset of Z^* is clear; that it is, moreover, $\mathbb{P}$-negligible if and only if it is nowhere dense is an immediate corollary of Theorem 3.1.8 or Theorem 3.1.14. ▮

3.2.24 Theorem *Let* $(\boldsymbol{\mathfrak{A}}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exists a semantic decomposition such that*

(XII-b) $\mathbb{P}(\zeta([F])^* \triangle B) = 0$ *if and only if* $B \in \zeta([F])$,

the probability of the symmetric difference of the open-closed representative $\zeta([F])^*$ *of the equivalence class* $\zeta([F])$ *with an arbitrary Baire set* B *being* 0 *if and only if* B *belongs to* $\zeta([F])$.

PROOF Indeed, combining Schemas (XII-a) and (VII), there certainly exists a semantic bridge, hence a semantic decomposition, such that $\mathbb{P}(\zeta([F])^* \triangle B) = 0$ if and only if $B \in \zeta([F])$ as claimed. ▮

Schema (XII-b) is the probabilistic version of Schema (VII). Likewise, at first glance, the reader may contend that it is inconsequential: $\zeta([F])^* \triangle B$ is $\mathbb{P}$-negligible, that is to say, $\zeta([F])^*$ and B are $\mathbb{P}$-impossible to discriminate from each other, if and only if $\zeta([F])^*$ and B belong to the same equivalence class of $\mathbb{P}$-measurable sets (in which case, $\mathbb{P}(\zeta([F])^*) = \mathbb{P}(B)$), and the equivalence relation defined on $\mathcal{B}a(Z^*)$ naturally inducing a congruence relation on $\mathcal{B}a(Z^*) \,/\, \mathcal{N}$, they are interchangeable *salva veritate.*

But the present state of affairs is, once again, more subtle. For recall that every statement of probability $\mathbb{P}(\zeta([F])^*)$, as a sentence of the metalanguage $\mathcal{L}^E$ of mathematical English, can be decomposed into a statement of probability and a statement of modal logic by combining Schemas (VIII)–(XI) and (III)–(VI). Then $\mathbb{P}(\zeta([F])^*)$ is a theorem of $\mathcal{L}^E$ if and only if $\zeta([F])^*$ is open-closed because Schemas (III)–(VI) are theorems of $\mathcal{L}^E$ if and only if $\zeta([F])^*$ is open-closed. Hence, although $\zeta([F])^* \triangle B$ is $\mathbb{P}$-negligible, that is to say, although $\zeta([F])^*$ and B are $\mathbb{P}$-impossible to discriminate from each other, they are not interchangeable *salva veritate.*

Schema (XII-b) uncovers, therefore, a probabilistic version of agnoia: Unbeknown to the probabilist, every statement of probability $\mathbb{P}(\zeta([F])^*)$, as a sentence of the metalanguage $\mathcal{L}^E$, can be false in the sense that its semantic decomposition is not a theorem of $\mathcal{L}^E$, in which case $\mathbb{P}(\zeta([F])^*)$, as a probability measure, is meaningless. For example, consider the following semantic decomposition

$$\begin{aligned}\mathbb{P}(\zeta([F])^*) = 0 \text{ if and only if } \operatorname{int}\operatorname{cl}\zeta([F])^* = \emptyset \\ \text{if and only if } \zeta([F])^* = \emptyset,\end{aligned}$$

and suppose that $\zeta([F])^*$ is closed. If $\zeta([F])^*$ is closed and empty (i.e. the empty set), then it is nowhere dense in Z^*, hence $\mathbb{P}$-negligible; whilst if $\zeta([F])^*$ is closed and $\mathbb{P}$-negligible, then it is nowhere dense in Z^*, although not necessarily empty, the homeomorphic copy of the Cantor set in Z^* being a classic example.

3.2.25 Definition Let $\boldsymbol{Z} = (Z^*, \mathcal{T}_S^*, \mathcal{B}a(Z^*), \mathbb{P})$ be the Stone probability space of a Lindenbaum–Tarski probability algebra $\boldsymbol{\mathfrak{A}} = (\mathfrak{A}^*, \mathbf{p})$ and P be a probability measure defined on $\mathcal{B}a(Z^*)$. Then

(i) P is *absolutely continuous with respect to* $\mathbb{P}$ if, for every Baire subset B of Z^*, $\mathrm{P}(B) = 0$ whenever $\mathbb{P}(B) = 0$;

(ii) P and $\mathbb{P}$ are *singular with respect to each other*, or *mutually singular*, if, for every Baire subset B_1, B_2 of Z^* such that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = Z^*$, $\mathrm{P}(B_1) = \mathbb{P}(B_2) = 0$.

3.2.26 Theorem *Let* $(\mathfrak{A}, \boldsymbol{Z}, \boldsymbol{\mathcal{E}}, \zeta)$ *be a model for the calculus of probability. Then there exist no finer measures of agnoia than* $\mathbb{P}$*, that is to say, if* P *is a probability measure on the* σ*-field* $\mathcal{B}a(Z^*)$ *of Baire subsets of the Stone space* Z^* *of a measurable Lindenbaum–Tarski algebra* $\mathfrak{A}^*$*, then* P *is absolutely continuous with respect to* $\mathbb{P}$*.*

PROOF Since P and $\mathbb{P}$ are probability measures, there exists a unique Lebesgue decomposition [1, Theorem 8.11, p. 88] $\mathrm{P} = \mathrm{P}_1 + \mathrm{P}_2$ such that P_1 is absolutely continuous with respect to $\mathbb{P}$, and P_2 is singular with respect to $\mathbb{P}$. In particular, if P_2 and $\mathbb{P}$ are mutually singular, then, for every Baire subset B of Z^*, $\mathrm{P}_2(B) = \mathbb{P}(Z^* \setminus B) = 0$. A contradiction because every Baire set in Z^* is negligible if and only if it is nowhere dense, and, in every topological space, the complement of a nowhere dense set is non-boundary. It follows that no singular probability measure P_2 with respect to $\mathbb{P}$ exists on $\mathcal{B}a(Z^*)$, whence $\mathrm{P} = \mathrm{P}_1$ is absolutely continuous with respect to $\mathbb{P}$ as claimed. The proof is complete. ∎

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