Topological-Holomorphic Field Theories and Integrability

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Abstract

Recent developments have revealed that various two-dimensional integrable and conformal field theories (CFTs) can be understood as descending from a common higher-dimensional origin: holomorphic Chern-Simons theory in six dimensions. Building on foundational ideas by Costello, the work of Bittleston and Skinner, described how two distinct approaches of deriving integrable models, namely from defects in four-dimensional Chern-Simons theory or via symmetry reductions of four-dimensional anti-self-dual Yang-Mills (ASDYM) equations, are in fact unified within a six-dimensional framework. This thesis provides a complete description of this framework for a broad class of deformed sigma-models, extending beyond previously studied Dirichlet boundary conditions.

By formulating holomorphic Chern-Simons theory on twistor space with a meromorphic three-form, we construct novel four-dimensional integrable field theories whose equations of motion can be identified with ASDYM. Subsequent symmetry reduction yields rich families of two-dimensional integrable models, including multi-parameter deformations of sigma-models. Additionally, we show that performing the reduction in reverse order—first obtaining four-dimensional Chern-Simons theory with generalised boundary conditions, then constructing defect theories—recovers the same integrable models. Importantly, we extend this correspondence to include models realised through gaugings, thereby providing a higher-dimensional origin for coset CFTs and homogeneous sine-Gordon models. This expanded framework not only unifies known constructions but also uncovers novel classes of integrable theories, offering new directions in the study of integrable systems.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere by the author for any other degree or qualification and it is all work undertaken by the author unless referenced to the contrary in the text.

§3 and §4 of the thesis are based on the publications the author produced with collaborators under the duration of the PhD candidature:

1. Lewis T. Cole, Ryan A. Cullinan, Ben Hoare, Joaquin Liniado, Daniel C. Thompson. “Integrable Deformations from Twistor Space”. (2024) [Col+24b] ,

2. Lewis T. Cole, Ryan A. Cullinan, Ben Hoare, Joaquin Liniado, Daniel C. Thompson. “Gauging The Diamond: Integrable Coset Models from Twistor Space”. (2024) [Col+24a] .

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Finally, to my mother and Scott, this thesis would not have been possible without your love and support, thank you.

In memory of Mark Andrew and Patrick Joseph Cullinan,

two gentlemen I knew only half as well as I should have.

"These peregrinations are incalculable."

The Painter to K.

Franz Kafka, The Trial.

Contents

Chapter 0

Introduction

Modern theoretical physics suffers from an issue with complexity. Mathematically describing even the most fundamental interactions of nature is a profoundly tough exercise given the non-linear nature of the equations describing such systems. It is for this reason that integrable systems provide a pragmatic opportunity for physicists. Such systems can at first appear intractable; however, behind this superficial exterior, integrable systems display large amounts of order. Such order often being manifested through the existence of infinitely many charges which in turn constrains their dynamics rendering the system solvable.

The study of integrable systems emerged from Newton's studies of celestial mechanics, which culminated in his solution to the famous Kepler problem. This problem is a special case of the two-body problem, where the potential between the two bodies is given by the gravitational potential between the two objects. The phase space of the Kepler problem defined in $\mathbb{R}^3$ is the 12 dimensional space, $T^*\mathbb{R}^3 \times T^*\mathbb{R}^3$, however, using the translational, rotational and Runge-Lenz symmetries present in the system one can reduce the problem to that of a first order ordinary differential equation,

$$\dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))} \ ,$$

which may be solved by quadratures. The Kepler problem thus presented the first known physical system which was integrable in the sense of Liouville, that is, a system with a $2n$-dimensional phase space possessing n independent, Poisson commuting conserved charges.

The works of Euler, Lagrange, Hamilton and Jacobi in the 18th and 19th centuries presented many more examples of integrable systems exhibiting Liouville integrability, in particular highlighting the efficacy of utilising a system's symmetries in solving its equations of motion.

Liouville integrability is a strong notion for systems with a finite-dimensional phase space. However, its analogue becomes less clear when we shift our focus to systems with infinite-dimensional phase spaces. After all, infinitely many conserved charges may not suffice if we have missed every second one, as required for Liouville integrability. As such, for systems described by partial differential equations, integrability can manifest itself in a different fashion. One such manifestation is the ability of a system to admit solitonic solutions.

The inception of endeavours in this direction is tied to the observations of John Scott Russell, who, while on horseback along the towpath of the famous Union Canal in 1834, made a remarkable discovery:

> "*I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.*"
>
> — *John Scott Russell*

What J.S. Russell had observed in the union canal that day was a solitonic solution to the Korteweg–De Vries (KdV) equation [KV95]. Building upon the preceding work of John William Strutt, 3rd Baron Rayleigh and Joseph Boussinesq, Korteweg and de

Vries derived the following equation

$$4u_t - u_{xxx} - 6uu_x = 0 \ ,$$

describing the behaviour of waves propagating in a shallow body of water, providing a mathematical framework aligning with the empirical observations of J.S. Russell in 1834. The solution describing a single soliton is given by

$$u = 2\kappa^2 \text{sech}^2(\kappa x + \kappa^3 t - \kappa x_0) \ ,$$

for κ and x_0 constants. This is exactly the aforementioned 'wave of translation' Russell observed.

In general, the term soliton refers to a solution of a differential equation characterized by a localized, non-singular concentration of energy. It transpires that the stability of such objects during propagation is maintained by constraints arising from the existence of infinitely many dynamically conserved currents.[1]

Another sense in which a model may be deemed integrable is if it possesses a Lax pair. The Lax pair method for constructing conserved charges relies on writing the equations of motion in a specific form, namely the Lax form. In the Lax formalism, the associated charges are conserved due to the equations of motion for our system being satisfied. Let us demonstrate the concept through the medium of an example, again with the KdV equation. Defining

$$L = -\partial_x^2 + u \text{ and } \quad M = 4\partial_x^3 - 6u\partial_x - 3u_x \ ,$$

we can write the KdV equation as

$$u_t = L_t = [L, M] \ .$$

The infinite tower of associated charges can then be constructed via the inverse scattering transform.

The concept of a Lax pair naturally leads to the notion of Lax integrability, which

[1]There are also 'topological solitons' whose stability is owing to the fact they are finite energy solutions to the equations of motion, which are in turn characterised by the classes of non-trivial homotopy groups. To deform a topological soliton from one configuration to another topologically distinct configuration requires infinite energy and as such the soliton is stable. Examples of such objects include instanton solutions to Yang-Mills theory and BPS monopoles.

can alternatively be framed in terms of a Lax connection, providing a more geometric perspective on a system's integrability.

Given the abundance of models deemed 'integrable' in one of the aforementioned senses, it becomes essential to ask what universal characteristics these models share that enable their integrability. An attempt at answering this question was given by Richard Ward,

> *... many (and perhaps all?) of the ordinary or partial differential equations that are regarded as being integrable or solvable may be obtained from the self-duality equations (or its generalizations) by reduction. In a sense, they are special cases of the self-duality equations.*
>
> — *Richard Ward*

The latter statement has become known as the 'Ward conjecture' [War85]. It postulates that perhaps there is a sense in which integrable models can be understood as coming from certain symmetry reductions of the anti-self-dual Yang-Mills (ASDYM) equations, given by

$$F(A) = - * F(A) \ .$$

The contemporary understanding is still marred by an uncertainty as to whether the ASDYM equations truly are universal in the manner specified in Ward's original conjecture however. For instance, the Kadomtsev–Petviashvili (KP) equations, an integrable system in three dimensions, escapes a neat description as a reduction of the ASDYM equations.[2]

What remains true is that a striking abundance of physically interesting integrable models are found as reductions of the ASDYM equations, the Bogomolny equations, the Toda field equations, the Non-linear Schödinger equation and the KdV equation, discussed above, to name just a few examples.

This research direction has been pursued with renewed vigour in recent years with the introduction of four-dimensional Chern-Simons theory by Costello. Indeed, a relationship between Chern-Simons theory and integrable systems had been suspected prior [Ati88],

[2]During the course of this work, the talk [Ski25], and subsequently the related article [Bit+25], appeared. These provide a clear description of the KP equation in terms of a local action on mini-twistor correspondence space. The authors remark that "a variant of the Ward conjecture, that all integrable models should arise as symmetry reductions of twistorial theories, therefore seems plausible."

... Atiyah advocated that a natural explanation of the three-dimensional invariance of the Jones polynomial should have an extension to explain the spectral parameter of integrable systems and the associated Yang–Baxter equation.

— *Edward Witten*

In the seminal works [Cos14; Cos13], Costello introduced the following action,

$$S_{4d\mathrm{CS}} = \frac{1}{2\pi i}\int_X \omega \wedge \mathrm{CS}(A) \ ,$$

for four-dimensional Chern-Simons (4dCS).[3] Here, Costello demonstrated that the partition function of the XXX Heisenberg spin chain can be elegantly recast as the vacuum expectation value of specific Wilson line configurations in 4D Chern-Simons theory. This striking result provided yet another vivid illustration of the deep and intricate connections between gauge theory and integrable systems. Notably, the spectral parameter, previously an emergent quantity in the study of integrable models, now takes centre stage as a fundamental element defining the system; the underlying Riemann surface is no longer a 'secondary construct' but an intrinsic part of the spacetime on which the theory is defined. In the subsequent articles [CWY18a; CWY18b], further details of this new paradigm were elucidated, using the more familiar machinery of perturbative Feynman diagrammatics.

A key observation was that Wilson lines in 4D Chern-Simons theory are not solely classified by their representation of the gauge group G but are also associated with representations of the infinite-dimensional algebra $\mathfrak{g}[[z]]$, the Lie algebra of power series in the formal parameter z with coefficients in $\mathfrak{g}$. This arises naturally via a computation: when evaluating quantum corrections to the operator product expansion (OPE) of Wilson line operators, one finds that closure of the OPE necessitates the inclusion of line operators associated with representations of $\mathfrak{g}[[z]]$. At higher orders, further corrections require deforming $\mathfrak{g}[[z]]$ into a mathematical object called a Yangian algebra $\mathcal{Y}$. The emergence of infinite-dimensional symmetry algebras, such as Yangians, is a hallmark of the existence of an integrable structure underpinning a system's solvability, an attribute exemplified by the XXX Heisenberg spin chain, 6-vertex model, and principal chiral model.

[3] An action of this form had also appeared in the PhD thesis of Nekrasov [Nek96].

It was also demonstrated that the deformation from $\mathfrak{g}[[z]]$ to $\mathcal{Y}$ is a fundamental requirement for anomaly cancellation. Specifically, an anomaly would otherwise emerge from the contribution of a two-loop Feynman diagram describing the interaction of two gauge fields with a Wilson line operator in the perturbative expansion, as displayed in fig.1. It was shown that ensuring the Wilson lines transform as a representation of the Yangian algebra is a sufficient condition to prevent this anomaly.

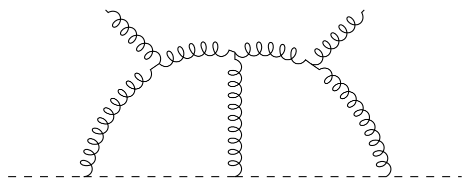


Figure 1: The two-loop Feynman diagram contributing to the gauge anomaly

Having established the link between integrable lattice theories and 4dCS, the details of how 4dCS describes 2d integrable field theories (IFTs) appeared in [CY19]. The 4dCS approach allowed for many exciting new insights. Costello and Yamazaki demonstrated that the action principle governing non-ultra-local integrable field theories emerge naturally from a certain class of defects, known as disorder defects, within the framework of 4dCS theory. In these instances, by permitting ω to be meromorphic, 2d theories arise at the poles of ω, with the constraints on permissible gauge transformations giving rise to dynamical fields, known as edge modes [BSV22], forming the 2d theory's field content.

This novel gauge-theoretic description elucidated many aspects of 2d IFTs. Most notably, giving a systematic approach to obtaining the Lax connection, the quantity underpinning a system's integrability. Four-dimensional Chern-Simons theory can in turn be understood as coming from a reduction of six-dimensional holomorphic Chern-Simons (6dhCS) on Euclidean twistor space, $\mathbb{PT}$,

$$S_{\mathrm{hCS}_6} = \frac{1}{2\pi i} \int_{\mathbb{PT}} \Omega \wedge \mathrm{CS}(\bar{\mathcal{A}}) \ .$$

This action was first considered in [Wit95] as the cubic open string field theory action for the type B topological string. In the context of type B topological string theory, the target spacetime is necessarily Calabi-Yau which ensures it is complemented with a trivial canonical bundle, admitting a globally holomorphic top form Ω. Twistor space however is not Calabi-Yau and as such does not possess a trivial canonical bundle. Therefore, to study 6dhCS on $\mathbb{PT}$ we instead require that Ω is a meromorphic $(3,0)$-form on $\mathbb{PT}$. Schematically, this process can be understood as defining a non-compact Calabi-Yau 3-fold by excising the poles of Ω from $\mathbb{PT}$, which we can now take to be a consistent target space of our type B topological string.

In [BS23], Bittleston and Skinner presented an incredibly aesthetic picture, one that linked the two ways of describing lower-dimensional IFTs via the two aforementioned four-dimensional gauge theories, as depicted in fig.2. It was shown that, just as integrable models can be obtained through symmetry reductions of the anti-self-dual Yang–Mills equations, four-dimensional Chern–Simons theory arises as a symmetry reduction of six-dimensional holomorphic Chern–Simons theory. On the other hand, by performing computations analogous to those employed in localising 4dCS to derive effective actions for 2dIFTs, for the more intricate setting of 6dhCS, it was demonstrated how one can localise 6dhCS, arriving at effective actions for 4dIFTs, the latter theories being integrable in the sense that they are classically equivalent to ASDYM.

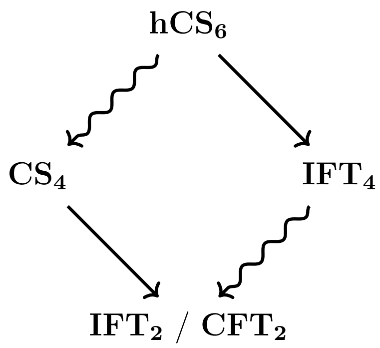


Figure 2: The diamond correspondence of integrable avatars, in which wavy arrows indicate a descent by reduction and straight arrows involve localisation.

Just as the quantisation of 4dCS had resulted in the emergence of objects such as the Yangian, which underpins the quantum integrability of 2d QIFTs, the analysis in [Cos21] investigates the quantum properties of 6dhCS, in particular, highlighting numerous previously obscure aspects of the quantisation of 4d IFTs. Costello showed that 6dhCS theory has a gauge anomaly originating from the one-loop four-point diagram as displayed in fig.3. To cancel the gauge anomaly from this diagram one can use a Green-Schwarz mechanism. This prescription cancels the anomaly by coupling the open and closed sectors of the string field theory. The cost is that this anomaly cancellation restricts the allowed gauge group of the theory, with Costello elucidating the details for the choice $G = SO(8)$.

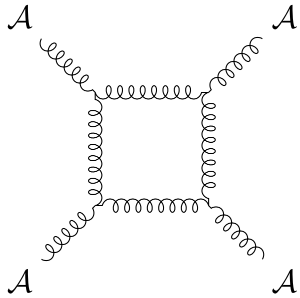


Figure 3: The gauge anomaly for holomorphic Chern-Simons comes from the one-loop four-point diagram

This gauge anomaly in twistor space has the interpretation on spacetime as the anomaly that arises when quantising classically integrable field theories that fail to be quantum integrable. The construction outlined above addresses this, with the closed-string field localising to an axionic field on spacetime that cancels the anomaly, such that the resulting theory is quantum integrable. Notably, the computation reveals a crucial distinction between 4D and 2D quantum integrability: generically, 4D theories require the presence of gravitational fields to preserve integrability at the quantum level.

The framework outlined above has been further developed in recent contributions [CL24; SV24]. In these works the authors undertake a systematic investigation of integrable systems through the lens of five-dimensional semi-holomorphic 2-Chern–Simons theory (5d2CS). Just as classical gauge theory is a theory studying connections, A, on principal G-bundles, 2-gauge theory offers a categorified analogue: it studies 2-connections, $\mathscr{A} = (A, B) \in \Omega^1(X, \mathfrak{g}) \oplus \Omega^2(X, \mathfrak{h})$, on principal 2-bundles, geometric structures whose fibres are 2-groups, subject to coherence and compatibility conditions intrinsic to higher group gauge symmetry. A neat analogy is drawn by using the 2-Chern-Simons 4-form given by

$$2\mathrm{CS}(\mathscr{A}) = \langle F(A) - \frac{1}{2} t_*(B), B\rangle - \frac{1}{2} d\langle A, B\rangle \ ,$$

and forming the action

$$S_{5d2\mathrm{CS}} = \int_{\mathbb{R}^3 \times \mathbb{CP}^1} \omega \wedge 2\mathrm{CS}(\mathscr{A}) \ .$$

By trivialising the 2-connection over $\mathbb{CP}^1$ in an analogous fashion to 4dCS and 6dhCS and localising the action to an effective theory on $\mathbb{R}^3$, the authors chart a new realm of novel integrable phenomena within this higher-dimensional, categorified framework. As one would expect by considering the admissible 2-gauge transformations at the boundaries of 5d2CS, many of the 3-dimensional theories exhibit global or holomorphic 2-group symmetries. The 2-connection provides a natural notion of higher Lax connection for 3-dimensional integrable field theories, by including a 2-form component B which can be integrated over Cauchy surfaces to produce conserved charges. The link between theories integrable in the preceding categorical sense and those theories integrable due to being attained as symmetry reductions of ASDYM is, at this moment, opaque, and an important direction for understanding the landscape of integrable field theories certainly includes illuminating this relationship.

Outline of the Thesis: In this thesis we will explore the following research direction: What is the landscape of integrable field theories that can be studied using this diamond of correspondences as displayed schematically in fig.2? This will turn out to be an even richer story than expected.

We begin in chapter §1 by reviewing foundational topics in two-dimensional integrability, introducing sigma-models and the conditions that such models satisfy when they are classically integrable. This introduces the notion of Lax integrability; Lax integrable models are incredibly symmetric, possessing an infinite number of conserved charges in involution. These symmetries severely constrain the dynamics, often making the model exactly solvable. We proceed in describing integrable deformations of sigma-models: performing an integrable deformation of a sigma-model involves modifying the action, breaking some of the global symmetries, but in such a way as to preserve its integrability. These deformations alter the target space geometry or interaction terms while maintaining the solvable structure, allowing exact methods such as the Bethe ansatz to still apply to solving the theory's spectrum. To conclude, we will introduce the broad notion of dualities of sigma-models. Dualities are a tool of great utility in physics, allowing for the possibility of translating difficult problems in one theory to a more tractable calculation in another dual theory. We will cover Abelian, non-Abelian and Poisson-Lie dualities.

In chapter §2 we explore how gauge theories in higher dimensions can be employed to describe integrable field theories. This involves introducing the main protagonists of the thesis, four-dimensional Chern-Simons, six-dimensional holomorphic Chern-Simons and anti-self-dual Yang-Mills. We begin by introducing Chern-Simons theory, a topological theory in three-dimensions, and then present a topological-holomorphic generalisation of the purely topological Chern-Simons theory, namely four-dimensional Chern-Simons. We describe how the latter can be utilised in the description of two-dimensional integrable field theories. We will highlight how ASDYM can be used to describe integrable systems in lower dimensions, by discussing its symmetry reductions. The concluding sections of the chapter are dedicated to introducing topics in twistor theory relevant to the succeeding components of the thesis. We will provide discourse covering the Ward transform, relating holomorphic vector bundles over twistor space to anti-self-dual connections on spacetime. In the peroration of the chapter we cover holomorphic Chern-Simons theory. We shall elucidate how, in much the same manner four-dimensional Chern-Simons theory describes two-dimensional integrable field theories, six-dimensional holomorphic Chern-Simons theory describes integrable field theories in four dimensions.

Chapter §3 is based on the article [Col+24b], a piece of work which the author produced in collaboration with Lewis Cole, Ben Hoare, Joaquin Liniado and Daniel Thompson. The content of this chapter contains an investigation into the nature of integrable deformed models in dimensions two and four. Starting from 6d holomorphic Chern-Simons theory on twistor space with a particular meromorphic 3-form, we construct the defect theory to find a novel 4d integrable field theory, whose equations of motion can be recast as the 4d anti-self-dual Yang-Mills equations. Symmetry reducing, we find a multi-parameter 2d integrable model, which specialises to the λ-deformation at a certain point in parameter space. The same model is recovered by first symmetry reducing, to give 4d Chern-Simons with generalised boundary conditions, and then constructing the defect theory.

Chapter §4, is based on the article [Col+24a], developed by the author in collaboration with Lewis Cole, Ben Hoare, Joaquin Liniado and Daniel Thompson. This chapter presents work extending the holomorphic Chern-Simons framework to incorporate models realised through gaugings. As well as describing a higher-dimensional origin of coset CFTs, by choosing the details of the reduction from higher dimensions, we obtain rich classes of two-dimensional integrable models including homogeneous sine-Gordon models and generalisations that are new to the literature.

In chapter §5 we conclude the thesis with a discussion and outlook section where we will expand on possible future directions that one could explore from the lines of endeavour presented earlier in this thesis.

Chapter 1

Aspects of 2d Integrability

1.1 Integrable Sigma-Models

The dynamical field configurations of a sigma model are mappings between two manifolds $X : \Sigma \to M$, which describes an embedding of the two-dimensional worldsheet Σ in the n-dimensional manifold M. We will use the field X^i as local coordinates on M, and the coordinate $\boldsymbol{\sigma}^\alpha = (\tau, \sigma)$ to coordinatise Σ. The generic form of an action for a sigma-model is given by,

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left[\sqrt{|h|} h^{\alpha\beta} G_{ij}(X) \partial_\alpha X^i \partial_\beta X^j + \epsilon^{\alpha\beta} B_{ij}(X) \partial_\alpha X^i \partial_\beta X^j \right] , \qquad (1.1.1)$$

where G_{ij} is a symmetric tensor playing the role of the metric of the target space manifold M and B_{ij} is an antisymmetric tensor called the Kalb-Ramond field [KR74]. In the case where Σ is a 2d worldsheet, the first term in the action is exactly the famous Polyakov action [Pol81], which describes strings propagating through a spacetime M.[1] Using local diffeomorphism invariance, the dynamic worldsheet metric $h^{\alpha\beta}(\boldsymbol{\sigma})$ is often gauged fixed to that of the flat Lorentzian metric, which then allows one to adopt the light-cone coordinates defined by $\sigma^\pm = (\tau \pm \sigma)/2$. With this choice, the above action

[1] In string theory, the worldsheet metric $h_{\alpha\beta}$ has no direct physical interpretation in the target space. Consequently, a faithful description of a string propagating on M should rely solely on the embedding X of the worldsheet into the target space. This requirement leads one to impose the Virasoro constraints, which enforce the vanishing of the worldsheet energy-momentum tensor,

$$T_{\alpha\beta} = 0,$$

thereby fixing the worldsheet metric and ensuring that all physical degrees of freedom are encoded in the embedding fields. In the discussion that follows, we will refrain from imposing such constraints.

reads

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, E_{ij}(X)\partial_- X^i \partial_+ X^j \ , \quad E_{ij}(X) = G_{ij}(X) + B_{ij}(X) \ , \tag{1.1.2}$$

where we have introduced the generalised metric E_{ij}.

The Principal Chiral Model:

Consider a sigma-model with target space given by a Lie group G, such that our field configurations are group-valued mappings $g : \Sigma \to G$. The Principal Chiral Model (PCM) [Poh76; ZM78a] describes the propagation of $g(\boldsymbol{\sigma})$ in G using the action

$$S_{\text{PCM}} = \frac{\tilde{K}}{2\pi}\int_\Sigma d^2\sigma \ \text{Tr}(g^{-1}\partial_- g\, g^{-1}\partial_+ g) \ . \tag{1.1.3}$$

Or rather, using the left invariant Maurer-Cartan forms $j = g^{-1}dg \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, we can write

$$S_{\text{PCM}} = \frac{\tilde{K}}{2\pi}\int_\Sigma d^2\sigma \ \text{Tr}(j_- j_+) \ . \tag{1.1.4}$$

The equations of motions of the action are given by

$$\partial_+ j_- + \partial_- j_+ = 0 \ . \tag{1.1.5}$$

The first thing to note is this action has the form of a sigma-model (1.1.2). If we parametrise $g(\boldsymbol{\sigma}) = g(X(\boldsymbol{\sigma}))$ where X^i are local coordinates on the group manifold G, we can expand $j_\pm$ in terms of the left invariant frame fields by

$$j_\pm = L^a_i(X)\partial_\pm X^i\, T_a \tag{1.1.6}$$

where the $T_a \in \mathfrak{g}$ constitute a basis of Lie algebra $\mathfrak{g}$ of G, such that $[T_a, T_b] = f_{ab}{}^c T_c$. Then we immediately see that the PCM is equivalent to (1.1.2) with $G_{ij} = L^a_i(X)L^b_j(X)\text{Tr}(T_a T_b)$ and $B_{ij} = 0$. The PCM is a highly symmetric model, with global isometry group given by $G_L \times G_R$ acting on the fields as

$$g \to h_l^{-1} g\, h_r \ . \tag{1.1.7}$$

The associated conserved currents are given by the left and right Maurer-Cartan forms, $j = g^{-1}dg$ and $k = -dg\, g^{-1}$. The PCM is also exceedingly symmetric in another sense, owing to the fact that it is Lax integrable. It will transpire that one can write down infinitely many non-local conserved charges associated to the model, each in turn constraining the dynamics.

The Lax formalism is indeed a powerful formulation of an integrable field theory. The idea is to introduce a spectral parameter dependent, $\mathfrak{g}$-valued connection over Σ, whose flatness implies the equations of motion of the corresponding theory. We can define a Lax connection for the PCM by

$$\mathcal{L}_\pm = \frac{j_\pm}{1 \mp z} \ , \tag{1.1.8}$$

where z is the local coordinate on the Riemann sphere $\mathbb{CP}^1$. The curvature of the connection $\mathcal{L}$ is defined by,

$$F(\mathcal{L}) = d\mathcal{L} + [\mathcal{L}, \mathcal{L}] \ . \tag{1.1.9}$$

One finds flatness of $\mathcal{L}$,

$$F(\mathcal{L}) = \frac{1}{1-z^2}\left[\partial_+ j_- - \partial_- j_+ + [j_+, j_-] - z(\partial_+ j_- + \partial_- j_+)\right] \ , \tag{1.1.10}$$

for all $z \in C$, implies that $j_\pm$ is both conserved and flat. Conversely, if $j_\pm$ is both flat and conserved, then the corresponding Lax is also flat. Once a connection with such properties is acquired, the construction of the monodromy matrix and conserved charges follows systematically. We define the monodromy matrix by

$$\mathcal{M}(\tau; z) = \mathcal{P}\overleftarrow{\exp}\left(-\int_{-\infty}^{\infty} d\sigma\, \mathcal{L}_\sigma(\sigma, \tau; z)\right) = \mathcal{P}\overleftarrow{\exp}\left(-\int_{-\infty}^{\infty} dx\, \frac{j_\sigma - z j_\tau}{1-z^2}\right) \ , \tag{1.1.11}$$

with $\mathcal{P}\overleftarrow{\exp}$ denoting the path-ordered exponential, where functions with argument $x' > x$ are placed to the left in the expression[2]. Given the flatness of the Lax, one can show the monodromy matrix satisfies

$$\partial_\tau \mathcal{M}(\tau; z) = -\mathcal{L}_\tau \mid_{\sigma\to\infty} \mathcal{M}(\tau; z) + \mathcal{M}(\tau; z)\mathcal{L}_\tau \mid_{\sigma\to-\infty} \ . \tag{1.1.12}$$

As such, imposing boundary conditions that ensure $\mathcal{L}_\tau$ vanishes at spatial infinity, we

[2]This gives rise to the useful mnemonic '**last** (or **later** for time ordering) to the **left**'

find that our monodromy matrix is conserved. Taylor expanding $\mathcal{M}(t;z)$ around $z=\infty$, we arrive at the following expression;

$$\mathcal{M}(\tau;z) = 1 + \frac{1}{z}\int_{-\infty}^{\infty} d\sigma\, j_\tau(\tau,\sigma) + \frac{1}{z^2}\int_{-\infty}^{\infty} d\sigma\, j_\sigma(\tau,\sigma) + \frac{1}{z^2}\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\sigma} d\sigma' j_\tau(\tau,\sigma') j_\tau(\tau,\sigma') + \cdots \ . \tag{1.1.13}$$

At first order in $1/z$, the conserved charge is exactly the Noether charge associated to the right acting symmetry. At higher orders, we generate less familiar terms, all non-local in nature. Such higher order terms generate the classical Yangian algebra of the model, an infinite dimensional algebra that underlies the classical integrability.[3]

The Wess-Zumino Term

Our discussion of sigma-models with target space a Lie group G has been limited to the case where the B-field is identically zero, such as in the case of the principal chiral model. Given the fact the PCM enjoyed the existence of a $G_L \times G_R$ symmetry, one can ask if it is possible to modify the action and add in a non-trivial B-field term whilst preserving this global symmetry? The answer is yes, however it will come in the odd shape of adding a 3-form term to the action. Let us zoom in momentarily on the B-field term in our sigma-model:

$$S_B = \frac{1}{4\pi\alpha'}\int_\Sigma \epsilon^{\alpha\beta} B_{ij}\partial_\alpha X^i \partial_\beta X^j \ . \tag{1.1.14}$$

One finds that under the local target-space transformations,

$$B_{ij} \to B_{ij} + \partial_i\Lambda_j - \partial_j\Lambda_i \ , \tag{1.1.15}$$

for $\Lambda_i = \Lambda_i(X)$, the action for S_B is invariant. As such, one can form the gauge-invariant field strength for B, a three form defined by $H = dB$, with components

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \ . \tag{1.1.16}$$

[3] In the literature, people often refer to integrable systems as those systems that possess a "hidden symmetry" algebra. It is often algebraic structures such as the Yangian algebra they are referring to.

The fact H is spacetime gauge invariant means that the spacetime physics of B will depend only its field strength. This certainly advocates the primacy of H and as such we should not be surprised that we have a 3-form term in our action. There is a glaring issue however, our worldsheet Σ is two dimensional, and as such 3-forms vanish identically on it. To proceed then, we introduce a 3 dimensional manifold $\mathcal{N}$ such that the boundary of $\mathcal{N}$ is given by Σ, i.e. $\partial\mathcal{N} = \Sigma$.[4]

This introduces another obstacle, in order to write down a meaningful theory, we will need to promote our group valued fields from maps $g : \Sigma \to M$ to maps $\hat{g} : \mathcal{N} \to M$, so that we extend g from the boundary Σ into the bulk $\mathcal{N}$ in a consistent fashion. We will require the physics to be completely agnostic to the exact choice of extension into the bulk, $\hat{g}$. In other words, if we were to deform $\hat{g}$ continuously by a small quantity, leaving its value on Σ unchanged, the integral of our 3-form should remain invariant.

We will thus consider the following term [WZ71; Nov82; Wit83]:

$$S_{\text{WZ}} = \frac{k}{2\pi}\int_{\mathcal{N}} \text{WZ}[\hat{g}] \;, \quad \text{WZ}[\hat{g}] = \frac{1}{3}\text{Tr}(\hat{g}^{-1}d\hat{g}\wedge\hat{g}^{-1}d\hat{g}\wedge\hat{g}^{-1}d\hat{g}) \;. \tag{1.1.17}$$

Clearly due to the ad-invariance of the inner product, Tr, we have that the term also has a manifest $G_L \times G_R$ global symmetry. Let us show that it is also invariant under small continuous deformations. Send $\hat{g} \to \hat{g}\exp(\varepsilon) \sim \hat{g}(1+\varepsilon)$, where ε vanishes on the boundary Σ. Under this deformation, the left current transforms as $\delta j = d\varepsilon + [j, \varepsilon]$. We thus have

$$\begin{aligned} \delta S_{\text{WZ}} &= \frac{k}{4\pi}\int_{\mathcal{N}} \text{Tr}((d\varepsilon + [j,\epsilon])\,[j,j]) \\ &= -\frac{k}{4\pi}\int_{\mathcal{N}} \text{Tr}\Big(\varepsilon(d[j,j] + [j,[j,j]])\Big) \\ &= -\frac{3k}{4\pi}\int \text{Tr}\left(\varepsilon[j,[j,j]]\right) \;= 0, \end{aligned} \tag{1.1.18}$$

where we have used integration by parts, ad-invariance of Tr, the flatness of j and finally the Jacobi identity. So indeed, continuous deformations of $\hat{g}$ do not change the WZ-term. One concludes that the WZ-term, despite superficially appearing to be a bona-fide three dimensional term, is in fact only sensitive to the physics on the two dimensional boundary!

It is worth noting at this point, there may exist extensions of the group-valued field into $\mathcal{N}$ that are not related by continuous deformations. Consider two such fields $\hat{g}$ and

[4] Or a disjoint union of such worldsheets $\partial\mathcal{N} = \sqcup\,\Sigma$.

$\hat{g}'$, defined on $\mathcal{N}$ and $\mathcal{N}'$, which agree on their mutual boundary $\partial\mathcal{N} = \partial\mathcal{N}' = \Sigma$, such that $\hat{g} \mid_\Sigma = \hat{g}' \mid_\Sigma$. The difference between the two distinct terms corresponding to each extension is given by

$$I = \int_{\mathcal{N}} \mathrm{WZ}[\hat{g}] - \int_{\mathcal{N}'} \mathrm{WZ}[\hat{g}'] \ . \tag{1.1.19}$$

We denote the reversing of the orientation of $\mathcal{N}'$ by $\bar{\mathcal{N}}'$. Gluing together the manifolds $\mathcal{N}$ and $\bar{\mathcal{N}}'$ forms a manifold homeomorphic to a 3-sphere $S^3 \cong \mathcal{N} \cup \bar{\mathcal{N}}'$. With this, we can identify I with the Brouwer degree [Bro12] of a mapping $\tilde{g}$ defined by,

$$\tilde{g} = \begin{cases} \hat{g} \text{ on } \mathcal{N} \ , \\ \hat{g}' \text{ on } \mathcal{N}' \ . \end{cases} \tag{1.1.20}$$

So I measures the winding number of the mapping $\tilde{g} : S^3 \to G$ defined as $\tilde{g} \mid_{\mathcal{N}} = \hat{g}$ and $\tilde{g} \mid_{\mathcal{N}'} = \hat{g}'$. Such mappings from $S^3 \to G$ are, by definition, characterised by the third homotopy group $\pi_3(G)$. For G a compact and simple Lie group, we have that $\pi_3(G) \cong \mathbb{Z}$. As such we have that,

$$\int_{\mathcal{N}} \mathrm{WZ}[\hat{g}] = \int_{\mathcal{N}'} \mathrm{WZ}[\hat{g}'] + 2\pi n \ , \quad n \in \mathbb{Z} \, . \tag{1.1.21}$$

Explicitly, this result tells us that extensions of g into the bulk with differing winding numbers (or in differing homotopy classes) will lead to actions differing by a constant of $2\pi n$. Now, when we come to quantise the action we will need the path integral to be single valued, this will require the constant k to be integer valued.
Let us consider the system described by the action

$$S_{\mathrm{PCM+WZ}} = \frac{\tilde{K}}{2\pi} \int_\Sigma \mathrm{Tr}(g^{-1}\partial_- g \, g^{-1}\partial_+ g) + \frac{k}{2\pi} \int_{\mathcal{N}} \mathrm{WZ}[g] \ . \tag{1.1.22}$$

We call such a theory the PCM with WZ term. Varying the action, the equations of motion of the theory are given by

$$(\tilde{K} - k)\partial_+ j_- + (\tilde{K} + k)\partial_- j_+ = 0 \ . \tag{1.1.23}$$

We thus find that the conserved current with the addition of the WZ term is given by

$$J_\pm = \frac{1}{\eta}(\tilde{K} \mp k) j_\pm \ , \tag{1.1.24}$$

where η is a normalisation constant that we will fix to likewise ensure flatness. Calculating $F(J)$, the choice $\eta = \tilde{K}$ ensures that $J_\pm$ is indeed flat. With this we can define the Lax connection of the PCM+WZ as

$$\mathcal{L}_\pm = \frac{J_\pm}{1 \mp z} = \frac{1 \mp a}{1 \mp z}\, g^{-1}\partial_\pm g \ , \tag{1.1.25}$$

where we have introduced the parameter $a = k/\tilde{K}$. As such, introducing a WZ term has preserved the Lax integrability of the PCM, allowing one to conclude that the PCM + WZ term shares many of the same integrable properties.

At a special point in the parameter space, namely $\tilde{K} = -k$, the equations of motion (1.1.23) take the form

$$\partial_+ j_- = 0 \ . \tag{1.1.26}$$

One can solve the above in closed form by

$$g(\sigma, \tau) = g(\sigma^-)\, \bar{g}(\sigma^+) \ . \tag{1.1.27}$$

In Euclidean signature this is nothing more than the statement that our fields factor into holomorphic and anti-holomorphic sectors. The result is that the theory is chiral in nature: it has two independently conserved currents generating independent affine Lie algebras. This chirality is a powerful constraint on the theory, by the Sugawara construction it can be shown that the (anti-)holomorphic energy momentum tensor of the theory is a product of the (anti-)holomorphic currents which one can then use to generate the Virasoro algebra familiar in the study of 2d CFTs. The CFT corresponding to the point $\tilde{K} = -k$ in parameter space is known as the Wess-Zumino-Witten (WZW) model [Nov82; Wit84].

Conserved Quantities and the Yang-Baxter equation

Before proceeding, we will emphasize the implications of a theory possessing infinitely many conserved quantities. At the classical level we discussed how the emergence of these conserved charges, in principle, allows one to constrain the dynamics of the system sufficiently, allowing one to perform explicit calculations. But what about at the quantum level? How do the hidden symmetries constrain the dynamics after performing quantisation? A priori there is no guarantee that such charges survive the quantisation process. In some famous classically integrable models (such as the $\mathbb{CP}^n$ model for example), the tower of conserved charges becomes anomalous and so cannot be employed to constrain the quantum S-matrix, the observable describing the quantum dynamics of the system. We will consider the constraints imposed on the S-matrix by those theories whose integrable properties admit a quantisation. For massive two-dimensional theories, the following properties are possessed by quantum integrable field theories:

- There are no processes in which particle production may occur.
- The collection of initial and final momenta are equal up to permutation.
- The $n \to n$ S-matrix can be factorised into a product of $2 \to 2$ S-matrices.

For a more pedagogical and complete discussion of the above, we refer the reader to [Dor96]. We will briefly highlight the implications of third point so as to motivate future discussions. The fact that all scattering processes in 2dQIFTs may be given in terms of the elementary $2 \to 2$ S-matrices suggests the primacy of the two particle S-matrix: once it is known, all other S-matrix elements follow. For consistency, we need our 2-particle S-matrices to solve the Yang-Baxter equation.

The S-matrix is the operator describing the change of basis between the in-state basis and the out-state basis, describing the evolution of the system as it goes from the asymptotic far-past to the asymptotic far-future. For a $2 \to 2$ process the S-matrix satisfies

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)\, A_k(\theta_2)A_l(\theta_1) \; . \tag{1.1.28}$$

Here $A_i(\theta_i)$ denotes a particle of type i, transforming in a representation V_i of the underlying global symmetry group, with rapidity θ_i, where an on-shell particle of mass m_i has rapidity given by

$$(p_i)_+ = m_i \exp(\theta_i) \; , \quad (p_i)_- = m_i \exp(-\theta_i) \; ,$$

where $(p_i)_\pm$ are the light-cone momenta of the ith particle. The Yang-Baxter equation is then given by the necessity that our $3 \to 3$ process be single valued when decomposed into the 2 distinct ways of factorising into multiple $2 \to 2$ processes. Referring to figure 1.1 one can see for an S-matrix given by the linear mapping

$$S(z_1, z_2) : V_1 \otimes V_2 \to V_1 \otimes V_2 \; ,$$

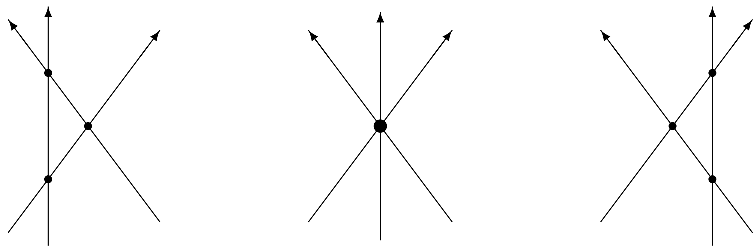


Figure 1.1: Possibilities for a $3 \to 3$ process. The Yang-Baxter equations ensures that all three diagrams are equal.

the Yang Baxter equation (YBe) reads

$$S_{12}(z_1, z_2)S_{13}(z_1, z_3)S_{23}(z_2, z_3) = S_{23}(z_2, z_3)S_{13}(z_1, z_3)S_{12}(z_1, z_2) \ , \tag{1.1.29}$$

where z_i are complex spectral parameters, generalising the dependence on rapidity, which the S-matrices depend on in a meromorphic fashion and the subscript denotes on which of the factors in the tensor product $V_1 \otimes V_2 \otimes V_3$ the S-matrix acts.

The YBe is an inherently quantum constraint. As such, one should consider whether it admits a classical counterpart. If we consider expanding $S(z_i, z_j)$ as a formal power series in $\hbar$ one has

$$S(z) = \mathrm{Id}_{V_1 \otimes V_2} + \hbar r(z_1, z_2) + \mathcal{O}(\hbar^2) \ . \tag{1.1.30}$$

Here $r(z_1, z_2)$ is again a meromorphic function of z_1 and z_2, valued in $\mathrm{End}(V_1 \otimes V_2)$, known as the classical r-matrix. Plugging in (1.1.30) into (1.1.29) one attains the classical Yang-Baxter equation (cYBe):

$$[r_{23}(z_2, z_3), r_{13}(z_1, z_3)] + [r_{13}(z_1, z_3), r_{12}(z_1, z_2)] + [r_{23}(z_2, z_3), r_{12}(z_1, z_2)] = 0 \ , \tag{1.1.31}$$

where once again the subscripts denote on which factors of the tensor product the classical r-matrix is acting.

A classical r-matrix is called non-degenerate if $\det r \neq 0$ identically. Remarkably, it was shown by the work of Belavin and Drinfeld [BD84], that non-degenerate solutions to the cYBe are classified. The solutions fall into three distinct families called rational, trigonometric and elliptic. As such, we likewise refer to the corresponding integrable systems as rational, trigonometric or elliptic depending on the class of their underlying classical r-matrix.

Given the existence of a constant and antisymmetric classical r-matrix, satisfying (1.1.31) for a Lie algebra $\mathfrak{g}$, one may define a new distinct Lie algebra over the underlying vector space of $\mathfrak{g}$. To show how, we introducing tensorial notation. Take $X, Y \in \mathfrak{g}$ we denote by $X_{\underline{\mathbf{1}}}$ and $X_{\underline{\mathbf{2}}}$ the two different embeddings of the element X in the tensor product of the universal enveloping algebra $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, defined so that

$$X_{\underline{\mathbf{1}}} := X \otimes \text{Id} \ , \ \text{and} \quad X_{\underline{\mathbf{2}}} := \text{Id} \otimes X \ . \tag{1.1.32}$$

Then given an r-matrix valued in $\mathfrak{g} \otimes \mathfrak{g}$ we can introduce an equivalent object $\mathcal{R} : \mathfrak{g} \to \mathfrak{g}$,

$$\mathcal{R}X = \text{Tr}_{\underline{\mathbf{2}}}(r_{\underline{\mathbf{12}}}(\text{Id} \otimes X)) \ , \tag{1.1.33}$$

where $\text{Tr}_{\underline{\mathbf{2}}}$ denotes taking the trace with respect to the second factor in the tensor product. We can use then use $\mathcal{R}$ to define the bracket

$$[\bullet, \bullet]_{\mathcal{R}} := [\mathcal{R} \bullet, \bullet] + [\bullet, \mathcal{R} \bullet] \ , \tag{1.1.34}$$

where the Jacobi identity of $[\bullet, \bullet]_{\mathcal{R}}$ is satisfied by virtue of $\mathcal{R}$ being a solution to the cYBe. This ensures $(\mathfrak{g}, [\bullet, \bullet]_{\mathcal{R}})$ indeed forms a Lie algebra.

The Lax connection and the Yang-Baxter equation

We have seen that certain emergent structures appear to underlie the remarkable properties of integrable systems. Thus far, we have introduced two notable instances: the Lax formalism and the Yang-Baxter equation. As such, a question one should naturally be inclined to pose is the following: what is the relation between the notions of integrability in the Lax sense and the cYBe? In this subsection, we will endeavour to answer this question by studying the canonical structure underlying integrable systems, closely following [Mai85].

Recall the definition of the monodromy matrix (1.1.11). Here we defined a quantity that was independent of the spatial coordinates of our worldsheet by integrating over an entire spatial slice at a fixed time τ. One can equally generalise the definition of the monodromy matrix, defined in such a fashion as to be dependent on the spatial coordinates of our worldsheet by

$$\mathcal{M}(\sigma,\sigma',\tau;z) = \mathcal{P}\overleftarrow{\exp}\left(\int_{\sigma}^{\sigma'} d\sigma''\mathcal{L}_\sigma(\sigma'',\tau;z)\right) . \tag{1.1.35}$$

Clearly, the monodromy matrix satisfies the properties,

$$\mathcal{M}(\sigma,\sigma',\tau;z)\mathcal{M}(\sigma',\sigma'',\tau;z) = \mathcal{M}(\sigma,\sigma'',\tau;z) \ , \ \text{where} \quad \mathcal{M}(\sigma,\sigma,\tau;z) = \mathrm{Id} \ . \tag{1.1.36}$$

To study the canonical structure of the integrable theory, it is desirable to compute the equal time Poisson brackets of quantities in the theory. We introduce Poisson brackets, computed such that

$$\{X_{\underline{\mathbf{1}}}, Y_{\underline{\mathbf{2}}}\} = \{X^a, Y^b\}\, T_a \otimes T_b \in \mathfrak{g}\otimes\mathfrak{g} \ . \tag{1.1.37}$$

In integrable field theories, G-invariant quantities formed from the monodromy matrices, and hence the non-local conserved charges extracted from them, mutually Poisson commute. If two such charges fail to Poisson commute, the flow generated by one does not preserve the other—i.e., one charge is not conserved along the Hamiltonian flow of the other. This mutual incompatibility implies that the associated symmetry flows interfere, obstructing the construction of a consistent collection of commuting flows. As a result, the charges cannot jointly generate an integrable foliation of the infinite-dimensional phase space. Using the identity,

$$\begin{aligned} \delta\mathcal{M}(\sigma,\sigma';z) &= \mathcal{P}\left(\int_{\sigma}^{\sigma'} d\sigma''\ \delta\mathcal{L}_\sigma(\sigma'';z)\mathcal{M}(\sigma,\sigma';z)\right) \\ &= \int_{\sigma}^{\sigma'} d\sigma''\, \mathcal{M}(\sigma',\sigma'';z)\, \delta\mathcal{L}_\sigma(\sigma'';z)\, \mathcal{M}(\sigma'',\sigma;z) \ , \end{aligned} \tag{1.1.38}$$

one can verify that the Poisson brackets read,

$$\begin{aligned} \left\{\mathcal{M}_{\underline{\mathbf{1}}}(\sigma,\sigma';z),\, \mathcal{M}_{\underline{\mathbf{2}}}(\rho,\rho';w)\right\} = \int_{\sigma}^{\sigma'} d\sigma'' \int_{\rho}^{\rho'} d\rho''\, &\mathcal{M}_{\underline{\mathbf{1}}}(\sigma,\sigma'';z)\, \mathcal{M}_{\underline{\mathbf{2}}}(\rho,\rho'';w) \\ &\times \left\{\mathcal{L}_{\sigma\underline{\mathbf{1}}}(\sigma'';z),\, \mathcal{L}_{\sigma\underline{\mathbf{2}}}(\rho'';w)\right\} \\ &\times \mathcal{M}_{\underline{\mathbf{1}}}(\sigma'',\sigma';z)\, \mathcal{M}_{\underline{\mathbf{2}}}(\rho'',\rho';w) \ , \end{aligned} \tag{1.1.39}$$

where we have suppressed explicitly displaying the dependence on τ, since all quantities will be evaluated at equal time. The above shows that the canonical algebra of the monodromy matrices is determined by the Poisson bracket of our Lax matrices. Consider the following form for the Poisson bracket of our Lax matrices,

$$\begin{aligned}\big\{\mathcal{L}_{\sigma\underline{\mathbf{1}}}(\sigma;z),\,\mathcal{L}_{\sigma\underline{\mathbf{2}}}(\rho;w)\big\} = \big(\,[R_{\underline{12}}(z,w),\mathcal{L}_{\sigma\underline{1}}(\sigma,z)] - [R_{\underline{12}}(w,z),\mathcal{L}_{\sigma\underline{2}}(\sigma,w)]\,\big)\delta(\sigma-\rho)\\ + \big(R_{\underline{12}}(z,w) + R_{\underline{21}}(w,z)\big)\,\partial_\sigma\delta(\sigma-\rho)\ ,\end{aligned} \tag{1.1.40}$$

with R a function of the spectral parameters z and w, valued in $\mathfrak{g}\otimes\mathfrak{g}$. The canonical algebra given in (1.1.40) is called the Maillet bracket. It is the most general choice with at most one derivative of a delta function term, $\partial_\sigma\delta$, that ensures the Poisson bracket between the trace of polynomials of the monodromy matrices vanishes [Mai85]. Many of the 2d IFTs of interest to us in the succeeding segments of the thesis exhibit such an algebraic structure in the Poisson structure of their Lax connections. We call such theories non-ultralocal due to the presence of the derivative of the delta function in the algebra. Complementarily, if a 2d IFT is such that R is antisymmetric, ensuring that the last term in (1.1.40) vanishes, we call the theory ultralocal. In the ultralocal setting, the bracket (1.1.40) reduces to the Skylanin bracket [Skl82].

One could consider the condition for this choice of canonical algebra to form a Lie algebra. In this case, it will be necessary for the Maillet bracket to satisfy the Jacobi identity. Calculating

$$\begin{aligned}\big\{\mathcal{L}_{\sigma\underline{\mathbf{1}}}(\sigma_1;z_1),\big\{\mathcal{L}_{\sigma\underline{\mathbf{2}}}(\sigma_2;z_2),\,\mathcal{L}_{\sigma\underline{\mathbf{3}}}(\sigma_3;z_3)\big\}\big\} + \big\{\mathcal{L}_{\sigma\underline{\mathbf{3}}}(\sigma_3;z_3),\big\{\mathcal{L}_{\sigma\underline{\mathbf{1}}}(\sigma_1;z_1),\,\mathcal{L}_{\sigma\underline{\mathbf{2}}}(\sigma_2;z_2)\big\}\big\}\\ + \big\{\mathcal{L}_{\sigma\underline{\mathbf{2}}}(\sigma_2;z_2),\big\{\mathcal{L}_{\sigma\underline{\mathbf{3}}}(\sigma_3;z_3),\,\mathcal{L}_{\sigma\underline{\mathbf{1}}}(\sigma_1;z_1)\big\}\big\} = 0\ ,\end{aligned} \tag{1.1.41}$$

one can show that this is equivalent to the condition,

$$\big[Y_{\underline{123}}(z_1,z_2,z_3),\mathcal{L}_{\sigma_{\underline{1}}}(\sigma_1;z_1)\big] + \big[Y_{\underline{312}}(z_1,z_2,z_3),\mathcal{L}_{\sigma_{\underline{3}}}(\sigma_3;z_3)\big] + \big[Y_{\underline{231}}(z_1,z_2,z_3),\mathcal{L}_{\sigma_{\underline{2}}}(\sigma_2;z_2)\big] = 0\ , \tag{1.1.42}$$

where

$$Y_{\underline{123}}(z_1,z_2,z_3) = \big[R_{\underline{12}}(z_1,z_2),R_{\underline{13}}(z_1,z_3)\big] + \big[R_{\underline{12}}(z_1,z_2),R_{\underline{23}}(z_2,z_3)\big] + \big[R_{\underline{32}}(z_3,z_2),R_{\underline{13}}(z_1,z_3)\big]\ . \tag{1.1.43}$$

As such, the consistency condition for our canonical algebra (1.1.40) to form a Lie algebra is that $R(z,w)$ satisfies the cYBE (1.1.31) for antisymmetric R-matrices. With this we have thus elucidated the relationship between the Lax for ultralocal theories and the cYBE.

1.1.1 Integrable Deformations

Integrable deformations provide a pragmatic approach as we try to understand the full landscape of integrable theories. The motivation behind deforming integrable models is the following; given that integrable systems are incredibly symmetric, how much symmetry can we break whilst still retaining the properties underlying a system's integrability? We will investigate this question through the lens of two seemingly unrelated models, namely the λ-deformed model of Sfetsos [Sfe14] and the Yang-Baxter deformed model of Klimcik [Kli02; Kli09].

The λ-Deformed Model

In what follows we will outline a procedure that allows one to derive a one parameter family of integrable models. Our starting point will be the PCM on a group manifold G, as given by (1.1.3):

$$S_{\mathrm{PCM}} = -\frac{\tilde{k}^2}{2\pi}\int_\Sigma d^2\sigma\ \mathrm{Tr}(g^{-1}\partial_- g\, g^{-1}\partial_+ g)\ ,$$

where have chosen a more convenient normalisation to expedite future calculations. We begin by gauging the G_L symmetry, which acts as $G_L : g \to h^{-1}g$, by promoting the partial derivatives in the PCM to covariant derivatives, $D_\pm = \partial_\pm + A_\pm$, with the gauge fields transforming as

$$A \to h^{-1}Ah + h^{-1}dh\ . \tag{1.1.44}$$

The action for the gauged PCM takes the form,

$$S_{\mathrm{gPCM}}(g, A) = -\frac{\tilde{k}^2}{2\pi}\int_\Sigma d^2\sigma\ \mathrm{Tr}(g^{-1}D_- g\, g^{-1}D_+ g)\ . \tag{1.1.45}$$

We will add a WZW-model on G at level k, with the dynamical field $\tilde{g} \in G$.

$$S_{\mathrm{WZW},k}[\tilde{g}] = \frac{k}{2\pi}\int_\Sigma d^2\sigma\ \mathrm{Tr}(\tilde{g}^{-1}\partial_-\tilde{g}\,\tilde{g}^{-1}\partial_+\tilde{g}) + \frac{k}{2\pi}\int_{\mathcal{N}} \mathrm{WZ}[\tilde{g}]\ . \tag{1.1.46}$$

We will also gauge the diagonal subgroup of the WZW model acting as $G_{\mathrm{diag}} : \tilde{g} \to h^{-1}\tilde{g}h$, where $h \in G$. It transpires that gauging the WZW model is a subtle task. The WZ term is a closed 3-form, however, when we perform a naive gauging of the Wess-Zumino term by minimal coupling, $d \to d + A$, one generically finds that the resulting

3-form is no longer closed. This is undesirable, since for our purposes, we require the WZ term to be closed so that we can write it locally as an exact 3-form ensuring our theory is two-dimensional. The resolution lies in the machinery of equivariant cohomology [FS94a]. We can write down the gauged-WZW term as

$$\chi(\tilde{g}, A) = \mathrm{WZ}(\tilde{g}) + d\left[\mathrm{Tr}(d\tilde{g}\tilde{g}^{-1}A_1) + \mathrm{Tr}(\tilde{g}^{-1}d\tilde{g}\, A_2) + \mathrm{Tr}(\tilde{g}^{-1}A_1\tilde{g}A_2)\right] \ , \tag{1.1.47}$$

and the covariant derivatives act on $\tilde{g}$ as

$$\nabla g = dg + A_1 g - gA_2 \ , \tag{1.1.48}$$

where we have gauged by the subgroup $H \subset G_L \times G_R$ such that $A = (A_1, A_2) \in \mathfrak{h} \subset \mathfrak{g}_L \oplus \mathfrak{g}_R$. In the case of gauging the diagonal subgroup $H = G_{\mathrm{diag}}$, we have $A_1 = A_2 := A$, transforming as

$$A \mapsto h^{-1}Ah + h^{-1}dh \ , \tag{1.1.49}$$

for $h \in G_{\mathrm{diag}}$. Proceeding by replacing $dg \to \nabla g$ and $\mathrm{WZ}[\tilde{g}] \to \chi(\tilde{g}, A)$, we can write down the action for the diagonal-gauged WZW model as

$$\begin{aligned} S_{\mathrm{gWZW},k}(\tilde{g}, A) = S_{\mathrm{WZW},k}(\tilde{g}) + \frac{k}{2\pi}\int_\Sigma d^2\sigma\ &\mathrm{Tr}(A_-\partial_+\tilde{g}\tilde{g}^{-1}) - \mathrm{Tr}(A_+\tilde{g}^{-1}\partial_-\tilde{g}) \\ &+ \mathrm{Tr}(A_-\tilde{g}A_+\tilde{g}^{-1}) + \mathrm{Tr}(A_-A_+) \ . \end{aligned} \tag{1.1.50}$$

Now here we will make a crucial step. We will identify the two gauge fields in (1.1.45) and (1.1.50). Doing so we have the following action

$$S_{k,\lambda}(g, \tilde{g}, A) = S_{\mathrm{gPCM}}(g, A) + S_{\mathrm{gWZW},k}(\tilde{g}, A) \ , \tag{1.1.51}$$

where the two terms are coupled through the gauging. We can fix this gauge symmetry by fixing $g = \mathrm{Id}$. Doing so, the only term that survives from the PCM action is

$$\frac{\tilde{K}}{2\pi}\int_\Sigma d^2\sigma\, \mathrm{Tr}(A_-A_+) \ , \tag{1.1.52}$$

and so we arrive at the action,

$$S_{k,\lambda}(g,A) = S_{\text{WZW},k}(g) - \frac{k}{2\lambda\pi}\int_\Sigma d^2\sigma\,\text{Tr}(A_+(\text{Id}-\lambda\text{Ad}_g^{-1})A_-) + \frac{k}{2\pi}\int_\Sigma d^2\sigma\,\text{Tr}(A_-\partial_+ g g^{-1}) - \text{Tr}(A_+ g^{-1}\partial_- g)\ , \tag{1.1.53}$$

where we have introduced the paramater $\lambda = \frac{k}{K^2+k^2}$ and relabelled $\tilde{g}\to g$ on aesthetic grounds. One notes at this stage that our gauge field, $A_\pm$, appears in the action in a non-dynamical fashion. As such, one can integrate out the gauge fields, replacing them with their on-shell values. A quick calculation gives,

$$A_+ = \lambda(1-\lambda\text{Ad}_g)^{-1}\partial_+ g g^{-1}\ ,\quad A_- = -\lambda(1-\lambda\text{Ad}_g^{-1})^{-1}g^{-1}\partial_- g\ , \tag{1.1.54}$$

so plugging the above expression back into the action we arrive at

$$S_{k,\lambda} = S_{\text{WZW}} + \frac{k\lambda}{2\pi}\int_\Sigma d^2\sigma\,\text{Tr}(\partial_+ g g^{-1}(1-\lambda\text{Ad}_g^{-1})^{-1}g^{-1}\partial_- g)\ . \tag{1.1.55}$$

The above action is that of the λ-deformed PCM [Sfe14]. We have shown how to arrive at the action of the theory, however there is no a-priori guarantee that after undergoing the outlined prescription the resulting theory is still Lax integrable. This we will show explicitly. Luckily for us we can employ the gauge fields $A_\pm$, as in (1.1.54), and show that the equations of motion of (1.1.55) can be written as

$$\partial_\pm A_\mp = \pm\frac{1}{1+\lambda}[A_+, A_-]\ . \tag{1.1.56}$$

With this, we can define the following Lax connection

$$\mathcal{L}_\pm = \frac{-2}{1+\lambda}\frac{A_\pm}{1\mp z}\ , \tag{1.1.57}$$

one finds that the flatness of $\mathcal{L}$ implies the equations of motion of the λ-model are satisfied. As such, we have found that indeed the λ-model is Lax integrable.

The Yang-Baxter Deformation:

Given a solution to the modified classical Yang-Baxter, i.e. $\mathcal{R} : \mathfrak{g} \to \mathfrak{g}$, satisfying,

$$[\mathcal{R}X, \mathcal{R}Y] - \mathcal{R}[\mathcal{R}X, Y] - \mathcal{R}[X, \mathcal{R}Y] + c^2[X, Y] = 0 \ , \quad \forall X, Y \in \mathfrak{g} \ , \tag{1.1.58}$$

one can define the following model,

$$S_\eta = \int_\Sigma d^2\sigma \, \mathrm{Tr}(g^{-1}\partial_+ g \, \frac{1}{1 - \eta\mathcal{R}_g} \, g^{-1}\partial_- g) \ , \tag{1.1.59}$$

where $\mathcal{R}_g := \mathrm{Ad}_g^{-1}\mathcal{R}\mathrm{Ad}_g$. This is the aptly named Yang-Baxter model [Kli02], and it is indeed integrable with the Lax connection given by

$$\mathcal{L}_\pm = \frac{1}{1 \mp z}\frac{1 - c^2\eta^2}{1 \pm \eta\mathcal{R}_g} j_\pm \ . \tag{1.1.60}$$

By introducing the operator, $(1-\eta\mathcal{R}_g)^{-1}$, we have naively deformed the original $G_L \times G_R$ isometry group to just the right acting symmetry G_R. It was shown in [DMV13], for the case where $c^2 = -1$, that the left-acting symmetry is modified, with the original local charges associated with G_L, now giving rise to non-local conserved charges, forming the algebraic structure of a quantum group. We call such a phenomenon a q-deformed symmetry.

1.1.2 Sigma-Model Dualities

In the landscape of physical theories, some models exhibit a striking feature: their dynamics can be fully captured by a seemingly different, non-equivalent theory. In such cases, objects and quantities in theory X correspond to objects and quantities in theory Y through a non-trivial mapping, referred to as a "duality dictionary." This raises an intriguing question: can computations in theory X, which may be intractable using contemporary methods, be reformulated in theory Y, where they are more accessible, and then translated back using this dictionary? It is to this end that dualities not only reveal deep structural connections between theories, but also provide powerful computational tools, allowing physicists to extract insights that would otherwise remain out of reach. In this subsection we will provide an introduction to dualities in sigma-models by presenting a discourse covering Abelian [Bus87], non-Abelian [OQ93] and

Poisson-Lie [KS95] dualities between sigma-models.

Abelian duality in non-linear sigma-models

Before discussing Abelian duality for a generic sigma-model, it will prove useful to consider the simple example of a scalar field $\Phi : \Sigma \to \mathbb{R}$. If we impose periodicity of our scalar field such that $\Phi \sim \Phi + 2\pi$, we end up with a periodic scalar field, whose target space is now S^1, with a radius of R. Such a field has an action given by,

$$S[\Phi] = \frac{R^2}{4\pi\alpha'} \int_\Sigma d^2\sigma \, \partial_+\Phi \, \partial_-\Phi \ , \tag{1.1.61}$$

where $d^2\sigma = d\tau \wedge d\sigma = 2d\sigma^+ \wedge d\sigma^-$, and $\sigma^\pm = (\tau \pm \sigma)/2$. This theory clearly has a global translational symmetry given by $\Phi \to \Phi + \lambda$. Let us gauge this symmetry so that now $\lambda \to \lambda(\sigma)$. Such a promotion is tantamount to introducing a gauge field B on our worldsheet and replacing $\partial_\pm\Phi \to D_\pm = \partial_\pm\Phi + B_\pm$, where the gauge field transforms as $B_\pm \to B_\pm - \partial_\pm\lambda$. Clearly after gauging we have fundamentally changed our theory, and the equations of motions for Φ now read,

$$2\partial_+\partial_-\Phi + \partial_+ B_- + \partial_- B_+ = 0 \ . \tag{1.1.62}$$

However, if we introduce a Lagrange multiplier $\tilde{\theta}$, that sets B to be gauge trivial on-shell, we can ensure that equations of motion of our theory are unchanged after gauge fixing. We have the following action

$$S[\Phi, B, \tilde{\theta}] = \frac{R^2}{4\pi\alpha'} \int_\Sigma d^2\sigma \, D_+\Phi \, D_-\Phi + \frac{i}{2\pi} \int_\Sigma d^2\sigma \, \tilde{\theta} \, F_{+-}(B) \ , \tag{1.1.63}$$

where $F_{+-} = \partial_+ B_- - \partial_- B_+$. On a topologically trivial choice of worldsheet Σ, integrating out $\tilde{\theta}$ indeed ensures that B is pure gauge globally. However, this is not the case for topologically non-trivial worldsheets. On non-trivial worldsheets the connection can have non-zero holonomies around non-contractible cycles, $\gamma \subset \Sigma$, defined by

$$Q_\gamma = \oint_\gamma B_\alpha d\sigma^\alpha \ . \tag{1.1.64}$$

Put plainly, this means if we take our gauge field, B, around a loop in Σ it can transform in a non-trivial manner. The gauge invariance of (1.1.64) requires our gauge transformations be single valued. As such, one can immediately see that if B is pure gauge along the curve γ, i.e. $B = d\lambda$, the right hand side of (1.1.64) vanishes by Stokes' theorem, and we have a contradiction. So we conclude holonomies, as global quantities, restrict our ability to globally gauge fix a quantity, and so, on such worldsheets, B can only be locally pure gauge. Therefore the actions (1.1.61) and (1.1.63) are not necessarily equivalent for Σ topologically non-trivial.

There is a fix however, the equations of motion for $B_\pm$ return the following;

$$B_\pm = -i\frac{2\alpha'}{R^2}\partial_\pm\tilde{\theta} \,. \tag{1.1.65}$$

So plugging the above into (1.1.64), we find that such holonomies vanish if we have that $\tilde{\theta}$ is periodic around these cycles. It is in this sense that the winding modes of $\tilde{\theta}$ act as Lagrange multipliers for the holonomies. We would like to derive an action for just the Lagrange multiplier field $\tilde{\theta}$. To do this, let us first consistently eliminate Φ from (1.1.63), we can do this in a rather expedient fashion by gauge fixing $\Phi = 0$. This then leaves the action

$$S[A,\tilde{\theta}] = -\frac{R^2}{4\pi\alpha'}\int_\Sigma d^2\sigma\, B_+B_- + \frac{i}{2\pi}\int_\Sigma d^2\sigma\, B_-\partial_+\tilde{\theta} - B_+\partial_-\tilde{\theta} \;, \tag{1.1.66}$$

where we have integrated the Lagrange multiplier term by parts. Finally integrating out B next leaves us with the action

$$S = -\frac{\tilde{R}^2}{4\pi\alpha'}\int_\Sigma d^2\sigma\, \partial_+\tilde{\theta}\,\partial_-\tilde{\theta} \;, \tag{1.1.67}$$

where $\tilde{R} = \alpha'/R$ is the radius of the dual circle. So what have we shown here? We have shown that a theory defined on S^1 with radius R is equivalent at the level of the classical action to another theory with target S^1 but now of radius $\tilde{R} = \alpha'/R$. This is an example of a T-duality transformation between two classical models. Of course, in the full quantum theory many more things must occur for consistency when going between corresponding T-dual models. For instance, one must exchange the winding and momentum modes, and if the theory is coupled to a dilaton, the duality will induce a shift in the dilaton due to the Jacobian of the path integral [RV92; Tho19].

Let us employ the intuition built from the previous example to the more general case. Beginning with the sigma-model

$$S = \int d^2\sigma\, E_{ij}(X)\partial_- X^i\partial_+ X^j \;, \quad E_{ij}(X) = G_{ij}(X) + B_{ij}(X) \;, \tag{1.1.68}$$

consider the case where the target space of our sigma-model admits a splitting of its coordinates such that $\{\tilde{X}^i\} = \{X^0, X^I\}$ where $X^0 \sim X^0 + 2\pi R$. We can once again gauge the translational symmetry $X^0 \to X^0 + \lambda$. To write an action that is symmetric under such local translations, we implement the replacement $\partial_\pm X^0 \to D_\pm X^0 = \partial_\pm X^0 + B_\pm$. Finally, to ensure new action is equivalent to (1.1.68), we again add a Lagrange multiplier imposing the flatness of of the gauge field, giving the action

$$\begin{aligned} S = \int_\Sigma d^2\sigma \Big[& E_{00}(\partial_- X^0 + B_-)(\partial_+ X^0 + B_+) + E_{0I}\,(\partial_- X^0 + B_-)\partial_+ X^I \\ & + E_{I0}\,\partial_- X^I(\partial_+ X^0 + B_+) + E_{IJ}\,\partial_- X^I \partial_+ X^J + \tilde{\theta} F_{+-}(B)\Big] \,. \end{aligned} \tag{1.1.69}$$

Using the gauge symmetry to fix $X^0 = 0$ we arrive at the gauge fixed action

$$\begin{aligned} S = \int_\Sigma d^2\sigma \Big[& E_{00} B_- B_+ + E_{0I}\, B_- \partial_+ X^I + E_{I0}\,\partial_- X^I B_+ \\ & + E_{IJ}\,\partial_- X^I \partial_+ X^J + \tilde{\theta} F_{+-}(B)\Big] \,. \end{aligned} \tag{1.1.70}$$

As before, integrating out the gauge fields $B_\pm$, one arrives at the dual model with action given by

$$\tilde{S} = \int_\Sigma d^2\sigma\, \tilde{E}_{ij}(\tilde{X}) \partial_- \tilde{X}^i \partial_+ \tilde{X}^j \,, \tag{1.1.71}$$

where $\{\tilde{X}^i\} = \{\tilde{\theta}, X^I\}$ and $\tilde{E}_{ij}(\tilde{X}) = \tilde{G}_{ij}(\tilde{X}) + \tilde{B}_{ij}(\tilde{X})$, for $\tilde{G}_{ij}(\tilde{X})$ and $\tilde{B}_{ij}(\tilde{X})$ given by

$$\begin{aligned} \tilde{G}_{00} = \frac{1}{G_{00}} \,, \quad \tilde{G}_{0I} = \frac{B_{0I}}{G_{00}} \,, \quad \tilde{G}_{IJ} = G_{IJ} - \frac{G_{I0} G_{0J} + B_{I0} B_{0J}}{G_{00}} \,, \\ \tilde{B}_{0I} = \frac{G_{0I}}{G_{00}} \,, \quad \tilde{B}_{IJ} = B_{IJ} + \frac{G_{I0} B_{0J} + B_{I0} G_{0J}}{G_{00}} \,. \end{aligned} \tag{1.1.72}$$

The above relationship between the old and new metric is profound; we have that two models with completely distinct target space geometries are classically equivalent! It turns out that, for Abelian dualities, this equivalence holds for the path integral also, and so is an inherently quantum duality [RV92].

Non-Abelian dualities in non-linear sigma-models

Let us write down the recipe one adopts to arrive from one theory to its non-Abelian T-dual theory. In these cases, we will look to undertake the dualisation procedure for non-Abelian isometry groups of the target space manifold, which will present further technicalities one needs to navigate than in the simpler Abelian case.

i. **Identify an isometry of the target space:** Recall that an isometry is a symmetry of the metric. Suppose we have a group action of $\mathcal{G}$ on the target space M of our sigma-model (1.1.68), generated by the vector field, $v_a = v_a^i(X)\partial_i$, acting as

$$X^i \to \tilde{X}^i = X^i + \epsilon^a v_a^i \ , \tag{1.1.73}$$

such that the generalised metric transforms as $E_{ij}(X) \to \tilde{E}_{ij}(\tilde{X})$, where

$$E_{ij}(X) = \frac{\partial \tilde{X}^k}{\partial X^i}\frac{\partial \tilde{X}^l}{\partial X^j}\tilde{E}_{kl}(\tilde{X}) \ . \tag{1.1.74}$$

This gives us the following expression for the metric,

$$E_{ij}(X) = \tilde{E}_{ij}(X) + \epsilon^a \mathcal{L}_{v_a} E_{ij}(X) \ . \tag{1.1.75}$$

One can see if $\mathcal{L}_{v_a}E_{ij} = 0$, then $E_{ij}(X) = \tilde{E}_{ij}(X)$ and the generalised metric is unchanged under the transformations generated by $\mathcal{G}$. We call such a group, $\mathcal{G}$, an isometry group of our target space, and $v_a = v_a^i\partial_i$ is its associated Killing vector. Using the above results and performing the classic Noetherian trick, one finds that the action (1.1.68) transforms as

$$\delta_\epsilon S = \int \epsilon^a \mathcal{L}_{v_a} L + \int d\epsilon^a \wedge *K_a \ , \tag{1.1.76}$$

where the Noether current is given by $*K_a = v_a^i E_{ij}\partial_- X^j d\sigma^- - v_a^j E_{ij}\partial_+ X^i d\sigma^+$. If G is an isometry then one has $\mathcal{L}_{v_a}L = 0$ and K_a satisfies the conservation equation $d * K_a = 0$ on-shell.

ii. **Promote the global isometry group to a local gauge symmetry:** We promote our derivatives $\partial_\pm$ to covariant derivatives $D_\pm$ such that $D_\pm \to \mathrm{Ad}_{g^{-1}}D_\pm$, under the gauged isometry transformations, where $g = \exp(\epsilon^a v_a) \in \mathcal{G}$. We achieve this by introducing a gauge field transforming as $B \to g^{-1}Bg + g^{-1}dg$, so that $D_\pm = \partial_\pm + B_\pm$.

iii. **Impose the flatness of the connection with a Lagrange multiplier:** To ensure we had classical equivalence at the level of the equation of motions between the models, we add a Lagrange multiplier $\tilde{\theta}$ that imposes the constraint that our connection B was flat, i.e. $F(B) = dB + \frac{1}{2}[B, B] = 0$. Or rather, that the gauge field can locally be written as pure gauge. Considerations of the holonomies around non-trivial cycles of the worldsheet will impose conditions on the Lagrange multiplier akin to the periodicity in the Abelian examples. It is important to state here that gauging non-Abelian isometry groups on non-trivial worldsheets is currently a poorly understood procedure. This is due to the fact that the holonomies of the gauge field are a more challenging adversary. The Lagrange multipliers (i.e. the T-dual coordinates) are now necessarily adjoint valued, and so a natural notion of periodicity is not so easily assigned. Even so, non-Abelian holonomies are path ordered exponentials and as such are non-local quantities, meaning the resulting action would likewise have the undesirable property of being non-local.

iv. **Fix the gauge symmetry and integrate out X and B:** As the final result, we desire an action principle for the Lagrange multipliers that will form the coordinates of our T-dual space. At this point, it will prove useful to perform an appropriate gauge fixing of our gauge symmetry. Then putting the non-gauged fixed quantities on-shell and plugging their on-shell value back in the action yields an action for the dual coordinates.

The above recipe is completely general and can be exercised for the case where our isometries are non-Abelian [OQ93]. There are however some technical differences in this case when compared to the Abelian case stemming from the non-commutative nature, which we will highlight below.

Consider with the non-linear sigma-model with vanishing Kalb-Ramond two-form, $B_{ij} = 0$,

$$S[X] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma\, G_{ij}(X)\partial_- X^i \partial_+ X^j \ , \tag{1.1.77}$$

such that now the metric G has a non-Abelian isometry group $\mathcal{G}$ and the worldsheet Σ possesses no non-trivial cycles. We will adopt notation that allows the action of this isometry group on our target space to be as transparent as possible by again splitting the coordinates as $\{X^i\} = \{X^\mu, X^I\}$ where the Greek indices shall denote the isometry directions, acted upon by our isometry group $\mathcal{G}$ as,

$$X^\mu \to g^\mu{}_\nu(X)\, X^\nu \tag{1.1.78}$$

where $g \in \mathcal{G}$. Starting with the action (1.1.77) and performing (i.-iii.) of the outlined procedure for the whole of the Lie group $\mathcal{G}$ will result in the following expression for the path integral:

$$Z = \int \hat{\mathcal{D}}X \int \mathcal{D}\tilde{\theta} \int \frac{\mathcal{D}B_+ \mathcal{D}B_-}{V_{\mathcal{G}}} \exp\left(-iS_{\text{gauged}}[X, B_\pm] - i\int_\Sigma d^2\sigma \,\text{Tr}(\,\tilde{\theta}F(B))\right) , \tag{1.1.79}$$

where

$$S_{\text{gauged}} = G_{ij}D_+X^iD_-X^j \ , \quad D_\pm X^\mu = \partial_\pm X^\mu + B^a_\pm(T_a)^\mu{}_\nu X^\nu \ , \tag{1.1.80}$$

after fixing a basis for the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$, given by $\{T_a\}$ such that $Tr(T_aT_b) = \delta_{ab}$, and

$$\hat{\mathcal{D}}X = DX\sqrt{\det(G)}\,e^{-\Phi} \tag{1.1.81}$$

is the covariant path integral measure with dilaton coupling. We attain the dual theory by proceeding to integrate out our gauge fields, starting with B_-. To do so, we note that B_- appears linearly in our action. As such, after elementary manipulations, such as integrating by parts, we will be able to make use of the identity;

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{ipx} \tag{1.1.82}$$

adapted for functional quantities, allowing us to write the path integral (1.1.79) as,

$$Z = \int \frac{\hat{\mathcal{D}}X}{V_{\mathcal{G}}}\mathcal{D}\tilde{\theta}\int \mathcal{D}B_+\delta(FB_+ + h)\exp\left(-iS[X] - \frac{i}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\text{Tr}(\bar{h}B_+)\right) , \tag{1.1.83}$$

where h, $\bar{h}$ and f are defined as

$$\begin{aligned} h_a &= -\partial_+\tilde{\theta}_a + (G_{i\mu}\partial_+X^i + G_{\rho\mu}\partial_+X^\rho)(T_a)^\mu{}_\nu X^\nu \ , \\ \bar{h}_a &= \partial_-\tilde{\theta}_a + (G_{i\mu}\partial_-X^i + G_{\rho\mu}\partial_-X^\rho)(T_a)^\mu{}_\nu X^\nu \ , \\ F_{ab} &= -f_{ab}{}^c\tilde{\theta}_c + X^\mu(T_b)^\nu{}_\mu G_{\nu\rho}(T_a)^\rho{}_\sigma X^\sigma \ . \end{aligned} \tag{1.1.84}$$

Performing the functional integral over B_+ one attains,

$$Z = \int \frac{\hat{\mathcal{D}}X}{V_{\mathcal{G}}}\mathcal{D}\tilde{\theta}\,\exp\bigl(-iS'[X,\tilde{\theta}]\bigr)\det\bigl(F^{-1}\bigr) \ , \tag{1.1.85}$$

with

$$S'[X,\tilde{\theta}] = S[X] - \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \mathrm{Tr}(\bar{h}F^{-1}h) \ . \tag{1.1.86}$$

We are now in the position where we would like to perform a choice of gauge fixing. It will prove useful to denote the choice of gauge

$$\hat{X} = lX \ , \quad \hat{\theta} = l\tilde{\theta}l^{-1} \ . \tag{1.1.87}$$

As such, using the Fadeev-Popov trick to fix the gauge in the path integral we have

$$Z = \int \hat{\mathcal{D}}X\mathcal{D}\tilde{\theta}\,\delta(\mathcal{Q})\det\frac{\delta\mathcal{Q}}{\delta\omega}\exp\bigl(-iS'[X,\tilde{\theta}]\bigr)\det\bigl(F^{-1}\bigr) \ , \tag{1.1.88}$$

where $\mathcal{Q}$ is the gauge fixing function and ω the parameters of the isometry group. Denoting the new coordinates $\hat{X}$ and $\hat{\theta}$ collectively by Y we can write the path integral as

$$Z = \int \hat{\mathcal{D}}Y \exp(-iS'[Y])\det\bigl(F(Y)^{-1}\bigr) \ , \tag{1.1.89}$$

where the Fadeev–Popov determinant in the path integral, $\delta Q/\delta\omega$, contributes to the measure such that the correct volume element for the dual manifold is

$$\hat{\mathcal{D}}Y = \mathcal{D}Y\sqrt{G'}e^{-\Phi'} \ . \tag{1.1.90}$$

Finally one still needs to compute the $\det(F^{-1})$ that appeared due to the delta function for B_+. This contributes a local term to the action given by

$$\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\ \alpha' R^{(2)}\,(\Delta\Phi) \ , \tag{1.1.91}$$

which corresponds to the change of the dilaton due to the duality transformation given by

$$\Phi' = \Phi - \log\det F \ . \tag{1.1.92}$$

In general, we cannot easily write the gauge fixed dual action out explicitly, with the new metric and antisymmetric tensor fields failing to be in a closed form, as for the Abelian case above. For special examples where the duality transformation may be performed explicitly, we refer the reader to the original article [OQ93].

Poisson-Lie via promoting Isometries

Having outlined some of the features of the dualities for the case of Abelian and non-Abelian dual models in a systematic fashion, we will now turn our attention to another type of duality.

Consider momentarily that the vector field, v, instead of generating isometries, is a group action such that the associated current obeys the following Maurer-Cartan equation,

$$d * K_c = \frac{1}{2}\tilde{f}_c^{ab} * K_a \wedge *K_b \ , \tag{1.1.93}$$

where we have introduced the structure constants $\tilde{f}_c^{ab}$, which will define for us the Lie algebra $\tilde{\mathfrak{g}}$. We call the symmetry associated to such an action of the group a Poisson-Lie symmetry. After inserting (1.1.93) into (1.1.76) one concludes that the variation vanishes if

$$\mathcal{L}_{v_c} L = \frac{1}{2}\tilde{f}_c^{ab} * K_a \wedge *K_b \ , \tag{1.1.94}$$

or equivalently

$$\mathcal{L}_{v_c} E_{ij} = \frac{1}{2}\tilde{f}_c^{ab} E_{il} E_{kj}\, v_a^k v_b^l \ . \tag{1.1.95}$$

A target space that satisfies (1.1.95) is said to admit generalised isometries. It turns out that the mathematical underpinnings of this duality are familiar. Lie derivatives satisfy

$$[\mathcal{L}_{v_a}, \mathcal{L}_{v_b}] = \mathcal{L}_{[v_a,v_b]} = f_{ab}{}^c \mathcal{L}_{v_c} \ ,$$

where we have used the fact the structure constants are constant. As such, acting with the Lie derivative on (1.1.95) we find the consistency condition

$$\tilde{f}_a^{ed} f_{bd}{}^c - \tilde{f}_b^{ed} f_{ad}{}^c + \tilde{f}_b^{cd} f_{ad}{}^e - \tilde{f}_a^{cd} f_{bd}{}^e - \tilde{f}_d^{ec} f_{ab}{}^d = 0 \ . \tag{1.1.96}$$

It turns out that the above relation is exactly the Jacobi identity for a Drinfeld double. Let us start with a $2n$-dimensional Lie group D. We equip the corresponding Lie algebra $\mathfrak{d}$, with the ad-invariant bilinear form $\langle \bullet , \bullet \rangle_{\mathfrak{d}} : \mathfrak{d} \times \mathfrak{d} \to \mathbb{R}$, of ultrahyperbolic signature $(+, \cdots, +, -, \cdots, -)$, i.e. with an equal number of positive and negative eigenvalues. We call such a Lie group a Drinfeld double if $\mathfrak{d}$ admits two maximally isotropic subalgebras, $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ such that $\mathfrak{d} = \mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$. Fixing a basis $\{T_a\}$ for $\mathfrak{g}$ and $\{\tilde{T}^a\}$

for $\tilde{\mathfrak{g}}$, we choose the inner product to be such that

$$\langle T_a, T_b \rangle_{\mathfrak{d}} = \langle \tilde{T}^a, \tilde{T}^b \rangle_{\mathfrak{d}} = 0 \ , \quad \langle T_a, \tilde{T}^b \rangle_{\mathfrak{d}} = \delta_a^b \ . \tag{1.1.97}$$

Taking $\mathbb{T}_A = (T_a, \tilde{T}^b)$ and

$$[\mathbb{T}_A, \mathbb{T}_B] = \mathbb{F}_{AB}{}^C \mathbb{T}_C \ , \tag{1.1.98}$$

we can determine the structure constants by using the ad-invariance property of our bilinear form. The result is the following

$$\left[T_a, \tilde{T}^b\right] = f_{ca}{}^b \tilde{T}^c + \tilde{f}_a^{bc} T_c \ . \tag{1.1.99}$$

The former along with

$$[T_a, T_b] = f_{ab}{}^c T_c \quad \text{and} \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}_c^{ab} \tilde{T}^c \tag{1.1.100}$$

determine the structure constants of the Drinfeld double, $\mathbb{F}_{AB}{}^C$. One can then check that the resulting Jacobi identity for the mixed indices of $\mathbb{F}_{AB}{}^C$ is indeed given by (1.1.96).

An aside on Poisson Manifolds

Before introducing action principles for sigma-models manifesting Poisson-Lie symmetry, it will be instructive to understand the etymology of the name. Firstly, a Poisson Manifold is a manifold M together with a Poisson structure. A Poisson structure is a mapping

$$\{\bullet, \bullet\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \ , \tag{1.1.101}$$

such that it endows the space of smooth functions of M with a Lie algebraic structure,

$$\{f, g\} = -\{g, f\} \ , \tag{1.1.102}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \ , \tag{1.1.103}$$

and acts as a derivation on the space of smooth functions, i.e. for all $f, g, h \in C^\infty(M)$,

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \ . \tag{1.1.104}$$

Vector fields are defined as being the collection of derivations of smooth functions. As such, given $f \in C^\infty(M)$ we have that this uniquely defines a vector field X_f such that

$$\{f, g\} = X_f g = -X_g f = dg(X_f) = -df(X_g) \, . \tag{1.1.105}$$

We call X_f the Hamiltonian vector field of f. Given a choice of coordinates $\{x^i\}$ on M, we can write this as

$$\{f, g\} = X_f^j \partial_j g = \Pi^{ij} \partial_i f \partial_j g = \Pi(df, dg) \, , \tag{1.1.106}$$

where Π is called the Poisson bivector of the Poisson manifold $(M, \{\bullet, \bullet\})$. The bivector Π is a bilinear mapping $\Pi : \Omega^2(M) \to C^\infty(M)$ and so can be canonically identified with $\Pi \in \bigwedge^2 TM$. Any smooth mapping between two Poisson manifolds, $\phi : M \to N$, preserving the Poisson structure is called a Poisson mapping. That is, $\phi : M \to N$ is Poisson if for any $f, g \in C^\infty(N)$ we have

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi \, . \tag{1.1.107}$$

Given two Poisson manifolds M and N, we would like a notion of being able to combine the pair to form a third Poisson manifold on the underlying topological space $M \times N$. We can form a Poisson structure on $M \times N$ as follows; the projection $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are Poisson mappings. Explicitly, this means that pulling back the functions from M or N and taking their Poisson bracket on $M \times N$ is tantamount to pulling back their Poisson bracket on M or N to $M \times N$:

$$\{f' \circ \pi_M, g' \circ \pi_M\}_{M \times N} = \{f', g'\}_M \text{ , and } \quad \{f \circ \pi_N, g \circ \pi_N\}_{M \times N} = \{f, g\}_N \, , \tag{1.1.108}$$

for all $f', g' \in C^\infty(M)$ and $f, g \in C^\infty(N)$. However, naturally we will also need the Poisson bracket between functions defined on M and N. We will choose the Poisson structure on $M \times N$ such that

$$\{f' \circ \pi_M, g \circ \pi_N\}_{M \times N} = 0 \, , \tag{1.1.109}$$

for $f' \in C^\infty(M)$ and $g \in C^\infty(N)$.

A Poisson-Lie group is a Lie group together with a Poisson structure such that the multiplication mapping $m : G \times G \to G$, is a Poisson map, where $G \times G$ is equipped with the product Poisson structure. In this instance, we describe the Poisson structure on G as multiplicative (or grouped). We call a multivector field K on a Lie group G multiplicative if it satisfies the following,

$$K(gh) = l_*(g)K(h) + r_*(h)K(g) \ , \tag{1.1.110}$$

where l_* denotes the pushforward by the left action and r_* the pushforward by the right action. A Poisson structure is multiplicative if and only if the Poisson bivector is multiplicative. It proves useful to introduce the mappings Π_r and Π_l defined by

$$\Pi_l(g) := l_*(g^{-1})\Pi(g) \ , \ \text{and} \quad \Pi_r(g) := r_*(g^{-1})\Pi(g) \ . \tag{1.1.111}$$

Then the condition of the Poisson structure being multiplicative becomes,

$$\Pi_l(gh) = \Pi_l(h) + \mathrm{Ad}_h^{-1}\Pi_l(g) \ , \ \text{or} \quad \Pi_r(gh) = \Pi_r(g) + \mathrm{Ad}_g\Pi_r(h) \tag{1.1.112}$$

for all $g, h \in G$.

Poisson-Lie groups are particularly flavourful in the sense that their dual algebra $\mathfrak{g}^*$ is naturally 'compatible' with $\mathfrak{g}$, satisfying (1.1.96). A Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ satisfying the doubled Jacobi identity (1.1.96) is called a Lie bialgebra. Lie bialgebras are actually familiar to us! They are an example of a Drinfeld double where $\tilde{\mathfrak{g}} = \mathfrak{g}^*$.

Poisson-Lie Sigma-Models:

Working in the Drinfeld double, we introduce the following notation for the adjoint action of the group on its corresponding algebra

$$\begin{aligned} g^{-1}T_a g &= a_a{}^b(g)\, T_b \ , \quad g^{-1}\tilde{T}^a g = b^{ab}(g)\, T_b + (a^{-1})_b{}^a(g)\, \tilde{T}^b \ , \\ \tilde{g}^{-1}\tilde{T}^a \tilde{g} &= \tilde{a}^a_b(\tilde{g})\, \tilde{T}^b \ , \quad \tilde{g}^{-1}T_a\tilde{g} = \tilde{b}_{ab}(\tilde{g})\, \tilde{T}^b + (\tilde{a}^{-1})^b{}_a(\tilde{g})\, T_b \ . \end{aligned} \tag{1.1.113}$$

Clearly, as $\mathrm{Ad}_g \circ \mathrm{Ad}_{g^{-1}} = \mathrm{Id}$ we have that $a(g^{-1}) = a^{-1}(g)$. Furthermore, using the Ad-invariance of the inner product one finds $b^{ab}(g) = b^{ba}(g^{-1})$, with analogous relations for $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$. We will want to form the combinations

$$\Pi^{ab}(g) = b^{ac}a_c{}^b \ , \quad \tilde{\Pi}_{ab}(\tilde{g}) = \tilde{b}_{ca}\tilde{a}^c_b \ . \tag{1.1.114}$$

One can show $\Pi^{ab}(g) = -\Pi^{ba}(g)$ and $\tilde{\Pi}^{ab}(\tilde{g}) = -\tilde{\Pi}^{ba}(\tilde{g})$. It turns out that Π and $\tilde{\Pi}$ form Poisson bivectors on G and $\tilde{G}$ respectively. To demonstrate this, we must establish that the bivector is inherently multiplicative in the aforementioned sense (1.1.112). Using the relations (1.1.113), and concluding

$$a_a{}^b(gh) = a_a{}^c(g)\, a_c{}^b(h) \ , \text{ and } \quad b^{ab}(gh) = b^{ac}(g)\, a_c{}^b(h) + (a^{-1})_c{}^a(g)\, b^{bc}(h) \ ,$$

one can show explicitly that, indeed, Π defines a multiplicative bivector.

With this, given a Drinfeld double, we can define a pair of sigma-models whose background admits generalised isometries, (1.1.95). Let E_0 be an invertible matrix with constant entries. The two dual actions are given by

$$S_G(g) = \int d^2\sigma\, (g^{-1}\partial_- g)^a F_{ab} (g^{-1}\partial_+ g)^b \ , \quad F = (E_0 + \Pi(g))^{-1} \ , \tag{1.1.115a}$$

$$S_{\tilde{G}}(\tilde{g}) = \int d^2\sigma\, (\tilde{g}^{-1}\partial_- \tilde{g})^a \tilde{F}_{ab} (\tilde{g}^{-1}\partial_+ \tilde{g})^b \ , \quad \tilde{F} = (E_0^{-1} + \tilde{\Pi}(\tilde{g}))^{-1} \ . \tag{1.1.115b}$$

Using the definition of the left invariant frames $(g^{-1}\partial_\pm g)^a = L^a_i(X)\partial_\pm X^i$, we find the relation to the generic form of a sigma-model by identifying that the generalised metric is given by $E_{ij}(X) = L^a_i(X)\, F_{ab}\, L^b_j(X)$. The two actions are classically equivalent in the sense they are related by a canonical transformation.

A duality manifest formulation on the Drinfeld double:

In the previous section we saw that Poisson-Lie duality makes explicit use of the Poisson structure of the Drinfeld double, with the Poisson bivectors of the two maximal isotropic subgroups featuring in the two dual actions. As such, one may thus find it unsurprising that a duality manifest formulation of Poisson-Lie duality is best expressed as a Hamiltonian system on the (loop group of the) Drinfeld double, making explicit use of the associated symplectic form. For this discussion, let us assume that the worldsheet is periodic in the spatial direction such that the worldsheet is given by the cylinder, $\Sigma = \mathbb{R}_\tau \times S^1$. With this, the phase space of our theory at a fixed time τ consists of mappings from $l(\sigma) : S^1 \to D$, we call such a space the loop space of D, denoted by LD.

To describe the Hamiltonian dynamics of the associated system we will need to define a symplectic form on LD. If we have a particle propagating on the manifold X, the phase space is given by the cotangent bundle T^*X, with coordinates (X, P) and the symplectic structure given by the two-form of $\omega = \delta X \wedge \delta P$. Indeed, given any $2n$-dimensional manifold admitting a symplectic structure one has such a partition of the local coordinates is always possible by Darboux's theorem. The Drinfeld double D is by definition a $2n$-dimensional manifold and so one ponders if we can allow the two isotropic subspaces to define such a splitting.

We can achieve this in a rather agnostic fashion, by defining a linear operator $\mathcal{E}$: $\mathfrak{d} \to \mathfrak{d}$ such that $\mathcal{E}^2 = \text{Id}$, $\mathcal{E}$ is self adjoint with respect to the bilinear form $\langle \bullet , \bullet \rangle_{\mathfrak{d}}$, i.e. $\langle \mathcal{E}\bullet , \bullet \rangle_{\mathfrak{d}} = \langle \bullet , \mathcal{E}\bullet \rangle_{\mathfrak{d}}$, and the bracket $\langle \bullet , \mathcal{E}\bullet \rangle_{\mathfrak{d}}$ is positive definite [Kli02; Kli15; KS96]. The operator $\mathcal{E}$, by squaring to the identity and defining a positive definite inner product, must have eigenvalues of ± 1, defining two complementary eigenspaces. The self-adjoint condition then ensures the two subspaces of $\mathfrak{d}$ are orthogonal with respect to $\langle \bullet , \bullet \rangle_{\mathfrak{d}}$. Due to the hyperbolic signature of $\langle \bullet , \bullet \rangle_{\mathfrak{d}}$ it turns out these eigenspaces are both n-dimensional, as such we have achieved the splitting of our double into the positive and negative eigenspaces of $\mathcal{E}$, which we denote by $\mathfrak{e}_+$ and $\mathfrak{e}_-$ respectively.

With the operator $\mathcal{E}$ introduced and defining a choice of splitting of the Drinfeld double, one must now consider the dynamics. Ultimately, given a choice of splitting, we will want the model on D to reduce to two dual theories on G and $\tilde{G}$ in such a fashion as to depend on our choice of operator $\mathcal{E}$. The group-valued fields $l(\sigma) \in LD$ define a current $j(\sigma) = l^{-1}\partial_\sigma l$, so that

$$j^A(\sigma) := \langle j(\sigma), \mathbb{T}^A \rangle_{\mathfrak{d}} \; . \tag{1.1.116}$$

We can use the $\mathfrak{d}$-valued functions $j^A(\sigma)$ to coordinatise the loop group LD. We will now make an imposition on the currents, namely let $j^A(\sigma)$ satisfy the symplectic current algebra, i.e.

$$\{j^A(\sigma_{)}, j^B(\sigma')\} = \mathbb{F}^{AB}{}_C \, j^C(\sigma)\, \delta(\sigma - \sigma') + \eta^{AB}\, \partial_\sigma \delta(\sigma - \sigma') \, , \tag{1.1.117}$$

where

$$\eta^{AB} = \langle \mathbb{T}^A, \mathbb{T}^B \rangle_{\mathfrak{d}} \; . \tag{1.1.118}$$

The bilinear product η^{AB} is non-degenerate and hence we can invert the Poisson bivector to define a symplectic structure. Doing so, one indeed finds that the symplectic form Ω is given by

$$\Omega = -\frac{1}{2} \oint_{S^1} \langle l^{-1}\delta l, \partial_\sigma (l^{-1}\delta l) \rangle_{\mathfrak{d}} \; , \tag{1.1.119}$$

where

$$\delta j = \partial_\sigma(l^{-1}\delta l) + [\, j \,,\, l^{-1}\delta l\,] \;. \tag{1.1.120}$$

One should note the similarities between this expression for the symplectic form Ω and expression for the symplectic form on the space of solutions to the WZW model [Gaw91]. This is to be expected since we imposed that our current j satisfies the symplectic current algebra and so the calculation to invert the Poisson bracket follows in an identical fashion. The dynamical system of interest will be defined by the quadratic Hamiltonian

$$H_{\mathcal{E}}[j] = \frac{1}{2}\oint_{S^1} d\sigma \,\langle\, j(\sigma) \,,\, \mathcal{E}\, j(\sigma)\,\rangle_{\mathfrak{d}} \;. \tag{1.1.121}$$

Using the identity

$$\partial_\tau f = \{H, f\} \;, \tag{1.1.122}$$

one derives the following equation of motion of zero-curvature form for j,

$$\partial_\tau j = \partial_\sigma(\mathcal{E}j) + [\mathcal{E}j, j] \;. \tag{1.1.123}$$

Now we have defined a Hamiltonian system putatively describing the desired dynamics on the loop group of the D; the question may be asked of how we attain our dual theories. The procedure is somewhat simplified in the case where our Drinfeld double is perfect, or rather every element $l \in D$ is such that $l = \tilde{g}g$, for $g \in G$ and $\tilde{g} \in \tilde{G}$. In this case, we see that the Hamiltonian decomposes as

$$H_{\mathcal{E}}[g,\tilde{g}] = \frac{1}{2}\oint_{S^1} \langle g^{-1}\partial_\sigma g + g^{-1}(\tilde{g}^{-1}\partial_\sigma \tilde{g})g, \mathcal{E}\left(g^{-1}\partial_\sigma g + g^{-1}(\tilde{g}^{-1}\partial_\sigma\tilde{g})g\right)\rangle_{\mathfrak{d}} \;. \tag{1.1.124}$$

To arrive at the Poisson-Lie dual actions (1.1.115a) and (1.1.115b), we will need to pass from the Hamiltonian formulation of the above theory to its Lagrangian formulation.

From the Hamiltonian to an Action:

To pass from the Hamiltonian formulation of a field theory to a Lagrangian formulation one uses the familiar Legendre transformation,

$$S = \int_\gamma \Theta - H dt \;, \tag{1.1.125}$$

where Θ is the symplectic potential of the symplectic form Ω, a one-form on our phase space, satisfying $\delta\Theta = \Omega$. We denote by γ a curve in phase space $\gamma(t) : \mathbb{R} \to$ phase space (in our case LD), describing how our field configurations evolve through time. Notice that by integrating over the curve we are pulling back our symplectic potential by γ, so as to attain a function of time to integrate over. For our choice of symplectic form (1.1.119) and Hamiltonian (1.1.121) one attains the following first-order action,

$$\begin{aligned} S_{\mathcal{E}} = \frac{1}{2}\int_{\Sigma} d^2\sigma \, \langle l^{-1}\partial_\sigma l, l^{-1}\partial_\tau l\rangle_{\mathfrak{d}} + \frac{1}{6}\int_{\mathcal{N}} d^2\sigma dt \, \epsilon^{ijk}\langle l^{-1}\partial_i l, [l^{-1}\partial_j l, l^{-1}\partial_k l]\rangle_{\mathfrak{d}} \\ - \frac{1}{2}\int_{\Sigma} d^2\sigma \, \langle l^{-1}\partial_\sigma l, \mathcal{E} l^{-1}\partial_\sigma l\rangle_{\mathfrak{d}} \; . \end{aligned} \tag{1.1.126}$$

The above is the action principle for the so-called $\mathcal{E}$-model describing Poisson-Lie dual models [KS96]. To describe the passage to a sigma-model on the isotropic subgroup G, consider transforming our field l by $l \mapsto \tilde{g}l$, where $\tilde{g} \in \tilde{G}$. The action correspondingly transforms as

$$\begin{aligned} S_{\mathcal{E}} = \frac{1}{2}\int_{\Sigma} d^2\sigma \, \langle l^{-1}\partial_\tau l, \mathcal{E} l^{-1}\partial_\tau l\rangle_{\mathfrak{d}} - \langle l^{-1}\partial_\tau l, l^{-1}\partial_\sigma l\rangle_{\mathfrak{d}} \\ - \int_{\Sigma} \langle \mathrm{Ad}_l^{-1}(\tilde{g}^{-1}\partial_\sigma \tilde{g}) + l^{-1}\partial_\sigma l - \mathcal{E} l^{-1}\partial_\tau l, \mathcal{E}\left((\mathrm{Ad}_l^{-1}\tilde{g}^{-1}\partial_\sigma\tilde{g}) + l^{-1}\partial_\sigma l - \mathcal{E} l^{-1}\partial_\tau l\right)\rangle_{\mathfrak{d}} \\ + \frac{1}{6}\int_{\mathcal{N}} d^2\sigma dt \, \epsilon^{ijk}\langle l^{-1}\partial_i l, [l^{-1}\partial_j l, l^{-1}\partial_k l]\rangle_{\mathfrak{d}} \; , \end{aligned} \tag{1.1.127}$$

where we have used the fact that $\tilde{G}$ is isotropic and the Polyakov-Wiegmann identity,

$$\mathrm{WZ}[\tilde{g}l] = \mathrm{WZ}[\tilde{g}] + \mathrm{WZ}[l] - \int_{\Sigma} \mathrm{Tr}(\tilde{g}^{-1}d\tilde{g} \wedge dl\, l^{-1}) \; . \tag{1.1.128}$$

As such, one concludes that the action depends only on $\tilde{g}$ through the combination $\tilde{\Lambda} := \tilde{g}^{-1}\partial_x\tilde{g} \in \tilde{\mathfrak{g}}$, with the action being quadratic in $\tilde{\Lambda}$. This is an important observation, as it now opens up the possibility of being able to classically integrate out the $\tilde{g}$ field in much the same manner as one would integrate out an auxiliary-type field in a Lagrangian. This comes with a subtlety however, the equations of motion for $\tilde{\Lambda}$ possess a redundancy unless we have that

$$\mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}} \cap \mathcal{E}\mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}} = \{0\} \; . \tag{1.1.129}$$

Constraining the moduli of $\mathcal{E}$ to be such that the above relation is satisfied, we will now endeavour to integrate out $\tilde{\Lambda}$. A particularly convenient tool for doing such calculations is to introduce the projector $\mathcal{P}$ [Hoa22], defined such that $\mathrm{im}\mathcal{P} = \mathcal{E}\mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}}$ and $\ker\mathcal{P} =$

$\mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}}$. Similarly, the quantity $1-\mathcal{P}$ is also a projector with $\mathrm{im}(1-\mathcal{P}) = \mathrm{ker}\mathcal{P} = \mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}}$ and $\mathrm{ker}(1-\mathcal{P}) = \mathrm{im}\mathcal{P} = \mathcal{E}\mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}}$. Given the isotropic property of the Lagrangian subalgebra $\tilde{\mathfrak{g}}$ and the defining properties of the self-adjoint operator $\mathcal{E}$ the introduction of such projectors proves useful due to the following properties holding

$$\langle \mathcal{P}\,\mathfrak{d}, \mathcal{P}\,\mathfrak{d}\rangle_{\mathfrak{d}} = \langle (1-\mathcal{P})\,\mathfrak{d}, (1-\mathcal{P})\,\mathfrak{d}\rangle_{\mathfrak{d}} = 0\ . \tag{1.1.130}$$

Introducing the projectors and running through the computation for integrating out $\tilde{\Lambda}$ we arrive at the following relativistic second-order action,

$$\begin{aligned} S_{\mathcal{E}_{\tilde{G}}} = \frac{1}{2}\int_\Sigma d^2\sigma\ \big(\langle l^{-1}\partial_+ l, \mathcal{E}\mathcal{P}(\mathcal{E}+1)l^{-1}\partial_- l\rangle_{\mathfrak{d}} - \langle l^{-1}\partial_- l, \mathcal{E}\mathcal{P}(\mathcal{E}-1)l^{-1}\partial_+ l\rangle_{\mathfrak{d}}\big) \\ + \frac{1}{6}\int_{\mathcal{N}} d^2\sigma dt\ \epsilon^{ijk}\langle l^{-1}\partial_i l, [l^{-1}\partial_j l, l^{-1}\partial_k l]\rangle_{\mathfrak{d}}\ . \end{aligned} \tag{1.1.131}$$

The operators $\mathcal{E}\mathcal{P}(\mathcal{E}\pm 1)$ are projectors with $\mathrm{im}\,\mathcal{E}\mathcal{P}(\mathcal{E}\pm1) = \mathrm{Ad}_l^{-1}\tilde{\mathfrak{g}}$ and $\mathrm{ker}\,\mathcal{E}\mathcal{P}(\mathcal{E}\pm1) = \mathfrak{e}_\mp$, where $\mathfrak{e}_\pm$ are the eigenspaces of $\mathcal{E}$ with eigenvalues ±1. At this point, we should clarify what we have done. Taking the first-order action (1.1.127), we shifted our D-valued field l by a $\tilde{G}$-valued field, $\tilde{g}$, immediately finding that we could integrate out the field $\tilde{g}$ from the action. After doing so, we superficially appear to have a theory on $\mathcal{D}$. However, if we parametrised l such that $l = \tilde{g}g$ for $g \in G$ and repeated the above analysis we would have attained a theory on G. With this one can conclude that (1.1.131) must possess a $\tilde{G}$-gauge symmetry, $l \to \tilde{g}l$ and therefore defines a theory on the left-quotient $\tilde{G}/D \cong G$. With this, we can perform a gauge fixing such that we attain the action (1.1.115a), where by comparison we have,

$$\mathcal{E}(\mathcal{P}-\mathcal{P}^t)\mathcal{E} + \mathcal{E}\mathcal{P} + \mathcal{P}^t\mathcal{E} = F\ . \tag{1.1.132}$$

Likewise we could have equally parametrised l as $l = g\tilde{g}$ and repeated the above analysis, defining new projectors and thus attaining a theory on $G/D \cong \tilde{G}$ given by (1.1.115b).

It turns out that many of the integrable sigma-models familiar to us do indeed fall under the $\mathcal{E}$-model framework, including the λ- and Yang-Baxter deformations introduced in §1.1.1. In fact, it turns out that using the $\mathcal{E}$-model formulation of such theories allows one to relate the λ-model on a simple compact Lie group G to the Poisson-Lie T-dual of the Yang-Baxter model by analytic continuation [Kli15].

Chapter 2

Gauge Theories and Integrable Systems

2.1 3d Chern-Simons Theory

The space of all gauge invariant local operators of dimension four has dimension two,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4\pi g^2}\text{Tr}(F \wedge *F) \ , \tag{2.1.1a}$$

$$\mathcal{L}_{\theta} = \frac{\theta}{32\pi^2}\text{Tr}(F \wedge F) \ . \tag{2.1.1b}$$

The first term is the Lagrangian density for Yang-Mills theory [YM54], the much celebrated gauge theory underpinning the dynamics of the gauge boson in the standard model. The second term may be mysterious to physicists, but familiar to those versed in characteristic classes. The so-called 'theta-term' is exactly the second Chern Character of the underlying principal bundle, describing topological properties of the fibration [Che46]. Saving any further dialogue on the mathematical origins of the term for now, we will now study the theta-term through the physicist's lens, and indeed, it will turn out to source some incredibly beautiful physics. The first thing to note about $\mathcal{L}_\theta$ is that, due to the Bianchi identity, it is closed. As such, locally one can write it as an exact form

$$\text{Tr}(F \wedge F) = d\,\text{CS}(A) \ , \tag{2.1.2}$$

with $\mathrm{CS}(A) \in \Omega^3(M)$, being the Chern-Simons 3-form, first introduced in the seminal work [CS74] and given by

$$\mathrm{CS}(A) = \mathrm{Tr}(A \wedge dA + \frac{1}{3} A \wedge [A, A]) \ . \tag{2.1.3}$$

While (2.1.1b) is clearly gauge invariant, what is less clear is the gauge invariance of the Chern-Simons 3-form defined above. If gauge invariant, one could introduce a gauge invariant action principle in three dimensions with the Lagrangian given by (2.1.3). This would certainly define an interesting theory, worthy of study in its own right. However, the Chern-Simons 3-form fails to be gauge invariant, if we perform the transformation

$$A \to A^g = g^{-1} A g + g^{-1} dg \ , \tag{2.1.4}$$

one finds that $\mathrm{CS}(A) \to \mathrm{CS}(A^g)$ where

$$\begin{aligned} \mathrm{CS}(A^g) = \mathrm{CS}(A) &- d\,\mathrm{Tr}(dg g^{-1} \wedge \ A) \\ &+ \frac{1}{3} \mathrm{Tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg) \ . \end{aligned} \tag{2.1.5}$$

Imagining for a moment the underlying 3-manifold is such that the boundary is empty $\partial \mathcal{M} = \emptyset$, then one finds the obstruction to gauge invariance is in the final term,

$$\mathrm{WZ}[g] = \frac{1}{3} \mathrm{Tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg) \ , \tag{2.1.6}$$

which we recognise as the Wess-Zumino term [WZ71], familiar to us from section 1.1. This term is again a familiar object in mathematics, measuring the Brouwer degree of the mapping $g : \mathcal{M} \to G$ [Bro12]. In particular if G is a connected, simply connected, compact group, we have that

$$\int_{\mathcal{M}} \mathrm{WZ}[g] = 2\pi n \ , \quad n \in \mathbb{Z} \ , \tag{2.1.7}$$

where n is the aforementioned Brouwer degree of $g(x)$. The hope of defining a gauge invariant action is not lost however. We can define the Chern-Simons action by

$$S_{\mathrm{CS}}[A] = k \int_{\mathcal{M}} \mathrm{CS}(A) \ . \tag{2.1.8}$$

The corresponding measure in the path integral given by $\exp(iS_{\text{CS}}[A])$ is indeed invariant if the level $k \in \mathbb{Z}$. As such, Chern-Simons theory is fully gauge invariant at the quantum level [DJT82; Wit89].

Gauge invariance established, let us momentarily concentrate on the dynamics of (2.1.8). Varying the action one arrives at

$$\delta S[A] = \int_{\mathcal{M}} \text{Tr}(F(A) \wedge \delta A) - \frac{1}{2} \int_{\partial\mathcal{M}} \text{Tr}(A \wedge \delta A) \ = 0. \tag{2.1.9}$$

Here the variation separates into a bulk term and a boundary term. For the variation to define a well-posed differential equation for the equations of motion, we must first impose boundary conditions on our gauge field that ensure the boundary term

$$\theta = \int_{\partial\mathcal{M}} \text{Tr}(A \wedge \delta A) \tag{2.1.10}$$

vanishes. For the case that $\partial\mathcal{M} \neq \emptyset$, a sufficient choice of boundary condition is given by Dirichlet boundary conditions,

$$A \mid_{\partial\mathcal{M}} = 0 \ . \tag{2.1.11}$$

However, more generally we can make the boundary variation vanish by choosing a Lagrangian subspace of the phase space symplectic form ω form given by $\omega = \delta\Theta$. Having imposed our boundary conditions, we find that the bulk equations of motion are given by

$$F(A) = dA + A \wedge A = 0 \ . \tag{2.1.12}$$

As such one finds that the on-shell configurations of Chern-Simons theory are given by the moduli of flat connections of the principal G-bundles over $\mathcal{M}$.

The topological nature of 3dCS

3d-Chern-Simons theory is perhaps the most famous example of a topological field theory (TFT), but what do we mean by a TFT? One definition is that the theory is independent of the metric, i.e.

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \ . \tag{2.1.13}$$

A consequence of this is that the classical energy-momentum tensor defined by

$$T^{\mu\nu} = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{2.1.14}$$

vanishes identically. Topological theories, being independent of the metric, are agnostic to local geometrical properties, and instead sensitive only to the global topological properties of spacetime. Looking back at (2.1.8), it is transparent that 3dCS is topological in the aforementioned sense, given the clear absence of the metric in the defining Lagrangian. However, a theory can also topological in a more sophisticated sense. Given a QFT with a collection of operators $\{\mathcal{O}_{i_j}\}$, one can consider a theory such that all correlation functions are independent of the metric, or rather,

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0 \,. \tag{2.1.15}$$

It is in this sense that a QFT can define a topological quantum field theory (TQFT), the metric independence of the Lagrangian surviving at the quantum level. One should note this definition is independent of the Lagrangian and instead only dependent on the correlators, allowing for a Lagrangian free definition of TQFTs.

Naturally, one should consider whether the 3dCS theory falls into this class. Witten showed in [Wit89], that 3dCS is indeed a TQFT, modulo a subtlety, the so-called framing anomaly. We will omit the details of this calculation, but let us explore the implications.

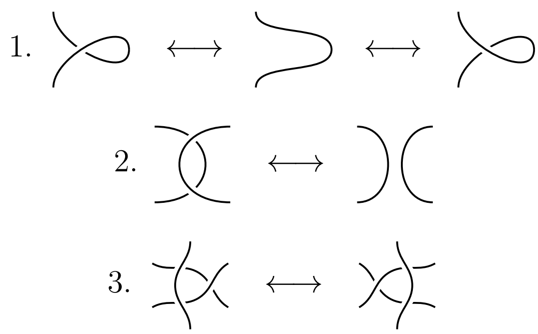
A consequence of a field theory being topological is the lack of local degrees of freedom within the theory. Local operators, which depend on specific points in spacetime, are sensitive to the metric and local geometry. As such, local operators are irrelevant in describing the quantum dynamics of a TQFT. Instead, the natural observables are non-local in nature. Let us consider Wilson line operators within our theory, these are operators defined by

$$W_\rho(\gamma) = \mathrm{Tr}_\rho \mathcal{P} \exp\left(\int_\gamma A \right) \,, \tag{2.1.16}$$

for a given choice of representation, ρ, of the Lie group G and path γ. We use $\mathcal{P}$ here to denote the path-ordered exponential. Witten argued that in 3dCS Wilson line operators and their correlation functions give rise to topological invariants of the underlying 3-manifold. We can combine a collection of closed paths $\gamma_i \subset S^3$ to form a link $L = \cup_i \gamma_i \subset S^3$ in the 3-sphere. If we then consider the partition function, defined by

$$Z(L) = \int \mathcal{D}A \, e^{\frac{ik}{2\pi} \int_{S^3} \mathrm{CS}(A)} \prod_i W_\rho(\gamma_i) \,, \tag{2.1.17}$$

and take the gauge group to be $G = SU(N)$ with the representation of our Wilson lines ρ in the fundamental representation, one finds that $Z(L)$ is the famous Jones polynomial. This fact is certainly exciting; the Jones polynomial of a knot is invariant under the Reidemeister moves, given by



the third of which looks tantalisingly close to the physical form of the Yang-Baxter equation, the relation underpinning 2d integrability! This likeness motivated many researchers to ponder the nature of the exact relationship between 3dCS theory and the field of 2d integrable systems; is the similarity a profound observation or is it a purely superficial facade? It will turn out that there is, in fact, an incredibly deep and intimate relation between Chern-Simons-type theories and 2d Integrability. However, for 3dCS there is a notable absence halting immediate progress, the lack of a spectral parameter for our Wilson lines operators.

2.2 4d Chern-Simons Theory

In the previous section, we highlighted the visual similarities between the Yang-Baxter equation and the third Reidemeister move. The optical parallels are undoubtedly intriguing and the latter being a manifest property of Wilson line operators in 3dCS seems to hint at a close connection between Chern-Simons and Integrable systems. However, the lack of spectral parameter is fatal to the premonition that there exists a direct link between the two, and motivates one to search for new ideas. With this observation in mind we will now introduce the first protagonist in this thesis.

Four-dimensional Chern-Simons Theory (4dCS), introduced by Costello in [Cos14], is an example of a topological-holomorphic theory. Defined on a four manifold $X_4 = \Sigma \times C$, where Σ is a two dimensional worldsheet and C is a Riemann curve. Much like how topological field theories are theories that are only sensitive to the underlying topological properties of the manifold, holomorphic field theories are theories for which the data of the theory is dependent only on the complex structure of the underlying manifold. A topological-holomorphic theory is a theory exhibiting both properties simultaneously, dependent only on topological properties along some directions in the underlying manifold and only on the holomorphic properties along the other directions. It turns out this state of matrimony between the topological and the holomorphic is exactly the recipe we need to describe 2dIFTs systematically.

2.2.1 An action for 4dCS

Our starting point is the following action,

$$S[A] = \frac{1}{2\pi i}\int_{X_4} \omega \wedge \mathrm{CS}(A) \ , \tag{2.2.1}$$

where ω is a meromorphic one form on C

$$\omega = \varphi(z)dz \ , \tag{2.2.2}$$

and $\mathrm{CS}(A)$ is the Chern-Simons three-form. The action has an obvious redundancy given by shifting our gauge field by a one-form with legs along only dz. As such we have established that our action is independent of the z-component of our gauge field A_z. Similarly, the dz-leg of the de-Rham differential d drops out, and so along C only the anti-holomorphic derivative appears in the action. It is in this sense we say the theory is sensitive to the complex structure of C through ω. Varying the action we arrive at

$$\delta S = \frac{1}{2\pi i}\int \omega \wedge \delta\,\mathrm{CS}(A) = \frac{1}{\pi i}\int_{X_4} \omega \wedge \mathrm{Tr}(F\wedge \delta A) - \frac{1}{2\pi i}\int_{X_4} d\omega \wedge \mathrm{Tr}(A \wedge \delta A) \ . \tag{2.2.3}$$

The first term once again provides the bulk equations of motion, given by

$$\omega \wedge F = 0 \ . \tag{2.2.4}$$

The second term is at first glance rather odd. When integrating by parts we were aware of the fact that ω need not necessarily be closed. As such, we have attained a 'boundary like' term which instead of being generated by Stokes' theorem and localising us to the boundary of our four manifold, is instead localising us to the poles of ω. It is in this sense that the poles of ω define 'the boundaries' of our theory. Let us study this fact more concretely; suppose that ω has a collection of poles at $\{z_i\}$ of order m_i and simple zeroes $\{y_i\}$. Choosing $C = \mathbb{CP}^1$ we can write

$$\omega = C\frac{(z-y_1)\cdots(z-y_k)}{(z-z_1)^{m_1}\cdots(z-z_n)^{m_n}}\,dz\ . \tag{2.2.5}$$

The relation between the number of zeroes and number and number of poles (counted with multiplicity) is constrained by the Riemann-Roch theorem,

$$\text{number of zeroes} - \text{number of poles} = 2g-2\ , \tag{2.2.6}$$

where g is the genus of the Riemann curve C. For the rest of this thesis we will fix our Riemann curve C to be the Riemann sphere $\mathbb{CP}^1$. We refer the reader to [CY19; LW24; MYZ25] for considerations of 4dCS theory on higher genus curves.

For ease in computations we will write out the one-form ω in terms of partial fractions

$$\omega = \sum_i \sum_{p=1}^{m_i} \frac{L_{i,p}}{(z-z_i)^p} + \cdots\ , \tag{2.2.7}$$

where the coefficients in the expansion $L_{i,p}$ are called the 'levels' and the additional terms are those that are regular at each of poles z_i. Employing the identity

$$\frac{\partial}{\partial\bar{z}}\left(\frac{1}{z}\right) = 2\pi i\delta^{(2)}(z) \tag{2.2.8}$$

we have that

$$d\omega = 2\pi i\sum_i\sum_{p=1}^{m_i}\frac{(-1)^{p+1}L_{i,p}}{(p-1)!}\,\partial_z^{p-1}\delta^{(2)}(z-z_i)\ d\bar{z}\wedge dz\ . \tag{2.2.9}$$

To evaluate the boundary term in (2.2.3), we will perform the integral over C, to do this we will need to integrate by parts the z derivatives off of the Dirac distribution.

This will allow us to localise the integral to the support of the distribution,

$$\delta S_{\text{boundary}} = \sum_i \sum_{p=1}^{m_i} \frac{L_{i,p}}{p!} \int_\Sigma \partial_z^{p-1} \operatorname{Tr}(A \wedge \delta A) \ . \tag{2.2.10}$$

As such, to ensure our action principle is well defined we need to choose boundary conditions for the gauge field at the poles of ω that ensure the boundary variation (2.2.10) vanishes. Having imposed boundary conditions upon our gauge field, we now endeavour to solve our bulk equations of motion (2.2.4), which in components read

$$\varphi(z)\epsilon^{ij} F_{ij}(A) = 0 \ , \tag{2.2.11a}$$

$$\varphi(z) F_{i\bar{z}}(A) = 0 \ , \tag{2.2.11b}$$

where $i = t, x$. Now here comes the crucial assumption in all that follows in this thesis, we shall restrict ourselves to consider gauge fields A of the form

$$A = \hat{g}^{-1} A' g + \hat{g}^{-1} d\hat{g} \tag{2.2.12}$$

where $A'_{\bar{z}} = 0$. What have we done here? It looks suspiciously like we have just performed a gauge transformation. We have not, instead we have restricted ourselves to the collection of gauge fields that are gauge trivial along their $d\bar{z}$-leg and this triviality is parametrised by the group valued field $\hat{g}$ or rather,

$$A_{\bar{z}} = \hat{g}^{-1} \partial_{\bar{z}} \hat{g} \ . \tag{2.2.13}$$

The attentive reader should perhaps feel uneasy since we have traded a single field for two fields, and one should not be surprised that in doing so we have introduced a further redundancy. Consider the transformation

$$A' \to A'^{\check{\gamma}} \ . \tag{2.2.14}$$

Then so long as we have

$$\check{\gamma}^{-1} \partial_{\bar{z}} \check{\gamma} = 0 \ ,$$

preserving the condition ${A'}_{\bar{z}} = 0$, one can perform the transformation

$$\hat{g} \to \check{\gamma}^{-1} \hat{g} \tag{2.2.15}$$

and the gauge field A is unchanged. We call such a transformation an internal gauge transformation. Internal gauge transformations, in their ability to leave our gauge field A fixed, will naturally be compatible with any choice of boundary conditions we choose.

So, internal gauge transformations are a redundancy in our choice of parameterisation, but what became of the gauge transformations we are used to? How do they fall into this framework? Performing a gauge transformation on (2.2.12) one arrives at

$$A \to A^{\hat{\gamma}} = \hat{\gamma}^{-1}\hat{g}^{-1}A'\hat{g}\hat{\gamma} + \hat{\gamma}^{-1}\hat{g}^{-1}(d\hat{g})\hat{\gamma} + \hat{\gamma}^{-1}d\hat{\gamma} \,, \tag{2.2.16}$$

which can be written as

$$A^{\hat{\gamma}} = (\hat{g}\hat{\gamma})^{-1}A'(\hat{g}\hat{\gamma}) + (\hat{g}\hat{\gamma})^{-1}d(\hat{g}\hat{\gamma}) \,. \tag{2.2.17}$$

So one finds that performing a gauge transformation is tantamount to sending

$$\hat{g} \to \hat{g}\hat{\gamma}$$

and leaving A' fixed. We shall call such a transformation an external gauge transformation. External gauge transformations change the value of A and as such one must ensure they are compatible with the choice of boundary conditions we impose. This has profound consequences that we will discuss in detail later.

Leveraging the above gauge transformations to perform tactful gauge fixings is a common feature in this work, making tough calculations much simpler to perform. We will show this explicitly through examples later in the discussion.

Reality Conditions

In what follows we will want the action (2.2.1) to be real valued. Let us briefly discuss how we impose consistent reality conditions to ensure this. We will require the meromorphic one-form $\omega = \varphi(z)dz$ to be equivariant with respect to conjuagation on $\mathbb{CP}^1$,

$$\overline{\varphi(z)} = \varphi(\bar{z}) \,. \tag{2.2.18}$$

This condition will require that the poles and zeroes of ω are either real or come in pairs along with their complex conjugate.

In this section, we have implicitly assumed that our gauge group is a complex Lie group $G^{\mathbb{C}}$ with corresponding complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. In order to make our action real valued, one will need to choose a real form of the gauge group and corresponding Lie algebra by restricting to the collection of fixed points of an involutive anti-linear automorphism $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, namely $\mathfrak{g} = \{X \in \mathfrak{g}^{\mathbb{C}} \,:\, \tau(X) = X\}$. As such, we require that the components of our gauge field are equivariant functions of z under complex conjugation on $\mathbb{CP}^1$ and the action of τ

$$\tau\left(A_\mu(t,x,z)\right) = A_\mu(t,x,\bar{z}) \ , \tag{2.2.19}$$

where $\mu = t, x, z$. To ensure gauge transformations preserve the above reality condition we will also impose the same equivariance condition on the collection of gauge transformations. To do this we lift the antilinear automorphism to the group $G^{\mathbb{C}}$ and impose

$$\tau\left(\hat{\gamma}(t,x,z)\right) = \hat{\gamma}(t,x,\bar{z}) \ .$$

The combination of (2.2.19) and (2.2.18), along with restricting the gauge transformations as above ensures that (2.2.1) is indeed real.

The appearance of the Lax

In terms of the parameterisation (2.2.12) the bulk equations of motion (2.2.11) read

$$\varphi(z)\left(\partial_t A'_x - \partial_x A'_t + [A'_t, A'_x]\right) = 0 \ , \tag{2.2.20a}$$

$$\varphi(z)\partial_{\bar{z}} A'_i = 0 \ . \tag{2.2.20b}$$

The latter of these equations is telling us that the components of A' are holomorphic away from the zeroes of $\varphi(z)$, however at the zeroes, (2.2.20b) is also satisfied without any conditions on A', this leaves the possibility that the components need not be holomorphic on C but instead meromorphic, with poles at the zeroes of $\varphi(z)$. Away from the zeroes of $\varphi(z)$, (2.2.20a) tells us that A' defines a flat connection on Σ, satisfying the zero curvature equation. As such, if we interpret the holomorphic coordinate of C, z, as a spectral parameter; the bulk equations of motion are telling us that A' is a spectral parameter dependent flat connection on Σ, paralleling exactly the definition of a Lax connection!

This discovery is rather profound. Previously, finding a Lax connection whose flatness yielded the equations of motion for a 2D theory was an arbitrary process, often relying on seemingly clairvoyant choices to determine the final form. On the contrary, 4dCS allows one to systematically find the Lax connection given a choice of ω and boundary conditions for A, all whilst impressively elucidating its geometrical origin in the context of gauge theory.

Solving for the Lax

To find the form of A' one must make some careful considerations, which we outline now. The generic choice of ω given in (2.2.5) possesses simple zeroes at the points $\{y_i\}$, as such we stand to have a kinetic term of the form

$$\mathcal{L}_{\text{kinetic}} = \omega \wedge \text{Tr}(A \wedge dA) \ ,$$

which vanishes at each y_i. This is an issue, currently we would be unable to invert this term at these points and define a sensible propagator. The solution is to allow the gauge field A to have simple poles that cancel with the zeroes of ω. We will choose the following form

$$A' = \sum_i \frac{U'^{(i)}}{z - y_i} + V \ . \tag{2.2.21}$$

This ansatz is not quite the full story, we should check if it is consistent with (2.2.20a). Looking at the second term, one finds that the term proportional to $A' \wedge A'$ will have a second order pole as opposed to the first order pole in the dA' term. We will thus require that each term vanishes on its own. A particularly transparent way to achieve this is to ensure that the $U'^{(i)}$ has legs along a single direction so that $U'^{(i)} \wedge U'^{(i)}$ is zero. In the interest of working with relativistic models later, we choose to allow our zeroes to be split into two: those that have legs along lightcone coordinate σ^+, which we denote by $\{y_{i^+}\}$, and those whose legs point along σ^-, $\{y_{i^-}\}$. As such, we will have

$$A'_+ = \sum_{i^+} \frac{U_+^{(i^+)}}{z - y_{i^+}} + V_+ \ , \quad A'_- = \sum_{i^-} \frac{U_-^{(i^-)}}{z - y_{i^-}} + V_- \ . \tag{2.2.22}$$

To solve for the exact form of our Lax components, $U_+^{(i^+)}$, V_+, $U_-^{(i^-)}$, and V_- in terms of the fundamental fields we simply solve the boundary conditions. We will make this more transparent when we tackle our first example.

An Action for the 2d Theory

Defining $\hat{\mathbf{j}} = -d\hat{g}\hat{g}^{-1}$ and using the identity

$$\mathrm{CS}(X+Y) = \mathrm{CS}(X) + 2\mathrm{Tr}(F(X)Y) - d\mathrm{Tr}(XY) + 2\mathrm{Tr}(XY^2) + \mathrm{CS}(Y)\ , \tag{2.2.23}$$

we have

$$\mathrm{CS}(A) = \frac{1}{3}\mathrm{Tr}(\hat{\mathbf{j}}^3) + d\,\mathrm{Tr}(\mathbf{j}\wedge A') + \mathrm{CS}(A')\ . \tag{2.2.24}$$

We can thus write (2.2.1) as

$$S[A',\hat{g}\,] = \frac{1}{2\pi i}\int_{X_4}\omega\wedge\left(\mathrm{Tr}(A'\wedge dA') + d\,\mathrm{Tr}(\hat{\mathbf{j}}\wedge A') + \frac{1}{3}\mathrm{Tr}(\hat{\mathbf{j}}^3)\right)\ , \tag{2.2.25}$$

where we have used the fact that $A'_{\bar{z}} = 0$. One can go a few steps further; first consider an extension of X_4 to $X_5 = [0,1]_t \times X_4$, with the interval coordinatised by t. We will accordingly extend our group valued field $\hat{g} : X_4 \to G$ to the field $\tilde{g} : X_5 \to G$, by a smooth homotopy such that at $t = 0$, $\tilde{g} = \mathrm{Id}$ and at $t = 1$ $\tilde{g} = \hat{g}$. Why have we done this? Well consider now the final term in (2.2.25), one has by Stokes' theorem

$$\int_{X_4}\omega\wedge\mathrm{Tr}(\hat{\mathbf{j}}^3) = \int_{X_5} d\left(\omega\wedge\mathrm{Tr}(\tilde{\mathbf{j}}^3)\right)\ . \tag{2.2.26}$$

One can show $d\mathrm{Tr}(\tilde{\mathbf{j}}^3)$ vanishes allowing us to write

$$\int_{X_4}\omega\wedge\mathrm{Tr}(\hat{\mathbf{j}}^3) = \int_{X_4} d\omega\wedge\int_{[0,1]_t}\mathrm{Tr}(\tilde{\mathbf{j}}^3)\ . \tag{2.2.27}$$

Bringing this all together we have

$$S[A',\hat{g}\,] = \frac{1}{2\pi i}\int_{X_4}\omega\wedge\mathrm{Tr}(A'\wedge dA') + \frac{1}{2\pi i}\int_{X_4} d\omega\wedge\left(\mathrm{Tr}(\hat{\mathbf{j}}\wedge A') + \frac{1}{3}\int_{[0,1]_t}\mathrm{Tr}(\tilde{\mathbf{j}}^3)\right)\ . \tag{2.2.28}$$

The first term is also zero on-shell. To see this note that the only term that can give $A'\wedge dA'$ a non-zero $d\bar{z}$ leg is the term $\sim \epsilon^{ij}A'_i\partial_{\bar{z}}A'_j$. But we know that A' is meromorphic, with simple poles at the zeroes of ω. So all the distributional terms generated by the derivative will evaluate the product $\omega\wedge\mathrm{Tr}(A'\wedge dA')$ at the zeroes of ω and will therefore vanish. Finally we have the expression

$$S[A',\hat{g}\,] = \frac{1}{2\pi i}\int_{X_4} d\omega\wedge\left(\mathrm{Tr}(\hat{\mathbf{j}}\wedge A') + \frac{1}{3}\int_{[0,1]_t}\mathrm{Tr}(\tilde{\mathbf{j}}^3)\right)\ . \tag{2.2.29}$$

The above expression, with A' given by its on shell value in terms of $\hat{g}$, is in fact a 2d action in disguise as a 4d action; the term $d\omega$ will localise the integral over C to the poles over ω, and so only the values of the fields and their derivatives with respect to z at the poles of C will contribute to the final expression.

At this point, we have touched on enough of the technical details of 4dCS to allow us to follow with an example.

2.2.2 PCM + WZ Term

Let us now work towards making the previous discussion concrete. From now on we will be explicit in our choice of manifold X_4, choosing $\Sigma = \mathbb{R}^2$ and $C = \mathbb{CP}^1$. We will start our story by choosing to work with the meromorphic one-form

$$\omega = k\frac{1-z^2}{(z-a)^2}\,dz \; . \tag{2.2.30}$$

Our choice has two simple zeroes at $z = \pm 1$ and two double poles at $z = a$ and $z = \infty$. In terms of partial fractions

$$\omega = k\left(\frac{1-a^2}{(z-a)^2} - \frac{2a}{z-a} - 1\right) dz \; . \tag{2.2.31}$$

Boundary Conditions and Admissible Gauge transformations

The first errand to run is to identify an appropriate choice of boundary conditions. The space of admissible boundary conditions is completely dependent on our choice of merormorphic one-form. Given,

$$\partial_{\bar{z}}\varphi(z) = -2k\pi i\left((1-a)^2\partial_z\delta^{(2)}(z-a) + 2a\,\delta^{(2)}(z-a)\right) + \text{“contributions at } z=\infty\text{”}\,, \tag{2.2.32}$$

it follows the boundary term (2.2.9) is of the form

$$\begin{aligned}\delta S_{\text{boundary}} &= k(1-a)^2\int_{X_4} dz\wedge d\bar{z}\wedge\,\delta^{(2)}(z-a)\left(\partial_z\text{Tr}(A\wedge\delta A) - \frac{2a}{(1-a)^2}\text{Tr}(A\wedge\delta A)\right) + \\ &\qquad\text{“contributions at } z=\infty\text{”} \\ &= k(1-a)^2\int_\Sigma \partial_z\text{Tr}(A\wedge\delta A)|_{z=a} - \frac{2a}{(1-a)^2}\text{Tr}(A\wedge\delta A)|_{z=a} + \cdots \; .\end{aligned} \tag{2.2.33}$$

A choice that ensures (2.2.33) vanishes is given by

$$A \mid_{z=a} = A \mid_{z=\infty} = 0 \ . \tag{2.2.34}$$

Proceeding with this choice, let us consider what collection of gauge transformations leave these boundary conditions fixed. As stated in the preceding subsection, internal gauge transformations are compatible with any choice, but what subset of the full set of external gauge transformations can we use? Requiring

$$A^{\hat{\gamma}} \mid_{z=a} = \hat{\gamma}^{-1} A \hat{\gamma} \mid_{z=a} + \hat{\gamma}^{-1} d\hat{\gamma} \mid_{z=a} = \hat{\gamma}^{-1} d\hat{\gamma} \mid_{z=a} = 0 \ , \tag{2.2.35}$$

implies that $\hat{\gamma} = \text{const.}$ at $z = a$, and similarly for $z = \infty$. As such we will only have the ability to employ our external gauge symmetry to fix the degrees of freedom away from the poles. We use our internal gauge symmetry to fix a gauge such that $\hat{g} \mid_{z=\infty} = \text{Id}$, this will expedite the calculations considerably, as now all the degrees of freedom in the 2d theory will be exclusively from the edge modes at the pole $z = a$, which we will denote by $\hat{g} \mid_{z=a} = g$.

Before continuing we should ask ourselves if these degrees of freedom are truly the only ones that can be generated at the poles. Naively the answer is no! The appearance of a second order pole ensures the distribution $d\omega$ contains a term of the form $\partial_z \delta^{(2)}(z - a)$. As such, when we integrate by parts before then evaluating the integral, the expression will not just depend on the value of $\hat{g}$, but also $\hat{u} = \hat{g}^{-1}\partial_z \hat{g}$ at $z = a$ and $z = \infty$.[1] Under an external gauge transformation

$$\hat{u} = \hat{g}^{-1}\partial_z \hat{g} \to \hat{\gamma}^{-1}(\hat{g}^{-1}\partial_z \hat{g})\hat{\gamma} + \hat{\gamma}^{-1}\partial_z \hat{\gamma} \ . \tag{2.2.36}$$

In the case at hand $\hat{\gamma}^{-1}\partial_z\hat{\gamma}$ is completely unconstrained by the boundary conditions at the poles of ω, so can be used to set $\hat{u}$ to zero at the poles. One can then conclude that the degree of freedom $\hat{u}$ will not appear in this example. However, one should note if our boundary conditions involved constraining $\partial_z A$ at the poles this argument would no longer hold, allowing $\hat{u}$ to be non-zero at the poles and appear in our 2d theory.

[1] This can be easily seen by explicitly calculating (2.2.29).

The Lax Connection

Having acquired our boundary conditions, we can now solve them for the components of the Lax connection, A'. Our choice of meromorphic one-form (2.2.30) contained zeroes at $z = \pm 1$. As such, one of our Lax components will have a pole at $z = 1$, and the other at $z = -1$. Using the ansatz (2.2.22) for zeroes at $z = \pm 1$ and solving for our boundary conditions (2.2.34), a quick calculation gives

$$A'_{\pm} = \frac{1 \mp a}{1 \mp z} g^{-1} \partial_{\pm} g \ . \tag{2.2.37}$$

With this we have found that our Lax connection exactly coincides with that of the PCM+WZ term (1.1.25).

The Action

The final thing to compute is the form of the action (2.2.29). A priori, we know that the contribution to the action from the distributions at $z = \infty$ will all be zero, this is due to the fact $\hat{\mathbf{j}} \,|_{z=\infty} = 0$ and $\hat{u} = 0$ identically. Using (2.2.32), this leaves us with following form of the action

$$\begin{aligned} S = k\,(1-a)^2 \int_{X_4} dz \wedge d\bar{z} \wedge \, \delta^{(2)}(z-a)\, \partial_z \left[\mathrm{Tr}(\hat{\mathbf{j}} \wedge A') + \int_{[0,1]_t} \frac{1}{3} \mathrm{Tr}(\tilde{\mathbf{j}}^3) \right] \\ + 2ak \int_{X_4} dz \wedge d\bar{z} \wedge \, \delta^{(2)}(z-a) \, \left[\mathrm{Tr}(\hat{\mathbf{j}} \wedge A') + \frac{1}{3} \int_{[0,1]_t} \mathrm{Tr}(\tilde{\mathbf{j}}^3) \right] \ . \end{aligned} \tag{2.2.38}$$

To evaluate this expression we note that $\partial_z \hat{\mathbf{j}} = 0$ and one arrives at the following action

$$S[g] = k \int_{\Sigma} d\sigma^+ \wedge d\sigma^- \, \mathrm{Tr}(g^{-1}\partial_+ g g^{-1} \partial_- g) + 2ak \int_{\Sigma \times [0,1]_t} \mathrm{WZ}[\tilde{g}] \ , \tag{2.2.39}$$

where $\mathrm{WZ}[\tilde{g}]$ is the Wess-Zumino term defined as in (2.1.6). We have found that for our choice of meromorphic one-form (2.2.30) and boundary conditions (2.2.34), 4d Chern-Simons gives us the action of the PCM+WZ term (1.1.22). One should note that had we decided to centre our double pole at $a = 0$ then the WZ term drops from the action leaving us with the action for the principal chiral model. As such one concludes that by 'turning on a residue' and shifting our pole away from zero, we introduce the WZ term into the action.

The non-Abelian T-dual of the PCM

Consider now (2.2.30) with $a = 0$. The boundary term now looks like

$$k \int_\Sigma \partial_z \mathrm{Tr}(A \wedge \delta A)|_{z=0} + \text{ contributions from } z = \infty \ . \tag{2.2.40}$$

In this case, making the choice $\partial_z A|_{z=0} = A|_{z=\infty} = 0$, also ensures the boundary variation vanishes. The compatible gauge transformations at $z = \infty$ are familiar to us. We again use the internal symmetry to set $\hat{g}\ |_{z=\infty}= \mathrm{Id}$ and use the fact $\hat{\Gamma} := \hat{\gamma}^{-1}\partial_z\hat{\gamma}$ is unconstrained to fix $\hat{u}\ |_{z=\infty}:= \hat{g}^{-1}\partial_z\hat{g}\ |_{z=\infty}= 0$. However, at $z = 0$ we have the condition that $\partial_z A^{\hat{\gamma}}\ |_{z=0}= 0$ which is solved by ensuring $\hat{\Gamma}\ |_{z=0}:= \hat{\gamma}^{-1}\partial_z\hat{\gamma}\ |_{z=0}= 0$. Crucially, $\hat{\gamma}$ is unconstrained at $z = 0$, so we can use an external gauge transformation to fix $\hat{g}\ |_{z=0}= \mathrm{Id}$. The same is not true for $\hat{\Gamma}$ however, and so we lack the gauge freedom to gauge fix $\hat{u}\ |_{z=0}:= \hat{g}^{-1}\partial_z\hat{g}\ |_{z=0}= u$. As such u will appear as an edge mode in our theory, localising to the 2d degree of freedom that acts as the fundamental field for our 2dIFT. So keeping tabs, after gauge fixing, we have the following:

$$\hat{g}\ |_{z=0}= \mathrm{Id}\ , \quad \hat{u}\ |_{z=0}= u\ , \quad \hat{g}\ |_{z=\infty}= \mathrm{Id}\ , \quad \hat{u}\ |_{z=\infty}= 0\ . \tag{2.2.41}$$

Now we need to use the boundary conditions to solve for the form of our Lax. The boundary condition at $z = 0$ imposes

$$\partial_z A\ |_{z=0}= \partial_z \left(\hat{g}^{-1}A'\hat{g} + \hat{g}^{-1}d\hat{g}\right)\ |_{z=0}= 0\ . \tag{2.2.42}$$

So, using the same ansatz for the form of our Lax, given by (2.2.22), a brief calculation gives $V_\pm = 0$ using the boundary conditions at $z = \infty$, and

$$U_+ = (\mathrm{ad}_u - 1)^{-1}\partial_+ u\ , \qquad U_- = (\mathrm{ad}_u + 1)^{-1}\partial_- u\ , \tag{2.2.43}$$

using the boundary condition at $z = 0$, where $\mathrm{ad}_u \bullet := [u, \bullet]$. As such, the Lax reads

$$A'_\pm = \frac{(\mathrm{ad}_u \mp 1)^{-1}}{1 \mp z}\partial_\pm u\ . \tag{2.2.44}$$

This Lax is familiar from the literature, it is the Lax for the non-Abelian T-dual (NATD) of the PCM. Indeed, if we then calculate the action for the localised theory we find

$$S[u] = k \int_{\Sigma} d\sigma^{+} \wedge d\sigma^{-} \operatorname{Tr}\left(\partial_{+} u \, \frac{1}{\mathrm{ad}_u + 1} \partial_{-} u\right) \, , \tag{2.2.45}$$

which is exactly the action for the NATD of the PCM [Tho19].

Here we have displayed clearly that the choice of boundary conditions we impose on our gauge field is indeed a defining part of the data that forms the 2dIFT. In choosing 'Neumann' boundary conditions instead of Dirichlet we have localised our 4d action to a completely different theory. However, one may wonder what the exact relationship is between two theories that share the same meromorphic one form but vary in the choice of boundary conditions imposed at the poles. In the above case, we have seen that the two models are T-dual models.

2.2.3 Integrable Deformations and T-duality from 4dCS

Let us introduce some more features about 4dCS through the medium of another example [Del+20].

The λ-deformed Principal Chiral Model

Consider the meromorphic one-form given by

$$\omega = \frac{k}{1-\alpha^2} \frac{1-z^2}{z^2-\alpha^2} \, dz \, , \tag{2.2.46}$$

where $k, \alpha \in \mathbb{R}$. In this choice we have two simple zeroes at $z = \pm 1$, two simple poles at $z = \pm\alpha$ and a double pole at $z = \infty$. Performing a partial fraction decomposition

$$\omega = \frac{k}{2\alpha} \left(\frac{1}{z-\alpha} - \frac{1}{z+\alpha} - 1 \right) dz \, . \tag{2.2.47}$$

This will generate a boundary term through

$$\partial_{\bar{z}} \varphi(z) = \frac{i\pi k}{\alpha} \left(\delta^{(2)}(z-\alpha) - \delta^{(2)}(z+\alpha) \right) + \text{“contributions at } z = \infty\text{”} \tag{2.2.48}$$

given by

$$\delta S_{\text{boundary}} = \frac{k}{2\alpha}\int_\Sigma \text{Tr}(A\wedge\delta A)\mid_{z=\alpha} - \text{Tr}(A\wedge\delta A)\mid_{z=-\alpha} + \text{“contributions at } z=\infty\text{”} . \tag{2.2.49}$$

We know from the PCM+WZ term case that the contributions at $z=\infty$ can be dealt with by choosing $A\mid_{z=\infty}=0$ and using the internal gauge symmetry to fix $\hat{g}\mid_\infty = \text{Id}$. One may be naturally compelled to deal with the contributions at $z=\pm\alpha$ in a similar vein, making the choice such that the gauge field vanishes at these points. This would be a poor choice, however, as it is over-constraining. The next most transparent choice of boundary conditions[2] are the so called ‘diagonal’ boundary conditions,

$$A\mid_{z=\alpha} = A\mid_{z=-\alpha} . \tag{2.2.50}$$

Curiously these boundary conditions are non-local, relying on equating the gauge field at two distinct points but they certainly ensure that (2.2.49) vanishes and importantly are not over-constraining, allowing for dynamics. Performing a gauge transformation on the boundary conditions (2.2.50) one finds, the gauge transformations satisfying $\hat{\gamma}\mid_{z=\alpha} = \hat{\gamma}\mid_{z=-\alpha}$ preserve the boundary conditions and so form our admissible external transformations. As a result, when we gauge fix the value of our parametrising field $\hat{g}$ at $z=-\alpha$ we are no longer able to fix its value at $z=\alpha$. With this we shall fix $\hat{g}\mid_{z=-\alpha} = \text{Id}$ leaving $\hat{g}\mid_{z=\alpha} = g$ having used up our allowed gauge transformations.

To find the Lax connection, we need to solve our boundary conditions. Given the double pole at $z=\infty$ we deduce,

$$A'_\pm = \frac{\alpha+1}{z\mp 1}\tilde{U}_\pm . \tag{2.2.51}$$

Then solving (2.2.50) we find

$$A'_+ = \frac{\alpha+1}{z-1}\frac{\lambda\,\text{Ad}_g}{\lambda\,\text{Ad}_g - 1}g^{-1}\partial_+ g , \quad A'_- = \frac{\alpha+1}{z+1}\frac{\text{Ad}_g}{\text{Ad}_g-\lambda}g^{-1}\partial_- g , \tag{2.2.52}$$

where $\lambda = (1+\alpha)/(1-\alpha)$. After some quick manipulations, that make use of the definition of the Ad-operator, we have that the above Lax is exactly that of the λ-deformed PCM (1.1.57). To show that our choice of ω, (2.2.46), with diagonal boundary

[2] This or the chiral boundary conditions.

conditions, (2.2.50), does indeed correspond to the λ-model we will need to show that the effective 2d action coincides with (1.1.55). This is a quick computation using (2.2.29) and (2.2.52),

$$S[A', \hat{g}] = \frac{k}{2\alpha} \int_{X_4} dz \wedge d\bar{z} \wedge \left(\delta^{(2)}(z-\alpha) - \delta^{(2)}(z+\alpha) \right) \left(\mathrm{Tr}(\hat{\mathbf{j}} \wedge A') + \frac{1}{3} \int_{[0,1]_t} \mathrm{Tr}(\tilde{\mathbf{j}}^3) \right) + \text{“contributions at } z = \infty\text{”} \,. \tag{2.2.53}$$

After some manipulations one can finesse the above into the form

$$\begin{aligned} S[g] = \frac{\tilde{k}}{2} \int_\Sigma d\sigma^+ \wedge d\sigma^- \, \mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \tilde{k}\, \mathrm{WZ}[\tilde{g}] \\ + \tilde{k} \int_\Sigma d\sigma^+ \wedge d\sigma^- \, \mathrm{Tr}\left(\frac{\lambda}{1 - \lambda \mathrm{Ad}_g} g^{-1}\partial_+ g \, g^{-1}\partial_- g \right) \,, \end{aligned} \tag{2.2.54}$$

where $\tilde{k} = -k/4\alpha$. After using the Ad-invariance of the inner product, one can see this is exactly the action for the λ-deformation of the PCM.

A brief afterthought

Schematically it appears as if we are able to obtain integrable deformed models by splitting the double pole at $z = 0$ into two simple poles at $z = \pm\alpha$. This is certainly an interesting thought, however the λ-deformed PCM is not the only integrable deformation we know of. There is also the Yang-Baxter deformation of the PCM, which is related to the λ-deformed PCM via Poisson-Lie T duality and then analytically continuing the deformation parameter. If 4dCS is as robust a framework as we imagine for describing 2dIFTs, one should conjecture that this relation between the 2d sigma-model should be manifest through a 4dCS lens: rather excitingly, it is.

The Yang-Baxter deformed Principal Chiral Model

Again, as our starting point we will have the same choice of ω as in the λ-deformed model (2.2.46). Let us look again at the boundary term (2.2.49) and ask what other solutions exist? We will deal with the contribution from the double pole at $z = \infty$ in an identical fashion as the previous examples, setting $A \mid_{z=\infty} = 0$ and using the internal gauge symmetry to set $\hat{g} \mid_{z=\infty} = \mathrm{Id}$. To address the contributions from the simple poles we introduce a constant linear operator on the Lie algebra $\mathfrak{g}$, given by $\mathcal{R} : \mathfrak{g} \to \mathfrak{g}$.

Setting

$$A\mid_{z=\pm\alpha}=(\mathcal{R}\mp 1)X \tag{2.2.55}$$

for some $X\in\mathfrak{g}$, we have that

$$(\mathcal{R}+1)A\mid_{z=\alpha}=(\mathcal{R}-1)A\mid_{z=-\alpha}\ . \tag{2.2.56}$$

Due to the fact that the variation of A must also satisfy the boundary condition, one arrives at

$$\delta S_{\text{boundary}}=-\frac{k}{\alpha}\int_{\Sigma}\text{Tr}\left((\mathcal{R}+\mathcal{R}^t)X\wedge\delta X\right)\ , \tag{2.2.57}$$

where the transpose of $\mathcal{R}$ with respect to the bilinear form Tr, $\mathcal{R}^t$, is defined by $\text{Tr}(X\mathcal{R}Y)=\text{Tr}(\mathcal{R}^tXY)$ for all $X,Y\in\mathfrak{g}$. As such, if $\mathcal{R}$ is antisymmetric then the boundary variation vanishes. We should check how these boundary conditions act under external gauge transformations. Performing an infinitesimal external gauge transformation on (2.2.56) we find

$$(\mathcal{R}+1)([A,\hat{\Gamma}]+d\hat{\Gamma})\mid_{z=\alpha}=(\mathcal{R}-1)([A,\hat{\Gamma}]+d\hat{\Gamma})\mid_{z=-\alpha}\ . \tag{2.2.58}$$

To solve this first we see we can make the two differential terms match by setting

$$\hat{\Gamma}_{z=\pm\alpha}=(\mathcal{R}\mp 1)Y\ . \tag{2.2.59}$$

Using (2.2.55), we then plug this in resulting in the following equation that needs satisfying

$$(\mathcal{R}+1)\left[(\mathcal{R}-1)X,(R-1)Y\right]-(\mathcal{R}-1)\left[(\mathcal{R}+1)X,(R+1)Y\right]=0\ . \tag{2.2.60}$$

Expanding out and collecting like terms gives

$$[\mathcal{R}X,\mathcal{R}Y]-\mathcal{R}[\mathcal{R}X,Y]-\mathcal{R}[X,\mathcal{R}Y]+[X,Y]=0\ , \tag{2.2.61}$$

or rather for gauge invariance $\mathcal{R}$ must solve the modified classical Yang-Baxter equation (mcYBe). As such given $\mathcal{R}$ is indeed a solution of the mcYBe, we have found that the residual external gauge transformations preserving the boundary conditions are those generated by infinitesimal transformations satisfying (2.2.59). This motivates the following definition; we define the Lie algebra $\mathfrak{g}^{\mathcal{R}}$ as

$$\mathfrak{g}^{\mathcal{R}} \cong \{\big((\mathcal{R}-1)X, (\mathcal{R}+1)X\big) : X \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{g} \ , \tag{2.2.62}$$

so that the collection of permissible infinitesimal gauge transformations is such that $(\hat{\Gamma}_{z=-\alpha}, \hat{\Gamma}_{z=+\alpha}) \in \mathfrak{g}^{\mathcal{R}}$, which in turn lift to the group of permissible finite gauge transformations $G^{\mathcal{R}}$. Clearly we can see that $\dim(\mathfrak{g}^{\mathcal{R}}) = \dim(\mathfrak{g})$, and furthermore, as vector spaces, we have the decomposition,

$$\mathfrak{g} \times \mathfrak{g} \cong \mathfrak{g}^{\mathcal{R}} \times \mathfrak{g}^{\Delta}, \tag{2.2.63}$$

where $\mathfrak{g}^{\Delta} := \{(X, X) : X \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{g}$, is the diagonal subset of $\mathfrak{g} \times \mathfrak{g}$. We will assume that this decomposition lifts to the level of the corresponding Lie groups [Del+20], such that $G \times G \cong G^{\mathcal{R}} \bullet G^{\Delta}$. Here $\bullet$ defines the necessary multiplication rule between the elements of the two groups to ensure the decomposition relationship holds. With this, gauge fixing by the permissible gauge transformations valued in $G^{\mathcal{R}}$ effectively means considering the left quotient $G^{\mathcal{R}} \backslash (G \times G) = G^{\Delta}$. We will again look to make use of this residual gauge symmetry to gauge fix some of our degrees of freedom. Currently, our physical degrees of freedom, namely the edge modes $\hat{g} \mid_{z=\alpha}$ and $\hat{g} \mid_{z=-\alpha}$ form a tuple $(\hat{g} \mid_{z=\alpha}, \hat{g} \mid_{z=-\alpha}) \in G \times G$. As such, we can gauge fix by the permissible gauge transformations valued in $G^{\mathcal{R}}$ such that

$$\hat{g} \mid_{z=\alpha} = \hat{g} \mid_{z=-\alpha} = g \ . \tag{2.2.64}$$

Following the now familiar method of solving for the Lax we arrive at,

$$A'_{\pm} = \frac{1}{1 \mp z} \frac{1-\alpha^2}{1 \pm \alpha \mathcal{R}_g} g^{-1} \partial_{\pm} g \ , \tag{2.2.65}$$

where we have defined the operator $\mathcal{R}_g$ by,

$$\mathcal{R}_g := \mathrm{Ad}_g^{-1} \mathcal{R} \mathrm{Ad}_g \ . \tag{2.2.66}$$

Localising the action to 2d we have

$$S_{\eta} = 2\alpha \int_{\Sigma} d\sigma^{+} \wedge d\sigma^{-} \, \mathrm{Tr}(g^{-1}\partial_{+} g \, \frac{1}{1 - \alpha \mathcal{R}_g} \, g^{-1} \partial_{-} g) \tag{2.2.67}$$

which upon redefining $\mathcal{R} \to c^{-1}\mathcal{R}$ and setting $\alpha = c\eta$ we can then identify as the Yang-Baxter deformed PCM (1.1.59).

2.3 Anti-Self-Dual Yang-Mills Theory

Having discussed at length how 4dCS theory allows us to describe 2dIFTs we turn to another 4d gauge theory that also gives rise to a plethora of lower-dimensional IFTs, namely anti-self-dual Yang-Mills. For a complete discussion of this vast topic, we refer the reader to [MW91].

In four dimensional spacetimes with Euclidean signature the curvature two-form has the property that it decomposes under the action of the Hodge star into self-dual (SD) and anti-self-dual (ASD) parts. That is

$$F(A) = dA + A \wedge A = F^+ + F^- \ , \ \text{where} \quad * F^{\pm} = \pm F \ , \tag{2.3.1}$$

where F^+ is the self-dual part and F^- the anti-self-dual part of the curvature. A connection A is called anti-self-dual if it satisfies

$$*F(A) = -F(A) \ . \tag{2.3.2}$$

Due to the Bianchi identity, $DF = 0$, any ASD connection also satisfies the Yang-Mills equations $D * F = 0$. It is important to note that the anti-self-duality equations are an algebraic condition on the curvature. As such, when we perform a conformal transformation $g_{\mu\nu} \to \Lambda^2(x) g_{\mu\nu}$, both sides of (2.3.2) scale with a factor of $\Lambda^2(x)$, cancelling, and allowing us to conclude the ASDYM equations are conformally invariant. This observation will be a key component of our discussion when introducing symmetry reductions of the ASDYM equations to attain integrable models in lower dimensions.

In this thesis, we will see that the ASDYM equations can assume many distinct, but ultimately equivalent guises, let us briefly discuss some of those important for future discussion. We adopt double null coordinates $(z, \bar{z}, w, \bar{w})$ for $\mathbb{R}^4$, with metric

$$ds^2 = 2(dz d\bar{z} + dw d\bar{w}) \ .$$

Such a choice of coordinates is useful as they allow for a particularly transparent choice of basis for the self-dual two forms, given by

$$\Omega = dz \wedge dw \ , \quad \bar{\Omega} = d\bar{z} \wedge d\bar{w} \ , \quad \omega = dz \wedge d\bar{z} + dw \wedge d\bar{w} \ . \tag{2.3.3}$$

We recognise that the first and second basis two-forms are the holomorphic and anti-holomorphic volume forms on $\mathbb{R}^4$, denoted Ω and $\bar{\Omega}$ respectively, and the third basis two-form is exactly the Kähler form for $\mathbb{R}^4$, which we denote by ω. As such, the above basis has an interpretation of defining a complex structure on $\mathbb{R}^4$. Expanding the curvature with respect to this choice of self-dual two-forms, the anti-self-duality condition (2.3.2) reads:

$$\Omega^{\mu\nu} F_{\mu\nu} = 0 \ , \quad \bar{\Omega}^{\mu\nu} F_{\mu\nu} = 0 \ , \quad \omega^{\mu\nu} F_{\mu\nu} = 0 \ , \tag{2.3.4}$$

that is, the self dual components of F vanish, where explicitly

$$\Omega_{\mu\nu} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad \bar{\Omega}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} , \text{ and } \quad \omega_{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} . \tag{2.3.5}$$

One can also write the above more explicitly in terms of the components of the connection,

$$\begin{aligned} &\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0 \ , \\ &\partial_{\bar{z}} A_{\bar{w}} - \partial_{\bar{w}} A_{\bar{z}} + [A_{\bar{z}}, A_{\bar{w}}] = 0 \ , \\ &\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z - \partial_w A_{\bar{w}} + \partial_{\bar{w}} A_w + [A_z, A_{\bar{z}}] - [A_w, A_{\bar{w}}] = 0 \ . \end{aligned} \tag{2.3.6}$$

2.3.1 Yang's Equation:

Using the above choice of complex structure on $\mathbb{R}^4$, we define the corresponding Dolbeault operators as $\partial = dz\, \partial_z + dw\, \partial_w$ and $\bar{\partial} = d\bar{z}\, \partial_{\bar{z}} + d\bar{w}\, \partial_{\bar{w}}$. With this, one finds the first two equations of (2.3.4) are solved by

$$A^{(1,0)} = \tilde{h}^{-1} \partial \tilde{h} \ , \quad A^{(0,1)} = h^{-1} \bar{\partial} h \ , \tag{2.3.7}$$

where h, $\tilde{h} \in C^\infty(\mathbb{R}^4, G)$. Now if we perform a gauge transformation by $\tilde{h}$, acting as $h \to h\tilde{h}^{-1}$, we transform the gauge field to

$$A = g^{-1}\bar{\partial}g \ , \text{ where } \quad g = h\,\tilde{h}^{-1} \ . \tag{2.3.8}$$

In this choice of gauge the first two equations are satisfied identically and the third equation of (2.3.4) is equivalent to

$$\omega \wedge \partial(g^{-1}\bar{\partial}g) = 0 \ . \tag{2.3.9}$$

This particular manifestation of the ASDYM equations we call Yang's equation [Yan77]. Although completely equivalent to ASDYM, Yang's equation is invariant under conformal isometries that preserve the Kähler form ω, as opposed to the full conformal group in four dimensions.

2.3.2 The 4d Wess-Zumino-Witten Model

In this section we will introduce a four-dimensional avatar of the WZW model [Los+96], whose equations of motion are exactly Yang's equation (2.3.9). Consider the action given by

$$S_{4d\mathrm{WZW}}[g] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}(g^{-1}dg \wedge *g^{-1}dg) + \frac{1}{3}\int_{\mathbb{R}^4\times[0,1]} \omega \wedge \mathrm{WZ}[g] \ , \tag{2.3.10}$$

where ω is necessarily a closed two-form, independent of the coordinate $t \in [0,1]$, thus ensuring that the 5 dimensional term is locally exact. Given the optical resemblance between the WZW model in two dimensions and (2.3.10), we refer to the above theory as the 4d WZW model (or WZW_4).

A particularly interesting point in the moduli of possible choices of ω is the so called Kähler point. Here ω is the Kähler form for $\mathbb{R}^4$ and we can write the action as

$$S_\omega = \frac{1}{2}\int_{\mathbb{R}^4} \omega \wedge \mathrm{Tr}(g^{-1}\partial g \wedge g^{-1}\bar{\partial}g) + \frac{1}{3}\int_{\mathbb{R}^4\times[0,1]} \omega \wedge \mathrm{WZ}[g] \tag{2.3.11}$$

and the equations of motion are exactly (2.3.9). One immediate consideration is the quantisation condition of the WZ-esque term. In WZW_2, we had a condition on the level k, here we require instead a four dimensional generalisation, namely

$$\omega \wedge \mathrm{WZ}[g] \in H^5(\mathbb{R}^4 \times [0,1]; 2\pi\mathbb{Z}) \tag{2.3.12}$$

which explicitly forces the cohomology class $[\omega]$ to be valued in

$$[\omega] \in H^2(\mathbb{R}^4; \mathbb{Z}) \ . \tag{2.3.13}$$

The class $[\omega]$ is the four-dimensional analogue of the quantised constant k in WZW_2.

The WZW_4 is a theory describing the ASD sector of Yang-Mills. Naively, it is profound that such a manifestation of the 2d WZW model in four dimensions manages to relate to ASDYM theory at all; we will motivate this from another perspective using twistor theory later on, showing that this action has a rather natural origin from considering a local action on twistor space. For now, let us consider the properties of the action S_ω which mirror those of WZW_2.

As previously discussed, WZW_2 can be thought of as a special point in the moduli space of parameters of the PCM+WZ action. It is the point where the model has vanishing β-function for the coupling at one-loop and becomes a 2d CFT. This is owed to the fact that it admits a chiral current algebra; the otherwise globally conserved currents of the PCM+WZ theory become (anti-)holomorphic currents at the WZW point, giving rise to an emergent chiral symmetry, enriching the algebraic structure of the theory. The story for WZW_4 at the Kähler point is similar in the sense that it too possesses a holomorphic symmetry. The corresponding conserved currents of this holomorphic symmetry form an algebra generalising the 2d current algebra of the WZW model [Nek96].

2.3.3 Symmetry Reductions of ASDYM

The anti-self-dual Yang Mills equations can be used to derive the equations of motion of integrable models in lower dimensions through the medium of symmetry reduction (see [MW91] for an in-depth treatment of the topic). Earlier we discussed the fact that the ASDYM equations are conformally invariant. The conformal transformations are generated by conformal Killing vectors, which, by definition, satisfy the conformal Killing equation

$$\partial_{(\mu} X_{\nu)} = \frac{1}{4}\eta_{\mu\nu}\partial_\sigma X^\sigma \ . \tag{2.3.14}$$

A generic solution of the conformal Killing equation is given by

$$X_\mu = T_\mu + L_{\mu\nu}x^\nu + Rx_\mu + x^\nu x_\nu S_\mu - 2S_\nu x^\nu x_\mu \ , \tag{2.3.15}$$

where the constant coefficients T_μ, $L_{\mu\nu} = -L_{\nu\mu}$, R and S_μ, parametrise translations, rotations, dilatations, and special conformal transformations, respectively.

An important observation is that a solution to the equations need not be invariant under conformal transformations: symmetry transformations will map one solution to another distinct solution, or rather will map one point to another point in the moduli space of solutions to a given equation.

However, let us consider solutions that are invariant under a subgroup $H \subset SO(1,5)$ of the conformal group in 4-dimensional Euclidean space. Given a conformal Killing vector field, X, generating the transformations of H on $\mathbb{R}^4$ the solutions will satisfy the constraints,

$$\mathcal{L}_X A = 0 \ , \tag{2.3.16}$$

where $\mathcal{L}_X$ is the Lie derivative. Clearly the constraint (2.3.16) is not gauge invariant, in fact, it will only be gauge invariant if we impose the same constraint on the collection of gauge transformations. So if A is constant along the flows generated by X, the collection of gauge transformations $\hat{\gamma}$ must also be constant along the flows of X, so that

$$\mathcal{L}_X A^{\hat{\gamma}} = 0 \ . \tag{2.3.17}$$

So in effect the reduction proceeds by imposing the constraint that the components of the gauge field A are independent of one or more variables that parametrise the orbits generated by a chosen subset of the conformal generators.

Another important consideration is the following: there may exist gauge fields which are gauge equivalent to a gauge field satisfying (2.3.16), as such it would appear that the reduction is gauge dependent. Therefore, one can conclude that the lower-dimensional theory one attains from performing a symmetry reduction is dependent on the choice of gauge of our connection. This turns out to be the case for the Heisenberg ferromagnet equation and non-linear Schrödinger equation. Here, reducing by the same subgroup of the conformal group but with the anti-self-dual connection being expressed in different gauges leads to two distinct 2d IFTs.

2.4 Twistor Space

Twistor theory, introduced by Roger Penrose in the 1960s [Pen67; Pen68], emerged as an elegant mathematical framework with the ambitious aim of unifying quantum mechanics and general relativity. At its core, twistor theory allows us to reformulate physical theories in spacetime in terms of geometric information on complex manifolds [War77; Pen76]. The key ingredient that proves the foundation of twistor theory's efficacy is the twistor correspondence, a mapping which takes points in space time to non-local objects in an auxiliary space called 'twistor space'.

In this subsection, we present the essential technical foundations of twistor theory, allowing us to engage with the problems where it shines brightest: the twistorial description of integrable systems. This section follows [Ada18; Dun10; Sha22].

2.4.1 Complexified Minkowski Space and 2-Spinors

To begin, let $x^\mu = (x^0, x^1, x^2, x^3)$ be holomorphic coordinates on $\mathbb{C}^4$. We will endow $\mathbb{C}^4$ with the holomorphic metric whose line element is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \ . \tag{2.4.1}$$

We call the combination of $\mathbb{C}^4$ with the metric (2.4.1) complexified Minkowski space. Starting with complexified Minkowski space is an advantageous choice. Indeed, working with a complexified spacetime allows us to treat all real spacetime signatures simultaneously, arriving at any choice we wish by imposing different choices of reality conditions.

On any given manifold, tangent vectors transform in the fundamental representation of the structure group of the tangent bundle. For complexified Minkowski spacetime, the structure group is given by $SO(4, \mathbb{C})$. The relation between $SO(4, \mathbb{C})$ and its spin group is given by the short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(4, \mathbb{C}) \to SO(4, \mathbb{C}) \to 1 \ . \tag{2.4.2}$$

One can show the group $\mathrm{Spin}(4, \mathbb{C})$ enjoys the following isomorphism,

$$\mathrm{Spin}(4, \mathbb{C}) \cong SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \,, \tag{2.4.3}$$

and so we have that

$$SO(4,\mathbb{C}) \cong (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\mathbb{Z}_2 \ . \tag{2.4.4}$$

As such the tangent vectors in the fundamental representation of $SO(4,\mathbb{C})$ must be related to certain representations of $(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\mathbb{Z}_2$. Or, representations of $(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\mathbb{Z}_2$ give representations of $SO(4,\mathbb{C})$. We denote by $(\mathbf{\frac{1}{2}},\mathbf{0})$ the fundamental representation of the $SL(2,\mathbb{C})$ factor on the left and $(\mathbf{0},\mathbf{\frac{1}{2}})$ the fundamental representation of the $SL(2,\mathbb{C})$ factor on the right. The objects living in these representations we will call left and right-handed Weyl spinors respectively, and we will adopt different indices to differentiate their origin,

$$\text{left-handed Weyl spinor} \rightarrow \eta^a \ , \quad \text{right-handed Weyl spinor} \rightarrow \chi^{\dot{a}} \ , \tag{2.4.5}$$

where the indices take the values $a = 0,1$ and $\dot{a} = 0,1$. We would now like to establish the relationship between the fundamental representation of $SO(4,\mathbb{C})$ and its corresponding form in $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Let us consider a tangent vector $v = v^\mu \partial_\mu$. On dimensionality grounds we know a vector should correspond to a 4-dimensional object on the right hand side of (2.4.4). One such choice of representation is $(\mathbf{\frac{1}{2}},\mathbf{0}) \otimes (\mathbf{0},\mathbf{\frac{1}{2}})$. Let us then consider a linear mapping of the form

$$v^{a\dot{a}} = M^{a\dot{a}}_\mu v^\mu \ , \tag{2.4.6}$$

where M is a four vector of 2×2 matrices with indices reflecting the fact it transforms in the representation $(\mathbf{\frac{1}{2}},\mathbf{0}) \otimes (\mathbf{0},\mathbf{\frac{1}{2}}) := (\mathbf{\frac{1}{2}},\mathbf{\frac{1}{2}})$. So long as $M^{a\dot{a}}_\mu$ is invertible, it defines an isomorphism at the level of the underlying vector spaces. The choice we will use to proceed is the Pauli four-vector,

$$\sigma^{a\dot{a}}_\mu = (I_{2\times 2}, \sigma_i) \ , \tag{2.4.7}$$

such that

$$v^{a\dot{a}} = \frac{1}{\sqrt{2}} \sigma^{a\dot{a}}_\mu v^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} \ . \tag{2.4.8}$$

One immediate consequence of choosing the above relation is that

$$\det\left(v^{a\dot{a}}\right) = \frac{1}{2}\,\eta_{\mu\nu}v^{\mu}v^{\nu}\ . \tag{2.4.9}$$

The fundamental representation of $SL(2,\mathbb{C})$ has an invariant tensor given by the Levi-Civita symbol ϵ_{ab}, $\epsilon_{\dot{a}\dot{b}}$. We will work with the conventions:

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \quad \epsilon^{ab}\epsilon_{bc} = \delta^{a}_{b}\ ,\ \epsilon^{\dot{a}\dot{b}}\epsilon_{\dot{b}\dot{c}} = \delta^{\dot{a}}_{\dot{b}}. \tag{2.4.10}$$

The invariant tensors ϵ_{ab}, $\epsilon_{\dot{a}\dot{b}}$ can be used to raise and lower spinor indices as follows

$$\lambda^{a} = \epsilon^{ab}\lambda_{b}\ , \quad \lambda_{a} = \lambda^{b}\epsilon_{ab} \tag{2.4.11}$$

and similarly for the right-handed spinors. We also will make ample use of the associated $SL(2,\mathbb{C})$-invariant antisymmetric brackets

$$\langle\lambda\kappa\rangle = \lambda^{a}\kappa_{a}\ , \quad [\omega\chi] = \omega^{\dot{a}}\chi_{\dot{a}}\ . \tag{2.4.12}$$

Given a vector one can find the relation between the Levi-Civita symbols ϵ_{ab}, $\epsilon_{\dot{a}\dot{b}}$ and the metric $\eta_{\mu\nu}$,

$$v_{\mu} = \eta_{\mu\nu}v^{\nu} \quad\Longleftrightarrow\quad v_{a\dot{a}} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}v^{b\dot{b}}\ . \tag{2.4.13}$$

As such the line element of the holomorphic metric (2.4.1) is given by

$$ds^{2} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}\,dx^{a\dot{a}}dx^{b\dot{b}}\ . \tag{2.4.14}$$

2.4.2 Euclidean Reality Conditions

In the discussion so far we have enjoyed the flexibility of working with complexified Minkowski space, a space containing all metric signatures of the real spacetime $\mathbb{R}^{4}$. Clearly, to recover the Euclidean metric from (2.4.1) we require that firstly all $x^{\mu} \in \mathbb{R}$, then we Wick rotate $x^{i} \mapsto ix^{i}$, $i = 1, 2, 3$. This leaves us with the desired metric

$$ds^{2} = (dx^{0})^{2} + (dx^{1})^{2} + (dx^{2})^{2} + (dx^{3})^{2}\ . \tag{2.4.15}$$

Performing the above manipulations on our complex coordinate $x^{a\dot{a}}$ amounts to the following form for our coordinates on $\mathbb{R}^4$,

$$x_{\mathbb{E}}^{a\dot{a}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^0 - ix^3 \end{pmatrix} , \tag{2.4.16}$$

where now $x^0, x^i \in \mathbb{R}$. Picking out the Euclidean slice in this way is equivalent to imposing the complex coordinate $x^{a\dot{a}}$ be invariant under the action of an involution-esque operation, defining an action $x \mapsto \hat{x}$ by

$$\hat{x}^{a\dot{a}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 - \bar{x}^3 & -\bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{pmatrix} , \tag{2.4.17}$$

and demanding the invariance of our complex coordinate under the 'hat' operation gives coordinates of the from (2.4.16). We call such an operation Euclidean conjugation. The action of Euclidean conjugation on vectors gives an induced action on spinors, with spinors λ_a and $\chi_{\dot{a}}$ transforming as

$$\lambda_a \mapsto \hat{\lambda}_a = (-\bar{\lambda}_1, \bar{\lambda}_0) , \quad \omega_{\dot{a}} \mapsto \hat{\omega} = (-\bar{\omega}_1, \bar{\omega}_0) . \tag{2.4.18}$$

An interesting fact about Euclidean conjugation is that it is not quite an involution but instead a quartic-involution, that is $\hat{\hat{\lambda}}_a = -\lambda_a$. For this reason the reality structure associated with Euclidean signature is called 'quarternionic'. An immediate consequence of this is the following; if we take a null vector in $\mathbb{C}^4$ given by $\lambda^a\tilde{\lambda}^{\dot{a}}$, the only null vector preserved by the action of Euclidean conjugation is 0. This is equivalent to the statement that there are no non-zero real null vectors in Euclidean space.

2.4.3 Homogeneous and Inhomogeneous coordinates

So far we have used inhomogeneous coordinates to coordinatise $\mathbb{CP}^1$. Inhomogeneous coordinates are, in a sense, completely standard. They locally allow us to describe points in a certain chart of $\mathbb{CP}^1$ by points in the complex plane $\mathbb{C}$ in much the same fashion as we would any space satisfying the definition of a Riemann surface. A particularly useful choice of atlas for $\mathbb{CP}^1$ is given by using two open sets, one open set excluding the north pole, $U_0 = \mathbb{CP}^1 \setminus \{0\}$, and coordinatised by z and another that excludes the south pole, $U_1 = \mathbb{CP}^1 \setminus \{\infty\}$, coordinatised by $\tilde{z}$. On the overlap the two holomorphic coordinates are related by $z = \tilde{z}^{-1}$.

Due to it being a purely local construction, using inhomogeneous coordinates has an anticipated deficiency when it comes to describing the global properties of the geometrical objects on our manifold. For instance, when checking if our meromorphic one-form ω had poles at $z = \infty$ we had to flip from one chart to the other, changing coordinates, and explicitly check if ω had a pole at $\tilde{z} = 0$.

This motivates the introduction of homogeneous coordinates, which have the advantage of being global on $\mathbb{CP}^1$. Instead of describing points of $\mathbb{CP}^1$ in terms of a single complex number, homogenous coordinates use a pair complex numbers, so that one can embed $\mathbb{CP}^1$ into $\mathbb{C}^2$ as

$$\mathbb{CP}^1 := \{(\pi_0, \pi_1) \in \mathbb{C}^2/\{\mathbf{0}\} \ : \ (\pi_0, \pi_1) \sim \lambda\,(\pi_0, \pi_1)\ ,\ \lambda \in \mathbb{C}^*\}\ . \tag{2.4.19}$$

Homogeneous coordinates on $\mathbb{CP}^1$ will be denoted by $\pi_a = (\pi_1, \pi_2)$, which are defined up to the equivalence relation $\pi_a \sim \lambda\,\pi_a$ for any non-zero $\lambda \in \mathbb{C}^*$. We can relate homogenous coordinates to inhomogeneous coordinates on the two patches covering $\mathbb{CP}^1$ using the following prescription: introducing an arbitrary spinor γ_a that satisfies $\langle\gamma\hat{\gamma}\rangle = 1$, the two patches covering $\mathbb{CP}^1$ will be defined as

$$U_0 = \{\pi_a \mid \langle\pi\hat{\gamma}\rangle \neq 0\}\ , \qquad U_1 = \{\pi_a \mid \langle\pi\gamma\rangle \neq 0\}\ . \tag{2.4.20}$$

Inhomogeneous coordinates may be defined on each patch by

$$z = \frac{\langle\gamma\pi\rangle}{\langle\pi\hat{\gamma}\rangle}\ , \qquad \tilde{z} = \frac{\langle\pi\hat{\gamma}\rangle}{\langle\gamma\pi\rangle}\ , \qquad z = \tilde{z}^{-1}\ . \tag{2.4.21}$$

In this section, we restrict our attention to U_0 and the inhomogeneous coordinate z, knowing that an analogous discussion holds for the other patch. The complex conjugate of the inhomogeneous coordinate z is

$$\bar{z} = -\frac{\langle\hat{\pi}\hat{\gamma}\rangle}{\langle\gamma\hat{\pi}\rangle}\ . \tag{2.4.22}$$

The adoption of homogeneous coordinates when describing objects on projective spaces comes with some subtleties. One is the scaling redundancy which comes with using coordinates on $\mathbb{C}^2$ to represent points in $\mathbb{CP}^1$. As such, we should ensure this scaling redundancy which equates points in $\mathbb{C}^2$ remains a redundancy. Let us first consider which functions descend consistently from $\mathbb{C}^2$ to $\mathbb{CP}^1$.

Clearly, to avoid being multivalued at a point in $\mathbb{CP}^1$, a representative function on $\mathbb{C}^2$ should be constant on every equivalence class in $\mathbb{C}^2$. That is for every $(\pi_0, \pi_1) \in \mathbb{C}^2 \backslash \{\mathbf{0}\}$ we have $f(\lambda\pi) = f(\pi)$ for every $\lambda \in \mathbb{C}^*$. Such functions are called homogenous functions of weight zero. One can conversely conclude that homogeneous functions of weight k on $\mathbb{C}^2$ defined by $f(\lambda\pi) = \lambda^k f(\pi)$ for all $\lambda \in \mathbb{C}^*$, cannot descend to functions on $\mathbb{CP}^1$. Instead, they describe sections of line bundles over $\mathbb{CP}^1$ which we denote by $\mathcal{O}(k)$.

2.4.4 Holomorphic line bundles over $\mathbb{CP}^1$

Given that complex projective spaces, $\mathbb{CP}^n$ are defined as the collection of straight lines through the origin of $\mathbb{C}^{n+1}$, one has the ability to define a natural notion of line bundle. Working with the case of $n = 1$, we can pick a point in $\pi_a \in \mathbb{CP}^1$ and simply attach to it the line that we 'projected away'

$$l := \{t\ (\pi_0, \pi_1) \subset \mathbb{C}^2 : t \in \mathbb{C}\}\ . \tag{2.4.23}$$

Given its natural appearance, we call such a bundle over $\mathbb{CP}^1$ the tautological bundle.

To explore the properties of line bundles over $\mathbb{CP}^1$ it will help to discuss holomorphic vector bundles more generally. The data defining holomorphic vector bundles is given by two complex manifolds E and M together with a holomorphic mapping $\pi : E \to M$. Given a point $x \in M$ we will denote the fibre over x by $E_x := \pi^{-1}(x) \subset E$. Rank-$k$ holomorphic vector bundles have the property that given $U_i \subset M$, we have the biholomorphism $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}^k$. Such maps are called 'local trivialisations', denoted by ψ_i. On any given fibre above $x \in M$, we require that the trivialisation is a $\mathbb{C}$-linear mapping from E_x to $\mathbb{C}^k$, thus inducing the fibre E_x with a vector space structure. Naturally one should ask what happens when the trivialising neighbourhoods overlap. For $x \in U_i \cap U_j$ one has the following choice of two trivialisations

$$\{x\} \times \mathbb{C}^k \xleftarrow{\psi_i} E_x \xrightarrow{\psi_j} \{x\} \times \mathbb{C}^k\ . \tag{2.4.24}$$

As such we have a vector space isomorphism defined by

$$\psi_{ij}(x) := \psi_i \circ \psi_j^{-1}(x, \bullet) : \mathbb{C}^k \to \mathbb{C}^k\ , \tag{2.4.25}$$

so clearly our transition functions are valued in the space of invertible linear mappings of $\mathbb{C}^k$, $GL(k,\mathbb{C})$,

$$\psi_{ij}(x) : U_i \cap U_j \to GL(k,\mathbb{C}) \ , \tag{2.4.26}$$

where the transition function is the same for all points $x \in U_i \cap U_j$. By construction we have that

$$\psi_{ij} = \psi_{ji}^{-1} \tag{2.4.27}$$

and on triple intersections, $U_i \cap U_j \cap U_k$, consistency also dictates that

$$\psi_{ij} \circ \psi_{jk} = \psi_{ik} \ . \tag{2.4.28}$$

We call the previous two conditions, (2.4.27) and (2.4.28), the cocycle conditions.

Such maps ψ_{ij} are called transition functions; they find efficacy in showing how a local trivialisation in one open neighbourhood varies from another local trivialisation in another open neighbourhood, effectively describing how the fibres 'twist' around the base manifold. Because of this property, one may wish to describe a vector bundle using the collection of open sets of M and their transition functions, 'patching the fibres together', as initial data instead, $(\{U_i\}, \psi_{ij})$.

Let us investigate what the sections of the tautological line bundle look like. The total space is given by

$$L = \{(\pi_a, l) \subset \mathbb{CP}^1 \times \mathbb{C}^2 : l = \lambda\pi_a \ , \ \lambda \in \mathbb{C}\} \ . \tag{2.4.29}$$

The projection mapping acts as $\pi\,(\pi_a, l) = \pi_a$. Given the local nature of the data defining the line bundles, it will prove beneficial to adopt inhomogeneous coordinates at this point in the discussion. Covering $\mathbb{CP}^1$ with the open sets given by

$$U_0 = \{(\pi_0, \pi_1) \in \mathbb{CP}^1 : \pi_0 \neq 0\} \quad \text{and} \quad U_1 = \{(\pi_0, \pi_1) \in \mathbb{CP}^1 : \pi_1 \neq 0\} \ , \tag{2.4.30}$$

we use the inhomogeneous coordinates $z = \pi_1/\pi_0$ on U_0 and $\tilde{z} = \pi_0/\pi_1$ on U_1. Clearly on the overlap, $U_0 \cap U_1$, we have that $z = \tilde{z}^{-1}$. Along the fibres, we will adopt the coordinates (Z_0, Z_1). By the nature of the definition of the fibres (2.4.23) of the tautological bundle, we have that these coordinates satisfy the constraint

$$\frac{Z_0}{Z_1} = \frac{\pi_0}{\pi_1} \quad \text{or rather} \quad \pi_1 Z_0 = \pi_0 Z_1 \ . \tag{2.4.31}$$

So that the fibres indeed form a 1 complex dimensional subset of $\mathbb{C}^2$. On U_0 $\pi_0 \neq 0$, and we can trivialise the bundle by

$$\pi^{-1}(U_0) = \{(z, Z_1) \in U_0 \times \mathbb{C} : Z_1 = zZ_0\} \subset L \,. \tag{2.4.32}$$

Similarly on U_1 we have

$$\pi^{-1}(U_1) = \{(\tilde{z}, Z_0) \in U_1 \times \mathbb{C} : Z_0 = \tilde{z}Z_1\} \subset L \,. \tag{2.4.33}$$

Clearly on the overlap $U_0 \cap U_1$, where $z = \tilde{z}^{-1}$, the transition function for the fibre coordinate is given by $\psi_{01} = \pi_0/\pi_1$. The sections of L, by definition, satisfy the property $s_0 = \psi_{01}s_1$. Given the fact that $\psi_{01} = z^{-1}$, we denote the tautological bundle by $\mathcal{O}(-1)$.

In a sense, $\mathcal{O}(-1)$ is fundamental, since one is able to construct all other holomorphic line bundles over $\mathbb{CP}^1$ starting from the tautological line bundle. Consider the line bundle, $\mathcal{O}(1)$, defined as the 'dual' of $\mathcal{O}(-1)$ in the following sense; if we take the tensor product of the two bundles, it gives the trivial line bundle over $\mathbb{CP}^1$, or rather $\mathcal{O}(-1) \otimes \mathcal{O}(1) \cong \mathbb{CP}^1 \times \mathbb{C}$. The trivial line bundle is defined by necessarily having $\psi_{01} = 1$. We conclude that $\mathcal{O}(1)$ must have transition functions given by $\psi_{01} = z$. The line bundle $\mathcal{O}(1)$ is often referred to as the hyperplane bundle. Taking tensor products between k copies of these bundles, we define the line bundle $\mathcal{O}(k)$ as the line bundle with transition function $\psi_{01} = z^k$. In all that follows we will take $\psi_{01} = \pi_1/\pi_0$, as in $\mathcal{O}(1)$.

At this point, we need to make an important clarification. Above, we used $\mathcal{O}(k)$ to denote the line bundle whose sections allowed us to describe homogeneous functions of weight k over $\mathbb{CP}^1$ in terms of a representing function defined on $\mathbb{C}^2$. This was done by design, as any homogenous polynomial of degree k in $\mathbb{CP}^1$ can be canonically identified with holomorphic sections of $\mathcal{O}(k)$.

Explicitly, consider a degree k polynomial in $\pi_a \in \mathbb{C}^2$ given by

$$P_k(\pi) = \sum_n a_n \, {\pi_0}^{k-n} \, \pi_1^n \,, \tag{2.4.34}$$

where a_n are the polynomial coefficients, then we can see that P_k does not descend to a polynomial on $\mathbb{CP}^1$, for $k \neq 0$, since it isn't homogenous of degree 0. However, the polynomial can be used to define functions on the standard affine charts U_0 and U_1, which can be glued together on the overlap $U_0 \cap U_1$, to form a global section of $\mathcal{O}(k)$.

We can see this as follows, in U_0 with the affine coordinate z we have

$$s_0 = \frac{P_k(\pi)}{\pi_0^k} = \sum_n a_n \left(\frac{\pi_1}{\pi_0}\right)^n = \sum_n a_n z^n \ . \tag{2.4.35}$$

Now to proceed in an analogous fashion on U_1 with the affine coordinate $\tilde{z}$,

$$s_1 = \frac{P_k(\pi)}{\pi_1^k} = \sum_n a_n \left(\frac{\pi_0}{\pi_1}\right)^k \left(\frac{\pi_1}{\pi_0}\right)^n = (\psi_{01})^{-k} s_0 \ . \tag{2.4.36}$$

Or rather we have that,

$$s_0 = (\psi_{01})^k s_1 \ ,$$

and so for $k \geq 0$ a homogenous degree k polynomial defines a unique holomorphic section of $\mathcal{O}(k)$, as claimed. One should also note that for $k < 0$ the polynomial (2.4.34) is not, in fact, holomorphic in U_1 but meromorphic at $\pi_0 = 0$, hence we conclude that there are no nonzero global holomorphic sections of $\mathcal{O}(k)$, with $k < 0$. The above discussion is often stated economically as [Wel80],

$$H^0\big(\mathbb{CP}^1, \mathcal{O}(k)\big) \cong \begin{cases} \mathbb{C}^{k+1}, & \text{if } k \geq 0, \\ 0, & \text{if } k < 0, \end{cases}$$

with H^0 denoting the 0^{th} cohomology class of $\mathcal{O}(k)$-valued sections over $\mathbb{CP}^1$ and hence the space of global holomorphic sections of $\mathcal{O}(k)$.

2.4.5 The Twistor Correspondence

With a familiarity of 2-spinors established we can now begin our discussion of the twistor correspondence. Understanding the twistor correspondence is tantamount to being well acquainted with the following double fibration:

$$\begin{array}{ccc} & \mathbb{PS} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathbb{PT} & & \mathbb{C}^4 \end{array} \tag{2.4.37}$$

The projective spin bundle, $\mathbb{PS}$ is defined such that

$$\mathbb{PS} = \mathbb{C}^4 \times \mathbb{CP}^1 \tag{2.4.38}$$

with coordinates given by $(x^{a\dot{a}}, \pi_b)$, where $\pi_a \sim r\,\pi_a$ for $r \in \mathbb{C}^*$ is the homogenous coordinate for the projective line $\mathbb{CP}^1$. Projective twistor space, $\mathbb{PT}$ is defined as the following open subset of $\mathbb{CP}^3$

$$\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1 = \{Z^\alpha = (\omega^{\dot{a}}, \pi_a) \,:\, \pi_a \neq 0 \;, Z^\alpha \sim rZ^\alpha \quad r \in \mathbb{C}^\times\} \;. \tag{2.4.39}$$

Finally $\mathbb{C}^4$ is complexified Minkowski space, coordinatised by $x^{a\dot{a}}$. The projections in (2.4.46) are given by

$$\begin{aligned} \pi_1(x^{a\dot{a}}, \pi_b) &= (x^{b\dot{a}}\pi_b, \pi_a) \;, \\ \pi_2(x^{a\dot{a}}, \pi_b) &= x^{a\dot{a}} \;. \end{aligned} \tag{2.4.40}$$

We call the following mapping,

$$\omega^{\dot{a}} = x^{a\dot{a}}\pi_a \;, \tag{2.4.41}$$

the incidence relations. It is this mapping that generates the non-local correspondences that make twistor theory such a strong tool. Let us discuss its immediate consequences. Consider fixing a point x in space time, $\mathbb{C}^4$, what does this correspond to in $\mathbb{PT}$? Or rather formulating this using the fibration (2.4.37), we can ask what the image of $\pi_1(\pi_2^{-1}(\{x\}))$ is in $\mathbb{PT}$. Ignoring projective scaling momentarily, let us pretend $Z^\alpha = (x^{a\dot{a}}\pi_a, \pi_b) \in \mathbb{C}^4$. Then with $x^{a\dot{a}}$ fixed one can see Z^α is completely defined by a sub-plane $\mathbb{C}^2 \subset \mathbb{C}^4$ coordinatised by π_α. Bringing back the projective scaling one can see this defines a $\mathbb{CP}^1 \subset \mathbb{CP}^3$ subset of $\mathbb{PT}$, or rather

$$\pi_1(\pi_2^{-1}(\{x\})) \cong \mathbb{CP}^1_x \;, \tag{2.4.42}$$

where $\mathbb{CP}^1_x$ denotes the Riemann sphere subset of $\mathbb{PT}$ corresponding to the space-time point x. With this we have established that points in $\mathbb{C}^4$ correspond to linearly holomorphic embedded Riemann spheres in $\mathbb{PT}$. It is worth noting here that in the literature such $\mathbb{CP}^1$ subsets are regularly called 'complex-projective lines'. This name makes sense in parallel with our familiar notions. Just as a line in Euclidean space is a 1-dimensional subset whose points are a linearly embedded curve, complex projective lines are the complex dimension 1 subsets whose points are a linearly and holomorphically embedded complex curve.

Conversely, fixing a point $Z^\alpha \in \mathbb{PT}$, what is the image $\pi_1(\pi_2^{-1}(\{Z\}))$? Using (2.4.41), one arrives at the following simultaneous equations,

$$\omega^{\dot{a}} = x^{a\dot{a}}\pi_a \text{ and } \quad \omega^{\dot{a}} = y^{a\dot{a}}\pi_a \quad \Rightarrow \quad (x-y)^{a\dot{a}}\pi_a = 0 \; . \tag{2.4.43}$$

This is solved by

$$(x-y)^{a\dot{a}} = \pi^a \tilde{\pi}^{\dot{a}} \; , \tag{2.4.44}$$

displaying that $x^{a\dot{a}}$ and $y^{a\dot{a}}$ are null separated points in $\mathbb{C}^4$, with their relative position parametrised by the right-handed Weyl spinor $\tilde{\pi}^{\dot{a}}$. Such a collection of points forms a self-dual, null two-plane in spacetime called an α-plane. As such we have established that, given a point $Z^\alpha = (\omega^{\dot{a}}, \pi_a) \in \mathbb{PT}$,

$$\pi_1(\pi_2^{-1}(\{Z\})) \cong \left\{ V^{a\dot{a}} \in \mathbb{C}^4 \; : \; V^{a\dot{a}} = \pi^a \tilde{\pi}^{\dot{a}} \right\} \; . \tag{2.4.45}$$

2.4.6 The Euclidean Twistor Correspondence

Let us consider the twistor correspondence for Euclidean spacetime. We obtain the Euclidean twistor correspondence by picking out the slice of each space in the fibration (2.4.37) which is invariant under the action of Euclidean conjugation. In doing so, we obtain the following double fibration

$$\begin{array}{ccccc} & & \mathbb{PS}_{\mathbb{E}} = \mathbb{R}^4 \times \mathbb{CP}^1 & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathbb{PT}_{\mathbb{E}} & & & & \mathbb{R}^4 \end{array} \tag{2.4.46}$$

The only unfamiliar space that needs explaining here is that of $\mathbb{PT}_{\mathbb{E}}$, Euclidean twistor space. As mentioned above, $\mathbb{PT}_{\mathbb{E}}$ is the slice of $\mathbb{PT}$ that is invariant under the action of Euclidean conjugation $Z^\alpha = \hat{Z}^\alpha$, where

$$\hat{Z} = (\hat{\omega}^{\dot{a}}, \hat{\pi}_a) \; . \tag{2.4.47}$$

As $\hat{\hat{Z}}^\alpha = -Z^\alpha$, clearly no points in twistor space are invariant under this quartic involution, and so one can conclude that there are no Euclidean real points in $\mathbb{PT}_{\mathbb{E}}$. This is to be expected, points in twistor space correspond to null planes in complexified Minkowski spacetime. As there are no non-zero null vectors in Euclidean space, there is no subset of the α-plane in $\mathbb{C}^4$ that can correspond to a point in $\mathbb{PT}_{\mathbb{E}}$. This motivates looking at the other side of the correspondence. If points in $\mathbb{PT}$ corresponding to α-planes cannot be Euclidean real, we can certainly look at which subset of complex projective lines, $\mathbb{CP}^1 \subset \mathbb{PT}$, are Euclidean real.

Consider the antisymmetric bi-twistor

$$Z^{\alpha\beta} = Z_1^{[\alpha} Z_2^{\beta]} \,, \tag{2.4.48}$$

where Z_1 and Z_2 are distinct points in $\mathbb{PT}$. Let us assume that both of the points lie on the same complex projective line, corresponding to the point $x^{a\dot{a}}$ in complexified Minkowski spacetime, i.e. $Z_1^\alpha = (x^{a\dot{a}}\pi_a, \pi_b)$ and $Z_2^\alpha = (x^{a\dot{a}}\lambda_\alpha, \lambda_b)$, then explicitly the bi-vector reads

$$Z^{\alpha\beta} = \langle \pi\lambda \rangle \begin{pmatrix} \frac{1}{2}\epsilon^{\dot{a}\dot{b}}x^2 & x_b^{\dot{a}} \\ -x_a^{\dot{b}} & \epsilon_{ab} \end{pmatrix} \,. \tag{2.4.49}$$

From this, one sees the bi-vector exactly encodes the information of the projective lines between the two points, up to a scaling factor $\langle \pi\lambda \rangle$. The complex projective lines described by $Z^{\alpha\beta}$ generically will not be invariant under the action of Euclidean conjugation. However, we can define the following quantity that is invariant under Euclidean conjugation:

$$Z_{\mathbb{E}}^{\alpha\beta} = Z^{[\alpha} \hat{Z}^{\beta]} \,. \tag{2.4.50}$$

These lines are ‘real’ with respect to Euclidean conjugation and will form the collection of lines in twistor space that correspond to the points in Euclidean spacetime $\mathbb{R}^4$. As such we have found that Euclidean twistor space is obtained by picking out the ‘Euclidean real’ complex projective lines in $\mathbb{PT}$. By the twistor correspondence, the collection of these Riemann spheres are equivalent to $\mathbb{R}^4$. One concludes that to describe a point in Euclidean twistor space amounts to picking which ‘real’ sphere you are on in $\mathbb{PT}$ (i.e. your point on $\mathbb{R}^4$) and your point on that sphere, i.e.

$$\mathbb{PT}_{\mathbb{E}} \cong \mathbb{R}^4 \times \mathbb{CP}^1 \,. \tag{2.4.51}$$

We have also shown that $\mathbb{PT}_{\mathbb{E}}$ is a $\mathbb{CP}^1$ fibration over $\mathbb{R}^4$, with the $\mathbb{CP}^1_x$ fibre being the collection of points in $\mathbb{PT}_{\mathbb{E}}$ corresponding to $x \in \mathbb{R}^4$ via the twistor correspondence. One can show the explicit form of the fibration by solving the incidence relations for Euclidean reality conditions

$$x^{a\dot{a}} = \frac{\hat{\omega}^{\dot{a}}\pi^a - \omega^{\dot{a}}\hat{\pi}^a}{\langle \pi\hat{\pi} \rangle} \,. \tag{2.4.52}$$

An interesting feature of the Euclidean twistor correspondence is the fact that the projective spin bundle and twistor space are diffeomorphic to one another. The equivalence is not true at the level of the complex category however. As a complex manifold, $\mathbb{PT}_{\mathbb{E}}$, has coordinates given by $(\omega^{\dot{a}}, \pi_a) = (x^{b\dot{a}}\pi_b, \pi_a)$ and as such is equivalent to the holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$. Given this one can equally use the holomorphic variables $(\omega^{\dot{a}}, \pi_\alpha)$ or the non-holomorphic variables $(x^{a\dot{a}}, \pi_b)$. We will work primarily with the latter as it allows us to easily relate our calculations on twistor space to their implications for the spacetime physics. We use the following choice of frame for $(0,1)$-vectors fields in $\mathbb{PT}$ adapted to the non-holomorphic coordinates $(x^{a\dot{a}}, \pi_b)$,

$$\bar{\partial}_0 = -\langle \pi\hat{\pi}\rangle\, \pi_a \frac{\partial}{\partial \hat{\pi}_a} \ , \quad \bar{\partial}_{\dot{a}} = \pi^a \partial_{a\dot{a}} \ , \tag{2.4.53}$$

where $\partial_{a\dot{a}} = \partial/\partial x^{a\dot{a}}$. Using the incidence relations $\omega^{\dot{a}} = x^{a\dot{a}}\pi_a$, one can check that these are in the span of the canonical choice $\{\partial/\partial\hat{\omega}^{\dot{a}}, \partial/\partial\hat{\pi}_a\}$. One may question why we have factors of $\langle \pi\hat{\pi}\rangle$ appearing in the definition of our frame. Such factors allow us to work with a basis of vectors that have homogeneity 0 in $\hat{\pi}_a$. That is, instead of using bona fide vector fields on $\mathbb{PT}_{\mathbb{E}}$, we adopt $(0,1)$-vector fields valued in the line bundles $\mathcal{O}(k)$ to give a convenient global representative of $T^{(0,1)}\mathbb{PT}_{\mathbb{E}}$. Clearly $\bar{\partial}_0 \in T^{(0,1)}\mathbb{PT}_{\mathbb{E}} \otimes \mathcal{O}(2)$ and $\bar{\partial}_{\dot{a}} \in T^{(0,1)}\mathbb{PT}_{\mathbb{E}} \otimes \mathcal{O}(1)$. In a similar vein the $(1,0)$-vector fields are spanned by

$$\partial_0 = \frac{\hat{\pi}_a}{\langle \pi\hat{\pi}\rangle} \frac{\partial}{\partial \pi_a} \ , \quad \partial_{\dot{a}} = -\frac{\hat{\pi}^a}{\langle \pi\hat{\pi}\rangle}\partial_{a\dot{a}} \ , \tag{2.4.54}$$

valued in $\mathcal{O}(-2)$ and $\mathcal{O}(-1)$ respectively. This choice of frame is perhaps peculiar as the vectors no longer commute with one another, in particular

$$[\bar{\partial}_0, \partial_{\dot{a}}] = \bar{\partial}_{\dot{a}} \ , \qquad [\bar{\partial}_{\dot{a}}, \partial_0] = \partial_{\dot{a}} \ . \tag{2.4.55}$$

We also introduce the dual basis of (1,0)-forms and (0,1)-forms as

$$\begin{aligned} e^0 &= \langle \pi d\pi\rangle \ , \qquad & e^{\dot{a}} &= \pi_a dx^{a\dot{a}} \ , \\ \bar{e}^0 &= \frac{\langle \hat{\pi} d\hat{\pi}\rangle}{\langle \pi\hat{\pi}\rangle^2} \ , \qquad & \bar{e}^{\dot{a}} &= \frac{\hat{\pi}_a dx^{\dot{a}a}}{\langle \pi\hat{\pi}\rangle} \ . \end{aligned} \tag{2.4.56}$$

The one form e^0, being a nowhere vanishing global section, trivialises the canonical bundle of $\mathbb{CP}^1$ and identifies it with $\mathcal{O}(-2)$.

Introducing the Dolbeault operators defined by

$$\partial = e^0\partial_0 + e^{\dot{\alpha}}\partial_{\dot{\alpha}} \ , \quad \bar{\partial} = \bar{e}^0\bar{\partial}_0 + \bar{e}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}} \ . \tag{2.4.57}$$

One can show that this basis of one-forms satisfies the following structure equations

$$\bar{\partial}e^{\dot{\alpha}} = e^0 \wedge \bar{e}^{\dot{\alpha}} \ , \quad \partial\bar{e}^{\dot{\alpha}} = e^{\dot{\alpha}} \wedge \bar{e}^0 \ . \tag{2.4.58}$$

These relations are vital to note when performing calculations in this choice of $\mathbb{CP}^1$-dependent frame on $\mathbb{PT}$.

2.4.7 The Ward Transform

Having set up the necessary twistorial machinery, let us now watch it pay dividends in the form of the Ward correspondence. The Ward correspondence establishes a one-to-one relationship between anti-self-dual connections on $\mathbb{R}^4$ and special holomorphic vector bundles on $\mathbb{PT}$ which are topologically trivial upon restriction to each projective line. The observation at the heart of the correspondence is that the curvature of an ASD connection necessarily vanishes upon restriction to an α-plane. Let us start with the following facts: an α-plane is the space spanned by those tangent vectors at a point in $\mathbb{C}^4$ of the form $V^{a\dot{a}} = \lambda^{\dot{a}}\pi^a$, for an arbitrary spinor $\lambda^{\dot{a}}$. As such, translations between points within an α-plane are of the form,

$$x^{a\dot{a}} \to x^{a\dot{a}} + \lambda^{\dot{a}}\pi^a \ ,$$

and are generated by the operator

$$\partial_{\dot{a}} = \pi^a\partial_{a\dot{a}} \ . \tag{2.4.59}$$

The curvature two-form has components

$$F_{a\dot{a}b\dot{b}} = \partial_{a\dot{a}}A_{b\dot{b}} - \partial_{b\dot{b}}A_{a\dot{a}} + [A_{a\dot{a}}, A_{b\dot{b}}]. \tag{2.4.60}$$

Due to the antisymmetry when exchanging the pair of indices $(a\dot{a}) \leftrightarrow (b\dot{b})$, one can show the components of F decompose as

$$F_{a\dot{a}b\dot{b}} = \epsilon_{ab}\tilde{\Phi}_{\dot{a}\dot{b}} + \epsilon_{\dot{a}\dot{b}}\Phi_{ab} \ . \tag{2.4.61}$$

Similarly the 4d Levi-Civita symbol is translated into 2-spinor notation as

$$\epsilon^{\mu\nu\rho\sigma} \leftrightarrow \epsilon^{ac}\epsilon^{bd}\epsilon^{\dot{a}\dot{d}}\epsilon^{\dot{b}\dot{c}} - \epsilon^{ad}\epsilon^{bc}\epsilon^{\dot{a}\dot{c}}\epsilon^{\dot{b}\dot{d}} \ , \tag{2.4.62}$$

where the spinor indices correspond to spacetime indices as $(a\dot{a}) \to \mu$, $(b\dot{b}) \to \nu$, $(c\dot{c}) \to \rho$ and $(d\dot{d}) \to \sigma$. Using this we can show that taking the Hodge star of F results in

$$(*F)_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu}{}_{\rho\sigma}F_{\mu\nu} \leftrightarrow \epsilon_{cd}\tilde{\Phi}_{\dot{c}\dot{d}} - \epsilon_{\dot{c}\dot{d}}\Phi_{cd} \ . \tag{2.4.63}$$

We conclude $\tilde{\Phi}_{\dot{a}\dot{b}}$ has eigenvalue +1 under Hodge duality and so corresponds to the self-dual component of F and Φ_{ab} has eigenvalue -1 under Hodge duality and so corresponds to the anti-self-dual component of F. As such, an ASD connection is such that the self-dual component $\tilde{\Phi}_{\dot{a}\dot{b}}$ of its curvature vanishes, i.e. $F_{a\dot{a}b\dot{b}} = \epsilon_{\dot{a}\dot{b}}\Phi_{ab}$.

Defining the gauge covariant derivative along an α-plane in an analogous fashion as

$$D_{\dot{a}} = \pi^a D_{a\dot{a}} = \pi^a(\partial_{a\dot{a}} + A_{a\dot{a}}) \ , \tag{2.4.64}$$

for an ASD connection one has

$$[D_{\dot{a}}, D_{\dot{b}}] = \pi^a\pi^b\left[\partial_{a\dot{a}} + A_{a\dot{a}}, \partial_{b\dot{b}} + A_{b\dot{b}}\right] = \pi^a\pi^b F_{a\dot{a}b\dot{b}} = \pi^a\pi^b\epsilon_{ab}\Phi_{\dot{a}\dot{b}} = 0 \ , \tag{2.4.65}$$

which confirms that the curvature of an ASD connection vanishes on restriction to an α-plane.

Now, consider a complex vector bundle E, with a connection ∇. For a complex vector bundle to be holomorphic we require the (0,1)-component of the connection $\pi_{(0,1)}\nabla = \bar{\partial}_{\mathcal{A}} = \bar{\partial} + \bar{\mathcal{A}}$ to satisfy the integrability condition

$$\bar{\partial}^2_{\mathcal{A}} = 0 = \bar{\partial}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \bar{\mathcal{A}} := \bar{\mathcal{F}}(\bar{\mathcal{A}}) \ . \tag{2.4.66}$$

Working explicitly with $\mathbb{PT}_{\mathbb{E}}$ our (0,1)-connection decomposes into the basis given by

(2.4.56) as

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_0 \bar{e}^0 + \bar{\mathcal{A}}_{\dot{a}} \bar{e}^{\dot{a}} \ . \tag{2.4.67}$$

The partial connection $\bar{\mathcal{A}}$ is a section of $\mathcal{O}(0)$, as such one concludes the components $\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}_{\dot{a}}$ are valued in $\mathcal{O}(2)$ and $\mathcal{O}(1)$ respectively. The components of the (0,2)-curvature are given by

$$\bar{\mathcal{F}}_{0\dot{a}} = \bar{\partial}_0 \bar{\mathcal{A}}_{\dot{a}} - \bar{\partial}_{\dot{a}} \bar{\mathcal{A}}_0 + [\bar{\mathcal{A}}_0, \bar{\mathcal{A}}_{\dot{a}}] \ , \tag{2.4.68}$$

$$\bar{\mathcal{F}}_{\dot{a}\dot{b}} = \bar{\partial}_{\dot{a}} \bar{\mathcal{A}}_{\dot{b}} - \bar{\partial}_{\dot{b}} \bar{\mathcal{A}}_{\dot{a}} + [\bar{\mathcal{A}}_{\dot{a}}, \bar{\mathcal{A}}_{\dot{b}}] \ , \tag{2.4.69}$$

with both components vanishing when E is a holomorphic vector bundle.

Consider now that our bundle E, further to being holomorphic is also holomorphically trivial upon restriction to any complex line in $\mathbb{PT}$. We can introduce a holomorphic frame that trivialises the bundle over the $\mathbb{CP}^1$. What do we mean by this? Well firstly, restricting to a complex line corresponding to the point $x \in \mathbb{R}^4$ is executed by setting $\omega^{\dot{a}} = x^{a\dot{a}} \pi_a$ in all our expressions. Secondly, the fact that $E \mid_{\mathbb{CP}^1_x}$ is holomorphically trivial is tantamount to finding a gauge transformation $\hat{g}(\pi, x) \in C^\infty(\mathbb{CP}^1_x, G)$ such that

$$\hat{g}^{-1} \circ (\bar{\partial} + \bar{\mathcal{A}}) \mid_{\mathbb{CP}^1_x} \circ \, \hat{g} = \bar{\partial} \mid_{\mathbb{CP}^1_x} \ , \tag{2.4.70}$$

where we have used $\circ$ to emphasise the equality of the left and right hand side when acting on sections of E. On the left, we first act on a section by $\hat{g}$, then $\bar{\partial} + \bar{\mathcal{A}}$, then finally by $\hat{g}^{-1}$. Explicitly calculating the right hand side of (2.4.70) by acting on a generic section, one finds $\hat{g}(\pi, x)$ needs to satisfy

$$(\bar{\partial} + \bar{\mathcal{A}}) \, \hat{g}|_{\mathbb{CP}^1_x} = 0 \ . \tag{2.4.71}$$

Given this, the above tells us that, under this gauge transformation, covariantly holomorphic objects become simply holomorphic objects. It is in this sense that $\hat{g}$ is said to define a choice of holomorphic frame for $E \mid_{\mathbb{CP}^1_x}$. We note that (2.4.71) tells us our holomorphic frame is uniquely defined up to a gauge transformation $\hat{g}(\pi, x) \to h(\pi, x)\hat{g}(\pi, x)$, where h is globally holomorphic on $\mathbb{CP}^1_x$ and hence a constant by Liouville's theorem.

Solutions to equations of the form (2.4.71) are well known, and are given by path-ordered exponentials. In this case the solution is of the form

$$\hat{g}(x,\pi) = \mathcal{P}\exp\left(-\int_{\mathbb{CP}^1_x} \theta \wedge \bar{\mathcal{A}}\right) , \tag{2.4.72}$$

where θ is an $\mathcal{O}(0)$-valued meromorphic $(1,0)$-form on $\mathbb{CP}^1$. The quantity (2.4.72) is called a holomorphic Wilson line [BS11], an object analogous to the familiar Wilson line in gauge theory, however adapted for the complex category. This raises an interesting consideration, complex numbers have no notion of ordering, so how is the path ordering in (2.4.72) defined? For the real category, we have the following formal expression for path ordering exponentials,

$$\begin{aligned}
&\mathcal{P}\exp\left(-\int_{x_0}^{x} A\right) \\
&\quad = 1_R - \int_{x_0}^{x} \mathrm{d}y^\mu\, A_\mu(y) + \int_{x_0}^{x} \mathrm{d}y^\mu\, A_\mu(y) \int_{x_0}^{y} \mathrm{d}y'^\nu\, A_\nu(y') - \cdots \\
&\quad = 1_R + \sum_{m=1}^{\infty} (-1)^m \int_{([0,1])^m} \mathrm{d}s_m \cdots \mathrm{d}s_1\, A_\mu(s_m) \frac{\mathrm{d}x^\mu}{\mathrm{d}s_m} \Theta(s_m - s_{m-1}) \cdots \Theta(s_2 - s_1)\, A_\nu(s_1) \frac{\mathrm{d}x^\nu}{\mathrm{d}s_1} ,
\end{aligned} \tag{2.4.73}$$

where we have introduced the Heaviside step function Θ defined by,

$$\Theta(s - s') = \begin{cases} 1, & s > s', \\ 0, & s < s' . \end{cases} \tag{2.4.74}$$

The above are Green's functions for the de-Rham differential satisfying

$$\frac{\partial}{\partial s}\,\Theta(s - s') = \delta(s - s') . \tag{2.4.75}$$

To proceed in analogy to the real Wilson line above, let us consider the following form for the meromorphic $(1,0)$-form θ,

$$\theta = \frac{\langle \alpha\beta \rangle}{\langle \pi\alpha \rangle \langle \pi\beta \rangle} \frac{e^0}{2\pi i}. \tag{2.4.76}$$

θ has residue $+1$ at $\pi = \alpha$ and -1 at $\pi = \beta$, given this one can then show

$$\bar{\partial}\theta = \left(\delta^{(2)}(\langle \pi\alpha \rangle) - \delta^{(2)}(\langle \pi\beta \rangle)\right)\, \bar{e}^0 \wedge e^0 . \tag{2.4.77}$$

We have thus established that θ provides a holomorphic generalisation of the Green's function Θ. As such, we can define path ordering in our holomorphic Wilson lines using the following formula,

$$\mathcal{P}\exp\left(-\int_{\mathbb{CP}^1}\theta\wedge\mathcal{A}\right)\equiv 1_R+\sum_{m=1}^{\infty}(-1)^m\int_{(\mathbb{CP}^1)^m}\bigwedge_{i=1}^{m}\left(\theta(\lambda_i)\wedge\mathcal{A}(\lambda_i)\right), \tag{2.4.78}$$

where λ_i is the homogenous coordinate for the i^{th} factor of $\mathbb{CP}^1$ in the integral. For a holomorphic Wilson line parallel transporting from α to π the path ordering gives

$$\bigwedge_{i=1}^{m}\theta(\lambda_i)=\frac{\langle\pi\alpha\rangle}{\langle\pi\lambda_m\rangle\cdots\langle\lambda_2\lambda_1\rangle\langle\lambda_1\alpha\rangle}\frac{\langle\lambda_m d\lambda_m\rangle}{2\pi i}\wedge\cdots\wedge\frac{\langle\lambda_1 d\lambda_1\rangle}{2\pi i}\ . \tag{2.4.79}$$

One can equivalently write (2.4.71) as

$$\bar{\mathcal{A}}\mid_{\mathbb{CP}^1_x}=-\bar{\partial}\hat{g}\hat{g}^{-1}\mid_{\mathbb{CP}^1_x}\ . \tag{2.4.80}$$

As such, we have demonstrated that for connections on E we can find a gauge transformation for our connections such that

$$\bar{\partial}+\bar{\mathcal{A}}=\hat{g}(\bar{\partial}+\bar{\mathcal{A}}')\hat{g}^{-1}\ ,\ \text{where}\quad \iota_x^*\bar{\mathcal{A}}'=0\ , \tag{2.4.81}$$

with $\iota_x:\mathbb{CP}^1_x\hookrightarrow\mathbb{PT}_{\mathbb{E}}$ being the inclusion mapping, defined such that, when composed with the projection mapping $\pi:\mathbb{PT}_{\mathbb{E}}\to\mathbb{E}^4$ we have $\pi\circ\iota_x=x\in\mathbb{E}^4$. It then follows that in components $\bar{\mathcal{A}}'=\bar{\mathcal{A}}'_{\dot{a}}\bar{e}^{\dot{a}}$. In the 'gauge' defined by the holomorphic frame, the curvature satisfies

$$\bar{\mathcal{F}}(\bar{\mathcal{A}})=\hat{g}\bar{\mathcal{F}}(\bar{\mathcal{A}}')\hat{g}^{-1}\ . \tag{2.4.82}$$

The holomorphicity condition given by equation (2.4.68) vanishing becomes

$$\bar{\partial}_0\bar{\mathcal{A}}'_{\dot{a}}=0\ . \tag{2.4.83}$$

Now as $\bar{\mathcal{A}}'_{\dot{a}}$ is valued in $\mathcal{O}(1)$ and holomorphic we have, that

$$\bar{\mathcal{A}}'_{\dot{a}}=\pi^a A_{a\dot{a}}(x), \tag{2.4.84}$$

where $A_{a\dot{a}}$ is a function of $\mathbb{R}^4$ only. With this one finds that by plugging in (2.4.84) into (2.4.69) one attains the condition (2.4.65), and so we have shown that $A_{a\dot{a}}$ is indeed an ASD-connection on $\mathbb{R}^4$. So to conclude, given a holomorphic vector bundle, trivial upon each $\mathbb{CP}^1$, we can construct solutions to the anti-self-dual Yang-Mills equations. It can be shown that this is indeed a one-to-one correspondence, with anti-self-dual connections defining holomorphic vector bundles over $\mathbb{PT}$ [War77].

2.5 Holomorphic Chern-Simons on $\mathbb{PT}_{\mathbb{E}}$

In this section we introduce 6d holomorphic Chern-Simons theory (6dhCS). In much the same way many integrable systems can be understood as reductions of the ASDYM equations, 4dCS theory can be understood as a reduction of 6dhCS [BS23]. This observation suggests an intriguing paradigm: both 4dCS and ASDYM are gauge theories in four dimensions that give rise to two-dimensional integrable field theories. This naturally raises the question; can six-dimensional holomorphic Chern–Simons theory serve as a unifying framework that connects these two approaches, thereby revealing a deeper equivalence in their capacity to describe two-dimensional integrable systems?

The action of holomorphic Chern-Simons is given by

$$S_{\mathrm{hCS}_6} = \frac{1}{2\pi i}\int_{\mathbb{PT}_{\mathbb{E}}} \Omega \wedge \mathrm{hCS}(\bar{\mathcal{A}}) \tag{2.5.1}$$

where the holomorphic-Chern-Simons (0,3)-form is given by,

$$\mathrm{hCS}(\bar{\mathcal{A}}) = \mathrm{Tr}(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{1}{3}\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}})\ , \tag{2.5.2}$$

the $(3,0)$-form Ω is defined by,

$$\Omega = \Phi\, e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}}\ , \ \text{for} \quad \Phi \in \mathcal{O}(-4)\ ,$$

and Tr denotes the ad-invariant bilinear form on $\mathfrak{g}$. In much of the previous literature where an action of the form (2.5.1) has appeared [Wit04; BMS07], Ω was typically defined as a nowhere-vanishing holomorphic $(3,0)$-form—or, more precisely, as a nowhere-vanishing section of the holomorphic canonical bundle. Since the existence of a global holomorphic section of a bundle implies the holomorphic triviality of the bundle itself, the existence of such an Ω is equivalent to the underlying manifold being Calabi–Yau. However, twistor space is not Calabi–Yau, as its canonical bundle is isomorphic to $\mathcal{O}(-4)$. Consequently, to adapt (2.5.1) for $\mathbb{PT}$, we require Ω to be a meromorphic $(3,0)$-form on $\mathbb{PT}$ which, in this work, is assumed to be nowhere vanishing [Cos20]. Schematically, this process can be understood as defining a non-compact Calabi-Yau 3-fold by excising the poles of Ω from $\mathbb{PT}$.

The equations of motion of (2.5.1) are given by

$$\bar{\mathcal{F}}(\bar{\mathcal{A}}) = \bar{\partial}\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \bar{\mathcal{A}} = 0 \ . \tag{2.5.3}$$

That is the equations of motion define a holomorphic G-bundle over $\mathbb{PT}$. In section 2.4.7, we highlighted the fact that holomorphic vector bundles, trivial over each $\mathbb{CP}^1$ subspace of $\mathbb{PT}$, are equivalent to solutions to the ASDYM equations. As such, given we can find a holomorphic frame $\hat{g}$ such that

$$\hat{g} \circ (\bar{\partial} + \bar{\mathcal{A}}) \,|_{\mathbb{CP}^1_x} \circ \hat{g}^{-1} = \bar{\partial} \,|_{\mathbb{CP}^1_x} \ , \tag{2.5.4}$$

we should expect that the action (2.5.1) could be used to describe anti-self-dual connections. We will explore this notion through an example.

2.5.1 WZW$_4$ action from hCS$_6$

Consider the case where the meromorphic $(3,0)$-form, Ω, has two double poles at the points $\alpha, \beta \in \mathbb{CP}^1$, so that

$$\Omega = \frac{1}{2}\Phi(\pi)\, \epsilon_{\dot{a}\dot{b}}\, \pi_a dx^{a\dot{a}} \wedge \pi_b dx^{b\dot{b}} \wedge \langle \pi d\pi \rangle \ , \qquad \Phi = \frac{\langle \alpha\beta \rangle^2}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} \ . \tag{2.5.5}$$

The poles of Ω in $\mathbb{CP}^1$ play the role of boundaries in hCS$_6$ because total derivatives pick up a contribution from $\bar{\partial}\Omega$ which is a distribution with support at these poles. To ensure a well-defined variational principle, we impose boundary conditions on the gauge field at these poles given by

$$\bar{\mathcal{A}}|_{\pi=\alpha} = 0 \ , \qquad \bar{\mathcal{A}}|_{\pi=\beta} = 0 \ . \tag{2.5.6}$$

Turning to the symmetries of this model, the theory is invariant under gauge transformations acting as

$$\hat{\gamma} : \quad \bar{\mathcal{A}} \mapsto (\bar{\mathcal{A}})^{\hat{\gamma}} = \hat{\gamma}^{-1}\bar{\mathcal{A}}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma} \ , \tag{2.5.7}$$

so long as they preserve the boundary conditions. This implies restrictions on the allowed external transformations at the poles of Ω, which are given by

$$\pi^a \partial_{a\dot{a}} \hat{\gamma}|_{\pi=\alpha} = 0 \ , \quad \pi^a \partial_{a\dot{a}} \hat{\gamma}|_{\pi=\beta} = 0 \ . \tag{2.5.8}$$

2.5.2 Localisation of hCS$_6$ with double poles to WZW$_4$

Proceeding in analogy to 4dCS, we will parametrise our holomorphic connection in such a way that highlights the fact that the holomorphic frame $\hat{g}$ trivialises our holomorphic vector bundle over $\mathbb{CP}^1$,

$$\bar{\mathcal{A}} = (\bar{\mathcal{A}}')^{\hat{g}} = \hat{g}^{-1} \bar{\mathcal{A}}' \hat{g} + \hat{g}^{-1} \bar{\partial} \hat{g} \ . \tag{2.5.9}$$

As such, one would expect the holomorphic frame $\hat{g}$ to act as the edge modes in our theory, localising to the dynamical fields of the theory in $\mathbb{R}^4$. Expressing the action $S_{\mathrm{hCS}_6}[\bar{\mathcal{A}}]$ in terms of the fields $\mathcal{A}'$ and $\hat{g}$ one obtains

$$\begin{aligned} S_{\mathrm{hCS}_6}[\bar{\mathcal{A}}] = S_{\mathrm{hCS}_6}[\bar{\mathcal{A}}'] &+ \frac{1}{2\pi i} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\bar{\mathcal{A}}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) \\ &- \frac{1}{6\pi i} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1} d\hat{g} \wedge \hat{g}^{-1} d\hat{g} \wedge \hat{g}^{-1} d\hat{g}\big) \ , \end{aligned} \tag{2.5.10}$$

where, with a slight abuse of notation, we are also denoting by $\hat{g}$ a smooth homotopy to a constant map in the last term (this will be an indiscretion we continue to make). Notably, the edge mode $\hat{g}$ only appears in this action against the 4-form $\bar{\partial}\Omega$, which is a distribution with support at the poles of Ω. This means that the action only depends on $\hat{g}$ and its $\mathbb{CP}^1$-derivative $\hat{u} := \hat{g}^{-1}\partial_0 \hat{g}$ through their values at the poles of Ω, which we denote by

$$\hat{g}|_{\pi=\alpha} = g \ , \qquad \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\alpha} = u \ , \qquad \hat{g}|_{\pi=\beta} = \tilde{g} \ , \qquad \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\beta} = \tilde{u} \ . \tag{2.5.11}$$

Let us consider the symmetries of the theory in this new parametrisation. The gauge transformation (2.5.7) acts trivially on $\mathcal{A}'$ while $\hat{g}$ transforms with a right action as

$$\hat{\gamma}: \quad \bar{\mathcal{A}}' \mapsto \bar{\mathcal{A}}' \ , \qquad \hat{g} \mapsto \hat{g}\hat{\gamma} \ . \tag{2.5.12}$$

In addition to the external gauge symmetry, the new parametrisation has introduced a redundancy, which we dub an internal gauge symmetry, acting as

$$\check{\gamma}: \quad \bar{\mathcal{A}}' \mapsto \check{\gamma}^{-1}\bar{\mathcal{A}}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma} \ , \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g} \ . \tag{2.5.13}$$

We can exploit these symmetries to impose gauge fixing conditions on the fields $\mathcal{A}'$ and $\hat{g}$. Let us fix $\bar{\mathcal{A}}'$ such that it has no $\mathbb{CP}^1$ leg, and fix the value of $\hat{g}$ at $\pi = \beta$ to the identity.[3] The surviving edge mode at the other pole $g = \hat{g}|_{\pi=\alpha}$ will become the fundamental field of WZW$_4$.

Returning to the action (2.5.10), the first term is a genuine six-dimensional bulk term that we eliminate by going on-shell. We find the bulk equation of motion $\bar{\partial}_0 \bar{\mathcal{A}}'_{\dot{a}} = 0$, which implies that these components are holomorphic. Using the fact that $\bar{\mathcal{A}}'_{\dot{a}} \in \mathcal{O}(1)$ this may be solved in terms of $\mathbb{CP}^1$-independent components $A'_{a\dot{a}}$ as

$$\bar{\mathcal{A}}' = \pi^a A_{a\dot{a}} \bar{e}^{\dot{a}} \ , \qquad \bar{e}^{\dot{a}} = \frac{\hat{\pi}_a dx^{a\dot{a}}}{\langle \pi\hat{\pi}\rangle} \ . \tag{2.5.14}$$

This completely specifies the $\mathbb{CP}^1$-dependence of $\bar{\mathcal{A}}'$, and the boundary conditions (2.5.6) may be solved to determine $A'_{a\dot{a}}$ in terms of g,

$$A_{a\dot{a}} = -\frac{\beta_a \alpha^b}{\langle \alpha\beta\rangle} \partial_{b\dot{a}} g g^{-1} \ . \tag{2.5.15}$$

From these components, we can construct a 4d connection $A = A_{a\dot{a}} dx^{a\dot{a}}$. This parametrisation of $A_{a\dot{a}}$ in terms of g we identify as Yang's parametrisation (2.3.8) with g being identified with Yang's matrix. This solution for $\bar{\mathcal{A}}'$ may now be substituted into the action and the integral over $\mathbb{CP}^1$ can be computed explicitly. The second and third terms of (2.5.10) localise to a four-dimensional action and we land on the WZW$_4$ theory defined by

$$S_{\mathrm{WZW}_4} = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}\big(g^{-1}dg \wedge \star g^{-1}dg\big) + \int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathrm{WZ}[g] \ . \tag{2.5.16}$$

[3] At this point, we may further fix the $\mathbb{CP}^1$-derivative of $\hat{g}$ at both $\pi = \alpha$ and $\pi = \beta$ to zero. However, these fields drop out of the action anyway without specifying this.

In the second term, we have introduced a 2-form defined by

$$\omega_{\alpha,\beta} = \frac{1}{\langle\alpha\beta\rangle}\alpha_a\beta_b\,\epsilon_{\dot{a}\dot{b}}\,dx^{a\dot{a}}\wedge dx^{b\dot{b}}\ , \tag{2.5.17}$$

and the WZ 3-form

$$\mathrm{WZ}[g] = \frac{1}{3}\mathrm{Tr}\big(\bar{g}^{-1}d\bar{g}\wedge\bar{g}^{-1}d\bar{g}\wedge\bar{g}^{-1}d\bar{g}\big)\ , \tag{2.5.18}$$

defined, as usual, using a suitable extension $\bar{g}$ of g.

The equations of motion of this theory are given by

$$d\left(\star-\omega_{\alpha,\beta}\wedge\right)dgg^{-1}=0 \quad\Leftrightarrow\quad \epsilon^{\dot{a}\dot{b}}\beta^a\partial_{a\dot{a}}\left(\alpha^b\partial_{b\dot{b}}gg^{-1}\right)=0\ . \tag{2.5.19}$$

The six-dimensional gauge transformations (constrained by boundary conditions) descend to semi-local symmetries of the action (2.5.16), which act as

$$g\to\gamma_L^{-1}\cdot g\cdot\gamma_R\ ,\qquad \alpha^a\partial_{a\dot{a}}\gamma_R=0\ ,\qquad \beta^a\partial_{a\dot{a}}\gamma_L=0\ , \tag{2.5.20}$$

where $\gamma_L=\hat{\gamma}|_\beta$ and $\gamma_R=\hat{\gamma}|_\alpha$. Of particular interest is the case where $\beta=\hat{\alpha}$, i.e. the poles of Ω are antipodal on $\mathbb{CP}^1$, in which case $\omega_{\alpha,\hat{\alpha}}=\varpi$ is proportional to the Kähler form on $\mathbb{R}^4$. Here, we are referring to the Kähler form with respect to the complex structure $\mathcal{J}_\alpha$ that is defined by the point $\alpha\in\mathbb{CP}^1$. In this case, the semi-local symmetries can be interpreted as a holomorphic left action and anti-holomorphic right action (akin to the 2d WZW current algebra).

2.5.3 Interpretation as ASDYM

A 4d Yang-Mills connection A' with curvature $F[A']=\mathrm{d}A'+A'\wedge A'$ is said to be anti-self dual if it obeys $F=-\star F$. After converting to bi-spinor notation, the anti-self-dual Yang-Mills (ASDYM) equations can be expressed as

$$\pi^a\pi^bF_{a\dot{a}b\dot{b}}=0\qquad\forall\,\pi_a\in\mathbb{CP}^1\ . \tag{2.5.21}$$

This contains three independent equations that can be extracted by introducing basis spinors α_a and β_a satisfying $\langle\alpha\beta\rangle\neq0$. The three independent equations are then expressed in terms of contractions with these basis spinors as

$$\alpha^a\alpha^b F_{a\dot{a}b\dot{b}} = 0 \;, \tag{2.5.22}$$

$$\beta^a\beta^b F_{a\dot{a}b\dot{b}} = 0 \;, \tag{2.5.23}$$

$$(\alpha^a\beta^b + \beta^a\alpha^b) F_{a\dot{a}b\dot{b}} = 0 \;. \tag{2.5.24}$$

The six-dimensional origin of WZW_4 (and indeed all IFT_4 constructed in this way) ensures that the connection A' introduced in the previous section satisfies the ASDYM equation when evaluated on solutions to the WZW_4 equations of motion. This follows from the six-dimensional equation $\Omega\wedge\mathcal{F}[\mathcal{A}'] = 0$, which encodes both the holomorphicity of $\mathcal{A}'$ and eq. (2.5.21). To see this explicitly for WZW_4 where the connection A' is given by eq. (2.5.15), we note that the β-contracted eq. (2.5.23) holds because $\langle\beta\beta\rangle = 0$, and the α-contracted eq. (2.5.22) holds due to the Maurer-Cartan identity. The remaining equation (2.5.24) yields the equations of motion of WZW_4 (2.5.19).

2.5.4 Reduction of WZW_4 to WZW_2

Next, we will apply a two-dimensional reduction to WZW_4 specified by two vector fields V_i on $\mathbb{R}^4$ with $i = 1, 2$. The idea of reduction is to restrict to field configurations that are invariant under the flow of these vector fields. The two-dimensional dynamics of the reduced theory will be specified by the Lagrangian $\mathcal{L}_{\mathrm{IFT}_2} = (V_1 \wedge V_2) \vee \mathcal{L}_{\mathrm{IFT}_4}$ where $\mathcal{L}_{\mathrm{IFT}_4}$ is the Lagrangian of the parent theory and we denote the contraction of a vector field V with a differential form X by $V \vee X$.

Let us introduce a pair of unit norm spinors γ_a and $\kappa_{\dot{a}}$ and define the basis of 1-forms on $\mathbb{R}^4$

$$\mathrm{d}z = \gamma_a\kappa_{\dot{a}}\mathrm{d}x^{a\dot{a}} \;, \qquad \mathrm{d}\bar{z} = \hat{\gamma}_a\hat{\kappa}_{\dot{a}}\mathrm{d}x^{a\dot{a}} \;, \qquad \mathrm{d}w = \gamma_a\hat{\kappa}_{\dot{a}}\mathrm{d}x^{a\dot{a}} \;, \qquad \mathrm{d}\bar{w} = -\hat{\gamma}_a\kappa_{\dot{a}}\mathrm{d}x^{a\dot{a}} \;. \tag{2.5.25}$$

These are adapted to the complex structure $\mathcal{J}_\gamma$ defined by $\gamma_a \in \mathbb{CP}^1$. We choose to reduce along the vector fields dual to $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ by demanding that $\partial_z g = \partial_{\bar{z}} g = 0$.[4] Contracting the WZW_4 Lagrangian with these vector fields results in the two-

[4]In this case for reality we have $\mathcal{L}_{\mathrm{IFT}_2} = \mathrm{i}(\partial_z \wedge \partial_{\bar{z}}) \vee \mathcal{L}_{\mathrm{IFT}_4}$.

dimensional action of a principal chiral model (PCM) plus Wess-Zumino (WZ) term:

$$S_{\mathrm{PCM}+k\mathrm{WZ}_2}[g] = \frac{1}{2}\int_\Sigma \mathrm{Tr}\big(g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big) \ + \ \frac{\mathrm{i}k}{3}\int_{\Sigma\times[0,1]} \mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g}\wedge\hat{g}^{-1}\mathrm{d}\hat{g}\wedge\hat{g}^{-1}\mathrm{d}\hat{g}) \ . \tag{2.5.26}$$

In this action, the relative coefficient between the WZ term and the PCM term is given by

$$k = \frac{\alpha+\beta}{\alpha-\beta}\ , \qquad \alpha = \frac{\langle\gamma\alpha\rangle}{\langle\alpha\hat{\gamma}\rangle}\ , \qquad \beta = \frac{\langle\gamma\beta\rangle}{\langle\beta\hat{\gamma}\rangle}\ . \tag{2.5.27}$$

Varying the basis spinor γ_a in these expressions changes the choice of reduction vector fields and parametrises a family of two-dimensional theories interpolating between WZW_2 and the PCM. The WZW_2 CFT limit is obtained when $k\to 1$ with $\alpha\beta$ held fixed. This can be achieved by starting at the Kähler point in 4d, with $\beta=\hat{\alpha}$, and choosing the reduction to be aligned with the complex structure, i.e. setting $\gamma=\alpha$. An alternative reduction that turns off the WZ term and recovers the PCM is achieved by setting $\beta=-\alpha$.

For general choices of reduction, the four-dimensional semi-local symmetries descend to a global $G_L\times G_R$ symmetry. This is because, for example, the conditions $\alpha^a\partial_{a\dot{a}}\gamma_R=0$ and $\partial_z\gamma_R=\partial_{\bar{z}}\gamma_R=0$ generically contain four independent constraints leaving only constant solutions. However, when the reduction is taken to the CFT point, this system of four constraints is not linearly independent, and chiral symmetries emerge satisfying $\partial_w\gamma_R=0$ (and vice versa for γ_L).

Lax connection. A virtue of this approach is that a $\mathfrak{g}^{\mathbb{C}}$-valued Lax connection for the dynamics of the resultant IFT_2 may be derived from the 4d connection A':

$$\begin{aligned}\mathcal{L}_{\bar{w}} &= \frac{1}{\langle\pi\hat{\gamma}\rangle}\hat{\kappa}^{\dot{a}}\pi^a(\partial_{a\dot{a}}+A'_{a\dot{a}}) = \partial_{\bar{w}} + \frac{(\beta-\zeta)}{(\alpha-\beta)}\partial_{\bar{w}}gg^{-1}\ ,\\ \mathcal{L}_{w} &= \frac{1}{\langle\pi\gamma\rangle}\kappa^{\dot{a}}\pi^a(\partial_{a\dot{a}}+A'_{a\dot{a}}) = \partial_{w} + \frac{\alpha(\beta-\zeta)}{\zeta(\alpha-\beta)}\partial_{w}gg^{-1}\ ,\end{aligned} \tag{2.5.28}$$

where the spectral parameter is given by $\zeta=\frac{\langle\gamma\pi\rangle}{\langle\pi\hat{\gamma}\rangle}$. Flatness of this connection for all values of ζ is equivalent to the equations of motion of the PCM plus WZ term

$$\alpha\partial_{\bar{w}}(\partial_w gg^{-1}) - \beta\partial_w(\partial_{\bar{w}}gg^{-1}) = 0 \quad \Leftrightarrow \quad \mathrm{d}(\star-\mathrm{i}k)\mathrm{d}gg^{-1}=0\ . \tag{2.5.29}$$

Notice that in the CFT limit $k \to 1$ with $\beta \to \infty$, $\alpha \to 0$ the Lax connection becomes chiral and spectral parameter independent.

2.5.5 Reduction of hCS$_6$ to CS$_4$

Instead of first integrating over $\mathbb{CP}^1$ and then reducing to two dimensions, we could instead directly apply the reduction to hCS$_6$. This gives the action of CS$_4$,

$$S_{\text{CS}_4}[A] = \frac{1}{2\pi \text{i}} \int_{\Sigma \times \mathbb{CP}^1} \omega \wedge \text{Tr}\left(A \wedge \text{d}A + \frac{2}{3} A \wedge A \wedge A \right) . \tag{2.5.30}$$

Here Σ is the $\mathbb{R}^2 \subset \mathbb{R}^4$ with coordinates $w, \bar{w}$, and the meromorphic 1-form ω is given by

$$\omega = \text{i}(\partial_z \wedge \partial_{\bar{z}}) \vee \Omega . \tag{2.5.31}$$

A crucial feature here is that this contraction introduces zeroes in ω to complement its poles, as required by the Riemann-Roch theorem. For the case at hand, ω is given explicitly by

$$\omega = \text{i} \frac{\langle \alpha\beta \rangle^2 \langle \pi\gamma \rangle \langle \pi\hat{\gamma} \rangle}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2} \langle \pi \text{d}\pi \rangle , \tag{2.5.32}$$

and the zeroes are introduced at the points $\pi_a = \gamma_a, \hat{\gamma}_a$. The details of the reduction show that, while our six-dimensional gauge field was regular, the connection A entering in CS$_4$ develops poles at the zeroes of ω. In particular, the component A_w will have a simple pole at $\pi_a = \gamma_a$ and $A_{\bar{w}}$ will have a simple pole at $\pi_a = \hat{\gamma}_a$. The four-dimensional Chern-Simons connection is subject to the same boundary conditions as its parent, namely it vanishes at the points $\alpha, \beta \in \mathbb{CP}^1$. The subsequent localisation of CS$_4$ then gives the same PCM plus WZ term derived by reducing WZW$_4$ as we have shown in subsection 2.2.2.

2.5.6 LMP action from hCS$_6$

Let us review the localisation of hCS$_6$ with a fourth-order pole. We start with the action and $(3,0)$-form defined by

$$S_{\text{hCS}_6}[\mathcal{A}] = \frac{1}{2\pi i} \int_{\mathbb{PT}} \Omega \wedge \text{CS}(\bar{\mathcal{A}}) , \qquad \Omega = k \, \frac{e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}}}{\langle \pi\alpha \rangle^4} . \tag{2.5.33}$$

As is usual in hCS$_6$, we impose boundary conditions on the gauge field $\mathcal{A}$ to ensure the vanishing of the boundary variation

$$\delta S_{\mathrm{hCS}_6}\big|_{\mathrm{bdry}} = \frac{1}{2\pi i}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\delta\bar{\mathcal{A}} \wedge \bar{\mathcal{A}}) \ . \tag{2.5.34}$$

Evaluating this integral is achieved by making use of the following localisation formula

$$\frac{1}{2\pi i}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge Q = \frac{k}{6}\int_{\mathbb{R}^4} \alpha_a\alpha_b\Sigma^{ab} \wedge \partial_0^3 Q\big|_{\alpha} \ . \tag{2.5.35}$$

Then, we find that the boundary variation vanishes if we impose the boundary conditions

$$\bar{\mathcal{A}}|_{\pi=\alpha} = 0 \quad \text{and} \quad \partial_0\bar{\mathcal{A}}|_{\pi=\alpha} = 0 \ . \tag{2.5.36}$$

Admissible gauge transformations. We now check which residual gauge symmetries survive once we impose our choice of boundary conditions. We proceed in the familiar fashion, introducing a new parametrisation of our gauge field $\bar{\mathcal{A}}$ as

$$\bar{\mathcal{A}} = \hat{g}^{-1}\bar{\mathcal{A}}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \ , \qquad \bar{\mathcal{A}}'_0 = 0 \ . \tag{2.5.37}$$

This parametrisation has both external and internal gauge symmetries, which act as

$$\begin{aligned} &\text{External} \quad \hat{\gamma}: \qquad \bar{\mathcal{A}} \mapsto \bar{\mathcal{A}}^{\hat{\gamma}} \ , \qquad \bar{\mathcal{A}}' \mapsto \bar{\mathcal{A}}' \ , \qquad \hat{g} \mapsto \hat{g}\hat{\gamma} \ , \\ &\text{Internal} \quad \check{\gamma}: \qquad \bar{\mathcal{A}} \mapsto \bar{\mathcal{A}} \ , \qquad \bar{\mathcal{A}}' \mapsto \bar{\mathcal{A}}'^{\check{\gamma}} \ , \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g} \ . \end{aligned} \tag{2.5.38}$$

The internal gauge transformations must satisfy $\bar{\partial}_0\check{\gamma} = 0$ to preserve the condition $\bar{\mathcal{A}}'_0 = 0$. These transformations leave $\bar{\mathcal{A}}$ invariant and as such they are fully compatible with the boundary conditions. We use the internal gauge symmetry to fix $\hat{g}|_{\pi=\alpha} = \mathrm{id}$. The story for the external gauge symmetries is slightly different; under external gauge transformations $\bar{\mathcal{A}} \mapsto \bar{\mathcal{A}}^{\hat{\gamma}}$ and so the value of $\bar{\mathcal{A}}$ at the poles is not necessarily invariant. Requiring our boundary conditions to be invariant under external gauge transformations imposes constraints on the admissible symmetries at $\pi = \alpha$. This limits the amount of symmetry available for gauge fixing. The gauge transformation of the first boundary condition reads

$$0 = \bar{\mathcal{A}}^{\hat{\gamma}}|_{\pi=\alpha} = \left(\hat{\gamma}^{-1}\bar{\mathcal{A}}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\right)|_{\pi=\alpha} \quad \Longrightarrow \quad \gamma^{-1}\alpha^a\partial_{a\dot{a}}\gamma = 0 \ , \tag{2.5.39}$$

where we have defined

$$\hat{\gamma}|_{\pi=\alpha} = \gamma \ .$$

Here, we have shown that at $\pi = \alpha$ the gauge transformations are restricted such that they are holomorphic on $\mathbb{R}^4$ with respect to the complex structure given by the point $\pi = \alpha$. Another way of stating this is that our admissible external gauge symmetries on $\mathbb{PT}$ localise to semi-local symmetries in the effective theory on $\mathbb{R}^4$. However, this restriction is derived from only one half of the boundary conditions. Introducing the notation

$$\hat{\mathbf{\Gamma}} := \hat{\gamma}^{-1}\partial_0\hat{\gamma} \ ,$$

the gauge transformation of the second boundary condition reads

$$\begin{aligned} 0 = \partial_0\bar{\mathcal{A}}^{\hat{\gamma}}|_{\pi=\alpha} &= \partial_0\left(\hat{\gamma}^{-1}\bar{\mathcal{A}}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}\right)|_{\pi=\alpha} \\ &= \left(\left[\hat{\gamma}^{-1}\bar{\mathcal{A}}\hat{\gamma}, \hat{\mathbf{\Gamma}}\right] + \hat{\gamma}^{-1}\partial_0\bar{\mathcal{A}}\hat{\gamma} + \bar{\partial}\hat{\mathbf{\Gamma}} + \left[\hat{\gamma}^{-1}\bar{\partial}\hat{\gamma}, \hat{\mathbf{\Gamma}}\right] + \hat{\gamma}^{-1}\partial_{\dot{a}}\hat{\gamma}\,\bar{e}^{\dot{a}}\right)|_{\pi=\alpha} \ . \end{aligned} \tag{2.5.40}$$

Imposing the original boundary conditions we arrive at the constraint equation

$$\alpha^a\partial_{a\dot{a}}\mathbf{\Gamma} + \gamma^{-1}\hat{\alpha}^a\partial_{a\dot{a}}\gamma = 0 \ , \tag{2.5.41}$$

where we have used $\langle\alpha\hat{\alpha}\rangle = 1$ and defined

$$\hat{\mathbf{\Gamma}}|_{\pi=\alpha} = \mathbf{\Gamma} \ .$$

One solution to (2.5.39) and (2.5.40) is that the external gauge transformations are global symmetries of the localised effective theory $d_{\mathbb{R}^4}\gamma = 0$, and $\mathbf{\Gamma}$ is holomorphic on $\mathbb{R}^4$ with respect to the choice of complex structure given by the point $\alpha \in \mathbb{CP}^1$.

Tentatively, our localised theory should have 4 degrees of freedom, known as 'edge modes',

$$\underline{\mathbf{u}} := (g, \mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3) \ , \tag{2.5.42}$$

where

$$g = \hat{g}|_{\pi=\alpha} \ , \quad \mathbf{u}^1 := \hat{g}^{-1}\partial_0\hat{g}|_{\pi=\alpha} \ , \quad \mathbf{u}^2 := \hat{g}^{-1}\partial_0^2\hat{g}|_{\pi=\alpha} \ , \quad \mathbf{u}^3 := \hat{g}^{-1}\partial_0^3\hat{g}|_{\pi=\alpha} \ . \tag{2.5.43}$$

However, some of these fields are spurious and can be gauged fixed away using the admissible gauge symmetries. We have already used the internal gauge symmetry to fix $g = \mathrm{Id}$. Furthermore, the second and third ∂_0-derivatives of the external gauge transformations are unconstrained by the boundary conditions, so they can be used to gauge fix $\mathbf{u}^2 = \mathbf{u}^3 = 0$. This leaves us with one dynamical degree of freedom in the localised theory on $\mathbb{R}^4$, namely $\mathbf{u}^1 : \mathbb{R}^4 \to \mathfrak{g}$, which we will now denote by $\mathbf{u}$ for brevity. In conclusion, after gauge fixing we have

$$\underline{\mathbf{u}} = (\mathrm{Id}, \mathbf{u}, 0, 0) \ . \tag{2.5.44}$$

Solving the boundary conditions. Using the boundary conditions, we will solve for $\bar{\mathcal{A}}'$ in the parametrisation (2.5.37) in terms of the edge modes. The first boundary condition tells us

$$\bar{\mathcal{A}}'|_{\pi=\alpha} = 0 \quad \Rightarrow \quad \alpha^a A_{a\dot{a}} = 0 \quad \Rightarrow \quad A_{a\dot{a}} = \alpha_a C_{\dot{a}} \ . \tag{2.5.45}$$

The second boundary condition equation is then written as

$$\big(\partial_0 \bar{\mathcal{A}}' + \bar{\partial}(\hat{g}^{-1}\partial_0 \hat{g})\big)|_{\pi=\alpha} = 0 \quad \Rightarrow \quad \frac{\hat{\alpha}^a}{\langle \alpha \hat{\alpha} \rangle} A_{a\dot{a}} + \alpha^a \partial_{a\dot{a}} \mathbf{u} = 0 \ , \tag{2.5.46}$$

which allows us to conclude that

$$C_{\dot{\alpha}} = \alpha^a \partial_{\alpha\dot{\alpha}} \mathbf{u} \ . \tag{2.5.47}$$

We now have all the ingredients to localise the hCS$_6$ action to $\mathbb{R}^4$.

Localisation to $\mathbb{R}^4$. We can write the action (2.5.33) in the new variables as

$$S = \frac{1}{2\pi i} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}(\bar{\mathcal{A}}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}) - \frac{1}{6\pi i} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\left((\hat{g}^{-1} d\hat{g})^3\right) \ , \tag{2.5.48}$$

where in the second term we have extended $\mathbb{PT}$ to the 7-manifold $\mathbb{PT} \times [0,1]$, whose boundary is a disjoint union of two copies of $\mathbb{PT}$. We have also extended our fields via a smooth homotopy $\hat{g} \to \hat{g}(t)$ so that $\hat{g}(0) = \mathrm{Id}$ and $\hat{g}(1) = \hat{g}$. Applying the localisation formula (2.5.35) and the choice of gauge fixing (2.5.44), we arrive at the spacetime action

$$S_{\mathrm{LMP}}[\mathbf{u}] = \frac{k}{3} \int_{\mathbb{R}^4} \frac{1}{2} \mathrm{Tr}(d\mathbf{u} \wedge \star d\mathbf{u}) + \frac{1}{3} \alpha_a \alpha_b \Sigma^{ab} \wedge \mathrm{Tr}(\mathbf{u}\,[d\mathbf{u}, d\mathbf{u}]) \ . \tag{2.5.49}$$

The action (2.5.49) is the LMP model for ASDYM [LM87; Par92], which upon reduction to $\mathbb{R}^2$ becomes the pseudo-dual of the PCM.

In summary

In this section we have outlined a systematic procedure describing how one can descend from holomorphic Chern-Simons theory in six-dimensions, we can picture this using fig.2.1. The solid arrows represent the procedure of localising the theory by integrating

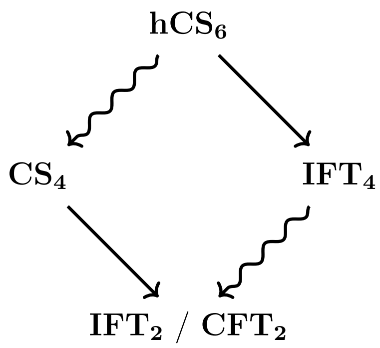


Figure 2.1: The diamond correspondence of integrable field theories as outlined in §2.5.

over the $\mathbb{CP}^1$ factor, and the wavy arrows representing performing a symmetry reduction of the theory. In the following chapter, we will continue to explore this paradigm, beginning from a novel choice of divisor on twistor space,

$$\Phi = \frac{K}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2} \ ,$$

having two simple poles and a double pole. From this starting point, we will present a rich story that widens the landscape of known four-dimensional integrable field theories, and allows us to recover integrable deformations of two-dimensional integrable field theories from this six-dimensional starting point.

Chapter 3

Integrable Deformations from Twistor Space

In this chapter we present a discourse on a diamond of integrable theories for a family of deformed sigma models. In particular, we will consider alternate boundary conditions for hCS_6, distinct from the Dirichlet boundary conditions that we have been primarily considering thus far in the thesis. Starting from 6d holomorphic Chern-Simons theory on twistor space with a particular meromorphic 3-form Ω, we construct the defect theory to find a novel 4d integrable field theory, whose equations of motion can be recast as the 4d anti-self-dual Yang-Mills equations. Symmetry reducing, we find a multi-parameter 2d integrable model, which specialises to the λ-deformation at a certain point in parameter space. The same model is recovered by first symmetry reducing, to give 4d Chern-Simons with generalised boundary conditions, and then constructing the defect theory.

3.1 6d Holomorphic Chern-Simons

Our primary interest in this chapter will be the hCS_6 diamond containing the λ-deformed IFT_2 originally constructed in [Sfe14]. By proposing a carefully chosen set of boundary conditions, we will be able to find a diamond of theories that arrives at a multi-parametric class of integrable λ-deformations between coupled WZW models.

To this end we restrict our study of hCS_6, defined by the action

$$S_{\mathrm{hCS}_6} = \frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\Big(\bar{\mathcal{A}} \wedge \bar{\partial}\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\Big) \ , \tag{3.1.1}$$

to the case where the $(3,0)$-form is given by, in the basis of $(1,0)$-forms, as defined by (2.4.56)

$$\Omega = \frac{1}{2}\,\Phi\, e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}} \ , \qquad \Phi = \frac{K}{\langle\pi\alpha\rangle\langle\pi\tilde{\alpha}\rangle\langle\pi\beta\rangle^2} \ . \tag{3.1.2}$$

The constant spinors α, $\tilde{\alpha}$ and β should be understood as part of the definition of the model. The gauge field is similarly written in the basis of $(0,1)$-forms as

$$\bar{\mathcal{A}} = \bar{\mathcal{A}}_0\, \bar{e}^0 + \bar{\mathcal{A}}_{\dot{a}}\, \bar{e}^{\dot{a}} \ . \tag{3.1.3}$$

The action (3.1.1) is invariant under shifts of $\bar{\mathcal{A}}$ by any $(1,0)$-form, i.e. $\bar{\mathcal{A}} \mapsto \bar{\mathcal{A}} + \rho$ where $\rho \in \Omega^{(1,0)}(\mathbb{PT})$.

The first step in studying the 6-dimensional theory is to impose conditions ensuring the vanishing of the 'boundary' term that appears in the variation of the action

$$0 = \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\bar{\mathcal{A}} \wedge \delta\bar{\mathcal{A}}\big) \ . \tag{3.1.4}$$

Since Ω is meromorphic, as opposed to holomorphic, this receives contributions from the poles at α, $\tilde{\alpha}$, and β. Following on from our discussion in § 2.5.1, we will again impose Dirichlet boundary conditions $\bar{\mathcal{A}}_{\dot{a}}|_{\pi=\beta} = 0$ at the second-order pole. At the first-order poles, we can then evaluate the integral over $\mathbb{CP}^1$ to obtain[1] the condition

$$\frac{1}{\langle\alpha\tilde{\alpha}\rangle\langle\alpha\beta\rangle^2}\int_{\mathbb{E}^4} \mathrm{vol}_4\, \epsilon^{\dot{a}\dot{b}}\mathrm{Tr}\big(\bar{\mathcal{A}}_{\dot{a}}\delta\bar{\mathcal{A}}_{\dot{b}}\big)\big|_{\pi=\alpha} = \frac{1}{\langle\alpha\tilde{\alpha}\rangle\langle\tilde{\alpha}\beta\rangle^2}\int_{\mathbb{E}^4} \mathrm{vol}_4\, \epsilon^{\dot{a}\dot{b}}\mathrm{Tr}\big(\bar{\mathcal{A}}_{\dot{a}}\delta\bar{\mathcal{A}}_{\dot{b}}\big)\big|_{\pi=\tilde{\alpha}} \ . \tag{3.1.5}$$

Using the Schouten identity we can decompose a 2-spinor $X^{\dot{a}}$ using the 2-spinors

[1]To compute the boundary variation of the action, we have used the identities $e^{\dot{c}} \wedge e_{\dot{c}} \wedge \bar{e}^{\dot{a}} \wedge \bar{e}^{\dot{b}} = -2\,\mathrm{vol}_4\, \epsilon^{\dot{a}\dot{b}}$ (where $\mathrm{vol}_4 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$) and

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{CP}^1} e^0 \wedge \bar{e}^0\, \bar{\partial}_0\Big(\frac{1}{\langle\pi\alpha\rangle}\Big) f(\pi) = f(\alpha) \ .$$

$\{\mu^{\dot{a}}, \hat{\mu}^{\dot{a}}\}$

$$X^{\dot{a}} = [X\hat{\mu}]\mu^{\dot{a}} - [X\mu]\hat{\mu}^{\dot{a}} \ , \ \text{where} \quad [\mu\hat{\mu}] = 1 \ . \tag{3.1.6}$$

Expanding the gauge field components in terms of the basis $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$, and solving locally pointwise on $\mathbb{E}^4$, this condition may be written as

$$\frac{1}{\langle\alpha\beta\rangle^2}\mathrm{Tr}\big([\bar{\mathcal{A}}\mu][\delta\bar{\mathcal{A}}\hat{\mu}] - [\bar{\mathcal{A}}\hat{\mu}][\delta\bar{\mathcal{A}}\mu]\big)\big|_{\pi=\alpha} = \frac{1}{\langle\tilde{\alpha}\beta\rangle^2}\mathrm{Tr}\big([\bar{\mathcal{A}}\mu][\delta\bar{\mathcal{A}}\hat{\mu}] - [\bar{\mathcal{A}}\hat{\mu}][\delta\bar{\mathcal{A}}\mu]\big)|_{\pi=\tilde{\alpha}} \ . \tag{3.1.7}$$

The boundary conditions we are led to consider are

$$[\bar{\mathcal{A}}\mu]|_{\pi=\alpha} = \sigma\frac{\langle\alpha\beta\rangle}{\langle\tilde{\alpha}\beta\rangle}[\bar{\mathcal{A}}\mu]|_{\pi=\tilde{\alpha}} \ , \qquad [\bar{\mathcal{A}}\hat{\mu}]|_{\pi=\alpha} = \sigma^{-1}\frac{\langle\alpha\beta\rangle}{\langle\tilde{\alpha}\beta\rangle}[\bar{\mathcal{A}}\hat{\mu}]|_{\pi=\tilde{\alpha}} \ , \tag{3.1.8}$$

where we have introduced the free parameter σ, which will play the role of the deformation parameter in the IFT_4.

Let us note that these boundary conditions are invariant under the following discrete transformations

$$\alpha \leftrightarrow \tilde{\alpha} \ , \qquad \sigma \mapsto \sigma^{-1} \ , \tag{3.1.9}$$

$$\mu \mapsto \hat{\mu} \ , \qquad \sigma \mapsto \sigma^{-1} \ . \tag{3.1.10}$$

These will descend to transformations that leave the IFT_4 invariant.

3.1.1 Residual Symmetries and Edge Modes

A general feature of Chern-Simons theory with a boundary is the emergence of propagating edge modes as a consequence of the violation of gauge symmetry by boundary conditions. A similar effect underpins the emergence of the dynamical field content of the lower dimensional theories that descend from hCS_6. Generally, group-valued degrees of freedom, here denoted by h and $\tilde{g}$, would be sourced at the locations of the poles of Ω. If however, the boundary conditions (3.1.8) admit residual symmetries, then these will result in symmetries of the IFT_4 potentially mixing the h and $\tilde{g}$ degrees of freedom. These may be global symmetries, gauge symmetries, or semi-local symmetries depending on the constraints imposed by the boundary conditions. It is thus important to understand the nature of any residual symmetry preserved by the boundary conditions (3.1.8).

Gauge transformations act on the hCS_6 gauge field as

$$\hat{\gamma} : \quad \bar{\mathcal{A}} \mapsto \hat{\gamma}^{-1}\bar{\mathcal{A}}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma} \ . \tag{3.1.11}$$

In the bulk, i.e. away from the poles of Ω, these are unconstrained, but at the poles they will only leave the action invariant if they preserve the boundary conditions. For later convenience, we will denote the values of the gauge transformation parameters at the poles by

$$\hat{\gamma}|_{\alpha} = r \ , \quad \hat{\gamma}|_{\tilde{\alpha}} = \tilde{r} \ , \quad \hat{\gamma}|_{\beta} = \ell^{-1} \ . \tag{3.1.12}$$

Firstly, the transformation acting at β must preserve the constraint $\bar{\mathcal{A}}_{\dot{a}}|_{\beta} = 0$. Initially, one might suppose that only constant ℓ would preserve this boundary condition, but in fact it is sufficient for ℓ to be holomorphic with respect to the complex structure defined by β

$$\beta^a \partial_{a\dot{a}}\ell = 0 \quad \Rightarrow \quad \frac{1}{\langle\alpha\beta\rangle}\alpha^a\partial_{a\dot{a}}\ell = \frac{1}{\langle\tilde{\alpha}\beta\rangle}\tilde{\alpha}^a\partial_{a\dot{a}}\ell \ . \tag{3.1.13}$$

These differential constraints arise from the fact that the anti-holomorphic vector fields $\bar{\partial}_{\dot{a}} = \pi^a\partial_{a\dot{a}}$ are valued in $\mathcal{O}(1)$, depending explicitly on the $\mathbb{CP}^1$ coordinate.

Secondly, the transformations acting at α and $\tilde{\alpha}$ must preserve the boundary conditions (3.1.8), implying the constraints

$$\begin{aligned} \tilde{r} &= r \ , \\ \frac{1}{\langle\alpha\beta\rangle}\mu^{\dot{a}}\alpha^a\partial_{a\dot{a}}r &= \frac{\sigma}{\langle\tilde{\alpha}\beta\rangle}\mu^{\dot{a}}\tilde{\alpha}^a\partial_{a\dot{a}}r \ , \\ \frac{1}{\langle\alpha\beta\rangle}\hat{\mu}^{\dot{a}}\alpha^a\partial_{a\dot{a}}r &= \frac{\sigma^{-1}}{\langle\tilde{\alpha}\beta\rangle}\hat{\mu}^{\dot{a}}\tilde{\alpha}^a\partial_{a\dot{a}}r \ . \end{aligned} \tag{3.1.14}$$

These residual symmetries are neither constant (i.e. global symmetries) nor fully local (i.e. gauge symmetries). Instead, we expect that our IFT_4 should exhibit two semi-local symmetries subject to the above differential constraints, akin to the semi-local symmetries of the 4d WZW model first identified in [NS90; NS92][2].

Symmetry reduction As we progress around the diamond, we will perform 'symmetry reduction'. In essence, this will mean we restrict to fields and gauge parameters that are independent of two directions, i.e. they obey the further differential constraints

[2]Complementary to this perspective, the WZW_4 algebra can also be obtained as a global symmetry of five-dimensional Kähler Chern-Simons on a manifold with boundary [BGH96].

(where ι^a is some constant spinor)

$$\mu^{\dot{a}}\iota^a\partial_{a\dot{a}}\hat{\gamma} = 0\,, \quad \hat{\mu}^{\dot{a}}\hat{\iota}^a\partial_{a\dot{a}}\hat{\gamma} = 0\,. \tag{3.1.15}$$

We can then predict some special points in the lower dimensional theories by considering how these differential constraints interact with those imposed by the boundary conditions. Generically, these four differential constraints (two from the boundary conditions and two from symmetry reduction) will span a copy of $\mathbb{E}^4$ at each pole, meaning that only constant transformations (i.e. global symmetries) will survive. However, if the symmetry reduction is carefully chosen, the two sets of constraints may partially or entirely coincide. In the case that they entirely coincide, the lower dimensional symmetry parameter will be totally unconstrained, meaning that the IFT_2 will possess a gauge symmetry. Alternatively, if the constraints partially coincide then the lower dimensional theory will have a symmetry with free dependence on half the coordinates, e.g. the chiral symmetries of the 2d WZW model.

3.2 Localisation of hCS_6 to IFT_4

The localisation analysis is naturally presented in terms of new variables $\bar{\mathcal{A}}'$ and $\hat{g}$, which are related to the fundamental field by

$$\bar{\mathcal{A}} = \hat{g}^{-1}\bar{\mathcal{A}}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}\ . \tag{3.2.1}$$

However, there is some redundancy in this new parametrisation. There are internal gauge transformations, leaving $\bar{\mathcal{A}}$ invariant, given by

$$\check{\gamma}: \quad \bar{\mathcal{A}}' \mapsto \check{\gamma}^{-1}\bar{\mathcal{A}}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma}\ , \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g}\,. \tag{3.2.2}$$

These transformations allow us to impose the constraint $\bar{\mathcal{A}}'_0 = 0$, i.e. it has no leg in the $\mathbb{CP}^1$-direction.

There are still internal gauge transformations that are $\mathbb{CP}^1$-independent, and we can use these to fix the value of $\hat{g}$ at one pole. We will therefore impose the additional constraint $\hat{g}|_\beta = \mathrm{Id}$ so that we have resolved this internal redundancy. The values of $\hat{g}$ at the remaining poles

$$\hat{g}|_\alpha = g\ , \qquad \hat{g}|_{\tilde{\alpha}} = \tilde{g}\ , \tag{3.2.3}$$

will be dynamical edge modes as a consequence of the violation of gauge symmetry by boundary conditions. As we will now see, the entire action localises to a theory on $\mathbb{E}^4$ depending only on these edge modes.

The hCS$_6$ action is written in these new variables as

$$\begin{aligned} S_{\mathrm{hCS}_6} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\big(\bar{\mathcal{A}}' \wedge \bar{\partial}\bar{\mathcal{A}}'\big) + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\bar{\mathcal{A}}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) \\ - \frac{1}{6\pi \mathrm{i}} \int_{\mathbb{PT}} \Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1}\bar{\partial}\hat{g} \wedge \hat{g}^{-1}\bar{\partial}\hat{g} \wedge \hat{g}^{-1}\bar{\partial}\hat{g}\big) \ . \end{aligned} \tag{3.2.4}$$

The cubic term in $\bar{\mathcal{A}}'$ has dropped out since we have imposed $\bar{\mathcal{A}}'_0 = 0$. Inspecting the terms in our action involving $\hat{g}$, we see that the second term localises to the poles due to the anti-holomorphic derivative acting on Ω. The third term similarly localises to the poles. For this, we consider a manifold whose boundary is $\mathbb{PT}$.[3] We take the 7-manifold $\mathbb{PT} \times [0, 1]$ and extend our field $\hat{g}$ over this interval. We do this by choosing a smooth homotopy to a constant map, such that it's restriction to $\mathbb{PT} \times \{0\}$ coincides with $\hat{g}$. Denoting this extension with the same symbol, we see that the third term in our action may be equivalently written as

$$-\frac{1}{6\pi i} \int_{\mathbb{PT}\times[0,1]} d\Big[\Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1}d\hat{g} \wedge \hat{g}^{-1}d\hat{g} \wedge \hat{g}^{-1}d\hat{g}\big)\Big] \ . \tag{3.2.5}$$

Then, using the closure of the Wess-Zumino 3-form and the fact that all of the holomorphic legs on $\mathbb{PT}$ are saturated by Ω, this is equal to

$$S_{\mathrm{WZ}_4} = -\frac{1}{6\pi i} \int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\hat{g}^{-1}d\hat{g} \wedge \hat{g}^{-1}d\hat{g} \wedge \hat{g}^{-1}d\hat{g}\big) \ . \tag{3.2.6}$$

Therefore, this contribution also localises, meaning that the only information contained in the field $\hat{g} : \mathbb{PT} \to G$ are its values[4] at the poles of Ω. Explicitly, this contribution is

[3] More generally, a manifold whose boundary is a disjoint union of copies of $\mathbb{PT}$.

[4] For higher order poles in Ω, the $\mathbb{CP}^1$-derivatives of $\hat{g}$ would also contribute to the action.

given by[5]

$$S_{\mathrm{WZ}_4} = \frac{K}{6\langle\alpha\tilde{\alpha}\rangle}\int_{\mathbb{E}^4\times[0,1]}\left[\frac{1}{\langle\alpha\beta\rangle^2}\mu_\alpha\wedge\mathrm{Tr}\big(g^{-1}dg\big)^3 - \frac{1}{\langle\tilde{\alpha}\beta\rangle^2}\mu_{\tilde{\alpha}}\wedge\mathrm{Tr}\big(\tilde{g}^{-1}d\tilde{g}\big)^3\right] , \qquad (3.2.7)$$

where

$$\mu_\alpha = \epsilon_{\dot{a}\dot{b}}\alpha_a\alpha_b\,\mathrm{d}x^{a\dot{a}}\wedge\mathrm{d}x^{b\dot{b}} , \qquad \mu_{\tilde{\alpha}} = \epsilon_{\dot{a}\dot{b}}\tilde{\alpha}_a\tilde{\alpha}_b\,\mathrm{d}x^{a\dot{a}}\wedge\mathrm{d}x^{b\dot{b}} , \qquad (3.2.8)$$

are the $(2,0)$-forms with respect to the complex structure on $\mathbb{E}^4$ defined by the spinors α_a and $\tilde{\alpha}_a$ respectively.

Knowing that the latter two terms in the action (3.2.4) localise to the poles, we are one step closer to deriving the IFT$_4$. There are two unresolved problems: the first term is still a genuine bulk term; and the second term contains $\bar{\mathcal{A}}'$, rather than being written exclusively in terms of the fields g and $\tilde{g}$. Both of these problems will be resolved by invoking the bulk equations of motion for $\bar{\mathcal{A}}'$. This will completely specify its $\mathbb{CP}^1$-dependence, and, combined with the boundary conditions, we will then be able to solve for $\bar{\mathcal{A}}'$ in terms of the edge modes g and $\tilde{g}$.

Varying the first term in the action, which is the only bulk term, we find the equation of motion $\bar{\partial}_0\bar{\mathcal{A}}'_{\dot{a}} = 0$, which implies that these components are holomorphic. Combined with the knowledge that $\bar{\mathcal{A}}'_{\dot{a}}$ has homogeneous weight 1, we deduce that the $\mathbb{CP}^1$-dependence is given by $\bar{\mathcal{A}}'_{\dot{a}} = \pi^a A_{a\dot{a}}$ where $A_{a\dot{a}}$ is $\mathbb{CP}^1$-independent.

Turning our attention to the boundary conditions, we first consider the double pole where we have imposed $\bar{\mathcal{A}}_{\dot{a}}|_\beta = 0$. Recalling that $\hat{g}|_\beta = \mathrm{Id}$, this simply translates to $\bar{\mathcal{A}}'_{\dot{a}}|_\beta = 0$. This tells us that $\bar{\mathcal{A}}'_{\dot{a}} = \langle\pi\beta\rangle B_{\dot{a}}$ for some $B_{\dot{a}}$, hence $A_{a\dot{a}} = \beta_a B_{\dot{a}}$. Therefore, we have that

$$\bar{\mathcal{A}}_{\dot{a}} = \langle\pi\beta\rangle\mathrm{Ad}_{\hat{g}}^{-1}B_{\dot{a}} + \pi^a\hat{g}^{-1}\partial_{a\dot{a}}\hat{g} . \qquad (3.2.9)$$

The solution for $B_{\dot{a}}$ found by solving the remaining two boundary conditions (3.1.8) is written more concisely if we introduce some notation. We will make extensive use of the operators

$$U_\pm = \big(1-\sigma^{\pm1}\Lambda\big)^{-1} , \qquad \Lambda = \mathrm{Ad}_{\tilde{g}}^{-1}\mathrm{Ad}_g , \qquad (3.2.10)$$

[5]In principle there are also contributions from the double pole at β both in this term and the second term in the action (3.2.4). Since this is a double pole, these contributions may depend on $\partial_0\hat{g}|_\beta$, which is unconstrained. However, one can check that they vanish using just the boundary conditions $\mathcal{A}_{\dot{a}}|_\beta = 0$ and internal gauge-fixing $\hat{g}|_\beta = \mathrm{Id}$. Alternatively, we may use part of the residual external gauge symmetry to fix $\partial_0\hat{g}|_\beta = 0$, which ensures such contributions vanish.

which enjoy the useful identities

$$U_+^T + U_- = \mathrm{Id} \ , \qquad U_\pm \Lambda = -\sigma^{\mp 1} U_\mp^T \ , \tag{3.2.11}$$

where transposition is understood to be with respect to the ad-invariant inner product on $\mathfrak{g}$. In terms of the components of $\hat{g}^{-1}\partial_{a\dot{a}}\hat{g}$, defined with useful normalisation factors,

$$\begin{aligned} j &= \langle \alpha\beta\rangle^{-1}\mu^{\dot{a}}\alpha^a g^{-1}\partial_{a\dot{a}}g \ , \qquad & \widehat{\jmath} &= \langle \alpha\beta\rangle^{-1}\hat{\mu}^{\dot{a}}\alpha^a g^{-1}\partial_{a\dot{a}}g \ , \\ \widetilde{\jmath} &= \langle \tilde{\alpha}\beta\rangle^{-1}\mu^{\dot{a}}\tilde{\alpha}^a \tilde{g}^{-1}\partial_{a\dot{a}}\tilde{g} \ , \qquad & \widehat{\widetilde{\jmath}} &= \langle \tilde{\alpha}\beta\rangle^{-1}\hat{\mu}^{\dot{a}}\tilde{\alpha}^a \tilde{g}^{-1}\partial_{a\dot{a}}\tilde{g} \ , \end{aligned} \tag{3.2.12}$$

we find that the solutions to the remaining boundary conditions may be written as

$$\mathrm{Ad}_g^{-1} B_{\dot{a}} = \widehat{b}\mu_{\dot{a}} - b\hat{\mu}_{\dot{a}} \ , \qquad b = U_+(j - \sigma\,\widetilde{\jmath}) \ , \qquad \widehat{b} = U_-(\widehat{\jmath} - \sigma^{-1}\widehat{\widetilde{\jmath}}) \ , \tag{3.2.13}$$

$$\mathrm{Ad}_{\tilde{g}}^{-1} B_{\dot{a}} = \widehat{\widetilde{b}}\mu_{\dot{a}} - \widetilde{b}\hat{\mu}_{\dot{a}} \ , \qquad \widetilde{b} = U_-^T(\widetilde{\jmath} - \sigma^{-1}j) \ , \qquad \widehat{\widetilde{b}} = U_+^T(\widehat{\widetilde{\jmath}} - \sigma\,\widehat{\jmath}) \ . \tag{3.2.14}$$

Note that $b = \mathrm{Ad}_g^{-1}[B\mu]$, $\widehat{b} = \mathrm{Ad}_g^{-1}[B\hat{\mu}]$, etc., and that b, $\widetilde{b}$, $\widehat{b}$ and $\widehat{\widetilde{b}}$ are related as

$$\widetilde{b} - \widetilde{\jmath} = \sigma^{-1}(b - j) \ , \qquad \widehat{\widetilde{b}} - \widehat{\widetilde{\jmath}} = \sigma(\widehat{b} - \widehat{\jmath}) \ . \tag{3.2.15}$$

Recovering the IFT$_4$ is then simple. The first term in the action (3.2.4) vanishes identically on shell, and we can substitute in our solution for $\bar{\mathcal{A}}'$ in terms of g and $\tilde{g}$ to get a 4d theory only depending on these edge modes. This results in the action

$$\begin{aligned} S_{\mathrm{IFT}_4} &= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\bar{\mathcal{A}}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1}\big) + S_{\mathrm{WZ}_4} \\ &= \frac{K}{\langle \alpha\tilde{\alpha}\rangle}\int_{\mathbb{E}^4} \mathrm{vol}_4\, \mathrm{Tr}\big(b(\widehat{\jmath} - \Lambda^T\widehat{\widetilde{\jmath}}) - \widehat{b}(j - \Lambda^T\widetilde{\jmath})\big) + S_{\mathrm{WZ}_4} \\ &= \frac{K}{\langle \alpha\tilde{\alpha}\rangle}\int_{\mathbb{E}^4} \mathrm{vol}_4\, \mathrm{Tr}\Big(j(U_+^T - U_-)\,\widehat{\jmath} + \widetilde{\jmath}(U_+^T - U_-)\,\widehat{\widetilde{\jmath}} - \\ &\qquad\qquad 2\sigma\,\widetilde{\jmath}\,U_+^T\,\widehat{\jmath} + 2\sigma^{-1} j\,U_-\,\widehat{\widetilde{\jmath}}\Big) + S_{\mathrm{WZ}_4} \ , \end{aligned} \tag{3.2.16}$$

where

$$S_{\mathrm{WZ}_4} = \frac{K}{\langle \alpha\tilde{\alpha}\rangle}\int_{\mathbb{E}^4\times[0,1]} \mathrm{vol}_4 \wedge \mathrm{d}\rho\, \mathrm{Tr}\big(g^{-1}\partial_\rho g \cdot [j, \widehat{\jmath}] - \tilde{g}^{-1}\partial_\rho\tilde{g} \cdot [\widetilde{\jmath}, \widehat{\widetilde{\jmath}}]\big) \ . \tag{3.2.17}$$

Observe that the 4d IFT (3.2.16) with (3.2.17) is mapped into itself under the formal transformation

$$g \leftrightarrow \tilde{g} \ , \qquad \alpha \leftrightarrow \tilde{\alpha} \ , \qquad \sigma \mapsto \sigma^{-1} \ , \tag{3.2.18}$$

interchanging the positions of the two poles. This directly follows from the invariance (3.1.9) of the hCS_6 boundary conditions. On the other hand, looking at how the transformation (3.1.10) descends to the IFT_4, we find[6]

$$j \mapsto \widehat{\jmath} \ , \qquad \widehat{\jmath} \mapsto -j \ , \qquad \widetilde{\jmath} \mapsto \widehat{\widetilde{\jmath}} \ , \qquad \widehat{\widetilde{\jmath}} \mapsto -\widetilde{\jmath} \ , \qquad \sigma \mapsto \sigma^{-1} \ . \tag{3.2.19}$$

It is then straightforward to check that the action (3.2.16) with (3.2.17) is invariant under this transformation.

Let us emphasise that, to our knowledge, the IFT_4 described by the action (3.2.16) with (3.2.17) has not been considered in the literature before. In the following subsections we will study some properties of this model starting with its symmetries, and moving onto its equations of motion and their relation to the 4d ASDYM equations.

3.2.1 Symmetries of the IFT_4

Having derived the action functional for the IFT_4, we will now examine those symmetries that leave this action invariant. While they may not be obvious from simply looking at the action, in § 3.1.1 we leveraged the hCS_6 description to predict the symmetries of the IFT_4. These may then be verified by explicit computation.

To this end, we recall that the hCS_6 gauge transformations act as

$$\hat{\gamma}: \quad \bar{\mathcal{A}} \mapsto \hat{\gamma}^{-1}\bar{\mathcal{A}}\hat{\gamma} + \hat{\gamma}^{-1}\bar{\partial}\hat{\gamma} \ , \tag{3.2.20}$$

and we denoted the value of this transformation parameter at the poles by

$$\hat{\gamma}|_{\alpha} = r \ , \quad \hat{\gamma}|_{\tilde{\alpha}} = \tilde{r} \ , \quad \hat{\gamma}|_{\beta} = \ell^{-1} \ . \tag{3.2.21}$$

Tracing through the derivation above, we find that these result in an induced action on the IFT_4 fields,

$$(\ell, r, \tilde{r}): \quad g \mapsto \ell g r \ , \quad \tilde{g} \mapsto \ell \tilde{g} \tilde{r} \ , \tag{3.2.22}$$

where ℓ, r and $\tilde{r}$ obey the constraints (3.1.13) and (3.1.14) respectively. One can

[6] Note that to derive this we use that $\hat{\mu} = -\mu$

explicitly verify that the IFT$_4$ is indeed invariant under these transformations. Key to this is exploiting a Polyakov-Wiegmann identity such that the variation of S_{WZ_4} in eq. (3.2.17) produces a total derivative. This gives a contribution on $\mathbb{E}^4$ that cancels the variation of the remainder of eq. (3.2.16). Useful intermediate results to this end are

$$\begin{gathered}\mathrm{Ad}_g \mapsto \mathrm{Ad}_\ell \mathrm{Ad}_g \mathrm{Ad}_r \ , \quad \mathrm{Ad}_{\tilde{g}} \mapsto \mathrm{Ad}_\ell \mathrm{Ad}_{\tilde{g}} \mathrm{Ad}_r \ , \quad U_\pm \mapsto \mathrm{Ad}_r^{-1} U_\pm \mathrm{Ad}_r \ , \\ b \mapsto \mathrm{Ad}_r^{-1}(b + \langle\alpha\beta\rangle^{-1} \mathrm{Ad}_g^{-1} \mu^{\dot{a}} \alpha^a \ell^{-1} \partial_{a\dot{a}} \ell) \ , \quad \hat{b} \mapsto \mathrm{Ad}_r^{-1}(\hat{b} + \langle\alpha\beta\rangle^{-1} \mathrm{Ad}_g^{-1} \hat{\mu}^{\dot{a}} \alpha^a \ell^{-1} \partial_{a\dot{a}} \ell) \ ,\end{gathered} \tag{3.2.23}$$

in which the constraints (3.1.13) and (3.1.14) have been used.

We can also derive the Noether currents corresponding to these residual semi-local symmetries directly from hCS$_6$. The variation of the action under an infinitesimal gauge transformation $\delta\bar{\mathcal{A}} = \bar{\partial}\hat{\epsilon} + [\bar{\mathcal{A}}, \hat{\epsilon}]$ is

$$\delta S_{\mathrm{6dCS}} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\bar{\mathcal{A}} \wedge \bar{\partial}\hat{\epsilon}\big) \ . \tag{3.2.24}$$

First let us consider a transformation that descends to the ℓ-symmetry, i.e. one for which

$$\hat{\epsilon}|_\alpha = \hat{\epsilon}|_{\tilde{\alpha}} = 0 \ , \quad \hat{\epsilon}|_\beta = \epsilon^{(\ell)} \ .$$

The only contribution to the variation localises to β and is given by

$$\delta_\ell S_{\mathrm{6dCS}} \propto \int_{\mathbb{E}^4} \mathrm{vol}_4\, \partial_0 \bigg(\frac{1}{\langle\pi\alpha\rangle\langle\pi\tilde{\alpha}\rangle} \epsilon^{\dot{a}\dot{b}} \mathrm{Tr}\big(\bar{\mathcal{A}}_{\dot{a}} \bar{\partial}_{\dot{b}} \hat{\epsilon}\big)\bigg)\bigg|_\beta \ . \tag{3.2.25}$$

Since $\bar{\mathcal{A}}_{\dot{a}}|_\beta \propto \langle\pi\beta\rangle$ (recall that we fix $\hat{g}|_\beta = \mathrm{id}$) the only way the integrand will be non-vanishing is for the ∂_0 operator to act on $\bar{\mathcal{A}}_{\dot{a}}$. Noting that $\partial_0\langle\pi\beta\rangle|_\beta = 1$ we have that $\partial_0\bar{\mathcal{A}}_{\dot{a}}|_\beta = B_{\dot{a}}$, and hence the variation becomes

$$\delta_\ell S_{\mathrm{6dCS}} \propto \int_{\mathbb{E}^4} \mathrm{vol}_4\, \mathrm{Tr}\Big(B^B \beta^b \partial_{Bb} \epsilon^{(\ell)}\Big) \ . \tag{3.2.26}$$

If we think of the ℓ-symmetry as a global transformation, then this would provide the conservation law associated to the Noether current, i.e.

$$\beta^b \partial_{b\dot{b}} B^B = 0 \ , \tag{3.2.27}$$

and indeed we will see later that this conservation law follows from the equations of motion of the IFT_4. As the parameter $\epsilon^{(\ell)}$ is allowed to be holomorphic with respect to the complex structure defined by β, the interpretation is more akin to a Kac-Moody current.

For the case corresponding to the r-symmetry we have

$$\hat{\epsilon}|_{\alpha} = \hat{\epsilon}|_{\tilde{\alpha}} = \epsilon^{(r)} \,, \quad \hat{\epsilon}|_{\beta} = 0 \,.$$

In this case the variation receives two contributions with an opposite sign

$$\delta_r S_{6\mathrm{dCS}} \propto \int_{\mathbb{E}^4} \mathrm{vol}_4 \, \epsilon^{\dot{a}\dot{b}} \, \mathrm{Tr}\Big(\frac{1}{\langle \alpha\beta\rangle^2} \bar{\mathcal{A}}_{\dot{a}} \bar{\partial}_{\dot{b}} \hat{\epsilon}\big|_{\alpha} - \frac{1}{\langle \tilde{\alpha}\beta\rangle^2} \bar{\mathcal{A}}_{\dot{a}} \bar{\partial}_{\dot{b}} \hat{\epsilon}\big|_{\tilde{\alpha}} \Big) \,. \tag{3.2.28}$$

Integrating by parts gives

$$\delta_r S_{6\mathrm{dCS}} \propto \int_{\mathbb{E}^4} \mathrm{vol}_4 \, \mathrm{Tr}\Big(\epsilon^{(r)} \Big(\frac{\alpha^a}{\langle \alpha\beta\rangle^2} \partial_{a\dot{a}} \bar{\mathcal{A}}^{\dot{a}}|_{\alpha} - \frac{\tilde{\alpha}^a}{\langle \tilde{\alpha}\beta\rangle^2} \partial_{a\dot{a}} \bar{\mathcal{A}}^{\dot{a}}|_{\tilde{\alpha}} \Big) \Big) \,. \tag{3.2.29}$$

Introducing new currents defined by

$$\langle \alpha\beta\rangle \, C_{\dot{a}} = \bar{\mathcal{A}}_{\dot{a}}|_{\alpha} \,, \quad \langle \tilde{\alpha}\beta\rangle \, \widetilde{C}_{\dot{a}} = \bar{\mathcal{A}}_{\dot{a}}|_{\tilde{\alpha}} \,, \tag{3.2.30}$$

the conservation law associated to the r-symmetry is given by

$$\frac{1}{\langle \alpha\beta\rangle} \alpha^a \partial_{a\dot{a}} C^{\dot{a}} - \frac{1}{\langle \tilde{\alpha}\beta\rangle} \tilde{\alpha}^a \partial_{a\dot{a}} \widetilde{C}^{\dot{a}} = 0 \,. \tag{3.2.31}$$

Recalling from eq. (3.2.9) that $\bar{\mathcal{A}}_{\dot{a}} = \langle \pi\beta\rangle \mathrm{Ad}_{\hat{g}}^{-1} B_{\dot{a}} + \pi^a \hat{g}^{-1} \partial_{a\dot{a}} \hat{g}$, we can relate the B current to the C and $\widetilde{C}$ currents as follows

$$C_{\dot{a}} = \mathrm{Ad}_g^{-1} B_{\dot{a}} + \frac{1}{\langle \alpha\beta\rangle} \alpha^a g^{-1} \partial_{a\dot{a}} g = (\widehat{b} - \widehat{\jmath}) \mu_{\dot{a}} - (b - j) \hat{\mu}_{\dot{a}} \,, \tag{3.2.32}$$

$$\widetilde{C}_{\dot{a}} = \mathrm{Ad}_{\tilde{g}}^{-1} B_{\dot{a}} + \frac{1}{\langle \tilde{\alpha}\beta\rangle} \tilde{\alpha}^a \tilde{g}^{-1} \partial_{a\dot{a}} \tilde{g} = \sigma(\widehat{b} - \widehat{\jmath}) \mu_{\dot{a}} - \sigma^{-1} (b - j) \hat{\mu}_{\dot{a}} \,, \tag{3.2.33}$$

where we have used the identities (3.2.15). The transformation of these currents under the (ℓ, r)-symmetries is given by

$$
\begin{aligned}
(\ell, r): &\quad B_{\dot{a}} \mapsto \mathrm{Ad}_\ell B_{\dot{a}} - \langle \alpha\beta\rangle^{-1}\alpha^a \partial_{a\dot{a}}\ell\ell^{-1} \ , \\
(\ell, r): &\quad C_{\dot{a}} \mapsto \mathrm{Ad}_r^{-1} C_{\dot{a}} + \langle \alpha\beta\rangle^{-1}\alpha^a r^{-1}\partial_{a\dot{a}} r \ , \\
(\ell, r): &\quad \widetilde{C}_{\dot{a}} \mapsto \mathrm{Ad}_r^{-1} \widetilde{C}_{\dot{a}} + \langle \tilde{\alpha}\beta\rangle^{-1}\tilde{\alpha}^a r^{-1}\partial_{a\dot{a}} r \ .
\end{aligned}
\tag{3.2.34}
$$

As a consequence notice that the 4d ($\mathbb{CP}^1$-independent) gauge field introduced above, $A_{a\dot{a}} = \beta_a B_{\dot{a}}$, is invariant under the right action, whereas the left action acts as a conventional gauge transformation

$$
(\ell, r): \quad A_{a\dot{a}} \mapsto \mathrm{Ad}_\ell A_{a\dot{a}} - \partial_{a\dot{a}}\ell\ell^{-1} \ , \tag{3.2.35}
$$

albeit semi-local rather than fully local since ℓ is constrained as in eq. (3.1.13). The transformation of these currents and the 4d ASD connection also follows from the hCS_6 description. While the original gauge transformations act on $\bar{\mathcal{A}}$, we observe that r and $\tilde{r}$ become right-actions on $\hat{g}$, leaving $\bar{\mathcal{A}}'$ and $A_{a\dot{a}}$ invariant. By comparison, after fixing $\hat{g}|_\beta = \mathrm{id}$, it is only a combination of the 'internal' transformations and the original gauge transformations that preserve this constraint. In particular, ℓ has an induced action on g, $\tilde{g}$ and $\bar{\mathcal{A}}'$, thus leading to the above transformations of $B_{\dot{a}}$ and $A_{a\dot{a}}$.

As we will show momentarily, the equations of motion of the theory correspond to anti-self duality of the field strength of the connection $A_{a\dot{a}}$, hence it immediately follows that the equations of motion are preserved by the symmetry transformations (3.2.35). To close the section we note that the action is concisely given in terms of the currents as

$$
S_{\mathrm{IFT}_4} = \frac{K}{\langle \alpha\tilde{\alpha}\rangle} \int_{\mathbb{E}^4} \mathrm{vol}_4 \, \epsilon^{\dot{a}\dot{b}} \mathrm{Tr}\big(\mathrm{Ad}_g^{-1} B_{\dot{a}} (C_{\dot{b}} - \Lambda^T \widetilde{C}_{\dot{b}})\big) + S_{\mathrm{WZ}_4} \ . \tag{3.2.36}
$$

3.2.2 Equations of Motion, 4d ASDYM and Lax Formulation

The equations of motion of the IFT_4 can be obtained in a brute force fashion by performing a variation of the action (3.2.16). This calculation requires the variation of the operators $U_\pm$

$$
\delta U_\pm(X) = U_\pm(\delta X) + U_\pm\left([\tilde{g}^{-1}\delta\tilde{g}, U_\mp^T(X)]\right) - U_\mp^T\left([g^{-1}\delta g, U_\pm(X)]\right) \ , \tag{3.2.37}
$$

but is otherwise straightforward. The outcome is that the equations of motion can be written as

$$\begin{aligned}
&-\frac{\mu^{\dot a}\alpha^a}{\langle\alpha\beta\rangle}\partial_{a\dot a}\widehat{b}+\frac{\hat\mu^{\dot a}\alpha^a}{\langle\alpha\beta\rangle}\partial_{a\dot a}b+[\widehat{\jmath},b]-[j,\widehat{b}]-[\widehat{b},b]=0\ ,\\
&-\frac{\mu^{\dot a}\tilde\alpha^a}{\langle\tilde\alpha\beta\rangle}\partial_{a\dot a}\widehat{\widetilde{b}}+\frac{\hat\mu^{\dot a}\tilde\alpha^a}{\langle\tilde\alpha\beta\rangle}\partial_{a\dot a}\widetilde{b}+[\widehat{\widetilde{\jmath}},\widetilde{b}]-[\widetilde{\jmath},\widehat{\widetilde{b}}]-[\widehat{\widetilde{b}},\widetilde{b}]=0\ ,
\end{aligned} \tag{3.2.38}$$

in which we invoke the definitions of b, $\widetilde{b}$, $\widehat{b}$ and $\widehat{\widetilde{b}}$ above in eqs. (3.2.13) and (3.2.14). These equations can be written in terms of the current $B_{\dot a}$ as

$$\begin{aligned}
&\alpha^a\partial_{a\dot a}B^{\dot a}+\frac{1}{2}\langle\alpha\beta\rangle\big[B_{\dot a},B^{\dot a}\big]=0\ ,\\
&\tilde\alpha^a\partial_{a\dot a}B^{\dot a}+\frac{1}{2}\langle\tilde\alpha\beta\rangle\big[B_{\dot a},B^{\dot a}\big]=0\ .
\end{aligned} \tag{3.2.39}$$

Taking a weighted sum and difference equations gives

$$\begin{aligned}
\big(\langle\tilde\alpha\beta\rangle\alpha^a-\langle\alpha\beta\rangle\tilde\alpha^a\big)\partial_{a\dot a}B^{\dot a}&=-\langle\alpha\tilde\alpha\rangle\beta^a\partial_{a\dot a}B^{\dot a}=0\ ,\\
\big(\langle\tilde\alpha\beta\rangle\alpha^a+\langle\alpha\beta\rangle\tilde\alpha^a\big)\partial_{a\dot a}B^{\dot a}&+\langle\tilde\alpha\beta\rangle\langle\alpha\beta\rangle\big[B_{\dot a},B^{\dot a}\big]=0\ ,
\end{aligned} \tag{3.2.40}$$

the first of which is the anticipated conservation equation for the ℓ-symmetry. Making use of the definitions of C and $\widetilde{C}$ in (3.2.32), the equations of motion are equivalently expressed as

$$\begin{aligned}
&\alpha^a\partial_{a\dot a}C^{\dot a}+\frac{1}{2}\langle\alpha\beta\rangle[C_{\dot a},C^{\dot a}]=0\ ,\\
&\tilde\alpha^a\partial_{a\dot a}\widetilde{C}^{\dot a}+\frac{1}{2}\langle\tilde\alpha\beta\rangle[\widetilde{C}_{\dot a},\widetilde{C}^{\dot a}]=0\ .
\end{aligned} \tag{3.2.41}$$

Noting that $[C_{\dot a},C^{\dot a}]=[\widetilde{C}_{\dot a},\widetilde{C}^{\dot a}]$ we can take the difference of these equations to obtain

$$\frac{1}{\langle\alpha\beta\rangle}\alpha^a\partial_{a\dot a}C^{\dot a}-\frac{1}{\langle\tilde\alpha\beta\rangle}\tilde\alpha^a\partial_{a\dot a}\widetilde{C}^{\dot a}=0\ , \tag{3.2.42}$$

which is the anticipated conservation law for the r-symmetry.

ASDYM. We will now justify the claim that this theory is integrable by constructing explicit Lax pair formulations of the dynamics in two different fashions. First we will show the equations of motion (3.2.39) can be recast as the anti-self-dual equation for a Yang-Mills connection. Before demonstrating that this holds for our particular model, let us highlight that this follows from the construction of hCS$_6$ by briefly reviewing the Penrose-Ward correspondence [War77]. Recalling that we have resolved one of the hCS$_6$ equations of motion $\mathcal{F}'_{0\dot{a}} = 0$ to find $\bar{\mathcal{A}}'_{\dot{a}} = \pi^a A_{a\dot{a}}$, it follows that the remaining system of equations should be equivalent to the vanishing of the other components of the field strength $\mathcal{F}'_{\dot{a}\dot{b}} = 0$. We may express this in terms of the anti-holomorphic covariant derivative $\bar{D}'_{\dot{a}} = \bar{\partial}_{\dot{a}} + \bar{\mathcal{A}}'_{\dot{a}}$ as $\left[\bar{D}'_{\dot{a}}, \bar{D}'_{\dot{b}}\right] = 0$, which may also be written as

$$\pi^a \pi^b [D_{a\dot{a}}, D_{b\dot{b}}] = 0 \ . \tag{3.2.43}$$

This is equivalent to the vanishing of $\pi^a \pi^b F_{a\dot{a}Bb}$ where F is the field strength of the 4d connection $A_{a\dot{a}}$. To make contact with the anti-self-dual Yang-Mills equation, note that an arbitrary tensor that is anti-symmetric in Lorentz indices, e.g. $F_{\mu\nu}$, can be expanded in spinor indices as

$$F_{a\dot{a}b\dot{b}} = \epsilon_{\dot{a}\dot{b}}\, \phi_{ab} + \epsilon_{ab}\, \tilde{\phi}_{\dot{a}\dot{b}} \ . \tag{3.2.44}$$

Here, ϕ and $\tilde{\phi}$ are both symmetric and correspond to the self-dual and anti-self-dual components of the field strength respectively. Explicitly computing the contraction (3.2.43), we find that the remaining equation is simply $\phi = 0$ which is indeed the anti-self-dual Yang-Mills equation. In effect, this argument demonstrates that a holomorphic gauge field on twistor space (which is gauge-trivial in $\mathbb{CP}^1$) is in bijection with a solution to the 4-dimensional anti-self-dual Yang-Mills equation – this statement is the Penrose-Ward correspondence.

Now, returning to the case at hand, recall that our connection is of the form $A_{a\dot{a}} = \beta_a B_{\dot{a}}$, so the anti-self-dual Yang-Mills equation specialises to

$$\langle \pi\beta \rangle \big(\pi^a \partial_{a\dot{a}} B_B - \pi^b \partial_{Bb} B_{\dot{a}} + \langle \pi\beta \rangle [B_{\dot{a}}, B_B] \big) = 0 \ . \tag{3.2.45}$$

This should hold for any $\pi^a \in \mathbb{CP}^1$ and we can extract the key information by expanding π^a in the basis formed by α^a and $\tilde{\alpha}^a$, that is

$$\pi^a = \frac{1}{\langle \alpha\tilde{\alpha} \rangle} \Big(\langle \pi\tilde{\alpha} \rangle \alpha^a - \langle \pi\alpha \rangle \tilde{\alpha}^a \Big) \ . \tag{3.2.46}$$

Substituting into (3.2.45), we find two independent equations with $\mathbb{CP}^1$-dependent coefficients $\langle \pi\beta \rangle \langle \pi\tilde{\alpha} \rangle$ and $\langle \pi\beta \rangle \langle \pi\alpha \rangle$ respectively. These are explicitly given by the two equations in eq. (3.2.39). Therefore, as expected, the equations of motion for this IFT$_4$

are equivalent to the anti-self-dual Yang-Mills equation for $A_{a\dot{a}} = \beta_a B_{\dot{a}}$.

Let us comment on the relation to Ward's conjecture which postulates that many integrable models arise as reductions of the ASDYM equations. It is clear that that the equations of motion for the λ-deformations explored in this paper arise as symmetry reductions of the ASDYM equations for the explicit form of the connection given above. On the other hand, a generic ASDYM connection can be partially gauge-fixed such that the remaining degrees of freedom are completely captured by the so-called Yang's matrix, which is the fundamental field of the WZW_4 model. In this case, the equations of motion of the WZW_4 model, known as Yang's equations, are the remaining ASDYM equations. It is natural to ask whether a generic ASDYM connection can also be partially gauge-fixed to take the explicit form found in this paper. This would provide a 1-to-1 correspondence between solutions of the ASDYM equations and solutions to our 4d IFT.

B-Lax. The anti-self-dual Yang-Mills equation is also 'integrable' in the sense that it admits a Lax formalism. Using the basis of spinors $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$, we define the Lax pair L and M by

$$L^{(B)} = \langle \pi \hat{\iota} \rangle^{-1} \hat{\mu}^{\dot{a}} \pi^a D_{a\dot{a}} \ , \qquad M^{(B)} = \langle \pi \iota \rangle^{-1} \mu^{\dot{a}} \pi^a D_{a\dot{a}} \ , \tag{3.2.47}$$

where the normalisations are for later convenience. It is important to emphasise that here π^a is not just an *ad hoc* spectral parameter. It is introduced directly as a result of the hCS_6 equations of motion and is the coordinate on $\mathbb{CP}^1 \hookrightarrow \mathbb{PT}$. The vanishing of $\left[L^{(B)}, M^{(B)}\right] = 0$ for any $\pi^a \in \mathbb{CP}^1$ is equivalent to the anti-self-dual Yang-Mills equation.

C-Lax. Let us now turn to the equations of motion cast in terms of the C-currents in eq. (3.2.41). Evidently, looking at the definition of these currents eq. (3.2.32), we see that when $\sigma = 1$ we have $\widetilde{C} = C$ and the equations of motion have the same form as the B-current equations (3.2.39). Therefore, when $\sigma = 1$, we can package the C-equations in terms of a ASDYM connection $A^{(C)}_{a\dot{a}} = \beta_a C_{\dot{a}}$. Away from this point, when $\widetilde{C} \neq C$ it is not immediately evident if these equations follow from an ASDYM connection. Regardless, we can still give these equations a Lax pair presentation as follows.

Letting $\varrho \in \mathbb{C}$ denote a spectral parameter we define

$$\begin{aligned} L^{(C)} &= \frac{1}{n_L}\hat{\mu}^{\dot{a}}\left(\frac{\alpha^a}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\sigma^{-1}\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle}(1-\varrho)\right)\partial_{a\dot{a}} + \frac{1}{n_L}\hat{\mu}^{\dot{a}}C_{\dot{a}} \ , \\ M^{(C)} &= \frac{1}{n_M}\mu^{\dot{a}}\left(\frac{\alpha^a}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\sigma\tilde{\alpha}^a}{\langle\tilde{\alpha}\beta\rangle}(1-\varrho)\right)\partial_{a\dot{a}} + \frac{1}{n_M}\mu^{\dot{a}}C_{\dot{a}} \ . \end{aligned} \tag{3.2.48}$$

Noting that $\mu^B\widetilde{C}_B = \sigma^{-1}\mu^B C_B$ and $\hat{\mu}^B\widetilde{C}_B = \sigma\hat{\mu}^B C_B$ one immediately sees that the terms inside $[L^{(C)}, M^{(C)}]$ linear in ϱ yield the conservation law eq. (3.2.42). The contributions independent of ϱ, combined with eq. (3.2.42), give either of eq. (3.2.41). It will be convenient to fix the overall normalisation of these Lax operators to be

$$n_L = \frac{\langle\alpha\hat{\iota}\rangle}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\tilde{\alpha}\beta\rangle}\sigma^{-1}(1-\varrho) \ , \qquad n_M = \frac{\langle\alpha\iota\rangle}{\langle\alpha\beta\rangle}(1+\varrho) + \frac{\langle\tilde{\alpha}\iota\rangle}{\langle\tilde{\alpha}\beta\rangle}\sigma(1-\varrho) \ . \tag{3.2.49}$$

Unlike the spectral parameter π_a entering the B-Lax, there is no clear way to associate the spectral parameter of the C-Lax, ϱ, with the $\mathbb{CP}^1$ coordinate alone. Indeed, under a natural assumption, we will see that ϱ has origins from both the $\mathbb{CP}^1$ geometry *and* the parameters that enter the boundary conditions.

The existence of a second Lax formulation of the theory, distinct from the ASDYM equations encoded via the B-Lax, is an unexpected feature. We will see shortly that, upon symmetry reduction, this twin Lax formulation is inherited by the IFT_2.

3.2.3 Reality Conditions and Parameters

To understand how the reality of the action of the IFT_4 (3.2.16) with (3.2.17), as well as the dependence on the parameters K, σ, α_a, $\tilde{\alpha}_a$, β_a, $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$, let us denote our coordinates

$$\begin{aligned} \mathsf{w} &= \frac{\langle\alpha\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\hat{\mu}_{\dot{a}}\tilde{\alpha}_a x^{a\dot{a}} \ , \qquad & \hat{\mathsf{w}} &= -\frac{\langle\alpha\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\mu_{\dot{a}}\tilde{\alpha}_a x^{a\dot{a}} \ , \\ \mathsf{z} &= -\frac{\langle\tilde{\alpha}\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\hat{\mu}_{\dot{a}}\alpha_a x^{a\dot{a}} \ , \qquad & \hat{\mathsf{z}} &= \frac{\langle\tilde{\alpha}\beta\rangle}{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}\mu_{\dot{a}}\alpha_a x^{a\dot{a}} \ , \end{aligned} \tag{3.2.50}$$

such that

$$j = g^{-1}\partial_{\mathsf{w}} g \ , \qquad \widehat{\jmath} = g^{-1}\partial_{\hat{\mathsf{w}}} g \ , \qquad \widetilde{\jmath} = \tilde{g}^{-1}\partial_{\mathsf{z}}\tilde{g} \ , \qquad \widehat{\widetilde{\jmath}} = \tilde{g}^{-1}\partial_{\hat{\mathsf{z}}}\tilde{g} \ . \tag{3.2.51}$$

In this subsection we let $\mu^{\dot{a}}$ and $\hat{\mu}^{\dot{a}}$ be an unconstrained basis of spinors, i.e., not related by Euclidean conjugation or of fixed norm. This means the action (3.2.16) with (3.2.17) comes with an extra factor of $[\mu\hat{\mu}]^{-1}$. Writing the volume element $\mathrm{vol}_4 = \frac{1}{12}\epsilon_{\dot{a}\dot{b}}\epsilon_{\dot{c}\dot{d}}\epsilon_{ac}\epsilon_{bd}dx^{a\dot{a}} \wedge dx^{BB'} \wedge dx^{c\dot{c}} \wedge dx^{d\dot{d}}$ in terms of the coordinates $\{\mathsf{w}, \hat{\mathsf{w}}, \mathsf{z}, \hat{\mathsf{z}}\}$ we find

$$\mathrm{vol}_4 = \frac{\langle\alpha\tilde{\alpha}\rangle^2[\mu\hat{\mu}]^2}{\langle\alpha\beta\rangle^2\langle\tilde{\alpha}\beta\rangle^2}\mathrm{d}\mathsf{w} \wedge \mathrm{d}\hat{\mathsf{w}} \wedge \mathrm{d}\mathsf{z} \wedge \mathrm{d}\hat{\mathsf{z}} = \frac{\langle\alpha\tilde{\alpha}\rangle^2[\mu\hat{\mu}]^2}{\langle\alpha\beta\rangle^2\langle\tilde{\alpha}\beta\rangle^2}\mathrm{vol}_4' \ . \tag{3.2.52}$$

Substituting into the action (3.2.16) with (3.2.17), we see that the IFT_4 now only depends explicitly on two parameters

$$K' = \frac{\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}{\langle\alpha\beta\rangle^2\langle\tilde{\alpha}\beta\rangle^2}K \quad \text{and} \quad \sigma \ . \tag{3.2.53}$$

Moreover, the action is invariant under the following $GL(1;\mathbb{C})$ space-time symmetry

$$\mathsf{z} \to e^{\vartheta}\mathsf{z} \ , \qquad \mathsf{w} \to e^{\vartheta}\mathsf{w} \ , \qquad \hat{\mathsf{z}} \to e^{-\vartheta}\hat{\mathsf{z}} \ , \qquad \hat{\mathsf{w}} \to e^{-\vartheta}\hat{\mathsf{w}} \ , \tag{3.2.54}$$

where $\vartheta \in \mathbb{C}$.

To find a real action we should impose reality conditions on the coordinates $\{\mathsf{w}, \hat{\mathsf{w}}, \mathsf{z}, \hat{\mathsf{z}}\}$, the fields g and $\tilde{g}$, and the parameters K' and σ. We start by observing four sets of admissible reality conditions simply found by inspection of the 4d IFT. Note that, implicitly, we will not assume Euclidean reality conditions for $x^{a\dot{a}}$. Starting from Euclidean reality conditions we complexify and take different split signature real slices. We will then turn to the hCS_6 origin of the different reality conditions.

Introducing Θ, the lift of an antilinear involutive automorphism θ of the Lie algebra $\mathfrak{g}$ to the group G, the four sets of reality conditions are as follows:

1. In the first case, the coordinates are all real, $\bar{\mathsf{w}} = \mathsf{w}$, $\bar{\hat{\mathsf{w}}} = \hat{\mathsf{w}}$, $\bar{\mathsf{z}} = \mathsf{z}$, $\bar{\hat{\mathsf{z}}} = \hat{\mathsf{z}}$; K' and σ are real; and the group-valued fields are elements of the real form, $\Theta(g) = g$ and $\Theta(\tilde{g}) = \tilde{g}$. Under conjugation we have $U_\pm \to U_\pm$.

2. In the second case, the coordinates conjugate as $\bar{\mathsf{w}} = \hat{\mathsf{w}}$, $\bar{\mathsf{z}} = \hat{\mathsf{z}}$; K' is imaginary and σ is a phase factor; and the group-valued fields are elements of the real form,

$\Theta(g) = g$ and $\Theta(\tilde{g}) = \tilde{g}$. Under conjugation we have $U_\pm \to U_\mp$.

3. In the third case, the coordinates conjugate as $\bar{\mathsf{w}} = \hat{\mathsf{z}}$, $\bar{\mathsf{z}} = \hat{\mathsf{w}}$; K' and σ are real; and the group-valued fields are related by conjugation $\Theta(g) = \tilde{g}$. Under conjugation we have $U_\pm \to U_\pm^T$.

4. In the final case, the coordinates conjugate as $\bar{\mathsf{w}} = \mathsf{z}$, $\bar{\hat{\mathsf{w}}} = \hat{\mathsf{z}}$; K' is imaginary and σ is a phase factor; and the group-valued fields are related by conjugation $\Theta(g) = \tilde{g}$. Under conjugation we have $U_\pm \to U_\mp^T$.

The action (3.2.16) with (3.2.17) is real for each of these sets of reality conditions. To determine the signature for each set of reality conditions, we note that[7]

$$\epsilon_{\dot{a}\dot{b}}\epsilon_{ab}\mathrm{d}x^{a\dot{a}}\mathrm{d}x^{BB'} = \frac{2\langle\alpha\tilde{\alpha}\rangle[\mu\hat{\mu}]}{\langle\alpha\beta\rangle\langle\tilde{\alpha}\beta\rangle}(\mathrm{d}\mathsf{w}\mathrm{d}\hat{\mathsf{z}} - \mathrm{d}\mathsf{z}\mathrm{d}\hat{\mathsf{w}}) \ . \tag{3.2.55}$$

It is then straightforward to see that the four sets of reality conditions above all correspond to split signature. Note that for the metric to be real, we require the prefactor to be real in cases 1 and 4 and imaginary in cases 2 and 3. We will see that this is indeed the case when we comment on the hCS$_6$ origin.

In cases 1 and 4 the reality conditions are preserved by an $SO(1,1)$ space-time symmetry (3.2.54) with $\vartheta \in \mathbb{R}$. On the other hand, in cases 2 and 3, the reality conditions are preserved by an $SO(2)$ space-time symmetry with $|e^{\vartheta}| = 1$. In § 3.4, we will be interested in symmetry reducing while preserving the space-time symmetry, recovering an action on $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ that is invariant under the Euclidean or Poincaré groups respectively. We have freedom in how we do this since the action is not invariant under $SO(2)$ rotations acting on (z, w) and $(\hat{\mathsf{z}}, \hat{\mathsf{w}})$. Therefore, we can choose to symmetry reduce along different directions in each of these planes, in principle introducing an additional two parameters. We should note that in the Euclidean case, since the two planes are related by conjugation, we will break the reality properties of the action unless

[7] Conjugating in Euclidean signature we find the reality conditions

$$\bar{\mathsf{w}} = \frac{\langle\hat{\alpha}\hat{\beta}\rangle}{\langle\hat{\alpha}\hat{\tilde{\alpha}}\rangle}\left(\frac{\langle\alpha\hat{\tilde{\alpha}}\rangle}{\langle\alpha\beta\rangle}\hat{\mathsf{w}} + \frac{\langle\tilde{\alpha}\hat{\tilde{\alpha}}\rangle}{\langle\tilde{\alpha}\beta\rangle}\hat{\mathsf{z}}\right) \ , \qquad \bar{\mathsf{z}} = \frac{\langle\hat{\tilde{\alpha}}\hat{\beta}\rangle}{\langle\hat{\alpha}\hat{\tilde{\alpha}}\rangle}\left(\frac{\langle\hat{\alpha}\tilde{\alpha}\rangle}{\langle\tilde{\alpha}\beta\rangle}\hat{\mathsf{z}} - \frac{\langle\alpha\hat{\alpha}\rangle}{\langle\alpha\beta\rangle}\hat{\mathsf{w}}\right) \ .$$

As an example, let us take $\hat{\alpha} = \tilde{\alpha}$, in which case the reality conditions simplify to $\bar{\mathsf{w}} = \frac{\langle\hat{\alpha}\hat{\beta}\rangle}{\langle\hat{\alpha}\beta\rangle}\hat{\mathsf{z}}$ and $\bar{\mathsf{z}} = \frac{\langle\alpha\hat{\beta}\rangle}{\langle\alpha\beta\rangle}\hat{\mathsf{w}}$. Substituting into the metric we find $\frac{2\langle\alpha\hat{\alpha}\rangle[\mu\hat{\mu}]}{\langle\alpha\beta\rangle\langle\hat{\alpha}\hat{\beta}\rangle}\left(\mathrm{d}\mathsf{w}\mathrm{d}\bar{\mathsf{w}} + \psi\bar{\psi}\mathrm{d}\mathsf{z}\mathrm{d}\bar{\mathsf{z}}\right)$ where $\psi = \frac{\langle\alpha\beta\rangle}{\langle\alpha\hat{\beta}\rangle}$. Since the prefactor is real and positive, this indeed has Euclidean signature. Note that these reality conditions are distinct from case 3 above, and they do not imply reality of the 4d IFT.

the two symmetry reduction directions are also related by conjugation, reducing the number of parameters by one for a real action. This is not an issue in the Lorentzian case since the coordinates are real, hence we expect to find a four-parameter real Lorentz-invariant IFT_2. We will indeed see that this is the case in § 3.4.

Origin of reality conditions from $\mathbf{hCS_6}$. Let us now briefly describe the origin of the different sets of reality conditions from 6 dimensions. It is shown in [BS23] that for the hCS_6 action to be real we require that

$$\overline{C(\Phi)} = C(\Phi) \ , \tag{3.2.56}$$

where Φ is defined in eq. (3.1.2) and C is a conjugation that acts on the coordinates (x, π), not on the fixed spinors α, $\tilde{\alpha}$ and β.[8] In Euclidean signature this constraint becomes

$$\frac{\bar{K}}{\langle \pi\hat{\alpha}\rangle\langle \pi\hat{\tilde{\alpha}}\rangle\langle \pi\hat{\beta}\rangle^2} = \frac{K}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2} \ . \tag{3.2.57}$$

We immediately see that this has no solutions since the double pole at β is mapped to $\hat{\beta}$ and $\hat{\beta} = \beta$ has no solutions. On the other hand, in split signature we have

$$\frac{\bar{K}}{\langle \pi\bar{\alpha}\rangle\langle \pi\bar{\tilde{\alpha}}\rangle\langle \pi\bar{\beta}\rangle^2} = \frac{K}{\langle \pi\alpha\rangle\langle \pi\tilde{\alpha}\rangle\langle \pi\beta\rangle^2} \ . \tag{3.2.58}$$

This can be solved by taking K and β to be real, and α and $\tilde{\alpha}$ to either both be real or to form a complex conjugate pair.

We also need to ask that the boundary conditions (3.1.8) are compatible with the reality conditions. Imposing $C^*(\mathcal{A}_{\dot{a}}) = \theta(\mathcal{A}_{\dot{a}})$, we can either take μ and $\hat{\mu}$ to either both be real or to form a complex conjugate pair. The two choices of reality conditions for $(\alpha, \tilde{\alpha})$ and the two for $(\mu, \hat{\mu})$ give a total of four sets of reality conditions, which we anticipate will recover those in the list presented above. With the same ordering, we have the following:

1. In the first case, we take real $(\alpha, \tilde{\alpha})$ and real $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ is valued in the real form at the poles, implying that

[8] Conjugation in Euclidean signature can be defined as $C(\mu_{\dot{a}}) = \hat{\mu}_{\dot{a}} = \epsilon_{\dot{a}}{}^{B}\bar{\mu}_B$, $C(\iota'_{\dot{a}}) = \hat{\iota}_a = \epsilon_a{}^{B'}\bar{\iota}_{B'}$ and $C(x^{a\dot{a}}) = (\epsilon^T)^{\dot{a}}{}_B\bar{x}^{BB'}\epsilon_{B'}{}^{a}$ with ${\epsilon_1}^2 = -1$, while in split signature, we take $C(\mu_{\dot{a}}) = \bar{\mu}_{\dot{a}}$, $C(\iota_a) = \bar{\iota}_a$ and $C(x^{a\dot{a}}) = \bar{x}^{a\dot{a}}$. We will restrict our attention to Euclidean and split signatures since there are no ASD connections in Lorentzian signature [BS23].

g and $\tilde{g}$ are as well, and that σ is real. Since both $\langle\alpha\tilde{\alpha}\rangle$ and $[\mu\hat{\mu}]$ are real, real K implies that K' is real using eq. (3.2.53).

2. In the second case, we take real $(\alpha, \tilde{\alpha})$ and complex conjugate $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ is valued in the real form at the poles, implying that g and $\tilde{g}$ are as well, and that σ is a phase factor. Since $\langle\alpha\tilde{\alpha}\rangle$ is real and $[\mu\hat{\mu}]$ is imaginary, real K implies that K' is imaginary using eq. (3.2.53).

3. In the third case, we take complex conjugate $(\alpha, \tilde{\alpha})$ and complex conjugate $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ at α is the conjugate of $\mathcal{A}_{\dot{a}}$ at $\tilde{\alpha}$, implying that g and $\tilde{g}$ are also conjugate, and that σ is real. Since both $\langle\alpha\tilde{\alpha}\rangle$ and $[\mu\hat{\mu}]$ are imaginary, real K implies that K' is real using eq. (3.2.53).

4. In the final case, we take complex conjugate $(\alpha, \tilde{\alpha})$ and real $(\mu, \hat{\mu})$. Analysing the boundary conditions we find that $\mathcal{A}_{\dot{a}}$ at α is the conjugate of $\mathcal{A}$ at $\tilde{\alpha}$, implying that g and $\tilde{g}$ are also conjugate, and that σ is a phase factor. Since $\langle\alpha\tilde{\alpha}\rangle$ is imaginary and $[\mu\hat{\mu}]$ is real, real K implies that K' is imaginary using eq. (3.2.53).

Finally, one can also check that in split signature, the different reality conditions for $(\alpha, \tilde{\alpha})$ and $(\mu, \hat{\mu})$ imply the different reality conditions for the coordinates $\{\mathsf{w}, \hat{\mathsf{w}}, \mathsf{z}, \hat{\mathsf{z}}\}$ given above.

As implied above, see also [BS23], a real action in split signature in 4 dimensions is useful for symmetry reducing and constructing real 2d IFTs since both Euclidean and Lorentzian signature in 2 dimensions can be accessed. However, the lack of a real action in Euclidean signature raises questions about the quantisation of the IFT_4 itself.

3.2.4 Equivalent Forms of the Action and its Limits

In this section we describe alternative, but equivalent ways of writing the action of the 4d IFT (3.2.16) with (3.2.17), and consider two interesting limits of the theory. These constructions are motivated by analogous ones that are important in the context of the 2d λ-deformed WZW model.

First, let us note that the IFT_4 (3.2.16) with (3.2.17) can be written in the following two equivalent forms

$$\begin{aligned} S_{\mathrm{IFT}_4} &= K' \int_{\mathbb{E}^4} \mathrm{vol}_4' \, \mathrm{Tr}\Big((j - \sigma \widetilde{\jmath})(U_+^T - U_-)(\widehat{\jmath} + \sigma^{-1} \widehat{\widetilde{\jmath}}) - \sigma \widetilde{\jmath}\, \widehat{\jmath} + \sigma^{-1} j \, \widehat{\widetilde{\jmath}} \Big) + S_{\mathrm{WZ}_4} \\ &= K' \int_{\mathbb{E}^4} \mathrm{vol}_4' \, \mathrm{Tr}\Big((\mathrm{Ad}_g j - \mathrm{Ad}_{\tilde{g}} \widetilde{\jmath})(\widetilde{U}_+^T - \widetilde{U}_-)(\mathrm{Ad}_g \widehat{\jmath} - \mathrm{Ad}_{\tilde{g}} \widehat{\widetilde{\jmath}}) \\ &\qquad + \mathrm{Ad}_{\tilde{g}} \widetilde{\jmath} \, \mathrm{Ad}_g \widehat{\jmath} - \mathrm{Ad}_g j \, \mathrm{Ad}_{\tilde{g}} \widehat{\widetilde{\jmath}} \Big) + S_{\mathrm{WZ}_4} \ , \end{aligned} \tag{3.2.59}$$

where

$$\begin{aligned} U_\pm &= \big(1 - \sigma^{\pm 1} \Lambda\big)^{-1} \ , \qquad \Lambda = \mathrm{Ad}_{\tilde{g}}^{-1} \mathrm{Ad}_g \ , \\ \widetilde{U}_\pm &= \big(1 - \sigma^{\pm 1} \widetilde{\Lambda}\big)^{-1} \ , \qquad \widetilde{\Lambda} = \mathrm{Ad}_g \mathrm{Ad}_{\tilde{g}}^{-1} \ . \end{aligned} \tag{3.2.60}$$

Written in this way, it is straightforward to see that the symmetries of the 4d IFT are given by transformations of the form (3.2.22) with

$$(\partial_{\mathsf{w}} - \sigma \partial_{\mathsf{z}}) r = (\partial_{\hat{\mathsf{w}}} - \sigma^{-1} \partial_{\hat{\mathsf{z}}}) r = 0 \ , \qquad (\partial_{\mathsf{w}} - \partial_{\mathsf{z}}) \ell = (\partial_{\hat{\mathsf{w}}} - \partial_{\hat{\mathsf{z}}}) \ell = 0 \ , \tag{3.2.61}$$

which, as expected, coincide with (3.1.13) and (3.1.14) upon using the definitions (3.2.50).

We can also introduce auxiliary fields $B^{\dot{a}}$, $C^{\dot{a}}$ and $\widetilde{C}^{\dot{a}}$ to rewrite the action as

$$\begin{aligned} S_{\mathrm{IFT}_4} = K' \int_{\mathrm{E}_4} \mathrm{vol}_4' \, \mathrm{Tr}\big(& j \, \widehat{\jmath} - 2 j \mathrm{Ad}_g^{-1} [B\hat{\mu}] + 2 \widehat{\jmath} [C\mu] - 2 [C\mu] \mathrm{Ad}_g^{-1} [B\hat{\mu}] \\ & + \widetilde{\jmath} \, \widehat{\widetilde{\jmath}} - 2 \widehat{\widetilde{\jmath}} \mathrm{Ad}_{\tilde{g}}^{-1} [B\mu] + 2 \widetilde{\jmath} [\widetilde{C}\hat{\mu}] - 2 [\widetilde{C}\hat{\mu}] \mathrm{Ad}_{\tilde{g}}^{-1} [B\mu] \\ & + 2 [B\mu][B\hat{\mu}] + 2 \sigma^{-1} [C\mu][\widetilde{C}\hat{\mu}] \big) + S_{\mathrm{WZ}_4} \ . \end{aligned} \tag{3.2.62}$$

Here we take the auxiliary fields $B^{\dot{a}}$, $C^{\dot{a}}$ and $\widetilde{C}^{\dot{a}}$ to be undetermined. Varying the action and solving their equations of motion, we find that on-shell, they are given by the expressions introduced above in eqs. (3.2.13) and (3.2.33). Moreover, substituting their on-shell values back into (3.2.62) we recover the 4d IFT. Using the symmetry (3.1.9), we note that the action can also be written in a similar equivalent form, in which tilded and untilded quantities are swapped, $\sigma \to \sigma^{-1}$, $K' \to -K'$ and $\widetilde{B} = B$. This can also be seen by making the off-shell replacements $[B\mu] \to [B\mu]$, $[B\hat{\mu}] \to \mathrm{Ad}_g([C\hat{\mu}] + \widehat{\jmath})$, $[C\mu] \to \sigma[\widetilde{C}\mu]$ and $[\widetilde{C}\hat{\mu}] \to \mathrm{Ad}_{\tilde{g}}^{-1}[B\hat{\mu}] - \widehat{\widetilde{\jmath}}$, all of which are compatible with the on-shell values of the auxiliary fields.

The first limit we consider is $\sigma \to 0$, in which the action becomes

$$S_{\mathrm{IFT}_4}|_{\sigma\to 0} = \mathring{S}_{\mathrm{IFT}_4} = K' \int_{\mathbb{E}^4} \mathrm{vol}_4' \, \mathrm{Tr}\big(j\,\widehat{\jmath} + \widetilde{\jmath}\,\widehat{\widetilde{\jmath}} - 2\mathrm{Ad}_g j\, \mathrm{Ad}_{\tilde{g}} \widehat{\widetilde{\jmath}}\big) + S_{\mathrm{WZ}_4} \ . \tag{3.2.63}$$

This has the form of a current-current coupling between two building blocks that could be described as 'holomorphic WZW$_4$' of the form

$$S_{\mathrm{hWZW4}}[g,\alpha] = \int_{\mathbb{E}^4} \mathrm{vol}_4' \, \mathrm{Tr}\big(j\,\widehat{\jmath}\big) - \int_{\mathbb{E}^4\times[0,1]} \mathrm{vol}_4' \wedge d\rho \, \mathrm{Tr}\big(g^{-1}\partial_\rho g[j,\widehat{\jmath}]\big) \ . \tag{3.2.64}$$

This somewhat unusual theory has derivatives only in the holomorphic two-plane singled out by the complex structure on $\mathbb{E}^4$ defined by α (i.e. only ∂_{w} and $\partial_{\hat{\mathsf{w}}}$ enter), although the field depends on all coordinates of $\mathbb{E}^4$. This structure is quite different (both in the kinetic term and Wess-Zumino term) from the conventional WZW$_4$ [Don85; Los+96] for which the action[9] is

$$\begin{aligned} S_{\mathrm{WZW}_4}[h,\alpha,\beta] &= \int_{\mathbb{E}^4} \mathrm{Tr}\big(g^{-1}dg \wedge \star g^{-1}dg\big) + \frac{1}{6}\int_{\mathbb{E}^4\times[0,1]} \varpi_{\alpha,\beta} \wedge \mathrm{Tr}\big((g^{-1}dg)^3\big) \ , \\ \varpi_{\alpha,\beta} &= \epsilon_{\dot{a}\dot{b}}\alpha_a\beta_b \mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}} \ . \end{aligned} \tag{3.2.65}$$

The Kähler point of the theory is achieved when $\beta = \hat{\alpha}$ such that $\varpi_{\alpha,\beta}$ is the Kähler form associated to the complex structure defined by α. In fact, the WZ term of our holomorphic WZW$_4$ is of this general form with $\beta = \alpha$ such that $\varpi_{\alpha,\beta}$ defines a $(2,0)$-form. However even in the $\beta = \alpha$ case, the kinetic term does not match that of the holomorphic WZW$_4$ action.

Returning to the holomorphic WZW$_4$, we can establish that the theory is invariant under a rather large set of symmetries. Since only w and $\hat{\mathsf{w}}$ derivatives enter, it is immediate that the transformation $g \mapsto l(\hat{\mathsf{z}},\mathsf{z})gr(\hat{\mathsf{z}},\mathsf{z})$ leaves the action eq. (3.2.64) invariant. These are further enhanced, as in a WZW$_2$, to

$$(\ell, r): \quad g \mapsto \ell(\mathsf{z},\hat{\mathsf{z}},\mathsf{w})\, g\, r(\mathsf{z},\hat{\mathsf{z}},\hat{\mathsf{w}}) \ . \tag{3.2.66}$$

From this perspective holomorphic WZW$_4$ can be considered the embedding of a WZW$_2$

[9] This is also the 4d IFT that was found in [BS23; Pen21] from hCS$_6$ with two double poles at $\pi = \alpha$ and $\pi = \beta$, with Dirichlet boundary conditions.

in 4 dimensions. Similarly, we have a symmetry for the holomorphic WZW_4 for $\tilde{g}$

$$(\tilde{\ell},\tilde{r}): \quad \tilde{g} \mapsto \tilde{\ell}(\hat{\mathsf{z}},\hat{\mathsf{w}},\mathsf{w})\,\tilde{g}\,\tilde{r}(\mathsf{z},\hat{\mathsf{w}},\mathsf{w}) \ . \tag{3.2.67}$$

The interaction term, $\mathrm{Ad}_g \jmath\, \mathrm{Ad}_{\tilde{g}} \widehat{\tilde{\jmath}}$, in the action (3.2.63) preserves the right actions, but breaks the enhanced independent $\ell, \tilde{\ell}$ left actions. Instead a new 'diagonal' left action emerges

$$(\ell, r, \tilde{r}): \quad g \mapsto \ell(\mathsf{z}+\mathsf{w},\hat{\mathsf{z}}+\hat{\mathsf{w}})\,g\,r(\mathsf{z},\hat{\mathsf{z}},\hat{\mathsf{w}}) \ , \quad \tilde{g} \mapsto \ell(\mathsf{z}+\mathsf{w},\hat{\mathsf{z}}+\hat{\mathsf{w}})\,\tilde{g}\,\tilde{r}(\mathsf{z},\mathsf{w},\hat{\mathsf{w}}) \ . \tag{3.2.68}$$

It is important to emphasise that here the right actions on g and $\tilde{g}$ are independent (r and $\tilde{r}$ are not the same). This stems from the enlargement of the residual symmetries of the 6-dimensional boundary conditions. The constraints of eq. (3.1.14) are relaxed such that gauge parameters at different poles are unrelated but are chiral.

In this limit the currents associated to the left and right action become

$$\begin{aligned} B_{\dot{a}}|_{\sigma\to 0} &= \mathring{B}_{\dot{a}} = \mathrm{Ad}_{\tilde{g}} \widehat{\tilde{\jmath}}\, \mu_{\dot{a}} - \mathrm{Ad}_g \jmath\, \hat{\mu}_{\dot{a}} \ , \\ C_{\dot{a}}|_{\sigma\to 0} &= \mathring{C}_{\dot{a}} = -\big(\widehat{\jmath} - \Lambda^{-1}\widehat{\tilde{\jmath}}\big)\mu_{\dot{a}} \ , \\ \widetilde{C}_{\dot{a}}|_{\sigma\to 0} &= \mathring{\widetilde{C}}_{\dot{a}} = \big(\tilde{\jmath} - \Lambda \jmath\big)\hat{\mu}_{\dot{a}} \ . \end{aligned} \tag{3.2.69}$$

The conservation laws become

$$\begin{gathered} \partial_{\mathsf{w}}(\widehat{\jmath} - \Lambda^{-1}\widehat{\tilde{\jmath}}) = 0 \ , \qquad \partial_{\hat{\mathsf{z}}}(\tilde{\jmath} - \Lambda \jmath) = 0 \ , \\ \partial_{\hat{\mathsf{w}}}(\mathrm{Ad}_g \jmath) - \partial_{\mathsf{w}}(\mathrm{Ad}_{\tilde{g}} \widehat{\tilde{\jmath}}) + \partial_{\mathsf{z}}(\mathrm{Ad}_{\tilde{g}} \widehat{\tilde{\jmath}}) - \partial_{\hat{\mathsf{z}}}(\mathrm{Ad}_g \jmath) = 0 \ . \end{gathered} \tag{3.2.70}$$

To compute the $\mathcal{O}(\sigma)$ correction to the action (3.2.63) we first note that

$$B_{\dot{a}} = \mathring{B}_{\dot{a}} + \sigma\left(\mathrm{Ad}_g \mathring{\widetilde{C}}_{\dot{a}} + \mathrm{Ad}_{\tilde{g}} \mathring{C}_{\dot{a}}\right) + \mathcal{O}(\sigma^2) \ , \tag{3.2.71}$$

and that the combination $C_{\dot{a}} - \Lambda^T \widetilde{C}_{\dot{a}} = \mathring{C}_{\dot{a}} - \Lambda^T \mathring{\widetilde{C}}_{\dot{a}}$ is independent of σ. Then from the expression of the IFT_4 action in terms of currents (3.2.36), we see that the leading correction to $\mathring{S}_{\mathrm{IFT}_4}$ is given by

$$2\sigma K' \int_{\mathbb{E}^4} \mathrm{vol}_4'\, \epsilon^{\dot{a}\dot{b}} \mathrm{Tr}\big(\mathring{\widetilde{C}}_{\dot{a}} \mathring{C}_{\dot{b}}\big) = -2\sigma K' \int_{\mathbb{E}^4} \mathrm{vol}_4\, \mathrm{Tr}\big((\tilde{\jmath} - \Lambda \jmath)(\widehat{\jmath} - \Lambda^{-1}\widehat{\tilde{\jmath}})\big) \ , \tag{3.2.72}$$

i.e. the perturbing operator is the product of two currents associated to the right-acting symmetries.

The second limit we consider is $\sigma \to 1$. Recall that in this limit, we have that $\widetilde{C} = C$ from eqs. (3.2.32) and (3.2.33), and a symmetry emerges interchanging B and C, as well as g and $\tilde{g}^{-1}$. This is also evident if we set $\sigma = 1$ in (3.2.62). An alternative way to take $\sigma \to 1$ is to first set $g = \exp\left(\frac{\nu}{K'}\right)$ and $\tilde{g} = \exp\left(\frac{\tilde{\nu}}{K'}\right)$, along with $\sigma = e^{\frac{1}{K'}}$ and take $K' \to \infty$. In this limit the 4d IFT becomes

$$S_{\mathrm{IFT}_4}|_{K'\to\infty} = -\int_{\mathbb{E}^4} \mathrm{vol}'_4 \,\mathrm{Tr}\big((\partial_{\mathsf{w}}\nu - \partial_{\mathsf{z}}\tilde{\nu})\frac{1}{1-\mathrm{ad}_\nu + \mathrm{ad}_{\tilde{\nu}}}(\partial_{\hat{\mathsf{w}}}\nu - \partial_{\hat{\mathsf{z}}}\tilde{\nu})\big) \,, \tag{3.2.73}$$

which is reminiscent of a 4d version of the non-abelian T-dual of the principal chiral model, albeit with two fields instead of one. If instead we take the limit in the action with auxiliary fields (3.2.62), also setting $[C\mu] = [B\mu] + \mathcal{O}(K'^{-1})$ and $[\widetilde{C}\hat{\mu}] = [B\hat{\mu}] + \mathcal{O}(K'^{-1})$, we find

$$\begin{aligned} S_{\mathrm{IFT}_4}|_{K'\to\infty} = \int_{\mathrm{E}_4} \mathrm{vol}'_4 \,\mathrm{Tr}\Big(& 2\nu\big(\partial_{\mathsf{w}}[B\hat{\mu}] - \partial_{\hat{\mathsf{w}}}[B\mu] + [[B\hat{\mu}],[B\mu]]\big) \\ & + 2\tilde{\nu}\big(\partial_{\hat{\mathsf{z}}}[B\mu] - \partial_{\mathsf{z}}[B\hat{\mu}] + [[B\mu],[B\hat{\mu}]]\big) - 2[B\mu][B\hat{\mu}]\Big) \,, \end{aligned} \tag{3.2.74}$$

after integrating by parts. Integrating out the auxiliary field $B^{\dot{a}}$, we recover the action (3.2.73). It would be interesting to instead integrate out the fields ν and $\tilde{\nu}$ to give a 4d analogue of 2d non-abelian T-duality. However, note that, unlike in 2 dimensions, ν and $\tilde{\nu}$ do not enforce the flatness of a 4d connection, hence there is no straightforward way to parametrise the general solution to their equations.

3.3 Symmetry Reduction of hCS$_6$ to CS$_4$

The idea of symmetry reduction is to take a truncation of a d-dimensional theory specified by a d-form Lagrangian $\mathcal{L}^d$ depending on a set of fields $\{\Phi\}$ to obtain a lower dimensional theory. We assume here that we are reducing along two directions singled out by vector fields V_i, $i = 1, 2$. The reduction procedure imposes that all fields are invariant, $L_{V_i}\Phi = 0$, with dynamics now specified by the $d-2$-form Lagrangian $\mathcal{L}^{d-2} = V_1 \vee V_2 \vee \mathcal{L}^d$. While similar in spirit to a dimensional reduction, there is no requirement that V_i span a compact space, hence there is no scale separation in this truncation.

In order to perform the symmetry reduction, we will introduce a unit norm spinor ι_a about which we can expand any spinor X_a as

$$X_a = \langle X\hat{\iota}\rangle \iota_a - \langle X\iota\rangle \hat{\iota}_a \ . \tag{3.3.1}$$

The spinor ι_a defines another complex structure on $\mathbb{E}^4$ which coincides with the complex structure on $\mathbb{E}^4 \subset \mathbb{PT}$ at the point $\pi_a = \iota_a$. It coincides with the opposite complex structure – swapping holomorphic and anti-holomorphic – at the antipodal point $\pi_a = \hat{\iota}_a$. Using the spinor $\mu^{\dot{a}}$, we can define a basis of one-forms adapted to this complex structure,

$$\begin{aligned} \mathrm{d}z &= \mu_{\dot{a}}\iota_a \mathrm{d}x^{a\dot{a}} \ , \qquad & \mathrm{d}\bar{z} &= \hat{\mu}_{\dot{a}}\hat{\iota}_a \mathrm{d}x^{a\dot{a}} \ , \\ \mathrm{d}w &= \hat{\mu}_{\dot{a}}\iota_a \mathrm{d}x^{a\dot{a}} \ , \qquad & \mathrm{d}\bar{w} &= -\mu_{\dot{a}}\hat{\iota}_a \mathrm{d}x^{a\dot{a}} \ . \end{aligned} \tag{3.3.2}$$

We will perform symmetry reduction along the vector fields dual to $\mathrm{d}z$ and $\mathrm{d}\bar{z}$,

$$\partial_z = \hat{\mu}^{\dot{a}}\hat{\iota}^a \partial_{a\dot{a}} \ , \quad \partial_{\bar{z}} = \mu^{\dot{a}}\iota^a \partial_{a\dot{a}} \ . \tag{3.3.3}$$

The symmetry reduction along the ∂_z and $\partial_{\bar{z}}$ directions takes us from a theory on $\mathbb{PT}$ to a theory on $\Sigma \times \mathbb{CP}^1$ in which $w, \bar{w}$ are coordinates on the worldsheet Σ.

To perform this reduction, it is expedient [Bit22] to make use of the invariance of the action (3.1.1) under the addition of any $(1,0)$-form to $\mathcal{A} \mapsto \hat{\mathcal{A}} = \mathcal{A} + \rho_0 e^0 + \rho_{\dot{a}} e^{\dot{a}}$. By choosing $\rho_{\dot{a}}$ appropriately, we can ensure that $\hat{\mathcal{A}}$ has no $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs and is given by

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}_w \mathrm{d}w + \hat{\mathcal{A}}_{\bar{w}} \mathrm{d}\bar{w} + \mathcal{A}_0 \bar{e}^0 \ , \tag{3.3.4}$$

where these components are related to those of $\mathcal{A}$ by

$$\hat{\mathcal{A}}_w = -\frac{[\bar{\mathcal{A}}\mu]}{\langle \pi\iota\rangle} \ , \qquad \hat{\mathcal{A}}_{\bar{w}} = -\frac{[\bar{\mathcal{A}}\hat{\mu}]}{\langle \pi\hat{\iota}\rangle} \ . \tag{3.3.5}$$

An important feature to note is that $\hat{\mathcal{A}}$ necessarily has singularities at ι and $\hat{\iota}$. While at the 6-dimensional level this is a mere gauge-choice artefact, it plays a crucial role in the construction of the CS_4 theory.

In these variables, the boundary variation and boundary condition of hCS_6 are restated as

$$r_+ \mathrm{Tr}\big(\hat{\mathcal{A}}_w \delta \hat{\mathcal{A}}_{\bar{w}} - \hat{\mathcal{A}}_{\bar{w}} \delta \hat{\mathcal{A}}_w\big)\big|_{\pi=\alpha} = -r_- \mathrm{Tr}\big(\hat{\mathcal{A}}_w \delta \hat{\mathcal{A}}_{\bar{w}} - \hat{\mathcal{A}}_{\bar{w}} \delta \hat{\mathcal{A}}_w\big)\big|_{\pi=\tilde{\alpha}} \ , \tag{3.3.6}$$

$$\hat{\mathcal{A}}_w|_{\pi=\alpha} = ts\hat{\mathcal{A}}_w|_{\pi=\tilde{\alpha}} \ , \qquad \hat{\mathcal{A}}_{\bar{w}}|_{\pi=\alpha} = t^{-1}s\hat{\mathcal{A}}_{\bar{w}}|_{\pi=\tilde{\alpha}} \ , \tag{3.3.7}$$

where we have introduced the combinations

$$\begin{aligned} r_+ &= K\frac{\langle\alpha\iota\rangle\langle\alpha\hat{\iota}\rangle}{\langle\alpha\tilde{\alpha}\rangle\langle\alpha\beta\rangle^2} \ , & r_- &= -K\frac{\langle\tilde{\alpha}\iota\rangle\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\alpha\tilde{\alpha}\rangle\langle\tilde{\alpha}\beta\rangle^2} \ , \\ s &= \sqrt{-\frac{r_-}{r_+}} = \frac{\langle\alpha\beta\rangle}{\langle\tilde{\alpha}\beta\rangle}\sqrt{\frac{\langle\tilde{\alpha}\iota\rangle\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\alpha\iota\rangle\langle\alpha\hat{\iota}\rangle}} \ , & t &= \sigma s\frac{\langle\tilde{\alpha}\beta\rangle\langle\alpha\hat{\iota}\rangle}{\langle\alpha\beta\rangle\langle\tilde{\alpha}\hat{\iota}\rangle} \ . \end{aligned} \tag{3.3.8}$$

Upon symmetry reduction to CS$_4$, $r_\pm$ will correspond to the residues of the 1-form ω.

Since the shifted gauge field $\hat{\mathcal{A}}$ manifestly has no $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs, and we impose $L_z\hat{\mathcal{A}} = L_{\bar{z}}\hat{\mathcal{A}} = 0$, the contraction by ∂_z and $\partial_{\bar{z}}$ only hits Ω. It then follows that the symmetry reduction yields

$$S_{\mathrm{CS}_4} = \frac{1}{2\pi\mathrm{i}}\int_{\Sigma\times\mathbb{CP}^1} \omega\wedge\mathrm{Tr}\Big(\hat{\mathcal{A}}\wedge\mathrm{d}\hat{\mathcal{A}} + \frac{2}{3}\hat{\mathcal{A}}\wedge\hat{\mathcal{A}}\wedge\hat{\mathcal{A}}\Big) \ , \tag{3.3.9}$$

in which the meromorphic 1-form on $\mathbb{CP}^1$ is given by

$$\omega = \partial_{\bar{z}} \vee \partial_z \vee \Omega = \Phi\,\epsilon_{\dot{a}\dot{b}}\left(\partial_{\bar{z}} \vee e^{\dot{a}}\right)\left(\partial_z \vee e^{B}\right)\, e^0 = -K\,\frac{\langle\pi\iota\rangle\langle\pi\hat{\iota}\rangle}{\langle\pi\alpha\rangle\langle\pi\tilde{\alpha}\rangle\langle\pi\beta\rangle^2}\, e^0 \ . \tag{3.3.10}$$

To compare with the literature, we will also translate to inhomogeneous coordinates on $\mathbb{CP}^1$. The $\mathbb{CP}^1$ coordinate itself will be given by $\zeta = \pi_{2'}/\pi_{1'}$ on the patch $\pi_{1'} \neq 0$. We also specify representatives for the various other spinors in our theory. Without loss of generality we can choose

$$\alpha_a = (1, \alpha_+) \ , \qquad \tilde{\alpha}_a = (1, \alpha_-) \ , \qquad \beta_a = (0, 1) \ , \tag{3.3.11}$$

such that

$$\langle\tilde{\alpha}\beta\rangle = \langle\alpha\beta\rangle = 1 \ , \qquad \langle\tilde{\alpha}\alpha\rangle = \alpha_+ - \alpha_- = \Delta\alpha \ . \tag{3.3.12}$$

We also denote the inhomogeneous coordinates for ι_a and $\hat{\iota}_a$ by

$$\iota_+ = \frac{\iota_{2'}}{\iota_{1'}} \ , \qquad \iota_- = \frac{\hat{\iota}_{2'}}{\hat{\iota}_{1'}} = -\frac{\overline{\iota_{1'}}}{\overline{\iota_{2'}}} \ , \qquad \iota_{1'}\overline{\iota_{2'}} = \frac{1}{\iota_+ - \iota_-} = \frac{1}{\Delta\iota} \ . \tag{3.3.13}$$

Then, the meromorphic 1-form ω is written in inhomogeneous coordinates as

$$\omega = \frac{K}{\Delta\iota}\frac{(\zeta-\iota_+)(\zeta-\iota_-)}{(\zeta-\alpha_+)(\zeta-\alpha_-)}\mathrm{d}\zeta = \varphi(\zeta)\,\mathrm{d}\zeta \ . \tag{3.3.14}$$

To complete the specification of the theory we simply note that the 6d boundary conditions immediately descend to

$$\hat{\mathcal{A}}_w\mid_{\pi=\alpha} = ts\hat{\mathcal{A}}_w\mid_{\pi=\tilde{\alpha}} \ , \qquad \hat{\mathcal{A}}_{\bar{w}}\mid_{\pi=\alpha} = t^{-1}s\hat{\mathcal{A}}_{\bar{w}}\mid_{\pi=\tilde{\alpha}} \ . \tag{3.3.15}$$

Before we discuss the residual symmetries of the CS_4 models, let us make two related observations. First, fixing the shift symmetry to ensure $\hat{\mathcal{A}}$ is horizontal with respect to the symmetry reduction introduces poles into our gauge field $\hat{\mathcal{A}}$ at ι and $\hat{\iota}$. Thus, despite starting with potentially smooth field configurations in 6 dimensions we are forced to consider singular ones in 4 dimensions. We can understand the origin of these singularities by recalling the holomorphic coordinates on $\mathbb{E}^4$ with respect to the complex structure on $\mathbb{PT} = \mathbb{CP}^3\backslash\mathbb{CP}^1$. As noted in §2.4.6, twistor space $\mathbb{PT}$ is only diffeomorphic to $\mathbb{E}^4 \times \mathbb{CP}^1$, and the complex structure is more involved in these coordinates. The holomorphic coordinates on $\mathbb{E}^4$ with respect to this complex structure are given by $v^{\dot{\alpha}} = \pi_a x^{a\dot{\alpha}}$, which align with our coordinates $\{z, w\}$ at $\pi \sim \iota$ and $\{\bar{z}, \bar{w}\}$ at $\pi \sim \hat{\iota}$. It is precisely at these points that we are forced to introduce poles by the symmetry reduction procedure.

Second, in line with the singular behaviour in the gauge field, we have also introduced zeroes in ω at $\pi \sim \iota$ and $\pi \sim \hat{\iota}$ whereas Ω in 6 dimensions was nowhere vanishing. Of course, given the pole structure of Ω, the introduction of two zeroes was inevitable given the Riemann-Roch theorem.

3.3.1 Residual Symmetries and the Defect Algebra

Let us take a moment to consider the residual symmetries of these CS_4 models. Here the residual symmetry preserving the boundary condition (3.3.15) is generically constrained to only include constant modes,

$$r = \tilde{r} \ , \qquad (1-ts)\partial_w r = 0 \ , \qquad (1-ts^{-1})\partial_{\bar{w}} r = 0 \ . \tag{3.3.16}$$

At the special 'diagonal' point in parameter space where $t = s = 1$, notice these differential equations are identically solved and we find a local gauge symmetry. This enhancement of residual gauge freedom matches with previous considerations in the context of CS$_4$, where diagonal boundary conditions of the form $A|_\alpha = A|_{\tilde{\alpha}}$ are known to give rise to the λ-deformed WZW as an IFT$_2$, a theory that depends on a single field g. The residual gauge symmetries are those satisfying $\hat{\gamma}|_\alpha = \hat{\gamma}|_{\tilde{\alpha}}$ and can be used to reduce the number of fields appearing in the resulting IFT$_2$ to one (see § 5.4 in [Del+20]).

Another interesting point occurs when we take $t = s$ or $t = s^{-1}$ in which case we retain a chiral residual symmetry. When $t = 0$ the boundary conditions admit an enlarged residual symmetry as there is no requirement that $r = \hat{g}|_\alpha$ and $\tilde{r} = \hat{g}|_{\tilde{\alpha}}$ match. Instead they must be chiral and of opposite chiralities i.e. $\partial_w r = \partial_{\bar{w}} \tilde{r} = 0$. As mentioned earlier, for more generic values of t and s the residual symmetries will be constrained, preventing them from being used to eliminate any degrees of freedom. While these boundary conditions have not been yet considered for $t, s \neq 1$ and $t \neq 0$, we will see that they give rise to the multi-parametric class of λ-deformations between coupled WZW models introduced in [GS17].

To make further contact with the literature, it is helpful to rephrase the boundary conditions (3.3.15) in terms of a defect algebra, which in the case at hand is simply the Lie algebra $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$ equipped with an ad-invariant pairing

$$\langle\langle \mathbb{X}, \mathbb{Y} \rangle\rangle = r_+ \mathrm{Tr}(x_1\, y_1) + r_- \mathrm{Tr}(x_2\, y_2) \ , \qquad \mathbb{X} = (x_1, x_2) \ , \mathbb{Y} = (y_1, y_2) \ . \tag{3.3.17}$$

We map our boundary conditions into this algebra by defining $\mathbb{A}_w = (\hat{\mathcal{A}}_w|_{\pi=\alpha}, \hat{\mathcal{A}}_w|_{\pi=\tilde{\alpha}})$ and $\mathbb{A}_{\bar{w}} = (\hat{\mathcal{A}}_{\bar{w}}|_{\pi=\alpha}, \hat{\mathcal{A}}_{\bar{w}}|_{\pi=\tilde{\alpha}})$ such that the requirement that the boundary variation vanishes locally can be recast as

$$0 = \langle\langle \mathbb{A}_w, \delta \mathbb{A}_{\bar{w}} \rangle\rangle - \langle\langle \mathbb{A}_{\bar{w}}, \delta \mathbb{A}_w \rangle\rangle \ . \tag{3.3.18}$$

The boundary conditions (3.3.15) read

$$\begin{aligned} \mathbb{A}_w \in \mathfrak{l}_t &= \mathrm{span}\{(tsx, x) \mid x \in \mathfrak{g}\} \ , \\ \mathbb{A}_{\bar{w}} \in \mathfrak{l}_{t^{-1}} &= \mathrm{span}\{\big(t^{-1} sx, x\big) \mid x \in \mathfrak{g}\} \ . \end{aligned} \tag{3.3.19}$$

Since $\langle\langle \mathfrak{l}_t, \mathfrak{l}_{t^{-1}} \rangle\rangle = 0$ the boundary conditions are suitable, however it should be noted that $\mathfrak{l}_t$ is itself neither a subalgebra nor an isotropic subspace of $\mathfrak{d}$. This is more general than boundary conditions previously considered[10] in the context of 4-dimensional Chern Simons theory. In particular, we might expect that generalising [BSV22; LV21; LV23] to boundary conditions defined by subspaces that are neither a subalgebra nor an isotropic subspace of $\mathfrak{d}$ will lead to novel families of 2-dimensional integrable field theories.

It is worth highlighting that these boundary conditions still define maximal isotropic subspaces, but now inside the space of algebra-valued 1-forms, rather than just the defect algebra. Consider the space of $\mathfrak{g}$-valued 1-forms on $\Sigma \times \mathbb{CP}^1$, equipped with the symplectic structure[11]

$$\mathcal{W}(X,Y) = \int_{\Sigma\times\mathbb{CP}^1} \mathrm{d}\omega \wedge \mathrm{Tr}\big(X \wedge Y\big) \ , \quad X, Y \in \Omega^1(\Sigma \times \mathbb{CP}^1) \otimes \mathfrak{g} \ . \tag{3.3.20}$$

The boundary conditions above define maximal isotropic subspaces with respect to this symplectic structure, that is half-dimensional subspaces $\mathcal{Y} \subset \Omega^1(\Sigma \times \mathbb{CP}^1) \otimes \mathfrak{g}$ such that $\mathcal{W}(X,Y) = 0$ for all $X, Y \in \mathcal{Y}$. Indeed, this is required for them to be 'good' boundary conditions. The isotropic subspaces of the defect algebra described earlier are then special cases of these subspaces.

3.4 Symmetry Reduction of IFT$_4$ to IFT$_2$

Recalling that the reduction requires that the fields g and $\tilde{g}$ depend only on $w, \bar{w}$ and not on $z, \bar{z}$, we can simply set $\partial_z = \partial_{\bar{z}} = 0$ in the action eq. (3.2.16). To compare with the literature, when discussing 2-dimensional theories we will define $\partial_+ \equiv \partial_w$ and $\partial_- \equiv \partial_{\bar{w}}$ (implicitly rotating to 2d Minkowski space where the action is rendered real for real parameters) and denote

$$J_\pm = g^{-1}\partial_\pm g \ , \qquad \widetilde{J}_\pm = \tilde{g}^{-1}\partial_\pm \tilde{g} \ . \tag{3.4.1}$$

[10] Of course in the limit $t, s \to 1$ $\mathfrak{l}_t$ revert to defining the diagonal isotropic subalgebra. In the special case where $t \to 0, \infty$ we recover chiral Dirichlet boundary conditions considered in [CY19; ASY23].

[11] As defined, this is not quite a symplectic structure since it is degenerate – for example, it vanishes on the subspace of 1-forms which only have legs along $\mathbb{CP}^1$. A more careful treatment would involve restricting the symplectic form to a subspace where it is non-degenerate, but we will neglect this for the purpose of our brief discussion.

To evaluate the symmetry reduction, denoted by $\rightsquigarrow$, of the IFT$_4$ action we first note that

$$j \rightsquigarrow \frac{\langle\alpha\iota\rangle}{\langle\alpha\beta\rangle} J_+ \ , \quad \widehat{\jmath} \rightsquigarrow \frac{\langle\alpha\hat{\iota}\rangle}{\langle\alpha\beta\rangle} J_- \ , \quad \widetilde{\jmath} \rightsquigarrow \frac{\langle\tilde{\alpha}\iota\rangle}{\langle\tilde{\alpha}\beta\rangle} \widetilde{J}_+ \ , \quad \widehat{\widetilde{\jmath}} \rightsquigarrow \frac{\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\tilde{\alpha}\beta\rangle} \widetilde{J}_- \ . \tag{3.4.2}$$

The resulting 2-dimensional action is given by

$$\begin{aligned} S_{\mathrm{IFT}_2} \ = \int_\Sigma \mathrm{vol}_2 \, \mathrm{Tr}\big(& r_+ \, J_+(U_+^T - U_-)J_- - r_- \, \widetilde{J}_+(U_+^T - U_-)\widetilde{J}_- + r_+ \, \mathcal{L}_{\mathrm{WZ}}(h) \\ & + r_- \, \mathcal{L}_{\mathrm{WZ}}(\tilde{h}) - 2\, t\sqrt{-r_+ \cdot r_-} \, \widetilde{J}_+ U_+^T J_- + 2\, t^{-1}\sqrt{-r_+ \cdot r_-} \, J_+ U_- \widetilde{J}_- \big) \ , \end{aligned} \tag{3.4.3}$$

where $\mathrm{vol}_2 = \mathrm{d}\bar{w}\wedge\mathrm{d}w = \mathrm{d}\sigma^-\wedge\mathrm{d}\sigma^+$. This theory, depending on two G-valued fields, g and $\tilde{g}$, and four independent parameters, $r_\pm$, t and σ, exactly matches a theory introduced in [GS17] as a multi-field generalisation of the λ-deformed WZW model [Sfe14]. To make a precise match with [GS17] we relate their fields (g_1, g_2) to our fields $(g, \tilde{g}^{-1})$. The model in [GS17] is defined by two WZW levels $k_{1,2}$ and by two deformation matrices which we take to be proportional to the identity with constants of proportionality $\lambda_{1,2}$. The mapping of parameters is then

$$\begin{gathered} \lambda_1 = \sigma t^{-1} \ , \qquad \lambda_2 = t \ , \qquad k_1 = r_+ \ , \qquad k_2 = -r_- \ , \\ \lambda_0 = \sqrt{k_1/k_2} = \sqrt{-r_-/r_+} = s^{-1} \ . \end{gathered} \tag{3.4.4}$$

In 2d Minkowski space, the Lagrangian (3.4.3) is real if the parameters $r_\pm$, s t and σ are all real. Assuming K and σ are real, this is the case if r_+ and r_- have the opposite sign and the parameters $\alpha_\pm$ and $\iota_\pm$ lie on the same line in $\mathbb{C}$, which we can take to be the real line without loss of generality. This follows since $r_\pm$, s and t are all expressed as ratios of differences of $\alpha_\pm$ and $\iota_\pm$, hence are invariant under translations and scalings.[12]

3.4.1 Limits

The four-parameter model has a number of interesting limits, many of which are discussed in [GS17]. Here, we briefly summarise some key ones. First, let us take $t \to 0$. In order to have a well-defined limit we keep $\sigma t^{-1} = \lambda$ finite, implying $\sigma \to 0$ as well.[13]

[12] Note that this is the subgroup of $SL(2,\mathbb{C})$ transformations that preserves the choice $\beta_a = (0, 1)$.

[13] An analogous limit is to take $\sigma \to 0$ and keep t finite.

The resulting 2d Lagrangian is given by

$$S_{\mathrm{IFT}_2}|_{t,\sigma\to 0} = \int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\big(r_+\,J_+J_- - r_-\,\widetilde{J}_+\widetilde{J}_- - 2\,\lambda s r_+ J_+\Lambda^{-1}\widetilde{J}_- + r_+\,\mathcal{L}_{\mathrm{WZ}}(g) + r_-\,\mathcal{L}_{\mathrm{WZ}}(\tilde{g})\big)\ . \tag{3.4.5}$$

This current-current deformation preserves half the chiral symmetry of the $G_{r_+}\times G_{-r_-}$ WZW model, which corresponds to the UV fixed point $\lambda = 0$. Indeed, this model can be found by taking chiral Dirichlet boundary conditions in 4d CS [CY19; ASY23], corresponding to the special case $t = 0$ in the boundary conditions we find from symmetry reduction (3.3.19). Assuming $-r_- > r_+$, in the IR we have that $\lambda = s^{-1}$. At this point the Lagrangian can be written as

$$\begin{aligned} S_{\mathrm{IFT}_2}|_{t,\sigma\to 0,t\sigma^{-1}=s} = \int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\big(&r_+(\mathrm{Ad}_g J_+ - \mathrm{Ad}_{\tilde{g}}\tilde{J}_+)(\mathrm{Ad}_g J_- - \mathrm{Ad}_{\tilde{g}}\tilde{J}_-) - (r_- + r_+)\,\widetilde{J}_+\widetilde{J}_- \\ &+ r_+\,\mathcal{L}_{\mathrm{WZ}}(\tilde{g}^{-1}g) + (r_- + r_+)\,\mathcal{L}_{\mathrm{WZ}}(\tilde{g})\big)\ . \end{aligned} \tag{3.4.6}$$

Redefining $g \to \tilde{g}g$, we find the $G_{r_+}\times G_{-r_- - r_+}$ WZW model. In the case of equal levels $r_- = -r_+$ this reduces to the G_{r_+} WZW model.

The equal-level, $r_- = -r_+$, version of (3.4.5), whose classical integrability was first shown in [BBS97], is canonically equivalent [GSS17] and related by a path integral transformation [HLT19a] to the λ-deformed WZW model. Indeed, from the point of view of 4d CS, these two models have the same twist function. To recover (3.4.5) with equal levels, we take chiral Dirichlet boundary conditions, $t = 0$, $s = 1$ in (3.3.19), while to recover the λ-deformed WZW model we take diagonal boundary conditions $t = s = 1$.

It follows that if we take $t = s = 1$ in eq. (3.4.3), we expect to recover the λ-deformed WZW model. Indeed, setting $r_- = -r_+$ and $t = 1$, the Lagrangian (3.4.3) becomes

$$S_{\mathrm{IFT}_2}|_{t=s=1} = \int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\big(r_+\,(J_+ - \widetilde{J}_+)(U_+^T - U_-)(J_- - \widetilde{J}_-) + r_+\,\mathcal{L}_{\mathrm{WZ}}(g\tilde{g}^{-1})\big)\ . \tag{3.4.7}$$

As explained in subsection 3.1.1, at this point in parameter space the symmetry reduction directions are aligned such that the constrained symmetry transformations (3.1.14) become a gauge symmetry of the IFT_2. This allows us to fix $\tilde{g} = 1$, recovering the standard form of the λ-deformed WZW model [Sfe14] with σ playing the role of λ. Further taking $\sigma \to 0$, we recover the G_{r_+} WZW model.

Another point in parameter space where we expect a gauge symmetry to emerge is when the symmetry reduction preserves the left-acting symmetry. This corresponds to setting $t = \sigma$ and $s = 1$, i.e. $r_- = -r_+$. Doing so we find

$$S_{\mathrm{IFT}_2}|_{t=\sigma,s=1} = \int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\big(r_+(\mathrm{Ad}_g J_+ - \mathrm{Ad}_{\tilde{g}}\widetilde{J}_+)(\widetilde{U}_+^T - \widetilde{U}_-)(\mathrm{Ad}_g J_- - \mathrm{Ad}_{\tilde{g}}\widetilde{J}_-) + r_+\,\mathcal{L}_{\mathrm{WZ}}(\tilde{g}^{-1}g)\big)\ , \tag{3.4.8}$$

where we recall that $\widetilde{U}_\pm$ are defined in eq. (3.2.60). This Lagrangian is invariant under a left-acting gauge symmetry as expected, which can be used to fix $\tilde{g} = 1$. We again recover the standard form of the λ-deformed WZW model with σ playing the role of λ.

Before we move onto the integrability of the 2d IFT and its origin from the 4d IFT, let us briefly note the symmetry reduction implications of the formal transformations (3.2.18) and (3.2.19), which in turn descended from the discrete invariances of the hCS$_6$ boundary conditions (3.1.9) and (3.1.10). The first (3.2.18) implies that the 2d IFT is invariant under

$$r_+ \leftrightarrow r_-\ , \qquad \sigma \to \sigma^{-1}\ , \qquad t \to t^{-1}\ , \qquad g \leftrightarrow \tilde{g}\ , \tag{3.4.9}$$

recovering the 'duality' transformation of [GS17]. Since the second involves interchanging w and z, it tells us the parameters are transformed if we symmetry reduce requiring that the fields g and $\tilde{g}$ only depend on $z, \bar{z}$, instead of $w, \bar{w}$. We find that $\sigma \to \sigma^{-1}$ and $t \to t\sigma^{-2}$.

3.4.2 Integrability and Lax Formulation

The analysis of [GS17] shows that the equations of motion of (3.4.3) are best cast in terms of auxiliary fields[14] $B_\pm, C_\pm$ which are related to the fundamental fields by

$$\begin{aligned} J_- &= \mathrm{Ad}_g^{-1}B_- - \lambda_0^{-1}\lambda_2^{-1}C_-\ , & \tilde{J}_- &= \lambda_0\lambda_1^{-1}\mathrm{Ad}_{\tilde{g}}^{-1}B_- - C_-\ ,\\ J_+ &= \lambda_0^{-1}\lambda_1^{-1}\mathrm{Ad}_g^{-1}B_+ - C_+\ , & \tilde{J}_+ &= \mathrm{Ad}_{\tilde{g}}^{-1}B_+ - \lambda_0\lambda_2^{-1}C_+\ . \end{aligned} \tag{3.4.10}$$

The equations of motion for g and $\tilde{g}$, together with the Bianchi identities obeyed by their associated Maurer-Cartan forms, can be repackaged into the flatness of two Lax connections with components

$$\mathcal{L}^1_\pm = \frac{2\zeta_{GS}}{\zeta_{GS}\mp 1}\frac{1-\lambda_0^{\mp 1}\lambda_1}{1-\lambda_1^2}B_\pm\ , \qquad \mathcal{L}^2_\pm = \frac{2\zeta_{GS}}{\zeta_{GS}\mp 1}\frac{1-\lambda_0^{\pm 1}\lambda_2}{1-\lambda_2^2}C_\pm\ . \tag{3.4.11}$$

[14]To avoid conflict of notation $B_\pm, C_\pm$ here correspond to $A_\pm, B_\pm$ of [GS17].

Here ζ_{GS} is the spectral parameter used in [GS17]. Taken together, the flatness of this pair of Lax connections implies both the Bianchi identities and the equations of motions. However, if one is prepared to enforce the definition (3.4.10) of auxiliary fields in terms of fundamental fields (such that the Bianchi equations are automatically satisfied) then either Lax will generically (i.e. away from special points in parameter space such as $\lambda_i = 1$) imply the equations of motion[15] of the theory.

We can relate this discussion to the construction above by symmetry reducing the 4d Lax operators (3.2.47) and (3.2.48). First we note that the currents corresponding to the (ℓ, r)-symmetries reduce to simple combinations of the auxiliary fields introduced in eq. (3.4.10)

$$\begin{aligned} B_{\dot{a}} &\rightsquigarrow \frac{\langle\alpha\hat{\iota}\rangle}{\langle\alpha\beta\rangle} B_- \mu_{\dot{a}} - \frac{\langle\tilde{\alpha}\iota\rangle}{\langle\tilde{\alpha}\beta\rangle} B_+ \hat{\mu}_{\dot{a}} \ , \\ C_{\dot{a}} &\rightsquigarrow \frac{\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\tilde{\alpha}\beta\rangle} \sigma^{-1} C_- \mu_{\dot{a}} - \frac{\langle\alpha\iota\rangle}{\langle\alpha\beta\rangle} C_+ \hat{\mu}_{\dot{a}} \ . \end{aligned} \tag{3.4.12}$$

Notice that all explicit appearances of the operators $U_\pm$ have dropped out such that these currents reduce exactly to the 2-dimensional auxiliary gauge fields.

Using the complex coordinates adapted for symmetry reduction defined in eq. (3.3.2), and introducing a specialised inhomogeneous coordinate on $\mathbb{CP}^1$ given by $\varsigma = \langle\pi\hat{\iota}\rangle / \langle\pi\iota\rangle$, the 4d B-Lax pair (3.2.47) may be written as

$$L^{(B)} = D_{\bar{w}} - \varsigma^{-1} D_z \ , \quad M^{(B)} = D_w + \varsigma D_{\bar{z}} \ . \tag{3.4.13}$$

We can symmetry reduce the 4d Lax pairs, $L^{(B/C)}, M^{(B/C)}$ of eqs. (3.2.47) and (3.2.48) to obtain

$$\begin{aligned} L^{(B)} &\rightsquigarrow \partial_- + (\langle\beta\iota\rangle - \varsigma^{-1}\langle\beta\hat{\iota}\rangle) \frac{\langle\alpha\hat{\iota}\rangle}{\langle\alpha\beta\rangle} B_- \ , \quad M^{(B)} \rightsquigarrow \partial_+ + (\varsigma\langle\beta\iota\rangle - \langle\beta\hat{\iota}\rangle) \frac{\langle\tilde{\alpha}\iota\rangle}{\langle\tilde{\alpha}\beta\rangle} B_+ \ , \\ L^{(C)} &\rightsquigarrow \partial_- - \frac{1}{(1-\varrho) + \lambda_0\lambda_2(1+\varrho)} C_- \ , \quad M^{(C)} \rightsquigarrow \partial_+ - \frac{1}{(1+\varrho) + \lambda_0^{-1}\lambda_2(1-\varrho)} C_+ \ . \end{aligned} \tag{3.4.14}$$

Now using the inhomogeneous coordinates introduced in eqs. (3.3.11) and (3.3.13), and the relations between parameters (3.4.4), the 4d Lax operators immediately descend upon symmetry reduction to the 2d Lax connections (3.4.11), provided the 4d and 2d

[15] It is less evident in contrast if *all* the non-local conserved charges of the theory can be obtained from a single Lax.

spectral parameters are related as

$$
\begin{aligned}
&L^{(B)} \rightsquigarrow \partial_- + \mathcal{L}^1_- \,, \quad M^{(B)} \rightsquigarrow \partial_+ + \mathcal{L}^1_+ \,, \quad \zeta_{GS} = \frac{\bar{\iota}_2 + \iota_1 \varsigma}{-\bar{\iota}_2 + \iota_1 \varsigma} \,, \\
&L^{(C)} \rightsquigarrow \partial_- + \mathcal{L}^2_- \,, \quad M^{(C)} \rightsquigarrow \partial_+ + \mathcal{L}^2_+ \,, \quad \zeta_{GS} = \frac{1-\lambda_2^2}{(\lambda_0 - \lambda_0^{-1})\lambda_2 - (1-\lambda_0\lambda_2)(1-\lambda_0^{-1}\lambda_2)\varrho} \,.
\end{aligned} \tag{3.4.15}
$$

The relation between ζ_{GS} and ς can be recast in the standard $\mathbb{CP}^1$ homogeneous coordinate $\pi \sim (1, \zeta)$ as

$$
\zeta_{GS} = \frac{\iota_+ - \iota_-}{2\zeta - (\iota_+ + \iota_-)} \,, \tag{3.4.16}
$$

such that if we choose to fix $\iota_\pm = \pm 1$ then $\zeta_{GS} = \zeta^{-1}$. If we make the assumption that the ζ_{GS} entering in the two different Lax formulations have the same origin then we can map between between ϱ and the $\mathbb{CP}^1$ homogeneous coordinate

$$
\varrho = -\frac{1+\zeta}{2}\frac{1+ts}{1-ts} + \frac{1-\zeta}{2}\frac{1+ts^{-1}}{1-ts^{-1}} \,. \tag{3.4.17}
$$

Therefore, under this assumption, we see that ϱ depends on the parameter t, which is part of the specification of boundary conditions and not just geometric data of $\mathbb{CP}^1$. Indeed ϱ becomes constant when $t \to 1$, hence there is no spectral parameter dependence left. In contrast, when $t \to 0$, we have $\varrho \to -\zeta$.

3.5 Localisation of $\mathbf{CS}_4$ to $\mathbf{IFT}_2$

Having descended via symmetry reduction from hCS_6, in this subsection we will perform the localisation of CS_4. Recall the action (3.3.9)

$$
S_{\mathrm{CS}_4} = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma \times \mathbb{CP}^1} \omega \wedge \mathrm{Tr}\Big(\hat{\mathcal{A}} \wedge \mathrm{d}\hat{\mathcal{A}} + \frac{2}{3} \hat{\mathcal{A}} \wedge \hat{\mathcal{A}} \wedge \hat{\mathcal{A}} \Big) \,, \tag{3.5.1}
$$

with the meromorphic one form of eq.(3.3.10) and boundary conditions as per eq.(3.3.7).

We parametrise our gauge field $\hat{\mathcal{A}}$ as

$$
\hat{\mathcal{A}}_{\bar{\zeta}} = \hat{g}^{-1}\partial_{\bar{\zeta}}\hat{g} \,, \quad \hat{\mathcal{A}}_I = \hat{g}^{-1}\mathcal{L}_I \hat{g} + \hat{g}^{-1}(\partial_I \hat{g}) \,, \quad I = w, \bar{w} \,. \tag{3.5.2}
$$

Viewing $\hat{\mathcal{A}} = \mathcal{L}^{\hat{g}}$ as the formal gauge transform of $\mathcal{L}$ by $\hat{g}$, we use the following identity satisfied by the Chern-Simons density

$$\mathrm{CS}(\hat{\mathcal{A}}) = \mathrm{CS}(\mathcal{L}^{\hat{g}}) = \mathrm{CS}(\mathcal{L}) - \mathrm{d}\,\mathrm{Tr}\big(\hat{J}\wedge\hat{g}^{-1}\mathcal{L}\,\hat{g}\big) - \frac{1}{6}\mathrm{Tr}\big(\hat{J}\wedge[\hat{J},\hat{J}\,]\big)\ , \tag{3.5.3}$$

in which $\hat{J} = \hat{g}^{-1}\mathrm{d}\hat{g}$. Noting that on shell $\mathrm{CS}(\mathcal{L}) = \mathcal{L}\wedge\mathrm{d}\mathcal{L}$ and $\omega\wedge(\partial_\varsigma\mathcal{L})\wedge\mathcal{L} = 0$ we then arrive at the following action

$$S_{\mathrm{CS}_4} = -\frac{1}{2\pi\mathrm{i}}\int \mathrm{d}\omega\wedge\mathrm{Tr}\big(\hat{J}\wedge\hat{g}^{-1}\mathcal{L}\,\hat{g}\big) - \frac{1}{12\pi\mathrm{i}}\int \omega\wedge\big(\hat{J}\wedge[\hat{J},\hat{J}\,]\big)\ . \tag{3.5.4}$$

In this form we see how our action will localise at the poles of ω giving a 2d theory

$$S = r_+\int_\Sigma \mathrm{Tr}\big(\hat{J}\wedge\hat{g}^{-1}\mathcal{L}\,\hat{g}\big)|_\alpha + r_-\int_\Sigma \mathrm{Tr}\big(\hat{J}\wedge\hat{g}^{-1}\mathcal{L}\,\hat{g}\big)|_{\tilde{\alpha}} + \mathrm{WZ\ terms}\ , \tag{3.5.5}$$

where we recall

$$r_+ = K\frac{\langle\alpha\iota\rangle\langle\alpha\hat{\iota}\rangle}{\langle\alpha\tilde{\alpha}\rangle\langle\alpha\beta\rangle^2} \qquad \text{and} \qquad r_- = -K\frac{\langle\tilde{\alpha}\iota\rangle\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\alpha\tilde{\alpha}\rangle\langle\tilde{\alpha}\beta\rangle^2}\ .$$

To complete the construction we need to specify the meromorphic structure of $\mathcal{L}$ that ensures the theory is well defined given the form of ω and is compatible with the boundary conditions. This requires that

$$\mathcal{L}_w = \frac{\langle\pi\beta\rangle}{\langle\pi\iota\rangle}\mathcal{M}_w + \mathcal{N}_w\ \ , \qquad \mathcal{L}_{\bar{w}} = \frac{\langle\pi\beta\rangle}{\langle\pi\hat{\iota}\rangle}\mathcal{M}_{\bar{w}} + \mathcal{N}_{\bar{w}}\ , \tag{3.5.6}$$

where $\mathcal{M}_I, \mathcal{N}_I \in C^\infty(\Sigma, \mathfrak{g})$. The boundary conditions in this parametrisation read,

$$\begin{aligned} &\hat{g}|_\beta = \mathrm{id}\,, \quad \mathcal{L}|_\beta = 0\ ,\\ &\mathrm{Ad}_g^{-1}\mathcal{L}_w|_\alpha + J_w \ = ts\left(\mathrm{Ad}_{\tilde{g}}^{-1}\mathcal{L}_w|_{\tilde{\alpha}} + \tilde{J}_w\right),\\ &\mathrm{Ad}_g^{-1}\mathcal{L}_{\bar{w}}|_\alpha + J_{\bar{w}} \ = t^{-1}s\left(\mathrm{Ad}_{\tilde{g}}^{-1}\mathcal{L}_{\bar{w}}|_{\tilde{\alpha}} + \tilde{J}_{\bar{w}}\right), \end{aligned} \tag{3.5.7}$$

in which we use the definitions,

$$\hat{g}|_\alpha = g\ , \qquad \hat{g}|_{\tilde{\alpha}} = \tilde{g}\ , \qquad \hat{J}|_\alpha = J\ , \qquad \hat{J}|_{\tilde{\alpha}} = \tilde{J}\ . \tag{3.5.8}$$

Solving these conditions uniquely determines the Lax connection

$$\begin{aligned}
\mathcal{M}_w &= \frac{\langle\alpha\iota\rangle}{\langle\alpha\beta\rangle}\mathrm{Ad}_g\left[1-\sigma\,\mathrm{Ad}_{\tilde{g}}^{-1}\mathrm{Ad}_g\right]^{-1}\left(ts\tilde{J}_w - J_w\right)\ , \qquad & \mathcal{N}_w = 0\ ,\\
\mathcal{M}_{\bar{w}} &= \frac{\langle\alpha\hat{\iota}\rangle}{\langle\alpha\beta\rangle}\mathrm{Ad}_g\left[1-\sigma^{-1}\mathrm{Ad}_{\tilde{g}}^{-1}\mathrm{Ad}_g\right]^{-1}\left(t^{-1}s\tilde{J}_{\bar{w}} - J_{\bar{w}}\right)\ , \qquad & \mathcal{N}_{\bar{w}} = 0\ ,
\end{aligned} \tag{3.5.9}$$

where we have introduced the parameter $\sigma = t\sqrt{\frac{\langle\alpha\iota\rangle\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\tilde{\alpha}\iota\rangle\langle\alpha\hat{\iota}\rangle}}$. It will also be useful to state the alternative forms

$$\begin{aligned}
\mathcal{M}_w &= \frac{\langle\tilde{\alpha}\iota\rangle}{\langle\tilde{\alpha}\beta\rangle}\mathrm{Ad}_{\tilde{g}}\left[1-\sigma^{-1}\mathrm{Ad}_g^{-1}\mathrm{Ad}_{\tilde{g}}\right]^{-1}\left(t^{-1}s^{-1}J_w - \tilde{J}_w\right)\ ,\\
\mathcal{M}_{\bar{w}} &= \frac{\langle\tilde{\alpha}\hat{\iota}\rangle}{\langle\tilde{\alpha}\beta\rangle}\mathrm{Ad}_{\tilde{g}}\left[1-\sigma\mathrm{Ad}_g^{-1}\mathrm{Ad}_{\tilde{g}}\right]^{-1}\left(ts^{-1}J_{\bar{w}} - \tilde{J}_{\bar{w}}\right)\ .
\end{aligned} \tag{3.5.10}$$

Inserting (3.5.9) and (3.5.10) into (3.5.5) we obtain

$$\begin{aligned}
S = r_+ &\int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\Big(J_w\left[1-\sigma^{-1}\mathrm{Ad}_{\tilde{g}}^{-1}\mathrm{Ad}_g\right]^{-1}\left(t^{-1}s\tilde{J}_{\bar{w}} - J_{\bar{w}}\right)\Big)\\
- r_+ &\int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\Big(J_{\bar{w}}\left[1-\sigma\mathrm{Ad}_{\tilde{g}}^{-1}\mathrm{Ad}_g\right]^{-1}\left(ts\tilde{J}_w - J_w\right)\Big)\\
- r_- &\int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\Big(\tilde{J}_w\left[1-\sigma\mathrm{Ad}_g^{-1}\mathrm{Ad}_{\tilde{g}}\right]^{-1}\left(ts^{-1}J_{\bar{w}} - \tilde{J}_{\bar{w}}\right)\Big)\\
+ r_- &\int_\Sigma \mathrm{vol}_2\,\mathrm{Tr}\Big(\tilde{J}_{\bar{w}}\left[1-\sigma^{-1}\mathrm{Ad}_g^{-1}\mathrm{Ad}_{\tilde{g}}\right]^{-1}\left(t^{-1}s^{-1}J_w - \tilde{J}_w\right)\Big)\\
+ &\,\text{WZ terms}\ ,
\end{aligned} \tag{3.5.11}$$

where $\mathrm{vol}_2 = d\bar{w}\wedge dw$. Expanding out this action, collecting together terms, and Wick rotating to Minkowski space, we arrive at the action (3.4.3), thus demonstrating the diamond of correspondences.

Let us note that this IFT_2 has also been constructed from CS_4 in a two-step process in [BL20]. First, a more general 2-field model based on a twist function with additional poles and zeroes, and the familiar isotropic subalgebra boundary conditions, is constructed. Second, a special decoupling limit is taken, where a subset of these poles and zeroes collide. It remains to understand how to recover our boundary conditions (3.3.15) from those considered in [BL20].

Another important point to discuss is the apparent lack of a systematic derivation, in the 4dCS spirit, of the other Lax $\mathcal{L}^2$. This is in contrast to the derivation of this model from CS4 in [BL20] where the extra data associated to the additional poles means that both of the Lax connections can be directly constructed. More generally, this highlights an interesting question that we leave for future work about the integrability and the counting of conserved charges, beyond the existence of a Lax connection, when we consider boundary conditions not based on isotropic subalgebras.

3.6 Renormalisation Group Flow

Before discussing the renormalisation group (RG) flow of the 2d IFT through the lens of 4dCS theory, let us begin by introducing the RG flow in a more general capacity. In a generic quantum field theory, the behaviour of the theory is explicitly dependent on the energy scale it is defined up to. We can implement a change in scale via a dilatation transformation acting on our world-sheet coordinates as

$$\sigma^\alpha \to \lambda\, \sigma^\alpha \ , \tag{3.6.1}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ which has the effect of changing the world-sheet metric as

$$h_{\alpha\beta} \to \lambda^2 h_{\alpha\beta} \ . \tag{3.6.2}$$

In the case where $\lambda > 1$ we are changing the scale of our theory to longer distances, and hence smaller energies and vice-versa for $\lambda < 1$ we are changing the scale to shorter distances and hence higher energies. Dilatations are a symmetry of the classical action (1.1.1). This symmetry is generically anomalous at the quantum level however, and it is precisely this anomaly that gives rise to the phenomena captured by the renormalisation group. We will want to see how quantities in our theory such as couplings and operators change at higher order in perturbation theory as a function of the effective scale we regulate our theory at. Generically, it proves convenient to choose a regularisation scheme such that we have an ultraviolet cut off at $k = \mu_{\text{UV}}$ and an infrared cutoff at $k = \Lambda_{\text{IR}}$. As such, our momenta integrals are evaluated for momenta values $\Lambda_{\text{IR}} \leq |k| \leq \mu_{\text{UV}}$, rendering the resulting integrals finite.

One can ask how changing the energy scale infinitesimally $\mu_{\text{UV}} \to \mu_{\text{UV}} + \delta\mu_{\text{UV}}$ changes the couplings g_i in our theory. This behaviour is captured by the beta-function defined by

$$\beta_i = \mu \frac{\partial}{\partial \mu} g_i \ . \tag{3.6.3}$$

For sigma-models, the couplings of the theory are defined by the target-space metric and B-field. In the case with vanishing B-field we can calculate the one-loop β-function (see e.g. [Hor+03])

$$B_{ij}^{(G)} := \frac{d}{d\ln\mu} G_{ij}(\mu) = R_{ij} \tag{3.6.4}$$

where R_{ij} is the Ricci curvature tensor. This is an intriguing prospect, firstly if the Ricci curvature of the target space manifold vanishes the theory is scale invariant at one-loop! Of course, the β-function may receive non-vanishing contributions at higher loops order, spoiling this. In the case where $R_{ij} > 0$, the sigma model is increasingly weakly coupled at higher and higher energies, as the coupling is inversely proportional to the size of the target space. This property is called asymptotic freedom. When $R_{ij} < 0$ the coupling will increase as $\mu \to \infty$, and so our perturbation expansion in the coupling becomes meaningless in the UV. However, in the IR, the theory can be completely consistent, and the IR dynamics can be described perturbatively.

3.6.1 RG Flow for IFT$_2$ from CS$_4$

Let us recall the RG equations given in [GS17]

$$\dot{\lambda}_i = -\frac{c_G}{2\sqrt{k_1 k_2}} \frac{\lambda_i^2(\lambda_i - \lambda_0)(\lambda_i - \lambda_0^{-1})}{(1-\lambda_i^2)^2} \ , \quad i = 1, 2 \ , \tag{3.6.5}$$

where dot indicates the derivative with respect to RG 'time' $\frac{d}{d\log\mu}$ and c_G is the dual Coxeter number. The levels k_1 and k_2 and $\lambda_0 = \sqrt{k_1/k_2}$ are RG invariants. In this section we will interpret this flow in terms of the data that is more natural from the perspective of 4d CS, namely the poles and zeroes of the differential

$$\omega = \frac{K}{\Delta\iota} \frac{(\zeta - \iota_+)(\zeta - \iota_-)}{(\zeta - \alpha_+)(\zeta - \alpha_-)} \mathrm{d}\zeta = \varphi(\zeta)\, \mathrm{d}\zeta \ , \tag{3.6.6}$$

and the boundary conditions of the theory.

Using the map between parameters given in eq. (3.4.4) we can infer from eq. (3.6.5) a flow on the parameters $\{t, \alpha_\pm, \iota_\pm, K\}$. Let us first consider the parameter $t = \lambda_2$. As discussed in [GS17], there is a flow from $t = 0$ in the UV to $t = \lambda_0$ in the IR (assuming that $\lambda_0 < 1$). Explicitly the flow equation

$$\dot{t} = \frac{c_G}{2k_2\lambda_0}\frac{t^2}{(1-t^2)^2}(t-\lambda_0)(t-\lambda_0^{-1}) \ , \tag{3.6.7}$$

has the solution

$$f(\lambda_0,t)+f(\lambda_0^{-1},t)+t+t^{-1} = \frac{c_G}{2\sqrt{k_1k_2}}\log\mu/\mu_{t_0} \ , \quad f(x,t) = x\log\left(\frac{t^{-1}-x}{t-x}\right) \ . \tag{3.6.8}$$

The interesting observation is that the boundary conditions

$$\begin{aligned} \mathbb{A}_w \in \mathfrak{l}_t &= \text{span}\{\big(t\lambda_0^{-1}x, x\big) \mid x \in \mathfrak{g}\} \ , \\ \mathbb{A}_{\bar{w}} \in \mathfrak{l}_{t^{-1}} &= \text{span}\{\big(\lambda_0^{-1}x, tx\big) \mid x \in \mathfrak{g}\} \ , \end{aligned} \tag{3.6.9}$$

display algebraic enhancements at the fixed points. In the UV, $t = 0$ limit, these boundary conditions become chiral, $\mathbb{A}_w \in \mathfrak{g}_R \subset \mathfrak{d}$ and $\mathbb{A}_{\bar{w}} \in \mathfrak{g}_L \subset \mathfrak{d}$. While $\mathfrak{g}_{L,R}$ are now subalgebras, neither are isotropic with respect to the inner product (4.2.26). In non-doubled notation the UV limit becomes

$$\hat{\mathcal{A}}_w|_\alpha = 0 \ , \qquad \hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}} = 0 \ . \tag{3.6.10}$$

On the other hand in the IR limit, $t = \lambda_0$, we see that $\mathbb{A}_w \in \mathfrak{g}_{\text{diag}} \subset \mathfrak{d}$, again a subalgebra, but only an isotropic one for $k_1 = k_2$, i.e. $r_+ = -r_-$. In non-doubled notation the IR limit becomes[16]

$$\hat{\mathcal{A}}_w|_\alpha = \hat{\mathcal{A}}_w|_{\tilde{\alpha}} \ , \qquad k_1\hat{\mathcal{A}}_{\bar{w}}|_\alpha = k_2\hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}} \ . \tag{3.6.11}$$

While in general, there are no residual gauge transformations preserving the boundary conditions, in the UV and IR limits we notice chiral boundary symmetries emerging. For example, in the IR these are those satisfying $g^{-1}\partial_{\bar{w}}g = 0$, which corresponds to $t = s^{-1}$ in eq. (3.3.16).

Let us now turn to the action of RG on the differential ω. An immediate observation

[16]The seemingly more democratic boundary condition of $t = 1$,

$$\sqrt{k_1}\hat{\mathcal{A}}_w|_\alpha = \sqrt{k_2}\hat{\mathcal{A}}_w|_{\tilde{\alpha}} \ , \quad \sqrt{k_1}\hat{\mathcal{A}}_{\bar{w}}|_\alpha = \sqrt{k_2}\hat{\mathcal{A}}_{\bar{w}}|_{\tilde{\alpha}}$$

which does define an isotropic space of $\mathfrak{d}$ (not a subalgebra however) is *not* attained along this flow.

is that the RG invariant WZW levels are given by monodromies about simple poles[17]

$$\pm k_{1,2} = r_\pm = \frac{1}{2\pi i}\oint_{\alpha_\pm} \omega = \mathrm{res}_{\zeta=\alpha_\pm}\varphi(\zeta) \ , \tag{3.6.12}$$

exactly in line with the conjecture of Costello (reported and supported by Derryberry [Der21]). While there are more parameters in ω than there are RG equations, we can form the ratios of poles and zeroes

$$q_\pm = \frac{\alpha_\pm - \iota_+}{\alpha_\pm - \iota_-} = \frac{v_\pm}{u_\pm} \ , \tag{3.6.13}$$

in terms of which the RG system of [GS17] translates to

$$\dot{q}_\pm = -\frac{c_G}{2K}\frac{(1+q_\mp)}{(-1+q_\mp)}q_\pm \ , \qquad \dot{K} = -\frac{c_G}{2}\frac{q_- + q_+}{(1-q_-)(1-q_+)} \ . \tag{3.6.14}$$

The RG invariants are given by

$$k_1 k_2 = \frac{K^2 q_- q_+}{(q_+ - q_-)^2} \ , \qquad \frac{k_1}{k_2} = \lambda_0^2 = \frac{q_+}{(1-q_+)^2}\frac{(1-q_-)^2}{q_-} \ , \tag{3.6.15}$$

which allows us to retain either of $q_\pm$ as independent variables. We can directly solve these equations

$$\sqrt{k_1 k_2}\frac{q_+ - q_-}{\sqrt{q_+ q_-}} + k_1 \log q_+ - k_2 \log q_- = \frac{c_G}{2}\log \mu/\mu_{q_0} \ , \tag{3.6.16}$$

and a remarkable feature, also conjectured by Costello, is that this quantity is precisely the contour integral between zeroes

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\int_{\iota_-}^{\iota_+}\omega = \frac{c_G}{2} \ . \tag{3.6.17}$$

To best understand the action of the RG flow on the locations of the poles directly, we replace K with the RG invariant k_2 (or k_1), and fix the zeroes to be located at $\iota_\pm = \pm 1$. This yields the RG invariant relation

$$1 - \alpha_+^2 - \lambda_0^2(1-\alpha_-^2) = 0 \ , \tag{3.6.18}$$

[17]The monodromy about the double pole at infinity is trivially RG invariant since the sum of all the residues vanishes.

and a flow equation

$$\dot{\alpha}_- = \frac{c_G}{8k_2}\frac{\alpha_+(1-\alpha_-^2)^2}{\alpha_- - \alpha_+} \ , \tag{3.6.19}$$

the solution of which is

$$\frac{\alpha_+ - \alpha_-}{1-\alpha_+^2} + \frac{1}{2}\log\frac{\alpha_+ + 1}{\alpha_+ - 1} - \frac{1}{2\lambda_0^2}\log\frac{\alpha_- + 1}{\alpha_- - 1} = \frac{c_G}{4k_1}\log\mu/\mu_{\alpha_0} \ . \tag{3.6.20}$$

As illustrated in fig. 3.1, this system displays a finite RG trajectory linking fixed points. In the UV limit the poles accumulate to different zeroes, and in the IR the poles accumulate to the same zero. Let us consider the upper red trajectory of fig. 3.1 in which we choose $\lambda_0 < 1$ and pick the positive branch of the solution $\alpha_+ = +\sqrt{1-\lambda_0^2(1-\alpha_-^2)}$. With this choice we see that there are finite fixed points[18] such that the right hand side of eq. (3.6.19) vanishes at

$$\text{UV}: \quad (\alpha_-, \alpha_+) = (-1, 1) \ , \quad \lambda_1 = 0 \ , \qquad \text{IR}: \quad (\alpha_-, \alpha_+) = (1, 1) \ , \quad \lambda_1 = \lambda_0 \ , \tag{3.6.21}$$

in which we recall the map

$$\lambda_1 = \left(\frac{(1+\alpha_-)(-1+\alpha_+)}{(-1+\alpha_-)(1+\alpha_+)}\right)^{\frac{1}{2}} \ . \tag{3.6.22}$$

One of the appealing features of the IFT$_2$ (3.4.3) is that it provides a classical Lagrangian interpolation that includes its own UV and IR limits [GS17]. That is to say these CFTs can be obtained directly from the Lagrangian eq. (3.4.3) by tuning the parameters of the theory to their values at the end points of the RG flow. Given the interpretation of these RG flows as describing poles colliding with zeroes it is natural to expect that a similar interpolation can be obtained directly in 4d by taking limits of the differential ω in eq. (3.6.6).

Here we will explore how this works for the IFT$_2$ (3.4.3) in the IR. The limit we will consider is to collide the poles at $\alpha_\pm$ with the zero at ι_+, following the upper red trajectory in fig. 3.1. This corresponds to taking $q_\pm \to 0$. In order to be consistent with

[18] There are also fixed points to the RG flow at $\alpha_+ = 0$ with $\alpha_-^2 = 1 - \frac{k_2}{k_1}$ however by assumption $k_2 > k_1$, and so these do not correspond to real values of α_- and consequently λ_1 is imaginary. We do not consider such complex limits here.

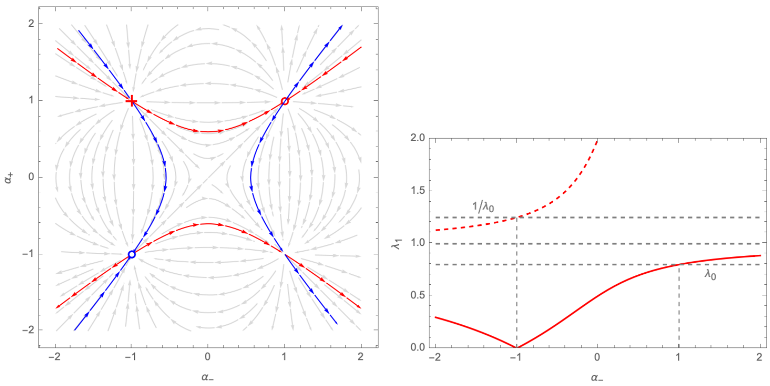


Figure 3.1: Left: RG flow across the α_+, α_- plane with arrows directed to the IR. The highlighted parabola are the solutions that lie on the locus of the RG invariant quantity $\lambda_0^2 = k_1/k_2$, plotted here for $\lambda_0 = 0.8$ (red) and $\lambda_0 = 1.2$ (blue). Right: The value of λ_1 plotted along the red loci of the left panel (upper branch solid and lower branch dotted). In both cases $\lambda_1 \to 1$ asymptotically as $\alpha_- \to \pm\infty$. Of note is the flow displayed by the upper red branch between the UV fixed point $(\alpha_-, \alpha_+) = (-1, 1)$ with $\lambda_1 = 0$ and the IR fixed point $(\alpha_-, \alpha_+) = (1, 1)$ with $\lambda_1 = \lambda_0$.

the RG invariants (3.6.15), we take this limit as

$$q_+ = k_1\epsilon + \mathcal{O}(\epsilon^2) \ , \qquad q_- = k_2\epsilon + \mathcal{O}(\epsilon^2) \ , \qquad K = k_1 - k_2 + \mathcal{O}(\epsilon) \ , \qquad \epsilon \to 0 \ . \quad (3.6.23)$$

Taking this limit in (3.6.6), and redefining the spectral parameter such that the remaining pole and zero are fixed to 1 and -1 respectively, yields

$$\omega \to \frac{k_1 - k_2}{2}\frac{\zeta + 1}{\zeta - 1}d\zeta \ . \quad (3.6.24)$$

Let us consider the implication of this limit from the CS_4 perspective. Given that the pole structure of ω is modified in this limit, so will the double $\mathfrak{d}$, and thus we should be careful in our interpretation of the boundary conditions. If we take ω to be given by (3.6.24) and consider the boundary conditions (3.6.9) with $t = 0$, the condition $\mathbb{A}_w \in \mathfrak{l}_t$ becomes $\hat{\mathcal{A}}_w|_\alpha = 0$. From eq. (3.3.5) we know that $\hat{\mathcal{A}}_w$ has a pole at ι_+. In other words, we can write

$$\hat{\mathcal{A}}_w = \frac{(\zeta - \alpha_+)}{(\zeta - \iota_+)}\Xi(\zeta) \ , \quad (3.6.25)$$

with $\Xi(\zeta)$ regular as $\alpha_+ \to \iota_+$. Hence, in the IR there is no boundary condition for $\hat{\mathcal{A}}_w$ at $\zeta = 1$. On the other hand, the boundary condition $\mathbb{A}_{\bar{w}} \in \mathfrak{l}_{t^{-1}}$ for $t = 0$ is $\hat{\mathcal{A}}_{\bar{w}}|_{\bar{\alpha}} = 0$

which in the limit $\alpha_+, \iota_+ \to 1$ becomes a chiral boundary condition for the $\bar{w}$ component

$$\hat{\mathcal{A}}_{\bar{w}}|_{\zeta=1} = 0 \ . \tag{3.6.26}$$

For this choice of boundary condition one can localise the CS_4 action following the procedure described in § 3.3, and the resulting two dimensional IFT is the WZW model at level $k_2 - k_1$.

In contrast, from the 2-dimensional perspective it is known that the full result at this IR fixed point is actually a product WZW model on $G_{k_2} \times G_{k_1 - k_2}$ [GS17]. This indicates that there is some delicacy in taking the IR limit directly as a Lagrangian interpolation in 4d even when it is possible in 2d. One reason for this is there is also the freedom to perform redefinitions of the spectral parameter, which can, in general, produce non-equivalent limits of ω. Such limits are known as decoupling limits [Del+19; BL20] in the literature, and have been investigated for the UV fixed point of the bi-Yang-Baxter model in [KLT24].

Chapter 4

Gauging The Diamond

In this chapter we present the findings of an investigation intending to extend the framework of the diamond correspondence of integrable theories to incorporate models realised through gaugings. As well as describing a higher-dimensional origin of coset CFTs, by choosing the details of the reduction from higher dimensions, we obtain rich classes of two-dimensional integrable models including homogeneous sine-Gordon models and novel generalisations.

4.1 Introduction

Given an IFT_2 or CFT_2 it is sometimes possible to obtain another $\text{IFT}_2/\text{CFT}_2$ via gauging. Perhaps the most famous examples are the GKO G/H coset CFTs [GKO85], which can be given a Lagrangian description by taking a WZW_2 CFT on G and gauging a (vectorially acting) H subgroup [GK88; Kar+89; BCR90; Wit92]. This motivates the core question of this work:

How can the diamond correspondence be gauged?

Resolving this question dramatically expands the scope of theories that can be given a higher-dimensional avatar. A significant clue is given by the rather remarkable Polyakov-Wiegmann (PW) identity, which shows that the G/H gauged WZW_2 model is actually equivalent to the difference of a G WZW_2 model and an H WZW_2 model. This points towards a general resolution that certain integrable gauged models might be obtained as differences of ungauged models. This is less obvious than it might first seem; it was noted in [Los+96] that for a PW identity to apply for WZW_4 it is necessary for the gauging to be performed by connections with field strength restricted to type $(1,1)$. The six-dimensional origin of such a constraint is rather intriguing and will be elucidated in this paper. In the context of CS_4, Stedman recently proposed [Ste21] considering the difference of CS_4 to give rise to gaugings of IFT_2. We will recover this construction as a reduction of hCS_6 theory in the present work, as well as uncovering some additional novelties in the CS_4 description.

At the top of the diamond, we will consider a theory of two connections, $\mathcal{A} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathcal{B} \in \Omega^1(\mathbb{PT}) \otimes \mathfrak{h}$ for a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The action of this theory is

$$S_{\text{ghCS}_6} = S_{\text{hCS}_6}[\mathcal{A}] - S_{\text{hCS}_6}[\mathcal{B}] + S_{\text{int}}[\mathcal{A}, \mathcal{B}] \ , \tag{4.1.1}$$

in which the term S_{int} couples the two gauge fields. We will develop this story by means of two explicit examples: choosing Ω to have two double poles, we will study the diamond relevant to the gauged WZW theory, and with Ω containing a single fourth-order pole we will study the gauged LMP model. This seemingly simple setup gives rise to a rich story whose results we now summarise:

1. Starting from the holomorphic theory on twistor space (4.1.1), we localise to arrive at an action for a gauged version of WZW_4 (denoted gWZW_4). After localising, the gauge field B is constrained to satisfy two of the three anti-self-dual Yang-Mills equations, namely $F^{2,0}[B] = 0$ and $F^{0,2}[B] = 0$, and the resulting gWZW_4 is an IFT_4.[1]

2. The two gauge fields $\mathcal{A}$ and $\mathcal{B}$ of the gauged hCS_6 theory (denoted ghCS_6) source various degrees of freedom in gWZW_4. In particular, as well as the fundamental field g and the 4d gauge field B, auxiliary degrees of freedom enter as Lagrange multipliers for the constraints $F^{2,0}[B] = 0$ and $F^{0,2}[B] = 0$.

3. Reducing by two dimensions, we recover a variety of IFT_2 including the special case of the gauged WZW_2 model (denoted gWZW_2). In general, we find a model coupling a gauged IFT_2 and a Hitchin system [Hit87] involving the gauge field B and a pair of adjoint scalar fields. These scalars may source a potential for the gWZW_2 in which case we recover the complex sine-Gordon model and more

[1] This indicates that general unconstrained gaugings of WZW_4 will break integrability in the sense outlined above.

broadly the homogeneous sine-Gordon models [Fer+97]. At the special point associated to the 2d PCM, Lagrange multipliers ensure that the gauge field is flat and hence trivial — this is essential as the gauged PCM is not generically integrable.

4. We also use this formalism to perform an integrable gauging of the LMP model. Just as in the gauging of WZW_4, the field strength of the gauge field must be constrained to obey two of the anti-self-dual Yang-Mills equations, this time $F^{2,0}[B]=0$ and $\varpi\wedge F^{1,1}[B]=0$. It is noteworthy that the two equations that are enforced by Lagrange multipliers agree with the two equations that are identically solved in the ungauged case. This is true for both WZW_4 and the LMP model. In addition, we show that the gauged LMP model obeys a PW-like identity such that it may be expressed as the difference of two LMP models on $\mathfrak{g}$ and $\mathfrak{h}$.

In section 4.2, we introduce the gauging of the WZW_4 diamond concentrating in particular on the right hand side. We recover the gauged IFT_4 and demonstrate that its equations of motion may be rewritten as the ASDYM equations. The wide array of IFT_2 are explored in section 4.3 where we also show that they are integrable and provide the associated Lax connection. Following the gauging of WZW_4, section 4.4 elaborates on the left hand side of the diamond, connecting to CS_4 by first reducing, and then to the IFT_2 by localisation. Section 4.5 describes the diamond for the gauged LMP theory.

4.2 The gauged WZW diamond

In this section, we will construct a diamond correspondence of theories which realises the gauged WZW_2 model, i.e. the G/H coset CFT.

4.2.1 Gauged WZW Models

First, let us review the gauging of the WZW model and the crucial Polyakov-Wiegmann identity. Letting G be a Lie group and $g\in C^\infty(\Sigma,G)$ a smooth G-valued field, the WZW_2 action is[2]

$$S_{\mathrm{WZW}_2}[g]=\frac{1}{2}\int_\Sigma \mathrm{Tr}_{\mathfrak{g}}\big(g^{-1}\mathrm{d}g\wedge\star g^{-1}\mathrm{d}g\big)\;+\;\frac{1}{3}\int_{\Sigma\times[0,1]}\mathrm{Tr}_{\mathfrak{g}}(\hat{g}^{-1}\mathrm{d}\hat{g}\wedge\hat{g}^{-1}\mathrm{d}\hat{g}\wedge\hat{g}^{-1}\mathrm{d}\hat{g})\;. \quad (4.2.1)$$

[2]To minimise factors of imaginary units we momentarily adopt Lorentzian signature. Schematically, we have $S_{\mathrm{Lorentz}}=-\mathrm{i}S_{\mathrm{Euclid}}|_{\star\to\mathrm{i}\star}$.

Gauging a vectorial H-action of the PCM term is straightforward. We introduce an $\mathfrak{h}$-valued connection $B \in \Omega^1(\Sigma) \otimes \mathfrak{h}$ transforming as

$$\ell \in C^\infty(\Sigma, H): \quad B \mapsto \ell^{-1} B \ell + \ell^{-1} \mathrm{d}\ell \ , \quad g \mapsto \ell^{-1} g \ell \ , \tag{4.2.2}$$

with field strength $F[B] = \mathrm{d}B + B \wedge B$. The PCM term is then gauged by replacing the exterior derivatives with covariant derivatives $\mathrm{d}g \to \mathrm{D}g = \mathrm{d}g + [B, g]$. Less trivially, the gauge completion of the WZ 3-form is [Wit92; FS94a; FS94b; FM05]

$$\mathcal{L}_{\mathrm{gWZ}}[g, B] = \mathcal{L}_{\mathrm{WZ}}[g] + \mathrm{d}\, \mathrm{Tr}_{\mathfrak{g}}(g^{-1}\mathrm{d}g \wedge B + \mathrm{d}g g^{-1} \wedge B + g^{-1} B g \wedge B) \ . \tag{4.2.3}$$

Adding these two pieces together gives the gauged WZW_2 action,

$$\begin{aligned} S_{\mathrm{gWZW}_2}[g, B] = S_{\mathrm{WZW}_2}[g] + \int_\Sigma \mathrm{Tr}_{\mathfrak{g}}(g^{-1} dg \wedge (1 - \star) B + dg g^{-1} \wedge (1 + \star) B \\ + B \wedge \star B + g^{-1} B g \wedge (1 - \star) B) \ . \end{aligned} \tag{4.2.4}$$

Notice that chiral couplings between currents and gauge fields emerge from combinations of the PCM and WZ contributions. The identity

$$\mathcal{L}_{\mathrm{WZ}}[g_1 g_2] = \mathcal{L}_{\mathrm{WZ}}[g_1] + \mathcal{L}_{\mathrm{WZ}}[g_2] + \mathrm{d}\, \mathrm{Tr}_{\mathfrak{g}} \left(\mathrm{d}g_2 g_2^{-1} \wedge g_1^{-1} \mathrm{d}g_1 \right) \ , \tag{4.2.5}$$

ensures that (4.2.4) can be recast as the difference of two WZW_2 models. To see this we choose a parametrisation of the gauge field B in terms of two smooth H-valued fields

$$B = \frac{1 + \star}{2} a^{-1} \mathrm{d}a + \frac{1 - \star}{2} b^{-1} \mathrm{d}b \ , \quad a, b \in C^\infty(\Sigma, H) \ . \tag{4.2.6}$$

In two dimensions, this is not a restriction on the field content of the gauge field, but simply a way of parametrising the two independent components of B. With such a parametrisation, if we then further define $\tilde{g} = a g b^{-1} \in C^\infty(\Sigma, G)$ and $\tilde{h} = a b^{-1} \in C^\infty(\Sigma, H)$ the gauged model (4.2.4) can be written as the difference of two WZW_2 models:

$$S_{\mathrm{gWZW}_2}[g, B] = S_{\mathrm{WZW}_2}[\tilde{g}] - S_{\mathrm{WZW}_2}[\tilde{h}] \ . \tag{4.2.7}$$

This is known as the Polyakov-Wiegmann (PW) identity [PW83].

4.2.2 Gauging of the $\mathbf{WZW}_4$ model

Let us now consider the four-dimensional WZW model, given by eq. (2.5.16). The gauging procedure follows in the exact same manner, producing an analogous gauged WZW_4 action,

$$S^{(\alpha,\beta)}_{\text{gWZW}_4}[g,B] = \frac{1}{2}\int_{\mathbb{R}^4} \text{Tr}(g^{-1}\nabla g \wedge \star g^{-1}\nabla g) + \int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathcal{L}_{\text{gWZ}}[g,B] \ . \tag{4.2.8}$$

Here, we denote the covariant derivative by $\nabla g = \mathrm{d}g + [B,g]$. A critical difference between two and four dimensions is the applicability of the PW identity as was pointed out in [Los+96]. In two dimensions, this mapping relies on the relation (4.2.6). To extend it to four dimensions, we consider the operator on 1-forms

$$\mathcal{J}_{\alpha,\beta}(\sigma) = -\mathrm{i} \star (\omega_{\alpha,\beta} \wedge \sigma) \ . \tag{4.2.9}$$

Checking that $\mathcal{J}^2_{\alpha,\beta} = -\text{id}$, we can introduce the useful projectors

$$P = \frac{1}{2}(\text{id} - \mathrm{i}\mathcal{J}) \ , \qquad \bar{P} = \frac{1}{2}(\text{id} + \mathrm{i}\mathcal{J}) \ , \tag{4.2.10}$$

which furnish a range of identities detailed in appendix A. With these in mind, we can write a four-dimensional analogue to (4.2.6),

$$B = P\left(a^{-1}\mathrm{d}a\right) + \bar{P}\left(b^{-1}\mathrm{d}b\right) \ , \quad a,b \in C^\infty(\mathbb{R}^4, H) \ . \tag{4.2.11}$$

With this parametrisation of the gauge field, it is indeed possible to use the composite fields $\tilde{g} = agb^{-1} \in C^\infty(\mathbb{R}^4, G)$ and $\tilde{h} = ab^{-1} \in C^\infty(\mathbb{R}^4, H)$ to express the gauged WZW_4 action in a fashion akin to eq. (4.2.7) as

$$S^{(\alpha,\beta)}_{\text{gWZW}_4}[g,B] = S^{(\alpha,\beta)}_{\text{WZW}_4}[\tilde{g}] - S^{(\alpha,\beta)}_{\text{WZW}_4}[\tilde{h}] \ . \tag{4.2.12}$$

However, unlike in two dimensions, the parametrisation of the gauge field in eq. (4.2.11) is not generic. It implies a restriction on the connection, namely that its curvature satisfies

$$\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}[B] = 0 \ , \quad \beta^a\beta^b F_{a\dot{a}b\dot{b}}[B] = 0 \ . \tag{4.2.13}$$

This can be thought of as an analogue of imposing that F is strictly a $(1,1)$-form, which indeed is the case when $\beta = \hat{\alpha}$ and the WZW_4 is taken at the Kähler point. It is noteworthy that these constraints on the background gauge field agree with two of the three ASDYM equations; the same two equations that were identically satisfied by the Yang parametrisation of the connection A'. In the forthcoming analysis, we will see how this arises from the hCS_6 construction.

4.2.3 A six-dimensional origin

We now turn to the six-dimensional holomorphic Chern-Simons theory on twistor space that will descend to the above gauged WZW models in two and four dimensions. Given the factorisation of gWZW_2 to the difference of WZW_2 models, a natural candidate is to simply consider the difference of hCS_6 theories to generalise the six-dimensional action introduced in [Cos20; BS23; Pen21; Cos21]. Indeed, a similar idea was proposed in [Ste21] to construct 2d coset models from the difference of CS_4 theories. However, how this should work in six dimensions is less clear as the factorisation of gWZW_4 requires the curvature of the gauge field to be constrained.

The fundamental fields of our theory are two connections $\mathcal{A} \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathcal{B} \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{h}$, which appear in the six-dimensional action

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] - \frac{1}{2\pi \mathrm{i}} \int \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\mathcal{A} \wedge \mathcal{B}\big) \ , \tag{4.2.14}$$

where the functional S_{hCS_6} is defined in eq. (3.1.1), and

$$\Omega = \frac{1}{2}\Phi(\pi)\, \epsilon_{\dot{a}\dot{b}}\, \pi_a dx^{a\dot{a}} \wedge \pi_b dx^{b\dot{b}} \wedge \langle \pi d\pi \rangle \ , \qquad \Phi = \frac{\langle \alpha\beta\rangle^2}{\langle \pi\alpha\rangle^2 \langle \pi\beta\rangle^2} \ . \tag{4.2.15}$$

As well as the bulk hCS_6 functionals, we have also included a coupling term between the two connections that contributes on the support of $\bar{\partial}\Omega$, i.e. at the poles of Ω. We will shortly provide a motivation for this boundary term related to the boundary conditions we impose on the theory.

This definition is slightly imprecise; strictly speaking, the inner product denoted by 'Tr' should be defined separately for each algebra, i.e. $\mathrm{Tr}_{\mathfrak{g}}$ and $\mathrm{Tr}_{\mathfrak{h}}$. In the coupling term, where $\mathcal{B}$ enters inside $\mathrm{Tr}_{\mathfrak{g}}$, we should first act on $\mathcal{B}$ with a Lie algebra homomorphism from $\mathfrak{h}$ to $\mathfrak{g}$, and, in principle, this homomorphism could be chosen differently at each pole of Ω. We discuss more general gaugings, beyond the vectorial gauging considered here, in appendix C. It will also be useful for us to assume that we have an orthogonal decomposition of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{k} \ , \qquad \mathrm{Tr}\big(X \cdot Y\big) = \mathrm{Tr}\big(X^{\mathfrak{h}} \cdot Y^{\mathfrak{h}}\big) + \mathrm{Tr}\big(X^{\mathfrak{k}} \cdot Y^{\mathfrak{k}}\big) \ , \tag{4.2.16}$$

and that the homogeneous space G/H is reductive,

$$[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k} \ . \tag{4.2.17}$$

To complete the specification of the theory, we must supply boundary conditions that ensure the vanishing of the boundary term in the variation of (4.2.14),

$$\delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = \frac{1}{2\pi \mathrm{i}} \int \bar{\partial}\Omega \wedge \mathrm{Tr}\big((\delta\mathcal{A} + \delta\mathcal{B}) \wedge (\mathcal{A} - \mathcal{B})\big) \ . \tag{4.2.18}$$

Since $\bar{\partial}\Omega$ only has support at the poles of Ω, the integral over $\mathbb{CP}^1$ may be computed explicitly in this term. As well as contributions proportional to delta-functions on $\mathbb{CP}^1$, this will also include $\mathbb{CP}^1$-derivatives of delta-functions since the poles in Ω are second order. Using the localisation formula in the appendix B, we find

$$\begin{aligned} \delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = - \int_{\mathbb{R}^4} \bigg[& \frac{\alpha_a \beta_b \Sigma^{ab}}{\langle \alpha\beta \rangle} \wedge \mathrm{Tr}\big((\delta\mathcal{A} + \delta\mathcal{B}) \wedge (\mathcal{A} - \mathcal{B})\big) \\ & + \frac{1}{2} \alpha_a \alpha_b \Sigma^{ab} \wedge \partial_0 \mathrm{Tr}\big((\delta\mathcal{A} + \delta\mathcal{B}) \wedge (\mathcal{A} - \mathcal{B})\big) \bigg] + \alpha \leftrightarrow \beta \ . \end{aligned} \tag{4.2.19}$$

In this expression, we have introduced a basis for the self-dual 2-forms defined by $\Sigma^{ab} = \varepsilon_{\dot{a}\dot{b}} \mathrm{d}x^{a\dot{a}} \wedge \mathrm{d}x^{b\dot{b}}$. To attain the vanishing of the boundary variation, we consider the boundary conditions

$$\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0 \ , \qquad \mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \mathcal{B}\big|_{\alpha,\beta} \ , \qquad \partial_0 \mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \partial_0 \mathcal{B}\big|_{\alpha,\beta} \ , \tag{4.2.20}$$

where the superscripts $\mathfrak{k}$ and $\mathfrak{h}$ denote projections corresponding to the decomposition (4.2.16). This completes our definition of the gauged hCS$_6$ theory.

We might choose to think of the boundary term in the variation as being a potential for a ‘symplectic’ form[3,4]

$$\Theta = \delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} \ , \qquad \Omega = \delta\Theta = -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega\wedge\Big(\mathrm{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge\delta\mathcal{A}\big) - \mathrm{Tr}_{\mathfrak{h}}\big(\delta\mathcal{B}\wedge\delta\mathcal{B}\big)\Big) \ , \tag{4.2.21}$$

such that our boundary conditions define a Lagrangian (i.e. maximal isotropic) subspace. We would like to interpret this as a symplectic form on an appropriate space of fields defined over $\mathbb{R}^4$. Evaluating the integral over $\mathbb{CP}^1$ and writing $\Omega = \Omega_{\mathcal{A}} - \Omega_{\mathcal{B}}$, this symplectic form is given by

$$\Omega_{\mathcal{A}} = \int_{\mathbb{R}^4}\left[\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle}\wedge\mathrm{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge\delta\mathcal{A}\big)\big|_{\alpha} + \frac{1}{2}\alpha_a\alpha_b\Sigma^{ab}\wedge\partial_0\mathrm{Tr}_{\mathfrak{g}}\big(\delta\mathcal{A}\wedge\delta\mathcal{A}\big)\big|_{\alpha}\right] + \alpha\leftrightarrow\beta \ , \tag{4.2.22}$$

with an analogous expression for $\Omega_{\mathcal{B}}$. Since our boundary conditions are identical at each pole, we concentrate only on the contributions associated to the pole at α. The symplectic form is not sensitive to the entire field configuration $\mathcal{A}\in\Omega^1(\mathbb{PT})\otimes\mathfrak{g}$, but rather to the evaluation of $\mathcal{A}$ at the poles and its first $\mathbb{CP}^1$-derivative,

$$\vec{\mathcal{A}} = \big(\mathcal{A}|_{\alpha}, \partial_0\mathcal{A}|_{\alpha}\big) \ . \tag{4.2.23}$$

This data may be interpreted as defining a 1-form (or more precisely a $(0,1)$-form with respect to the complex structure defined by α) on $\mathbb{R}^4$ valued in the Lie algebra[5] $\vec{\mathfrak{g}} = \mathfrak{g}\ltimes\mathbb{R}^{\dim(G)}$. With this in mind, it is more accurate to say that the contribution from the pole at α in Ω is a symplectic form on the space of configurations

$$\big(\vec{\mathcal{A}},\vec{\mathcal{B}}\big)\in\Omega^{0,1}(\mathbb{R}^4)\otimes\big(\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}\big) \ . \tag{4.2.24}$$

This symplectic form may be succinctly written by introducing an inner product on the Lie algebra $\vec{\mathfrak{g}}\oplus\vec{\mathfrak{h}}$, and our boundary conditions describe an isotropic subspace with respect to this inner product.[6]

[3]Precedent in the literature dictates that we denote the symplectic form by Ω; we trust that context serves to disambiguate from the meromorphic differential Ω.

[4]This is slightly loose as the 2-form is degenerate; strictly speaking we should restrict to symplectic leaves.

[5]The dimension of $\vec{\mathfrak{g}}$ is $2\dim(G)$, so it must be isomorphic to $\mathbb{R}^{\dim(G)}\oplus\mathbb{R}^{\dim(G)}$ as a vector space. The Lie algebra structure may be derived by considering consecutive infinitesimal gauge transformations. In the CS_4 literature, these structures have been studied under the name ‘defect Lie algebras’ [BSV22; LV21].

[6]This need not be the case since our boundary conditions could generically intertwine constraints on the algebra and spacetime components, meaning they would not be captured by a subspace of the

To be explicit, we associate $\mathbb{R}^{\dim G}$ with the dual $\mathfrak{g}^*$ and denote the natural pairing of the algebra and its dual by $\tilde{x}(x) \in \mathbb{R}$ for $x \in \mathfrak{g}$ and $\tilde{x} \in \mathfrak{g}^*$. We let $\vec{X} = (x, \tilde{x})$ and $\vec{Y} = (y, \tilde{y})$ be elements of $\vec{\mathfrak{g}}$ such that the bracket on $\vec{\mathfrak{g}}$ is defined by

$$[\vec{X}, \vec{Y}]_{\vec{\mathfrak{g}}} = ([x, y], \mathrm{ad}^*_x \tilde{y} - \mathrm{ad}^*_y \tilde{x}) \ , \tag{4.2.25}$$

where the co-adjoint action is $\mathrm{ad}^*_y \tilde{x}(x) = \tilde{x}([x, y])$. We equip $\vec{\mathfrak{g}}$ with the inner product

$$\langle \vec{X}, \vec{Y} \rangle_{\vec{\mathfrak{g}}} = \frac{\langle \beta \hat{\alpha} \rangle}{\langle \alpha \beta \rangle \langle \alpha \hat{\alpha} \rangle} \mathrm{Tr}_{\mathfrak{g}}(x \cdot y) + \frac{1}{2}\Big(\tilde{y}(x) + \tilde{x}(y)\Big) \ , \tag{4.2.26}$$

such that the contribution from the pole at α to $\Omega_{\mathcal{A}}$is given by

$$\Omega_{\mathcal{A}} = \int_{\mathbb{R}^4} \mu_\alpha \wedge \langle \delta \vec{\mathcal{A}}, \delta \vec{\mathcal{A}} \rangle_{\vec{\mathfrak{g}}} \ , \tag{4.2.27}$$

where $\mu^\alpha = \alpha_a \alpha_b \Sigma^{ab}$ is the $(2, 0)$-form defined by the complex structure associated to $\alpha \in \mathbb{CP}^1$.

In a similar fashion we let $\vec{U} = (u, \tilde{u})$ and $\vec{V} = (v, \tilde{v})$ be elements of $\vec{\mathfrak{h}}$, which is equipped with a bracket and pairing via the same recipe. We consider the commuting direct sum $\vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}}$ equipped with pairing and bracket

$$\langle\!\langle (\vec{X}, \vec{U}), (\vec{Y}, \vec{V}) \rangle\!\rangle = \langle \vec{X}, \vec{Y} \rangle_{\vec{\mathfrak{g}}} - \langle \vec{U}, \vec{V} \rangle_{\vec{\mathfrak{h}}} \ , \qquad [\![(\vec{X}, \vec{U}), (\vec{Y}, \vec{V})]\!] = \Big(\big[\vec{X}, \vec{Y}\big]_{\vec{\mathfrak{g}}} , \big[\vec{U}, \vec{V}\big]_{\vec{\mathfrak{h}}} \Big) \ , \tag{4.2.28}$$

such that the total symplectic form coming from the pole at α is just

$$\Omega = \int_{\mathbb{R}^4} \mu_\alpha \wedge \langle\!\langle (\delta \vec{\mathcal{A}}, \delta \vec{\mathcal{B}}), (\delta \vec{\mathcal{A}}, \delta \vec{\mathcal{B}}) \rangle\!\rangle \ . \tag{4.2.29}$$

Then, our boundary conditions can be expressed as $\big(\vec{\mathcal{A}}, \vec{\mathcal{B}}\big) \in \Omega^{0,1}(\mathbb{R}^4) \otimes L$ where we introduce a subspace

$$L = \big\{ (\vec{X}, \vec{U}) \in \vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}} \mid x = u \ , \ P^*_{\mathfrak{h}} \tilde{x} = \tilde{u} \big\} \ , \tag{4.2.30}$$

in which $P^*_{\mathfrak{h}}$ is dual to the projector $P_{\mathfrak{h}}$ onto the subalgebra, i.e. $P^*_{\mathfrak{h}} \tilde{x}(x) = \tilde{x}(P_{\mathfrak{h}} x)$. As L is defined by $\dim \mathfrak{g} + \dim \mathfrak{h}$ constraints, it is half-dimensional and it is also isotropic

algebra alone. They would always, however, define an isotropic subspace of $\Omega^{0,1}(\mathbb{R}^4) \otimes \big(\vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}}\big)$ by definition. Examples of this more general type of boundary condition can be found in [Col+24b].

with respect to $\langle\!\langle \cdot , \cdot \rangle\!\rangle$, hence defines a Lagrangian subspace. Moreover, assuming that G/H is reductive, L is a subalgebra.[7] Pre-empting the following section, this analysis indicates that there will be a residual $\vec{\mathfrak{h}}$ gauge symmetry associated to the pole at α, and similarly at β.

We can make one further observation[8] on the role of the boundary term from a symplectic perspective, which is best illustrated by a finite-dimensional analogy. Recall that the cotangent bundle $\mathcal{M} = T^*X$ is a symplectic manifold; if we let $\{x^i\}$ be local coordinates on X and $\{\xi_i\}$ the components of a 1-form $\xi = \xi_i dx^i \in T^*_x X$, then $p = (x^i, \xi_i)$ provide local coordinates for $\mathcal{M}$ in terms of which the canonical symplectic form is $\Omega = \mathrm{d}\xi_i \wedge \mathrm{d}x^i$. The tautological potential, which admits a coordinate free definition in terms of the projection $\pi : T^*X \to X$, for this is given by $\Theta = \xi_i \mathrm{d}x^i$. The zero section, i.e. points $p = (x^i, \xi_i = 0)$ of T^*X, is a Lagrangian submanifold and we note that Θ vanishes trivially here. Now Weinstein's tubular neighbourhood theorem ensures that in the vicinity of a Lagrangian submanifold L, any symplectic manifold $\mathcal{M}$ locally looks like T^*L with L given by the zero section. In the case at hand, our boundary conditions are of the schematic form $\xi = \mathcal{A} - \mathcal{B} = 0$, and the effect of including the additional boundary contribution in the action (4.2.14) ensures that the resultant symplectic potential is the tautological one.

To close this section, let us comment that at the special point $\alpha = \hat{\beta}$, one of the terms in the inner product (4.2.26) vanishes. This allows for a larger class of admissible boundary conditions, even in the ungauged model, including the examples

$$\mathcal{A}\big|_{\hat{\alpha}} = 0 \ , \qquad \partial_0 \mathcal{A}\big|_{\alpha} = 0 \qquad \text{or} \qquad \partial_0 \mathcal{A}\big|_{\hat{\alpha}} = 0 \ , \qquad \partial_0 \mathcal{A}\big|_{\alpha} = 0 \ . \tag{4.2.31}$$

We leave these for future development.

4.2.4 Localisation to gWZW$_4$

The localisation procedure follows in a similar fashion to the ungauged model. However, given that there are now two gauge fields $\mathcal{A}$ and $\mathcal{B}$, some care is required to account for degrees of freedom and residual symmetries.

[7] If $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ is not assumed to be reductive then the stabiliser of L consists of elements of the form

$$\mathrm{stab}_L = \left\{ (\vec{X}, \vec{U}) \in \vec{\mathfrak{g}} \oplus \vec{\mathfrak{h}} \mid x = u \ , \ P^*_{\mathfrak{h}} \tilde{x} = \tilde{u} \ , \ [u, \mathfrak{k}] = 0 \ , \ ([\mathfrak{h}, \mathfrak{k}], \tilde{u}) = 0 \right\} \ .$$

[8] We thank A. Arvanitakis for this suggestion.

We introduce a new pair of connections $\mathcal{A}' \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{g}$ and $\mathcal{B}' \in \Omega^{0,1}(\mathbb{PT}) \otimes \mathfrak{h}$, along with group-valued fields $\hat{g} \in C^\infty(\mathbb{PT}, G)$ and $\hat{h} \in C^\infty(\mathbb{PT}, H)$ related to the original gauge fields by

$$\begin{aligned} \mathcal{A} &= \hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g} \equiv \mathcal{A}'^{\hat{g}} \ , \\ \mathcal{B} &= \hat{h}^{-1}\mathcal{B}'\hat{h} + \hat{h}^{-1}\bar{\partial}\hat{h} \equiv \mathcal{B}'^{\hat{h}} \ . \end{aligned} \tag{4.2.32}$$

The redundancy in this parametrisation is given by the action of $\check{\gamma} \in C^\infty(\mathbb{PT}, G)$ and $\check{\eta} \in C^\infty(\mathbb{PT}, H)$:

$$\mathcal{A}' \mapsto \check{\gamma}^{-1}\mathcal{A}'\check{\gamma} + \check{\gamma}^{-1}\bar{\partial}\check{\gamma} \ , \qquad \hat{g} \mapsto \check{\gamma}^{-1}\hat{g} \ , \tag{4.2.33}$$

$$\mathcal{B}' \mapsto \check{\eta}^{-1}\mathcal{B}'\check{\eta} + \check{\eta}^{-1}\bar{\partial}\check{\eta} \ , \qquad \hat{h} \mapsto \check{\eta}^{-1}\hat{h} \ , \tag{4.2.34}$$

which leave $\mathcal{A}$ and $\mathcal{B}$ invariant. As before, this is partially used to fix away the $\mathbb{CP}^1$ legs

$$\mathcal{A}'_0 = \mathcal{B}'_0 = 0 \ . \tag{4.2.35}$$

The localisation procedure will produce a four-dimensional boundary theory with fields given by the evaluations of $\hat{g}, \hat{h}$ and their $\mathbb{CP}^1$-derivatives at the poles α and β of Ω. Since the $\mathbb{CP}^1$-derivatives will play an important role, we denote them

$$\hat{u} = \hat{g}^{-1}\partial_0\hat{g} \ , \quad \hat{v} = \hat{h}^{-1}\partial_0\hat{h} \ . \tag{4.2.36}$$

After fixing (4.2.35), we note that there is still some remaining symmetry given by internal gauge transformations (4.2.33) and (4.2.34) that are $\mathbb{CP}^1$-independent. We use this residual symmetry to fix

$$\hat{g}|_\beta = \mathrm{id} \ , \quad \hat{h}|_\beta = \mathrm{id} \ . \tag{4.2.37}$$

On the other hand, the action (4.2.14) is invariant under gauge transformations acting on $\mathcal{A}$ and $\mathcal{B}$ that preserve the boundary conditions. These are given by smooth maps $\hat{\gamma} \in C^\infty(\mathbb{PT}, G)$ and $\hat{\eta} \in C^\infty(\mathbb{PT}, H)$ satisfying[9]

$$\hat{\gamma}|_{\alpha,\beta} = \hat{\eta}|_{\alpha,\beta} \ , \quad \partial_0\hat{\gamma}|_{\alpha,\beta} = \partial_0\hat{\eta}|_{\alpha,\beta} \ . \tag{4.2.38}$$

[9]Here we use that the homogeneous space G/H is reductive (4.2.17) to ensure that the boundary conditions are preserved.

The induced action of these gauge transformations on the new field content is

$$\mathcal{A}' \mapsto \mathcal{A}' \ , \qquad \hat{g} \mapsto \hat{g}\hat{\gamma} \ , \qquad \hat{u} \mapsto \hat{\gamma}^{-1}\hat{u}\hat{\gamma} + \hat{\gamma}^{-1}\partial_0\hat{\gamma} \ , \tag{4.2.39}$$

$$\mathcal{B}' \mapsto \mathcal{B}' \ , \qquad \hat{h} \mapsto \hat{h}\hat{\eta} \ , \qquad \hat{v} \mapsto \hat{\eta}^{-1}\hat{v}\hat{\eta} + \hat{\eta}^{-1}\partial_0\hat{\eta} \ . \tag{4.2.40}$$

We would like to use this symmetry to further fix degrees of freedom. Note that, while the right action on the fields $\hat{g}$ and $\hat{h}$ at α is entirely unconstrained, the action at β should preserve the gauge fixing condition (4.2.37). This is achieved by performing both an internal and external gauge transformation simultaneously, and requiring $\hat{\gamma}|_\beta = \check{\gamma}$ and $\hat{\eta}|_\beta = \check{\eta}$. This results in an induced left action on the fields $\hat{g}$ and $\hat{h}$ at α. In summary, introducing some notation for simplicity, we have our boundary degrees of freedom

$$\hat{g}|_\alpha \coloneqq g \ , \qquad \hat{g}|_\beta = \mathrm{id} \ , \qquad \hat{u}|_\alpha \coloneqq u \ , \qquad \hat{u}|_\beta \coloneqq \tilde{u} \ , \tag{4.2.41}$$

$$\hat{h}|_\alpha \coloneqq h \ , \qquad \hat{h}|_\beta = \mathrm{id} \ , \qquad \hat{v}|_\alpha \coloneqq v \ , \qquad \hat{v}|_\beta \coloneqq \tilde{v} \ , \tag{4.2.42}$$

and boundary gauge transformations

$$\hat{\gamma}|_\alpha = \hat{\eta}|_\alpha \coloneqq r \ , \qquad \hat{\gamma}^{-1}\partial_0\hat{\gamma}|_\alpha = \hat{\eta}^{-1}\partial_0\hat{\eta}|_\alpha \coloneqq \epsilon \ , \tag{4.2.43}$$

$$\hat{\gamma}|_\beta = \hat{\eta}|_\beta \coloneqq \ell^{-1} \ , \qquad \hat{\gamma}^{-1}\partial_0\hat{\gamma}|_\beta = \hat{\eta}^{-1}\partial_0\hat{\eta}|_\beta \coloneqq \tilde{\epsilon} \ , \tag{4.2.44}$$

which act on the boundary fields as

$$g \mapsto \ell g r \ , \quad u \mapsto r^{-1}ur + \epsilon \ , \quad \tilde{u} \mapsto \ell\tilde{u}\ell^{-1} + \tilde{\epsilon} \ , \tag{4.2.45}$$

$$h \mapsto \ell h r \ , \quad v \mapsto r^{-1}vr + \epsilon \ , \quad \tilde{v} \mapsto \ell\tilde{v}\ell^{-1} + \tilde{\epsilon} \ , \tag{4.2.46}$$

with $\ell, r \in C^\infty(\mathbb{R}^4, H)$ and $\epsilon, \tilde{\epsilon} \in C^\infty(\mathbb{R}^4, \mathfrak{h})$. Based on our expectation of a gauge theory containing a G-valued field and a vectorial H-gauge symmetry, we use the above symmetries to fix

$$h = \mathrm{id} \ , \qquad v = \tilde{v} = 0 \ . \tag{4.2.47}$$

We are thus left with a residual symmetry $r = \ell^{-1}$ acting as

$$g \mapsto \ell g \ell^{-1} \ , \quad u \mapsto \ell u \ell^{-1} \ , \quad \tilde{u} \mapsto \ell\tilde{u}\ell^{-1} \ , \quad B \mapsto \ell B \ell^{-1} - \mathrm{d}\ell\ell^{-1} \ , \tag{4.2.48}$$

which will become the H-gauge symmetry of our 4d theory.

We now proceed with the localisation of the six-dimensional action. As with the ungauged model, the first step is to write the action in terms of $\mathcal{A}', \mathcal{B}'$ and $\hat{g}, \hat{h}$. Given that the localisation formula (B.11) introduces at most one ∂_0 derivative, all dependence on $\hat{h}$ will drop out due to our gauge fixing choices (4.2.42) and (4.2.47). Hence there will be no contribution from $S_{\mathrm{hCS}_6}[\mathcal{B}]$ to the four-dimensional action. As per eq. (2.5.10), we find that the bulk equations of motion (i.e. contributions to the variation of the action that are not localised at the poles of Ω) enforce $\bar{\partial}_0\mathcal{A}'_{\dot{a}} = \bar{\partial}_0\mathcal{B}'_{\dot{a}} = 0$. This implies that the components $\mathcal{A}'_{\dot{a}}, \mathcal{B}'_{\dot{a}}$ are holomorphic, which (combined with the fact that they have homogeneous weight 1) allows us to deduce that

$$\mathcal{A}'_{\dot{a}} = \pi^a A'_{a\dot{a}} \ , \quad \mathcal{B}'_{\dot{a}} = \pi^a B'_{a\dot{a}} \ , \tag{4.2.49}$$

where $A'_{a\dot{a}}, B'_{a\dot{a}}$ are $\mathbb{CP}^1$-independent. Imposing the bulk equations of motion and the gauge fixings described above, the remaining contributions in the action (4.2.14) are given by

$$\begin{aligned} S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = \frac{1}{2\pi\mathrm{i}}\int &\bar{\partial}\Omega\wedge\mathrm{Tr}\big(\mathcal{A}'\wedge\bar{\partial}\hat{g}\hat{g}^{-1} - (\hat{g}^{-1}\mathcal{A}'\hat{g}+\hat{g}^{-1}\bar{\partial}\hat{g})\wedge\mathcal{B}'\big) \\ &- \frac{1}{6\pi\mathrm{i}}\int_{\times[0,1]}\bar{\partial}\Omega\wedge\mathrm{Tr}\big(\hat{g}^{-1}\mathrm{d}\hat{g}\wedge\hat{g}^{-1}\mathrm{d}\hat{g}\wedge\hat{g}^{-1}\mathrm{d}\hat{g}\big) \ . \end{aligned} \tag{4.2.50}$$

In the ungauged model, the next step was to solve the boundary conditions for $\mathcal{A}'$ in terms of $\hat{g}$. Here, the boundary conditions on $\mathcal{A}$ and $\mathcal{B}$, i.e. excluding those relating the $\mathbb{CP}^1$-derivatives of the gauge fields, do not fully determine $A'_{a\dot{a}}, B'_{a\dot{a}}$ and instead relate them as[10]

$$A'_{a\dot{a}} = B'_{a\dot{a}} + \Theta_{a\dot{a}} \coloneqq B'_{a\dot{a}} - \frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\nabla_{b\dot{a}}gg^{-1} \ , \tag{4.2.51}$$

where the covariant derivative is given by $\nabla_{a\dot{a}}gg^{-1} = \partial_{a\dot{a}}gg^{-1} + B'_{a\dot{a}} - \mathrm{Ad}_g B'_{a\dot{a}}$. The relation (4.2.51) allows us to express (4.2.50) entirely in terms of $\mathcal{B}'$, $\Theta = \pi^a\Theta_{a\dot{a}}\bar{e}^{\dot{a}}$ and $\hat{g}$. Many of the terms combine to produce a gauged Wess-Zumino contribution (4.2.3)

[10]The boundary conditions relating the $\mathbb{CP}^1$-derivatives of the gauge fields impose

$$\frac{\alpha^a}{\langle\alpha\beta\rangle}\big(\nabla_{a\dot{a}}gg^{-1}\big)^{\mathfrak{h}} = -\beta^a\nabla_{a\dot{a}}\tilde{u}^{\mathfrak{h}} \ , \qquad \frac{\beta^a}{\langle\alpha\beta\rangle}\big(g^{-1}\nabla_{a\dot{a}}g\big)^{\mathfrak{h}} = -\alpha^a\nabla_{a\dot{a}}u^{\mathfrak{h}} \ ,$$

which, in principle, can be solved for $B'_{a\dot{a}}$. However, we will not invoke these since they will follow as equations of motion of the 4d theory due to the addition of the boundary term in the gauged hCS_6 action (4.2.14). See appendix C for more details.

with the result

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = \frac{1}{2\pi \mathrm{i}}\int \bar{\partial}\Omega\wedge \mathrm{Tr}\big(\Theta\wedge\big(\nabla\hat{g}\hat{g}^{-1} - \mathcal{B}'\big)\big) - \frac{1}{2\pi \mathrm{i}}\int_{\times[0,1]} \bar{\partial}\Omega\wedge\mathcal{L}_{\mathrm{gWZ}}[\hat{g},\mathcal{B}'] \ . \quad (4.2.52)$$

Given that both $B'_{a\dot{a}}$ and $\Theta_{a\dot{a}}$ are $\mathbb{CP}^1$-independent, we have that

$$\int \bar{\partial}\Omega \wedge \mathrm{Tr}(\Theta\wedge\mathcal{B}') = 0 \ , \quad (4.2.53)$$

with cancelling contributions from the two end points of the integral. Hence we are left with a manifestly covariant result

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = \frac{1}{2\pi \mathrm{i}}\int \bar{\partial}\Omega \wedge \mathrm{Tr}\big(\Theta\wedge\big(\nabla\hat{g}\hat{g}^{-1}\big) - \mathcal{L}_{\mathrm{gWZ}}[\hat{g},\mathcal{B}']\big) \ . \quad (4.2.54)$$

Applying the localisation formula (B.15) in appendix B yields the four-dimensional action

$$\begin{aligned} S_{\mathrm{IFT}_4} =& \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}\big(\nabla g g^{-1}\wedge\star\nabla g g^{-1}\big) + \int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta}\wedge\mathcal{L}_{\mathrm{gWZ}}[g,B'] \\ & - \int_{\mathbb{R}^4} \mu_\alpha\wedge \mathrm{Tr}(u\cdot F[B']) + \mu_\beta\wedge\mathrm{Tr}(\tilde{u}\cdot F[B']) \ . \end{aligned} \quad (4.2.55)$$

At this point only the $\mathfrak{h}$-components of u and $\tilde{u}$ contribute to the action, and so henceforth, to ease notation and without loss of generality, we set their projection onto $\mathfrak{k}$ to zero.

Something rather elegant has happened; we have found that the localisation of the six-dimensional theory returns not only the gauging of the WZW_4 model, but also residual edge modes serving as Lagrange multipliers constraining the field strength to obey exactly those conditions (4.2.13) that ensure the theory can be written as the difference of WZW_4 models. The constraints $F^{2,0} = 0$ and $F^{0,2} = 0$ have also been imposed by Lagrange multipliers in the context of 5d Kähler Chern-Simons theory [NS90; NS92]. This theory bears a similar relationship to WZW_4 as 3d Chern-Simons theory bears to WZW_2. This poses a natural question: what is the direct relationship between this 5d Kähler Chern-Simons theory and 6d holomorphic Chern-Simons theory? We suspect the mechanism here is rather similar to that which relates CS_4 and CS_3 [Yam19] and comment on this further in the outlook.

4.2.5 Equations of motion and ASDYM

Making use of the projectors previously introduced in eq. (4.2.10), the equations of motion following from the action (4.2.55) read

$$\begin{aligned} \delta B' : \quad & 0 = \bar{P}\nabla g g^{-1}|_{\mathfrak{h}} - P g^{-1}\nabla g|_{\mathfrak{h}} + \star(\mu_\alpha \wedge \nabla u + \mu_\beta \wedge \nabla \tilde{u}) \ , \\ \delta g : \quad & 0 = \nabla \star \nabla g g^{-1} - \omega_{\alpha,\beta} \wedge \nabla(\nabla g g^{-1}) + 2\omega_{\alpha,\beta} \wedge F[B'] \ , \\ \delta u : \quad & 0 = \mu_\alpha \wedge F[B'] \ , \\ \delta \tilde{u} : \quad & 0 = \mu_\beta \wedge F[B'] \ . \end{aligned} \tag{4.2.56}$$

We can exploit the projectors to extract two independent contributions from the B' equation of motion:

$$\delta B' : \quad \begin{aligned} & 0 = \bar{P}\left(\nabla g g^{-1}|_{\mathfrak{h}} + \star(\mu_\beta \wedge \nabla \tilde{u})\right) \ , \\ & 0 = P\left(g^{-1}\nabla g|_{\mathfrak{h}} - \star(\mu_\alpha \wedge \nabla u)\right) \ . \end{aligned} \tag{4.2.57}$$

As expected from the discussion in appendix C, these are exactly the conditions that arise from the boundary conditions relating the $\mathbb{CP}^1$-derivatives of the gauge fields, $\partial_0 \mathcal{A}^{\mathfrak{h}}|_{\alpha,\beta} = \partial_0 \mathcal{B}|_{\alpha,\beta}$, justifying a posteriori why we did not impose them in the localisation procedure.

Making use of the identity

$$\nabla(\omega_{\alpha,\beta} \wedge \star(\mu_\beta \wedge \nabla \tilde{u})) = \nabla(\mu_\beta \wedge \nabla \tilde{u}) = \mu_\beta \wedge F[B'] \cdot \tilde{u} \ , \tag{4.2.58}$$

we obtain an on-shell integrability condition for the first equation in (4.2.57), namely that

$$\nabla(\omega_{\alpha,\beta} \wedge \bar{P}(\nabla g g^{-1}|_{\mathfrak{h}})) = 0 \ . \tag{4.2.59}$$

Hence, using the projection of the δg equation of motion onto $\mathfrak{h}$, we have that $\omega_{\alpha,\beta} \wedge F[B'] = 0$ follows on-shell.

Let us return to the ASDYM equations, which we can recast as

$$\mu_\alpha \wedge F = 0 \ , \quad \mu_\beta \wedge F = 0 \ , \quad \omega_{\alpha,\beta} \wedge F = 0 \ . \tag{4.2.60}$$

In differential form notation, the relation (4.2.51) can be written as

$$A' = B' - \bar{P}(\nabla g g^{-1}) \ . \tag{4.2.61}$$

By virtue of the identities obeyed by the projectors in eqs. (A.3) to (A.5) and the covariant Maurer-Cartan identity obeyed by $R^\nabla = \nabla g g^{-1}$,

$$\nabla R^\nabla - R^\nabla \wedge R^\nabla = (1 - \mathrm{Ad}_g) F[B'] \ , \tag{4.2.62}$$

we can readily establish

$$\begin{aligned} \mu_\beta \wedge F[A'] &= \mu_\beta \wedge F[B'] \ , && (4.2.63)\\ \mu_\alpha \wedge F[A'] &= \mu_\alpha \wedge \mathrm{Ad}_g F[B'] \ , && (4.2.64)\\ 2\omega_{\alpha,\beta} \wedge F[A'] &= 2\omega_{\alpha,\beta} \wedge F[B'] + 2\omega_{\alpha,\beta} \wedge \nabla \bar{P}(R^\nabla) \\ &= 2\omega_{\alpha,\beta} \wedge F[B'] - \nabla(\star \nabla g g^{-1}) + \omega_{\alpha,\beta} \wedge \nabla(\nabla g g^{-1}) \ . && (4.2.65) \end{aligned}$$

Hence we conclude that the $\delta g, \delta u, \delta \tilde{u}$ equations of motion are equivalent to the ASDYM equations for the connection A'. Demanding that the B' connection is also ASD requires in addition that $\omega_{\alpha,\beta} \wedge F[B'] = 0$, which is indeed a consequence of the B' equations of motion as shown above.

As we have seen, the equations of motion of gWZW$_4$ (4.2.55) are equivalent to the ASDYM equations for the two connections, A' and B'. This ensures that the gauging of the WZW$_4$ is compatible with integrability.

4.2.6 Constraining then reducing

We now proceed to the bottom of the diamond by reduction of the IFT$_4$. In this section, we shall first implement the constraints imposed by the Lagrange multipliers $u, \tilde{u}$ in the 4d theory and then reduce. While not the most general reduction, this will allow us to directly recover the gauged WZW coset CFT. In section 4.3, we will investigate more general reductions, in particular, what happens if we reduce without first imposing constraints.

Imposing the reduction ansatz that $\partial_z = \partial_{\bar{z}} = 0$ in the complex coordinates of eq. (3.3.2), we have that the solution to the constraints $B = P(a^{-1}\mathrm{d}a) + \bar{P}(b^{-1}\mathrm{d}b)$

becomes

$$B' = B'_{a\dot{a}}\mathrm{d}x^{a\dot{a}} = \frac{1}{\alpha-\beta}\left(\alpha b^{-1}\partial_w b - \beta a^{-1}\partial_w a\right)\mathrm{d}w - \frac{1}{\alpha-\beta}\left(\beta b^{-1}\partial_{\bar{w}} b - \alpha a^{-1}\partial_{\bar{w}} a\right)\mathrm{d}\bar{w} + \frac{1}{\alpha-\beta}\left(b^{-1}\partial_{\bar{w}} b - a^{-1}\partial_{\bar{w}} a\right)\mathrm{d}z + \frac{\alpha\beta}{\alpha-\beta}\left(b^{-1}\partial_w b - a^{-1}\partial_w a\right)\mathrm{d}\bar{z} \ . \tag{4.2.66}$$

For simplicity, let us first consider the Kähler point and align the reduction to the complex structure (implemented by taking $\alpha \to 0$ and $\beta \to \infty$). In this scenario, the reduction ansatz enforces that $B'_z = B'_{\bar{z}} = 0$ with the remaining components of B' parametrising a generic two-dimensional gauge field. Effectively, we can simply ignore the constraints altogether but impose $B'_z = B'_{\bar{z}} = 0$ as part of the reduction ansatz. This could be interpreted as demanding $D_z = D_{\bar{z}} = 0$ acting on fields. In this case, it is immediate that the 4d gauged WZW reduces to a 2d gauged WZW.

Away from the Kähler point and aligned reduction, i.e. not fixing α and β, we need to keep track of contributions coming from B'_z and $B'_{\bar{z}}$. We can still view the B'_w and $B'_{\bar{w}}$ components of eq. (4.2.66) as a parametrisation of a generic 2d gauge field, but there is no way in which we can view the B'_z and $B'_{\bar{z}}$ as a local combination of B'_w and $B'_{\bar{w}}$. We are forced to work with the variables a and b rather than a 2d gauge field. Fortunately, however, the reduction can still be performed immediately if we use the composite fields $\tilde{g} = agb^{-1}$ and $\tilde{h} = ab^{-1}$. These composite fields are invariant under the H-gauge symmetry, but a new semi-local symmetry emerges given by $a \to \ell a$, $b \to br^{-1}$ with $\alpha^b\partial_{b\dot{b}}r = \beta^b\partial_{b\dot{b}}\ell = 0$. These leave B', g, h invariant but act as $\tilde{g} \to \ell\tilde{g}r$ and $\tilde{h} \to \ell\tilde{h}r$. At the Kähler point and aligned reduction, these symmetries descend to affine symmetries, but in general descend only to global transformations. Recall that in terms of the composite fields gWZW_4 becomes

$$S^{(\alpha,\beta)}_{\mathrm{gWZW}_4}[g, B'] = S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{g}] - S^{(\alpha,\beta)}_{\mathrm{WZW}_4}[\tilde{h}] \ . \tag{4.2.67}$$

It is then immediate that this reduces to the difference of PCM plus WZ term theories with action (2.5.26) and WZ coefficient k:

$$S_{\mathrm{IFT}_2}[\tilde{g}, \tilde{h}] = S_{\mathrm{PCM+WZ}_2}[\tilde{g}] - S_{\mathrm{PCM+WZ}_2}[\tilde{h}] \ . \tag{4.2.68}$$

Away from the CFT point, $k = 1$, this cannot be recast in terms of a deformation of gWZW_2 expressed as a local functional of B', g.

Lax formulation. To obtain the Lax connection of the resulting IFT_2 we first note that the four-dimensional gauge fields, upon solving the constraints on B', are gauge equivalent to

$$A'_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\partial_{b\dot{a}}\tilde{g}\tilde{g}^{-1} \ , \quad B'_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\partial_{b\dot{a}}\tilde{h}\tilde{h}^{-1} \ .$$

Therefore, we may simply follow the construction of the Lax connection from the ungauged model in eq. (2.5.28), with the connection A' producing a Lax for the $S_{\mathrm{PCM+WZ}_2}[\tilde{g}]$ and B' producing one for $S_{\mathrm{PCM+WZ}_2}[\tilde{h}]$.

4.3 More general IFT_2 from IFT_4: reducing then constraining

In the previous section, we reduced from the gauged WZW_4 model to an IFT_2, but prior to reduction we enforced the constraints imposed by the Lagrange multiplier fields. These constraints determine implicit relations between the components of the gauge field as per eq. (4.2.66). In the simplest case, where we work at the Kähler point and align the reduction directions with the complex structure, the constraints enforce $B'_z = B'_{\bar{z}} = 0$. However, if we do not impose the constraints in 4d, the standard reduction ansatz would only require that B'_z and $B'_{\bar{z}}$ are functionally independent of z and $\bar{z}$, a weaker condition.

In this section, we explore the consequences of reducing without first constraining. Denoting the reduction by $\rightsquigarrow$ we anticipate that the lower-dimensional description will include additional fields as[11]

$$\begin{aligned} B'_w(w,\bar{w},z,\bar{z}) &\rightsquigarrow B_w(w,\bar{w}) \ , \qquad & B'_{\bar{w}}(w,\bar{w},z,\bar{z}) &\rightsquigarrow B_{\bar{w}}(w,\bar{w}) \ , \\ B'_z(w,\bar{w},z,\bar{z}) &\rightsquigarrow \bar{\Phi}(w,\bar{w}) \ , \qquad & B'_{\bar{z}}(w,\bar{w},z,\bar{z}) &\rightsquigarrow \Phi(w,\bar{w}) \ , \end{aligned} \tag{4.3.1}$$

where Φ and $\bar{\Phi}$ will be adjoint scalars in the lower-dimensional theory (sometimes called Higgs fields in the literature). These will enter explicitly in the lower-dimensional theory through the reduction of covariant derivatives

$$\nabla_z g g^{-1} \rightsquigarrow \bar{\Phi} - g\bar{\Phi}g^{-1} \ , \qquad \nabla_{\bar{z}} g g^{-1} \rightsquigarrow \Phi - g\Phi g^{-1} \ . \tag{4.3.2}$$

[11] Note, we are dropping the prime on the 2d gauge field B.

On-shell, the 4d gauge field B' is ASD and couples to matter in the gWZW$_4$ model. It is well-known that the reduction of an ASDYM connection leads to the Hitchin system, and we will see this feature in the lower-dimensional dynamics below.

The two-dimensional Lagrangian that arises from reducing eq. (4.2.55) without first constraining is[12]

$$\begin{aligned} L_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar{w}}g\big) + \frac{1}{2}\frac{\alpha+\beta}{\alpha-\beta}L_{\mathrm{gWZ}} + \mathrm{Tr}\Big(\Phi\bar{\Phi} + \frac{\alpha}{\alpha-\beta}\Phi\mathrm{Ad}_g\bar{\Phi} - \frac{\beta}{\alpha-\beta}\Phi\mathrm{Ad}_g^{-1}\bar{\Phi}\Big) \\ + \frac{1}{\alpha-\beta}\mathrm{Tr}\big(\Phi\,(g^{-1}D_{\bar{w}}g + D_{\bar{w}}gg^{-1}) + \alpha\beta\,\bar{\Phi}\,(g^{-1}D_w g + D_w g g^{-1})\big) \\ + \mathrm{Tr}\big(\tilde{u}(F_{\bar{w}w} - \beta^{-1}D_{\bar{w}}\Phi - \beta D_w\bar{\Phi} - [\bar{\Phi},\Phi])\big) + \\ + \mathrm{Tr}\big(u(F_{\bar{w}w} - \alpha^{-1}D_{\bar{w}}\Phi - \alpha D_w\bar{\Phi} - [\bar{\Phi},\Phi])\big)\ , \end{aligned} \tag{4.3.3}$$

where we denote the 2d covariant derivative as $D = \mathrm{d} + \mathrm{ad}_B$ and note that we have rescaled

$$\tilde{u} \to \frac{\tilde{u}}{\langle\beta\gamma\rangle\langle\beta\hat{\gamma}\rangle}\ , \text{ and } \quad u \to \frac{u}{\langle\alpha\gamma\rangle\langle\alpha\hat{\gamma}\rangle}\ .$$

The fields of the IFT$_2$ are $g \in G$ and $B_{w,\bar{w}}, \Phi, \bar{\Phi}, u, \tilde{u} \in \mathfrak{h}$. In addition to the overall coupling, the IFT$_2$ (4.3.3) only depends on a single parameter. This can be seen by introducing[13]

$$k = \frac{\alpha+\beta}{\alpha-\beta}\ , \qquad k' = -\frac{2\sqrt{\alpha\beta}}{\alpha-\beta}\ , \qquad k^2 - k'^2 = 1\ , \tag{4.3.4}$$

rescaling $\Phi \to \sqrt{\alpha\beta}\Phi$ and $\bar{\Phi} \to \frac{1}{\sqrt{\alpha\beta}}\bar{\Phi}$, and defining $X^- = k'^{-1}(u+\tilde{u})$ and $\tilde{X}^+ =$

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[12]The 2d Lagrangians are defined as $S_{\mathrm{IFT}_2} = 2\mathrm{i}\int_{\mathbb{R}^2} \mathrm{d}w \wedge \mathrm{d}\bar{w}\, L_{\mathrm{IFT}_2}$. We have also introduced the scalar densities L_{WZ} and L_{gWZ} where $\int \mathrm{d}w \wedge \mathrm{d}\bar{w}\, L_{(\mathrm{g})\mathrm{WZ}}(g) = \int_{\mathbb{R}^2\times[0,1]} \mathcal{L}_{(\mathrm{g})\mathrm{WZ}}(\hat{g})$ and the 3-forms $\mathcal{L}_{\mathrm{WZ}}$ and $\mathcal{L}_{\mathrm{gWZ}}$ are defined in eqs. (2.5.18) and (4.2.3) respectively. Explicitly, we have

$$L_{\mathrm{gWZ}} = L_{\mathrm{WZ}}(g) + \mathrm{Tr}\big((g^{-1}\partial_w g + \partial_w g g^{-1})B_{\bar{w}} - (g^{-1}\partial_{\bar{w}}g + \partial_{\bar{w}}gg^{-1})B_w + B_w\mathrm{Ad}_g B_{\bar{w}} - B_w\mathrm{Ad}_g^{-1}B_{\bar{w}}\big)\ ,$$

which we include for convenience.

[13]Here, we have implicitly assumed that $\alpha\beta \geq 0$, which implies that $|k| \geq 1$. The other regime of interest, $\alpha\beta \leq 0$ and $|k| \leq 1$ is related by the analytic continuation $k' \to -\mathrm{i}k'$.

$k'^{-1}(u-\tilde{u})$. The Lagrangian (4.3.3) can then be rewritten as

$$\begin{aligned} L_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1} D_{\bar{w}} g\big) + \frac{k}{2} L_{\mathrm{gWZ}} + \mathrm{Tr}\big(\Phi\mathcal{O}\bar{\Phi} + \Phi\, V_{\bar{w}} + \bar{\Phi}\, V_w\big) \\ + \mathrm{Tr}\big(X^-(k'(F_{\bar{w}w} - [\bar{\Phi},\Phi]) + k(D_w\bar{\Phi} + D_{\bar{w}}\Phi))\big) + \mathrm{Tr}\big(\tilde{X}^+(D_w\bar{\Phi} - D_{\bar{w}}\Phi)\big) \ , \end{aligned} \tag{4.3.5}$$

where

$$\mathcal{O} = 1 - \frac{k+1}{2}\mathrm{Ad}_g + \frac{k-1}{2}\mathrm{Ad}_g^{-1} \ , \qquad V_{w,\bar{w}} = -\frac{k'}{2}(g^{-1}D_{w,\bar{w}}g + D_{w,\bar{w}}g g^{-1}) \ . \tag{4.3.6}$$

Note that the CFT points $k=1$ or $k=-1$ correspond to taking $\gamma \to \beta$ or $\gamma \to \alpha$, i.e. when the zeroes of the twist function coincide with the poles.

By construction, as the reduction of gWZW$_4$, the equations of motion of this theory are expected to be equivalent to the zero curvature of Lax connections, whose components are given by the dw and d$\bar{w}$ legs of the 4d gauge fields. Explicitly, these Lax connections are given by

$$\begin{aligned} \mathcal{L}_w^{(A)} &= \partial_w + B_w - \frac{k+1}{2}K_w - \frac{1}{\zeta}\big(\Phi + \frac{k'}{2}K_w\big) \ , \\ \mathcal{L}_{\bar{w}}^{(A)} &= \partial_{\bar{w}} + B_{\bar{w}} + \frac{k-1}{2}K_{\bar{w}} + \zeta\big(\bar{\Phi} + \frac{k'}{2}K_{\bar{w}}\big) \ , \end{aligned} \tag{4.3.7}$$

$$\mathcal{L}_w^{(B)} = \partial_w + B_w - \frac{1}{\zeta}\Phi \ , \qquad \mathcal{L}_{\bar{w}}^{(B)} = \partial_{\bar{w}} + B_{\bar{w}} + \zeta\bar{\Phi} \ , \tag{4.3.8}$$

where we have also redefined the spectral parameter $\zeta \to \sqrt{\alpha\beta}\zeta$ compared to section 2.5.4 and introduced the currents

$$K_w = D_w g g^{-1} + \frac{k-1}{k'}(1-\mathrm{Ad}_g)\Phi \ , \qquad K_{\bar{w}} = D_{\bar{w}} g g^{-1} - \frac{k+1}{k'}(1-\mathrm{Ad}_g)\bar{\Phi} \ . \tag{4.3.9}$$

It is natural to ask whether first reducing and then constraining leads to a consistent truncation since we are dropping certain parts of the 4d constraints. In the language of Kaluza-Klein compactifications these would be the higher-mode constraints. Since every term in a higher-mode constraint will depend on the higher modes of some field, it follows that setting all the higher modes to zero is expected to be a consistent truncation. Another viewpoint is that of symmetry reduction; since the action is invariant under shifts of z and $\bar{z}$ we can consistently set all fields to be independent of them. If the truncation is consistent then we expect the resulting 2d theory to be integrable, which we now check explicitly.

4.3.1 Lax formulation

Before analysing the Lagrangian (4.3.5) in more detail, let us show explicitly that its equations of motion are indeed equivalent to the zero-curvature condition for the Lax connections (4.3.7) and (4.3.8). The equations of motion that follow from the Lagrangian (4.3.5) by varying $\tilde{X}^+$, X^- and g are

$$\begin{aligned}
\delta\tilde{X}^+ :\quad & \mathcal{E}_+ \equiv D_w\bar{\Phi} - D_{\bar{w}}\Phi = 0 \ , \\
\delta X^- :\quad & \mathcal{E}'_-\big(F_{\bar{w}w} - [\bar{\Phi},\Phi]\big) + k\big(D_w\bar{\Phi} + D_{\bar{w}}\Phi\big) = 0 \ , \\
\delta gg^{-1} :\quad & \mathcal{E}_g \equiv \frac{k-1}{2}\Big(D_wK_{\bar{w}} + \frac{k+1}{k'}[\bar{\Phi},K_w]\Big) - \\
& \qquad \frac{k+1}{2}\Big(D_{\bar{w}}K_w - \frac{k-1}{k'}[\Phi,K_{\bar{w}}]\Big) + \frac{k}{k'}\mathcal{E}_- - \frac{1}{k'}\big(D_w\bar{\Phi} + D_{\bar{w}}\Phi\big) = 0 \ .
\end{aligned} \tag{4.3.10}$$

We also have the Bianchi identity following from the zero-curvature of the Maurer-Cartan form $\mathrm{d}gg^{-1}$

$$\begin{aligned}
\mathcal{Z} \equiv D_wK_{\bar{w}} + \frac{k+1}{k'}[\bar{\Phi},K_w] - D_{\bar{w}}K_w + \frac{k-1}{k'}[\Phi,K_{\bar{w}}] \\
+ [K_{\bar{w}},K_w] + \frac{1}{k'}(1-\mathrm{Ad}_g)(\mathcal{E}_- + \mathcal{E}_+) = 0 \ .
\end{aligned} \tag{4.3.11}$$

The zero curvature of the A-Lax eq. (4.3.7) gives rise to three equations that are linear combinations of the equations of motion eq. (4.3.10) and the Bianchi identity eq. (4.3.11):

$$\begin{aligned}
0 &= \frac{k-1}{2}\mathcal{Z}' - \mathcal{E}_g + \frac{k}{k'}\mathcal{E}_- - \frac{1}{k'}\mathcal{E}_+ \ , \\
0 &= k'^2\mathcal{Z}' - 2k\mathcal{E}_g + 2k'\mathcal{E}_- \ , \\
0 &= \frac{k+1}{2}\mathcal{Z}' - \mathcal{E}_g + \frac{k}{k'}\mathcal{E}_- + \frac{1}{k'}\mathcal{E}_+ \ ,
\end{aligned} \tag{4.3.12}$$

where we have defined $\mathcal{Z}' \equiv \mathcal{Z} - \frac{1}{k'}(1-\mathrm{Ad}_g)(\mathcal{E}_- + \mathcal{E}_+)$. On the other hand, the zero curvature of the B-Lax (4.3.8) defines the Hitchin equations:

$$0 = D_{\bar{w}}\Phi \ , \qquad 0 = F_{\bar{w}w} - [\bar{\Phi},\Phi] \ , \qquad 0 = D_w\bar{\Phi} \ , \tag{4.3.13}$$

which can be rewritten as the three equations $\mathcal{E}_\pm = 0$ and $\mathcal{E}_0 \equiv D_w\bar{\Phi} + D_{\bar{w}}\Phi = 0$. Therefore, the two Lax connections give rise to five independent equations, which are linear combinations of the equations of motion (4.3.10), the Bianchi identity (4.3.11), and the additional equation $\mathcal{E}_0 = 0$.

To recover this final equation from the equations of motion, let us consider the variational equations for $B_w, B_{\bar{w}}, \bar{\Phi}, \Phi$:

$$\begin{aligned}
\delta B_w : \mathcal{E}_B \equiv k' D_{\bar{w}} X^- - [\bar{\Phi}, \tilde{X}^+ + kX^-] + \frac{k-1}{2} P_{\mathfrak{h}} K_{\bar{w}} + \frac{k+1}{2} P_{\mathfrak{h}} \mathrm{Ad}_g^{-1} K_{\bar{w}} \\
- \frac{k+1}{k'} P_{\mathfrak{h}} (1 - \mathrm{Ad}_g^{-1}) \bar{\Phi} = 0 \ , \\
\delta B_{\bar{w}} : \mathcal{E}_{\bar{B}} \equiv k' D_w X^- - [\Phi, \tilde{X}^+ - kX^-] + \frac{k+1}{2} P_{\mathfrak{h}} K_w + \frac{k-1}{2} P_{\mathfrak{h}} \mathrm{Ad}_g^{-1} K_w \\
- \frac{k-1}{k'} P_{\mathfrak{h}} (1 - \mathrm{Ad}_g^{-1}) \Phi = 0 \ , \\
\delta\Phi : \mathcal{E}_\Phi \equiv D_{\bar{w}}(\tilde{X}^+ - kX^-) + k'[\bar{\Phi}, X^-] - \frac{k'}{2} P_{\mathfrak{h}} (1 + \mathrm{Ad}_g^{-1}) K_{\bar{w}} + P_{\mathfrak{h}} (1 - \mathrm{Ad}_g^{-1}) \bar{\Phi} = 0 \ , \\
\delta\bar{\Phi} : \mathcal{E}_{\bar{\Phi}} \equiv D_w(\tilde{X}^+ + kX^-) + k'[\Phi, X^-] + \frac{k'}{2} P_{\mathfrak{h}} (1 + \mathrm{Ad}_g^{-1}) K_w - P_{\mathfrak{h}} (1 - \mathrm{Ad}_g^{-1}) \Phi = 0 \ .
\end{aligned} \tag{4.3.14}$$

These can be understood as a first-order system of equations for $\tilde{X}^+$ and X^-. Consistency of the system implies that they should satisfy the integrability conditions $[D_{\bar{w}}, D_w]\tilde{X}^+ = [F_{\bar{w}w}, \tilde{X}^+]$ and $[D_{\bar{w}}, D_w]X^- = [F_{\bar{w}w}, X^-]$. We find that

$$k'[D_{\bar{w}}, D_w]X^- - k'[F_{\bar{w}w}, X^-] = [X^+, \mathcal{E}_+] + [X^-, \mathcal{E}_-] + P_{\mathfrak{h}}(1 - \mathrm{Ad}_g^{-1})\mathcal{E}_g + kP_{\mathfrak{h}}\mathrm{Ad}_g^{-1}\mathcal{Z} \ , \tag{4.3.15}$$

hence, using the Bianchi identity (4.3.11), this vanishes on the equations of motion for $\tilde{X}^+$, X^- and g (4.3.10). On the other hand, we have

$$\begin{aligned}
k'[D_{\bar{w}}, D_w]\tilde{X}^+ - k'[F_{\bar{w}w}, \tilde{X}^+] = [X^+, \mathcal{E}_-] + [X^-, \mathcal{E}_+] + \frac{2k}{k'}\mathcal{E}_- \\
- \frac{2}{k'}\mathcal{E}_0 - P_{\mathfrak{h}}(1 + \mathrm{Ad}_g^{-1})\mathcal{E}_g + kP_{\mathfrak{h}}\mathrm{Ad}_g^{-1}\mathcal{Z} \ .
\end{aligned} \tag{4.3.16}$$

Here, we see that in addition to the Bianchi identity (4.3.11) and equations of motion (4.3.10), we also require $\mathcal{E}_0 = 0$, recovering the final equation of the Lax system.

4.3.2 Relation to known models

As we will shortly see, if we take H to be abelian, the Lagrangian (4.3.5) can be related to known models, including the homogeneous sine-Gordon models and the PCM plus WZ term. However, for non-abelian H this model has not been considered before, and defines a new integrable field theory in two dimensions. Moreover, by integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$, it leads to an integrable sigma model for the fields g, $\tilde{X}^+$ and X^-. We leave the study of these models for future work.

To recover a sigma model from the Lagrangian (4.3.5) for abelian H, after integrating out $B_w, B_{\bar{w}}$, we have two options. The first is to integrate out $\Phi, \bar{\Phi}$. The second is to solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$. For abelian H the Lagrangian (4.3.5) simplifies to

$$\begin{aligned} L^{\mathrm{ab}}_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar{w}}g\big) + \frac{k}{2}L_{\mathrm{gWZ}} + \mathrm{Tr}\big(\Phi\mathcal{O}\bar{\Phi} + \Phi\, V_{\bar{w}} + \bar{\Phi}\, V_w\big) \\ + \mathrm{Tr}\big((X^-(k'F_{\bar{w}w} + k(\partial_w\bar{\Phi} + \partial_{\bar{w}}\Phi))\big) + \mathrm{Tr}\big(\tilde{X}^+(\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi)\big) \ . \end{aligned} \tag{4.3.17}$$

This takes the form of a first-order action in the Büscher procedure, and it follows that the two sigma models will be T-dual to each other with dual fields X^+ and $\tilde{X}^+$. Explicitly the Lagrangians, before integrating out $B_w, B_{\bar{w}}$, are

$$\begin{aligned} L^{\tilde{X}}_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar{w}}g\big) + \frac{k}{2}L_{\mathrm{gWZ}} + k'\mathrm{Tr}\big(X^-F_{\bar{w}w}\big) \\ + \mathrm{Tr}\big((\partial_w\tilde{X}^+ - V_w + k\partial_w X^-)\mathcal{O}^{-1}(\partial_{\bar{w}}\tilde{X}^+ + V_{\bar{w}} - k\partial_{\bar{w}}X^-)\big) \ , \end{aligned} \tag{4.3.18}$$

and

$$\begin{aligned} L^{X}_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1}D_{\bar{w}}g\big) + \frac{k}{2}L_{\mathrm{gWZ}} + k'\mathrm{Tr}\big(X^-F_{\bar{w}w}\big) \\ + \frac{1}{4}\mathrm{Tr}\big(\partial_w X^+\mathcal{O}\partial_{\bar{w}}X^+ + 2\partial_w X^+(V_{\bar{w}} - k\partial_{\bar{w}}X^-) + 2\partial_{\bar{w}}X^+(V_w - k\partial_w X^-)\big) \ , \end{aligned} \tag{4.3.19}$$

where in the second Lagrangian we have locally solved the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ by setting

$$\Phi = \frac{1}{2}\partial_w X^+ \ , \qquad \bar{\Phi} = \frac{1}{2}\partial_{\bar{w}}X^+ \ , \qquad X^+ \in \mathfrak{h} \ . \tag{4.3.20}$$

As mentioned above, the first approach can also be straightforwardly applied for non-abelian H. Generalising the second approach is more subtle. The constraint imposed by the Lagrange multiplier $\tilde{X}^+$ in the Lagrangian (4.3.5) implies that

$$D_w\bar{\Phi} - D_{\bar{w}}\Phi = 0 \ . \tag{4.3.21}$$

Typically the full solution to this equation would be expressed in terms of path-ordered exponentials of B_w and $B_{\bar{w}}$. To avoid non-local expressions, we can restrict Φ and $\bar{\Phi}$ to be valued in the centre of $\mathfrak{h}$, denoted $\mathcal{Z}(\mathfrak{h})$. Note that this is not a restriction if H is abelian. With this restriction, the Lagrangian (4.3.5) again simplifies to (4.3.17), and the constraint (4.3.21) becomes $\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi = 0$, which we can again locally solve by (4.3.20) now with $X^+ \in \mathcal{Z}(\mathfrak{h})$, similarly leading to the Lagrangian (4.3.19).

Relation to PCM plus WZ term. Taking H to be abelian, we can relate the Lagrangian (4.3.17) to that of the PCM plus WZ term for $G \times H$ through a combination of T-dualities and field redefinitions. We start by parametrising

$$g \to e^{\frac{1}{2}\tau} g e^{\frac{1}{2}\tau} \ , \qquad \tau \in \mathfrak{h} \ , \tag{4.3.22}$$

and setting $\partial_{w,\bar{w}}\tau \to 2C_{w,\bar{w}}$. We also integrate by parts and set $\partial_w X^- \to 2\Psi$ and $\partial_{\bar{w}} X^- \to 2\bar{\Psi}$. To maintain equivalence with the Lagrangian we started with, we add $\mathrm{Tr}\big(\tilde{\tau}(\partial_w C_{\bar{w}} - \partial_{\bar{w}} C_w)\big) + \mathrm{Tr}\big(\tilde{X}^-(\partial_w\bar{\Psi} - \partial_{\bar{w}}\Psi)\big)$, i.e. the Lagrange multipliers $\tilde{\tau}$ and $\tilde{X}^-$ locally impose $C_{w,\bar{w}} = \frac{1}{2}\partial_{\bar{w},w}\tau$, $\Psi = \frac{1}{2}\partial_w X^-$ and $\bar{\Psi} = \frac{1}{2}\partial_{\bar{w}} X^-$. We can then redefine the fields as[14]

$$\begin{aligned}
&B_w \to B_w - \frac{k}{k'}\Phi \ , && C_w \to C_w - \frac{1}{k'}\Phi \ , && \Psi \to \Psi + \frac{k}{k'^2}\Phi \ , \\
&B_{\bar{w}} \to B_{\bar{w}} + \frac{k}{k'}\bar{\Phi} \ , && C_{\bar{w}} \to C_{\bar{w}} - \frac{1}{k'}\bar{\Phi} \ , && \bar{\Psi} \to \bar{\Psi} + \frac{k}{k'^2}\bar{\Phi} \ , \\
&\tilde{X}^+ \to \frac{1}{k'}\tilde{X}^+ - \frac{k}{k'}\tilde{X}^- + \frac{1}{k'}\tilde{\tau} \ , && \tilde{X}^- \to k'\tilde{X}^- \ , && \tilde{\tau} \to \tilde{\tau} \ .
\end{aligned} \tag{4.3.23}$$

[14] To arrive at this field redefinition, we first look for the shifts of $B_{w,\bar{w}}$, $C_{w,\bar{w}}$, Ψ and $\bar{\Psi}$ that decouple Φ and $\bar{\Phi}$ from all other fields apart from $\tilde{X}^+$. Since both C_w and $C_{\bar{w}}$ transform in the same way, as do Ψ and $\bar{\Psi}$, we can then easily compute the transformation of $\tilde{\tau}$, $\tilde{X}^-$ and $\tilde{X}^+$ by demanding that the triplet of terms $\mathrm{Tr}\big(\tilde{\tau}F_{w\bar{w}}(C) + \tilde{X}^-F_{w\bar{w}}(\Psi) + \tilde{X}^+F_{w\bar{w}}(\Phi)\big)$ is invariant up to a simple rescaling, i.e. it becomes $\mathrm{Tr}\big(\tilde{\tau}F_{w\bar{w}}(C) + k'\tilde{X}^-F_{w\bar{w}}(\Psi) + \frac{1}{k'}\tilde{X}^+F_{w\bar{w}}(\Phi)\big)$.

Doing so, we arrive at the following Lagrangian

$$\begin{aligned}L^{\mathrm{ab}}_{\mathrm{IFT}_2} = {} & \frac{1}{2}\mathrm{Tr}\big(g^{-1}\partial_w g g^{-1}\partial_{\bar{w}} g\big) + \frac{k}{2}L_{\mathrm{WZ}}(g) \\ & + \frac{1-k}{2}\mathrm{Tr}\big(g^{-1}\partial_w g(C_{\bar{w}} - B_{\bar{w}}) + \partial_{\bar{w}} g g^{-1}(C_w + B_w) + (C_w + B_w)\mathrm{Ad}_g(C_{\bar{w}} - B_{\bar{w}})\big) \\ & + \frac{1+k}{2}\mathrm{Tr}\big(g^{-1}\partial_{\bar{w}} g(C_w - B_w) + \partial_w g g^{-1}(C_{\bar{w}} + B_{\bar{w}}) + (C_w - B_w)\mathrm{Ad}_g^{-1}(C_{\bar{w}} + B_{\bar{w}})\big) \\ & + \mathrm{Tr}\big(B_w B_{\bar{w}} + C_w C_{\bar{w}} + kC_w B_{\bar{w}} - kB_w C_{\bar{w}}\big) \\ & + \mathrm{Tr}\big(\tilde{\tau}(\partial_w C_{\bar{w}} - \partial_{\bar{w}} C_w)\big) + k'\mathrm{Tr}\big(\tilde{X}^-(\partial_w \bar{\Psi} - \partial_{\bar{w}}\Psi)\big) + 2k'\mathrm{Tr}\big(\Psi B_{\bar{w}} - B_w\bar{\Psi}\big) \\ & + \frac{1}{k'}\mathrm{Tr}\big(\tilde{X}^+(\partial_w\bar{\Phi} - \partial_{\bar{w}}\Phi)\big) - \frac{2}{k'^2}\mathrm{Tr}\big(\Phi\bar{\Phi}\big) \ .\end{aligned} \tag{4.3.24}$$

The final steps are to integrate out $\tilde{\tau}, \Psi, \bar{\Psi}$, and $\Phi, \bar{\Phi}$, leading us to set

$$C_{w,\bar{w}} = \frac{1}{2}\partial_{w,\bar{w}}\tau \ , \qquad B_{w,\bar{w}} = -\frac{1}{2}\partial_{w,\bar{w}}\tilde{X}^- \ , \qquad \Phi = -\frac{k'}{2}\partial_w\tilde{X}^+ \ , \quad \bar{\Phi} = \frac{k'}{2}\partial_{\bar{w}}\tilde{X}^+ \ . \tag{4.3.25}$$

Redefining $g \to e^{-\frac{1}{2}(\tau+\tilde{X}^-)} g e^{-\frac{1}{2}(\tau-\tilde{X}^-)}$, we find the difference of PCM plus WZ term Lagrangians for G and H

$$L_{\mathrm{PCM}+k\mathrm{WZ}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}\partial_w g g^{-1}\partial_{\bar{w}} g\big) + \frac{k}{2}L_{\mathrm{WZ}}(g) - \frac{1}{2}\mathrm{Tr}\big(\partial_w\tilde{X}^+\partial_{\bar{w}}\tilde{X}^+\big) \ , \tag{4.3.26}$$

where we recall that for abelian H the WZ term vanishes.

To summarise, starting from the sigma model (4.3.19) we T-dualise in τ, X^+ and X^-, we then perform a $GL(3)$ transformation on the dual coordinates, and finally T-dualise back in $\tilde{\tau}$ to recover (4.3.26), the difference of the PCM plus WZ term Lagrangians for G and H. This relation through dualities may have been anticipated since this is the model we would expect to find starting from the ghCS$_6$ action (4.2.14) and instead imposing the boundary conditions $\mathcal{A}|_{\alpha,\beta} = \mathcal{B}|_{\alpha,\beta} = 0$.

$k \to 1$ **limit.** As we have seen, the $k \to 1$ limit is special since if we first constrain and then reduce we recover the gauged WZW coset CFT. By first reducing and then constraining, we can recover massive integrable perturbations of these theories. We consider the setup where Φ and $\bar{\Phi}$ are restricted to lie in $\mathcal{Z}(\mathfrak{h})$ and solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ by (4.3.20). Taking $k \to 1$ the Lagrangian (4.3.19) simplifies further to

$$\begin{aligned}L_{\mathrm{IFT}_2} = &\frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1} D_{\bar w} g\big) + \frac{1}{2} L_{\mathrm{gWZ}} \\ &+ \frac{1}{4}\mathrm{Tr}\big(\partial_w X^+ (1-\mathrm{Ad}_g)\partial_{\bar w} X^+ - 2\partial_w X^+ \partial_{\bar w} X^- - 2\partial_{\bar w} X^+ \partial_w X^-\big) \ .\end{aligned} \tag{4.3.27}$$

This is reminiscent of a sigma model for a pp-wave background, with the kinetic terms for the transverse fields described by the gauged WZW model for the coset G/H, except that the would-be light-cone coordinates X^+ and X^- have $\dim \mathcal{Z}(\mathfrak{h})$ components. Nevertheless, we still have the key property that the equation of motion for X^- is $\partial_w \partial_{\bar w} X^+ = 0$, whose general solution is $X^+ = Y(w) + \bar Y(\bar w)$. Substituting into the Lagrangian (4.3.27) we find

$$L_{\mathrm{IFT}_2} = \frac{1}{2}\mathrm{Tr}\big(g^{-1}D_w g g^{-1} D_{\bar w} g\big) + \frac{1}{2} L_{\mathrm{gWZ}} + \frac{1}{4}\mathrm{Tr}\big(\partial_w Y \partial_{\bar w} \bar Y - \partial_w Y \mathrm{Ad}_g \bar\partial_{\bar w} \bar Y\big) \ . \tag{4.3.28}$$

In the special case that $Y = w\Lambda$ and $\bar Y = \bar w \bar\Lambda$, which is the most general solution preserving the translational invariance of the action, this is the gauged WZW model for the coset G/H perturbed by a massive integrable potential $V = \mathrm{Tr}(\Lambda \mathrm{Ad}_g \bar\Lambda) - \mathrm{Tr}(\Lambda\bar\Lambda)$ as studied in [Par94]. Taking the limit $k \to 1$ directly at the level of the Lax connection given in eq. (4.3.7), keeping track of the definitions of the currents $K_w, K_{\bar w}$, which depend on k, we find

$$\mathcal{L}_w \to \partial_w + B_w - D_w g g^{-1} + \frac{1}{2\zeta}\Lambda \ , \qquad \mathcal{L}_{\bar w} = \partial_{\bar w} + B_{\bar w} - \frac{\zeta}{2}\mathrm{Ad}_g \bar\Lambda \ , \tag{4.3.29}$$

recovering the Lax given in [Par94; Fer+97].

When G is compact and $H = U(1)^{\mathrm{rk}_G}$, Λ and $\bar\Lambda$ can be chosen such that these models have a positive-definite kinetic term and a mass gap. These are known as the homogeneous sine-Gordon models [Fer+97]. For $G = SU(2)$ and $H = U(1)$ the homogeneous sine-Gordon model becomes the complex sine-Gordon model after integrating out $B_w, B_{\bar w}$. Note that if $\mathcal{Z}(\mathfrak{h})$ is one-dimensional and $Y(w)$ and $\bar Y(\bar w)$ are both non-constant then we can always use the classical conformal symmetry of the sigma model to reach $Y = w\Lambda$ and $\bar Y = \bar w \Lambda$, hence recovering a constant potential. This is not the case for higher-dimensional $\mathcal{Z}(\mathfrak{h})$.

4.3.3 Example: $SL(2)/U(1)_V$

To illustrate the features of this construction, let us consider the example of $SL(2)/U(1)_V$ for which the 2d gauged WZW describes the trumpet CFT. To be explicit we use $\mathfrak{sl}(2)$ generators

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad T_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{4.3.30}$$

and parametrise the group element as

$$g = \begin{pmatrix} \cos(\theta)\sinh(\rho) + \cosh(\rho)\cos(\tau) & \sin(\theta)\sinh(\rho) + \cosh(\rho)\sin(\tau) \\ \sin(\theta)\sinh(\rho) - \cosh(\rho)\sin(\tau) & \cosh(\rho)\cos(\tau) - \cos(\theta)\sinh(\rho) \end{pmatrix} . \tag{4.3.31}$$

We choose the $U(1)$ vector action generated by T_3 such that

$$\delta g = \epsilon[g, T_3] \quad \Rightarrow \quad \delta\rho = \delta\tau = 0 \ , \quad \delta\theta = \epsilon \ , \tag{4.3.32}$$

hence we gauge fix by setting $\theta = 0$. The analysis here is simplified by the observation that there is no WZ term since there are no 3-forms on the two-dimensional target space.

The CFT point. For orientation, we first work at the CFT point corresponding to $k = 1$. Recall from the discussion in section 4.2 that first constraining in 4d and then reducing enforces $\bar{\Phi} = \Phi = 0$ and the Lagrange multiplier sector vanishes. This gives the conventional gauged WZW model described by the target space geometry

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \coth^2\rho\, \mathrm{d}\tau^2 \ . \tag{4.3.33}$$

Let us now consider the IFT$_2$ that results from taking the same reduction that would lead to the CFT, but now in our reduction ansatz set $\Phi = \frac{m}{2}T_3$ and $\bar{\Phi} = -\frac{m}{2}T_3$. The Lagrangian that follows is

$$L_{\mathrm{CsG}} = \partial_w\rho\partial_{\bar{w}}\rho + \coth^2\rho\, \partial_w\tau\partial_{\bar{w}}\tau - m^2\sinh^2\rho \ . \tag{4.3.34}$$

This theory is well known as the complex sinh-Gordon model, a special case of the integrable massive perturbations of G/H gauged WZW models known as the homogeneous sine-Gordon models [Par94; Fer+97].

Unconstrained reduction: integrating out Φ, $\bar{\Phi}$ and $B_{w,\bar{w}}$. We now turn to the more general story, away from the CFT point, by considering the reduction without first imposing constraints. Taking the IFT_2 (4.3.17) and integrating out Φ, $\bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$ while retaining X^- and $\tilde{X}^+$, results in the sigma model with target space metric and B-field

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 + \mathrm{csch}^2\rho\left(\mathrm{d}\tilde{X}^{+2} - \mathrm{d}X^{-2}\right)\ , \\ B_2 &= \mathcal{V}\wedge\mathrm{d}\tilde{X}^+\ , \qquad \mathcal{V} = k\,\mathrm{csch}^2\rho\,\mathrm{d}X^- + k'\coth^2\rho\,\mathrm{d}\tau\ . \end{aligned} \tag{4.3.35}$$

Unconstrained reduction: the dual. On the other hand, if we solve the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ setting $\Phi = \frac{1}{2}\partial_w X^+$ and $\bar{\Phi} = \frac{1}{2}\partial_{\bar{w}} X^+$, we find the sigma model with target space geometry

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 - \mathrm{csch}^2\rho\,\mathrm{d}X^{-2} + \sinh^2\rho\,(\mathrm{d}X^+ + \mathcal{V})^2\ , \\ B_2 &= 0\ . \end{aligned} \tag{4.3.36}$$

This can of course be recognised as the T-dual of (4.3.35) along $\tilde{X}^+$. In the limit $k \to 1$ (4.3.36) becomes the pp-wave background

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + \coth^2\rho\,\mathrm{d}\tau^2 + \sinh^2\rho\,\mathrm{d}X^{+2} + 2\mathrm{d}X^+\mathrm{d}X^-\ , \\ B_2 &= 0\ , \end{aligned} \tag{4.3.37}$$

and if we light-cone gauge fix, $X^+ = m(w - \bar{w})$, in the associated sigma model we recover the complex sinh-Gordon Lagrangian (4.3.34) as expected.

Relation to PCM plus WZ term. Finally we demonstrate a relation between the models above and the PCM plus WZ term. Let us start with the metric and B-field for the PCM plus WZ term for $G = GL(2)$

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\tilde{X}^{+2} + \mathrm{d}\rho^2 - \cosh^2\rho\mathrm{d}\tau^2 + \sinh^2\rho\mathrm{d}\tilde{X}^{-2}\ , \\ B &= k\cosh^2\rho\,\mathrm{d}\tau\wedge\mathrm{d}\tilde{X}^-\ . \end{aligned} \tag{4.3.38}$$

Note that $\mathrm{d}B = k \sinh 2\rho \, \mathrm{d}\rho \wedge \mathrm{d}\tau \wedge \mathrm{d}\tilde{X}^-$, which is proportional to the volume for $SL(2)$. We first T-dualise $\tau \to \tilde{\tau}$, and then perform the following field redefinition

$$\tilde{X}^+ \to k'\tilde{X}^+ + \frac{k}{k'}\tilde{X}^- - \tilde{\tau} \ , \qquad \tilde{X}^- \to \frac{1}{k'}\tilde{X}^- \ . \tag{4.3.39}$$

It is straightforward to check that this is the inverse transformation to (4.3.23). Finally, T-dualising back, $\tilde{X}^+ \to X^+$, $\tilde{X}^- \to X^-$ and $\tilde{\tau} \to \tau$, we precisely recover the background (4.3.36), demonstrating that it can be understood as a generalised TsT transformation of the PCM plus WZ term.

4.3.4 The LMP limit

The PCM plus WZ term admits a limit in which it becomes the 2d analogue of the LMP model, otherwise known as the pseudodual of the PCM [ZM78b], see, e.g. [HLT19b]. It is possible to generalise this limit to the gauged model (4.3.5) by setting $g = \exp(\varepsilon U)$, $k = \varepsilon^{-1}\ell$, $\tilde{X}^+ \to \varepsilon^2 \tilde{X}^+$, $X^- \to \varepsilon^3 X^- - \varepsilon P_{\mathfrak{h}} U$, rescaling the Lagrangian by ε^{-2}, and taking $\varepsilon \to 0$. Implementing this limit in (4.3.5) we find

$$\begin{aligned} L^{\mathrm{LMP}}_{\mathrm{IFT}_2} = & \frac{1}{2}\mathrm{Tr}\big(D_w U D_{\bar{w}} U + [\Phi, U][\bar{\Phi}, U]\big) - \frac{\ell}{6}\mathrm{Tr}\big((D_w U + [\Phi, U][U, (D_{\bar{w}} U - [\bar{\Phi}, U]]\big) \\ & + \ell \mathrm{Tr}\big(X^-(F_{\bar{w}w} - [\bar{\Phi}, \Phi] + D_w \bar{\Phi} + D_{\bar{w}} \Phi)\big) + \mathrm{Tr}\big((\tilde{X}^+(D_w \bar{\Phi} - D_{\bar{w}} \Phi)\big) \\ & + \frac{1}{2\ell}\mathrm{Tr}\big(U(F_{\bar{w}w} - [\bar{\Phi}, \Phi] - D_w \bar{\Phi} - D_{\bar{w}} \Phi)\big) \ . \end{aligned} \tag{4.3.40}$$

Similarly we can take the limit in the Lax connections (4.3.7) and (4.3.8). The B-Lax (4.3.8) is unchanged, while the A-Lax (4.3.7) becomes

$$\begin{aligned} \mathcal{L}^{(A)}_w &= \partial_w + B_w - \frac{\ell}{2}K^{\mathrm{LMP}}_w - \frac{1}{\zeta}\big(\Phi + \frac{\ell}{2}K^{\mathrm{LMP}}_w\big) \ , \\ \mathcal{L}^{(A)}_{\bar{w}} &= \partial_{\bar{w}} + B_{\bar{w}} + \frac{\ell}{2}K^{\mathrm{LMP}}_{\bar{w}} + \zeta\big(\bar{\Phi} + \frac{\ell}{2}K^{\mathrm{LMP}}_{\bar{w}}\big) \ , \end{aligned} \tag{4.3.41}$$

where

$$K^{\mathrm{LMP}}_w = D_w U + [\Phi, U] \ , \qquad K^{\mathrm{LMP}}_{\bar{w}} = D_{\bar{w}} U - [\bar{\Phi}, U] \ . \tag{4.3.42}$$

As we will see in section 4.5 this model can also be found directly from gauged hCS$_6$ and CS$_4$ by considering a twist function with a single fourth-order pole.

As for the gauged WZW case, we can again find an integrable sigma model from (4.3.40) by integrating out $\Phi, \bar{\Phi}$ and the gauge field $B_{w,\bar{w}}$. For abelian H we can also construct the dual model by solving the constraint imposed by the Lagrange multiplier $\tilde{X}^+$ and integrating out $B_w, B_{\bar{w}}$. For $SL(2)/U(1)_V$ the resulting backgrounds can be found by taking the LMP limit

$$\begin{aligned}
&\rho \to \varepsilon\rho - \frac{1}{6}\varepsilon^3\rho\tau^2 , \qquad \tau \to \varepsilon\tau - \frac{1}{3}\varepsilon^3\rho^2\tau , \qquad (\mathrm{d}s^2, B_2) \to \varepsilon^{-2}(\mathrm{d}s^2, B_2) , \qquad k \to \varepsilon^{-1}\ell , \\
&X^- \to \varepsilon^3 X^- - \varepsilon\tau , \qquad \tilde{X}^+ \to \varepsilon^2\tilde{X}^+ , \qquad X^+ \to X^+ , \qquad \varepsilon \to 0 ,
\end{aligned} \tag{4.3.43}$$

in eqs. (4.3.35) and (4.3.36). This limit breaks the manifest global symmetry given by shifts of the coordinate τ. This is in agreement with the fact that the Lagrangian (4.3.40) is not invariant under $U \to U + H_0$ ($H_0 \in \mathfrak{h}$), while its gauged WZW counterpart (4.3.5) is invariant under $g \to h_0 g h_0$ ($h_0 \in H$) for abelian H.

Curiously, we can actually take a simplified LMP limit

$$\begin{aligned}
&\rho \to \varepsilon\rho , \qquad \tau \to \varepsilon\tau , \qquad (\mathrm{d}s^2, B_2) \to \varepsilon^{-2}(\mathrm{d}s^2, B_2) , \qquad k \to \varepsilon^{-1}\ell , \\
&X^- \to \varepsilon^3 X^- - \varepsilon\tau , \qquad \tilde{X}^+ \to \varepsilon^2\tilde{X}^+ , \qquad X^+ \to X^+ , \qquad \varepsilon \to 0 ,
\end{aligned} \tag{4.3.44}$$

in the backgrounds (4.3.35) and (4.3.36) that preserves this global symmetry. Taking this limit in eq. (4.3.35) we find

$$\begin{aligned}
&\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}\tau^2 + \frac{1}{\rho^2}\mathrm{d}\tilde{X}^{+2} + \frac{2}{\rho^2}\mathrm{d}X^-\mathrm{d}\tau , \\
&B_2 = \mathcal{V} \wedge \mathrm{d}\tilde{X}^+ , \qquad \mathcal{V} = \frac{\ell}{\rho^2}\mathrm{d}X^- + \left(\ell - \frac{1}{2\ell\rho^2}\right)\mathrm{d}\tau ,
\end{aligned} \tag{4.3.45}$$

while the limit of eq. (4.3.36) is

$$\begin{aligned}
&\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}\tau^2 + \rho^2(\mathrm{d}\tilde{X}^+ + \mathcal{V})^2 + \frac{2}{\rho^2}\mathrm{d}X^-\mathrm{d}\tau , \\
&B_2 = 0 .
\end{aligned} \tag{4.3.46}$$

As for the gauged WZW case these two backgrounds can also be constructed as a generalised TsT transformation of the background for the LMP model on $GL(2)$

$$\begin{aligned}
&\mathrm{d}s^2 = \mathrm{d}\tilde{X}^{+2} + \mathrm{d}\rho^2 - \mathrm{d}\tau^2 + \rho^2\mathrm{d}\tilde{X}^{-2} , \\
&B_2 = \ell\rho^2 d\tau \wedge \mathrm{d}\tilde{X}^- .
\end{aligned} \tag{4.3.47}$$

Explicitly, if we first T-dualise $\tau \to \tilde{\tau}$, then perform the following field redefinition

$$\tilde{X}^+ \to \ell \tilde{X}^+ + \frac{1}{2\ell^2}\tilde{X}^- - \tilde{\tau} \ , \qquad \tilde{X}^- \to \frac{1}{\ell}\tilde{X}^- \ , \qquad \tilde{\tau} \to \tilde{\tau} - \frac{1}{2\ell^2}\tilde{X}^- \ , \tag{4.3.48}$$

and finally T-dualise back,[15] $\tilde{X}^+ \to X^+$, $\tilde{X}^- \to X^-$ and $\tilde{\tau} \to \tau$, we recover the background (4.3.45).

4.4 Reduction to gCS$_4$ and localisation

Having discussed the right hand side of the diamond, we briefly describe the left hand side that follows from first reducing to obtain a gauged 4d Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{CP}^1$ and then integrating over $\mathbb{CP}^1$ to localise to a two-dimensional field theory on $\mathbb{R}^2$. We show that the resulting IFT$_2$ matches (4.3.3).

We recall the six-dimensional coupled action

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] - \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}\,(\mathcal{A}\wedge\mathcal{B}) \ , \tag{4.4.1}$$

and note that the three terms in the action are invariant under the transformations $\mathcal{A} \mapsto \hat{\mathcal{A}} = \mathcal{A} + \rho^{\mathcal{A}}_{\dot{a}} e^{\dot{a}} + \rho^{\mathcal{A}}_0 e^0$ and $\mathcal{B} \mapsto \hat{\mathcal{B}} = \mathcal{B} + \rho^{\mathcal{B}}_{\dot{a}} e^{\dot{a}} + \rho^{\mathcal{B}}_0 e^0$, given that both Ω and $\bar{\partial}\Omega$ are top forms in the holomorphic directions. By choosing $\rho^{\mathcal{A}}$ and $\rho^{\mathcal{B}}$ appropriately, we can ensure that neither $\hat{\mathcal{A}}$ nor $\hat{\mathcal{B}}$ have $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs, so

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}_w \mathrm{d}w + \hat{\mathcal{A}}_{\bar{w}} \mathrm{d}\bar{w} + \mathcal{A}_0 \bar{e}^0 \quad \text{with} \quad \hat{\mathcal{A}}_w = -\frac{[\mathcal{A}\kappa]}{\langle \pi\gamma\rangle} \ , \ \hat{\mathcal{A}}_{\bar{w}} = -\frac{[\mathcal{A}\hat{\kappa}]}{\langle \pi\hat{\gamma}\rangle} \ , \tag{4.4.2}$$

$$\hat{\mathcal{B}} = \hat{\mathcal{B}}_w \mathrm{d}w + \hat{\mathcal{B}}_{\bar{w}} \mathrm{d}\bar{w} + \mathcal{B}_0 \bar{e}^0 \quad \text{with} \quad \hat{\mathcal{B}}_w = -\frac{[\mathcal{B}\kappa]}{\langle \pi\gamma\rangle} \ , \ \hat{\mathcal{B}}_{\bar{w}} = -\frac{[\mathcal{B}\hat{\kappa}]}{\langle \pi\hat{\gamma}\rangle} \ . \tag{4.4.3}$$

To perform the reduction we follow the procedure outlined in section 2.5.4. Namely, we contract the six-dimensional Lagrangian of (4.4.1) with the vector fields ∂_z and $\partial_{\bar{z}}$, and restrict to gauge connections that are invariant under the flow of these vector fields. Thus, since the shifted gauge fields $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ manifestly have no $\mathrm{d}z$ or $\mathrm{d}\bar{z}$ legs, and we are restricting to field configurations satisfying $L_{\partial_z}\hat{\mathcal{A}} = L_{\partial_z}\hat{\mathcal{B}} = L_{\partial_{\bar{z}}}\hat{\mathcal{A}} = L_{\partial_{\bar{z}}}\hat{\mathcal{B}} = 0$, the contraction by ∂_z and $\partial_{\bar{z}}$ only hits Ω in the first two terms and $\bar{\partial}\Omega$ in the third. In

[15]Note that here the order of T-dualities matters. In particular, we cannot first T-dualise $\tilde{\tau}$ after the coordinate redefinition since it turns out to be a null coordinate.

particular, we find

$$(\partial_z\wedge\partial_{\bar z})\vee\Omega = \frac{\langle\alpha\beta\rangle^2}{2}\frac{\langle\pi\gamma\rangle\langle\pi\hat\gamma\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}e^0 \ , \qquad (\partial_z\wedge\partial_{\bar z})\vee\bar\partial\Omega = -\frac{\langle\alpha\beta\rangle^2}{2}\bar\partial_0\Big(\frac{\langle\pi\gamma\rangle\langle\pi\hat\gamma\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}\Big)e^0\wedge\bar e^0 \ . \tag{4.4.4}$$

Hence the six-dimensional action reduces to a four-dimensional coupled Chern-Simons action

$$S_{\mathrm{gCS}_4}[\hat A,\hat B] = \int_X \omega\wedge \mathrm{CS}[\hat A] - \int_X \omega\wedge\mathrm{CS}[\hat B] - \frac{1}{2\pi i}\int_X \bar\partial\omega\wedge\mathrm{Tr}(\hat A\hat B) \ , \tag{4.4.5}$$

where $X = \mathbb{CP}^1\times\mathbb{R}^2$,

$$\omega = \frac{\langle\alpha\beta\rangle^2}{2}\frac{\langle\pi\gamma\rangle\langle\pi\hat\gamma\rangle}{\langle\pi\alpha\rangle^2\langle\pi\beta\rangle^2}e^0 \ , \tag{4.4.6}$$

and $\hat A$ and $\hat B$ are the restrictions of $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ to X. Similarly, the boundary conditions (4.2.20) descend to analogous boundary conditions on $\hat A$ and $\hat B$. The action (4.4.5) has been considered before in [Ste21], albeit not with the choice of ω discussed here.

With the gauged 4d Chern-Simons action at hand, we may now localise. The procedure is entirely analogous to the one described in section 4.2.4 so we shall omit some of the details. We begin by reparametrising our four-dimensional gauge fields $\hat A$ and $\hat B$ in terms of a new pair of connections $\hat A', \hat B'$ and smooth functions $\hat g\in C^\infty(X,G)$ and $\hat h\in C^\infty(X,H)$. We use the redundancy in the reparametrisation to fix $\hat A'_0 = \hat B'_0 = 0$. The boundary degrees of freedom of the resulting IFT_2 will a priori be given by the evaluation of $\hat g$, $\hat h$, $\hat u$ and $\hat v$ at α and β. However, as in the 6d setting, we have some residual symmetry we can use to fix $\hat g|_\beta = \mathrm{id}$, $\hat h|_{\alpha,\beta} = \mathrm{id}$, and similarly, $\hat v|_{\alpha,\beta} = 0$. We are thus left with

$$\hat g|_\alpha \coloneqq g \ , \quad \hat u|_\alpha \coloneqq u \ , \quad \hat u|_\beta = \tilde u \ . \tag{4.4.7}$$

In terms of these variables, the bulk equations of motion of gCS_4 imply

$$\bar\partial_0\hat A'_i = 0 \ , \qquad \bar\partial_0\hat B'_i = 0 \ , \tag{4.4.8}$$

away from the zeroes of ω, namely γ and $\hat\gamma$. The on-shell gCS_4 action can be thus written as

$$\begin{aligned}S_{\text{gCS}_4}[\hat{A}',\hat{B}'] = \frac{1}{2\pi\text{i}}\int_X \bar{\partial}\omega\wedge\text{Tr}\big(\hat{A}'\wedge\bar{\partial}\hat{g}\hat{g}^{-1} - (\hat{g}^{-1}\hat{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g})\wedge\hat{B}'\big) \\ - \frac{1}{6\pi\text{i}}\int_{X\times[0,1]}\bar{\partial}\omega\wedge\text{Tr}\big(\hat{g}^{-1}\text{d}\hat{g}\wedge\hat{g}^{-1}\text{d}\hat{g}\wedge\hat{g}^{-1}\text{d}\hat{g}\big)\ .\end{aligned} \tag{4.4.9}$$

To obtain the IFT$_2$ we begin by looking at the bulk equations of motion (4.4.8). Liouville's theorem shows that the only bounded, holomorphic functions on $\mathbb{CP}^1$ are constant functions. However, we are after something more general than this since we do not require the components of our gauge field to be bounded at the zeroes of ω. Indeed, we allow the w-component to have a pole at $\pi\sim\gamma$ and the $\bar{w}$-component to have a pole at $\pi\sim\hat{\gamma}$. With this analytic structure in mind, we can parametrise the solution of the bulk equation for $\hat{B}'$ as

$$\hat{B}'_w = B_w + \frac{\langle\pi\hat{\gamma}\rangle}{\langle\pi\gamma\rangle}\,\Phi\ , \qquad \hat{B}'_{\bar{w}} = B_{\bar{w}} - \frac{\langle\pi\gamma\rangle}{\langle\pi\hat{\gamma}\rangle}\,\bar{\Phi}\ , \tag{4.4.10}$$

where we have conveniently used the field variables introduced in (4.3.1) to ease comparison with (4.3.3) after localisation to the IFT$_2$. In particular, under π-independent gauge transformations $B_w, B_{\bar{w}}$ transform as the components of a 2d gauge field, while Φ and $\bar{\Phi}$ transform as adjoint scalars.

Note that in the singular part of these solutions, we have chosen to align the zero of each with the pole of the other. This choice is completely general since moving the zeroes in the singular parts amounts to field redefinitions relating B_w and Φ or $B_{\bar{w}}$ and $\bar{\Phi}$. However, it is a convenient choice since the flatness condition on $\hat{B}'$ immediately reproduces Hitchin's equations,

$$F_{w\bar{w}}[\hat{B}'] = F_{w\bar{w}}[B] - [\Phi,\bar{\Phi}] - \frac{\langle\pi\gamma\rangle}{\langle\pi\hat{\gamma}\rangle}D_w\bar{\Phi} - \frac{\langle\pi\hat{\gamma}\rangle}{\langle\pi\gamma\rangle}D_{\bar{w}}\Phi\ . \tag{4.4.11}$$

On the other hand, for the $\hat{A}'$ gauge field a convenient choice of parametrisation when solving the bulk equation of motion (4.4.8) is

$$\begin{aligned}\hat{A}'_w &= \frac{\langle\pi\alpha\rangle}{\langle\pi\gamma\rangle}\frac{\langle\beta\gamma\rangle}{\langle\beta\alpha\rangle}\,U_w + \frac{\langle\pi\beta\rangle}{\langle\pi\gamma\rangle}\frac{\langle\alpha\gamma\rangle}{\langle\alpha\beta\rangle}\,V_w\ , \\ \hat{A}'_{\bar{w}} &= \frac{\langle\pi\alpha\rangle}{\langle\pi\hat{\gamma}\rangle}\frac{\langle\beta\hat{\gamma}\rangle}{\langle\beta\alpha\rangle}\,U_w + \frac{\langle\pi\beta\rangle}{\langle\pi\hat{\gamma}\rangle}\frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle}\,V_{\bar{w}}\ .\end{aligned} \tag{4.4.12}$$

This parametrisation, in which we have chosen the coefficients such that one term vanishes at $\pi \sim \alpha$ while the other vanishes at $\pi \sim \beta$, is adapted to the boundary conditions, which can be solved for U_i and V_i to yield

$$\begin{aligned}\hat{A}'_w &= \hat{B}'_w - \frac{\langle\pi\beta\rangle}{\langle\pi\gamma\rangle}\frac{\langle\alpha\gamma\rangle}{\langle\alpha\beta\rangle}\Big(D_w g g^{-1} + \frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\gamma\rangle}(1-\mathrm{Ad}_g)\Phi\Big)\ ,\\ \hat{A}'_{\bar{w}} &= \hat{B}'_{\bar{w}} - \frac{\langle\pi\beta\rangle}{\langle\pi\hat{\gamma}\rangle}\frac{\langle\alpha\hat{\gamma}\rangle}{\langle\alpha\beta\rangle}\Big(D_{\bar{w}} g g^{-1} - \frac{\langle\alpha\gamma\rangle}{\langle\alpha\hat{\gamma}\rangle}(1-\mathrm{Ad}_g)\bar{\Phi}\Big)\ .\end{aligned} \tag{4.4.13}$$

Replacing (4.4.10) and (4.4.13) in (4.4.9) and integrating along the $\mathbb{CP}^1$ fibre using the localisation formula given by:

$$\frac{1}{2\pi\mathrm{i}}\int_X \bar{\partial}\omega\wedge Q = -\frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\langle\alpha\gamma\rangle\langle\beta\hat{\gamma}\rangle+\langle\alpha\hat{\gamma}\rangle\langle\beta\gamma\rangle}{\langle\alpha\beta\rangle}Q|_\alpha + \langle\alpha\gamma\rangle\langle\alpha\hat{\gamma}\rangle(\partial_0 Q)|_\alpha\right] \quad + \quad \alpha\leftrightarrow\beta\ , \tag{4.4.14}$$

for any $Q \in \Omega^2(X)$, we recover the IFT$_2$ given in (4.3.3).

4.5 Gauged LMP action

In the previous sections, we analysed the ghCS$_6$ action (4.2.14) where the meromorphic $(3,0)$-form Ω had two double poles, showing that such a theory leads to a gauged WZW$_4$ upon localisation to $\mathbb{R}^4$. To highlight some of the universal features of this procedure, we will now focus on another example in which the meromorphic $(3,0)$-form has a single fourth-order pole. For the ungauged hCS$_6$, such a configuration was shown in [Bit22] to lead to the LMP action for ASDYM [LM87], [Par92].

4.5.1 Gauged LMP action from ghCS$_6$

Earlier in the thesis in section §2.5.6, we derived the LMP action from hCS$_6$. Next, we consider the same fourth-order pole structure for gauged hCS$_6$. The starting point is to compute the boundary variation and choose boundary conditions to ensure that it vanishes.

Boundary conditions. Starting from the action

$$S_{\mathrm{ghCS}_6}[\mathcal{A},\mathcal{B}] = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] - \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge\mathrm{Tr}\big(\mathcal{A}\wedge\mathcal{B}\big)\ , \tag{4.5.1}$$

where

$$\Omega = k\,\frac{e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}}}{\langle \pi\alpha\rangle^4}\ , \tag{4.5.2}$$

the boundary variation is given by

$$\delta S_{\mathrm{ghCS}_6}\big|_{\mathrm{bdry}} = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{PT}} \bar\partial\Omega \wedge \mathrm{Tr}\big(\delta\mathcal{A}\wedge(\mathcal{A}-\mathcal{B}) - \delta\mathcal{B}\wedge(\mathcal{B}-\mathcal{A})\big)\ . \tag{4.5.3}$$

Following in a parallel fashion to the hCS_6 case, we find that a suitable choice of boundary conditions is given by

$$\begin{aligned} \mathcal{A}|_{\pi=\alpha} &= \mathcal{B}|_{\pi=\alpha}\ , & \partial_0\mathcal{A}|_{\pi=\alpha} &= \partial_0\mathcal{B}|_{\pi=\alpha}\ , \\ \partial_0^2\mathcal{A}^{\mathfrak{h}}|_{\pi=\alpha} &= \partial_0^2\mathcal{B}|_{\pi=\alpha}\ , & \partial_0^3\mathcal{A}^{\mathfrak{h}}|_{\pi=\alpha} &= \partial_0^3\mathcal{B}|_{\pi=\alpha}\ . \end{aligned} \tag{4.5.4}$$

Gauge fixing. Gauge fixing will once again prove helpful to proceed with the localisation calculation. As such, we will consider the set of admissible gauge transformations respecting our boundary conditions. Performing a gauge transformation on the first boundary condition gives

$$\big(\hat\gamma^{-1}\mathcal{A}\hat\gamma + \hat\gamma^{-1}\bar\partial\hat\gamma\big)\,|_{\pi=\alpha} = \big(\hat\eta^{-1}\mathcal{B}\hat\eta + \hat\eta^{-1}\bar\partial\hat\eta\big)\,|_{\pi=\alpha}\ , \tag{4.5.5}$$

from which we conclude that the admissible gauge transformations should obey $\hat\gamma|_\alpha = \hat\eta|_\alpha$. Running through systematically, the second boundary condition requires

$$\begin{aligned} &\Big(\big[\hat\gamma^{-1}\mathcal{A}\,\hat\gamma + \hat\gamma^{-1}\bar\partial\hat\gamma,\, \hat{\mathbf{\Gamma}}\big] + \hat\gamma^{-1}\partial_0\mathcal{A}\hat\gamma + \bar\partial\hat{\mathbf{\Gamma}} + \hat\gamma^{-1}\partial_{\dot a}\hat\gamma\ \bar e^{\dot a}\Big)\,|_{\pi=\alpha} \\ &\qquad = \Big(\big[\hat\eta^{-1}\mathcal{B}\,\hat\eta + \hat\eta^{-1}\bar\partial\hat\eta,\, \hat{\mathbf{N}}\big] + \hat\eta^{-1}\partial_0\mathcal{B}\hat\eta + \bar\partial\hat{\mathbf{N}} + \hat\eta^{-1}\partial_{\dot a}\hat\eta\ \bar e^{\dot a}\Big)\,|_{\pi=\alpha}\ , \end{aligned} \tag{4.5.6}$$

where we have denoted $\hat{\mathbf{\Gamma}} = \hat\gamma^{-1}\partial_0\hat\gamma$ and $\hat{\mathbf{N}} = \hat\eta^{-1}\partial_0\hat\eta$. Making use of the original boundary condition and the constraint $\hat\gamma|_\alpha = \hat\eta|_\alpha$, we conclude that admissible gauge transformations should also obey $\hat{\mathbf{\Gamma}}|_{\pi=\alpha} = \hat{\mathbf{N}}|_{\pi=\alpha}$. In a similar fashion, from the third boundary condition we conclude that $\hat{\mathbf{\Gamma}}^{(2)}_{\mathfrak{h}}|_\alpha = \hat{\mathbf{N}}^{(2)}|_\alpha$ where $\hat{\mathbf{\Gamma}}^{(2)} := \hat\gamma^{-1}\partial_0^2\hat\gamma$ and $\hat{\mathbf{N}}^{(2)} := \hat\eta^{-1}\partial_0^2\hat\eta$. Finally, from the fourth boundary condition we find $\hat{\mathbf{\Gamma}}^{(3)}_{\mathfrak{h}}|_\alpha = \hat{\mathbf{N}}^{(3)}|_\alpha$ where $\hat{\mathbf{\Gamma}}^{(3)} := \hat\gamma^{-1}\partial_0^3\hat\gamma$ and $\hat{\mathbf{N}}^{(3)} := \hat\eta^{-1}\partial_0^3\hat\eta$. It is important to note here that the boundary conditions do not constrain $\hat{\mathbf{\Gamma}}^{(2)}_{\mathfrak{k}}$ and $\hat{\mathbf{\Gamma}}^{(3)}_{\mathfrak{k}}$.

Now we know the admissible gauge symmetries of our theory, we can gauge fix the degrees of freedom. Initially, there are 8 degrees of freedom in our theory,

$$\begin{aligned} \underline{\mathbf{u}} &:= \left(g, \mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3\right) \ , \\ \underline{\mathbf{v}} &:= \left(h, \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\right) \ . \end{aligned} \tag{4.5.7}$$

We first consider the internal gauge symmetries of $\mathcal{A}$ and $\mathcal{B}$, which we can use to set both g and h to the identity. Next, we note that the H-valued external gauge transformations of $\mathcal{B}$ parametrised by $\hat{\eta}$ are unconstrained at the point $\pi = \alpha$. As such, we can gauge fix $\mathbf{v}^i = 0$ for $i = 1, 2, 3$. After this choice of fixing we note that the external gauge transformations of $\mathcal{A}$ parametrised by $\hat{\gamma}$ are constrained to coincide with $\hat{\eta}$ at $\pi = \alpha$, and we have used these symmetries in our choice of gauge fixing, we find that we are unable to gauge fix $\mathbf{u}, \mathbf{u}^2_{\mathfrak{h}}$ or $\mathbf{u}^3_{\mathfrak{h}}$. As such, each of these degrees of freedom will appear as fields in our effective theory on $\mathbb{R}^4$. After gauge fixing $\mathbf{u}^2_{\mathfrak{k}} = \mathbf{u}^3_{\mathfrak{k}} = 0$ and renaming $\mathbf{u}^1$ to $\mathbf{u}$, we have

$$\begin{aligned} \underline{\mathbf{u}} &= \left(\mathrm{id}, \mathbf{u}, \mathbf{u}^2_{\mathfrak{h}}, \mathbf{u}^3_{\mathfrak{h}}\right) \ , \\ \underline{\mathbf{v}} &= (\mathrm{id}, 0, 0, 0) \ . \end{aligned} \tag{4.5.8}$$

Solving the boundary conditions. The first boundary condition reads

$$\left(\hat{g}^{-1}\mathcal{A}'\hat{g} + \hat{g}^{-1}\bar{\partial}\hat{g}\right)|_{\pi=\alpha} = (\hat{h}^{-1}\mathcal{B}'\hat{h} + \hat{h}^{-1}\bar{\partial}\hat{h})|_{\pi=\alpha} \ . \tag{4.5.9}$$

Given our choice of gauge fixing (4.5.8) and the bulk solutions $\mathcal{A}'_{\dot{a}} = \pi^a A_{a\dot{a}}$ and $\mathcal{B}'_{\dot{a}} = \pi^a B_{a\dot{a}}$, this implies

$$\mathcal{A}'|_{\pi=\alpha} = \mathcal{B}'|_{\pi=\alpha} \quad \Rightarrow \quad \alpha^a A_{a\dot{a}} = \alpha^a B_{a\dot{a}} \quad \Rightarrow \quad A_{a\dot{a}} = B_{a\dot{a}} - \alpha_a Q_{\dot{a}} \ . \tag{4.5.10}$$

We can then use the second boundary condition to solve for $Q_{\dot{a}}$,

$$\partial_0\mathcal{A}|_{\pi=\alpha} = \partial_0\mathcal{B}|_{\pi=\alpha} \ , \qquad \Longrightarrow \qquad Q_{\dot{a}} = -\alpha^a\left([B_{a\dot{a}}, \mathbf{u}] + \partial_{a\dot{a}}\mathbf{u}\right) = -\alpha^a\nabla_{a\dot{a}}\mathbf{u} \ . \tag{4.5.11}$$

These two boundary conditions are sufficient to solve for $A_{a\dot{a}}$ in terms of the other degrees of freedom,

$$A_{a\dot{a}} = B_{a\dot{a}} + \alpha_a\alpha^b\nabla_{b\dot{a}}\mathbf{u} \ . \tag{4.5.12}$$

Localisation to $\mathbb{R}^4$. Writing the action (4.5.1) in terms of the new field variables, the only terms that contribute to the effective action given our choice of gauge (4.5.8) will be

$$S_{\text{ghCS}_6} = \frac{1}{2\pi i}\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \text{Tr}(\mathcal{A}' \wedge \bar{\partial}\hat{g}\hat{g}^{-1} - \hat{g}^{-1}\mathcal{A}'\hat{g} \wedge \mathcal{B}' - \hat{g}^{-1}\bar{\partial}\hat{g} \wedge \mathcal{B}') \\ - \frac{1}{6\pi i}\int_{\mathbb{PT}\times[0,1]} \bar{\partial}\Omega \wedge \text{Tr}\left((\hat{g}^{-1}\text{d}\hat{g})^3\right) \ . \tag{4.5.13}$$

The localisation calculation of the gauged model is slightly more involved than the ungauged case due to the additional degrees of freedom appearing. However, in analogy with calculations in previous sections, we expect $\mathbf{u}^2_{\mathfrak{h}}$ and $\mathbf{u}^3_{\mathfrak{h}}$ to appear only as Lagrange multipliers, in particular, imposing self-duality type constraints for our gauge field B. With this in mind, we can show that the 4d theory is given by

$$S_{\text{gLMP}}[\mathbf{u}, B] = k\int_{\mathbb{R}^4} \text{vol}_4 \ \frac{1}{2}\text{Tr}\left(\nabla^{a\dot{a}}\mathbf{u}\nabla_{a\dot{a}}\mathbf{u}\right) + \frac{1}{3}\epsilon^{\dot{a}\dot{b}}\ \text{Tr}\left(\mathbf{u}\left[\alpha^a\nabla_{a\dot{a}}\mathbf{u}, \alpha^b\nabla_{b\dot{b}}\mathbf{u}\right]\right) \\ + \mathbf{u}\ \epsilon^{\dot{a}\dot{b}}\hat{\alpha}^a\hat{\alpha}^b F_{a\dot{a}b\dot{b}}(B) + \frac{1}{2}\mathbf{u}^2_{\mathfrak{h}}\epsilon^{\dot{a}\dot{b}}\left(\alpha^a\hat{\alpha}^b + \hat{\alpha}^a\alpha^b\right) F_{a\dot{a}b\dot{b}}(B) \\ + \tilde{\mathbf{u}}^3_{\mathfrak{h}}\epsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}(B) \ , \tag{4.5.14}$$

where we have performed a field redefinition $\mathbf{u}^3_{\mathfrak{h}} \to \tilde{\mathbf{u}}^3_{\mathfrak{h}} := \frac{1}{6}(\mathbf{u}^3_{\mathfrak{h}} + 2\left[\mathbf{u}, \mathbf{u}^2_{\mathfrak{h}}\right])$. Reducing along a particular $\mathbb{R}^2$, and appropriately redefining fields and parameters, we find that the gauged LMP action gives the IFT_2 (4.3.40).

Implementing the Lagrange multipliers. In section 2.5.3 we reviewed how solutions to the ASDYM can be formulated in terms of Yang's matrix after a partial gauge fixing of the ASD connection. We conclude this section by integrating out the Lagrange multiplier fields present in the action (4.5.14) by solving the self duality constraints they impose in a similar fashion. Indeed, the LMP equations of motion can be understood as the remaining ASDYM equation once these two constraints have been solved. This is analogous to the statement that the WZW_4 equations of motion are the remaining ASDYM equation for Yang's matrix.

The equation of motion found by varying $\tilde{\mathbf{u}}^3_{\mathfrak{h}}$ is an integrability condition along the 2-plane defined by α^a, and it can be solved by

$$\epsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b F_{a\dot{a}b\dot{b}}(B) = 0 \qquad \Longrightarrow \qquad \alpha^a B_{a\dot{a}} = h^{-1}\alpha^a\partial_{a\dot{a}}h \ , \tag{4.5.15}$$

where $h \in C^\infty(\mathbb{R}^4) \otimes H$. It is helpful to parametrise the remaining degrees of freedom in $B_{a\dot{a}}$ in terms of a new field $C_{\dot{a}}$, defined by the relation

$$B_{a\dot{a}} = h^{-1}\partial_{a\dot{a}}h - \alpha_a\, h^{-1}C_{\dot{a}}h \ . \tag{4.5.16}$$

Then, the $\mathbf{u}^2$ equation of motion becomes

$$\epsilon^{\dot{a}\dot{b}}\left(\alpha^a\hat{\alpha}^b + \hat{\alpha}^a\alpha^b\right)F_{a\dot{a}b\dot{b}}(B) = 0 \qquad \Longleftrightarrow \qquad \epsilon^{\dot{a}\dot{b}}\alpha^a\partial_{a\dot{a}}C_{\dot{b}} = 0 \ . \tag{4.5.17}$$

This may be solved explicitly by $C_{\dot{a}} = \alpha^a\partial_{a\dot{a}}f$ for $f \in C^\infty(\mathbb{R}^4)\otimes\mathfrak{h}$, such that the gauge field B is given by

$$B_{a\dot{a}} = h^{-1}\partial_{a\dot{a}}h + h^{-1}X_{a\dot{a}}h \quad \text{ where } \quad X_{a\dot{a}} = -\alpha_a\alpha^b\partial_{b\dot{a}}f \ . \tag{4.5.18}$$

Reinserting this expression into the action (4.5.14), the resulting theory may be written as a difference of two LMP actions. This follows after performing a field redefinition $h\mathbf{u}h^{-1} = v - f$, for $v \in C^\infty(\mathbb{R}^4)\otimes\mathfrak{g}$, such that we arrive at the action

$$\begin{aligned} S_{\mathrm{gLMP}}[\mathbf{u}, B] = k\int_{\mathbb{R}^4}\frac{1}{2}\mathrm{Tr}(\mathrm{d}v\wedge\star\mathrm{d}v) + \frac{1}{3}\alpha_a\alpha_b\Sigma^{ab}\wedge\mathrm{Tr}(v\,[\mathrm{d}v,\mathrm{d}v]) \\ - k\int_{\mathbb{R}^4}\frac{1}{2}\mathrm{Tr}(\mathrm{d}f\wedge\star\mathrm{d}f) + \frac{1}{3}\alpha_a\alpha_b\Sigma^{ab}\wedge\mathrm{Tr}(f\,[\mathrm{d}f,\mathrm{d}f]) \ . \end{aligned} \tag{4.5.19}$$

This demonstrates the conclusion

$$S_{\mathrm{gLMP}}[\mathbf{u}, B] = S_{\mathrm{LMP}}[v] - S_{\mathrm{LMP}}[f] \ , \tag{4.5.20}$$

in much a similar vein as (4.2.67) for the case of gWZW_4.

Chapter 5

Discussion and Outlook

In this thesis we have presented work that has attempted to illuminate the rich landscape of integrable field theories. We will provide an outlook and discussion for §3 and then similarly for §4, before concluding with further directions of study via the medium of higher gauge theories.

Integrable Deformations from Twistor Space

In §3 we constructed a diamond of integrable models related by localisation and symmetry reduction. Starting from holomorphic Chern-Simons theory with the meromorphic (3,0)-form (3.1.2), we found a new choice of admissible boundary conditions, which leads to a well-defined 6-dimensional theory. This generalised the analysis carried out in [BS23; Pen21] to a new class of boundary conditions not of Dirichlet type.

This work opens up a range of interesting further directions. There are a selection of direct generalisations that can be made to incorporate the wide variety of integrable deformations known in the literature. Perhaps the most interesting outcome of this would be the construction of swathes of new four-dimensional integrable field theories. Our work focused on the case where Ω was nowhere vanishing; it would be interesting to explore the relaxation of this condition together with its possible boundary conditions, and how the ASDYM equations are modified. Moreover, one might hope that the study of boundary conditions in hCS_6 could lead to a full classification of the landscape of integrable sigma-models in 2d, and perhaps result in novel theories not yet encountered in the literature.

From the perspective of the IFT_2, there is a close relationship between the notions of Poisson-Lie symmetry, duality and integrability [Vic15; HT15; SST15; Kli15]. This poses an interesting question as to the implications of such dualities for both the IFT_4 and hCS_6. For the model considered here, we might seek to understand the semi-local symmetries of IFT_4 in the context of the q-deformed symmetries expected to underpin the IFT_2.

By coupling the open and closed string sectors [CL15; CG18; CL20; Cos21] one can find an anomaly free quantization to all loop orders in perturbation theory of the coupled hCS_6-BCOV action. This mechanism has already proven a powerful tool in the context of the top-down approach to celestial holography [CPS23; BSS23]. This could provide an angle of attack to address the important questions of when the IFT_4 can be quantised, if the IFT_4 is *quantum* integrable, and if there is a higher dimensional origin of IFT_2 as quantum field theories.

Gauging the Diamond

The construction presented in §4 led to the discovery of novel integrable field theories in both four and two dimensions, opening up a plethora of interesting future directions to be explored.

Motivated by the observation that the gauged WZW model on the coset G/H in two dimensions can be written as the difference of WZW models for the groups G and H, we took the difference of two hCS_6 theories as our starting point. The boundary conditions (4.2.20) led us to add a boundary term resulting in the action (4.2.14). It is worth highlighting that the boundary variation vanishes on the boundary conditions (4.2.20) whether or not the boundary term is included, and the contribution of the boundary term to the IFT_4 vanishes if we invoke all the boundary conditions. However, while the algebraic boundary conditions $\mathcal{A}^{\mathfrak{k}}|_{\alpha,\beta} = 0$ and $\mathcal{A}^{\mathfrak{h}}|_{\alpha,\beta} = \mathcal{B}|_{\alpha,\beta}$ can be straightforwardly solved, this is not the case for the differential one $\partial_0\mathcal{A}^{\mathfrak{h}}|_{\alpha,\beta} = \partial_0\mathcal{B}|_{\alpha,\beta}$. Therefore, we relaxed this condition such that the contribution of the boundary term no longer vanishes. Importantly, for the specific boundary term added in (4.2.14), the constraints implied by the differential boundary condition now follow as on-shell equations of motion, leading to fully consistent IFT_4 and IFT_2.

There are compelling reasons to follow this strategy, including that the symplectic potential becomes tautological upon including the boundary term. However, a systematic interpretation of when boundary conditions can be consistently dropped for particular choices of boundary term is an open question. To address this, it would be appropriate to pursue a more formal study, complementing a homotopic analysis (along the lines done for CS_4 in [BSV22]) with a symplectic/Hamiltonian study of the 6d holomorphic Chern-Simons theory (similar to [Vic21] in the context of CS_4).

A second arena for formal development is the connection between 6d holomorphic Chern-Simons and five-dimensional Kähler Chern-Simons (KCS_5) theory [NS90; NS92]. This should mirror the relationship between CS_4 and CS_3 theories described by Yamazaki [Yam19]. To make this suggestion precise in the present context one may consider a Kaluza-Klein expansion around the $U(1)$ rotation in the $\mathbb{CP}^1$ that leaves the location of the double poles fixed, retaining the transverse coordinate as part of the bulk five-manifold of KCS_5. The details of this are left for future study.

It would also be interesting to explore the new integrable IFT_4 and IFT_2 that we have constructed. G/H coset CFTs in two dimensions have a rich spectrum of parafermionic operators [BCR90; BCH91]. It would be very interesting to establish the lift or analogue of these objects in the context of the IFT_4. The natural framework for this is likely to involve the study of co-dimension one defects and associated higher-form symmetries.

For abelian H we find IFT_2 that, in the $k \to 1$ limit, are related to massive integrable perturbations of the G/H gauged WZW models known as homogeneous sine-Gordon models [Par94; Fer+97]. These include the sine-Gordon and complex sine-Gordon models as special cases, two of the most well-understood IFT_2. There is nothing in our construction that prohibits non-abelian H and it would be interesting to study the resulting models in more detail. The homogeneous sine-Gordon models before gauging are closely related to the non-abelian Toda equations [LS83; HMP95], for which an alternative derivation from CS_4 involving both order and disorder defects was presented in [FSY22]. It would be instructive to understand the relationship between the two approaches.

An important class of IFT_2 are the symmetric space sigma models. These can be constructed either by restricting fields to parametrise G/H directly or by gauging a left action of H in the PCM. These theories have been realised in CS_4 through branch cut defects [CY19] and recently in hCS_6 [CW24]. One might explore the realisation of the gauging construction of such models within the current framework, and generalise to $\mathbb{Z}_4$-graded semi-symmetric spaces (relevant for applications of CS_4 to string worldsheet theories [CS20; BP24]).

When G/H is a symmetric space, an alternative class of massive integrable perturbations of the G/H gauged WZW model are known as the symmetric space sine-Gordon models [BPS96; Fer+97]. In the landscape of IFT_2 these are related to the $\lambda \to 0$ limit [HMS14; HT15] of the λ-deformation of the symmetric space sigma model [Sfe14]. Note that $k \to 1$ and $\lambda \to 0$ both correspond to conformal limits and it would be instructive to explore the relation between the two constructions. More generally, it would be interesting to generalise the construction in this chapter to deformed models, in particular, splitting one or both double poles in the meromorphic (3,0)-form Ω into simple poles, or dual models, for example, considering the alternative boundary conditions (4.2.31).

Higher-Dimensional Integrability from Higher Gauge Theory

In this thesis, we have investigated novel classically integrable systems whose equations of motion are equivalent to the ASDYM equations. This equivalence, in principle, allows for the complete solvability of these models via the ADHM construction [Ati+78]. The manifestation of integrability by such models has its roots in the fact that they descend from a local holomorphic action on twistor space [Cos21]; however, when compared against the notion of Lax integrability in two-dimensional integrable systems the parallels between the two mechanisms that allow solvability are opaque. As such, a natural question is the following: Is there a class of higher-dimensional integrable field theories, whose integrability is given by the flatness of a higher Lax connection?

As briefly described in the introduction, this is the ambition of a new programme, which studies a higher-categorical generalisation of 4dCS theory, named five-dimensional 2-Chern-Simons (5d2CS) [SV24; CL24; CL25]. Using the 2-categorified notion of a Lie group, namely a Lie 2-group, one can define a 2-Chern-Simons four-form for a 2-connection, $\mathscr{A} = (A, B) \in \Omega^1(X, \mathfrak{g}) \oplus \Omega^2(X, \mathfrak{h})$, on a principal 2-bundle [BS04], by

$$2\text{CS}(\mathscr{A}) = \langle F(A) - \frac{1}{2} t_*(B), B\rangle - \frac{1}{2} d\langle A, B\rangle \ .$$

The idea is to proceed in analogy to 4dCS, defining the action

$$S_{5d2\text{CS}} = \int_{\mathbb{R}^3 \times \mathbb{CP}^1} \omega \wedge 2\text{CS}(\mathscr{A}) \ ,$$

where ω is a meromorphic (1,0)-form on $\mathbb{CP}^1$.

This landscape is currently sparsely charted, and numerous promising directions for future research remain open, a few of which we outline below.

The emergence of the Yang-Baxter equation from considering Wilson line operators in 4dCS was a remarkable success story [Cos13; CWY18a; CWY18b], giving an elegant gauge-theoretic account of the algebraic structures underlying two-dimensional integrable lattice models. In the context of three-dimensional lattice models, the algebraic relations underpinning integrability are given by the Zamolodchikov tetrahedron equation (ZTE) [Zam80; Zam81]. It is therefore natural to investigate the extent to which 5d2CS may illuminate the origin of the ZTE. One may conjecture that the topological–holomorphic character of 5d2CS theory, together with its intrinsic 2-group structure [BC10], renders it a particularly suitable framework in which such algebraic behaviour might be realised.

An important open question concerns charting the landscape of theories obtained by localising 5d 2-Chern–Simons (5d2CS) theory. While known integrable field theories in three dimensions admit elegant twistorial formulations [MW91; Bit+25], it remains unclear whether these theories can also be realised from the data of 5d2CS theory. This line of inquiry has two significant implications. Should these established models possess a 5d2CS description, their dynamics could be encoded in a Lax 2-connection, thereby offering a framework that more transparently parallels Lax integrability in two dimensions. Conversely, if the theories arising from 5d2CS lie beyond the scope of twistorial constructions, this would suggest the existence of two genuinely distinct notions of integrability in three dimensions, opening up a range of compelling directions investigating the structure and properties of such novel models.

Appendices

Appendix A

Projector technology

We consider the operator on 1-forms on $\mathbb{R}^4$ given by

$$\mathcal{J}_{\alpha,\beta}(\sigma) = -\mathrm{i} \star (\omega_{\alpha,\beta} \wedge \sigma) \ , \quad \mathcal{J}^2_{\alpha,\beta} = -\mathrm{id} \ , \tag{A.1}$$

where $\omega_{\alpha,\beta} = \frac{1}{\langle\alpha\beta\rangle}\alpha_a\beta_b\,\epsilon_{\dot{a}\dot{b}}dx^{a\dot{a}}dx^{b\dot{b}}$, which allows us to define projectors

$$P = \frac{1}{2}\,(\mathrm{id} - \mathrm{i}\mathcal{J}) \qquad \bar{P} = \frac{1}{2}\,(\mathrm{id} + \mathrm{i}\mathcal{J}) \ . \tag{A.2}$$

For this to define a complex structure on the real Euclidean slice of $\mathbb{R}^4 \subset \mathbb{C}^4$ we require that $\mathcal{J}$ maps Euclidean-real 1-forms to Euclidean-real 1-forms. While not true for general α and β, this is the case if we take $\alpha = \gamma$ and $\beta = \hat{\gamma}$. Then $\mathcal{J}_{\gamma,\hat{\gamma}}$ is the complex structure $\mathcal{J}_\gamma$. The projectors P and $\bar{P}$ project onto the $(1,0)$ and $(0,1)$ components thus realising the Dolbeault complex.

These projectors satisfy a range of useful identities:

$$\bar{P}(\star(\mu_\alpha \wedge \sigma)) = 0 \ , \quad P(\star(\mu_\beta \wedge \sigma)) = 0 \ , \quad \mu_\beta \wedge \bar{P}(\sigma) = 0 \ , \quad \mu_\alpha \wedge P(\sigma) = 0 \ , \tag{A.3}$$

$$\omega_{\alpha,\beta} \wedge \bar{P}(\sigma) = -\star\bar{P}(\sigma) \ , \qquad \omega_{\alpha,\beta} \wedge P(\sigma) = \star P(\sigma) \ , \tag{A.4}$$

$$\omega_{\alpha,\beta} \wedge \bar{P}(\sigma) \wedge \tau = \omega_{\alpha,\beta} \wedge \sigma \wedge P(\tau) \ , \qquad \omega_{\alpha,\beta} \wedge \bar{P}(\sigma) \wedge \bar{P}(\tau) = 0 \ , \tag{A.5}$$

where

$$\mu_\alpha = \alpha_a\alpha_b\,\epsilon_{\dot{a}\dot{b}}dx^{a\dot{a}}dx^{b\dot{b}} \ , \quad \text{and} \quad \mu_\beta = \beta_a\beta_b\,\epsilon_{\dot{a}\dot{b}}dx^{a\dot{a}}dx^{b\dot{b}} \ . \tag{A.6}$$

To move between form and component notation it is useful to observe that

$$P(\sigma)_{a\dot{a}} = -\frac{1}{\langle\alpha\beta\rangle}\alpha_a\beta^b\sigma_{b\dot{a}} \ , \qquad \bar{P}(\sigma)_{a\dot{a}} = \frac{1}{\langle\alpha\beta\rangle}\beta_a\alpha^b\sigma_{b\dot{a}} \ . \tag{A.7}$$

Further relations, useful for analysing the $\mathbb{CP}^1$-derivative boundary conditions, are

$$2\alpha^a\sigma_{a\dot{a}}e^{\dot{a}}|_\alpha = \star(\mu_\alpha\wedge\sigma) \ , \quad \beta^a\tau_{a\dot{a}}e^{\dot{a}}|_\alpha = -\langle\alpha\beta\rangle P(\tau) \ , \tag{A.8}$$

$$2\beta^a\sigma_{a\dot{a}}e^{\dot{a}}|_\beta = \star(\mu_\beta\wedge\sigma) \ , \quad \alpha^a\tau_{a\dot{a}}e^{\dot{a}}|_\beta = \langle\alpha\beta\rangle \bar{P}(\tau) \ . \tag{A.9}$$

As an application of this projector technology let us consider the (ungauged) WZW_4 model, for which the equations of motion can be cast in terms of the right-invariant Maurer-Cartan form $R = \mathrm{d}gg^{-1}$, which obeys $\mathrm{d}R = R\wedge R$, as

$$\mathrm{d}\star\bar{P}(R) = \frac{1}{2}\mathrm{d}\left(\star - \omega_{\alpha,\beta}\wedge\right)\mathrm{d}gg^{-1} = 0 \ . \tag{A.10}$$

We now consider a Yang-Mills connection $A = -\bar{P}(X)$. The equations for this to be anti-self dual are

$$\mu_\beta\wedge F[A] = 0 \ , \qquad \mu_\alpha\wedge F[A] = 0 \ , \qquad \omega_{\alpha,\beta}\wedge F[A] = 0 \ . \tag{A.11}$$

The first of these vanishes identically by virtue of the fact that $\mu_\beta\wedge A = 0$. Since $\mu_\alpha\wedge A = -\mu_\alpha\wedge X$, the second yields a Bianchi identity

$$\mu_\alpha\wedge F[A] = -\mu_\alpha\wedge\left(\mathrm{d}X - X\wedge X\right) \ , \tag{A.12}$$

hence is solved by $X = R$. The final equation returns the equations of motion as

$$\omega_{\alpha,\beta}\wedge F[A] = -\mathrm{d}(\omega_{\alpha,\beta}\wedge\bar{P}(R)) + \omega_{\alpha,\beta}\wedge\bar{P}(R)\wedge\bar{P}(R) = \mathrm{d}\star\bar{P}(R) \ . \tag{A.13}$$

At the Kähler point $\beta = \hat{\alpha} = \hat{\gamma}$, we can simply write the ASDYM equations as

$$F^{2,0} = 0 \ , \qquad F^{0,2} = 0 \ , \qquad \omega\wedge F^{1,1} = 0 \ . \tag{A.14}$$

In this case, the connection given by $A = -\bar{\partial}gg^{-1}$ is of type $(0,1)$, hence $F^{2,0} = 0$ automatically, $F^{0,2} = 0$ is zero by the Bianchi identity and the equations of motion of

WZW_4 are

$$\omega \wedge \partial(\bar{\partial} g g^{-1}) = 0 \ . \tag{A.15}$$

Appendix B

Derivation of localisation formulae

In this work we are required to evaluate integrals of the form

$$I = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge Q \ , \qquad Q \in \Omega^{0,2}(\mathbb{PT}) \ . \tag{B.1}$$

In this appendix, we will derive general formulae for these integrals for the cases in which Ω has either two double poles or a single fourth-order pole. To compute these integrals efficiently we will work in inhomogeneous coordinates and make use of the identities

$$\partial_{\bar{\zeta}}\Big(\frac{1}{\zeta - \alpha}\Big) = 2\pi\mathrm{i}\, \delta^{(2)}(\zeta - \alpha) \ , \qquad \int_{\mathbb{CP}^1} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\, \delta^{(2)}(\zeta - \alpha)\, f(\zeta) = f(\alpha) \ . \tag{B.2}$$

Introducing an arbitrary spinor γ_a, the inhomogenous coordinate ζ may be related to the homogenous coordinate π_a by

$$\zeta = \frac{\langle \gamma\pi \rangle}{\langle \pi\hat{\gamma} \rangle} \ . \tag{B.3}$$

B.1 Two double poles

We consider the $(3,0)$-form given by

$$\Omega = \frac{1}{2}\frac{\langle \alpha\beta \rangle^2}{\langle \pi\alpha \rangle^2 \langle \pi\beta \rangle^2}\, e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}} = \frac{1}{2}\frac{(\alpha - \beta)^2}{(\zeta - \alpha)^2(\zeta - \beta)^2}\, \mathrm{d}\zeta \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}} \ . \tag{B.4}$$

Here we have introduced the inhomogenous one-forms, $\mathrm{d}\zeta$ and $\theta^{\dot{a}}$ defined by

$$\mathrm{d}\zeta = \frac{e^0}{\langle\pi\gamma\rangle^2}\,, \quad \text{and} \quad \theta^{\dot{a}} = \frac{e^{\dot{a}}}{\langle\pi\gamma\rangle}\,. \tag{B.5}$$

Substituting this into the integral gives

$$I = -\frac{1}{2}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\,\partial_{\bar{\zeta}}\left(\frac{(\alpha-\beta)^2}{(\zeta-\alpha)^2(\zeta-\beta)^2}\right)\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\,. \tag{B.6}$$

Then, using the identity (B.2) gives

$$I = -\frac{(\alpha-\beta)^2}{2}\int_{\mathbb{PT}} \mathrm{d}\zeta\wedge\mathrm{d}\bar{\zeta}\left[\frac{\partial_\zeta\delta^{(2)}(\zeta-\alpha)}{(\zeta-\beta)^2}+\frac{\partial_\zeta\delta^{(2)}(\zeta-\beta)}{(\zeta-\alpha)^2}\right]\wedge\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q\,. \tag{B.7}$$

Since the integral is symmetric under $\alpha\leftrightarrow\beta$ we will only compute the first term explicitly. Integrating by parts and evaluating the integral over $\mathbb{CP}^1$ gives

$$I = \frac{(\alpha-\beta)^2}{2}\int_{\mathbb{R}^4}\partial_\zeta\left(\frac{\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q}{(\zeta-\beta)^2}\right)\bigg|_\alpha \quad + \quad \alpha\leftrightarrow\beta\,. \tag{B.8}$$

We first distribute the ∂_ζ derivative, leaving the 2-form Q completely general, resulting in

$$\begin{aligned} I = \frac{(\alpha-\beta)^2}{2}\int_{\mathbb{R}^4}\Bigg[&\frac{-2}{(\zeta-\beta)^3}\theta^{\dot{a}}\wedge\theta_{\dot{a}}\wedge Q + \frac{2}{(\zeta-\beta)^2}\hat{\gamma}_a\mathrm{d}x^{a\dot{a}}\wedge\theta_{\dot{a}}\wedge Q \\ &+\frac{\theta^{\dot{a}}\wedge\theta_{\dot{a}}}{(\zeta-\beta)^2}\wedge\partial_\zeta Q\Bigg]\bigg|_\alpha \quad + \quad \alpha\leftrightarrow\beta\,. \end{aligned} \tag{B.9}$$

The overall factor of $(\alpha-\beta)^2$ outside the integral cancels with the denominators in the integrand. Returning to spinor notation and introducing the self-dual 2-forms defined by $\Sigma^{ab} = \varepsilon_{\dot{a}\dot{b}}\mathrm{d}x^{a\dot{a}}\wedge\mathrm{d}x^{b\dot{b}}$ to write

$$\begin{aligned} I = \frac{1}{2}\int_{\mathbb{R}^4}\Bigg[&\frac{-2\langle\hat{\gamma}\beta\rangle}{\langle\alpha\beta\rangle\langle\alpha\hat{\gamma}\rangle}\alpha_a\alpha_b\Sigma^{ab}\wedge Q|_\alpha + \frac{2}{\langle\alpha\hat{\gamma}\rangle}\hat{\gamma}_a\alpha_b\Sigma^{ab}\wedge Q|_\alpha \\ &+\alpha_a\alpha_b\Sigma^{ab}\wedge\frac{\partial_\zeta Q}{\langle\pi\hat{\gamma}\rangle^2}\bigg|_\alpha\Bigg] \quad + \quad \alpha\leftrightarrow\beta\,. \end{aligned} \tag{B.10}$$

Expanding α_a in the basis formed by $\hat{\gamma}_a$ and β_a, we see that one component of the first term cancels the second term, and only a term proportional to $\alpha_a\beta_b\Sigma^{ab}$ survives. In the third term of the integral, we recognise the combination ∂_0 acting on Q and make this replacement. In conclusion, we have the general formula

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\frac{\alpha_a\beta_b\Sigma^{ab}}{\langle\alpha\beta\rangle}\wedge Q|_\alpha+\frac{1}{2}\alpha_a\alpha_b\Sigma^{ab}\wedge(\partial_0 Q)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ ,\quad\text{(B.11)}$$

or in differential form notation

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\omega_{\alpha,\beta}\wedge Q|_\alpha+\frac{1}{2}\mu_\alpha\wedge(\partial_0 Q)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ .\qquad\text{(B.12)}$$

It is also helpful to specialise to 2-forms of the form $Q=\pi^a\pi^b Q_{a\dot{a}b\dot{b}}\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}$, which we will often encounter. In this case, we may make use of the identity

$$e^{\dot{c}}\wedge e_{\dot{c}}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\varepsilon^{\dot{a}\dot{b}}\ ,\qquad\text{(B.13)}$$

and its generalisation valid for any spinors α_a and β_a

$$\alpha_a\beta_b\Sigma^{ab}\wedge\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}=-2\,\mathrm{vol}_4\,\frac{\langle\alpha\hat{\pi}\rangle\langle\beta\hat{\pi}\rangle}{\langle\pi\hat{\pi}\rangle^2}\,\varepsilon^{\dot{a}\dot{b}}\ .\qquad\text{(B.14)}$$

Using these identities on the above formula for $Q=\pi^a\pi^b Q_{a\dot{a}b\dot{b}}\bar{e}^{\dot{a}}\wedge\bar{e}^{\dot{b}}$ gives

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=-\int_{\mathbb{R}^4}\mathrm{vol}_4\left[\frac{\varepsilon^{\dot{a}\dot{b}}(\alpha^a\beta^b+\beta^a\alpha^b)}{\langle\alpha\beta\rangle}Q_{a\dot{a}b\dot{b}}|_\alpha+\varepsilon^{\dot{a}\dot{b}}\alpha^a\alpha^b(\partial_0 Q_{a\dot{a}b\dot{b}})|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ .\qquad\text{(B.15)}$$

Finally, we specialise to the case when $Q_{a\dot{a}b\dot{b}}=X_{a\dot{a}}Y_{b\dot{b}}$, for which the answer can again be recast in differential form notation as

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge Q=\int_{\mathbb{R}^4}\left[\omega_{\alpha,\beta}\wedge X\wedge Y|_\alpha+\frac{1}{2}\mu_\alpha\partial_0\wedge(X\wedge Y)|_\alpha\right]\quad+\quad\alpha\leftrightarrow\beta\ .\quad\text{(B.16)}$$

To apply these formulae we also need the following $\mathbb{CP}^1$-derivatives:

$$\partial_0(\mathrm{d}\hat{g}\hat{g}^{-1})=\hat{g}\mathrm{d}\hat{u}\hat{g}^{-1}\ ,\qquad\text{(B.17)}$$

$$\partial_0(\hat{g}^{-1}\mathrm{d}\hat{g})=\mathrm{d}\hat{u}+[\hat{g}^{-1}\mathrm{d}\hat{g},\hat{u}]\ ,\qquad\text{(B.18)}$$

$$\partial_0(A)=\partial_0(B)=0\ ,\qquad\text{(B.19)}$$

$$\partial_0(\hat{g}^{-1}A\hat{g})=[\hat{g}^{-1}A\hat{g},\hat{u}]\ ,\qquad\text{(B.20)}$$

$$\partial_0\mathrm{Tr}(\hat{g}^{-1}\mathrm{d}\hat{g})^3=3\,\mathrm{d}\,\mathrm{Tr}(\hat{u}(\hat{g}^{-1}\mathrm{d}\hat{g})^2)\ ,\qquad\text{(B.21)}$$

where we have defined $\hat{u} = \hat{g}^{-1}\partial_0\hat{g}$.

B.2 Fourth-order pole

In section 4.5, we consider the $(3,0)$-form given by

$$\Omega = k\frac{e^0 \wedge e^{\dot{a}} \wedge e_{\dot{a}}}{\langle\pi\alpha\rangle^4} = \frac{k'}{\langle\hat{\gamma}\alpha\rangle^4}\frac{d\zeta \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}}}{(\zeta-\alpha)^4} \ . \tag{B.22}$$

Substituting this into the general integral expression above gives

$$I = -\frac{k}{\langle\hat{\gamma}\alpha\rangle^4}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{PT}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\,\partial_{\bar{\zeta}}\left(\frac{1}{(\zeta-\alpha)^4}\right) \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q \ . \tag{B.23}$$

Then, using the identity (B.2), we find

$$I = -\frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{PT}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}\left(\partial^3_\zeta\,\delta^{(2)}(\zeta-\alpha)\right) \wedge \theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q \ . \tag{B.24}$$

Integrating by parts and evaluating the integral over $\mathbb{CP}^1$ gives

$$I = \frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{R}^4} \partial^3_\zeta\left(\theta^{\dot{a}} \wedge \theta_{\dot{a}} \wedge Q\right)\Big|_\alpha \ . \tag{B.25}$$

In order to distribute the ∂_ζ derivatives, it is helpful to use the identities

$$\theta^a\Big|_\alpha = \frac{\alpha_a \mathrm{d}x^{a\dot{a}}}{\langle\hat{\gamma}\alpha\rangle} \ , \qquad \partial_\zeta\theta^a\Big|_\alpha = \hat{\gamma}_a\mathrm{d}x^{a\dot{a}} \ , \qquad \partial^2_\zeta\theta^a\Big|_\alpha = 0 \ . \tag{B.26}$$

Distributing the three ∂_ζ derivatives gives

$$I = \frac{k}{6\langle\hat{\gamma}\alpha\rangle^4}\int_{\mathbb{R}^4}\left[\frac{\alpha_a\alpha_b\Sigma^{ab}}{\langle\alpha\hat{\gamma}\rangle^2} \wedge \partial^3_\zeta Q\Big|_\alpha + 6\,\frac{\alpha_a\hat{\gamma}_b\Sigma^{ab}}{\langle\alpha\hat{\gamma}\rangle} \wedge \partial^2_\zeta Q\Big|_\alpha + 6\,\hat{\gamma}_a\hat{\gamma}_b\Sigma^{ab} \wedge \partial_\zeta Q\Big|_\alpha\right] \ . \tag{B.27}$$

Converting this expression back into homogeneous coordinates (and using the fact that Q is a $(0,2)$-form on twistor space hence $\hat{\alpha}_a\mathrm{d}x^{a\dot{a}} \wedge Q|_\alpha = 0$) this integral becomes

$$I = \frac{k}{6}\int_{\mathbb{R}^4} \alpha_a\alpha_b\Sigma^{ab} \wedge \partial^3_0 Q\Big|_\alpha \ . \tag{B.28}$$

Appendix C

Localisation derivation with general gaugings

In this appendix we describe in more detail the derivation of the gauged WZW_4 model from the gauged hCS_6 theory and the application of the localisation formulae in appendix B.1. We will do this in a more general manner, allowing the gauging of an H subgroup that acts as

$$g \mapsto \rho_\beta(\ell) g \rho_\alpha(\ell^{-1}) \ , \quad B \mapsto \ell B \ell^{-1} - \mathrm{d}\ell\ell^{-1} \ , \quad \ell \in H \subset G \ , \tag{C.1}$$

where $\rho_i : H \to G$ are group homomorphisms (algebra homomorphisms will be denoted by the same symbol). The covariant derivative is then given by

$$\nabla g g^{-1} = \mathrm{d} g g^{-1} + B_\beta - g B_\alpha g^{-1} \mapsto \rho_\beta(\ell)(\nabla g g^{-1})\rho_\beta(\ell^{-1}) \ , \tag{C.2}$$

in which we ease the notation by setting $B_i = \rho_i(B)$.

The starting point is the six-dimensional theory

$$S_{\mathrm{ghCS}_6}[\mathcal{A}, \mathcal{B}] = S_{\mathrm{hCS}_6}[\mathcal{A}] - S_{\mathrm{hCS}_6}[\mathcal{B}] + S_{\mathrm{bdry}}[\mathcal{A}, \mathcal{B}] \ , \tag{C.3}$$

where we take the boundary interaction term to be

$$S_{\mathrm{bdry}}[\mathcal{A}, \mathcal{B}] = -\frac{q}{2\pi \mathrm{i}} \int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \mathrm{Tr}_{\mathfrak{g}} \left(\mathcal{A} \wedge \rho(\mathcal{B}) \right) \ . \tag{C.4}$$

Here we have introduced a parameter q, which will ultimately be set to one, to keep track of the contributions from this boundary term. To specify this term we include an algebra homomorphism ρ that only needs to be defined piecewise on the components of the support of $\bar{\partial}\Omega$. We could choose to dispense with the higher-dimensional covariance and simply add different boundary terms specified only at the location of the poles, but it is convenient to formally consider ρ to be a defined as a piecewise map that takes values $\rho|_{\pi=\alpha,\beta} = \rho_{\alpha,\beta}$.

To define a six-dimensional theory requires imposing conditions that ensure the vanishing of the boundary variation

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left(\mathrm{Tr}_{\mathfrak{g}}\left(\delta\mathcal{A} \wedge (\mathcal{A} - q\rho(\mathcal{B})) + q\rho(\delta\mathcal{B}) \wedge \mathcal{A}\right) - \mathrm{Tr}_{\mathfrak{h}}(\delta\mathcal{B} \wedge \mathcal{B})\right) \ . \tag{C.5}$$

We are required to cancel a term involving the inner product on the algebra $\mathfrak{h}$ with one on $\mathfrak{g}$, which can be achieved by demanding

$$\mathrm{Tr}_{\mathfrak{g}}(\rho(x)\rho(y))|_{\alpha,\beta} = \mathrm{Tr}_{\mathfrak{h}}(xy) \quad \forall x, y \in \mathfrak{h} \ . \tag{C.6}$$

Note that as a consequence this implies

$$\mathrm{Tr}_{\mathfrak{g}}(\rho_\alpha(x)\rho_\alpha(y)) = \mathrm{Tr}_{\mathfrak{h}}(xy) = \mathrm{Tr}_{\mathfrak{g}}(\rho_\beta(x)\rho_\beta(y)) \ , \tag{C.7}$$

which is the familiar anomaly-free condition required to construct a gauge-invariant extension to the WZW model with the gauge symmetry (C.1). With this condition satisfied, the boundary term produced by variation is given by

$$\int_{\mathbb{PT}} \bar{\partial}\Omega \wedge \left(\mathrm{Tr}_{\mathfrak{g}}\left(\delta\mathcal{A} \wedge (\mathcal{A} - q\rho(\mathcal{B})) + q\rho(\delta\mathcal{B}) \wedge (\mathcal{A} - q^{-1}\rho(\mathcal{B}))\right)\right) \ , \tag{C.8}$$

and is set to zero by the conditions

$$\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0 \ , \qquad \mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathcal{B})\big|_{\alpha,\beta} \ , \qquad \partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\partial_0\mathcal{B})\big|_{\alpha,\beta} \ . \tag{C.9}$$

If we impose all of these conditions from the outset, the contribution from the explicit boundary term $S_{\text{bdry}}[\mathcal{A}, \mathcal{B}]$ would vanish. However, from a four-dimensional perspective the $\mathbb{CP}^1$-derivative boundary conditions lead to constraints relating derivatives of the fundamental fields to the 4-dimensional gauge field B that comes from $\mathcal{B}$. While these can be formally solved for B, our aim is to construct a gauged IFT$_4$ with a gauge field. Therefore, we only impose the conditions $\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta} = 0$ and $\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta} = \rho(\mathcal{B})\big|_{\alpha,\beta}$, which can be solved for the 4-dimensional gauge field A that comes from $\mathcal{A}$ and substituted into the Lagrangian without concern. Doing this, we find that $S_{\text{bdry}}[\mathcal{A},\mathcal{B}]$ does contribute, and when $q=1$ in particular, it provides a gauge invariant completion of the action. Importantly, the $\mathbb{CP}^1$-derivative boundary conditions that we have not imposed have not been forgotten, instead when $q=1$ they are recovered as on-shell equations in this four-dimensional theory. This provides an alternative view of the procedure; when $q=1$ the explicit boundary term (C.4) is serving to implement the constraints arising from $\partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta}=\rho(\partial_0\mathcal{B})\big|_{\alpha,\beta}$ at the Lagrangian level. We can see this explicitly by observing that if we just impose $\mathcal{A}^{\mathfrak{k}}\big|_{\alpha,\beta}=0$ and $\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta}=\rho(\mathcal{B})\big|_{\alpha,\beta}$ then

$$\begin{aligned}&\Big(\delta\mathcal{A}\wedge(\mathcal{A}-q\rho(\mathcal{B}))+q\rho(\delta\mathcal{B})\wedge\mathcal{A}-\rho(\delta\mathcal{B})\wedge\rho(\mathcal{B})\Big)|_{\alpha,\beta}=0\ ,\\&\partial_0\Big(\delta\mathcal{A}\wedge(\mathcal{A}-q\rho(\mathcal{B}))+q\rho(\delta\mathcal{B})\wedge\mathcal{A}-\rho(\delta\mathcal{B})\wedge\rho(\mathcal{B})\Big)|_{\alpha,\beta}\\&\qquad=(1-q)\delta(\partial_0\mathcal{A}-\rho(\partial_0\mathcal{B}))\wedge\rho(\mathcal{B})|_{\alpha,\beta}+(1+q)\rho(\delta\mathcal{B})\wedge(\partial_0\mathcal{A}-\rho(\partial_0\mathcal{B}))|_{\alpha,\beta}\ .\end{aligned}\tag{C.10}$$

Therefore, for $q=1$ we see that the boundary equations of motion for $\mathcal{B}$ are precisely the $\mathbb{CP}^1$-derivative boundary conditions $\partial_0\mathcal{A}^{\mathfrak{h}}\big|_{\alpha,\beta}=\rho(\partial_0\mathcal{B})\big|_{\alpha,\beta}$.

The localisation proceeds as follows. First, we change parametrisation $\mathcal{A}=\mathcal{A}'^{\hat{g}}$ and $\mathcal{B}=\mathcal{B}'^{\hat{h}}$ fixing some of the redundancy by demanding that $\mathcal{A}'$ and $\mathcal{B}'$ have no $\mathbb{CP}^1$ legs. Second, we fix some of the residual symmetry preserved by the boundary conditions to set $\hat{g}|_\beta=\hat{h}|_{\alpha,\beta}=\text{id}$ and $\partial_0\hat{h}|_{\alpha,\beta}=0$. The remaining fields are $\hat{g}|_\alpha=g$, $\hat{g}^{-1}\partial_0\hat{g}|_\alpha=u$, $\hat{g}^{-1}\partial_0\hat{g}|_\beta=\tilde{u}$ and the four-dimensional gauge fields A and B that arise from $\mathcal{A}'$ and $\mathcal{B}'$ once their holomorphicity is imposed.

We may now directly apply the localisation formulae (B.16) to show that the hCS$_6$ terms localise, before imposing boundary conditions, to give

$$\begin{aligned}S_{\text{hCS}_6}[\mathcal{A}]\simeq\int_{\mathbb{R}^4}&\omega_{\alpha,\beta}\wedge\text{Tr}_{\mathfrak{g}}(A^g\wedge g^{-1}\text{d}g)-\omega_{\alpha,\beta}\wedge\mathcal{L}_{\text{WZ}}[g]\\&+\frac{1}{2}\mu_\alpha\wedge\text{Tr}_{\mathfrak{g}}(A^g\wedge\text{d}u)+\frac{1}{2}\mu_\beta\wedge\text{Tr}_{\mathfrak{g}}(A\wedge\text{d}\tilde{u})\ ,\end{aligned}\tag{C.11}$$

while $S_{\text{hCS}_6}[\mathcal{B}]$ yields zero in this gauge. Let us first consider the terms involving $\omega_{\alpha,\beta}$. Since the gauge completion of the WZ term is

$$\mathcal{L}_{\text{gWZ}}[g,B]=\mathcal{L}_{\text{WZ}}[g]+\text{Tr}_{\mathfrak{g}}\left(g^{-1}\text{d}g\wedge B_\alpha+\text{d}gg^{-1}\wedge B_\beta+g^{-1}B_\beta gB_\alpha\right)\ ,\tag{C.12}$$

we may express them (trace implicit) as

$$\begin{aligned} &\omega_{\alpha,\beta}\wedge\big(A^g\wedge g^{-1}\mathrm{d}g-\mathcal{L}_{\mathrm{WZ}}[g]\big)\\ =\ &\omega_{\alpha,\beta}\wedge\big(A^g\wedge g^{-1}\mathrm{d}g-\mathcal{L}_{\mathrm{gWZ}}[g,B]+g^{-1}\mathrm{d}g\wedge B_\alpha+\mathrm{d}gg^{-1}B_\beta+g^{-1}B_\beta gB_\alpha\big)\\ =\ &\omega_{\alpha,\beta}\wedge\big((A^g-B_\alpha)\wedge g^{-1}\nabla g-\mathcal{L}_{\mathrm{gWZ}}[g,B]+A^g\wedge B_\alpha-A\wedge B_\beta\big)\ . \end{aligned} \tag{C.13}$$

In differential form notation, the algebraic boundary conditions of eq. (C.9) become

$$A=B_\beta-\bar{P}(\nabla gg^{-1})\ ,\qquad A^g=P(g^{-1}\nabla g)+B_\alpha\ . \tag{C.14}$$

It follows that

$$\begin{aligned} &\omega_{\alpha,\beta}\wedge\big(A^g\wedge g^{-1}\mathrm{d}g-\mathcal{L}_{\mathrm{WZ}}[g]\big)\\ =\ &\omega_{\alpha,\beta}\wedge\big(P(g^{-1}\nabla g)\wedge g^{-1}\nabla g-\mathcal{L}_{\mathrm{gWZ}}[g,B]+A^g\wedge B_\alpha-A\wedge B_\beta\big)\\ =\ &-\frac{1}{2}g^{-1}\nabla g\wedge\star(g^{-1}\nabla g)-\omega_{\alpha,\beta}\wedge\left(\mathcal{L}_{\mathrm{gWZ}}[g,B]-A^g\wedge B_\alpha+A\wedge B_\beta\right)\ . \end{aligned} \tag{C.15}$$

Here, in the last line, we made use of the identity $\omega\wedge P(\sigma)\wedge\sigma=-\frac{1}{2}\sigma\wedge\star\sigma$ for a 1-form σ. To treat the terms involving μ_α and μ_β we combine the algebraic boundary conditions (C.14) with the properties $\mu_\alpha\wedge P(X)=\mu_\beta\wedge\bar{P}(X)=0$ such that $\mu_a\wedge A^g=\mu_\alpha B_\alpha$ and $\mu_\beta\wedge A=\mu_\beta B_\beta$. In summary, we find

$$\begin{aligned} S_{\mathrm{hCS}_6}[\mathcal{A}]\simeq\int_{\mathbb{R}^4}-\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}\left(g^{-1}\nabla g\wedge\star g^{-1}\nabla g\right)-\omega_{\alpha,\beta}\wedge\left(\mathcal{L}_{\mathrm{gWZ}}[g,B]+\mathrm{Tr}_{\mathfrak{g}}(A\wedge B_\beta-A^gB_\alpha)\right)\\ +\frac{1}{2}\mu_\alpha\wedge\mathrm{Tr}(B_\alpha\wedge\mathrm{d}u)+\frac{1}{2}\mu_\beta\wedge\mathrm{Tr}(B_\beta\wedge\mathrm{d}\tilde{u})\ . \end{aligned} \tag{C.16}$$

The localisation of the explicit boundary term yields, after using $\mu_a\wedge A^g=\mu_\alpha B_\alpha$,

$$\begin{aligned} S_{\mathrm{bdry}}[\mathcal{A},\mathcal{B}]\simeq -\,q\int_{\mathbb{R}^4}\omega_{\alpha,\beta}\wedge\mathrm{Tr}_{\mathfrak{g}}(A^gB_\alpha-AB_\beta)\\ +\frac{1}{2}\mu_\alpha\wedge\mathrm{Tr}_{\mathfrak{g}}((\mathrm{d}u+[B_\alpha,u])B_\alpha)+\frac{1}{2}\mu_\beta\wedge\mathrm{Tr}_{\mathfrak{g}}((\mathrm{d}\tilde{u}+[B_\beta,\tilde{u}])B_\beta)\ . \end{aligned} \tag{C.17}$$

The significance of the boundary term now becomes clear. It serves to ensure manifest gauge invariance when we do not impose the $\mathbb{CP}^1$-derivative boundary conditions. When $q=1$ the terms $\omega_{\alpha,\beta}\wedge\mathrm{Tr}(A^gB_\alpha-AB_\beta)$ directly cancel. The contributions of the entire localised action that are wedged against μ_α sum to

$$\mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}} \left((1-q)\ \mathrm{d}u \wedge B_\alpha + 2q\ uF[B]_\alpha - 2q\ \mathrm{d}(B_\alpha u) \right) . \tag{C.18}$$

We see that for $q = 1$ we find a gauge-invariant field strength together with a total derivative term that we discard. The terms wedged against μ_β give a similar contribution. Hence the fully localised action becomes

$$\begin{aligned} S \simeq \int_{\mathbb{R}^4} -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}} \left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g \right) - \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}[g,B] \\ + \mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{g}}(uF[B]_\alpha) + \mu_\beta \wedge \mathrm{Tr}_{\mathfrak{g}}(\tilde{u}F[B]_\beta) . \end{aligned} \tag{C.19}$$

Noting that the components of u and $\tilde{u}$ in the complement of $\mathfrak{h}$ decouple, we can view u and $\tilde{u}$ as $\mathfrak{h}$-valued and write

$$\begin{aligned} S \simeq \int_{\mathbb{R}^4} -\frac{1}{2}\mathrm{Tr}_{\mathfrak{g}} \left(g^{-1}\nabla g \wedge \star g^{-1}\nabla g \right) - \omega_{\alpha,\beta} \wedge \mathcal{L}_{\mathrm{gWZ}}[g,B] \\ + \mu_\alpha \wedge \mathrm{Tr}_{\mathfrak{h}}(uF[B]) + \mu_\beta \wedge \mathrm{Tr}_{\mathfrak{h}}(\tilde{u}F[B]) . \end{aligned} \tag{C.20}$$

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