On Fourier–Jacobi Dirichlet Series for Hermitian and Orthogonal Modular Forms

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Analytic Properties and Euler Products

Rafail Psyroukis

A Thesis presented for the degree of
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Department of Mathematical Sciences
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Abstract: This thesis is concerned with the study of analytic and arithmetic properties of Dirichlet series involving Fourier-Jacobi coefficients of Hermitian and orthogonal modular forms. It is naturally divided into two main parts.

In the first part, motivated by a work of B. Heim, we consider a Dirichlet series associated with three Hermitian cuspidal eigenforms of degrees $2, 2$ and 1 over $\mathbb{Q}(i)$ and study its p-factor for every rational prime p. Using factorisation methods in parabolic Hecke rings, we show that for inert primes, this factor can be identified with the GL_2-twist of the degree 6 Euler factor attached to a Hermitian modular form of degree 2 by Gritsenko. For split primes, we obtain a rational expression for the local factor, allowing us to show that the Dirichlet series has an Euler product. Moreover, we show that this Dirichlet series arises as part of a Rankin-Selberg integral representation.

In the second part, we consider, in the spirit of Kohnen and Skoruppa, a Dirichlet series involving the Fourier-Jacobi coefficients of a pair of orthogonal modular forms of real signature $(2, n+2)$, $n \geq 1$. First, we obtain an integral representation of Rankin-Selberg type and use theta correspondence to deduce its analytic properties for certain orthogonal groups. Next, using results of Sugano and Shimura, we obtain, for certain orthogonal groups, an Euler product for the Dirichlet series and relate it to the standard L-function for $\mathrm{SO}(2, n+2)$.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Parts of this thesis have been published/accepted for publication in journals or are available as preprints in public repositories – the relevant publications are listed below.

- T. Bouganis and R. Psyroukis. "On a Rankin-Selberg integral of three Hermitian cusp forms". *arXiv Preprint* (2025) [Both authors contributed equally to this work].
- R. Psyroukis. "A Fourier-Jacobi Dirichlet series for cusp forms on orthogonal groups". In: *Research in Number Theory* 90.11 (2025)
- R. Psyroukis. "Analytic Properties of an orthogonal Fourier-Jacobi Dirichlet Series". *arXiv Preprint* (2024)

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Contents

Chapter 1

Introduction

1.1 Background and Motivation

L-functions have been a subject of extensive study in the literature. In order to define what exactly we mean by the term L-function, we need the notion of a **Dirichlet series**.

Definition 1.1.1. A Dirichlet series is a series of the form

$$\sum_{n\geq 1} a_n n^{-s},$$

where $s \in \mathbb{C}$ and a_n is a series of complex numbers, which grow at most polynomially as $n \to \infty$. This growth condition is there so that there is some $c > 0$, so that this series converges absolutely and uniformly on compact subsets and thus defines a holomorphic function in the right half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > c\}$.

When dealing with such series, we are interested in two main properties:

- The possibility of the series having a **meromorphic continuation** to the complex plane and admitting a **functional equation**.
- The possibility of the series having an **Euler product** expansion, i.e., if it can be written as an infinite product of some factors over (some) primes.

In the case when a Dirichlet series has the above two properties, it is usually referred to as an **L-function** (see, for example [Bum97, p. 1]).

The most common example of an L-function is the **Riemann zeta function**:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

This converges absolutely and uniformly on compact subsets of $\mathbb{C}$ for $\mathrm{Re}(s) > 1$. It is a well-known result that $\zeta(s)$ admits a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$. Moreover, if $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$, then

$$\xi(s) = \xi(1-s).$$

Finally, from the Fundamental Theorem of Arithmetic, $\zeta(s)$ has an Euler product expansion of the form

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$

Another important class of L-functions is the L-functions attached to **classical modular forms**, which are also Hecke eigenforms. Let f be a cusp form of integer weight $k \geq 0$ and level $\mathrm{SL}_2(\mathbb{Z})$ and consider its Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

We then define its L-function as

$$L(f, s) := \sum_{n=1}^{\infty} a_n n^{-s}. \tag{1.1.1}$$

This converges absolutely and uniformly on compact subsets of the usual upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ for $\mathrm{Re}(s) > 1 + k/2$ (cf. [DS06, Proposition 5.9.1]). The following Theorem is classical (e.g., see [DS06, Theorems 5.9.2, 5.10.2]).

Theorem 1.1.2. *Assume f is a normalised (i.e. $a_1 = 1$) Hecke eigenform of weight $k \geq 0$. Then $L(f, s)$ admits a meromorphic continuation to the complex plane and if $\Lambda(f, s) := (2\pi)^{-s}\Gamma(s)L(f, s)$, we have*

$$\Lambda(f, s) = i^k \Lambda(f, k - s).$$

Moreover, $L(f, s)$ has an Euler product expansion of the form

$$L(f, s) = \prod_{p \text{ prime}} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

The analytic properties of $L(f, s)$ are therefore established due to the above Theorem.

Another key aspect of studying L-functions is their values at critical points, called **special values**. In particular, assume we complete some L-function $L(s)$ by some Gamma factors, so that the completed L-function satisfies a functional equation $s \longmapsto k - s$. The critical points are the set of integers m for which both the gamma factors of $L(s)$ and $L(k - s)$ **do not** have a pole at $s = m$. Deligne in [Del79] conjectured that these special values are algebraic up to certain prescribed transcendental periods. For the case of classical modular forms, results on the algebraicity of the special values of their L-functions have been established by the work of Shimura in [Shi76] and [Shi77].

Nevertheless, the subject matter of this thesis is modular forms of **higher rank**. The most natural (both from an analytic and an arithmetic point of view) such objects are the so-called **Siegel modular forms**. As a rough description, Siegel modular forms of degree $n \geq 1$ are holomorphic functions from a generalised upper half plane, consisting of $n \times n$ complex matrices, to the complex numbers, that satisfy a modularity condition under the action of the symplectic group $\mathrm{Sp}_n(\mathbb{Z})$. Siegel discovered these, motivated by his classical investigations on the problem of integral representations of quadratic forms (e.g. [Sie35]), and they have numerous links with arithmetic.

Analogous to classical modular forms but with key differences, one can also attach L-functions to Siegel modular forms of any degree $n \geq 1$, which are Hecke eigenforms for specific Hecke algebras. There are two main L-functions associated with a cuspidal Siegel eigenform: the standard and the spin L-function. The standard is well-understood due to the so-called **doubling method** (see Section 3.1 for a discussion). The spin L-function, however, has been much more difficult to study. This perhaps comes as a surprise, as this is the formal analogue of (1.1.1) for Siegel modular forms. Remarkably, for $n \geq 4$, its meromorphic continuation and functional equation are still open conjectures. For $n = 3$, they were only proven recently by A. Pollack in [Pol17], subject to a certain non-vanishing condition on a Fourier coefficient. This was established by S. Böcherer and S. Das in [BD22].

The degree 2 case, however, is much more approachable. The following classical result can be used to establish the analytic properties of the spin L-function.

Result 1. *(Kohnen and Skoruppa, [KS89]) Given two degree 2 Siegel cusp forms F, G of integral weight k, with Fourier-Jacobi coefficients $\{\phi_m\}_{m=1}^{\infty}, \{\psi_m\}_{m=1}^{\infty}$, let*

$$D_{F,G}(s) := \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle m^{-s}, \ \mathrm{Re}(s) \gg 0, \tag{1.1.2}$$

where $\langle\ ,\ \rangle$ denotes an inner product on the space of Fourier-Jacobi forms of weight k and index m. The authors prove two main theorems:

Theorem 1.1.3. *The function*

$$D^*_{F,G}(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s - k + 2)D_{F,G}(s)$$

has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \longmapsto 2k - 2 - s$.

Theorem 1.1.4. *If F is a Hecke eigenform and G is in the Maass space, then*

$$D_{F,G}(s) = \langle \phi_1, \psi_1 \rangle Z_F(s),$$

*where $Z_F(s)$ is the **spin L-function** attached to F.*

The analytic properties of $Z_F(s)$ therefore follow from the above two theorems, subject to the condition $\phi_1 \not\equiv 0$. This condition is always satisfied, as Manickam in [Man21] has recently shown. This result, however, cannot be used to obtain algebraicity properties of $Z_F(s)$, mainly because $D_{F,G}(s)$ admits an integral representation of Rankin-Selberg type with an Eisenstein series of zero weight, and hence non-holomorphic (see [KS89, Theorem 1]).

There are other integral representations of the spin L-function attached to a degree 2 Siegel cuspidal eigenform. In fact, Andrianov in [And74] was the first one to give such a representation, using factorisation methods in parabolic Hecke rings. However, his result cannot be used (or at least it is not known how) to obtain algebraicity results either. The difficulty in this case seems to be related to the fact that the integral representation involves Eisenstein series defined over symmetric spaces which do not have the structure of a Shimura variety.

The closest algebraic result is the following, due to B. Heim.

Result 2. *(Heim, [Hei99]) Let F, G be two degree 2 Siegel cuspidal eigenforms and h a classical normalised Hecke eigenform, all having the same even weight $k \geq 0$. Assume F is the Maass lift of a classical modular form f of weight $2k-2$, and let*

$$D_{F,G,h}(s) := \sum_{\substack{m,\epsilon,\ell=1 \\ (\epsilon,\ell)=1}}^{\infty} \langle \phi_m \mid U_\ell,\ \psi_{m\ell^2} \rangle a_{m\epsilon^2} \epsilon^{-2(k+s-1)} \ell^{-2(k+s-2)} m^{-(2k+s-3)}. \tag{1.1.3}$$

Here, $\{\phi_m\}, \{\psi_m\}$ are the Fourier-Jacobi coefficients of F, G respectively, $\{a_m\}$ the Fourier coefficients of h, and U_ℓ is an index-raising operator acting on Fourier-Jacobi forms. Heim proved the following two Theorems.

Theorem 1.1.5.

$$D_{F,G,h}(s) = \langle \phi_1, \psi_1 \rangle \zeta(2s+k-2)^{-1} L(f, 2s+2k-3)^{-1} Z_{G\otimes h}(s),$$

*where $Z_{G\otimes h}(s)$ is the **twist** of the spin L-function attached to G by the Satake parameters of h, and $L(f,s)$ the classical L-function attached to f.*

Theorem 1.1.6. *Let $E^k_{5,0}(Z,s)$ denote the weight k and degree 5 Eisenstein series of Siegel type. Then*

$$\langle\langle\langle E^k_{5,0}(\mathrm{diag}[z_1,z_2,z_3]), F(z_1)\rangle, G(z_2)\rangle, h(z_3)\rangle = b(2,s,k) L(F, 2s+k-2) D_{F,G,h}(s),$$

*where $b(2,s,k)$ is a product of zeta and gamma factors, $L(F,s)$ is the **standard** L-function attached to F and $\langle\ ,\ \rangle$ denotes the inner product on the space of Siegel modular forms of certain weight and degree.*

Note that in contrast to Result 1, here we take F instead of G to be in the Maass space. We decided to use this notation throughout the thesis as this is compatible with the way the results in [KS89] and [Hei99] are formulated.

To prove Theorem 1.1.5, Heim's approach was to use factorisation methods in parabolic Hecke rings, in the same spirit as Andrianov. By combining Theorems 1.1.5 and 1.1.6, he managed to connect a seemingly more complicated object, $Z_{G\otimes h}$, with $E^k_{5,0}$ (again subject to $\psi_1 \not\equiv 0$). The presence of the Eisenstein series $E^k_{5,0}$, which is holomorphic at $s = 0$, allows one to study algebraicity properties of the special values of $Z_{G\otimes h}$. B. Heim and S. Böcherer exploited this integral representation to lift various restrictions on the weights of the modular forms considered ([BH00]) and prove parts of Deligne's conjectures ([BH06]).

We should note here that Furusawa in [Fur93] also gave an integral representation for the twisted spinor L-function, using an Eisenstein series over a unitary group and its restriction to the symplectic group of degree two. There is a series of works based on this idea, most notably by Saha in [Sah09], generalising the work of Furusawa.

In this thesis, our aim is twofold:

1) To generalise **Result 2** to the case of **Hermitian modular forms** of degree 2 over $\mathbb{Q}(i)$. Shortly after the work of Andrianov on the spinor L-function in [And74], Gritsenko, in a series of papers, extended Andrianov's approach of the use of parabolic Hecke algebras to the study of a **degree 6 L-function** attached to a cuspidal Hermitian eigenform of degree 2, where the underlying imaginary quadratic field is taken to be the field of Gaussian numbers $K := \mathbb{Q}(i)$. Indeed, in [Gri88b], Gritsenko first defined such an L-function, and in the later work of [Gri92a], he obtained the analogue construction of Kohnen and Skoruppa using the factorisation approach. Both integral representations allowed him to obtain a functional equation and study the analytic properties. However, as in the case of the symplectic group, neither of the above integral representations could be used to derive algebraicity properties, due to the Eisenstein series involved (only of real analytic nature). In Chapter 3, we consider the exact Hermitian analogue of the Dirichlet series (1.1.3) and study its arithmetic and analytic properties. Our aim is to demonstrate a connection with the **twisted Gritsenko's L-function**. The possibility of obtaining algebraicity results for this L-function has been the main motivation for this work.

2) To extend **Result 1** to the case of **orthogonal modular forms** of real signature $(2, n+2)$, $n \geq 1$. In particular, in Chapter 5, we investigate the analytic properties of the Dirichlet series, and in Chapter 6, we consider the question of obtaining Euler products. Our aim is to show that the Dirichlet series has good analytic properties and demonstrate a connection with the standard L-function for the orthogonal group.

This work is motivated by the existence of accidental isogenies between orthogonal groups of low rank and classical groups, e.g., $\mathrm{SO}(2,3)$ and Sp_2 and $\mathrm{SO}(2,4)$ and $\mathrm{U}(2,2)$. Such isogenies make it possible to relate modular forms for classical groups with orthogonal modular forms, and also obtain a correspondence between their L-functions (see, for example, [Shi04, p. 241] for a discussion in the case of Siegel modular forms of degree 2). Therefore, the results of Kohnen and Skoruppa in [KS89] and Gritsenko in [Gri92a] make this question natural to consider.

1.2 Statement of Main Results

Naturally, this thesis is divided into two main parts, namely the one addressing the Hermitian case and the one addressing the orthogonal case. We will now state our main results.

1.2.1 Hermitian Case

The setting is as follows. Assume F, G, h are Hermitian cuspidal eigenforms over $\mathbb{Q}(i)$ of degrees $2, 2, 1$ respectively, all having weight $k \equiv 0 \pmod 4$ and real Fourier coefficients. We consider the exact Hermitian analogue of (1.1.3), namely:

$$D_{F,G,h}(s) := \sum_{p,q} \sum_{m=1}^{\infty} \langle \phi_m \mid U_p, \psi_{mN(p)} \rangle a_{mN(q)} N(p)^{-(k+s-3)} N(q)^{-(k+s-1)} m^{-(2k+s-4)}, \tag{1.2.1}$$

where $p, q \in \mathbb{Z}[i]$ coprime, $\{\phi_m\}, \{\psi_m\}$ are the Fourier-Jacobi coefficients of F, G, $\{a_m\}$ the Fourier coefficients of h, U_p is a certain operator acting on Fourier-Jacobi forms, and $N(z) := z\overline{z}$. By assuming now that F is the Maass lift of a classical modular form f of weight $k-1$ and of a certain character (see Proposition 2.5.5), we obtain the following Theorem.

Theorem 1.2.1. *If p is an inert prime, we have for the p-factor (i.e. taking all summations over the rational prime p, see* (3.3.4)*)*

$$D^{(p)}_{F,G,h}(s) = \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1 \rangle_{\mathcal{A}} L_p(f, k+s-2) L_p\left(f, k+s-2, \left(\frac{-4}{p}\right)\right)(1 - p^{-k-2s+2})}{Q^{(2)}_{p,G}(X_1) Q^{(2)}_{p,G}(X_2)},$$

where X_i depend on the Satake parameters of h and $Q^{(2)}_{p,G}$ are the Euler factors of $Z_G(s)$, Gritsenko's L-function (see also Definition 2.5.2). Also, $L_p(f, s)$ denotes the p-factor of the L-function attached to f (see Definition 2.5.4), and $\langle\ ,\ \rangle_{\mathcal{A}}$ denotes the inner product of Definition 2.2.10.

Moreover, we make considerable progress in connecting $D^{(p)}_{F,G,h}(s)$ with the p-factor of $Z_{G\otimes h}(s)$, the twisted Gritsenko's L-function, for the case of the split primes p. Our progress is summarised in **Theorem 3.4.21**. Along the way, we prove several important results. In particular, in Theorem 3.4.3, we give the factorisation of the p-Euler factor of the standard L-function in the parabolic Hecke ring $H_p^{1,1}$ (see Section 2.4), deduce the rationality Proposition 3.4.8, and also obtain important relations between Hecke operators in the parabolic Hecke ring $H_p^{1,1}$ (see Table 3.1).

In fact, it is those results that allow us to prove in **Theorem 3.5.1** that the Dirichlet series $D_{F,G,h}(s)$ possesses an Euler product.

Theorem 1.2.2. *$D_{F,G,h}(s)$ possesses an Euler product of its p-factors (see* (3.4.6) *for a definition in the case of split primes).*

Finally, in Section 3.6, we aim to obtain an integral representation for $D_{F,G,h}(s)$. In **Theorem 3.6.5**, we show that $D_{F,G,h}(s)$ originates as part of a Rankin-Selberg integral with an Eisenstein series of Klingen type (see Definition 2.1.6).

Theorem 1.2.3. *For $k + 2\mathrm{Re}(s) > 10$, we have*

$$\left\langle\left\langle E^k_{3,2}\left(\begin{pmatrix} W & 0 \\ 0 & Z\end{pmatrix}, F; s\right), G(Z)\right\rangle, h(W)\right\rangle = (4\pi)^{-(2k+s-4)}\times$$
$$\times\,\frac{\Gamma(2k+s-4)\Gamma(k+s-3)\Gamma(k+s-1)}{\Gamma(2k+2s-4)} D_{F,G,h}(s) + R_{F,G,h}(s),$$

where $R_{F,G,h}(s)$ is an additional residue term (see Theorem 3.6.5).

In particular, by using the doubling method for the unitary group (see (3.6.1)), we can obtain a Theorem analogous to Theorem 1.1.6, involving, of course, $R_{F,G,h}(s)$. This additional term is very interesting and is special to our setting. It is related to the fact that not every vector is isotropic with respect to a Hermitian bilinear form. In particular, it does not appear in Heim's work. However, we do not study it in this thesis, and we hope it will be the subject matter of an upcoming work.

1.2.2 Orthogonal Case

For the second part of the thesis, the setting is as follows. Let S denote an even symmetric positive definite matrix of rank $n \geq 1$. Even here means $S[x] \in 2\mathbb{Z},\ \forall x \in \mathbb{Z}^n$. We then set

$$S_0 := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix},\ S_1 := \begin{pmatrix} & & 1 \\ & S_0 & \\ 1 & & \end{pmatrix}$$

of real signature $(1, n+1)$ and $(2, n+2)$ respectively. If now K is a field containing $\mathbb{Q}$, we define the corresponding special orthogonal groups of K-rational points via

$$G_K^* := \{g \in \mathrm{SL}_{n+2}(K) \mid g^t S_0 g = S_0\},\ G_K := \{g \in \mathrm{SL}_{n+4}(K) \mid g^t S_1 g = S_1\}.$$

Let $G_{\mathbb{R}}^0$ denote the connected component of the identity in $G_{\mathbb{R}}$. There is a well-defined action of $G_{\mathbb{R}}^0$ on a suitable tube domain, which we will call $\mathcal{H}_S \subset \mathbb{C}^{n+2}$. Let also $\Gamma_S := G_{\mathbb{R}}^0 \cap \mathrm{Mat}_{n+4}(\mathbb{Z})$.

Consider now two orthogonal cusp forms $F, G : \mathcal{H}_S \longrightarrow \mathbb{C}$ with respect to Γ_S with Fourier-Jacobi coefficients $\{\phi_m\}, \{\psi_m\}$ respectively. The object of interest is then

$$\mathcal{D}_{F,G}(s) := \sum_{m \geq 1} \langle \phi_m, \psi_m \rangle m^{-s},\ \mathrm{Re}(s) \gg 0,$$

where $\langle\ ,\ \rangle$ is a suitable inner product defined on the space of Fourier-Jacobi forms of certain weight and (lattice) index.

In Chapter 5, our aim is to obtain the analytic properties of $\mathcal{D}_{F,G}(s)$, i.e. its meromorphic continuation to $\mathbb{C}$ and a functional equation. Through an orthogonal Eisenstein series of Klingen-type $E(W, s)$ (see Definition 5.1.3), we obtain the following integral representation.

Proposition 1.2.4. *For $W \in \mathcal{H}_S$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n+2$, we have*

$$\langle F(W)E(W,s), G(W)\rangle = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})}(4\pi)^{-(s+k-n-1)}\Gamma(s+k-n-1)\mathcal{D}_{F,G}(s+k-n-1),$$

where $\langle\ ,\ \rangle$ denotes the inner product of Definition 4.2.3 and $\mathrm{SO}(S;\mathbb{Z})$ is the finite integral orthogonal group of S.

Therefore, the analytic properties of $\mathcal{D}_{F,G}(s)$ reduce to the ones of the Eisenstein series. Our aim is to produce an explicit **theta-correspondence** between $E(W, s)$ and an Eisenstein series of Siegel type for Sp_2. The first step towards that is to write $E(W, s)$ in the form of an Epstein zeta function. We are able to do this when the underlying lattice has one 1-dimensional cusp (see Definition 5.2.1).

Proposition 1.2.5. *Let S be such that Γ_S has only one 1-dimensional cusp. Then, for each $W \in \mathcal{H}_S$, there is a R_W in the space of majorants (see Definition 5.2.2) such that*

$$E(W,s) = \sum_{\ell \in X/\mathrm{GL}_2(\mathbb{Z})} \left(\det(R_W[\ell])\right)^{-s/2},$$

where

$$X := \left\{\begin{pmatrix} l & m \end{pmatrix} \mid l, m \in \mathbb{Z}^{n+4},\ \begin{pmatrix} l & m \end{pmatrix} \text{ primitive},\ S_1\left[\begin{pmatrix} l & m \end{pmatrix}\right] = 0\right\}.$$

Here, a matrix being primitive means that its elementary divisors are all 1*.*

We now consider the real-analytic theta series $\Theta(Z, W)$, as defined in Definition 5.3.3 ($Z \in \mathbb{H}_2$, Siegel's upper half-space). This transforms with weight $k = -n/2$ and some character χ under the action of (a congruence subgroup) of Sp_2 (see Proposition 5.3.5).

Assume now $4 \mid n$ and $r = n/4$. After applying the so-called **Maass-Shimura operator** δ_k^r (see Definition 5.4.1) and the operator R of Proposition 5.4.2, in order to compensate for divergent terms, we arrive at the main Theorem.

Theorem 1.2.6. *Let* $\widetilde{E}(Z, \chi, s)$ *denote the degree* 2 *symplectic Siegel-type Eisenstein series (see* (5.5.1)*). Assume* $4 \mid n$ *and that there is only one* 1*-dimensional cusp. Let also* $k = -n/2$ *and* $r = n/4$*. We then have for* $Re(s) > n + 1$

$$\left\langle \widetilde{E}(Z, \chi, (s+1)/2 - r), R[\delta_k^{(r)}\Theta](Z, W) \right\rangle = \xi(s)\xi(s-1)\gamma_S(s)E(W, s),$$

where $\gamma_S(s)$ *is an explicit gamma factor and* $\xi(s)$ *is the completed zeta function.*

The meromorphic continuation of $E(W, s)$ and hence $\mathcal{D}_{F,G}(s)$ to $\mathbb{C}$ then follows as a corollary. We note that, although the general Langlands' philosophy predicts the analytic properties of $E(W, s)$, here we obtain an explicit connection with a well-studied object: the symplectic Eisenstein series of degree two. This connection can be used to obtain further results on $E(W, s)$, including information on its poles and zeroes (see Corollary 5.5.3). Moreover, in the case of the E_8 lattice (see Section 5.6), we have the following Theorem.

Theorem 1.2.7. *Let* S *correspond to the* E_8 *lattice. Then, we can complete* $\mathcal{D}_{F,G}(s)$ *to* $\mathcal{D}^*_{F,G}(s)$*, so that the last one has a meromorphic continuation to* $\mathbb{C}$ *and is invariant under* $s \longmapsto 2k - 9 - s$*.*

In Chapter 6, we address the other direction of the problem, i.e., how the method of Kohnen and Skoruppa in [KS89] can be extended in order to obtain an Euler product for $\mathcal{D}_{F,G}(s)$ in this case too. Assume F is a Hecke eigenform for the corresponding Hecke algebra. Assume also that $k > n/2 + 2$ is even. Take G to be of the form

$$G(\tau', z, \tau) := \sum_{N \geq 1} (V_N P_{k,D,r})(\tau, z)e(N\tau'),$$

where $P_{k,D,r}$ is a **Poincaré series** depending on (D, r), which is in the support of the lattice $\mathbb{Z}^n$ with respect to S, and V_N is an index-raising operator acting on Fourier-Jacobi forms. Let $V_0 := \mathbb{Q}^{n+2}$ and ϕ_0 the bilinear form on $V_0 \times V_0$ corresponding to $S_0/2$. Let also $L_0 := \mathbb{Z}^{n+2}$ and $L_0^* := S_0^{-1}L_0$. Pick now any $\xi \in V_0$ such that

$\phi_0(\xi,\xi) = -D/q$ and $2\phi_0(\xi, L_0) = \mathbb{Z}$, where q denotes the level of S (see Definition 4.1.4). Consider then the algebraic subgroup of $G^*_{\mathbb{Q}}$, defined by

$$H(\xi)_{\mathbb{Q}} := \{g \in G^*_{\mathbb{Q}} \mid g\xi = \xi\}.$$

This is a negative definite orthogonal group of rank $n+1$. It is one of the main results by Shimura in [Shi04] that the congruence subgroup of $G^*_{\mathbb{Q}}$ which fixes the lattice L_0 acts on the set

$$\{x \in V_0 \mid \phi_0(x,x) = -D/q \text{ and } 2\phi_0(x, L_0) = \mathbb{Z}\},$$

and there are finitely many equivalence classes under this action. Let $\{\xi_i\}_{i=1}^h$ be representatives for this action. These representatives correspond to elements of the finite set

$$H(\xi)_{\mathbb{Q}} \backslash H(\xi)_{\mathbb{A}} / (H(\xi)_{\mathbb{A}} \cap C), \tag{1.2.2}$$

where $C = \{x \in G^*_{\mathbb{A}} \mid xL_0 = L_0\}$. This is the analogue of the classical theorem for binary quadratic forms of fixed discriminant. Assume $\{u_i\}_{i=1}^h$ are these elements.

Let also $\{f_j\}_{j=1}^h$ denote an orthonormal basis of simultaneous eigenforms on the set (1.2.2). Denote by $L(-,s)$ the standard L-function attached to either F or any of the $f_j's$. We then formulate our main Theorem.

Theorem 1.2.8. *Outside a finite set of primes $\mathcal{P}$, $\mathcal{D}_{F,G}(s)$ can be written as*

$$L_{\mathcal{P}}\left(F; s-k+(n+2)/2\right) \sum_{j=1}^{h} A_{f_j} L_{\mathcal{P}}\left(\overline{f_j}; s-k+(n+3)/2\right)^{-1} \times$$
$$\times \sum_{i=1}^{h} \zeta_{\xi_i,\mathcal{P}}(s-k+n+1) f_j(u_i) \times \begin{cases} 1 & \text{if } n \text{ odd} \\ \zeta_{\mathcal{P}}(2s-2k+n+2)^{-1} & \text{if } n \text{ even} \end{cases}.$$

Here, $\zeta_{\xi_i}(s)$ denote certain zeta functions counting number of congruences, A_{f_j} are expressions depending on f_j and the Fourier coefficients of F and for any zeta function, the subscript $\mathcal{P}$ means that we do not take into account the terms sharing factors with elements of $\mathcal{P}$.

A connection with L-functions therefore exists, but it is not clear how one can obtain an Euler product. Nevertheless, when we choose $D = -q$, $\xi = (1, \mathbf{0}, 1)^t$ and S such that $h = 1$, we can obtain a clear-cut result. We find all the cases when this happens in the $n = 1$ case (with these specific choices of D, ξ) and some cases when $n \in \{2, 4, 6, 8\}$. We give the Euler product expression of $\mathcal{D}_{F,G}(s)$ in **Theorems 6.6.3** and **6.6.9**. In particular, up to finitely many primes, we recover Theorem 1.1.5 of Kohnen and Skoruppa (see Remark 6.6.4).

1.3 Outline of the Thesis

In this **introductory** Chapter, we gave an overview of the thesis. We discussed how the classical results of Heim in [Hei99] and Kohnen and Skoruppa in [KS89] serve as the main motivation for this dissertation and naturally divide it into two main parts. We next described the ways we have generalised their results and presented our main theorems. Finally, we provided an outline of the thesis, which serves as a roadmap for the reader.

In **Chapter** 2, we develop the theory of Hermitian modular forms. After giving the main definitions and considering their Fourier–Jacobi expansions, we introduce Hecke operators and describe the associated Hecke rings. We then define the relevant L-functions and prepare the groundwork for the next Chapter.

In **Chapter** 3, we present the main results regarding the first part of the thesis. We begin the chapter by providing an overview of the so-called doubling method and Heim's results. We then consider a certain Dirichlet series attached to three Hermitian cuspidal eigenforms of weight $k \equiv 0 \pmod 4$ and degrees $2, 2$, and 1 over $\mathbb{Q}(i)$. In the case when F is in the Maass space, we obtain an Euler product for the Dirichlet series. Moreover, for an inert prime p, we identify its p-factor with the p-factor of the L-function attached to G by Gritsenko in [Gri88b], twisted by the Satake parameters of h. The question of whether the same holds for primes that split remains unanswered; however, a big part of this chapter is concerned with making progress towards that end. Most notably, the results we obtained on the parabolic Hecke rings are of independent interest. Finally, we show that this Dirichlet series originates as part of a Rankin-Selberg integral representation. This representation also produces an additional residue term that we do not study in this thesis. The material for this chapter is taken from our joint paper with T. Bouganis in [BP25].

The **fourth chapter** marks the beginning of the second part of the thesis. We aim to extend the paper [KS89] by Kohnen and Skoruppa in the orthogonal setting. We start by giving definitions of quadratic spaces and modular forms for orthogonal groups of real signature $(2, n+2)$, $n \geq 1$. We then discuss their Fourier–Jacobi coefficients, the corresponding Maass spaces, and finally define a Fourier–Jacobi Dirichlet series $\mathcal{D}_{F,G}(s)$ associated with a pair of orthogonal modular forms.

In **Chapter** 5, we consider the problem of obtaining the analytic properties of the Dirichlet series. Using an orthogonal Eisenstein series of Klingen type, we obtain an integral representation for this Dirichlet series. In the case when the underlying lattice has only one 1-dimensional cusp, we rewrite this Eisenstein series in the form of an Epstein zeta function. If additionally $4 \mid n$, we deduce a theta correspondence between this Eisenstein series and a Siegel-type Eisenstein series for the symplectic group of degree 2. As a consequence, we obtain the meromorphic continuation of the Dirichlet series to $\mathbb{C}$. In the case of the E_8 lattice, we can further deduce a precise functional equation for the Dirichlet series. The material for this chapter is taken from my paper [Psy24].

Chapter 6 is devoted to the investigation of the connection of this Dirichlet series with the standard L-function for the orthogonal group. In the case when F is a Hecke eigenform and G is a Maass lift of a specific Poincaré series, we establish a connection with the standard L-function attached to F. What is more, we find explicit choices of orthogonal groups, for which we obtain a clear-cut Euler product expression for this Dirichlet series. Through our considerations, we recover the classical result of Kohnen and Skoruppa, but also provide a range of new examples, which can be related to other kinds of modular forms, such as paramodular, Hermitian, and quaternionic. The material for this chapter is taken from my paper [Psy25].

Finally, in **Chapter** 7, we give the consequences of the main results in the previous Chapters. We discuss their importance and highlight points of interest, which provide directions for future work.

1.4 Notation

Below, we provide some standard notation that will be used throughout the Thesis.

- $e(z) := e^{2\pi i z},\ z \in \mathbb{C}$.
- $A[B] := \overline{B}^t AB$ for suitably sized complex matrices A, B.
- $A > B$ for matrices A, B: Denotes that the matrix $A - B$ is positive definite.
- $0_n, 1_n$ denote the $n \times n$ zero and identity matrices repsectively.
- $\det(M)$, $\mathrm{tr}(M)$ denote the determinant and trace of a matrix M, respectively.
- $M_{m,n}(R)$: Denotes the space of $m \times n$ matrices with coefficients in a ring R.
- $M_n(R)$: Denotes the space of $n \times n$ matrices with coefficients in a ring R.
- $\mathrm{GL}_n(R),\ \mathrm{SL}_n(R)$: Denote the matrices in $M_n(R)$ with non-zero determinant and determinant 1, respectively.
- $[A_1, A_2, \cdots, A_n]$ or $\mathrm{diag}(A_1, \cdots, A_n)$: Denotes the block diagonal matrix with the matrices $A_1, A_2, \cdots, A_n$ in the diagonal blocks.
- $\zeta(s)$: Denotes the usual Riemann zeta function.
- $N(q)$: Denotes the norm of a complex number q, i.e. $N(q) := q\overline{q}$.
- $\mathcal{C}^\infty$: Denotes the class of functions which are infinitely differentiable.

Chapter 2

Hermitian Modular Forms

In this Chapter, we collect background material for the theory of Hermitian modular forms. We state the main definitions, discuss the Fourier and Fourier-Jacobi expansions, define Eisenstein series, and develop the necessary Hecke theory. Throughout this Hermitian part of the thesis, we assume that $K = \mathbb{Q}(i)$, the field of Gaussian numbers, and $\mathcal{O}_K = \mathbb{Z}[i]$ its ring of integers.

2.1 Preliminaries

Everything below is standard and can be found in [Kri85].

Definition 2.1.1. Let R be either K, $\mathcal{O}_K$ or $\mathbb{C}$ and fix an embedding $R \hookrightarrow \mathbb{C}$. We write $U(n,n)(R)$ for the R-points of the **unitary group** of degree $n \geq 1$. That is,

$$U(n,n)(R) := \{g \in \mathrm{GL}_{2n}(R) \mid J_n[g] = J_n\},$$

where $J_n := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. The notation $J_n[g]$ means $\overline{g}^t J_n g$ (see Notation).

Hence, for an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)(R)$ with $n \times n$ matrices A, B, C, D, these satisfy the relations

$$\overline{A}^t C = \overline{C}^t A,\ \overline{D}^t B = \overline{B}^t D,\ A\overline{D}^t - \overline{B}^t C = 1_n.$$

Definition 2.1.2. The **Hermitian upper half-plane** of degree n is defined by

$$\mathbb{H}_n := \{Z \in \mathrm{M}_n(\mathbb{C}) | -i(Z - \overline{Z}^t) > 0\}.$$

For $n = 1$ we obtain the usual upper half plane, which we will just denote by $\mathbb{H}$.

We fix an embedding $K \hookrightarrow \mathbb{C}$. Then, an element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)(K) \hookrightarrow U(n,n)(\mathbb{C})$ of the unitary group acts on the above upper half plane via the action

$$Z \longmapsto g\langle Z\rangle := (AZ+B)(CZ+D)^{-1}.$$

The usual **factor of automorphy** is defined by $j(g,Z) := \det(CZ+D)$.

Let now Γ_n denote the **Hermitian modular group**, that is $\Gamma_n := U(n,n)(\mathcal{O}_K)$. Note that

$$\Gamma_1 = \mathrm{SL}_2(\mathbb{Z}) \cdot \{\alpha \cdot 1_2 \mid \alpha \in \mathcal{O}_K^\times\}. \tag{2.1.1}$$

We now define the **slash** operator.

Definition 2.1.3. Let $n \geq 1$ and k be any integer. Then, for any function F on $\mathbb{H}_n$ and a matrix $g \in U(n,n)(K)$, we define

$$(F\mid_k g)(Z) := j(g,Z)^{-k} F(g\langle Z\rangle).$$

We then have the following definition of a Hermitian modular form.

Definition 2.1.4. A function $F : \mathbb{H}_n \longrightarrow \mathbb{C}$ is called a **Hermitian modular form** of degree n and integer weight $k \geq 0$ if

- F is holomorphic,
- F satisfies
$$(F\mid_k g)(Z) = F(Z),$$
for all $g \in \Gamma_n$ and $Z \in \mathbb{H}_n$.

If $n = 1$, we further require that F is holomorphic at infinity.

It is well-known ([Kri85, Chapter III.2]) that the set of all such forms constitutes a finite-dimensional complex vector space, which we denote by M_n^k. Because of our assumption if $n = 1$ and of **Köcher's principle** ([Kri85, Lemma III.1.5]) for $n \geq 2$, each such F admits a Fourier expansion

$$F(Z) = \sum_N a(N) e(\mathrm{tr}(NZ)), \tag{2.1.2}$$

where $a(N) \in \mathbb{C}$ and N runs through all the semi-integral non-negative Hermitian matrices

$$N \in \left\{ (n_{ij})_{i,j=1}^n \geq 0 \mid n_{ii} \in \mathbb{Z}, n_{ij} = \overline{n_{ji}} \in \frac{1}{2}\mathcal{O}_K \right\}.$$

F is called a **cusp form** if $a(N) \neq 0$ only for N positive definite. We denote the space of cusp forms by S_n^k. We also have a notion of an inner product.

Definition 2.1.5. The **Petersson inner product** for two Hermitian modular forms F, G, is given by

$$\langle F, G\rangle := \int_{\Gamma_n\backslash\mathbb{H}_n} F(Z)\overline{G(Z)}(\det Y)^k \mathrm{d}^*Z,$$

whenever this converges. Here, $Z = X + iY$ and $\mathrm{d}^*Z = (\det Y)^{-2n}\mathrm{d}X\mathrm{d}Y$ is the $U(n,n)(\mathbb{C})$-invariant measure ([Kri85, Theorem II.1.10]).

Finally, we want to define the so-called **Eisenstein series**. To this end, we first need to define some **parabolic subgroups** of the unitary group.

For R as in Definition 2.1.1 and $0 \le r \le n$, we consider the following parabolic subgroups of $U(n,n)(R)$:

$$\begin{aligned} P_{n,r}(R) &= \left\{ \begin{pmatrix} * & * \\ 0_{n-r,n+r} & * \end{pmatrix} \in U(n,n)(R) \right\} \\ C_{n,r}(R) &= \left\{ \begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in U(n,n)(R) \right\} \end{aligned}. \tag{2.1.3}$$

When $R = \mathcal{O}_K$, we will just write $P_{n,r}$ and $C_{n,r}$.

Definition 2.1.6. Let $0 \le r \le n$ and $F \in S_r^k$, with $k \equiv 0 \pmod 4$. The **Klingen-type** Eisenstein series with respect to the parabolic subgroup $C_{n,r}$ attached to F is given by

$$E_{n,r}^k(Z, F; s) = \sum_{\gamma \in C_{n,r}\backslash\Gamma_n} F(\gamma\langle Z\rangle_*) j(\gamma, Z)^{-k} \left(\frac{\det \operatorname{Im} \gamma\langle Z\rangle}{\det \operatorname{Im} \gamma\langle Z\rangle_*} \right)^s, \tag{2.1.4}$$

where $Z \in \mathbb{H}_n$ and $*$ denotes the lower right $r \times r$ part of the matrix.

When $r = 0$, we omit $F \equiv \mathbf{1}$ and we call $E_{n,0}^k(Z; s)$ an Eisenstein series of **Siegel type**. When $r = n$, we have $E_{n,n}^k(Z; s) = F(Z)$.

Lemma 2.1.7. *This series is well-defined and converges absolutely and uniformly on compact subsets of $\mathbb{C}$ for $k + 2\mathrm{Re}(s) > 2(n + r)$. If $s = 0$ and $k > 2(n + r)$, $E_{n,r}^k(Z, F; 0) \in M_n^k$ for all $F \in S_r^k$.*

Proof. Let $\delta \in C_{n,r}$. We write

$$\delta = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ 0 & A_4 & B_3 & B_4 \\ 0 & 0 & D_1 & 0 \\ 0 & C_4 & D_3 & D_4 \end{pmatrix},$$

with $A_4, B_4, C_4, D_4 \in M_r(\mathcal{O}_K)$ and set $\delta_* := \begin{pmatrix} A_4 & B_4 \\ C_4 & D_4 \end{pmatrix}$.

We then have $\delta\langle Z\rangle_* = \delta_*\langle Z_*\rangle$ (cf. [Kri85, Proposition V.2.1]). Also

$$\det(\operatorname{Im} g\langle Z\rangle) = |j(g,Z)|^{-2}\det(\operatorname{Im}Z),\ \forall g\in U(n,n)(\mathbb{C}),\ Z\in\mathbb{H}_n. \tag{2.1.5}$$

Now, $j(\delta,Z) = \det(D_1)j(\delta_*,Z_*)$ and $\det(D_1)$ is a unit in $\mathcal{O}_K$. Using the above, the transformation condition for $F\in S_r^k$ and the fact that $k\equiv 0 \pmod 4$, we obtain that the series is well-defined (under $\gamma\longmapsto\delta\gamma$).

For the convergence, we use the fact that the function $(\det\operatorname{Im}\gamma\langle Z\rangle_*)^{k/2}F(\gamma\langle Z\rangle_*)$ is bounded on $\mathbb{H}_n$, say by a constant C (see [Kri85, Lemma III.2.4]) and that $\det(\operatorname{Im}\gamma\langle Z\rangle) = |j(\gamma,Z)|^{-2}\det(\operatorname{Im}Z)$. Hence, the series is bounded by

$$C(\det\operatorname{Im}Z)^s\sum_{\gamma\in C_{n,r}\backslash\Gamma_n}(\det\operatorname{Im}\gamma\langle Z\rangle_*)^{-\frac{1}{2}(k+2s)}|j(\gamma,Z)|^{-(k+2s)},$$

and the last series converges absolutely and uniformly on compact subsets of $\mathbb{C}$, whenever $k+2\mathrm{Re}(s)>2(n+r)$, from [Kri85, Theorem V.2.8].

Finally, the last assertion is in [Kri85, Theorem V.2.9]. We remark that for any s is the region of convergence, $E_{n,r}^k(Z,F;s)$ satisfies the modularity property of Definition 2.1.4, but is holomorphic in Z only when $s=0$. □

2.2 Hermitian Fourier-Jacobi Forms

In this Section, we introduce the notion of Hermitian Fourier-Jacobi forms. This is very similar to the classical case of Fourier-Jacobi forms, as developed by Eichler and Zagier in [EZ85]. The Hermitian case was first studied by Haverkamp in his thesis [Hav95]. His work has recently been generalised to Hermitian Jacobi forms of higher degree by Haight in [Hai24]. In this thesis, we follow Haverkamp's paper in [Hav96]. We should note here that in that paper, the case of $K=\mathbb{Q}(i)$ is excluded, in order to make the exposition simpler, due to the existence of non-trivial units. The results, however, are naturally transferred to the Gaussian case as well.

Definition 2.2.1. The **Hermitian Jacobi group** is defined by $\Gamma^J(\mathcal{O}_K) := \Gamma_1\rtimes\mathcal{O}_K^2$, with the multiplication of elements defined by

$$[\epsilon_1M_1,(\lambda_1,\mu_1)]\cdot[\epsilon_2M_2,(\lambda_2,\mu_2)] := [\epsilon_1\epsilon_2M_1M_2,(\lambda_1,\mu_1)\epsilon_2M_2+(\lambda_2,\mu_2)],$$

where $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathcal{O}_K$, $M_1,M_2\in\mathrm{SL}_2(\mathbb{Z})$ and $\epsilon_1,\epsilon_2\in\mathcal{O}_K^\times$. We remind the reader here that Γ_1 is given in (2.1.1).

We now define some more **slash operators**:

Definition 2.2.2. Let ϕ be a complex-valued function on $\mathbb{H}\times\mathbb{C}^2$. Let also $k, m \in \mathbb{Z}$ and $(\tau, z_1, z_2) \in \mathbb{H}\times\mathbb{C}^2$. Then:

- For $\epsilon \in \mathcal{O}_K^\times$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$(\phi\mid_{k,m} \epsilon M)(\tau, z_1, z_2) := \epsilon^{-k}(c\tau+d)^{-k} e\left(\frac{-mcz_1z_2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\epsilon z_1}{c\tau+d}, \frac{\overline{\epsilon}z_2}{c\tau+d}\right).$$

- For $\lambda, \mu \in \mathcal{O}_K$, we have

$$(\phi\mid_m [\lambda,\mu])(\tau, z_1, z_2) := e(m(N(\lambda)\tau + \overline{\lambda}z_1 + \lambda z_2))\phi(\tau, z_1+\lambda\tau+\mu, z_2+\overline{\lambda}\tau+\overline{\mu}).$$

We have the definition of a **Hermitian Fourier-Jacobi form**.

Definition 2.2.3. A holomorphic function $\phi : \mathbb{H}\times\mathbb{C}^2 \longrightarrow \mathbb{C}$ is called a (Hermitian) Fourier-Jacobi form of weight k and index m, where $k, m \in \mathbb{Z}_{\geq 0}$, if $\forall(\tau, z_1, z_2) \in \mathbb{H}\times\mathbb{C}^2$, we have

- $(\phi\mid_{k,m} \epsilon M)(\tau, z_1, z_2) = \phi(\tau, z_1, z_2),\ \forall\epsilon \in \mathcal{O}_K^\times,\ \forall M \in \mathrm{SL}_2(\mathbb{Z})$.
- $(\phi\mid_m [\lambda,\mu])(\tau, z_1, z_2) = \phi(\tau, z_1, z_2),\ \forall\lambda,\mu \in \mathcal{O}_K$.
- ϕ admits a Fourier expansion of the form

$$\phi(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathcal{O}_K^\# \\ mn\geq N(r)}} c_\phi(n,r)e(n\tau + rz_1 + \overline{r}z_2),$$

  where $c_\phi(n,r) \in \mathbb{C}$ and $\mathcal{O}_K^\# := \frac{i}{2}\mathcal{O}_K$.

We denote the complex vector space of Hermitian Fourier-Jacobi forms by $J_{k,m}$. It is a known fact that $J_{k,m}$ is finite dimensional ([Hav96, Theorem 3]). A Hermitian Fourier-Jacobi form is called a **cusp form** if $c_\phi(n,r) = 0$ for $nm = N(r)$.

In the following, we will drop the word Hermitian and just write Fourier-Jacobi forms. The main reason we are interested in the theory of Fourier-Jacobi forms is that they appear naturally in the context of Hermitian modular forms of degree two. Indeed, let $F \in S_2^k$. For $Z \in \mathbb{H}_2$, we can partition $Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix}$, with $\tau, \omega \in \mathbb{H}$ and $z_1, z_2 \in \mathbb{C}$. From (2.1.2), we can then write the Fourier expansion of F with respect to the variable ω as

$$F(Z) = \sum_{m=1}^{\infty} \phi_m(\tau, z_1, z_2)e(m\omega).$$

The functions $\phi_m : \mathbb{H}\times\mathbb{C}^2 \longrightarrow \mathbb{C}$ are then Fourier-Jacobi cusp forms in the sense of Definition 2.2.2 and are called the Fourier-Jacobi coefficients of F.

There is, however, an equivalent characterisation of Fourier-Jacobi forms, used by Gritsenko in [Gri92a]. In the following, it will be important for us to consider the Fourier-Jacobi forms as a special kind of modular forms under the action of some parabolic subgroups, as these are defined in (2.1.3).

Let $\Gamma_{n,1} := P_{n+1,n}(\mathcal{O}_K)$, the group of integral points of the parabolic $P_{n+1,n}(K)$. We then have the following definition ([Gri92a, p. 2887]):

Definition 2.2.4. Let $n \geq 1$. A holomorphic function F on $\mathbb{H}_{n+1}$ is a modular form of weight k with respect to the parabolic subgroup $\Gamma_{n,1}$ if the following conditions hold:

- $F \mid_k M = F$ for all $M \in \Gamma_{n,1}$,
- The function $F(Z)$ is bounded in the domain $\mathrm{Im}(Z) \geq c1_2$, for all $c > 0$.

We note here that we can omit the second condition if $n \geq 2$. This again follows by Köcher's principle. The space of all such forms will be denoted by $M^k_{n,1}$. Again, each such F has a Fourier expansion as in equation (2.1.2) and we call F a **cusp form** if $a(N) \neq 0$ only for positive definite matrices N.

We can now give the definition of Fourier-Jacobi forms, as in [Gri92a, p. 2887].

Definition 2.2.5. A complex-valued, holomorphic function ϕ on $\mathbb{H}_n \times \mathbb{C}^n \times \mathbb{C}^n$ is said to be a **Fourier-Jacobi form** of degree n, weight k and index m if the function

$$\widetilde{\phi}\left(\begin{pmatrix} \tau & z_1 \\ z_2^t & \omega \end{pmatrix}\right) := \phi(\tau, z_1, z_2)e(m\omega),$$

where $\omega \in \mathbb{H}$ is chosen so that $\begin{pmatrix} \tau & z_1 \\ z_2^t & \omega \end{pmatrix} \in \mathbb{H}_{n+1}$, is a modular form with respect to the group $\Gamma_{n,1}$. The space of such forms is denoted by $J^n_{k,m}$ and we will call $\widetilde{\phi}$ a P**-form**, as in [Hei99, Section 3.4].

Remark 2.2.6. For $n = 1$, this coincides with the space considered by Haverkamp above (cf. [Hav96, Remark 1]) and we will just write $J_{k,m}$ in this case.

Remark 2.2.7. We remark here that the approach taken by Gritsenko deals more naturally with the arbitrary degree $n \geq 1$ case. On the other hand, the formulas appearing in [Hai24] are sufficiently complicated for our purposes. Nevertheless, in this thesis, we are mainly interested in Hermitian modular forms of degree two and therefore both considerations mentioned above can be used (and are equivalent).

We will now indeed focus on Hermitian modular forms of degree two. We have the following notion of an inner product on $J_{k,m}$ ([Gri92a, (1.10)]):

Definition 2.2.8. The **Petersson inner product** of two Fourier-Jacobi forms $\phi, \psi \in J_{k,m}$ is defined as

$$\langle \phi, \psi \rangle := \int_{\mathcal{F}^J} \phi(\tau, z_1, z_2)\overline{\psi(\tau, z_1, z_2)} v^k e^{-\pi m |z_1 - \overline{z_2}|^2/v} \mathrm{d}\mu,$$

where $\mathrm{d}\mu = v^{-4}\mathrm{d}u\mathrm{d}v\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2$ with $\tau = u + iv$, $z_j = x_j + iy_j$ for $j = 1, 2$ and $\mathcal{F}^J$ is a fundamental domain for the action of $P_{2,1}$ on $\mathbb{H} \times \mathbb{C}^2$.

The reader should note that we are using the same symbol to denote the inner product for two Fourier-Jacobi forms as the one we use to denote the inner product for two Hermitian modular forms (see Definition 2.1.5). However, we will always use a Greek letter (ϕ or ψ) to denote a Fourier-Jacobi form and a Latin letter to denote a Hermitian modular form. This should help eliminate any possibility of confusion.

In the following, for $Z \in \mathbb{H}_2$ as above, we write

$$\mathrm{Re}(Z) = \begin{pmatrix} x_\tau & x_{z_1} \\ x_{z_2} & x_\omega \end{pmatrix}, \ \mathrm{Im}(Z) = \begin{pmatrix} y_\tau & y_{z_1} \\ y_{z_2} & y_\omega \end{pmatrix}, \tag{2.2.1}$$

for its real and imaginary parts, respectively. We now define an inner product on the space of P-forms, as this will be useful later.

Definition 2.2.9. Let $\phi_m, \psi_m \in J_{k,m}$ and denote by $\widetilde{\phi}_m, \widetilde{\psi}_m$ the P-forms obtained as in Definition 2.2.5. We then define

$$\langle \widetilde{\phi}_m, \widetilde{\psi}_m \rangle_{\mathcal{A}} := \int_{\mathcal{Q}_{1,1}} \widetilde{\phi}_m(Z)\widetilde{\psi}_m(Z)(\det Y)^k \mathrm{d}^*Z,$$

where $\mathrm{d}^*Z = (\det Y)^{-4}\mathrm{d}X\mathrm{d}Y$ is the invariant element for the action of the unitary group $U(2,2)$ on $\mathbb{H}_2$ and

$$\mathcal{Q}_{1,1} := \left\{ Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix} \in \mathbb{H}_2 \mid (\tau, z_1, z_2) \in \mathcal{F}^J \text{ and } |x_\omega| \leq 1/2 \right\}.$$

There is a relation between the two inner products above, given in the following Lemma.

Lemma 2.2.10. *Let $\phi_m, \psi_m \in J_{k,m}$ and denote by $\widetilde{\phi}_m, \widetilde{\psi}_m$ the corresponding P-forms. Then*

$$\langle \phi_m, \psi_m \rangle = \beta_k m^{k-3} \langle \widetilde{\phi}_m, \widetilde{\psi}_m \rangle_{\mathcal{A}},$$

where $\beta_k = (4\pi)^{k-3}\Gamma(k-3)^{-1}$.

Proof. We have

$$\langle \widetilde{\phi}_m, \widetilde{\psi}_m \rangle_{\mathcal{A}} = \int_{Q_{1,1}} \phi(\tau, z_1, z_2) e^{2\pi i m \omega} \overline{\psi(\tau, z_1, z_2)} e^{-2\pi i m \overline{\omega}} (\det Y)^{k-4} \mathrm{d}X\mathrm{d}Y.$$

Let now $\widetilde{y}_\omega := y_\omega - |z_1 - \overline{z_2}|^2/4y_\tau$. Then $\det Y = y_\tau \widetilde{y}_\omega$. Hence, the above integral can be written as

$$\int_{\widetilde{y}_\omega>0}\int_{\mathcal{F}^J}\int_{x_\omega \pmod 1} \phi_m(\tau,z_1,z_2)e^{-4\pi m\left(\widetilde{y}_\omega+|z_1-\overline{z_2}|^2/4y_\tau\right)}\overline{\psi_m(\tau,z_1,z_2)}(y_\tau\widetilde{y}_\omega)^{k-4}\times$$
$$\times\, \mathrm{d}\tau\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}\widetilde{y}_\omega\mathrm{d}x_\omega$$

$$= \langle\phi_m,\psi_m\rangle\int_{\widetilde{y}_\omega>0} e^{-4\pi m\widetilde{y}_\omega}\widetilde{y}_\omega^{k-4}\mathrm{d}\widetilde{y}_\omega = (4\pi m)^{3-k}\Gamma(k-3)\langle\phi_m,\psi_m\rangle,$$

so the result follows with $\beta_k = (4\pi)^{k-3}\Gamma(k-3)^{-1}$. □

2.3 Unitary Hecke Rings

In this Section, we give an account of the Hecke theory for Hermitian modular forms. We follow Gritsenko in [Gri92a]. We start with the Definition of Hecke pairs and Hecke rings.

Definition 2.3.1. A pair (Γ, G), where $\Gamma \le G$ is called a **Hecke pair** if for all $g \in G$, the double coset $\Gamma g\Gamma$ is a union of a finite number of left or, equivalently, right Γ-cosets. Let $V(\Gamma, G)$ denote the $\mathbb{Q}$-vector space of all formal finite linear combinations of left Γ-cosets with rational coefficients:

$$V(\Gamma,G) := \left\{X = \sum_i a_i\Gamma g_i \mid a_i\in\mathbb{Q},\ g_i\in G\right\}.$$

The group Γ acts on V via right multiplication:

$$X \longmapsto X\cdot\gamma := \sum_i a_i\Gamma(g_i\gamma).$$

The Γ-invariant subspace $H(\Gamma, G)$ of V is called the **Hecke ring** of (Γ, G). If now $X = \sum_i a_i\Gamma g_i,\ Y = \sum_j b_j\Gamma h_j$ are in $H(\Gamma, G)$, we define their product by

$$X\cdot Y := \sum_{i,j} a_ib_j\Gamma(g_ih_j).$$

This is independent of the choice of the representatives g_i, h_j, and $H(\Gamma, G)$ is an associative ring.

Let $n \ge 1$. We define the **groups of similitude**:

$$S^n := \{g\in M_{2n}(K) \mid J_n[g] = \mu(g)J_n,\ \text{for some } \mu(g)>0\},$$

$$S_p^n := \{g\in S^n\cap M_{2n}(\mathcal{O}_K[p^{-1}]) \mid \mu(g)=p^\delta, \delta\in\mathbb{Z}\},$$

where p is a rational prime. It is then well-known that the pairs $(\Gamma_n, S^n), (\Gamma_n, S_p^n)$ are Hecke pairs and we can define the corresponding Hecke rings, which we will also denote by H^n and H_p^n respectively. We start with the following Lemma on elementary divisors.

Lemma 2.3.2. *If $g \in S^n \cap M_{2n}(\mathcal{O}_K)$, we can write*

$$\Gamma_n g \Gamma_n = \Gamma_n \mathrm{diag}(a_1, \cdots, a_n, d_1, \cdots, d_n)\Gamma_n,$$

where the principal ideals, generated by the elements $a_i, d_i \in \mathcal{O}_K$, satisfy the relations $(a_1) \supseteq \cdots \supseteq (a_n) \supseteq (d_1) \supseteq \cdots \supseteq (d_n)$ and $a_i\overline{d_i} = \mu(g)$ for all $i = 1, \cdots, n$.

Proof. See [Gri92a, p. 2889]. Note the typo in Gritsenko's statement (we must have $a_i\overline{d_i} = \mu(g)$ because of the unitary setting). □

Assume now $F \in M_n^k$. For any $g \in S^n$, we define

$$(F \mid_k g)(Z) := \mu(g)^{nk-n^2} j(g, Z)^{-k} F(g\langle Z\rangle).$$

Then, if $\Gamma_n g \Gamma_n \in H^n$, we write $\Gamma_n g \Gamma_n = \sum_{i=1}^{m} \Gamma_n g_i$, with $g_i \in S^n$. We then define

$$F \mid_k \Gamma_n g \Gamma_n := \sum_{i=1}^{m} F \mid_k g_i. \tag{2.3.1}$$

We can now modify the Petersson inner product of Definition 2.1.5 so that it applies to congruence subgroups of Γ_n (see, for example, [Klo15, p. 808-809]). We have the following Lemma.

Lemma 2.3.3. *For any $g \in S^n \cap M_{2n}(\mathcal{O}_K)$ and $F, G \in S_n^k$, we have*

$$\langle F \mid_k \Gamma_n g \Gamma_n, G\rangle = \langle F, G \mid_k \Gamma_n \overline{g} \Gamma_n\rangle,$$

with the appropriate inner product on each side.

Proof. Let $g \in S^n \cap M_{2n}(\mathcal{O}_K)$. From the general setting of [Shi97, Lemma 11.4], we have that we can find representatives $g_i \in S^n$ such that

$$\Gamma_n g \Gamma_n = \bigsqcup_{i=1}^{\ell} \Gamma_n g_i = \bigsqcup_{i=1}^{\ell} g_i \Gamma_n, \tag{2.3.2}$$

for some ℓ. Moreover, by using standard arguments (see for example [And87, Theorem 2.5.3]), we have that for any $M \in S^n$ and $F, G \in S_n^k$,

$$\langle F \mid_k M, G \mid_k M\rangle = \mu(M)^{nk-2n^2}\langle F, G\rangle.$$

Hence,

$$\langle F \mid_k \Gamma_n g \Gamma_n, G\rangle = \left\langle \sum_{i=1}^{\ell} F \mid_k g_i, G \right\rangle = \sum_{i=1}^{\ell} \langle F \mid_k g_i, G\rangle =$$

$$= \sum_{i=1}^{\ell} \mu(g_i)^{-nk+2n^2} \langle F \mid_k g_i \mid_k g_i^{-1}, G \mid_k g_i^{-1}\rangle = \sum_{i=1}^{\ell} \langle F, G \mid_k (\mu(g_i) g_i^{-1})\rangle.$$

But from the second equality of (2.3.2), we have that $\Gamma_n \mu(g) g^{-1} \Gamma_n = \bigsqcup_{i=1}^{\ell} \Gamma_n \mu(g) g^{-1}$. Moreover, from Lemma 2.3.2, the elementary divisors of $\mu(g)g^{-1}$ are the complex conjugates of those of g. The Lemma follows. □

Remark 2.3.4. From [Gri92a, Proposition 2.1] we get that H^n is commutative. Moreover, since $g \in S^n \implies \overline{g} \in S^n$, each $\Gamma_n g \Gamma_n \in H^n$ acts as a normal operator on S_n^k.

From [Gri92a, Corollary 2.2], we can decompose the global Hecke ring into the tensor product of p-rings as follows:

$$H(\Gamma_n, S^n) = \bigotimes_p H(\Gamma_n, S_p^n).$$

Now, each p-ring is isomorphic to the Hecke ring over the corresponding local field, and the structure of these rings depends on the decomposition of the prime p in $\mathcal{O}_K$ (see [Gri92a, p. 2889]). In order to work locally, we give the following definitions:

$$K_p := K \otimes \mathbb{Q}_p,\ \mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p,\ \Phi_p := (2i)^{-1} \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}, \tag{2.3.3}$$

which denote the algebra over $\mathbb{Q}_p$, the maximal lattice, and a Hermitian form on the vector space K_p respectively. We also define the unitary group G_p^n and a maximal compact subgroup U_p^n by

$$G_p^n := \{g \in \mathrm{GL}_{2n}(K_p) \mid g^* \Phi_p g = \mu(g)\Phi_p,\ \text{for some } \mu(g) \in \mathbb{Q}_p^\times\}, \tag{2.3.4}$$

$$U_p^n := \{g \in G_p^n \cap M_{2n}(\mathcal{O}_p) \mid \mu(g) \in \mathbb{Z}_p^\times\}, \tag{2.3.5}$$

where $g^* := (g_{ji})^\sigma$, with σ is the canonical involution of the algebra K_p, determined by the behaviour of the prime p in K (split, inert or ramified). We now have the following Proposition:

Proposition 2.3.5. *For every prime p, the local Hecke ring $H(U_p^n, G_p^n)$ is isomorphic to the p-ring $H(\Gamma_n, S_p^n)$.*

Proof. See [Gri92a, Proposition 2.3]. □

The reason we would like to work with local Hecke rings is that these have been investigated by Satake in his paper [Sat63].

To that end, let us now recall the definition of the so-called spherical or Satake mapping. We again follow [Gri92a]. We need to distinguish between the cases: (i) p is inert or $p = 2$ and (ii) p splits.

In the first case, we know that given $g \in G_p^n$, we have the double coset decomposition

$$U_p^n g U_p^n = \sum_i U_p^n M^{m_i} N_i,$$

where N_i is a unipotent matrix, $m_i = (m_{i_1}, \cdots, m_{i_n}; m_{i_0})$ an integer tuple, and

$$M^{m_i} = \begin{pmatrix} p^{m_{i_0}}(\overline{D}^t)^{-1} & 0 \\ 0 & D \end{pmatrix}, \ D = \mathrm{diag}(\pi^{m_{i_1}}, \cdots, \pi^{m_{i_n}}),$$

with $\pi = p$ if p is inert or $\pi = (1+i)$ if $p = 2$. We then define

$$\Phi : H(U_p^n, G_p^n) \longrightarrow \mathbb{Q}^{W_n}[x_0^{\pm 1}, \cdots, x_n^{\pm 1}],$$

via

$$\Phi(U_p^n g U_p^n) = \sum_i x_0^{m_{i_0}} \prod_{j=1}^{n} (x_j q^{-j})^{m_{i_j}}, \tag{2.3.6}$$

where the ring $\mathbb{Q}^{W_n}[x_0^{\pm 1}, \cdots, x_n^{\pm 1}]$ denotes the ring of polynomials invariant with respect to the permutation of the variables $x_0, \cdots, x_n$ under the transformations $w^{(i)}, \ i = 1, \cdots, n$, defined by

$$x_0 \longmapsto p^{-1} x_0 x_i^e, \ x_i \longmapsto p^{2/e} x_i^{-1}, \ x_j \longmapsto x_j \ (j \neq 0, i),$$

with q denoting the number of elements in the residue field $K \otimes \mathbb{Q}_p$ and e is the ramification index of the prime p.

For the case of decomposable p, the definition of the spherical mapping is different. In particular, from [Gri92a, Proposition 2.4], there is an isomorphism

$$\rho : H(U_p^n, G_p^n) \longrightarrow H(\mathrm{GL}_{2n}(\mathbb{Z}_p), \mathrm{GL}_{2n}(\mathbb{Q}_p))[x^{\pm 1}].$$

We can then define the Satake mapping Ω for $H(\mathrm{GL}_{2n}(\mathbb{Z}_p), \mathrm{GL}_{2n}(\mathbb{Q}_p))$ in an analogous way as for the case p inert or $p = 2$, as in [Gri92b, p. 2873].

For the reader's convenience, let us describe it here: Given an element $X \in H(\mathrm{GL}_{2n}(\mathbb{Z}_p), \mathrm{GL}_{2n}(\mathbb{Q}_p))$, we know that we can write it as

$$X = \sum_i a_i \mathrm{GL}_{2n}(\mathbb{Z}_p) g_i,$$

where $g_i = \begin{pmatrix} p^{d_{i1}} & * & * & * \\ 0 & p^{d_{i2}} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & p^{d_{i2n}} \end{pmatrix}$ and $a_i \in \mathbb{Q}$. Then, the mapping Ω given by

$$\Omega(X) := \sum_i \prod_{j=1}^{2n} (x_j p^{-j})^{d_{ij}},$$

defines an isomorphism between $H(\mathrm{GL}_{2n}(\mathbb{Z}_p), \mathrm{GL}_{2n}(\mathbb{Q}_p))$ and the ring of symmetric polynomials $\mathbb{Q}^{\mathrm{sym}}[x_1^{\pm 1}, \cdots, x_{2n}^{\pm 1}]$. We then define the Satake mapping Φ in this case as the composition

$$\Phi := \Omega \circ \rho. \tag{2.3.7}$$

2.4 Parabolic Hecke Rings

In this Section, we define Hecke rings corresponding to parabolic subgroups of the unitary group, as these were defined in (2.1.3). These are necessary in order to develop a Hecke theory for Fourier-Jacobi forms, as these are defined as modular objects under the action of integral parabolic subgroups of the Hermitian modular group.

We start with a very general Lemma regarding embeddings of Hecke rings.

Lemma 2.4.1. *Let (Γ_0, S_0) and (Γ, S) be two Hecke pairs. We assume that*

$$\Gamma_0 \subset \Gamma, \;\; \Gamma S_0 = S, \;\; \Gamma \cap S_0 S_0^{-1} \subset \Gamma_0.$$

Then, given an arbitrary element $X \in H(\Gamma, S)$, according to the second condition, we can write it as

$$X = \sum_i a_i(\Gamma g_i),$$

with $g_i \in S_0$. Then, if we set

$$\epsilon(X) := \sum_i a_i(\Gamma_0 g_i),$$

then ϵ does not depend on the selection of the elements $g_i \in S_0$ and is an embedding (as a ring homomorphism) of the Hecke ring $H(\Gamma, S)$ to $H(\Gamma_0, S_0)$.

Proof. See [Gri92a, page 2890]. □

Let us now define the parabolic Hecke rings we will need. Let $S^{n,1}$, $S_p^{n,1}$, $\Gamma_{n,1}$ denote the intersection of the groups S^{n+1}, S_p^{n+1}, Γ_{n+1} with the parabolic subgroup

$P_{n+1,n}(K)$, respectively. Again, the pairs $(\Gamma_{n,1}, S^{n,1})$ and $(\Gamma_{n,1}, S_p^{n,1})$ are Hecke pairs (cf. [Gri92a, Section 3]) and we can then define the Hecke rings

$$H^{n,1} := H(\Gamma_{n,1}, S^{n,1}),\ H_p^{n,1} := H(\Gamma_{n,1}, S_p^{n,1}),$$

for any rational prime p. Since $\Gamma_{n+1}S_p^{n,1} = S_p^{n+1}$ and after writing an element $X \in H_p^{n+1}$ as

$$X = \sum_i a_i \Gamma_{n+1} g_i,$$

with $g_i \in S_p^{n+1}$, we can define an embedding

$$X \longmapsto \epsilon(X) = \sum_i a_i \Gamma_{n,1} g_i,$$

using Lemma 2.4.1. In this way, we obtain an embedding of H_p^{n+1} into $H_p^{n,1}$.

Moreover, we can embed $H(\Gamma_n, S^n) \hookrightarrow H(\Gamma_{n,1}, S^{n,1})$ in two ways, as follows:

If $X = \Gamma_n g \Gamma_n$ with $g = [A, D] \in S^n$, we define

$$j_-(X) := \Gamma_{n,1}\,[A, \mu(g), D, 1]\,\Gamma_{n,1},\ \ j_+(X) := \Gamma_{n,1}\,[A, 1, D, \mu(g)]\,\Gamma_{n,1}. \qquad (2.4.1)$$

These are related by an anti-homomorphism $* : H_p^{n,1} \longrightarrow H_p^{n,1}$, given by

$$\sum_i a_i \Gamma_{n,1} M_i \Gamma_{n,1} \longmapsto \sum_i a_i \Gamma_{n,1} \mu(M_i) M_i^{-1} \Gamma_{n,1}, \qquad (2.4.2)$$

as in [Gri92a, Lemma 3.1]. In particular, we have $j_-(X)^* = j_+\left(\overline{X}^t\right)$.

We now again restrict our discussion to the degree two case. We note that $H^{1,1}$ is not commutative and also does not split into the tensor product of the $H_p^{1,1}$ rings.

The structure of the parabolic Hecke rings $H_p^{1,1}$ again depends on the decomposition of the prime p in $\mathcal{O}_K$.

If p is inert or $p = 2$, then the structure of the parabolic Hecke ring is constructed in a similar way as the corresponding ring for the symplectic group of degree 2, see [Hei99, Section 3] or [Gri84, Section 2] for example.

In the case of a decomposable p, however, the situation is quite different. This follows from the fact that

$$H(U_p^2, G_p^2) \cong H(\mathrm{GL}_4(\mathbb{Z}_p), \mathrm{GL}_4(\mathbb{Q}_p))[x^{\pm 1}].$$

From this, the corresponding p-ring of the parabolic Hecke algebra is isomorphic to the ring of polynomials of one variable with coefficients from the Hecke ring of the parabolic subgroup

$$P_{1,2,1}(\mathbb{Z}_p) := \left\{ \begin{pmatrix} g_1 & * & * \\ 0 & g & * \\ 0 & 0 & g_2 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}_p) \mid g_1, g_2 \in \mathbb{Z}_p^\times,\ g \in \mathrm{GL}_2(\mathbb{Z}_p) \right\}.$$

Properties of this ring have been investigated in [Gri92b], and this is the ring where our calculations involving elements of the Hecke rings are going to occur. Finally, let us describe the action of elements of $H^{1,1}$ on Fourier-Jacobi forms.

Let F denote any modular form of weight k with respect to the parabolic subgroup $\Gamma_{1,1}$, as in Definition 2.2.4. If

$$X = \Gamma_{1,1} \begin{pmatrix} * & 0 & * & * \\ * & a & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & b \end{pmatrix} \Gamma_{1,1} = \sum_i \Gamma_{1,1} g_i \in H^{1,1},$$

for some $g_i \in S^{1,1}$, we define $F \mid_k X$ as in (2.3.1). Gritsenko gave the following very convenient definition of the signature.

Definition 2.4.2. The **signature** of X is defined as $s(X) := b/a$.

Lemma 2.4.3. *The signature is well-defined.*

Proof. We write $X = \Gamma_{1,1} g \Gamma_{1,1}$ for the above expression of X. We then want to show that if $\gamma_1, \gamma_2 \in \Gamma_{1,1}$ and $g' := \gamma_1 g \gamma_2$, then $g'_{44}/g'_{22} = b/a$. But, by the form of the elements γ_1, γ_2, we have $g'_{44} = b_1 b b_2$ and $g'_{22} = a_1 a a_2$, where we write a_1, b_1 for $(\gamma_1)_{22}, (\gamma_1)_{44}$ respectively and similarly for a_2, b_2. But now $b_1 \overline{a_1} = 1$ and $b_1, a_1 \in \mathcal{O}_K$. By going through the units in $\mathcal{O}_K$, we deduce $a_1 = b_1$ in any case. Similarly, $a_2 = b_2$ and from this the Lemma follows. □

Using the signature $s := s(X)$ of X, we can now define its action on Fourier-Jacobi forms.

Proposition 2.4.4. *Let $\phi \in J_{k,m}$ denote a Fourier-Jacobi form of weight k and index m. Then, for $Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix} \in \mathbb{H}_2$, we define the action of X on ϕ via*

$$(\phi \mid_k X)(\tau, z_1, z_2) := \left(\widetilde{\phi} \mid_k X\right)(Z) e\left(-\frac{m}{s}\omega\right),$$

with $\widetilde{\phi}(Z) := \phi(\tau, z_1, z_2) e(m\omega)$. Then $\phi \mid_k X$ belongs to $J_{k,m/s}$ if m/s is an integer and is 0 otherwise.

Proof. See [Gri92a, Lemma 4.1]. □

Note: Throughout this part of the thesis, we will often write $|$ instead of $|_k$ for the weight k-action on Fourier-Jacobi forms, as the weight is always fixed.

Now, if $F \in S_2^k$ is a Hermitian cusp form, we can write

$$F\left(\begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix}\right) = \sum_{m=1}^{\infty} \phi_m(\tau, z_1, z_2) e(m\omega).$$

For $X \in H^{1,1}$ as above, we have that $F\ |_k\ X$ is a modular form with respect to $\Gamma_{1,1}$ ([Gri92a, p. 2903]) and so we can write

$$(F\ |_k\ X)\left(\begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix}\right) = \sum_{m=1}^{\infty} \psi_m(\tau, z_1, z_2) e(m\omega).$$

Therefore, there is an action of Hecke operators from $H^{1,1}$ on the Fourier-Jacobi forms coming from a Hermitian modular form F via

$$\phi_m^{(F)} \mid\mid X := \psi_m^{(F|_k X)}. \tag{2.4.3}$$

Finally, we note here that this action is extended to P-forms in the obvious way.

2.5 *L*-functions and the Maass space

In this Chapter, we define the two main L-functions that we attach to a Hermitian cuspidal eigenform of degree two, the **standard** and **Gritsenko's *L*-function**.

Assume that $G \in S_2^k$ is a Hecke eigenform for H^2, i.e., it is an eigenfunction for all Hecke operators in H^2. For a polynomial $U[X] \in H^2[X]$ and G a Hecke eigenform, we denote by U_G the polynomial obtained by substituting the operators with their corresponding eigenvalues.

Definition 2.5.1. The **standard *L*-function** attached to G (see also [Shi00, Paragraph 20.6]) is defined as

$$Z_G^{(2)}(s) := \prod_{p \text{ inert or } p=2} Z_{p,G}^{(2)}(p^{-2s})^{-1} \prod_{p=\pi\overline{\pi}} Z_{\pi,G}^{(2)}(p^{-s})^{-1} Z_{\overline{\pi},G}^{(2)}(p^{-s})^{-1},$$

where for each inert prime p or $p = 2$, $Z_p^{(2)}(t) := \Phi^{-1}\left(z_p^{(2)}(t)\right)$ and for $2 \neq p = \pi\overline{\pi}$, $Z_\pi^{(2)}(t) := \Phi^{-1}\left(z_\pi^{(2)}(t)\right)$ and $Z_{\overline{\pi}}^{(2)}(t) := \Phi^{-1}\left(z_{\overline{\pi}}^{(2)}(t)\right)$, where

$$z_p^{(2)}(t) := \begin{cases} \prod_{i=1}^{2}(1 - p^2 x_{i,p} t)(1 - p^4 x_{i,p}^{-1} t) & \text{if } p \text{ inert} \\ \prod_{i=1}^{2}(1 - p x_i t)(1 - p^2 x_i^{-1} t) & \text{if } p = 2 \end{cases},$$

$$z_\pi^{(2)}(t) := \prod_{i=1}^{4}(1-p^{-1}x_{i,p}t),\ \ z_{\overline{\pi}}^{(2)}(t) := \prod_{i=1}^{4}(1-p^{4}x_{i,p}^{-1}t),$$

and Φ is the Satake mapping of equations (2.3.6) and (2.3.7).

Definition 2.5.2. The L-function attached to G by **Gritsenko** in [Gri88b, p. 2545] (for the case of p inert and $p=2$) and in the proof of [Gri88b, Lemma 2.1] (for the case of split prime p) is defined as

$$Q_G^{(2)}(s) := \prod_{p \text{ inert}}(1+p^{k-2-s})^{-2}Q_{p,G}^{(2)}(p^{-s})^{-1}\prod_{p \text{ splits or } p=2}Q_{p,G}^{(2)}(p^{-s})^{-1},$$

where $Q_p^{(2)}(t) := \Phi^{-1}\left(q_p^{(2)}(t)\right)$ with

$$q_p^{(2)}(t) := \begin{cases} (1-x_{0,p}t)\prod_{r=1}^{2}\prod_{1\le i_1<i_2\le 2}(1-p^{-r}x_{i_1,p}\cdots x_{i_r,p}x_{0,p}t) & \text{if } p \text{ is inert} \\ (1-x_{0,p}t)\prod_{r=1}^{2}\prod_{1\le i_1<i_2\le 2}(1-p^{-r}(x_{i_1,p}\cdots x_{i_r,p})^2x_{0,p}t) & \text{if } p=2 \\ \prod_{1\le i<j\le 4}(1-p^{-3}x_{i,p}x_{j,p}xt) & \text{if } p \text{ splits} \end{cases},$$

and Φ the Satake mapping of equations (2.3.6) and (2.3.7).

Let us now define the so-called Maass space for the case of Hermitian cusp forms. We mainly follow [Gri90] and for the Definition we will use [Gri90, Lemma 2.4].

Definition 2.5.3. The **Maass space** is the space

$$\left\{F\left(\begin{pmatrix}\tau & z_1\\ z_2 & \omega\end{pmatrix}\right) = \sum_{m=1}^{\infty}\left(\phi(\tau,z_1,z_2)\mid_k T_-(m)\right)e^{2\pi i m\omega}m^{3-k}\ \mid\ \phi\in J_{k,1}\right\},$$

where $T_-(m) := j_-(T(m)) \in H^{1,1}$, with

$$T(m) := \sum_{\substack{g\in S^1\cap \mathrm{M}_2(\mathbb{Z})\\ \mu(g)=m}}\Gamma_1 g\Gamma_1,$$

and j_- is the embedding of equation (2.4.1). In particular, this is the standard Hecke element of $\mathrm{SL}_2(\mathbb{Z})$, viewed as an element of H^1.

If now $F\in S_2^k$ is a Hecke eigenform in the Maass space, we can relate its Gritsenko L-function with the so-called symmetric square function of a classical modular form. Let us make this precise. First of all, for any $N\geq 1$, let

$$\Gamma_0(N) := \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})\mid c\equiv 0 \pmod N\right\}.$$

For any Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}$, let $S_k(\Gamma_0(N), \chi)$ denote the space of cusp forms of weight k and level $\Gamma_0(N)$ (see [DS06, p. 119] for a definition). Moreover, we associate to χ, the **Dirichlet L-function**

$$L(s,\chi) := \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1-\chi(p)p^{-s})^{-1}, \ \mathrm{Re}(s) > 1. \tag{2.5.1}$$

This can be extended meromorphically to the whole complex plane. For Chapters 2 and 3, we fix once and for all the Dirichlet character $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \longrightarrow \mathbb{C}$, with $\chi(3) = -1$. We then have the following Definition.

Definition 2.5.4. Let $f \in S_{k-1}(\Gamma_0(4), \chi)$ be a normalised Hecke eigenform with Fourier expansion $f(\tau) = \sum_{n\geq 1} a(n)e(n\tau)$. For each prime $p \neq 2$, we write

$$1 - a(p)t + \chi(p)p^{k-2}t^2 = (1-\alpha_p t)(1-\beta_p\chi(p)t), \tag{2.5.2}$$

where $\alpha_p, \beta_p \in \mathbb{C}$. The L-function attached to f and its twist by χ are given by:

$$L(f,s) := (1-a(2)2^{-s})^{-1}\prod_{p\neq 2}(1-\alpha_p p^{-s})^{-1}\left(1-\beta_p\chi(p)p^{-s}\right)^{-1} = \prod_p L_p(f,s)^{-1},$$

$$L(f,s,\chi) := (1-a(2)^{-1}2^{k-2-s})^{-1}\prod_{p\neq 2}(1-\alpha_p\chi(p)p^{-s})^{-1}\left(1-\beta_p p^{-s}\right)^{-1} = \\ = \prod_p L_p(f,s,\chi)^{-1}.$$

We also define the **symmetric square function** attached to f as follows:

$$R(f,s) := (1-a(2)^2 2^{-s})^{-1}(1-\overline{a(2)^2}2^{-s})^{-1}\times \\ \times \prod_{p\neq 2}\left[(1-\alpha_p^2 p^{-s})\left(1-\chi(p)\alpha_p\beta_p p^{-s}\right)(1-\beta_p^2 p^{-s})\right]^{-1}.$$

These converge in some right half plane and can be meromorphically continued to $\mathbb{C}$. The main property of the Maass space can then be stated as follows.

Proposition 2.5.5. *Let $F \in S_2^k$ belong in the Maass space (Definition 2.5.3) and assume F is an eigenfunction for the Hecke algebra H^2. Then, there exists a Hecke eigenform $f \in S_{k-1}(\Gamma_0(4), \chi)$, such that*

$$Q_F^{(2)}(s) = \zeta(s-k+1)L(s-k+2,\chi)\zeta(s-k+3)R(f,s),$$

We call F the ***Maass lift*** *of f.*

Proof. See [Gri90, Theorem, p. 69] or the Appendix in [Gri92a]. □

We end this Section by giving a Lemma regarding the correspondence of elliptic cusp forms and Hermitian cusp forms of degree 1, both as analytic objects as well as Hecke eigenforms.

Lemma 2.5.6. *A Hermitian cusp form of degree* 1 *and weight* k *with* $k \equiv 0 \pmod 4$ *can be considered as a classical cusp form of the same weight (i.e., for the group* $SL_2(\mathbb{Z})$*) and vice versa. Also, a classical cusp form which is a normalised eigenform for the Hecke algebra* $H(\mathrm{GL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Q}))$ *is also a normalised eigenform for* $H(\Gamma_1, S^1)$*, when considered as a Hermitian cusp form and vice versa.*

Proof. We have that $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z}) \cdot \{\alpha \cdot 1_2 \mid \alpha \in \mathcal{O}_K^\times\}$ and the corresponding upper half planes are the same. So, holomorphicity is equivalent (including infinity).

For the invariance condition, the one direction is trivial, as $\mathrm{SL}_2(\mathbb{Z}) \subseteq \Gamma_1$. For the other one, let $\gamma \in \Gamma_1$ and write $\gamma = \alpha\delta$ with $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\alpha \in \mathcal{O}_K^\times$. Then

$$(F \mid_k \gamma)(Z) = (\alpha cz + \alpha d)^{-k} F(Z) = (F \mid_k \delta)(Z),$$

as $k \equiv 0 \pmod 4$ and $\mathcal{O}_K^\times = \{\pm 1, \pm i\}$. Cuspidality is also clear from the definitions.

Assume now that we start with a normalised (i.e., $a(1) = 1$ in the Fourier expansion) Hermitian cusp form h of degree 1, which we further take to be an eigenform for H^1. The canonical embedding of $\mathrm{GL}_2^+(\mathbb{Q})$ into S^1, the group of similitudes of degree 1, induced from the embedding $\mathbb{Q} \hookrightarrow K$, allows us to see h as a normalised Hecke eigenform with respect to $H(\mathrm{GL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Q}))$, as we can always choose $g \in \mathrm{GL}_2^+(\mathbb{Q})$ as a representative for $\mathrm{GL}_2(\mathbb{Z}) g \mathrm{GL}_2(\mathbb{Z})$, and $\mathrm{GL}_2(\mathbb{Z}) \subset \Gamma_1$.

But the converse is also true, that is, if we start with h a classical normalised Hecke eigenform, then it is also a normalised Hermitian eigenform of degree 1. Indeed, since the Hecke operators of the Hermitian Hecke algebra are normal (see Remark 2.3.4), we know that the space of Hermitian cusp forms is diagonalizable with a finite basis $\{h_i\}$ of normalised eigenforms for H^1. But from the above, each h_i is a Hecke eigenform for the classical Hecke algebra as well. Hence, this basis has to coincide with the basis derived by diagonalising the action of the classical Hecke algebra, thanks to the multiplicity one theorem. Now, because of the normalisation, $h = h_i$ for some i, which shows that h is indeed a Hecke eigenform for H^1. □

Remark 2.5.7. From now on, we will use the terms "classical (or elliptic) cusp form" and "Hermitian cusp form of degree 1" interchangeably.

Chapter 3

A Dirichlet Series Associated With Three Hermitian Modular Forms

In this Chapter, we consider a Dirichlet series $D_{F,G,h}(s)$, analogous to the one considered by Heim in [Hei99, Section 2.4, (29)], attached to three Hermitian cuspidal eigenforms F, G, h, of degrees $2, 2$ and 1, respectively, all having weight $k \equiv 0 \pmod 4$. We take F in the Maass space, and we study the p-factor $D^{(p)}_{F,G,h}(s)$ of the Dirichlet series for each rational prime p. We show that in the case when p remains prime in $\mathcal{O}_K = \mathbb{Z}[i]$, $D^{(p)}_{F,G,h}(s)$ is identified with the p-factor of $Z_{G\otimes h}(s)$, the twist by h of Gritsenko's L-function attached to G. Moreover, for the case of a split prime p, we obtain a rational expression for $D^{(p)}_{F,G,h}(s)$, showing a relation with $Z_{G\otimes h}(s)$. By combining these results, we show that $D_{F,G,h}(s)$ has an Euler product.

Moreover, we show that this Dirichlet series arises as part of a Rankin-Selberg inner product of a Hermitian Eisenstein series of Siegel type (see Definition 2.1.6) on the unitary group $U(5,5)(K)$, diagonally-restricted on $U(2,2)(K) \times U(2,2)(K) \times U(1,1)(K)$, against F, G and h. This representation also produces an additional residue term, which is not studied in this thesis.

3.1 Overview of Heim's Results

The so-called **doubling method** has been a very powerful tool in the study of the standard L-function attached to Siegel and Hermitian modular forms.

The main idea, going back to Garrett in [Gar84] and Böcherer in [Böc85], is as follows: Let $1 \leq n \leq m$. Then, the Petersson inner product of a diagonally restricted Siegel-type Eisenstein series of degree $n+m$ against a Siegel cusp form F of degree n is proportional to the product of the standard L-function attached to F with a

Klingen-type Eisenstein series of degree m, attached to $\overline{F}$. Therefore, analytic (and algebraic) properties for the standard L-function can be studied using the Eisenstein series.

The powerful consequences of this method make the question of whether this generalises to more copies of the group natural to consider. Garrett in [Gar87] considered the inner product of a Siegel-type symplectic Eisenstein series of degree 3, diagonally restricted to $\mathbb{H} \times \mathbb{H} \times \mathbb{H}$, against three classical cusp eigenforms. He managed to relate this to the **triple product L-function** attached to them ([Gar87, Theorem 1.3]). Analytic and arithmetic properties of this L-function then follow from the ones of the Eisenstein series.

It seems, therefore, natural to investigate the following idea: Consider an Eisenstein series of Siegel-type of degree $n \geq 1$ and restrict it diagonally in k blocks of sizes $n_1, n_2, \cdots, n_k$. We could then ask if the inner product of the Eisenstein series against k cusp forms of degrees $n_1, n_2, \cdots, n_k$ affords an Euler product, which can be related to some known L-function.

The only known instance when that happens is the case of the symplectic group with $n = 5$ and $n_1 = n_2 = 2, n_3 = 1$. In particular, let F, G, h be Siegel modular forms of even weight $k \geq 0$ and degrees $2, 2$ and 1, respectively. Heim, in [Hei99, Theorem 2.7], by considering such an inner product, obtained an integral representation of the Dirichlet series

$$D_{F,G,h}(s) := \sum_{\substack{m,\epsilon,\ell=1 \\ (\epsilon,\ell)=1}}^{\infty} \langle \phi_m \mid U_\ell,\ \psi_{m\ell^2} \rangle a_{m\epsilon^2} \epsilon^{-2(k+s-1)} \ell^{-2(k+s-2)} m^{-(2k+s-3)}. \tag{3.1.1}$$

Here, $\{\phi_m\}, \{\psi_m\}$ are the Fourier-Jacobi coefficients of F, G respectively, $\{a_m\}$ the Fourier coefficients of h, and U_ℓ is an index-raising operator acting on Fourier-Jacobi forms (see [Hei99, p. 214, (24)]).

In the case when F, G, h are all Hecke eigenforms and F is in the Maass space (hence plays the role of an **auxiliary function**), Heim considered a Hecke-Jacobi theory in the context of parabolic Hecke rings in order to obtain an Euler product. In particular, he used factorisation methods, as introduced by Andrianov in [And74] and then developed by Gritsenko in a series of papers (e.g. [Gri84], [Gri95]).

It is then Heim's result in [Hei99, Theorem 5.1] that gives the relation of this inner product with the L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$. This integral expression was later exploited systematically by Böcherer and Heim in [BH00] and [BH06], in order to establish various algebraicity properties and lift restrictions on the weights of the Siegel and elliptic modular forms by the use of differential operators.

In our case, we consider the exact analogue of the Dirichlet series in (3.1.1) for the case of three Hermitian cuspidal eigenforms. Our motivation is the similarities between the symplectic and the unitary group of degree 2, as well as analogous factorisation methods that exist for Hermitian modular forms as well (see, for example, [Gri92a]). We investigate the arithmetic properties of $D_{F,G,h}(s)$ and relate it to the twist of Gritsenko's L-function by a classical L-function. Moreover, we also consider the analogous integral considered by Heim and show that it produces $D_{F,G,h}(s)$, together with an **additional residue term**. This is a fascinating phenomenon, special to the unitary setting. However, it is not investigated in this thesis, and we hope it will be the subject matter of a future work.

3.2 Hermitian Dirichlet Series

In this Section, we will define the Dirichlet series, which will be the main object of study for this Chapter.

Assume $k \equiv 0 \pmod 4$. Let $F, G \in S_2^k$ and $h \in S_1^k$ with **real** Fourier coefficients. This is a technical assumption that could be lifted. Write $Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix} \in \mathbb{H}_2,\ W \in \mathbb{H}_1$ and consider the Fourier-Jacobi expansions of F, G and the Fourier expansion of h as follows:

$$\begin{gathered} F(Z) = \sum_{m=1}^{\infty} \phi_m(\tau, z_1, z_2) e^{2\pi i m \omega},\ G(Z) = \sum_{m=1}^{\infty} \psi_m(\tau, z_1, z_2) e^{2\pi i m \omega}, \\ h(W) = \sum_{n=1}^{\infty} a_n e^{2\pi i n W}. \end{gathered} \tag{3.2.1}$$

Now, for any $p \in \mathcal{O}_K$, we define the operator U_p acting on Fourier-Jacobi forms:

$$\begin{aligned} U_p : J_{k,m} &\longrightarrow J_{k,mN(p)} \\ \phi_m(\tau, z_1, z_2) &\longmapsto \phi_m(\tau, \overline{p} z_1, p z_2). \end{aligned} \tag{3.2.2}$$

This is well-defined by [Das10, p. 427]. We now define the Dirichlet series of interest as follows:

$$D_{F,G,h}(s) := \sum_{p,q} \sum_{m=1}^{\infty} \langle \phi_m \mid U_p, \psi_{mN(p)} \rangle a_{mN(q)} N(p)^{-(k+s-3)} N(q)^{-(k+s-1)} m^{-(2k+s-4)}. \tag{3.2.3}$$

Here, $p, q \in \mathbb{Z}[i] \backslash \{0\}$ with $\gcd(p, q) = 1,\ q = u + iv,\ u > 0,\ v \geq 0$. The reason we sum like this will become clear when we consider the integral representation (see Corollary 3.6.3).

Lemma 3.2.1. *The Dirichlet series $D_{F,G,h}(s)$ converges absolutely for* $\mathrm{Re}(s) > 4$ *and represents a holomorphic function in this domain.*

Proof. We observe that

$$F\left(\begin{pmatrix} \tau & \overline{p}z_1 \\ pz_2 & N(p)\omega \end{pmatrix}\right) = \sum_{m=1}^{\infty} (\phi_m \mid U_p)(\tau, z_1, z_2) e^{2\pi i m N(p)\omega}.$$

In particular, as in the proof of [KS89, Lemma 1], we have $\langle \phi_m \mid U_p, \psi_{mN(p)} \rangle = \mathcal{O}\left((mN(p))^k\right)$ for each $m \geq 1$ (see also [Gri92a, (5.6)]). Moreover, $a_m = \mathcal{O}(m^{k/2})$, since h is a cuspidal Hecke eigenform. Hence, the Lemma follows. □

3.3 Inert primes

Our aim in this Section is to relate the p-factor of the Dirichlet series of (3.2.3) with L-functions, when F, G, h are Hecke eigenforms. This case is closer to the situation considered by Heim in [Hei99]. We first start by proving several results related to the Hecke theory.

3.3.1 Hecke operators and weak rationality theorems

Throughout this section, p is assumed to be a rational prime which remains prime in $\mathcal{O}_K$. Let us make a list of Hecke operators in $H(\Gamma_2, S_p^2)$ and $H(\Gamma_{1,1}, S_p^{1,1})$ and relations between them. We use the notation of Chapter 2. In particular, Γ_2 is the Hermitian modular group and $\Gamma_{1,1}$ the relevant parabolic subgroup.

- $T_p := \Gamma_2 \mathrm{diag}(1, 1, p, p)\Gamma_2$.
- $T_{1,p} := \Gamma_2 \mathrm{diag}(1, p, p^2, p)\Gamma_2$.
- $\Delta_p := \Gamma_2 \mathrm{diag}(p, p, p, p)\Gamma_2 = \Gamma_2 \mathrm{diag}(p, p, p, p)$.
- $T^J(p) := \Gamma_{1,1} \mathrm{diag}(1, p, p^2, p)\Gamma_{1,1}$.
- $\nabla_p := \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \Gamma_{1,1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & a \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \nabla a.$
- $\Delta_{p^\delta} := \Gamma_{1,1} \mathrm{diag}(p^\delta, p^\delta, p^\delta, p^\delta)\Gamma_{1,1} = \Gamma_{1,1} \mathrm{diag}(p^\delta, p^\delta, p^\delta, p^\delta),\ \delta \geq 1$.

It is known that $H(\Gamma_2, S_p^2)$ is generated by $T_p, T_{1,p}, \Delta_p$ and their inverses as a $\mathbb{Q}$-algebra (see for example [Klo15, Section 4.1.2]).

Also, for any operator $X(p)$, we write $X^r(p)$ to denote $\Delta_p^{-1} X(p)$. Note here that we use the same notation for Δ_p as an element of H_p^2 and as an element of $H_p^{1,1}$. Finally, we define

$$T_\pm(p^\delta) := j_\pm(T(p^\delta)),\ \Lambda_\pm(p^\delta) := j_\pm\left(\Gamma_1 \mathrm{diag}(p^\delta, p^\delta)\Gamma_1\right),\ \delta \geq 1, \tag{3.3.1}$$

where $j_\pm$ are the embeddings of equation (2.4.1) and $T(p^\delta)$ as in Definition 2.5.3. Therefore

- $\Lambda_-(p) = \Gamma_{1,1}\mathrm{diag}(p, p^2, p, 1)\Gamma_{1,1} = \Gamma_{1,1}\mathrm{diag}(p, p^2, p, 1)$.
- $\Lambda_+(p) = \Gamma_{1,1}\mathrm{diag}(p, 1, p, p^2)\Gamma_{1,1}$.
- $T_-(p) = \Gamma_{1,1}\mathrm{diag}(1, p, p, 1)\Gamma_{1,1}$.
- $T_+(p) = \Gamma_{1,1}\mathrm{diag}(1, 1, p, p)\Gamma_{1,1}$.

As $\Lambda_-(p)$ has only one right coset and $(\Lambda_-(p))^* = \Lambda_+(p)$, where $*$ is the anti-homomorphism of equation (2.4.2) (see [Gri92a, p. 2894]), we have

$$\Lambda_\pm(p^\delta) = \Lambda_\pm(p^{\delta-1})\Lambda_\pm(p),\ \forall\delta \geq 1. \tag{3.3.2}$$

This also implies $\Lambda_\pm(p^\delta) = \Lambda_\pm(p)^\delta$ for all $\delta \geq 1$.

Proposition 3.3.1. *Let Φ denote the Satake mapping of Section 2.3 for the inert prime p. We have*

- $\Phi(T_p) = x_0 + p^{-1}x_0x_1 + p^{-1}x_0x_2 + p^{-2}x_0x_1x_2 = x_0(1 + p^{-1}x_1)(1 + p^{-1}x_2)$.
- $\Phi(T_{1,p}) = p^{-2}x_0^2x_1 + p^{-2}x_0^2x_2 + p^{-4}x_0^2x_1^2x_2 + p^{-4}x_0^2x_1x_2^2 + p^{-6}(p^2+1)(p-1)x_0^2x_1x_2$.
- $\Phi(\Delta_p) = p^{-6}x_0^2x_1x_2$.

Proof. From Definition 2.5.2, we have

$$q_p^{(2)}(t) = (1 - x_0t)(1 - p^{-1}x_0x_1t)(1 - p^{-1}x_0x_2t)(1 - p^{-2}x_0x_1x_2t).$$

Also, from [Gri92a, Lemma 3.6], we have

$$\Phi^{-1}(q_p^{(2)}(t)) = 1 - T_pt + (pT_{1,p} + p(p^3 + p^2 - p + 1)\Delta_p)t^2 - p^4\Delta_pT_pt^3 + p^8\Delta_p^2t^4.$$

By comparing these two expressions, the Proposition follows. □

Let now $D_p^{(2)}(X) := Z_p^{(2)}(p^{-3}X)$, with $Z_p^{(2)}$ as in Definition 2.5.1.

Proposition 3.3.2. *We have*

$$D_p^{(2)}(X) = 1 - B_1X + B_2X^2 - B_1X^3 + X^4,$$

where

$$B_1 = p^{-3}\Delta_p^{-1}(T_{1,p} - (p^2 + 1)(p - 1)\Delta_p),$$
$$B_2 = p^{-4}\Delta_p^{-1}(T_p^2 - 2pT_{1,p} - 2p(p^2 - p + 1)\Delta_p).$$

Proof. This follows by direct verification, after applying the Satake isomorphism and using Proposition 3.3.1. We remind the reader here that $Z_p^{(2)}(X) = \Phi^{-1}(z_p^{(2)}(X))$, where

$$z_p^{(2)}(X) = \prod_{i=1}^{2}(1 - p^4 x_{i,p}^{-1}X)(1 - p^2 x_{i,p}X).$$

This also gives the Φ-image of $D_p^{(2)}$. □

We now have the following Proposition regarding the factorisation of $D_p^{(2)}$.

Proposition 3.3.3. *We have the following factorisation in $H_p^{1,1}[X]$:*

$$D_p^{(2)}(X) = (1 - p^{-3}\Delta_p^{-1}\Lambda_-(p)X)S^{(2)}(X)(1 - p^{-3}\Delta_p^{-1}\Lambda_+(p)X),$$

where

$$S^{(2)}(X) = S_0 - S_1X + S_2X^2 - S_3X^3,$$

with

- $S_0 = 1$.
- $S_1 = p^{-3}(T^{J,r}(p) + \nabla_p^r - p(p^2 - p + 1))$.
- $S_2 = p^{-4}\Delta_p^{-1}T_+(p)T_-(p) - p^{-3}T^{J,r}(p) - 2p^{-3}\nabla_p^r - p^{-2}(p - 2)$.
- $S_3 = p^{-3}(\nabla_p^r - p)$.

Proof. This can be verified directly by using the following relations, which can be found in the proof [Gri92a, Proposition 3.2], or can be proved directly.

- $\epsilon(T_{1,p}) = T^J(p) + \Lambda_-(p) + \Lambda_+(p) + \nabla_p - \Delta_p$.
- $\epsilon(T_p) = T_-(p) + T_+(p)$.
- $T_-(p)T_+(p) = pT^J(p) + (p^3 + p^4)\Delta_p$.
- $\Lambda_-(p)T_+(p) = p^3\Delta_pT_-(p)$.
- $T_-(p)\Lambda_+(p) = p^3\Delta_pT_+(p)$.
- $\Lambda_-(p)\Lambda_+(p) = p^6\Delta_p^2$.
- $\Lambda_-^r(p)\nabla_p^r = p\Lambda_-^r(p)$.
- $\nabla_p^r\Lambda_+^r(p) = p\Lambda_+^r(p)$.

Here, ϵ denotes the embedding of $H(\Gamma_2, S^2)$ to $H(\Gamma_{1,1}, S^{1,1})$, as described in Lemma 2.4.1. □

Now, if $F \in S_2^k$ has a Fourier-Jacobi expansion as in equation (3.2.1) and $Q_p^{(2)}$ denotes the p-factor of Gritsenko's L-function, as in Definition 2.5.2, we have the following weak rationality Propositions.

Proposition 3.3.4. *Let $F \in S_2^k$ be a Hecke eigenform for $H(\Gamma_2, S^2)$ and $m \geq 1$. Then*

$$Q_{p,F}^{(2)}(X)\sum_{\delta\geq 0}\phi_{mp^\delta} \mid T_+(p^\delta)X^\delta = \left(\phi_m - \phi_{m/p} \mid T_-(p)X + p\phi_{m/p^2} \mid \Lambda_-(p)X^2\right) \mid$$
$$\mid (1+p(\nabla_p - p\Delta_p)X^2),$$

where $\phi_m \mid (1+p(\nabla_p - p\Delta_p)X^2) = \begin{cases}\phi_m & \text{if } p \mid m \\ (1-p^{2k-6}X^2)\phi_m & \text{otherwise}\end{cases}.$

Proof. We follow the same proof as in [Gri95, Corollary 1]. Then, the result follows from [Gri92a, Proposition 3.2]. We will just show the computations for the last claim of our Proposition. We have

$$(F \mid \Delta_p)(Z) = (p^2)^{2k-4}(p^2)^{-k}F(Z) = p^{2k-8}F(Z),$$

and so $\phi_m \mid \Delta_p = p^{2k-8}\phi_m$. Also,

$$(F \mid \nabla a)(Z) = (p^2)^{2k-4}(p^2)^{-k}F\left(\begin{pmatrix}\tau & z_1 \\ z_2 & \tau' + a/p\end{pmatrix}\right) =$$
$$= p^{2k-8}\sum_{m=1}^{\infty}\phi_m(\tau, z_1, z_2)e^{2\pi i m\tau'}e^{2\pi i m a/p},$$

so

$$\phi_m \mid \nabla_p = \left(p^{2k-8}\sum_{a=0}^{p-1}e^{2\pi i m a/p}\right)\phi_m = \begin{cases}0 & \text{if } (m,p)=1 \\ p^{2k-7}\phi_m & \text{otherwise}\end{cases},$$

from which the result follows. □

Proposition 3.3.5. *Let $F \in S_2^k$ be a Hecke eigenform for $H(\Gamma_2, S^2)$ and $m \geq 1$. Then*

$$D_{p,F}^{(2)}(X)\sum_{\delta\geq 0}\phi_{mp^{2\delta}} \mid (\Delta_{p^\delta}^{-1}\Lambda_+(p^\delta))(p^{-3}X)^\delta = \phi_m \mid S^{(2)}(X) -$$
$$- \phi_{m/p^2} \mid (\Delta_p^{-1}\Lambda_-(p)S^{(2)}(X))p^{-3}X.$$

Proof. This follows from Proposition 3.3.3, using the same techniques as in [Gri95, Corollary 1]. □

In a similar fashion to Heim in [Hei99, page 227] now, we have that the action of the operators $T_+(p), \Lambda_+(p), \nabla_p^r(p)$ on Fourier-Jacobi forms of index coprime to p is identical to zero. This leads to the definition of the following polynomials:

$$S^{(2)}(X)^{\text{factor}} := 1 - (p^{-3}T^{J,r}(p) - p^{-2} + p^{-1})X + p^{-2}X^2,$$

$$S^{(2)}(X)^{\text{prim}} := 1 - (p^{-3}T^{J,r}(p) - p^{-2}(p^2 - p + 1))X +$$
$$(-p^{-3}T^{J,r}(p) - p^{-2}(p-2))X^2 + p^{-2}X^3$$

$$= S^{(2)}(X)^{\text{factor}}(1+X).$$

Hence, $\phi \mid S^{(2)}(X) = \phi \mid S^{(2)}(X)^{\text{prim}}$ if $\phi \in J_{k,m}$ with $\gcd(m,p)=1$. We now have the following Lemma.

Lemma 3.3.6. *Let $\phi \in J_{k,p}$. Then $\phi \mid S^{(2)}(X)T_+(p) = \phi \mid T_+(p)S^{(2)}(X)^{\text{factor}}$.*

Proof. The proof follows by the following results:

- $\phi \mid \nabla_p^r = p\phi$.
- $\phi \mid T_+(p)\nabla_p^r = 0$, because $\phi \mid T_+(p)$ will have index 1.
- $\phi \mid \left[T^{J,r}(p),\ T_+(p)\right] = \phi \mid (p^3T_+(p) - \nabla_p^r T_+(p) = (p^3-p)\phi \mid T_+(p)$, by the first point.

Here $\left[T^{J,r}(p),\ T_+(p)\right] := T^{J,r}(p)T_+(p) - T_+(p)T^{J,r}(p)$ denotes the commutator. We will now give the proof of the third point. As in the proof of Proposition 3.3.3, we have

- $\epsilon(T_{1,p}) = T^J(p) + \Lambda_-(p) + \Lambda_+(p) + \nabla_p - \Delta_p$,
- $\epsilon(T_p) = T_-(p) + T_+(p)$.

Now, H_p^2 is a commutative Hecke algebra and as ϵ is a ring homomorphism, we have

$$\epsilon(T_{1,p})\epsilon(T_p) = \epsilon(T_p)\epsilon(T_{1,p}).$$

By then considering the elements whose product has signature p (see Definition 2.4.2 and [Hei99, Section 3.3]), we obtain

$$T^J(p)T_+(p) + \Lambda_+(p)T_-(p) + (\nabla_p - \Delta_p)T_+(p) = T_+(p)T^J(p) + T_-(p)\Lambda_+(p) + \\ + T_+(p)(\nabla_p - \Delta_p),$$

from which the result follows, as $\phi \mid \Lambda_+(p) = 0$ for $\phi \in J_{k,p}$ and $T_-(p)\Lambda_+(p) = p^3\Delta_pT_+(p)$. □

3.3.2 Calculation of the Dirichlet Series

Let now F, G, h have Fourier expansions as in equation (3.2.1) with real Fourier coefficients. In what follows, we will assume that F, G, h are all Hecke eigenforms for their corresponding Hecke rings (h is assumed to be normalised) and F is in the Maass space, as we have defined in Definition 2.5.3. We can rewrite $D_{F,G,h}(s)$ of equation (3.2.3) as (we have $\phi_m = m^{3-k}\phi_1 \mid T_-(m)$ for all $m \geq 1$ from Definition 2.5.3)

$$D_{F,G,h}(s) = 4\sum_{l,\epsilon,m} \langle m^{3-k}\phi_1 \mid T_-(m)U_l, \psi_{mN(l)}\rangle a_{mN(\epsilon)} N(l)^{-(k+s-3)} N(\epsilon)^{-(k+s-1)} \times \times m^{-(2k+s-4)}$$
$$= 4\beta_k \sum_{l,\epsilon,m} \langle \widetilde{\phi}_1 \mid T_-(m)U_l, \widetilde{\psi}_{mN(l)}\rangle_{\mathcal{A}} a_{mN(\epsilon)} N(l)^{-s} N(\epsilon)^{-(k+s-1)} m^{-(2k+s-4)}, \tag{3.3.3}$$

with $l, \epsilon \in \mathbb{Z}[i]$ coprime with their real parts positive and imaginary parts non-negative and $m \in \mathbb{N}$. Also, β_k is the constant of Lemma 2.2.10. Now, if $\phi \in J_{k,m}$, we have from Proposition 2.4.4 and the fact that $\Lambda_-(p)$ has a single right coset representative, that

$$\phi \mid \Lambda_-(p) = p^{3k-8}\tilde{\phi}\left(\begin{pmatrix} \tau & pz_1 \\ pz_2 & p^2\tau' \end{pmatrix}\right) e^{-2\pi m p^2 \tau'} = p^{3k-8}\phi(\tau, pz_1, pz_2) = p^{3k-8}\phi \mid U_p,$$

with U_p the operator of (3.2.2). We now define the p-part of the Dirichlet series

$$D^{(p)}_{F,G,h}(s) := \sum_{l,\epsilon,m\geq 0} \langle \widetilde{\phi}_1 \mid T_-(p^m)U_{p^l}, \widetilde{\psi}_{p^{m+2l}}\rangle_{\mathcal{A}} a_{p^{m+2\epsilon}} p^{-2sl} p^{-2(k+s-1)\epsilon} p^{-(2k+s-4)m}$$
$$= \sum_{l,\epsilon,m\geq 0} \langle \widetilde{\phi}_1 \mid T_-(p^m)\Lambda_-(p^l), \widetilde{\psi}_{p^{m+2l}}\rangle_{\mathcal{A}} a_{p^{m+2\epsilon}} p^{-(3k+2s-8)l} p^{-2(k+s-1)\epsilon} \times \times p^{-(2k+s-4)m}, \tag{3.3.4}$$

together with the condition that $\min(l, \epsilon) = 0$. The last line is obtained using the relation between U_p and $\Lambda_-(p)$ (and hence of $\Lambda_-(p^l)$ and U_{p^l}). This series converges absolutely for $\mathrm{Re}(s) > 4$, by comparison with $D_{F,G,h}(s)$ (see Lemma 3.2.1).

Now, with respect to the inner product of Fourier-Jacobi forms, we have by [Gri92a, Proposition 5.1] that $\Lambda_-^{\mathrm{adj}}(p^l) = p^{(2k-6)l}\Lambda_+(p^l)$ and $T_-^{\mathrm{adj}}(p^l) = p^{(k-3)l}T_+(p^l)$ for any $l \geq 1$. This then gives that the adjoint of $\Lambda_-(p^l)$ is $\Lambda_+(p^l)$ for the inner product of P-forms and similarly the P-form adjoint for $T_-(p^l)$ is $T_+(p^l)$.

Let now $X := p^{-(k+s-1)}$ and $N := p^{k-1}$. Consider the Satake parameters α_1, α_2 of the modular form h such that $\alpha_1 + \alpha_2 = a_p$ and $\alpha_1\alpha_2 = p^{k-1}$. Let also $X_i := \alpha_i p^{-(2k+s-4)}$, $i = 1, 2$. We write

$$D^{(p)}_{F,G,h}(s) = D_{(\epsilon)}(s) + D_{(l)}(s) - D_{(\epsilon,l)}(s), \tag{3.3.5}$$

where the corresponding index means that this variable (or both) is 0. Using the fact that

$$a_{p^m} = \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2},$$

and properties for the adjoint operators we mentioned above, we obtain:

$$\begin{aligned} D_{(\epsilon)}(s)(\alpha_1 - \alpha_2) = \alpha_1 \sum_{l,m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{m+2l}} \mid T_+(p^m)\Lambda_+(p^l)\rangle_{\mathcal{A}} p^{-(3k+2s-8)l}(\alpha_1 p^{-(2k+s-4)})^m \\ - \alpha_2 \sum_{l,m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{m+2l}} \mid T_+(p^m)\Lambda_+(p^l)\rangle_{\mathcal{A}} p^{-(3k+2s-8)l}(\alpha_2 p^{-(2k+s-4)})^m. \end{aligned} \tag{3.3.6}$$

$$\begin{aligned} D_{(l)}(s)(\alpha_1 - \alpha_2) = \alpha_1 \sum_{\epsilon,m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^m} \mid T_+(p^m)\rangle_{\mathcal{A}} (\alpha_1 p^{-(k+s-1)})^{2\epsilon}(\alpha_1 p^{-(2k+s-4)})^m \\ - \alpha_2 \sum_{\epsilon,m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^m} \mid T_+(p^m)\rangle_{\mathcal{A}} (\alpha_2 p^{-(k+s-1)})^{2\epsilon}(\alpha_2 p^{-(2k+s-4)})^m. \end{aligned} \tag{3.3.7}$$

$$\begin{aligned} D_{(\epsilon,l)}(s)(\alpha_1 - \alpha_2) = \alpha_1 \sum_{m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^m} \mid T_+(p^m)\rangle_{\mathcal{A}} (\alpha_1 p^{-(2k+s-4)})^m \\ - \alpha_2 \sum_{m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^m} \mid T_+(p^m)\rangle_{\mathcal{A}} (\alpha_1 p^{-(2k+s-4)})^m. \end{aligned} \tag{3.3.8}$$

Remark 3.3.7. In the following, we want to show a relation of $D^{(p)}_{F,G,h}(s)$ with some other holomorphic function on an open subset of $\mathbb{C}$, namely for $\mathrm{Re}(s)$ large enough. By the Identity Theorem, it suffices to show equality when s is large enough and real. This is true because that part of the real line has accumulation points (in fact, every point is an accumulation point). Therefore, we will show the equalities below for $s \in \mathbb{R}$ large enough.

Proposition 3.3.8. *We have*

$$D_{(l)}(s) - D_{(\epsilon,l)}(s) = \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}}}{\alpha_1 - \alpha_2} \left(\frac{\alpha_1^3 X^2}{Q^{(2)}_{p,G}(X_1)} - \frac{\alpha_2^3 X^2}{Q^{(2)}_{p,G}(X_2)} \right).$$

Proof. This follows from equations (3.3.7), (3.3.8) and Proposition 3.3.4 with $m = 1$. We have (because we have a Hermitian inner product, we need to conjugate in the second argument)

$$\begin{aligned} \sum_{m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^m} \mid T_+(p^m)\rangle_{\mathcal{A}} X_1^m &= \sum_{m=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^m} \mid T_+(p^m) X_2^m\rangle_{\mathcal{A}} \\ &= \langle \tilde{\phi}_1, Q^{(2)}_{p,G}(X_2)^{-1}(1 - p^{2k-6}X_2^2)\tilde{\psi}_1\rangle_{\mathcal{A}} \end{aligned}$$

$$= (1 - p^{2k-6}X_1^2)Q_{p,G}^{(2)}(X_1)^{-1}\langle\tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}},$$

because X_1, X_2 are complex conjugates. Also,

$$\sum_{\epsilon=0}^{\infty}(\alpha_1 p^{-(k+s-1)})^{2\epsilon} = \frac{1}{1 - \alpha_1^2 p^{-2(k+s-1)}}.$$

So, the first part of the difference we are interested in is

$$\alpha_1 \frac{\langle\tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}}}{\alpha_1 - \alpha_2} Q_{p,G}^{(2)}(X_1)^{-1}(1 - p^{2k-6}X_1^2)\left(\frac{1}{1 - \alpha_1^2 p^{-2(k+s-1)}} - 1\right) = \frac{\langle\tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}}}{\alpha_1 - \alpha_2}\frac{\alpha_1^3 X^2}{Q_{p,G}^{(2)}(X_1)},$$

and similarly for the second part. □

Proposition 3.3.9. *Let $Y := p^2NX^2$ and $l \geq 0$. Then, for $i = 1, 2$, we have*

$$\sum_{m=0}^{\infty} \tilde{\psi}_{p^{m+2l}} \mid T_+(p^m)\Lambda_+(p^l)X_i^m(X^2N^{-1}p^5)^l = Q_{p,G}^{(2)}(X_i)^{-1}(\tilde{\psi}_{p^{2l}} - \tilde{\psi}_{p^{2l-1}} \mid T_-(p)X_i +$$
$$+ p\tilde{\psi}_{p^{2l-2}} \mid \Lambda_-(p)X_i^2) \mid \left((1 + p(\nabla_p - p\Delta_p)X_i^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l\right).$$

Proof. The proof follows from Proposition 3.3.4, together with the fact that $G \mid \Delta_{p^l} = (p^{2k-8})^l G$. □

In order to compute $D_\epsilon(s)$ of equation (3.3.6), we will compute each of the summands above. We note that we need to interchange the X_i's when we take them in/out of the inner products.

Proposition 3.3.10. *We have*

$$\alpha_1 Q_{p,G}^{(2)}(X_1)^{-1}\sum_{l=0}^{\infty}\langle\tilde{\phi}_1, \tilde{\psi}_{p^{2l}} \mid_k (1 + p(\nabla_p - p\Delta_p)X_2^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l\rangle_{\mathcal{A}} -$$
$$-\alpha_2 Q_{p,G}^{(2)}(X_2)^{-1}\sum_{l=0}^{\infty}\langle\tilde{\phi}_1, \tilde{\psi}_{p^{2l}} \mid_k (1 + p(\nabla_p - p\Delta_p)X_1^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l\rangle_{\mathcal{A}} =$$

$$= \left(\alpha_1 Q_{p,G}^{(2)}(X_1)^{-1} - \alpha_2 Q_{p,G}^{(2)}(X_2)^{-1}\right)\frac{\langle\tilde{\phi}_1, \tilde{\psi}_1 \mid_k S^{(2)}(Y)\rangle_{\mathcal{A}}}{D_{p,G}^{(2)}(Y)} +$$
$$+ \left(\frac{\alpha_2^3 X^2}{Q_{p,G}^{(2)}(X_2)} - \frac{\alpha_1^3 X^2}{Q_{p,G}^{(2)}(X_1)}\right)\langle\tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}}.$$

Proof. We first observe

$$\tilde{\psi}_{p^{2l}} \mid_k (1 + p(\nabla_p - p\Delta_p)X_2^2) = \begin{cases}(1 - p^{2k-6}X_2^2)\tilde{\psi}_1 & \text{if } l = 0\\ \tilde{\psi}_{p^{2l}} & \text{if } l \geq 1\end{cases},$$

by using the result of Proposition 3.3.4. Hence, by Proposition 3.3.5 we obtain

$$\sum_{l=0}^{\infty} \tilde{\psi}_{p^{2l}} \mid_k (1 + p(\nabla_p - p\Delta_p)X_2^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l = (1 - p^{2k-6}X_2^2)\tilde{\psi}_1 + \\ + \sum_{l=1}^{\infty} \tilde{\psi}_{p^{2l}} \mid_k \Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l$$

$$= \sum_{l=0}^{\infty} \tilde{\psi}_{p^{2l}} \mid_k \Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l - p^{2k-6}X_2^2\tilde{\psi}_1 = \tilde{\psi}_1 \mid_k S^{(2)}(Y)D_{p,G}^{(2)}(Y)^{-1} - \alpha_2^2 X^2 \tilde{\psi}_1.$$

After taking the inner product with $\tilde{\phi}_1$ and keeping in mind the conjugation happening, we get the expression for the first term. Similarly for the other term and from this, the result follows. □

Proposition 3.3.11. *We have*

$$\alpha_1 Q_{p,G}^{(2)}(X_1)^{-1} \sum_{l=0}^{\infty} p\langle \tilde{\phi}_1, \tilde{\psi}_{p^{2l-2}} \mid_k \Lambda_-(p)X_2^2(1 + p(\nabla_p - p\Delta_p)X_2^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l\rangle_{\mathcal{A}} -$$

$$-\alpha_2 Q_{p,G}^{(2)}(X_2)^{-1} \sum_{l=0}^{\infty} p\langle \tilde{\phi}_1, \tilde{\psi}_{p^{2l-2}} \mid_k \Lambda_-(p)X_1^2(1 + p(\nabla_p - p\Delta_p)X_1^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l\rangle_{\mathcal{A}} =$$

$$= \left(\frac{\alpha_1^3 N p^4 X^4}{Q_{p,G}^{(2)}(X_1)} - \frac{\alpha_2^3 N p^4 X^4}{Q_{p,G}^{(2)}(X_2)} \right) \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1 \mid_k S^{(2)}(Y)\rangle_{\mathcal{A}}}{D_{p,G}^{(2)}(Y)}.$$

Proof. We use the identities $\Lambda_-(p)(\nabla_p - p\Delta_p) = 0$ and $\Lambda_-(p)\Lambda_+(p) = p^6(\Delta_p)^2$, from the proof of Proposition 3.3.3. We have, for $i = 1, 2$:

$$\sum_{l=0}^{\infty} p\tilde{\psi}_{p^{2l-2}} \mid_k \Lambda_-(p)X_i^2(1 + p(\nabla_p - p\Delta_p)X_i^2)\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l =$$

$$= p\sum_{l=1}^{\infty} \tilde{\psi}_{p^{2l-2}} \mid_k \Lambda_-(p)X_i^2\Delta_{p^l}^{-1}\Lambda_+(p^l)(p^{-3}Y)^l =$$

$$= p\sum_{l=1}^{\infty} \tilde{\psi}_{p^{2l-2}} \mid_k p^6(\Delta_p)^2\Delta_{p^l}^{-1}\Lambda_+(p^{l-1})(p^{-3}Y)^l X_i^2 =$$

$$= p^4\sum_{l=1}^{\infty} \tilde{\psi}_{p^{2l-2}} \mid \Delta_{p^{l-1}}^{-1}\Lambda_+(p^{l-1})(p^{-3}Y)^{l-1}\Delta_p X_i^2 Y = p^{2k-4}D_{p,G}^{(2)}(Y)^{-1}\tilde{\psi}_1 \mid S^{(2)}(Y)X_i^2 Y,$$

from which the result then follows. The last equality follows from Proposition 3.3.5. We also used equation (3.3.2) as well as the facts that $\Delta_p^l = \Delta_{p^l}$ and that Δ_p and $\Lambda_+(p)$ commute, because of the fact that Δ_p has a single right coset representative, which is $p1_4$. □

Now, from the proof of Proposition 3.3.3, we have $\epsilon(T_p) = T_-(p) + T_+(p)$. Since G is a Hecke eigenform, we can write $G \mid T_p = \lambda_p G$ for some $\lambda_p \in \mathbb{C}$. Hence,

$$\tilde{\psi}_1 \mid\mid T_p = \tilde{\psi}_p \mid T_+(p) + \tilde{\psi}_{1/p} \mid T_-(p) = \tilde{\psi}_p \mid T_+(p), \tag{3.3.9}$$

and so $\tilde{\psi}_p \mid T_+(p) = \lambda_p \tilde{\psi}_1$. Moreover, from Lemma 2.3.3 and the fact that T_p has real elementary divisors, we have that it is in fact self-adjoint. Hence $\lambda_p \in \mathbb{R}$.

Proposition 3.3.12. *Let λ_p denote the eigenvalue given by $\tilde{\psi}_p \mid T_+(p) = \lambda_p \tilde{\psi}_1$. We then have*

$$\alpha_1 Q^{(2)}_{p,G}(X_1)^{-1} \sum_{l=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{2l-1}} \mid_k T_-(p) X_2 (1 + p(\nabla_p - p\Delta_p) X_2^2) \Delta_{p^l}^{-1} \Lambda_+(p^l)(p^{-3}Y)^l \rangle_{\mathcal{A}} -$$

$$-\alpha_2 Q^{(2)}_{p,G}(X_2)^{-1} \sum_{l=0}^{\infty} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{2l-1}} \mid_k T_-(p) X_1 (1 + p(\nabla_p - p\Delta_p) X_1^2) \Delta_{p^l}^{-1} \Lambda_+(p^l)(p^{-3}Y)^l \rangle_{\mathcal{A}} =$$

$$= \left(\frac{\alpha_1^2 p^4 X^3 \lambda_p}{Q^{(2)}_{p,G}(X_1)(1+Y)} - \frac{\alpha_2^2 p^4 X^3 \lambda_p}{Q^{(2)}_{p,G}(X_2)(1+Y)} \right) \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1 \mid S^{(2)}(Y)^{\mathrm{prim}} \rangle_{\mathcal{A}}}{D^{(2)}_{p,G}(Y)}.$$

Proof. The proof is exactly the same as the proof in [Hei99, Proposition 4.5] (using also the fact that $\phi \mid_k S^{(2)}(X) T_+(p) = \phi \mid_k T_+(p) S^{(2)}(X)^{\mathrm{factor}}$, if $\phi \in J_{k,p}$ as in Lemma 3.3.6.) □

Now, Proposition 3.3.8 gives us a way to compute $D_{(l)}(s) - D_{(\epsilon,l)}(s)$. Propositions 3.3.9, 3.3.10, 3.3.11, 3.3.12 can be used to compute $D_{(l)}(s)$. Moreover, the coefficients of $S^{(2)}(X)^{\mathrm{prim}}$ are self-adjoint (see [Gri92a, Lemma 4.3]) and $\tilde{\phi}_1$ is an eigenform for these operators. Indeed, this can be seen from [Gri92a, Theorem, p. 2911], as F is an eigenform in the Maass space and the element $T^J(p)$ has signature 1 (see Definition 2.4.2). Hence

$$\langle \tilde{\phi}_1, \tilde{\psi}_1 \mid S^{(2)}(Y) \rangle_{\mathcal{A}} = \langle \tilde{\phi}_1, \tilde{\psi}_1 \mid S^{(2)}(Y)^{\mathrm{prim}} \rangle_{\mathcal{A}} = \langle \tilde{\phi}_1 \mid S^{(2)}(Y)^{\mathrm{prim}}, \tilde{\psi}_1 \rangle_{\mathcal{A}} =$$
$$= S^{(2)}_F(Y)^{\mathrm{prim}} \langle \tilde{\phi}_1, \tilde{\psi}_1 \rangle_{\mathcal{A}},$$

where we write $S^{(2)}_F(Y)^{\mathrm{prim}}$ for the polynomial obtained when we substitute the eigenvalue of $\tilde{\phi}_1$ with respect to the action of $T^{J,r}(p)$. Hence, from equation (3.3.5), we obtain

$$D^{(p)}_{F,G,h}(s) = \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1 \rangle_{\mathcal{A}} S^{(2)}_F(Y)^{\mathrm{prim}}}{(\alpha_1 - \alpha_2) D^{(2)}_{p,G}(Y)} \times \left(\frac{\alpha_1}{Q^{(2)}_{p,G}(X_1)} - \frac{\alpha_2}{Q^{(2)}_{p,G}(X_2)} + \right.$$
$$\left. + \frac{\alpha_1^3 N p^4 X^4}{Q^{(2)}_{p,G}(X_1)} - \frac{\alpha_2^3 N p^4 X^4}{Q^{(2)}_{p,G}(X_2)} - \frac{\alpha_1^2 p^4 X^3 \lambda_p}{Q^{(2)}_{p,G}(X_1)(1+Y)} + \frac{\alpha_2^2 p^4 X^3 \lambda_p}{Q^{(2)}_{p,G}(X_2)(1+Y)} \right).$$

Let us now look at the expression in the big bracket. The numerator equals

$$((\alpha_1 + \alpha_1^3 N p^4 X^4)(1+Y) - \alpha_1^2 p^4 X^3 \lambda_p) Q^{(2)}_{p,G}(X_2) -$$
$$- ((\alpha_2 + \alpha_2^3 N p^4 X^4)(1+Y) - \alpha_2^2 p^4 X^3 \lambda_p) Q^{(2)}_{p,G}(X_1).$$

Here

$$Q_{p,G}^{(2)}(t) = 1 - \lambda_p t + (p\lambda_{T_{1,p}} + p(p^3 + p^2 - p + 1)p^{2k-8})t^2 - p^4 p^{2k-8}\lambda_p t^3 + p^{4k-8}t^4,$$

where $\lambda_{T_{1,p}}$ is the eigenvalue corresponding to the operator $T_{1,p}$. Let then

$$A_2 := p\lambda_{T_{1,p}} + p(p^3 + p^2 - p + 1)p^{2k-8}.$$

By then performing the very lengthy calculation, and grouping in powers of Y, we obtain that the above numerator equals

$$\begin{aligned}(\alpha_1 - \alpha_2)(1-Y)(1 - Y(A_2p^2N^{-2} - 2) + Y^2(p^2N^{-2}\lambda_p^2 - 2A_2p^2N^{-2} + 2) - \\ - Y^3(A_2p^2N^{-2} - 2) + Y^4) \\ = (\alpha_1 - \alpha_2)(1-Y)D_{p,G}^{(2)}(Y),\end{aligned}$$

using Proposition 3.3.2. Hence, we obtain

$$D_{F,G,h}^{(p)}(s) = \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}} S_F^{(2)}(Y)^{\text{factor}}(1-Y)}{Q_{p,G}^{(2)}(X_1)Q_{p,G}^{(2)}(X_2)}. \tag{3.3.10}$$

Let us now explore the connection of $S_F^{(2)}(Y)^{\text{factor}}$ with known L-functions. We recall that χ is the character we have fixed right before Definition 2.5.4.

Proposition 3.3.13. *We have*

$$S_F^{(2)}(Y)^{\text{factor}} = L_p(f, k+s-2)L_p\left(f, k+s-2, \chi\right),$$

where $f \in S_{k-1}\left(\Gamma_0(4), \chi\right)$ is the modular form whose Maass lift is F, as in Proposition 2.5.5. Here, L_p denotes the p-factor of the L-functions appearing in Definition 2.5.4.

Proof. Assume f has a Fourier expansion as in Definition 2.5.4. Let $T(a,b) := \Gamma_0(4)\mathrm{diag}(a,b)\Gamma_0(4)$ for $a, b \geq 1$ such that $a \mid b$. Then, write

$$f\mid_{k-1} T(p) = a(p)f,$$

for the standard operator $T(p) := \Gamma_0(4)\mathrm{diag}(1,p)\Gamma_0(4)$, with $a(p) \in \mathbb{C}$. This is the same operator as $T(1,p)$. Here, the $|_{k-1}$ action is the usual GL_2-action. By standard relations between Hecke operators, we then have

$$T(p^2) = T(p)^2 - \chi(p)p^{k-2},$$

where $T(p^2) := T(1,p^2) + \chi(p)p^{k-3}$. This then implies

$$f\mid_{k-1} T(1,p^2) = (a(p)^2 + p^{k-2} + p^{k-3})f.$$

Using now [Gri90, Lemma 3.3], we obtain that

$$\tilde{\phi}_1 \mid_k T^J(p) = p^{k-4}(a(p)^2 + p^{k-2} + p^{k-3})\tilde{\phi}_1.$$

Hence

$$S_F^{(2)}(Y)^{\text{factor}} = 1 - p^{1-k}(a(p)^2 + 2p^{k-2})Y + p^{-2}Y^2.$$

But, $Y = p^2NX^2$, $N = p^{k-1}$ and $X = p^{-(k+s-1)}$. If α_p, β_p are as in Definition 2.5.4, we obtain:

$$\begin{aligned}
S_F^{(2)}(Y)^{\text{factor}} &= 1 - p^{4-2k}(\alpha_p^2 + \beta_p^2)p^{-2s} + p^{-2k+4-2s} \\
&= (1 - p^{4-2k-2s}\alpha_p^2)(1 - p^{4-2k-2s}\beta_p^2) \\
&= (1 - \alpha_p p^{2-k-s})\left(1 - \beta_p\chi(p)p^{2-k-s}\right)\left(1 - \alpha_p\chi(p)p^{2-k-s}\right) \times \\
&\qquad\qquad \times (1 - \beta_p p^{2-k-s}) \\
&= L_p(f, k+s-2)L_p\left(f, k+s-2, \chi\right).
\end{aligned}$$

□

From equation (3.3.10) and Proposition 3.3.13, we obtain the following Theorem.

Theorem 3.3.14. *Let $F, G \in S_2^k$ and $h \in S_1^k$ be Hecke eigenforms, all having real Fourier coefficients, h normalised, and F belonging in the Maass space, with corresponding $f \in S_{k-1}(\Gamma_0(4), \chi)$. Let also ϕ_1, ψ_1 denote the first Fourier-Jacobi coefficients of F, G, $X_i = \alpha_i p^{-(2k+s-4)}$, $i = 1, 2$, where α_i denote the Satake parameters of h and $Y = p^{-k-2s+2}$. We then have, for $\mathrm{Re}(s)$ large enough*

$$D_{F,G,h}^{(p)}(s) = \frac{\langle\tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}} L_p(f, k+s-2)L_p\left(f, k+s-2, \chi\right)(1-Y)}{Q_{p,G}^{(2)}(X_1)Q_{p,G}^{(2)}(X_2)}.$$

Here, $D_{F,G,h}^{(p)}$ denotes the p-part of the Dirichlet series, as in equation (3.3.4) and $Q_{p,G}^{(2)}$ denotes the p-factor of Gritsenko's L-function, as in Definition 2.5.2.

3.4 Split Primes

We will now consider the case where the odd rational prime p splits. That is, we have that $p = \pi\overline{\pi}$ for some prime element $\pi \in \mathcal{O}_K$. Our aim in this Section is to prove weak rationality theorems analogous to Propositions 3.3.4 and 3.3.5. In order to do that, we will first have to factorise the polynomials which serve as the p-factors of the standard and Gritsenko's L-function in the parabolic Hecke ring $H_p^{1,1}$, as defined in Definitions 2.5.1 and 2.5.2. The factorisation of the latter polynomial has been done by Gritsenko in [Gri92a, Proposition 3.2]. Our aim, therefore, is to factorise the standard Hecke polynomial. As we mentioned in Section 2.3, $H_p^{1,1}$ is isomorphic to the ring of polynomials of one variable with coefficients from the Hecke ring of the parabolic subgroup

$$P_{1,2,1}(\mathbb{Z}_p) = \left\{ \begin{pmatrix} g_1 & * & * \\ 0 & g & * \\ 0 & 0 & g_2 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}_p) \mid g_1, g_2 \in \mathbb{Z}_p^\times,\ g \in \mathrm{GL}_2(\mathbb{Z}_p) \right\}.$$

Hence, we will first investigate Hecke rings of the general linear group and then use this isomorphism to translate the relations back to $H_p^{1,1}$.

3.4.1 Hecke rings in GL_4 and factorisation

Let p be a prime that splits in $\mathcal{O}_K$ and let

$$P_{1,2,1}(\mathbb{Q}_p) = \left\{ \begin{pmatrix} g_1 & * & * \\ 0 & g & * \\ 0 & 0 & g_2 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Q}_p) \mid g_1, g_2 \in \mathbb{Q}_p^\times,\ g \in \mathrm{GL}_2(\mathbb{Q}_p) \right\}.$$

be a parabolic subgroup of $\mathrm{GL}_4(\mathbb{Q}_p)$. Denote by $\Gamma_{1,2,1} := P_{1,2,1}(\mathbb{Q}_p) \cap M_4(\mathbb{Z}_p)$, the group of $\mathbb{Z}_p$-points in $P_{1,2,1}(\mathbb{Q}_p)$. Let also $H_4 := H(\mathrm{GL}_4(\mathbb{Z}_p), \mathrm{GL}_4(\mathbb{Q}_p))$ be the full Hecke ring in this case and $H_{1,2,1} := H(\Gamma_{1,2,1}, P_{1,2,1}(\mathbb{Q}_p))$ denote the corresponding parabolic Hecke ring.

Let us now explicitly describe the isomorphism $H_p^2 \cong H(\mathrm{GL}_4(\mathbb{Z}_p),\ \mathrm{GL}_4(\mathbb{Q}_p))[x^\pm]$, which will yield $H_p^{1,1} \cong H_{1,2,1}[x^\pm]$, as Gritsenko does in [Gri92a, Proposition 2.4] (H_p^2 and $H_p^{1,1}$ are the corresponding unitary rings).

We are in the setting of (2.3.3), where p is a prime that splits in K. We fix an identification $K_p := K \otimes \mathbb{Q}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$ and denote by $(\mu, -\mu)$ the image of the element $(2i)^{-1}$. Let also $e := (1, 0)$, $e^\sigma := (0, 1) \in K_p$. We perform the change of variables $g \longmapsto C^{-1} g C$, where

$$C := \begin{pmatrix} eI_2 & -\mu e^\sigma I_2 \\ \mu e^\sigma 1_2 & e1_2 \end{pmatrix} = \left(\begin{pmatrix} I_2 & 0_2 \\ 0_2 & 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & -\mu I_2 \\ \mu 1_2 & 0_2 \end{pmatrix} \right) \in \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_4(\mathbb{Q}_p) \tag{3.4.1}$$

with $I_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We then have that $G_p^{(2)}$ (see (2.3.4)) is identified by

$$\widetilde{G}_p^{(2)} = \{(X, Y) \in \mathrm{GL}_4(\mathbb{Q}_p) \times \mathrm{GL}_4(\mathbb{Q}_p) \mid Y^t \cdot X = c1_4,\ c \in \mathbb{Q}_p^\times\},$$

and $U_p^{(2)}$ (see (2.3.5)) by

$$\widetilde{U}_p^{(2)} = \{(\gamma, \alpha(\gamma^{-1})^t) \mid \gamma \in \mathrm{GL}_4(\mathbb{Z}_p),\ \alpha \in \mathbb{Z}_p^\times\},$$

(see [Gri92a, p. 2890, 2891]). Define then pr : $\mathrm{GL}_4(K_p) \longrightarrow \mathrm{GL}_4(\mathbb{Q}_p)$ by $\mathrm{pr}(x,y) = x$, induced by $K_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$. Then (pr, c) gives an isomorphism $\widetilde{G}_p^{(2)} \cong \mathrm{GL}_4(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$. The double coset of $(M, c(M^{-1})^t) \in \widetilde{G}_p^{(2)}$ with respect to $\widetilde{U}_p^{(2)}$ is determined by the double coset of M with respect to $\mathrm{GL}_4(\mathbb{Z}_p)$ and by the order δ of the ideal $c\mathbb{Z}_p$. We will denote such a coset by $(M,\delta)_{\widetilde{U}_p^{(2)}}$. Note that here there is a choice, namely whether we have $\pi \longmapsto (pu, v)$ with $u, v \in \mathbb{Z}_p^\times$ or $\overline{\pi} \longmapsto (pu', v')$ with $u', v' \in \mathbb{Z}_p^\times$. In the following, we always choose the first identification.

We remark that

$$\begin{pmatrix} I_2 & 0 \\ 0 & 1_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 & 0 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} a_4 & a_3 & b_3 & b_4 \\ 0 & a_1 & b_1 & b_2 \\ 0 & c_1 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix}.$$

Therefore, via the map pr, we also obtain the isomorphism $H_p^{1,1} \cong H_{1,2,1}[x^\pm]$.

We now note that the conditions of Lemma 2.4.1 hold for the Hecke rings $H_4, H_{1,2,1}$, as explained in [Gri92b, page 2870]. The above identification makes the following diagram commutative:

$$\begin{array}{ccc} H_p^2 & \longrightarrow & H_4[x^\pm] \\ \epsilon \downarrow & & \downarrow \epsilon' \\ H_p^{1,1} & \longrightarrow & H_{1,2,1}[x^\pm], \end{array}$$

where ϵ, ϵ' are the corresponding embeddings of Lemma 2.4.1.

We will now give a Lemma regarding the decomposition of an element in $H_{1,2,1}$ into right cosets. Let $n \geq 1$. For a given square matrix $R \in M_n(\mathbb{Z}_p)$, we define $\Gamma_n^R := \Gamma_n \cap R^{-1}\Gamma_n R$. Also, if A, D are square matrices of sizes n_1, n_2 respectively, we define

$$V(A,D) := \{AY \mid Y \in M_{n_1,n_2}(\mathbb{Z}_p) \pmod{D^\times}\},$$

where $AY_1 \equiv AY_2 \pmod{D^\times}$ if and only if $AY_1D^{-1} - AY_2D^{-1} \in M_{n_1,n_2}(\mathbb{Z}_p)$.

A straightforward generalisation of [Gri88a, Lemma 2] gives the following Lemma.

Lemma 3.4.1. *Let $a, b \in \mathbb{Q}_p^\times$ and $A \in \mathrm{GL}_2(\mathbb{Q}_p)$. We then have*

$$\Gamma_{1,2,1} \begin{pmatrix} a & * & * \\ 0 & A & * \\ 0 & 0 & b \end{pmatrix} \Gamma_{1,2,1} = \sum \Gamma_{1,2,1} \begin{pmatrix} a & B & D \\ 0 & A & C \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $N \in \Gamma_2^A \backslash \Gamma_2$ and $B \in V(a,A)$, $D \in V(a,b)$, $C \in V(A,b)$.

We now denote by $T(a,b,c,d)$ the element of the Hecke ring $H(\mathrm{GL}_4(\mathbb{Q}_p), \mathrm{GL}_4(\mathbb{Z}_p))$

defined by

$$T(a,b,c,d) := \mathrm{GL}_4(\mathbb{Z}_p)\mathrm{diag}(a,b,c,d)\mathrm{GL}_4(\mathbb{Z}_p).$$

We then have the standard elements of H_4:

$$T_1 := T(1,1,1,p),\ T_2 := T(1,1,p,p),\ T_3 := T(1,p,p,p),\ \Delta := T(p,p,p,p). \quad (3.4.2)$$

The decomposition of these elements into right cosets can be found in [And87, Lemma 3.2.18]. Our aim is to now compute the images of the standard Hecke operators of equation (3.4.2) under the embedding of Lemma 2.4.1. In a similar fashion to H_4, we write

$$T_0(a,b,c,d) := \Gamma_{1,2,1}\mathrm{diag}(a,b,c,d)\Gamma_{1,2,1},$$

for an element of $H_{1,2,1}$. Let us now introduce some useful elements of $H_{1,2,1}$.

- $\Lambda_+^{1,3} := T_0(1,p,p,p)$.
- $\Lambda_+^{3,1} := T_0(1,1,1,p)$.
- $\Lambda_-^{1,3} := T_0(p,1,1,1)$.
- $\Lambda_-^{3,1} := T_0(p,p,p,1)$.
- $T_-(p) := T_0(p,1,p,1)$.
- $T_+(p) := T_0(1,1,p,p)$.
- $\Delta := T_0(p,p,p,p)$.

The right coset decompositions of these elements can now be computed by Lemma 3.4.1. We again note that we use the same symbol Δ in both H_4 and $H_{1,2,1}$. We can do that, as the embedding of Lemma 2.4.1 does not change this specific element.

Proposition 3.4.2. *Let ϵ denote the embedding of the Hecke ring H_4 into $H_{1,2,1}$, as described in Lemma 2.4.1. We then have the following images of the elements T_i:*

- $\epsilon(T_1) = \Lambda_-^{1,3} + T_0(1,1,p,1) + \Lambda_+^{3,1}$.
- $\epsilon(T_2) = T_-(p) + T_+(p) + T_0(1,p,p,1) + T_0(p,1,1,p)$.
- $\epsilon(T_3) = \Lambda_-^{3,1} + T_0(p,1,p,p) + \Lambda_+^{1,3}$.

Proof. These follow directly by the right coset decompositions and the definition of the embedding ϵ. We note here a typo in Gritsenko's paper [Gri92b, page 2879] in the Λ_+ component (it appears we need to swap $\Lambda_+^{1,3}$ and $\Lambda_+^{3,1}$). □

We are now in a position to give the factorisation of the standard Hecke polynomial Q_4, as this is defined in [Gri92b, Example 2].

Theorem 3.4.3. *Let $Q_4(t) := 1 - T_1t + pT_2t^2 - p^3T_3t^3 + p^6\Delta t^4 \in H^4[t]$. Then, in $H_{1,2,1}[t]$, we have the factorisation*

$$Q_4(t) = (1-\Lambda_-^{1,3}t)(1 - A_1t + pA_2t^2)(1-\Lambda_+^{3,1}t),$$

where $A_1 := T_0(1,1,p,1)$ and $A_2 := T_0(1,p,p,1)$.

Proof. The factorisation follows by the images of the elements in Proposition 3.4.2 as well as the following relations:

- $\Lambda_-^{1,3} A_1 = pT_-(p)$.
- $A_1 \Lambda_+^{3,1} = pT_+(p)$.
- $\Lambda_-^{1,3} \Lambda_+^{3,1} = pT_0(p,1,1,p)$.
- $A_2 \Lambda_+^{3,1} = p^2 \Lambda_+^{1,3}$.
- $\Lambda_-^{1,3} A_2 = p^2 \Lambda_-^{3,1}$.
- $\Lambda_-^{1,3} A_1 \Lambda_+^{3,1} = p^3 T_0(p,1,p,p)$.
- $\Lambda_-^{1,3} A_2 \Lambda_+^{3,1} = p^5 \Delta$.

These can be obtained by using the right coset decompositions of Lemma 3.4.1. □

3.4.2 Hecke Operators and weak rationality theorems

We will now translate the results above back to the Hecke rings H_p^2 and $H_p^{1,1}$ of the unitary group. We have the following correspondence between the standard elements of H_4 (see equation (3.4.2)) and of H_p^2:

- $T_1 \longleftrightarrow T_{\overline{\pi}} := \Gamma_2 \mathrm{diag}(1, \overline{\pi}, p, \overline{\pi}) \Gamma_2$.
- $T_2 \longleftrightarrow T_p := \Gamma_2 \mathrm{diag}(1, 1, p, p) \Gamma_2$.
- $T_3 \longleftrightarrow T_\pi := \Gamma_2 \mathrm{diag}(1, \pi, p, \pi) \Gamma_2$.

Also, for the correspondence between the Hecke operators of $H_p^{1,1}$ and $H_{1,2,1}$, we have:

- $\Lambda_+^{1,3} \longleftrightarrow \Lambda_+(\pi) := \Gamma_{1,1} \mathrm{diag}(\pi, 1, \pi, p) \Gamma_{1,1}$.
- $\Lambda_+^{3,1} \longleftrightarrow \Lambda_+(\overline{\pi}) := \Gamma_{1,1} \mathrm{diag}(\overline{\pi}, 1, \overline{\pi}, p) \Gamma_{1,1}$.
- $\Lambda_-^{3,1} \longleftrightarrow \Lambda_-(\pi) := \Gamma_{1,1} \mathrm{diag}(\pi, p, \pi, 1) \Gamma_{1,1}$.
- $\Lambda_-^{1,3} \longleftrightarrow \Lambda_-(\overline{\pi}) := \Gamma_{1,1} \mathrm{diag}(\overline{\pi}, p, \overline{\pi}, 1) \Gamma_{1,1}$.
- $T_0(1,1,p,1) \longleftrightarrow T(\overline{\pi}) := \Gamma_{1,1} \mathrm{diag}(1, \overline{\pi}, p, \overline{\pi}) \Gamma_{1,1}$.
- $T_0(p,1,p,p) \longleftrightarrow T(\pi) := \Gamma_{1,1} \mathrm{diag}(1, \pi, p, \pi) \Gamma_{1,1}$.
- $T_0(1,p,p,1) \longleftrightarrow T(\pi, \overline{\pi}) := \Gamma_{1,1} \mathrm{diag}(\pi, \overline{\pi}, \pi, \overline{\pi}) \Gamma_{1,1}$.
- $T_0(p,1,1,p) \longleftrightarrow T(\overline{\pi}, \pi) := \Gamma_{1,1} \mathrm{diag}(\overline{\pi}, \pi, \overline{\pi}, \pi) \Gamma_{1,1}$.
- $T_-(p) \longleftrightarrow T_-(p) := \Gamma_{1,1} \mathrm{diag}(1, p, p, 1) \Gamma_{1,1}$.
- $T_+(p) \longleftrightarrow T_+(p) := \Gamma_{1,1} \mathrm{diag}(1, 1, p, p) \Gamma_{1,1}$.

We denote by $\Delta_\pi := \Gamma_{1,1} \mathrm{diag}(\pi, \pi, \pi, \pi) \Gamma_{1,1}$ and similarly for $\Delta_{\overline{\pi}}$ and Δ_p. We again use the same notation for these as elements of H_p^2 as well. Finally, we also have the operator ∇_p as in Subsection 3.3.1. From [Klo15, Section 4.1.1], H_p^2 is generated by $T_p, T_\pi, T_{\overline{\pi}}, \Delta_\pi, \Delta_{\overline{\pi}}$ and their inveres as a $\mathbb{Q}$-algebra.

In order to make clear how the isomorphism described at the beginning of Subsection 3.4.1 works, let us describe it in the case of $\Lambda_+(\pi)$. We remark that

$$\begin{pmatrix} I_2 & 0 \\ 0 & 1_2 \end{pmatrix} \mathrm{diag}(a_1, a_2, a_3, a_4) \begin{pmatrix} I_2 & 0 \\ 0 & 1_2 \end{pmatrix}^{-1} = \mathrm{diag}(a_2, a_1, a_3, a_4).$$

We then have by sending $\pi \longmapsto p$

$$\Gamma_{1,1}\mathrm{diag}(\pi, 1, \pi, p)\Gamma_{1,1} \longmapsto \Gamma_{1,2,1}\mathrm{diag}(p, 1, p, p)\Gamma_{1,2,1} \longmapsto \Gamma_{1,2,1}\mathrm{diag}(1, p, p, p)\Gamma_{1,2,1},$$

where the second arrow is simply the swap of the first two diagonal elements induced by the matrix C, as described in (3.4.1). Also, since $\mu(\mathrm{diag}(\pi, 1, \pi, p)) = p$, $\Lambda_+(\pi)$ gets mapped to $(\Lambda_+^{1,3}, 1)_{\tilde{U}_p^{(2)}}$, but in general we will not keep account of the second coordinate. The only case in which this plays a difference is in the identification of Δ_π and Δ_p, which both get mapped to $\mathrm{diag}(p, p, p, p)$, but their factors of similitude are $1, 2$ respectively. The reason why factors of $\Delta_{\overline{\pi}}$ appear in the relations below is to compensate for the second coordinate, as $\mathrm{diag}(\overline{\pi}, \overline{\pi}, \overline{\pi}, \overline{\pi}) \longmapsto \mathrm{diag}(1, 1, 1, 1)$.

The table below shows some relations between the above Hecke operators. These can be obtained by translating back to $H_{1,2,1}$ and using the right coset decompositions. The way to read the table is that we first read an operator X in the first row, then an operator Y in the first column, and the result is XY. We write "comm" to mean that the operators commute.

	$\Lambda_+(\pi)$	$\Lambda_+(\overline{\pi})$	$\Lambda_-(\pi)$	$\Lambda_-(\overline{\pi})$	$T(\overline{\pi})$	$T(\pi)$	$T(\pi, \overline{\pi})$	$T(\overline{\pi}, \pi)$	$T_-(p)$	$T_+(p)$
$\Lambda_+(\pi)$		comm	$p\Delta_\pi T(\pi, \overline{\pi})$	$p^3\Delta_p$	comm	$p\Delta_\pi T_+(p)$	comm	$p^2\Delta_\pi\Lambda_+(\overline{\pi})$	$p^2\Delta_\pi T(\overline{\pi})$	comm
$\Lambda_+(\overline{\pi})$	comm		$p^3\Delta_p$	$p\Delta_{\overline{\pi}} T(\overline{\pi}, \pi)$	$p\Delta_{\overline{\pi}} T_+(p)$	comm	$p^2\Delta_{\overline{\pi}}\Lambda_+(\pi)$	comm	$p^2\Delta_{\overline{\pi}} T(\pi)$	comm
$\Lambda_-(\pi)$										
$\Lambda_-(\overline{\pi})$										
$T(\overline{\pi})$			comm	$p\Delta_{\overline{\pi}} T_-(p)$			comm			
$T(\pi)$			$p\Delta_\pi T_-(p)$	comm				comm		
$T(\pi, \overline{\pi})$			comm	$p^2\Delta_{\overline{\pi}}\Lambda_-(\pi)$						
$T(\overline{\pi}, \pi)$			$p^2\Delta_\pi\Lambda_-(\overline{\pi})$	comm						
$T_-(p)$										
$T_+(p)$			$p^2\Delta_\pi T(\overline{\pi})$	$p^2\Delta_{\overline{\pi}} T(\pi)$						

Table 3.1: Relations of Hecke Operators for split primes.

Proposition 3.4.4. *Let*

$$D_\pi^{(2)}(t) := 1 - T_{\overline{\pi}} t + p\Delta_{\overline{\pi}} T_p t^2 - p^3\Delta_{\overline{\pi}}^2 T_\pi t^3 + p^6\Delta_{\overline{\pi}}^3\Delta_\pi t^4 \in H_p^2[t],$$

and

$$D_{\overline{\pi}}^{(2)}(t) := 1 - T_\pi t + p\Delta_\pi T_p t^2 - p^3\Delta_\pi^2 T_{\overline{\pi}} t^3 + p^6\Delta_\pi^3\Delta_{\overline{\pi}} t^4 \in H_p^2[t].$$

Let also

$$S_\pi(t) := 1 - T(\overline{\pi})t + p\Delta_{\overline{\pi}} T(\pi,\overline{\pi})t^2 \in H_p^{1,1}[t],$$

and

$$S_{\overline{\pi}}(t) := 1 - T(\pi)t + p\Delta_{\pi} T(\overline{\pi},\pi)t^2 \in H_p^{1,1}[t].$$

We then have the following factorisations

$$D_\pi^{(2)}(t) = (1-\Lambda_-(\overline{\pi})t)S_\pi(t)(1-\Lambda_+(\overline{\pi})t),$$

$$D_{\overline{\pi}}^{(2)}(t) = (1-\Lambda_-(\pi)t)S_{\overline{\pi}}(t)(1-\Lambda_+(\pi)t).$$

Proof. This follows from Theorem 3.4.3 after pulling back to the parabolic Hecke ring $H_p^{1,1}$ of the unitary group. □

Remark 3.4.5. We remark here that $Z_\pi^{(2)}(t) = D_\pi^{(2)}(\Delta_{\overline{\pi}}^{-1}t)$ and $Z_{\overline{\pi}}^{(2)}(t) = D_{\overline{\pi}}^{(2)}(\Delta_\pi^{-1}t)$, where $Z_\pi^{(2)}, Z_{\overline{\pi}}^{(2)}$ are the standard polynomials defined in Section 2.3. This can be seen by computing the images under the Satake mapping of the above coefficients, as can be found in [Gri92a, Lemma 3.7].

We will need a Lemma regarding the decomposition of a Hecke operator in $H_p^{1,1}$ into right cosets. Let $\iota : \Gamma_1 \longrightarrow \Gamma_{1,1}$ denote the embedding

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Also, for a set of representatives of $\mathcal{O}_K/m^k$ with $m \in \mathcal{O}_K$ and $k \in \mathbb{Z}$, we understand that we only take 0 as the only representative if $k \leq 0$. We then have:

Lemma 3.4.6. *Let*

$$M = \operatorname{diag}(\pi^{a_1}\overline{\pi}^{b_1},\ \pi^{a_2}\overline{\pi}^{b_2},\ \pi^{a_3}\overline{\pi}^{b_3},\ \pi^{a_4}\overline{\pi}^{b_4}) \in S_p^2,$$

with a_i, b_i nonnegative integers. Then

$$\Gamma_{1,1}M\Gamma_{1,1} = \sum_{\substack{l,q,r \\ \gamma\in V}} \Gamma_{1,1}M \begin{pmatrix} 1 & 0 & 0 & l \\ -\overline{q} & 1 & \overline{l} & r - l\overline{q} \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \imath(\gamma),$$

where l, q, r run over elements in $\mathcal{O}_K$ that satisfy $r \in \mathbb{Z}$ and they give representatives of

$$l \in \mathcal{O}_K/\pi^{a_4-a_1}\overline{\pi}^{b_4-b_1},\ q \in \mathcal{O}_K/\pi^{a_4-a_3}\overline{\pi}^{b_4-b_3},\ r \in \mathbb{Z}/p^{a_4-a_2}.$$

Finally, γ runs over a set V such that

$$\Gamma_1 \mathrm{diag}(\pi^{a_1}\overline{\pi}^{b_1}, \pi^{a_3}\overline{\pi}^{b_3})\Gamma_1 = \sum_{\gamma \in V} \Gamma_1 \mathrm{diag}(\pi^{a_1}\overline{\pi}^{b_1}, \pi^{a_3}\overline{\pi}^{b_3})\gamma$$

is a decomposition into distinct right cosets relative to Γ_1.

Proof. We write

$$\mathcal{H}_{1,1} := \left\{ \begin{pmatrix} 1 & 0 & 0 & l \\ -\overline{q} & 1 & \overline{l} & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_{1,1} \mid l, q, r \in \mathcal{O}_K \right\}$$

for the (integral) Heisenberg part of the Klingen parabolic. We then claim that

$$\mathcal{H}_{1,1} M \mathcal{H}_{1,1} = \sum_{l,q,r} \mathcal{H}_{1,1} M \begin{pmatrix} 1 & 0 & 0 & l \\ -\overline{q} & 1 & \overline{l} & r - l\overline{q} \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.4.3}$$

where l, q, r are as in the statement of the Lemma.

To see this, we first set $h(l, q, r) := \begin{pmatrix} 1 & 0 & 0 & l \\ -\overline{q} & 1 & \overline{l} & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $M = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and calculate

$$h(l_1, q_1, r_1) M h(l_2, q_2, r_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_1 l_2 + l_1 \alpha_4 \\ -\overline{q}_1 \alpha_1 - \alpha_2 \overline{q}_2 & \alpha_2 & \alpha_2 \overline{l}_2 + \overline{l}_1 \alpha_3 & * \\ 0 & 0 & \alpha_3 & \alpha_3 q_2 + q_1 \alpha_4 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix},$$

where $* = -\overline{q}_1 \alpha_1 l_2 + \alpha_2 r_2 + \overline{l}_1 \alpha_3 q_2 + r_1 \alpha_4$. We first look at the upper right entry. We have

$$\alpha_1 l_2 + l_1 \alpha_4 = \pi^{a_1}\overline{\pi}^{b_1} l_2 + l_1 \pi^{a_4}\overline{\pi}^{b_4}.$$

If $a_1 \geq a_4$ and $b_1 \geq b_4$, we may write

$$\alpha_1 l_2 + l_1 \alpha_4 = \left(\pi^{a_1 - a_4}\overline{\pi}^{b_1 - b_4} l_2 + l_1\right) \pi^{a_4}\overline{\pi}^{b_4},$$

and so we do not need right cosets in this case. In the other cases, we write

$$l_2 = x + y\pi^{a_4 - a_1}\overline{\pi}^{b_4 - b_1},$$

with the understanding that we set $\pi^i = 1$ and $\overline{\pi}^j = 1$ if $i, j \leq 0$. We then have

$$\alpha_1 l_2 + l_1 \alpha_4 = \pi^{a_1}\overline{\pi}^{b_1} x + (l_1 + y)\pi^{a_4}\overline{\pi}^{b_4}.$$

For example, when $a_4 > a_1$ and $b_1 \geq b_4$, we write $l_2 = x + y\pi^{a_4 - a_1}$ and obtain

$$\alpha_1 l_2 + l_1 \alpha_4 = \pi^{a_1}\overline{\pi}^{b_1} x + y\pi^{a_4}\overline{\pi}^{b_1} + l_1 \pi^{a_4}\overline{\pi}^{b_4} = \pi^{a_1}\overline{\pi}^{b_1} x + (y\overline{\pi}^{b_1 - b_4} + l_1)\pi^{a_4}\overline{\pi}^{b_4}.$$

Similarly, looking at the entry

$$\alpha_2 \bar{l}_2 + \bar{l}_1 \alpha_3 = \pi^{a_2}\overline{\pi}^{b_2}\bar{l}_2 + \bar{l}_1 \pi^{a_3}\overline{\pi}^{b_3}$$

with $l_2 = x + y\pi^{a_4 - a_1}\overline{\pi}^{b_4 - b_1}$ as above, we obtain,

$$\alpha_2 \bar{l}_2 + \bar{l}_1 \alpha_3 = \pi^{a_2}\overline{\pi}^{b_2}\overline{x} + \overline{y}\pi^{a_2}\overline{\pi}^{b_2}\overline{\pi}^{a_4 - a_1}\pi^{b_4 - b_1} + \bar{l}_1 \pi^{a_3}\overline{\pi}^{b_3} = \pi^{a_2}\overline{\pi}^{b_2}\overline{x} + (\overline{y} + \overline{l_1})\pi^{a_3}\overline{\pi}^{b_3},$$

where we have used the fact that $a_2 + b_4 = b_1 + a_3$ and $a_4 + b_2 = a_1 + b_3$, since $M \in S_p^2$.

In particular, for these entries it is enough to consider the entry l_2 modulo $\pi^{a_4 - a_1}\overline{\pi}^{b_4 - b_1}$ (with our convention). Similarly, by looking at the entries $-\overline{q}_1\alpha_1 - \alpha_2\overline{q}_2$ and $\alpha_3 q_2 + q_1\alpha_4$, we obtain the corresponding result for q_2.

We are now left with the $*$ entry. Using the fact that we can write $r_1 = r_1' - l_1\overline{q}_1$ and $r_2 = r_2' - l_2\overline{q}_2$ for some $r_i' \in \mathbb{Z}$, we have that the only part of the $*$ entry which is not determined by our choices of q_1, q_2, l_1, l_2 is $\alpha_2 r_2' + r_1'\alpha_4$. But

$$\alpha_2 r_2' + r_1'\alpha_4 = \pi^{a_2}\overline{\pi}^{b_2} r_2' + r_1'\pi^{a_4}\overline{\pi}^{b_4}.$$

Arguing as above and using the fact that $a_4 - a_2 = b_4 - b_2$, we see that the element r_2' needs to be selected from integers modulo $\pi^{a_4 - a_2}\overline{\pi}^{b_4 - b_2} = p^{a_4 - a_2}$.

This establishes our claim of equation (3.4.3). The rest of the proof is identical to the symplectic case, as done by Gritsenko in [Gri84, Lemma 3.1]. □

We now define the elements

$$T_{\pm}(p^{\delta}) := j_{\pm}\left(T\left(p^{\delta}\right)\right),\ \Lambda_{\pm}(\pi^{\delta}) := j_{\pm}\left(\Gamma_1 \mathrm{diag}\left(\pi^{\delta}, \pi^{\delta}\right)\Gamma_1\right),\ \delta \geq 1,$$

as in the case of an inert prime (see equations (2.4.1), (3.3.1)) and similarly for $\Lambda_{\pm}(\overline{\pi}^{\delta})$. In particular, using Lemma 3.4.6, or translating back to the Hecke algebra of GL_4, we obtain

$$\Lambda_{\pm}(\pi^{\delta}) = \Lambda_{\pm}(\pi^{\delta - 1})\Lambda_{\pm}(\pi),\ \delta \geq 1. \tag{3.4.4}$$

This implies $\Lambda_{\pm}(\pi^{\delta}) = \Lambda_{\pm}(\pi)^{\delta},\ \delta \geq 1$. The same holds for $\Lambda_{\pm}(\overline{\pi})$.

We are now finally ready to obtain the rationality Theorems, as in the case of an inert prime. Assume $F \in S_2^k$ has a Fourier-Jacobi expansion as in equation (3.2.1) and $Q_p^{(2)}$ denotes the p-factor of Gritsenko's L-function, as in Definition 2.5.2.

Proposition 3.4.7. *Let $F \in S_2^k$ be a Hecke eigenform for $H(\Gamma_2, S^2)$ and $m \geq 1$. Then*

$$Q_{p,F}^{(2)}(X) \sum_{\delta \geq 0} \phi_{mp^\delta} \mid T_+(p^\delta) X^\delta = \left(\phi_m - \phi_{m/p} \mid T_-(p) X + p\phi_{m/p^2} \mid \Lambda_-(p) X^2\right) \mid B(X),$$

where we define B to be the middle polynomial of degree 4 in the factorisation of $Q_p^{(2)}(t)$ given in [Gri92a, Proposition 3.2, (3)]. In particular, we have

$$B(t) = 1 - B_1 t + B_2 t^2 - B_3 t^3 + B_4 t^4 \in H_p^{1,1}[t],$$

where

$$B_1 = T(\overline{\pi}, \pi) + T(\pi, \overline{\pi}),\ B_2 = p\left(\Lambda_+(\pi)\Lambda_-(\overline{\pi}) + \Lambda_+(\overline{\pi})\Lambda_-(\pi)\right) - p\nabla_p + (p^2 - p^4)\Delta_p,$$
$$B_3 = p^3\left(\Delta_\pi \Lambda_+(\overline{\pi})\Lambda_-(\overline{\pi}) + \Delta_{\overline{\pi}}\Lambda_+(\pi)\Lambda_-(\pi)\right) - p^4\Delta_p B_1,\ B_4 = p^5\Delta_p\nabla_p - p^6\Delta_p^2. \tag{3.4.5}$$

Moreover, from [Gri92a, Proposition 4.2], we have

$$\phi_m \mid B(X) = (1 - p^{k-3}X)^2(1 - p^{2k-4}X^2)\phi_m,$$

if $(m, p) = 1$.

Proof. The proof follows by [Gri92a, Proposition 4.1] and the fact that

$$Q_{p,-}^{(1)}(t) = 1 - T_-(p)t + p\Lambda_-(p)t^2,$$

as is defined in [Gri92a, Proposition 3.2, (1)]. □

Proposition 3.4.8. *Let $F \in S_2^k$ be a Hecke eigenform for $H(\Gamma_2, S^2)$ and $m \geq 1$. Then,*

$$D_{\pi,F}^{(2)}(X) \sum_{\delta \geq 0} \phi_{mp^\delta} \mid \Lambda_+(\overline{\pi}^\delta) X^\delta = \left(\phi_m - \phi_{m/p} \mid \Lambda_-(\overline{\pi}) X\right) \mid S_\pi(X).$$

Similarly,

$$D_{\overline{\pi},F}^{(2)}(X) \sum_{\delta \geq 0} \phi_{mp^\delta} \mid \Lambda_+(\pi^\delta) X^\delta = \left(\phi_m - \phi_{m/p} \mid \Lambda_-(\pi) X\right) \mid S_{\overline{\pi}}(X),$$

with notation obtained by exchanging π and $\overline{\pi}$. Here, $S_\pi, S_{\overline{\pi}}$ are the polynomials appearing in Proposition 3.4.4.

Proof. We have, in the same way as in [Gri84, Corollary, p. 264], the proof of which can be found in [Gri84, Proposition 5.2], that

$$\phi_m \mid\mid_k \Lambda_+(\overline{\pi}) = \phi_{mp} \mid_k \Lambda_+(\overline{\pi}).$$

Then, inductively, using equation (3.4.4), we obtain

$$\phi_m \mid\mid_k \Lambda_+(\overline{\pi}^\delta) = \phi_{mp^\delta} \mid_k \Lambda_+(\overline{\pi}^\delta), \ \forall \delta \geq 1.$$

Now, since F is an eigenfunction for $H(\Gamma_2, S^2)$, we have $D^{(2)}_{\pi,F}(t)\phi_m = \phi_m \mid\mid D^{(2)}_{\pi}(t)$. Then, we can write

$$\begin{aligned}
D^{(2)}_{\pi,F}(X)\sum_{\delta\geq 0}\phi_{mp^\delta} \mid \Lambda_+(\overline{\pi}^\delta)X^\delta &= D^{(2)}_{\pi,F}(X)\sum_{\delta\geq 0}(\phi_m \mid\mid \Lambda_+(\overline{\pi}^\delta))X^\delta \\
&= (\phi_m \mid\mid D^{(2)}_{\pi}(X)) \mid\mid \sum_{\delta\geq 0}\Lambda_+(\overline{\pi}^\delta)X^\delta \\
&= (\phi_m \mid\mid D^{(2)}_{\pi}(X)) \mid\mid \sum_{\delta\geq 0}\Lambda_+(\overline{\pi})^\delta X^\delta \\
&= \phi_m \mid\mid (1-\Lambda_-(\overline{\pi})X)S_\pi(X) \\
&= (\phi_m - \phi_{m/p} \mid \Lambda_-(\overline{\pi})X) \mid S_\pi(X),
\end{aligned}$$

as claimed. In the equalities above, we used that $\Lambda_+(\overline{\pi}^\delta) = \Lambda_+(\overline{\pi})^\delta$ from equation (3.4.4) and also Proposition 3.4.4. □

To end this Section, we will prove a couple of Lemmas, which will be useful later, when we are dealing with the Dirichlet series of interest.

Lemma 3.4.9. *Denote by $\Lambda_-(\pi)^{\mathrm{adj}}$ the adjoint of the operator $\Lambda_-(\pi)$ with respect to the inner product of Fourier-Jacobi forms (see Definition 2.2.8). Then*

$$\Lambda_-(\pi)^{\mathrm{adj}} = p^{k-3}\Lambda_+(\overline{\pi}).$$

Proof. Let ϕ_l, ψ_{lp} be two Fourier-Jacobi forms of weight k and of index l, lp respectively. We observe from Lemma 3.4.6 that $\Lambda_-(\pi) = \Gamma_{1,1}\mathrm{diag}(\pi, p, \pi, 1)$. We also note that the Jacobi form $\phi_l \mid_k \mathrm{diag}(\pi, p, \pi, 1)$ is of index lp for the group

$$\Gamma_- := \Gamma_1 \times \mathcal{O}_K \times \mathcal{O}_K = \left\{ \begin{pmatrix} a & 0 & b & \kappa \\ * & 1 & * & * \\ c & 0 & d & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_{1,1} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1, \ \kappa, \lambda \in \mathcal{O}_K \right\}.$$

In particular, if $\mathbb{H}_1^J := \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}$, we may write

$$\langle \phi_l |_k \Lambda_-(\pi), \psi_{lp}\rangle = \frac{p^{2k-4}\pi^{-k}}{[\Gamma_{1,1}:\Gamma_-]}\int_{\Gamma_-\backslash\mathbb{H}_1^J} \phi_l(\tau,\pi z_1,\overline{\pi} z_2)\overline{\psi_{lp}(\tau,z_1,z_2)}\times$$
$$\times \exp\left(-\pi l p\frac{|z_1-\overline{z_2}|^2}{v}\right) v^{k-4}\mathrm{d}\mu,$$

as in Definition 2.2.8. We now perform the change of variables $z_1 \longmapsto \pi^{-1}z_1$ and $z_2 \longmapsto \overline{\pi}^{-1}z_2$. This is equivalent to the action of the matrix $\mathrm{diag}(\overline{\pi},1,\overline{\pi},p)$ on $\mathbb{H}_1^J$. Now

$$\left(\psi_{lp} |_k \mathrm{diag}(\overline{\pi},1,\overline{\pi},p)\right)(\tau,z_1,z_2) = p^{k-4}\overline{\pi}^{-k}\psi_{lp}(\tau,\pi^{-1}z_1,\overline{\pi}^{-1}z_2)$$

is a Jacobi form of weight k and index l with respect to the group

$$\Gamma_+ := \Gamma_1\times\pi\mathcal{O}_K\times\pi\mathcal{O}_K = \left\{\begin{pmatrix} a & 0 & b & \kappa\\ * & 1 & * & *\\ c & 0 & d & \lambda\\ 0 & 0 & 0 & 1\end{pmatrix}\in\Gamma_{1,1} \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_1,\right.$$
$$\left.\kappa,\lambda\equiv 0 \pmod{\pi}\right\}.$$

This group is obtained by considering the group $\mathrm{diag}(\overline{\pi},1,\overline{\pi},p)^{-1}\Gamma_-\mathrm{diag}(\overline{\pi},1,\overline{\pi},p)$. We therefore have

$$\langle \phi_l |_k \Lambda_-(\pi), \psi_{lp}\rangle = \frac{p^{2k-6}\pi^{-k}}{[\Gamma_{1,1}:\Gamma_-]}\int_{\Gamma_+\backslash\mathbb{H}_1^J} \phi_l(\tau,z_1,z_2)\overline{\psi_{lp}(\tau,\pi^{-1}z_1,\overline{\pi}^{-1}z_2)}\times$$
$$\times \exp\left(-\pi l\frac{|z_1-\overline{z_2}|^2}{v}\right) v^{k-4}\mathrm{d}\mu.$$

On the other hand, we have by Lemma 3.4.6, that

$$\Lambda_+(\overline{\pi}) = \sum_{\substack{a,b\in\mathcal{O}_K/\pi,\\ c\in\mathbb{Z}/p}} \Gamma_{1,1}\begin{pmatrix} \overline{\pi} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \overline{\pi} & 0\\ 0 & 0 & 0 & p\end{pmatrix} h(a,b,c),$$

where $h(a,b,c) := \begin{pmatrix} 1 & 0 & 0 & a\\ -\overline{b} & 1 & \overline{a} & c-a\overline{b}\\ 0 & 0 & 1 & b\\ 0 & 0 & 0 & 1\end{pmatrix}$. By now using the fact that

$$\langle \phi_l |_k h(a,b,c)^{-1}, \psi_{lp}\rangle = \langle \phi_l, \psi_{lp} |_k h(a,b,c)\rangle,$$

we obtain

$$\langle \phi_l, \psi_{lp} |_k \Lambda_+(\overline{\pi})\rangle = p^3\frac{p^{k-4}\pi^{-k}}{[\Gamma_{1,1}:\Gamma_+]}\int_{\Gamma_+\backslash\mathbb{H}_1^J} \phi_l(\tau,z_1,z_2)\overline{\psi_{lp}(\tau,\pi^{-1}z_1,\overline{\pi}^{-1}z_2)}\times$$

$$\times \exp\left(-\pi l \frac{|z_1 - \overline{z_2}|^2}{v}\right) v^{k-4} \mathrm{d}\mu,$$

as $h(a,b,c)^{-1} \in \Gamma_{1,1}$ and hence they act trivially on ϕ_l. Therefore,

$$\langle \phi_l \mid_k \Lambda_-(\pi), \psi_{lp}\rangle = p^{k-5}\frac{[\Gamma_{1,1} : \Gamma_+]}{[\Gamma_{1,1} : \Gamma_-]}\langle \phi_l, \psi_{lp} \mid \Lambda_+(\overline{\pi})\rangle = p^{k-3}\langle \phi_l, \psi_{lp} \mid_k \Lambda_+(\overline{\pi})\rangle,$$

as the ratio of indices is p^2. The result now follows. □

Finally, knowing the action of the operators $T(\pi, \overline{\pi})$ and $T(\overline{\pi}, \pi)$ on $J_{k,1}$ will prove helpful in the following, so we give the following Lemma:

Lemma 3.4.10. *Let $\phi \in J_{k,1}$. We then have*

$$\phi \mid_k T(\overline{\pi}, \pi) = p^{k-3}\phi.$$

Proof. Using the decomposition in Lemma 3.4.6, we can write

$$T(\overline{\pi}, \pi) = \sum_{\gamma \in \mathcal{O}_K/\pi,\ \beta \in \mathcal{O}_K/\overline{\pi}} \Gamma_{1,1} \begin{pmatrix} \overline{\pi} & 0 & 0 & \overline{\pi}\overline{\beta} \\ -\overline{\gamma} & \pi & \pi\beta & -\overline{\gamma}\overline{\beta} \\ 0 & 0 & \overline{\pi} & \gamma \\ 0 & 0 & 0 & \pi \end{pmatrix}.$$

The result now follows from [Gri90, Lemma 3.2]. □

Remark 3.4.11. The same is true for the operator $T(\pi, \overline{\pi}) = \Gamma_{1,1}\mathrm{diag}(\pi, \overline{\pi}, \pi, \overline{\pi})\Gamma_{1,1}$.

3.4.3 Calculation of the Dirichlet series - First Part

Assume F, G, h satisfy the same assumptions as in the beginning of Subsection 3.3.2. We recall from equation (3.3.3) that

$$D_{F,G,h}(s) = 4\beta_k \sum_{l,\epsilon,m} \langle \tilde{\phi}_1 \mid T_-(m)U_l, \tilde{\psi}_{mN(l)}\rangle_{\mathcal{A}} a_{mN(\epsilon)} N(l)^{-s} N(\epsilon)^{-(k+s-1)} m^{-(2k+s-4)},$$

with $l, \epsilon \in \mathbb{Z}[i]$ coprime with their real parts positive and imaginary parts non-negative and $m \in \mathbb{N}$. In the case of a split prime $p = \pi\overline{\pi}$, we define the p-part of the Dirichlet series by

$$D^{(p)}_{F,G,h}(s) := \sum_{l_1,l_2,\epsilon_1,\epsilon_2,m \geq 0} \langle \widetilde{\phi}_1 \mid T_-(p^m)U_{\pi^{l_1}}U_{\overline{\pi}^{l_2}}, \widetilde{\psi}_{p^{m+l_1+l_2}}\rangle_{\mathcal{A}} a_{p^{m+\epsilon_1+\epsilon_2}} p^{-s(l_1+l_2)} \times$$
$$\times\, p^{-(k+s-1)(\epsilon_1+\epsilon_2)} p^{-(2k+s-4)m}, \quad (3.4.6)$$

together with the conditions $\min(\epsilon_1, l_1) = 0$ and $\min(\epsilon_2, l_2) = 0$. This series converges absolutely by comparison with $D^{(p)}_{F,G,h}(s)$ (see Lemma 3.2.1).

Consider now the Hecke operator

$$\Lambda_-(\pi) = \Gamma_{1,1}\mathrm{diag}(\pi, p, \pi, 1)\Gamma_{1,1} = \Gamma_{1,1}\mathrm{diag}(\pi, p, \pi, 1),$$

by Lemma 3.4.6. Then, if ϕ is a Fourier-Jacobi form of any index m, we get

$$\phi \mid_k \Lambda_-(\pi) = p^{2k-4}\pi^{-k}\tilde{\phi}\left(\begin{pmatrix} \tau & \pi z_1 \\ \overline{\pi} z_2 & p\tau' \end{pmatrix}\right) e^{-2\pi \frac{m}{p}\tau'} = p^{2k-4}\pi^{-k}\phi(\tau, \pi z_1, \overline{\pi} z_2) =$$
$$= p^{2k-4}\pi^{-k}\phi \mid_k U_\pi. \quad (3.4.7)$$

Hence, we can rewrite the series as:

$$D^{(p)}_{F,G,h}(s) = \sum_{\substack{l_1,l_2,\\ \epsilon_1,\epsilon_2,m\geq 0\\ \min(l_i,\epsilon_i)=0}} \langle \tilde{\phi}_1 \mid T_-(p^m)\Lambda_-(\pi^{l_1})\Lambda_-(\overline{\pi}^{l_2}), \tilde{\psi}_{p^{m+l_1+l_2}}\rangle_{\mathcal{A}} a_{p^{m+\epsilon_1+\epsilon_2}} \times$$
$$\times\, p^{(4-2k)l_1} p^{(4-2k)l_2} \pi^{l_1 k}\overline{\pi}^{l_2 k} p^{-s(l_1+l_2)} p^{-(k+s-1)(\epsilon_1+\epsilon_2)} p^{-(2k+s-4)m}.$$

By then using an inclusion-exclusion argument, we have that the above series can be written as

$$D^{(p)}_{F,G,h}(s) = D_{(\epsilon_1,\epsilon_2)}(s) + D_{(l_1,l_2)}(s) + D_{(\epsilon_1,l_2)}(s) + D_{(\epsilon_2,l_1)}(s) -$$
$$- D_{(\epsilon_1,\epsilon_2,l_1)}(s) - D_{(\epsilon_2,l_1,l_2)}(s) - D_{(\epsilon_1,l_1,l_2)}(s) - D_{(\epsilon_1,\epsilon_2,l_2)}(s) + D_{(\epsilon_1,\epsilon_2,l_1,l_2)}(s),$$

where we use the same notation as in Subsection 3.3.2, meaning that the corresponding index means the variables are 0. We can then deal with the "easy" parts first, i.e., when the operators Λ_- do not appear. We again consider $s \in \mathbb{R}$ big enough, as in Remark 3.3.7.

Proposition 3.4.12. *We have*

$$D_{(l_1,l_2)}(s) - D_{(l_1,l_2,\epsilon_1)}(s) - D_{(l_1,l_2,\epsilon_2)}(s) + D_{(l_1,l_2,\epsilon_1,\epsilon_2)}(s) =$$
$$= \frac{\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}}}{\alpha_1 - \alpha_2}\left[\frac{\alpha_1^3(1-p^{2k-4}X_1^2)X^2}{Q^{(2)}_{p,G}(X_1)} - \frac{\alpha_2^3(1-p^{2k-4}X_2^2)X^2}{Q^{(2)}_{p,G}(X_2)}\right],$$

where $X = p^{-(k+s-1)}$ *and* $X_i = \alpha_i p^{-(2k+s-4)}$, *as in Subsection 3.3.2.*

Proof. We have

$$D_{(l_1,l_2)}(s) = \sum_{\epsilon_1,\epsilon_2,m\geq 0} \langle \tilde{\phi}_1 \mid T_-(p^m), \tilde{\psi}_{p^m}\rangle_{\mathcal{A}} a_{p^{m+\epsilon_1+\epsilon_2}} p^{-(k+s-1)(\epsilon_1+\epsilon_2)} p^{-m(2k+s-4)},$$

$$D_{(l_1,l_2,\epsilon_1)}(s) = \sum_{\epsilon_2,m\geq 0} \langle \tilde{\phi}_1 \mid T_-(p^m), \tilde{\psi}_{p^m}\rangle_{\mathcal{A}} a_{p^{m+\epsilon_2}} p^{-(k+s-1)\epsilon_2} p^{-m(2k+s-4)},$$

$$D_{(l_1,l_2,\epsilon_2)}(s)=\sum_{\epsilon_1,m\geq 0}\langle\tilde{\phi}_1\mid T_-(p^m),\tilde{\psi}_{p^m}\rangle_{\mathcal{A}}a_{p^{m+\epsilon_1}}p^{-(k+s-1)\epsilon_1}p^{-m(2k+s-4)},$$

$$D_{(l_1,l_2,\epsilon_1,\epsilon_2)}(s)=\sum_{m\geq 0}\langle\tilde{\phi}_1\mid T_-(p^m),\tilde{\psi}_{p^m}\rangle_{\mathcal{A}}a_{p^m}p^{-m(2k+s-4)}.$$

Using now the fact that $a_{p^m}=(\alpha_1^{m+1}-\alpha_2^{m+1})/(\alpha_1-\alpha_2)$ and the fact that the adjoint of $T_-(p^m)$ is $T_+(p^m)$, when they are acting on P-forms (see also in [Gri92a, Proposition 5.1] and Subsection 3.3.2), we get

$$\begin{aligned}D_{(l_1,l_2)}(s)(\alpha_1-\alpha_2)=\alpha_1&\sum_{\epsilon_1,\epsilon_2,m\geq 0}\langle\tilde{\phi}_1,\tilde{\psi}_{p^m}\mid T_+(p^m)\rangle_{\mathcal{A}}(\alpha_1p^{-(k+s-1)})^{\epsilon_1+\epsilon_2}(\alpha_1p^{-(2k+s-4)})^m\\ -\alpha_2&\sum_{\epsilon_1,\epsilon_2,m\geq 0}\langle\tilde{\phi}_1,\tilde{\psi}_{p^m}\mid T_+(p^m)\rangle_{\mathcal{A}}(\alpha_2p^{-(k+s-1)})^{\epsilon_1+\epsilon_2}(\alpha_2p^{-(2k+s-4)})^m,\end{aligned}$$

and similarly for the others. Now, by Proposition 3.4.7, we obtain, as in Proposition 3.3.8:

$$\sum_{m=0}^{\infty}\langle\tilde{\phi}_1,\tilde{\psi}_{p^m}\mid T_+(p^m)\rangle_{\mathcal{A}}X_1^m=(1-p^{k-3}X_1)^2(1-p^{2k-4}X_1^2)\langle\tilde{\phi}_1,\tilde{\psi}_1\rangle_{\mathcal{A}}Q_{p,G}^{(2)}(X_1)^{-1}.$$

Also, $\sum_{\epsilon_2=0}^{\infty}(\alpha_1p^{-(k+s-1)})^{\epsilon_2}=\dfrac{1}{1-\alpha_1p^{-(k+s-1)}}$ and similarly for ϵ_1, and

$$\sum_{\epsilon_1,\epsilon_2=0}^{\infty}(\alpha_1p^{-(k+s-1)})^{(\epsilon_1+\epsilon_2)}=\left(\frac{1}{1-\alpha_1p^{-(k+s-1)}}\right)^2.$$

Hence, we obtain

$$D_{(l_1,l_2)}(s)-D_{(l_1,l_2,\epsilon_1)}(s)-D_{(l_1,l_2,\epsilon_2)}(s)+D_{(l_1,l_2,\epsilon_1,\epsilon_2)}(s)=$$

$$=\frac{\langle\tilde{\phi}_1,\tilde{\psi}_1\rangle_{\mathcal{A}}}{\alpha_1-\alpha_2}\left[\frac{\alpha_1^3(1-p^{2k-4}X_1^2)X^2}{Q_{p,G}^{(2)}(X_1)}-\frac{\alpha_2^3(1-p^{2k-4}X_2^2)X^2}{Q_{p,G}^{(2)}(X_2)}\right].\qquad\square$$

3.4.4 Calculation of the Dirichlet Series - Second Part

In the following, we define

$$Y_1:=\pi^kp^{-(2k+s-4)},\ Y_2:=\overline{\pi}^kp^{-(2k+s-4)},\ X:=p^{-(k+s-1)},\ X_i:=\alpha_ip^{-(2k+s-4)},$$

for $i=1,2$. Let us now consider the series

$$D_{(\epsilon_1,\epsilon_2,l_2)}(s)=\sum_{l_1,m\geq 0}\langle\tilde{\phi}_1\mid T_-(p^m)U_{\pi^{l_1}},\tilde{\psi}_{p^{m+l_1}}\rangle_{\mathcal{A}}a_{p^m}p^{-sl_1}p^{-(2k+s-4)m}.$$

Using the fact that $a_{p^m} = (\alpha_1^{m+1} - \alpha_2^{m+1})/(\alpha_1 - \alpha_2)$ and the relation between U_π and $\Lambda_-(\pi)$ of equation (3.4.7), we obtain that

$$(\alpha_1 - \alpha_2) D_{(\epsilon_1,\epsilon_2,l_2)}(s) = \alpha_1 S_1(s) - \alpha_2 S_2(s), \tag{3.4.8}$$

where

$$S_i(s) := \sum_{l,m\geq 0} \langle \tilde{\phi}_1 \mid T_-(p^m)\Lambda_-(\pi^l), \tilde{\psi}_{p^{m+l}} \rangle_{\mathcal{A}} p^{-(2k+s-4)l} \pi^{lk} (\alpha_i p^{-(2k+s-4)})^m.$$

Using now the fact that the adjoint (with respect to the inner product of P-forms) of $T_-(p)$ is $T_+(p)$ and of $\Lambda_-(\pi)$ is $\Lambda_+(\overline{\pi})$ (Lemma 3.4.9) and that $T_-(p)$ and $\Lambda_-(\pi)$ commute, we get

$$S_i(s) = \sum_{l,m\geq 0} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{m+l}} \mid T_+(p^m)\Lambda_+(\overline{\pi}^l) \rangle_{\mathcal{A}} X_i^m Y_1^l =$$
$$= \sum_{l,m\geq 0} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{m+l}} \mid T_+(p^m)\Lambda_+(\overline{\pi}^l) Y_2^l \rangle_{\mathcal{A}} X_i^m,$$

because we have a Hermitian inner product (and therefore we have to conjugate in the second component of the inner product). We remind the reader that we work with $s \in \mathbb{R}$ big enough.

Lemma 3.4.13. *For $i = 1, 2$, we have*

$$\sum_{l,m\geq 0} \tilde{\psi}_{p^{m+l}} \mid T_+(p^m)\Lambda_+(\overline{\pi}^l) X_i^m Y_2^l = \frac{1}{Q_{p,G}^{(2)}(X_i)} \sum_{l\geq 0} \Big[\tilde{\psi}_{p^l} - \tilde{\psi}_{p^{l-1}} | T_-(p) X_i +$$
$$+ p\tilde{\psi}_{p^{l-2}} | \Lambda_-(p) X_i^2\Big] \mid B(X_i)\Lambda_+(\overline{\pi}^l) Y_2^l,$$

with B the polynomial of Proposition 3.4.7.

Proof. The proof follows immediately from Proposition 3.4.7. □

Let us now compute each of the sums occurring above.

Proposition 3.4.14. *For $i = 1, 2$, we have*

$$\frac{1}{1 - \alpha_i X} \sum_{l\geq 0} \tilde{\psi}_{p^l} \mid B(X_i)\Lambda_+(\overline{\pi}^l) Y_2^l =$$

$$\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)}{D_{\pi,G}^{(2)}(Y_2)} - p^2 \frac{\Big[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi}) Y_2\Big] \mid S_\pi(Y_2)\Lambda_+(\pi)\Delta_{\overline{\pi}} Y_2 X_i}{D_{\pi,G}^{(2)}(Y_2)} +$$
$$+ \Big[(1 - \alpha_i X)(1 - p^{2k-4} X_i^2) - 1\Big] \tilde{\psi}_1.$$

Proof. Using the commutativity relations from Table 3.1, we obtain (using also equation (3.4.5))

$$B_2\Lambda_+(\overline{\pi}) = p^2\Delta_{\overline{\pi}}\Lambda_+(\pi)T(\overline{\pi},\pi),\ \ B_3\Lambda_+(\overline{\pi}) = B_4\Lambda_+(\overline{\pi}) = 0 \qquad (3.4.9)$$

Hence, from the rationality theorem given in Proposition 3.4.8, we obtain

$$\sum_{l\geq 0}\tilde{\psi}_{p^l} \mid \Lambda_+(\overline{\pi}^l)Y_2^l = \frac{\tilde{\psi}_1 \mid S_\pi(Y_2)}{D^{(2)}_{\pi,G}(Y_2)}.$$

$$\sum_{l\geq 0}\tilde{\psi}_{p^l} \mid T(\overline{\pi},\pi)\Lambda_+(\overline{\pi}^l)Y_2^l = \sum_{l\geq 0}\tilde{\psi}_{p^l} \mid \Lambda_+(\overline{\pi}^l)T(\overline{\pi},\pi)Y_2^l = \frac{\tilde{\psi}_1 \mid S_\pi(Y_2)T(\overline{\pi},\pi)}{D^{(2)}_{\pi,G}(Y_2)}.$$

$$\begin{aligned}\sum_{l\geq 0}\tilde{\psi}_{p^l} \mid T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^l)Y_2^l &= \tilde{\psi}_1 \mid T(\pi,\overline{\pi}) + p^2\sum_{l\geq 1}\tilde{\psi}_{p^l} \mid \Delta_{\overline{\pi}}\Lambda_+(\pi)\Lambda_+(\overline{\pi}^{l-1})Y_2^l \\ &= \tilde{\psi}_1 \mid T(\pi,\overline{\pi}) + p^2\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y)\Lambda_+(\pi)\Delta_{\overline{\pi}}Y_2}{D^{(2)}_{\pi,G}(Y_2)}.\end{aligned}$$

$$\begin{aligned}\sum_{l\geq 0}\tilde{\psi}_{p^l} \mid B_2\Lambda_+(\overline{\pi}^l)Y_2^l &= \tilde{\psi}_1 \mid B_2 + p^2\sum_{l\geq 1}\tilde{\psi}_{p^l} \mid \Lambda_+(\overline{\pi}^{l-1})\Lambda_+(\pi)T(\overline{\pi},\pi)\Delta_{\overline{\pi}}Y_2^l \\ &= \tilde{\psi}_1 \mid B_2 + p^2\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)\Lambda_+(\pi)\Delta_{\overline{\pi}}T(\overline{\pi},\pi)Y_2}{D^{(2)}_{\pi,G}(Y_2)}.\end{aligned}$$

$$\sum_{l\geq 0}\tilde{\psi}_{p^l} \mid B_3\Lambda_+(\overline{\pi}^l)Y_2^l = \tilde{\psi}_1 \mid B_3.$$

$$\sum_{l\geq 0}\tilde{\psi}_{p^l} \mid B_4\Lambda_+(\overline{\pi}^l)Y_2^l = \tilde{\psi}_1 \mid B_4.$$

By putting all these together and then using Lemma 3.4.10, together with the fact that $a_iX = p^{k-3}X_i$, we obtain the result. □

Let us now consider the third sum.

Proposition 3.4.15. *For $i = 1, 2$, we have*

$$\begin{aligned}&\frac{1}{1-\alpha_i X}\sum_{l\geq 0}\tilde{\psi}_{p^{l-2}} \mid \Lambda_-(p)X_i^2 \mid B(X_i)\Lambda_+(\overline{\pi}^l)Y_2^l = \\ &\quad = \frac{\tilde{\psi}_1 \mid S_\pi(Y_2)U_\pi(X_i)X_i^2Y_2^2}{D^{(2)}_{\pi,G}(Y_2)} = \frac{p^{2k-4}(p^{k-3} - p^{2k-4}X_i)\tilde{\psi}_1 \mid S_\pi(Y_2)\Delta_{\overline{\pi}}Y_2^2X_i^2}{D^{(2)}_{\pi,G}(Y_2)},\end{aligned}$$

where we define $U_\pi(t) := p^4\Delta_{\overline{\pi}}\Delta_p(T(\overline{\pi},\pi) - p^4\Delta_p t) \in H_p^{1,1}[t]$.

Proof. We will first simplify $\Lambda_-(p)B(X_i)$. But $\Lambda_-(p) = \Lambda_-(\pi)\Lambda_-(\overline{\pi}) = \Lambda_-(\overline{\pi})\Lambda_-(\pi)$, so from the relations of Table 3.1 and equation (3.4.5), we have:

- $\Lambda_-(p)B_1 = \Lambda_-(p)(T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)) = p^2\left(\Delta_{\overline{\pi}}\Lambda_-(\pi)^2 + \Delta_{\pi}\Lambda_-(\overline{\pi})^2\right)$.
- $\Lambda_-(p)B_2 = p^4\Delta_p\Lambda_-(p)$.
- $\Lambda_-(p)B_3 = \Lambda_-(p)B_4 = 0$.

Also, again from Table 3.1, we have that

- $\Lambda_-(\pi)\Lambda_+(\overline{\pi}) = p^3\Delta_p$.
- $\Lambda_-(\overline{\pi})\Lambda_+(\overline{\pi}) = p\Delta_{\overline{\pi}}T(\overline{\pi},\pi)$.
- $\Lambda_-(p)\Lambda_+(\overline{\pi}^2) = \Lambda_-(\overline{\pi})\Lambda_-(\pi)\Lambda_+(\overline{\pi})\Lambda_+(\overline{\pi}) = p^4\Delta_p\Delta_{\overline{\pi}}T(\overline{\pi},\pi)$.

Now, $T(\overline{\pi},\pi)$ and Δ_p commute with $\Lambda_+(\overline{\pi})$ and so for $l \geq 2$, we can write

$$\begin{aligned}\Lambda_-(p)B(X_i)\Lambda_+(\overline{\pi}^l) = \Lambda_+(\overline{\pi}^{l-2})[p^4\Delta_p\Delta_{\overline{\pi}}T(\overline{\pi},\pi)-\\ -(p^4\Delta_{\overline{\pi}}\Delta_pT(\overline{\pi},\pi)^2 + p^8\Delta_{\overline{\pi}}\Delta_p^2)X_i + p^8\Delta_p^2\Delta_{\overline{\pi}}T(\overline{\pi},\pi)X_i^2] =\\ = \Lambda_+(\overline{\pi}^{l-2})(1 - T(\overline{\pi},\pi)X_i)U_\pi(X_i).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{l\geq 0}\tilde{\psi}_{p^{l-2}} \mid (\Lambda_-(p)X_i^2) \mid B(X_i)\Lambda_+(\overline{\pi}^l)Y_2^l =\\ = \sum_{l\geq 2}\tilde{\psi}_{p^{l-2}} \mid \Lambda_+(\overline{\pi}^{l-2})Y_2^{l-2}(1 - T(\overline{\pi},\pi)X_i)U_\pi(X_i)X_i^2Y_2^2 =\\ = (1-\alpha_iX)\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)U_\pi(X_i)X_i^2Y_2^2}{D^{(2)}_{\pi,G}(Y_2)},\end{aligned}$$

by Proposition 3.4.8 and Lemma 3.4.10. Hence, the result follows. □

Finally, for the middle term, we have:

Proposition 3.4.16. *We have, for $i = 1,2$*

$$\begin{aligned}-\frac{1}{1-\alpha_iX}\sum_{l\geq 0}\tilde{\psi}_{p^{l-1}} \mid T_-(p)X_i \mid B(X_i)\Lambda_+(\overline{\pi}^l)Y_2^l = -p^2\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)T(\pi)\Delta_{\overline{\pi}}X_iY_2}{D^{(2)}_{\pi,G}(Y_2)}+\\ + p^5\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)\Delta_p\Delta_{\overline{\pi}}T_+(p)Y_2^2X_i^2}{D^{(2)}_{\pi,G}(Y_2)} + p^{2k-4}\tilde{\psi}_1 \mid T(\overline{\pi})Y_2X_i^2.\end{aligned}$$

Proof. Firstly, we have no terms for $l = 0$, so we consider $l \geq 1$. The idea is to pass $\Lambda_+(\overline{\pi}^m)$ to the left for some m, so that it acts on the Fourier-Jacobi coefficients, and then we will be able to apply the rationality Proposition 3.4.8. From equation (3.4.9), we have $B_3\Lambda_+(\overline{\pi}) = B_4\Lambda_+(\overline{\pi}) = 0$. Now, using Table 3.1, we have $T_-(p)\Lambda_+(\overline{\pi}) = p^2\Delta_{\overline{\pi}}T(\pi)$ and that $T(\pi)$ commutes with $\Lambda_+(\overline{\pi})$. Therefore, we can compute:

$$\sum_{l\geq 0}\tilde{\psi}_{p^{l-1}} \mid T_-(p)X_i \mid \Lambda_+(\overline{\pi}^l)Y_2^l = p^2\sum_{l\geq 1}\tilde{\psi}_{p^{l-1}} \mid \Lambda_+(\overline{\pi}^{l-1})Y_2^{l-1}T(\pi)\Delta_{\overline{\pi}}X_iY_2 =$$

$$= p^2\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)T(\pi)\Delta_{\overline{\pi}}Y_2X_i}{D^{(2)}_{\pi,G}(Y_2)}.$$

Let us now deal with $T_-(p)B_1\Lambda_+(\overline{\pi}^l)$. We remind the reader that $B_1 = T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)$. We will deal with each part separately. By Table 3.1, we have ($l \geq 1$)

$$T_-(p)T(\overline{\pi},\pi)\Lambda_+(\overline{\pi}^l) = T_-(p)\Lambda_+(\overline{\pi}^l)T(\overline{\pi},\pi) = p^2\Delta_{\overline{\pi}}\Lambda_+(\overline{\pi}^{l-1})T(\pi)T(\overline{\pi},\pi).$$

For the other part, if $l \geq 2$,

$$T_-(p)T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^l) = p^2\Delta_{\overline{\pi}}T_-(p)\Lambda_+(\pi)\Lambda_+(\overline{\pi}^{l-1}) = p^4\Delta_pT(\overline{\pi})\Lambda_+(\overline{\pi}^{l-1}) =$$

$$= p^5\Delta_{\overline{\pi}}\Delta_p\Lambda_+(\overline{\pi}^{l-2})T_+(p).$$

For $l = 1$, we have

$$T_-(p)T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}) = p^2T_-(p)\Lambda_+(\overline{\pi}) = p^4\Delta_pT(\overline{\pi}).$$

Finally, we will deal with the term $T_-(p)B_2\Lambda_+(\overline{\pi}^l)$. Using equation (3.4.9) and relations of Table 3.1, we have for $l \geq 2$,

$$T_-(p)B_2\Lambda_+(\overline{\pi}^l) = p^2\Delta_{\overline{\pi}}T_-(p)\Lambda_+(\pi)T(\overline{\pi},\pi)\Lambda_+(\overline{\pi}^{l-1}) = p^4\Delta_pT(\overline{\pi})\Lambda_+(\overline{\pi}^{l-1})T(\overline{\pi},\pi) =$$

$$= p^5\Delta_{\overline{\pi}}\Delta_p\Lambda_+(\overline{\pi}^{l-2})T_+(p)T(\overline{\pi},\pi).$$

Finally, for $l = 1$, we get

$$T_-(p)B_2\Lambda_+(\overline{\pi}) = p^4\Delta_pT(\overline{\pi})T(\overline{\pi},\pi).$$

Applying now Proposition 3.4.8 and using Lemma 3.4.10 as well, we obtain the stated result. □

3.4.5 Calculation of the Dirichlet Series - Third Part

We will now deal with the Dirichlet series

$$D_{(\epsilon_1,\epsilon_2)}(s) = \sum_{l_1,l_2,m\geq 0}\langle\tilde{\phi}_1 \mid T_-(p^m)\Lambda_-(\pi^{l_2})\Lambda_-(\overline{\pi}^{l_1}), \tilde{\psi}_{p^{m+l_1+l_2}}\rangle_{\mathcal{A}}a_{p^m}p^{(4-2k)l_2}p^{(4-2k)l_1}\times$$

$$\times\, \pi^{l_2k}\overline{\pi}^{l_1k}p^{-s(l_1+l_2)}p^{-(2k+s-4)m} :=$$

$$:= (\alpha_1V_1(s) - \alpha_2V_2(s))/(\alpha_1-\alpha_2), \tag{3.4.10}$$

where

$$V_i(s) := \sum_{l_1,l_2,m\geq 0} \langle \tilde{\phi}_1, \tilde{\psi}_{p^{m+l_1+l_2}} \mid T_+(p^m)\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})\rangle_{\mathcal{A}} X_i^m Y_2^{l_1} Y_1^{l_2},\ i=1,2.$$

Here, we remind that $X_i = \alpha_i p^{-(2k+s-4)}$, $Y_1 = \pi^k p^{-(2k+s-4)}$, $Y_2 = \overline{\pi}^k p^{-(2k+s-4)}$ and we keep in mind that the operators $T_+(p), \Lambda_+(\pi), \Lambda_+(\overline{\pi})$ all commute with each other. This follows from the fact that j_+ of equation (2.4.1) is a ring homomorphism and $H(\Gamma_1, S_p^1)$ is commutative.

Lemma 3.4.17. *For $i=1,2$, we have*

$$\sum_{l_1,l_2,m\geq 0} \tilde{\psi}_{p^{m+l_1+l_2}} \mid T_+(p^m)\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2}) X_i^m Y_1^{l_1} Y_2^{l_2} =$$

$$= Q^{(2)}_{p,G}(X_i)^{-1} \sum_{l_1,l_2\geq 0} \left[\tilde{\psi}_{p^{l_1+l_2}} - \tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)X_i + p\tilde{\psi}_{p^{l_1+l_2-2}} \mid \Lambda_-(p)X_i^2\right] \mid$$
$$\mid B(X_i)\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1}) Y_1^{l_1} Y_2^{l_2}.$$

Proof. The proof follows immediately from Proposition 3.4.7. □

We will now deal with each sum occurring above.

Proposition 3.4.18. *For $i=1,2$, we have*

$$\sum_{l_1,l_2\geq 0} \tilde{\psi}_{p^{l_1+l_2}} \mid B(X_i)\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1}) Y_1^{l_1} Y_2^{l_2} =$$

$$= (1-p^{2k-5}Y_1Y_2)(1-p^2Y_2Y_1^{-1}\lambda_{\overline{\pi}}X_i)(1-p^2Y_1Y_2^{-1}\lambda_{\pi}X_i)\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_{\pi}(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}$$

$$-\left[(p^{k-3}-p^2Y_1Y_2^{-1}\lambda_{\pi})X_i + p^{2k-4}X_i^2\right]\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)}{D^{(2)}_{\overline{\pi},G}(Y_1)} -$$
$$-\left[(p^{k-3}-p^2Y_2Y_1^{-1}\lambda_{\overline{\pi}})X_i + p^{2k-4}X_i^2\right]\frac{\tilde{\psi}_1 \mid S_{\pi}(Y_2)}{D^{(2)}_{\pi,G}(Y_2)} +$$
$$+p^2\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_{\pi}(Y_2)\Lambda_+(\pi)\Delta_{\overline{\pi}}T(\overline{\pi},\pi)Y_2X_i^2}{D^{(2)}_{\pi,G}(Y_2)} +$$
$$+p^2\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)\Lambda_+(\overline{\pi})\Delta_{\pi}T(\pi,\overline{\pi})Y_1X_i^2}{D^{(2)}_{\overline{\pi},G}(Y_1)} +$$
$$+\left[(1-\alpha_iX)^2(1-p^{2k-4}X_i^2) + 2\alpha_iX - (1-p^{2k-4}X_i^2)\right]\tilde{\psi}_1,$$

where $\lambda_{\pi}, \lambda_{\overline{\pi}}$ are the eigenvalues of $\Delta_{\pi}, \Delta_{\overline{\pi}}$ respectively.

Proof. Firstly, using Proposition 3.4.8 and commutativity relations of Table 3.1, we have

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_1^{l_1}Y_2^{l_2} = $$
$$= \sum_{l_1\geq 0}\frac{\left[\tilde{\psi}_{p^{l_1}} - \tilde{\psi}_{p^{l_1-1}} \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)\Lambda_+(\pi^{l_1})Y_1^{l_1}}{D^{(2)}_{\pi,G}(Y_2)} =$$
$$= \frac{(1-p^{2k-5}Y_1Y_2)\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}.$$

Now, from equation (3.4.5), $B_1 = T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)$. Hence, from Proposition 3.4.8 and Table 3.1:

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_2^{l_2}Y_1^{l_1} =$$
$$= \sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\pi^{l_1})T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^{l_2})Y_2^{l_2}Y_1^{l_1} =$$
$$= \sum_{l_1\geq 0}\tilde{\psi}_{p^{l_1}} \mid \Lambda_+(\pi^{l_1})T(\pi,\overline{\pi})Y_1^{l_1} + p^2\sum_{l_1\geq 0,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\pi^{l_1+1})\Lambda_+(\overline{\pi}^{l_2-1})\Delta_{\overline{\pi}}Y_1^{l_1}Y_2^{l_2} =$$
$$= \frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\pi,\overline{\pi})}{D^{(2)}_{\overline{\pi},G}(Y_1)} +$$
$$+ p^2Y_2Y_1^{-1}\left((1-p^{2k-5}Y_1Y_2)\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)\Delta_{\overline{\pi}}}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} - \frac{\tilde{\psi}_1 \mid S_\pi(Y_2)\Delta_{\overline{\pi}}}{D^{(2)}_{\pi,G}(Y_2)}\right),$$

and we get an analogous result for

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid T(\overline{\pi},\pi)\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_2^{l_2}Y_1^{l_1}.$$

Next, using Table 3.1, we observe that for $l_1, l_2 \geq 1$ we have

$$B_2\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1}) = p\Lambda_+(\pi)\Lambda_-(\overline{\pi})\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1}) =$$
$$= p\Lambda_+(\pi)\Lambda_-(\overline{\pi})\Lambda_+(\pi)\Lambda_+(\pi^{l_1-1})\Lambda_+(\overline{\pi}^{l_2}) =$$
$$= p^4\Lambda_+(\pi)\Delta_p\Lambda_+(\pi^{l_1-1})\Lambda_+(\overline{\pi}^{l_2}) = p^4\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})\Delta_p. \quad (3.4.11)$$

Hence,

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid B_2\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2} =$$

$$\sum_{l_1\geq 0}\tilde{\psi}_{p^{l_1}} \mid B_2\Lambda_+(\pi^{l_1})Y_1^{l_1} + \sum_{l_2\geq 0}\tilde{\psi}_{p^{l_2}} \mid B_2\Lambda_+(\pi^{l_2})Y_2^{l_2} +$$

$$+\sum_{l_1,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2}} \mid B_2\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_1^{l_1}Y_2^{l_2} - \tilde{\psi}_1 \mid B_2 =$$

$$=\tilde{\psi}_1 \mid B_2 + p^2\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})\right] \mid S_\pi(Y_2)\Lambda_+(\pi)\Delta_{\overline{\pi}}T(\overline{\pi},\pi)Y_2}{D^{(2)}_{\pi,G}(Y_2)}+$$

$$+p^2\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)\right] \mid S_{\overline{\pi}}(Y_1)\Lambda_+(\overline{\pi})\Delta_{\pi}T(\pi,\overline{\pi})Y_1}{D^{(2)}_{\overline{\pi},G}(Y_1)}+$$

$$+p^{2k-4}\left(\frac{(1-p^{2k-5}Y_1Y_2)\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} - \frac{\tilde{\psi}_1 \mid S_\pi(Y_2)}{D^{(2)}_{\pi,G}(Y_2)} - \frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)}{D^{(2)}_{\overline{\pi},G}(Y_1)} + \tilde{\psi}_1\right),$$

as the sum

$$\sum_{l_1,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})p^4\Delta_pY_1^{l_1}Y_2^{l_2}$$

can be computed to be

$$p^{2k-4}\left(\frac{(1-p^{2k-5}Y_1Y_2)\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} - \frac{\tilde{\psi}_1 \mid S_\pi(Y_2)}{D^{(2)}_{\pi,G}(Y_2)} - \frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)}{D^{(2)}_{\overline{\pi},G}(Y_1)} + \tilde{\psi}_1\right).$$

Finally,

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid B_3\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_1^{l_1}Y_2^{l_2} = \tilde{\psi}_1 \mid B_3,$$

and

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2}} \mid B_4\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_1^{l_1}Y_2^{l_2} = \tilde{\psi}_1 \mid B_4,$$

as $B_3\Lambda_+(\overline{\pi}) = B_4\Lambda_+(\overline{\pi}) = B_3\Lambda_+(\pi) = B_4\Lambda_+(\pi) = 0$ from equation (3.4.9). □

Proposition 3.4.19. *For $i = 1, 2$, we have*

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2-2}} \mid \Lambda_-(p)B(X_i)\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2}X_i^2 =$$

$$=(1-\alpha_iX)\left[\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)U_\pi(X_i)X_i^2Y_2^2}{D^{(2)}_{\pi,G}(Y_2)} + \frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)U_{\overline{\pi}}(X_i)X_i^2Y_1^2}{D^{(2)}_{\overline{\pi},G}(Y_1)}\right]+$$

$$+p^{4k-10}Y_1Y_2(1-p^{2k-5}Y_1Y_2)(1-p^2Y_2Y_1^{-1}\lambda_{\overline{\pi}}X_i)(1-p^2Y_1Y_2^{-1}\lambda_\pi X_i)X_i^2\times$$
$$\times\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}$$

$$-p^{4k-10}X_i^3Y_1Y_2(p^{k-3}-p^2Y_1Y_2^{-1}\lambda_\pi)\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)}{D^{(2)}_{\overline{\pi},G}(Y_1)}-$$
$$-p^{4k-10}X_i^3Y_1Y_2(p^{k-3}-p^2Y_2Y_1^{-1}\lambda_{\overline{\pi}})\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)}{D^{(2)}_{\pi,G}(Y_2)},$$

where again $U_\pi(t) := p^4\Delta_{\overline{\pi}}\Delta_p(T(\overline{\pi},\pi) - p^4\Delta_p t) \in H_p^{1,1}[t]$, as in Proposition 3.4.15.

Proof. For the proof, we rewrite the sum as follows:

$$\sum_{l_1,l_2\geq 0} = \sum_{l_1=0,\ l_2\geq 2} + \sum_{l_2=0,\ l_1\geq 2} + \sum_{l_1,l_2\geq 1}.$$

We know how to compute the first two sums by Proposition 3.4.15, so will now deal with the last one. We rewrite this as

$$\sum_{l_1,l_2\geq 1} \tilde{\psi}_{p^{l_1+l_2-2}} \mid \Lambda_-(p)B(X_i)\Lambda_+(p)\Lambda_+(\pi^{l_1-1})\Lambda_+(\overline{\pi}^{l_2-1})Y_1^{l_1}Y_2^{l_2}X_i^2.$$

But

$$\Lambda_-(p)B(X_i)\Lambda_+(p) = p^6\Delta_p^2(1 - B_1X_i + p^4\Delta_pX_i^2),$$

as we can obtain by Table 3.1 or the relations written in [Gri92b, p. 2881-2882]. Now, using Proposition 3.4.8, we get

$$\sum_{l_1,l_2\geq 1} \tilde{\psi}_{p^{l_1+l_2-2}} \mid \Lambda_+(\pi^{l_1-1})\Lambda_+(\overline{\pi}^{l_2-1})Y_1^{l_1}Y_2^{l_2}X_i^2 =$$

$$= Y_1Y_2\sum_{l_1,l_2\geq 0} \tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2}X_i^2 =$$

$$= (1 - p^{2k-5}Y_1Y_2)Y_1Y_2X_i^2\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}.$$

Also, $B_1 = T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)$ and we have

$$\sum_{l_1,l_2\geq 1} \tilde{\psi}_{p^{l_1+l_2-2}} \mid T(\pi,\overline{\pi})\Lambda_+(\pi^{l_1-1})\Lambda_+(\overline{\pi}^{l_2-1})Y_1^{l_1}Y_2^{l_2} =$$

$$= \sum_{l_1,l_2\geq 1} \tilde{\psi}_{p^{l_1+l_2-2}} \mid \Lambda_+(\pi^{l_1-1})T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^{l_2-1})Y_1^{l_1}Y_2^{l_2} =$$

$$= Y_1Y_2\sum_{l_1,l_2\geq 0} \tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\pi^{l_1})T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2} =$$

$$= Y_1Y_2\sum_{l_1\geq 0} \tilde{\psi}_{p^{l_1}} \mid \Lambda_+(\pi^{l_1})T(\pi,\overline{\pi})Y_1^{l_1} +$$

$$+ p^2Y_1Y_2\sum_{l_1\geq 0,l_2\geq 1} \tilde{\psi}_{p^{l_1+l_2}} \mid \Lambda_+(\pi^{l_1+1})\Lambda_+(\overline{\pi}^{l_2-1})\Delta_{\overline{\pi}}Y_1^{l_1}Y_2^{l_2}$$

$$= Y_1Y_2\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\pi,\overline{\pi})}{D^{(2)}_{\overline{\pi},G}(Y_1)} +$$

$$+ p^2Y_2^2\sum_{l_1\geq 0}\frac{\left[\tilde{\psi}_{p^{l_1+1}} - \tilde{\psi}_{p^{l_1}} \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)\Lambda_+(\pi^{l_1+1})\Delta_{\overline{\pi}}Y_1^{l_1+1}}{D^{(2)}_{\pi,G}(Y_2)}$$

$$= Y_1Y_2\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\pi,\overline{\pi})}{D^{(2)}_{\pi,G}(Y_1)}+$$

$$+ p^2Y_2^2\left((1-p^{2k-5}Y_1Y_2)\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)S_{\pi}(Y_2)\Delta_{\overline{\pi}}}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} - \frac{\tilde{\psi}_1 \mid S_{\pi}(Y_2)\Delta_{\overline{\pi}}}{D^{(2)}_{\pi,G}(Y_2)}\right),$$

We obtain a similar expression for $T(\overline{\pi},\pi)$ and then the result follows. □

Proposition 3.4.20. *For $i=1,2$, we have*

$$\sum_{l_1,l_2\geq 0}\tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)B(X_i)\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2} =$$

$$(1-\alpha_iX)\ \times$$

$$\times\left[p^2\frac{\tilde{\psi}_1 \mid S_{\pi}(Y_2)T(\pi)\Delta_{\overline{\pi}}X_iY_2}{D^{(2)}_{\pi,G}(Y_2)} - p^5\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_{\pi}(Y_2)\Delta_p\Delta_{\overline{\pi}}T_+(p)Y_2^2X_i^2}{D^{(2)}_{\pi,G}(Y_2)}-\right.$$

$$-p^{2k-4}\tilde{\psi}_1 \mid T(\overline{\pi})Y_2X_i^2 +$$

$$+p^2\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\overline{\pi})\Delta_{\pi}X_iY_1}{D^{(2)}_{\overline{\pi},G}(Y_1)} - p^5\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)\Delta_p\Delta_{\pi}T_+(p)Y_1^2X_i^2}{D^{(2)}_{\overline{\pi},G}(Y_1)}-$$

$$\left.-p^{2k-4}\tilde{\psi}_1 \mid T(\pi)Y_1X_i^2\right]+$$

$$+\frac{1}{2}p^{2k-5}(1+p^{2k-4}X_i^2)X_i\ \times$$

$$\times\left[(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_{\pi}(Y_2)T_+(p)Y_1Y_2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}-\right.$$

$$-\frac{\tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})S_{\pi}(Y_2)T_+(p)Y_1Y_2^2}{D^{(2)}_{\pi,G}(Y_2)}+$$

$$+(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_{\pi}(Y_2)S_{\overline{\pi}}(Y_1)T_+(p)Y_1Y_2}{D^{(2)}_{\pi,G}(Y_2)D^{(2)}_{\overline{\pi},G}(Y_1)}-$$

$$\left.-\frac{\tilde{\psi}_1 \mid \Lambda_-(\pi)S_{\overline{\pi}}(Y_1)T_+(p)Y_2Y_1^2}{D^{(2)}_{\overline{\pi},G}(Y_1)}\right]-$$

$$-X_i^2\left[p^5(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_{\pi}(Y_2)\Delta_{\overline{\pi}}\Delta_pT_+(p)Y_2^2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}-\right.$$

$$-p^5\frac{\tilde{\psi}_p \mid S_{\pi}(Y_2)T_+(p)\Delta_{\overline{\pi}}\Delta_pY_2^2}{D^{(2)}_{\pi,G}(Y_2)} +\ p^{2k-4}Y_2\left(\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\overline{\pi})}{D^2_{\overline{\pi},G}(Y_1)} - \tilde{\psi}_1 \mid T(\overline{\pi})\right)+$$

$$+p^5(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p-\tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)S_{\overline{\pi}}(Y_1)\Delta_\pi\Delta_p T_+(p)Y_1^2}{D^{(2)}_{\pi,G}(Y_2)D^{(2)}_{\overline{\pi},G}(Y_1)}-$$
$$-p^5\frac{\tilde{\psi}_p \mid S_{\overline{\pi}}(Y_1)T_+(p)\Delta_\pi\Delta_p Y_1^2}{D^{(2)}_{\overline{\pi},G}(Y_1)}+\ p^{2k-4}Y_1\left(\frac{\tilde{\psi}_1 \mid S_\pi(Y_2)T(\pi)}{D^2_{\pi,G}(Y_2)}-\tilde{\psi}_1 \mid T(\pi)\right)\Bigg].$$

Proof. If $l_1=0$ or $l_2=0$, then we know how to compute this by Proposition 3.4.16. So, assume $l_1,l_2\geq 1$. Now,

$$T_-(p)\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})=p^3\Lambda_+(\overline{\pi}^{l_2-1})\Lambda_+(\pi^{l_1-1})T_+(p)\Delta_p,$$

using that $T_-(p)\Lambda_+(p)=p^3\Delta_pT_+(p)$. Hence, we have

$$\sum_{l_1,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2}=$$
$$=p^3\sum_{l_1,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2-1}} \mid \Lambda_+(\pi^{l_1-1})\Lambda_+(\overline{\pi}^{l_2-1})T_+(p)\Delta_pY_1^{l_1}Y_2^{l_2}=$$
$$=p^3\sum_{l_1\geq 1}\frac{\left[\tilde{\psi}_{p^{l_1}}-\tilde{\psi}_{p^{l_1-1}} \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)\Lambda_+(\pi^{l_1-1})T_+(p)\Delta_pY_1^{l_1}Y_2}{D^{(2)}_{\pi,G}(Y_2)}=$$
$$=p^3\frac{\left[\tilde{\psi}_p-\tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p)\Delta_pY_1Y_2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}-$$
$$-p^3\frac{\tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})S_\pi(Y_2)T_+(p)\Delta_pY_1Y_2^2}{D^{(2)}_{\pi,G}(Y_2)}-$$
$$-p^6\frac{\left[\tilde{\psi}_p-\tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p)\Delta_p^2Y_1^2Y_2^2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}=$$

$$=p^3(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p-\tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p)\Delta_pY_1Y_2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}-$$
$$-p^3\frac{\tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})S_\pi(Y_2)T_+(p)\Delta_pY_1Y_2^2}{D^{(2)}_{\pi,G}(Y_2)}.$$

We note here that the last expression is not (visibly) symmetric when we interchange $\pi \longleftrightarrow \overline{\pi}$. In order to make it symmetric, we compute it by calculating the series involving the operator $\Lambda_-(\pi)$ first and hence we can write

$$\sum_{l_i\geq 1}\tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2})Y_1^{l_1}Y_2^{l_2}=$$

$$= \frac{1}{2}p^{2k-5}\left[(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p)Y_1Y_2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} - \right.$$

$$-\frac{\tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})S_\pi(Y_2)T_+(p)Y_1Y_2^2}{D^{(2)}_{\pi,G}(Y_2)} +$$

$$+ (1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)S_{\overline{\pi}}(Y_1)T_+(p)Y_1Y_2}{D^{(2)}_{\pi,G}(Y_2)D^{(2)}_{\overline{\pi},G}(Y_1)} -$$

$$\left. -\frac{\tilde{\psi}_1 \mid \Lambda_-(\pi)S_{\overline{\pi}}(Y_1)T_+(p)Y_2Y_1^2}{D^{(2)}_{\overline{\pi},G}(Y_1)}\right].$$

Moreover, as in equation (3.4.11), we have that

$$B_2\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1}) = p^4\Delta_p\Lambda_+(\pi^{l_1})\Lambda_+(\overline{\pi}^{l_2}),$$

and so

$$T_-(p)B_2\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1}) = p^7\Lambda_+(\overline{\pi}^{l_2-1})\Lambda_+(\pi^{l_1-1})T_+(p)\Delta_p^2.$$

Finally, for the last one, we note $B_1 = T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)$. Now

$$\sum_{l_1,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_1^{l_1}Y_2^{l_2} =$$

$$= p^2\sum_{l_1,l_2\geq 1}\tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)\Lambda_+(\pi^{l_1+1})\Lambda_+(\overline{\pi}^{l_2-1})\Delta_{\overline{\pi}}Y_1^{l_1}Y_2^{l_2} =$$

$$= p^2\sum_{l_1\geq 1,l_2=1} + p^2\sum_{l_1\geq 1,l_2\geq 2}.$$

For the first sum, we have

$$p^2\sum_{l_1\geq 1}\tilde{\psi}_{p^{l_1}} \mid T_-(p)\Lambda_+(\pi^{l_1+1})\Delta_{\overline{\pi}}Y_1^{l_1}Y_2 = p^4Y_2\sum_{l_1\geq 1}\tilde{\psi}_{p^{l_1}} \mid \Lambda_+(\pi^{l_1})T(\overline{\pi})\Delta_pY_1^{l_1} =$$

$$= p^4Y_2\sum_{l_1\geq 0}\tilde{\psi}_{p^{l_1}} \mid \Lambda_+(\pi^{l_1})T(\overline{\pi})\Delta_pY_1^{l_1} - p^4Y_2\tilde{\psi}_1 \mid T(\overline{\pi})\Delta_p =$$

$$= p^{2k-4}Y_2\left[\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\overline{\pi})}{D^2_{\overline{\pi},G}(Y_1)} - \tilde{\psi}_1 \mid T(\overline{\pi})\right].$$

For the second

$$p^2\sum_{l_1\geq 1,l_2\geq 2}\tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)\Lambda_+(\pi^{l_1+1})\Lambda_+(\overline{\pi}^{l_2-1})\Delta_{\overline{\pi}}Y_1^{l_1}Y_2^{l_2} =$$

$$= p^5\sum_{l_1\geq 1,l_2\geq 2}\tilde{\psi}_{p^{l_1+l_2-1}} \mid \Lambda_+(\overline{\pi})^{l_2-2}\Lambda_+(\pi^{l_1})\Delta_{\overline{\pi}}\Delta_pT_+(p)Y_1^{l_1}Y_2^{l_2} =$$

$$= p^5\sum_{l_1\geq 1}\frac{\left[\tilde{\psi}_{p^{l_1+1}} - \tilde{\psi}_{p^{l_1}} \mid \Lambda_-(\overline{\pi})Y_2\right] \mid S_\pi(Y_2)\Lambda_+(\pi^{l_1})\Delta_{\overline{\pi}}\Delta_pT_+(p)Y_2^2Y_1^{l_1}}{D^{(2)}_{\pi,G}(Y_2)}.$$

But

$$\sum_{l_1\geq 1} \tilde{\psi}_{p^{l_1+1}} \mid S_\pi(Y_2)\Lambda_+(\pi^{l_1})\Delta_{\overline{\pi}}\Delta_p T_+(p)Y_2^2 Y_1^{l_1} =$$

$$= \frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)\Delta_{\overline{\pi}}\Delta_p T_+(p)Y_2^2}{D^{(2)}_{\overline{\pi},G}(Y_1)} - \tilde{\psi}_p \mid S_\pi(Y_2)T_+(p)\Delta_{\overline{\pi}}\Delta_p Y_2^2,$$

and

$$\sum_{l_1\geq 1} \tilde{\psi}_{p^{l_1}} \mid \Lambda_-(\overline{\pi})Y_2 S_\pi(Y_2)\Lambda_+(\pi^{l_1})\Delta_{\overline{\pi}}\Delta_p T_+(p)Y_2^2 Y_1^{l_1} =$$

$$= p^3 \sum_{l_1\geq 1} \tilde{\psi}_{p^{l_1}} \mid \Lambda_+(\pi^{l_1-1})Y_1^{l_1} S_\pi(Y_2)T_+(p)\Delta_{\overline{\pi}}\Delta_p^2 Y_2^3 =$$

$$= p^3 \frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)\Delta_{\overline{\pi}}\Delta_p^2 T_+(p)Y_2^3 Y_1}{D^{(2)}_{\overline{\pi},G}(Y_1)}.$$

Hence, in total

$$\sum_{l_1,l_2\geq 1} \tilde{\psi}_{p^{l_1+l_2-1}} \mid T_-(p)T(\pi,\overline{\pi})\Lambda_+(\overline{\pi}^{l_2})\Lambda_+(\pi^{l_1})Y_1^{l_1}Y_2^{l_2} =$$

$$= p^5 \frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)\Delta_{\overline{\pi}}\Delta_p T_+(p)Y_2^2}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} - p^5 \frac{\tilde{\psi}_p \mid S_\pi(Y_2)T_+(p)\Delta_{\overline{\pi}}\Delta_p Y_2^2}{D^{(2)}_{\pi,G}(Y_2)}$$

$$- p^8 \frac{\left[\tilde{\psi}_p - \tilde{\psi}_1 \mid \Lambda_-(\pi)Y_1\right] \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)\Delta_{\overline{\pi}}\Delta_p^2 T_+(p)Y_2^3 Y_1}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)} +$$

$$+ p^{2k-4}Y_2 \left[\frac{\tilde{\psi}_1 \mid S_{\overline{\pi}}(Y_1)T(\overline{\pi})}{D^2_{\overline{\pi},G}(Y_1)} - \tilde{\psi}_1 \mid T(\overline{\pi})\right],$$

and the corresponding expression for $T(\overline{\pi},\pi)$. □

3.4.6 Final expression for the Dirichlet series

We recall that

$$D^{(p)}_{F,G,h}(s) = D_{(\epsilon_1,\epsilon_2)}(s) + D_{(l_1,l_2)}(s) + D_{(\epsilon_1,l_2)}(s) + D_{(\epsilon_2,l_1)}(s) -$$

$$-D_{(\epsilon_1,\epsilon_2,l_1)}(s) - D_{(\epsilon_2,l_1,l_2)}(s) - D_{(\epsilon_1,l_1,l_2)}(s) - D_{(\epsilon_1,\epsilon_2,l_2)}(s) + D_{(\epsilon_1,\epsilon_2,l_1,l_2)}(s).$$

Now, from (3.4.8), we have

$$(\alpha_1-\alpha_2)D_{(\epsilon_1,\epsilon_2,l_2)}(s) = \alpha_1 S_1(s) - \alpha_2 S_2(s).$$

Hence,

$$(\alpha_1-\alpha_2)D_{(\epsilon_1,l_2)}(s) = \frac{\alpha_1}{1-\alpha_1 X}S_1(s) - \frac{\alpha_2}{1-\alpha_2 X}S_2(s),$$

so

$$(\alpha_1-\alpha_2)[D_{(\epsilon_1,l_2)}(s)-D_{(\epsilon_1,\epsilon_2,l_2)}(s)]=\frac{\alpha_1^2X}{1-\alpha_1X}S_1(s)-\frac{\alpha_2^2X}{1-\alpha_2X}S_2(s).$$

We also recall from equation (3.4.10) that we have

$$(\alpha_1-\alpha_2)D_{(\epsilon_1,\epsilon_2)}(s)=\alpha_1V_1(s)-\alpha_2V_2(s).$$

We can now state:

Theorem 3.4.21. *Let $2\neq p=\pi\overline{\pi}$ be a split prime in $\mathcal{O}_K$. Let $F,G\in S_2^k$ and $h\in S_1^k$ be Hecke eigenforms, all having real Fourier coefficients, h normalised, and F belonging in the Maass space. Let also ϕ_1,ψ_1 be the first Fourier-Jacobi coefficients of F,G respectively and $X_i=\alpha_i p^{-(2k+s-4)}$, $Y_1=\pi^k p^{-(2k+s-4)}$, $Y_2=\overline{\pi}^k p^{-(2k+s-4)}$. We then have for $\mathrm{Re}(s)$ large enough*

$$(\alpha_1-\alpha_2)D^{(p)}_{F,G,h}(s)=\frac{1}{Q^{(2)}_{p,G}(X_1)}\langle\tilde{\phi}_1,P(\alpha_2,\overline{s};\ G)\rangle_{\mathcal{A}}-\frac{1}{Q^{(2)}_{p,G}(X_2)}\langle\tilde{\phi}_1,P(\alpha_1,\overline{s};\ G)\rangle_{\mathcal{A}},$$

where (keeping in mind the conjugation because of the inner product)

$$P(\alpha_i,s;\ G):=\alpha_iX_ip^{k-2}(1-p^{k-2}X_i)\Bigg[(1+p^{3k-8}X_iY_1Y_2)\frac{\tilde{\psi}_1\mid S_\pi(Y_2)}{D^{(2)}_{\pi,G}(Y_2)}-$$

$$-Y_1\frac{\tilde{\psi}_1\mid S_\pi(Y_2)T(\pi)}{D^{(2)}_{\pi,G}(Y_2)}+(1+p^{3k-8}X_iY_1Y_2)\frac{\tilde{\psi}_1\mid S_{\overline{\pi}}(Y_1)}{D^{(2)}_{\overline{\pi},G}(Y_1)}-Y_2\frac{\tilde{\psi}_1\mid S_{\overline{\pi}}(Y_1)T(\overline{\pi})}{D^{(2)}_{\overline{\pi},G}(Y_1)}\Bigg]-$$

$$-\frac{1}{2}\alpha_iX_ip^{2k-5}Y_1Y_2(1-p^{k-2}X_i)^2\times$$

$$\times\Bigg[(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p-\tilde{\psi}_1\mid\Lambda_-(\pi)Y_1\right]\mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)}-$$

$$-\frac{\tilde{\psi}_1\mid\Lambda_-(\overline{\pi})S_\pi(Y_2)T_+(p)Y_2}{D^{(2)}_{\pi,G}(Y_2)}+$$

$$+(1-p^{2k-5}Y_1Y_2)\frac{\left[\tilde{\psi}_p-\tilde{\psi}_1\mid\Lambda_-(\overline{\pi})Y_2\right]\mid S_\pi(Y_2)S_{\overline{\pi}}(Y_1)T_+(p)}{D^{(2)}_{\pi,G}(Y_2)D^{(2)}_{\overline{\pi},G}(Y_1)}-$$

$$-\frac{\tilde{\psi}_1\mid\Lambda_-(\pi)S_{\overline{\pi}}(Y_1)T_+(p)Y_1}{D^{(2)}_{\overline{\pi},G}(Y_1)}\Bigg]+$$

$$+\alpha_i(1-p^{2k-5}Y_1Y_2)(1+p^{4k-9}Y_1Y_2X_i^2)(1-p^{k-2}X_i)^2\frac{\tilde{\psi}_1\mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)}{D^{(2)}_{\overline{\pi},G}(Y_1)D^{(2)}_{\pi,G}(Y_2)},$$

with S_π, $S_{\overline{\pi}}$ the polynomials defined in Proposition 3.4.4 and $\Lambda_-(\pi)$, $\Lambda_-(\overline{\pi})$, $T(\pi)$, $T(\overline{\pi})$, $T_+(p)$ the operators defined in Subsection 3.4.2. Also, $Q_p^{(2)}$ and $D_\pi^{(2)},D_{\overline{\pi}}^{(2)}$ denote the p-factors of Gritsenko's and standard's

L-function respectively, as in Definitions 2.5.2 and 2.5.1 and $D^{(p)}_{F,G,h}(s)$ *is the p-factor of the Dirichlet series, as in equation* (3.4.6)*.*

Proof. We observe that both the left and right hand side of the claimed equation in the Theorem are holomorphic functions in s for $\mathrm{Re}(s)$ large enough. Hence, it is enough to prove the equality for $s \in \mathbb{R}$ (see also the Remark 3.3.7 before Proposition 3.3.8). But then this follows by putting together the results of the last three Subsections. □

We finally have the following Proposition about the relation of $S_{\overline{\pi}}(Y_1)S_{\pi}(Y_2)$ with known L-functions.

Proposition 3.4.22. *Assume* $2 \neq p = \pi\overline{\pi}$ *is a split prime in* $\mathcal{O}_K$*. We have*

$$S_{\overline{\pi},F}(Y_1)S_{\pi,F}(Y_2) = L_p(s+k-2,f)L_p\left(s+k-2,f,\chi\right),$$

where $f \in S_{k-1}\left(\Gamma_0(4),\chi\right)$ *is the modular form whose Maass lift is* F*, as in Proposition 2.5.5. We recall here that* χ *is the quadratic character we fixed right before Definition 2.5.4.*

Proof. Assume f has a Fourier expansion as in Definition 2.5.4. Let us first consider $S_{\pi,F}(Y_2)$. We have (here, $|_{k-1}$ is the usual GL_2-action)

$$f\mid_{k-1} T(p) = a(p)f,$$

for the standard Hecke element $T(p) := \Gamma_0(4)\mathrm{diag}(1,p)\Gamma_0(4)$. Using now [Gri90, Lemma 3.3], we obtain that

$$\tilde{\phi}_1 \mid_k T(\overline{\pi}) = p^{k-2}(\overline{\pi})^{-k}a(p)\tilde{\phi}_1.$$

Using now the fact that $Y_2 = \overline{\pi}^k p^{-(2k+s-4)}$ and that

$$S_{\pi}(Y_2) = 1 - T(\overline{\pi})Y_2 + p\Delta_{\overline{\pi}}T(\pi,\overline{\pi})Y_2^2,$$

we get

$$S_{\pi,F}(Y_2) = 1 - p^{-k-s+2}a(p) + p^{-k-2s+2} = L_p(s+k-2,f),$$

and similarly for $S_{\overline{\pi},F}(Y_1)$. Given that $\chi(p) = 1$ in this case, the result follows. □

3.5 Euler Product

We can now use the above calculations in order to deduce the following Theorem:

Theorem 3.5.1. *Assume F, G, h satisfy the same assumptions as in the beginning of Subsection 3.3.2, with $\psi_1 \not\equiv 0$, i.e. not identically equal to zero. We then have that the series $D_{F,G,h}(s)$ of Theorem* (3.2.3) *has an Euler product of the form*

$$D_{F,G,h}(s) = 4\beta_k \langle \tilde{\phi}_1, \tilde{\psi}_1 \rangle_{\mathcal{A}} \prod_{p \text{ prime}} \frac{D^{(p)}_{F,G,h}(s)}{\langle \tilde{\phi}_1, \tilde{\psi}_1 \rangle_{\mathcal{A}}},$$

where $D^{(p)}_{F,G,h}(s)$ has been defined in equations (3.3.4) *and* (3.4.6) *for $p \neq 2$ and for $p = 2$, we define*

$$D^{(2)}_{F,G,h}(s) := \sum_{l,\epsilon,m \geq 0} \langle \tilde{\phi}_1 \mid T_-(2^m) U_{\pi^l}, \tilde{\psi}_{2^{m+l}} \rangle_{\mathcal{A}} a_{2^{m+\epsilon}} 2^{-sl} 2^{-(k+s-1)\epsilon} 2^{-(2k+s-4)m},$$

with $\pi := (1 + i)$, together with the condition $\min(l, \epsilon) = 0$. Also, β_k is the quanitity defined in Lemma 2.2.10.

The proof of this Theorem is the subject matter of this Section. We first need to define some elements of the global Hecke ring $H^{1,1}$. Let $m \geq 1$ and $l \in \mathcal{O}_K$. We then define

$$T_-(m) := j_-(T(m)),\ \Lambda_-(l) := j_-\left(\Gamma_1 \mathrm{diag}(l, l) \Gamma_1\right),$$

where j_- is the embedding of equation (2.4.1). Here, $T(m)$ is the standard Hecke element in H^1, as in Definition 2.5.3. We then observe that

$$T_-(m_1 m_2) = T_-(m_1) T_-(m_2),\ \Lambda_-(l_1 l_2) = \Lambda_-(l_1) \Lambda_-(l_2) \tag{3.5.1}$$

when $m_1, m_2 \in \mathbb{N}$ and $l_1, l_2 \in \mathcal{O}_K$ are co-prime. This follows from the corresponding statements for H^1 and the fact that the j_- embedding is a ring homomorphism. We also claim that these elements commute with our known Hecke elements when we allow co-prime arguments.

Lemma 3.5.2. *Let $p \neq 2$ be any rational prime. Assume that $m \in \mathbb{N}$ and $l \in \mathcal{O}_K$ are co-prime to p. Then, the elements $T_-(m)$ and $\Lambda_-(l)$ commute with all the elements listed in Subsection 3.3.1 (if p is inert) and all the elements listed in Subsection 3.4.2 (if p splits).*

Proof. The proof is done case by case. By the multiplicative property of equation (3.5.1), it suffices to consider m, l prime powers, co-prime to p. Assume first p is inert. Let then X be either T_- or Λ_- with the corresponding argument being prime co-prime to p. By [Gri92a, Lemma 3.8], we have

$$\epsilon(T_{1,p}) X = X \epsilon(T_{1,p}),\ \epsilon(T_p) X = X \epsilon(T_p). \tag{3.5.2}$$

The first equation now gives (from the proof of Proposition 3.3.3)

$$\left(T^J(p) + \Lambda_-(p) + \Lambda_+(p) + \nabla_p - \Delta_p\right) X = X \left(T^J(p) + \Lambda_-(p) + \Lambda_+(p) + \nabla_p - \Delta_p\right).$$

By then looking at the different signatures of the elements (see Definition 2.4.2) and using [Gri92a, Proposition 3.3] or [Hei99, Section 3.3], we obtain the relations

$$\begin{aligned}
\Lambda_-(p)X &= X\Lambda_-(p). \\
\Lambda_+(p)X &= X\Lambda_+(p). \\
\left(T^J(p) + (\nabla_p - \Delta_p)\right) X &= X \left(T^J(p) + (\nabla_p - \Delta_p)\right).
\end{aligned}$$

Now, X commutes with Δ_p, and we can show commutativity with ∇_p using coset decompositions. Then, commutativity with $T^J(p)$ follows from the third equation above. Finally, commutativity with $T_+(p), T_-(p)$ follows from the second equation in 3.5.2, as $\epsilon(T_p) = T_+(p) + T_-(p)$.

For the split case, we proceed similarly, using the embeddings of the standard elements $T_\pi, T_{\overline{\pi}}$ and T_p (which follow from Proposition 3.4.2). The only relation we do not obtain immediately is the commutativity with each of $T(\pi, \overline{\pi})$ and $T(\overline{\pi}, \pi)$. Instead, we get the commutativity with their sum (from the ϵ-embedding of T_p). But, from Table 3.1, we have $\Lambda_-(\pi)\Lambda_+(\pi) = p\Delta_\pi T(\pi, \overline{\pi})$ and then commutativity follows from the commutativity of X with the $\Lambda_\pm(\pi)$ elements and the fact that Δ_π is a unit in $H^{1,1}$. □

Let us now focus on the proof of Theorem 3.5.1. We need to distinguish cases when p is inert or splits in $\mathbb{Z}[i]$. We have the following two Propositions, the proof of which is essentially the same.

Proposition 3.5.3. *Let p be an inert prime. Let $m' \in \mathbb{N}$ and $l', \epsilon' \in \mathbb{Z}[i]$ all relative prime to p. Then, we claim*

$$\sum_{\substack{l,\epsilon,m \geq 0 \\ \min(l,\epsilon)=0}} \langle \tilde{\phi}_1 \mid T_-(m'p^m)\Lambda_-(l'p^l), \tilde{\psi}_{m'N(l')p^{m+2l}} \rangle_{\mathcal{A}} a_{m'N(\epsilon')p^{m+2\epsilon}} p^{-(3k+2s-8)l} p^{-2(k+s-1)\epsilon} \times$$
$$\times\, p^{-(2k+s-4)m} =$$
$$= \langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')} \rangle_{\mathcal{A}} a_{m'N(\epsilon')} \left(\frac{D^{(p)}_{F,G,h}(s)}{\langle \tilde{\phi}_1, \tilde{\psi}_1 \rangle_{\mathcal{A}}} \right).$$

Proposition 3.5.4. *Let $2 \neq p = \pi\overline{\pi}$ be a prime that splits in $\mathbb{Z}[i]$. Let $m' \in \mathbb{N}$ and $l', \epsilon' \in \mathbb{Z}[i]$ all relative prime to p (or equivalently coprime to both $\pi, \overline{\pi}$). Then, we* *claim*

$$\sum_{\substack{l_1,l_2,\\ \epsilon_1,\epsilon_2,m\geq 0\\ \min(l_i,\epsilon_i)=0}} \langle \tilde{\phi}_1 \mid T_-(m'p^m)\Lambda_-(l'\pi^{l_1}\overline{\pi}^{l_2}), \tilde{\psi}_{m'N(l')p^{m+l_1+l_2}}\rangle_{\mathcal{A}} a_{m'N(\epsilon')p^{m+\epsilon_1+\epsilon_2}} \times$$

$$\times\, p^{(4-2k)l_1} p^{(4-2k)l_2} \pi^{l_1 k} \overline{\pi}^{l_2 k} p^{-s(l_1+l_2)} p^{-(k+s-1)(\epsilon_1+\epsilon_2)} p^{-(2k+s-4)m}$$

$$= \langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')}\rangle_{\mathcal{A}} a_{m'N(\epsilon')} \left(\frac{D^{(p)}_{F,G,h}(s)}{\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}}} \right).$$

Proof. The proof is analogous to the proof of the results in Sections 3.3 and 3.4. By the multiplicative property of equation (3.5.1), we rewrite the sum in the inert case as

$$a_{m'N(\epsilon')} \sum_{l,\epsilon,m\geq 0} \langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')p^{m+2l}} \mid T_+(p^m)\Lambda_+(p^l)\rangle_{\mathcal{A}} a_{p^{m+2\epsilon}} p^{-(3k+2s-8)l} \times$$

$$\times\, p^{-2(k+s-1)\epsilon} p^{-(2k+s-4)m},$$

using the multiplicativity property of the Fourier coefficients of h as well. Similarly, we rewrite the sum for the split case in an analogous way.

We can now apply the rationality Propositions, as in Sections 3.3 and 3.4. The difference is that every time we previously had the term $\tilde{\psi}_1$, we will instead now have $\tilde{\psi}_{m'N(l')}$. This follows from fact that $m'N(l')$ is co-prime to p (so terms of the form $\tilde{\psi}_{m'N(l')/p}$ vanish). Similarly, we now have terms of the form $\tilde{\psi}_{pm'N(l')} \mid T_+(p)$ instead of the terms $\tilde{\psi}_p \mid T_+(p)$.

By the calculations leading to Theorems 3.3.14 and 3.4.21, we claim that the expressions involving the Fourier-Jacobi coefficients of G (i.e., before taking the inner product with $\tilde{\phi}_1 \mid T_-(m')\Lambda_-(l')$), can be written in the form $\tilde{\psi}_{m'N(l')} \mid R(Y, Y_1, Y_2, X_1, X_2)$, where R is a polynomial with coefficients involving the operators $T^J(p)$, $T(\pi)$, $T(\overline{\pi})$, $T(\pi,\overline{\pi})$, $T(\overline{\pi},\pi)$ and is independent of m', l'.

Let us first deal with the inert case. The only expressions that are not in the form claimed above are these of the form $\tilde{\psi}_{pm'N(l')} \mid T_+(p)$ (see Proposition 3.3.12). But we can write

$$\tilde{\psi}_{pm'N(l')} \mid T_+(p) = \lambda_p \tilde{\psi}_{m'N(l')},$$

where λ_p is the eigenvalue of the operator $T_p \in H_p^2$, when it acts on G, i.e. $G \mid_k T_p = \lambda_p G$. This is true because of the embedding

$$\epsilon(T_p) = T_+(p) + T_-(p),$$

as in the proof of Proposition 3.3.3. So, we get

$$\tilde{\psi}_{m'N(l')} \,||\, T_p = \lambda_p \tilde{\psi}_{m'N(l')},$$

and

$$\tilde{\psi}_{m'N(l')} \,||\, T_p = \tilde{\psi}_{pm'N(l')} \mid T_+(p) + \tilde{\psi}_{m'N(l')/p} \mid T_-(p) = \tilde{\psi}_{pm'N(l')} \mid T_+(p).$$

The same can be said for the split case as well. From Theorem 3.4.21, we will have terms of the form

$$\tilde{\psi}_{pm'N(l')} \mid S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p),\; \tilde{\psi}_{m'N(l')} \mid \Lambda_-(\pi)S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p),$$
$$\tilde{\psi}_{m'N(l')} \mid \Lambda_-(\overline{\pi})S_\pi(Y_2)T_+(p),$$

(and the corresponding expressions for $\overline{\pi}$). But, from Table 3.1, we have the relations:

$$\begin{aligned} S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p) = T_+(p) &\left[1 - T(\pi)Y_1 + p^3\Delta_p Y_1 Y_2\right][1 - T(\overline{\pi})Y_2] + \\ &+ \Lambda_+(\pi)\left[T(\overline{\pi},\pi)Y_1 - p^2\Delta_{\overline{\pi}}Y_2\right][1 - T(\overline{\pi})Y_2] - \\ &- p^2\Delta_\pi Y_1 \Lambda_+(\overline{\pi})\left[1 - T(\pi)Y_1 + p^3\Delta_p Y_1Y_2\right][1 - T(\overline{\pi})Y_2] + \\ &+ \Lambda_+(\overline{\pi})S_{\overline{\pi}}(Y_1)Y_2. \end{aligned}$$

$$\begin{aligned} \Lambda_-(\pi)S_{\overline{\pi}}(Y_1)S_\pi(Y_2)T_+(p) = p^2\Delta_\pi T(\overline{\pi}) &\left[1 - T(\pi)Y_1 + p^3\Delta_p Y_1Y_2\right][1 - T(\overline{\pi})Y_2] + \\ &+ p\Delta_\pi T(\pi,\overline{\pi})\left[T(\overline{\pi},\pi)Y_1 - p^2\Delta_{\overline{\pi}}Y_2\right][1 - T(\overline{\pi})Y_2] - \\ &- p^5\Delta_\pi\Delta_p Y_1\left[1 - T(\pi)Y_1 + p^3\Delta_pY_1Y_2\right][1 - T(\overline{\pi})Y_2] + \\ &+ p^3\Delta_p S_{\overline{\pi}}(Y_1)Y_2. \end{aligned}$$

$$\begin{aligned} \Lambda_-(\overline{\pi})S_\pi(Y_2)T_+(p) = p^2\Delta_{\overline{\pi}}T(\pi) - &\Big[p^5\Delta_{\overline{\pi}}\Delta_p - p\Delta_{\overline{\pi}}T(\pi,\overline{\pi})T(\overline{\pi},\pi) + \\ &p^2\Delta_{\overline{\pi}}T(\pi)T(\overline{\pi})\Big]Y_2 + p^5\Delta_{\overline{\pi}}\Delta_pT(\overline{\pi})Y_2^2. \end{aligned}$$

Now, from Proposition 3.4.2, we have

$$\epsilon(T_p) = T_-(p) + T_+(p) + T(\pi,\overline{\pi}) + T(\overline{\pi},\pi),$$

and so we get

$$\begin{aligned} \tilde{\psi}_{m'N(l')} \,||\, T_p &= \tilde{\psi}_{m'N(l')} \,||\, (T_+(p) + T_-(p) + T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)) \\ &= \tilde{\psi}_{pm'N(l')} \mid T_+(p) + 0 + \tilde{\psi}_{m'N(l')} \mid (T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)) \\ &= \tilde{\psi}_{pm'N(l')} \mid T_+(p) + \tilde{\psi}_{m'N(l')} \mid (T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)). \end{aligned}$$

But $\tilde{\psi}_{m'N(l')} \,||\, T_p = \lambda_p\tilde{\psi}_{m'N(l')}$, where λ_p is the eigenvalue of G corresponding to T_p, and so we obtain

$$\tilde{\psi}_{pm'N(l')} \mid T_+(p) = \lambda_p\tilde{\psi}_{m'N(l')} - \tilde{\psi}_{m'N(l')} \mid (T(\pi,\overline{\pi}) + T(\overline{\pi},\pi)).$$

Moreover, again from Proposition 3.4.2, we have

$$\epsilon(T_{\overline{\pi}}) = \Lambda_-(\overline{\pi}) + T(\overline{\pi}) + \Lambda_+(\overline{\pi}).$$

Hence, by a similar argument as above, we get

$$\tilde{\psi}_{pm'N(l')} \mid \Lambda_+(\overline{\pi}) = \lambda_{T_{\overline{\pi}}}\tilde{\psi}_{m'N(l')} - \tilde{\psi}_{m'N(l')} \mid T(\overline{\pi}),$$

and similarly for $\Lambda_+(\pi)$.

In particular, our claim now follows for both the inert and split case and therefore, the expression involving the Fourier-Jacobi coefficients of G can be written in the form $\tilde{\psi}_{m'N(l')} \mid R$, where $R = R(Y, Y_1, Y_2, X_1, X_2)$ is a polynomial with coefficients involving the operators $T^J(p)$, $T(\pi)$, $T(\overline{\pi})$, $T(\pi, \overline{\pi})$, $T(\overline{\pi}, \pi)$. These are all self-adjoint operators (see [Gri92a, Lemma 4.3]). Moreover, since F is in the Maass space, from [Gri92a, Theorem, p. 2911], $\tilde{\phi}_1$ is an eigenform for these operators, as these all have signature 1. By now writing R_F for the polynomial obtained when we substitute the eigenvalues of $\tilde{\phi}_1$ with respect to the above operators and using the commutativity of Lemma 3.5.2, we can write

$$\langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')} \mid R\rangle_{\mathcal{A}} = \langle \tilde{\phi}_1 \mid \overline{R}T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')}\rangle_{\mathcal{A}} =$$

$$= \overline{R_F}\langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')}\rangle_{\mathcal{A}} =$$

$$= \frac{1}{\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle}\langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')}\rangle_{\mathcal{A}}\overline{R_F}\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle_{\mathcal{A}} =$$

$$= \frac{1}{\langle \tilde{\phi}_1, \tilde{\psi}_1\rangle}\langle \tilde{\phi}_1 \mid T_-(m')\Lambda_-(l'), \tilde{\psi}_{m'N(l')}\rangle_{\mathcal{A}}\langle \tilde{\phi}_1, \tilde{\psi}_1 \mid R\rangle_{\mathcal{A}},$$

where $\overline{R_F}$ is the polynomial obtained by taking the complex conjugate. The result now follows by comparing with the initial expression for $D^{(p)}_{F,G,h}(s)$, as the rightmost term is what we have originally (i.e., for $m' = l' = 1$). □

The proof of Theorem 3.5.1 now follows from the above two Propositions by working prime by prime and factoring from the initial Dirichlet series the corresponding expression for each prime.

3.6 Integral Representation

In this Section, we will show how $D_{F,G,h}(s)$ originates as part of a Rankin-Selberg integral and how this compares to the integral representation given by Heim in

[Hei99, Theorem 2.6]. We consider the triple inner product

$$\Phi(F,G,h;s) := \left\langle \left\langle \left\langle E^k_{5,0}\left(\begin{pmatrix} z_1 & & \\ & z_2 & \\ & & z_3 \end{pmatrix}; s\right), F(z_3)\right\rangle, G(z_2)\right\rangle, h(z_1)\right\rangle.$$

The main reason we consider this is the following algebraic result.

Proposition 3.6.1. *Assume that F, G and h have algebraic Fourier coefficients. For $k > 10$ we have*

$$\frac{\Phi(F,G,h;0)}{\langle F,F\rangle\langle G,G\rangle\langle h,h\rangle} \in \overline{\mathbb{Q}}.$$

Proof. This can be shown exactly as [Hei99, Theorem 1.9]. In the proof there, a result of Böcherer is used on the algebraic decomposition of the space of modular forms as an orthogonal product of the space of cusp forms and of the Eisenstein series, i.e.

$$M^k_2\left(\overline{\mathbb{Q}}\right) = S^k_2\left(\overline{\mathbb{Q}}\right) \oplus \mathrm{Eis}^k_2\left(\overline{\mathbb{Q}}\right).$$

Such a result is also available for unitary groups in [Shi00, Theorem 27.14]. □

Actually, one can give an even stronger statement of the proposition above, namely, establish even a reciprocity law on the action of the absolute Galois group. The statement is similar to [Hei99, Theorem 1.9]. The main point here is that we can establish an algebraicity result for special values of L-functions if we can relate the expression above to an Euler product expression. This is our main motivation.

By now using the doubling method for unitary groups, as for example is studied in [Shi00, Equation 24.29 (a)], we know that the first inner product is related to a Klingen-type Eisenstein series attached to F (recall F has real Fourier coefficients), as in Definition 2.1.6. That is,

$$\left\langle E^k_{5,0}\left(\begin{pmatrix} z_1 & & \\ & z_2 & \\ & & z_3 \end{pmatrix}; s\right), F(z_3)\right\rangle = \nu(s)\frac{Z^{(2)}_F(s+k/2)}{\prod_{i=0}^3 L(2s+k-i,\chi^i)}\times$$
$$\times E^k_{3,2}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, F; s\right), \quad (3.6.1)$$

where $Z^{(2)}_F$ is the standard L-function attached to F (see Definition 2.5.1), χ is the non-trivial quadratic character attached to the extension $K/\mathbb{Q}$ and $\nu(s)$ is an expression involving Gamma factors (the explicit expression can be computed by [Shi00, Equation 24.29 (a)]).

So, our focus shifts to computing

$$\left\langle\left\langle E^k_{3,2}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2\end{pmatrix}, F; s\right), G(z_2)\right\rangle, h(z_1)\right\rangle \tag{3.6.2}$$

Given the Definition 2.1.6 of the Eisenstein series, we will start by finding representatives for $C_{3,2}\backslash\Gamma_3$. We begin by first finding representatives for $C_{3,2}(K)\backslash U(3,3)(K)$. Surprisingly, this decomposition differs in a significant way from the one obtained by Heim in [Hei99, Proposition 2.1]. For now, let us write $U_n(K)$ for $U(n,n)(K)$.

Before we proceed, let us briefly introduce the notion of **isotropic vectors** and **isotropic spaces** in $U_n(K)$. We say a vector $x \in K^{2n}$ is isotropic if $\overline{x}^t J_n x = 0$. A subspace U of K^{2n} will be called isotropic if $\overline{x}^t J_n y = 0$ for all $x, y \in U$.

For any $m, n \geq 1$, there is an embedding $U_m(K) \times U_n(K) \hookrightarrow U_{m+n}(K)$ given by

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1\end{pmatrix} \times \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2\end{pmatrix} \hookrightarrow \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2\end{pmatrix}.$$

For each $r \in \mathbb{Q}^\times$, we consider the following subgroups of $U_1(K)$ and $U_2(K)$, respectively:

$$H_{1,r}(K) := \left\{\begin{pmatrix} a & b \\ -br^2 & a\end{pmatrix} \in U_1(K)\right\},$$

$$H_{2,r}(K) := \left\{\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ irb_3 & a_4 & b_3 & b_4 \\ -r^2 b_1 & ira_2 & a_1 & irb_2 \\ ird_3 & c_4 & d_3 & d_4\end{pmatrix} \in U_2(K)\right\}. \tag{3.6.3}$$

We then have the following Proposition.

Proposition 3.6.2. *The right coset space $C_{3,2}(K)\backslash U_3(K)$ has representatives*

$$S_1 = C_{1,0}(K)\backslash U_1(K) \times 1_4,$$

$$S_2 = \pi \cdot (1_2 \times C_{2,1}(K)\backslash U_2(K)),$$

$$S_3 = \xi \cdot \left(C_{1,0}(K)\backslash U_1(K) \times ((T \times 1_2) \cdot C_{2,1}(K)\backslash U_2(K))\right),$$

$$W_r = \xi_r \cdot (D \cdot H_{1,r}(K)\backslash U_1(K) \times H_{2,r}(K)\backslash U_2(K)),\ r \in \mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times),$$

where

$$\pi := \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \end{pmatrix}, \ \xi := \begin{pmatrix} 1 & 0 & & & & \\ 1 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & -1 & \\ & & & 0 & 1 & \\ & & & & & 1 \end{pmatrix},$$

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \mid a \in K^\times \right\},$$

$$\xi_r := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ ir & ir & 1 & -1 \\ -ir & 0 & 0 & 1 \end{pmatrix} \times 1_2, \ D := \left\{ \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \mid d \in K, \, N(d) = 1 \right\}.$$

Proof. Let $e_1, e_2, e_3, f_1, f_2, f_3$ denote the standard basis for K^6, viewed as row vectors. The map $g \longmapsto f_1 g$ induces a bijection between $C_{3,2}(K)\backslash U_3(K)$ and the set X of one-dimensional isotropic subspaces in K^6 (this is a standard fact, consequence of Witt's Theorem, as in [Shi97, Lemma 2.1] for example). Let V be such a subspace. We decompose $K^6 = K^2 \oplus K^4$ according to the embedding $U_1(K) \times U_2(K) \hookrightarrow U_3(K)$ (i.e., $K^2 = Ke_1 \oplus Kf_1$). There are three possibilities: V is contained in K^2, V is contained in K^4 or V is not contained in either.

In the first two cases, V is an isotropic subspace of K^2 or K^4, respectively, and hence we obtain the same set of representatives S_1, S_2 as in [Hei99, Proposition 2.1].

For the last case, assume V is spanned by the isotropic vector v. We decompose $v = v_1 \oplus v_2$ and we have two possibilities: v_1 is isotropic or v_1 is not isotropic. In the first case, in analogy with [Hei99, Proposition 2.1], we obtain the set S_3.

Assume now v_1 is not isotropic. Then v_2 will not be isotropic either (since v is isotropic). Let us write $\overline{v_1}^t J_1 v_1 = 2ir = -\overline{v_2}^t J_2 v_2$, with $r \in \mathbb{Q}^\times$. By Witt's Theorem ([Shi97, Theorem 1.2]), we have that an isotropic vector $w = w_1 \oplus w_2 \in K^6$ will be in the same orbit as in v under the action of $U_1(K) \times U_2(K)$ if and only if $\overline{w_1}^t J_1 w_1 = 2ir = -\overline{w_2}^t J_2 w_2$. This shows that the isotropic vectors $v = v_1 \oplus v_2$, with the same norm on the first component (hence the second too), form a single orbit under the action of $U_1(K) \times U_2(K)$. Since we are free to scale v by some $\lambda \in K^\times$, (because we work with the subspace spanned by v), we must consider $r \in \mathbb{Q}^\times / N_{K/\mathbb{Q}}(K^\times)$. Here, $N_{K/\mathbb{Q}}(a + ib) := a^2 + b^2$ for $a, b \in \mathbb{Q}$.

Now, for each such r, we consider $v_r := \begin{pmatrix} ir & ir & 0 & 1 & -1 & 0 \end{pmatrix}$ as a representative of its orbit. We then observe that the matrix ξ_r defined in the statement of the Proposition satisfies $f_1 \xi_r = v_r$. Hence, we deduce that the double quotient

$C_{3,2}(K)\backslash U_3(K)/(U_1(K)\times U_2(K))$ has the following irredundant representatives:

$$1_6, \pi, \xi, \{\xi_r, r \in \mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times)\}.$$

We now have that for $g_1 \in U_1(K)$, $g_2 \in U_2(K)$ and r as above, $C_{3,2}(K)\xi_r(g_1 \times g_2) = C_{3,2}(K)\xi_r$ if and only if $\xi_r(g_1\times g_2)\xi_r^{-1} \in C_{3,2}(K)$. This then implies that $g_1 \in H_{1,r}(K)$, $g_2 \in H_{2,r}(K)$, and $a + irb = a_1 - irb_1$, where we write g_1, g_2 as in (3.6.3).

For $i = 1, 2$, if $g_i \in U_i(K)$, we write $g_1 = h_1e$ and $g_2 = h_2f$, where $h_i \in H_{i,r}(K)$ and e, f belong to a set of proper representatives for $H_{i,r}(K)\backslash U_i(K)$, respectively. By writing h_i as in (3.6.3), we let $d := (a_1 - irb_1)(a + irb)^{-1}$. This is well-defined as $N(a + irb) = N(a_1 - irb_1) = 1$ by unitarity, so in particular $a + irb$, $a_1 + irc_1$ are non-zero. Moreover, $N(d) = 1$ and if $D := \mathrm{diag}(d, d) \in U_1(K)$, we have that $Dh_1 \times h_2 \in \xi_r^{-1}C_{3,2}(K)\xi_r$. Hence,

$$\begin{aligned} C_{3,2}(K)\xi_r(g_1\times g_2) = C_{3,2}(K)\xi_r(h_1e\times h_2f) &= C_{3,2}(K)\xi_r(Dh_1\times h_2)(D^{-1}\times 1)(e\times f)\\ &= C_{3,2}(K)\xi_r(D^{-1}e\times f). \end{aligned}$$

This gives us the set of representatives W_r. Hence, the Proposition follows. □

We now want to pull these representatives back to representatives for $C_{3,2}\backslash\Gamma_3$.

Corollary 3.6.3. *The right coset space $C_{3,2}\backslash\Gamma_3$ has representatives*

$$T_1 = C_{1,0}\backslash\Gamma_1 \times 1_4,$$

$$T_2 = \pi\cdot(1_2 \times C_{2,1}\backslash\Gamma_2),$$

$$T_3 = \bigsqcup_{p,q} (\xi^{p,q}\times 1_2)\cdot(C_{1,0}\backslash\Gamma_1 \times C_{2,1}\backslash\Gamma_2),$$

$$V_r,\ r \in \mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times)$$

Here, $p, q \in \mathbb{Z}[i]\backslash\{0\}$ with $\gcd(p, q) = 1$, $q = u + iv$, $u > 0$, $v \geq 0$, and

$$\xi^{p,q} := \begin{pmatrix} * & * & 0 & 0\\ q & p & 0 & 0\\ 0 & 0 & \overline{p} & -\overline{q}\\ 0 & 0 & * & * \end{pmatrix},$$

with $\xi^{p,q}\times 1_2 \in \Gamma_3$. The sets V_r correspond to the representatives obtained by pulling W_r of Proposition 3.6.2 back to $\mathcal{O}_K$.

Proof. We first observe that there is a one-to-one correspondence between $C_{n,r}(K)\backslash U_n(K)$ and $C_{n,r}\backslash\Gamma_n$, for any n, r with $0 \leq r \leq n$. Indeed, since K has class number one, that is $\mathbb{Z}[i]$ is a principal ideal domain, we have from [Shi97, Proposition 7.2 (2), p. 48], that $U_n(K) = C_{n,r}(K)\Gamma_n$, and hence

$$C_{n,r}(K)\backslash U_n(K) \cong (\Gamma_n \cap C_{n,r}(K))\backslash\Gamma_n = C_{n,r}\backslash\Gamma_n.$$

Now, each V_r is obtained by pulling W_r of Proposition 3.6.2 back to $\mathcal{O}_K$, thanks to this correspondence. Moreover, T_1 and T_2 are obtained by the sets S_1, S_2 of Proposition 3.6.2. In order to obtain the set T_3, it suffices to pull S_3 back to $\mathcal{O}_K$. We therefore need to find a matrix M_a in $C_{3,2}(K)$ such that

$$M_a \cdot \xi \cdot \left(1_2 \times \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \times 1_2\right) \in \Gamma_3, \tag{3.6.4}$$

with ξ the matrix of Proposition 3.6.2. We parametrize K as

$$\left\{\frac{p}{q} \mid p, q \in \mathbb{Z}[i],\ \gcd(p,q) = 1,\ q = u + iv,\ u > 0,\ v \geq 0\right\}.$$

All these elements are different as p, q are in $\mathbb{Z}[i]$ with the above conditions and their union is K. For $a = p/q$ as above, we define

$$M_{p,q} := \begin{pmatrix} p^{-1} & y & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \overline{p} & 0 \\ 0 & 0 & l & \overline{q}^{-1} \end{pmatrix} \times 1_2,$$

with l, q chosen so that $l\overline{q} \equiv 1 \pmod{\overline{p}}$ and $y = -q\bar{l}/p$. Then $M_{p,q} \in C_{3,2}(K)$ and we can then see that the product of equation (3.6.4) belongs to Γ_3 and has the claimed form. □

Given the above decomposition, we can now appropriately split the Eisenstein series in order to compute the integral of equation (3.6.2). For any $n \geq 1$, if $M \in \Gamma_n,\ Z \in \mathbb{H}_n$, we define the following quantities:

$$\chi^{k,s}(M,Z) := j(M,Z)^{-k}|j(M,Z)|^{-2s},\ \delta(Z) := \det(\operatorname{Im} Z).$$

From [Kri85, Theorem II.1.7, (c)], we have $\delta(M\langle Z\rangle) = |j(M,Z)|^{-2}\delta(Z)$.

Proposition 3.6.4. *Let $k \equiv 0 \pmod 4$ and $k + 2\operatorname{Re}(s) > 10$. Let also $F \in S_2^k$, $z_1 \in \mathbb{H}_1$ and $z_2 \in \mathbb{H}_2$. We then have*

$$E^k_{3,2}([z_1, z_2], F; s) = E^k_{1,0}(z_1; s)F(z_2) + E^k_{2,1}(z_2, F_{z_1}; s) +$$

$$+\ \delta(z_1)^s\delta(z_2)^s \sum_{p,q} \sum_{\substack{\gamma_1\in C_{1,0}\backslash\Gamma_1\\ \gamma_2\in C_{2,1}\backslash\Gamma_2}} \chi^{k,s}(\gamma_1,z_1)\chi^{k,s}(\gamma_2,z_2)\times$$

$$\times F\left(\begin{pmatrix} N(q)\gamma_1\langle z_1\rangle & 0\\ 0 & 0\end{pmatrix} + (\gamma_2\langle z_2\rangle)\left[\begin{pmatrix}\overline{p} & 0\\ 0 & 1\end{pmatrix}\right]\right)\times$$

$$\times\ \delta\left(\begin{pmatrix} N(q)\gamma_1\langle z_1\rangle & 0\\ 0 & 0\end{pmatrix} + (\gamma_2\langle z_2\rangle)\left[\begin{pmatrix}\overline{p} & 0\\ 0 & 1\end{pmatrix}\right]\right)^{-s} + \mathcal{E}^k\left([z_1,z_2],F;s\right),$$

where

$$\mathcal{E}^k\left([z_1,z_2],F;s\right) :=$$

$$= \sum_{r\in\mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times)} \sum_{\gamma\in V_r} F(\gamma\langle[z_1,z_2]\rangle_*)j(\gamma,[z_1,z_2])^{-k}\left(\frac{\det\operatorname{Im}\gamma\langle[z_1,z_2]\rangle}{\det\operatorname{Im}\gamma\langle[z_1,z_2]\rangle_*}\right)^s.$$

Here, for $\tau\in\mathbb{H}_1$, $F_{z_1}(\tau) := F\left([z_1,\tau]\right)\in S_1^k$ and p,q are summed as in Corollary 3.6.3. We also remind the reader here that $N(q)$ denotes the norm of q and $[a,b]$ the block diagonal matrix with diagonal blocks a,b as in Notation.

Proof. As we have shown after Definition 2.1.6, for $k+2\mathrm{Re}(s)>10$, the Eisenstein series $E^k_{3,2}\left(Z,F;s\right)$ is absolutely and uniformly convergent on compact subsets of $\mathbb{C}$. We split the Eisenstein series according to the representatives of Corollary 3.6.3. We can write

$$E^k_{3,2}\left([z_1,z_2],F;s\right) = \delta(z_1)^s\delta(z_2)^s\sum_{i=1}^{3}\sum_{M\in T_i}\chi^{k,s}(M,[z_1,z_2])F(M\langle[z_1,z_2]\rangle_*)\times$$
$$\times\ \delta\left(M\langle[z_1,z_2]\rangle_*\right)^{-s} + \mathcal{E}^k\left([z_1,z_2],F;s\right).$$

For the representatives of T_1,T_2 in Corollary 3.6.3, the proof is exactly the same as in [Hei99, Theorem 2.3].

For a representative M of T_3, write $M=(\xi^{p,q}\times 1_2)(\gamma_1\times\gamma_2)$ with $\gamma_1\in C_{1,0}\backslash\Gamma_1$ and $\gamma_2\in C_{2,1}\backslash\Gamma_2$. We write $\gamma_2\langle z_2\rangle = \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$ and we then have

$$\begin{aligned}(M\langle[z_1,z_2]\rangle)_* &= ((\xi^{p,q}\times 1_2)[\gamma_1 z_1,\gamma_2 z_2])_*\\ &= \left(\begin{pmatrix} * & * & 0\\ q & p & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\langle z_1\rangle & 0 & 0\\ 0 & x_1 & x_2\\ 0 & x_3 & x_4\end{pmatrix}\begin{pmatrix} * & \overline{q} & 0\\ 0 & \overline{p} & 0\\ 0 & 0 & 1\end{pmatrix}\right)_*\\ &= \begin{pmatrix} N(q)\gamma_1\langle z_1\rangle + N(p)x_1 & px_2\\ \overline{p}x_3 & x_4\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} N(q)\gamma_1\langle z_1\rangle & 0 \\ 0 & 0 \end{pmatrix} + (\gamma_2\langle z_2\rangle)\left[\begin{pmatrix} \overline{p} & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Also,

$$j((\xi^{p,q}\times 1_2)(\gamma_1\times\gamma_2),[z_1,z_2]) = j(\xi^{p,q}\times 1_2,[\gamma_1 z_1,\gamma_2 z_2])j(\gamma_1\times\gamma_2,[z_1,z_2]),$$

and $j(\xi^{p,q}\times 1_2,[\gamma_1 z_1,\gamma_2 z_2]) = \det(D)$, where we denote $\xi^{p,q} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$.

By unitarity, we have $D\overline{A}^t = 1_2$ so $\det(D)\cdot\overline{\det(A)} = 1_2$, so $N(\det(D)) = 1$ and $\det(D)$ is in $\mathbb{Z}[i]$, which shows that $\det(D)\in\{\pm 1,\pm i\}$. As $k\equiv 0 \pmod 4$, we get

$$\chi^{k,s}((\xi\times 1_2)(\gamma_1\times\gamma_2),[z_1,z_2]) = \chi^{k,s}(\gamma_1,z_1)\chi^{k,s}(\gamma_2,z_2),$$

and so the Proposition follows. □

We can now use this decomposition in order to show how $D_{F,G,h}(s)$ originates as part of a Rankin-Selberg integral.

Theorem 3.6.5. *Let $k\equiv 0 \pmod 4$. Let $F,G\in S_2^k$ and $h\in S_1^k$. Then, for $k+2\mathrm{Re}(s)>10$, we have*

$$\left\langle\left\langle E^k_{3,2}\left(\begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}, F; s\right), G(Z)\right\rangle, h(W)\right\rangle = (4\pi)^{-(2k+s-4)}\times$$
$$\times\frac{\Gamma(2k+s-4)\Gamma(k+s-3)\Gamma(k+s-1)}{\Gamma(2k+2s-4)}D_{F,G,h}(s) + R_{F,G,h}(s),$$

where $R_{F,G,h}(s) := \langle\langle\mathcal{E}^k([W,Z],F;s), G(Z)\rangle, h(W)\rangle$.

Proof. From Proposition 3.6.4, we can rewrite the Eisenstein series as a sum involving four summands. Clearly, $R_{F,G,h}(s)$ corresponds to the summand $\mathcal{E}^k([Z,W],F;s)$.

We will now deal with the third one. This can be written as (the summations are as in Corollary 3.6.3):

$$I_3 := \int_{\Gamma_1\backslash\mathbb{H}_1}\int_{\Gamma_2\backslash\mathbb{H}_2}\delta(W)^{k+s}\delta(Z)^{k+s}\sum_{p,q}\sum_{\gamma_1,\gamma_2}\chi^{k,s}(\gamma_1,W)\chi^{k,s}(\gamma_2,Z)\times$$
$$\times F\left(\begin{pmatrix} N(q)\gamma_1\langle W\rangle & 0 \\ 0 & 0 \end{pmatrix} + (\gamma_2\langle Z\rangle)\left[\begin{pmatrix} \overline{p} & 0 \\ 0 & 1 \end{pmatrix}\right]\right)\times$$
$$\times\delta\left(\begin{pmatrix} N(q)\gamma_1\langle W\rangle & 0 \\ 0 & 0 \end{pmatrix} + (\gamma_2\langle Z\rangle)\left[\begin{pmatrix} \overline{p} & 0 \\ 0 & 1 \end{pmatrix}\right]\right)^{-s}\overline{G(Z)h(W)}\mathrm{d}^*W\mathrm{d}^*Z.$$

Now, using the automorphy condition for G and h, we have

$$G(Z) = (G|_k\gamma_2)(Z) = j(\gamma_2, Z)^{-k} G(\gamma_2\langle Z\rangle),$$

$$h(W) = (h|_k\gamma_1)(W) = j(\gamma_1, W)^{-k} h(\gamma_1\langle W\rangle).$$

Also $\delta(\gamma_2\langle Z\rangle) = |j(\gamma_2, Z)|^{-2}\delta(Z)$, $\delta(\gamma_1\langle W\rangle) = |j(\gamma_1, W)|^{-2}\delta(W)$, so

$$\begin{aligned}\overline{G(Z)}\delta(Z)^{k+s}\chi^{k,s}(\gamma_2, Z) &= \overline{j(\gamma_2, Z)}^{-k}\overline{G(\gamma_2\langle Z\rangle)}\delta(\gamma_2\langle Z\rangle)^{k+s}|j(\gamma_2, Z)|^{2(k+s)}\times \\ &\times j(\gamma_2, Z)^{-k}|j(\gamma_2, Z)|^{-2s} = \overline{G(\gamma_2\langle Z\rangle)}\delta(\gamma_2\langle Z\rangle)^{k+s},\end{aligned}$$

and similarly for h. Hence, by the usual "unfolding" trick, we obtain:

$$\begin{aligned} I_3 = \int_{C_{1,0}\backslash\mathbb{H}_1}\int_{C_{2,1}\backslash\mathbb{H}_2}\sum_{p,q} F\left(\begin{pmatrix} N(q)W & 0 \\ 0 & 0\end{pmatrix} + Z\left[\begin{pmatrix}\overline{p} & 0 \\ 0 & 1\end{pmatrix}\right]\right)\times \\ \times\,\delta\left(\begin{pmatrix} N(q)W & 0 \\ 0 & 0\end{pmatrix} + Z\left[\begin{pmatrix}\overline{p} & 0 \\ 0 & 1\end{pmatrix}\right]\right)^{-s}\overline{G(Z)h(W)}\delta(Z)^{k+s}\delta(W)^{k+s}\mathrm{d}^*Z\mathrm{d}^*W.\end{aligned}$$

We now consider the matrix $M := \begin{pmatrix} I & \\ & I\end{pmatrix}$, where $I := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. We then check that $M \in \Gamma_2$ and from Definition 2.1.4, we have

$$F\left(Z\left[\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right]\right) = F(Z)\ \forall Z \in \mathbb{H}_2,$$

as k is even and $M\langle Z\rangle = Z[I]$. The same also holds for G. In particular, this shows

$$F\left(\begin{pmatrix} N(q)W & 0 \\ 0 & 0\end{pmatrix} + Z\left[\begin{pmatrix}\overline{p} & 0 \\ 0 & 1\end{pmatrix}\right]\right) = F\left(\begin{pmatrix} 0 & 0 \\ 0 & N(q)W\end{pmatrix} + Z\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{p}\end{pmatrix}\right]\right).$$

What is more, we can directly compute

$$\delta\left(\begin{pmatrix} N(q)W & 0 \\ 0 & 0\end{pmatrix} + Z\left[\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right]\left[\begin{pmatrix}\overline{p} & 0 \\ 0 & 1\end{pmatrix}\right]\right) = \delta\left(\begin{pmatrix} 0 & 0 \\ 0 & N(q)W\end{pmatrix} + Z\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{p}\end{pmatrix}\right]\right).$$

By now setting $Z \longmapsto M\langle Z\rangle$ and using the fact that $M^{-1}C_{2,1}M = P_{2,1}$, we can rewrite I_3 as ($C_{1,0} = P_{1,0}$ as well):

$$\begin{aligned} I_3 = \int_{P_{1,0}\backslash\mathbb{H}_1}\int_{P_{2,1}\backslash\mathbb{H}_2}\sum_{p,q} F\left(\begin{pmatrix} 0 & 0 \\ 0 & N(q)W\end{pmatrix} + Z\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{p}\end{pmatrix}\right]\right)\times \\ \times\,\delta\left(\begin{pmatrix} 0 & 0 \\ 0 & N(q)W\end{pmatrix} + Z\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{p}\end{pmatrix}\right]\right)^{-s}\overline{G(Z)h(W)}\delta(Z)^{k+s}\delta(W)^{k+s}\mathrm{d}^*Z\mathrm{d}^*W.\end{aligned}$$

Now, a fundamental domain for the action of $P_{1,0}$ on $\mathbb{H}_1$ is

$$\mathcal{F} = \{z \in \mathbb{H}_1 \mid z = x + iy,\ 0 \leq x \leq 1\},$$

while a fundamental domain for the action of $P_{2,1}$ on $\mathbb{H}_2$ is (cf. [Gri92a, p. 2907])

$$\left\{ \begin{pmatrix} \tau & z_1 \\ z_2 & \omega \end{pmatrix} \mid (\tau, z_1, z_2) \in \mathcal{F}^J,\ y_\omega > |z_1 - \overline{z_2}|^2/4y_\tau,\ 0 \leq x_\omega \leq 1 \right\},$$

using the notation of equation (2.2.1) for the real and imaginary parts. Hence, we have

$$I_3 = \sum_{p,q} \int_{\mathcal{F}^J} \mathrm{d}\tau \mathrm{d}z_1 \mathrm{d}z_2 \int_{y_\omega > |z_1 - \overline{z_2}|^2/4y_\tau} \mathrm{d}y_\omega \int_0^1 \mathrm{d}x_\omega \int_0^1 \mathrm{d}x_W \times$$
$$\times \int_0^\infty \mathrm{d}y_W \delta(Z)^{k+s-4} \sum_{m=1}^\infty \phi_m(\tau, \overline{p}z_1, pz_2) e^{2\pi i m (N(q)W + N(p)\omega)} \times$$
$$\times \sum_{n=1}^\infty \overline{a_n} e^{-2\pi i n \overline{W}} \sum_{l=1}^\infty \overline{\psi_l(\tau, z_1, z_2)} e^{-2\pi i l \overline{\omega}} \delta\left(\begin{pmatrix} 0 & 0 \\ 0 & N(q)W \end{pmatrix} + Z\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{p} \end{pmatrix} \right] \right)^{-s}.$$

We first perform the integration over x_ω and x_W. For x_ω, we have

$$\int_0^1 e^{2\pi i m N(p) x_\omega - 2\pi i l x_\omega} \mathrm{d}x_\omega = \begin{cases} 1 & \text{if } l = mN(p) \\ 0 & \text{otherwise} \end{cases}.$$

Similarly for x_W, we have

$$\int_0^1 e^{2\pi i m N(q) x_W - 2\pi i n x_W} \mathrm{d}x_W = \begin{cases} 1 & \text{if } n = mN(q) \\ 0 & \text{otherwise} \end{cases}.$$

These are the only terms we need to integrate as the real parts of ω and W do not appear as arguments of δ by definition. We now substitute $t = y_\omega - |z_1 - \overline{z_2}|^2/4y_\tau$ and compute

$$\delta(Z) = \det\left(\frac{1}{2i} \left(Z - \overline{Z}^t \right) \right) = y_\tau t,$$

and

$$\delta\left(\begin{pmatrix} 0 & 0 \\ 0 & N(q)W \end{pmatrix} + Z\left[\begin{pmatrix} 1 & 0 \\ 0 & \overline{p} \end{pmatrix} \right] \right) = y_\tau (N(q) y_W + N(p) t).$$

So, the integral I_3 becomes

$$\sum_{p,q} \int_{\mathcal{F}^J} \sum_{m=1}^\infty \phi_m(\tau, \overline{p}z_1, pz_2) \overline{\psi_{mN(p)}(\tau, z_1, z_2)} \overline{a_{mN(q)}} y_\tau^{k-4} e^{-\pi m (|z_1 - \overline{z_2}|^2/y_\tau)} \mathrm{d}\tau \mathrm{d}z_1 \mathrm{d}z_2 \times$$
$$\times \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}y_W t^{k+s-4} (N(q) y_W + N(p) t)^{-s} e^{-4\pi m (N(q) y_W + N(p) t)} y_W^{s+k-2} =$$

$$= \sum_{p,q} \sum_{m=1}^{\infty} \langle \phi_m | U_p, \psi_{mN(p)} \rangle \overline{a_{mN(q)}} \int_0^{\infty} \int_0^{\infty} \frac{\mathrm{d}t}{t} \frac{\mathrm{d}y_W}{y_W} t^{k+s-3} y_W^{s+k-1} (N(q)y_W + N(p)t)^{-s} \times \\ \times e^{-4\pi m (N(q)y_W + N(p)t)} =$$

$$= (4\pi)^{-(2k+s-4)} \frac{\Gamma(2k+s-4)\Gamma(k+s-3)\Gamma(k+s-1)}{\Gamma(2k+2s-4)} \times \\ \times \sum_{p,q} \sum_{m=1}^{\infty} \langle \phi_m | U_p, \psi_{mN(p)} \rangle \overline{a_{mN(q)}} N(p)^{-(k+s-3)} N(q)^{-(k+s-1)} m^{-(2k+s-4)}, \quad (3.6.5)$$

by using the fact that (cf. [Hei99, Theorem 2.6])

$$\int_0^{\infty} \int_0^{\infty} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} x^{\alpha} y^{\beta} \left(\frac{xy}{x+y} \right)^{\gamma} e^{-(x+y)} = \frac{\Gamma(\alpha+\beta+\gamma)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+2\gamma)},$$

and substituting $x = 4\pi m N(p) t$, $y = 4\pi m N(q) y_W$, $\alpha = k-3$, $\beta = k-1$, $\gamma = s$. This formula follows after setting $u = x+y$ and then $t = x/u$ and using the Euler integral of the first kind

$$\int_{t=0}^{1} t^{z_1-1}(1-t)^{z_2-1} \mathrm{d}t = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}, \ \forall z_1, z_2 \in \mathbb{C}.$$

Let us now consider the second summand in the decomposition of Proposition 3.6.4. We want to compute

$$I_2 := \int_{\Gamma_1 \backslash \mathbb{H}_1} \int_{\Gamma_2 \backslash \mathbb{H}_2} \sum_{\gamma \in C_{2,1} \backslash \Gamma_2} j(\gamma, Z)^{-k} F_W(\gamma \langle Z \rangle_*) \left(\frac{\delta(\gamma \langle Z \rangle)}{\delta(\gamma \langle Z \rangle_*)} \right)^s \overline{G(Z) h(W)} \delta(Z)^k \times \\ \times \delta(W)^k \mathrm{d}^* Z \mathrm{d}^* W.$$

Using again the automorphy condition for G, we obtain, by unfolding the integral, that

$$I_2 = \int_{\Gamma_1 \backslash \mathbb{H}_1} \int_{C_{2,1} \backslash \mathbb{H}_2} F_W(Z_*) \overline{G(Z) h(W)} \delta(Z)^{k+s} \delta(Z_*)^{-s} \mathrm{d}^* Z \mathrm{d}^* W.$$

Using the same reasoning for the interchange of the parabolic subgroups $C_{2,1}$, $P_{2,1}$ as in the case of I_3, we rewrite the inner integral as

$$J_2 := \int_{P_{2,1} \backslash \mathbb{H}_2} F\left(\begin{pmatrix} \tau & 0 \\ 0 & W \end{pmatrix} \right) \overline{G(Z)} \delta(Z)^{k+s} y_{\tau}^{-s} \mathrm{d}^* Z,$$

and by using the Fourier-Jacobi expansions and the fundamental domains mentioned before, we have

$$J_2 = \int_{\mathcal{F}^J} \mathrm{d}\tau \mathrm{d}z_1 \mathrm{d}z_2 \int_{y_{\omega} > |z_1 - \overline{z_2}|^2/4y_{\tau}} \int_0^1 \mathrm{d}x_{\omega} \sum_{m=1}^{\infty} \phi_m(\tau, 0, 0) e^{2\pi i m W} \sum_{n=1}^{\infty} \overline{\psi_n(\tau, z_1, z_2)} \times \\ \times e^{-2\pi i n \overline{\omega}} \delta(Z)^{k+s-4} y_{\tau}^{-s}.$$

We can then see that this is zero by calculating the integral over x_ω, i.e.,

$$\int_0^1 e^{-2\pi i n x_\omega} dx_\omega = 0.$$

Therefore, $I_2 = 0$. Finally, we will show that the integral involving the first summand of Proposition 3.6.4 is zero. We have, after a direct computation

$$\langle\langle E_{1,0}(W;s)F(Z), G(Z)\rangle, h(W)\rangle = \langle F(Z), G(Z)\rangle\langle E_1(W;s), h(W)\rangle,$$

and we will show that the second inner product is zero. But

$$\langle E_{1,0}(W;s), h(W)\rangle = \int_{\Gamma_1\backslash\mathbb{H}_1} \sum_{\gamma\in C_{1,0}\backslash\Gamma_1} j(\gamma,W)^{-k}\delta(\gamma W)^s\overline{h(W)}\delta(W)^k \mathrm{d}^*W.$$

By the usual unfolding trick, the above integral equals

$$\int_{C_{1,0}\backslash\mathbb{H}_1} \delta(W)^{k+s}\overline{h(W)}\mathrm{d}^*W = \int_{x=0}^1\int_{y=0}^\infty \sum_{n=1}^\infty \overline{a_n}e^{-2\pi i n(x-iy)}y^{k-2}\mathrm{d}x\mathrm{d}y = 0,$$

by looking at the integral

$$\int_0^1 e^{-2\pi i n x}\mathrm{d}x = 0,$$

for all $n \geq 1$. Hence, I_3 is the only integral that has a non-zero contribution and the Theorem now follows from equation (3.6.5). □

It is therefore clear that Theorem 3.6.5 has an important difference to the Theorem obtained by Heim in [Hei99, Theorem 2.6], namely the term $R_{F,G,h}(s)$. To calculate this, one needs to obtain a better understanding of the representatives V_r of Corollary 3.6.3, which in turn requires understanding the groups $H_{1,r}(K)$ and $H_{2,r}(K)$ of (3.6.3). We conclude this Chapter with the following Proposition, regarding the relation of these groups with the respective unitary groups.

Proposition 3.6.6. *For $i = 1, 2$ and $r \in \mathbb{Q}^\times/N_{K/\mathbb{Q}}(K^\times)$, we have*

$$U(i,i)(K) = H_{i,r}(K)C_{i,0}(K),$$

where $C_{i,0}(K)$ denote the Siegel parabolics (recall Definition 2.1.3).

Proof. We will first show $U(1,1)(K) = H_{1,r}(K)C_{1,0}(K)$. Let $\phi := 2/r$ and consider the unitary group with respect to ϕ, i.e., $G^\phi := \{a \in K^\times \mid \overline{a}\phi a = \phi\}$. Moreover, we set $\omega := \mathrm{diag}(\phi, -\phi)$ and consider the unitary group with respect to ω, i.e. $G^\omega := \{a \in \mathrm{GL}_2(K) \mid \overline{a}\omega a = \omega\}$.

Consider now the ω-isotropic subspace of K^2 given by $U := \{(v,v) \mid v \in K\}$ and let P_U^ω be the parabolic subgroup of G^ω defined by $P_U^\omega := \{a \in G^\omega \mid Ua = U\}$. From

[Shi97, Proposition 2.4], we then have that

$$G^\omega = P_U^\omega (G^\phi \times G^\phi) = (G^\phi \times G^\phi) P_U^\omega.$$

The second equality follows from taking inverses. Now, if we let

$$S_1 := \begin{pmatrix} 1 & -i/r \\ -1 & -i/r \end{pmatrix},$$

we have from [Shi97, (21.1.8)], that $S_1^{-1} G^\omega S_1 = U(1,1)(K)$ and $S_1^{-1} P_U^\omega S_1 = C_{1,0}(K)$. Hence

$$U(1,1)(K) = S_1^{-1} G^\omega S_1 = S_1^{-1}(G^\phi \times G^\phi) P_U^\omega S_1 = S_1^{-1}(G^\phi \times G^\phi) S_1 C_{1,0}(K).$$

But, for $a, b \in K^\times$, we can compute

$$S_1^{-1} \mathrm{diag}(a,b) S_1 = \begin{pmatrix} (a+b)/2 & -i(a-b)/2r \\ i(a-b)r/2 & (a+b)/2 \end{pmatrix}.$$

Now, if $x = (a+b)/2,\ y = -i(a-b)/2r$, we get $a = x + iry,\ b = x - iry$ and therefore, $N(a) = N(b) = 1$ is equivalent to $\overline{x}y = \overline{y}x$ and $N(x) + r^2 N(y) = 1$. Hence, $S^{-1}(G^\phi \times G^\phi) S = H_{1,r}(K)$. Therefore, $U(1,1)(K) = H_{1,r}(K) C_{1,0}(K)$, as claimed.

For the second group, we consider $\phi := 2/r$ and $\psi := \begin{pmatrix} & & 1 \\ & 2/r & \\ 1 & & \end{pmatrix}$ and set $\omega :=$ $\mathrm{diag}(\psi, -\phi)$. We then consider the ω-isotropic subspace $U := \{(0, v, i, v) \mid v, i \in K\}$ and the respective parabolic subgroup P_U^ω of G^ω. Again, from [Shi97, Proposition 2.4], we have

$$G^\omega = P_U^\omega (G^\psi \times G^\phi) = (G^\psi \times G^\phi) P_U^\omega.$$

Similarly to before, if we let

$$S_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i/r \\ 0 & 0 & -i & 0 \\ 0 & -1 & 0 & -i/r \end{pmatrix},$$

we have from [Shi97, (21.1.8)], that $S_2^{-1} G^\omega S_2 = U(2,2)(K)$ and $S_2^{-1} P_U^\omega S_2 = C_{2,0}(K)$. Now, as before, by computing $S_2^{-1} \mathrm{diag}(A, a) S_2$, with $A \in G^\psi,\ a \in G^\phi$, we have $S_2^{-1}(G^\psi \times G^\phi) S_2 = M H_{2,r}(K) M^{-1}$, where $M = \mathrm{diag}(I, I)$, with $I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. But since $M^{-1} U(2,2)(K) M = U(2,2)(K)$ and $M^{-1} C_{2,0}(K) M = C_{2,0}(K)$, we get $U(2,2)(K) = H_{1,r}(K) C_{2,0}(K)$, as claimed. $\square$

Chapter 4

A Dirichlet series for Orthogonal Modular Forms

This Section marks the beginning of the second part of the thesis. As we said in the introduction, for this part, we work with orthogonal modular forms of real signature $(2, n+2),\ n \geq 1$. We aim to extend the work of Kohnen and Skoruppa in [KS89] for this case.

4.1 Quadratic Spaces

In this Section, we will prepare the ground for the theory of orthogonal modular forms. Our main reference is [Shi04].

Let F be a field. Let V denote a finite dimensional vector space over F, with $\dim(V) = m$. Define also an F-bilinear symmetric form $\varphi : V \times V \longrightarrow F$, which is **non-degenerate**, i.e., there is no $0 \neq x \in V$ such that $\varphi(x, V) = 0$. For all $x \in V$, we put $\varphi[x] := \varphi(x, x)$. We call φ **isotropic** on V if $\phi[x] = 0$ for some $0 \neq x \in V$ and **anisotropic** on V if $\varphi[x] = 0$ only for $x = 0$. We call a subspace U of V **totally isotropic** if $\varphi(x, y) = 0$ for all $x, y \in U$. We define

$$\mathrm{O}^{\varphi}(V) := \{g \in \mathrm{GL}(V) \mid \varphi[gx] = \varphi[x],\ \forall x \in V\},$$

$$\mathrm{SO}^{\varphi}(V) := \mathrm{O}^{\varphi}(V) \cap \mathrm{SL}(V),$$

for the **orthogonal group** and **special orthogonal group** of φ. We call (V, φ) a **quadratic space**.

Let now X be a subspace of V and ψ the restriction of φ on X. If ψ is non-degenerate on X, then the symbols $\mathrm{O}^{\psi}(X)$ and $\mathrm{SO}^{\psi}(X)$ are meaningful. By abuse of notation, we will just refer to those spaces by $\mathrm{O}^{\varphi}(X)$ and $\mathrm{SO}^{\varphi}(X)$.

Now, from [Shi04, Lemma 1.4, (i)], we can find elements $\{e_i\}_{i=1}^r, \{f_i\}_{i=1}^r$ of V and a subspace Z of V such that

$$V = Z + \sum_{i=1}^{r}(Fe_i + Ff_i), \tag{4.1.1}$$

$$\varphi(e_i, e_j) = \varphi(f_i, f_j) = 0 \text{ and } 2\varphi(e_i, f_j) = \delta_{ij} \text{ for every } i, j,$$

$$Z := \{v \in V \mid \varphi(e_i, v) = \varphi(f_i, v) = 0, \text{ for every } i\},$$

and the restriction of φ on Z to be anisotropic. We call the decomposition (4.1.1) a **Witt decomposition**. We also call Z the **core subspace** of (V, φ) and $t = \dim(Z)$ the **core dimension** of (V, φ).

In this thesis, we will be interested in the cases $F = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Q}_p$, with p a rational prime. If $F = \mathbb{Q}$ or $\mathbb{Q}_p$, we denote by $\mathfrak{g}$ the maximal order of F, i.e. $\mathfrak{g} = \mathbb{Z}$ or $\mathfrak{g} = \mathbb{Z}_p$ respectively. We then have the following definition of a $\mathfrak{g}$-lattice.

Definition 4.1.1. A $\mathfrak{g}$-lattice Λ in (V, φ) is a free, finitely generated $\mathfrak{g}$-module, which spans V over F.

We also have the following notion of integral and maximal lattices.

Definition 4.1.2. A $\mathfrak{g}$-lattice Λ in (V, φ) is called $\mathfrak{g}$-integral if $\varphi[x] \in \mathfrak{g}$ for all $x \in \Lambda$. It is called maximal if it is maximal among all $\mathfrak{g}$-integral lattices.

Moreover, we define the dual lattice and the level.

Definition 4.1.3. The **dual lattice** of a $\mathfrak{g}$-lattice Λ in (V, φ) is defined by

$$\Lambda^* := \{x \in V \mid 2\varphi(x, y) \in \mathfrak{g} \ \forall y \in \Lambda\}.$$

Definition 4.1.4. The **level** of the lattice Λ is the least positive integer q such that $q\varphi[x] \in \mathfrak{g}$ for every $x \in \Lambda^*$.

We also define the integral orthogonal groups via

$$\mathrm{O}^{\phi}(\Lambda) := \{\alpha \in \mathrm{SO}^{\phi}(V) \mid \alpha\Lambda = \Lambda\}. \tag{4.1.2}$$

$$\mathrm{SO}^{\phi}(\Lambda) := \{\alpha \in \mathrm{SO}^{\phi}(V) \mid \alpha\Lambda = \Lambda\}. \tag{4.1.3}$$

Assume now we work over the local field $\mathbb{Q}_p$ and assume L is a maximal $\mathbb{Z}_p$-lattice in V. From [Shi04, Lemma 6.5], we have a Witt decomposition as in (4.1.1), such that additionally

$$L = M + \sum_{i=1}^{r}(\mathbb{Z}_p e_i + \mathbb{Z}_p f_i), \ M = \{z \in Z \mid \varphi[z] \in \mathbb{Z}_p\}.$$

From [Shi04, Theorem 7.6], we have that over $\mathbb{Q}_p$, $t = \dim(Z) \leq 4$.

In general, if (V, φ) is a quadratic space over $\mathbb{Q}$, we can define its localisation (V_p, φ_p) for all primes p, by setting $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and extending φ to a $\mathbb{Q}_p$-valued bilinear form φ_p on V_p in a natural way.

Finally, we will express things by matrices. Assume we choose the standard basis for V over F and write $V = F^m$. We then have that φ is given by $\varphi(x, y) = x^t \varphi_0 y,\ \forall x, y \in V$, for some $\varphi_0^t = \varphi_0 \in \mathrm{GL}_n(F)$. In that case, we have

$$\mathrm{O}^{\varphi}(V) = \{g \in \mathrm{GL}_n(F) \mid g^t \varphi_0 g = \varphi_0\},$$

and similarly for $\mathrm{SO}^{\varphi}(V)$.

Finally, we denote by $\delta(\varphi)$ the coset of $F^\times/(F^\times)^2$, represented by $(-1)^{m(m-1)/2} \det(\varphi_0)$, where $(F^\times)^2 := \{a^2 \mid a \in F^\times\}$.

4.2 Orthogonal Modular Forms

In this Section, we will give the basic definitions for orthogonal modular forms of signature $(2, n+2),\ n \geq 1$. Our main reference is [Sch22].

In the following, let $V := \mathbb{Q}^n$ and $L := \mathbb{Z}^n$ with $n \geq 1$. Assume S is an even integral positive definite symmetric matrix of rank n. Here, even means $S[x] \in 2\mathbb{Z}$ for all $x \in L$. We define

$$S_0 := \begin{pmatrix} & & 1 \\ & -S & \\ 1 & & \end{pmatrix},\ S_1 := \begin{pmatrix} & & 1 \\ & S_0 & \\ 1 & & \end{pmatrix}$$

of real signatures $(1, n+1)$ and $(2, n+2)$ respectively. Let also $V_0 := \mathbb{Q}^{n+2}$ and $V_1 := \mathbb{Q}^{n+4}$ and consider the quadratic spaces $(V_0, \phi_0),\ (V_1, \phi_1)$, where

$$\begin{aligned} \phi_i : V_i \times V_i &\longmapsto \mathbb{Q} \\ (x, y) &\longmapsto \frac{1}{2} x^t S_i y, \end{aligned}$$

for $i = 0, 1$. We then have that $\phi := \phi_0 \mid_{V \times V}$ is just $(x, y) \longmapsto -x^t S y/2$, and we make the assumption that $L = \mathbb{Z}^n$ is a maximal $\mathbb{Z}$-lattice with respect to ϕ.

From [Shi04, Lemma 6.3], we then obtain that $L_0 := \mathbb{Z}^{n+2}$ is a $\mathbb{Z}$-maximal lattice in V_0. If now $K \supset \mathbb{Q}$ is a field, we define the corresponding special orthogonal groups of K-rational points via

$$G_K^* := \{g \in \mathrm{SL}_{n+2}(K) \mid g^t S_0 g = S_0\}, \tag{4.2.1}$$

$$G_K := \{g \in \mathrm{SL}_{n+4}(K) \mid g^t S_1 g = S_1\}. \tag{4.2.2}$$

We view G_K^* as a subgroup of G_K via the embedding

$$g \longmapsto \begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix}. \tag{4.2.3}$$

Let now $\mathcal{H}_S$ denote one of the connected components of $\{Z \in V_0 \otimes_{\mathbb{Q}} \mathbb{C} \mid \phi_0[\mathrm{Im} Z] > 0\}$. In particular, let

$$\mathcal{P}_S := \{y' = (y_1, y, y_2) \in \mathbb{R}^{n+2} \mid y_1 > 0,\ \phi_0[y'] > 0\}. \tag{4.2.4}$$

We then choose

$$\mathcal{H}_S := \{z = u + iv \in V_0 \otimes_{\mathbb{R}} \mathbb{C} \mid v \in \mathcal{P}_S\}. \tag{4.2.5}$$

For a matrix $g \in \mathrm{M}_{n+4}(\mathbb{R})$, we write it as

$$g = \begin{pmatrix} \alpha & a^t & \beta \\ b & A & c \\ \gamma & d^t & \delta \end{pmatrix},$$

with $A \in \mathrm{M}_{n+2}(\mathbb{R}), \alpha, \beta, \gamma, \delta \in \mathbb{R}$ and a, b, c, d real column vectors. Now the map

$$Z \longmapsto g\langle Z \rangle = \frac{-\frac{1}{2}S_0[Z]b + AZ + c}{-\frac{1}{2}S_0[Z]\gamma + d^t Z + \delta} \tag{4.2.6}$$

gives a well-defined **transitive** action of $G_{\mathbb{R}}^0$ on $\mathcal{H}_{\mathcal{S}}$, where $G_{\mathbb{R}}^0$ denotes the identity component of $G_{\mathbb{R}}$. The denominator of the above expression is the factor of automorphy

$$j(g, Z) := -\frac{1}{2}S_0[Z]\gamma + d^t Z + \delta.$$

Let now $L_1 := \mathbb{Z}^{n+4}$ and define the groups

$$\Gamma(L_0) := \{g \in G_{\mathbb{Q}}^* \mid gL_0 = L_0\},$$

$$\Gamma(L_1) := \{g \in G_{\mathbb{Q}} \mid gL_1 = L_1\}.$$

Let also $\Gamma^+(L_0) := \Gamma(L_0) \cap G_{\mathbb{R}}^{*,0}$. Moreover, let

$$\Gamma_S := G_{\mathbb{R}}^0 \cap \Gamma(L_1),$$

and

$$\widetilde{\Gamma}_S := \{M \in \Gamma_S \mid M \in 1_{n+4} + \mathrm{M}_{n+4}(\mathbb{Z})S_1\}, \tag{4.2.7}$$

the **discriminant kernel**.

Definition 4.2.1. Let $k \in \mathbb{Z}$ and $\widetilde{\Gamma}_S \leq \Gamma \leq \Gamma_S$ a subgroup of finite index. A holomorphic function $F : \mathcal{H}_S \longrightarrow \mathbb{C}$ is called a modular form of weight k with respect to Γ if it satisfies the equation

$$(F|_k\gamma)(Z) := j(\gamma, Z)^{-k} F(\gamma\langle Z\rangle) = F(Z)$$

for all $\gamma \in \Gamma$ and $Z \in \mathcal{H}_S$. We will denote the set of such forms by $M_k(\Gamma)$.

Now, if $\widetilde{\Gamma}_S \leq \Gamma \leq \Gamma_S$, $F \in M_k(\Gamma)$ admits a Fourier expansion of the form (see [Sug95, (5.10)])

$$F(Z) = \sum_{r \in L_0^*} A(r) e(r^t S_0 Z), \tag{4.2.8}$$

where $Z \in \mathcal{H}_S$. It is then Koecher's principle that gives us that $A(r) = 0$ unless $r \in L_0^* \cap \overline{\mathcal{P}_\mathcal{S}}$ ($\overline{\mathcal{P}_S}$ denotes the closure of $\mathcal{P}_\mathcal{S}$, see [Sch22, Theorem 1.5.2]). By [Sch22, Theorem 1.6.23], and because L is maximal, we have the following definition for cusp forms.

Definition 4.2.2. If $\widetilde{\Gamma}_S \leq \Gamma \leq \Gamma_S$, $F \in M_k(\Gamma)$ is called a **cusp form** if it admits a Fourier expansion of the form

$$F(Z) = \sum_{r \in L_0^* \cap \mathcal{P}_\mathcal{S}} A(r) e(r^t S_0 Z),$$

We denote the space of cusp forms by $S_k(\Gamma)$.

We finally define a Petersson inner product, as in [Sch22, Remark 1.6.25].

Definition 4.2.3. If $\widetilde{\Gamma}_S \leq \Gamma \leq \Gamma_S$, let $\mathcal{Q}_\Gamma$ denote a fundamental domain for the action of Γ on $\mathcal{H}_S$. Assume $F, G \in M_k(\Gamma)$, with at least one belonging in $S_k(\Gamma)$. We define their **Petersson inner product** as

$$\langle F, G\rangle_\Gamma := \frac{1}{[\Gamma_S : \Gamma]} \int_{\mathcal{Q}_\Gamma} F(Z)\overline{G(Z)} \left(Q_0[\mathrm{Im} Z]\right)^k \mathrm{d}^* Z,$$

where $\mathrm{d}^*Z = (Q_0[\mathrm{Im}Z])^{-(n+2)}\, \mathrm{d}Z$ denotes the $G^0_\mathbb{R}$-invariant volume element on $\mathcal{H}_S$. Here, $Q_0 := S_0/2$. This is independent of the choice of the fundamental domain, or in fact, of the subgroup Γ, so in the following, we drop the subscript.

4.3 Fourier-Jacobi Forms of Lattice Index

In this Section, we will define Fourier-Jacobi forms of **lattice** index. We follow Mocanu's thesis [Moc19], and Krieg in [Kri96].

For now, assume that V is a vector space of dimension $n < \infty$ over $\mathbb{Q}$, together with a positive definite symmetric bilinear form σ and an even lattice Λ in V, i.e. $\sigma(\lambda, \lambda) \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$. We start with the following definitions:

Definition 4.3.1. We define the **Heisenberg group** to be:

$$H^{(\Lambda,\sigma)}(\mathbb{R}) = \{(x, y, \zeta) \mid x, y \in \Lambda \otimes \mathbb{R}, \zeta \in S^1\},$$

where $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, equipped with the following composition law:

$$(x_1, y_1, \zeta_1)(x_2, y_2, \zeta_2) := (x_1 + x_2, y_1 + y_2, \zeta_1\zeta_2 e(\sigma(x_1, y_2))).$$

The **integral Heisenberg group** is defined to be $H^{(\Lambda,\sigma)}(\mathbb{Z}) := \{(x, y, 1) \mid x, y \in \Lambda\}$ and in the following we drop the last coordinate for convenience.

Proposition 4.3.2. *The group* $\mathrm{SL}_2(\mathbb{R})$ *acts on* $H^{(\Lambda,\sigma)}(\mathbb{R})$ *from the right, via*

$$((x, y, \zeta), A) \longmapsto (x, y, \zeta)^A := \left((x, y)A, \zeta e\left(\sigma[(x, y)A] - \frac{1}{2}\sigma(x, y)\right)\right).$$

where $(x, y)A$ *denotes the formal multiplication of the vector* (x, y) *with* A*, i.e. if* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*, we have* $(x, y)A := (ax + cy, bx + dy)$*.*

Definition 4.3.3. The **real Jacobi** group associated with (Λ, σ), denoted by $J^{(\Lambda,\sigma)}(\mathbb{R})$, is defined to be the semi-direct product of $\mathrm{SL}_2(\mathbb{R})$ and $H^{(\Lambda,\sigma)}(\mathbb{R})$. The composition law is then

$$(A, h) \cdot (A', h') := (AA', h^{A'}h').$$

We also define the **integral Jacobi group** to be the semi-direct product of $\mathrm{SL}_2(\mathbb{Z})$ and $H^{(\Lambda,\sigma)}(\mathbb{Z})$ and we will denote it by $J^{(\Lambda,\sigma)}$.

We are now going to define some slash operators, acting on holomorphic, complex-valued functions on $\mathbb{H} \times (\Lambda \otimes \mathbb{C})$.

Definition 4.3.4. Let k be a positive integer and $f : \mathbb{H} \times (\Lambda \otimes \mathbb{C}) \longrightarrow \mathbb{C}$ a holomorphic function. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, we define:

$$\left(f|_{k,(\Lambda,\sigma)}[M]\right)(\tau, z) := (c\tau + d)^{-k} e^{-\pi i c\sigma(z,z)/(c\tau+d)} f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right).$$

In the case when $M \in \mathrm{GL}_2^+(\mathbb{R})$, we use $\det(M)^{-1/2}M$ instead of M. For $h = (x, y, \zeta) \in H^{(\Lambda,\sigma)}(\mathbb{R})$:

$$\left(f \mid_{k,(\Lambda,\sigma)} h\right)(\tau, z) := \zeta \cdot e^{\pi i \tau\sigma(x,x) + 2\pi i\sigma(x,z)} f(\tau, z + x\tau + y).$$

Finally, for the action of $J^{(\Lambda,\sigma)}(\mathbb{R})$ on complex-valued, holomorphic functions on $\mathbb{H} \times (L \otimes \mathbb{C})$, we have:

$$(f, (A, h)) \longmapsto \left(f \mid_{k,(\Lambda,\sigma)} (A, h)\right)(\tau, z) := \left((f \mid_{k,(\Lambda,\sigma)} A) \mid_{k,(\Lambda,\sigma)} h\right)(\tau, z).$$

We now have the following Definition ([Moc19, Definition 1.23]):

Definition 4.3.5. Let $V_{\mathbb{C}} := V \otimes \mathbb{C}$ and extend σ to $V_{\mathbb{C}}$ by $\mathbb{C}$-linearity. For k a positive integer, a holomorphic function $f : \mathbb{H} \times V_{\mathbb{C}} \to \mathbb{C}$ (where $\mathbb{H}$ denotes the usual upper half plane) is called a Jacobi form of weight k with respect to (Λ, σ) if the following hold:

- For all $\gamma \in J^{(\Lambda,\sigma)}$ and $(\tau, z) \in \mathbb{H} \times V_{\mathbb{C}}$, we have

$$\left(f \mid_{k,(\Lambda,\sigma)} \gamma\right)(\tau, z) = f(\tau, z).$$

- f has a Fourier expansion of the form

$$f(\tau, z) = \sum_{m \in \mathbb{Z}, r \in \Lambda^*, 2n \geq \sigma[r]} c_f(m, r) e(m\tau + \sigma(r, z)).$$

We denote the space of such forms by $J_k(\Lambda, \sigma)$. We say f is a **Jacobi cusp form** if $c_f(m, r) = 0$, when $2m = \sigma[r]$. We denote the space of Fourier-Jacobi cusp forms by $S_k(\Lambda, \sigma)$.

We now have a notion of a scalar product for elements of $S_k(\Lambda, \sigma)$, see [Moc19, Definition 1.33].

Definition 4.3.6. Let $\phi, \psi \in S_k(\Lambda, \sigma)$. If $U \leq J^{(\Lambda,\sigma)}$ of finite index, we define the **Petersson inner product** via:

$$\langle \phi, \psi \rangle_U := \frac{1}{[J^\Lambda : U]} \int_{U \backslash \mathcal{H} \times (\Lambda \otimes \mathbb{C})} \phi(\tau, z) \overline{\psi(\tau, z)} v^k e^{-2\pi\sigma(y,y)v^{-1}} \mathrm{d}V,$$

where $\tau = u + iv$, $z = x + iy$ and $\mathrm{d}V := v^{-n-2} \mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}y$. This inner product does not depend on the choice of U, so in what follows, we drop the subscript.

We now specify to our case by taking $\Lambda = L = \mathbb{Z}^n$ and $\sigma(x, y) = x^t S y$ for all $x, y \in V$. We also write J_S for the integral Jacobi group in this case, i.e. $J_S := \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)$.

Let us discuss the Fourier-Jacobi expansion of orthogonal cusp forms of weight k with respect to Γ_S. If we write $Z = (\omega, z, \tau) \in \mathcal{H}_S$ with $\omega, \tau \in \mathbb{C}, z \in \mathbb{C}^n$, we have that for any $m \in \mathbb{Z}$ ([Gri91, page 244]):

$$F(\omega + m, z, \tau) = F(\omega, z, \tau).$$

Hence, we can write

$$F(Z) = \sum_{m\geq 1} \phi_m(\tau, z) e^{2\pi i m\omega}, \tag{4.3.1}$$

and we call the functions $\phi_m(\tau, z)$ the Fourier-Jacobi coefficients of F. We note that then $\phi_m \in S_k(\mathbb{Z}^n, m\sigma)$ (see [Sch22, Theorem 1.7.16]).

4.4 Maass Space

In this Section, we will give an account of the analogue of the Maass space in the orthogonal setting. This has been defined by Krieg in [Kri96], Gritsenko in [Gri91], and Sugano in [Sug95]. We have the following definition, due to Krieg in [Kri96].

Definition 4.4.1. Let $\widetilde{\Gamma}_S \leq \Gamma \leq \Gamma_S$ be a subgroup of finite index. The **Maass space** $M_k^*(\Gamma)$ consists of all $F \in M_k(\Gamma)$, so that if their Fourier expansion is

$$F(Z) = \sum_{r\in L_0^*} A(r) e(r^t S_0 Z),$$

we have

$$A(r) = \sum_{d|\gcd(\rho)} d^{k-1} A\left(\begin{pmatrix} lm/d^2 \\ -S^{-1}\lambda/d \\ 1 \end{pmatrix}\right),$$

where $r = S_0^{-1}\rho$, with $\rho = \begin{pmatrix} m \\ \lambda \\ l \end{pmatrix}$.

There is an important connection between the Maass space and the space of Fourier-Jacobi forms. In order to make this specific, we first need to define various subgroups of Γ_S. Consider the following elements of Γ_S:

$$J := \begin{pmatrix} 0 & 0 & -I \\ 0 & 1_n & 0 \\ -I & 0 & 0 \end{pmatrix},\ I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad T_\lambda := \begin{pmatrix} 1 & -\lambda^t S_0 & -\frac{1}{2}S_0[\lambda] \\ 0 & 1_{n+2} & \lambda \\ 0 & 0 & 1 \end{pmatrix},\ \lambda \in \mathbb{Z}^{n+2},$$

$$R_K := \begin{pmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{pmatrix}, K \in \Gamma^+(L_0),\quad K_U := \begin{pmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix}, U \in \mathrm{SO}(L).$$

We have the following Theorem.

Theorem 4.4.2. *([Kri16, Theorem 1]) Γ_S is generated by the matrices*

$$J,\ (T_\lambda,\ \lambda \in \mathbb{Z}^{n+2}),\ (R_K,\ K \in \Gamma^+(L_0)).$$

We now recall that we have defined the so-called **discriminant kernel** in equation (4.2.7). It is easy to show that it is a normal subgroup of Γ_S and from [Sch22, p. 24], it is generated by the matrices J, $(T_\lambda,\ \lambda \in \mathbb{Z}^{n+2})$, $(R_K,\ K \in \widetilde{\Gamma}^+(L_0))$. Here

$$\widetilde{\Gamma}^+(L_0) := \{M \in \Gamma^+(L_0) \mid M \in 1_{n+2} + \mathrm{M}_{n+2}(\mathbb{Z})S_0\}.$$

Finally, we define the groups

$$\widetilde{\Gamma}_S^\bullet := \langle J,\ (T_\lambda, \lambda \in \mathbb{Z}^{n+2})\rangle, \tag{4.4.1}$$

and

$$\Gamma_S^\bullet := \langle \widetilde{\Gamma}_S^\bullet,\ (R_{K_U} : U \in \mathrm{SO}(L))\rangle. \tag{4.4.2}$$

The connection between the Maass space and Fourier-Jacobi forms is now given in the following Theorem.

Theorem 4.4.3. *([Kri96, Theorem 3]) The mapping $F \longmapsto \phi_1$ gives an isomorphism between the Maass space $M_k^*(\widetilde{\Gamma}_S^\bullet)$ and $J_k(L,\sigma)$, as vector spaces. Here $J_k(L,\sigma)$ is the space defined in Definition 4.3.5 with $L = \mathbb{Z}^n$, $\sigma(x,y) = x^tSy$ for all $x,y \in V$ and ϕ_1 is the first Fourier-Jacobi coefficient of F (see* (4.3.1)*).*

The above mapping restricted to cusp forms also gives an isomorphism of $S_k^*(\widetilde{\Gamma}_S^\bullet)$ and $S_k(L,\sigma)$. Here $S_k^*(\widetilde{\Gamma}_S^\bullet)$ is the subspace of $M_k^*(\widetilde{\Gamma}_S^\bullet)$, consisting of cusp forms.

The inverse of the above mapping is sometimes referred to as **Gritsenko lift** (cf. [Gri91, Proposition 5]) and in the case of cusp forms is given as follows:

$$\sum_{\substack{m\in\mathbb{Z}, r\in L^*,\\ 2n>\sigma[r]}} c_\phi(m,r)e(m\tau+\sigma(r,z)) \longmapsto \sum_{\lambda\in L_0^*\cap\mathcal{P}_S}\ \sum_{d\mid\gcd(S_0\lambda)} d^{k-1}c_\phi\left(\frac{mN}{d^2},\frac{r}{d}\right)e(\lambda^tS_0Z), \tag{4.4.3}$$

where $\lambda = (m,r,N)^t \in L_0^*$.

Gritsenko in [Gri91] has defined the above via the action of a T_- operator on Fourier-Jacobi forms, in analogy with the Hermitian Maass space of Definition 2.5.3. The action on the Fourier coefficients turns out to be the same as the one in (4.4.3).

Finally, Sugano in [Sug95] has defined the same mapping via an operator V_N, which corresponds to T_- of Gritsenko (see [Sug95, Section 6]). We will discuss this in detail in Section 6.3. It should be noted that Sugano proves in [Sug95, Corollary 6.7] that his mapping gives an isomorphism between $S_k(L,\sigma)$ and $S_k^*(\widetilde{\Gamma}_S)$ (see equation (4.2.7)). It turns out that this is no different to Krieg, because if $F \in S_k^*(\widetilde{\Gamma}_S^\bullet)$, then $F \in S_k^*(\widetilde{\Gamma}_S)$ because of the special relations between the Fourier coefficients of F. Also, clearly $S_k^*(\widetilde{\Gamma}_S) \subseteq S_k(\widetilde{\Gamma}_S^\bullet)$ as $\widetilde{\Gamma}_S^\bullet \subseteq \widetilde{\Gamma}_S$ and so the two spaces are the same.

4.5 Dirichlet Series

In this short Section, we will define the Dirichlet series of interest, in analogy with Kohnen and Skoruppa, in their seminal paper [KS89]. We will also show that it is well-defined.

Let $\widetilde{\Gamma}_S \leq \Gamma \leq \Gamma_S$ be a subgroup of finite index. We then have $T_\lambda \in \Gamma$ for all $\lambda \in \mathbb{Z}^{n+2}$. Therefore, if $F, G \in S_k(\Gamma)$, they admit a Fourier-Jacobi expansion of the form (4.3.1). Assume their Fourier-Jacobi coefficients are $\{\phi_m\}_{m=1}^{\infty}, \{\psi_m\}_{m=1}^{\infty}$ respectively. We then define the following **Dirichlet series**:

$$\mathcal{D}_{F,G}(s) := \sum_{m=1}^{\infty} \langle \phi_m, \psi_m \rangle m^{-s}. \tag{4.5.1}$$

Here, $\langle \ , \ \rangle$ denotes the inner product of Definition 4.3.6.

Lemma 4.5.1. *$\mathcal{D}_{F,G}(s)$ converges absolutely for $\mathrm{Re}(s) > k+1$ and represents a holomorphic function on this domain.*

Proof. The proof is similar to [KS89, Lemma 1]. We will show for $N \geq 1$ that

$$\langle \phi_N, \psi_N \rangle = \mathcal{O}(N^k),$$

with the constant depending only on F, G. Indeed, fix $(\tau, z) \in \mathbb{H} \times \mathbb{C}^n$ and write $\tau = u + iv$, $z = x + iy$. If $F(q) = \sum_{N=1}^{\infty} \phi_N(\tau, z) q^N$, with $q = e^{2\pi i \tau'}$, we have by Cauchy's integral formula that

$$\phi_N(\tau, z) = \oint_{|q|=r} \frac{F(q)}{q^{N+1}} \mathrm{d}q,$$

for any $0 < r < e^{-\pi S[y]/v}$. The bounds follow from the fact that $S_0[\mathrm{Im} Z] > 0$ (here $Z = (\tau', z, \tau) \in \mathcal{H}_S$). If now $\tau' = u' + iv'$, the integral can be written as

$$\phi_N(\tau, z) = \int_0^1 F(Z) e^{-2\pi i N \tau'} \mathrm{d}u',$$

for any $v' > S[y]/2v$. But now $|F(Z)| \left(S_0[\mathrm{Im} Z]/2\right)^{k/2}$ is bounded on $\mathcal{H}_S$ from [Hau21, II, Lemma 3.28], say by a constant $C > 0$. Therefore, after choosing $v' = S[y]/2v + 1/N$, we have

$$|\phi_N(\tau, z)| \leq C e^{2\pi} \int_0^1 \left(S_0[\mathrm{Im} Z]/2\right)^{-k/2} e^{\pi N S[y]/v} \mathrm{d}u' = C e^{2\pi} \left(\frac{v}{N}\right)^{-k/2} e^{\pi N S[y]/v}.$$

Similarly for ψ_N and then the claim follows from the definition of the inner product in Definition 4.3.6. □

Chapter 5

Analytic Properties of the Dirichlet series

In this Chapter, we will discuss the analytic properties, i.e., meromorphic continuation to $\mathbb{C}$ and functional equation of the Dirichlet series defined in (4.5.1). This is the first consideration of Kohnen and Skoruppa in their paper [KS89] (see Theorem 1.1.3).

Let us analyse their result in greater detail. Their method of proof can be summarised in two steps: The first (and easier one) is to obtain an integral representation for $D_{F,G}$, using a non-holomorphic Eisenstein series of Klingen type. The second is to prove the meromorphic continuation and functional equation for this Eisenstein series. The proof of that involves writing the Eisenstein series in the form of an Epstein zeta function and then proving that it is a Mellin transform of a specific theta series.

This general method of proof has been successful in a number of other cases as well. For example, Raghavan and Sengupta in [RS91] and Gritsenko in [Gri92a] considered the same problem, but in the case when F, G are Hermitian cusp forms of degree 2 over $\mathbb{Q}(i)$. In both papers, the authors managed to deduce the analytic properties of $D_{F,G}$ by applying a very similar idea; however, there are two key differences regarding the second step above. The first one is that the Eisenstein series of Klingen type arises as the inner product of a theta series and a classical Eisenstein series for SL_2. The second and more important one is that it is now necessary to apply some differential operators to the theta series first. The reason for this is that there are terms that cause the inner-product integral to diverge, so we need to eliminate them with the use of differential operators.

It should be noted here that the degree of the modular objects considered is not important. Yamazaki in [Yam90] generalised the (analytic) result of Kohnen and

Skoruppa for Siegel cusp forms of arbitrary degree $n \geq 1$. However, his method of deducing the analytic properties of the Eisenstein series of Klingen type is derived from the general Langlands' theory. Krieg, on the other hand, in [Kri91], used theta correspondence in the arbitrary degree n case for Siegel, Hermitian (over $\mathbb{Q}(i)$) and quaternionic (over the Hamiltonian quaternions) cusp forms. The use of differential operators was again essential for the last two cases. Even more remarkably, Deitmar and Krieg in [DK91, Section 4] managed to prove theta correspondence between Eisenstein series of Klingen and Siegel type for the groups $\mathrm{Sp}_n(\mathbb{Z})$ and $\mathrm{O}(m,m)$, for arbitrary $m, n \geq 1$. Their proof is based on proving the existence of an invariant differential operator R, which, when applied to the suitable theta series, eliminates the terms that cause the divergence of the inner-product integral. This was a significant advance because up until then, all the operators had to be found explicitly, and this could only be done in a handful of cases. Moreover, using such an explicit correspondence, the authors deduce finer information regarding the poles and zeros of the Eisenstein series considered.

In our case, we essentially combine all the techniques we mentioned above. We first obtain an integral representation for the Dirichlet series, and then demonstrate, under certain restrictions, an explicit theta correspondence between an Eisenstein series of Klingen type for the orthogonal group and a Siegel-type Eisenstein series for the symplectic group of degree 2.

5.1 Integral Representation

In this Section, we will give a Rankin-Selberg integral for the Dirichlet series (4.5.1). We need several preparations. We will first view Fourier-Jacobi forms as modular forms under the action of a parabolic subgroup, very much in the same way as we have done for the Hermitian case in Section 2.2. We begin by defining a special parabolic subgroup of Γ_S.

Definition 5.1.1. The **parabolic subgroup** of Γ_S fixing the two-dimensional isotropic subspace spanned by e_1, e_2 (standard basis vectors) is defined by

$$\Gamma_{S,J} := \left\{ \begin{pmatrix} * & * \\ 0 & D \end{pmatrix} \in \Gamma_S \mid D \in M_2(\mathbb{Z}) \right\}. \tag{5.1.1}$$

We will now embed the (integral) Jacobi group J_S (defined just after Definition 4.3.6) into $\Gamma_{S,J}$. Consider the embedding

$$\iota : J_S \longrightarrow \Gamma_{S,J}$$

$$(D,[x,y]) \longmapsto M_D \cdot H_{x,y},$$

where

$$H_{x,y} := \begin{pmatrix} 1 & 0 & y^tS & 0 & \frac{1}{2}S[y] \\ 0 & 1 & x^tS & \frac{1}{2}S[x] & x^tSy \\ 0 & 0 & 1_n & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; M_D := \begin{pmatrix} D^* & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & D \end{pmatrix},$$

with $D^* := D\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]$ (cf. [Sch22, page 44]). We denote by $\Gamma^{\bullet}_{S,J} := \iota(J_S)$. By [Ajo15, Proposition 2.2.7], we have that the action of J_S on $\mathbb{H} \times \mathbb{C}^n$ is given by

$$((D,[x,y]),(\tau,z)) \longmapsto \left(D\tau, \frac{z + x\tau + y}{j'(D,\tau)}\right),$$

where j' denotes the usual factor of automorphy for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Now, if we take an element $M = \iota((D,[x,y])) \in \Gamma^{\bullet}_{S,J}$, we can see that its action on an element $(\omega,z,\tau) \in \mathcal{H}_S$ (see (4.2.6))

$$M\langle(\omega,z,\tau)\rangle = \left(*, \frac{z + x\tau + y}{j(D,\tau)}, D\tau\right)$$

is the same as the action of J_S on $\mathbb{H} \times \mathbb{C}^n$.

We have the following Proposition regarding the fundamental domain of the action of $\Gamma^{\bullet}_{S,J}$ on $\mathcal{H}_S$, which will be useful later.

Proposition 5.1.2. *For $Z = (\omega,z,\tau) \in \mathcal{H}_S$, we write $\omega = x_1 + iy_1, z = u + iv, \tau = x_2 + iy_2$. A valid choice for the fundamental domain of the action of $\Gamma^{\bullet}_{S,J}$ on $\mathcal{H}_S$ is*

$$\mathcal{F}^J := \left\{ Z = (\omega,z,\tau) \in \mathcal{H}_S \mid (z,\tau) \in \mathcal{F}, y_1y_2 - \frac{1}{2}S[v] > 0, -\frac{1}{2} \le x_1 \le \frac{1}{2} \right\},$$

where $\mathcal{F}$ is a fundamental domain of the action of J_S on $\mathbb{H} \times \mathbb{C}^n$.

Proof. Let $Z = (\omega,z,\tau) \in \mathcal{H}_S$. We can then pick $g \in \Gamma^{\bullet}_{S,J}$ such that $\iota^{-1}(g) \in J_S$ and that $Z' := g\langle Z\rangle = (\omega',z,\tau)$ with $(z,\tau) \in \mathcal{F}$ and $\omega' \in \mathbb{H}$ arbitrary. This follows from the fact that the actions are the same, as we have shown above. Now, if $\lambda \in \mathbb{Z}^{n+2}$, we have $T_\lambda\langle Z'\rangle = Z' + \lambda$ (see Section 4.4 for the definition of T_λ) and so we can act with a suitable T_λ, so that $-1/2 \le x_1 \le 1/2$. Finally, the condition $y_1y_2 - S[v]/2 > 0$ follows from the definition of $\mathcal{P}_S$ in 4.2.4. □

Let now $F, G \in S_k(\Gamma_S)$. In order to give an integral representation for $\mathcal{D}_{F,G}(s)$, we need to define an appropriate Eisenstein series of Klingen type, in an analogous way to [Sch22, Chapter 3].

Definition 5.1.3. Let $Z \in \mathcal{H}_S$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n+1$. We define the real analytic **Eisenstein series of Klingen type** to be

$$E(Z,s) := \sum_{\gamma \in \Gamma_{S,J} \backslash \Gamma_S} \left(\frac{Q_0[\mathrm{Im}(\gamma Z)]}{\mathrm{Im}((\gamma Z)_2)} \right)^s,$$

where for $Z = (\omega, z, \tau) \in \mathcal{H}_S$, we write $Z_2 := \tau$. Also, $Q_0 = S_0/2$.

Proposition 5.1.4. *$E(Z,s)$ is well defined, is invariant under the action of Γ_S and converges absolutely and uniformly whenever Z belongs to a compact subset of $\mathcal{H}_S$ and s satisfies* $\mathrm{Re}(s) > n+1$.

Proof. Let $\gamma \in \Gamma_{S,J}$. We write

$$\gamma = \begin{pmatrix} * & * \\ 0 & D \end{pmatrix}, \ D \in \mathrm{SL}_2(\mathbb{Z}),$$

because γ is in the connected component of the identity. Now, for $Z = (\omega, z, \tau) \in \mathcal{H}_S$, we have $\gamma\langle Z \rangle_2 = D\langle \tau \rangle$, where the action on the right denotes the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (see also [Sch22, page 115]). By [Bru97, Lemma 3.20], we have

$$Q_0[\mathrm{Im}(\gamma Z)] = |j(\gamma, Z)|^{-2} Q_0[\mathrm{Im} Z], \tag{5.1.2}$$

and so

$$\frac{Q_0[\mathrm{Im}(\gamma Z)]}{\mathrm{Im}((\gamma Z)_2)} = \frac{1}{|j(\gamma,Z)|^2} \frac{Q_0[\mathrm{Im} Z]}{\mathrm{Im}(D\tau)} = \frac{|j'(D,\tau)|^2}{|j(\gamma,Z)|^2} \frac{Q_0[\mathrm{Im} Z]}{\mathrm{Im}(Z_2)},$$

where again j' denotes the usual factor of automorphy for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. But we have $j(\gamma, Z) = j'(D, \tau)$ and hence $E(Z,s)$ is well-defined.

The invariance under Γ_S follows from the fact that for fixed $\delta \in \Gamma_S$, the map $\gamma \longmapsto \gamma\delta$ induces a bijection between $\Gamma_{S,J} \backslash \Gamma_S$ to itself.

For the convergence, we can write from (5.1.2)

$$E(Z,s) = \sum_{\gamma \in \Gamma_{S,J} \backslash \Gamma_S} (Q_0[\mathrm{Im} Z])^s \, (\mathrm{Im}(\gamma Z)_2)^{-s} \, |j(\gamma, Z)|^{-2s}.$$

But by the proof of [Sch22, Theorem 3.1.1], the sum

$$\sum_{\gamma \in \Gamma_{S,J} \backslash \Gamma_S} (\mathrm{Im}(\gamma Z)_2)^{-k/2} \, |j(\gamma, Z)|^{-k}$$

converges locally uniformly whenever $k > 2n+2$. From this, the claim follows. □

We are now ready to give the main Proposition of this Section.

Proposition 5.1.5. *Let $F, G \in S_k(\Gamma_S)$. For $Z \in \mathcal{H}_S$ and $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > n+2$, *we have*

$$\langle F(Z)E(Z,s), G(Z)\rangle = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})}(4\pi)^{-(s+k-n-1)}\Gamma(s+k-n-1)\mathcal{D}_{F,G}(s+k-n-1),$$

where $\langle\ ,\ \rangle$ denotes the inner product of Definition 4.2.3,

$$\mathrm{SO}(S;\mathbb{Z}) := \{g \in \mathrm{SL}_n(\mathbb{Z}) \mid g^t S g = S\},$$

which is finite and $\mathcal{D}_{F,G}(s)$ is the Dirichlet series of (4.5.1).

Proof. Let $I(s) := \langle F(Z)E(Z,s),\, G(Z)\rangle$. If we denote by $\mathcal{Q}$ a fundamental domain for the action of Γ_S on $\mathcal{H}_S$, by using the standard unfolding argument, we have for $\mathrm{Re}(s) > n+1$

$$\begin{aligned} I(s) &= \int_{\mathcal{Q}} F(Z)\overline{G(Z)} \sum_{\gamma \in \Gamma_{S,J}\backslash\Gamma_S} \left(\frac{Q_0[\mathrm{Im}(\gamma Z)]}{\mathrm{Im}((\gamma Z)_2)}\right)^s (Q_0[\mathrm{Im} Z])^k \,\mathrm{d}^* Z \\ &= \frac{1}{[\Gamma_{S,J} : \Gamma^\bullet_{S,J}]} \int_{\mathcal{F}^J} F(Z)\overline{G(Z)} \left(\frac{Q_0[Y]}{\mathrm{Im}(\tau)}\right)^s (Q_0[Y])^{k-n-2}\,\mathrm{d}X\mathrm{d}Y, \end{aligned}$$

where $\mathcal{F}^J$ is a fundamental domain for the action of $\Gamma^\bullet_{S,J} = \iota(J_S)$ on $\mathcal{H}_S$ and $Z = (\omega, z, \tau) = X + iY$. We note here

$$[\Gamma_{S,J} : \Gamma^\bullet_{S,J}] = \#\mathrm{SO}(S;\mathbb{Z}) < \infty,$$

because S is positive definite (cf. [Sch22, Section 1.7]). Hence, from Proposition 5.1.2, with the same notation as there, we get

$$\begin{aligned} I(s) = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})} \int_{\mathcal{F}} \int_{y_1y_2 - \frac{1}{2}S[v] > 0} \int_{-\frac{1}{2} \le x_1 \le \frac{1}{2}} F(Z)\overline{G(Z)} y_2^{-s} \times \\ \times \left(y_1y_2 - \frac{1}{2}S[v]\right)^{k-n-2+s} \mathrm{d}X\mathrm{d}Y, \end{aligned}$$

where $\mathcal{F}$ is a fundamental domain of the action of J_S on $\mathbb{H} \times \mathbb{C}^n$. We now write

$$F(Z) = \sum_{m=1}^{\infty} \phi_m(\tau, z)e^{2\pi i m\omega},\ \ G(Z) = \sum_{m=1}^{\infty} \psi_m(\tau, z)e^{2\pi i m\omega}.$$

Using the fact that for integers m_1, m_2, we have

$$\int_{-1/2}^{1/2} e^{2\pi i(m_1-m_2)x_1}\mathrm{d}x_1 = \begin{cases} 1 & \text{if } m_1 = m_2 \\ 0 & \text{otherwise} \end{cases},$$

we get

$$I(s) = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})} \int_{\mathcal{F}} \int_{y_1y_2 - \frac{1}{2}S[v]>0} \sum_{m=1}^{\infty} \phi_m(\tau,z)\overline{\psi_m(\tau,z)} e^{-4\pi m y_1} y_2^{-s} \times \\ \times \left(y_1y_2 - \frac{1}{2}S[v]\right)^{k-n-2+s} \mathrm{d}X\mathrm{d}Y.$$

We now set $t := y_1 - \dfrac{S[v]}{2y_2}$, or equivalently $y_2 t = y_1 y_2 - \dfrac{1}{2}S[v]$. We then get

$$I(s) = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})} \int_{\mathcal{F}} \int_{t=0}^{\infty} \sum_{m=1}^{\infty} \phi_m(\tau,z)\overline{\psi_m(\tau,z)} e^{-4\pi m t} e^{-2\pi m \frac{S[v]}{y_2}} y_2^{-s} \times \\ \times (ty_2)^{k-n-2+s} \mathrm{d}t\mathrm{d}u\mathrm{d}v\mathrm{d}x_2\mathrm{d}y_2.$$

But

$$\int_{t=0}^{\infty} e^{-4\pi m t} t^{k-n-2+s} \mathrm{d}t = \Gamma(s+k-n-1)(4\pi)^{-(s+k-n-1)} m^{-(s+k-n-1)}.$$

Moreover, in this case, the inner product of Definition 4.3.6 reads as:

$$\langle \phi_m, \psi_m \rangle = \int_{\mathcal{F}} \phi_m(\tau,z)\overline{\psi_m(\tau,z)} y_2^{k-n-2} e^{-\frac{2\pi m}{y_2}S[v]} \mathrm{d}u\mathrm{d}v\mathrm{d}x_2\mathrm{d}y_2.$$

Putting all the above together, we obtain

$$I(s) = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})} (4\pi)^{-(s+k-n-1)} \Gamma(s+k-n-1) \mathcal{D}_{F,G}(s+k-n-1),$$

or equivalently

$$\langle F(Z)E(Z,s), G(Z) \rangle = \frac{1}{\#\mathrm{SO}(S;\mathbb{Z})} (4\pi)^{-(s+k-n-1)} \Gamma(s+k-n-1) \mathcal{D}_{F,G}(s+k-n-1),$$

as claimed. □

5.2 Eisenstein Series as an Epstein Zeta Function

It is now clear from the above that the analytic properties of the Dirichlet series of interest reduce to the ones of the Klingen Eisenstein series, as given in Definition 2.1.6. This Section is devoted to writing this Eisenstein series in the form of an Epstein zeta function, similar to [Kri91, equation (7)]. In our case, because of the form of the Eisenstein series, we cannot use the method of Krieg in [Kri91] with the minors of the determinant. It turns out we can write the Eisenstein series in such a form, provided that the number of one-dimensional cusps is 1.

Definition 5.2.1. The set of Γ_S-orbits of one-dimensional cusps of $\mathcal{H}_S$ is defined by

$$\mathcal{C}^1(\Gamma_S) := \{\Gamma_S W \mid W \text{ is isotropic in } V_1\}. \tag{5.2.1}$$

A two-dimensional **isotropic plane**, or isotropic plane in V_1, is defined by two linearly independent vectors $g, h \in V_1$. We normalise g, h such that $g, h \in L_1^*$ and such that $\gcd(S_1 g) = \gcd(S_1 h) = 1$. The isotropy condition means $S_1\left[\begin{pmatrix} g & h \end{pmatrix}\right] = 0$.

We also have the following Definition of the majorants for S_1.

Definition 5.2.2. The space of **majorants** for S_1 is defined by

$$\mathfrak{H} := \{R \in M_{n+4}(\mathbb{R}) \mid R = R^t > 0,\ RS_1^{-1}R = S_1\}.$$

The main Proposition of the Section is the following:

Proposition 5.2.3. *Let S be such that $\#\mathcal{C}^1(\Gamma_S) = 1$. Then, for each $Z \in \mathcal{H}_S$, there is a $R_Z \in \mathfrak{H}$ such that*

$$E(Z,s) = \sum_{\gamma \in \Gamma_{S,J}\backslash\Gamma_S} \left(\frac{\operatorname{Im}(\gamma Z)_2}{Q_0[\operatorname{Im}(\gamma Z)]}\right)^{-s} = \sum_{\ell \in X/\mathrm{GL}_2(\mathbb{Z})} \left(\det(R_Z[\ell])\right)^{-s/2},$$

where

$$X := \left\{\begin{pmatrix} l & m \end{pmatrix} \mid l, m \in \mathbb{Z}^{n+4},\ \begin{pmatrix} l & m \end{pmatrix} \text{ primitive},\ S_1\left[\begin{pmatrix} l & m \end{pmatrix}\right] = 0\right\}.$$

Here, a matrix being primitive means that its elementary divisors are all 1 *(see [Shi97, Section 3]).*

The rest of the Section is devoted to proving this Proposition. We start with two lemmas regarding the elements of $G_{\mathbb{R}}$, i.e., the special orthogonal group attached to S_1.

Lemma 5.2.4. *Let $\gamma \in G_{\mathbb{R}}$ and write $\gamma = \begin{pmatrix} * & l & m \end{pmatrix}^t$ with $l, m \in \mathbb{R}^{n+4}$. Then*

$$\gamma^{-1} = \begin{pmatrix} S_1^{-1} m & S_1^{-1} l & * \end{pmatrix}.$$

Proof. The proof follows from the fact that if

$$\gamma = \begin{pmatrix} \alpha & a^t & \beta \\ b & A & c \\ \gamma & d^t & \delta \end{pmatrix} \in G_{\mathbb{R}},$$

then from the relation $S_1[\gamma] = S_1$, we have

$$\gamma^{-1} = \begin{pmatrix} \delta & c^t S_0 & \beta \\ S_0^{-1} d & S_0^{-1} A^t S_0 & S_0^{-1} a \\ \gamma & b^t S_0 & \alpha \end{pmatrix}. \qquad \square$$

Lemma 5.2.5. *Let $\gamma \in G_{\mathbb{R}}$ and write* $\gamma = \begin{pmatrix} * \\ l^t \\ m^t \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ l_1 & l_2 & x^t & l_{n+3} & l_{n+4} \\ m_1 & m_2 & y^t & m_{n+3} & m_{n+4} \end{pmatrix}$.
Then, if

$$\ell := S_1^{-1} \begin{pmatrix} m & l \end{pmatrix} = \begin{pmatrix} m_{n+4} & l_{n+4} \\ m_{n+3} & l_{n+3} \\ -S^{-1} y & -S^{-1} x \\ m_2 & l_2 \\ m_1 & l_1 \end{pmatrix},$$

we have $S_1[\ell] = 0$.

Proof. Let $M = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$, where e_1, e_2 are the two standard basis vectors of $\mathbb{R}^{n+4}$. From Lemma 5.2.4, we have $\ell = \gamma^{-1} M$. But

$$S_1[\gamma^{-1} M] = S_1[\gamma^{-1}][M] = S_1[M] = 0,$$

because $\gamma^{-1} \in G_{\mathbb{R}}$ and the subspace generated by M is totally isotropic. Hence, the result follows. $\square$

Now, for any $Z \in \mathcal{H}_S$, we want to choose a specific majorant for S_1, so that the terms of the Eisenstein series take the form in Proposition 5.2.3. We start with defining the majorant for the element $I := (i, 0 \cdots, 0, i)^t \in \mathcal{H}_S$.

Lemma 5.2.6. *Let $I := (i, 0, \cdots, 0, i)^t$ and $R_I = \operatorname{diag}(1, 1, S, 1, 1)$. Then, if $\gamma \in G_{\mathbb{R}}$, with $\gamma = \begin{pmatrix} * & l & m \end{pmatrix}^t$, we have*

$$\left(\frac{\operatorname{Im}((\gamma I)_2)}{Q_0[\operatorname{Im}(\gamma I)]} \right)^2 = \det(R_I[\ell]),$$

where $\ell = S_1^{-1} \begin{pmatrix} m & l \end{pmatrix}$, as in Lemma 5.2.5.

Proof. We have $S_0[I] = -2$ and therefore

$$\operatorname{Im}((\gamma I)_2) = \operatorname{Im}\left(\frac{l_1 + (l_2 + l_{n+3}) i + l_{n+4}}{m_1 + (m_2 + m_{n+3}) i + m_{n+4}} \right) = \\ = \frac{(l_2 + l_{n+3})(m_1 + m_{n+4}) - (l_1 + l_{n+4})(m_2 + m_{n+3})}{(m_1 + m_{n+4})^2 + (m_2 + m_{n+3})^2}.$$

Also,

$$Q_0[\mathrm{Im}(\gamma I)] = |j(\gamma, I)|^{-2} Q_0[\mathrm{Im} I] = \frac{1}{(m_1 + m_{n+4})^2 + (m_2 + m_{n+3})^2}.$$

Hence,

$$\frac{\mathrm{Im}((\gamma I)_2)}{Q_0[\mathrm{Im}(\gamma I)]} = (l_2 + l_{n+3})(m_1 + m_{n+4}) - (l_1 + l_{n+4})(m_2 + m_{n+3}).$$

Now, from Lemma 5.2.5, with ℓ written as there, we have

$$\begin{cases} S^{-1}[x] = 2(l_{n+4} l_1 + l_{n+3} l_2). \\ S^{-1}[y] = 2(m_{n+4} m_1 + m_{n+3} m_2). \\ x^t S^{-1} y = l_{n+4} m_1 + l_{n+3} m_2 + l_2 m_{n+3} + l_1 m_{n+4}. \end{cases}$$

By then computing $R_I[\ell]$ and taking the determinant, the result follows. □

After defining the majorant for I, we can use the transitivity of the action in (4.2.6), in order to define a majorant for every Z in $\mathcal{H}_S$.

Proposition 5.2.7. *Let $Z \in \mathcal{H}_S$. Then, $\exists R_Z \in \mathfrak{H}$ such that for all $\gamma \in G_{\mathbb{R}}$,*

$$\left(\frac{\mathrm{Im}((\gamma Z)_2)}{Q_0[\mathrm{Im}(\gamma Z)]} \right)^2 = \det(R_Z[\ell]),$$

where ℓ is the matrix formed by the first two columns of γ^{-1}.

Proof. We start the proof by constructing such an R_Z. We denote by $I = (i, 0, \cdots, 0, i)^t$. By transitivity, $\exists \delta \in G^0_{\mathbb{R}}$ such that $\delta\langle I \rangle = Z$. We then define $R_Z := R_I[\delta^{-1}]$ and we claim this is well-defined. We prove this in Lemma 5.2.8. Then

$$\left(\frac{\mathrm{Im}((\gamma Z)_2)}{Q_0[\mathrm{Im}(\gamma Z)]} \right)^2 = \left(\frac{\mathrm{Im}((\gamma \delta I)_2)}{Q_0[\mathrm{Im}(\gamma \delta I)]} \right)^2 = \det \left(R_I[\ell] \right),$$

where if $\gamma\delta = \begin{pmatrix} * & l & m \end{pmatrix}^t$, we have $\ell = S_1^{-1} \begin{pmatrix} m & l \end{pmatrix}$, by Lemma 5.2.6. Now,

$$R_Z = R_I[\delta^{-1}] \implies R_I = R_Z[\delta].$$

Hence

$$\left(\frac{\mathrm{Im}((\gamma Z)_2)}{Q_0[\mathrm{Im}(\gamma Z)]} \right)^2 = \det(R_Z[\delta\ell]).$$

Now, if $\gamma = \begin{pmatrix} * & l' & m' \end{pmatrix}^t$, we want to show that the above quotient equals

$\det\left(R_Z[S_1^{-1}\begin{pmatrix} m' & l'\end{pmatrix}]\right)$. But

$$\gamma\delta = \begin{pmatrix} * \\ l^t \\ m^t \end{pmatrix} \implies \gamma = \begin{pmatrix} * \\ l^t \\ m^t \end{pmatrix}\delta^{-1} = \begin{pmatrix} * \\ l^t\delta^{-1} \\ m^t\delta^{-1} \end{pmatrix} = \begin{pmatrix} * \\ ((\delta^{-1})^t l)^t \\ ((\delta^{-1})^t m)^t \end{pmatrix},$$

so

$$\begin{pmatrix} m' & l' \end{pmatrix} = \begin{pmatrix} (\delta^{-1})^t m & (\delta^{-1})^t l \end{pmatrix}.$$

Suffices to then show that (we remind here that $\ell = S_1^{-1}\begin{pmatrix} m & l\end{pmatrix}$)

$$S_1^{-1}(\delta^{-1})^t = \delta S_1^{-1} \iff \delta S_1^{-1}\delta^t = S_1^{-1} \iff (\delta^{-1})^t S_1 \delta^{-1} = S_1 \iff S_1[\delta^{-1}] = S_1,$$

which is true.

The only thing remaining to show is that $R_Z \in \mathfrak{H}$. But as R_I is symmetric and positive definite, the same holds for $R_Z = R_I[\delta^{-1}]$. Finally, it is easy to show that $R_I S_1^{-1} R_I = S_1$ and then

$$R_Z S_1^{-1} R_Z = (\delta^{-1})^t R_I \delta^{-1} S_1^{-1} (\delta^{-1})^t R_I \delta^{-1} = (\delta^{-1})^t R_I S_1^{-1} R_I \delta^{-1} = \\ = (\delta^{-1})^t S_1 \delta^{-1} = S_1[\delta^{-1}] = S_1.$$

Hence, $R_Z \in \mathfrak{H}$, as required. □

We now give the final Lemma, showing that R_Z is well-defined.

Lemma 5.2.8. *For each $Z \in \mathcal{H}_S$, R_Z, as defined in the proof of Proposition 5.2.7, is well-defined.*

Proof. We want to show that for $\delta_1, \delta_2 \in G_{\mathbb{R}}$,

$$\delta_1\langle I\rangle = \delta_2\langle I\rangle \implies R_I[\delta_1^{-1}] = R_I[\delta_2^{-1}].$$

For that, it suffices to show that if $g \in G_{\mathbb{R}}$, such that $g\langle I\rangle = I$, then $R_I[g^{-1}] = R_I$. Let us now write

$$g = \begin{pmatrix} \alpha & a_1 & x^t & a_{n+2} & \beta \\ b_1 & A_{1,1} & E^t & A_{1,n+2} & c_1 \\ y & F & K & G & z \\ b_{n+2} & A_{n+2,1} & H^t & A_{n+2,n+2} & c_{n+2} \\ \gamma & d_1 & w^t & d_{n+2} & \delta \end{pmatrix},$$

with $\alpha, \beta, \gamma, \delta, A_{1,1}, A_{1,n+2}, A_{n+2,1}, A_{n+2,n+2} \in \mathbb{R}$, $E, F, G, H, x, y, z, w \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n,n}$. By the definition of the action and the fact that $g\langle I\rangle = I$, we obtain (cf.

[Sug85, (1.3)]):

$$g\begin{pmatrix}1\\I\\1\end{pmatrix} = j(g,I)\begin{pmatrix}1\\I\\1\end{pmatrix}.$$

But we have $j(g,I) = \gamma + \delta + (d_1 + d_{n+2})i$. Hence, we obtain the following relations:

$$\begin{cases} \alpha + \beta + i(a_1 + a_{n+2}) = \gamma + \delta + i(d_1 + d_{n+2}). & (5.2.2)\\ b_1 + c_1 + i(A_{1,1} + A_{1,n+2}) = i(\gamma + \delta) - (d_1 + d_{n+2}). & (5.2.3)\\ y + z + i(F + G) = 0. & (5.2.4)\\ b_{n+2} + c_{n+2} + i(A_{n+2,1} + A_{n+2,n+2}) = i(\gamma + \delta) - (d_1 + d_{n+2}). & (5.2.5) \end{cases}$$

Now, $g\langle I\rangle = I$ implies $g^{-1}\langle I\rangle = I$ as well. But, as in Lemma 5.2.4, we have

$$g^{-1} = \begin{pmatrix} \delta & c_{n+2} & -z^tS & c_1 & \beta\\ d_{n+2} & A_{n+2,n+2} & -G^tS & A_{1,n+2} & a_{n+2}\\ -S^{-1}w & -S^{-1}H & S^{-1}K^tS & -S^{-1}E & -S^{-1}x\\ d_1 & A_{n+2,1} & -F^tS & A_{1,1} & a_1\\ \gamma & b_{n+2} & -y^tS & b_1 & \alpha \end{pmatrix},$$

and therefore, we also obtain the relations

$$\begin{cases} \delta + \beta + i(c_{n+2} + c_1) = \gamma + \alpha + i(b_1 + b_{n+2}). & (5.2.6)\\ a_{n+2} + d_{n+2} + i(A_{n+2,n+2} + A_{1,n+2})) = i(\gamma + \alpha) - (b_1 + b_{n+2}). & (5.2.7)\\ S^{-1}w + S^{-1}x + i(S^{-1}H + S^{-1}E) = 0. & (5.2.8)\\ a_1 + d_1 + i(A_{n+2,1} + A_{1,1}) = i(\gamma + \alpha) - (b_1 + b_{n+2}). & (5.2.9) \end{cases}$$

Using now equations (5.2.3), (5.2.5), (5.2.7), (5.2.9), we obtain

$$A_{1,1} = A_{n+2,n+2},\ A_{1,n+2} = A_{n+2,1} \text{ and } \alpha = \delta.$$

Using (5.2.2) we also get $\alpha + \beta = \gamma + \delta \implies \beta = \gamma$ as well. From (5.2.4), (5.2.8) we get

$$y = -z,\ w = -x,\ F = -G \text{ and } H = -E.$$

Finally, from (5.2.2), (5.2.7), (5.2.9), we have the equations

$$a_1 + d_1 = -(b_{n+2} + b_1),\ a_{n+2} + d_{n+2} = -(b_1 + b_{n+2}),\ a_1 + a_{n+2} = d_1 + d_{n+2}.$$

These give $a_1 + d_1 = a_{n+2} + d_{n+2}$, which together with $a_1 + a_{n+2} = d_1 + d_{n+2}$ gives

$$a_1 = d_{n+2} \text{ and } a_{n+2} = d_1.$$

Similarly, $b_1 = c_{n+2}$ and $c_1 = b_{n+2}$ and these relations are enough to check $g^t R_I = R_I g^{-1}$, i.e. what we wanted to prove. □

We are now ready to give the proof of the main Proposition 5.2.3. For convenience, we restate it here.

Proposition 5.2.9. *Let S be such that $\#\mathcal{C}^1(\Gamma_S) = 1$. Then, for each $Z \in \mathcal{H}_S$, there is a $R_Z \in \mathfrak{H}$ such that*

$$E(Z,s) = \sum_{\gamma \in \Gamma_{S,J}\backslash\Gamma_S} \left(\frac{\mathrm{Im}(\gamma Z)_2}{Q_0[\mathrm{Im}(\gamma Z)]}\right)^{-s} = \sum_{\ell \in X/\mathrm{GL}_2(\mathbb{Z})} (\det(R_Z[\ell]))^{-s/2},$$

where

$$X := \left\{ \begin{pmatrix} l & m \end{pmatrix} \mid l, m \in \mathbb{Z}^{n+4},\ \begin{pmatrix} l & m \end{pmatrix} \text{ primitive},\ S_1\left[\begin{pmatrix} l & m \end{pmatrix}\right] = 0 \right\}.$$

Here, a matrix being primitive means that its elementary divisors are all 1 *(see [Shi97, Section 3]).*

Proof. Let $M = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$. We claim that the map

$$\begin{aligned} \Gamma_S &\longrightarrow X \\ \gamma &\longmapsto \gamma^{-1} M \end{aligned}$$

induces a bijection $\Gamma_{S,J}\backslash\Gamma_S \xrightarrow{\sim} X/\mathrm{GL}_2(\mathbb{Z})$.

- Well-Defined: Firstly, $\gamma^{-1}M \in X$ as the first two columns of γ^{-1} are integer vectors and as $\gamma^{-1} \in \Gamma_S \subset \mathrm{GL}_{n+4}(\mathbb{Z})$, we get that $\gamma^{-1}M$ is primitive. Moreover, $S_1[\gamma^{-1}M] = 0$, as we have already shown in Lemma 5.2.5.

  Now, if $\delta = p\gamma$ for some $p \in \Gamma_{S,J}$, then we can write $p^{-1}M = \begin{pmatrix} a & b \\ c & d \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ (because of the form of the parabolic). Hence

  $$\delta^{-1}M = \gamma^{-1}p^{-1}M = \gamma^{-1}M \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

  which gives the map is well-defined.
- Injective: If $\gamma^{-1}M = \delta^{-1}MN$ for some $N \in \mathrm{GL}_2(\mathbb{Z})$, then

  $$\gamma\delta^{-1}M = MN^{-1},$$

which shows that $\gamma\delta^{-1} \in \Gamma_{S,J}$ (the first two columns of $\gamma\delta^{-1}$ belong in the $\mathbb{Z}$-span of the vectors e_1, e_2 and $\gamma, \delta \in \Gamma_S \subset \mathrm{SL}_{n+4}(\mathbb{Z})$).

- Surjective: Consider now $\begin{pmatrix} l & m \end{pmatrix} \in X/\mathrm{GL}_2(\mathbb{Z})$. The plane generated by the vectors l, m can be checked to be isotropic. Indeed,

$$\mathrm{span}\{l, m\} = \mathrm{span}\left\{ \frac{l}{\gcd(S_1 l)}, \frac{m}{\gcd(S_1 m)} \right\}$$

in V_1. These vectors both belong in L_1^* (can be checked directly) and also $\gcd\left(\frac{S_1 l}{\gcd(S_1 l)}\right) = 1$ and similarly for the other vector. The isotropy condition still holds for the two new vectors (as we only change them by a scalar), and therefore the claim follows.
Now, because $\#\mathcal{C}^1(\Gamma_S) = 1$ by assumption, if U is the plane generated by the basis vectors e_1, e_2 (U is an isotropic plane too), there is an element $K \in \Gamma_S$ such that K maps U to W. Hence, there exist $x, y, z, w \in \mathbb{Q}$ such that

$$\begin{pmatrix} Ke_1 & Ke_2 \end{pmatrix} = \begin{pmatrix} l & m \end{pmatrix} \begin{pmatrix} x & z \\ y & w \end{pmatrix}.$$

But as $\begin{pmatrix} l & m \end{pmatrix}$ is assumed to be primitive, we have by [Shi97, Lemma 3.3] that $\exists A \in \mathrm{Mat}_{2,n+4}(\mathbb{Z})$ such that $A\begin{pmatrix} l & m \end{pmatrix} = 1_2$. Hence, we obtain

$$A\begin{pmatrix} Ke_1 & Ke_2 \end{pmatrix} = \begin{pmatrix} x & z \\ y & w \end{pmatrix},$$

and so $x, y, z, w \in \mathbb{Z}$. Now $\begin{pmatrix} Ke_1 & Ke_2 \end{pmatrix}$ is also primitive by [Shi97, Lemma 3.3], as it can be completed to an element of $\mathrm{GL}_{n+4}(\mathbb{Z})$, namely K. Hence, $\exists B \in \mathrm{Mat}_{2,n+4}(\mathbb{Z})$ such that $B\begin{pmatrix} Ke_1 & Ke_2 \end{pmatrix} = 1_2$. Hence,

$$\begin{pmatrix} x & z \\ y & w \end{pmatrix}^{-1} = B\begin{pmatrix} l & m \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}).$$

Hence, as the inverse of that matrix also has integer entries, we must have that its determinant is ± 1, i.e., $xw - yz = \pm 1$. Therefore, K^{-1} gets mapped to $\begin{pmatrix} l & m \end{pmatrix} \mathrm{GL}_2(\mathbb{Z})$, as wanted.

The rest of the proof now follows from Proposition 5.2.7. □

Remark 5.2.10. We would like to make a few comments here regarding the condition $\#\mathcal{C}^1(\Gamma_S) = 1$. This condition is discussed in [Sch22, Section 1.6.5]. However, the whole discussion there and the examples given (basically the Euclidean lattices in the sense of [Sch22, Definition 1.6.13], a complete list of the 70 that exist is known, see [Neb03]) is about when Γ_S is the integral connected component of the identity of $\mathrm{O}(2, n+2)$ and not of $\mathrm{SO}(2, n+2)$, as in our case. However, in the case when $\mathrm{O}(L)$ (see (4.1.2)) contains an element K of determinant -1, we can show that there is a single orbit under the action of the SO-group too. Indeed, assume there is a single orbit under the action of the O-group. This means, that given an isotropic plane W in V_1, $\exists \gamma \in \mathrm{O}^+(L_1)$ (i.e. in the connected component of $\mathrm{O}(L_1)$) such that $\gamma W = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$. If now $\gamma \in \Gamma_S$, we are done. If not, consider the element

$$\delta := \mathrm{diag}(1, 1, K, 1, 1) \in \mathrm{O}^+(L_1)$$

of determinant -1. Then $\delta\gamma W = \delta \begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \end{pmatrix}$ and $\delta\gamma \in \Gamma_S$.

However, the question of whether such a K exists is non-trivial, at least when n is even (if n is odd, -1_n always works). A condition we can impose so that we ensure the existence of such an element can be found in [Shi04, Lemma 9.23, (iii)]. That is, the existence of some $x \in \mathbb{Z}^n$ so that $S[x] = 2$. In our examples in practice, this is usually satisfied.

Finally, we have the following Lemma, where we replace the condition of primitivity of the elements of X in Proposition 5.2.9 with the condition that the elements have maximal rank.

Lemma 5.2.11. *With the notation as above, we have*

$$\zeta(s)\zeta(s-1)E(Z,s) = \sum_{\substack{\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{rank}\ \ell = 2, S_1[\ell]=0}} \left(\det(R_Z[\ell])\right)^{-s/2}.$$

Proof. The proof is analogous to [DK91, Lemma 3.1]. In particular, every matrix $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$ with rank $\ell = 2$ and $S_1[\ell] = 0$ can be written as $\ell = N \cdot M$ with N being primitive and $S_1[N] = 0$ and $M \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{Mat}_{2,2}(\mathbb{Z})$. The proof then follows from the fact that

$$\sum_{M \in (\mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{Mat}_{2,2}(\mathbb{Z}))/\mathrm{GL}_2(\mathbb{Z})} |\det(M)|^{-s} = \zeta(s)\zeta(s-1).$$

This can be found in [DK91] or its local version in [Shi97, Lemma 3.13]. □

5.3 Theta Series and Transformation Properties

In order now to prove the analytic properties of the Klingen Eisenstein series $E(Z, s)$, we want to prove a theta correspondence between $\mathrm{SO}(2, n+2)$ and Sp_2. That is, to integrate a Siegel Eisenstein series for Sp_2 against an appropriately defined theta series and end up with $E(Z,s)$. However, as in most cases, the inner-product integrals will diverge, so we need to apply an appropriate differential operator first. In this Section, we recall the action of the symplectic group on Siegel's upper half plane, define the appropriate theta series, and prove some important transformation properties.

The (real) symplectic group of degree $m \geq 1$ is defined by

$$\mathrm{Sp}_m(\mathbb{R}) := \left\{ g \in \mathrm{GL}_{2m}(\mathbb{R}) \mid g^t \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} g = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \right\}.$$

The Siegel's upper half plane is defined by

$$\mathbb{H}_m := \left\{ Z = X + iY \in M_m(\mathbb{C}) \mid X = X^t,\ Y = Y^t > 0 \right\}.$$

Now $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_m(\mathbb{R})$ acts on $\mathbb{H}_m$ via

$$(g, Z) \longmapsto g\langle Z \rangle := (AZ + B)(CZ + D)^{-1}.$$

This defines a transitive action of $\mathrm{Sp}_m(\mathbb{R})$ on $\mathbb{H}_m$. We also define the factor of automorphy $j(g,Z) := \det(CZ + D)$. We call $\mathrm{Sp}_m(\mathbb{Z}) := \mathrm{Sp}_m(\mathbb{R}) \cap \mathrm{GL}_{2m}(\mathbb{Z})$ the **full modular group**. For any integer $N > 0$, we define the following congruence subgroup of $\mathrm{Sp}_m(\mathbb{Z})$:

$$\Gamma_0^{(m)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_m(\mathbb{Z}) \mid C \equiv 0 \pmod{N} \right\}. \tag{5.3.1}$$

We then have the following Definition (cf. [CP91, Section 2.4.1]):

Definition 5.3.1. Let $k \in \mathbb{Z}$ and $m \geq 2$. A function $F : \mathbb{H}_m \longrightarrow \mathbb{C}$, with $F \in \mathcal{C}^\infty(\mathbb{H}_m)$ (see Notation), is called a $\mathcal{C}^\infty$-modular form of weight k on the group $\Gamma_0^{(m)}(N)$ with a Dirichlet character $\psi \pmod{N}$, if for all

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(m)}(N),$$

we have

$$F(\gamma\langle Z \rangle) = \psi(\det D) \det(CZ + D)^k F(Z),$$

for all $Z \in \mathbb{H}_m$. We denote the space of such functions by $\widetilde{M}_k(N, \psi)$.

Finally, in some cases, we can define a suitable inner product on the above space.

Definition 5.3.2. The Petersson inner product of a pair of modular functions

$F, G \in \widetilde{M}_k(N, \psi)$ is

$$\langle F, G \rangle := \frac{1}{\left[\mathrm{Sp}_m(\mathbb{Z}) : \Gamma_0^{(m)}(N)\right]} \int_{\Gamma_0^{(m)}(N)\backslash \mathbb{H}_m} F(Z)\overline{G(Z)}(\det Y)^k \mathrm{d}^* Z,$$

whenever the integral converges. Here, d^*Z is the $\mathrm{Sp}_m(\mathbb{R})$-invariant measure $(\det Y)^{-(m+1)}\mathrm{d}X\mathrm{d}Y$.

We are now ready to give the definition of the theta series of interest, which is related to Sp_2. Such theta series were first studied by Siegel.

Definition 5.3.3. Let $Z = X + iY \in \mathbb{H}_2$ and $W \in \mathcal{H}_S$. We then define the theta series

$$\theta(Z, W) := \sum_{\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})} \theta_\ell(Z, W),$$

where for any $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$, we define

$$\theta_\ell(Z, W) := e^{\pi i \mathrm{tr}(S_1[\ell]X) - \pi \mathrm{tr}(R_W[\ell]Y)}.$$

Let also $\Theta(Z, W) := \det(Y)^{\frac{n+2}{2}} \theta(Z, W)$. Because R_W is positive definite, it follows that Θ converges absolutely and uniformly in any region of the form $\mathbb{H}_2(\epsilon) = \{Z = X + iY \mid Y \geq \epsilon 1_2\}$ with $\epsilon > 0$ and therefore defines a real analytic function of the matrices X, Y with $X + iY \in \mathbb{H}_2$ (cf. [And89, page 291]).

We start with the following Lemma regarding the invariance of Θ under Γ_S.

Lemma 5.3.4. *Let $W \in \mathcal{H}_S$. We then have*

$$\Theta(Z, M\langle W \rangle) = \Theta(Z, W)$$

for all $M \in \Gamma_S$.

Proof. Let $M \in \Gamma_S$. Then

$$\begin{aligned}
\Theta(Z, M\langle W\rangle) &= \det(Y)^{\frac{n+2}{2}} \sum_{\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})} e^{\pi i \mathrm{tr}(S_1[\ell]X) - \pi \mathrm{tr}(R_{M\langle W\rangle}[\ell]Y)} \\
&= \det(Y)^{\frac{n+2}{2}} \sum_{\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})} e^{\pi i \mathrm{tr}(S_1[M^{-1}\ell]X) - \pi \mathrm{tr}(R_W[M^{-1}\ell]Y)} \\
&= \Theta(Z, W),
\end{aligned}$$

because $R_{M\langle W\rangle} = R_W[M^{-1}]$ (this follows by the way R_W is defined in the proof of Proposition 5.2.7) and $S_1[M^{-1}] = S_1$. These relations, together with the fact that $M \in \Gamma_S \subset \mathrm{GL}_{n+4}(\mathbb{Z}) \implies M^{-1} \in \mathrm{GL}_{n+4}(\mathbb{Z})$, gives the invariance under Γ_S. $\square$

The situation, however, is quite different when it comes to the transformation of Θ with respect to $\mathrm{Sp}_2(\mathbb{Z})$. In particular, it is necessary to consider a specific congruence subgroup of $\mathrm{Sp}_2(\mathbb{Z})$ and even then Θ transforms as a modular form of a non-trivial weight and character.

Proposition 5.3.5. *Let q denote the level of S, equivalently S_1 (see Definition 4.1.4). Then, for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(q)$ (see (5.3.1)), we have*

$$\Theta(\gamma Z, W) = \chi_S(\gamma) j(\gamma, Z)^{-n/2} \Theta(Z, W),$$

where $\chi_S(\gamma)$ is an eighth root of unity which does not depend on Z and W. In the case when the rank n of S is even, we have $\chi_S(\gamma) = \psi_S(\det D)$, with ψ_S a Dirichlet character modulo q such that

$$\psi_S(p) = \left(\frac{(-1)^{n/2} \det S}{p} \right)$$

for all odd primes p (Legendre symbol) and $\psi_S(-1) = (-1)^{n/2}$. In particular, this implies $\Theta \in \widetilde{M}_k(q, \psi_S)$ with $k = -n/2$.

Proof. From [And89, Theorem 1], we obtain that if $\gamma \in \Gamma_0^{(2)}(q)$, then

$$\theta(\gamma Z, W) = \chi_S(\gamma) j(\gamma, Z) j(\gamma, \overline{Z})^{(n+2)/2} \theta(Z, W)$$

for some character as described in the statement of the Proposition. Multiplying by $(\det Y)^{(n+2)/2}$ gives the first part of the Proposition. The second part follows immediately from [And89, Theorem 2]. □

5.4 Differential Operators

In this Section, we prepare the ground for the theta-correspondence. In the same fashion as Krieg in [Kri91], Gritsenko in [Gri92a], and Raghavan and Sengupta in [RS91], we need to apply some differential operators to the theta series first, so that the integral converges. Our first step is to make Θ invariant under the action of $\mathrm{Sp}_2(\mathbb{Z})$, up to the character χ_S. This is essential because the differential operators that will eliminate the terms of the theta series that cause divergence are $\mathrm{Sp}_2(\mathbb{R})$-invariant.

Assumption: From now on, we assume that $4 \mid n$.

For $m \geq 1$, let $Z = X + iY \in \mathbb{H}_m$. We denote by ∂_Z the matrix

$$(\partial_Z)_{ij} := \frac{1+\delta_{ij}}{2}\frac{\partial}{\partial Z_{ij}},$$

where δ_{ij} denotes the Kronecker's delta, for $1 \leq i,j \leq m$. Here $\partial/\partial Z_{ij} := (\partial/\partial X_{ij} - i\partial/\partial Y_{ij})/2$. Due to Maass in [Maa71], we have the operator ($k \in \mathbb{Z}$)

$$\delta_k := \det(Y)^{-k+\frac{m-1}{2}} \det(\partial_Z) \det(Y)^{k-\frac{m-1}{2}}, \tag{5.4.1}$$

which sends functions from $\widetilde{M}_k(N,\psi_S)$ to $\widetilde{M}_{k+2}(N,\psi_S)$ ([CP91, Section 3.3.1]).

For any integer $r \geq 1$, one then defines the Shimura differential operator as the composition

$$\delta_k^{(r)} := \delta_{k+2r-2} \cdots \delta_{k+2}\delta_k.$$

This sends functions from $\widetilde{M}_k(N,\psi_S)$ to $\widetilde{M}_{k+2r}(N,\psi_S)$ ([CP91, Section 3.3.1]).

Therefore, in the case $m = 2$, $\delta_k^{(r)}\Theta \in \widetilde{M}_0(q,\psi_S)$ with $k = -n/2$ and $r = -k/2 = n/4$, because of Proposition 5.3.5.

However, we still need to apply an invariant differential operator R, so that we remove the singular terms that cause the integrals to diverge. The existence of a suitable such operator is guaranteed by a result of Deitmar and Krieg in [DK91]. Let us now describe it here.

For any dimension $m \geq 1$, we consider the algebras $\mathbb{D}(\mathbb{H}_m)$ and $\mathbb{D}(\mathcal{P}_m)$ of invariant differential operators with respect to $\mathrm{Sp}_m(\mathbb{R})$ and $\mathrm{GL}_m(\mathbb{R})$. Here, $\mathcal{P}_m$ denotes the symmetric space $\mathrm{GL}_m(\mathbb{R})/\mathrm{O}_m(\mathbb{R})$ and can be identified with the set

$$\mathcal{P}_m = \left\{ Y \in M_m(\mathbb{R}) \mid Y = Y^t > 0 \right\},$$

by means of the action of $\mathrm{GL}_m(\mathbb{R})$ on $\mathcal{P}_m$ given by

$$M \cdot Y = Y[M^t] = MYM^t.$$

Here, $\mathrm{O}_m(\mathbb{R}) = \{M \in \mathrm{GL}_m(\mathbb{R}) \mid M \cdot M^t = 1_m\}$. Next, if we consider the injective map

$$\phi : \mathcal{C}^\infty(\mathcal{P}_m) \longrightarrow \mathcal{C}^\infty(\mathbb{H}_m)$$

defined by

$$\phi(f)(X+iY) := f(Y),$$

we may associate to ϕ a well-defined map

$$\phi^* : \mathbb{D}(\mathbb{H}_m) \longrightarrow \mathbb{D}(\mathcal{P}_m),$$

which sends $D \longmapsto \phi^{-1} \circ D \circ \phi$. Then, from [DK91, Theorem 1.1], we obtain that

ϕ^* is an injective algebra homomorphism.

Now, for any $l \in \mathbb{R}$, we define the operator $D_l = D_{l,m} \in \mathbb{D}(\mathcal{P}_m)$ by

$$D_{l,m}(Y) := (\det Y)^l \det(\partial_Y) \det(Y)^{1-l}.$$

From [DK91, Theorem 1.2], we have that $D_{m+2-l,m} D_{l,m}$ belongs in the image of ϕ^*. Define then

$$R(m,l) := (\phi^*)^{-1}(D_{m+2-l,m} D_{l,m}) \in \mathbb{D}(\mathbb{H}_m). \tag{5.4.2}$$

In the case $m = 2$, we write for $Z = X + iY \in \mathbb{H}_2$,

$$Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}, \ X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}, \ Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}.$$

We first have the following Lemma regarding the behaviour of Θ under the action of the Maass-Shimura operator.

Lemma 5.4.1. *Let $k = -n/2$ and $r = n/4$. For any $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$, we have*

$$\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}}\theta_\ell(Z,W)\right] = (\det Y)^{1+r} p(\det((S_1 + R_W)[\ell]Y), \mathrm{tr}((S_1 + R_W)[\ell]Y))\times \\ \times \theta_\ell(Z,W), \tag{5.4.3}$$

where $p = p(U,V) \in \mathbb{R}[U,V]$ is a polynomial in two variables, and its coefficients do not depend on ℓ. In particular, if $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$ is such that $S_1[\ell] = 0$, this becomes

$$\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}}\theta_\ell(Z,W)\right] = (\det Y)^{1+r} p\left(\det(R_W[\ell]Y), \mathrm{tr}\left(R_W[\ell]Y\right)\right) e^{-\pi \mathrm{tr}(R_W[\ell]Y)}. \tag{5.4.4}$$

Proof. For any $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$, we have by [Sat86, Lemma 1.1]

$$\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}}\theta_\ell(Z,W)\right] = (\det Y)^{\frac{n+2}{2}} \delta_1^{(r)} \theta_\ell(Z,W).$$

Define now the operator $\sigma \in \mathbb{D}(\mathbb{H}_2)$ by

$$\sigma := i \sum_{j=1}^{3} y_j \frac{\partial}{\partial z_j}. \tag{5.4.5}$$

By [Sat86, Proposition 1.2, (a)], $\delta_1^{(r)}\theta_\ell(Z,W)$ is a $\mathbb{Z}[1/2]$-linear combination of functions of the form

$$\det(Y)^{-b}\sigma^c \left(\det(\partial_Z)\right)^d \theta_\ell(Z,W), \tag{5.4.6}$$

where b, c, d are integers with $0 \le b \le c \le r,\ 0 \le d \le r$ and $b + d = r$. Note that the additional requirements stated in [Sat86, Proposition 1.2] are not needed for the proof of this part. We compute

$$\frac{\partial}{\partial z_1}\theta_\ell(Z,W) = A_\ell\theta_\ell(Z,W),\ \frac{\partial}{\partial z_2}\theta_\ell(Z,W) = B_\ell\theta_\ell(Z,W),$$
$$\frac{\partial}{\partial z_3}\theta_\ell(Z,W) = 2C_\ell\theta_\ell(Z,W), \quad (5.4.7)$$

where for $\ell = \begin{pmatrix} l & m\end{pmatrix}$, with $l, m \in \mathbb{Z}^{n+4}$ we have

$$A_\ell = \frac{1}{2}\pi i\left(S_1[l] + R_W[l]\right),\ B_\ell = \frac{1}{2}\pi i\left(S_1[m] + R_W[m]\right),\ C_\ell = \frac{1}{2}\pi i\left(l^t S_1 m + l^t R_W m\right).$$

We now set $T_\ell = \frac{1}{2}\pi i(S_1 + R_W)[\ell]$ and write $T_\ell = \begin{pmatrix} t_1 & t_3/2 \\ t_3/2 & t_2\end{pmatrix}$, with $t_1 = A_\ell, t_2 = B_\ell$ and $t_3 = 2C_\ell$.

We first compute $\det(\partial_Z)\theta_\ell = \det(T_\ell)\theta_\ell$. Moreover, if $B := \mathrm{tr}(T_\ell Y)$, we claim that for any $0 \le c \le r$, $\sigma^c\theta_\ell = f_c(B)\theta_\ell$ for some polynomial f of degree c. We show this by induction. Note that for $j = 1, 2, 3$, we have $\partial B/\partial z_j = -it_j/2$.

If $c = 0$, then the claim is clear with $f_0(B) = 1$. If now $\sigma^c\theta_\ell = f_c(B)\theta_\ell$, we have

$$\sigma^{c+1}\theta_\ell = i\sum_{j=1}^{3} y_j\left[f_c'(B)\left(\frac{-it_j}{2}\right)\theta_\ell + f_c(B)t_j\theta_\ell\right] = B\left(\frac{1}{2}f_c'(B) + if_c(B)\right)\theta_\ell,$$

from which the claim follows with $f_{c+1}(B) = B\left(\frac{1}{2}f_c'(B) + if_c(B)\right)$ of degree $c+1$.

Now, each term in (5.4.6) can be written as

$$(\det Y)^{-b}\sigma^c\det(\partial_Z)^d\theta_\ell = (\det Y)^{-b}\sigma^c\det(T_\ell)^d\theta_\ell = (\det Y)^{-r}[\det(T_\ell Y)^d\sigma^c\theta_\ell] =$$
$$= (\det Y)^{-r}[\det(T_\ell Y)^d f_c(\mathrm{tr}(T_\ell Y))\theta_\ell],$$

because $r - b = d$. Equation (5.4.3) now follows after absorbing the $\pi i/2$ factor of T_ℓ and observing (by induction) that $f_c(B)$ has purely imaginary coefficients in the odd powers of B and real coefficients in the even powers. Hence p will have real coefficients. Equation (5.4.4) now follows immediately from (5.4.3). □

We are now ready to give the main Proposition of this Section, regarding the elimination of the terms that will cause the divergence of the integral.

Proposition 5.4.2. *Let $k = -n/2$ and $r = -k/2 = n/4$. Let also $R := R(2, 2+r)$. We then have*

$$R\left[\delta_k^{(r)}\Theta\right](Z,W) = R\left[\sum_{\substack{\ell\in\mathrm{Mat}_{n+4,2}(\mathbb{Z})\\ \mathrm{rank}\ \ell=2}}\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}}e^{\pi i\mathrm{tr}(S_1[\ell]X + iR_W[\ell]Y)}\right]\right].$$

Proof. Due to Maass in [Maa71] and Shimura in [Shi90], R can be written in the form

$$R = u(H_1, H_2) \tag{5.4.8}$$

where H_1, H_2 are some "generalised" Laplacians and $u \in \mathbb{C}[X, Y]$ a polynomial (see also [Yan15, Example 3.2] and [BC06, Proposition 6] for an explicit description of H_1, H_2). Therefore, because Θ is absolutely and uniformly convergent on compact subsets of $\mathbb{H}_2$, we can apply the differential operators term by term. Hence, it suffices to show that for any $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$ with rank $\ell < 2$, we have

$$R\left[\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}} e^{\pi i \mathrm{tr}(S_1[\ell]X + iR_W[\ell]Y)}\right]\right] = 0.$$

Fix now $\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})$ such that rank $\ell < 2$. We may assume that

$$S_1[\ell] = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \text{ and } R_W[\ell] = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix},$$

for some $x, y \in \mathbb{R}$. This is true because we can find $U \in \mathrm{SL}_2(\mathbb{R})$ such that $\ell U = \begin{pmatrix} a & 0 \end{pmatrix}$ for some $a \in \mathbb{R}^{n+4}$. But then

$$\widetilde{U} := \begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})$$

and the action of this matrix on $\mathbb{H}_2$ is $Z \longmapsto UZU^t$. Since R is Sp_2-invariant and $j(\widetilde{U}, Z) = 1$, $\psi_S(\det U) = 1$ (so the action $Z \longmapsto \widetilde{U}\langle Z\rangle$ does not change $\delta_k^{(r)}\Theta$), we can change variables $Z \longmapsto UZU^t$. But $\det(UYU^t) = \det(Y)$ and

$$\mathrm{tr}(S_1[\ell]UXU^t) = \mathrm{tr}(U^t S_1[\ell]UX) = \mathrm{tr}(S_1[\ell U]X),$$

so we can replace ℓ with ℓU, which will then give the form of $S_1[\ell]$ wanted. Similarly for $R_W[\ell]$. By [Sat86, Lemma 1.1], we have (we remind here that $k = -n/2$)

$$\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}} \theta_\ell(Z, W)\right] = (\det Y)^{\frac{n+2}{2}} \delta_1^{(r)}\left[\theta_\ell(Z, W)\right]$$

and by [Sat86, Proposition 1.2, (a)], we have that the quantity $\delta_1^{(r)}\theta_\ell(Z, W)$ will be a $\mathbb{Z}[1/2]$-linear combination of functions of the form

$$\det(Y)^{-b}\sigma^c(\det(\partial_Z))^d\theta_\ell(Z, W),$$

where b, c, d are integers with $0 \le b \le c \le r$, $0 \le d \le r$ and $b + d = r$ (the operator σ here is as in (5.4.5)). Now, in the case rank $\ell < 2$, we have that $B_\ell = C_\ell = 0$ in (5.4.7) because of the form of $S_1[\ell]$ and $R_W[\ell]$. Hence,

$$\det(\partial_Z)\left[\theta_\ell(Z, W)\right] = 0.$$

So, we only need to consider the case $b = c = r,\ d = 0$. But from the proof of Lemma 5.4.1, we have $\sigma^r\theta_\ell(Z,W) = f_r(B)\theta_\ell(Z,W)$, with $B = \mathrm{tr}(T_\ell Y)$ and T_ℓ as in Lemma 5.4.1. But, since $B_\ell = C_\ell = 0$, we have that B, hence $\sigma^r\theta_\ell$ depends only on y_1 and θ_ℓ. Consider now the map

$$\tau : \mathcal{C}^\infty(\mathbb{H}_1 \times \mathbb{H}_1) \longrightarrow \mathcal{C}^\infty(\mathbb{H}_2),$$

defined by

$$\tau(h)\left(\begin{pmatrix} Z_1 & a \\ a & Z_2 \end{pmatrix}\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right]\right) = h(Z_1, Z_2)\ \forall a, b \in \mathbb{R}.$$

Now, if we write

$$Z = \begin{pmatrix} Z_1 & a \\ a & Z_2 \end{pmatrix}\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right] = X + iY,$$

we observe that

$$Y = \begin{pmatrix} Y_1 & Y_1 b \\ bY_1 & b^2Y_1 + Y_2 \end{pmatrix},$$

where $Z_j = X_j + iY_j,\ j = 1,2$. Hence, $\det(Y) = Y_1Y_2$ and $\sigma^r\theta_\ell(Z,W)$ depends only on $\theta_\ell(Z,W)$ and Y_1. But

$$\theta_\ell(Z,W) = e\left(\frac{1}{2}\left(xX_1 + iyY_1\right)\right),$$

where $S_1[\ell] = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $R_W[\ell] = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$. Therefore,

$$\delta_k^{(r)}\left[(\det Y)^{\frac{n+2}{2}}\theta_\ell\right](Z,W)$$

is independent of a, b and so belongs in the image of τ. But by [DK91, Proposition 1.1], $\tau^{-1} \circ R \circ \tau$ is a simple tensor in $\mathbb{D}(\mathbb{H}_1 \times \mathbb{H}_1)$, so the problem is reduced in the one-dimensional case and $S_1[\ell] = R_W[\ell] = 0$, i.e., suffices to show

$$R(1, 2 + n/4)[\delta_k^{(r)} y^{\frac{n+2}{2}}] = 0,$$

where now we have $\delta_k = \dfrac{k}{2iy} + \dfrac{\partial}{\partial z}$ in the one-dimensional case. But we have

$$\delta_k^{(r)} y^{\frac{n+2}{2}} = (\mathrm{const}) \times y^{n/4+1}.$$

Also, from [BC08, page 807], we have that in the one-dimensional case

$$R(1, 2 + n/4) = 4y^2 \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} - \frac{n}{4}\left(\frac{n}{4} + 1\right)$$

(note that $R(m,k) = R^{\text{Böch}}(m, k-1)$, where $R^{\text{Böch}}$ is the operator defined in [BC08, Theorem 2.1]). Therefore, the result follows. □

Finally, combining the two above results, we obtain the following Lemma.

Lemma 5.4.3. *Let $k=-n/2$ and $r=n/4$, as before. Given $l\in\mathbb{R}$, $\epsilon>0$, and a compact subset $\mathcal{C}$ of $\mathcal{H}_S$, there exists a constant $C>0$ such that*

$$\left|R\left[\delta_k^{(r)}\Theta\right](Z,W)\right|\leq C(\det Y)^l$$

holds for all $W\in\mathcal{C}$ and $Z=X+iY\in\mathbb{H}_2$, with $Y\geq\epsilon 1_2$.

Proof. We can write

$$R\left[\delta_k^{(r)}\Theta\right](Z,W)=(\det Y)^{\frac{n+2}{2}}\sum_{\substack{\ell\in\mathrm{Mat}_{n+4,2}(\mathbb{Z})\\ \mathrm{rank}\ \ell=2}} g(S_1[\ell],R_W[\ell],Y)\theta_\ell(Z,W),\qquad(5.4.9)$$

for some polynomial g in the entries of the matrices $S_1[\ell], R_W[\ell]$ and Y. This follows from Proposition 5.4.2, the fact that R is well-behaved (see (5.4.8)), (5.4.6) and the relations we have obtained in the proof of Lemma 5.4.1. The rest of the proof now follows in the same way as in [DK91, Proposition 2.1, (b)]. □

5.5 Theta Correspondence

In this Section, we finally give the theta correspondence between the Klingen-type Eisenstein series of $\mathrm{SO}(2,n+2)$ and the Siegel-type Eisenstein series for Sp_2. We start with the following definition.

Definition 5.5.1. Let $\chi=\chi_S$ denote the character of Proposition 5.3.5. Let also

$$P_{2,0}:=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}_2(\mathbb{Z})\mid C=0\right\}$$

denote the Siegel parabolic subgroup of Sp_2. Notice that $P_{2,0}\cap\Gamma_0^{(2)}(q)=P_{2,0}$. For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>3/2$, we then define the Siegel Eisenstein series with respect to $P_{2,0}$ and with character χ as

$$\widetilde{E}(Z,\chi,s):=\sum_{\gamma\in P_{2,0}\backslash\Gamma_0^{(2)}(q)}\overline{\chi(\gamma)}(\det(\mathrm{Im}(\gamma Z)))^s.\qquad(5.5.1)$$

If $\delta=\begin{pmatrix}A&B\\0&D\end{pmatrix}\in P_{2,0}$, we have $\det D=\pm1$, so $\chi(\delta)=\psi(\det D)=1$ by Proposition 5.3.5. Therefore, the Eisenstein series is well-defined. This Eisenstein series has a meromorphic continuation to $\mathbb{C}$ (see [Kal77, Theorem 1, (3)]). Finally, using the fact that χ is a character, we have

$$\widetilde{E}(\gamma Z, \chi, s) = \chi(\gamma)\widetilde{E}(Z, \chi, s),$$

for all $\gamma \in \Gamma_0^{(2)}(q)$. The main Theorem of the paper can now be stated as follows:

Theorem 5.5.2. *Let S have rank n, with $4 \mid n$ and be such that $\#\mathcal{C}^1(\Gamma_S) = 1$. Let also $k = -n/2$ and $r = n/4$, as before. Define*

$$\phi_2(s) := s\left(s - \frac{1}{2}\right), \ \xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

for $s \in \mathbb{C}$. We then have for $Re(s) > n + 1$

$$\left\langle \widetilde{E}(Z, \chi, (s+1)/2 - r), R[\delta_k^{(r)}\Theta](Z, W) \right\rangle_{\Gamma_0^{(2)}(q)} = \xi(s)\xi(s-1)\gamma_S(s)E(W, s),$$

where $\gamma_S(s) := \dfrac{1}{[\mathrm{Sp}_2(\mathbb{Z}) : \Gamma_0^{(2)}(q)]}(-4)^{-r}\phi_2(s/2 - 2r)\phi_2(s/2)\prod_{j=1}^{r}\phi_2\left(\dfrac{s-(2j-1)}{2}\right).$

Proof. Let

$$I := [\mathrm{Sp}_2(\mathbb{Z}) : \Gamma_0^{(2)}(q)] \left\langle \widetilde{E}(Z, \chi, (s+1)/2 - r), R[\delta_k^{(r)}\Theta](Z, W) \right\rangle_{\Gamma_0^{(2)}(q)}.$$

First of all, this integral is well-defined because of Lemma 5.4.3. We then have

$$\begin{aligned}
I &= \int_{\Gamma_0^{(2)}(q)\backslash\mathbb{H}_2} \left[\sum_{\gamma \in P_{2,0}\backslash\Gamma_0^{(2)}(q)} \overline{\chi(\gamma)}(\det(\mathrm{Im}(\gamma Z)))^{(s+1)/2-r}\overline{R[\delta_k^{(r)}\Theta](Z, W)} \right] \mathrm{d}^*Z \\
&= \int_{\Gamma_0^{(2)}(q)\backslash\mathbb{H}_2} \left[\sum_{\gamma \in P_{2,0}\backslash\Gamma_0^{(2)}(q)} \overline{\chi(\gamma)}(\det(\mathrm{Im}(\gamma Z)))^{(s+1)/2-r}\chi(\gamma)\overline{R[\delta_k^{(r)}\Theta](\gamma Z, W)} \right] \mathrm{d}^*Z \\
&= \int_{P_{2,0}\backslash\mathbb{H}_2} (\det Y)^{(s+1)/2-r}\overline{R[\delta_k^{(r)}\Theta](Z, W)}\mathrm{d}^*Z \\
&= \int_{\mathcal{C}(2,\mathbb{R})}\int_{\mathcal{R}(2,\mathbb{R})} (\det Y)^{(s+1)/2-r}\overline{R[\delta_k^{(r)}\Theta](Z, W)}\mathrm{d}^*Z,
\end{aligned}$$

where in the second equation we used the invariance up to χ of $R[\delta_k^{(r)}\Theta]$ and in the third equation we used the usual unfolding trick. Here, $C(2, \mathbb{R}) + i\mathcal{R}(2, \mathbb{R})$ is a fundamental domain for the action of $P_{2,0}$ on $\mathbb{H}_2$, where $C(2, \mathbb{R})$ denotes a fundamental parralepiped of $\mathrm{Sym}(2, \mathbb{Z})$ in $\mathrm{Sym}(2, \mathbb{R})$ and $\mathcal{R}(2, \mathbb{R})$ the Minkowski reduced matrices, as in [Kri85, p. 29]. In the following, we write $\mathrm{Sym}_2(\mathbb{R}/\mathbb{Z})$ for $\mathcal{C}(2, \mathbb{R})$.

Now, from Proposition 5.4.2, Lemma 5.4.1 and the proof of Lemma 5.4.3, we have

$$\int_{\mathrm{Sym}_2(\mathbb{R}/\mathbb{Z})} \overline{R[\delta_k^{(r)}\Theta](Z,W)}\mathrm{d}X =$$
$$= \sum_{\substack{\ell\in \mathrm{Mat}_{n+4,2}(\mathbb{Z}) \\ \mathrm{rank}\ \ell=2, S_1[\ell]=0}} R_0^*\left[p\left(\det(R_W[\ell]Y), \mathrm{tr}\left(R_W[\ell]Y\right)\right)e^{-\pi\mathrm{tr}(R_W[\ell]Y)}\right],$$

where $R_0^* = (\det Y)^{2-r}\det(\partial_Y)(\det Y)^{1+2r}\det(\partial_Y)$. The fact that only terms with $S_1[\ell]=0$ remain follows from (5.4.9). The rest of the expression of this integral follows from (5.4.4) and from the fact that $R = (\phi^*)^{-1}\left(R_0^*(\det Y)^{-(1+r)}\right)$ (note that from Lemma 5.4.1, the polynomial p has real coefficients). Hence, from [Kri85, Proposition I.4.4, I.4.5], the integral I becomes

$$\sum_{\substack{\ell\in \mathrm{Mat}_{n+4,2}(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{rank}\ell=2, S_1[\ell]=0}} \int_{\mathcal{P}_2}(\det Y)^{s/2}R_0\left[p\left(\det(R_W[\ell]Y), \mathrm{tr}\left(R_W[\ell]Y\right)\right)e^{-\pi\mathrm{tr}(R_W[\ell]Y)}\right]\mathrm{d}^*Y,$$

where now $R_0 = (\det Y)^{-(1+r)}R_0^*$ and $\mathrm{d}^*Y = (\det Y)^{-\frac{3}{2}}\mathrm{d}Y$ is the invariant measure for the action of GL_2 on $\mathcal{P}_2$. Now, for every ℓ in our sum, we have that $R_W[\ell]$, hence $(R_W[\ell])^{-1}$, is positive definite. So, we can write $(R_W[\ell])^{-1} = AA^t$ with A lower triangular and change variables $Y \longmapsto Y[A^t]$. Then $\det(R_W[\ell]Y)$ and $\mathrm{tr}\,(R_W[\ell]Y)$ become $\det Y$ and tr Y respectively and the measure remains invariant. Now, from [BC08, equation (2.4)], R_0 is GL_2-invariant ($n=2$ and $m=2+2r$ in the notation there). Hence, the integral becomes

$$I = \sum_{\substack{\ell\in \mathrm{Mat}_{n+4,2}(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{rank}\ \ell=2,\ S_1[\ell]=0}} (\det R_W[\ell])^{-s/2}\int_{\mathcal{P}_2}(\det Y)^{s/2}R_0[p(\det Y, \mathrm{tr}Y)e^{-\pi\mathrm{tr}Y}]\mathrm{d}^*Y.$$

Let now $M = \det Y\det\partial_Y$. For any $t\in\mathbb{R}$, we have for its adjoint operator $\widehat{M}$ (see [Maa71, page 57] for a definition) that

$$\widehat{M}[(\det Y)^t] = \phi_2(t)(\det Y)^t,$$

where $\phi_2(t) = t\left(t-\dfrac{1}{2}\right)$ (see [BC08, (3.3)]). Therefore, we have

$$\begin{aligned} I = &\sum_{\substack{\ell\in \mathrm{Mat}_{n+4,2}(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{rank}\ \ell=2, S_1[\ell]=0}} (\det R_W[\ell])^{-s/2}\times \\ &\times\int_{\mathcal{P}_2}(\det Y)^{s/2-2r}M\left[(\det Y)^{1+2r}\det(\partial_Y)\left[p(\det Y, \mathrm{tr}Y)e^{-\pi\mathrm{tr}Y}\right]\right]\mathrm{d}^*Y \\ = &\sum_{\substack{\ell\in \mathrm{Mat}_{n+4,2}(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{rank}\ \ell=2, S_1[\ell]=0}} (\det R_W[\ell])^{-s/2}\phi_2(s/2-2r)\times \\ &\times\int_{\mathcal{P}_2}(\det Y)^{s/2}M[p(\det Y, \mathrm{tr}Y)e^{-\pi\mathrm{tr}Y}]\mathrm{d}^*Y \end{aligned}$$

$$= \sum_{\substack{\ell \in \mathrm{Mat}_{n+4,2}(\mathbb{Z})/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{rank}\ \ell = 2, S_1[\ell]=0}} (\det R_W[\ell])^{-s/2} \phi_2(s/2-2r)\phi_2(s) \times$$

$$\times \int_{\mathcal{P}_2} (\det Y)^{s/2} p(\det Y, \mathrm{tr} Y) e^{-\pi \mathrm{tr} Y} \mathrm{d}^* Y$$

$$= \zeta(s)\zeta(s-1)E(W,s)\phi_2(s/2-2r)\phi_2(s/2) \int_{\mathcal{P}_2} (\det Y)^{s/2} p(\det Y, \mathrm{tr} Y) e^{-\pi \mathrm{tr} Y} \mathrm{d}^* Y,$$

where the second and third equality follow after transferring M to its adjoint. The last equality follows from Proposition 5.2.9 and Lemma 5.2.11. We now just need to compute the last integral. It is true that (cf. [Maa71, page 80, 81])

$$\int_{\mathcal{P}_2} (\det Y)^s e^{-\mathrm{tr}(TY)} \mathrm{d}^* Y = \pi^{1/2} \Gamma_2(s) (\det T)^{-s},$$

for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$ and $T \in \mathcal{P}_2$. Here $\Gamma_2(s) := \Gamma(s)\Gamma(s-1/2)$. Setting now $T \longmapsto \pi T$ and then applying the operator $\delta_1^{(r)}(S,T)$, where S is symmetric, so that $U := S + iT \in \mathbb{H}_2$, we obtain from Lemma 5.4.1 and (5.4.4)

$$\int_{\mathcal{P}_2} (\det Y)^{s/2} (\det T)^{-r} p(\det(YT),\ \mathrm{tr}(YT)) e^{-\pi \mathrm{tr}(YT)} \mathrm{d}^* Y =$$

$$= \pi^{1/2-s} \Gamma_2(s/2) \delta_1^{(r)} [(\det T)^{-s/2}].$$

But for any $\alpha \in \mathbb{Z}$ and $w \in \mathbb{C}$, we have from (5.4.1)

$$\begin{aligned} \delta_\alpha \left[(\det T)^w \right] &= (\det T)^{1/2-\alpha} \det(\partial_U) (\det T)^{\alpha - 1/2 + w} \\ &= -\frac{1}{4} (\det T)^{1/2-\alpha} \det(\partial_T) (\det T)^{\alpha-1/2+w} \\ &= -\frac{1}{4} (\det T)^{1/2-\alpha} (\alpha - 1/2 + w)(\alpha + w) (\det T)^{\alpha - 3/2 + w} \\ &= -\frac{1}{4} \phi_2(w+\alpha) (\det T)^{w-1}, \end{aligned}$$

because $\det(\partial_U) B(T) = -\det(\partial_T) B(T)/4$ for a function $B = B(T)$ depending only on T and the third equality follows from a well-known formula (see [CSS13, Theorem 2.2] for example). Hence, by successively applying the above and using the fact that $\phi_2(s) = \phi_2(1/2 - s)$ for any $s \in \mathbb{C}$, we get

$$\delta_1^{(r)} \left[(\det T)^{-s/2} \right] = (-4)^{-r} \prod_{j=1}^{r} \phi_2 \left(\frac{s - (2j-1)}{2} \right) (\det T)^{-s/2-r}.$$

By evaluating at $T = 1_2$, the proof is complete. □

Corollary 5.5.3. *Assume S satisfies the assumptions of Theorem 5.5.2. Then, $E(W,s)$ admits a meromorphic continuation to the complex plane and*

$$\Gamma\left(\frac{s+1}{2} - r\right) \Gamma(s-2r) L_q(s+1-2r, \chi) \zeta_q(2s-4r) \xi(s) \xi(s-1) \gamma_S(s) E(W,s)$$

has only possible simple poles at $s \in \{(n+2)/2,\ (n+4)/2\}$ if $q \neq 1$ and at $s \in \{(n-2)/2,\ n/2,\ (n+2)/2,\ (n+4)/2\}$ if $q = 1$. Here, the subscript $_q$ means that we omit the Euler factors sharing prime factors with q.

Proof. From [Shi97, Theorem 19.3], we have that

$$\Gamma(s)\Gamma(2s-1)L_q(2s,\chi)\zeta_q(4s-2)\widetilde{E}(Z,\chi,s)$$

has a meromorphic continuation to the complex plane with possible simple poles only at $s \in \{1, 3/2\}$ if $q \neq 1$ and at $s \in \{0, 1/2, 1, 3/2\}$ (see [Shi97, (19.3.1), (19.3.2)] because our character has order two). Hence, the corollary follows from Theorem 5.5.2. □

The main point here is that our method not only gives the meromorphic continuation to the complex plane for $E(W,s)$ (something that is expected to be true from the general Langlands' philosophy) but can also be used to extract finer information on the poles and zeroes of the Eisenstein series. Moreover, we get:

Corollary 5.5.4. *From Proposition 5.1.5 and Corollary 5.5.3, we obtain the meromorphic continuation of $\mathcal{D}_{F,G}(s)$ to $\mathbb{C}$, due to the one of $\widetilde{E}(Z,\chi,s)$, as we noted in the beginning of Section 5.5.*

Remark 5.5.5. The conditions of Theorem 5.5.2 are satisfied when S corresponds to at least the A_4, D_4 and E_8 lattices (see [Sch22, Section 1.6.2] for a description). These are Euclidean in the sense of [Sch22, Section 1.6.4] (see [Sch22, Example 1.6.14]). By looking at their Grammian matrix (see [Sch22, Section 1.6.2]), we can take x to be the first standard basis vector and then $S[x] = 2$. Hence, from Remark 5.2.10, the condition $\#\mathcal{C}^1(\Gamma_S) = 1$ is true in our setting.

5.6 The E_8 lattice

As an application, we obtain a precise result regarding the functional equation of $\mathcal{D}_{F,G}(s)$ in the case when

$$S = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

This is a positive definite, even matrix with $\det S = 1$ and the lattice it corresponds to is the so-called E_8 lattice (cf. [Sch22, Example 1.2.10]). This is the unique unimodular lattice with only one-dimensional cusp (and no other higher-dimensional cusps) ([Sch22, Example 1.6.20]). As this is unimodular, the level q is 1 and so the character χ_S of Proposition 5.3.5 is trivial. Therefore, in this case, $\Gamma_0^{(2)}(q)$ is the whole $\mathrm{Sp}_2(\mathbb{Z})$ and the symplectic Eisenstein series (5.5.1) is

$$\widetilde{E}(Z,s) = \sum_{\gamma \in P_{2,0}\backslash \mathrm{Sp}_2(\mathbb{Z})} (\det \mathrm{Im}(\gamma Z))^s.$$

Now, if $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, it is well-known (see for example [Kal77, Theorem 2]), that the modified Eisenstein series $\widetilde{\mathcal{E}}(Z,s) := \xi(2s)\xi(4s-2)\widetilde{E}(Z,s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 3/2$ and satisfies the functional equation $\widetilde{\mathcal{E}}(Z, 3/2 - s) = \widetilde{\mathcal{E}}(Z,s)$. Now, for $W \in \mathcal{H}_S$, let

$$E^*(W,s) := \xi(s-3)\xi(2s-8)\xi(s)\xi(s-1)\gamma_S(s)E(W,s).$$

From our main Theorem 5.5.2, we have in this case

$$\left\langle \widetilde{E}(Z,(s-3)/2), R[\delta_{-4}^{(2)}\Theta](Z,W) \right\rangle = \xi(s)\xi(s-1)\gamma_S(s)E(W,s).$$

Hence, we obtain that $E^*(W,s)$ has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \longmapsto 9 - s$.

Remark 5.6.1. We note here that we can rewrite $E^*(W,s)$ to have the form of a completed Eisenstein series using just gamma and zeta factors, using the relations $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s)$, valid for $\mathrm{Re}(s)$ large enough.

Now, from Proposition 5.1.5, we have

$$(4\pi)^{-s}\Gamma(s)\mathcal{D}_{F,G}(s) = \#\mathrm{SO}(S;\mathbb{Z}) \cdot \langle F(W) \cdot E(W, s-k+9), G(W)\rangle.$$

Hence, if we define

$$\mathcal{D}^*_{F,G}(s) := (4\pi)^{-s}\Gamma(s)\xi(s-k+6)\xi(2s-2k+10)\xi(s-k+9)\xi(s-k+8)\gamma_S(s-k+9)\times \\ \times \mathcal{D}_{F,G}(s), \quad (5.6.1)$$

we have

$$\mathcal{D}^*_{F,G}(s) = \#\mathrm{SO}(S;\mathbb{Z}) \cdot \langle F(W) \cdot E^*(W, s-k+9), G(W)\rangle.$$

Therefore, we arrive at the following Theorem.

Theorem 5.6.2. *Let S be as above, corresponding to the E_8 lattice. With the notation as above, $\mathcal{D}^*_{F,G}(s)$ has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \longmapsto 2k - 9 - s$.*

Chapter 6

Relation to L-functions

In this Chapter, we will investigate the other direction of the problem, i.e., how the method of Kohnen and Skoruppa can be extended in order to establish a connection of the Dirichlet series of Section 4.5 to the standard L-function of the orthogonal group.

Kohnen and Skoruppa prove their main result using a fundamental identity by Andrianov in [And74], which gives an Euler product expression for a Dirichlet series involving the Fourier coefficients of a Siegel cuspidal eigenform of degree 2, twisted by ideal class characters. Gritsenko, in [Gri87], initiated the study of a Dirichlet series involving the Fourier coefficients of an orthogonal modular form and its connection to the standard L-function attached to it, following factorisation methods similar to [Gri92a]. Sugano in [Sug85] extended Gritsenko's result by proving an Euler product relation for a Dirichlet series involving twists of the Fourier coefficients by modular forms on a definite orthogonal group of lower rank. His work can be seen as an extension of Andrianov's work in [And74] in the orthogonal setting. As we mentioned, our methods here can be considered as an extension of the methods employed by Kohnen and Skoruppa in [KS89], and therefore, Sugano's result is pivotal. Finally, results proved by Shimura in [Shi04] on the sets of solutions $\phi(x, x) = q$, where ϕ is a bilinear form and $q \in \mathbb{Q}^\times$, turn out to be crucial.

Below, we generalise all the main ingredients of the proof of Kohnen and Skoruppa and establish a relation to the standard L-function for certain orthogonal groups.

6.1 An operator on Fourier-Jacobi forms

To establish such a connection, we take, in a similar fashion to [KS89], G a specific element of the Maass space (cf. 4.4.1). To define G, we need some preparations. We use the notation of Chapter 4. We have $\sigma(x,y) = x^t S y$ for $x, y \in V$. Let $N \geq 1$ and define $M_2(\mathbb{Z})_N := \{g \in M_2(\mathbb{Z}) | \det g = N\}$. For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})_N$ and $\tau \in \mathbb{H}$ (usual upper half plane), we define

$$M\langle\tau\rangle := \frac{a\tau + b}{c\tau + d}.$$

Given a Jacobi cusp form $\phi \in S_k(\mathbb{Z}^n, \sigma)$, we define the operator

$$V_N : S_k(\mathbb{Z}^n, \sigma) \longrightarrow S_k(\mathbb{Z}^n, N\sigma),$$

given by

$$(V_N\phi)(\tau, z) = N^{k-1} \sum_{M \in \mathrm{SL}_2(\mathbb{Z}) \backslash M_2(\mathbb{Z})_N} (c\tau + d)^{-k} e^{-\pi i c N S[z]/(c\tau+d)} \phi\left(M\langle\tau\rangle, \frac{Nz}{c\tau + d}\right),$$

This is well-defined by [Sug95, Lemma 6.1] or [Moc19, Definition 4.25].

Our aim is to compute its adjoint with respect to the scalar product of Fourier-Jacobi forms (Definition 4.3.6). This is the analogue of the main Proposition in [KS89].

Now, if $\phi \in S_k(\mathbb{Z}^n, N\sigma)$, we will write its Fourier expansion in a form similar to Kohnen and Skoruppa in [KS89, Section 2].

By Definition 4.3.5, we can write

$$\phi(\tau, z) = \sum_{\substack{m \in \mathbb{Z}, r \in \mathbb{Z}^n \\ 2Nm > r^t S^{-1} r}} c'_\phi(m, r) e(m\tau + r^t z),$$

for some $c'_\phi(m, r) \in \mathbb{C}$. Now, from the condition $2Nm > r^t S^{-1} r$ and the definition of the level (see Definition 4.1.4), we can write

$$Nmq - \frac{1}{2} q S^{-1}[r] = -D \implies m = \frac{\frac{1}{2} q S^{-1}[r] - D}{qN},$$

for some integer $D < 0$. Therefore, we can write

$$\phi(\tau, z) = \sum_{\substack{D \in \mathbb{Z}_{<0}, r \in \mathbb{Z}^n \\ D \equiv \frac{1}{2} q S^{-1}[r] \mod qN}} c_\phi(D, r)\, e\left(\frac{\frac{1}{2} q S^{-1}[r] - D}{qN} \tau + r^t z\right). \tag{6.1.1}$$

Remark 6.1.1. By adjusting [Moc19, Proposition 1.25] to the notation above, we have that

$$D = D' \text{ and } S^{-1} r \equiv S^{-1} r' \pmod{N\mathbb{Z}^n} \implies c_\phi(D, r) = c_\phi(D', r').$$

We are now ready to give the result concerning the adjoint of V_N. We use the notation that $s \equiv s' \pmod{NS\mathbb{Z}^n} \iff s - s' = NSu$ for some $u \in \mathbb{Z}^n$.

Proposition 6.1.2. *The action of V_N^*, defined as the adjoint of V_N with respect to the scalar product of Fourier-Jacobi forms defined in 4.3.6, is given by (for $\phi \in S_k(\mathbb{Z}^n, N\sigma)$)*

$$\sum_{\substack{D\in\mathbb{Z}_{<0}, r\in\mathbb{Z}^n \\ D\equiv \frac{1}{2}qS^{-1}[r] \mod qN}} c_\phi(D,r)\, e\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{qN}\tau + r^t z\right) \longmapsto$$

$$\longmapsto \sum_{\substack{D<0, r\in\mathbb{Z}^n \\ D\equiv \frac{1}{2}qS^{-1}[r] \mod q}} \left(\sum_{d|N} d^{k-(n+1)} \sum_{\substack{s \mod dS\mathbb{Z}^n \\ D\equiv \frac{1}{2}qS^{-1}[s] \mod qd}} c_\phi\left(\left(\frac{N}{d}\right)^2 D, \frac{N}{d}s\right)\right) \times$$
$$\times\, e\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{q}\tau + r^t z\right).$$

Remark 6.1.3. The (D,r) coefficient of $V_N^*\phi$ is independent of r.

Proof. We start by writing

$$V_N\phi = N^{k/2-1} \sum_{A\in \mathrm{SL}_2(\mathbb{Z})\backslash M_2(\mathbb{Z})_N} \phi_{\sqrt{N}}|_{k,(\mathbb{Z}^n,N\sigma)}\left(\frac{1}{\sqrt{N}}A\right),$$

where we define $\phi_{\sqrt{N}}(\tau,z) := \phi\left(\tau, \sqrt{N}z\right)$.

We remind here that if $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, we use $(\det A)^{-1/2}A$ in the $|_k$ −action defined in Definition 4.3.4.

We can then follow the proof in [KS89, pages 554-556] line by line and arrive at the result that the adjoint $V_N^* : S_k(\mathbb{Z}^n, N\sigma) \longrightarrow S_k(\mathbb{Z}^n,\sigma)$ is given by

$$\phi \longmapsto N^{k/2-2n-1} \sum_{X \pmod{N\mathbb{Z}^{2n}}} \sum_{A\in \mathrm{SL}_2(\mathbb{Z})\backslash M_2(\mathbb{Z})_N} \phi_{\sqrt{N}^{-1}}|_{k,(\mathbb{Z}^n,\sigma)}\left(\frac{1}{\sqrt{N}}A\right)|_{k,(\mathbb{Z}^n,\sigma)}X.$$

Let us now compute the action on the Fourier coefficients. We choose a set of representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash M_2(\mathbb{Z})_N$ of the form $\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$ with $ad = N$ and $0 \leq b < d$. We then obtain that the above expression can be written as

$$N^{k/2-2n-1} \sum_{\lambda,\mu\in\mathbb{Z}^n/N\mathbb{Z}^n} \sum_{\substack{ad=N \\ b\in\mathbb{Z}/d\mathbb{Z}}} \left(\frac{d}{\sqrt{N}}\right)^{-k} \phi\left(\frac{a\tau+b}{d}, \frac{z+\lambda\tau+\mu}{d}\right) e^{\pi i\tau S[\lambda]+2\pi i\lambda^t Sz}.$$

By then substituting the Fourier expansion

$$\phi(\tau,z)=\sum_{\substack{D\in\mathbb{Z}_{<0},r\in\mathbb{Z}^n\\ D\equiv\frac{1}{2}qS^{-1}[r]\mod qN}} c_\phi\left(D,r\right)e\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{qN}\tau+r^tz\right),$$

for ϕ, the above can be written as

$$N^{k/2-2n-1}\sum_{\lambda,\mu\in\mathbb{Z}^n/N\mathbb{Z}^n}\sum_{\substack{ad=N\\ b\in\mathbb{Z}/d\mathbb{Z}}}\left(\frac{d}{\sqrt{N}}\right)^{-k}\sum_{D,r}c_\phi(D,r)\times \\ \times e\left(\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{qN}\cdot\frac{a}{d}+\frac{S[\lambda]}{2}+\frac{r^t\lambda}{d}\right)\tau+\left(\frac{r^tz}{d}+\lambda^tSz\right)+ \right. \\ \left. +\frac{\frac{1}{2}qS^{-1}[r]-D}{qN}\cdot\frac{b}{d}+\frac{r^t\mu}{d}\right).$$

Now, the term

$$\sum_{\substack{b\in\mathbb{Z}/d\mathbb{Z}\\ \mu\in\mathbb{Z}^n/N\mathbb{Z}^n}} e\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{qN}\cdot\frac{b}{d}+\frac{r^t\mu}{d}\right)$$

is zero unless

$$d\mid\frac{\frac{1}{2}qS^{-1}[r]-D}{qN}\text{ and }d\mid r,$$

meaning it divides all of its components. In that case, the sum equals dN^n. The conditions then imply that we can replace $r\longmapsto dr$ and $D\longmapsto Dd^2$. The last one follows from the fact that $N=ad$ and so $d\mid N$ as well. Hence, we obtain that the expression equals

$$N^{k-n-1}\sum_{\lambda\in\mathbb{Z}^n/N\mathbb{Z}^n}\sum_{d\mid N}d^{1-k}\sum_{\substack{D<0,r\in\mathbb{Z}^n\\ D\equiv\frac{1}{2}qS^{-1}[r]\ (\mathrm{mod}\ \frac{qN}{d})}}c_\phi(d^2D,dr)\times \\ \times e\left(\left(\frac{\frac{1}{2}qS^{-1}[r]+\frac{1}{2}qS[\lambda]+qr^t\lambda-D}{q}\right)\tau+r^tz+\lambda^tSz\right)=$$

$$=N^{k-n-1}\sum_{d\mid N}d^{1-k}\sum_{\lambda\in\mathbb{Z}^n/N\mathbb{Z}^n}\sum_{\substack{D<0,r\in\mathbb{Z}^n\\ D\equiv\frac{1}{2}qS^{-1}[r-S\lambda]\ (\mathrm{mod}\ \frac{qN}{d})}}c_\phi(d^2D,d(r-S\lambda))\times \\ \times e\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{q}\tau+r^tz\right),\quad(6.1.2)$$

after setting $r\longmapsto r-S\lambda$. Now, as in [KS89, p. 557], we set

$$\lambda\equiv t+\frac{N}{d}t'\ (\mathrm{mod}\ N\mathbb{Z}^n),\qquad(6.1.3)$$

with $t \pmod{\frac{N}{d}\mathbb{Z}^n}$ and $t' \pmod{d\mathbb{Z}^n}$. We then have

$$S^{-1}(d(r-S\lambda)) \equiv S^{-1}(d(r-St)) \mod N\mathbb{Z}^n,$$

and

$$D \equiv \frac{1}{2}qS^{-1}[r-St] \mod \frac{qN}{d}.$$

The first property can be seen easily and the second follows from the fact that we already have $D \equiv \frac{1}{2}qS^{-1}[r-S\lambda] \mod \frac{qN}{d}$ and also we can check

$$\frac{1}{2}qS^{-1}[r-St] - \frac{1}{2}qS^{-1}[r-S\lambda] = \frac{1}{2}q[2r^t(\lambda - t) + S[t] - S[\lambda]].$$

So, it suffices to show that

$$\frac{1}{2}S[t] \equiv \frac{1}{2}S[\lambda] \mod \frac{N}{d}.$$

But this follows after we write $\lambda = t + N\alpha/d$ with $\alpha \in \mathbb{Z}^n$, from (6.1.3).

Hence, because of Remark 6.1.1, expression (6.1.2) becomes (after replacing d with N/d)

$$\sum_{d|N} d^{k-(n+1)} \sum_{t \in \mathbb{Z}^n/d\mathbb{Z}^n} \sum_{\substack{D<0, r\in\mathbb{Z}^n \\ D \equiv \frac{1}{2}qS^{-1}[r-St] \mod qd}} c_\phi\left(\left(\frac{N}{d}\right)^2 D, \frac{N}{d}(r-St)\right) \times$$
$$\times e\left(\frac{\frac{1}{2}qS^{-1}[r]-D}{q}\tau + r^t z\right).$$

What is left to prove now (after fixing D, r with the appropriate conditions) is

$$\sum_{d|N} d^{k-(n+1)} \sum_{\substack{t\in\mathbb{Z}^n/d\mathbb{Z}^n \\ D \equiv \frac{1}{2}qS^{-1}[r-St] \mod qd}} c_\phi\left(\left(\frac{N}{d}\right)^2 D, \frac{N}{d}(r-St)\right) =$$

$$= \sum_{d|N} d^{k-(n+1)} \sum_{\substack{t \mod dS\mathbb{Z}^n \\ D \equiv \frac{1}{2}qS^{-1}[t] \mod qd}} c_\phi\left(\left(\frac{N}{d}\right)^2 D, \frac{N}{d}t\right).$$

This follows by setting $u = r - St$ and then observing that $D \equiv \frac{1}{2}qS^{-1}[u] \mod qd$,

$$r - St \equiv r - St' \mod dS\mathbb{Z}^n \iff t \equiv t' \mod d\mathbb{Z}^n$$

and by using the fact that $c_\phi(D,s) = c_\phi(D,s')$ if $s \equiv s' \mod NS\mathbb{Z}^n$ (see Remark 6.1.1). So we can consider the entries $r - St \pmod{NS\mathbb{Z}^n}$ and all of these are different $\pmod{dS\mathbb{Z}^n}$. □

6.2 Poincaré Series

In this Section, we define a very special class of Fourier-Jacobi forms, called Poincaré series, which reproduce the Fourier coefficients of Jacobi forms under the Petersson scalar product. What is more, they generate the space of cusp forms and, in turn, the Maass space $S_k^*(\widetilde{\Gamma}_S)$, as we will show in the next Section.

Definition 6.2.1. We define the **support** of the lattice $L := \mathbb{Z}^n$ with respect to the bilinear form $\sigma(x, y) = x^t S y$ for $x, y \in V$ to be

$$\operatorname{supp}(L, \sigma) := \left\{ (D, r) \mid D \in \mathbb{Q}_{\leq 0}, r \in L^*, D \equiv \frac{1}{2} S[r] \pmod{\mathbb{Z}} \right\}.$$

Now, if we write $r \longmapsto S^{-1} r$ with $r \in L$, we get

$$qD \equiv \frac{1}{2} q S^{-1}[r] \pmod{q\mathbb{Z}},$$

which then implies $D \in \dfrac{1}{q}\mathbb{Z}$. Hence, by writing $D \longmapsto D/q$, we get that equivalently

$$\operatorname{supp}(L, \sigma) = \left\{ (D/q, S^{-1} r) \mid D \in \mathbb{Z}_{\leq 0}, r \in L, D \equiv \frac{1}{2} q S^{-1}[r] \pmod{q} \right\}.$$

Let then

$$\widetilde{\operatorname{supp}}(L, \sigma) := \{ (D, r) \in \mathbb{Z}_{\leq 0} \times L \mid (D/q, S^{-1} r) \in \operatorname{supp}(L, \sigma) \},$$

and in the following, this is the set we will use.

Definition 6.2.2. Let $(D, r) \in \widetilde{\operatorname{supp}}(L, \sigma)$. We then define the following complex valued function on $\mathbb{H} \times (L \otimes \mathbb{C})$:

$$g_{D,r}(\tau, z) := e\left(\frac{\frac{1}{2} q S^{-1}[r] - D}{q} \tau + r^t z \right),$$

where q is the level of L (see Definition 4.1.4).

Definition 6.2.3. Let $(D, r) \in \widetilde{\operatorname{supp}}(L, \sigma)$ and set

$$J_\infty^{(L,\sigma)} := \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \mid n \in \mathbb{Z}, \mu \in L \right\}.$$

The **Poincaré series** of weight k for the lattice (L, σ) is defined by

$$P_{k,D,r}(\tau, z) := \sum_{\gamma \in J_\infty^{(L,\sigma)} \backslash J^{(L,\sigma)}} g_{D,r} \mid_{k,(L,\sigma)} \gamma(\tau, z).$$

If $k > n/2+2$, then $P_{k,D,r}$ is absolutely and uniformly convergent on compact subsets of $\mathbb{H} \times (L \otimes \mathbb{C})$ and defines an element of $S_k(L, \sigma)$ (see [Moc19, Theorem 2.3, (i)]). Moreover, from the same Theorem, it only depends on $S^{-1}r \pmod{L}$.

We now have the following very important property, which again can be found in [Moc19, Theorem 2.3, (i)].

Proposition 6.2.4. *Let $(D, r) \in \widetilde{\operatorname{supp}}(L, \sigma)$. Then for any $f \in S_k(L, \sigma)$ with*

$$f(\tau, z) = \sum_{\substack{D' \in \mathbb{Z}_{<0}, r' \in L \\ \frac{1}{2}qS^{-1}[r'] \equiv D' \pmod{q}}} c_f(D', r')\, e\left(\frac{\frac{1}{2}qS^{-1}[r'] - D'}{q}\tau + (r')^t z\right),$$

we have

$$\langle f, P_{k,D,r} \rangle = \lambda_{k,D} c_f(D, r),$$

for some constant $\lambda_{k,D} \in \mathbb{C}$ depending on k and D.

6.3 Relation of the Dirichlet series to the Fourier coefficients

For $k > n/2 + 2$ an **even** integer, let $F \in S_k(\Gamma_S)$ and write ϕ_N for its Fourier-Jacobi coefficients. For $(D, r) \in \widetilde{\operatorname{supp}}(L, \sigma)$, let

$$\mathcal{P}_{k,D,r}(\tau', z, \tau) := \sum_{N \geq 1} (V_N P_{k,D,r})(\tau, z) e(N\tau'). \tag{6.3.1}$$

By [Sug95, Corollary 6.7], we have that $\mathcal{P}_{k,D,r} \in S_k^*(\widetilde{\Gamma}_S)$ (see Definition 4.4.1). Here, by abusing notation, we write $P_{k,D,r}$ to actually denote the Poincaré series $P_{k,D,r}/\lambda_{k,D}$, with the quantities defined in Proposition 6.2.4.

Remark 6.3.1. Even though F and $\mathcal{P}_{k,D,r}$ are taken invariant with respect to different modular groups, we note here that the proof of Lemma 4.5.1 is still valid, because $F \in S_k(\Gamma_S) \subset S_k(\widetilde{\Gamma}_S)$.

Remark 6.3.2. A corollary of [Moc19, Theorem 2.3 (i)] is that $S_k(L, \sigma)$ is generated by $\left\{P_{k,D,r} \mid (D, r) \in \widetilde{\operatorname{supp}}(L, \sigma),\ S^{-1}r \pmod{L}\right\}$ (cf. [Moc19, Corollary 2.4]). This observation, together with the fact that the map

$$\phi(\tau, z) \longmapsto \sum_{N=1}^{\infty} (V_N \phi)(\tau, z) e(N\tau')$$

is an isomorphism between $S_k(\Lambda, \sigma)$ and $S_k^*(\widetilde{\Gamma}_S)$ (see [Sug95, Corollary 6.7]) gives us that the Maass space $S_k^*(\widetilde{\Gamma}_S)$ is generated by $\left\{\mathcal{P}_{k,D,r} \mid (D, r) \in \widetilde{\operatorname{supp}}(L, \sigma),\ S^{-1}r \pmod{L}\right\}$. It is therefore enough to consider G to be a Poincaré series in (4.5.1).

Remark 6.3.3. In the case when the lattice is Euclidean ([Sch22, Definition 1.6.13]) and $r \in L$, we have that actually $\mathcal{P}_{k,D,r} \in S_k^*(\Gamma_S)$. This follows from the fact that the Poincaré series depends only on $r \pmod{L}$ (see [Moc19, Theorem 2.3]) and therefore we can take $r = \mathbf{0}$. Then $P_{k,D,r}$ is invariant under $z \longmapsto Uz$ for $U \in \mathrm{SO}(L)$ by comparison with [Moc19, (2.8)]. By [Sch22, Theorem 1.9.2], we then have that $\mathcal{P}_{k,D,r} \in S_k^*(\Gamma_S^\bullet)$ (see (4.4.2)). Because S is Euclidean, $\Gamma_S^\bullet = \Gamma_S$, by [Kri16, Theorem 2].

Now, for the Nth term of the Dirichlet series (see (4.5.1)), we can write

$$\langle \phi_N, V_N P_{k,D,r} \rangle = \langle V_N^* \phi_N, P_{k,D,r} \rangle.$$

We will relate this with the Fourier coefficients of F.

Proposition 6.3.4. *With the notation as above and $N \geq 1$, we have:*

$$\langle V_N^* \phi_N, P_{k,D,r} \rangle = \sum_{d|N} d^{k-(n+1)} \sum_{\substack{s \pmod{dS\mathbb{Z}^n} \\ D \equiv \frac{1}{2}qS^{-1}[s] \pmod{qd}}} A\left(\frac{N}{d}\left(\frac{\frac{1}{2}qS^{-1}[s] - D}{qd}, S^{-1}s, d\right)\right).$$

Proof. F admits a Fourier expansion of the form (see equation (4.2.8))

$$F(Z) = \sum_{\tilde{r} \in L_0^*} A(\tilde{r}) e(\tilde{r}^t S_0 Z) = \sum_{N=1}^{\infty} \phi_N(\tau, z) e(N\tau'),$$

where $Z = (\tau', z, \tau) \in \mathcal{H}_S$ and $\tilde{r} = (m, r, N)$ with $r \in L^*$. We can then write

$$\phi_N(\tau, z) = \sum_{m \in \mathbb{Z}, r \in L^*} A(m, r, N) e(m\tau - r^t S z) = \sum_{m \in \mathbb{Z}, r \in \mathbb{Z}^n} A(m, S^{-1}r, N) e(m\tau - r^t z).$$

But now $\begin{pmatrix} -1_2 & & \\ & 1_n & \\ & & -1_2 \end{pmatrix} \in \Gamma_S$ and therefore if $Z = (\tau', z, \tau) \in \mathcal{H}_S$, we have

$$F\left((\tau', -z, \tau)\right) = (-1)^k F((\tau', z, \tau)) = F((\tau', z, \tau)),$$

which then implies $A(m, r, N) = A(m, -r, N)$ for all $m, N \in \mathbb{Z},\ r \in L^*$. Therefore, after setting $r \longmapsto -r$, we can re-write the above as

$$\phi_N(\tau, z) = \sum_{m \in \mathbb{Z}, r \in \mathbb{Z}^n} A(m, S^{-1}r, N) e(m\tau + r^t z). \tag{6.3.2}$$

Now, a priori, we can write

$$\phi_N(\tau, z) = \sum_{D,r} c_{\phi_N}(D, r)\, e\left(\frac{\frac{1}{2}qS^{-1}[r] - D}{qN}\tau + r^t z\right), \tag{6.3.3}$$

with (D, r) as in equation (6.1.1). Hence, from Proposition 6.1.2 and Proposition 6.2.4, we have

$$\begin{aligned}\langle V_N^*\phi_N, P_{k,D,r}\rangle &= \sum_{d|N} d^{k-(n+1)} \sum_{\substack{s \pmod{dS\mathbb{Z}^n} \\ D\equiv \frac{1}{2}qS^{-1}[s] \pmod{qd}}} c_{\phi_N}\left(\left(\frac{N}{d}\right)^2 D, \frac{N}{d}s\right) \\ &= \sum_{d|N} d^{k-(n+1)} \sum_{\substack{s \pmod{dS\mathbb{Z}^n} \\ D\equiv \frac{1}{2}qS^{-1}[s] \pmod{qd}}} A\left(\frac{N}{d}\left(\frac{\frac{1}{2}qS^{-1}[s]-D}{qd}, S^{-1}s, d\right)\right),\end{aligned}$$

because after setting $r \longmapsto Ns/d, D \longmapsto N^2D/d^2$ in (6.3.3), we obtain from (6.3.2)

$$c_{\phi_N}\left(\frac{N^2}{d^2}D, \frac{N}{d}s\right) = A\left(\frac{N}{d}\left(\frac{\frac{1}{2}qS^{-1}[s]-D}{qd}, S^{-1}s, d\right)\right). \qquad \square$$

6.4 Relation to the class number

In this Section, our goal is to bring $\mathcal{D}_{F,\mathcal{P}_{k,D,r}}(s)$ into a form similar to the one in [KS89, page 553]. We will need to exclude some terms. We first need some definitions of the adelized groups and of the genus and class of a lattice.

Let V denote any finite-dimensional vector space over $\mathbb{Q}$ of dimension $n \geq 1$. For each prime p (including infinity), we define $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$. For a $\mathbb{Z}$-lattice L in V (see Definition 4.1.1), we denote by $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This coincides with the $\mathbb{Z}_p$-linear span of L in V_p.

Proposition 6.4.1. *Let L be a fixed $\mathbb{Z}$-lattice in V. Then, the following are true:*

- *If M is another $\mathbb{Z}$-lattice, then $L_p = M_p$ for almost all p. Moreover, $L \subset M$ if $L_p \subset M_p$ for all p and $L = M$ if $L_p = M_p$ for all p.*
- *For all $p < \infty$, let $N_p \in V_p$ denote a $\mathbb{Z}_p$-lattice such that $N_p = L_p$ for almost all $p < \infty$. Then, there is a $\mathbb{Z}$-lattice M in V such that $M_p = N_p$ for all $p < \infty$.*

Proof. See [Shi04, Lemma 9.2]. $\square$

Definition 6.4.2. We define the **adelizations** V and $\mathrm{GL}(V)$, as follows:

$$V_{\mathbb{A}} = \left\{v \in \prod_{p\leq\infty} V_p \mid v_p \in L_p \text{ for almost all } p < \infty\right\},$$

$$\mathrm{GL}(V)_{\mathbb{A}} = \left\{\alpha \in \prod_{p\leq\infty} \mathrm{GL}(V_p) \mid \alpha_p L_p = L_p \text{ for almost all } p < \infty\right\}.$$

Here L is an arbitrary $\mathbb{Z}$-lattice, and the above definitions do not depend on the choice of it, by virtue of Proposition 6.4.1.

Remark 6.4.3. When we identify V with $\mathbb{Q}^n$, we identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(\mathbb{Q})$. Then, we identify $\mathrm{GL}_n(\mathbb{Q})_{\mathbb{A}}$ with $\mathrm{GL}_n(\mathbb{Q}_{\mathbb{A}})$, where $\mathbb{Q}_{\mathbb{A}}$ is the usual ring of adeles.

Remark 6.4.4. Given $x \in \mathrm{GL}(V)_{\mathbb{A}}$, we have that $x_pL_p = L_p$ for almost all $p < \infty$. By Proposition 6.4.1, there is a $\mathbb{Z}$-lattice M such that $M_p = x_pL_p$ for all p. We denote this lattice by xL. Hence, xL is the lattice which has the property $(xL)_p = x_pL_p$ for all $p < \infty$.

Definition 6.4.5. Let $G \subset \mathrm{GL}_n(\mathbb{Q})$ any algebraic linear group. For any field K containing $\mathbb{Q}$, we denote by G_K the group of K-rational points in G. Now, for any prime number p, we abbreviate by G_p the group $G_{\mathbb{Q}_p}$. Let also G_∞ denote $G_{\mathbb{R}}$. We then define

$$G_{\mathbb{A}} := \left\{ x \in \prod_{p \leq \infty} G_p \mid x_pL_p = L_p \text{ for almost all } p \right\}.$$

Moreover, we use the notation $G_{\mathbb{A},f}$ to denote the finite part of $G_{\mathbb{A}}$.

Definition 6.4.6. Let L denote a $\mathbb{Z}$-lattice in V and $G \subset \mathrm{GL}_n(\mathbb{Q})$ any algebraic group. Then, for any $x \in G$, xL is also a $\mathbb{Z}$-lattice in V. We define the **class** of L to be $\{xL \mid x \in G\}$. Similarly, for any $x \in G_{\mathbb{A}}$, xL is again a $\mathbb{Z}$-lattice in V. The set $\{xL \mid x \in G_{\mathbb{A}}\}$ is called the **genus** of L.

Now, the genus of L can be decomposed into a disjoint union of classes. If $C := \{x \in G_{\mathbb{A}} \mid xL = L\}$, the map $xC \longmapsto xL$ gives a bijection between $G_{\mathbb{A}}/C$ and the genus of L, so gives a bijection between $G\backslash G_{\mathbb{A}}/C$ and the set of classes contained in the genus of L. In general, if U is an open subgroup of $G_{\mathbb{A}}$, we call the number $\#(G\backslash G_{\mathbb{A}}/U)$ the **class number** of G with respect to U.

Now, with all the main definitions out of the way, we want to deal with the vectors appearing in Proposition 6.3.4. We use the notation of Chapter 4. For a $\mathbb{Z}$-lattice Λ in V_0, $\mathfrak{b}$ a fractional ideal of $\mathbb{Q}$ and $q \in \mathbb{Q}^\times$, we define

$$\Lambda[q, \mathfrak{b}] := \{x \in V_0 \mid \phi_0[x] = q \text{ and } \phi_0(x, \Lambda) = \mathfrak{b}\}.$$

We then have the following Lemma.

Lemma 6.4.7. *Let $d \geq 1$, $D \in \mathbb{Z}_{<0}$ and define the set*

$$\Xi_d := \left\{ \xi = \left(\frac{\frac{1}{2}qS^{-1}[s] - D}{qd}, S^{-1}s, d \right)^t \mid s \pmod{dS\mathbb{Z}^n},\ D \equiv \frac{1}{2}qS^{-1}[s] \pmod{qd} \right\}.$$

We then have

$$\Xi_d \subset L_0 \left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z} \right]$$

for all $d \geq 1$ coprime to D.

Proof. Firstly, for any vector $\xi \in \Xi_d$, we can directly compute $\phi_0[\xi] = -D/q$. It now remains to show that the $\mathbb{Z}$-ideal

$$\{\xi^t S_0 x \mid x \in \mathbb{Z}^{n+2}\}$$

equals $\mathbb{Z}$ (because the bilinear form is $\phi_0 = S_0/2$). But for any basis vector e_i of the lattice $\mathbb{Z}^{n+2}$, we have

$$\xi^t S_0 e_i = \begin{cases} d & \text{if } i = 1 \\ -s_{i-1} & \text{if } 2 \leq i \leq n+1 \\ \dfrac{\frac{1}{2}qS^{-1}[s] - D}{qd} & \text{if } i = n+2 \end{cases},$$

so the above ideal is contained in $\mathbb{Z}$ by the conditions that define the set Ξ_d. Now, if it were equal to $k\mathbb{Z}$ for some $k \geq 1$, then we would have

$$k \mid d, \;\; k \mid s, \;\; \text{and } k \mid \frac{\frac{1}{2}qS^{-1}[s] - D}{qd}.$$

This would then imply $k \mid D$, and so $k = 1$, as $d, \; D$ are assumed to be coprime. □

We now have the following very important Lemma.

Lemma 6.4.8. *The class number of $G^*_{\mathbb{Q}}$ (see (4.2.1)), defined as $\#(G^*_{\mathbb{Q}}\backslash G^*_{\mathbb{A}}/C)$, where*

$$C := \{x \in G^*_{\mathbb{A}} \mid xL_0 = L_0\}, \tag{6.4.1}$$

is 1.

Proof. This is shown in [Shi06b, Remark 2.4, (5)], which is an improvement of [Shi04, Theorem 9.26], as the technical assumptions are weakened. □

Fix now an element $\xi \in L_0\left[-\dfrac{D}{q}, \dfrac{1}{2}\mathbb{Z}\right]$ and consider the algebraic subgroup of $G^*_{\mathbb{Q}}$

$$H(\xi)_{\mathbb{Q}} = \left\{g \in G^*_{\mathbb{Q}} \mid g\xi = \xi\right\}.$$

We note that $H(\xi)_{\mathbb{Q}} = \mathrm{SO}^{\psi}(W)$, where $W := \{x \in V_0 \mid \phi_0(x, \xi) = 0\}$ and $\psi := \phi_0 \mid_W$.

We now have the following Proposition, which is a special case of [Shi06b, Theorem 2.2].

Proposition 6.4.9. *There is a bijection*

$$L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]/\Gamma(L_0) \longleftrightarrow H(\xi)_{\mathbb{Q}}\backslash H(\xi)_{\mathbb{A}}/(H(\xi)_{\mathbb{A}} \cap C),$$

which is given as follows:

If $k \in L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]$, *then by Witt's Theorem ([Shi04, Lemma 1.5 (ii)]) there is some* $\alpha \in G^*_{\mathbb{Q}}$ *such that*

$$k = \alpha\xi.$$

We then assign the $H(\xi)_{\mathbb{Q}}$*-class of* $\alpha^{-1}L_0$ *to* k*. In particular, this then gives*

$$\#\left(L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]/\Gamma(L_0)\right) = \#(H(\xi)_{\mathbb{Q}}\backslash H(\xi)_{\mathbb{A}}/(H(\xi)_{\mathbb{A}} \cap C)).$$

Proof. First of all, $\Gamma(L_0)$ acts on $L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]$. This can be seen because if $\gamma \in \Gamma(L_0),\ x \in L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]$, then $\phi_0[\gamma x] = \phi_0[x] = -D/q$ and

$$\phi_0(\gamma x, L_0) = \phi_0(\gamma x, \gamma L_0) = \phi_0(x, L_0) = \frac{1}{2}\mathbb{Z}.$$

The remaining assertions follow from [Shi06b, Theorem 2.2, (iv)], because $G^*_{\mathbb{Q}}$ has class number one (Lemma 6.4.8) and $G^*_{\mathbb{Q}} \cap C = \Gamma(L_0)$. $\square$

Let us now write

$$L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right] = \bigsqcup_{i=1}^{h} \Gamma(L_0)\xi_i. \tag{6.4.2}$$

with some $\xi_i \in L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]$. This in particular implies that $\xi_i \in L_0^*$ for all i.

We now assert we can take

$$\xi_i \in \mathcal{P}_S = \{y' = (y_1, y, y_2) \in \mathbb{R}^{n+2} \mid y_1 > 0, \phi_0[y'] > 0\}$$

of Section 4.1, for all $i = 1, \cdots, h$. The second condition is clear, as $S_0[\xi_i] = -2D/q > 0$, because we take $D < 0$. For the first one, we can always mutliply with $\mathrm{diag}(-1, 1_n, -1) \in \Gamma(L_0)$ and the assertion follows.

Now, if $\xi \in \Xi_d$ of Lemma 6.4.7, we can write $\xi = \gamma\xi_j$ for some $1 \le j \le h$ and $\gamma \in \Gamma(L_0)$. But, $\xi \in \mathcal{P}_S$ as well, so we must have $\gamma \in \Gamma(L_0) \cap G^{*,0}_{\mathbb{R}} = \Gamma^+(L_0)$. Indeed, if $\gamma \in G^*_{\mathbb{R}}\backslash G^{*,0}_{\mathbb{R}}$, then $\tilde{\gamma} := \mathrm{diag}(1, \gamma, 1) \in G_{\mathbb{R}}\backslash G^0_{\mathbb{R}}$. But then $\tilde{\gamma}\langle i\xi_j\rangle = i\xi$ and $i\xi,\ i\xi_j \in \mathcal{H}_S$. This is a contradiction, because $\tilde{\gamma}$ sends $\mathcal{H}_S$ to $-\mathcal{H}_S := \{z = x - iy \in V_0 \otimes_{\mathbb{R}} \mathbb{C} \mid y \in \mathcal{P}_S\}$ (see [Sch22, p. 18]).

Therefore, from [Sch22, p. 26], we have $A(\xi) = A(\gamma\xi_i) = A(\xi_i)$. Define now

$$n(\xi_i; d) := \#\left\{s \in \mathbb{Z}^n/dS\mathbb{Z}^n \mid D \equiv \frac{1}{2}qS^{-1}[s] \pmod{qd} \text{ and}\right.$$

$$\left(\frac{\frac{1}{2}qS^{-1}[s]-D}{qd}, S^{-1}s, d\right)^t = \gamma\xi_i, \gamma \in \Gamma^+(L_0)\Bigg\}.$$

From the above considerations, Proposition 6.3.4 and Lemma 6.4.7, we may write, for $(N, D) = 1$

$$\langle V_N^*\phi_N, P_{k,D,r}\rangle = \sum_{i=1}^{h}\sum_{0<d|N} d^{k-(n+1)}n(\xi_i;d)A\left(\frac{N}{d}\xi_i\right).$$

In particular, we arrive at the following Proposition.

Proposition 6.4.10. *Let* $(D,r) \in \widetilde{\operatorname{supp}}(L,\sigma)$. *Let* $\mathcal{P}$ *be a finite set of primes, which includes the prime factors of* D. *Let*

$$\mathcal{D}_{F,\mathcal{P}_{k,D,r},\mathcal{P}}(s) := \sum_{\substack{N=1\\(N,p)=1\forall p\in\mathcal{P}}}^{\infty} \langle V_N^*\phi_N, P_{k,D,r}\rangle N^{-s},$$

which converges absolutely for $\operatorname{Re}(s) > k+1$ *by comparison with* $D_{F,\mathcal{P}_{k,D,r}}(s)$ *(see Lemma 4.5.1). Let also*

$$\zeta_{\xi_i,\mathcal{P}}(s) := \sum_{\substack{N=1\\(N,p)=1\forall p\in\mathcal{P}}}^{\infty} n(\xi_i;N)N^{-s}.$$

We then have that

$$\mathcal{D}_{F,\mathcal{P}_{k,D,r},\mathcal{P}}(s) = \sum_{i=1}^{h}\zeta_{\xi_i,\mathcal{P}}(s-k+n+1)D_{F,\xi_i,\mathcal{P}}(s), \tag{6.4.3}$$

where $D_{F,\xi_i,\mathcal{P}}(s) := \sum_{\substack{N=1\\(N,p)=1\forall p\in\mathcal{P}}}^{\infty} A(N\xi_i)N^{-s}.$

6.5 Relation to Sugano's Theorem

In order to now obtain an Euler product, we will make use of the main Theorem of Sugano in his paper [Sug85]. We first need some setup.

For each prime number $p < \infty$, we define $K_p := G_p \cap \mathrm{SL}_{n+4}(\mathbb{Z}_p) = G(\mathbb{Z}_p)$ and let

$$K_f := \prod_{p<\infty} K_p.$$

We remind ourselves here that G^0_∞ acts transitively on $\mathcal{H}_S$ (cf. Section 4.1). Let $\mathcal{Z}_0$ denote any point of $\mathcal{H}_S$ with real part 0 and denote by K_∞ its stabiliser in G^0_∞. Then, we have that $G^0_\infty/K_\infty \cong \mathcal{H}_S$.

Definition 6.5.1. Let $k \geq 0$. A function $\boldsymbol{F} : G_{\mathbb{A}} \longrightarrow \mathbb{C}$ is called a holomorphic cusp form of weight k with respect to K_f if the following conditions hold:

1. $\boldsymbol{F}(\gamma g u) = \boldsymbol{F}(g)\ \forall \gamma \in G_{\mathbb{Q}}, u \in K_f$.
2. For any $g = g_\infty g_f$ with $g_\infty \in G^0_\infty$ and $g_f \in G_{\mathbb{A},f}$, $\boldsymbol{F}(g_\infty g_f) j(g_\infty, \mathcal{Z}_0)^k$ depends only on g_f and $\mathcal{Z} = g_\infty\langle \mathcal{Z}_0\rangle$ and is holomorphic on $\mathcal{H}_S$ as a function of $\mathcal{Z}$.
3. $\boldsymbol{F}$ is bounded on $G_{\mathbb{A}}$.

Denote the above space by $\mathfrak{S}_k(K_f)$.

For each $g_f \in G_{\mathbb{A},f}$ and $\mathcal{Z} \in \mathcal{H}_S$, we define

$$\boldsymbol{F}(g_f; \mathcal{Z}) := \boldsymbol{F}(g_\infty g_f) j(g_\infty, \mathcal{Z}_0)^k, \tag{6.5.1}$$

where $g_\infty \in G^0_\infty$ is chosen so that $\mathcal{Z} = g_\infty\langle \mathcal{Z}_0\rangle$. Let now

$$\Gamma(g_f) = G_{\mathbb{Q}} \cap (G^0_\infty \times g_f K_f g_f^{-1}),$$

which is a discrete subgroup of G^0_∞. We then have

$$\boldsymbol{F}(g_f; \gamma\langle \mathcal{Z}\rangle) = j(\gamma, \mathcal{Z})^k \boldsymbol{F}(g_f; \mathcal{Z})$$

for all $\gamma \in \Gamma(g_f)$ and $\mathcal{Z} \in \mathcal{H}_S$. Now, if $X \in V_0$, define the element $\gamma_X \in G$ by

$$\gamma_X := \begin{pmatrix} 1 & -X^t S_0 & -\frac{1}{2} S_0[X] \\ 0 & 1_{n+2} & X \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, the holomorphic function $\boldsymbol{F}(g_f; \mathcal{Z})$ is invariant under $\mathcal{Z} \longmapsto \mathcal{Z} + X$ for X in the lattice

$$L(g_f) := \{X \in V_0 \mid \gamma_X \in \Gamma(g_f)\}.$$

Hence, every such function then admits a Fourier expansion of the form

$$\boldsymbol{F}(g_f; \mathcal{Z}) = \sum_{\substack{r \in \hat{L}(g_f) \\ ir \in \mathcal{H}_S}} a(g_f; r) e[2\phi_0(r, Z)],$$

where

$$\hat{L}(g_f) := \{X \in V_0 \mid 2\phi_0(X, Y) \in \mathbb{Z} \text{ for all } Y \in L(g_f)\}$$

is the dual lattice of $L(g_f)$.

Finally, let us introduce adelic Fourier coefficients. Let $\chi = \prod_{p \leq \infty} \chi_p$ be a character of $\mathbb{Q}_{\mathbb{A}}$ such that $\chi_{|\mathbb{Q}} = 1$ and $\chi_\infty(x) = e(x)$ for all $x \in \mathbb{R}$. For $\eta \in V_0$ and $g \in G_{\mathbb{A}}$, we define

$$\boldsymbol{F}_\chi(g;\eta) := \int_{V_0\backslash V_{\mathbb{A}}} \boldsymbol{F}(\gamma_\chi g)\chi\left(-2\phi_0(\eta, X)\right) \mathrm{d}X.$$

Now, for $g_\infty \in G_\infty^0$ and $g_f \in G_{\mathbb{A},f}$ we have

$$\boldsymbol{F}_\chi(g_\infty g_f;\eta) = a(g_f;\eta)j(g_\infty;\mathcal{Z}_0)^{-k}e\left(2\phi_0(\eta, g_\infty\langle\mathcal{Z}_0\rangle)\right). \tag{6.5.2}$$

We then have from the above definitions (see [Sug85, (1.14)])

(i) $\boldsymbol{F}_\chi(\gamma_X gu;\eta) = \chi(2\phi_0(\eta, X))\boldsymbol{F}_\chi(g;\eta)$ for all $X \in V_{\mathbb{A}}, u \in K_f$.

(ii) $\boldsymbol{F}_\chi\left(\begin{pmatrix}\alpha & & \\ & \beta & \\ & & \alpha^{-1}\end{pmatrix} g;\eta\right) = \boldsymbol{F}_\chi(g;\beta^{-1}\eta\alpha),$ for all $\alpha \in \mathbb{Q}^\times, \beta \in G_{\mathbb{Q}}^*$.

(iii) $\boldsymbol{F}(\gamma_X g) = \sum_{\eta\in V_0} \boldsymbol{F}_\chi(g;\eta)\chi(2\phi_0(\eta, X))$ for all $X \in V_{0,\mathbb{A}}$.

We now want to show that there is a bijection between the spaces $\mathfrak{S}(K_f)$ and $S_k(\Gamma_S)$. This is true because of the following Lemma:

Lemma 6.5.2. *We have* $G_{\mathbb{A}} = G_{\mathbb{Q}}G_\infty^0 K_f$.

Proof. From the proof of [Sug85, Lemma 1], we have $G_{\mathbb{A}} = G_{\mathbb{Q}}G_{\mathbb{A},f}^* G_\infty^0 K_f$, where we view G^* as a subgroup of G via (4.2.3). But, from Lemma 6.4.8, we have $G_{\mathbb{A}}^* = G_{\mathbb{Q}}^* G_\infty^* K_f^*$, where we now define

$$K_f^* := \prod_{p<\infty} G^*(\mathbb{Z}_p).$$

From this, we obtain $G_{\mathbb{A},f}^* = G_{\mathbb{Q}}^* K_f^*$. Therefore, because elements of K_f^* and G_∞^0 commute, we obtain

$$G_{\mathbb{A}} = G_{\mathbb{Q}}G_{\mathbb{Q}}^* G_\infty^0 K_f^* K_f = G_{\mathbb{Q}}G_\infty^0 K_f,$$

as required. □

The bijection is now given by $\boldsymbol{F} \longmapsto F(\mathcal{Z}) := \boldsymbol{F}(g_\infty)j(g_\infty, \mathcal{Z}_0)^k$, where $g_\infty \in G_\infty^0$ is chosen so that $g_\infty\langle\mathcal{Z}_0\rangle = \mathcal{Z}$.

Let now $g_f = (id, id, \cdots)$, which we denote by just id. It then follows that $F(\mathcal{Z}) = \boldsymbol{F}(id, \mathcal{Z})$, $\forall\mathcal{Z} \in \mathcal{H}_S$. In that case, we have $L(g_f) = L_0 = \mathbb{Z}^{n+2}$. Now, $a(id, r) = A(r)$ for all $r \in L_0^*$. Hence

$$a(id, \xi_i) = A(\xi_i),$$

for all i, where ξ_i's are the representatives for $L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]/\Gamma(L_0)$, as in (6.4.2).

Fix now a complete system of representatives $\{u_i\}_{i=1}^h$ for $H(\xi)_{\mathbb{Q}}\backslash H(\xi)_{\mathbb{A}}/(H(\xi)_{\mathbb{A}}\cap C)$, corresponding to the $\xi_i' s$ of (6.4.2), in the sense of Proposition 6.4.9. Assume also that $u_{i,\infty} = 1$ for all $i = 1, \cdots, h$. We then prove the following Lemma.

Lemma 6.5.3. *Let* $\xi \in L_0\left[-\frac{D}{q}, \frac{1}{2}\mathbb{Z}\right]$ *be the element we have fixed right after Lemma 6.4.8. We have*

$$a(u_i, \xi) = a(id, \xi_i),$$

for all $i = 1, \cdots, h$. *Therefore,* $a(u_i, \xi) = A(\xi_i)$ *for all* $i = 1, \cdots, h$.

Proof. By the definition of the ξ_i's and Witt's theorem ([Shi04, Lemma 1.5 (ii)]), there is $\alpha \in G^*_{\mathbb{Q}}$ such that $\xi_i = \alpha\xi$. By the correspondence given by Shimura in Proposition 6.4.9, we get $\alpha^{-1}L_0 = u_iL_0$. This then implies $\alpha u_i L_0 = L_0$, so $\alpha u_i \in C$ (we remind ourselves that $C = \{x \in G^*_{\mathbb{A}} \mid xL_0 = L_0\}$). By definition, we then get $\alpha u_i \in K_f$. Hence, we obtain

$$\boldsymbol{F}_\chi(u_i;\xi) = \boldsymbol{F}_\chi(u_i;\alpha^{-1}\xi_i) = \boldsymbol{F}_\chi\left(\mathrm{diag}(1,\alpha,1)u_i;\xi_i\right) = \boldsymbol{F}_\chi\left(\left(\mathrm{diag}(1,\alpha,1), id\right);\xi_i\right),$$

where $(\mathrm{diag}(1,\alpha,1), id)$ denotes the element of $G_{\mathbb{A}}$ with infinity part $\mathrm{diag}(1,\alpha,1)$. The second equality above follows from the second bullet point and the third equality from the first bullet point, just before Lemma 6.5.2. By the property of $\boldsymbol{F}_\chi$ in (6.5.2), we obtain

$$a(u_i;\xi)e\left(2\phi_0(\xi,\mathcal{Z}_0)\right) = a(id,\xi_i)j\left(\mathrm{diag}(1,\alpha,1),\mathcal{Z}_0\right)^{-k} e\left(2\phi_0\left(\xi_i, \mathrm{diag}(1,\alpha,1)\langle\mathcal{Z}_0\rangle\right)\right).$$

But $j\left(\mathrm{diag}(1,\alpha,1),\mathcal{Z}_0\right) = 1$ and $\mathrm{diag}(1,\alpha,1)\langle\mathcal{Z}_0\rangle = \alpha\mathcal{Z}_0$ which then gives that the right hand side above equals

$$a(id;\xi)e\left(2\phi_0(\xi_i,\alpha\mathcal{Z}_0)\right) = a(id;\xi)e\left(2\phi_0(\alpha\xi,\alpha\mathcal{Z}_0)\right) = a(id;\xi)e\left(2\phi_0(\xi,\mathcal{Z}_0)\right),$$

because $\xi_i = \alpha\xi$. This then gives the result. □

For each prime p and $g_f \in G_{\mathbb{A},f}$, let

$$M(g_f;\xi)_p := H(\xi)_p \cap g_f K_f g_f^{-1} \text{ and } M(g_f;\xi)_f := \prod_p M(g_f;\xi)_p.$$

Define then $e(\xi)_i := \#\{H(\xi)_{\mathbb{Q}} \cap M(u_i g_f;\xi)_f\}$ for $1 \le i \le h$ and $\mu(\xi) := \sum_{i=1}^h e(\xi)_i^{-1}$.

Let also $V(g_f;\xi)$ the space of functions on $H(\xi)_{\mathbb{A}}$, which are left $H(\xi)_{\mathbb{Q}}$ and right $H(\xi)_\infty M(g_f;\xi)_f$ invariant. We now have the following Theorem, which follows from [Sug85, Theorem 1].

Theorem 6.5.4. *Assume $F \in S_k(\Gamma_S)$ corresponds to $\boldsymbol{F} \in \mathfrak{S}_k(K_f)$, as above. Assume $\boldsymbol{F}$ is a simultaneous eigenfunction of the Hecke pairs $\mathcal{H}_p = (G_p, K_p)$ for all p. We also assume $A(\xi) \neq 0$. Then, there is a finite set of primes $\mathcal{P}$, such that if $f \in V(g_f;\xi)$ is a simultaneous eigenfunction of the Hecke algebras $\mathcal{H}'_p := (H(\xi)_p, M(g_f;\xi)_p)$ for all $p \notin \mathcal{P}$, we have*

$$\sum_{\substack{N=1\\(N,p)=1\forall p\in\mathcal{P}}}^{\infty} \mu(\xi)^{-1}\sum_{i=1}^{h} A(N\xi_i)\frac{\overline{f}(u_i)}{e(\xi)_i}N^{-(s+k-\frac{n+2}{2})} = \left(\mu(\xi)^{-1}\sum_{i=1}^{h} A(\xi_i)\frac{\overline{f}(u_i)}{e(\xi)_i}\right)\times$$

$$\times L_{\mathcal{P}}(F;s)L_{\mathcal{P}}\left(\overline{f};s+1/2\right)^{-1}\times(\text{zeta})^{-1}(s),$$

where $(\text{zeta})(s) := \begin{cases} 1 & \text{if } n \text{ odd} \\ \zeta_{\mathcal{P}}(2s) & \text{if } n \text{ even}\end{cases}.$

The definitions of the Hecke algebras can be found in [Sug85, Section 2]. $L(-,s)$ denotes the standard L-function attached to orthogonal modular forms, as this is defined in [Sug85, Paragraph 4-1, (4.4), (4.7)]. Also, for any L-function, we write $L_{\mathcal{P}}$ for the Euler product not containing the primes in $\mathcal{P}$.

Proof. This follows from Sugano's main Theorem in [Sug85, Theorem 1]. In our setting, we take $g_f = (id, id, \cdots)$ and then substitute $a(u_i;N\xi)$ with $A(N\xi_i)$ in the original form of [Sug85, Theorem 1], due to Lemma 6.5.3. We also note that in this case $H(\xi)_\infty M(g_f;\xi)_f = H(\xi)_{\mathbb{A}} \cap C$, where C is defined in (6.4.1). The set of primes $\mathcal{P}$ contains all the primes contained in the set $\mathcal{P}_2$ of [Sug85, Theorem 1] and the finite set of primes p for which $\partial_p \neq 0$ or $\partial'_p \neq 0$, where ∂_p, ∂'_p are defined in [Sug85, Theorem 1]. We note that in our case, L_p is maximal for all p, so $\mathcal{P}_1$ in [Sug85, Theorem 1] is empty. □

From now on, we fix $g_f = id$. Now, for any $f \in V(g_f;\xi)$, we set

$$\widetilde{f}(u_i) := \frac{\overline{f}(u_i)}{e(\xi)_i}\mu(\xi)^{-1},\ i = 1,\cdots,h \text{ and } A_f := \sum_{i=1}^{h}\widetilde{f}(u_i)A(\xi_i). \tag{6.5.3}$$

The formula in Theorem 6.5.4 then becomes

$$(\text{zeta})(s)\times L_{\mathcal{P}}\left(\overline{f};s+1/2\right)\sum_{i=1}^{h}\widetilde{f}(u_i)D_{F,\xi_i,\mathcal{P}}\left(s+k-(n+2)/2\right) = A_f L_{\mathcal{P}}(F;s),$$

where $D_{F,\xi_i,\mathcal{P}}(s)$ is the Dirichlet series appearing in Proposition 6.4.10. This is true for any simultaneous eigenfunction f of the Hecke algebras $H'_p = (H(\xi)_p, M(g_f;\xi)_p)$ for all $p \notin \mathcal{P}$. Our aim is to invert it so that we solve for $D_{F,\xi_i,\mathcal{P}}(s)$.

From the definition of the $u_i's$ just before Lemma 6.5.3, we have that

$$H(\xi)_{\mathbb{A}} = \bigsqcup_{i=1}^{h} H(\xi)_{\mathbb{Q}} u_i D,$$

where $D := H(\xi)_\infty M(g_f;\xi)_f$. We note D is an open subgroup of $H(\xi)_{\mathbb{A}}$ and $D \cap H(\xi)_f$ is compact. By [Shi04, Lemma 17.6], there is a correspondence

$$f \longleftrightarrow \{f^{(i)} \mid i = 1, \cdots, h\},$$

with each $f^{(i)} \in \mathbb{C}$, because $H(\xi)_\infty \cong \mathrm{SO}(n+1)$ is compact. We also note here that $f(u_i) = f^{(i)}$ for all $i = 1, \cdots, h$, as we can see by the way these $f^{(i)}$ are defined in the proof of [Shi97, Lemma 10.8].

Now, for any two simultaneous eigenfunctions f_i, f_j of the Hecke algebras defined by the pairs $\mathcal{H}'_p = (H(\xi)_p, M(g_f;\xi)_p)$ for all $p \notin \mathcal{P}$, their inner product is defined via the formula

$$\langle f_i, f_j \rangle := \left\{ \sum_{k=1}^{h} \nu\left(\Gamma^k\right) \right\}^{-1} \sum_{k=1}^{h} \nu\left(\Gamma^k\right) \overline{f_i^{(k)}} f_j^{(k)} = \\ = \left\{ \sum_{k=1}^{h} \nu\left(\Gamma^k\right) \right\}^{-1} \sum_{k=1}^{h} \nu\left(\Gamma^k\right) \overline{f_i(u_k)} f_j(u_k),$$

where $\Gamma^k := H(\xi)_{\mathbb{Q}} \cap u_k D u_k^{-1}$ and $\nu\left(\Gamma^k\right) = \#(\Gamma^k)^{-1}$, as in [Shi04, (17.23)] (here we again use the fact that $H(\xi)_\infty$ is compact).

As $e(\xi)_i = \#\{H(\xi)_{\mathbb{Q}} \cap M(u_i g_f;\xi)_f\}$, we have that $e(\xi)_i = \nu\left(\Gamma^i\right)^{-1}$, which also gives

$$\mu(\xi) = \sum_{k=1}^{h} \nu\left(\Gamma^k\right).$$

It is now possible to choose a basis of orthonormal Hecke eigenforms $\{f_1, f_2, \cdots, f_h\}$ for $V(g_f;\xi)$ with respect to the above inner product. This is true because the Hecke algebra defined by $\mathcal{H}'_p$ is commutative and consists of self-adjoint operators for all $p \notin \mathcal{P}$ (see proof of [Shi04, Proposition 17.14]). Also, by [Shi04, Lemma 17.6, (1)], there is an isomorphism between $V(g_f;\xi)$ and $\mathbb{C}^h$. Therefore, this basis must consist of h eigenforms. Hence, we get the expression

$$D_{F,\xi_i,\mathcal{P}}\left(s + k - (n+2)/2\right) = \mu(\xi)^{-1} (\mathrm{zeta})^{-1}(s) L_{\mathcal{P}}(F;s) \sum_{j=1}^{h} \nu\left(\Gamma^i\right) \frac{\mu(\xi)}{e(\xi)_i^{-1}} f_j(u_i) \times \\ \times L_{\mathcal{P}}\left(\overline{f_j}; s + 1/2\right)^{-1} A_{f_j},$$

which, after the simplifications, becomes

$$D_{F,\xi_i,\mathcal{P}}(s) = (\text{zeta})^{-1}\,(s-k+(n+2)/2)\,L_{\mathcal{P}}\,(F; s-k+(n+2)/2)\times \\ \times \sum_{j=1}^{h} f_j(u_i) L_{\mathcal{P}}\left(\overline{f_j}; s-k+(n+3)/2\right)^{-1} A_{f_j}. \quad (6.5.4)$$

Hence, we arrive at the following Theorem.

Theorem 6.5.5. *Let $(D,r)\in \widetilde{\operatorname{supp}}(L,\sigma)$. Let $\mathcal{P}$ be a finite set of primes, containing the primes described in Theorem 6.5.4 and all the prime divisors of D. Let $F \in S_k(\Gamma_S)$ corresponding to $\boldsymbol{F}\in \mathfrak{S}(K_f)$ with $A(\xi)\neq 0$. Assume $\boldsymbol{F}$ is a simultaneous eigenfunction for the Hecke algebra $\mathcal{H}_p$, defined by the pair (G_p, K_p) for all p and let $\mathcal{P}_{k,D,r}$ denote the Poincaré series of (6.3.1). Let also $\{f_j\}_{j=1}^h$ denote an orthonormal basis of simultaneous eigenfunctions for the pairs $\mathcal{H}'_p = (H(\xi)_p, M(g_f;\xi)_p)$ for all $p\notin\mathcal{P}$, A_{f_j} as in (6.5.3), and denote with $L_{\mathcal{P}}(-,s)$ the standard L-function attached to either F or any f_j, by ignoring the p-factors for $p\in\mathcal{P}$. We then have*

$$\mathcal{D}_{F,\mathcal{P}_{k,D,r},\mathcal{P}}(s) = L_{\mathcal{P}}\,(F; s-k+(n+2)/2)\sum_{j=1}^{h} A_{f_j} L_{\mathcal{P}}\left(\overline{f_j}; s-k+(n+3)/2\right)^{-1}\times \\ \times \sum_{i=1}^{h}\zeta_{\xi_i,\mathcal{P}}(s-k+n+1) f_j(u_i)\times \begin{cases} 1 & \text{if } n \text{ odd} \\ \zeta_{\mathcal{P}}(2s-2k+n+2)^{-1} & \text{if } n \text{ even}\end{cases},$$

where $\zeta_{\xi_i,\mathcal{P}}(s)$ are as in Proposition 6.4.10.

Proof. By substituting the expression we deduced in (6.5.4) into (6.4.3), we obtain (here we denote by "zeta" the function of Theorem 6.5.4 after $s\longmapsto s-k+(n+2)/2$)

$$\sum_{\substack{N=1\\(N,p)=1\forall p\in\mathcal{P}}}^{\infty} \langle V_N^*\phi_N, P_{k,D,r}\rangle N^{-s} = \sum_{i=1}^{h}\zeta_{\xi_i,\mathcal{P}}(s-k+n+1) D_{F,\xi_i,\mathcal{P}}(s) = \\ = (\text{zeta})^{-1}\times\sum_{i=1}^{h}\zeta_{\xi_i,\mathcal{P}}(s-k+n+1) L_{\mathcal{P}}\left(F; s-k+\frac{n+2}{2}\right)\times \\ \times\sum_{j=1}^{h} f_j(u_i) L_{\mathcal{P}}\left(\overline{f_j}; s-k+\frac{n+3}{2}\right)^{-1} A_{f_j} = \\ = (\text{zeta})^{-1}\times L_{\mathcal{P}}\left(F; s-k+\frac{n+2}{2}\right)\sum_{j=1}^{h} A_{f_j} L_{\mathcal{P}}\left(\overline{f_j}; s-k+\frac{n+3}{2}\right)^{-1}\times \\ \times\sum_{i=1}^{h}\zeta_{\xi_i,\mathcal{P}}(s-k+n+1) f_j(u_i). \quad \square$$

Hence, we would like to explore the connection between $\sum_{i=1}^{h}\zeta_{\xi_i,\mathcal{P}}(s-k+n+1) f_j(u_i)$ and $L\left(\overline{f_j}; s-k+(n+3)/2\right)$.

It turns out we can now obtain a clear-cut Euler product expression in the case $h = 1$ and when D is a specifically chosen number. The question is how we can pick S, so that we can get $h = 1$. This is the theme of the next Section.

6.6 Explicit Examples

We will now focus our attention on some specific examples of matrices S and corresponding Poincaré series. In particular, we set $D = -q$ and choose matrices S, so that the number of representatives for $L_0\left[1, \frac{1}{2}\mathbb{Z}\right]/\Gamma(L_0)$ is 1. We therefore take G to be the Poincaré series $\mathcal{P}_{k,-q,r}$, with $r \in L$ such that $(-q, r) \in \widetilde{\text{supp}}(L, \sigma)$. In particular, those choices imply that we can take

$$\xi = (1, 0, \cdots, 0, 1)^t$$

as an element of $L_0\left[1, \frac{1}{2}\mathbb{Z}\right]$. Therefore, $\zeta_{\xi,\mathcal{P}}(s)$ of Proposition 6.4.10 can be written as

$$\zeta_{\xi,\mathcal{P}}(s) = \sum_{\substack{N=1 \\ (N,p)=1 \forall p \in \mathcal{P}}}^{\infty} n(\xi; N) N^{-s},$$

where this time

$$n(\xi; d) = \#\left\{ s \in \mathbb{Z}^n / dS\mathbb{Z}^n \mid D \equiv \frac{1}{2} q S^{-1}[s] \pmod{qd} \right\}.$$

In these cases, we are able to deduce an exact Euler product expression, connecting the Dirichlet series of interest and the standard L-function of the orthogonal group.

6.6.1 Examples with rank 1

We consider the case where $S = 2t$ for some $t \geq 1$ with t square-free. This condition is needed so that the lattice $L = \mathbb{Z}$ (and therefore L_0 and L_1) is maximal (cf. [Sch22, Example 1.6.6(ii)]). Now $V_0 = \mathbb{Q}^3$ and the quadratic form of interest is then

$$\phi_0(x, y) = \frac{1}{2} x^t S_0 y,$$

for $x, y \in V_0$, where $S_0 = \begin{pmatrix} & & 1 \\ & -2t & \\ 1 & & \end{pmatrix}$. Hence, ϕ_0 is represented by $S_0/2$ with respect to the standard basis e_1, e_2, e_3 of V_0.

By [Shi04, Paragraph 7.3] we have that there exists a quaternion algebra B

over $\mathbb{Q}$ such that we can put $V_0 = B^\circ\zeta$, $\phi_0[x\zeta] = dxx^\iota$, $2\phi_0(x\zeta, y\zeta) = d\mathrm{Tr}_{B/\mathbb{Q}}(xy^\iota)$, where $B^\circ = \{x \in B \mid x^\iota = -x\}$ with ι the main involution of B and $\zeta \in A(V_0)$ such that $\zeta^2 = -d$. Here, by $A(V_0)$ we mean the Clifford algebra of (V_0, ϕ_0) (see [Shi04, Chapter 2]). Also, in general, we define $\mathrm{Tr}_{B/\mathbb{Q}}(x) := x + x^\iota$ for $x \in B$.

Now, from [Shi04, Paragraph 7.3], we have a way to compute ζ and d. We first need a basis h_1, h_2, h_3 of V_0 such that $\phi_0(h_i, h_j) = c_i\delta_{ij}$ for all $1 \le i, j \le 3$. In our case, we can make a choice

$$h_1 = e_1 + e_3, h_2 = e_2, h_3 = e_1 - e_3.$$

Then, we get that the condition above is satisfied with $c_1 = 1, c_2 = -t, c_3 = -1$ and therefore $d = c_1c_2c_3 = t$. Also, even though it is not needed in what follows, $\zeta = h_1h_2h_3$.

We now remind ourselves that $\xi = (1, 0, 1)^t$ and $W = (\mathbb{Q}\xi)^\perp$. This then implies that $\phi_0[\xi] = 1$. From [Shi06b, Paragraph 5.2], we get that $\exists k \in B^\circ$ such that $\xi = k\zeta$. Then, if $K := \mathbb{Q} + \mathbb{Q}k$, we get that $K = \mathbb{Q}(\sqrt{-t})$ because $-t$, which is $-d\phi_0[h]$ in the notation there, cannot be a square in $\mathbb{Q}^\times$.

Using [Shi06b, Theorem 5.7] and the formula (5.11) given there, tailored to our situation, we have the following Theorem.

Theorem 6.6.1. *We define the following quantities:*

- c_K *denotes the class number of* K.
- $\mathfrak{c}$ *denotes the ideal of* $\mathbb{Z}$*, determined by the local conditions*

$$\mathfrak{c}_p^2 N_{K/\mathbb{Q}}(\mathfrak{d}_{K/\mathbb{Q}})_p = \mathfrak{a}_p\phi_0[\xi]\phi_0(\xi, L_0)_p^{-2}, \tag{6.6.1}$$

  for all primes p*, where* $\mathfrak{a} = t\mathbb{Z}$ *and* $\mathfrak{d}_{K/\mathbb{Q}}$ *is the different ideal.*
- *For a prime* p *dividing* $\mathfrak{c}$*, we define* $[K/\mathbb{Q},\ p]$ *to be* $-1, 0$ or 1*, according to whether* p *remains prime, ramifies or splits in* K.
- *Let* p *be a rational prime. Pick* $\epsilon_p \in \delta(\phi_{0,p})$ *(see Section 4), which is either a unit or a prime element of* $\mathbb{Q}_p$ *and choose an element* $\beta_p \in \mathbb{Q}_p$ *such that* $\phi_0(\xi, L_{0,p}) = \beta_p\mathbb{Z}_p$*. Define then* $r_p(\xi) := \epsilon_p^{-1}\phi_0[\xi]\beta_p^{-2}$*. Define also*

$$\mathfrak{C}_p := \{u^2 + 4w \mid u, w \in \mathbb{Z}_p\}. \tag{6.6.2}$$

- $\mathfrak{a}^*$ *is the product of the prime factors* p *of* t *such that* $r_p(\xi) \in p^{-1}\mathbb{Z}_p$ *and* $r_p(\xi) \notin \mathfrak{C}_p$.
- μ *is the number of prime ideals dividing* $\mathfrak{a}^*$ *and ramified in* K.
- $U := \mathcal{O}_K^\times$ *and* $U' := \{x \in \mathcal{O}_K^\times \mid x - 1 \in \mathfrak{c}_p(\mathfrak{d}_{K/\mathbb{Q}})_p\ \forall p \nmid \mathfrak{a}^*\}$.

We then have

$$[H(\xi)_{\mathbb{A}} : H(\xi)_{\mathbb{Q}}(H(\xi)_{\mathbb{A}} \cap C)] = c_K 2^{1-\mu}[U : U']^{-1} N(\mathfrak{c}) \prod_{p|\mathfrak{c}} \left\{1 - \frac{1}{p}[K/\mathbb{Q},\ p]\right\}, \quad (6.6.3)$$

Proof. This follows from [Shi06b, Theorem 5.7]. In our case, we have, in the notation of the Theorem:

- The base field F is $\mathbb{Q}$, which has class number 1.
- The product of all the prime ideals in $\mathbb{Q}$ for which B ramifies, $\mathfrak{e}$, equals $\mathbb{Z}$. This is true because for each prime p, ϕ_0, viewed as a bilinear form over $\mathbb{Q}_p$, is isotropic (the Witt index has to be 1 for all primes p). Therefore by [Shi04, Paragraph 7.3], B over $\mathbb{Q}_p$ is not a division algebra, hence is isomorphic to $M_2(\mathbb{Q}_p)$, i.e. B splits or is unramified over p. Hence, $\mathfrak{e} = \mathbb{Z}$.
- Because t is square-free, $\mathfrak{a} = t\mathbb{Z}$.
- Because K is imaginary quadratic, ∞ ramifies, so $\nu = 1$ and $N_{K/\mathbb{Q}}(\mathcal{O}_K^\times) = \{1\}$, so $\left[\mathbb{Z}^\times : N_{K/\mathbb{Q}}(\mathcal{O}_K^\times)\right] = 2$. □

The above Theorem gives us the number $\#\left(L_0\left[1, \frac{1}{2}\mathbb{Z}\right]/\Gamma(L_0)\right)$ from Proposition 6.4.9 and the fact that now $H(\xi)$ is commutative (see proof of [Shi06b, Theorem 5.10]). We are interested in the cases when this number is 1.

For the different $\mathfrak{d}_{K/\mathbb{Q}}$, we have

$$\mathfrak{d}_{K/\mathbb{Q}} = \begin{cases} 2\sqrt{-t}\mathcal{O}_K & \text{if } -t \not\equiv 1 \pmod 4 \\ \sqrt{-t}\mathcal{O}_K & \text{if } -t \equiv 1 \pmod 4 \end{cases}.$$

We now want to determine the ideal $\mathfrak{c}$. But, by the above

$$N_{K/\mathbb{Q}}(\mathfrak{d}_{K/\mathbb{Q}}) = \begin{cases} 4t\mathbb{Z} & \text{if } -t \not\equiv 1 \pmod 4 \\ t\mathbb{Z} & \text{if } -t \equiv 1 \pmod 4 \end{cases},$$

and $\phi_0[\xi] = 1$, $\phi_0(\xi, L_0) = \frac{1}{2}\mathbb{Z}$. So, from (6.6.1), we get the equation

$$\mathfrak{c}_p^2 \cdot \begin{cases} 4t\mathbb{Z}_p & \text{if } -t \not\equiv 1 \pmod 4 \\ t\mathbb{Z}_p & \text{if } -t \equiv 1 \pmod 4 \end{cases} = 4t\mathbb{Z}_p.$$

Therefore,

$$\mathfrak{c} = \begin{cases} \mathbb{Z} & \text{if } -t \not\equiv 1 \pmod 4 \\ 2\mathbb{Z} & \text{if } -t \equiv 1 \pmod 4 \end{cases}. \quad (6.6.4)$$

We will now consider specific cases in order to determine when the index in (6.6.3) is 1.

- $t = 1$. Then, $\mathfrak{a} = \mathfrak{e} = \mathfrak{a}^* = \mathbb{Z}$ and from (6.6.4) $\mathfrak{c} = \mathbb{Z}$ as well. Now

$$U' = \{x \in \mathcal{O}_K^\times \mid x - 1 \in 2\mathcal{O}_{K,p}\ \forall p\}.$$

Clearly $\pm 1 \in U'$ but $\pm i \notin U'$ because $2\mathcal{O}_K = (1+i)^2 = (1-i)^2$. So, $[U : U'] = 2$. Finally, $\mu = 0$ because $\mathfrak{a}^*\mathfrak{e} = \mathbb{Z}$ and therefore, we get

$$[H(\xi)_\mathbb{A} : H(\xi)_\mathbb{Q}(H(\xi)_\mathbb{A} \cap C)] = 1 \cdot 2^{1-0} \cdot 2^{-1} = 1.$$

- $t = 3$. Here $\mathfrak{a} = 3\mathbb{Z}$ and $\mathfrak{a}^* = 3\mathbb{Z}$ because $r_3(\xi) = -\frac{4}{3} \in \frac{1}{3}\mathbb{Z}_3$ but $r_3(\xi) \notin \mathfrak{C}_3$, as $\mathfrak{C}_3 = \mathbb{Z}_3$ (see (6.6.2)). By (6.6.4), we get $\mathfrak{c} = 2\mathbb{Z}$. In this case, $\mathcal{O}_K^\times = \{\pm 1, \pm\omega, \pm\omega^2\}$, where $\omega = \frac{1}{2}(1 + \sqrt{-3})$. Then,

$$U' = \{x \in \mathcal{O}_K^\times \mid x - 1 \in 2\sqrt{-3}\mathcal{O}_{K,p}\ \forall p \neq 3\}.$$

But for $p \neq 3$, $\sqrt{-3}$ is a unit in $\mathcal{O}_{K,p}$. So, the condition becomes $x - 1 \in 2\mathcal{O}_{K,p}\ \forall p \neq 3$. We can then check that this is true only for $\pm 1 \in U$. Therefore, $[U : U'] = 3$. Also, $\mu = 1$ in this case, because 3 ramifies in K. Also, as $-3 \equiv 5 \pmod 8$, we have that 2 remains prime in K, hence

$$[H(\xi)_\mathbb{A} : H(\xi)_\mathbb{Q}(H(\xi)_\mathbb{A} \cap C)] = 1 \cdot 2^{1-1} \cdot 3^{-1} \cdot 2 \cdot (1 + 1/2) = 1.$$

- $t = 2$ or $t > 3$ with $-t \not\equiv 1 \pmod 4$. We write $t = p_1 \cdots p_k$, where p_i are distinct prime factors. In this case, $\mathfrak{e} = \mathfrak{c} = \mathbb{Z}$ and $\mathfrak{a} = t\mathbb{Z}$. Let us now compute the quantities $r_{p_i}(\xi)$. Let p be one of the $p_i's$. We have that $\delta(\phi_{0,p})$ can be represented by $-t$. Pick ϵ_p such that $\epsilon_p(\mathbb{Q}_p^\times)^2 = -t(\mathbb{Q}_p^\times)^2$ such that ϵ_p is either unit or a prime element. Write $\epsilon_p = -tu_p^2$, with $u_p \in \mathbb{Q}_p^\times$. By considering valuations, we must have that $u \in \mathbb{Z}_p^\times$, because ϵ_p has valuation 0 or 1. Moreover, we pick $\beta_p = 1/2$. Now

$$r_p(\xi) = \epsilon_p^{-1}\phi_0[\xi]\beta_p^{-2} = -\frac{4}{tu_p^2} \in p^{-1}\mathbb{Z}_p.$$

Moreover, if $p \neq 2$, $\mathfrak{C}_p = \mathbb{Z}_p$, so $r_p(\xi) \notin \mathfrak{C}_p$. If $p = 2$, then assume

$$-\frac{4}{tu_2^2} = a^2 + 4b^2,$$

where $a, b \in \mathbb{Z}_2$. We can write $-4/tu_2^2 = 2w_2$, with $w_2 \in \mathbb{Z}_2^\times$. By taking valuations, we must have that the valuation of a is at least one. But then we get a contradiction, because the valuation on the right is at least 2, but on the left exactly 1. Therefore $\mathfrak{a}^* = t\mathbb{Z}$. We have $\mu = k$ as each p_i is ramified in K.

Now $U = \mathcal{O}_K^\times = \{\pm 1\}$ and

$$U' = \{x \in \mathcal{O}_K^\times \mid x - 1 \in 2\sqrt{-t}\mathcal{O}_{K,p} \ \forall p \neq p_i, i = 1, \cdots, k\}.$$

But for all $p \neq p_i$, $\sqrt{-t}$ is a unit in $\mathcal{O}_{K,p}$ and therefore $\pm 1 \in U'$, i.e. $[U : U'] = 1$. We then obtain:

$$[H(\xi)_{\mathbb{A}} : H(\xi)_{\mathbb{Q}}(H(\xi)_{\mathbb{A}} \cap C)] = c_K \cdot 2^{1-k}.$$

Therefore, this is 1 if $c_K = 2^{k-1}$. Hence, the answer in this case is the number fields $K = \mathbb{Q}(\sqrt{-t})$ that satisfy $c_K = 2^{k-1}$, where k is the number of prime factors of t and $-t \not\equiv 1 \pmod 4$. For example, when $c_K = 2$, t must have 2 prime factors, and examples would be $t = 6, 10$, etc.

- $t > 3$ with $-t \equiv 1 \pmod 4$. We write $t = p_1 \cdots p_k$, where p_i are distinct prime factors. In these cases, similarly to the case $t = 3$, we have $\mathfrak{a} = t\mathbb{Z}$, $\mathfrak{c} = 2\mathbb{Z}$ and $\mathfrak{a}^* = t\mathbb{Z}$, as above. Also, $\mu = k$ because each prime p_i is ramified in K. Now, $U = \mathcal{O}_K^\times = \{\pm 1\}$ and then

$$U' = \{x \in \mathcal{O}_K^\times \mid x - 1 \in 2\sqrt{-t}\mathcal{O}_{K,p} \ \forall p \neq p_i, i = 1, \cdots, k\}.$$

But for $p \neq p_i$, $\sqrt{-t}$ is a unit in $\mathcal{O}_{K,p}$ and therefore $\pm 1 \in U'$, which means $[U : U'] = 1$. Now, if $-t \equiv 5 \pmod 8$, we get that 2 is inert in K and if $-t \equiv 1 \pmod 8$, 2 splits in K. Therefore

$$[H(\xi)_{\mathbb{A}} : H(\xi)_{\mathbb{Q}}(H(\xi)_{\mathbb{A}} \cap C)] = \\ = \begin{cases} c_K \cdot 2^{1-k} \cdot 2 \cdot (1 + 1/2) = 3c_K \cdot 2^{1-k} & \text{if } -t \equiv 5 \pmod 8 \\ c_K \cdot 2^{1-k} \cdot 2 \cdot (1 - 1/2) = c_K \cdot 2^{1-k} & \text{if } -t \equiv 1 \pmod 8 \end{cases}.$$

In the first case, the index cannot be 1, while in the second case, we must have $c_K = 2^{k-1}$. Therefore, the answer in this case is t such that $-t \equiv 1 \pmod 8$ so that if $K = \mathbb{Q}(\sqrt{-t})$ we have $c_K = 2^{k-1}$, where k is the number of distinct prime factors of t. For example, the only example for $k = 2$ is $t = 15$.

Hence, we arrive at the following Proposition:

Proposition 6.6.2. *Let t be one of the following:*

- $t = 1, 3$.
- $t \not\equiv 3 \pmod 4$ *and if* $t = p_1 \cdots p_k$, $K := \mathbb{Q}(\sqrt{-t})$, *we have* $c_K = 2^{k-1}$.
- $t \equiv 7 \pmod 8$ *and if* $t = p_1 \cdots p_k$, $K := \mathbb{Q}(\sqrt{-t})$, *we have* $c_K = 2^{k-1}$.

Then, with the notation as above, we have $[H(\xi)_{\mathbb{A}} : H(\xi)_{\mathbb{Q}}(H(\xi)_{\mathbb{A}} \cap C)] = 1$.

Let now t be one of the above. The set of primes $\mathcal{P}$ is as described in Theorem 6.5.5 and we include the prime 2. We aim to give an Euler product expression for $\mathcal{D}_{F,\mathcal{P}_{k,-q,r},\mathcal{P}}(s)$. In particular, from Theorem 6.5.5, we need to give an Euler product expression for

$$\zeta_{\xi,\mathcal{P}}(s) := \sum_{\substack{N=1\\(N,p)=1\forall p\in\mathcal{P}}}^{\infty} n(\xi;N)N^{-s},$$

where now $n(\xi;N) = \#\{s \in \mathbb{Z}/2tN\mathbb{Z} \mid s^2 \equiv -4t \pmod{4tN}\}$, as we have explained in the beginning of Section 6.6.

Now, as t is square-free, we obtain from $s^2 \equiv -4t \pmod{4tN}$, that $2t \mid s$, so it suffices to look for the number of solutions of the congruence

$$ts^2 \equiv -1 \pmod{N},$$

with $s \pmod{N}$. The last number of solutions is multiplicative in N, so we can write

$$\zeta_{\xi,\mathcal{P}}(s) = \prod_{p\notin\mathcal{P}}\left(\sum_{k=0}^{\infty} n(\xi;p^k)p^{-ks}\right).$$

Now, for all $p \notin \mathcal{P}$, we have $(p,t) = 1$, so by [Tót14, Proposition 14] (as we assume $2 \in \mathcal{P}$), we get that

$$n(\xi;p^k) = 1 + \left(\frac{-t}{p}\right),$$

for all $k \geq 1$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Therefore, if we define $\chi_t(p) := \left(\frac{-t}{p}\right)$ for $p \notin \mathcal{P}$, we deduce that (bear in mind that $\chi_t^2 = 1$)

$$\zeta_{\xi,\mathcal{P}}(s) = \zeta_{\mathcal{P}}(s)\zeta_{\mathcal{P}}(2s)^{-1}\zeta_{\mathcal{P}}(s,\chi_t), \tag{6.6.5}$$

where $\zeta_{\mathcal{P}}(s,\chi_t) := \prod_{p\notin\mathcal{P}}(1-\chi_t(p)p^{-s})^{-1}$. Therefore, we have the following Theorem.

Theorem 6.6.3. *Let $S = 2t$, with t being chosen as in Proposition 6.6.2. Assume $\mathcal{P}$ is a finite set of primes, containing the primes described in Theorem 6.5.5, the prime 2, and the primes so that the conditions of [Shi99, Proposition 5.13] are satisfied for all $p \notin \mathcal{P}$. Moreover, for all $p \notin \mathcal{P}$, we define*

$$\chi_t(p) := \left(\frac{-t}{p}\right),\ \psi(p) := \left(\frac{-1}{p}\right).$$

If F and $\mathcal{P}_{k,-q,r}$ are as in Theorem 6.5.5 and $\xi = (1,0,1)^t$ (in particular $A(\xi) \neq 0$), we have

$$\mathcal{D}_{F,\mathcal{P}_{k,-q,r},\mathcal{P}}(s) = A(\xi)L_{\mathcal{P}}\left(F;s-k+3/2\right)\zeta_{\mathcal{P}}(2s-2k+4)^{-1}\frac{\zeta_{\mathcal{P}}(s-k+2,\chi_t)}{\zeta_{\mathcal{P}}(s-k+2,\psi)}.$$

Proof. The proof follows from Theorem 6.5.5 after choosing f to be the constant $\mathbf{1}$ on $H(\xi)_{\mathbb{A}}$. We have computed $\zeta_{\xi,\mathcal{P}}(s)$ in (6.6.5) and

$$L_{\mathcal{P}}(\mathbf{1}, s-k+2) = \zeta_{\mathcal{P}}(s-k+2, \psi)\zeta_{\mathcal{P}}(s-k+2),$$

as this can be computed by [Shi99, Proposition 5.15], because $\mathcal{P}$ is chosen so that conditions of [Shi99, Proposition 5.13] are satisfied. □

Remark 6.6.4. In the case $t = 1$, we recover (partly) the result of Kohnen and Skoruppa in [KS89]. In particular, it is clear that with the above approach, some Euler factors might be missing. However, the benefit is that we also obtain results for $t > 1$. These could be interpreted as results on modular forms for a paramodular group (cf. [Kri16, Corollary 6], [GK18]).

6.6.2 The rank $n \geq 2$ case

In the rank $n \geq 2$ case, a Theorem like [Shi06b, Theorem 5.7] is not available. For this reason, we seek examples of matrices S so that the following conditions hold:

1. The lattice $L = \mathbb{Z}^n$ is $\mathbb{Z}$-maximal.
2. With the notation as in Section 6.4, $L_0 \cap W$ is a $\mathbb{Z}$-maximal lattice in W and if $D = \{\alpha \in H(\xi)_{\mathbb{A}} \mid \alpha(L_0 \cap W) = L_0 \cap W\}$, we have $D = H(\xi)_{\mathbb{A}} \cap C$.
3. The number of classes in the genus of maximal lattices (this is independent of the choice of the maximal lattice, see [Shi04, Paragraph 9.7]) in $H(\xi)_{\mathbb{Q}} = \mathrm{SO}^{\psi}(W)$ is 1. Here, $\psi := \phi_0 \mid_W$.

We will show that for rank $n = 2, 4, 6, 8$, there is at least one positive definite even symmetric matrix S of rank n, so that the above conditions are satisfied. We start with the following Lemma:

Lemma 6.6.5. *We have $H(\xi)_{\mathbb{Q}} = \mathrm{SO}^{\psi}(W)$ and ψ can be represented by the matrix*

$$T = \frac{1}{2}\begin{pmatrix} -2 & \\ & -S \end{pmatrix}. \tag{6.6.6}$$

Proof. We have $W = \{x \in V_0 \mid \phi_0(x, \xi) = 0\}$. Now $\xi \in U := \mathbb{Q}e_1 + \mathbb{Q}e_{n+2}$ and $W = (W \cap U) \oplus U^{\perp}$. But on $U^{\perp}$, ϕ is represented by $-S$. Moreover, $W \cap U$ has dimension 1 and if we write $x = \lambda e_1 + \mu e_{n+2} \in W \cap U$, we have $\phi_0(x, \xi) = (\lambda + \mu)/2$ and so $W \cap U$ is spanned by $e_1 - e_{n+2}$. By evaluating $\phi_0[e_1 - e_{n+2}] = -1$, the Lemma follows. □

We claim the following choices for the matrix S satisfy the conditions (1)-(3) above.

- $n = 2$: $S = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$ with determinants $3, 15$ respectively.
- $n = 4$: $S = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ with determinants $5, 9$ respectively.
- $n = 6$: $S = \begin{pmatrix} 2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 0 & 1 & -1 & 1 \\ -1 & 0 & 2 & -1 & 1 & 0 \\ 1 & 1 & -1 & 2 & -1 & 0 \\ -1 & -1 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$ with determinant 3.
- $n = 8$: $S = \begin{pmatrix} 2 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & 2 & 0 & -1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 & -1 & -1 & 1 & -1 \\ -1 & 0 & -1 & -1 & 2 & 1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\ -1 & 1 & 0 & -1 & 1 & 1 & -1 & 2 \end{pmatrix}$ with determinant 1.

Remark 6.6.6. The matrices S of rank 2 correspond to the unitary groups of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-15})$ respectively (cf. [Sch22, Example 1.6.6, (v)]).

Let us first check the conditions (1) and (3), right before Lemma 6.6.5. Condition (1) follows by [Sch22, Proposition 1.6.12] for the matrices with square-free determinant and [Sch22, Lemma 1.6.5, (ii)] for the matrix of determinant 9.

Condition (3) follows by [Han11, Section 8], because for the above choices of S, the matrix T of (6.6.6) corresponds to the following quadratic forms:

- $n = 2$: Examples 4, 26 in matrices of 3 variables in [Han11, Section 8].
- $n = 4$: Examples $3, 5$ in matrices of 5 variables in [Han11, Section 8].
- $n = 6$: Example 3 in matrices of 7 variables in [Han11, Section 8].
- $n = 8$: Example 1 in matrices of 9 variables in [Han11, Section 8].

All these examples correspond to one class in the genus of the standard lattice $\mathbb{Z}^{n+1}$, in the cases when it is maximal. This can also be seen by a simple computation in MAGMA, for example. As we mentioned above, this shows that every maximal lattice has one class in its genus.

Finally, we need to check condition (2). We will use [Shi04, Proposition 11.12]. We have the following local result:

Proposition 6.6.7. *Let S denote any of the matrices above. Let $D_p = \{\alpha \in H(\xi)_p \mid \alpha(L_{0,p} \cap W_p) = L_{0,p} \cap W_p\}$ for each prime p. We then have that $L_{0,p} \cap W_p$ is a $\mathbb{Z}_p$-maximal lattice in W_p and also*

$$D_p := H(\xi)_p \cap C_p,$$

where $C_p = \{x \in G_p^* \mid xL_{0,p} = L_{0,p}\}$.

Proof. Our proof is based on [Shi04, Proposition 11.12]. We will first establish the following claim:

$$L_{0,p}^* \neq L_{0,p} \iff p \mid \det(S).$$

This follows from the fact that $L_{0,p}^* = S_0^{-1} L_{0,p}$ and that

$$S_0^{-1} = \begin{pmatrix} & & 1 \\ & -S^{-1} & \\ 1 & & \end{pmatrix}, \quad S^{-1} = \frac{1}{\det(S)} \operatorname{adj}(S).$$

We remind ourselves that $\phi_0[\xi] = 1$ and $\phi_0(\xi, L_{0,p}) = \frac{1}{2}\mathbb{Z}_p$ for all primes p. Therefore, $\phi_0(\xi, L_{0,p})^2 = \phi_0[\xi]\mathbb{Z}_p$ for all $p \neq 2$. Moreover, $L_{0,2}^* = L_{0,2}$ for every choice of S, as $2 \nmid \det(S)$ for any S. This means [Shi04, Proposition 11.12] is applicable in every case.

Let now t_p denote the dimension of the maximal anisotropic subspace of $(\mathbb{Q}_p^{n+2}, \phi_0)$ (see Section 4.1). For $p \nmid \det(S)$, we have $L_{0,p}^* = L_{0,p}$, $t_p \neq 1$ as n is even and $4\phi_0[\xi]^{-1}\phi_0(\xi, L_{0,p})^2 = \mathbb{Z}_p$. So, by [Shi04, Proposition 11.12, (iii), (2)], $L_{0,p} \cap W_p$ is $\mathbb{Z}_p$-maximal.

If now $p \mid \det(S)$, we claim $t_p > 1$ and therefore [Shi04, Proposition 11.12, (iii), (1)] will be applicable. We show this on a case-by-case basis. Define $K_0 := \mathbb{Q}_p(\sqrt{\delta})$, where $\delta := (-1)^{(n+2)(n+1)/2} \det(\phi_0)$. We note that from the proof of [Shi06a, Lemma 3.3], we have $t_p = 2$ if and only if $K_0 \neq \mathbb{Q}_p$.

- $S = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then, we claim that $t_3 = 2$. Now, $\det(\phi_0) = -3/2^4$ and then $K_0 = \mathbb{Q}_3(\sqrt{\det(\phi_0)}) = \mathbb{Q}_3(\sqrt{-3}) \neq \mathbb{Q}_3$.

- $S = \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$. In this case, $\det(\phi_0) = -15/2^4$ and we claim $t_3 = t_5 = 2$. But again, if $p \in \{3, 5\}$, we have $K_0 = \mathbb{Q}_p(\sqrt{\det \phi_0}) = \mathbb{Q}_p(\sqrt{-15}) \neq \mathbb{Q}_p$ by taking valuations (for example, if $\sqrt{-15} \in \mathbb{Q}_3$, then $-15 = u^2$ for some $u \in \mathbb{Q}_3$ and so $2v_3(u) = 1$, contradiction).

- $S = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$. In this case, $\det(\phi_0) = -5/2^6$ and so $K_0 = \mathbb{Q}_5(\sqrt{5}) \neq \mathbb{Q}_5$, so $t_5 = 2$.

- $S = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$. In this case, $K_0 = \mathbb{Q}_3$ and we can compute $t_3 = 4$ in SAGE.

- $S = \begin{pmatrix} 2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 0 & 1 & -1 & 1 \\ -1 & 0 & 2 & -1 & 1 & 0 \\ 1 & 1 & -1 & 2 & -1 & 0 \\ -1 & -1 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}$. In this case, $\det(\phi_0) = -3/2^8$ and so again $K_0 = \mathbb{Q}_3(\sqrt{-3}) \neq \mathbb{Q}_3$. Therefore, $t_3 = 2$.

- In the $n = 8$ case, we have that $\det(S) = 1$, so there are no primes to check ($L_{0,p}^* = L_{0,p}$ for all primes p).

Finally, the fact that $D_p = H(\xi)_p \cap C_p$ follows from [Shi04, Proposition 11.12, (iv)], as $t_p \neq 1$ always, because n is even. □

We are now ready to give the global statement.

Proposition 6.6.8. *With S be any of the matrices above, we have that $L_0 \cap W$ is $\mathbb{Z}$-maximal in W and if $D := \{\alpha \in H(\xi)_{\mathbb{A}} \mid \alpha(L_0 \cap W) = L_0 \cap W\}$, then $D = H(\xi)_{\mathbb{A}} \cap C$.*

Proof. The first claim follows by [Sch22, Proposition 1.6.9], as all the localisations are maximal by Proposition 6.6.7. For the second one, we have

$$H(\xi)_{\mathbb{A}} \cap C = \{x \in H(\xi)_{\mathbb{A}} \mid xL_0 = L_0\}.$$

But the lattice xL_0 is the lattice which is defined by $(xL_0)_p = x_pL_{0,p}$ for all primes p. Now, if $x \in H(\xi)_{\mathbb{A}}$ with $xL_0 = L_0$, we have $x(L_0 \cap W) = L_0 \cap W$ (see [Shi04, page 104]), so $H(\xi)_{\mathbb{A}} \cap C \subset D$.

On the other hand, for all primes p, let $D_p := D \cap H(\xi)_p$. Then, if $x \in D$, then $x_p \in D_p$, so $x_p \in H(\xi)_p \cap C_p$ by Proposition 6.6.7. Therefore, $x_pL_{0,p} = L_{0,p}$ for all primes p. This means that $x \in H(\xi)_{\mathbb{A}} \cap C$, as wanted. □

6.6.3 Euler product expression for the Dirichlet series

The question of this Section is to obtain an Euler product expression for the Dirichlet series of interest in each of the above cases and relate it to the standard L-function attached to F. Again, let $\mathcal{P}$ be as in Theorem 6.5.5, containing also the prime 2. In particular, $\mathcal{P}$ contains the prime factors of q, which are also the prime factors of $\det S$. By Theorem 6.5.5, the first step is to determine $\zeta_{\xi,\mathcal{P}}(s)$. Hence, we need to compute the quantity

$$n(\xi; d) = \#\left\{ s \in \mathbb{Z}^n/dS\mathbb{Z}^n \mid \frac{1}{2}qS^{-1}[s] \equiv -q \pmod{qd} \right\},$$

with $\xi = (1, \mathbf{0}, 1)^t$, as we have explained in the beginning of Section 6.6. The steps we follow are:

1. Find unimodular integer matrices P, Q such that $PSQ = \operatorname{diag}(a_1, \cdots, a_n)$, for some positive integers a_i.
2. We then substitute $t = Ps \implies s = P^{-1}t$. Then, we have

   $$s - s' \in dS\mathbb{Z}^n \iff t - t' \in dPS\mathbb{Z}^n \iff t - t' \in dPSQ\mathbb{Z}^n,$$

   because Q is unimodular. Hence, if $t = (t_1, \cdots, t_n)^t$, we consider each t_i $\pmod{da_i}$.
3. We then solve the congruence

   $$\frac{1}{2}qS^{-1}[P^{-1}t] \equiv -q \pmod{qd}.$$

Let us now deal with the specific examples we have. In the following, let for $p \notin \mathcal{P}$

$$\chi_S(p) := \left(\frac{(-1)^{n/2} \det S}{p} \right).$$

1. $S = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Here, we have $PSQ = \operatorname{diag}(1, 3)$, with $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. We

then have

$$P^{-1}t = \begin{pmatrix} 2t_1 - t_2 \\ -t_1 + t_2 \end{pmatrix}$$

and after substituting, the congruence of interest becomes ($q = 3$ here)

$$3t_1^2 - 3t_1t_2 + t_2^2 \equiv -3 \pmod{3d}$$

with $t_1 \pmod{d}$ and $t_2 \pmod{3d}$. Now, by the form of the equation, we get $3 \mid t_2$, so the congruence becomes

$$t_1^2 - 3t_1t_2 + 3t_2^2 \equiv -1 \pmod{d}$$

with $t_1, t_2 \pmod{d}$. Now, we have $n(\xi; d) = N(T; d)$, where $N(T; d)$ is defined as the number of solutions to the congruence $T[t] \equiv -1 \pmod{d}$, with

$$T = \begin{pmatrix} 1 & -3/2 \\ -3/2 & 3 \end{pmatrix}.$$

But $N(T; d)$ is multiplicative in d. Let now $p \notin \mathcal{P}$, so that 2 has a multiplicative inverse $\pmod{p}$ and $p \nmid \det T$ ($\mathcal{P}$ contains the prime factors of $\det S$ by assumption). Then, by [Hak11, Corollary 1], we know that for each $k \geq 1$, there is a non-singular $\pmod{p}$ matrix U_k such that $T[U_k] \equiv R \pmod{p^k}$ with some diagonal matrix R. By then setting $t \longmapsto U_k t$, we can still consider $t_i \pmod{p^k}$ for all i (as the determinant of U_k is non-zero $\pmod{p}$). We then have $N(T; p^k) = N(R; p^k)$, with R diagonal. We can now count the number of solutions $N(R; p^k)$ by [Tót14, Proposition 4]. In particular, if $R = \operatorname{diag}(a_1, a_2)$, we have

$$N(R; p^k) = p^k \left[1 - \frac{1}{p}\left(\frac{-a_1a_2}{p}\right)\right].$$

But $\det R = a_1a_2$, and $\det R \equiv (\det U_k)^2 \det T \pmod{p}$, so

$$\left(\frac{-a_1a_2}{p}\right) = \left(\frac{-\det T}{p}\right) = \left(\frac{-3u^2}{p}\right) = \left(\frac{-3}{p}\right) = \chi_S(p),$$

where $2u \equiv 1 \pmod{p}$. Therefore, we obtain

$$n(\xi; p^k) = N(R; p^k) = p^k \left[1 - \frac{\chi_S(p)}{p}\right].$$

2. $S = \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$. In this case, we have $PSQ = \operatorname{diag}(1, 15)$ with $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. By following the above steps, the congruence becomes

$$15t_1^2 - 15t_1t_2 + 4t_2^2 \equiv -15 \pmod{15d}$$

with $t_1 \pmod{d}$ and $t_2 \pmod{15d}$. But now $15 \mid t_2$ and so after $t_2 \longmapsto 15t_2$, we get

$$t_1^2 - 15t_1t_2 + 60t_2^2 \equiv -1 \pmod{d}$$

with $t_1, t_2 \pmod{d}$ and with exact same reasoning as before, we get

$$n(\xi; p^k) = p^k \left[1 - \frac{\chi_S(p)}{p}\right].$$

for all $p \notin \mathcal{P}$ with $n(\xi; d)$ multiplicative in d.

3. $S = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$. In this case, we have $PSQ = \mathrm{diag}(1,1,1,5)$ with

$$P = \begin{pmatrix} 3 & 1 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 4 \end{pmatrix}.$$

By then substituting $s = P^{-1}t$ the congruence becomes

$$5t_1^2 + 5t_2^2 + 5t_3^2 + 2t_4^2 - 5t_1t_2 - 5t_1t_3 - 5t_1t_4 + 5t_2t_3 \equiv -5 \pmod{5d},$$

with $t_1, t_2, t_3 \pmod{d}$ and $t_4 \pmod{5d}$. But, again, $5 \mid t_4$, so after setting $t_4 \longmapsto 5t_4$, we have

$$t_1^2 + t_2^2 + t_3^2 + 10t_4^2 - t_1t_2 - t_1t_3 - 5t_1t_4 + t_2t_3 \equiv -1 \pmod{d},$$

with $t_i \pmod{d}$ for all i. As before, this then gives

$$n(\xi; p^k) = p^{3k} \left[1 - \frac{\chi_S(p)}{p^2}\right],$$

for all $p \notin \mathcal{P}$ with $n(\xi; d)$ multiplicative in d.

4. $S = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$. In this case, using the same method, we obtain

$$3t_1^2 + 12t_2^2 + t_3^2 + t_4^2 - 9t_1t_2 + 3t_1t_3 - 6t_2t_3 + 3t_2t_4 \equiv -3 \pmod{3d},$$

with $t_1, t_2 \pmod{d}$ and $t_3, t_4 \pmod{3d}$. This then implies $3 \mid t_3^2 + t_4^2$ and so

$3 \mid t_3, t_4$. Therefore, as previously, we obtain

$$n(\xi; p^k) = p^{3k}\left[1 - \frac{\chi_S(p)}{p^2}\right],$$

for all $p \notin \mathcal{P}$ with $n(\xi; d)$ multiplicative in d.

5. In the last two cases, we omit the calculations, but by applying the exact same reasoning as above, we will again get

$$n(\xi; p^k) = p^{k(n-1)}\left[1 - \frac{\chi_S(p)}{p^{n/2}}\right],$$

for all $p \notin \mathcal{P}$ and with $n(\xi; d)$ multiplicative in d. Here $n = 6, 8$.

In all of the above cases, we therefore obtain (here $\mathcal{P}$ can be possibly enlarged but still finite)

$$\zeta_{\xi,\mathcal{P}}(s) = \prod_{p\notin\mathcal{P}} \zeta_{\mathcal{P}}(s-(n-1))\zeta_{\mathcal{P}}(s-n/2+1, \chi_S)^{-1}, \tag{6.6.7}$$

where $\zeta_{\mathcal{P}}(s, \chi_S) := \prod_{p\notin\mathcal{P}}(1-\chi_S(p)p^{-s})^{-1}$. We therefore arrive at the following Theorem.

Theorem 6.6.9. *Let S denote any of the matrices of Section 6.6.2 with even rank n and $\mathcal{P}$ a finite set of primes, containing the primes described in the proof of Theorem 6.5.5, the prime 2, and the primes so that the conditions of [Shi99, Proposition 5.13] are satisfied for all $p \notin \mathcal{P}$. For all $p \notin \mathcal{P}$, we define*

$$\chi_S(p) := \left(\frac{(-1)^{n/2}\det S}{p}\right).$$

If F and $\mathcal{P}_{k,-q,r}$ are as in Theorem 6.5.5 and $\xi = (1, \mathbf{0}, 1)^t$ (in particular $A(\xi) \neq 0$), we have

$$\begin{aligned}\mathcal{D}_{F,\mathcal{P}_{k,-q,r},\mathcal{P}}(s) = A(\xi)\zeta_{\mathcal{P}}(2s-2k+n+2)^{-1}\zeta_{\mathcal{P}}\left(s-k+(n+4)/2, \chi_S\right)^{-1} \times \\ \times L_{\mathcal{P}}\left(F; s-k+(n+2)/2\right)\prod_{i=1}^{n-1}\zeta_{\mathcal{P}}(s-k+(n+2)-i)^{-1}. \end{aligned} \tag{6.6.8}$$

Proof. The proof follows from Theorem 6.5.5 after we choose f to be the constant $\mathbf{1}$ on $H(\xi)_{\mathbb{A}}$. From [Shi99, Proposition 5.15], we have that (since $\mathcal{P}$ is chosen so that conditions of [Shi99, Proposition 5.13] are satisfied for $p \notin \mathcal{P}$):

$$L_{\mathcal{P}}\left(\mathbf{1}, s-k+(n+3)/2\right) = \prod_{i=1}^{n}\zeta_{\mathcal{P}}(s-k+(n+2)-i),$$

as in general $L_{\text{Sug}}\left(\mathbf{1}; s-(n-1)/2\right) = L_{\text{Shi}}(s)$. Here, the notation means the L-function we encounter in [Sug85] and [Shi99] respectively, as these are normalised differently. We can see this relation by comparing the expressions [Sug85, (2.22)] and [Shi99, (5.13.2)] (we remind the reader here that **1** is a function on a definite orthogonal group of rank $n+1$). Finally, the computation of $\zeta_{\xi,\mathcal{P}}(s)$ has been performed in (6.6.7). $\square$

Chapter 7

Conclusions

In this final Chapter, we will briefly recap our results and highlight points of particular interest, which provide directions for future work.

7.1 Hermitian Case

In the **first part** of the thesis, our main goal was to obtain a relation between the twisted Gritsenko's L-function and a certain Dirichlet series, analogous to Heim in [Hei99], for the case of Hermitian modular forms. To that end, we studied this Dirichlet series, both arithmetically and analytically.

In the arithmetic part, we showed that for inert primes p, the p-factor of the Dirichlet series is identified with the desired factor of the twisted Gritsenko's L-function. For split primes p, it is still uncertain whether the p-factor is related to the L-function of $\mathrm{GU}(2,2) \times \mathrm{GL}_2$. We essentially obtained all the necessary tools in the context of parabolic Hecke rings in order to show this; however, the last few computations turned out to be quite complicated, meaning we could not identify this factor with the twisted Gritsenko's Euler factor, or any other known factor for that matter. It should be noted that the only ramified prime, 2, is not considered here; however, our work for inert primes contains all the necessary ideas to deal with this case as well.

In the analytic part, we showed that this Dirichlet series arises as part of a triple Rankin-Selberg inner product; however, this integral produces an additional residue term, which we have not yet investigated. This is something that does not appear in Heim's work and is special to our situation. In particular, with the notation of Section 3.6, the double quotient $C_{3,2}(K)\backslash U_3(K)/(U_1(K) \times U_2(K))$ has infinitely many representatives. The reason for this is that not every vector is isotropic in the unitary setting, contrary to the symplectic setting, in which Heim works. We believe that this additional term might be responsible for the complicated expression we obtain in the case of split primes. It is our hope to revisit this problem in future work.

Our main motivation for considering this problem is to obtain algebraicity results on the special values of the twisted Gritsenko's L-function. Such results are not available even for Gritsenko's L-function (i.e., not twisted). The integral representation considered here will allow us to obtain algebraicity results, due to the presence of the Eisenstein series, which is holomorphic at $s = 0$ (see Proposition 3.6.1).

On another note, an important restriction in order to obtain such results is that the first Fourier-Jacobi coefficient of the Hermitian eigenform of degree 2 is not identically zero (see Theorem 3.5.1 for example). Such a restriction exists for Siegel modular forms too; for example, in [KS89], [Hei99], and other papers working on characteristic twists of similar Fourier-Jacobi Dirichlet series (e.g. [KSK95]). As we mentioned in the introduction, Manickam in [Man21] recently showed that the first Fourier-Jacobi coefficient of a Siegel cuspidal eigenform of degree 2 is indeed not identically zero. It is an interesting question to investigate this in the case of Hermitian modular forms of degree 2 as well.

Finally, we note here that as long as the underlying number field has class number 1, the results of this part of the thesis should transfer without much difficulty. The case of number fields with class number larger than 1 is of quite a different nature. The main reason is that the Hecke algebra can no longer be written as the tensor product of its p-components (see, for example, [HK20]). One would have to work adelically to deal with this issue.

7.2 Orthogonal Case

In the second part of the thesis, we focused on generalising the method of Kohnen and Skoruppa in [KS89] to the case of orthogonal modular forms of real signature $(2, n+2), n \geq 1$.

In **Chapter 5**, we obtained a Rankin-Selberg integral representation of the Dirichlet series through an orthogonal Eisenstein series of Klingen-type. Note that orthogonal Eisenstein series associated with zero-dimensional cusps are well-studied ([Sch22, Chapter 2], [Kie23]), but it seems that little explicit work has been done for 1-dimensional cusps.

Our goal was then to obtain an explicit theta correspondence with a Siegel-type Eisenstein series for the symplectic group of degree 2, the analytic properties of which are well-known. We used the classical method of rewriting the Eisenstein series as an Epstein zeta function, as has been done in a series of papers dealing with such Dirichlet series (e.g. [KS89], [Gri92a] and [Kri91]). In our case, we were able to do this only when the underlying lattice has one 1-dimensional cusp. The main difficulty seems to arise from the general form of the symmetric space in the orthogonal setting. In particular, with the notation of Chapter 4, elements of $\mathcal{H}_S$ are vectors in $\mathbb{C}^{n+2}$, contrary to the symplectic and Hermitian cases, where the symmetric space consists of square matrices. Krieg in [Kri91, p. 248] exploits this fact and, by using a method involving minors of determinants, manages to write this Dirichlet series as an Epstein zeta function uniformly for the symplectic, Hermitian and quaternionic case.

Nevertheless, under this restriction and through the use of differential operators, we managed to make the theta series of zero weight and remove the terms that cause the inner-product integrals to diverge, thus obtaining our result. An interesting question that might be worth exploring is whether we can remove the condition $4 \mid n$ that appears in Theorem 5.5.2. This amounts to asking the question of whether an analogue of the differential operator R of (5.4.2) that transforms with non-zero weight exists.

In **Chapter 6**, we considered the problem of obtaining Euler products for the Dirichlet series. By taking G in the Maass space and F a Hecke eigenform, we managed to prove Euler products for some specific orthogonal groups and connect the Dirichlet series with the standard L-function attached to F.

In particular, our strategy was to generalise the method of Kohnen and Skoruppa to the orthogonal setting. First, we obtained an explicit form of the adjoint of the operator V_N (see Section 6.1) and its action on the Fourier coefficients of a Fourier-Jacobi form, in the style of the main Proposition in [KS89, p. 549]. To our knowledge, this is the first time such a formula appears in the orthogonal case, even though the operator V_N is well-known (see, for example, [Moc19], [Sug85], [Gri92a]).

As a next step, Kohnen and Skoruppa compute the Nth term of their Dirichlet series and obtain an expression involving the Fourier coefficients of F. Crucially for them, the determinant of each matrix appearing in the Fourier coefficients of this expression is fixed. Hence, they can group terms together using the well-known theorem on binary quadratic forms of fixed discriminant. We can also obtain a similar grouping of terms, thanks to the main theorem of Shimura in [Shi06b, Theorem 2.2]. Note, however, that it is necessary to omit a finite set of primes in order to apply this Theorem. In this way, we obtain our Proposition 6.4.10.

To then obtain a relation to L-functions, Kohnen and Skoruppa use the well-known formula of Andrianov in [And74, Theorem 2.4.1], which gives a relation between a Dirichlet series involving the Fourier coefficients of a degree two Siegel cuspidal eigenform, twisted by ideal class characters, with the spin L-function attached to F. In our case, we use the formula of Sugano in [Sug85], which serves as a generalisation of Andrianov's formula to the orthogonal setting. In particular, Sugano's formula involves twists by definite orthogonal forms of rank $n+1$ (here, we work with a quadratic space of real signature $(2, n+2)$).

To finally obtain clear-cut Euler products, the main difficulty arises from the fact that, in general, there does not seem to exist an easy connection between

$$\sum_{i=1}^{h} \zeta_{\xi_i}(s-k+n+1) f_j(u_i) \text{ and } L\left(\overline{f_j}; s-k+(n+3)/2\right), \qquad (7.2.1)$$

as we mentioned before Section 6.6. Kohnen and Skoruppa can establish such a connection in the $n=1$ case by using the correspondence between equivalence classes of binary quadratic forms with fixed discriminant and ideal classes in the class group of a quadratic extension. This is because the functions f_j correspond to ideal class characters in their setting.

This has led us to restrict our attention to the case where $h=1$. In order to find explicit examples, we need to pick ξ such that the size of $H(\xi)_{\mathbb{Q}} \backslash H(\xi)_{\mathbb{A}} / (H(\xi)_{\mathbb{A}} \cap C)$ is 1. The reason we choose $\xi = (1, \mathbf{0}, 1)^t$ is that then $\phi_0[\xi] = 1$, which does not have prime factors, hence makes it possible to use the results of Shimura in order to compute the size of this quotient (see for example (6.6.1) and the proof of Proposition 6.6.7). In the rank 1 case, Shimura in [Shi06b] has given an explicit formula for the index $[H(\xi)_{\mathbb{A}} : H(\xi)_{\mathbb{Q}}(H(\xi)_{\mathbb{A}} \cap C)]$, which we use to find all the cases for which this index is 1 (see Section 6.6.1). In the rank $n \geq 2$, such a formula is not available, so we need an explicit description of $H(\xi)_{\mathbb{Q}}$ as a definite orthogonal group. It is well-known that there is a finite number of definite orthogonal groups with class number 1 and their rank is at most 10 (cf. [Wat63]). Using, therefore, the enumeration of definite orthogonal groups with class number 1 in [Han11], as well as [Shi04, Proposition 11.12], we arrive at the specific examples of Section 6.6 for rank $n \geq 2$.

A fascinating question that clearly arises is how one can remove the condition $h=1$. This is especially interesting when we consider the so-called "accidental" isogenies of low rank orthogonal groups with classical groups. Examples are $\mathrm{SO}(2,3)$ and Sp_2, $\mathrm{SO}(2,4)$ and $\mathrm{U}(2,2)$, $\mathrm{SO}(2,6)$ and $\mathrm{Sp}(2,\mathbb{H})$, where $\mathbb{H}$ is a quaternion algebra over $\mathbb{R}$. It may be possible to use these isogenies in order to obtain results without the $h=1$ restriction, at least for these cases. In a joint work with T. Bouganis in preparation, we are able to actually remove this condition for the $\mathrm{SO}(2,4)$ case. Crucially, we used the isogenies of orthogonal groups of signature $(2,4)$ with unitary groups of degree 2 and also the correspondence of Hermitian binary quadratic forms with ideals in a quaternion algebra. Answering question (7.2.1) is interesting in its own right; it means that we can relate the L-function of a definite orthogonal modular form to some concrete zeta functions counting numbers of solutions of equations. This can potentially have computational applications to these L-functions.

7.3 The case of the E_8 lattice

In this final Section, we would like to give a few remarks on how one could combine the results of Chapters 5 and 6 for the case of the E_8 lattice and obtain analytic properties for the standard L-function in this case. The two matrices in Sections 5.6 and 6.6.2 both correspond to the E_8 lattice as these are even, unimodular, and positive definite. It is well-known that there is a unique such lattice up to isometry ([Sch22, Example 1.2.10]).

Assume now $F \in S_k(\Gamma_S)$ is a Hecke eigenform. Let also $\xi = (1, \mathbf{0}, 1)^t$ and assume $A(\xi) \neq 0$. Because of Remark 6.3.3, we have that if we take $r = \mathbf{0}$, we have that $\mathcal{P}_{k,D,r} \in S_k^*(\Gamma_S) \subset S_k(\Gamma_S)$. Therefore, from Theorem 5.6.2, we obtain that $\mathcal{D}^*_{F,\mathcal{P}_{k,D,r}}(s)$ has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \longmapsto 2k-9-s$, where $\mathcal{D}^*_{F,\mathcal{P}_{k,D,r}}(s)$ denotes the completed Dirichlet series of (5.6.1).

Moreover, in the case of the E_8 lattice, we have a partial Euler product for $\mathcal{D}_{F,\mathcal{P}_{k,D,r},\mathcal{P}}(s)$, where $\mathcal{P}$ is a finite set of primes. In this case, and because the matrix S is unimodular, we have $q = 1$ and $D = -q$, so by checking the primes in Theorem 6.6.9, we can see $\mathcal{P} = \{2\}$.

We will describe a couple of ways one could try in order to compensate for the missing Euler factors. Unfortunately, none of these is fully developed yet.

7.3.1 Characteristic Twists

Let $N := \prod_{p \in \mathcal{P}} p$ (we know in this case $N = 2$, but this method could work for other lattices too). Let χ denote a Dirichlet character $\pmod N$ such that $\chi(m) = 1$ for all $(m, N) = 1$. We then define

$$F_\chi(Z) := \sum_{m=1}^{\infty} \chi(m)\phi_m(\tau, z)e^{2\pi i m\omega},$$

where $Z = (\omega, z, \tau) \in \mathcal{H}_S$. We consider the congruence subgroup of Γ_S given by

$$\Gamma_S(N, N^2, 1) := \left\{ M = \begin{pmatrix} A & X & B \\ Y & L & Z \\ C & W & D \end{pmatrix} \in \Gamma_S \mid Y \equiv W \equiv 0_2 \pmod N \right),$$

$$C \equiv 0_2 \pmod{N^2},\ A \equiv D \equiv 1_2 \pmod{N}\Big\}.$$

In analogy to [KSK95, Proposition 1], we have the following Lemma.

Lemma 7.3.1. *We have*

$$F_\chi(Z) = \frac{1}{N}\sum_{\nu(N)}\sum_{\mu(N)} \chi(\nu)e^{-2\pi i\nu\mu/N}(F\mid_k T_{(\mu/N,0,0)})(Z),$$

where T_λ is defined in Section 4.4 and $\nu(N)$ means that N runs through a set of representatives $\pmod{N}$*. We then have that F_χ is an orthogonal cusp form of weight k with respect to $\Gamma_S(N, N^2, 1)$.*

Proof. The proof of this is analogous to the proof of [KSK95, Proposition 1] after we check that $T_{(\mu/N,0,0)} \cdot M \cdot T^{-1}_{(\mu/N,0,0)} \in \Gamma_S$ for all $M \in \Gamma_S(N, N^2, 1)$. □

Now, for $Z \in \mathcal{H}_S$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n+1$, we define

$$E_{N,N^2,1}(Z, s) := \sum_{\gamma\in\Gamma_{S,J}(N,N^2,1)\backslash\Gamma_S(N,N^2,1)} \left(\frac{Q_0[\mathrm{Im}(\gamma Z)]}{\mathrm{Im}((\gamma Z)_2)}\right)^s,$$

where for $Z = (\omega, z, \tau) \in \mathcal{H}_S$, we write $Z_2 := \tau$. Also, $\Gamma_{S,J}(N, N^2, 1) := \Gamma_{S,J} \cap \Gamma_S(N, N^2, 1)$ and $Q_0 = S_0/2$. In analogy with Proposition 5.1.4, $E_{N,N^2,1}(Z, s)$ converges absolutely and uniformly in compact subsets of $\mathcal{H}_S$ and is invariant under the action of $\Gamma_{S,J}(N, N^2, 1)$.

By performing an integral analogous to Proposition 5.1.5, we can obtain an integral representation of $\mathcal{D}_{F,\mathcal{P}_{k,D,r},\mathcal{P}}(s)$ via $\langle E_{N,N^2,1}(Z, s)F_\chi(Z), G(Z)\rangle$.

One can now study $E_{N,N^2,1}(Z, s)$ in order to obtain the analytic properties of $\mathcal{D}_{F,\mathcal{P}_{k,D,r},\mathcal{P}}(s)$. This could be done in a similar way as in Chapter 5 (see also [KSK95, Proposition 2]). Alternatively, in an analogous way to [Shi00, (23.13a)], we have the relation

$$[\Gamma_{S,J} : \Gamma_{S,J}(N, N^2, 1)]E(Z, s) = \sum_{\delta\in\Gamma_S(N,N^2,1)\backslash\Gamma_S} E_{N,N^2,1}(\delta\langle Z\rangle, s).$$

Hence, one could attempt to deduce properties of $E_{N,N^2,1}(Z, s)$ from those of $E(Z, s)$.

7.3.2 Euler Product

This approach is based on the following unproven assumption. Assume $F \in S_k(\Gamma_S)$, with $\phi_1 \not\equiv 0$.

Assumption: $\mathcal{D}_{F,\mathcal{P}_{k,D,r}}(s)$ admits an Euler product, i.e. it can be written as the product of its p-factors (i.e. summing over p).

Remark 7.3.2. We write $\{\psi_m\}_{m=1}^{\infty}$ for the Fourier-Jacobi coefficients of $\mathcal{P}_{k,D,r}$. It suffices to show that $m \longmapsto \langle \phi_m, \psi_m \rangle / \langle \phi_1, \psi_1 \rangle$ is multiplicative. But by definition, $\psi_m = V_m \psi_1$. Moreover, from [Moc19, Corollary 4.35], V_m, hence its adjoint V_m^* is multiplicative. Hence, for $(m, n) = 1$, we can write

$$\langle \phi_{mn}, \psi_{mn} \rangle = \langle \phi_{mn}, V_{mn}\psi_1 \rangle = \langle V_{mn}^* \phi_{mn}, \psi_1 \rangle.$$

Now, V_m is the same as the operator $T_-(m)$ defined by Gritsenko in [Gri91] in a parabolic Hecke algebra. We hope that $T_-(m)$ is actually in the image of a "global" Hecke operator through an embedding of the same type as the one in Lemma 2.4.1, as is the case for the symplectic and Hermitian case. Then, the multiplicativity of the Hecke operators in the global Hecke algebra would give us the required result.

Given this assumption, one could proceed as follows. We write G for $\mathcal{P}_{k,D,r}$. We then have

$$\mathcal{D}_{F,G}(s) = \mathcal{D}_{F,G,\mathcal{P}}(s) \widetilde{\mathcal{D}}_{F,G,\mathcal{P}}(s), \tag{7.3.1}$$

where $\widetilde{\mathcal{D}}_{F,G,\mathcal{P}}(s) := \prod_{p \in \mathcal{P}} \widetilde{\mathcal{D}}_{F,G,p}(s)$ with

$$\widetilde{\mathcal{D}}_{F,G,p}(s) = \sum_{m=1}^{\infty} \langle \phi_{p^m}, \psi_{p^m} \rangle p^{-ms}.$$

Then, $\mathcal{D}^*_{F,G}(s)$ of (5.6.1) also admits an Euler product expansion and hence a decomposition of the form of (7.3.1). Now, by using the expression of (6.6.8), we have

$$\begin{aligned}
\mathcal{D}^*_{F,G,\mathcal{P}}(s) &= 4^{-s} \pi^{\alpha(s)} \Gamma(s) \Gamma\left(\frac{s-k+6}{2}\right) \Gamma(s-k+5) \Gamma\left(\frac{s-k+9}{2}\right) \times \\
&\quad \times \Gamma\left(\frac{s-k+8}{2}\right) \gamma_S(s-k+9) \prod_{i=3}^{7} \zeta_{\mathcal{P}}(s-k+10-i)^{-1} \times \\
&\quad \times L_{\mathcal{P}}(F; s-k+5) \\
&= 4^{-s} \pi^{-(s+4)} \Gamma(s) \Gamma\left(\frac{s-k+6}{2}\right) \Gamma(s-k+5) \Gamma\left(\frac{s-k+9}{2}\right) \times \\
&\quad \times \Gamma\left(\frac{s-k+8}{2}\right) \gamma_S(s-k+9) \times \\
&\quad \times \prod_{i=3}^{7} \left[\Gamma\left(\frac{s-k+i}{2}\right) \xi_{\mathcal{P}}(s-k+i)^{-1} \right] L_{\mathcal{P}}(F; s-k+5) \\
&= 4^{-s} \pi^{-(s+4)} \Gamma(s) \Gamma\left(\frac{s-k+6}{2}\right) \Gamma(s-k+5) \Gamma\left(\frac{s-k+9}{2}\right) \times \\
&\quad \times \Gamma\left(\frac{s-k+10}{2}\right) \Gamma\left(\frac{s-k+9}{2}\right) \Gamma\left(\frac{s-k+8}{2}\right) \times
\end{aligned}$$

$$\times\Gamma\left(\frac{s-k+7}{2}\right)\Gamma\left(\frac{s-k+4}{2}\right)\Gamma\left(\frac{s-k+3}{2}\right)\phi_2\left(\frac{s-k+1}{2}\right)\times$$
$$\times\phi_2\left(\frac{s-k+9}{2}\right)L_{\mathcal{P}}(F;s-k+5)\prod_{i=3}^{7}\xi_{\mathcal{P}}(s-k+i)^{-1},$$

where in the first equality, $\alpha(s)=-(7s-5k+33)/2$. Also, to obtain the last equality, we used the relation $\Gamma(s+1)=s\Gamma(s)$. Now, we observe that the expression

$$\pi^{-4}\phi_2\left(\frac{s-k+1}{2}\right)\phi_2\left(\frac{s-k+9}{2}\right)\prod_{i=3}^{7}\xi_{\mathcal{P}}(s-k+i)^{-1}$$

is invariant under $s\longmapsto 2k-9-s$. Therefore, we arrive at the following Theorem.

Theorem 7.3.3. *Let the assumptions be as in the beginning of the Section. We define the **completed** L-function attached to F, via*

$$\Lambda_{\mathcal{P}}(F;s):=(4\pi)^{-(s+k-5)}\Gamma(s)\Gamma(s+k-5)\Gamma\left(\frac{s-2}{2}\right)\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\times$$
$$\times\Gamma\left(\frac{s+2}{2}\right)\Gamma\left(\frac{s+3}{2}\right)\Gamma^2\left(\frac{s+4}{2}\right)\Gamma\left(\frac{s+5}{2}\right)L_{\mathcal{P}}(F;s)$$

Then, $\Lambda_{\mathcal{P}}(F;s)$ admits a meromorphic continuation to $\mathbb{C}$ and satisfies

$$\Lambda_{\mathcal{P}}(2k-9-s)=\Lambda_{\mathcal{P}}(s)\frac{\widetilde{\mathcal{D}}^{*}_{F,G,\mathcal{P}}(s)}{\widetilde{\mathcal{D}}^{*}_{F,G,\mathcal{P}}(2k-9-s)}.$$

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