A Review of Linear Rational Expectations Models: Solution Methods, Existence, Uniqueness and Discontinuity

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Abstract

This paper reviews solution methods for solving linear rational expectations models divided into three categories, then discusses the existence, uniqueness and discontinuity of the model solutions, and reviews a regularization method to solve the discontinuity problems of the solution to linear rational expectations model.

Keywords: Linear rational expectations model, solution methods, uniqueness, discontinuity, regularization.

1 Introduction

When policy rules change, economic agents adjust their behavior accordingly. Lucas (1976) points out that traditional empirical regression models fail to incorporate such behavioral adjustment mechanisms and therefore fail to be used as reliable and stable tools for policy analysis. This critique spurred the development of rational expectations theory, with its core idea, as Muth (1961) indicates, that the economic agents use their understanding of the economic structure when forming expectations about the future, and their expectations are consistent with reality in a statistical sense. Such models built on this foundation are known as rational expectations models, requiring expectations to be generated endogenously within the model rather than being imposed exogenously. However, since rational expectations introduce intertemporal variables (such as inflation and output), the resulting model structures become highly complex. To preserve the rational expectations assumption while maintaining operational tractability, researchers typically linearize the model structure, giving rise to linear rational expectations model (LREM).

As an important subclass of dynamic stochastic general equilibrium (DSGE) models, LREMs are widely used in economics to describe the dynamic evolution of economic variables over time. LREMs can be represented in various forms, but their fundamental assumption is consistent: under the rational expectations hypothesis, all such models incorporate the behavior of economic agents, who optimize their decisions based on expectations of future economic variables. Typically, as Whiteman (1984) suggested, an LREM is usually expressed by a system of linear dynamic equations that include lagged variables and forward-looking expectations to describe the evolution of variables over time. The model also includes shock terms modeled as white noise or other stochastic processes, to capture the impact of exogenous shocks on the economic system. LREM generally includes two types of endogenous variables: predetermined variables, which are determined by past decisions and reflect the "inertia" of the economy (such as capital deposit); non-predetermined variables (or jump variables), which can be adjusted immediately by expectations about the future to satisfy the equilibrium conditions (such as prices or interest rates). The exogenous variables in LREM are given externally and are independent of the current state of the model, such as random shocks (e.g., technology) or policy variables.

The solution to an LREM is a dynamic path whereby the endogenous variables both satisfy the model structure and form their rational expectations in a mutually consistent manner at each step. This solution is not determined by a single equation but rather by the constraints of the entire system, the expectation structure, stability requirements, and initial conditions. An LREM may have multiple solutions, meaning that under a given set of parameters, multiple distinct trajectories can satisfy the model equations. Alternatively, an LREM may have a unique solution, where a well-defined mechanism for expectation formation ensures that the behavior of endogenous variables is uniquely determined given the exogenous shocks. In this case, economic agents do not deviate from equilibrium due to incorrect expectations, and the unique solution serves as a determinate and stable tool for policy analysis. Policymakers are therefore particularly concerned with the conditions for uniqueness. Simultaneously, some LREM may have no solutions at all. Since Blanchard and Kahn (1980) proposed the classic stability conditions for LREMs, numerous alternative methods have been developed to solve these models such as Anderson and Moore (1985), King and Watson (1998) , Klein (2000), Sims (2002), Binder and Pesaran (1999) and Uhlig (1995). Currently, there are few papers that systematically review solution methods for LREMs. A representative paper is Anderson (2008), but it primarily focuses on emphasizing the advantages of the Anderson and Moore (1985).

For models that satisfy the Blanchard-Kahn conditions, these methods produce equivalent solutions in the time domain, the difference lies in their respective core ideas. Despite their respective advantages, these methods invariably expose a fundamental challenge in LREM solutions: non-uniqueness. If the solution to LREM is not unique, there exist multiple (or even infinite many) dynamic paths that simultaneously satisfy all the equations constraints of the given model. Using the thought of Shell (1989), different paths may correspond to different macroeconomic dynamics, and the equilibrium state my depend on "beliefs" beyond expectations (known as "sunspot equilibria", see p.274 of Shell (1989)). Therefore, policymakers fails to uniquely predict which equilibrium the economic agents will coordinate on, leading the policy analysis ambiguous and even ineffective.

When there are multiple solutions for an LREM, these solutions form a high-dimensional linear subspace. One must select a solution from this subspace for further analysis. If this selection is arbitrary, then even small changes in the model parameters may cause the solution to jump along other directions within the subspace, which is called discontinuity of the solution. In empirical work for solving LREMs, researchers typically need to estimate model parameters using methods such as maximum likelihood or Bayesian inference. For maximum likelihood estimation, if the solution to the linear rational expectations model is discontinuous, then even minor changes in the parameters may cause the equilibrium of the model to jump to an entirely different region; moreover, a discontinuous objective function can prevent optimization algorithms from finding a maximum or cause them to oscillate among multiple local optima. Therefore, for frequentist methods, it is generally required that the objective function be at least continuous. For Bayesian methods, if the solution to the linear rational expectations model is discontinuous, then the likelihood function may exhibit sudden jumps at certain parameter values, which could result in the posterior distribution placing substantial probability mass on certain "isolated" points or "atoms" (see Kingman (1975)). In such cases, the posterior distribution would no longer be a continuous density but rather a mixture that includes discrete mass points. Thus, for Bayesian inference, it is desirable that the posterior remains a continuous density, without unexpected atoms appearing at unknown locations. Al-Sadoon (2020) proposed a method named regularization, which allows the users to select a solution form the solution subspace which is continuous with respect to the model parameters by incorporating the prior information or the "preferences" of the users. The regularized solution ensures both the uniqueness and economic interpret-ability of the solution, while also allows the condition of using both frequentis and Bayesian analysis.

This paper is organized as follows. Section 2 reviews LREM solution methods by three categories; Section 3 discusses the existence, uniqueness of the LREM solutions, revealing how these issues manifest in solution methods; Section 4 discusses the discontinuity of the LREM solution and reviews the regularization. Section 5 concludes.

2 Review of the LREM Solution Methods

Before starting the review of LREM solution methods, it is important to understand the basic structure of these models and what it means for a solution to be economically and mathematically valid. LREMs are characterized by the current decisions that depend not only on past and present conditions, but on expectations of the economic agents about the future. This forward-looking behavior, while economically realistic, can create mathematical instability in the model. Thus, the key problems of any solution methods is to determine whether a solution exist, whether it is stable, and is unique. This paper next will have a quick look into stochastic difference equations to ensure that one can understand the essential ideas hidden in the LREM solutions by these methods. The letters and Latin symbols used below are defined independently within each subsection, one can freely choose their preferred notation when applying these methods.

Consider a model with single variable:

$$y_t = x_t + bE_t[y_{t+1}]$$

where
y_t is the current decision variable (e.g., consumption, price and output);
x_t is a fundamental (while x_{t-1} is predetermined) variable known at time t;
$E_t[y_{t+1}]$ is the expectation of y_{t+1} (non-predetermined variable) formed at time t;
$b \in \mathbb{R}$ measures the weight of the forward-looking channel.

This equation of the model shows that current decisions y_t depend on current fundamentals x_t and beliefs about the future y_{t+1}. Iterating this equation forward:

$$y_t = x_t + bE_t[x_{t+1}] + b^2 E_t[x_{t+2}] + \cdots + b^n E_t[y_{t+n}],$$

then add a transversality condition to ensure this expression converges:

$$\lim_{n\to\infty} b^n E_t[y_{t+n}] = 0,$$

if such condition holds, the solution can be written as

$$y_t = \sum_{k=0}^{n-1} b^k E_t[x_{t+k}].$$

It is noticed that this forward-looking solution is valid and stable (mathematically converges) only if $|b| < 1$, since the distant future effects will vanish with the time increases. If $|b| > 1$, the forward-looking solution diverges, in such case, the model can be rewritten as

$$y_t = x_t + by_{t+1} + b\varepsilon_{t+1}$$

where ε_{t+1} is the expectational error term. Iterating backward yields:

$$y_t = -\sum_{k=0}^{\infty} b^{-k} \varepsilon_{t-k} - \sum_{k=1}^{\infty} b^{-k} x_{t-k},$$

which converges only if $|b| > 1$.

Therefore, the conclusion is that if $|b| < 1$, one should use the forward-looking solution; if $|b| > 1$, one should use the backward-looking solution. This is the very essential idea of the Blanchard-Kahn condition in the following review part.

Blanchard-Kahn Condition

Blanchard and Kahn (1980) proposed the well-known Blanchard-Kahn condition denotes that if the number of the unstable roots (eigenvalues outside the unit circle) in the model equal to the number of jump (forward-looking) variables, there is a unique and stable solution to LREM; if the number of unstable roots is fewer than the jump variables, there are infinite many solutions to the model; if the number of unstable roots exceeds the jump variables, there is no solution.

A linear rational expectation model can be expressed as:

$$Bx_{t+1} = Ax_t + G\varepsilon_t$$

$$BE_t y_{t+1} = Ay_t + G\varepsilon_t,$$

or in matrix form

$$B \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G\varepsilon_t,$$

where
x_t is $n \times 1$ vector of predetermined variable at time t;
y_t is $m \times 1$ vector of non-predetermined (jump) variables at time t;
ε_t is $k \times 1$ vector of stochastic shocks;
A, B are $(n+m) \times (n+m)$ square matrices and G is $(n+m) \times k$ matrix due to the dimension of ε_t.

The expectation of y_{t+1} is essentially a conditional expectation denoted as $E_t(y_{t+1}|\Omega_t)$, where Ω_t is information set at time t, which may include past and current values of x, y and ε. The difference between predetermined and non-predetermined variable is a that predetermined variables is a function of variables known at $t+1$ in Ω_{t+1} while non-predetermined variable may be a function of any variable in Ω_t.

Suppose B is non-singular (invertible), the model can be rewritten as

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B^{-1}A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B^{-1}G\varepsilon_t,$$

replace $B^{-1}A$ by Z, and $B^{-1}G$ by W, there is

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = Z \begin{bmatrix} x_t \\ y_t \end{bmatrix} + W\varepsilon_t,$$

which is a difference equation system in matrix form. Blanchard and Kahn (1980) then performs diagonalisation on Z to obtain PDP^{-1}, where eigenvalues orders from the smallest to largest on the diagonal of matrix D, the corresponding eigenvalues withe same order forms P. Blanchard-Kahn condition depends on the number of eigenvalues which are outside the unit circle. If the number of eigenvalues larger than one equals m, which is the dimension of y, then the Blanchard-Kahn condition holds, there is a stable solution.

Taking the diagonalisation,

$$Z = PDP^{-1} \Rightarrow P^{-1} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = DP^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + P^{-1}W\varepsilon_t$$

partition the matrices P^{-1} and D as

$$P^{-1} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix}, \quad D = \begin{bmatrix} \hat{\Lambda}_{11} & 0 \\ 0 & \hat{\Lambda}_{22} \end{bmatrix}, \quad \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix} = P^{-1}W = P^{-1}B^{-1}G$$

Substituting back to obtain:

$$\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{\Lambda}_{11} & 0 \\ 0 & \hat{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix} \varepsilon_t$$

Multiplying the matrices gives the system of equations

$$\hat{P}_{11}xt + 1 + \hat{P}_{12}E_t yt + 1 = \hat{\Lambda}_{11}(\hat{P}_{11}x_t + \hat{P}_{12}y_t) + \hat{G}_1\varepsilon_t$$

$$\hat{P}_{21}xt + 1 + \hat{P}_{22}E_t yt + 1 = \hat{\Lambda}_{22}(\hat{P}_{21}x_t + \hat{P}_{22}y_t) + \hat{G}_2\varepsilon_t$$

Focusing on the second equation above, which involves the explosive dynamics (since eigenvalues in $\hat{\Lambda_{22}}$ are outside the unit circle), define:

$$\lambda_t = \hat{P}_{21}x_t + \hat{P}_{22}y_t,$$

then the second equation becomes

$$E_t \lambda_{t+1} = \hat{\Lambda}_{22} \lambda_t + \hat{G}_2 \varepsilon_t,$$

solving forward, since all eigenvalues in $\hat{\Lambda_{22}}$ are greater than one in modulus:

$$\lambda_t = -\sum_{i=0}^{\infty} \hat{\Lambda}_{22}^{-i-1} \hat{G}_2 E_t \varepsilon_{t+i}.$$

Suppose future shocks are identically distributed (i.i.d.) with zero expectation, then there is

$$\lambda_t = -\hat{\Lambda}_{22}^{-1} \hat{G} 2 \varepsilon_t \Rightarrow \hat{P}_{21} x_t + \hat{P}_{22} y_t = -\hat{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t$$

Solving for y_t,

$$\hat{P}_{22} y_t = -\hat{P}_{21} x_t - \hat{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t \Rightarrow y_t = -\hat{P}_{22}^{-1} \hat{P}_{21} x_t - \hat{P}_{22}^{-1} \hat{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t,$$

taking expectation of y_{t+1},

$$E_t y_{t+1} = -\hat{P}_{22}^{-1} \hat{P}_{21} x_{t+1}$$

Substituting both above two equations into

$$\hat{P}_{11} x t + 1 + \hat{P}_{12} E_t y t + 1 = \hat{\Lambda}_{11} (\hat{P}_{11} x_t + \hat{P}_{12} y_t) + \hat{G}_1 \varepsilon_t,$$

then there is

$$\hat{P}_{11} x_{t+1} - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21} x_{t+1} = \hat{\Lambda}_{11} [\hat{P}_{11} x_t - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21} x_t - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t] + \hat{G}_1 \varepsilon_t,$$

after rearranging,

$$[\hat{P}_{11} - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21}] x_{t+1} = \hat{\Lambda}_{11} [\hat{P}_{11} - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21}] x_t - \hat{\Lambda}_{11} [\hat{P}_{12} \hat{P}_{22}^{-1} \hat{\Lambda}_{22}^{-1} \hat{G}_2 + \hat{G}_1] \varepsilon_t,$$

and the non-predetermined variables are obtained:

$$x_{t+1} = [\hat{P}_{11} - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21}]^{-1} \hat{\Lambda}_{11} [\hat{P}_{11} - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21}] x_t - [\hat{P}_{11} - \hat{P}_{12} \hat{P}_{22}^{-1} \hat{P}_{21}]^{-1} \hat{\Lambda}_{11} [\hat{P}_{12} \hat{P}_{22}^{-1} \hat{\Lambda}_{22}^{-1} \hat{G}_2 + \hat{G}_1] \varepsilon_t,$$

this is the equilibrium law of motion under stochastic shocks. Since all eigenvalues in $\hat{\Lambda}_{11}$ are stable (inside the unite circle), this solution is dynamic stable.

The above derivation demonstrates that in order to prevent explosive from the eigenvalues in $\hat{\Lambda}_{22}$, the model must impose exactly m constraints on the jump variables y_t corresponding to the m eigenvalues outside the unit circle, which is:

(i) if the number of unstable roots equals to m, LREM has a unique stable solution;
(ii) if the number of unstable roots are more than m, LREM has no solution, the model system is overdetermined;
(iii) if the number of unstable roots are fewer than m, LREM has multiple or even infinite many solutions, the model is under-determined.

Blanchard-Kahn gives the necessary and sufficient condition for a unique, stable solution. Note that it never discusses some special cases such as B is singular when forming $Z = B^{-1}A$. But it provides a clear standard for determining the nature of the model solutions and becomes the fundamental basis for understanding and solving the LREMs. Since Blanchard and Kahn (1980) proposed these classic stability conditions for linear rational expectations models, a variety of alternative methods have been developed to solve such models based on distinct underlying ideas, such as Anderson and Moore (1985) and King and Watson (1998) apply eigenvalue system; Sims (2002) and Klein (2000) use QZ decomposition; Binder and Pesaran (1999) and Uhlig (1995) apply matrix polynomials. Researchers use the outputs of these solution techniques to estimate models, compute impulse response functions, calculate asymptotic covariances, solve infinite-horizon linear-quadratic control problems, and construct terminal constraints for nonlinear models.

This paper next reviews three solution methods for LREM divided into the above three categories of underlying ideas respectively.

Applying Eigenvalue System: AIM

Anderson and Moore (1985) provides a systematic and efficient procedure (denoted as AIM in this paper) to solve the linear rational expectations models. The core idea of the AIM is transforming the original model into a structured matrix problem and applying a sequence of linear algebra techniques to determine the existence, uniqueness and the form of the solution. AIM is easy to implement by MATLAB, code implementing AIM can be found on the official website of the Federal Reserve. This section states the mathematical process that outlines the solution strategy entailed in AIM.

AIM solves models of the form:

$$\sum_{i=-\tau}^{0} G_i x_{t+i} + \sum_{i=1}^{\theta} F_i E_t[x_{t+i}] = \epsilon_t, \quad \tau > 0, \ \theta > 0$$

where
$x_t \in \mathbb{R}^n$ contains all the variables, irrespective of whether they have an endogenous, an exogenous or a predetermined nature;

G_i, $F_i \in \mathbb{R}^{n\times n}$ are given coefficient matrices on current, lagged and lead values;
$\epsilon_t \in \mathbb{R}^n$ is a mean-zero shock;
the number of the lags τ and leads θ are greater than 1.

The model allows for arbitrary lags and leads. The solution sought takes the following form:

$$x_t = \sum_{i=1}^{\tau} B_i x_{t-i} + B_0 \epsilon_t$$

where B_i are reduced form coefficient matrices that represent the dynamics of the endogenous variables. AIM computes these B_i from the original structural system. First, by forward iterating the model and taking expectations, AIM constructs a homogeneous system:

$$\sum_{i=-\tau}^{0} H_i x_{t+k+i} = 0, \quad k \geq 0$$

where if $i \leq 0$, $H_i = G_i$, and $i > 0$, $H_i = F_i$. These equations are stacked into a system over time t, using the block vector

$$X_t = \begin{bmatrix} x_{t+\theta} \\ x_{t+\theta-1} \\ \vdots \\ x_{t-\tau+1} \end{bmatrix} \in \mathbb{R}^{n(\tau+\theta)},$$

which leads to the formation of a generalized first-order linear system:

$$AX_{t+1} = BX_t,$$

where A, B are $n(\tau+\theta) \times n(\tau+\theta)$ matrices constructed by shifting the H_i across the block structure, which capture the dynamic structure of the system with all lags and leads preserved.

AIM second determines the existence and uniqueness of a stable solution by solving a generalized eigenvalue problem (in MATLAB, using $eig(A, B)$):

$$\lambda A s = B s$$

to obtain $n(\tau+\theta)$ eigenvalue–eigenvector pairs $(\lambda_1, s_1), (\lambda_2, s_2), \ldots, (\lambda_{n(\tau+\theta)}, s_{n(\tau+\theta)})$. Let m denotes the forward-looking (or jump) variables outside the unit circle ($|\lambda| > 1$), which must match the number of jump variables m in the model to satisfy the Blanchard-Kahn condition.

Define the set of stable eigenvectors associated with eigenvalues inside the unite circle

$$\mathcal{S}\text{stab} = \{s_j \in \mathbb{C}^{n(\tau+\theta)} : |\lambda_j| < 1\}.$$

Let Sstab be the matrix formed by column-wise stacking the vectors in $\mathcal{S}$stab

$$S\text{stab} = [s_{j_1}, s_{j_2}, \ldots, s_{j_k}],$$

where each $|\lambda_{j_\ell}| < 1$.

AIM next impose a convergence condition, which is the full stacked state vector X_t must lie in the column space (or image) of Sstab

$$X_t \in \mathrm{Im}(S_{\text{stab}}),$$

this ensures that the trajectory of the system remains bounded and excludes the explosive solutions. To construct X_t, let the known vector of initial conditions or the historical states of the system be:

$$\tilde{x}t = \begin{bmatrix} xt - \tau + 1 \\ x_{t-\tau+2} \\ \vdots \\ x_t \end{bmatrix} \in \mathbb{R}^{n\tau}.$$

If such an X_t can be uniquely written as a linear combination of the stable eigenvectors in Sstab, then the system has a unique stable solution; if no such X_t exists, the model has no solution; if more than one X_t are found, the model has multiple solutions or even infinite many solutions.

In some cases, the structural system contains redundant or linear dependent equations, especially when some H_i are rank-deficient (not in full rank). AIM solves this issue by using a QR decomposition on the constraint matrices. For any rank-deficient H_i, compute (in MATLAB, $qr(H_i)$):

$$H_i = Q_i \times R_i, \quad Q_i, R_i \in \mathbb{R}^{n\times n}$$

where Q_i is an orthogonal matrix with $Q_i^\top Q_i = I$, and R_i is an upper triangular matrix with possible zero rows. Left-multiplying the system by $Q_i^\top$ gives

$$Q_i^\top H_i x_{t+i} = Q_i^\top \varepsilon_t,$$

which shifts zero rows (redundant constraints) to the top of the matrix, leaving the non-zero constraint block at the bottom. By repeating this decomposition across all H_i, AIM ensures that the system of constraints has full rank, thus the matrix pencil (A, B) is well-posed. If the effective number of constraints does not match the number of jump variables after the QR decomposition, AIM reports either no solution (too many constraints) or multiple solutions (too few constraints).

Once the convergence condition is satisfied and the expectation term are eliminated sing the solution subspace, AIM recovers the observable structure

$$\sum_{i=-\tau}^{0} S_i x_{t+i} = \varepsilon_t,$$

which is a purely backward-looking representation consistent with the data-generating process. By left-multiplying with $-S_0^{-1}$ (when it is invertible), the reduced form is:

$$x_t = \sum_{i=1}^{\tau} B_i x_{t-i} + B_0 \varepsilon_t,$$

which is the solution to the LREM showed above.

AIM requires no process that transforming the model into a special form with only one lag or lead, and no expression explicitly distinguishing between predetermined and non-predetermined variables, thus simplifying the coding implementation. As the model size increases, the computational advantages of AIM become more pronounced. Coenen et al. (2021) applies AIM to solve a simplified NAWM II (New Area-Wide Model II), a model designed to support monetary policy decisions for the Euro area while analyzing the impact of financial frictions and policy tools on the economy, significantly improving the efficiency and robustness of the solution. The application of AIM provides NAWM II with a powerful technical foundation to support the monetary policy research and decision-making of the European central bank in a low-interest-rate environment.

The key note of using AIM is that, since it does not distinguish explicitly between predetermined and forward-looking variables, the number of jump variables must be specified to assess determinacy by users. What is more, although avoiding the explicit distinction simplifies the coding, AIM limits the ability to analyze models where this distinction carries important economic meanings. If assuming that historical data fully determine all variables date $t-1$ or earlier, then for the models where the distinction between predetermined and non-predetermined variables is crucial, AIM may not be appropriate.

Using QZ Decomposition: Sims

Sims (2002) provides a method (denoted as Sims) applies the generalized eigenvalue decomposition and matrix analysis to transform dynamic systems with lead and lag variables into a recursive relationship at a stable state. It begins with the standard form of a dynamic system, which can be expressed as:

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + \Psi z(t) + \Pi \eta(t)$$

where
y_t are the $L \times 1$ dimensional state variables;
z_t are the $M_1 \times 1$ dimensional exogenous variables, independent and i.i.d., with $E[z_t z_t'] > 0$;
η_t is the $M_2 \times 1$ dimensional expectational error, a k-dimensional martingale difference sequence with respect to z and is measurable with respect to z_t, z_{t-1}, ... with $E_t \eta_{t+1} = 0$;
Γ_0 is the $L \times L$ dimensional structural coefficients matrix;
Γ_1 is the $L \times L$ dimensional structural coefficients matrix;
Ψ is the $L \times M_1$ dimensional structural exogenous variables coefficients matrix;
Π is the $L \times M_2$ dimensional structural exogenous errors coefficients matrix, in which designation of expectational errors identifies the predetermined variables.

Sims uses the QZ decomposition. The QZ decomposition leverages the generalized Schur decomposition (see Golub and Van (2013)) to decompose the matrix pair, for example (A, B), in an LREM into two upper triangular matrices (T, S), thus rewriting the model into a standard recursive form. This decomposition clearly reveals the nature of the dynamic of the model system: by examining the eigenvalues $\lambda(A, B) = t_{ii}/s_{ii}$, the QZ decomposition distinguishes between stable and unstable components of the dynamics and solves for the path of the model system which satisfies the Blanchard-Kahn condition. Sims expresses the model in a form which is convenient for solving "forward", suggesting that it can determine the path of the endogenous variable consistent with arbitrary future values of the exogenous variables, this facilitates numerical computation while keeping theoretical soundness and transparency.

Under the assumption that the determinant $det(\Gamma_0 + \Gamma_1 x) \neq 0$ for all $x \in \mathbb{C}$ with $|x| = 1$ to ensure that y will not be canceled out by elementary algebraic operations and there is no unit root ($\lambda = 1$) in the model system, Sims uses two orthogonal matrices $Q, Z \in \mathbb{R}^{L \times L}$ such that $Q\Gamma_0 Z$ and $Q\Gamma_1 Z$ are block upper triangular with either 1×1 or 2×2 blocks on the diagonal (the following steps are followed by Al-Sadoon (2020)), the matrices are partitioned as:

$$Q\Gamma_0 Z = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix},$$

$\mathrm{Q}\Gamma_1 Z = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix}$ where Q and Z are $n \times n$ orthogonal matrices; all the zeros of the polynomial $det(\Lambda_{11} + \Omega_{11} x)$ outside the unit circle, such that Λ_{11} is non-singular; all the zeros of the polynomial $det(\Lambda_{22} + \Omega_{22} x)$ inside the unit circle, such that Ω_{22} is non-singular. Then introduce the new variable

$$w_t = Z' y_t = \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix}$$

where $w_{1,t}$ is the component in the unstable subspace, $w_{2,t}$ is the component in the stable subspace, then rewrite the model system as:

$$\Lambda w_t = \Omega w_{t-1} + Q\Psi z_t + Q\Pi\eta_t, \quad t \in \mathbb{Z}$$

where the stable part is

$$\Lambda_{22} w_{2,t} = \Omega_{22} w_{2,(t-1)} + Q_{2.}\Psi z_t + Q_{2.}\Pi\eta_t, \quad t \in \mathbb{Z}$$

the stable part of the system must exhibit covariance stationarity and thus cannot have explosive solutions. Moreover, since w_2 lies in the stable subspace, the associated dynamic matrix has all its eigenvalues within the unit circle. Consequently, from the expectation relationships, one obtain that

$$w_{2,(t-1)} = \Omega_{22}^{-1}\Lambda_{22}E_{t-1}w_{2,t}, \quad t \in \mathbb{Z},$$

which is

$$w_{2,t} = (\Omega_{22}^{-1}\Lambda_{22})^{s-t}E_t w_{2,s}, \quad s \geq t.$$

To avoid the drift or explosion in w_2, set $w_{2,t} = 0$. Substituting it back into the system:

$$Q_{2.}\Psi z_t + Q_{2.}\Pi\eta_t = 0, \quad t \in \mathbb{Z}.$$

Then multiplying right by z_t' and taking the expectations, using the joint covariance stationarity of η and z, there is

$$Q_{2.}\Psi E(z_0 z_0') + Q_{2.}\Pi E(\eta_0\eta_0') = 0.$$

Also, for arbitrary $t \in \mathbb{Z}$, the vector $(Q_2\Pi)^{\dagger}Q_2\Psi z_t + \eta_t$ lies in the kernel (null space) of $Q_2\Pi$, where the $Q_2\Pi)^{\dagger}$ is the Moore-Penrose generalized inverse (see the guidance of Barata and Hussein (2012)) of $Q_2\Pi$, and $E_{t-1}((Q_2\Pi)^{\dagger}Q_2\Psi z_t + \eta_t) = 0$. Therefore, given arbitrary matrix denoted as K whose columns form a basis for the kernel of $Q_2\Pi$, there is a martingale difference sequence with respect to z denoted as ν, such that

$$K\nu_t = (Q_2\Pi)^{\dagger}Q_2\Psi z_t + \eta_t,$$

then the solution of the model is a pair of (y_t, η_t), where

$$y_t = \Theta_1 y_{t-1} + \Theta_z z_t + \Theta_\nu \nu_t$$

$$\eta_t = K\nu - (Q_2\Pi)^{\dagger}Q_2\Psi z_t$$

with

$$\Theta_1 = Z \begin{bmatrix} \Lambda_{11}^{-1}\Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} Z', \quad \Theta_z = Z \begin{bmatrix} \Lambda_{11}^{-1}(Q_{1.}\Psi - Q_{1.}\Pi(Q_{2.}\Pi)^{\dagger}Q_{2.}\Psi) \\ 0 \end{bmatrix},$$

$$\Theta_\nu = Z \begin{bmatrix} \Lambda_{11}^{-1} Q_1.\Pi) \\ 0 \end{bmatrix} K.$$

Sims provides a clear framework for solving dynamic system under uncertainty and expectational conditions, distinguishing between predetermined and non-predetermined variables while simultaneously incorporating expectational error terms. It reformulates the problem of solving LREMs as a generalized eigenvalue problem and extends the application of the QZ decomposition in this field. Lubik and Schorfheide (2004) uses the sims method to test the determinacy or indeterminacy in monetary policy rules under a New-Keynesian DSGE model. Similarly, Klein (2000) also uses the QZ decomposition to solve LREMs, this method decouple backward and forward variables of the transformed system by QZ decomposition. However, Sims explicitly eliminates both dynamic and non-dynamic jump variables by imposing the horizontal conditions, which contributes to ensure the uniqueness and stability of LREMs solutions.

However, the dynamic relationships in the model are implicit, requiring matrix decomposition, which leads to high computational costs, and the expectational variables are not explicitly represented in the model equations. Although the Sims method appears recursive in form, it actually requires iterative computation to obtain a stable solution, as it relies on generalized eigenvalue decomposition combined with stability conditions and initial values. The expression of the model by Sims must be in a form with one lag and no leads, which is not efficient for solving the models with more than a couple of equations. Empirical tests in Anderson (2008) show that the Sims method incurs a floating point operation cost approximately 30 times higher than that of the AIM method, and the numerical precision of the Sims method is about five times worse than AIM. Therefore, although Sims is capable of handling predetermined and non-predetermined variables as well as expectational errors, its drawbacks include complex matrix decomposition, high computational burden and inferior numerical accuracy, which makes the model more susceptible to ill-posed matrices.

Applying Matrix Polynomials: Uhligs

Uhlig (1995) provides a solution method (denoted as Uhligs) built on the model expression of Binder and Pesaran (1999), solving LREMs by expressing them in terms of expectations over both endogenous and exogenous variables. Uhligs explicitly incorporates expectations and solves for a time-invariant, convergent law of motion. It is well suited for applications that require the forecastability of future variables like consumption and output.

Uhligs expresses the LREM as:

$$E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t] = 0$$

$$z_{t+1} = Nz_t + \mu_{t+1}; \quad E_t[\mu_{t+1}] = 0$$

where
$x_t \in \mathbb{R}^n$ is the vector of endogenous state variable at time t;
$z_t \in \mathbb{R}^m$ is the vector of exogenous variables following a stable VAR(1) process;
$F, G, H, L, M \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}$, are the given coefficient matrices;
$N \in \mathbb{R}^{m\times m}$ is a matrix associates with the exogenous variables which only contains the stable eigenvalues (all the eigenvalues are in the unit circle), ensuring stability of z_t.
μ_t is an i.i.d. shock vector with $E[\mu_t\mu_t'] > 0$.

Uhligs seeks a solution of the recursive equilibrium law of motion form:

$$x_t = Px_{t-1} + Qz_t$$

where $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are matrices of undetermined coefficients. Uhligs aims to solve for a pair of (P, Q) such that this law of motion satisfies the LREM under the rational expectations operator.

Uhligs first iterating the law of motion,

$$x_{t+1} = Px_t + Qz_{t+1},$$

and substituting them with the exogenous process into the model

$$z_{t+1} = Nz_t + \mu_{t+1},$$

taking expectations at time t:

$$E_t[x_{t+1}] = Px_t + QNz_t = P(Px_{t-1} + Qz_t) + QNz_t = P^2x_{t-1} + PQz_t + QNz_t,$$

then substituting $x_{t+1}, x_t, x_{t-1}, z_{t+1}, z_t$ into the original model to obtain:

$$F(P^2x_{t-1} + PQz_t + QNz_t) + G(Px_{t-1} + Qz_t) + Hx_{t-1} + LNz_t + Mz_t = 0,$$

grouping terms for x_{t-1},

$$(FP^2 + GP + H)x_{t-1}$$

for z_t,

$$(FPQ + GQ + FQN + LN + M)z_t.$$

Since the model must hold for all x_{t-1} and z_t, their coefficients must vanish:

$$FP^2 + GP + H = 0$$

$$(FP + G)Q + FQN + LN + M = 0.$$

Uhligs next solves $FP^2 + GP + H = 0$ for P, proposes to linearize this quadratic equation by a generalized eigenvalue problem. Define two $2n \times 2n$ matrices

$$\Xi = \begin{bmatrix} -G & -H \\ I_m & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} F & 0 \\ 0 & I_m \end{bmatrix}$$

consider the generalized eigenvalue problem

$$\lambda \Delta s = \Xi s,$$

where

$$s = \begin{bmatrix} \lambda x \\ x \end{bmatrix} \Longrightarrow (\lambda^2 F + \lambda G + H)x = 0.$$

This transforms the problem into seeking the eigenvalues λ and eigenvectors x satisfying:

$$det(\lambda^2 F + \lambda G + H) = 0,$$

from the solutions to it, construct eigenvectors

$$s_i = \begin{bmatrix} \lambda_i x_i \\ x_i \end{bmatrix}, \quad i = 1, ..., m.$$

Stack the stable ones into matrices:

$$\Omega = [x_1, ..., x_m], \quad \Lambda = diag(\lambda_1, ..., \lambda_m).$$

The stability selection here is critical, users should select only the number of n eigenvalues λ_i with $|\lambda_i| < 1$. If fewer than n stable eigenvalues are found, no stable solution exists; if more, there are multiple solutions even infinite many solutions, users can choose a subset that spans a stable subspace.

Then recover P by:

$$F\Omega\Lambda^2 + G\Omega\Lambda + H\Omega = 0,$$

right multiplying both sides by Ω^{-1} yields

$$F(\Omega\Lambda\Omega^{-1})^2 + G(\Omega\Lambda\Omega^{-1}) + H = 0.$$

Thus, a solution to the quadratic matrix equation is

$$P = \Omega\Lambda\Omega^{-1},$$

Uhlig (1995) proved that any diagonalizable solution can be expressed in above form through the transformation. In particular, when $m = 1$, the matrix reduces to a scalar, then the solution to the equation becomes:

$$P = \frac{-G \pm \sqrt{G^2 - 4FH}}{2F}, \quad F \neq 0.$$

If $F = 0$ but $G \neq 0$, the equation reduces to a linear form

$$GP + H = 0, \quad P = -\frac{H}{G}.$$

After P is solved, Uhligs solves for Q. Given P, equation

$$(FP + G)Q + FQN + LN + M = 0$$

becomes a linear matrix equation in Q, but note that Q appears twice including inside FQN, there is no way to factor out Q in term FQN, thus take vec operation. By the properties of the vec operation:

Let vec$(\cdot)$ be the column stacking operator, there is

$$vec(ABC) = (B^T \otimes A)vec(Q)$$

Then the equation solving for Q becomes

$$[N^T \otimes F + I_m \otimes (FP + G)]vec(Q) + vec(LN + M) = 0,$$

let

$$V = N^T \otimes F + I \otimes (FP + G),$$

then

$$vec(Q) = -V^{-1}vec(LN + M).$$

This equation can be solved using standard numerical linear algebra routines (for example, using the backslash operator in MATLAB). Consider the condition of V is invertible or ill-posed, the Moore-Penrose pseudo inverse may be required.

Once P and Q are obtained, the model solution is:

$$x_t = Px_{t-1} + Qz_t$$

with $z_t = Nz_{t-1} + \mu_t$, which implies

$$x_t = P^t x_0 + \sum_{i=0}^{t-1} P^i Q z_{t-i},$$

the influence of the initial condition x_0 on x_t is governed by $P^t x_0$. If $|\lambda_i| < 1$ for all eigenvalues λ_i of P, $\lim_{t\to\infty} P^t = 0$, then $\lim_{t\to\infty} P^t x_0 = 0$, which indicates that the influence of the initial condition vanishes over time t, and the state variable x_t converges to a path driven by the exogenous variables (essentially driven only by the exogenous shocks μ_t):

$$x_t = \sum_{i=0}^{t-1} P^i Q z_{t-i}.$$

The core of Uhligs lies in the explicit incorporation of future expectations ($E_t[x_{t+1}]$ and $E_t[z_{t+1}]$) into the dynamic equations of the model. In this expression to linear rational expectations model, the expected values of future variables are explicitly present and visible in the equations. Uhligs explicitly expresses the expectation terms, the clear model structure is well suited for studies which require analyzing expectation shocks. Matrix polynomials can better express the temporal dependencies between the variables. For example, in the matrix polynomial above, λ^2 corresponds to distant future expectations, λ reflects current term effects, and the constant term represents the contemporaneous shocks. This structure is more logically align with how LREM form expectations about the future, especially regarding forecasts of variables such as future consumptions or output. Barbier-Gauchard et al. (2023) applies Uhligs to log-linearize and solve a behavioral macroeconomic DSGE model, the results indicate that using Uhlig to rewrite and solve the DSGE model helps investigate the issue of fiscal policy credibility, as well as how the expectations of agents about the output gap, public debt, and taxation influence the fiscal multiplier and debt stability.

However, the solution to the matrix polynomial is highly complex and not unique, and it is sensitive to the matrix conditions, such as the invertibility of the F matrix. The results in Anderson (2008) indicate that, although Uhligs is more computationally efficient compared to Sims, it failed in few given models, and in general, still requires more than twice the computational cost of AIM. The results also suggest that Uhligs strikes a balance between accuracy and efficiency, but it is only suitable for cases where a VAR of the exogenous process is required and the model size is moderate (e.g., a small size DSGE model).

To summarize, for models that satisfy the Blanchard-Kahn conditions, these methods produce equivalent solutions in the time domain, the difference lies in their respective underlying ideas. Researchers is suggested to select the most suitable method depending on their proficiency with a given approach, and the size of the model to be studied, as Anderson (2008) shown, the AIM method distinguishes itself with exceptional computational speed and precision, making it highly suitable for large-scale empirical models, though it is less expressive than other methods in terms of theoretical structure and expectation modeling. Sims expresses model structure with explaining the dynamic mechanisms, and delineating the roles of different variables. Uhligs explicitly expresses and solves for expectations that enhancing structural transparency and better serving policy-oriented research where clear analysis of expectation-driven shocks is essential.

However, there not always exists a solution for an LREM, and the selected method may not always yield a unique solution, which makes policy analysis difficult or even infeasible. The next section reviews the problems of the existence and uniqueness of the LREMs solutions, and discusses their implications for the application of LREMs solutions.

3 Existence and Uniqueness of the LREM Solution

Macroeconomic theory typically assumes that the economy is always in some form of equilibrium, which implies that, given the current policy rules, shocks, and parameters, the economic system should follow a stable path. If no solution exists, the effects of exogenous shocks fail to be absorbed or offset by appropriately choosing the endogenous variables and martingale difference sequences, and the economic system would fail to reach any equilibrium state. In such a case, the model would not be suitable for policy analysis and forecasting. At the same time, under the framework of rational expectations theory, economic agents are assumed to know the structure of the model and to form expectations that are consistent and rational. If the model admits multiple solutions, then a given economic environment may correspond to more than one equilibrium path. In other words, different agents may coordinate on different equilibria, resulting in distinct dynamic behaviours in variables, such as inflation, interest rates and output. This not only contradicts the assumption of rational expectations, where all agents share the same equilibrium expectations, but also renders policy analysis ambiguous and infeasible. Thus, it is crutial to clarify the conditions for the existence and uniqueness of the LREM solution.

Taking the Sims methods as example, as shown in the previous section, to solve the linear equation

$$Q_{2.}\Psi z_t + Q_{2.}\Pi\eta_t = 0,$$

the column space of $Q_2\Psi$ must be spanned by the column space of $Q_2\Pi$. In other words, the impact of exogenous shocks within the stable subspace must be fully absorbed by the martingale difference term. Otherwise, no stationary solution exists. Then there is the condition for the existence of a solution to the model:

$$im(Q_{2.}\Psi) \subseteq im(Q_{2.}\Pi)$$

Once the existence condition is satisfied, the next step is to establish the uniqueness condition of the solution. Assuming that η_t is not unique, then there exists a vector x such that

$$Q_2\Pi x = 0.$$

However,

$$Q_1\Pi x \neq 0$$

then the adjustments to η_t along the vector x will not affect the stable subspace but will affect the unstable subspace, rendering the solution non-unique. Therefore, it is necessary that the martingale difference term be ineffective not only in the stable subspace but also in the unstable subspace, that is, all directions mapped to zero by $Q_2\Pi$ must also be mapped to zero by $Q_1\Pi$:

$$ker(Q_{2.}\Pi) \subseteq ker(Q_{1.}\Pi)$$

Therefore, following Sims (2002), assuming that $det(\Gamma_0 - \Gamma_1 x) \neq 0$ for all complex number x with $|x| = 1$, there exist a solution for linear rational expectations model if $im(Q_{2.}\Psi) \subseteq im(Q_{2.}\Pi)$, there is a unique solution if $ker(Q_{2.}\Pi) \subseteq ker(Q_{1.}\Pi)$. When using Sims, a unique solution is ensured if the given model satisfied the above conditions.

If the existence condition is satisfied but the uniqueness condition is not satisfied, meaning the solution set constitutes a high-dimensional linear subspace, along some directions within this subspace, the values of η_t can be chosen freely without violating the stability of the system. Such degree of "freedom" implies that one can select any point as the solution in the solution space. However, the "arbitrary" selection may lead to the chosen solution suddenly jumping to another point in the solution space when the model parameters experience even slight changes. Consequently, as the parameters vary, both the dimension of the solution space and the position of the solution within that space may shift abruptly. This phenomenon manifests as the discontinuity, in which the solution depending discontinuously on the parameters. The discontinuity of the solution will affect both frequentis and Bayesian analysis.

The next section clarifies the discontinuity by the Sims example and review a method for dealing the cases where the model solution is non-unique and discontinuous. This method selects a unique solution from the infinite many solutions of the LREM that aligns with prior information, therefore addressing the impact of discontinuity on frequentis and Bayesian analysis.

4 Addressing the Discontinuity: Regularization

When the uniqueness condition is not satisfied in the example of Sims, the solution is not unique, and there exists a linear subspace of solutions:

$$y_t = a\ particular\ solution\ +\ arbitrary\ linear\ combination\ of\ free\ direction.$$

This linear subspace is akin to a high-dimensional plane, on which one can arbitrarily choose a point. However, each point corresponds to a different set of economic dynamics, and these free directions may shift abruptly as the model parameters change, resulting in a discontinuous solution.

In a recent study, Al-Sadoon (2020) proposed a regularization method, successfully applied the regularized solution within the LREM in Sims form: by introducing a weighted objective function, one can select a solution from the infinite solution space that minimizes a convex and continuous cost function, incorporating prior knowledge or "preferences." This solution is both continuous and economically meaningful. The idea of the regularization under the previous work of Al-Sadoon (2017) is explicitly making the indeterminacy in the solution space, then penalizing the undesirable solution directions by minimizing a weighted quadratic loss function. The weight matrix, denoted as W, specified by the researcher to reflect economic preferences regarding the solution, for instance, emphasizing fit or smoothness at particular frequencies.

As discussed above, the general solution of LREM by Sims takes the form:

$$y_t = \Theta_1 y_{t-1} + \Theta_z z_t + \Theta_\nu \nu_t$$

$$\eta_t = K\nu_t - (Q_2\Pi)^\dagger Q_2 \Psi z_t$$

where ν_t is a martingale difference sequence freely chosen on $ker(Q_2\Pi)$, along the directions of non-uniqueness, representing the indeterminacy of the solution. Without any restrictions, ν_t can vary arbitrarily along these directions, resulting in an infinite number of solutions.

Regularization defines a weighted quadratic loss objective function:

$$\mathcal{L} = \frac{1}{2} tr(WE[y_0 y_0'])$$

where W is a given semi-definite weight matrix (selected by users) that reflects preferences for certain properties of the solution, such as smoother dynamics or solutions that better align with business cycle frequencies. The aim of regularization is to minimize $\mathcal{L}$ subject to the model constraints.

The indeterminacy term ν_t is decomposed as

$$\nu_t = Bz_t + \zeta_t, \quad E[z_t\zeta_t] = 0$$

where B represents the deterministic component and ζ is martingale difference sequence, ζ_t is a martingale difference sequence serving as the residual with respect to z_t. Express covariance $E[y_0y_0']$ as a function of B and $C = CC'$:

$$E[y_0y_0'] = \sum_{j=0}^{\infty}\Theta_1^j(\Theta_z\Sigma_{zz}\Theta_z' + \Theta_z\Sigma_{zz}B'\Theta_\nu'$$

$$+\Theta_\nu B\Sigma_{zz}\Theta_z' + \Theta_\nu B\Sigma_{zz}B'\Theta_\nu' + \Theta_\nu CC'\Theta_\nu')\Theta_1^{j'}$$

and the objective function can be written as:

$$\mathcal{L} = \frac{1}{2}tr(W\sum_{j=0}^{\infty}\Theta_1^j(...)\Theta_1^{j'}) = \frac{1}{2}tr((...)\Xi),$$

where

$$\Xi = \sum_{j=0}^{\infty}(\Theta_1')^{j'}W\Theta_1^j$$

is the unique solution of the equation:

$$\Xi = \Theta_1'\Xi\Theta_1 + W,$$

taking the derivative of $\mathcal{L}$ with respect to B and C, then there are

$$\Theta_\nu'\Xi(\Theta_z + \Theta_\nu B) = 0, \quad \Theta_\nu'\Xi\Theta_\nu C = 0$$

if $\Theta_\nu'\Xi\Theta_\nu$ is invertible, then the unique solution is

$$B^* = -(\Theta_\nu'\Xi\Theta_\nu)^{-1}\Theta_\nu'\Xi\Theta_z, \quad C^* = 0$$

if $\Theta_\nu'\Xi\Theta_\nu$ is not invertible, then there is no unique solution, but a class of solution can still be obtained by minimizing the objective function.

$$B^* = -(\Theta_\nu'\Xi\Theta_\nu)^\dagger\Theta_\nu'\Xi\Theta_z + X, \quad C^* = Y$$

where $im(X)$, $im(Y) \subseteq ker(\Theta_\nu'\Xi\Theta_\nu)$.

Substituting B^* and C^* into y_t and η_t yields the unique or a class of regularized solutions:

$$y_t = \Theta_1 y_{t-1} + \Theta_{reg} z_t,$$

$$\eta_t = -(K(\Theta_\nu' \Xi \Theta_\nu)^{-1} \Theta_\nu' \Xi \Theta_z + (Q_2 \Pi)^\dagger Q_2 \Psi) z_t$$

where

$$\Theta_{reg} = (I - \Theta_\nu (\Theta_\nu' \Xi \Theta_\nu)^{-1} \Theta_\nu' \Xi) \Theta_z.$$

To incorporate frequency-specific weights, such as emphasizing business cycle frequencies, the objective function can be expressed as an integral over frequencies:

$$\mathcal{L} = \frac{1}{2} tr(\int_{-\pi}^{\pi} W_\omega f_\omega d\omega)$$

where f_w is the spectral density matrix of y, W_ω is the frequency-dependent weight matrix. By choosing an appropriate W_ω, undesirable frequency components can be penalized.

The idea of regularization is analogous to adding a penalty term in linear regression (such as ridge regression), selecting the solution with the smallest norm from an infinite set of possible solutions. Regularization addresses the discontinuity and indeterminacy of LREM solutions in theory, and it allows researchers to incorporate prior information to guide the selection of a solution by their preference. This feature is particularly important for practical applications in the frequency domain or Bayesian analysis, as it ensures both the stability of the solution and the economic interpret-ability of the results.

However, when using regularization, the choice of the weight matrix from the user has a decisive impact on the results, which requires the user to processes strong economic sense and mathematical analytical foundation. Applying regularization to the other methods may expand their applicability, if encounters discontinuity problems during the solution process, one can first find the sets of all solutions under the selected method and then use regularization to select a "preferred" unique solution.

5 Conclusion

This paper reviews three solution methods for solving linear rational expectations models divided into three categories, uses Sims (2002) method to discuss the existence, uniqueness and the discontinuity of the LREM solution. This paper also reviews a recent method solving for the discontinuity problems of the LREMs, regularization, which not only addresses the discontinuity issues but allows users to select economically meaningful solutions tailored to business cycle considerations and research preferences. Future works could expand the use of regularization to the other solution method such as Uhlig (1995).

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