Word maps, random permutations and random graphs

CASSIDY, EWAN,GEORGE

Word Maps, Random Permutations and Random Graphs

A thesis submitted to Durham University for the degree of Doctor of Philosophy in the Department of Mathematical Sciences

Ewan George Cassidy

2025

Abstract

The aim of this thesis is to study word maps on the symmetric group, with applications in the study of spectral properties of random regular graphs.

We establish that, if $w \in F_r$ is not a proper power, then $\mathbb{E}_{\phi_n \in \hom(F_r, S_n)} [\chi(\phi_n(w))] = O\left(\frac{1}{\dim \chi}\right)$ as $n \to \infty$, where χ is any stable irreducible character of S_n.

We use this to prove that random sequences of representations of F_r that factor through non–trivial irreducible representations of S_n converge strongly to the left regular representation $\lambda : F_r \to U\left(\ell^2(F_r)\right)$, for any non–trivial irreducible representation of dimension $\leq Cn^{n^{\frac{1}{20}-\delta}}$.

An immediate consequence is that a random $2r$–regular Schreier graph depicting the action of r random permutations on $n^{\frac{1}{20}-\delta}$–tuples of distinct elements in $[n]$ has a near optimal spectral gap, with probability $\to 1$ as $n \to \infty$.

Contents

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Statement of copyright

Acknowledgements

I would first like to thank my supervisor Michael Magee for all of his support throughout my studies, and for his willingness to share his time and ideas with me. I have enjoyed our many conversations and I have learned a lot from him over the last 4 years – without Michael, this thesis would likely look very different. I very much hope that we can have further discussions in the future.

I would also like to thank anybody else that I have been able to discuss mathematics with since I began my PhD. I am especially grateful to Benoit Collins for inviting me to Kyoto and for hosting me during my stay, as well as Jonathan Novak and Brendon Rhoades for their hospitality in San Diego. I thoroughly enjoyed both of these visits and took a lot away from them.

I am also grateful to Will, Joe, Irving and Anitej from whom I have learned a lot through our reading group and general discussions. Thank you also to Doron Puder and Jack Shotton for agreeing to examine this thesis, and thank you to Ramon van Handel, Benoit Collins and Doron again for kindly agreeing to support my post–PhD applications.

Thank you to my family for supporting me in many ways over the years and for helping me get to where I am today.

Finally, a big thank you to Charlotte for all of the love, support and encouragement she has given me.

This thesis is part of a project that received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 949143).

1 Introduction

A d–regular graph on n vertices $G_{n,d}$ with adjacency matrix $A_{G_{n,d}}$ has n real eigenvalues,

$$d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -d,$$

with $\lambda_1 = \lambda_2$ if and only if $G_{n,d}$ is disconnected and $\lambda_n = -d$ if and only if $G_{n,d}$ is bipartite. We denote by $\lambda(G_{n,d}) = \max_{|\lambda_i| \neq d} \lambda_i$, the second largest eigenvalue and $d - \lambda(G_{n,d})$ is the spectral gap. The Alon–Boppana bound [Alo86, Fri03, Nil91] dictates that, for fixed d and growing n,

$$\lambda(G_{n,d}) \geq 2\sqrt{d-1} - o(1),$$

so that $d - 2\sqrt{d-1}$ is the asymptotically optimal spectral gap for graphs of fixed degree d. Graphs $G_{n,d}$ that satisfy $\lambda(G_{n,d}) \leq 2\sqrt{d-1}$ are called *Ramanujan* and graphs $G_{n,d}$ that satisfy $\lambda(G_{n,d}) \leq 2\sqrt{d-1} + \epsilon$ are called *weakly Ramanujan*.

The main aim of this thesis is to address the following question.

Question 1.1. Are random regular graphs on $N = N(n)$ vertices weakly Ramanujan with high probability? That is, as $n \to \infty$, with $N(n) \geq n$, does

$$\mathbb{P}\left[\text{a random regular graph of fixed degree on } N(n) \text{ vertices is weakly Ramanujan}\right] \to 1?$$

The model of random regular graphs that we study is that of Schreier graphs of the symmetric group acting on k–tuples. Given an action of a finite group G on a finite set X and elements $g_1, \ldots, g_r \in G$, one obtains the $2r$–regular Schreier graph, denoted $\mathrm{Sch}(G \curvearrowright X, g_1, \ldots, g_r)$, with vertex set X and, for each vertex u, an edge between u and $g_i u$ for each $i \in [r] \stackrel{\text{def}}{=} \{1, \ldots, r\}$. Choosing $g_1, \ldots, g_r$ i.i.d. uniformly random yields a random $2r$–regular graph. We are particularly interested in analysing the spectral gap of the random Schreier graphs in the case of $G = S_n$ and whereby X is the set of k–tuples of distinct elements of $[n]$. The main outcome of this thesis, Theorem 1.13, is that in this random model, the answer to Question 1.1 is *yes*, for $N(n) = n^{n^{\frac{1}{20}-\delta}}$.

Our approach is a study of word maps on the symmetric group, where results obtained can be used as input to a new method detailed in [CGVTvH24] for proving strong convergence of random permutations in high–dimensional representations, from which one can deduce statements regarding the spectral gap of the related random Schreier graphs.

1.1 Motivation

1.1.1 Spectral gaps

In general, we are interested in regular graphs with spectral gap as large as possible. Such graphs have some desirable properties which we describe here. Given any graph $G = (V, E)$ and a subset $S \subseteq V$, define $e(S)$ to be the number of edges between S and $V \backslash S$. A d–regular graph on n vertices, $G_{n,d}$, is an ϵ–expander if

$$h\left(G_{n,d}\right) \stackrel{\text{def}}{=} \min_{S \subseteq V,\ |S| \leq \frac{|V|}{2}} \frac{e(S)}{|S|} \geq \epsilon.$$

The number $h\left(G_{n,d}\right)$ is called the Cheeger constant of the graph $G_{n,d}$. Intuitively, a large Cheeger constant means that *every* subset of the vertices is well connected to its complement. Graphs of fixed degree on a large number of vertices with a large Cheeger constant are therefore sparse (they have few edges) but well connected, which is a desirable property for applications in e.g. computer science. Observe that $e(S) \leq d|S|$, so that $h\left(G_{n,d}\right) \leq d$. Graphs with Cheeger constant similar to d are therefore good expanders and the following inequality, which can be seen in e.g. [AM85], relates the Cheeger constant with the spectral gap.

Theorem 1.1. *For any d–regular graph on n vertices, $G_{n,d}$, we have*

$$h\left(G_{n,d}\right) \geq \frac{1}{2}\left(d - \lambda\left(G_{n,d}\right)\right).$$

So the spectral gap can, in some sense, be viewed as a measure of how well connected a regular graph is and one sees that a larger spectral gap yields a larger lower bound on the Cheeger constant. Indeed, if $G_{n,,d}$ is Ramanujan, then $h\left(G_{n,d}\right) \geq \frac{d}{2} - \sqrt{d-1}$, which is not far from the best possible expansion. Therefore, explicit constructions of graphs with a large spectral gap are of interest and group theoretic methods have proven fruitful in this endeavour.

1.1.2 Previous results

We further motivate this thesis by highlighting some important previous results. We begin by discussing previous results in spectral graph theory, with a focus on constructions of graphs using the symmetric group, before discussing some important previous results in the context of strong convergence of representations of free groups, particularly those that factor through permutations.

Expander graphs using the symmetric group

It was conjectured by Alon [Alo86] that 'most' d–regular graphs on a large number of vertices are weakly Ramanujan i.e. a randomly chosen d–regular graph on a large number of vertices is weakly Ramanujan with high probability, although it was not specified how to choose a random d–regular graph. Friedman [Fri08] proved Alon's conjecture in a breakthrough monograph making use of random Schreier graphs.

Theorem 1.2 (Friedman)**.** *Let $G_{n,d} = \mathrm{Sch}\left(S_n \curvearrowright [n], \sigma_1, \ldots, \sigma_r\right)$ be a random $2r$–regular Schreier graph. Then $G_{n,d}$ is weakly Ramanujan* asymptotically almost surely *(a.a.s.), that is, for any $\epsilon > 0$,*

$$\mathbb{P}\left[\lambda\left(G_{n,d}\right) \leq 2\sqrt{2r-1} + \epsilon\right] \overset{n\to\infty}{\longrightarrow} 1.$$

Remark 1.3. Friedman's theorem thus confirms that the answer to Question 1.1 is *yes*, for $N(n) = n$.

A simpler proof to Friedman's theorem was later given by Bordenave [Bor20] and a substantially simpler proof of a very similar statement ($\epsilon = 1$) was also given by Puder [Pud15]. In fact, Puder's approach has the added benefit that it is applicable to the generalised conjecture of Friedman for non–regular graphs (see [Fri03, Section 5]). This conjecture concerns random n–lifts of a fixed base graph Ω, and Puder's approach is used to prove a nearly optimal upper bound on the second largest eigenvalue of such a random graph. This generalised conjecture was later resolved in full by Bordenave and Collins in [BC19] as an application of the strong convergence theorem proved therein (we will discuss strong convergence in more detail very shortly).

Questions remain about expansion properties of random *Cayley graphs* of the symmetric group – these are denoted $\mathrm{Cay}\left(S_n, S\right)$ where $S = \{s_1, \ldots, s_r\} \subseteq S_n$ is some random generating set for S_n and they are equivalent to $\mathrm{Sch}\left(S_n \curvearrowright S_n, s_1, \ldots, s_r\right)$. One can also consider these random Cayley graphs as the random Schreier graphs $\mathrm{Sch}\left(S_n \curvearrowright [n]_n, s_1, \ldots, s_r\right)$, using the notation

$$[n]_k \stackrel{\text{def}}{=} \left\{k\text{–tuples of distinct elements in } [n]\right\}.$$

Friedman's theorem is the case $k = 1$. In [FJR+98], it was shown that for any fixed $k \in \mathbb{Z}_{>0}$, a random $2r$–regular Schreier graph $G_{n,d} = \mathrm{Sch}\left(S_n \curvearrowright [n]_k, \sigma_1, \ldots, \sigma_r\right)$ has a uniform (not necessarily near–optimal) spectral gap asymptotically almost surely. Later, using a remarkable novel approach to strong convergence, Chen, Garza–Vargas, Tropp and van Handel [CGVTvH24] gave a shorter proof of Friedman's theorem and they also extended their method to show for each fixed k, these random $2r$–regular Schreier graphs are weakly Ramanujan asymptotically almost surely[1], thus confirming that the answer to Question 1.1 is *yes*, for $N(n) = n^k$ with k any fixed positive integer.

As we mentioned before, to answer such a question for random Cayley graphs of the symmetric group, one would need to prove Friedman's theorem in the case of S_n acting on n–tuples. As far as we know, the best result in this direction is the following, see [Kas05].

Theorem 1.4 (Kassabov). *There exist $L, \epsilon > 0$ such that, for any n, there exists an explicit generating set $X_n \subseteq S_n$, of size $\leq L$, such that the Cayley graphs $\left\{\mathrm{Cay}\left(S_n, X_n\right)\right\}_{n\geq 1}$ form a family of ϵ–spectral expanders.*

In the case of a random generating set of fixed size, it is not known whether these graphs are weakly Ramanujan – it is not even known if they have a uniform spectral gap. Our Theorem 1.13 can, in some sense, be viewed as an advancement in the effort to bridge the gap between the result in [CGVTvH24] for $\mathrm{Sch}\left(S_n \curvearrowright [n]_k, \sigma_1, \ldots, \sigma_r\right)$ with k fixed and the result for $\mathrm{Sch}\left(S_n \curvearrowright [n]_n, \sigma_1, \ldots, \sigma_r\right)$ result that would resolve these unanswered questions for random Cayley graphs of the symmetric group.

[1] The statement for the spectral gap of random graphs in their case follows immediately from their result on the strong convergence of random representations that factor through permutations, in the same way as our Theorem 1.13 follows immediately from 1.14, as described in §1.2.3.

Strong convergence

Before giving the definition, we describe the intuition behind strong convergence, which can also be described as 'strong asymptotic freeness' for the random matrix models described here. What strong convergence implies is that, as the size of the random system grows, the interactions between the images of the generators in the random matrix model become increasingly decorrelated. In the limit, the algebra they generate no longer exhibits a group–like structure (in this thesis, this would be that of the symmetric group) with some degree of commutativity, but actually approximates the freely independent behaviour of the generators in the left regular representation. In our case, the idea is to try and show that large, randomly chosen, independent permutations behave freely with a high probability.

Definition 1.5. Given a sequence of random, finite dimensional, unitary representations $\{\pi_n : F_r \to U(N_n)\}_{n\geq 1}$, we say that π_n strongly converges to the left regular representation $\lambda : F_r \to U\left(\ell^2(F_r)\right)$ a.a.s. if, for any $z \in \mathbb{C}[F_r]$, for any $\epsilon > 0$, we have

$$\mathbb{P}\left[\left|\,\|\pi_n(z)\| - \|\lambda(z)\|\,\right| < \epsilon\right] \xrightarrow{n\to\infty} 1. \tag{1}$$

The norms here are operator norms, see §5.1.1. If this condition holds, we will write

$$\pi_n \xrightarrow{\text{strong}} \lambda.$$

Strong convergence for random (and non–random) representations of discrete groups Γ can be analogously defined but, in this thesis, we are particularly interested in strong convergence of random representations of free groups, particularly those that factor through permutations. The relevance of such representations to analysing the spectral gap of random graphs will be discussed later. There are results regarding strong convergence of other discrete groups which we will also touch on briefly later and, for a survey on strongly convergent representations of discrete groups, we direct the reader to [Mag25b].

The first instance of strong convergence of random representations of F_r that factor through S_n was given by Bordenave and Collins [BC19]. They showed that

$$\text{std} \circ \phi_n \xrightarrow{\text{strong}} \lambda, \tag{2}$$

where $\phi_n \in \hom(F_r, S_n)$ is chosen uniformly at random (i.e. by mapping each generator to a uniformly random permutation, see §1.2.2) and std is the $(n-1)$–dimensional standard representation of S_n. As we mentioned before, to illustrate the power of this result, they use this to resolve the generalised conjecture of Friedman on random lifts of fixed base graphs, see [BC19, Section 1.5]. In [HM23], Hide and Magee use this in the context of random covers of hyperbolic surfaces, showing that a random n–degree cover of a finite–area, non–compact (i.e. 'cusped') hyperbolic surface X has no eigenvalues below $\frac{1}{4} - \epsilon$ a.a.s. (other than the eigenvalues of X), an analogue of Friedman's theorem and further illustrating the power of strong convergence of random representations of F_r factoring through even relatively low–dimensional representations of S_n.

The work of Bordenave and Collins followed the breakthrough paper of Haagerup and Thorbjornsen [HT05], who gave the first example of a strongly convergent random sequence of representations of F_r by use of GUE matrices and Collins and Male [CM14], who proved the strong convergence (almost surely) of random representations of F_r factoring through unitaries (via mapping each x_i to a Haar random unitary u_i).

1.2 Main results

The main original results of this thesis are presented in this section. They can be divided into three subsections, each of a different flavour. The first of these subsections contain the main results of the author in [Cas25a] and the latter two contain the main results of the author in [Cas25b]. The main results of this thesis are Theorem 1.6, Theorem 1.12, Theorem 1.13 and Theorem 1.14.

1.2.1 Projection formulas

Classic Schur–Weyl duality due to Schur [Sch27, Sch01] asserts that, in $\text{End}\left((\mathbb{C}^n)^{\otimes k}\right)$, $\text{GL}_n(\mathbb{C})$ and S_k generate full mutual centralizers of one another, when GL_n acts diagonally and S_k permutes tensor coordinates. This gives a decomposition

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash k,\ l(\lambda) \leq n} V^{\lambda} \otimes S^{\lambda}, \tag{3}$$

where each V^{λ} is the irreducible representation of S_k corresponding to $\lambda \vdash k$ and S^{λ} is the irreducible representation of GL_n corresponding to λ. This relates to a formula obtained earlier by Frobenius [Fro00], giving the Schur polynomial expansion of the power sum symmetric polynomials. Indeed, if $g \in \mathrm{GL}_n$ has eigenvalues $x_1, \ldots, x_n$ and $\sigma \in S_k$ has cycle type μ, then the bitrace of (g, σ) on $(\mathbb{C}^n)^{\otimes k}$, that is, the trace the endomorphism of $(\mathbb{C}^n)^{\otimes k}$ obtained by applying σ and then g, is given by

$$\mathrm{btr}_{(\mathbb{C}^n)^{\otimes k}}(g, \sigma) = p_{\mu}(x_1, \ldots, x_n), \tag{4}$$

where p_{μ} is a power sum symmetric polynomial,

$$p_{\mu}(x_1, \ldots, x_n) = \prod_{i=1}^{l(\mu)} \left(\sum_{j=1}^{n} x_j^{\mu_i} \right).$$

Combining (4) with (3) yields the expansion

$$p_{\mu} = \sum_{\lambda \vdash k,\ l(\lambda) \leq n} \chi^{\lambda}(\mu)\, s_{\lambda},$$

where χ^{λ} is the character of V^{λ} and s_{λ} is the Schur polynomial corresponding to λ.

The analogous results in the case of S_n acting diagonally on $(\mathbb{C}^n)^{\otimes k}$ were obtained by Jones [Jon94] and can be expressed using the partition algebra, $P_k(n)$. When $n \geq 2k$, S_n and $P_k(n)$ generate full mutual centralizers of one another in $\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$, leading to the decomposition

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \vdash l,\ 0 \leq l \leq k} V^{\lambda^+(n)} \otimes R^{\lambda},$$

where R^{λ} is the irreducible $P_k(n)$ representation corresponding to λ and $V^{\lambda^+(n)}$ is the irreducible S_n representation corresponding to

$$\lambda^+(n) = (n - |\lambda|, \lambda) \vdash n,$$

see §2.1.2.

We use a refinement of this existing 'Schur–Weyl–Jones duality' due to Sam and Snowden [SS15] (considered also by Littlewood [Lit58] in a slightly different context), constructing a subspace

$$A_k(n) \subseteq (\mathbb{C}^n)^{\otimes k}$$

on which the inherited action of $P_k(n)$ descends to an action of $\mathbb{C}[S_k]$ (via the natural restriction map $R : P_k(n) \to \mathbb{C}[S_k]$, see (17)) and whereby $\sigma \in S_k$ permutes tensor coordinates. $A_k(n)$ also inherits an action of S_n from $(\mathbb{C}^n)^{\otimes k}$ and we denote the associated representations by θ and ρ respectively. These actions commute, making $A_k(n)$ a $\mathbb{C}[S_n \times S_k]$–representation, with decomposition

$$A_k(n) \cong \bigoplus_{\lambda \vdash k} V^{\lambda^+(n)} \otimes V^{\lambda} \tag{5}$$

for $n \geq 2k$. Our main result below is an explicit formula for the orthogonal projection $\mathcal{Q}_{\lambda,n}$, from $(\mathbb{C}^n)^{\otimes k}$ to each irreducible block $\mathcal{U}_{\lambda^+(n)} \cong V^{\lambda^+(n)} \otimes V^{\lambda}$, for each $\lambda \vdash k$. For each $\lambda \vdash k$, we will write $d_\lambda \overset{\text{def}}{=} \dim V^{\lambda}$ (and similarly $d_\mu \overset{\text{def}}{=} \dim V^{\mu}$ for any irreducible representation V^{μ} of $S_{|\mu|}$).

Theorem 1.6 ([Cas25a, Theorem 1.1])**.** *For any $k \in \mathbb{Z}_{>0}$, for any $\lambda \vdash k$ and for any $n \geq 2k$, there exists a $S_n \times S_k$–subrepresentation*

$$\mathcal{U}_{\lambda^+(n)} \subseteq A_k(n)$$

such that

a) $\mathcal{U}_{\lambda^+(n)}$ is irreducible and satisfies

$$\mathcal{U}_{\lambda^+(n)} \cong V^{\lambda^+(n)} \otimes V^{\lambda},$$

and

b) the orthogonal projection

$$\mathcal{Q}_{\lambda,n} : (\mathbb{C}^n)^{\otimes k} \to \mathcal{U}_{\lambda^+(n)}$$

is given by

$$d_{\lambda^+(n)}(-1)^k \sum_{\tau\in S_k} \chi^\lambda(\tau) \sum_{\pi\le\iota(\tau)} \frac{(-1)^{|\pi|}}{(n)_{|\pi|}} P_\pi^{\text{strict}}, \tag{6}$$

where

$$\left\langle P_\pi^{\text{strict}}\left(e_{i_1}\otimes\cdots\otimes e_{i_k}\right), e_{i_{k+1}}\otimes\cdots\otimes e_{i_{2k}}\right\rangle = \begin{cases} 1 & \text{if } j\sim k \text{ in } \pi \iff i_j = i_k \\ 0 & \text{otherwise,} \end{cases}$$

$\iota : \mathbb{C}[S_k] \to P_k(n)$ is the natural inclusion map (as in (18)) and '$\le$' is the natural partial ordering on set partitions (as in §2.1.5).

Theorem 1.6 has the following corollary, detailing how one can compute the irreducible character $\chi^{\lambda^+(n)}(g)$ by instead computing the trace in $(\mathbb{C}^n)^{\otimes k}$ of $g \circ \mathcal{Q}_{\lambda,n}$.

Corollary 1.7. *For any $g \in \mathbb{C}[S_n]$, $\sigma \in \mathbb{C}[S_k]$,*

$$\text{btr}_{\mathcal{U}_{\lambda^+(n)}}(g,\sigma) = \chi^{\lambda^+(n)}(g)\chi^\lambda(\sigma). \tag{7}$$

In particular, taking $\sigma = \text{Id}$, we have

$$\begin{aligned} d_\lambda \chi^{\lambda^+(n)}(g) &= \text{btr}_{\mathcal{U}_{\lambda,n}}(g,\text{Id}) \\ &= \text{tr}_{\mathcal{U}_{\lambda,n}}(g) \\ &= \text{btr}_{(\mathbb{C}^n)^{\otimes k}}\left(g, \mathcal{Q}_{\lambda,n}\right). \end{aligned}$$

1.2.2 Word maps

The intended application of Theorem 1.6 is the study of word maps on the symmetric group. This topic has been addressed in e.g. [Pud14, PP15, HP23] (as well as in [MP19, MP22]) for other compact groups). Given a word $w \in F_r = \langle x_1,\dots,x_r\rangle$ and a compact group G, one obtains a word map

$$w : \underbrace{G\times\cdots\times G}_{r} \to G$$

by substitutions. For example, if $w = [x_1, x_2]$ and $g, h \in G$, then $w(g,h) = ghg^{-1}h^{-1} \in G$. We are particularly interested in the case $G = S_n$.

One can equivalently think of the image of a word map in S_n as the image of w under some homomorphism $\phi_n \in \hom(F_r, S_n)$, whereby one fixes a basis $x_1, \ldots, x_r$ for F_r and sets $w(\sigma_1, \ldots, \sigma_r) = \phi_n(w)$ by choosing $\phi_n(x_i) = \sigma_i$. Where we refer to a w–random permutation, we refer to a random permutation obtained by choosing i.i.d. uniformly random permutations $\sigma_1, \ldots, \sigma_r \in S_n$ and evaluating $w(\sigma_1, \ldots, \sigma_r)$ (equivalently, evaluating $\phi_n(w)$ – this is the perspective we will adopt when we consider random representations of S_n obtained by composing such a random ϕ_n with a fixed representation of S_n).

Determining the distribution of w–random permutations is, in most cases, highly non–trivial and a natural starting point is to consider, for each $w \in F_r$,

$$\mathbb{E}_w\left(\#\text{fix}\right) \stackrel{\text{def}}{=} \mathop{\mathbb{E}}_{\sigma_1,\ldots,\sigma_m \in S_n}\left[\#\text{fixed points of } \left(w\left(\sigma_1, \ldots, \sigma_m\right)\right)\right].$$

Puder and Parzanchevski [PP15] give sharp asymptotic bounds for the expected number of fixed points in terms of the *primitivity rank* of w, an algebraic invariant of w introduced by Puder in [Pud14]. A word $w \in F_r$ is said to be primitive in a free group if it belongs to some basis of that group and the primitivity rank $\pi(w)$ of a word in F_r is defined by

$$\pi(w) \stackrel{\text{def}}{=} \min\left\{\text{rk}H : \ H \leq F_m, \ w \in H, \ w \text{ not primitive in } H\right\}.$$

If no such subgroup exists, then we set $\pi(w) = \infty$. The quantity $\text{Crit}(w)$ is defined as the number of subgroups $H \leq F_r$ with $w \in H$, w not primitive in H and $\text{rk}H = \pi(w)$.

Theorem 1.8 ([PP15, Theorem 1.8])**.** *For any $w \in F_r$,*

$$\mathbb{E}_w\left(\#\text{fix}\right) = 1 + \frac{|\text{Crit}(w)|}{n^{\pi(w)-1}} + O\left(\frac{1}{n^{\pi(w)}}\right).$$

In [HP23], Hanany and Puder generalize this to all stable irreducible characters of S_n. These are the family of irreducible representations of S_n corresponding to Young diagrams $\lambda^+(n)$ where $\lambda \vdash k$ is fixed. It is now well–known that the expected stable irreducible character of a w–random permutation is a rational expression in n, this follows by combining [Nic94, LP10] with [HP23, Proposition B.2], see also §2.1.2. Alternatively, this fact can also be seen immediately from (57).

Theorem 1.9 ([HP23, Theorem 1.3])**.** *For any $k \in \mathbb{Z}_{\geq 2}$, for any $\lambda \vdash k$ and for any $w \in F_r$ that is not a proper power or the identity,*

$$\mathbb{E}_w\left(\chi^{\lambda^+(n)}\right) = O\left(\frac{1}{n^{\pi(w)}}\right).$$

Moreover, they conjecture[2] the much stronger bound:

Conjecture 1.10. *For any $k \in \mathbb{Z}_{\geq 0}$, for any $\lambda \vdash k$ and for any $w \in F_r$*

$$\mathbb{E}_w\left(\chi^{\lambda^+(n)}\right) = O\left(\frac{1}{\left(d_{\lambda^+(n)}\right)^{\pi(w)-1}}\right).$$

Remark 1.11. That this conjecture holds for $k = 0$ is completely trivial and the case for $k = 1$ (i.e. where $V^{\lambda^+(n)} = V^{(n-1,1)}$ is the $(n-1)$–dimensional standard representation) follows from Theorem 1.8. The conjecture is also known to be true for words w with $\pi(w) = 1$ (i.e. for proper powers) and this follows from [Nic94] and [LP10, Section 4].

The remaining cases of interest are therefore $k \geq 2$ and $\pi(w) \geq 2$. The main result of this section is below, obtained using the projection formula in Theorem 1.6 and a method referred to as *combinatorial integration*. Theorem 1.12 solves one aspect of Conjecture 1.10, confirming the conjecture for $\pi(w) = 2$ and thus for F_2. Perhaps more importantly, it is powerful enough to be used to prove the results stated in §1.2.3.

Theorem 1.12 ([Cas25b, Theorem 1.5])**.** *For any $k \in \mathbb{Z}_{\geq 2}$, for any $\lambda \vdash k$ and for any $w \in F_r$ that is not a proper power or the identity,*

$$\mathbb{E}_w\left(\chi^{\lambda^+(n)}\right) = O\left(\frac{1}{d_{\lambda^+(n)}}\right) = O\left(\frac{1}{n^k}\right).$$

1.2.3 Strong convergence and spectral gaps

This section details the main results to be proved in §5. We use that $\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right]$ is both rational in n and is $= O\left(\frac{1}{d_{\lambda^+(n)}}\right)$ for large n to show that random representations of F_r

[2] In fact, it has recently been conjectured by Puder and Shomroni [PS23, Conjecture 1.2] that the sharpest bound should be given by $O\left(\frac{1}{\left(\dim V^{\lambda^+(n)}\right)^{s\pi(w)}}\right)$, where $s\pi(w)$ is a property of w called the stable primitivity rank, as introduced by Wilton [Wil24, Definition 10.6], and it is further conjectured by Wilton that $s\pi(w) = \pi(w) - 1$, so that the conjecture of Hanany and Puder is a combination of these two conjectures.

that factor through high–dimensional representations of S_n (via a random homomorphism $\phi_n \in \hom(F_r, S_n)$) strongly converge towards the left regular representation of F_r. It then follows that a large family of random Schreier graphs are a.a.s. weakly Ramanujan. Some motivation for these results is described in §1.1. For any integers $r, k > 0$, define $\mathcal{G}_r(n,k)$ to be the collection of $2r$–regular Schreier graphs $\mathrm{Sch}(S_n \curvearrowright [n]_k, \sigma_1, \dots, \sigma_r)$, where $[n]_k$ denotes the set of all k–tuples of distinct elements in $[n]$.

Theorem 1.13 ([Cas25b, Theorem 1.2])**.** *Fix any integer $r > 1$ and let $\alpha < \frac{1}{20}$. For any sequence of positive integers $(k_n)_{n\geq 1}$ with $k_n \leq n^{\alpha}$, let $(G_n)_{n\geq 1}$ be a sequence of random $2r$–regular Schreier graphs, where for each $n \geq 1$, $G_n \in \mathcal{G}_r(n, k_n)$ is obtained by choosing r i.i.d. uniformly random permutations, $\sigma_1, \dots, \sigma_r \in S_n$.*

Then, for any $\epsilon > 0$,

$$\mathbb{P}\left[\lambda(G_n) \leq 2\sqrt{2r-1} + \epsilon\right] \overset{n\to\infty}{\longrightarrow} 1.$$

The case $\alpha = 0$ is exactly Friedman's theorem (see §1.1) and the collection of Schreier graphs $\mathcal{G}_r(n,n)$ is the collection of Cayley graphs of S_n, so proving the above result for $\alpha = 1$ would show that random Cayley graphs of S_n are a.a.s. weakly Ramanujan. Nevertheless, Theorem 1.13 is the first result showing that the Schreier graphs constructed using random permutations acting on tuples of distinct elements have near–optimal spectral gap with high probability, in the case where the size of the tuple is allowed to grow with n.

Our proof of Theorem 1.13 relies on the remarkable new approach to *strong convergence* detailed in [CGVTvH24], as well as the additional criterion for temperedness of *arbitrary* functions on finitely generated groups Γ with a finite fixed generating set (an adaptation of the classical notion of a tempered representation) given by Magee and de la Salle in [MdlS24].

We are interested in the cases whereby the representations $\pi_n = \rho \circ \phi_n$, where $\phi_n \in \hom(F_r, S_n)$ is random and $\rho : S_n \to \mathrm{End}(V)$ is a representation of S_n of dimension $N(n)$. The power of strong convergence in this setting is that, since the convergence must hold for *every* $z \in \mathbb{C}[F_r]$, it allows us to prove results like Theorem 1.13, which require (1) to hold only for specific elements of $\mathbb{C}[F_r]$.

For example, proving Friedman's Theorem is equivalent to showing that

$$\mathbb{P}\left[\left|\,\|\pi_n(z)\| - 2\sqrt{2r-1}\right| < \epsilon\right] \overset{n\to\infty}{\longrightarrow} 1 \tag{8}$$

where

$$z = x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1} \tag{9}$$

and $\pi_n = \mathrm{std} \circ \phi_n$, with $\phi_n \in \hom(F_r, S_n)$ uniformly random (i.e. obtained by uniformly randomly choosing a permutation $\sigma_i \in S_n$ for each i and mapping $x_i \mapsto \sigma_i$). This is because, if $\rho : S_n \to \mathrm{End}(\mathbb{C}^n)$ is the defining representation of S_n, then

$$\rho\left(\sigma_1 + \sigma_1^{-1} + \cdots + \sigma_r + \sigma_r^{-1}\right)$$

is the adjacency matrix of the graph $\mathrm{Sch}(S_n \curvearrowright [n], \sigma_1, \ldots, \sigma_r)$ and $2\sqrt{2r-1} = \|\lambda(z)\|$ in this case. We consider the orthogonal complement to the S_n–invariant vectors in $\mathbb{C}^n$ to account for the trivial eigenvalue $2r$, which, in the case of ρ above, is the standard representation.

More generally, for any $k \in \mathbb{Z}_{>0}$ we define

$$\bar{\rho}_{n,k} : \mathbb{C}[S_n] \to \mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right),$$

the k^{th} tensor power of the defining representation. We then define

$$\rho_{n,k} : \mathbb{C}[S_n] \to \mathrm{End}(V_{n,k})$$

to be the restriction of $\bar{\rho}_{n,k}$ to the orthocomplement to the S_n–invariant vectors, $V_{n,k} \subseteq (\mathbb{C}^n)^{\otimes k}$ and define a random sequence of unitary representations of F_r, $\{\pi_{n,k} \overset{\text{def}}{=} \rho_{n,k} \circ \phi_n\}_{n\geq 1}$, with $\phi_n \in \hom(F_r, S_n)$ uniformly random as before.

In this language, Bordenave and Collins [BC19] proved strong convergence to the left regular representation $\lambda : F_r \to U\left(\ell^2(F_r)\right)$ a.a.s. for the sequence of random representations of F_r, $\{\pi_{n,1}\}_{n\geq 1}$, i.e. where the random representation factors through std. They extend their results to show strong convergence a.a.s. for $\{\pi_{n,2}\}_{n\geq 1}$. With the new approach in [CGVTvH24], strong convergence a.a.s. for $\{\pi_{n,k}\}_{n\geq 1}$ for any fixed k was proved but using a very different approach. We adopt and extend this new approach to prove our final main theorem below.

Theorem 1.14 ([Cas25b, Theorem 1.9]). *For any* $\alpha < \frac{1}{20}$*, for any* $z \in \mathbb{C}[F_r]$ *and for any* $\epsilon > 0$,

$$\mathbb{P}\left[\sup_{k_n \le n^\alpha} \left| \|\pi_{n,k_n}(z)\| - \|\lambda(z)\| \right| < \epsilon\right] \xrightarrow{n\to\infty} 1.$$

Corollary 1.15. [3] *For any* $\alpha < \frac{1}{20}$*, for any* $z \in \mathbb{C}[F_r]$ *and for any* $\epsilon > 0$,

$$\mathbb{P}\left[\sup_{\substack{\rho \\ \dim \rho \le C n^{n^\alpha}}} \left| \|\rho \circ \phi_n(z)\| - \|\lambda(z)\| \right| < \epsilon\right] \xrightarrow{n\to\infty} 1.$$

where the supremum is over all non–trivial irreducible representations of S_n*.*

Remark 1.16. Theorem 1.13 follows immediately from Theorem 1.14 by taking $z = x_1 + x_1^{-1} + \cdots + x_r + x_r^{-1}$. Theorem 1.14 is a much stronger statement, and it is certainly not necessary to prove such a statement to prove Theorem 1.13.

1.3 Other related works

As we have previously discussed, the most closely related works with respect to the word maps section of this thesis are the results of Puder, Puder–Parzanchevski and Hanany–Puder [Pud14, PP15, HP23], as well as earlier works of [Nic94, LP10], all of which consider word maps on the symmetric group. Some of these results were obtained through somewhat algebraic arguments, whilst the approach in this thesis is of a more analytic/combinatorial nature.

Compact groups

Word maps on other finite/compact groups can be considered in a similar context and there are many related results to be seen in e.g. [Voi91, MŚS07, EWPS24, MP19, MP22, PS23]. One takeaway from these works is that, in many cases, the character statistics are rational functions of the parameter n, a fact which is exploited in [CGVTvH24], in which it is detailed how one can prove the strong convergence of random representations of free

[3] This corollary and its proof in §5.2 were suggested to me by Ramon van Handel, for which I am especially grateful.

groups that factor through representations of compact groups using this simple fact and some additional 'soft' arguments. To demonstrate the power of their method, prior to the results of this thesis, they use the asymptotic bound given by Hanany and Puder [HP23] (Theorem 1.9 in §1.2.2) to prove that the analogue of Theorem 1.14 holds for any fixed k. In [MdlS24], Magee and de la Salle extend this method and, using as an input an asymptotic bound for the expected stable irreducible character of a w–random unitary following from [MP19, Mag22, Mag25a], they prove an analogue of our Theorem 1.14 for random representations of F_r factoring through unitary matrices in non–trivial irreducible representations of quasi–exponential dimension, with a slightly worse constant ($\alpha < \frac{1}{42}$[4]), but which holds almost surely.

Discrete groups

In addition to the asymptotic statistics of $\phi_n \in \hom(F_r, S_n)$ that are considered in this thesis, one can equally consider the asymptotic statistics of $\phi_n \in \hom(\Gamma, G)$ for discrete groups Γ and compact groups G. Of particular interest is, for example,

$$\mathop{\mathbb{E}}_{\phi_n \in \hom(\Gamma_g, S_n)} \left(\#\mathrm{fix}(\phi_n(w))\right),$$

where Γ_g is the fundamental group of a surface of genus g and $w \in \Gamma_g$. Although the expression in this case is not a rational expression in n, precise asymptotics can still be obtained for the Laurent expansion (see [MP23]) and, very recently, Magee, Puder and van Handel in [MPvH25] have extended the methods of [CGVTvH24] to prove strong convergence of the related random representation of Γ_g using the Laurent expansion. Results for other discrete groups are, as yet, unclear and similarly for other compact groups, more on this in §6. We include this discussion here to highlight the importance of such expressions for asymptotic statistics of w–random group elements and the role such expansions play in these new methods for proving strong convergence, as well as their relevance in geometric settings.

[4] This was later improved anyway to $\alpha < \frac{1}{3}$ in [CGVvH24].

2 Background

2.1 Preliminaries and notation

2.1.1 Representation Theory

For a thorough introduction, we direct the reader to the textbooks by Fulton and Harris [FH04] or Serre [Ser77], for example. Given a representation (ρ, V) of a finite group G, we will denote by $V^\vee$ its *dual representation*, where if $\{v_i\}$ is a basis for V, $V^\vee$ is the vector space with basis $\{\check{v}_i\}$, where $\check{v}_i(v_j) = \delta_{ij}$. We will write the decomposition of a representation into irreducible representations as

$$V \cong \bigoplus_{i=1}^{k} V_i^{\oplus a_i}, \tag{10}$$

where the V_i are the distinct (non–isomorphic) irreducible representations of G. This decomposition is unique up to isomorphisms of the isotypic components, $V_i^{\oplus a_i}$. Given a decomposition $V \cong \bigoplus V_i^{\oplus a_i}$ of an arbitrary representation V of a finite group G, we define

$$\mathscr{P}_{V_i} \stackrel{\text{def}}{=} \frac{\dim(V_i)}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)} g \in \mathbb{C}[G], \tag{11}$$

a central idempotent in the group algebra whose image $\rho\left(\mathscr{P}_{V_i}\right) \in \operatorname{End}(V)$ is the projection from V on to the V_i–isotypic subspace.

We denote $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$ and throughout this thesis, $e_1, \ldots, e_n \in \mathbb{C}^n$ will be the standard orthonormal (with respect to the standard Hermitian inner product $\langle ., . \rangle$) basis of $\mathbb{C}^n$ so that the set

$$\left\{ e_{i_1} \otimes \cdots \otimes e_{i_k} : i_j \in [n] \text{ for } j \in [k] \right\}$$

is the standard basis for $(\mathbb{C}^n)^{\otimes k}$. Given a multi–index $I = (i_1, \ldots, i_k)$, we will write

$$e_I \stackrel{\text{def}}{=} e_{i_1} \otimes \cdots \otimes e_{i_k} \in (\mathbb{C}^n)^{\otimes k}$$

and similarly $\check{e}_I$ denotes $\check{e}_{i_1} \otimes \cdots \otimes \check{e}_{i_k} \in \left((\mathbb{C}^n)^\vee\right)^{\otimes k}$. Given a representation $\rho : G \to \operatorname{End}(V)$ and a subgroup $H \leq G$, we will write $\operatorname{Res}_H^G V$ or $V \downarrow_H$ for the restriction of V to H and, given a representation U of H, we will denote by $\operatorname{Ind}_H^G U$ the induced representation of G.

2.1.2 Symmetric Group

We denote the symmetric group on n elements by $S_n = \{\text{bijections } [n] \to [n]\}$ together with function composition. Where we refer to an inclusion of the form $S_m \subseteq S_n$, for $m \leq n$, unless specified otherwise, we refer to the subgroup consisting of permutations in S_n which fix the elements $\{m+1, \ldots, n\}$. In this case, we may also refer to an inclusion $S'_{n-m} \subseteq S_n$, which refers to the subgroup consisting of permutations that fix the elements $\{1, \ldots, m\}$. It can be shown (see [Hum96] for example) that S_n is generated by transpositions of adjacent elements,

$$S_n \cong \langle (12), (23), \ldots, (n-1\ n) \rangle .$$

These are the *Coxeter generators,* denoted $s_i = (i\ i+1)$ for $i = 1, \ldots, n-1$.

A *Young Diagram* (YD) λ is an arrangement of rows of boxes, where the number of boxes in each row is non–increasing as the row index increases. If λ is a YD with n boxes and $l(\lambda)$ non–empty rows of boxes, then we write $\lambda \vdash n$ (or $|\lambda| = n$) and we say that the length of λ is $l(\lambda)$. We can write this as $\lambda = \left(\lambda_1, \ldots, \lambda_{l(\lambda)}\right)$, with $\lambda_1 \geq \cdots \geq \lambda_{l(\lambda)} > 0$ and $\lambda_1 + \cdots + \lambda_{l(\lambda)} = n$. There is a notion of *inclusion* for Young diagrams λ and μ, where $|\mu| \leq |\lambda|$. Informally, we say μ is contained inside λ if we can obtain λ by adding boxes to μ (equivalently, removing boxes from λ to obtain μ). More formally, if $\mu = \left(\mu_1, \ldots, \mu_{l(\mu)}\right)$ and $\lambda = \left(\lambda_1, \ldots, \lambda_{l(\lambda)}\right)$, then μ is contained in λ if $l(\mu) \leq l(\lambda)$ and for each $i \in [p]$, we have $\mu_i \leq \lambda_i$. Where μ is contained in λ, we define the *skew diagram* $\lambda \backslash \mu$ to be the diagram consisting of the boxes that are in λ, but not in μ. There is a natural bijection between distinct isomorphism classes of irreducible representations of S_n and Young diagrams $\lambda \vdash n$, for example see [FH04, Chapter 4] or [VO05]. As such, we label every irreducible representation of S_n by its corresponding YD λ, and denote this representation by V^λ. With this and (10), we can write every representation V of S_n uniquely in the following way

$$V \cong \bigoplus_{\lambda \vdash n} \left(V^\lambda\right)^{\oplus a_\lambda},$$

where $a_\lambda \in \mathbb{Z}_{\geq 0}$ for each $\lambda \vdash n$. We will denote the character of V^λ by χ^λ.

Remark 2.1. The characters of the symmetric group are integer valued (see [Ser77, p. 103] for example). Consequently, all representations V of the symmetric group are *self dual,* meaning that $V \cong V^\vee$, since

$$\chi_{V^\vee}(g) = \overline{\chi_V(g)} = \chi_V(g).$$

Recall that we denote by $d_\lambda \stackrel{\text{def}}{=} \chi^\lambda(\text{Id}) = \dim V^\lambda$. A *standard tableau* or *Young tableau of shape* $\lambda \vdash n$ is a labeling of the boxes of λ with the integers $1, \ldots, n$, in which every integer appears exactly once and the numbers are increasing both along the rows and down the columns. The set of standard tableau of shape λ is denoted by $\text{Tab}(\lambda)$ and it is a fact that $|\text{Tab}(\lambda)| = d_\lambda$. The content, $\text{cont}(\square)$, of a box $\square$ in a YD λ is defined by

$$\text{cont}(\square) = \text{column index of } \square \; - \; \text{row index of } \square.$$

Given $T \in \text{Tab}(\lambda)$, we define a *content vector* $(c_1, \ldots, c_n) \in \mathbb{Z}^n$, where c_i is the content of the box labeled i. Then we have the following proposition of Vershik and Okounkov [VO05, Proposition 6.2].

Proposition 2.2. *There exists an orthonormal basis* $\{v_T\}_{T \in \text{Tab}(\lambda)}$ *of* V^λ *in which the Coxeter generators* s_i *act according to the following rules:*

- *If the boxes labeled* i *and* $i+1$ *are in the same row of* T*, then* $s_i v_T = v_T$;
- *If the boxes labeled* i *and* $i+1$ *are in the same column of* T*, then* $s_i v_T = -v_T$*;*
- *If the boxes labeled* i *and* $i+1$ *are in neither the same row or the same column of* T*, then* s_i *acts on the two dimensional space spanned by* v_T *and* $v_{T'}$ *(the Young tableau obtained from* T *by swapping* i *and* $i+1$*) by the following matrix*

$$\begin{pmatrix} r^{-1} & \sqrt{1-r^{-2}} \\ \sqrt{1-r^{-2}} & -r^{-1} \end{pmatrix},$$

where $r = c_{i+1} - c_i$*.*

From now on, for each λ we will fix such a basis of V^λ, and refer to this as the *Gelfand–Tsetlin* basis.

Stable irreducible representations

For any fixed $k \in \mathbb{Z}_{>0}$ and any $\lambda \vdash k$, we define a Young diagram $\lambda^+(n) = (n-k, \lambda) \vdash n$ for any $n \geq k + \lambda_1$. This gives rise to a family of irreducible representations indexed by $\{\lambda^+(n)\}_{n \geq k+\lambda_1}$ and we denote the collection of all such families by $\widehat{S_\infty}$.

These representations appear naturally in the representation theory of S_n since their characters form a linear basis for the polynomial ring $\mathbb{Q}[\eta_1, \eta_2, \dots]$, whereby $\eta_i : S_n \to \mathbb{Z}$ satisfies $\eta_i(\sigma) = \#\text{fix}\left(\sigma^i\right)$. For example, std, which corresponds to $(n-1,1) \vdash n$, is expressed as $\eta_1 - 1$. A more thorough discussion of this can be found in [HP23, Appendix B].

2.1.3 Hyperoctahedral Group

We give an overview of the hyperoctahedral group, H_k, viewed as a subgroup of S_{2k}, and its representation theory. This brief introduction comprises of a summary similar to that of Koike and Terada in [KT87]. A very thorough introduction to this topic can be found in [Mus93]. There is an injective group homomorphism

$$\psi : S_k \to S_{2k} \tag{12}$$

whereby each Coxeter generator $s_i = (i\ i{+}1)$ is mapped to $\psi(s_i) = \big(2i{-}1\ 2i{+}1\big)\big(2i\ 2i{+}2\big)$. We will write

$$S_k^{\psi} = \psi\left(S_k\right) \leq S_{2k}$$

and the hyperoctahedral group is then defined to be the following subgroup of S_{2k}:

$$H_k = \Big\langle \psi\left(S_k\right),\ s_1,\ s_3, \dots,\ s_{2k-1} \Big\rangle \cong \psi\left(S_k\right) \ltimes \mathcal{D}, \tag{13}$$

where

$$\mathcal{D} = \big\langle s_1,\ s_3, \dots,\ s_{2k-1} \big\rangle \leq S_{2k}.$$

For each $i = 0,\ 1,\ 3, \ldots,\ 2k-1$, define a representation ρ_i of $\mathcal{D}$ by

$$\rho_i\left(s_j\right) = \begin{cases} 1 & \text{if } j \leq i \\ -1 & i < j. \end{cases} \tag{14}$$

Then, given an irreducible representation V^{λ} of S_k, define an irreducible representation $W^{\lambda,\emptyset}$ of H_k via (13) – Given $h \in H_k$ and we have $h = \psi(\sigma)\delta$ and the character $\chi^{\lambda,\emptyset}$ of $W^{\lambda,\emptyset}$ is given by $\chi^{\lambda,\emptyset}(h) = \chi^{\lambda,\emptyset}(\psi(\sigma)\delta) = \chi^{\lambda}(\sigma)$. We can also define an irreducible representation of H_k from ρ_0, by extending ρ_0 by letting $\psi\left(S_k\right)$ act trivially. We denote this representation of H_k by $W^{\emptyset,(k)}$ and we define

$$W^{\emptyset,\lambda} = W^{\lambda,\emptyset} \otimes W^{\emptyset,(k)}, \tag{15}$$

with character given by $\chi^{\emptyset,\lambda} = \operatorname{sign}(\delta)\chi^{\lambda}(\sigma)$.[5]

Given any $i = 1,\ 3, \ldots,\ 2k-1$, define $j_i \stackrel{\text{def}}{=} \frac{i+1}{2}$ and define two subgroups of H_k,

$$H_{j_i} = \left\langle \psi\left(S_{j_i}\right),\ s_1, s_3, \ldots,\ s_i \right\rangle$$

and

$$H'_{k-j_i} = \left\langle \psi\left(S'_{k-j_i}\right), s_{i+2},\ s_{i+4}, \ldots, s_{2k-1} \right\rangle,$$

where $S'_{k-\frac{i+1}{2}}$ is defined as in Section 2.1.2. These are obviously isomorphic to smaller index hyperoctahedral groups, and we get the following theorem (see e.g. [Mus93, Theorem 4.7.7]) using the 'Wigner–Mackey method of little groups'.

Theorem 2.3. *With the subgroups and representations defined as above,*

a) for any $i = 1,\ 3, \ldots,\ 2k-1$ and Young diagrams $\mu \vdash j_i$, $\nu \vdash k - j_i$, the representation

$$W^{\mu,\nu} = \operatorname{Ind}_{H_{j_i} \times H'_{k-j_i}}^{H_k} \left[W^{\mu,\emptyset} \times \left(W^{\nu,\emptyset} \otimes W^{\emptyset,(k-j_i)} \right) \right]$$

is an irreducible representation of H_k;

[5] This construction only works with ρ_0 since it is $\psi\left(S_k\right)$ invariant, but does not work for more general ρ_i.

b) the set

$$\{W^{\mu,\nu} : \ (\mu,\nu) \models k\}$$

constitutes a complete set of representatives of the equivalence classes of irreducible representations of H_k*, where* $(\mu,\nu) \models k$ *denotes an ordered pair of Young diagrams* μ, ν *with* $|\mu| + |\nu| = k$*.*

Remark 2.4. In the case of $\mu = \emptyset$, it is easily seen that $\dim\left(W^{\emptyset,\nu}\right) = d_\nu$, the dimension of the irreducible S_k representation, V^ν.

2.1.4 Möbius Inversion

Here we give definitions and results that are needed for our description of the Weingarten calculus in Section 2.1.6. Most of the details can be found in the foundational paper of Rota [Rot64]. A *poset* $(P, \leq)$ is a set P with a partial order $\leq$. In general, we will simply write P in place of $(P, \leq)$ in reference to a poset, unless it is necessary to be explicit. A *lattice* is a poset in which the maximum and minimum of two elements is defined and they will be called the *join* and *meet* respectively. We denote the join of two elements x, y by $x \vee y$ and we denote the meet of these two elements by $x \wedge y$. Let P be a poset and $x, y \in P$. A segment $[x, y]$ is defined as follows:

$$[x, y] \stackrel{\text{def}}{=} \{z \in P : \ x \leq z \leq y\}.$$

Open and half open segments are defined similarly. We say that a poset P is locally finite if every segment contains finitely many elements. The Möbius function $\mu(x, y)$ of a locally finite poset P is defined inductively in [Rot64, Proposition 1]. For a segment $[x, y]$ of a poset P, we first set $\mu(x, x) = 1$. Then, assuming $\mu(x, z)$ is defined for all $z \in [x, y)$, we inductively define

$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z). \tag{16}$$

The Möbius inversion formula below is given in [Rot64, Corollary 1].

Theorem 2.5 (Möbius Inversion Formula)**.** *Let* $(P, \leq)$ *be a locally finite poset and let*

$r : P \to \mathbb{C}$. *Suppose there exists a* $p \in P$ *such that* $x \nleq p \implies r(x) = 0$. *Then, if*

$$s(x) = \sum_{y \geq x} r(y)$$

we have

$$r(x) = \sum_{y \geq x} \mu(x, y) s(y).$$

2.1.5 The Partition Algebra

Jones [Jon94], and independently Martin [Mar94], initially developed the partition algebra in relation to statistical mechanics and it has since been used to develop various formulations of Schur–Weyl duality. We write

$$\operatorname{Part}([n]) = (\{\text{set partitions of } [n]\},\ \leq),$$

where $\pi_1 \leq \pi_2$ if π_1 is a refinement of π_2, meaning every block of π_1 is contained in a block of π_2. In this setting, $\operatorname{Part}([n])$ is a lattice. Given $\pi \in \operatorname{Part}([n])$, we write $i \sim j$ to indicate that i and j belong to the same block of the partition.

Remark 2.6. Given this partial ordering, one can find the Möbius function for partitions using the inductive definition (16). Suppose $\pi_1 = \left\{\mathcal{S}_1, \dots, \mathcal{S}_l\right\}$ consists of l subsets and that π_2 is a refinement of π_1, with each subset $\mathcal{S}_i$ of π_1 splitting into a further m_i subsets, $\mathcal{T}_1, \dots, \mathcal{T}_{m_i}$. So π_2 consists of $\sum_i m_i = m$ subsets. Then

$$\mu(\pi_1, \pi_2) = (-1)^{m-l} \prod_{i=1}^{l} (m_i - 1)!,$$

see [Rot64, Section 7].

The number $1 \leq s \leq 2k$ of blocks of $\pi \in \operatorname{Part}([2k])$ is the *size* of the partition, denoted $|\pi|$. The partition algebra $P_k(n)$ is the $\mathbb{C}$–linear span of $\operatorname{Part}([2k])$, with a multiplication described using partition diagrams as follows. For each $\pi \in \operatorname{Part}([2k])$, we construct a diagram with $2k$ vertices, drawn in two rows of k, labeled from $1, \dots, k$ on the top row and from $k+1, \dots, 2k$ on the bottom row. An edge is drawn between two vertices whenever they are in the same subset of π.[6] Obviously, any such diagram also defines some $\pi \in \text{Part}([2k])$. The product $\pi_1\pi_2$ is computed as follows, let x be an indeterminate.

1. Identify the bottom row of vertices in π_1 with the top row of vertices in π_2 to obtain a diagram with 3 rows of k vertices.

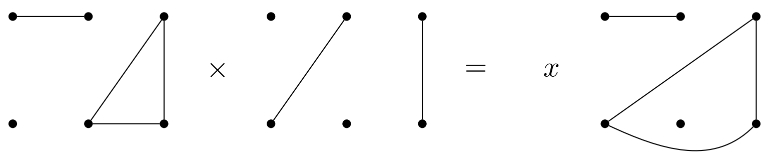
2. Let γ be the number of connected components of this diagram with vertices in only the middle row.

3. Add edges between any two vertices in the same connected component, if there is not already an edge.

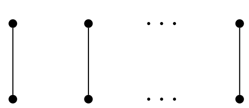
4. Remove the middle row of vertices and any adjacent edges to vertices in the middle row to obtain a new partition diagram. Label this diagram π_3.

5. Define $\pi_1\pi_2 = x^\gamma \pi_3$.

Example. For $k = 3$, $\pi_1 = \Big\{\{1,2\},\{3,5,6\},\{4\}\Big\}$ and $\pi_2 = \Big\{\{1\},\{2,4\},\{3,6\},\{5\}\Big\}$, then $\pi_1\pi_2 = x\Big\{\{1,2\},\{3,4,6\},\{5\}\Big\}$. Diagrammatically,



If $\mathbb{C}(x)$ is the field of rational functions with complex coefficients, the *partition algebra* $P_k(x)$ is the $\mathbb{C}(x)$–linear span of $\text{Part}([2k])$. With the multiplication as described above, this is an associative algebra, with identity element:



For each $n \in \mathbb{C}$, we define the partition algebra $P_k(n)$ over $\mathbb{C}$ as the linear span of $\text{Part}([2k])$, with x replaced by n in the multiplication described. For most choices of n, this is a semisimple algebra.

[6] The diagram for $\pi_1 \wedge \pi_2$ is obtained by removing any edges in the diagram of π_1 that are not present in that of π_2. The diagram for the join $\pi_1 \vee \pi_2$ is obtained by adding all edges of π_2 to π_1, and then completing every connected component.

Theorem 2.7 ([MS93])**.** *The partition algebra $P_k(n)$ is semisimple for any $n \in \mathbb{C}$, unless $n \in \mathbb{Z} \cap [0, 2k-1]$.*

Proposition 2.8 ([Mar96, Proposition 1])**.** *The following elements generate $P_k(n)$:*

- $\mathrm{Id} = \Big\{\{1, k+1\}, \dots, \{k, 2k\}\Big\}$
- $\mathcal{S}_{i,j} = \Big\{\{1, k+1\}, \dots \{i, k+j\}, \dots, \{j, k+i\}, \dots, \{k, 2k\}\Big\}$ *for* $i, j = 1, \dots, k$
- $\mathcal{A}_i = \Big\{\{1, k+1\}, \dots, \{i\}, \{k+i\}, \dots, \{k, 2k\}\Big\}$ *for* $i = 1, \dots, k$ *and*
- $\mathcal{A}_{i,j} = \Big\{\{1, k+1\}, \dots, \{i, j, k+i, k+j\}, \dots, \{k, 2k\}\Big\}$ *for* $i, j = 1, \dots, k.$

There is a surjection

$$R : P_k(n) \to \mathbb{C}\left[S_k\right], \tag{17}$$

whereby $\mathcal{S}_{i,i+i} \mapsto s_i$ and $\mathcal{A}_i, \mathcal{A}_{i,j} \mapsto 0$ and a corresponding algebra injection

$$\iota : \mathbb{C}\left[S_k\right] \to P_k(n), \tag{18}$$

where, for $\sigma \in S_k$,

$$\iota(\sigma) = \Big\{\{1, k + \sigma^{-1}(1)\}, \dots, \{k, k + \sigma^{-1}(k)\}\Big\}.$$

It is corresponding in the sense that $R \circ \iota$ is the identity map on $\mathbb{C}\left[S_k\right]$. Indeed, if $\sigma = s_{i_1} \dots s_{i_m}$, then it is not hard to see that $\iota(\sigma) = \mathcal{S}_{i_1, i_1+1} \dots \mathcal{S}_{i_m, i_m+1}$, so that

$$R(\iota(\sigma)) = s_{i_1} \dots s_{i_m} = \sigma.$$

Intuitively, each permutation in S_k corresponds to some matching of the two rows of vertices in the diagram. Henceforth, any reference to the inclusion or restriction between $\mathbb{C}\left[S_k\right]$ and $P_k(n)$ will reference maps (17) and (18).

2.1.6 The Weingarten Calculus for the Symmetric Group

Here, we describe the method for integrating over S_n outlined by Collins, Matsumoto and Novak in the short survey [CMN21], in the languages to be used in this thesis. The first goal is to explicitly compute

$$w \stackrel{\text{def}}{=} \int_{S_n} e_{g(i_1)} \otimes \cdots \otimes e_{g(i_k)} dg \in (\mathbb{C}^n)^{\otimes k},$$

with respect to the Haar measure. We write

$$w = \sum_{1 \le j_1, \ldots, j_k \le n} (\alpha_{j_1,\ldots,j_k})\, e_{j_1} \otimes \cdots \otimes e_{j_k}$$

and define, for each $\pi \in \mathrm{Part}([k])$, a linear functional $\pi^{\text{strict}} : (\mathbb{C}^n)^{\otimes k} \to \mathbb{C}$ where, for each $I = (i_1, \ldots, i_k)$,

$$\pi^{\text{strict}}(e_I) = \begin{cases} 1 & \text{if } j \sim l \text{ in } \pi \iff i_j = i_l \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

In general, there exists a $g \in S_n$ such that

$$e_{g(i_1)} \otimes \cdots \otimes e_{g(i_k)} = e_{j_1} \otimes \cdots \otimes e_{j_k}, \tag{20}$$

if and only if the multi–index $I = (i_1, \ldots, i_k)$ and the multi–index $J = (j_1, \ldots, j_k)$ 'define the same partition' i.e. there is exactly one partition $\pi \in \mathrm{Part}([k])$ for which $\pi^{\text{strict}}(e_I)\, \pi^{\text{strict}}(e_J) = 1$. In this case, there are exactly $(n - |\pi|)!$ permutations $g \in S_n$ satisfying (20), leading to the following proposition.

Proposition 2.9. *The coefficient $\alpha_{j_1,\ldots,j_k}$ of $e_{j_1} \otimes \cdots \otimes e_{j_k}$ in w is*

$$\sum_{\pi \in \mathrm{Part}([k])} \pi^{\text{strict}}(e_{i_1} \otimes \cdots \otimes e_{i_k})\, \pi^{\text{strict}}(e_{j_1} \otimes \cdots \otimes e_{j_k}) \frac{1}{(n)_{|\pi|}}, \tag{21}$$

where $(n)_{|\pi|} = n(n-1)\ldots(n - |\pi| + 1)$ is the Pochhammer symbol.

This can be alternatively formulated using Möbius inversion, as in [BC10, Theorem 1.3 and Proposition 1.4]. Define the linear functional $\pi^{\text{weak}} : (\mathbb{C}^n)^{\otimes k} \to \mathbb{C}$, where, for each $I = (i_1, \ldots, i_k)$,

$$\pi^{\text{weak}}(e_I) = \begin{cases} 1 & \text{if } j \sim l \text{ in } \pi \implies e_{i_j} = e_{i_l} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Obviously,

$$\pi^{\text{strict}}(e_{i_1}\otimes\cdots\otimes e_{i_k})\neq 0 \implies \pi^{\text{weak}}(e_{i_1}\otimes\cdots\otimes e_{i_k})\neq 0$$

and in fact,

$$\pi^{\text{weak}} = \sum_{\pi_1\geq\pi} \pi_1^{\text{strict}}. \tag{23}$$

Using Theorem 2.5, we obtain the formula

$$\pi^{\text{strict}} = \sum_{\pi_1\geq\pi} \mu(\pi,\pi_1)\pi_1^{\text{weak}} \tag{24}$$

and can thus rewrite (21) as

$$\sum_{\pi_1,\pi_2\in\text{Part}([k])} \pi_1^{\text{weak}}(e_I)\,\pi_2^{\text{weak}}(e_J)\,\text{Wg}_{\text{n,k}}(\pi_1,\pi_2),$$

where $\text{Wg}_{n,k}$ is the Weingarten function for S_n, given by

$$\text{Wg}_{n,k}(\pi_1,\pi_2) = \sum_{\pi\leq\pi_1\wedge\pi_2} \mu(\pi,\pi_1)\mu(\pi,\pi_2)\frac{1}{(n)_{|\pi|}}.$$

It is a simple observation that

$$\text{Wg}_{n,k}(\pi_1,\pi_2) = O\left(\frac{1}{n^{|\pi_1\wedge\pi_2|}}\right). \tag{25}$$

Equivalently, one can reformulate this as follows: for any multi–index $I=(i_1,\ldots,i_k)$ and multi index $J=(j_1,\ldots,j_k)$, we have

$$\int_{S_n} g_{i_1j_1}\cdots g_{i_kj_k}dg = \sum_{\pi_1,\pi_2\in\text{Part}([k])} \delta_{\pi_1}(I)\delta_{\pi_2}(J)\text{Wg}_{n,k}(\pi_1,\pi_2), \tag{26}$$

where $\delta_{\pi_1}(I)=\pi_1^{\text{weak}}(e_I)$ and g_{ij} is the matrix coefficient of g acting diagonally on $(\mathbb{C}^n)^{\otimes k}$ (i.e. $\hat{\rho}(g)_{ij}$, with $\hat{\rho}$ defined as in §2.2.1). This is the form of the Weingarten calculus that we use in §4.3.

2.2 Schur–Weyl–Jones duality

2.2.1 Duality between $P_k(n)$ and S_n

Given $\pi \in \mathrm{Part}\left([2k]\right)$, define a right action of π on $(\mathbb{C}^n)^{\otimes k}$ by

$$\left\langle (e_{i_1} \otimes \cdots \otimes e_{i_k})\,\pi,\; e_{i_{k+1}} \otimes \cdots \otimes e_{i_{2k}} \right\rangle = \begin{cases} 1 & \text{if } j \sim k \text{ in } \pi \implies i_j = i_k \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Extending linearly gives $(\mathbb{C}^n)^{\otimes k}$ a right $P_k(n)$–module structure and defines, for each $\pi \in P_k(n)$, an element

$$P_\pi^{\text{weak}} \in \mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right),$$

where $P_\pi^{\text{weak}}(v) = v\pi$. We will denote the map $\pi \mapsto P_\pi^{\text{weak}}$ by

$$\hat{\theta} : P_k(n) \to \mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)^{\text{op}}.$$

Remark 2.10. We have

$$P_\pi^{\text{weak}} = \sum_{J,I} \pi^{\text{weak}}\left(e_J \otimes e_I\right)\left(\check{e}_J \otimes e_I\right)$$

via the canonical isomorphism $\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right) \cong \left(\check{\mathbb{C}}^n\right)^{\otimes k} \otimes (\mathbb{C}^n)^{\otimes k}$, where the sum is over all multi–indices J, I of size k. We also define $P_\pi^{\text{strict}} \in \mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$ by

$$\left\langle P_\pi^{\text{strict}}\left(e_{i_1} \otimes \cdots \otimes e_{i_k}\right),\; e_{i_{k+1}} \otimes \cdots \otimes e_{i_{2k}} \right\rangle = \begin{cases} 1 & \text{if } j \sim k \text{ in } \pi \iff i_j = i_k \\ 0 & \text{otherwise,} \end{cases}$$

noting that

$$P_\pi^{\text{strict}} = \sum_{J,I} \pi^{\text{strict}}\left(e_J \otimes e_I\right)\left(\check{e}_J \otimes e_I\right).$$

This is the form used in the statement of our main theorem.

Now, $(\mathbb{C}^n)^{\otimes k}$ is also a left S_n–module, as the k^{th} tensor power of the defining representation of S_n. Denote this representation

$$\hat{\rho} : \mathbb{C}[S_n] \to \mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right),$$

so that, for any $\tau \in S_n$ and basis vector $e_{i_1} \otimes \cdots \otimes e_{i_k} \in (\mathbb{C}^n)^{\otimes k}$, we have

$$\hat{\rho}(\tau)\left(e_{i_1} \otimes \cdots \otimes e_{i_k}\right) = e_{\tau(i_1)} \otimes \cdots \otimes e_{\tau(i_k)}.$$

Schur–Weyl–Jones duality asserts that these actions generate full mutual centralizers of one another in $\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right)$.

Theorem 2.11 ([Jon94]). *For $n \geq 2k$, where $P_k(n)$ acts via $\hat{\theta}$ and S_n acts via $\hat{\rho}$,*

1. *$P_k(n)$ generates $\mathrm{End}_{S_n}\left((\mathbb{C}^n)^{\otimes k}\right)$*
2. *S_n generates $\mathrm{End}_{P_k(n)}\left((\mathbb{C}^n)^{\otimes k}\right)$.*

2.2.2 Simple Modules for the Partition Algebra

For each $\lambda \vdash k$, recall the notation $\lambda^+(n) \vdash n$ for the Young diagram given by $(n-k, \lambda)$, defining a family of irreducible representations of S_n for each $n \geq \lambda_1$. Given any YD μ, we denote by μ^* the YD obtained by removing the first row of boxes (with this notation, $(\lambda^+(n))^* = \lambda$). Martin and Saleur [MS93, Corollary 4.1] showed that the simple modules over $P_k(n)$ are parametrized by Young diagrams of size $\leq k$. We define

$$\Lambda_{k,n} \stackrel{\text{def}}{=} \{\lambda \vdash i : i = 0, \dots, k\}.$$

Theorem 2.12. *When $P_k(n)$ is semisimple*[7]*, the set*

$$\left\{R^\lambda : \ \lambda \in \Lambda_{k,n}\right\}$$

constitutes a full set of representatives of the isomorphism classes of simple $P_k(n)$–modules.

So, for each $\lambda \in \Lambda_{k,n}$, there is a simple $P_k(n)$–module R^λ. These can be constructed inductively as is described in e.g. [Mar96, Section 1.3]. Using double centralizer theory,

[7]In fact, they subsequently show that this classification holds in the non–semisimple case.

the following decomposition follows from Theorem 2.11.

Theorem 2.13. *For each $k \in \mathbb{Z}_{>0}$, for every $n \geq 2k$, as a $(S_n, P_k(n))$–bimodule,*

$$(\mathbb{C}^n)^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_{k,n}} V^{\lambda^+(n)} \otimes R^\lambda.$$

2.2.3 Schur–Weyl Duality for S_n and S_k

We now present the refinement of Schur–Weyl–Jones duality due to Sam and Snowden [SS15, Section 6.1.3.] which is important for our construction of the projection $\mathcal{Q}_{\lambda,n}$. This construction was also considered by Littlewood [Lit58] in a somewhat different language to the one used in this thesis. For each $1 \leq j \leq k$, define the j^{th} linear contraction map

$$T_j : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k-1},$$

where

$$T_j\left(e_I\right) = e_{i_1} \otimes \cdots \otimes \dot{e}_{i_j} \otimes \cdots \otimes e_{i_k},$$

using the notation

$$e_{i_1} \otimes \cdots \otimes \dot{e}_{i_j} \otimes \cdots \otimes e_{i_k} \stackrel{\text{def}}{=} e_{i_1} \otimes \cdots \otimes e_{i_{j-1}} \otimes e_{i_{j+1}} \otimes \cdots \otimes e_{i_k}.$$

We also define

$$D_k(n) \stackrel{\text{def}}{=} \langle e_{i_1} \otimes \cdots \otimes e_{i_k} : \ i_1, \ldots, i_k \text{ pairwise distinct} \rangle \subseteq (\mathbb{C}^n)^{\otimes k}.$$

Definition 2.14. We define a vector subspace

$$A_k(n) \stackrel{\text{def}}{=} D_k(n) \cap \bigcap_{j=1}^{k} \operatorname{Ker}\left(T_j\right) \subseteq (\mathbb{C}^n)^{\otimes k}.$$

This space is clearly invariant under the inherited action of S_n and we denote this representation

$$\rho : \mathbb{C}\left[S_n\right] \to \operatorname{End}\left(A_k(n)\right).$$

Definition 2.14 can be reformulated as follows. Define a function

$$\mathrm{pn}:\mathrm{Part}\left([2k]\right)\to\mathbb{Z}_{\geq 0}$$

where $\mathrm{pn}(\pi)$ is the *propagating number* (see [Mar96, Definition 5]) of π – it is the number of elements of π that contain at least one element $i\leq k$ *and* at least one element j with $k+1\leq j\leq 2k$. Such elements correspond to connected components in the diagram of π with at least one vertex from each row. Two obvious properties are:

- For any $\pi\in\mathrm{Part}\left([2k]\right),\ 0\leq\mathrm{pn}(\pi)\leq k$;
- If $\mathrm{pn}(\pi)=k$ then $\pi=\iota(\tau)$ for some $\tau\in S_k$.

Let

$$I_k(n)\overset{\mathrm{def}}{=}\langle\pi:\pi\in\mathrm{Part}\left([2k]\right),\ \mathrm{pn}(\pi)\leq k-1\rangle_{\mathbb{C}}\subseteq P_k(n) \tag{28}$$

be the ideal generated by all $\pi\in\mathrm{Part}\left([2k]\right)$ with propagating number $\leq k-1$. So, $I_k(n)$ is the kernel of the map $R:P_k(n)\to\mathbb{C}\left[S_k\right]$ (recall (17)) which yields the isomorphism

$$\mathbb{C}\left[S_k\right]\cong P_k(n)/I_k(n).$$

Then we also have

$$A_k(n)=\bigcap_{\pi\in I_k(n)}\ker\left(P_\pi^{\mathrm{weak}}\right),$$

the subspace of $\left(\mathbb{C}^n\right)^{\otimes k}$ annihilated by $I_k(n)$. The inherited action of $P_k(n)$ on $A_k(n)$ thus descends to an action of $\mathbb{C}\left[S_k\right]$ and this action permutes tensor coordinates. We denote the associated representation

$$\theta:\mathbb{C}\left[S_k\right]\to\mathrm{End}\left(A_k(n)\right),$$

i.e. $\theta(\sigma)(w_1\otimes\cdots\otimes w_k)=\hat{\theta}\left(\iota\left(\sigma^{-1}\right)\right)(w_1\otimes\cdots\otimes w_k)=w_{\sigma^{-1}(1)}\otimes\cdots\otimes w_{\sigma^{-1}(k)}$. Defining the representation

$$\Delta:\mathbb{C}\left[S_n\times S_k\right]\to\mathrm{End}\left(A_k(n)\right)$$

by $\Delta(g,\sigma)\overset{\mathrm{def}}{=}\rho(g)\theta(\sigma)=\theta(\sigma)\rho(g)$, we get a decomposition of $A_k(n)$ into irreducible subrepresentations[8]

$$A_k(n) \cong \bigoplus_{\lambda \vdash k} V^{\lambda^+(n)} \otimes V^\lambda. \tag{29}$$

3 Obtaining projection formulas

The goal of this section is to prove Theorem 1.6, the material is largely unaltered from [Cas25a, Sections 3 and 4.].

3.1 Overview of Section 3

Given the decomposition

$$A_k(n) \cong \bigoplus_{\lambda \vdash k} V^{\lambda^+(n)} \otimes V^\lambda,$$

we use the central idempotent $\mathscr{P}_{V^\lambda} \in \mathbb{C}[S_k]$ (see (11)) to project

$$\xi \stackrel{\text{def}}{=} (e_1 - e_2) \otimes \cdots \otimes (e_{2k-1} - e_{2k})$$

to the irreducible block $V^{\lambda^+(n)} \otimes V^\lambda$. Writing

$$\xi_\lambda^{\text{norm}} \stackrel{\text{def}}{=} \frac{\theta\left(\mathscr{P}_{V^\lambda}\right)(\xi)}{\left\|\theta\left(\mathscr{P}_{V^\lambda}\right)(\xi)\right\|},$$

in §3.2, with $\theta : \mathbb{C}[S_k] \to \operatorname{End}(A_k(n))$ as defined in §2.2.3, we show that $\theta\left(\mathscr{P}_{V^\lambda}\right)(\xi)$ is non–zero for any $\lambda \vdash k$, so that the definition of $\xi_\lambda^{\text{norm}}$ makes sense and implying that the $S_n \times S_k$ representation generated by $\xi_\lambda^{\text{norm}}$, denoted by $\mathcal{U}_{\lambda^+(n)}$, is isomorphic to $V^{\lambda^+(n)} \otimes V^\lambda$ itself.

What did not work?

As is the case in the analogous setting of $U(n)$ and $S_k \times S_l$ acting on the mixed tensor space $(\mathbb{C}^n)^{\otimes k} \otimes \left((\mathbb{C}^n)^\vee\right)^{\otimes l}$ (see the construction given by Koike [Koi89] and further detailed in

[8] This decomposition follows from the discussion in [SS15, Section 6.1.3], detailing that the only irreducible representations appearing in the decomposition of $A_k(n)$ are those annihilated by $I_k(n)$. From the construction detailed in [Mar96, Section 1.2 and Section 1.3], these are exactly those indexed by Young diagrams of size k.

[Mag25a, Section 2.2]) one may expect to be able to show that

$$\mathcal{V}_{\lambda^+(n)} \stackrel{\text{def}}{=} \langle \rho(g)\,(\xi_\lambda^{\text{norm}}) :\ g \in S_n \rangle \cong V^{\lambda^+(n)}. \tag{30}$$

Were this to be the case, one could consider $\check{\xi}_\lambda^{\text{norm}} \otimes \xi_\lambda^{\text{norm}} \in \left(\mathcal{V}_{\lambda^+(n)}\right)^\vee \otimes \mathcal{V}_{\lambda^+(n)} \cong \text{End}\left(\mathcal{V}_{\lambda^+(n)}\right)$ and project this to $\text{End}_{S_n}\left(\mathcal{V}_{\lambda^+(n)}\right)$. This projection is exactly

$$\int_{S_n} \rho(g)\,(\xi_\lambda^{\text{norm}})^\vee \otimes \rho(g)\,(\xi_\lambda^{\text{norm}})\,dg \tag{31}$$

and, by Schur's lemma, we observe that this must be some multiple of the identity map on $\mathcal{V}_{\lambda^+(n)}$. Extending this by 0 on the orthocomplement of $\mathcal{V}_{\lambda^+(n)}$ in $(\mathbb{C}^n)^{\otimes k}$ then yields a scalar multiple of the orthogonal projection from $(\mathbb{C}^n)^{\otimes k} \to \mathcal{V}_{\lambda^+(n)}$.

One could then compute (31) explicitly using the Weingarten calculus for the symmetric group, yielding an explicit formula for this projection.The main obstacle to this approach is that, in general, $\mathcal{V}_{\lambda^+(n)}$ is not irreducible, with multiplicity dependent on λ and expressed in terms of Kostka numbers and Littlewood–Richardson coefficients, which are not easily computed.

The work around

To work around this difficulty, we use two key properties of ξ:

- ξ belongs to the sign–isotypic subspace of $A_k(n)$ for the action of $\mathcal{D} \cong S_2 \times \cdots \times S_2$ (see §3.3.2 for the details) and,
- For any $\sigma \in S_k$, $\theta\,(\sigma)\,\rho\,(\psi(\sigma))\,(\xi_\lambda^{\text{norm}}) = \xi_\lambda^{\text{norm}}$ (i.e. that $\psi(\sigma)\xi = \xi\sigma$). Here, $\psi : S_k \to S_{2k}$ is as defined in §2.1.3, we note that this is *not* the obvious inclusion obtained by adding k fixed points.

In §3.3.1 and §3.3.2, we use the first observation to show that, in the restriction $\mathcal{U}_{\lambda^+(n)} \downarrow_{S_{2k} \times S_k}$, the irreducible block $V^{\lambda^+(2k)} \otimes V^\lambda$ has multiplicity 1 and that $\xi_\lambda^{\text{norm}}$ is contained in this block. This implies that the $S_{2k} \times S_k$ representation generated by $\xi_\lambda^{\text{norm}}$, denoted $\mathcal{U}_{\lambda^+(2k)}$, is exactly isomorphic to $V^{\lambda^+(2k)} \otimes V^\lambda$.

We use the first observation again to show that $\xi_\lambda^{\text{norm}}$ is contained in the $\left(W^{\emptyset,\lambda} \otimes V^\lambda\right)$–isotypic subspace in the restriction $\mathcal{U}_{\lambda^+(2k)} \downarrow_{H_k \times S_k}$. One of the main technical challenges here is to show that this isotypic subspace always has multiplicity 1. This is done in §3.3.3 using the Gelfand–Tsetlin basis of $V^{\lambda^+(2k)} \otimes V^\lambda$.

Once we have this, it is relatively straightforward to show that the $\psi(S_k) \times S_k$ representation generated by $\xi_\lambda^{\text{norm}}$ is isomorphic to $V^\lambda \otimes V^\lambda$ and we use the second observation to show that (31) is a multiple of the orthogonal projection $\mathcal{Q}_{\lambda,n} : (\mathbb{C}^n)^{\otimes k} \to \mathcal{U}_{\lambda^+(n)}$. This is done in §3.3.4.

To complete the proof of Theorem 1.6, it remains to evaluate (31) explicitly using the Weingarten calculus and we do this in §3.4.

3.2 Dimension of $A_k(n)$

We digress briefly to find a recursive formula for the dimension of $A_k(n)$, when $n \geq 2k+1$.

Lemma 3.1. *For any $\lambda \vdash k$ and for any $n \geq 2k$, the projection ξ_λ of ξ to the V^λ–isotypic component of $A_k(n)$ is $\neq 0$.*

Proof. The projection of ξ to the V^λ–isotypic component of $A_k(n)$ is

$$\begin{aligned}\xi_\lambda &\stackrel{\text{def}}{=} \theta\left(\mathscr{P}_{V^\lambda}\right)(\xi) \\ &= \frac{d_\lambda}{k!} \sum_{\sigma \in S_k} \chi^\lambda(\sigma) \left[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}\right].\end{aligned}$$

Observe that, for $\sigma \neq \tau$, we have

$$\langle \theta(\sigma)(\xi),\ \theta(\tau)(\xi)\rangle = \prod_{i=1}^{k} \left\langle v_{\sigma(i)},\ v_{\tau(i)}\right\rangle = 0,$$

since at least one of these factors must be zero. So, $\{\theta(\sigma)(\xi) : \sigma \in S_k\}$ is a linearly independent set of vectors and, as a sum of these vectors with non–zero coefficients, ξ_λ is non–zero itself. □

This yields the proposition below, which details how to construct a subspace of $(\mathbb{C}^n)^{\otimes k}$ that is isomorphic to $V^{\lambda^+(n)} \otimes V^\lambda$.

Proposition 3.2. *When $n \geq 2k$, for any $\lambda \vdash k$, we have an isomorphism of representa-*

tions of $S_n \times S_k$,

$$\mathcal{U}_{\lambda^+(n)} \stackrel{\text{def}}{=} \left\langle \Delta\left(g,\sigma\right)\left(\xi_\lambda\right) : \ g \in S_n, \ \sigma \in S_k \right\rangle_{\mathbb{C}} \cong V^{\lambda^+(n)} \otimes V^{\lambda}.$$

Since

$$A_k(n) = \bigoplus_{\lambda \vdash k} \mathcal{U}_{\lambda^+(n)},$$

then, alongside Lemma 3.1, it is clear that

$$\langle \Delta(g,\sigma)\left(\xi\right) : \ g \in S_n, \ \sigma \in S_k \rangle_{\mathbb{C}} = A_k(n).$$

Now, consider the subspace $B_{k+1}(n)$ of $(\mathbb{C}^n)^{\otimes k+1}$,

$$B_{k+1}(n) \stackrel{\text{def}}{=} D_{k+1}(n) \cap \bigcap_{j=1}^{k} \ker(T_j) \subseteq (\mathbb{C}^n)^{\otimes k+1}.$$

Restricting

$$T_{k+1} : (\mathbb{C}^n)^{\otimes k+1} \to (\mathbb{C}^n)^{\otimes k}$$

to

$$\hat{T}_{k+1} : B_{k+1} \to (\mathbb{C}^n)^{\otimes k},$$

it is obvious that $\text{Im}\left(\hat{T}_{k+1}\right) \subseteq A_k(n)$. It is also clear that $A_k(n) \subseteq \text{Im}\left(\hat{T}_{k+1}\right)$, since

$$\hat{T}_{k+1}\left((e_1 - e_2) \otimes \cdots \otimes (e_{2k-1} - e_{2k}) \otimes e_{2k+1}\right) = \xi.$$

The inclusion $A_{k+1}(n) \hookrightarrow B_{k+1}(n)$ gives an exact sequence

$$0 \to A_{k+1}(n) \hookrightarrow B_{k+1}(n) \stackrel{\hat{T}_{k+1}}{\to} A_k(n) \to 0,$$

from which we obtain a recursive formula for the dimension of $A_{k+1}(n)$:

$$\dim\left(A_{k+1}(n)\right) = \dim\left(B_{k+1}(n)\right) - \dim\left(A_k(n)\right).$$

Now, $B_{k+1}(n) \cong \bigoplus_{i=1}^{n} A_k(n-1) \otimes e_i$. To see this, consider the vector space

$$A_k^i(n) = \langle e_{i_1} \otimes \cdots \otimes e_{i_k} : \; i_j \in \{1, \ldots, i-1, i+1, \ldots, n\}, \; i_j \text{ all distict} \rangle \cap \bigcap_{j=1}^{k} \ker(T_j).$$

Then, for each $i = 1, \ldots, n$,

$$A_k^i(n) \cong A_k(n-1)$$

and there is an obvious isomorphism

$$\bigoplus_{i=1}^{n} A_k^i(n) \otimes e_i \cong B_{k+1}(n).$$

From this observation we see that $\dim(B_{k+1}(n)) = n\dim(A_k(n-1))$, which yields the formula

$$\dim A_{k+1}(n) = n \dim A_k(n-1) - \dim A_k(n). \qquad (32)$$

The dimensions of $A_k(n)$ for $k = 0, \ldots, 10$ are in the table below, expressed as a polynomial in n.

Table 1: The dimension of $A_k(n)$ as a polynomial in n for fixed k

k	$\dim(A_k(n))$
1	$n-1$
2	$n^2 - 3n + 1$
3	$n^3 - 6n^2 + 8n - 1$
4	$n^4 - 9n^3 + 22n^2 - 13n + 1$
5	$n^5 - 12n^4 + 43n^3 - 49n^2 + 18n - 1$
6	$n^6 - 15n^5 + 71n^4 - 122n^3 + 87n^2 - 23n + 1$
7	$n^7 - 18n^6 + 106n^5 - 245n^4 + 265n^3 - 136n^2 + 28n - 1$
8	$n^8 - 21n^7 + 148n^6 - 431n^5 + 630n^4 - 491n^3 + 196n^2 - 33n + 1$
9	$n^9 - 24n^8 + 197n^7 - 693n^6 + 1281n^5 - 1351n^4 + 819n^3 - 267n^2 + 38n - 1$
10	$n^{10} - 27n^9 + 253n^8 - 1044n^7 + 2338n^6 - 3122n^5 + 2562n^4 - 1268n^3 + 349n^2 - 43n + 1$

3.3 Identifying a Projection Map

We first normalize ξ_λ – the norm$||\xi_\lambda|| = \langle \xi_\lambda, \xi_\lambda \rangle^{\frac{1}{2}}$ is easily computed:

$$
\begin{aligned}
\langle \xi_\lambda, \xi_\lambda \rangle &= \left\langle \sum_{\sigma \in S_k} \frac{d_\lambda}{k!} \chi^\lambda(\sigma) \left[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right], \sum_{\tau \in S_k} \frac{d_\lambda}{k!} \chi^\lambda(\tau) \left[v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)} \right] \right\rangle \\
&= \left(\frac{d_\lambda}{k!} \right)^2 \sum_{\sigma \in S_k} \sum_{\tau \in S_k} \chi^\lambda(\sigma) \chi^\lambda(\tau) \left\langle v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}, v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)} \right\rangle \\
&= \left(\frac{d_\lambda}{k!} \right)^2 \sum_{\sigma \in S_k} \chi^\lambda(\sigma)^2 \left\langle v_{\sigma(1)}, v_{\sigma(1)} \right\rangle \dots \left\langle v_{\sigma(k)}, v_{\sigma(k)} \right\rangle \\
&= \left(\frac{d_\lambda}{k!} \right)^2 2^k \sum_{\sigma \in S_k} \chi^\lambda(\sigma)^2 \\
&= \frac{2^k d_\lambda^2}{k!}.
\end{aligned}
$$

We will write

$$
\begin{aligned}
\xi_\lambda^{\text{norm}} \stackrel{\text{def}}{=} \frac{\xi_\lambda}{||\xi_\lambda||} &= \left(\frac{k!}{2^k d_\lambda^2} \right)^{\frac{1}{2}} \sum_{\sigma \in S_k} \frac{d_\lambda}{k!} \chi^\lambda(\sigma) \left[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right] \\
&= \left(\frac{1}{2^k k!} \right)^{\frac{1}{2}} \sum_{\sigma \in S_k} \chi^\lambda(\sigma) \left[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right].
\end{aligned}
$$

3.3.1 Restricting from S_n to S_{2k}

We write the decomposition of $V^{\lambda^+(n)} \downarrow_{S_{2k}}$ into irreducible representations of S_{2k} as

$$
V^{\lambda^+(n)} \downarrow_{S_{2k}} \cong \bigoplus_{\mu \vdash 2k} (V^\mu)^{\oplus c_\mu}. \tag{33}
$$

There are 'branching rules' for irreducible representations of the symmetric group, see [FH04, p. 59] for example. One can observe that the multiplicity of V^μ in (33) is zero, unless the YD μ is contained inside the YD $\lambda^+(n)$. In this case, the multiplicity is the number of ways of labeling the skew YD $\lambda^+(n)\backslash\mu$ with the numbers $1, \dots, n-2k$, so that no number is repeated and the numbers are increasing both along each row and down each column. Using this rule:

- if $\mu \vdash 2k$ with $\mu_1 < k$, then V^μ has multiplicity 0 in (33), since then $|\mu^*| > k$, meaning μ^* is not contained within λ;
- if $\mu \vdash 2k$ has $\mu_1 = k$, then V^μ has multiplicity zero, unless $\mu = \lambda^+(2k)$ *and*

- the multiplicity of $V^{\lambda^+(2k)}$ is 1 – the skew diagram $\lambda^+(n)\backslash\lambda^+(2k)$ is exactly one row of $n-2k$ boxes.

These observations imply the following proposition.

Proposition 3.3. *For any $\lambda \vdash k$ and $n \geq 2k$, we have*

$$\mathcal{U}_{\lambda^+(n)} \downarrow_{S_{2k}\times S_k} = \mathcal{U}_{\lambda^+(2k)} \oplus \mathcal{U}^{\perp}_{\lambda^+(2k)},$$

where

$$\mathcal{U}_{\lambda^+(2k)} \cong V^{\lambda^+(2k)} \otimes V^{\lambda} \tag{34}$$

and

$$\mathcal{U}^{\perp}_{\lambda^+(2k)} \cong \bigoplus_{\mu\vdash 2k,\ \mu_1>k} \left(V^{\mu}\otimes V^{\lambda}\right)^{\oplus c_\mu}.$$

3.3.2 Finding the $S_{2k}\times S_k$ representation generated by $\xi_\lambda^{\mathrm{norm}}$

We will use the Gelfand–Tsetlin basis of $\mathcal{U}_{\lambda^+(n)}$ to show that $\xi_\lambda^{\mathrm{norm}} \in \mathcal{U}_{\lambda^+(2k)}$, which leads to Proposition 3.5. We need to show that $\xi_\lambda^{\mathrm{norm}}$ is orthogonal to the subspace $\mathcal{U}^{\perp}_{\lambda^+(2k)}$. Consider the subgroup

$$\mathcal{D} = \left\langle s_1,\ s_3, \dots,\ s_{2k-1}\right\rangle \leq S_{2k} \leq S_n,$$

which appeared in §2.1.3 and notice that, for each $i = 1, 3, \dots, 2k-1$,

$$\rho(s_i)\left(\xi_\lambda^{\mathrm{norm}}\right) = -\xi_\lambda^{\mathrm{norm}}, \tag{35}$$

so that

$$\Delta\left(s_i, \mathrm{Id}\right)\left(\xi_\lambda^{\mathrm{norm}}\right) = -\xi_\lambda^{\mathrm{norm}}.$$

We write $\xi_\lambda^{\mathrm{norm}}$ in the Gelfand–Tsetlin basis:

$$\xi_\lambda^{\mathrm{norm}} = \sum_{T_1,T_2} \beta_{T_1,T_2}\left(v_{T_1}\otimes v_{T_2}\right), \tag{36}$$

where

$$T_1 \in \mathrm{Tab}\left(\lambda^+(2k)\right) \text{ and } T_2 \in \mathrm{Tab}\left(\lambda\right)$$

or $v_{T_1} \otimes v_{T_2}$ represents a Gelfand–Tsetlin basis vector of one of the subspaces

$$V^{\mu} \otimes V^{\lambda}$$

where $\mu \vdash 2k$ and $\mu_1 > k$. In this case we will have

$$T_1 \in \bigcup_{\mu \vdash 2k,\ \mu_1 > k} \mathrm{Tab}(\mu)$$

and

$$T_2 \in \mathrm{Tab}(\lambda).$$

Lemma 3.4. *For all $\mu \vdash 2k$ with $\mu_1 > k$ and for any $\tilde{T}_2 \in \mathrm{Tab}(\lambda)$, if $\tilde{T}_1 \in \mathrm{Tab}(\mu)$ then, in (36), we have*

$$\beta_{\tilde{T}_1, \tilde{T}_2} = 0.$$

Proof. Fix any $\tilde{T}_2 \in \mathrm{Tab}(\lambda)$ and consider any $v_{\tilde{T}_1} \otimes v_{\tilde{T}_2}$ where

$$\tilde{T}_1 \in \bigcup_{\mu \vdash 2k,\ \mu_1 > k} \mathrm{Tab}(\mu).$$

Since $\mu_1 > k$, out of each of the k pairs $\{1, 2\},\ \{3, 4\},\ \ldots,\ \{2k-1, 2k\}$, there must be *at least one* pair, say i and $i+1$, in which both elements appear in the first row of boxes of $\tilde{T}_1$. So by Proposition 2.2,

$$(s_i, \mathrm{Id}) \left(v_{\tilde{T}_1} \otimes v_{\tilde{T}_2} \right) = v_{\tilde{T}_1} \otimes v_{\tilde{T}_2}. \tag{37}$$

We look at the coefficient of $v_{\tilde{T}_1} \otimes v_{\tilde{T}_2}$ in

$$(s_i, \mathrm{Id}) \sum_{T_1, T_2} \beta_{T_1, T_2} \left(v_{T_1} \otimes v_{T_2} \right).$$

If $T_2 \neq \tilde{T}_2$, then, for any choice of T_1,

$$\begin{aligned}
&\left\langle (s_i, \mathrm{Id}) \left(v_{T_1} \otimes v_{T_2}\right), \ v_{\tilde{T}_1} \otimes v_{\tilde{T}_2} \right\rangle \\
&= \left\langle \left(s_i v_{T_1}\right), v_{\tilde{T}_1} \right\rangle \underbrace{\left\langle v_{T_2}, v_{\tilde{T}_2} \right\rangle}_{=0} \\
&= 0.
\end{aligned}$$

So now suppose that $T_2 = \tilde{T}_2$ and let

$$T_1 \in \mathrm{Tab}(\lambda^+(2k)) \cup \bigcup_{\mu \vdash 2k, \ \mu_1 > k} \mathrm{Tab}(\mu),$$

with $T_1 \neq \tilde{T}_1$. If T_1 is not of the same shape as $\tilde{T}_1$, then clearly

$$\left\langle s_i v_{T_1}, v_{\tilde{T}_1} \right\rangle = 0,$$

which implies

$$\left\langle (s_i, \mathrm{Id}) \left(v_{T_1} \otimes v_{\tilde{T}_2} \right), \ v_{\tilde{T}_1} \otimes v_{\tilde{T}_2} \right\rangle = 0.$$

If T_1 and $\tilde{T}_1$ are of the same shape, then there are 3 possibilities for the positions of the boxes labeled i and $i+1$ in T_1:

1. If the boxes labeled i and $i+1$ are in the same row of T_1, then $s_i v_{T_1} = v_{T_1}$.

2. If the boxes labeled i and $i+1$ are in the same column of T_1 then $s_i v_{T_1} = -v_{T_1}$.

3. If the boxes labeled i and $i+1$ are in neither the same row or column then we have $s_i v_{T_1} = \left(r^{-1} v_{T_1}\right) + \left(\sqrt{1 - r^{-2}}\right) v_{T_1'}$, where r and T_1' are as defined as in Proposition 2.2.

In any of the above cases, we have

$$\left\langle s_i v_{T_1}, v_{\tilde{T}_1} \right\rangle = 0.$$

This is because $T_1 \neq \tilde{T}_1$ and, in the final case, we also have $T_1' \neq \tilde{T}_1$. The above observations imply

$$\begin{aligned}
&\left\langle \Delta\left(s_i, \mathrm{Id}\right) \xi_\lambda^{\mathrm{norm}},\ v_{\tilde{T}_1} \otimes v_{\tilde{T}_2} \right\rangle \\
&= \sum_{T_1, T_2} \beta_{T_1, T_2} \left\langle \left(s_i, \mathrm{Id}\right)\left(v_{T_1} \otimes v_{T_2}\right),\ v_{\tilde{T}_1} \otimes v_{\tilde{T}_2} \right\rangle \\
&= \beta_{\tilde{T}_1, \tilde{T}_2} \left\langle \left(s_i, \mathrm{Id}\right)\left(v_{\tilde{T}_1} \otimes v_{\tilde{T}_2}\right),\ v_{\tilde{T}_1} \otimes v_{\tilde{T}_2} \right\rangle .
\end{aligned}$$

Using (37), this is exactly

$$\beta_{\tilde{T}_1, \tilde{T}_2}.$$

But (35) implies that the coefficient of $v_{\tilde{T}_1} \otimes v_{\tilde{T}_2}$ in $\Delta(s_i, \mathrm{Id})\xi_\lambda^{\mathrm{norm}}$ is $-\beta_{\tilde{T}_1, \tilde{T}_2}$, implying that $\beta_{\tilde{T}_1, \tilde{T}_2} = 0$. □

Proposition 3.5. *When $n \geq 2k$, given any $\lambda \vdash k$, we have*

$$\left\langle \Delta(g, \sigma)\left(\xi_\lambda^{\mathrm{norm}}\right) :\ g \in S_{2k},\ \sigma \in S_k \right\rangle_{\mathbb{C}} = \mathcal{U}_{\lambda + (2k)}.$$

Proof. This follows from Proposition 3.3 and Lemma 3.4, since Lemma 3.4 implies that

$$\xi_\lambda^{\mathrm{norm}} \in \mathcal{U}_{\lambda + (2k)}.$$

□

3.3.3 Constructing $W^{\emptyset, \lambda} \otimes V^\lambda$ inside $(\mathbb{C}^n)^{\otimes k}$

We will show that $\xi_\lambda^{\mathrm{norm}}$ is in the $\left(W^{\emptyset, \lambda} \otimes V^\lambda\right)$–isotypic subrepresentation of $\mathcal{U}_{\lambda + (2k)} \downarrow_{H_k \times S_k}$ and that this subrepresentation has multiplicity one, so that

$$\left\langle \Delta(g, \sigma)\left(\xi_\lambda^{\mathrm{norm}}\right) : g \in H_k,\ \sigma \in S_k \right\rangle_{\mathbb{C}} \cong W^{\emptyset, \lambda} \otimes V^\lambda.$$

We write the decomposition

$$\mathcal{U}_{\lambda + (2k)} \downarrow_{H_k \times S_k} \cong \bigoplus_{(\mu, \pi) \models 2k, \nu \vdash k} \left(W^{\mu, \pi} \otimes V^\nu\right)^{\oplus c(\mu, \pi, \nu)}. \tag{38}$$

By definition, $\xi_\lambda^{\mathrm{norm}}$ must be orthogonal to any component of this decomposition with $\nu \neq \lambda$. Moreover, by (35), for any generator s_i of $\mathcal{D}$, the element $(s_i, \mathrm{Id}) \in H_k \times S_k$ and

$\Delta(s_i, \mathrm{Id})\left(\xi_\lambda^{\mathrm{norm}}\right) = -\xi_\lambda^{\mathrm{norm}}$. The only irreducible representations of $H_k \times S_k$ for which this property holds *for every generator* s_i are representations of the form

$$W^{\emptyset,\pi} \otimes V^\nu,$$

where $\pi \vdash k$ and $\nu \vdash k$. Combining these observations implies that $\xi_\lambda^{\mathrm{norm}}$ is orthogonal to any subrepresentation that is *not* isomorphic to

$$\left(W^{\emptyset,\pi} \otimes V^\lambda\right)^{\oplus c(\emptyset,\pi,\lambda)},$$

where $\pi \vdash k$. With the observation that for any $\sigma \in S_k$,

$$\theta(\sigma)\left(\xi_\lambda^{\mathrm{norm}}\right) = \rho\left(\psi\left(\sigma^{-1}\right)\right)\left(\xi_\lambda^{\mathrm{norm}}\right),$$

it follows that $\xi_\lambda^{\mathrm{norm}}$ belongs to the $\left(W^{\emptyset,\lambda} \otimes V^\lambda\right)$–isotypic component in the decomposition. It remains to show that this isotypic component has multiplicity one. A first attempt would be to use the branching rule given by Koike and Terada in [KT87].

Proposition 3.6. *Denote by s_μ a Schur polynomial, f_i the elementary symmetric polynomial of degree i and p_j the j^{th}complete symmetric polynomial. Then, given $\pi \vdash 2k$,*

$$\mathrm{Res}_{H_k}^{S_{2k}} V^\pi \cong \bigoplus_{(\mu,\nu)\models k} \left(W^{\mu,\nu}\right)^{\oplus d_{\mu,\nu}^\pi},$$

where the multiplicity $d_{\mu,\nu}^\pi$ of each $W^{\mu,\nu}$ coincides exactly with the coefficient of s_π in the product $(s_\mu \circ p_2)(s_\nu \circ f_2)$. That is, $d_{\mu,\nu}^\pi$ satisfies

$$(s_\mu \circ p_2)(s_\nu \circ f_2) = \sum_\pi d_{\mu,\nu}^\pi s_\pi.$$

Following this proposition, we write

$$\mathcal{U}_{\lambda^+(2k)} \downarrow_{H_k \times S_k} \cong \bigoplus_{(\mu,\nu)\models k} \left(W^{\mu,\nu} \otimes V^\lambda\right)^{\oplus d_{\mu,\nu}^{\lambda^+(2k)}} \tag{39}$$

and the following lemma is immediate from the preceding discussion.

Lemma 3.7. *With $d^{\pi}_{\mu,\nu}$ defined as in Proposition 3.6 , for some $\kappa^{\lambda^+(2k)}_{\emptyset,\lambda}$ satisfying $1 \leq \kappa^{\lambda^+(2k)}_{\emptyset,\lambda} \leq d^{\lambda^+(2k)}_{\emptyset,\lambda}$,*

$$\left\langle \Delta(g,\sigma)\left(\xi^{\mathrm{norm}}_{\lambda}\right) : \ g \in H_k, \ \sigma \in S_k \right\rangle_{\mathbb{C}} \cong \left(W^{\emptyset,\lambda} \otimes V^{\lambda}\right)^{\oplus \kappa^{\lambda^+(2k)}_{\emptyset,\lambda}}, \tag{40}$$

as representations of $H_k \times S_k$.

The aim is to prove the following proposition.

Proposition 3.8. *The multiplicity $d^{\lambda^+(2k)}_{\emptyset,\lambda}$ of $W^{\emptyset,\lambda}$ in the restriction of $V^{\lambda^+(2k)}$ from S_{2k} to H_k is exactly 1.*

We do not know how to use Proposition 3.6 effectively, since, in general, it is difficult to compute

$$(s_{\emptyset} \circ p_2)\,(s_{\lambda} \circ f_2)\,.$$

Using that $s_{\emptyset}$ is the constant function and $f_2 = s_{(1,1)}$, our task reduces to showing that the coefficient of $s_{\lambda^+(2k)}$ in the Schur polynomial expansion of

$$s_{\lambda} \circ s_{(1,1)} \tag{41}$$

is indeed one. This plethysm can be evaluated in the special cases whereby $|\lambda| \leq 3$, see for example [COS+22, Theorem 5.3] and in some other special cases of $\lambda \vdash k$ (for example, when $\lambda = (k)$ or $\lambda = (1,\ldots,1)$, see [COS+22, Section 5.3]). To our knowledge, there is no simple expression for (41) for every λ, and even the task at hand of evaluating just one specific coefficient in the Schur expansion does not appear to have an obvious straightforward approach.

Instead, we will construct the $W^{\emptyset,\lambda}$–isotypic subspace of $V^{\lambda^+(2k)} \downarrow_{H_k}$ using the Gelfand–Tsetlin basis and, in doing so, we will see that the multiplicity must be one. Recall that the character of $W^{\emptyset,\lambda}$ is $\chi^{\lambda}\chi_{\rho_0}$, so that the generators of $\mathcal{D}$ act on $W^{\emptyset,\lambda}$ by multiplying elements by -1. Which is to say that the $W^{\emptyset,\lambda}$–isotypic subspace of $V^{\lambda^+(2k)} \downarrow_{H_k}$ must be contained in the sign–isotypic component of $\mathcal{D}$ in the vector space $V^{\lambda^+(2k)}$. Denote by

$$V^{\lambda^+(2k)}_{\mathcal{D},\mathrm{sign}} \stackrel{\mathrm{def}}{=} \mathrm{sign} - \mathrm{isotypic\ component\ of\ } \mathcal{D} \mathrm{\ in\ } V^{\lambda^+(2k)},$$

so that

$$\left(W^{\emptyset,\lambda}\right)^{d_{\emptyset,\lambda}^{\lambda^+(2k)}} \subseteq V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}. \tag{42}$$

Our method is as follows:

1. We will show that $V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$ is a H_k–submodule of $V^{\lambda^+(2k)}\downarrow_{H_k}$;

2. Then we will construct $V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$ using the Gelfand–Tsetlin basis of $V^{\lambda^+(2k)}$;

3. In doing the above step, we will see that $V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$ has dimension d_λ, so that (42) implies that $V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)} = W^{\emptyset,\lambda}$ and that $d_{\emptyset,\lambda}^{\lambda^+(2k)} = 1$.

Define

$$\begin{aligned} &\text{Tab}_{\mathcal{D},\text{sign}}\left(\lambda^+(2k)\right) \\ &\stackrel{\text{def}}{=}\Big\{T \in \text{Tab}\left(\lambda^+(2k)\right) \;:\; T \text{ has exactly one representative from each of } \{1,2\}, \\ &\qquad\qquad \{3,4\},\ldots,\ \{2\text{k}-1,2\text{k}\} \text{ in the first row of boxes}\Big\} \end{aligned} \tag{43}$$

and the associated subspace

$$\left\langle v_{\hat{T}} :\ \hat{T} \in \text{Tab}_{\mathcal{D},\text{sign}}\left(\lambda^+(2k)\right)\right\rangle \subseteq V^{\lambda^+(2k)}.$$

Proposition 3.9. *For any* $\lambda \vdash k$,

$$V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)} \subseteq \left\langle v_{\hat{T}} :\ \hat{T} \in \text{Tab}_{\mathcal{D},\text{sign}}\left(\lambda^+(2k)\right)\right\rangle.$$

Proof. Let

$$\hat{T} \in \text{Tab}\left(\lambda^+(2k)\right) \setminus \text{Tab}_{\mathcal{D},\text{sign}}\left(\lambda^+(2k)\right)$$

and suppose, towards a contradiction, that $u \in V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$ is such that

$$\langle u, v_{\hat{T}}\rangle = \alpha \neq 0.$$

Since $\lambda^+(2k)$ has k boxes in the top row and $\hat{T} \notin \text{Tab}_{\mathcal{D},\text{sign}}\left(\lambda^+(2k)\right)$, there must be some $i \in \{1,\ 3,\ldots,\ 2k-1\}$ for which the boxes labeled i and $i+1$ are both in the top row. Then $s_i = (i\ i+1)$ acts trivially on $v_{\hat{T}}$ by Proposition 2.2. Using the same reasoning as

Lemma 3.4, if $\hat{T}_1 \neq \hat{T}$, then $\left\langle s_i v_{\hat{T}_1}, v_{\hat{T}} \right\rangle = 0$. So then

$$\begin{aligned} \left\langle s_i u, v_{\hat{T}} \right\rangle &= \left\langle u, v_{\hat{T}} \right\rangle \left\langle s_i v_{\hat{T}}, v_{\hat{T}} \right\rangle \\ &= \left\langle u, v_{\hat{T}} \right\rangle \\ &= \alpha. \end{aligned}$$

However, since $u \in V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$ and $s_i \in \mathcal{D}$, we must have

$$s_i u = -u,$$

which obviously implies that

$$\left\langle s_i u, v_{\hat{T}} \right\rangle = -\alpha,$$

contradicting the fact that $\alpha \neq 0$. □

Proposition 3.10. *For each $\lambda \vdash k$, the space $V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)} \subset V^{\lambda^+(2k)}$ is a H_k–submodule.*

Proof. Let $h \in H_k$, $u \in V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$ and $\delta \in \mathcal{D}$. Since $\mathcal{D}$ is normal in H_k, we have

$$\begin{aligned} \delta(hu) &= hh^{-1}\delta(hu) \\ &= h(h^{-1}\delta h)u \\ &= \text{sign}\left(h^{-1}\delta h\right)(hu) \\ &= \text{sign}(\delta)(hu), \end{aligned}$$

so that $hu \in V_{\mathcal{D},\text{sign}}^{\lambda^+(2k)}$. □

For any μ of any given size, say $\mu \vdash l$, we define an injective map

$$\Psi : V^{\mu} \to V^{\mu^+(2l)},$$

where $\Psi(v_T)$ corresponds to the Young tableau of shape $\mu^+(2l)$, in which the labels of the boxes in the positions of $\mu^+(2l)^*$ are double the corresponding label in T, and the top row of boxes contains the labels (in order) $1, 3, \ldots, 2l-1$. By a slight abuse of notation, we will label this Young tableau by $\Psi(T)$, so that $\Psi(v_T) = v_{\Psi(T)}$. For example, with $\mu = (2,1)$,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \overset{\Psi}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} \overset{\Psi}{\mapsto} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}$$

For each $\lambda \vdash k$ and for each $T \in \mathrm{Tab}(\lambda)$, define

$$Z_T \stackrel{\text{def}}{=} \langle \mathcal{D}\Psi\left(v_T\right)\rangle \subseteq V^{\lambda^+(2k)}. \tag{44}$$

We must establish some extra notation for the remainder of this section. For a given Young tableau T of any shape, if the boxes labeled i and $i+1$ are in different rows and columns, then we will now denote by $T'_{\{i,i+1\}}$ the Young tableau of the same shape obtained by swapping the labels i and $i+1$. If we also have $j \neq i, i+1$ and $j+1 \neq i, i+1$, with the boxes labeled j and $j+1$ in different rows and columns, then we denote by $T''_{\{i,i+1\},\{j,j+1\}}$ the Young tableau of the same shape obtained by swapping the labels i and $i+1$ and then the labels j and $j+1$. The Young tableau

$$T^{(m)}_{\{i_1,i_1+1\},\ldots,\{i_m,i_m+1\}}$$

is defined in the obvious way. If T is a Young tableau of shape $\mu \vdash l$, we will write $T\backslash\{l\}$ for the Young tableau of shape $\mu' \vdash (l-1)$, the YD obtained from μ by removing the box labeled by l in T and keeping all other boxes and labels the same. We define $T\backslash\{l, l-1, \ldots, l-j\}$ inductively in the obvious way.

Lemma 3.11. *Let $\lambda \vdash k$ and $T, \dot{T} \in \mathrm{Tab}(\lambda)$. Then, if $T \neq \dot{T}$, $Z_T \perp Z_{\dot{T}}$.*

Proof. By Proposition 2.2, any $v \in Z_T$ must be written as a linear combination of $v_{\tilde{T}}$ such that, if a box $\square \in \lambda$ has label $x \in [k]$ in T, then the corresponding box $\tilde{\square} \in (\lambda^+(2k))^*$ has label either $2x$ or $2x-1$ in $\tilde{T}$. The same is true for any element $u \in Z_{\dot{T}}$. Since $T \neq \dot{T}$, there is a box $\square \in \lambda$ labeled x in T and y in $\dot{T}$ with $x \neq y$. So any $v \in Z_T$ is written as a linear combination of $v_{\tilde{T}}$ as described and then, since $x \neq y$, we cannot have $v \in Z_{\dot{T}}$. $\square$

Lemma 3.12. *For every $\lambda \vdash k$,*

$$\left\langle v_{\hat{T}} : \ \hat{T} \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}\left(\lambda^+(2k)\right)\right\rangle \subseteq \bigoplus_{T \in \mathrm{Tab}(\lambda)} Z_T.$$

Proof. We will prove this (for each basis vector of the LHS) by induction on the number of *even* elements in the top row of $\hat{T} \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}(\lambda^+(2k))$. Denote by

$$q_{E,\hat{T}}$$

the number of boxes with an even label in the top row of any Young tableau $\hat{T} \in \mathrm{Tab}(\lambda^+(2k))$ and suppose that

$$\hat{T} \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}\left(\lambda^+(2k)\right)$$

is such that $q_{E,\hat{T}} = 0$. Then, in $\hat{T}$, all of the boxes in $(\lambda^+(2k))^*$ *must* have an even label. Let $T^* \in \mathrm{Tab}(\lambda)$ be such that

$$\Psi\left(v_{T^*}\right) = v_{\hat{T}},$$

which implies that

$$v_{\hat{T}} \in Z_{T^*} \subseteq \bigoplus_{T \in \mathrm{Tab}(\lambda)} Z_T.$$

Now suppose that

$$\hat{T} \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}\left(\lambda^+(2k)\right)$$

is such that $q_{E,\hat{T}} > 0$ and that, for any $T \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}(\lambda^+(2k))$ with $0 \leq q_{E,T} < q_{E,\hat{T}}$, we have $v_T \in \bigoplus_{T \in \mathrm{Tab}(\lambda)} Z_T$. Suppose that the first *even* label in the top row of boxes of $\hat{T}$ is j and let $\tilde{T} = \hat{T}'_{\{j-1,j\}}$. Then we also have

$$\tilde{T} \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}\left(\lambda^+(2k)\right)$$

so that

$$v_{\tilde{T}} \in \left\langle v_{\hat{T}} : \ \hat{T} \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}\left(\lambda^+(2k)\right)\right\rangle.$$

Moreover, $q_{E,\tilde{T}} = q_{E,\hat{T}} - 1$, so, by the inductive hypothesis, $v_{\tilde{T}} \in \bigoplus_{T \in \mathrm{Tab}(\lambda)} Z_T$.

Since the Z_T are pairwise orthogonal, there is exactly one $T^* \in \mathrm{Tab}(\lambda)$ such that $v_{\tilde{T}} \in Z_{T^*}$. By Proposition 2.2, we have

$$s_{j-1} v_{\tilde{T}} = \left(r^{-1}\right) v_{\tilde{T}} + \left(\sqrt{1 - r^{-2}}\right) v_{\tilde{T}'_{\{j-1,j\}}},$$

where r is as defined in the proposition. This shows that

$$\left(r^{-1}\right) v_{\tilde{T}} + \left(\sqrt{1-r^{-2}}\right) v_{\tilde{T}'_{\{j-1,j]}} \in Z_{T^*},$$

which implies that

$$v_{\tilde{T}'_{\{j-1,j]}} \in Z_{T^*}.$$

But $\tilde{T}'_{\{j-1,j\}} = \hat{T}$, so we have $v_{\hat{T}} \in Z_{T^*} \subseteq \bigoplus_{T\in\mathrm{Tab}(\lambda)} Z_T$. □

Lemma 3.13. *For any $\lambda \vdash k$, the sign–isotypic component of $\mathcal{D}$ in $\bigoplus_{T\in\mathrm{Tab}(\lambda)} Z_T$ is exactly $V^{\lambda^+(2k)}_{\mathcal{D},\mathrm{sign}}$.*

Proof. Since $\bigoplus_{T\in\mathrm{Tab}(\lambda)} Z_T \subseteq V^{\lambda^+(2k)}$, the sign–isotypic component of $\mathcal{D}$ in

$$\bigoplus_{T\in\mathrm{Tab}(\lambda)} Z_T$$

is obviously contained in $V^{\lambda^+(2k)}_{\mathcal{D},\mathrm{sign}}$. On the other hand, by Proposition 3.9, $V^{\lambda^+(2k)}_{\mathcal{D},\mathrm{sign}}$ must be contained in the sign–isotypic component of $\mathcal{D}$ in

$$\left\langle v_T : \ T \in \mathrm{Tab}_{\mathcal{D},\mathrm{sign}}\left(\lambda^+(2k)\right)\right\rangle,$$

which, by the previous lemma, must be contained inside the sign–isotypic component of $\mathcal{D}$ in $\bigoplus_{T\in\mathrm{Tab}(\lambda)} Z_T$. □

For each $T \in \mathrm{Tab}(\lambda)$, we denote by $Z_T^{\mathcal{D},\mathrm{sign}}$ the sign–isotypic subspace of $\mathcal{D}$ in Z_T.

Proposition 3.14. *For any $\lambda \vdash k$ and for any $T \in \mathrm{Tab}(\lambda)$, $Z_T^{\mathcal{D},\mathrm{sign}}$ is one dimensional.*

To prove Proposition 3.14 we introduce some additional notation and prove three intermediate lemmas. For each $\lambda \vdash k$ and for each $T \in \mathrm{Tab}\,(\lambda)$, for each $i \in \{1,\ldots,k-1\}$, define

$$Z_T^{(i)} \stackrel{\mathrm{def}}{=} \left\langle \mathcal{D}^{(i)}\Psi\left(v_{T\setminus\{k,\ldots,k-i+1\}}\right)\right\rangle,$$

where

$$\mathcal{D}^{(i)} \stackrel{\mathrm{def}}{=} \left\langle s_1,\ldots,s_{2k-2i-1}\right\rangle.$$

Define

$$\mathcal{D}_{\perp}^{(i)} \stackrel{\text{def}}{=} \langle s_{2k-2i+1}, \ldots, s_{2k-1} \rangle$$

and also

$$\text{Tab}\left(Z_T^{(i)}\right) = \left\{\hat{T} \in \text{Tab}(\mu) : \mu \vdash (2k-2i),\ \langle z, v_{\hat{T}} \rangle \neq 0 \text{ for some } z \in Z_T^{(i)}\right\}.$$

Then, for each $i \in \{1, \ldots, k\}$, we introduce a map

$$\Phi_{T,i}^{\lambda} : Z_T^{(i-1)} \to \bigoplus_{\mu \vdash 2k-2i} V^{\mu}$$

whereby

$$\Phi_{T,i}^{\lambda}\left(v_{\hat{T}}\right) = v_{\hat{T} \setminus \{2k-2i+2, 2k-2i+1\}}$$

for any $\hat{T} \in \text{Tab}\left(Z_T^{(i)}\right)$ and $Z_T^{(0)}$ is understood to be Z_T.

Lemma 3.15. *For any $\lambda \vdash k$, $T \in \text{Tab}(\lambda)$ and for any $i \in \{1, \ldots, k-1\}$, the map $\Phi_{T,i}^{\lambda}$ is a $\mathcal{D}^{(i)}$–module homomorphism.*

Proof. To prove this lemma, we need to show that $\Phi_{T,i}^{\lambda}\left(s_j v_{\hat{T}}\right) = s_j \Phi_{T,i}^{\lambda}\left(v_{\hat{T}}\right)$ for any $\hat{T} \in \text{Tab}\left(Z_T^{(i-1)}\right)$ and for any $j \in \{1, 3, \ldots, 2k-2i-1\}$. To this end, fix any such $\hat{T}$ and j.

There are three possibilities for the positions of the boxes labeled j and $j+1$ in $\hat{T}$. If they are in the same row/same column/neither the same row or column, then this is the exact same relationship between the boxes labeled j and $j+1$ in $\hat{T} \setminus \{2k-2i+2, 2k-2i+1\}$. In the first case we have

$$\Phi_{T,i}^{\lambda}\left(s_j v_{\hat{T}}\right) = \Phi_{T,i}^{\lambda}\left(v_{\hat{T}}\right) = s_j \Phi_{T,i}^{\lambda}\left(v_{\hat{T}}\right)$$

and, in the second case,

$$\Phi_{T,i}^{\lambda}\left(s_j v_{\hat{T}}\right) = \Phi_{T,i}^{\lambda}\left(-v_{\hat{T}}\right) = -\Phi_{T,i}^{\lambda}\left(v_{\hat{T}}\right) = s_j \Phi_{T,i}^{\lambda}\left(v_{\hat{T}}\right).$$

In the case where these two boxes are in neither the same row or column, the contents of the boxes labeled j and $j+1$ are always the same in both $\hat{T}$ and $\hat{T} \setminus \{2k-2i, 2k-2i-1\}$.

Also in this case,

$$\Phi^{\lambda}_{T,i}\left(v_{\hat{T}'_{\{j,j+1\}}}\right) = v_{\left(\hat{T}\setminus\{2k-2i+2,2k-2i+1\}\right)'_{\{j,j+1\}}}.$$

So then we have

$$\begin{aligned}
\Phi^{\lambda}_{T,i}\Big(s_j v_{\hat{T}}\Big) &= \Phi^{\lambda}_{T,i}\Big(\left(r^{-1}\right)v_{\hat{T}} + \left(\sqrt{1-r^{-2}}\right)v_{\hat{T}'_{\{j,j+1\}}}\Big)\\
&= \left(r^{-1}\right)\Phi^{\lambda}_{T,i}\left(v_{\hat{T}}\right) + \left(\sqrt{1-r^{-2}}\right)\Phi^{\lambda}_{T,i}\left(v_{\hat{T}'_{\{j,j+1\}}}\right)\\
&= \left(r^{-1}\right)v_{\hat{T}\setminus\{2k-2i+2,2k-2i+1\}} + \left(\sqrt{1-r^{-2}}\right)v_{\left(\hat{T}\setminus\{2k-2i+2,2k-2i+1\}\right)'_{\{j,j+1\}}}\\
&= s_j v_{\hat{T}\setminus\{2k-2i+2,2k-2i+1\}}\\
&= s_j\Phi^{\lambda}_{T,i}\left(v_{\hat{T}}\right).
\end{aligned}$$

□

The next lemma asserts that $\Phi^{\lambda}_{T,i}$ is injective on the -1 eigenspace of $s_{2k-2i+1}$ in $Z_T^{(i-1)}$, so that $\Phi^{\lambda}_{T,i}$ always defines a $\mathcal{D}^{(i)}$–module isomorphism between this eigenspace and its image.

Lemma 3.16. *For any $\lambda \vdash k$, $T \in \mathrm{Tab}(\lambda)$ and for any $i \in \{1,\ldots,k\}$, the map $\Phi^{\lambda}_{T,i}$ is injective on the -1 eigenspace of $s_{2k-2i+1}$ in $Z_T^{(i-1)}$.*

Proof. It is easy to see that

$$\ker\left(\Phi^{\lambda}_{T,i}\right) = \bigoplus_{\hat{T}}\left\langle v_{\hat{T}} - v_{\hat{T}'_{\{2k-2i+2,2k-2i+1\}}}\right\rangle,$$

where the direct sum is over all $\hat{T} \in \mathrm{Tab}\left(Z_T^{(i-1)}\right)$ for which $\hat{T}'_{\{2k-2i+2,2k-2i+1\}}$ is defined (i.e. still a valid Young tableaux). The -1 eigenspace of $s_{2k-2i+1}$ in $Z_T^{(i-1)}$ is exactly

$$\bigoplus_{\hat{T}_1}\left\langle v_{\hat{T}_1}\right\rangle \oplus \bigoplus_{\hat{T}}\left\langle v_{\hat{T}} - \sqrt{\frac{1+r^{-1}}{1-r^{-1}}}\,v_{\hat{T}'_{\{2k-2i+2,2k-2i+1\}}}\right\rangle,$$

where the first direct sum is over all $\hat{T}_1 \in \mathrm{Tab}\left(Z_T^{(i-1)}\right)$ for which $2k-2i+1$ and $2k-2i+2$ are in the same column and the second direct sum is over all $\hat{T} \in \mathrm{Tab}\left(Z_T^{(i-1)}\right)$ for which $2k-2i+1$ and $2k-2i+2$ are in neither the same row or the same column (the case where they are both in the same row is not possible, since this cannot be the case in $\Psi(T)$ and in $\hat{T}$, the boxes labeled $2k-2i+1$ and $2k-2i+2$ are either in the same boxes as in $\Psi(T)$ or they have swapped with one another).

Clearly, this does not intersect $\ker\left(\Phi^{\lambda}_{T,i}\right)$ other than at 0, so $\Phi^{\lambda}_{T,i}$ is injective when restricted to the -1 eigenspace of $s_{2k-2i+1}$ in $Z_T^{(i-1)}$. □

The final intermediate lemma determines the image of $\Phi^{\lambda}_{T,i}$ when restricted to this eigenspace.

Lemma 3.17. *The image of the -1 eigenspace of $s_{2k-2i+1}$ in $Z_T^{(i-1)}$ under $\Phi^{\lambda}_{T,i}$ is $Z_T^{(i)}$.*

Proof. We have

$$\begin{aligned}
&\Phi^{\lambda}_{T,i}\left(\bigoplus_{\hat{T}_1}\left\langle v_{\hat{T}_1}\right\rangle \oplus \bigoplus_{\hat{T}}\left\langle v_{\hat{T}} - \sqrt{\frac{1+r^{-1}}{1-r^{-1}}}v_{\hat{T}'_{\{2k-2i+2,2k-2i+1\}}}\right\rangle\right)\\
&= \left\langle v_{\hat{T}\setminus\{2k-2i+2,2k-2i+1\}} : \hat{T} \in \operatorname{Tab}\left(Z_T^{(i-1)}\right)\right\rangle\\
&= \left\langle \mathcal{D}^{(i)}\Psi\left(v_{T\setminus\{k,\ldots,k-i+1\}}\right)\right\rangle\\
&= Z_T^{(i)}.
\end{aligned}$$

□

Proof of Proposition 3.14. We can apply the previous three lemmas iteratively to determine $Z_T^{\mathcal{D},\text{sign}}$. Applying them for $i=1$ shows that the -1 eigenspace of s_{2k-1} in Z_T is isomorphic to $Z_T^{(1)}$. Repeating this for $i=2$ shows that the -1 eigenspace of s_{2k-3} in $Z_T^{(1)}$ is isomorphic$Z_T^{(2)}$. But the -1 eigenspace of s_{2k-3} in $Z_T^{(1)}$ is the sign–isotypic subspace of $\mathcal{D}_{\perp}^{(2)}$ in Z_T. Repeating for each $j=3,,\ldots,j=k-1$ shows that the sign–isotypic subspace of

$$\mathcal{D}_{\perp}^{(k-1)} = \langle s_3,\ldots,s_{2k-1}\rangle$$

in Z_T is isomorphic to

$$Z_T^{(k-1)} = \left\langle s_1\Psi\left(v_{T\setminus\{k,\ldots,2\}}\right)\right\rangle.$$

Thus, the sign–isotypic subspace of $\mathcal{D}$ in Z_T, $Z_T^{\mathcal{D},\text{sign}}$, is just the -1 eigenspace of s_1 in $Z_T^{(k-1)}$. But $Z_T^{(k-1)}$ is clearly the one–dimensional sign representation of $\langle s_1\rangle$, so the

-1 eigenspace of s_1 is just the whole space and we conclude that $Z_T^{\mathcal{D},\mathrm{sign}}$ has dimension one. $\square$

By Lemma 3.13,

$$\bigoplus_{T\in\mathrm{Tab}(\lambda)} Z_T^{\mathcal{D},\mathrm{sign}} = V_{\mathcal{D},\mathrm{sign}}^{\lambda^+(2k)},$$

so that Proposition 3.14 implies that $V_{\mathcal{D},\mathrm{sign}}^{\lambda^+(2k)}$ has dimension d_λ. This completes the proof of Proposition 3.8, which has the following corollary.

Corollary 3.18. *For any $\lambda \vdash k$, we have an isomorphism of representations of $S_k^\psi \times S_k \cong S_k \times S_k$,*

$$\mathcal{U}_\lambda \stackrel{\mathrm{def}}{=} \left\langle \Delta(g,\sigma)\left(\xi_\lambda^{\mathrm{norm}}\right):\ g \in S_k^\psi,\ \sigma \in S_k \right\rangle \cong V^\lambda \otimes V^\lambda.$$

Proof. This follows since

$$W^{\emptyset,\lambda}\downarrow_{S_k^\psi} \cong V^\lambda.$$

$\square$

3.3.4 Identifying the Projection

Since the irreducible characters of S_k are integer valued, we have an isomorphism of S_k representations, $V^\lambda \cong \left(V^\lambda\right)^\vee$ and subsequent isomorphisms

$$V^\lambda \otimes V^\lambda \cong \left(V^\lambda\right)^\vee \otimes V^\lambda \cong \mathrm{End}\left(V^\lambda\right). \tag{45}$$

Noting that

$$\psi(\sigma)\xi_\lambda^{\mathrm{norm}} = \xi_\lambda^{\mathrm{norm}}\sigma$$

then implies that

$$\xi_\lambda^{\mathrm{norm}} \in \mathrm{End}_{S_k}\left(V^\lambda\right) \subseteq \mathrm{End}\left(V^\lambda\right),$$

when $\xi_\lambda^{\mathrm{norm}}$ is interpreted as an element of $\mathrm{End}\left(V^\lambda\right)$ through the isomorphisms presented in Corollary 3.18 and (45). Then, by Schur's lemma, $\xi_\lambda^{\mathrm{norm}}$ must correspond to a complex scalar multiple of Id_{V^λ}, so that

$$\xi_\lambda^{\mathrm{norm}} = c_\lambda \sum_i w_i \otimes w_i \in \mathcal{U}_\lambda \subseteq \mathcal{U}_{\lambda^+(n)} \tag{46}$$

for some complex scalar constant c_λ and any orthonormal basis $\{w_i\}$ of V^λ. By Proposition 3.2,

$$\check{\xi}_\lambda^{\mathrm{norm}} \otimes \xi_\lambda^{\mathrm{norm}} \in \left(\mathcal{U}_{\lambda+(n)}\right)^\vee \otimes \mathcal{U}_{\lambda+(n)}.$$

There are natural isomorphisms

$$\left(\mathcal{U}_{\lambda+(n)}\right)^\vee \otimes \mathcal{U}_{\lambda+(n)} \cong \left(V^{\lambda+(n)}\right)^\vee \otimes \left(V^\lambda\right)^\vee \otimes V^{\lambda+(n)} \otimes V^\lambda \cong \left(V^{\lambda+(n)}\right)^\vee \otimes V^{\lambda+(n)} \otimes \left(V^\lambda\right)^\vee \otimes V^\lambda,$$

where the second isomorphism is given by permuting the tensor coordinates. Through these isomorphisms and (46), we write

$$\check{\xi}_\lambda^{\mathrm{norm}} \otimes \xi_\lambda^{\mathrm{norm}} = c_\lambda^2 \sum_{i,j} \check{w}_i \otimes w_j \otimes \check{w}_i \otimes w_j. \tag{47}$$

Let

$$\tilde{\mathcal{Q}}_{\lambda,n} \stackrel{\mathrm{def}}{=} \int_{g \in S_n} g\check{\xi}_\lambda^{\mathrm{norm}} \otimes g\xi_\lambda^{\mathrm{norm}} dg = \frac{1}{n!} \sum_{g \in S_n} g\check{\xi}_\lambda^{\mathrm{norm}} \otimes g\xi_\lambda^{\mathrm{norm}},$$

where $g\xi_\lambda^{\mathrm{norm}} = \rho(g)\left(\xi_\lambda^{\mathrm{norm}}\right)$ and $g\check{\xi}_\lambda^{\mathrm{norm}} = \rho^*(g)\left(\check{\xi}_\lambda^{\mathrm{norm}}\right) = \left(\rho(g)\left(\xi_\lambda^{\mathrm{norm}}\right)\right)^\vee$. Then we claim that

$$\tilde{\mathcal{Q}}_{\lambda,n} = \frac{1}{d_{\lambda+(n)} d_\lambda} \mathcal{Q}_{\lambda,n},$$

where $\mathcal{Q}_\lambda$ is the orthogonal projection map defined in Theorem 1.6. Indeed, using (47), we have

$$\tilde{\mathcal{Q}}_{\lambda,n} = \frac{c_\lambda^2}{n!} \sum_{g \in S_n} \sum_{i,j} g\check{w}_i \otimes gw_j \otimes \check{w}_i \otimes w_j.$$

Since $\tilde{\mathcal{Q}}_{\lambda,n}$ commutes with S_n, then by Schur's lemma this corresponds to an element of

$$\mathbb{C}\mathrm{Id}_{V^{\lambda+(n)}} \otimes \left(V^\lambda\right)^\vee \otimes V^\lambda,$$

forcing $i = j$ so that the image of $\mathcal{Q}_{\lambda,n}$ in $\mathrm{End}\left(\mathcal{U}_{\lambda+(n)}\right)$ is

$$\frac{b_\lambda}{n!}\left[\mathrm{Id}_{V^{\lambda+(n)}} \otimes \mathrm{Id}_{V^\lambda}\right],$$

which, when we extend by zero on $\mathcal{U}_{\lambda+(n)}^\perp$ in $(\mathbb{C}^n)^{\otimes k}$, is a scalar multiple of the orthogonal projection map $\mathcal{Q}_\lambda : (\mathbb{C}^n)^{\otimes k} \to \mathcal{U}_{\lambda+(n)}$. Comparing traces in $(\mathbb{C}^n)^{\otimes k}$ shows that

$d_{\lambda^+(n)} d_\lambda \tilde{\mathcal{Q}}_\lambda = \mathcal{Q}_\lambda$. This is summarised below.

Theorem 3.19. *For any $\lambda \vdash k$, the orthogonal projection*

$$\mathcal{Q}_\lambda : (\mathbb{C}^n)^{\otimes k} \to \mathcal{U}_{\lambda^+(n)}$$

is equal to $d_{\lambda^+(n)} d_\lambda \tilde{\mathcal{Q}}_\lambda$, where

$$\tilde{\mathcal{Q}}_\lambda = \int_{S_n} g\check{\xi}_\lambda^{\text{norm}} \otimes g\xi_\lambda^{\text{norm}} dg.$$

3.4 Evaluating the Projection

It remains only to compute

$$\tilde{\mathcal{Q}}_\lambda = \int_{S_n} g\check{\xi}_\lambda^{\text{norm}} \otimes g\xi_\lambda^{\text{norm}} dg$$

using the Weingarten calculus for the symmetric group (§2.1.6) and evaluate this as an endomorphism of $(\mathbb{C}^n)^{\otimes k}$. Let $z = \xi_\lambda^{\text{norm}} \otimes \xi_\lambda^{\text{norm}}$ and denote, for each $g \in S_n$, $gz = \rho(g)\xi_\lambda^{\text{norm}} \otimes \rho(g)\xi_\lambda^{\text{norm}}$, so that $\tilde{\mathcal{Q}}_\lambda$ and

$$\int_{S_n} gz dg$$

define the same element of $\text{End}\left((\mathbb{C}^n)^{\otimes k}\right)$ via the canonical isomorphisms. We begin by computing, for fixed $\sigma, \sigma' \in S_k$ and $I = (i_1, \ldots, i_{2k})$, the coefficient of e_I in

$$\int_{S_n} g\left[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right] dg. \tag{48}$$

Using Proposition 2.9, the coefficient of e_I in (48) is

$$\sum_{\pi \in \text{Part}([2k])} \pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) \pi^{\text{strict}}(e_I) \frac{1}{(n)_{|\pi|}} \tag{49}$$

and we evaluate these coefficients in the following sections.

3.4.1 Conditions for π^{strict} to be non–zero

Obviously, if $i \neq j$, we cannot have $v_{\sigma(i)} = v_{\sigma(j)}$ or $v_{\sigma'(i)} = v_{\sigma'(j)}$. So, if $\pi \in \text{Part}([2k])$ has any two elements in $[k]$ in the same subset, or any two elements in $\{k+1, \ldots, 2k\}$ in the same subset, then

$$\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) = 0.$$

This leaves only $\pi \in \text{Part}([2k])$ in which every element $i \in [k]$ is in a subset with *exactly one* element $j \in \{k+1, \ldots, 2k\}$, or is a singleton. Similarly, the elements $j \in [k+1, 2k]$ may only be in a subset with exactly one element $i \in [k]$ or they are singletons. That is, $\pi \leq \iota(\tau)$ for some $\tau \in S_k$. The only possible candidate for τ is detailed in the next lemma.

Lemma 3.20. *Given $\sigma, \sigma' \in S_k$ and $\pi \in \text{Part}([2k])$, let $\tau = \sigma^{-1}\sigma' \in S_k$. Then*

$$\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) = 0$$

unless

1. *$\pi = \iota(\tau)$, or*

2. *$\pi \leq \iota(\tau)$.*

Proof. Fix any $\sigma, \sigma' \in S_k$. Recall that $\iota(\tau)$ is the partition $\Big\{\{1, k+\tau^{-1}(1)\}, \ldots, \{k, k+\tau^{-1}(k)\}\Big\}$. Suppose, towards a contradiction, that $\pi \neq \iota(\tau)$ and $\pi \not\leq \iota(\tau)$, but that

$$\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) \neq 0.$$

Then, by definition, there exists a subset S in π that is not contained inside any of the subsets in $\iota(\tau)$. Suppose S contains $i \in [k]$ and $k+j$, where $j \in [k] \backslash \{\tau^{-1}(i)\}$ as well (by the previous discussion, if S contains another element in $[k]$, then

$$\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) = 0$$

and, if S contains only the element i or the element $k+\tau^{-1}(i)$, then S would be contained inside one of the subsets in $\iota(\tau)$). Then, for any $L = (l_1, \ldots, l_{2k})$,

$$\pi^{\text{strict}}(e_L) \neq 0 \implies l_i = l_{k+j}.$$

Observe that

$$\begin{aligned}
& v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)} \\
&= v_{\sigma(1)} \otimes \cdots \otimes e_{2\sigma(i)-1} - e_{2\sigma(i)} \otimes \cdots \otimes v_{\sigma(k)} \\
&\quad \otimes v_{\sigma'(1)} \otimes \cdots \otimes e_{2\sigma'(j)-1} - e_{2\sigma'(j)} \otimes \cdots \otimes v_{\sigma'(k)} \\
&= v_{\sigma(1)} \otimes \cdots \otimes e_{2\sigma(i)-1} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes e_{2\sigma'(j)-1} \otimes \cdots \otimes v_{\sigma'(k)} \\
&+ v_{\sigma(1)} \otimes \cdots \otimes e_{2\sigma(i)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes e_{2\sigma'(j)} \otimes \cdots \otimes v_{\sigma'(k)} \\
&- v_{\sigma(1)} \otimes \cdots \otimes e_{2\sigma(i)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes e_{2\sigma'(j)-1} \otimes \cdots \otimes v_{\sigma'(k)} \\
&- v_{\sigma(1)} \otimes \cdots \otimes e_{2\sigma(i)-1} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes e_{\sigma'(j)} \otimes \cdots \otimes v_{\sigma'(k)}.
\end{aligned}$$

So, if $\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) \neq 0$, then either $\sigma(i) = \sigma'(j)$, which implies $(\sigma')^{-1}\sigma(i) = \tau^{-1}(i) = j$, a contradiction **or** $2\sigma(i) = 2\sigma'(j)-1$ *or* $2\sigma(i)-1 = 2\sigma'(j)$, which is obviously not possible. So, in any case, we have a contradiction. □

Refinements of $\iota(\tau)$ are obtained by either splitting each subset $\{i, k+\tau^{-1}(i)\}$ into two subsets $\{i\}$ and $\{k+\tau^{-1}(i)\}$, or leaving them as they are, which corresponds to 'deleting' edges from the diagram, and $\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right)$ can be evaluated in terms of the number of edges deleted.

Remark. If π is obtained from $\iota(\tau)$ by deleting p edges, then $|\pi| = |\iota(\tau)| + p = k + p$.

3.4.2 Evaluating π^{strict}

Lemma 3.21. *Let $\sigma, \sigma' \in S_k$ and let $\tau = \sigma^{-1}\sigma'$. Then, if $\pi = \iota(\tau)$ or $\pi \leq \iota(\tau)$,*

$$\pi^{\text{strict}}\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) = (-1)^{|\pi|-k} 2^k.$$

Proof. We will prove this by induction on the number of edges deleted from $\iota(\tau)$. The base case is where $\pi = \iota(\tau)$. In this case, $|\pi| = k$. We can write $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}$ as a sum of 2^{2k} standard basis vectors $e_I \otimes e_J$, with coefficients either 1 or

-1. In this case, $\pi^{\text{strict}}(e_I \otimes e_J) = 1$ if and only if, for each $i \in [k]$, the i^{th} coordinate in I matches the $\tau^{-1}(i)$ coordinate in J exactly. If this is the case, the coefficient of $e_I \otimes e_J$ is 1 and there are 2^k possible pairings, since for each of the 2^k possible I, there is exactly one J such that $\pi^{\text{strict}}(e_I \otimes e_J) = 1$.

Suppose that the claim is true for all $\pi \leq \iota(\tau)$ of size $\leq p$ and let $\pi \leq \iota(\tau)$ have $|\pi| = p+1$. Add an edge e from a singleton j to $k + \tau^{-1}(j)$ to obtain π', so that $\pi' \leq \iota(\tau)$ and $|\pi'| = p$, implying that

$$(\pi')^{\text{strict}} \left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) = (-1)^{p-k} 2^k.$$

For every (I, J) such that $(\pi')^{\text{strict}} (e_I \otimes e_J) = 1$, there is exactly one (I, J^-) such that $\pi^{\text{strict}} (e_I \otimes e_{J^-}) = 1$, obtained by swapping the $(\tau^{-1}(j))^{\text{th}}$ coordinate in J^- so that it no longer matches the j^{th} coordinate in I. The coefficient of $e_I \otimes e_{J^-}$ in $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}$ is -1 multiplied by the coefficient of $e_I \otimes e_J$, so that

$$\pi^{\text{strict}} \left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) = -(-1)^{p-k} 2^k.$$

□

3.4.3 Evaluating the projection

Lemmas 3.20 and 3.21 and (49) imply that, for each fixed $\sigma, \sigma' \in S_k$,

$$\begin{aligned}&\int_{S_n} g\left(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right) dg \\ =&\sum_I \left[\sum_{\pi \leq \iota(\tau)} \pi^{\text{strict}} (e_I) \frac{1}{(n)_{|\pi|}} (-1)^{|\pi|-k} 2^k\right] e_I,\end{aligned}$$

where the sum is over all multi-indices I that correspond to a basis vector of $(\mathbb{C}^n)^{\otimes 2k}$, and $\tau = \sigma^{-1}\sigma'$. Since

$$z = \frac{1}{2^k k!} \sum_{\sigma, \sigma' \in S_k} \chi^\lambda(\sigma)\chi^\lambda(\sigma') \left[v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{\sigma'(1)} \otimes \cdots \otimes v_{\sigma'(k)}\right],$$

we have

$$\int_{S_n} gzdg = \frac{(-1)^{-k}}{k!} \sum_{\sigma,\sigma' \in S_k} \chi^\lambda(\sigma)\chi^\lambda(\sigma') \sum_{\pi \leq \iota(\sigma^{-1}\sigma')} (-1)^{|\pi|} \frac{1}{(n)_{|\pi|}} \sum_I \pi^{\text{strict}}(e_I)\, e_I. \tag{50}$$

Proof of Theorem 1.6. We take

$$\mathcal{U}_{\lambda^+(n)} = \left\langle \Delta(g,\sigma)(\xi_\lambda) : \; g \in S_n, \; \sigma \in S_k \right\rangle_{\mathbb{C}} \cong V^{\lambda^+(n)} \otimes V^\lambda.$$

By Theorem 3.19, the orthogonal projection $\mathcal{Q}_{\lambda,n}$ is the image of $d_\lambda d_{\lambda^+(n)} \tilde{\mathcal{Q}}_{\lambda,n}$ inside $\text{End}\left((\mathbb{C}^n)^{\otimes k}\right)$. Combining (50) with the character orthogonality formula

$$\frac{1}{k!} \sum_{\sigma \in S_k} \chi^\lambda(\sigma)\chi^\lambda(\sigma\tau) = \frac{\chi^\lambda(\tau)}{d_\lambda}$$

and Remark 2.10, one sees that this is exactly (6). □

For use in the next section, we make some observations about this formula for $\mathcal{Q}_{\lambda.n}$. We will denote

$$\text{Part}_{\leq S_k}([2k]) \stackrel{\text{def}}{=} \left\{\pi \in \text{Part}([2k]) : \; \pi \leq \iota(\tau), \; \tau \in S_k\right\}$$

and write

$$\sum_{\pi}^{\leq S_k}$$

to indicate that we are summing over $\pi \in \text{Part}_{\leq S_k}([2k])$. Then we can rewrite (6) as

$$\sum_{\pi}^{\leq S_k} c(n,k,\lambda\pi) P_\pi^{\text{strict}}, \tag{51}$$

where

$$c(n,k,\lambda,\pi) = \frac{d_{\lambda^+(n)}(-1)^{|\pi|+k}}{(n)_{|\pi|}} \sum_{\substack{\tau \in S_k \\ \iota(\tau) \geq \pi}} \chi^\lambda(\tau).$$

If $\pi \in \text{Part}_{\leq S_k}([2k])$, say $\pi \leq \iota(\tau)$, then π is obtained from $\iota(\tau)$ by splitting a number of subsets into singletons or, equivalently, by deleting edges from the diagram for $\iota(\tau)$. We denote the number of edges deleted by $\text{del}(\pi)$ and note that $\text{del}(\pi) = |\pi| - k$, so that if

$\pi \leq \iota(\tau_1)$ and $\pi \leq \iota(\tau_2)$, then $\mathrm{del}(\pi)$ is independent of whether we count the edges deleted from $\iota(\tau_1)$ or from $\iota(\tau_2)$.

Lemma 3.22. *If $\pi \leq \iota(\tau)$ for some $\tau \in S_k$, then*

$$c(n,k,\lambda,\pi) = O\left(\frac{1}{n^{\mathrm{del}(\pi)}}\right).$$

In particular, if $\pi = \iota(\tau)$, then this is $O(1)$.

Proof. This follows since $d_{\lambda^+(n)} = O\left(n^k\right)$ and $(n)_{|\pi|} = O\left(n^{|\pi|}\right)$. ∎

4 Expected characters of w–random permutations

The goal of this section is to prove Theorem 1.12 and the material is almost exactly the same as [Cas25b, Section 4].

4.1 Overview of proof

Rather than working directly with the character $\chi^{\lambda^+(n)}$, we use Corollary 1.7 so that instead our task is to compute the bitrace of $\mathcal{Q}_{\lambda,n} \circ w\left(g_1,\ldots,g_r\right)$ on $\left(\mathbb{C}^n\right)^{\otimes k}$.

Recall that $\mathcal{Q}_{\lambda,n}^2 = \mathcal{Q}_{\lambda,n}$ and that, since it is a linear combination of P_π^{strict}, the action of $\mathcal{Q}_{\lambda,n}$ commutes with the action of any $g \in S_n$ on $\left(\mathbb{C}^n\right)^{\otimes k}$. With this in mind, if $w = x_1x_2x_1^{-1}x_2^{-1} \in F_2$ for example, what we are actually interested in is

$$\mathrm{btr}_{(\mathbb{C}^n)^{\otimes k}}\left(g_1 \circ \mathcal{Q}_{\lambda,n} \circ g_2 \circ \mathcal{Q}_{\lambda,n} \circ g_1^{-1} \circ \mathcal{Q}_{\lambda,n} \circ g_2^{-1} \circ \mathcal{Q}_{\lambda,n}\right),$$

where each $g_i \in S_n$.

We compute the expected trace by using the Weingarten calculus for S_n. A key component is our refinement of the usual Weingarten calculus for S_n, which uses the fact that we are operating within $A_k(n)$ to show that the contribution to the trace from all but a specific family of partitions is zero.

We obtain a combinatorial formula for the trace in §4.3. In §4.4, we construct graphs from the combinatorial data formula for the expected character. From these graphs we construct new graphs in which the asymptotic bound for our trace formula is encoded by the Euler characteristic.

In §4.5, we prove a variant of a theorem of Louder and Wilton that relates the Euler characteristic of a graph with the number of immersed w–cycles (see [LW17, Theorem 1.2]) to obtain our final bound for the expected character.

4.2 Expected character as a ratio of polynomials in $\frac{1}{n}$

In addition to our main theorem on word maps, we give another formulation for the expected stable irreducible character of a w–random permutation as the ratio of two polynomials in $\frac{1}{n}$. This formulation is required for the methods in §5.

For any $L, k \in \mathbb{Z}_{>0}$, we define

$$g_{L,k}(x) \stackrel{\text{def}}{=} \prod_{c=1}^{kL}(1-cx)^L \left[\prod_{j=1}^{2k}(1-(j-1)x)\right]^L .$$

Proposition 4.1. *Let $w \in F_r = \langle x_1, \ldots, x_r \rangle$ be a word in the free group with r generators, $w \neq e$. Let $k \in \mathbb{Z}_{>0}$ and $\lambda \vdash k$. If $l(w) \leq q$, there is a polynomial $P_{w,\lambda,q} \in \mathbb{Q}[x]$ such that, for $n \geq l(w)k$,*

$$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = \frac{P_{w,\lambda,q}\left(\frac{1}{n}\right)}{g_{q,k}\left(\frac{1}{n}\right)},$$

with $\deg\left(P_{w,\lambda,q}\right) \leq 3kq + kq^2$.

The proof of Proposition 4.1 is given in §4.6 and it can be followed from (57).

4.3 Combinatorial integration

Fix a word $w \in F_r = \langle x_1, \ldots, x_r \rangle$. We will assume that w is not the identity and is not primitive and that w is cyclically reduced. We will also assume that every x_i appears at least twice in w. If x_j did not appear in w, then we could consider $w \in F_{r-1} = \langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r \rangle$ and then proceed. If any x_j appeared only once, then w would be primitive and so $\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = 0$.

We will write

$$w = f_1^{\epsilon_1} f_2^{\epsilon_2} \ldots f_{l(w)}^{\epsilon(w)},$$

where each $f_i \in \{x_1, \ldots, x_r\}$ and each $\epsilon_i \in \{1, -1\}$. We will write $|w|_{x_i}$ for the number of j such that $f_j = x_i$.

Suppose that $\{v_p\}$ is an orthonormal basis for $\mathcal{U}_{\lambda^+(n)} \cong V^{\lambda^+(n)} \otimes V^{\lambda}$. Then, for any $(g_{x_1}, \ldots, g_{x_r}) \in S_n^r$, we have

$$\chi^{\lambda^+(n)}\left(w\left(g_{x_1}, \ldots, g_{x_r}\right)\right) = \frac{1}{d_\lambda} \mathrm{tr}_{\mathcal{U}_{\lambda^+(n)}}\left(w\left(g_{x_1}, \ldots, g_{x_r}\right)\right),$$

so that

$$\begin{aligned} &\chi^{\lambda^+(n)}\left(w\left(g_{x_1}, \ldots, g_{x_r}\right)\right) \\ =&\frac{1}{d_\lambda} \sum_{p_i} \left\langle g_{f_1}^{\epsilon_1} v_{p_2},\ v_{p_1} \right\rangle \left\langle g_{f_2}^{\epsilon_2} v_{p_3},\ v_{p_2} \right\rangle \cdots \left\langle g_{f_{l(w)}}^{\epsilon_{l(w)}} v_{p_1},\ v_{p_{l(w)}} \right\rangle. \end{aligned} \tag{52}$$

For each p, write v_p in the standard orthonormal basis of $(\mathbb{C}^n)^{\otimes k}$:

$$v_p = \sum_I \beta_{p,I} e_I.$$

Recall that $\mathcal{U}_{\lambda^+(n)} \subset D_k(n)$, so that we may assume that the above sum is over all $I = (i_1, \ldots, i_k)$ *with all indices distinct.* This is an important observation.

We introduce some new notation to avoid the cumbersome general expression for $\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right]$. For each $f \in \{x_1, \ldots, x_r\}$, we will write

$$\sum_{I^f}$$

in place of

$$\sum_{I_f^1, \ldots I_f^{|w|_f}}.$$

For any pair I, J of multi–indices and $\epsilon \in \{1, -1\}$, we define

$$(I/J)(\epsilon) = \begin{cases} J & \text{if } \epsilon = 1 \\ I & \text{if } \epsilon = -1. \end{cases}$$

In our expression, we will simply write (I/J) in place of $(I/J)(\epsilon)$ where it is clear which epsilon we are considering. To be even more clear, in the sum below, for each f_i, if $\epsilon_i = 1$, then the corresponding inner product has $e_{I_{f_i}^z}$ on the LHS, where $z \in \{1, \ldots, |w|_{f_i}\}$ denotes the number of times f_i has appeared in the subword $f_1^{\epsilon_1} \ldots f_i^{\epsilon_i}$. The corresponding pair of β–terms has $\beta_{p,I}$, with $\beta_{p,J}$ conjugated. If $\epsilon_i = -1$, we swap the positions of I and J.[9]

We will write

$$\prod_{f,w} \beta = \left(\beta_{p_2,\left(I^1_{f_1}/J^1_{f_1}\right)}\right)\left(\bar{\beta}_{p_1,\left(J^1_{f_1}/I^1_{f_1}\right)}\right)\cdots \cdots \left(\beta_{p_1,\left(I^{|w|_{f_{l(w)}}}_{f_{l(w)}}/J^{|w|_{f_{l(w)}}}_{f_{l(w)}}\right)}\right)\left(\bar{\beta}_{p_{l(w)},\left(J^{|w|_{f_{l(w)}}}_{f_{l(w)}}/I^{|w|_{f_{l(w)}}}_{f_{l(w)}}\right)}\right).$$

With this notation, (52) is equal to

$$\frac{1}{d_\lambda}\sum_{p_i}\sum_{I^f,J^f}\left(\prod_{f,w}\beta\right)\left\langle g_{f_1}^{\epsilon_1} e_{\left(I^1_{f_1}/J^1_{f_1}\right)},\ e_{\left(J^1_{f_1}/I^1_{f_1}\right)}\right\rangle \cdots \cdots \left\langle g_{f_{l(w)}}^{\epsilon_{l(w)}} e_{\left(I^{|w|_{f_{l(w)}}}_{f_{l(w)}}/J^{|w|_{f_{l(w)}}}_{f_{l(w)}}\right)}, e_{\left(J^{|w|_{f_{l(w)}}}_{f_{l(w)}}/I^{|w|_{f_{l(w)}}}_{f_{l(w)}}\right)}\right\rangle. \tag{53}$$

See the example below. *Without losing generality, we will always assume that $f_1 = x_1$ and that $\epsilon_1 = 1$.*

Example 4.2. Suppose $w = x_1x_2x_1^{-1}x_2^{-1}$. Then $\chi^{\lambda^+(n)}\left(w\left(g_{x_1}, g_{x_2}\right)\right)$ is equal to

$$\frac{1}{d_\lambda}\sum_{p_i}\sum_{\substack{I^1_{x_1}, I^2_{x_1}, I^1_{x_2}, I^2_{x_2},\\ J^1_{x_1}, J^2_{x_1}, J^1_{x_2}, J^2_{x_2}}}\left(\beta_{p_2,I^1_{x_1}}\right)\left(\bar{\beta}_{p_1,J^1_{x_1}}\right)\left(\beta_{p_3,I^1_{x_2}}\right)\left(\bar{\beta}_{p_2,J^1_{x_2}}\right)\left(\beta_{p_4,J^2_{x_1}}\right)\left(\bar{\beta}_{p_3,I^2_{x_1}}\right)$$
$$\left(\beta_{p_1,J^2_{x_2}}\right)\left(\bar{\beta}_{p_4,I^2_{x_2}}\right)\left\langle g_{x_1}e_{J^1_{x_1}}, e_{I^1_{x_1}}\right\rangle\left\langle g_{x_2}e_{J^1_{x_2}}, e_{I^1_{x_2}}\right\rangle\left\langle g_{x_1}^{-1}e_{I^2_{x_1}}, e_{J^2_{x_1}}\right\rangle\left\langle g_{x_2}^{-1}e_{I^2_{x_2}}, e_{J^2_{x_2}}\right\rangle.$$

We rewrite the inner product terms as products of matrix coefficients – for example,

$$\left\langle g_{x_1}e_{J^1_{x_1}}, e_{I^1_{x_1}}\right\rangle = \left(g_{x_1}\right)_{\left(I^1_{x_1}\right)_1\left(J^1_{x_1}\right)_1}\cdots\cdot\left(g_{x_1}\right)_{\left(I^1_{x_1}\right)_k\left(J^1_{x_1}\right)_k}.$$

Rearranging and taking the expectation over $g_{x_1}, \ldots, g_{x_r} \in S_n$, we obtain from (53) that

[9] We use this notation so that, in the graph construction detailed in Section 4.4, $\left(J^i_f\right)_j$ always represents the *initial* vertex of an f–edge and $\left(I^i_f\right)_j$ always represents the *terminal* vertex.

$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right]$ is equal to

$$\frac{1}{d_\lambda}\sum_{p_i}\sum_{I^f,J^f}\left(\prod_{f,w}\beta\right)\prod_{f\in\{x_1,\dots,x_r\}}\int_{g_{I^f,J^f}}, \tag{54}$$

where

$$\int_{g_{I^f,J^f}} \stackrel{\text{def}}{=} \int_{S_n}\prod_{i=1}^{|w|_f}(g_f)_{(I_f^i)_1(J_f^i)_1}\cdots\cdot(g_f)_{(I_f^i)_k(J_f^i)_k}\,dg_f.$$

For each $f\in\{x_1,\dots,x_r\}$ and for each fixed collection of multi–indices $I_f^1,\dots,I_f^{|w|_f},J_f^1,\dots,J_f^{|w|_f}$, this integral can be computed using the Weingarten calculus for the symmetric group. Using (26) we have

$$\begin{aligned}&\int_{S_n}\prod_{i=1}^{|w|_f}(g_f)_{(I_f^i)_1(J_f^i)_1}\cdots\cdot(g_f)_{(I_f^i)_k(J_f^i)_k}\,dg_f\\ =&\sum_{\sigma_f,\tau_f\in\mathrm{Part}\left(\left[|w|_f k\right]\right)}\delta_{\sigma_f}\left(J_f^1\sqcup J_f^2\sqcup\cdots\sqcup J_f^{|w|_f}\right)\delta_{\tau_f}\left(I_f^1\sqcup I_f^2\sqcup\cdots\sqcup I_f^{|w|_f}\right)\\ &\mathrm{Wg}_{n,\left(|w|_f k\right)}(\sigma_f,\tau_f).\end{aligned}$$

Now we give an improvement over the usual Weingarten calculus that simplifies the above equation greatly. The benefit of the improvement is that, instead of summing over *all* set partitions $\sigma_f,\tau_f\in\mathrm{Part}\left([|w|_f k]\right)$, we show that we need only sum over set partitions σ_f,τ_f that have a specific structure, since the contribution to (54) from the set partitions without this structure is zero.

To each partition we associate a diagram (in a similar way to §2.1.5) – the diagram has $|w|_f$ rows, each containing exactly k vertices. The vertices in row R are labeled from $(R-1)k+1$ to Rk, and two vertices are connected if and only if the corresponding vertex labels are in the same block of of the partition. We will also refer to the diagrams corresponding to σ_f and τ_f as σ_f and τ_f where this does not cause confusion.

Lemma 4.3. *For each $f\in\{x_1,\dots,x_r\}$, if σ_f has any two vertices from the same row connected, then*

$$\delta_{\sigma_f}\left(J_f^1\sqcup J_f^2\sqcup\cdots\sqcup J_f^{|w|_f}\right)=0.$$

The same is true for τ_f and $I_f^1\sqcup I_f^2\sqcup\cdots\sqcup I_f^{|w|_f}$.

Proof. Since $e_{J_f^1} \in D_k(n) = \langle e_{i_1} \otimes \cdots \otimes e_{i_k} : i_1, \ldots, i_k \text{ pairwise distinct}\rangle$, if any of the vertices in the top row of σ_f are in the same block, then

$$\delta_{\sigma_f}\left(J_f^1 \sqcup J_f^2 \sqcup \cdots \sqcup J_f^{|w|_f}\right) = 0.$$

Repeating this argument for $J_f^2, \ldots J_f^{|w|_f}$ proves the claim for σ_f and then repeating it for $I_f^1 \sqcup I_f^2 \sqcup \cdots \sqcup I_f^{|w|_f}$ proves the claim for τ_f. $\square$

The next lemma asserts that every vertex of σ_f and τ_f must be connected to at least one other vertex.

Lemma 4.4. *For any $f \in \{x_1, \ldots, x_r\}$, if σ_f has any singletons, then*

$$\frac{1}{d_\lambda} \sum_{p_i} \sum_{I^f, J^f} \left(\prod_{f,w} \beta\right) \left[\sum_{\tau_f \in \mathrm{Part}\left(\left[|w|_f k\right]\right)} \delta_{\sigma_f}\left(J_f^1 \sqcup J_f^2 \sqcup \cdots \sqcup J_f^{|w|_f}\right) \delta_{\tau_f}\left(I_f^1 \sqcup I_f^2 \sqcup \cdots \sqcup I_f^{|w|_f}\right) \mathrm{Wg}_{n,\left(|w|_f k\right)}\left(\sigma_f, \tau_f\right)\right] = 0.$$

The same is true when we swap σ_f and τ_f.

Proof. Without loss of generality, suppose that the vertex q in the first row of vertices of σ_f is a singleton. For all variables in the sum fixed except for $J_f^1, J_f^2, \ldots, J_f^{|w|_f}$, assuming

$$\delta_{\tau_f}\left(I_f^1 \sqcup I_f^2 \sqcup \cdots \sqcup I_f^{|w|_f}\right) \neq 0,$$

then the sum is equal to

$$\sum_{J_f^1, \ldots, J_f^{|w|_f}} \left(\bar{\beta}_{p_1, J_f^1}\right) \cdots \left(\beta_{p, J_f^{|w|_f}}\right) \delta_{\sigma_f}\left(J_f^1 \sqcup J_f^2 \sqcup \cdots \sqcup J_f^{|w|_f}\right),$$

multiplied by some constant coming from the other (fixed) β–terms and the Weingarten function $\mathrm{Wg}_{n,\left(|w|_f k\right)}\left(\sigma_f, \tau_f\right)$.

Then, for every fixed index $\left(J_f^1\right)_1, \ldots, \left(J_f^1\right)_{q-1}, \left(J_f^1\right)_{q+1}, \ldots, \left(J_f^1\right)_k$ and every fixed multi–index $J_f^2, \ldots, J_f^{|w|_f}$ satisfying σ_f, this is (some constant multiplied by)

$$\sum_{\left(J_f^1\right)_q=1}^{n} \bar{\beta}_{p_1, J_f^1}.$$

This is exactly the conjugate of the coefficient of $e_{(J_f^1)_1} \otimes \cdots \otimes e_{(J_f^1)_{q-1}} \otimes e_{(J_f^1)_{q+1}} \otimes \ldots e_{(J_f^1)_k}$ in $T_q\left(v_{p_1}\right)$. But then, v_{p_1} belongs to an orthonormal basis for $\mathcal{U}_{\lambda^+(n)}$, and $\mathcal{U}_{\lambda^+(n)} \subset A_k(n)$. In particular, $v_p \in \ker\left(T_q\right)$, so that

$$\sum_{\left(J_f^1\right)_q=1}^{n} \bar{\beta}_{p_1, J_f^1} = 0.$$

$\square$

Henceforth, we will denote by $\overset{\star}{\text{Part}}\left([|w|_f k]\right)$ *the set of set partitions of* $[|w|_f k]$ *for which the corresponding diagram has no singletons and has no two vertices in the same row in the same connected component.*

We will write

$$\sum_{\sigma_f, \tau_f}^{\star}$$

to indicate that the sum is over $\sigma_f, \tau_f \in \overset{\star}{\text{Part}}\left([|w|_f k]\right)$.

For any given $\sigma_f \in \overset{\star}{\text{Part}}\left([|w|_f k]\right)$ (and similarly for τ_f), and any collection of multi–indices $J_f^1, \ldots, J_f^{|w|_f}$, we will write

$$J_f^{\Sigma} \leftrightarrow \sigma_f$$

to indicate that

$$\delta_{\sigma_f}\left(J_f^1 \sqcup J_f^2 \sqcup \cdots \sqcup J_f^{|w|_f}\right) = 1.$$

Therefore, our new expression for $\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right]$ is

$$\frac{1}{d_\lambda} \sum_{\sigma_f, \tau_f}^{\star} \left(\prod_{f \in \{x_1, \ldots, x_r\}} \mathrm{Wg}_{n, \left(|w|_f k\right)}\left(\sigma_f, \tau_f\right) \right) \sum_{p_i} \sum_{\substack{J_f^{\Sigma} \leftrightarrow \sigma_f \\ I_f^{\Sigma} \leftrightarrow \tau_f}} \left(\prod_{f,w} \beta \right). \tag{55}$$

This expression can be further simplified using the projection $\mathcal{Q}_{\lambda,n}$. Consider Example 4.2 – for all terms fixed in the expression except for e.g. p_2, the ensuing sum is

$$\begin{aligned}
&\sum_{p_2} \left(\beta_{p_2, I^1_{x_1}}\right) \left(\bar{\beta}_{p_2, J^1_{x_2}}\right) \\
&= \sum_{p_2} \left\langle e_{I^1_{x_1}}, v_{p_2} \right\rangle \left\langle v_{p_2}, e_{J^1_{x_2}} \right\rangle \\
&= \left\langle \mathcal{Q}_{\lambda,n} \left(e_{I^1_{x_1}}\right), e_{J^1_{x_2}} \right\rangle \\
&= \sum_{\pi}^{\leq S_k} c(n,k,\lambda,\pi) \left\langle P^{\text{strict}}_{\pi} \left(e_{I^1_{x_1}}\right), e_{J^1_{x_2}} \right\rangle .
\end{aligned}$$

Repeating this and computing the sums over each p_i, the expression in Example 4.2 becomes

$$\begin{aligned}
&\frac{1}{d_\lambda} \sum_{\sigma_f, \tau_f}^{\star} \sum_{\pi_1, \ldots, \pi_{l(w)}}^{\leq S_k} \left(\prod_{i=1}^{4} c(n,k,\lambda,\pi_i) \right) \left(\prod_{f \in \{x_1, x_2\}} \mathrm{Wg}_{n, (|w|_f k)} \left(\sigma_f, \tau_f\right) \right) \\
&\sum_{\sigma_f \leftrightarrow J^{\Sigma}_f,\ \tau_f \leftrightarrow I^{\Sigma}_f} \left\langle P^{\text{strict}}_{\pi_1} \left(e_{I^1_{x_1}}\right), e_{J^1_{x_2}} \right\rangle \left\langle P^{\text{strict}}_{\pi_2} \left(e_{I^1_{x_2}}\right), e_{I^2_{x_1}} \right\rangle \\
&\left\langle P^{\text{strict}}_{\pi_3} \left(e_{J^2_{x_1}}\right), e_{I^2_{x_2}} \right\rangle \left\langle P^{\text{strict}}_{\pi_4} \left(e_{J^2_{x_2}}\right), e_{J^1_{x_1}} \right\rangle .
\end{aligned} \tag{56}$$

The same argument applies for any non–identity, non–primitive, cyclically reduced word $w \in F_r$.

Definition 4.5. Given, for each $f \in \{x_1, \ldots, x_r\}$, a collection of set partitions $\sigma_f, \tau_f \in \overset{\star}{\mathrm{Part}} \left([|w|_f k]\right)$ and a collection of set partitions $\pi_1, \ldots, \pi_{l(w)} \in \mathrm{Part}_{\leq S_k} \left([2k]\right)$, we write

$$\mathcal{N} \left(\sigma_{x_1}, \tau_{x_1}, \ldots, \sigma_{x_r}, \tau_{x_r}, \pi_1, \ldots, \pi_{l(w)}\right) = \mathcal{N} \left(\sigma_f, \tau_f, \pi_i\right),$$

for the number of multi–indices $I^1_f, \ldots I^{|w|_f}_f, J^1_f, \ldots, J^{|w|_f}_f$, with all indices distinct, satisfying:

- $\sigma_f \leftrightarrow J^{\Sigma}_f$,
- $\tau_f \leftrightarrow I^{\Sigma}_f$,
- $\left\langle P^{\text{strict}}_{\pi_1} \left(e_{I^1_{x_1}}\right),\ e_{(I/J)_{f_2}} \right\rangle = 1$,

$\vdots$

- $\left\langle P^{\text{strict}}_{\pi_{l(w)}}\left(e_{(I/J)_{f_{l(w)}}}\right),\ e_{J^1_{x_1}}\right\rangle = 1,$

where (I/J) is used to denote the correct multi–index arising from the β–terms.

With this notation, we have proved the following theorem.

Theorem 4.6. *Suppose $k \in \mathbb{Z}_{>0}$. For any $\lambda \vdash k$ and every cyclically reduced, non–identity, non–primitive word $w \in F_r$,*

$$\begin{aligned}&\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right]\\ =&\frac{1}{d_\lambda}\sum_{\sigma_f,\tau_f}^{\star}\sum_{\pi_1,\ldots,\pi_{l(w)}}^{\leq S_k}\left(\prod_{i=1}^{l(w)}c(n,k,\lambda,\pi_i)\right)\left(\prod_{f\in\{x_1,\ldots,x_r\}}\mathrm{Wg}_{n,\left(|w|_f k\right)}\left(\sigma_f,\tau_f\right)\right)\mathcal{N}\left(\sigma_f,\tau_f,\pi_i\right).\end{aligned} \tag{57}$$

Combining (57) with Lemma 3.22 and (25), the following bound is immediate.

Corollary 4.7. *Suppose $k \in \mathbb{Z}_{>0}$. For any $\lambda \vdash k$ and every cyclically reduced, non–identity, non–primitive word $w \in F_r$,*

$$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] \ll_{k,l(w)} \sum_{\sigma_f,\tau_f}^{\star}\sum_{\pi_1,\ldots,\pi_{l(w)}}^{\leq S_k} n^{-\sum_{i=1}^{l(w)}\mathrm{del}(\pi_i)}n^{-\sum_f|\sigma_f\wedge\tau_f|}\mathcal{N}\left(\sigma_f,\tau_f,\pi_i\right).$$

4.4 Graphical Interpretation

We construct a graph for each collection of set partitions $\sigma_f, \tau_f, \pi_1, \ldots, \pi_{l(w)}$. This is similar to the construction of a surface from a matching datum by Magee in [Mag25a]. Indeed, the graph is essentially the same as the 1–skeleton of the surfaces constructed there.

The graph we construct will be denoted $G(\sigma_f, \tau_f, \pi_i)$. The information in (57) will be encoded in the properties of $G(\sigma_f, \tau_f, \pi_i)$, allowing us to simplify further by analyzing the graph rather than dealing with tricky combinatorial arguments. Ultimately, from $G(\sigma_f, \tau_f, \pi_i)$ we will construct a new graph, from which we derive Theorem 1.12.

$G(\sigma_f, \tau_f, \pi_i)$ is a graph with $2kl(w)$ vertices, separated in to k distinct subsets, each containing exactly $2l(w)$ vertices. We number the subsets from 1 to k. For each $i \in \{1, \ldots, k\}$, the vertices in subset i are labeled $\left(I^1_f\right)_i, \ldots \left(I^{|w|_f}_f\right)_i, \left(J^1_f\right)_i, \ldots, \left(J^{|w|_f}_f\right)_i$ for each $f \in \{x_1, \ldots, x_r\}$.

Within each subset i, for each $f \in \{x_1, \ldots, x_r\}$ and for each $j \in \{1, \ldots, |w|_f\}$, we draw a directed, f–labeled edge from the vertex labeled $\left(J_f^j\right)_i$ to the vertex labeled $\left(I_f^j\right)_i$. This gives a total of $kl(w)$ directed edges. We refer to the i^{th} subset of vertices as the i^{th} w–loop, see Figure 1.

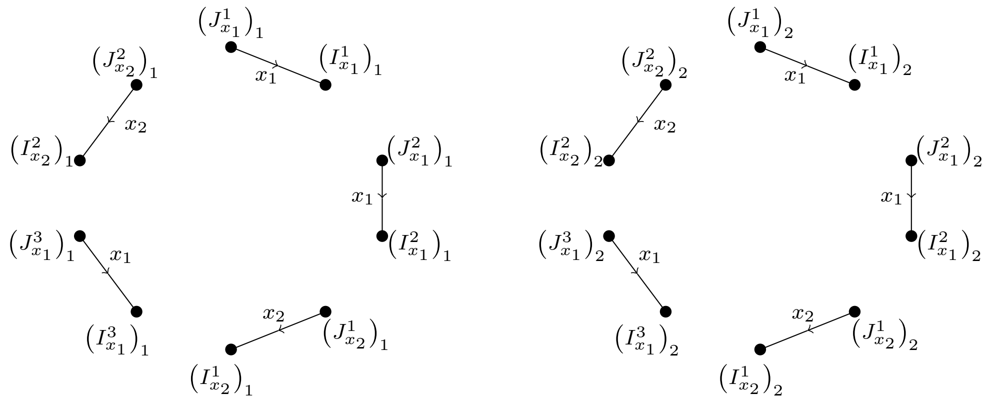


Figure 1: Here we have shown how to begin constructing $G(\sigma_f, \tau_f, \pi_i)$ with $w = x_1 x_1 x_2 x_1^{-1} x_2^{-1}$ and $k = 2$. There are $2kl(w) = 20$ vertices, split in to $k = 2$ distinct subsets, each containing $2l(w) = 10$ vertices each. The vertices on the left (i.e. with the outer subscript "1") are the 1^{st} w–loop and the vertices on the right are the 2^{nd} w–loop. There are $k|w|_{x_1} = 6$ directed x_1–edges and $k|w|_{x_2} = 4$ directed x_2–edges.

For each $f \in \{x_1, \ldots, x_r\}$, we add an undirected σ_f–edge between any two vertices $\left(J_f^j\right)_i$ and $\left(J_f^{j'}\right)_{i'}$ whenever σ_f dictates that the corresponding indices must be equal in order for $\delta_{\sigma_f}\left(J_f^1 \sqcup J_f^2 \sqcup \cdots \sqcup J_f^{|w|_f}\right)$ to be non–zero. This is illustrated in Figure 2.

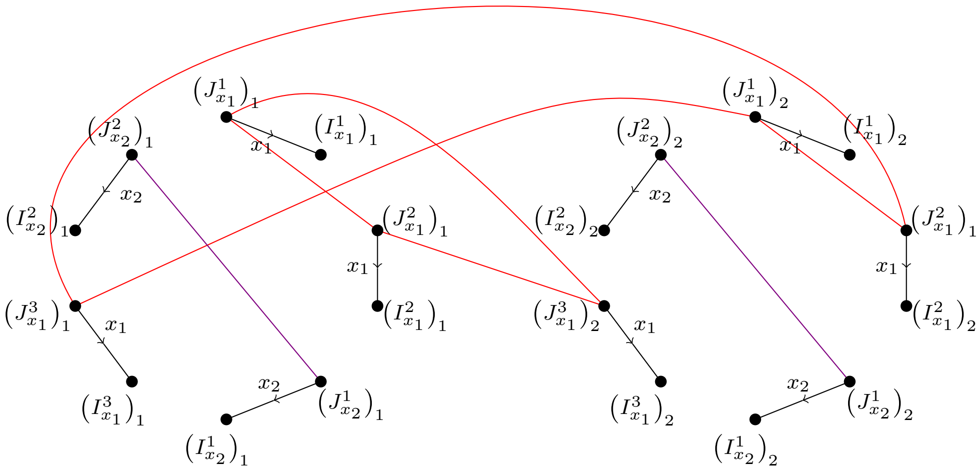


Figure 2: Here we continue the construction of the graph from Figure 1 by adding in the σ_{x_1}–edges (in red) and the σ_{x_2}–edges (in purple). In this example, we have $\sigma_{x_1} = \{\{1,3,6\},\ \{2,4,5\}\} \in \text{Part}\,([3k])$ and $\sigma_{x_2} = \{\{1,3\},\ \{2,4\}\} \in \text{Part}\,([2k])$.

For each $f \in \{x_1, \ldots, x_r\}$, we add undirected τ_f–edges similarly, illustrated in Figure 3. Finally, for each $i \in \{1, \ldots, l(w)\}$, we add an undirected π_i–edge between any two vertices whenever π_i dictates that the corresponding indices must be equal in order for the corresponding inner product term in Definition 4.5 to be non–zero.

In short, for each collection of set partitions $\sigma_f, \tau_f, \pi_1, \ldots, \pi_{l(w)}$, we draw an undirected edge between any two vertices for which the indices with the same label are necessarily equal for the conditions in Definition 4.5 to hold.

Figure 4 shows a complete example of $G(\sigma_f, \tau_f, \pi_i)$, with each σ_f, τ_f and π_i as described in the examples throughout this section.

Definition 4.8. We will write $\hat{G}(\sigma_f, \tau_f, \pi_i)$ for the subgraph of $G(\sigma_f, \tau_f, \pi_i)$ consisting of every vertex and *only the undirected edges.*

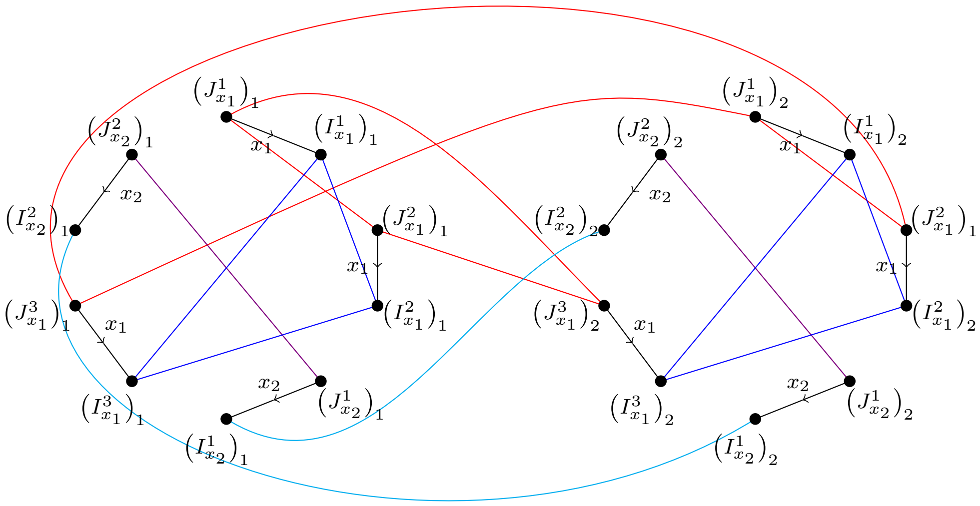


Figure 3: We continue the construction from Figures 1 and 2. We have added in the τ_{x_1}–edges (in dark blue) and the τ_{x_2}–edges (in light blue). In this example, $\tau_{x_1} = \big\{\{1,3,5\}\ \{2,4,6\}\big\}$ and $\tau_{x_2} = \big\{\{1,4\}\ \{2,3\}\big\}$.

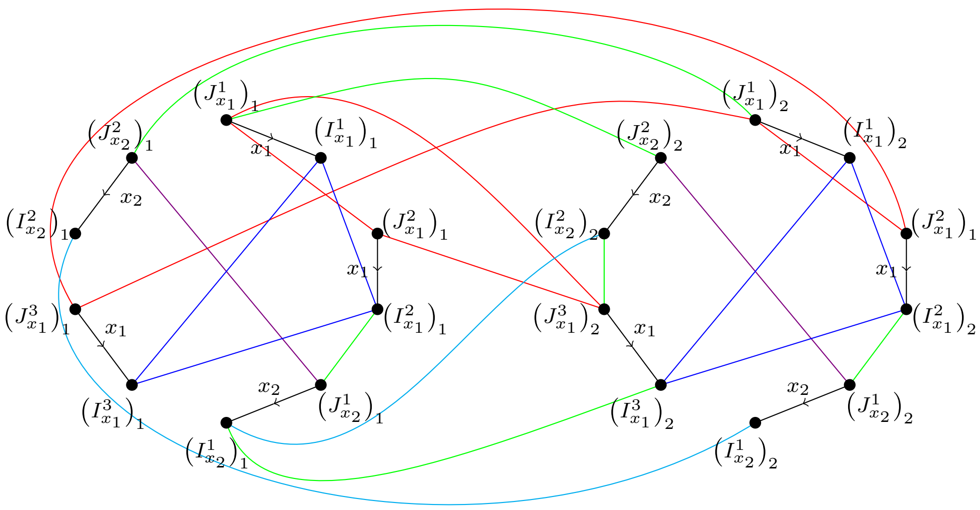


Figure 4: This depicts $G(\sigma_f, \tau_f, \pi_i)$, where σ_f and τ_f are as described in Figures 2 and 3 and the π_i are as follows: $\pi_1 = \big\{\{1\},\ \{2\},\ \{3\}\ \{4\}\big\}$, $\pi_2 = \big\{\{1,3\},\ \{2,4\}\big\}$, $\pi_3 = \big\{\{1,4\},\ \{2\},\ \{3\}\big\}$, $\pi_4 = \big\{\{1\},\ \{3\},\ \{2,4\}\big\}$ and $\pi_5 = \big\{\{1,4\},\ \{2,3\}\big\}$.

4.5 Obtaining the bound

With all $\pi_1, \ldots, \pi_{l(w)}$ corresponding to permutations (i.e. with $\sum_i \operatorname{del}(\pi_i) = 0$), we can apply [LW17, Theorem 1.2] to obtain the bound of n^{-k}. For the other cases, we need to prove a variant of this theorem that applies in our case. This section is dedicated to this task.

4.5.1 Stackings

Given graph morphisms $\rho_1 : \Gamma_1 \to G$ and $\rho_2 : \Gamma_2 \to G$, the fibre product $\Gamma_1 \times_G \Gamma_2$ is the graph with vertex set

$$V(\Gamma_1 \times_G \Gamma_2) = \left\{(v_1, v_2) \in V(\Gamma_1) \times V(\Gamma_2) : \ \rho_1(v_1) = \rho_2(v_2)\right\}$$

and edge set

$$E(\Gamma_1 \times_G \Gamma_2) = \left\{(e_1, e_2) \in E(\Gamma_1) \times E(\Gamma_2) : \ \rho_1(e_1) = \rho_2(e_2)\right\},$$

where $t(e_1, e_2) = (t(e_1), t(e_2))$ and $h(e_1, e_2) = (h(e_1), h(e_2))$.

Louder and Wilton [LW17] first developed the notion of a stacking of a graph immersion and we adapt their definitions slightly to suit our case.

Definition 4.9. Let Γ be a finite graph and $\mathbb{S}$ a 1–complex with an immersion $\Lambda : \mathbb{S} \to \Gamma$. A *stacking* is an embedding $\hat{\Lambda} : \mathbb{S} \to \Gamma \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$, where $\pi : \Gamma \times \mathbb{R} \to \Gamma$ is the trivial $\mathbb{R}$–bundle.

Let $\eta : \Gamma \times \mathbb{R} \to \mathbb{R}$ be the projection to $\mathbb{R}$.

Definition 4.10. Given a stacking $\hat{\Lambda} : \mathbb{S} \to \Gamma \times \mathbb{R}$ of an immersion $\Lambda : \mathbb{S} \to \Gamma$, define

$$\mathcal{A}_{\hat{\Lambda}} \stackrel{\text{def}}{=} \left\{x \in \mathbb{S} : \ \forall y \neq x, \text{ if } \Lambda(x) = \Lambda(y) \text{ then } \eta\left(\hat{\Lambda}(x)\right) > \eta\left(\hat{\Lambda}(y)\right)\right\}$$

and

$$\mathcal{B}_{\hat{\Lambda}} \stackrel{\text{def}}{=} \left\{x \in \mathbb{S} : \ \forall y \neq x, \text{ if } \Lambda(x) = \Lambda(y) \text{ then } \eta\left(\hat{\Lambda}(x)\right) < \eta\left(\hat{\Lambda}(y)\right)\right\}.$$

4.5.2 Bounding expected character

By Remark 1.11, we only need to prove the bound in Theorem 1.12 in the case where w is not a proper power. *From now, assume that w is not a proper power.*

For each σ_f, τ_f, π_i, construct the graph $G(\sigma_f, \tau_f, \pi_i)$ from Section 4.4, consisting of $kl(w)$ disjoint, directed f edges organised into k w–loops and then marking on the blocks of the partitions σ_f, τ_f, π_i using undirected edges.

We construct a new graph $\Gamma(\sigma_f, \tau_f, \pi_i)$ by:

- gluing together any vertices that are connected by a σ_f, τ_f or π_i–edge (and then deleting the σ_f, τ_f or π_i–edges),
- if σ_f and τ_f connect both the initial and terminal vertices of some collection of f–edges, we merge these into a single f–edge, see Figure 5.

Lemma 4.11. *Given $\sigma_f, \tau_f \in \overset{\star}{\mathrm{Part}}([|w|_f k])$ and $\pi_i \in \mathrm{Part}_{\leq S_k}([2k])$,*

$$\sum_f |\sigma_f \wedge \tau_f| = \Big| E(\Gamma(\sigma_f, \tau_f, \pi_i)) \Big|.$$

Proof. Every block of $\sigma_f \wedge \tau_f$ of size p corresponds to some collection of f–edges in $G(\sigma_f, \tau_f, \pi_i)$ of size p, whose initial and terminal vertices have been glued together and whose edges have been merged in the construction of $\Gamma(\sigma_f, \tau_f, \pi_i)$. Every f–edge in $\Gamma(\sigma_f, \tau_f, \pi_i)$ then corresponds to a block of $\sigma_f \wedge \tau_f$. □

Lemma 4.12. *Given $\sigma_f, \tau_f \in \overset{\star}{\mathrm{Part}}([|w|_f k])$ and $\pi_i \in \mathrm{Part}_{\leq S_k}([2k])$,*

$$\mathcal{N}(\sigma_f, \tau_f, \pi_i) \ll n^{\left|V\left(\Gamma\left(\sigma_f, \tau_f, \pi_i\right)\right)\right|}.$$

Proof. Each vertex of $\Gamma(\sigma_f, \tau_f, \pi_i)$ corresponds to a connected component of $\hat{G}(\sigma_f, \tau_f, \pi_i)$. Any collection of indices must be equal if their corresponding vertices are in the same connected component of $\hat{G}(\sigma_f, \tau_f, \pi_i)$. Clearly, if there were no more restrictions, we would have

$$\mathcal{N}(\sigma_f, \tau_f, \pi_i) = n^{\left|V\left(\Gamma\left(\sigma_f, \tau_f, \pi_i\right)\right)\right|}.$$

The lemma follows since we have the additional restriction that within each multi–index, each index must be distinct. ☐

So (57) becomes

$$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] \ll_{k,l(w)} \sum_{\sigma_f,\tau_f}^{\star} \sum_{\pi_1,\dots,\pi_{l(w)}}^{\leq S_k} n^{-\sum_i \mathrm{del}(\pi_i)} n^{\chi\left(\Gamma\left(\sigma_f,\tau_f,\pi_i\right)\right)}. \tag{58}$$

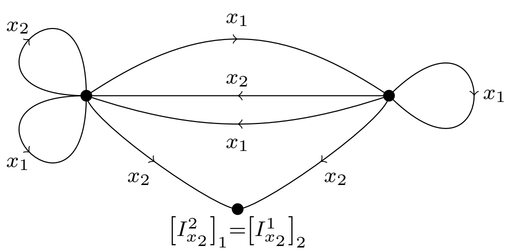


Figure 5: Above we show how to construct $\Gamma(\sigma_f, \tau_f, \pi_i)$ from the graph $G(\sigma_f, \tau_f, \pi_i)$ constructed in Figure 4. We have labeled one vertex to show which vertices of $G(\sigma_f, \tau_f, \pi_i)$ have been glued together in the construction.

Remark 4.13. It is worth noting here why the most obvious approach to obtaining the required bound does not work. We can 'complete' each π_i to a permutation $\hat{\pi}_i$ by gluing together vertices of $\Gamma(\sigma_f, \tau_f, \pi_i)$ according to the additional identifications dictated by the edges that must be added to π_i to construct $\hat{\pi}_i$. Then, one may observe that

$$n^{-\sum_i \mathrm{del}(\pi_i)} n^{\chi\left(\Gamma\left(\sigma_f,\tau_f,\pi_i\right)\right)} \ll n^{\chi\left(\Gamma\left(\sigma_f,\tau_f,\hat{\pi}_i\right)\right)}$$

and seek to apply [LW17, Theorem 1.2] directly. Unfortunately, in gluing the vertices of $\Gamma(\sigma_f, \tau_f, \pi_i)$ together, one may lose the distinctness property that no two vertices of $G(\sigma_f, \tau_f, \pi_i)$ labeled, say, $\left(J^i_f\right)_{j_1}$ and $\left(J^i_f\right)_{j_2}$ with $j_1 \neq j_2$, can be glued together, which would then weaken the bound achieved by applying [LW17, Theorem 1.2]. Therefore, we proceed with our extension of their result.

Define $\mathcal{B}_r \stackrel{\text{def}}{=} \bigvee_{i=1}^r S^1_i$, the bouquet of r oriented circles, labeled $x_1, \dots, x_r$ and with wedge point labeled o. Define W to be the graph consisting of a single cycle of oriented f–labeled edges that, when traversed in an appropriate direction from an appropriate vertex, reads out the word w. Then W comes equipped with an obvious primitive (since w is not a proper power) immersion

$$\Lambda : W \to \mathcal{B}_r.$$

By [LW17, Lemma 3.4], this immersion has a stacking

$$\hat{\Lambda} : W \to \mathcal{B}_r \times \mathbb{R}.$$

For each σ_f, τ_f, π_i, write $\Gamma = \Gamma(\sigma_f, \tau_f, \pi_i)$. We have a map

$$\rho : \Gamma \to \mathcal{B}_r$$

and we can form the fibre product graph

$$\Gamma \times_{\mathcal{B}_r} W.$$

We denote by $\mathbb{S}$ the components of the fibre product graph defined by following the paths in $G(\sigma_f, \tau_f, \pi_i)$ which alternate between f–edges and π_i–edges. This defines some partition of w^k. The connected components of $\mathbb{S}$ are either *circles* that read out w^{k_i} or *pieces,* which consist of two closed endpoints and a chain of vertices of valence two in between them. Each piece reads out some subword of w^k. The concatenation of the words spelled out by all the connected components of $\mathbb{S}$ is exactly w^k and the number of pieces of $\mathbb{S}$ is exactly $\sum_i \operatorname{del}(\pi_i)$.

We have an immersion

$$\Lambda' : \mathbb{S} \to \Gamma$$

and an immersion

$$\delta : \mathbb{S} \to W.$$

By the same argument as [LW17, Lemma 2.5], we have a stacking,

$$\hat{\Lambda}' : \mathbb{S} \to \Gamma \times \mathbb{R}.$$

Lemma 4.14. *Every vertex in Γ is covered at least twice by $\Lambda' : \mathbb{S} \to \Gamma$.*

Proof. Every vertex v of Γ is constructed by gluing together *at least two* vertices in $G(\sigma_f, \tau_f, \pi_i)$, and v is covered by Λ' exactly once for each vertex in $G(\sigma_f, \tau_f, \pi_i)$ that has been glued in the construction of v. □

The connected components of $\mathbb{S}$ are then one of the following:

- a circle
- a *closed arc*: a connected and simply connected union of vertices and interiors of edges, with both ends closed (i.e. a piece).

If we also define the following:

- an *open arc*: a connected and simply connected union of vertices and interiors of edges, with both ends open
- a *half–open arc*: a connected and simply connected union of vertices and interiors of edges, with one end open and one end closed,

then the following lemma is immediate.[10]

Lemma 4.15. *Every connected component of $\mathcal{A}_{\hat{\Lambda}'}$ or $\mathcal{B}_{\hat{\Lambda}'}$ is either a circle, an open arc, a half–open arc or a closed arc.*

Lemma 4.16. *Neither $\mathcal{A}_{\hat{\Lambda}'}$ or $\mathcal{B}_{\hat{\Lambda}'}$ contain a circle, a closed arc reading out a word of length $\geq l(w)$ or an open/half–open arc reading out a word of length $> l(w)$.*

Proof. Let $C \subseteq \mathbb{S}$ be a circle, a closed arc reading out a word of length $\geq l(w)$, or an open/half–open arc reading out a word of length $> l(w)$, which is contained inside $\mathcal{A}_{\hat{\Lambda}'}$. Then the map

$$\delta\big|_C : C \to W$$

is surjective.

W must contain a vertex x belonging to $\mathcal{B}_{\hat{\Lambda}}$. Then, the preimage $\left(\delta\big|_C\right)^{-1}(x)$ contains a vertex x' belonging to $\mathcal{B}_{\hat{\Lambda}'}$, so that $x' \in \mathcal{A}_{\hat{\Lambda}'} \cap \mathcal{B}_{\hat{\Lambda}'}$. This implies that $\Lambda'(x') \in V(\Gamma)$ is covered exactly once by Λ', contradicting Lemma 4.14. To prove the lemma for $\mathcal{B}_{\hat{\Lambda}'}$, we can swap $\mathcal{A}_{\hat{\Lambda}'}$ and $\mathcal{B}_{\hat{\Lambda}'}$ in the proof. □

[10] We are especially indebted to Noam Ta Shma for pointing out an error in our original argument for proving the extension of Wise's w–cycles conjecture and for his feedback on the correction presented from this point onward.

So the connected components of $\mathcal{A}_{\hat{\Lambda}'}$ are one of the following:

- open arcs that read out words of length $\leq l(w)$,
- half–open arcs that read out words of length $\leq l(w)$ or
- closed arcs that read out words of length $< l(w)$.

The main points here are that every connected component has two ends (that are either open or closed) and that none of the connected components can be too long. Moreover, we observe that the only closed ends that can be in $\mathcal{A}_{\hat{\Lambda}'}$ or $\mathcal{B}_{\hat{\Lambda}'}$ are those that are the closed ends of pieces in $\mathbb{S}$.

Write

$$\mathcal{A}_{\text{end}} \stackrel{\text{def}}{=} \#\{\text{ends of connected components of } \mathcal{A}_{\hat{\Lambda}'}\}$$

and

$$\mathcal{B}_{\text{end}} \stackrel{\text{def}}{=} \#\{\text{ends of connected components of } \mathcal{B}_{\hat{\Lambda}'}\}.$$

Lemma 4.17. *We have* $\mathcal{A}_{\text{end}} \geq 2k$ *and* $\mathcal{B}_{\text{end}} \geq 2k$.

Proof. Pick any vertex $w \in W$ such that $w \in \mathcal{A}_{\hat{\Lambda}}$. Then w has at least k preimages $\delta^{-1}(w)$ in $\mathbb{S}$, all of which must be contained in $\mathcal{A}_{\hat{\Lambda}'}$. If two such preimages belong to the same connected component C of $\mathcal{A}_{\hat{\Lambda}'}$, then C must either be a circle, a closed arc reading out a word of length $\geq l(w)$ or an open/half–open arc reading out a word of length $> l(w)$, contradicting Lemma 4.16. Thus, there must be at least k connected components in $\mathcal{A}_{\hat{\Lambda}'}$, so there must be at least $2k$ ends. Repeating the argument with $w \in \mathcal{B}_{\hat{\Lambda}}$ proves the lemma for $\mathcal{B}_{\text{end}}$. □

Write

$$\mathcal{A}_{\text{closed}} \stackrel{\text{def}}{=} \#\{\text{closed ends of connected components of } \mathcal{A}_{\hat{\Lambda}'}\}$$

and

$$\mathcal{B}_{\text{closed}} \stackrel{\text{def}}{=} \{\text{closed ends of connected components of } \mathcal{B}_{\hat{\Lambda}'}\}.$$

Lemma 4.18. *We have*

$$\mathcal{A}_{\text{end}} = -2\chi + 2\mathcal{A}_{\text{closed}}$$

and

$$\mathcal{B}_{\text{end}} = -2\chi + 2\mathcal{B}_{\text{closed}}.$$

Proof. We prove the lemma for $\mathcal{A}_{\hat{\Lambda}'}$. Each vertex $v \in V(\Gamma)$ is covered by $\mathcal{A}_{\hat{\Lambda}'}$ once. If v is covered by a connected component of $\mathcal{A}_{\hat{\Lambda}'}$ passing *through* one of v's preimages in $\mathbb{S}$, then there must be

$$\deg(v) - 2$$

open ends of $\mathcal{A}_{\hat{\Lambda}'}$ that end at the other preimages of v in $\mathbb{S}$, to cover the other edges of Γ that are incident at v.

Otherwise, v is covered by $\mathcal{A}_{\hat{\Lambda}'}$ by a closed end of a connected component of $\mathcal{A}_{\hat{\Lambda}'}$ that *ends* at one of v's preimages. In this case, we count exactly 1 closed end of $\mathcal{A}_{\hat{\Lambda}'}$ that ends at one of v's preimages and exactly

$$\deg(v) - 1$$

open ends of $\mathcal{A}_{\hat{\Lambda}'}$ that end at the other preimages of v in $\mathbb{S}$, to cover the other edges of Γ that are incident at v.

In total, we see that

$$\mathcal{A}_{\text{end}} = \sum_{v} (\deg(v) - 2) + \sum_{u} \deg(u),$$

where the first sum is over $v \in V(\Gamma)$ that are covered by $\mathcal{A}_{\hat{\Lambda}'}$ by a connected component passing through one of their preimages and the second sum is over $u \in V(\Gamma)$ that are covered by $\mathcal{A}_{\hat{\Lambda}'}$ by a closed end at one of their preimages. This can be rewritten as

$$\sum_{v \in V(\Gamma)} (\deg(v) - 2) + 2\sum_{u} = -2\chi(\Gamma) + 2\mathcal{A}_{\text{closed}}.$$

□

Combining Lemma 4.17 with Lemma 4.18 implies that

$$k \leq -\chi(\Gamma) + \mathcal{A}_{\text{closed}}$$

and also that

$$k \leq -\chi(\Gamma) + \mathcal{B}_{\text{closed}}.$$

This implies that

$$2k \leq -2\chi(\Gamma) + \mathcal{A}_{\text{closed}} + \mathcal{B}_{\text{closed}} \leq -2\chi(\Gamma) + 2\sum_i \operatorname{del}(\pi_{\mathrm{i}}), \tag{59}$$

since the total number of closed ends in $\mathbb{S}$ is $2\sum_i \operatorname{del}(\pi_i)$ and each closed end in $\mathbb{S}$ can be, at most, a closed end in $\mathcal{A}_{\hat{\Lambda}'}$ or a closed end in $\mathcal{B}_{\hat{\Lambda}'}$, but not both, since that would imply that its image in Γ is covered only once by the immersion $\Lambda' : \mathbb{S} \to \Gamma$, contradicting Lemma 4.14. Combining (59) with (58) proves Theorem 1.12.

4.6 Proof of Proposition 4.1

Let $w \in F_r = \langle x_1, \dots, x_r \rangle$ for r fixed. Let $m \leq r$ be the minimum number of generators $x_{i_1}, \dots, x_{i_m}$ such that w can be written using the alphabet $\{x_{i_1}, x_{i_1}^{-1}, \dots, x_{i_m}, x_{i_m}^{-1}\}$. So, up to relabeling the generators, $w \in F_m = \langle x_1, \dots, x_m \rangle$, with $m \leq r$.

Beginning with (57), we have

$$\begin{aligned} &\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] \\ =&\frac{1}{d_\lambda} \sum_{\sigma_f,\tau_f}^{\star} \sum_{\pi_1,\dots,\pi_{l(w)}}^{\leq S_k} \left(\prod_{i=1}^{l(w)} c(n,k,\lambda,\pi_i)\right) \left(\prod_{f\in\{x_1,\dots,x_m\}} \mathrm{Wg}_{n,(|w|_f k)}(\sigma_f,\tau_f)\right) \\ &\mathcal{N}(\sigma_f,\tau_f,\pi_i). \end{aligned}$$

For each $f \in \{x_1, \dots, x_m\}$, for each $\sigma_f, \tau_f \in \overset{\star}{\operatorname{Part}}([|w|_f k])$,

$$\mathrm{Wg}_{n,(|w|_f k)}(\sigma_f,\tau_f) = \sum_{\rho_f \leq \sigma_f \wedge \tau_f} \mu(\rho_f,\sigma_f)\mu(\rho_f,\tau_f)\frac{1}{(n)_{|\rho_f|}}.$$

This is equal to

$$\begin{aligned} &\frac{C_1}{(n)_{|w|_f k}} + \frac{C_2}{(n)_{|w_f|k-1}} + \dots + \frac{C_{|w|_f k - |\sigma_f\wedge\tau_f|+1}}{(n)_{|\sigma_f\wedge\tau_f|}} \\ =&\frac{g_{\sigma_f,\tau_f}(n)}{(n)_{|w|_f k}}, \end{aligned}$$

where $g_{\sigma_f,\tau_f}(n)$ is a polynomial in n of maximum degree $|w|_f k - |\sigma_f \wedge \tau_f|$.

For each $\pi \in \mathrm{Part}_{\leq S_k}([2k])$, we have

$$c(n,k,\lambda,\pi) = \frac{d_{\lambda^+(n)}(-1)^{|\pi|+k}}{(n)_{|\pi|}} \sum_{\substack{\tau\in S_k \\ \iota(\tau)\geq\pi}} \chi^\lambda(\tau).$$

Using the hook–length formula, this is equal to

$$\frac{\left[\frac{(-1)^{k+|\pi|}}{k!}\sum_{\substack{\tau\in S_k \\ \iota(\tau)\geq\pi}} \chi^\lambda(\tau)\right](n-|\pi|)(n-|\pi|-1)\dots(n-2k+1)}{(n)_\lambda},$$

where $(n)_\lambda = \prod_{j=1}^k \left(n-k+1-\check{\lambda}_j - j\right)$. It follows that

$$\prod_{i=1}^{l(w)} c(n,k,\lambda,\pi_i) = \frac{\prod_{i=1}^{l(w)} h_{k,\lambda,\pi_i}(n)}{(n)_\lambda^{l(w)}},$$

where

$$h_{k,\lambda,\pi_i}(n) = \left[\frac{(-1)^{k+|\pi|}}{k!}\sum_{\substack{\tau\in S_k \\ \iota(\tau)\geq\pi}} \chi^\lambda(\tau)\right](n-|\pi|)(n-|\pi|-1)\dots(n-2k+1)$$

is a polynomial in n of degree $2k-|\pi_i| = k-\mathrm{del}(\pi_i)$. Hence,

$$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = \frac{p_{k,\lambda,w}(n)}{(n)_\lambda^{l(w)}\prod_f (n)_{|w|_f k}}, \tag{60}$$

where

$$p_{k,\lambda,w}(n) = \frac{1}{d_\lambda}\sum_{\sigma_f}^{\star}\sum_{\pi_1,\dots,\pi_{l(w)}}^{\leq S_k}\left(\prod_f g_{\sigma_f,\tau_f}(n)\right)\left(\prod_{i=1}^{l(w)} h_{k,\lambda,\pi_i}(n)\right)\mathcal{N}(\sigma_f,\tau_f,\pi_i).$$

For each collection of σ_f, τ_f, π_i:

- $\left(\prod_f g_f(n)\right)$ has maximum degree

$$\sum_f (|w|_f k - |\sigma_f \wedge \tau_f|) = kl(w) - \sum_f |\sigma_f\wedge\tau_f|,$$

- $\left(\prod_{i=1}^{l(w)} h_{k,\lambda,\pi_i}(n)\right)$ has maximum degree

$$\sum_{i=1}^{l(w)} (k - \mathrm{del}(\pi_i)) = kl(w) - \sum_i \mathrm{del}(\pi_\mathrm{i}),$$

- $\mathcal{N}(\sigma_f, \tau_f, \pi_i)$ has maximum degree[11]

$$\sum_i \mathrm{del}(\pi_i) + \sum_f |\sigma_f \wedge \tau_f|,$$

so that the degree of $p_{k,\lambda,w}(n)$ is less than or equal to

$$2kl(w).$$

We can rewrite the denominator of (60) using reciprocal polynomials. We have

$$\begin{aligned}
&\prod_f (n)_{|w|_f k} \\
=&\prod_{d=1}^{m} \prod_{c=1}^{|w|_{x_d} k-1} (n-c) \\
=&n^{kl(w)} \prod_{d=1}^{m} \prod_{c=1}^{|w|_{x_d} k-1} \left(1 - c\frac{1}{n}\right).
\end{aligned}$$

Similarly,

$$(n)_\lambda^{l(w)} = n^{kl(w)} \left[\prod_{j=1}^{k} \left(1 + \left(1 + \check{\lambda}_j - j - k\right)\frac{1}{n}\right)\right]^{l(w)}.$$

It follows that

$$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = \frac{\frac{1}{n^{2kl(w)}} p_{k,\lambda,w}(n)}{\tilde{g}\left(\frac{1}{n}\right)} = \frac{\hat{P}_{w,k,\lambda}\left(\frac{1}{n}\right)}{\tilde{g}\left(\frac{1}{n}\right)},$$

where

$$\tilde{g}(x) = \prod_{d=1}^{m} \prod_{c=0}^{|w|_{x_d} k-1} (1 - cx) \left[\prod_{j=1}^{k} \left(1 + \left(1 + \check{\lambda}_j - j - k\right) x\right)\right]^{l(w)}.$$

The numerator $\hat{P}_{w,k,\lambda}\left(\frac{1}{n}\right)$ is clearly a polynomial in $\frac{1}{n}$ of maximum degree $2kl(w)$.

[11]This follows from the simple observation that $\Gamma(\sigma_f, \tau_f, \pi_i)$ has Euler characteristic $\leq \sum_i \mathrm{del}(\pi_i)$, regardless of if w is a proper power or not.

Now assume that $l(w) \leq q$, which also implies that $m \leq q$. Then $\tilde{g}(x)$ always divides

$$\hat{g}(x) = \prod_{c=1}^{kq} (1-cx)^q \left[\prod_{j=1}^{k} \left(1 + \left(1 + \check{\lambda}_j - j - k\right) x\right) \right]^q .$$

The sequence $\check{\lambda}_1 - 1, \ldots, \check{\lambda}_k - k$ is strictly decreasing. Moreover, for any $\lambda \vdash k$ and any $j \in [k]$,

$$1 + \check{\lambda}_j - j - k \in [1-2k, 0],$$

so that

$$\hat{g}(x) \Bigg| \prod_{c=1}^{kq} (1-cx)^q \left[\prod_{j=1}^{2k} (1 + (1-j)\, x) \right]^q = g_{q,k}(x).$$

It follows that $\mathbb{E}_w \left[\chi^{\lambda^+(n)} \right]$ can be written

$$\frac{P_{w,k,\lambda}\left(\frac{1}{n}\right)}{g_{q,k}\left(\frac{1}{n}\right)},$$

where

$$\begin{aligned} &P_{w,k,\lambda}\left(\frac{1}{n}\right) \\ =&\hat{P}_{w,k,\lambda}\left(\frac{1}{n}\right) \frac{g_{q,k}\left(\frac{1}{n}\right)}{\tilde{g}\left(\frac{1}{n}\right)}. \end{aligned}$$

This is a polynomial in $\frac{1}{n}$ of maximum degree

$$\begin{aligned} &2kl(w) + kq^2 + 2kq - kl(w) - kl(w) + m \\ \leq &3kq + kq^2. \end{aligned}$$

5 Strong convergence for random permutations

The goal of this section is to prove Theorem 1.14 (immediately proving Theorem 1.13), from Theorem 1.12. The material is entirely based on [Cas25b, Sections 2 and 3].

5.1 Analysis Background

We first present a number of analytic results of Magee and de la Salle [MdlS24] that are necessary for this section. They will allow for us to extend the methods of [CGVTvH24] in order to use Theorem 1.12 to prove Theorem 1.14. The key result in the present section is Proposition 5.7, see [MdlS24, Proposition 5.2]. This is a new criterion for strong convergence based on

- the new approach to strong convergence, 'differentiation with respect to $\frac{1}{n}$' introduced by Chen, Garza–Vargas, Tropp and van Handel in [CGVTvH24],
- a new criterion for temperdness of arbitrary functions on free groups (see 5.8).

This approach allows one to bypass Pisier's linearization trick [Pis19] used in previous arguments for strong convergence by instead considering random walks on the free group. Our proof of Theorem 1.14 relies on this new criterion for strong convergence, and so §5.2 is devoted to showing that the conditions detailed in Proposition 5.7 are satisfied.

5.1.1 C^*–algebras of free groups

We refer to [NS06] for a thorough introduction.

Definition 5.1. A C^*–algebra is a complex algebra $\mathcal{A}$ together with:

1. an involution $* : \mathcal{A} \to \mathcal{A}$ that satisfies, $\forall a, b \in \mathcal{A}$,
   (a) $(a^*)^* = a$,
   (b) $(a+b)^* = a^* + b^*$,
   (c) $(Ca)^* = \bar{C}a^*$ for any $C \in \mathbb{C}$,
   (d) $(ab)^* = b^*a^*$
2. a norm $\| - \| : \mathcal{A} \to \mathbb{R}_{\geq 0}$ such that, $\forall a, b \in \mathcal{A}$,
   (a) $\mathcal{A}$ is a Banach space with respect to this norm (i.e. complete)
   (b) $\|ab\| \leq \|a\|.\|b\|$
   (c) $\|a^*\| = \|a\|$ and
   (d) the C^* identity holds, that is, $\|a^*a\| = \|a^2\|$.

To the free group F_r we associate its group algebra $\mathbb{C}[F_r]$, the collection of elements of the form

$$x = \sum_{w \in F_r} x(w)w,$$

where each $x(w) \in \mathbb{C}$ and only finitely many are non–zero.

We define a norm on $\mathbb{C}[F_r]$ by

$$\|x\|_{C^*(F_r)} = \sup\left\{\|\pi(x)\|_{\mathrm{op}} : \ \pi \text{ a representation of } F_r\right\},$$

where $\| - \|_{\mathrm{op}}$ is the operator norm on $\ell^2(F_r)$. The completion of $\mathbb{C}[F_r]$ in this norm is the C^*–algebra of F_r, denoted $C^*(F_r)$. The left regular representation of F_r is denoted λ and admits the *reduced norm* on $\mathbb{C}[F_r]$ by

$$\|x\|_{\mathrm{C}^*_\lambda(\mathrm{F_r})} = \|\lambda(x)\|_{\mathrm{op}}.$$

The completion of $\mathbb{C}[F_r]$ in this norm is the reduced C^*–algebra of F_r, denoted $C^*_\lambda(F_r)$.

We define a trace τ on $\mathbb{C}[F_r]$ to be the map

$$\tau(x) = x(e).$$

This extends continuously to both the C^*–algebras defined above and remains a trace in both cases.

5.1.2 Polynomials and random walks

Lemma 5.2 ([MdlS24, Lemma 4.2])**.** *For every polynomial P in one variable with bounded degree,* $\deg(P) \le D$ *and for every integer* $k \le D$*,*

$$\sup_{\left[0,\frac{1}{2D^2}\right]} \left|P^{(k)}\right| \le \frac{2^{2k+1}D^{4k}}{(2k-1)!!} \sup_{n \ge D^2} \left|P\left(\frac{1}{n}\right)\right|,$$

where $(2k-1)!! = (2k-1)(2k-3)\dots(3)1.$

5.1.3 Random walks and free groups

Definition 5.3. We call a symmetric (ie $\mu(g) = \mu\left(g^{-1}\right)$) probability measure μ on the free group F_r *reasonable* if its support is finite, contains the identity element and generates F_r.

We denote by $(g_n)_{n\geq 0}$ the corresponding random walk in F_r. So we write

$$g_n = s_1 \dots s_n$$

where $s_i \in F_r$ are i.i.d. according to μ.

The spectral radius $\rho = \rho(\mu)$ measures how fast the probability that a random walk returns to where it started decays. It is equal to the norm of $\lambda(\mu)$ on $\ell^2(F_r)$, where λ is the left regular representation of F_r.

Recall that a proper power in F_r is an element of the form u^d for $u \in F_r$ and $d \geq 2$.

Proposition 5.4 ([MdlS24, Proposition 6.1])**.** *For any reasonable probability measure μ, there is a constant $C = C_\mu$ such that*

$$\mathbb{P}\left(g_n \text{ is a proper power}\right) \leq Cn^5\rho^n.$$

Definition 5.5. We denote the subspace of elements of $\mathbb{C}[F_r]$ supported in the ball of radius q by $\mathbb{C}_{\leq q}[F_r]$.

5.1.4 Temperedness and strong convergence

The following definition and proposition can be found in [MdlS24, Section 5]. Fix a generating set of size r for the free group F_r.

Definition 5.6. A function $u : F_r \to \mathbb{C}$ is called tempered if

$$\limsup_{n\to\infty} \left|u\left((x^*x)^n\right)\right|^{\frac{1}{2n}} \leq \|\lambda(x)\|_{\mathrm{op}}$$

for every $x \in \mathbb{C}(F_r)$, where λ is the left regular representation.

The above property is a part of a criterion for strong convergence of random representations.

Proposition 5.7. *Let $u_n : F_r \to \mathbb{C}$ be functions, π_n a sequence of random unitary representations of F_r with finite and non–random dimension. Let $\epsilon_n > 0$. If the following conditions are satisfied:*

- $\lim_{n\to\infty} \epsilon_n = 0$,
- *we have* $\left|\mathbb{E}\mathrm{Tr}\left(\pi_n(x)\right) - u_n(x)\right| \leq \epsilon_n \exp\left(\frac{q}{\log(2+q)^2}\right) \|x\|_{C^*(F_r)}$ *for every* q *and every* $x \in \mathbb{C}_{\leq q}\left[F_r\right]$, *and*
- u_n *is tempered and there is a polynomial* P_n *such that, for every* q *and every* $x \in \mathbb{C}_{\leq q}\left[F_r\right]$, $|u_n(x)| \leq P_n(q)\|x\|_{C^*(F_r)}$.

Then, for every $y \in \mathbb{C}\left[F_r\right]$ *and for every* $\delta > 0$,

$$\mathbb{P}\left[\|\pi_n(y)\| > \|\lambda(y)\| + \delta\right] \leq C\left(y, \delta\right) \epsilon_n$$

for some constant $C(y, \delta)$. *In particular,*

$$\lim_n \mathbb{P}\left[\|\pi_n(y)\| > \|\lambda(y)\| + \delta\right] = 0. \tag{61}$$

To prove that a function is tempered, we can use the following proposition (stated here for free groups only), from [MdlS24, Proposition 6.3].

Proposition 5.8. *Let* $u : F_r \to \mathbb{C}$ *and assume that, for every reasonable probability measure* μ *on* F_r, *if* $(g_n)_n$ *is the associated random walk on* F_r,

$$\limsup_{n\to\infty} \left(\mathbb{E}\left|u\left(g_n\right)\right|\right)^{\frac{1}{n}} \leq \rho(\mu).$$

Then u *is tempered.*

Remark 5.9. That this proposition holds for free groups is a result of Haagerup's inequality [Haa78, Lemma 1.5], which asserts that free groups with their standard generating sets have the *rapid decay property.*

5.2 Proof of Theorem 1.14

The results of this section are almost completely analogous to those of [MdlS24, Section 7]. Throughout, C will denote a constant that does not depend on anything, but that may change from line to line. We require our bound on the expected irreducible stable character of a w–random permutation from Theorem 1.12. Fix any integer $K > 0$ and define

$$\Sigma_{n,K} \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \vdash K} V^{\lambda^+(n)}.$$

We view $\Sigma_{n,K} \subseteq (\mathbb{C}^n)^{\otimes k}$, so that

$$\dim\left(\Sigma_{n,K}\right) \leq n^K. \tag{62}$$

The following theorem follows immediately by combining Theorem 1.12 and Proposition 4.1 – recall that, for each $\lambda \vdash K$, we can write

$$\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = \frac{P_{w,\lambda,q}\left(\frac{1}{n}\right)}{g_{q,k}\left(\frac{1}{n}\right)},$$

where $\deg\left(P_{w,\lambda,q}\right) \leq 3Kq + Kq^2$ and that, as a rational function in n, $\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = O(1)$ for proper powers and $\mathbb{E}_w\left[\chi^{\lambda^+(n)}\right] = O\left(\frac{1}{n^K}\right)$ for non–powers.

Theorem 5.10. *For every word $w \in F_r$, there is a rational function $\phi_w \in \mathbb{Q}[x]$ such that*

1. *For $n \geq K \max\left(l(w), 2\right)$,*

$$\phi_w\left(\frac{1}{n}\right) = \frac{1}{n^K}\mathbb{E}_w\left[\mathrm{Tr}_{\Sigma_{n,K}}\right].$$

2. *If w is not the identity and $l(w) \leq q$, then $g_{q,K}\phi_w$ is a polynomial of degree $\leq D_q = 3Kq + Kq^2 + K$.*

3. *If w is not a proper power then, for all $i < 2K$,*

$$\phi_w^{(i)}(0) = 0.$$

Otherwise, this holds for all $i < k$.

We collect some facts about the polynomial

$$g_{L,K}(x) = \prod_{c=1}^{KL}(1-cx)^L\left[\prod_{j=1}^{2K}(1-(j-1)x)\right]^L$$

in the following lemma.

Lemma 5.11. *Fix an integer $L > 0$. Then, for every t satisfying*

$$0 \le t \le \frac{1}{2} \frac{1}{KL^2(KL+1) + L(2K-1)(2K)}$$

and every integer $i \ge 0$:

1. $\frac{1}{2} \le g_{L,K}(t) \le 1$
2. $|g_{L,K}^{(i)}(t)| \le \left(K^2L^3 + L(2K-1)^2\right)^i$
3. $\left|\left(\frac{1}{g_{L,K}}\right)^{(i)}(t)\right| \le (2i)!!2^{i+1}\left(K^2L^3 + L(2K-1)^2\right)^i.$

Proof. Fix any $0 \le t \le \frac{1}{2}\frac{1}{KL^2(KL+1)+L(2K-1)(2K)}$. That $g_{L,K}(t) \le 1$ follows immediately from the fact that

$$|t| < \min\left\{\frac{1}{KL}, \frac{1}{2K-1}\right\}.$$

To see that $\frac{1}{2} \le g_{L,K}(t)$, observe that, in the regime $0 < z < \frac{1}{2}$, we have $(1-z) \ge e^{-2z}$. Since

$$t < \min\left\{\frac{1}{2KL}, \frac{1}{2(2K-1)}\right\},$$

we have

$$\begin{aligned}
&g_{L,K}(t) \\
&\ge \exp\left(-2\left(\sum_{c=1}^{KL} Lct + \sum_{j=1}^{2K} L(j-1)t\right)\right) \\
&= \exp\left(-2Lt\left(1 + \cdots + KL + 1 + \cdots + 2K - 1\right)\right) \\
&= \exp\left(-2Lt\left(\frac{KL(KL+1)}{2} + \frac{(2K-1)2K}{2}\right)\right) \\
&= \exp\left(-t\left(KL^2(KL+1) + L(2K-1)(2K)\right)\right) \\
&\ge \frac{1}{2},
\end{aligned}$$

where the final inequality follows from the fact that

$$t \le \frac{1}{2}\frac{1}{KL^2(KL+1) + L(2K-1)(2K)} < \frac{-\log\left(\frac{1}{2}\right)}{KL^2(KL+1) + L(2K-1)(2K)}.$$

For Part 2, we begin by rewriting

$$g_{L,K}(t) = \prod_{z=1}^{KL^2+2KL} (1 - b_z t)$$

and then differentiate using the Leibniz rule. The i^{th} derivative is a sum of terms in which i terms are derived once (so are equal to $-b_z$) and the others are not derived. Each factor that is not derived is of the form $1 - b_z t$, which belongs to $(0, 1]$. So we can bound the derivative using only the terms that are derived:

$$\begin{aligned}
&\left|g^{(i)}_{L,K}(t)\right| \\
&\leq \sum_{\substack{u_1,\dots,u_i \\ \text{distinct}}} i! \prod_{\alpha=1}^{i} b_{u_\alpha} \\
&\leq \left(\sum_{z=1}^{KL^2+2KL} b_z\right)^i \\
&= \left(\sum_{c=1}^{KL} Lc + \sum_{j=1}^{2K} L(j-1)\right)^i \\
&\leq \left(K^2L^3 + L(2K-1)^2\right)^i .
\end{aligned}$$

To prove Part 3, we use that the i^{th} derivative of $\frac{1}{g_{L,K}}$ can be written as a product of $(2i)!!$ terms of the form

$$\frac{\pm g^{(\alpha_1)}_{L,K} \cdots g^{(\alpha_i)}_{L,K}}{g^{i+1}_{L,K}},$$

where $\alpha_1, \dots, \alpha_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_1 + \cdots + \alpha_i = i$. By Part 2, each term is bounded by

$$\frac{\prod_j \left(K^2L^3 + L(2K-1)^2\right)^{\alpha_j}}{g^{i+1}_{L,K}}$$

and, by Part 1, this is bounded by

$$2^{i+1} \prod_j \left(K^2L^3 + L(2K-1)^2\right)^{\alpha_j} = 2^{i+1} \left(K^2L^3 + L(2K-1)^2\right)^i ,$$

from which Part 3 follows. □

Each $w \in F_r$ defines a map $w \mapsto \phi_w$. We extend this by linearity to define a map

$x \mapsto \phi_x$ for each

$$x = \sum_{w \in F_r} x(w)w \in \mathbb{C}[F_r].$$

Then we can prove the following.

Lemma 5.12. *For any $x \in \mathbb{C}_{\leq q}[F_r]$, for any $i \leq 2K$ and for $n \geq Kq$,*

$$\sup_{t \in \left[0, \frac{1}{2D_q^2}\right]} \frac{\left|\phi_x^{(i)}(t)\right|}{i!} \leq h(i,q) \|x\|_{C^*(F_r)},$$

where

$$h(i,q) = 4\left(\frac{CD_q^4}{i^2}\right)^i.$$

Proof. Let $P = g_{q,K}\phi_x$. Then P is a polynomial of bounded degree $d \leq D_q$ and so we can bound its derivatives using Lemma 5.2 if we can bound P itself. For $n \geq Kq$, we have $0 < g_{q,K}\left(\frac{1}{n}\right) < 1$, so that

$$\begin{aligned}
&\left|P\left(\frac{1}{n}\right)\right| \\
\leq &\left|\phi_x\left(\frac{1}{n}\right)\right| \\
= &\frac{1}{n^K}\left|\mathbb{E}_x\left[\mathrm{Tr}_{\Sigma_{\mathrm{n,K}}}\right]\right| \\
\underset{(\dagger)}{\leq} &\frac{1}{n^K}\dim\left(\Sigma_{n,K}\right)\|x\|_{C^*(F_r)} \\
\underset{(62)}{\leq} &\|x\|_{C^*(F_r)}.
\end{aligned}$$

The inequality $(\dagger)$ follows since the map $w \mapsto \Sigma_{n,K}(w(\sigma_1,\ldots,\sigma_r))$ is a unitary representation of F_r for every $(\sigma_1,\ldots,\sigma_r) \in S_n^r$. So we have

$$\|\Sigma_{n,K}(x(\sigma_1,\ldots,\sigma_r))\| \leq \|x\|_{C^*(F_r)}$$

almost surely, and since we can bound the trace of a matrix by its norm multiplied by its size, we can bound

$$\left|\mathbb{E}_x\left[\mathrm{Tr}_{\Sigma_{\mathrm{n,K}}}\right]\right| \leq \dim\left(\Sigma_{n,K}\right)\|x\|_{C^*(F_r)}.$$

By Lemma 5.2, for any integer j, we can bound

$$\sup_{t\in\left[0,\frac{1}{2D_q^2}\right]} \frac{\left|P^{(j)}(t)\right|}{j!} \leq \frac{2^{2j+1}D_q^{4j}}{j!(2j-1)!!} \|x\|_{C^*(F_r)} = \frac{2^{3j+1}D_q^{4j}}{(2j)!} \|x\|_{C^*(F_r)},$$

which, using Stirling's formula, is bounded above by

$$2\left(\frac{CD_q^4}{j^2}\right)^j \|x\|_{C^*(F_r)}. \tag{63}$$

We then have

$$\frac{\phi_x^{(i)}}{i!} = \frac{1}{i!}\sum_{j=0}^{\min(i,D_q)} \begin{pmatrix} i \\ j \end{pmatrix} P^{(j)} \left(\frac{1}{g_{q,K}}\right)^{(i-j)} = \sum_{j=0}^{\min(i,D_q)} \frac{P^{(j)}}{j!} \frac{1}{(i-j)!} \left(\frac{1}{g_{q,K}}\right)^{(i-j)}.$$

Note that $i \leq 2K \leq D_q = Kq^2 + 3Kq + K$, so that $\min(i, D_q) = i$. By Lemma 5.11 Part 3 , for $t \in \left[0, \frac{1}{2D_q^2}\right]$,

$$\frac{1}{(i-j)!}\left|\left(\frac{1}{g_{q,K}}\right)^{(i-j)}(t)\right| \leq 2^{2(i-j)+1}\left(K^2q^3 + q(2K-1)^2\right)^{i-j}.$$

Combining this with (63) we obtain that, in the range $t \in \left[0, \frac{1}{2D_q^2}\right]$,

$$\begin{aligned}\left|\frac{\phi_x^{(i)}}{i!}\right| &\leq 4\|x\|_{C^*(F_r)} \sum_{j=0}^{i}\left(\frac{CD_q^4}{j^2}\right)^j 4^{(i-j)}\left(K^2q^3 + q(2K-1)^2\right)^{i-j} \\ &\leq 4\left(CD_q^2\right)^i \|x\|_{C^*(F_r)} \sum_{j=0}^{i}\left(\frac{D_q^2}{j^2}\right)^j,\end{aligned}$$

where the final inequality follows from the fact that $k^2q^3 + q(2K-1)^2 \leq D_q^2$. Then observe that, for each j,

$$\frac{\left(\frac{D_q^2}{j^2}\right)^j}{\left(\frac{D_q^2}{(j+1)^2}\right)^{j+1}} = \left(\frac{j+1}{j}\right)^{2j}\left(\frac{j+1}{D_q}\right)^2 \leq e^2,$$

so that $\sum_{j=0}^{i}\left(\frac{D_q^2}{j^2}\right)^j \leq (1+e^2)^i\left(\frac{D_q^2}{i^2}\right)^i$, which proves the lemma. ☐

For each integer $i \geq 0$, we define a map

$$\psi_i : x \in \mathbb{C}[F_r] \mapsto \frac{\phi_x^{(K+i)}(0)}{(K+i)!} \in \mathbb{C}.$$

We want to show that ψ_i is tempered for each $i < K$ and that it satisfies the polynomial bound property in Proposition 5.7.

Lemma 5.13. *For every integer i with $0 \leq i \leq K$, there is a polynomial P of degree $4K + 4i + 1$ such that $|\psi_i(x)| \leq P_n(q) \|x\|_{C^*(F_r)}$ for every q and for every $x \in \mathbb{C}_{\leq q}[F_r]$.*

Proof. By Lemma 5.12, for each q and for each $x \in \mathbb{C}_{\leq q}[F_r]$, we have

$$\left| \frac{\phi_x^{(K+i)}(0)}{(K+i)!} \right| \leq h(K+i, q) \|x\|_{C^*(F_r)} . \tag{64}$$

Moreover,

$$\sup_{q \geq 1} \frac{h(K+i, q)}{(1+q^2)^{4K+4i+1}} < \infty,$$

from which the lemma follows. □

Lemma 5.14. *For any $i < K$, the function ψ_i is tempered.*

Proof. We will use Proposition 5.8 by showing that, for any reasonable probability measure μ with associated random walk $(g_n)_n$,

$$\limsup_n \left(\mathbb{E}|\psi_i(g_n)|\right)^{\frac{1}{n}} \leq \rho(\mu).$$

So, let μ be a reasonable probability measure, $(g_n)_n$ the associated random walk on F_r. If μ is supported in $\mathbb{C}_{\leq q}[F_r]$, then $g_n \in \mathbb{C}_{\leq qn}[F_r]$. Then we have

$$\begin{aligned}
&\mathbb{E}|\psi_i(g_n)| \\
\leq &C_i(1+q^2n^2)^{4K+4i+1}\mathbb{P}\left(\psi_i(g_n) \neq 0\right) \\
= &C_i(1+q^2n^2)^{4K+4i+1}\mathbb{P}\left(g_n \text{ is a proper power}\right) \\
\leq &C_iC_\mu(1+q^2n^2)^{4K+4i+1}n^5\rho(\mu)^n.
\end{aligned}$$

The first inequality follows from Lemma 5.13. Indeed, $g_n \in \mathbb{C}_{\leq qn}[F_r]$, so there is some constant C_i for which $|\psi(g_n)| \leq C_i \left(1+q^2n^2\right)^{4K+4i+1}$. The final inequality follows from

Proposition 5.4. It follows that

$$\limsup_n \left(\mathbb{E}|\psi_i(g_n)|\right)^{\frac{1}{n}} \leq \rho(\mu),$$

so by Proposition 5.8, ψ_i is tempered. □

The next step is showing that the second condition in Proposition 5.7 holds. We will need the following lemma.

Lemma 5.15. *For any $\epsilon > 0$, we have $\sup_{q\geq 1} h(2K,q)\exp\left(-\frac{q}{\log(2+q)^2}\right) \leq \left(C^2K^{20+\epsilon}\right)^K$.*

Proof. Fix $\epsilon > 0$. Given any $b > 1$, there exists $a > 0$ such that

$$\log{(2+q)}^2 \leq q^{1/b}$$

for all $q > a$.

So, for $q > a$, we have

$$4\left(CD_q^4\right)^{2K}\exp\left(\frac{-q}{\log{(2+q)}^2}\right) \leq 4\left(C\left(Kq^2+3Kq+K\right)^4\right)^{2K}\exp\left(-q^{\frac{b-1}{b}}\right) \tag{65}$$

which is bounded above by

$$\left(C^2K^{20+\epsilon}\right)^K \tag{66}$$

for sufficiently large b.[12]

For $q \leq a$, we obtain the bound

$$\left(C'K^2\right)^{2K}$$

by evaluating $h(2K,a)$ and ignoring the exponential term (since it is ≤ 1). This is less than (66) (when C is large enough) from which the lemma follows. □

We will also need the following observation to be used in Lemma 5.16, as well as our

[12] To see this, differentiate the RHS of (65) to see that the maximum is obtained around $q = CK^{\frac{b}{b-1}}$ and substitute this into $h(2K,q)$.

proof of Theorem 1.14 – the map

$$u : x \in \mathbb{C}\left[F_r\right] \mapsto \tau(x)\frac{\phi_e^{(K)}(0)}{K!} \in \mathbb{C}$$

is tempered. Indeed, if we write $\dim \Sigma_{n,K} = a_0 + a_1 n + \cdots + a_k n^K$, one sees that $\frac{\phi_e^{(K)}(0)}{K!} = a_0$, a constant (in fact, this constant is always $(-1)^K$), and our observation follows from the fact that $x \mapsto \tau(x)$ is tempered. Moreover, this map obviously satisfies the polynomial bound $|u(x)| \leq \|x\|_{C^*(F_r)}$.

Lemma 5.16. *Let* $w(q) = \exp\left(\frac{q}{\log(2+q)^2}\right)$. *Then, for every* $q \geq 1$, *for every* $n \geq Kq$ *and for every* $x \in \mathbb{C}_{\leq q}\left[F_r\right]$, *and any* $\epsilon > 0$,

$$\left| \mathbb{E}\left[\mathrm{Tr}\big(\Sigma_{n,K}\left(x(\sigma_1, \ldots, \sigma_r)\right) - \tau(x)\mathrm{Id}\big)\right] + \tau(x)\frac{\phi_e^{(K)}(0)}{K!} - \sum_{i=0}^{K-1} \frac{\psi_i(x)}{n^i} \right| \tag{67}$$

is bounded by

$$\frac{\left(C^2 K^{20+\epsilon}\right)^K}{n^K} w(q) \left\|x\right\|_{C^*(F_r)}.$$

Proof. We have

$$\mathbb{E}\left[\mathrm{Tr}\left(\tau(x)\mathrm{Id}\right)\right] = \tau(x) \dim \Sigma_{n,K} = \tau(x) n^K \sum_{i=0}^{K-1} \frac{\phi_e^{(i)}(0)}{i! n^i} + \tau(x)\frac{\phi_e^{(K)}(0)}{K!}$$

and, since $n \geq Kq$, by Part 3 of Theorem 5.10, we have

$$\sum_{i=0}^{K-1} \frac{\psi_i(x)}{n^i} = n^K \sum_{i=K}^{2K-1} \frac{\phi_x^{(i)}(0)}{i! n^i} = n^K \left[\sum_{w \neq e} x(w) \sum_{i=0}^{2K-1} \frac{\phi_w^{(i)}(0)}{i! n^i} + \tau(x) \sum_{i=K}^{2K-1} \frac{\phi_e^{(i)}(0)}{i! n^i} \right].$$

Combining these observations with Part 1 of Theorem 5.10, we see that the LHS of (67) is equal to

$$n^K \left| \phi_x\left(\frac{1}{n}\right) - \sum_{i=0}^{2K-1} \frac{\phi_x^{(i)}(0)}{i! n^i} \right|.$$

By Taylor's inequality, this is less than or equal to

$$\frac{n^K}{n^{2K}(2K)!} \left| \phi_x^{(2K)}\left(\frac{1}{n}\right) \right|.$$

If we further assume that $n \geq 2D_q^2$, then by Lemma 5.12, this is bounded by

$$\frac{h(2K,q)}{n^K} \|x\|_{C^*(F_r)} . \tag{68}$$

If $n \leq 2D_q^2$, then we can still bound the left hand side of (67) by

$$(2 \dim (\Sigma_{n,K}) + 1) \|x\|_{C^*(F_r)} + \sum_{i=0}^{K-1} \frac{|\psi_i(x)|}{n^i}$$

by using the triangle inequality. By (62) and Lemma 5.12, this is bounded above by

$$n^K \left[h(0,q) + \sum_{i=0}^{K-1} \frac{1}{n^{K+i}} h(K+i,q) \right] \|x\|_{C^*(F_r)} .$$

This is less than (68) whenever the constant C is large enough.

By Lemma 5.15, we have

$$\sup_{q \geq 1} h(2K,q) \exp\left(-\frac{q}{\log(2+q)^2} \right) \leq \left(C^2 K^{20+\epsilon} \right)^K$$

and the lemma follows. □

Proof of Theorem 1.14. Fix $\alpha < \frac{1}{20}$, say $\alpha = \frac{1}{20} - \epsilon'$, with $0 < \epsilon' \leq \frac{1}{20}$. For each n, and for any $K \leq n^\alpha$, let $\Pi_{n,K}$ be a random representation of F_r given by

$$\Pi_{n,K}(w) = \Sigma_{n,K} \left(w \left(\sigma_1, \ldots, \sigma_r \right) \right) .$$

Then, by Lemma 5.16, for every q and every $x \in \mathbb{C}_{\leq q} [F_r]$, we have

$$\left| \mathbb{E}\mathrm{Tr} \left(\Pi_{n,K}(x) \right) - T_n(x) \right| \leq \epsilon_n w(q) \|x\|_{C^*(F_r)} ,$$

where, for some ϵ satisfying $0 < \epsilon < \frac{20\epsilon'}{\frac{1}{20} - \epsilon'}$, we have

$$\epsilon_n = \frac{\left(C^2 K^{20+\epsilon} \right)^K}{n^K} \leq \left(\frac{C^2}{n^{1-(20+\epsilon)\alpha}} \right)^K$$

and

$$T_n(x) = \dim(\Sigma_{n,K})\,\tau(x) - \tau(x)\frac{\phi_e^{(K)}(0)}{K!} + \sum_{i=0}^{K-1} \frac{\psi_i(x)}{n^i}.$$

By Lemma 5.14, T_n is tempered, since it is a finite sum of tempered functions. Additionally, T_n satisfies the polynomial bound

$$|T_n(x)| \leq P_n(q)\,\|x\|_{C^*(F_r)}\,.$$

So by Proposition 5.7, with T_n in place of u_n, $\forall \delta > 0$, and any $z \in \mathbb{C}[F_r]$,

$$\mathbb{P}\left[\|\Pi_{n,K}(z)\| > \|z\|_{C^*_\lambda(F_r)} + \delta\right] \leq C(z,\delta)\left(\frac{C^2}{n^{1-(20+\epsilon)\alpha}}\right)^K.$$

For $k_n \leq n^\alpha$, we have

$$\begin{aligned} &\mathbb{P}\left[\|\pi_{n,k_n}(z)\| > \|z\|_{C^*_\lambda(F_r)} + \delta\right] \\ \leq &\sum_{1\leq K\leq n^\alpha} \mathbb{P}\left[\|\Pi_{n,K}(z)\| > \|z\|_{C^*_\lambda(F_r)} + \delta\right]. \end{aligned} \tag{69}$$

Write $A = 1-(20+\epsilon)\alpha$, which is > 0 by our choice of ϵ. Then the RHS of (69) is equal to

$$C(z,\delta)\left(\frac{C^2}{n^A}\right)\left(\frac{1-\left(\frac{C^2}{n^A}\right)^{n^\alpha - 1}}{1-\left(\frac{C^2}{n^A}\right)}\right),$$

which $\to 0$ as $n \to \infty$. So, for any $k_n \leq n^\alpha$,

$$\mathbb{P}\left[\|\pi_{n,k_n}(z)\| > \|z\|_{C^*_\lambda(F_r)} + \delta\right] \overset{n\to\infty}{\longrightarrow} 0. \tag{70}$$

By [MdlS24, Lemma 5.14], there exist $y_1, \ldots, y_m \in \mathbb{C}[F_r]$ and $\delta' = \delta'(\delta)$, such that if, for every i,

$$\|\pi_{n,k_n}(y_i)\| \leq \|y_i\|_{C^*_\lambda(F_r)} + \min(\delta, \delta'), \tag{71}$$

then

$$\|\pi_{n,k_n}(z)\| \geq \|z\|_{C^*_\lambda(F_r)} - \delta. \tag{72}$$

By (70), we know that (71) holds with probability $\to 1$ as $n \to \infty$, so then (72) also holds with probability $\to 1$ as $n \to \infty$. Therefore,

$$\mathbb{P}\left[\sup_{k_n \le n^\alpha} \left| \left\|\pi_{n,k_n}(z)\right\| - \|z\|_{C^*_\lambda(F_r)} \right| > \delta\right] \overset{n\to\infty}{\longrightarrow} 0.$$

□

Proof of Corollary 1.15. By e.g. [Eti14, Lemma 2.8][13], if ρ is a non–trivial irreducible representation of S_n of dimension $\le Cn^{n^\alpha}$, then ρ is either a subrepresentation of our ρ_{n,k_n} in Theorem 1.14, in which case the corollary is trivial, or ρ' is a subrepresentation of ρ_{n,k_n} (where ρ' is the conjugate representation obtained by swapping the rows and columns of the Young diagram associated to ρ).

For any $z = \sum_w z(w)w \in \mathbb{C}[F_r]$ and any $b_w \in \{1, -1\}$, define $z_{b(w)} = \sum_w b_w z(w)w$. Since $z \in C[F_r]$, there are only finitely many $z(w)$ that are non–zero, say M. So then there are at most 2^M possible z_b and so by a uniform bound, $\pi_n = \rho \circ \phi_n \overset{\text{strong}}{\longrightarrow} \lambda$ a.a.s. by Theorem 1.14. Indeed, consider $\phi_n(w) = w(\sigma_1, \dots, \sigma_r)$ for $\sigma_1, \dots, \sigma_r \in S_n$. Then

$$\rho(z(\sigma_1, \dots, \sigma_r)) = \sum_w z(w)\text{sign}(w(\sigma_1, \dots, \sigma_r))\rho'(w(\sigma_1, \dots, \sigma_r)) = \rho'(z_{b(w)}(\sigma_1, \dots, \sigma_r))$$

where $b(w) = \text{sign}(w(\sigma_1, \dots, \sigma_r))$ and so, by Theorem 1.14, for any $\epsilon < 0$,

$$\mathbb{P}\left[\left| \|\rho \circ \phi_n(z)\| - \left\|\lambda\left(z_{b(w)}\right)\right\| \right| < \epsilon\right] \overset{n\to\infty}{\longrightarrow} 1\,.$$

Finally, $\left\|\lambda\left(z_{b(w)}\right)\right\| = \|\lambda(z)\|$ by the Fell absorption principle (see e.g. [Pis03, Proposition 8.1]), which proves the corollary. □

6 Further questions

We discuss here some interesting further problems on the topics of word maps and expansion properties of random graphs.

[13] This is a classical fact, but one must inspect the proof of this lemma in [Eti14] to determine that this still holds for $k = k_n \le n^\alpha$.

6.1 Word maps

6.1.1 Compact groups

An immediately obvious future direction is the pursuit of Conjecture 1.10 which posits that, for any stable irreducible character χ of the symmetric group and for any $w \in F_r$, then

$$\mathbb{E}_w [\chi] = O\left(\frac{1}{(\dim \chi)^{\pi(w)-1}}\right).$$

The full conjectural picture is discussed in significant detail in [PS23], where it is conjectured that we should actually have

$$\mathbb{E}_w [\chi] = \tilde{\Theta}\left(\frac{1}{(\dim \chi)^{\mathrm{s}\pi(w)}}\right), \tag{73}$$

where $\mathrm{s}\pi(w)$ is the stable primitivity rank as defined by Wilton (see e.g. [Wil24]) and $\tilde{\Theta}$ means that $\mathbb{E}_w [\chi] = O\left(\frac{1}{(\dim \chi)^{\mathrm{s}\pi(w)}}\right)$ and that, for every w, there exists stable irreducible χ such that $\mathbb{E}_w [\chi] = O\left(\frac{1}{(\dim \chi)^{\mathrm{s}\pi(w)}}\right)$ **and** $\frac{1}{(\dim \chi)^{\mathrm{s}\pi(w)}} = O\left(\mathbb{E}_w [\chi]\right)$, i.e. that this bound is tight for some stable character(s). It is an open question whether $\mathrm{s}\pi(w) = \pi(w) - 1$ (Theorem 1.12 asserts that $\mathrm{s}\pi(w) \geq 1$ for words that are not primitive or proper powers) and establishing such an equality, or indeed (73), or Conjecture 1.10, would all be worthwhile pursuits.

Similar phenomena can be observed for natural 'stable' families of irreducible representations of compact groups. For example, for $U(n)$, one may consider the irreducible representation indexed by the weight

$$\left(\lambda_1, \ldots, \lambda_{l(\lambda)}, \underbrace{0, \ldots, 0}_{n-|\lambda-|\mu|}, -\mu_{l(\mu)}, \ldots, -\mu_1\right)$$

where λ and μ are Young diagrams of some fixed size. It can be shown that the asymptotic bound for the expected *polynomial* stable character (in this language, where $\mu = \emptyset$, so that the weight is non–negative) of a w–random unitary is controlled by the dimension and an invariant called the stable commutator length of w, denoted $\mathrm{scl}(w)$. For other compact groups, e.g. orthogonal, symplectic, $GL_n(\mathbb{F}_q)$, natural families of stable irreducible characters can be defined, as well as other natural stable invariants of words $w \in F_r$, and there are partial results and a number of conjectures (again, we urge the reader to consult [PS23]) in which bounds of the form

$$\tilde{\Theta}\left(\frac{1}{(\dim\chi)^{\mathrm{StabInvar}(w)}}\right)$$

(for some appropriately defined stable invariant, denoted $\mathrm{StabInvar}(w)$) are either already proven or are conjectured.[14] Extending the methods of this thesis using Schur–Weyl duality type results for other compact groups could be a concrete starting point to establishing bounds of the form

$$O\left(\frac{1}{\dim\chi}\right)$$

for certain irreducible stable characters χ (ones that can be isolated through Schur–Weyl duality techniques) in the unresolved cases. Even developing a version of the Weingarten calculus for some of these examples for which a version does not already exist, for example where the group structure and representation theory is not so straightforward, could be an interesting problem.

6.1.2 Discrete Groups

As we mentioned earlier, in addition to computing $\mathbb{E}\left[\chi\left(w(\sigma_1,\ldots,\sigma_r)\right]\right.$, equivalently, computing $\mathbb{E}_{\phi_n\in\mathrm{hom}(F_r,S_n)}\left[\chi\left(\phi_n(w)\right)\right]$, one could equally consider the problem of computing statistics for $\phi_n(w)$ where Γ is some discrete group and $\phi_n\in\mathrm{hom}(\Gamma,G)$ is some uniformly random homomorphism. A natural starting point, of interest also due to the connection with the the number of lifts of a geodesic C_w (in the surface) to a closed geodesic in the degree n covering space, is to compute

$$\mathbb{E}_{\phi_n\in\mathrm{hom}(\Gamma_g,S_n)}\left[\#\mathrm{fix}\left(\phi_n(w)\right)\right], \tag{74}$$

where $\Gamma_g=\langle a_1,\ldots,a_g,b_1,\ldots,b_g:\ [a_1,b_1]\ldots[a_g,b_g]\rangle$ is the fundamental group of the surface (a surface group). Such an expression is not rational in n (see e.g. [MPvH25]) but one can still determine the large n asymptotics. This can be done by other means, see [MP23], but can also be done using the projection formula in Theorem 1.6, see [MPvH25].

[14] In most cases, these asymptotic bounds are conjectures with a lot of supporting evidence, but relatively few are known to be true.

Adapting this approach to more general (torsion–free) one relator groups $\Gamma = \langle x_1, \ldots, x_m : r\rangle$ seems viable. In [MPvH25], it is asserted that the expression for (74) needn't be rational for the methods of [CGVTvH24] to be extended to prove that a random representation of the form $\pi_n = \rho \circ \phi_n$ for $\phi_n \in \hom(\Gamma_g, S_n)$ strongly converges a.a.s., and this suggests that obtaining a suitable asymptotic expansion in the case of Γ in place of Γ_g could determine that such a random sequence of representations of Γ converge strongly as well, establishing a large family of groups that have a uniformly random, strongly convergent, permutation representation. The main caveat to this is, for such an expression to be useful, one would need to improve on the combinatorial bound obtained in §4.5 of this thesis for words w in the alphabet $\{x_1^{\pm 1}, \ldots, x_r^{\pm 1}\}$ with $\pi(w) > 2$. It is conjectured that this is possible, but as yet we do not know how to obtain even a small improvement using the combinatorial argument detailed in this thesis.

6.2 Random Cayley graphs

As we mentioned before, if Corollary 1.15 were proven to be true for any $\alpha \leq 1$, then one would have established that $\pi_n = \rho \circ \phi_n$ converges strongly a.a.s. for *any* non–trivial irreducible representation of S_n. This would be a very strong statement, however, in [MPvH25], it is discussed how the left regular representation of S_n fits a certain, reasonably specific, structural requirement for $\pi_n \xrightarrow{\text{strong}} \lambda$, so that it is not necessarily beyond the realms of possibility.

Note that, if true, this would imply that random fixed degree Cayley graphs of the symmetric group are a.a.s. weakly Ramanujan, which is a significantly simpler statement, but one that is also far from known. In fact, it remains an open question whether or not these random Cayley graphs have a uniform (not necessarily near–optimal) spectral gap. Kassabov's [Kas05] result, Theorem 1.4, asserts the existence of fixed degree Cayley graphs of S_n with a uniform spectral gap as $n \to \infty$, but this question remains an interesting open problem related to the works presented in this thesis.

References

[Alo86] Noga Alon, *Eigenvalues and expanders*, Combinatorica **6** (1986), no. 2, 83–96.

[AM85] Noga Alon and Vitali D Milman, λ_1*, isoperimetric inequalities for graphs, and superconcentrators*, Journal of Combinatorial Theory, Series B **38** (1985), no. 1, 73–88.

[BC10] Teodor Banica and Stephen Curran, *Decomposition results for Gram matrix determinants*, Journal of Mathematical Physics **51** (2010), 1–14.

[BC19] Charles Bordenave and Benoit Collins, *Eigenvalues of random lifts and polynomials of random permutation matrices*, Annals of Mathematics **190** (2019), 811–875.

[Bor20] Charles Bordenave, *A new proof of Friedman's second eigenvalue theorem and its extension to random lifts*, Annales Scientifiques de l'École Normale Supérieure, vol. 4, 2020, pp. 1393–1439.

[Cas25a] Ewan Cassidy, *Projection formulas and a refinement of Schur–Weyl–Jones duality for symmetric groups*, preprint, arXiv 2312.01839 (2025).

[Cas25b] ———, *Random permutations acting on k–tuples have near–optimal spectral gap for $k = \mathrm{poly}(n)$*, preprint, arXiv 2412.13941 (2025).

[CGVTvH24] Chi-Fang Chen, Jorge Garza-Vargas, Joel A Tropp, and Ramon van Handel, *A new approach to strong convergence*, Annals of Mathematics, to appear (2024).

[CGVvH24] Chi-Fang Chen, Jorge Garza-Vargas, and Ramon van Handel, *A new approach to strong convergence II. the classical ensembles*, preprint, arXiv 2412.00593 (2024).

[CM14] Benoit Collins and Camille Male, *The strong asymptotic freeness of Haar and deterministic matrices*, Annales scientifiques de l'École Normale Supérieure, vol. 47, 2014, pp. 147–163.

[CMN21] Benoit Collins, Sho Matsumoto, and Jonathan Novak, *The Weingarten calculus*, Notices of the American Math. Soc **69** (2021), 734–745.

[COS+22] Laura Colmenarejo, Rosa Orellana, Franco Saliola, Anne Schilling, and Mike Zabrocki, *The mystery of plethysm coefficients*, Open Problems in Algebraic Combinatorics **110** (2022), 275–292.

[Eti14] Pavel Etingof, *Representation theory in complex rank, I*, Transformation Groups **19** (2014), 359–381.

[EWPS24] Danielle Ernst-West, Doron Puder, and Matan Seidel, *Word measures on $GL_n(q)$ and free group algebras*, Algebra & Number Theory **18** (2024), no. 11, 2047–2090.

[FH04] William Fulton and Joe Harris, *Representation theory: A first course*, Springer Science and Business Media, Inc, New York, 2004.

[FJR+98] Joel Friedman, Antoine Joux, Yuval Roichman, Jacques Stern, and Jean-Pierre Tillich, *The action of a few permutations on r-tuples is quickly transitive*, Random Structures & Algorithms **12** (1998), no. 4, 335–350.

[Fri03] Joel Friedman, *Relative expanders or weakly relatively Ramanujan graphs*, Duke Mathematical Journal **118** (2003), no. 1, 19 – 35.

[Fri08] ______, *A proof of Alon's second eigenvalue conjecture and related problems*, American Mathematical Soc., 2008.

[Fro00] Ferdinand Georg Frobenius, *Über die charaktere der symmetrischen gruppe*, Sitzungberichte der Koniglich Preussichen Akademie der Wissenschaften zu Berlin (1900), 515–534.

[Haa78] Uffe Haagerup, *An example of a non nuclear C^*–algebra, which has the metric approximation property*, Inventiones Mathematicae **50** (1978), no. 3, 279–293.

[HM23] Will Hide and Michael Magee, *Near optimal spectral gaps for hyperbolic surfaces*, Annals of Mathematics **198** (2023), no. 2, 791–824.

[HP23] Liam Hanany and Doron Puder, *Word measures on symmetric groups*, International Mathematics Research Notices **11** (2023), 9221–9297.

[HT05] Uffe Haagerup and Steen Thorbjørnsen, *A new application of random matrices: Ext($C^*_{red}(F_2)$) is not a group*, Annals of Mathematics (2005), 711–775.

[Hum96] John . F. Humphreys, *A course in group theory*, Oxford University Press Inc, New York, 1996.

[Jon94] Vaughan Frederick Randal Jones, *The Potts model and the symmetric group*, Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras (1994), 259–267.

[Kas05] Martin Kassabov, *Symmetric groups and expander graphs*, Inventiones Mathematicae **170** (2005), 327–354.

[Koi89] Kazuhiko Koike, *On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters*, Advances in Mathematics **74** (1989), 57–86.

[KT87] Kazuhiko Koike and Itaru Terada, *Littlewood's formulas and their application to representations of classical Weyl groups*, Commutative Algebra and Combinatorics (1987), 147–160.

[Lit58] Dudley E Littlewood, *Products and plethysms of characters with orthogonal, symplectic and symmetric groups*, Canadian Journal of Mathematics **10** (1958), 17–32.

[LP10] Nathan Linial and Doron Puder, *Word maps and spectra of random graph lifts*, Random Structures & Algorithms **37** (2010), 100–135.

[LW17] Larsen Louder and Henry Wilton, *Stackings and the w–cycles conjecture*, Canadian Mathematical Bulletin **60** (2017), no. 3, 604–612.

[Mag22] Michael Magee, *Random unitary representations of surface groups I: Asymptotic expansions*, Communications in Mathematical Physics **391** (2022), no. 1, 119–171.

[Mag25a] Michael Magee, *Random unitary representations of surface groups II: The large n limit*, Geometry and Toplology (2025), 1237–1281.

[Mag25b] Michael Magee, *Strong convergence of unitary and permutation representations of discrete groups*, Proceedings 9ECM, to appear (2025).

[Mar94] Paul Martin, *Temperley–Lieb algebras for non–planar statistical mechanics – the partition algebra construction*, Journal of Knot Theory and Its Ramifications **03** (1994), 51–82.

[Mar96] ______, *The structure of the partition algebras*, J. Algebra **183** (1996), 319–358.

[MdlS24] Michael Magee and Mikael de la Salle, *Strong asymptotic freeness of Haar unitaries in quasi–exponential dimensional representations*, preprint, arXiv 2409.03626 (2024).

[MP19] Michael Magee and Doron Puder, *Matrix group integrals, surfaces and mapping class groups I: $U(n)$*, Inventiones Mathematicae **218** (2019), 341–411.

[MP22] ______, *Matrix group integrals, surfaces and mapping class groups II: $O(n)$ and $Sp(n)$*, Mathematische Annalen (2022), 1437–1494.

[MP23] ______, *The asymptotic statistics of random covering surfaces*, Forum of Mathematics, Pi **11** (2023), e15.

[MPvH25] Michael Magee, Doron Puder, and Ramon van Handel, *Strong convergence of uniformly random permutation representations of surface groups*, preprint, arXiv 2504.08988 (2025).

[MS93] Paul Martin and Herbert Saleur, *On an algebraic approach to higher dimensional statistical mechanics*, Communications in Mathematical Physics **158** (1993), 155–190.

[MŚS07] James A Mingo, Piotr Śniady, and Roland Speicher, *Second order freeness and fluctuations of random matrices: II. unitary random matrices*, Advances in Mathematics **209** (2007), no. 1, 212–240.

[Mus93] C. Musili, *Representations of finite groups*, vol. 8, Hindustan Book Agency, Gurgaon, 1993.

[Nic94] Alexandru Nica, *On the number of cycles of given length of a free word in several random permutations*, Random Structures & Algorithms **5** (1994), no. 5, 703–730.

[Nil91] Alon Nilli, *On the second eigenvalue of a graph*, Discrete Mathematics **91** (1991), no. 2, 207–210.

[NS06] Alexandru Nica and Roland Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, Cambridge University Press, 2006.

[Pis03] Gilles Pisier, *Introduction to operator space theory*, no. 294, Cambridge University Press, 2003.

[Pis19] ———, *On a linearization trick*, L'Enseignement Mathématique **64** (2019), no. 3, 315–326.

[PP15] Doron Puder and Ori Parzanchevski, *Measure preserving words are primitive*, Journal of the American Mathematical Society **28** (2015), 63–97.

[PS23] Doron Puder and Yotam Shomroni, *Stable invariants and their role in word measures on groups*, preprint, arXiv 2311.17733 (2023).

[Pud14] Doron Puder, *Primitive words, free factors and measure preservation*, Israel Journal of Mathematics **201** (2014), 25–73.

[Pud15] ———, *Expansion of random graphs: New proofs, new results*, Inventiones Mathematicae **201** (2015), no. 3, 845–908.

[Rot64] Gian-Carlo Rota, *On the foundations of combinatorial theory I. theory of Möbius functions*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **2** (1964), 340–368.

[Sch01] Issai Schur, *Ueber eine klasse von matrizen, die sich einer gegebenen matrix zuordnen lassen*, 1901.

[Sch27] ———, *Uber die rationalen darstellungen der allgemeinen linearen gruppe*, 1927.

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag New York, Inc, New York, 1977.

[SS15] Steven V Sam and Andrew Snowden, *Stability patterns in representation theory*, Forum of Mathematics, Sigma, vol. 3, Cambridge University Press, 2015, p. e11.

[VO05] Anatoly Vershik and Andrei Okounkov, *A new approach to the representation theory of the symmetric groups. II*, Journal of Mathematical Sciences **131** (2005), 5471–5494.

[Voi91] Dan Voiculescu, *Limit laws for random matrices and free products*, Inventiones Mathematicae **104** (1991), 201–220.

[Wil24] Henry Wilton, *Rational curvature invariants for 2–complexes*, Proceedings of the Royal Society A **480** (2024), no. 2296, 20240025.