A Representation Theory for Categorical Symmetries

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Für Mama und Papa.

ABSTRACT

Symmetries play a fundamental role in our understanding of nature. While traditionally described by the theory of groups and their representations, recent years have seen a vast generalisation of the notion of symmetry, leading to so-called generalised (or categorical) symmetries in quantum field theory. In this thesis, we examine the mathematical structure that underlies such generalised symmetries and develop a representation theory that captures their action on physical observables. We focus on the case of finite bosonic symmetries in low spacetime dimensions, where the appropriate mathematical framework is given by the theory of (higher) fusion categories. We construct higher-dimensional analogues of Ocneanu's tube algebra and classify their higher representations using the so-called sandwich construction (or Symmetry TFT) for categorical symmetries. We provide explicit examples that include both anomalous group-like symmetries as well as non-invertible symmetries.

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Lastly, I would like to express my deepest gratitude towards my family, in particular my parents, Marianne and Michael, for their love and continued support throughout my academic journey and for always giving me a warm welcome when going back home. I dedicate this thesis to you.

DECLARATION

This thesis is being submitted for the degree of ‘Doctor of Philosophy’. I declare that this thesis is the result of my own work and has been written by me in its entirety. Except where stated otherwise, the findings of this thesis are based on work done in collaboration and published in the papers

[1] T. Bartsch, M. Bullimore, A. Grigoletto : *Higher representations for extended operators*. Preprint (Apr. 2023). arXiv: 2304.03789 [hep-th],

[2] T. Bartsch, M. Bullimore, A. Grigoletto : *Representation theory for categorical symmetries*. Preprint (May 2023). arXiv: 2305.17165 [hep-th],

as well as my own original work published in the papers

[3] T. Bartsch : *On Unitary 2-Group Symmetries*. Preprint (Nov. 2024). arXiv: 2411.05067 [math-ph],

[4] T. Bartsch : *Unitary Categorical Symmetries*. Preprint (Feb. 2025). arXiv: 2502.04440 [hep-th].

I have acknowledged all other sources of information which have been used in this thesis. During my doctoral studies, I further contributed to the publications

[5] T. Bartsch, M. Bullimore, A. E. V. Ferrari, J. Pearson: *Non-invertible symmetries and higher representation theory I.* In: *SciPost Phys.* 17.1 (2024), p. 015. arXiv: 2208.05993 [hep-th],

[6] T. Bartsch, M. Bullimore, A. E. V. Ferrari, J. Pearson: *Non-invertible symmetries and higher representation theory II.* In: *SciPost Phys.* 17.2 (2024), p. 067. arXiv: 2212.07393 [hep-th].

I declare that no substantial part of this thesis has been previously submitted or is currently being submitted for any other degree, diploma, or similar qualification at the University of Durham or any other university or similar institution.

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Durham, July 2025

Thomas Bartsch

CONTENTS

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INTRODUCTION

Symmetries provide a powerful tool for our understanding of nature. They reveal the patterns and regularities that govern the behaviour of a physical system and thereby reduce the number of variables needed for its theoretical description. They constrain the ways a system can evolve over time, leading to predictions about future outcomes of physical processes. The exploitation of symmetry-based principles in the study and construction of new theoretical models has been one of the driving forces in advancing our understanding of nature over the past century.

While the concept of symmetry was known to the Greeks, its rigorous mathematical treatment began during the nineteenth century with the development of *group theory* [7]. The latter captures symmetries as certain types of transformations that leave a given object "invariant", allowing for the notion of symmetry to be applied not only to geometrical figures but also to more abstract objects such as mathematical equations describing the dynamics of a physical system. For example, Newton's laws of motion are invariant with respect to Galilei transformations, while Maxwell's equations of electrodynamics are invariant under Lorentz transformations. With the discovery of relativity and quantum theory at the beginning of the twentieth century, we arrived at our modern understanding of symmetries in fundamental physics [8], according to which they play the following twofold role in both classical and quantum theory:

1. Given a physical system, identifying its symmetries allows us to simplify the theoretical description and constrain the dynamics of the system. According to *Noether's theorem* [9], every continuous symmetry leads to a conservation law, which puts restrictions on the way the system can evolve over time. For example, if a system possesses a time translation symmetry (meaning that the laws governing its behaviour do not change over time), the associated conserved quantity is the *energy* of the system. In quantum theory, symmetries lead to conserved operators, which act on and thereby organise the spectrum of physical observables, leading to selection rules and constraints on possible quantum transitions.
2. In the search for new theoretical models and descriptions of nature, imposing symmetry principles often constrains and dictates the form that natural laws can take. This point of view was initiated by the discovery of special relativity

by Einstein in 1905 [10], who promoted the Lorentz symmetries of Maxwell's equations to isometries of spacetime itself. As a result, Maxwell's equations were fixed by the requirement to be compatible with the geometry of space and time. Ten years later, Einstein used the principle of equivalence (i.e. the assumption that the laws of nature are invariant under local changes of spacetime coordinates) to derive the dynamical laws of gravity, leading to his celebrated general theory of relativity [11]. On the quantum side, the implementation of local symmetries resulted in the development of non-abelian gauge theory and culminated in the construction of the standard model of particle physics in the 1970s, which dictates and unifies the structure of the strong, weak, and electromagnetic forces [12].

While group-like symmetries continue to serve as a guiding principle in the search for new physics beyond the standard model, recent years have seen the discovery of a new, generalised type of symmetry as described in the seminal work [13]. Although no longer captured by the theory of groups, these "generalised symmetries" act on and organise the spectrum of physical observables in a quantum system just like ordinary symmetries, leading to new selection rules and constraints on the dynamics of the system. This raises the following question: What is the appropriate mathematical structure that replaces group theory as a descriptor for generalised symmetries and how does it act on physical observables in a quantum theory? Addressing this question in simple cases is the aim of this thesis.

0.1 Motivation

The term *symmetry* derives from the Greek words *sun* (meaning 'with' or 'together') and *metron* ('measure') and initially referred to two things being measurable or comparable by a common standard [7]. More generally, the ancient notion of symmetry used by the Greeks and Romans referred to a relation between two things that turns their union into a harmonious and balanced whole. For example, the two sides of the left image in Figure 1 are mirror images of one another, which makes their union symmetric. On the other hand, the right image is perceived asymmetric due to the lack of such relation between its two sides.

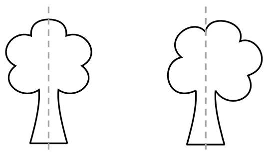


Figure 1

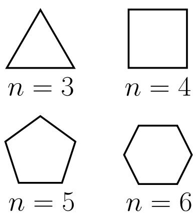
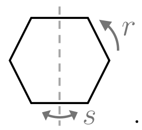


Figure 2

With the beginning of the seventeenth century, a new notion of symmetry (sometimes called the *crystallographic* notion of symmetry) developed, according to which symmetries correspond to certain types of transformations that leave a given object 'invariant' in an appropriate sense [7]. As an example, consider the regular polygon $\mathbb{P}_n$ with n edges (the cases $n = 3, ..., 6$ are illustrated in Figure 2). We denote by r the transformation that rotates $\mathbb{P}_n$

counterclockwise by an angle of $2\pi/n$ about its origin, and by s the transformation that reflects $\mathbb{P}_n$ about a fixed vertical axis as shown below for the case $n = 6$:



(0.1)

Clearly, applying either r or s to $\mathbb{P}_n$ leaves the appearance of the latter unaltered, which is why they are called symmetry transformations of $\mathbb{P}_n$. We can obtain other symmetry transformations by applying r and s repeatedly to $\mathbb{P}_n$ in different orders. However, the resulting transformations are not all independent of one another: Rotating by $2\pi/n$ for a total of n times is equivalent to not rotating at all; reflecting about a vertical axis twice is the same as not reflecting at all. Furthermore, one can convince oneself that a rotation followed by a reflection followed by a rotation is the same operation as a single reflection. Formally, we write these relations as

$$r^n = s^2 = 1\,, \qquad r \cdot s \cdot r = s\,, \tag{0.2}$$

where we denoted by 1 the transformation that does nothing to $\mathbb{P}_n$ at all. We denote by D_{2n} the set of all symmetry transformations of $\mathbb{P}_n$ that are generated by r and s subject to the relations (0.2). Formally, we define

$$D_{2n} := \langle\, r, s \mid r^n = s^2 = 1,\ rsr = s \,\rangle\,, \tag{0.3}$$

which, as one can check, is a finite set of cardinality $|D_{2n}| = 2n$. Furthermore, the set D_{2n} of symmetry transformations of $\mathbb{P}_n$ has the following properties:

- We can 'compose' any two elements g and h to obtain a new element $g \cdot h$ which as a symmetry transformation corresponds to the consecutive application of the transformations g and h.
- We can 'undo' the symmetry transformation associated to each element g in the sense that there exists an element g^{-1} which is such that the composition of g and g^{-1} gives the 'trivial' symmetry transformation 1.

The starting point for the development of our modern understanding of symmetries was the axiomatisation of the above structures underlying symmetry transformations during the early nineteenth century, which led to the mathematical notion of a *group*:

Definition: A *group* is a set G that is equipped with a binary operation $\cdot : G \times G \to G$ (called the *group multiplication* and denoted by $(g, h) \mapsto g \cdot h$) such that

1. the group multiplication is *associative*, i.e. $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for all $g, h, k \in G$,

2. there exists a distinguished element $1 \in G$ (called the *identity element*) such that $1 \cdot g \;=\; g \cdot 1 \;=\; g$ for all $g \in G$,

3. for every group element $g \in G$ there exists a unique $g^{-1} \in G$ (called the *inverse* of g) such that $g \cdot g^{-1} \;=\; g^{-1} \cdot g \;=\; 1$.

A group G is called *abelian* if its associated group multiplication is *commutative*, i.e. $g{\cdot}h \;=\; h{\cdot}g$ for all $g, h \in G$. A *group homomorphism* between groups G and G' is a map $f : G \to G'$ such that $f(g \cdot h) = f(g) \cdot f(h)$ for all $g, h \in G$. A group homomorphism is called a *group isomorphism* if it is a bijection. Two groups G and G' are said to be *isomorphic* ($G \cong G'$) if there exists a group isomorphism between them.

The above notion of a group captures the abstract properties that we expect symmetry transformations to have and puts them into a well-defined mathematical framework. In general, we distinguish between the following two types of groups:

- **Discrete:** A group G is said to be *discrete* if it does not contain any limit points, i.e. only consists of isolated group elements as illustrated in Figure 3. An important subset of discrete groups is given by *finite* groups, which only contain finitely many elements.

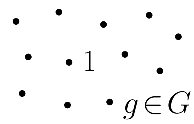


Figure 3

- **Continuous:** A group G is said to be *continuous* (or a *Lie group*) if its elements can be labelled by a set of real parameters λ_i that turn G into a smooth manifold. The *Lie algebra* $\mathfrak{g}$ of G is then defined to be the tangent space to G at the identity element; $\mathfrak{g} := T_1 G$ (see Figure 4). The usefulness of the latter is due to the existence of an exponential map

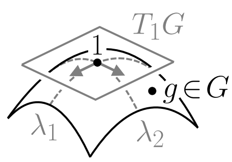


Figure 4

$$\exp : \; \mathfrak{g} \;\to\; G \; , \tag{0.4}$$

which is a local diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of $1 \in G$. As a result, we can write 'infinitesimal' group transformations $g \in G$ close to the identity as $g = e^{\varepsilon}$ for some unique $\varepsilon \in \mathfrak{g}$ close to 0, which allows us to *linearise* computations inside G. For example, the product of two group elements e^{ε} and e^{η} can be computed using the *Baker-Campbell-Hausdorff formula*

$$e^{\varepsilon} \cdot e^{\eta} \;=\; \exp\Big(\varepsilon \,+\, \eta \,+\, \frac{1}{2}\,[\varepsilon, \eta] \,+\, ...\Big) \; , \tag{0.5}$$

where $[.\,,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the *Lie bracket* on $\mathfrak{g}$. The *adjoint representation* of G is defined to be the map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ that maps a group element $g \in G$ to the differential $D_1({}^{g}(.)) : \mathfrak{g} \to \mathfrak{g}$ of the conjugation map ${}^{g}(.) : h \mapsto ghg^{-1}$.

Examples of (discrete and continuous) groups that will be used throughout this thesis repeatedly include the following:

- The set D_{2n} of symmetry transformations of $\mathbb{P}_n$ as defined in (0.3) forms a finite group called the *dihedral group of order* $2n$, which is non-abelian for $n > 2$. It contains as a subgroup the finite abelian group $\mathbb{Z}_n$, which has a single generator r subject to the relation $r^n = 1$.

- Given two groups G and G', we can construct their *direct product*, which as a set is given by $G \times G'$ with group multiplication $(g, g') \cdot (h, h') := (g \cdot h, g' \cdot h')$. According to the fundamental theorem of finite abelian groups, every finite abelian group A is of the form $A \cong \bigtimes_{i=1}^{n} \mathbb{Z}_{k_i}$ for some $n, k_i \in \mathbb{N}$ [14].

- Given a set X, we denote by $\mathrm{Aut}(X)$ the *automorphism group of* X, which consists of all bijections $f : X \to X$ with group multiplication given by composition. If X is equipped with additional structure, we often implicitly assume $\mathrm{Aut}(X)$ to consist of only those bijections f that are compatible with this structure in an appropriate sense. For example, if X is a vector space, we take f to be a *linear* automorphism of X, etc.

- For each $n \in \mathbb{N}$, we call $S_n := \mathrm{Aut}([n])$ the *symmetric group of degree* n, which consists of all permutations of the finite set $[n] := \{1, ..., n\}$. This group is abelian for $n = 2$ (where $S_2 \cong \mathbb{Z}_2$) and non-abelian for $n > 2$ (e.g. $S_3 \cong D_6$).

- Let M be a Riemannian manifold, i.e. a smooth manifold that is equipped with a symmetric, non-degenerate, and positive-definite 2-tensor $(.,.) \in T^2M$ (also called a *metric*). The latter allows us to compute the length of any smooth curve $\gamma : [a, b] \to M$ via the formula

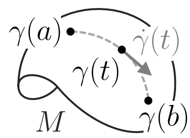


Figure 5

$$L[\gamma] \; := \; \int_a^b \|\dot{\gamma}(t)\| \; dt \; , \tag{0.6}$$

where we denoted by $\dot{\gamma}$ the tangent to the curve γ as illustrated in Figure 5 and $\|\dot{\gamma}(t)\| := (\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}$. A diffeomorphism $f : M \to M$ is called an *isometry* of M if it preserves the length of any smooth curve, i.e.

$$L[f \circ \gamma] \; = \; L[\gamma] \tag{0.7}$$

for all γ. We denote by $\mathrm{Iso}(M)$ the set of all isometries of M, which (for connected M) forms a Lie group under composition called the *isometry group* of M [15]. As an example, consider $M = \mathbb{R}^n$ with the standard metric tensor

$$(v, w) \; = \; \sum_{i=1}^{n} v_i \cdot w_i \tag{0.8}$$

for $v, w \in \mathbb{R}^n$. The isometries in this case consist of the following pieces:

1. *Translations:* For fixed $a \in \mathbb{R}^n$, we denote by $f_a : \mathbb{R}^n \to \mathbb{R}^n$ the isometry that maps $x \mapsto x + a$ corresponding to a 'translation by a'. Using $f_a \circ f_b = f_{a+b}$, we see that translations form a subgroup of $\mathrm{Iso}(\mathbb{R}^n)$ isomorphic to $\mathbb{R}^n$ with group multiplication given by addition.
2. *Orthogonal Transformations:* Given a square matrix[1] $R \in M_n(\mathbb{R})$, the map $f_R : \mathbb{R}^n \to \mathbb{R}^n$ that sends $x \mapsto R \cdot x$ is an isometry of $\mathbb{R}^n$ if and only if

$$R^T \cdot R \;=\; \mathbb{1}_n \;, \tag{0.9}$$

where ^{T} denotes the transpose of a matrix and $\mathbb{1}_n$ is the identity matrix in n dimensions. A matrix R satisfying (0.9) is called an *orthogonal matrix*[2]. The set of all orthogonal matrices forms a Lie group of dimension $n(n-1)/2$ under matrix multiplication which is called the *orthogonal group* $O(n)$ in n dimensions. It has two connected components distinguished by $\det(R) = \pm 1$, the first one of which forms the so-called *special orthogonal* (or *rotation*) *group* $SO(n)$ in n dimensions.

Putting the above together, every isometry of $\mathbb{R}^n$ can be written uniquely as $f_{(a,R)} := f_a \circ f_R$ for some $a \in \mathbb{R}^n$ and $R \in O(n)$. Their composition is given by

$$f_{(a,R)} \circ f_{(b,S)} \;=\; f_{(a+R\cdot b,\, R\cdot S)} \;, \tag{0.10}$$

which shows that the isometry group of $\mathbb{R}^n$ is the *semi-direct product*

$$\mathrm{Iso}(\mathbb{R}^n) \;=\; \mathbb{R}^n \rtimes O(n) \;. \tag{0.11}$$

More generally, we may consider pseudo-Riemannian manifolds such as $\mathbb{R}^{p,q}$, which as a smooth manifold is given by $\mathbb{R}^{p+q}$ together with the (not necessarily positive-definite) metric tensor

$$(v, w) \;=\; \sum_{i=1}^{p} v_i \cdot w_i \;-\; \sum_{j=p+1}^{p+q} v_j \cdot w_j \;. \tag{0.12}$$

[1] Given an arbitrary field $\mathbb{F}$, we denote by $M_{n\times m}(\mathbb{F})$ the set of $(n \times m)$-matrices with entries in $\mathbb{F}$. Furthermore, we set $M_n(\mathbb{F}) := M_{n\times n}(\mathbb{F})$.

[2] The term 'orthogonal matrix' stems from the fact that matrices obeying (0.9) map orthogonal vectors to orthogonal vectors. However, this terminology is somewhat unfortunate since there are matrices which do not obey (0.9) and still preserve orthogonality of vectors (e.g. $c \cdot \mathbb{1}_n$ for any $c \neq 1$). Condition (0.9) in fact ensures the stronger property that the matrix R preserves *any* inner product of two vectors, i.e. $(R\cdot v, R\cdot w) = (v, w)$ for all $v, w \in \mathbb{R}^n$.

As before, the corresponding isometry group[3] is given by a semi-direct product $\mathrm{Iso}(\mathbb{R}^{p,q}) = \mathbb{R}^{p+q} \rtimes O(p,q)$ of translations and generalised orthogonal transformations. In the case $p = d-1$ and $q = 1$, we call $\mathbb{M}^d := \mathbb{R}^{d-1,1}$ *d-dimensional Minkowski space*, whose isometry group is called the *Poincaré group*. Furthermore, the subgroup $O(d-1,1)$ is called the *Lorentz group*.

- Given a Hilbert space $\mathcal{H}$, i.e. a complex vector space equipped with an inner product[4] $\langle .,. \rangle : \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$ such that the induced metric turns $\mathcal{H}$ into a complete metric space, the *unitary group* $U(\mathcal{H})$ of $\mathcal{H}$ is the subgroup of $\mathrm{Aut}(\mathcal{H})$ consisting of all invertible (bounded) linear maps $U : \mathcal{H} \to \mathcal{H}$ obeying

$$\langle U(\Phi), U(\Psi) \rangle \;=\; \langle \Phi, \Psi \rangle \tag{0.14}$$

for all $\Phi, \Psi \in \mathcal{H}$. Equivalently, $U \in U(\mathcal{H})$ if and only if $U^\dagger \circ U = U \circ U^\dagger = \mathrm{id}_{\mathcal{H}}$, where the *adjoint* $U^\dagger : \mathcal{H} \to \mathcal{H}$ of U is the unique linear map that satisfies

$$\langle U(\Phi), \Psi \rangle \;=\; \langle \Phi, U^\dagger(\Psi) \rangle \tag{0.15}$$

for all $\Phi, \Psi \in \mathcal{H}$. We set $U(n) := U(\mathbb{C}^n)$, where $\mathbb{C}^n$ denotes the canonical n-dimensional Hilbert space equipped with the standard inner product. Then, $U(n)$ is a Lie group of dimension n^2, which contains as a hypersurface the (n^2-1)-dimensional Lie group $SU(n)$ consisting of those $U \in U(n)$ that have unit determinant, i.e. $\det(U) = 1$.

While the mathematical notion of a group elegantly captures the algebraic properties we expect symmetry transformations to have, it does not explicitly represent group elements as symmetry transformations of any specific object. This raises the following question: Given an abstract group G, in what sense do its elements correspond to symmetry transformations of some object X? Mathematically, this can be addressed using the notion of *group actions*:

Definition: A *(left) group action* of a group G on a set X is a map $\triangleright : G \times X \to X$ (denoted by $(g,x) \mapsto g \triangleright x$) such that

1. the identity element of G acts trivially on X, i.e. $1 \triangleright x = x$ for all $x \in X$,

---

[3] Since the length functional L from (0.6) is not well-defined for all curves on a pseudo-Riemannian manifold due to the lack of positive definiteness of the metric, we can define isometries on a pseudo-Riemannian manifold M to be diffeomorphisms $f : M \to M$ that leave the *energy*

$$E[\gamma] \;:=\; \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 \, dt \tag{0.13}$$

of any curve $\gamma : [a,b] \to M$ invariant, i.e. $E[f \circ \gamma] = E[\gamma]$ for all γ.

[4] Given a complex vector space V, an *inner product* on V is a bilinear map $\langle .,. \rangle : V^* \times V \to \mathbb{C}$ (where V^* denotes the complex conjugate of V) that is Hermitian (i.e. $\langle v, w \rangle = \langle w, v \rangle^*$ for all $v, w \in W$) and positive definite (i.e. $\langle v, v \rangle \geq 0$ for all $v \in V$ with equality if and only if $v = 0$).

2. the consecutive action of two group elements $g, h \in G$ is compatible with the group multiplication in G, i.e. $g \triangleright (h \triangleright x) = (g \cdot h) \triangleright x$ for all $x \in X$.

Put differently, the assignment $g \mapsto g \triangleright (.)$ defines a group homomorphism $G \to \mathrm{Aut}(X)$. We call a set X together with a G-action on it a *G-set* in what follows. A *morphism of G-sets* X and X' is a map $f : X \to X'$ such that

$$f(g \triangleright x) \;=\; g \triangleright' f(x) \tag{0.16}$$

for all $g \in G$ and $x \in X$. A morphism of G-sets is called an *isomorphism* if it is a bijection. Two G-sets X and X' are said to be *isomorphic* ($X \cong X'$) if there exists an isomorphism of G-sets between them.

The above notion of a group action puts the idea of a group G acting on an object X via symmetry transformations of X into a well-defined mathematical framework. To each group action $\triangleright : G \times X \to X$, we can further associate the following:

- **Orbits:** The *orbit* of an element $x \in X$ is given by the subset

$$\mathcal{O}_x \;:=\; \{g \triangleright x \mid g \in G\} \;\subset\; X\,. \tag{0.17}$$

We have that $\mathcal{O}_x = \mathcal{O}_y$ if and only if there exists a $g \in G$ such that $g \triangleright x = y$. The set of all orbits is denoted by X/G and induces a disjoint decomposition

$$X \;=\; \bigsqcup_{\mathcal{O} \in X/G} \mathcal{O}\,. \tag{0.18}$$

The *stabiliser* of an element $x \in X$ is defined to be the subgroup

$$G_x \;:=\; \{g \in G \mid g \triangleright x = x\} \;\subset\; G\,. \tag{0.19}$$

If $g \triangleright x = y$, then[5] $G_y = {}^g(G_x)$. The *orbit-stabiliser theorem* (see e.g. [16]) states that for each $x \in X$ there is an isomorphism of G-sets[6]

$$\mathcal{O}_x \;\cong\; G/G_x\,. \tag{0.20}$$

The group action is called *transitive* if it has a single orbit; $\mathcal{O}_x = X$ for all $x \in X$. This means that for any $x, y \in X$ there exists a $g \in G$ such that $g \triangleright x = y$.

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[5] Given a group G, a subgroup $H \subset G$, and an element $g \in G$, we denote by ${}^gH := gHg^{-1}$ the conjugation of H by g. Similarly, we denote $H^g := g^{-1}Hg$.

[6] Given a group G and a subgroup $H \subset G$, we denote by G/H the set of all *left H-cosets* in G. A left H-coset is a subset of G that is of the form $gH \equiv \{g \cdot h \,|\, h \in H\}$ with $g \in G$. Then, G/H is a G-set via multiplication from the left with elements in G.

- **Fixed Points:** The set of *fixed points* of an element $g \in G$ is given by

$$X^g \; := \; \{x \in X \mid g \triangleright x = x\} \; \subset \; X \; . \tag{0.21}$$

The fixed point sets X^g are dual to the stabiliser subgroups G_x in the sense that there is a bijection of sets

$$\bigsqcup_{g \in G} X^g \;\xrightarrow{\sim}\; \bigsqcup_{x \in X} G_x \; . \tag{0.22}$$

Plugging (0.20) into the above then yields *Burnside's lemma* [17]:

$$\sum_{g \in G} |X^g| \; = \; |G| \cdot |X/G| \; . \tag{0.23}$$

The group action is called *faithful* if $X^g = X$ implies $g = 1$. Equivalently, the group action is faithful if and only if the associated group homomorphism $G \to \mathrm{Aut}(X)$ is injective, meaning that its *kernel*

$$\ker(\triangleright) \; := \; \{k \in G \mid k \triangleright (.) = \mathrm{id}_X\} \tag{0.24}$$

comprises only the identity element $1 \in G$. The group action is called *free* if $g \triangleright x = x$ for some $x \in X$ implies $g = 1$. Clearly, every free group action is also faithful but not vice versa.

In general, the kernel $\ker(\triangleright)$ of a group action $\triangleright : G \times X \to X$ captures the subgroup of G that consists of all elements that act trivially on X. Intuitively, this means that we should *not* view g and $g \cdot k$ as distinct symmetry transformations of X if $k \in \ker(\triangleright)$. Rather, the set of *actual* symmetry transformations is given by the quotient

$$G \,/\, \ker(\triangleright) \; =: \; \widehat{G} \; , \tag{0.25}$$

which is again a group due to the fact that $\ker(\triangleright) \subset G$ is a *normal* subgroup[7]. The group action $\triangleright$ then induces a *faithful* group action $\widehat{\triangleright} : \widehat{G} \times X \to X$ of $\widehat{G}$ on X. This shows that, without loss of generality, we can always assume group actions to be faithful and to represent honest symmetry transformations of the associated object X. Simple examples of faithful group actions include the following:

- The dihedral group D_{2n} as defined in (0.3) acts on the n-gon $\mathbb{P}_n$ via rotations and reflections as illustrated in (0.1). This group action is faithful but not transitive.

[7] A subgroup $H \subset G$ is said to be *normal* if ${}^gH = H$ for all $g \in G$. It is easy to check that $H \subset G$ is normal if and only if there exists a group homomorphism $f : G \to G'$ such that $H = \ker(f)$.

- The symmetric group S_n of order n acts on the finite set $[n] = \{1, ..., n\}$ via permutations. This group action is both faithful and transitive.
- By construction, the isometry group $\mathrm{Iso}(M)$ of a Riemannian manifold M acts on the latter via diffeomorphisms in a faithful manner. When $M = \mathbb{R}^n$, this action is also transitive, since any two points $x, y \in \mathbb{R}^n$ are related by a translation.
- By construction, the unitary group $U(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ acts on the latter via linear transformations in a faithful manner. This action is not transitive in general. For instance, the multiplication action of $U(1) = \{e^{i\varphi} \,|\, \varphi \in [0, 2\pi)\}$ on $\mathbb{C}$ leaves the modulus of any complex number unchanged, so that $\mathbb{C}$ decomposes into a continuous union of orbits $\mathbb{C} = \bigsqcup_{r \in [0,\infty)} \mathcal{O}_r$ with $\mathcal{O}_r = \{z \in \mathbb{C} \,|\, |z| = r\}$.

While the theory of groups emerged as a rigorous mathematical framework for the description of symmetries during the early nineteenth century, its application to theoretical physics only began in the early twentieth century, when the idea of 'symmetry as invariance under certain transformations' was applied not only to geometric figures but also to mathematical expressions such as equations governing the dynamics of a physical system [7, 8]. As a result, the machinery of group theory could be applied to problems in both classical and quantum physics, leading to new insights and constraints on the dynamics of a variety of physical systems.

0.1.1 Classical Symmetries

Classical theories are *deterministic* in the sense that the future outcome of physical observables is uniquely determined by their initial conditions. In other words, classical theories exhibit no intrinsic randomness and allow for (in principle) exact predictions of physical quantities. Often, the fundamental observables are taken to be *fields*, which, broadly speaking, correspond to maps

$$\varphi : \; M \;\to\; N \tag{0.26}$$

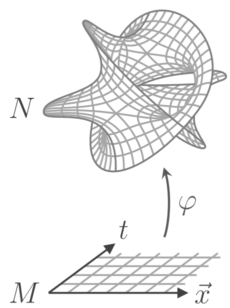

Figure 6

from d-dimensional *spacetime*[8] M into some target space N as illustrated in Figure 6. In other words, a field φ assigns to each point $m = (\vec{x}, t) \in M$ in space and time a value $\varphi(m)$ in the target space N. We denote the space of all such fields by

$$\mathcal{F} \;:=\; \{\varphi : M \to N\}\;. \tag{0.27}$$

[8] In what follows, we use the word 'spacetime' to refer to an oriented d-dimensional manifold M whose coordinates (locally) describe $d-1$ directions $\vec{x}$ of space and one direction t of time. While the distinction between space- and timelike coordinates only makes sense if M is equipped with a Lorentzian metric, the latter will not be essential for the following discussion.

A *(classical) field theory* is a mechanism that singles out those $\varphi_0 \in \mathcal{F}$ that correspond to 'physically observable' configurations of the field φ. Often, this is done using the *principle of least action*, according to which the physical field configurations correspond to local extrema (or *critical points*) of a given *action functional*

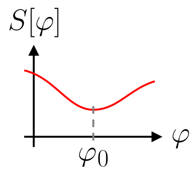

Figure 7

$$S : \mathcal{F} \to \mathbb{R} \tag{0.28}$$

as illustrated in Figure 7. In other words, the physical field configurations φ_0 are those that solve the *equations of motion*[9]

$$\left.\frac{\delta S}{\delta \varphi}\right|_{\varphi_0} \stackrel{!}{=} 0 \,. \tag{0.29}$$

We denote the set of all such solutions by $\mathcal{F}_0 \subset \mathcal{F}$ and call it the *phase space* of the theory. Examples of classical field theories include the following:

- Classical mechanics: The theory of classical mechanics is an example of a one-dimensional field theory that describes the motion of point particles in space as a function of time and in the presence of forces. The fields hence correspond to maps $\gamma : (a, b) \to N$ from a time interval (a, b) to a fixed Riemannian manifold N that we interpret as "space". Given a smooth function $V : N \to \mathbb{R}$ (called the *potential*), we can then define the action functional

$$S[\gamma] \;:=\; \int_a^b \left[\frac{m}{2}\,\|\dot{\gamma}(t)\|^2 - V\big(\gamma(t)\big)\right] dt \,, \tag{0.30}$$

where, as before, $\dot{\gamma}(t)$ denotes the tangent to the curve γ and the parameter $m \in \mathbb{R}_{>0}$ is called the *mass* of the point particle. The equations of motion that result from varying the action (0.30) w.r.t. γ are then given by

$$m \cdot \nabla_{\dot{\gamma}}\dot{\gamma} \;=\; -\mathrm{grad}(V) \,, \tag{0.31}$$

where ∇ denotes the Levi-Civita connection w.r.t. the Riemannian metric on N and $\mathrm{grad}(V)$ is the gradient vector field[10] of the function V. Upon identifying $a := \nabla_{\dot{\gamma}}\dot{\gamma}$ with the acceleration of the curve γ and $F := -\mathrm{grad}(V)$ with the force field on N, equation (0.31) simply becomes *Newton's law of motion* $F = ma$.

[9] Note that in order for the functional derivative in (0.29) and hence the equations of motion to be well-defined, one typically needs to impose suitable boundary conditions on the fields, which specify e.g. the value of φ and its differential $D\varphi$ on ∂M and which allow one to discard possible boundary terms in the variation of the action functional.

[10] Given a smooth function $f : N \to \mathbb{R}$ on N, its *gradient* is the vector field $\mathrm{grad}(f) \in \mathfrak{X}(N)$ on N that satisfies $(\mathrm{grad}(f), X) = (df)(X)$ for all vector fields $X \in \mathfrak{X}(N)$, where $(\,.\,,.\,) \in T^2N$ is the Riemannian metric on N and d denotes the exterior derivative.

As an example, consider $N = \mathbb{R}^3 \backslash \{0\}$ with the standard Euclidean metric and radial potential $V(\vec{r}) = -m \cdot M / \|\vec{r}\|$. The corresponding equations of motion are

$$\ddot{\vec{r}}(t) \;=\; -M \cdot \frac{\vec{r}(t)}{\|\vec{r}(t)\|^3} \;, \tag{0.32}$$

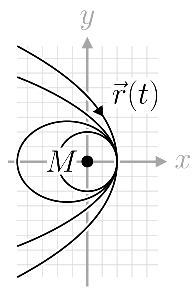

Figure 8

whose analytical solutions fall into four classes as illustrated in Figure 8: circles, ellipses, parabolae, and hyperbolae. These describe the possible orbits of a probe particle that moves in the gravitational field of a heavy object with mass $M \gg m$ centred at the origin of three-dimensional space.

- Maxwell theory: The theory of electrodynamics describes the behaviour of the electric and magnetic fields $\vec{E} = \vec{E}(\vec{x}, t)$ and $\vec{B} = \vec{B}(\vec{x}, t)$ as a function of space and time in the presence of a charge density ρ and a current density $\vec{J}$. It is convenient to express the electric and the magnetic field in terms of a scalar potential ϕ and a vector potential $\vec{A}$ as[11]

$$\vec{E} \;=\; -\,\partial_t \vec{A} - \mathrm{grad}(\phi) \qquad \text{and} \qquad \vec{B} \;=\; \mathrm{curl}(\vec{A}) \;, \tag{0.33}$$

which ensures that they obey the *homogeneous Maxwell equations*

$$\mathrm{div}(\vec{B}) \;=\; 0 \qquad \text{and} \qquad \mathrm{curl}(\vec{E}) + \partial_t \vec{B} \;=\; 0 \;. \tag{0.34}$$

The potentials ϕ and $\vec{A}$ are not unique, however, since redefining

$$\phi \;\to\; \phi - \partial_t \lambda \qquad \text{and} \qquad \vec{A} \;\to\; \vec{A} + \mathrm{grad}(\lambda) \tag{0.35}$$

for some arbitrary scalar function $\lambda(\vec{x}, t)$ leaves both the electric and the magnetic field in (0.33) unchanged[12]. We can combine the potentials ϕ and $\vec{A}$ into a single field living on four-dimensional Minkowski spacetime $\mathbb{M}^4$ by setting

$$A \;:=\; A_x\, dx + A_y\, dy + A_z\, dz - \phi\, dt \;, \tag{0.36}$$

[11] We denote by grad, div, and curl the gradient, the divergence, and the curl operator w.r.t. the three spatial coordinates x, y, and z, respectively.

[12] The transformations in (0.35) are often called 'gauge symmetries'. However, since the physical fields $\vec{E}$ and $\vec{B}$ are unaffected by these transformations, they do not correspond to genuine symmetries of the theory but should rather be viewed as redundancies of our chosen description.

which defines a 1-form $A \in \Omega^1(\mathbb{M}^4)$ called the *connection 1-form* (or *gauge field*). The electric and magnetic fields can be recovered from the latter via the *curvature 2-form* (or *field strength*) $F := dA \in \Omega^2(\mathbb{M}^4)$, whose components are given by

$$\begin{aligned} F \;&=\; \big[E_x\,dx \,+\, E_y\,dy \,+\, E_z\,dz\big] \wedge dt \\ &\quad\, + \, B_x\;dy \wedge dz \,+\, B_y\;dz \wedge dx \,+\, B_z\;dx \wedge dy\;. \end{aligned} \tag{0.37}$$

Here, we denoted by $d : \Omega^n(\mathbb{M}^4) \to \Omega^{n+1}(\mathbb{M}^4)$ the exterior derivative on differential forms, which satisfies $d^2 = d \circ d = 0$. This property ensures that the field strength F is invariant under 'gauge transformations' $A \to A + d\lambda$, which, using (0.36), reproduce the transformations in (0.35). Similarly, the homogeneous Maxwell equations (0.34) follow from $dF = d^2A = 0$. In order to obtain the *inhomogeneous Maxwell equations* that describe how the electric and magnetic fields are sourced by the charge and current densities ρ and $\vec{J}$, we combine the latter two into a *current 1-form*

$$J \;:=\; J_x\;dx \,+\, J_y\;dy \,+\, J_z\;dz \,-\, \rho\;dt \tag{0.38}$$

and define the following action for the connection 1-form A coupled to J:

$$S[A,J] \;:=\; \int_{\mathbb{M}^4} \Big[-\frac{1}{2}\,F \wedge \star F \,+\, A \wedge \star J\Big] \tag{0.39}$$

Here, we denoted by $\star : \Omega^n(\mathbb{M}^4) \to \Omega^{4-n}(\mathbb{M}^4)$ the *Hodge star operator* defined in terms of the Minkowski metric on $\mathbb{M}^4$. The equations of motion that result from varying the above action w.r.t. A are then given by

$$d \star F \;=\; \star J\;, \tag{0.40}$$

which, using (0.37) and (0.38), can be checked to be equivalent to the inhomogeneous Maxwell equations

$$\mathrm{div}(\vec{E}) \;=\; \rho \qquad \text{and} \qquad \mathrm{curl}(\vec{B}) \,-\, \partial_t\vec{E} \;=\; \vec{J}\;. \tag{0.41}$$

Furthermore, applying the exterior derivative d to (0.40) yields $d \star J = 0$, which is equivalent to the current conservation equation

$$\partial_t\rho \,+\, \mathrm{div}(\vec{J}) \;=\; 0\;. \tag{0.42}$$

- Gauge theory: Maxwell's theory of electromagnetism is an example of a *gauge theory*, where d-dimensional spacetime M forms the base space of a *principal G-bundle* associated to some Lie group G (called the *gauge group*). What this means is that M is the image of a smooth surjection $\pi : P \to M$, where P is a manifold (called the *total space*) that is equipped with a smooth (right) group action $\triangleleft : P \times G \to P$ satisfying $\pi(p \triangleleft g) = \pi(p)$ for all $p \in P$ and $g \in G$ and which locally looks like $U \times G$ as a G-set for small enough open neighbourhoods $U \subset M$. In other words, a principal G-bundle attaches to each point $m \in M$ in spacetime a fibre $P_m := \pi^{-1}(\{m\})$ that is a G-torsor in the sense that G acts freely and transitively on P_m as illustrated in Figure 9. We refer the reader to [18] for more background on mathematical gauge theory.

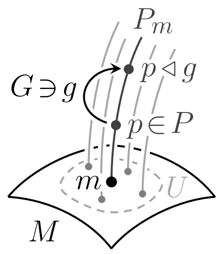

Figure 9

The 'fields' of gauge theory are (equivalence classes of) *gauge fields*, which are *connection 1-forms*[13] $A \in \Omega^1(P, \mathfrak{g})$ on the total space of the principal bundle with values in the Lie algebra $\mathfrak{g}$ of G. Two such gauge fields A and A' are considered equivalent if there exists a bundle automorphism[14] $f : P \to P$ (also called a *gauge transformation*) such that $A' = f^*(A)$. To each gauge field A we can associate a *field strength* $F \in \Omega^2(P, \mathfrak{g})$ via[15]

$$F \ := \ dA + \frac{1}{2}[A \wedge A] \ , \tag{0.43}$$

which obeys the *Bianchi identity* $dF + [A \wedge F] = 0$. Note that, as defined above, neither the gauge field A nor its field strength F are *fields* in the sense of (0.26), since their domain is the total space P rather than the spacetime manifold M. We can try to cure this by pulling back along a global *section* $s : M \to P$ obeying $\pi \circ s = \mathrm{id}_M$, whose existence, however, implies that the bundle P is trivial[16] (i.e. $P \cong M \times G$). In general, we may hence only be able to patch up $M = \bigcup_i U_i$ into a union of contractible open neighbourhoods with associated local sections $s_i : U_i \to \pi^{-1}(U_i) \subset P$, which allow us to define local connection 1-forms

$$A_i \ := \ s_i^*(A) \ \in \ \Omega^1(U_i, \mathfrak{g}) \ . \tag{0.44}$$

[13] A 1-form $A \in \Omega^1(P, \mathfrak{g})$ is called a *connection 1-form* if $R_g^*(A) = \mathrm{Ad}_{g^{-1}}(A)$ and $A(X_\varepsilon) = \varepsilon$ for all $g \in G$ and $\varepsilon \in \mathfrak{g}$, where $R_g : P \to P$ denotes the right action of a fixed group element $g \in G$ on P and $X_\varepsilon \in \mathfrak{X}(P)$ is the vector field on P defined by $(X_\varepsilon)_p := \frac{d}{dt}\big|_{t=0}[p \triangleleft \exp(t \cdot \varepsilon)]$ for all $p \in P$.

[14] A *bundle automorphism* of P is a diffeomorphism $f : P \to P$ that satisfies $\pi \circ f = \pi$ and $f(p \triangleleft g) = f(p) \triangleleft g$ for all $p \in P$ and $g \in G$.

[15] Here, the map $[\,.\wedge.\,] : \Omega^k(p, \mathfrak{g}) \times \Omega^l(p, \mathfrak{g}) \to \Omega^{k+l}(P, \mathfrak{g})$ is constructed using both the wedge product $\wedge$ of differential forms and the Lie bracket $[.\,,.]$ on $\mathfrak{g}$.

[16] If the base space M is contractible (such as $M = \mathbb{M}^4$), then every principal bundle P over M is trivial, and hence admits a global section $s : M \to P$.

However, these may not agree on local intersections $U_i \cap U_j$ for $i \neq j$, but are instead related by the formula

$$A_i \;=\; \mathrm{Ad}_{g_{ij}^{-1}}(A_j) \,+\, g_{ij}^*(\mu_G) \;, \tag{0.45}$$

where the smooth function $g_{ij} : U_i \cap U_j \to G$ is defined by

$$s_i(m) \;=:\; s_j(m) \triangleleft g_{ij}(m) \tag{0.46}$$

for $m \in U_i \cap U_j$ and $\mu_G \in \Omega^1(G, \mathfrak{g})$ is the *Maurer-Cartan form*[17] of G. On the other hand, the local curvature 2-forms $F_i := s_i^*(F) \in \Omega^2(U_i, \mathfrak{g})$ are related by

$$F_i \;=\; \mathrm{Ad}_{g_{ij}^{-1}}(F_j) \tag{0.48}$$

on $U_i \cap U_j$, so that upon choosing an Ad-invariant positive-definite symmetric bilinear form[18] $\langle . \, , . \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$, the following *(pure) Yang-Mills action* for the gauge field A is well-defined:

$$S[A] \;:=\; -\frac{1}{2} \int_M \langle F \wedge \star F \rangle \;. \tag{0.49}$$

Here, we regard the integrand as a 2-form on M by pulling F back along the local sections s_i, the choices of which do not matter due to (0.48) and the Ad-invariance of $\langle . \, , . \rangle$. The equations of motion that arise from varying the action (0.49) w.r.t. A are called the *Yang-Mills equations* and correspond to non-linear generalisations of Maxwell's equations for classical electrodynamics.

For applications in high energy physics, one often chooses four-dimensional Minkowski spacetime $M = \mathbb{M}^4$ to be the base space of a (necessarily trivial) principal G-bundle for some Lie group G. In the case of electromagnetism, this is $G = U(1)$, while for reasons nobody really understands the standard model of particle physics has $G = U(1) \times SU(2) \times SU(3)$. Upon choosing a global section of the G-bundle, we may then view the gauge field A and its field strength F as global forms on $\mathbb{M}^4$ as in (0.36) and (0.37), with the difference that their coefficients are now smooth maps $\mathbb{M}^4 \to \mathfrak{g}$ into the Lie algebra of G. For example, in the case of *chromodynamics* (where $G = SU(3)$), this yields a total of $\dim(\mathfrak{su}(3)) = 8$ so-called *gluon fields* that are distinguished by their "colour".

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[17] The *Maurer-Cartan form* of a Lie group G is the 1-form $\mu_G \in \Omega^1(G, \mathfrak{g})$ defined by

$$(\mu_G)_g \;:=\; D_g(L_{g^{-1}}) : \; T_g G \;\to\; T_1 G \cong \mathfrak{g} \;, \tag{0.47}$$

where $L_g : G \to G$ denotes left multiplication by $g \in G$.

[18] If G is a compact Lie group (such as $SO(N)$ or $U(N)$), there always exists a positive-definite symmetric bilinear form $\langle . \, , . \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on its Lie algebra which is Ad-*invariant* in the sense that $\langle \mathrm{Ad}_g(.), \mathrm{Ad}_g(.) \rangle = \langle . \, , . \rangle$ for all $g \in G$ [18].

As in the case of the examples above, we often consider field theories that are *local* in the sense that their action functional is of the form

$$S[\varphi] \;=\; \int_M \mathcal{L}(\varphi, D\varphi) \tag{0.50}$$

for some top-form $\mathcal{L} \in \Omega^d(M)$ (called the *Lagrangian*) constructed out of the fields $\varphi : M \to N$ and their differentials $D\varphi : TM \to TN$. The resulting equations of motion are then typically given by second-order (partial) differential equations for the fields φ, such as Newton's law of motion (0.31) for a particle's trajectory γ or Maxwell's equations (0.40) for the gauge field A. Solutions φ_0 to these equations are then usually constructed via a two-step process:

1. **Initial conditions:** Given a codimension-one hypersurface $X \subset M$ (interpreted as "space"), we prescribe the *initial conditions* of the fields to be given by some function $\phi : X_\varepsilon \to N$ that solves the restricted equations of motion on an infinitesimal neighbourhood $X_\varepsilon := X \times (-\varepsilon, \varepsilon)$ of X (see Figure 10).

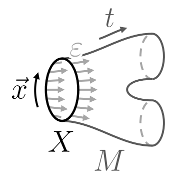


Figure 10

2. **Time evolution:** Given an initial condition ϕ, we find a solution φ_0 to the full equations of motion that restricts to $\varphi_0|_{X_\varepsilon} = \phi$ in an infinitesimal neighbourhood of X. The field φ_0 then describes how ϕ "evolves over time" on M.

In favourable circumstances, the restriction map that sends a solution φ_0 to its initial condition $\varphi_0|_{X_\varepsilon}$ is one-to-one, giving an isomorphism between the phase space $\mathcal{F}_0$ of physical field configurations and the space of their initial conditions in a neighbourhood of X. A hypersurface X with this property is called a *Cauchy hypersurface*[19] and embodies the idea of *classical determinism*: The time evolution of physical observables is completely determined by their initial conditions, allowing for (in principle) exact predictions of future outcomes.

In practice, our ability to describe the dynamics of a physical system by solving its equations of motion is limited by two factors: First, the system's initial conditions may be arbitrary and unpredictable and can only be determined with finite precision experimentally. Second, for a given set of initial conditions, the equations of motion may not admit analytical solutions, necessitating the use of numerical simulations and approximations. In this context, symmetries provide a powerful tool to obtain qualitative and quantitative constraints on a system's dynamics that are independent of initial conditions and that do not require solving the equations of motion explicitly. Here, by "symmetries of a field theory" we mean the following:

[19] If the equations of motion are given by a normally hyperbolic differential operator on spacetime, then the latter admits a Cauchy hypersurface [19].

Definition: A classical field theory is said to have a symmetry group G if there exists a faithful group action $\triangleright : G \times \mathcal{F} \to \mathcal{F}$ of G on the space of fields that leaves the action functional invariant, i.e. $S[g \triangleright \varphi] = S[\varphi]$ for all $g \in G$ and $\varphi \in \mathcal{F}$.

The invariance of S under the group action on the space of fields ensures that if φ_0 is a critical point of S, so is $g \triangleright \varphi_0$ for any $g \in G$. In other words, the group action has a well-defined restriction to the phase space $\mathcal{F}_0$ of physical field configurations and hence leaves the equations of motion (0.29) invariant. More generally, we may consider group actions that shift the action functional by a boundary term,

$$S[g \triangleright \varphi] \;=\; S[\varphi] + \text{boundary term}\,, \tag{0.51}$$

since (under suitable boundary conditions for the fields) the latter does not affect the equations of motion for the fields. Group actions satisfying (0.51) are often referred to as *quasi-symmetries* of a field theory. We say that a (quasi-)symmetry is *spontaneously broken* by a physical field configuration φ_0 if the associated orbit $\mathcal{O}_{\varphi_0}$ is non-trivial. This captures the idea that while the laws governing the dynamics of the fields may be invariant w.r.t. a certain symmetry group, a given solution to the equations of motion need not be. In this case, we call the stabiliser $H := G_{\varphi_0}$ the *unbroken subgroup* of G, which captures those symmetry transformations that leave the field configuration φ_0 invariant. We say that G is *fully spontaneously broken* by φ_0 if $H = 1$.

For a field theory with fundamental observables given by fields $\varphi : M \to N$ from spacetime M into some target manifold N, we typically further distinguish between the following two types of symmetries:

- **External:** Symmetries that are induced by symmetries of spacetime M are called *external* (or *spacetime*) symmetries. For example, every automorphism $f \in \mathrm{Aut}(M)$ induces an action $\varphi \mapsto f^*(\varphi)$ on the space of fields via pullbacks.
- **Internal:** Symmetries that are induced by symmetries of the target space N are called *internal* symmetries. For example, every automorphism $g \in \mathrm{Aut}(N)$ induces an action $\varphi \mapsto g \circ \varphi$ on the space of fields via post-composition.

The utility of (quasi-)symmetries stems from the fact that they are intimately tied to conservation laws, as captured by the famous *Noether theorem* [9]:

Theorem: For every continuous (quasi-)symmetry of a classical field theory there exists an associated conserved quantity Q satisfying $\frac{d}{dt}Q = 0$.

For local field theories with action functionals of the form (0.50), this result is easily justified using the following *Noether trick* (see e.g. [20]): Suppose that a local theory has a continuous Lie group symmetry G and consider an infinitesimal group action on the fields parameterised by a Lie algebra element $\varepsilon \in \mathfrak{g}$. Now promote ε to an arbitrary Lie algebra valued function on spacetime M. The resulting infinitesimal change of the action must then be of the form[20]

$$\delta S \;=\; \int_M \langle \varepsilon \wedge d \star k \rangle \;+\; \int_M \langle d\varepsilon \wedge \star \ell \rangle \tag{0.52}$$

for some 1-forms $k, \ell \in \Omega^1(M, \mathfrak{g}^\vee)$ valued in the dual space[21] $\mathfrak{g}^\vee$ of the Lie algebra and constructed out of the fields φ and their differentials $D\varphi$. This ensures that S shifts by at most a boundary term when ε is constant, as required for a (quasi-)symmetry transformation. Moreover, we must have $\delta S = 0$ for *any* ε once we substitute in a physical field configuration $\varphi_0 \in \mathcal{F}_0$ satisfying (0.29), which, after integrating by parts, implies the current conservation equation

$$d \star j|_{\varphi_0} \;=\; 0 \,, \tag{0.53}$$

where we defined the so-called *Noether current* $j := \ell - k$. This shows that for every continuous symmetry there exists an associated 1-form current j that is conserved *on-shell*, i.e. when the equations of motion are satisfied. In order to obtain from this a conserved quantity, we integrate the pairing of $\star j$ with a constant Lie algebra element $\varepsilon \in \mathfrak{g}$ against a codimension-one submanifold $\Sigma \subset M$, which yields

$$Q_\varepsilon(\Sigma) \;:=\; \int_\Sigma \langle \varepsilon, \star j \rangle \;. \tag{0.54}$$

Upon smoothly deforming Σ into a homologous codimension-one submanifold $\Sigma' \subset M$, the above quantity changes by

$$Q_\varepsilon(\Sigma') - Q_\varepsilon(\Sigma) \;=\; \int_{\Sigma'} \langle \varepsilon, \star j \rangle \;-\; \int_\Sigma \langle \varepsilon, \star j \rangle \;\equiv\; \int_U \langle \varepsilon, d \star j \rangle \;, \tag{0.55}$$

where $U \subset M$ is such that[22] $\partial U = \Sigma' \cup \overline{\Sigma}$ (see Figure 11) and we used Stokes' theorem for the integration of differential forms in the last step. Upon substituting in physical a field configuration, (0.55) together with (0.53) then implies that $Q_\varepsilon(\Sigma) = Q_\varepsilon(\Sigma')$ on-shell. In particular, for a spacetime of the form $M = X \times \mathbb{R}$, the quantity $Q_\varepsilon(t) := Q_\varepsilon(X \times \{t\})$ obeys $\frac{d}{dt} Q_\varepsilon = 0$. By varying ε over a basis $\{\varepsilon_i\}$ of $\mathfrak{g}$, this gives a total of $\dim(\mathfrak{g}) = \dim(G)$ independent conserved quantities

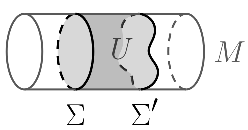


Figure 11

[20] Here, the appearance of the Hodge star operator $\star$ defined w.r.t. a spacetime metric on M is purely conventional to have k and ℓ correspond to 1-forms as opposed to $(d-1)$-forms.

[21] Given a vector space V over a field $\mathbb{K}$, we denote by $V^\vee := \mathrm{Hom}(V, \mathbb{K})$ the space of $\mathbb{K}$-linear maps $f : V \to \mathbb{K}$ and call it the *dual space* of V. We denote by $\langle .,. \rangle : V^\vee \times V \to \mathbb{K}$ the canonical pairing between V and its dual space given by $(f, v) \mapsto f(v)$. This is the pairing appearing in (0.52).

[22] Given an oriented manifold Σ, we denote by $\overline{\Sigma}$ its orientation reversal, i.e. the manifold Σ equipped with the opposite orientation.

$Q_i := Q_{\varepsilon_i}$ as claimed. Examples of internal and external (quasi-)symmetries and their associated conserved quantities include the following:

- Classical mechanics: The theory describing the motion of probe particles on a spatial manifold N via their trajectories $\gamma : \mathbb{R} \to N$ with action functional (0.30) has an external time translation symmetry that acts on the fields via $\gamma(t) \mapsto \gamma(t - \varepsilon)$ for fixed $\varepsilon \in \mathbb{R}$. The associated conserved quantity is given by

$$E \;=\; \frac{m}{2}\,\|\dot{\gamma}(t)\|^2 \,+\, V(\gamma(t)) \tag{0.56}$$

and is called the *energy* of the probe particle, where $V : N \to \mathbb{R}$ denotes the potential function and m is the mass of the particle as before.

For example, probe particles moving on trajectories $\vec{r}(t)$ in the gravitational potential $V(\vec{r}) = -m \cdot M/r$ (where $r := \|\vec{r}\|$) of a heavy object with mass $M \gg m$ centred at the origin of $N = \mathbb{R}^3 \backslash \{0\}$ have conserved energy

$$E \;=\; \frac{m}{2}\,\|\dot{\vec{r}}\|^2 \,-\, \frac{m \cdot M}{r}\;. \tag{0.57}$$

Without solving the equations of motion, this already allows us to establish a qualitative relationship between the particle's position and its speed: the closer the particle is to the heavy object at the origin, the greater the magnitude of its velocity. Moreover, the system has an internal $SO(3)$ rotation symmetry that acts on the fields via $\vec{r}(t) \mapsto R \cdot \vec{r}(t)$ for $R \in SO(3)$ and that yields $\dim(SO(3)) = 3$ independent conserved quantities

$$L_i \;=\; m \cdot [\,\vec{r} \times \dot{\vec{r}}\,]_i \tag{0.58}$$

(where $i = x, y, z$), which we identify as the components of the particle's *angular momentum* $\vec{L}$. Since the latter is perpendicular to both the particle's position $\vec{r}(t)$ and its velocity $\dot{\vec{r}}(t)$, the conservation of $\vec{L}$ implies that the particle's trajectory lies within a two-dimensional hyperplane $\mathbb{H}$ (e.g. the x-y-plane) that intersects the origin and has a normal vector parallel to $\vec{L}$. Without loss of generality, we may hence simplify the variational problem by restricting to those trajectories $\vec{r}(t)$ that are confined to $\mathbb{H}$ and that have conserved angular momentum $\vec{L}$. The action functional (0.30) then possesses three further quasi-symmetries, whose infinitesimal actions on the fields are given by

$$\delta_i \vec{r} \;=\; \frac{\varepsilon}{m} \cdot \vec{e}_i \times \vec{L}\;, \tag{0.59}$$

where $\varepsilon \ll 1$ and $\vec{e}_i$ $(i = x, y, z)$ denotes the standard basis of $\mathbb{R}^3$. The resulting infinitesimal shift of the action is given by the total derivative

$$\delta_i S \;=\; \varepsilon \cdot m \cdot M \cdot \int \frac{d}{dt}\left(\frac{r_i}{r}\right) \,, \tag{0.60}$$

which via Noether's theorem induces the conserved quantities

$$A_i \;=\; (\dot{\vec{r}} \times \vec{L})_i \,-\, \frac{m \cdot M}{r} \cdot r_i \tag{0.61}$$

that form the components of the so-called *Runge-Lenz vector* $\vec{A}$. The conservation of the latter then allows us to determine the shape of the particle's trajectory explicitly (see e.g. [21]): Consider the inner product

$$\vec{A} \cdot \vec{r} \;\equiv\; A \cdot r \cdot \cos(\theta) \;=\; \frac{L^2}{m} \,-\, m \cdot M \cdot r \,, \tag{0.62}$$

where $A := \|\vec{A}\|$, $L := \|\vec{L}\|$, and $\theta \in [0, 2\pi)$ labels the angle between $\vec{A}$ and $\vec{r}$[23]. Rearranging (0.62) and using the fact that

$$A^2 \;\equiv\; \vec{A} \cdot \vec{A} \;=\; \frac{2L^2}{m} \cdot E \,+\, (m \cdot M)^2 \tag{0.63}$$

then allows us to relate the particle's distance r from the origin to the angle θ by

$$r \;=\; \frac{C}{1 + \mu \cdot \cos(\theta)} \,, \tag{0.64}$$

where we defined the non-negative constants

$$C \;:=\; \frac{\ell^2}{M} \qquad \text{and} \qquad \mu \;:=\; \sqrt{1 + \frac{2\,\ell^2}{M^2} \cdot e} \tag{0.65}$$

with $\ell := L/m$ and $e := E/m$. Equation (0.64) famously parameterises *conic sections*, which fall into four classes depending on the value of the *eccentricity parameter* μ: circles $(\mu = 0)$, ellipses $(0 < \mu < 1)$, parabolae $(\mu = 1)$, and hyperbolae $(\mu > 1)$. This reproduces the classification of possible orbits of a probe particle moving in the gravitational field of a heavy object as illustrated in Figure 8. Note that we arrived at this result purely by symmetry considerations. We often say that orbits with eccentricity $0 \leq \mu < 1$ (or equivalently with energy $E < 0$) are *bound*, since they have a finite maximum distance to the origin.

[23] Note that since $\vec{A} \cdot \vec{L} = 0$, the Runge-Lenz vector $\vec{A}$ lies within the two-dimensional hyperplane that the particle's trajectory $\vec{r}(t)$ is confined to. The angle $\theta = \measuredangle(\vec{A}, \vec{r})$ hence captures the angle of the particle's position vector w.r.t. some fixed hyperplane axis labelled by $\vec{A}$.

- Gauge theory: Given a Lie group G, the theory of gauge fields living on a principal G-bundle P with action functional (0.49) has external symmetries given by the isometry group $\mathrm{Iso}(M)$ of spacetime M. For example, if $M = \mathbb{M}^4$ is Minkowski spacetime, this is the Poincaré group $\mathrm{Iso}(\mathbb{M}^4) = \mathbb{R}^4 \rtimes O(3,1)$ consisting of translations and generalised orthogonal transformations. The associated (canonical) Noether currents are *not* gauge-invariant in general. However, they can be improved by adding appropriate auxiliary terms as shown by Belinfante and Rosenfeld [22, 23]. For instance, the so-obtained Noether current associated to the translation symmetry $\mathbb{R}^4$ is given by the $(\mathbb{R}^4)^\vee$-valued 1-form

$$T(\,.\,) \;=\; \sum_b \langle F(\,.\,,b), F(-,b^\vee)\rangle \;-\; \frac{\langle F\wedge\star F\rangle}{2\cdot \mathrm{vol}}(\,.\,,-) \tag{0.66}$$

on $\mathbb{M}^4$ [24], where $F\in\Omega^2(P,\mathfrak{g})$ is the curvature 2-form of the gauge field and

  - $\langle .,.\rangle$ is the fixed Ad-invariant positive-definite symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of the gauge group G that appears in the action (0.49),
  - $\sum_b$ denotes a sum over elements of a fixed basis b of $\mathbb{R}^4$, where $b^\vee$ denotes the corresponding dual basis w.r.t. the Minkowski metric $(.,.)$,
  - $\mathrm{vol}\in\Omega^4(\mathbb{M}^4)$ is the metric-induced *volume form* on $\mathbb{M}^4$.

  Note that we can equivalently view (0.66) as defining a symmetric real-valued 2-tensor on $\mathbb{M}^4$, which is called the *energy-momentum* (or *stress-energy*) *tensor* associated to the gauge field.

  As an example, consider free Maxwell theory on $\mathbb{M}^4$, which is described by the gauge group $G = U(1)$. The components of the field strength F are given in terms of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ as in (0.37). The stress-energy tensor in this case can be computed to be

$$T \;=\; u\,dt^2 \;-\; 2\sum_i S_i\,dx_i\,dt \;-\; \sum_{ij}\sigma_{ij}\,dx_i\,dx_j\;, \tag{0.67}$$

  where u and $\vec{S}$ denote the *energy density* and the *Poynting vector*

$$u \;:=\; \frac{1}{2}(\vec{E}^2+\vec{B}^2) \qquad\text{and}\qquad S_i \;:=\; [\vec{E}\times\vec{B}]_i \tag{0.68}$$

  of the electromagnetic field and σ is the *Maxwell stress tensor*

$$\sigma_{ij} \;:=\; E_iE_j + B_iB_j - \delta_{ij}\,u\;. \tag{0.69}$$

  The conservation of T is then equivalent to the continuity equations

$$\partial_t u + \mathrm{div}(\vec{S}) \;=\; 0 \qquad\text{and}\qquad \partial_t\vec{S} - \mathrm{div}(\sigma) \;=\; 0\;, \tag{0.70}$$

which say that the rate of change of the electromagnetic energy $E(V) := \int_V u \, d^3x$ contained in a spatial volume $V \subset \mathbb{R}^3$ is given by the flux of the Poynting vector through the boundary ∂V of V. Similarly, the rate of change of the electromagnetic momentum $\vec{P}(V) := \int_V \vec{S} \, d^3x$ contained in V is given by the flux of the Maxwell stress tensor through ∂V. For instance, an electromagnetic wave propagating freely in the z-direction of space with wave profile $f : \mathbb{R} \to \mathbb{R}$ and non-trivial field components

$$E_x(\vec{x}, t) \;=\; B_y(\vec{x}, t) \;=\; f(z - t) \tag{0.71}$$

has associated conserved stress-energy tensor

$$T \;=\; \big[f(z-t) \cdot (dz - dt)\big]^2 \,, \tag{0.72}$$

which allows us to read off the energy density and the non-trivial components of the Poynting vector and the Maxwell stress tensor as $u = S_z = -\sigma_{zz} = f(z-t)^2$.

0.1.2 Quantum Symmetries

Quantum theories are *probabilistic* in the sense that the future outcomes of observables can be predicted only with certain likelihoods from their initial conditions. As a result, quantum theories exhibit an intrinsic randomness and constrain us to compute at most 'expectation values' of physical quantities. Heuristically, given a classical theory of fields φ, we can promote it to a quantum theory by introducing a 'measure' $\mathcal{D}\varphi$ on the space $\mathcal{F}$ of fields[24], which allows us to define and compute probability distributions on sets of field configurations by evolving suitable initial conditions over time (in analogy to the classical case). For this purpose, we will use the notation $\mathcal{F}(M)$ to denote the space of fields on a fixed spacetime M and $\mathcal{F}_X := \{\varphi|_X\}$ to denote the set of field configurations restricted to a spatial hypersurface $X \subset M$. Similarly, we will write $S_M : \mathcal{F}(M) \to \mathbb{R}$ for the action functional that evaluates field configurations living on M. The probabilistic evolution of fields can then be determined via the following axiomatic two-step process (also called *quantisation*):

1. **Initial conditions:** Given a codimension-one hypersurface $X \subset M$ (interpreted as "space"), we take the *initial conditions* of the fields to be given by some

[24] Since the space $\mathcal{F}$ of all field configurations is usually infinite-dimensional, the existence of a 'nice' measure $\mathcal{D}\varphi$ on $\mathcal{F}$ is far from guaranteed. While mathematically rigorous constructions exist in special cases (e.g. the Wiener measure in quantum mechanics), the measures used in generic quantum field theories are typically ill-defined (for instance, one can show that no locally-finite, strictly positive, and translation-invariant measure exists on an infinite-dimensional vector space [25]). The following discussion should hence only be viewed as a heuristic motivation for the general structure we expect quantum field theories to have.

probability distribution $p : \mathcal{F}_X \to \mathbb{R}_{\geq 0}$ on the space $\mathcal{F}_X$ of field configurations restricted to X. Concretely, we interpret[25]

$$P(\mathcal{S}) \;:=\; \int_{\mathcal{S}} \mathcal{D}\varphi|_X \cdot p[\varphi] \tag{0.73}$$

as the probability to measure a field configuration that lies in a given subset $\mathcal{S} \subset \mathcal{F}_X$. Since p (being a probability distribution) is everywhere non-negative, we can write it as[26] $p = |\Psi|^2$ for some complex functional $\Psi : \mathcal{F}_X \to \mathbb{C}$ called the *wave functional*, which needs to be normalised in a way such that

$$P(\mathcal{F}_X) \;=\; \int_{\mathcal{F}_X} \mathcal{D}\varphi|_X \cdot \left|\Psi[\varphi]\right|^2 \;\stackrel{!}{=}\; 1\,. \tag{0.74}$$

Consequently, Ψ may be viewed as an element of the Hilbert space

$$\mathcal{H}_X \;:=\; L^2(\mathcal{F}_X, \mathcal{D}\varphi|_X) \tag{0.75}$$

of square-integrable complex functionals on $\mathcal{F}_X$ obeying $\|\Psi\|^2 := \langle \Psi, \Psi \rangle < \infty$, where we denoted by $\langle .\,, . \rangle$ the Hermitian inner product on $\mathcal{H}_X$ given by

$$\langle \Phi, \Psi \rangle \;:=\; \int_{\mathcal{F}_X} \mathcal{D}\varphi|_X \cdot \Phi^*[\varphi] \cdot \Psi[\varphi]\;. \tag{0.76}$$

In fact, any non-zero $\Psi \in \mathcal{H}_X \backslash \{0\}$ induces a probability distribution on $\mathcal{F}_X$ via $p_\Psi := |\Psi|^2 / \|\Psi\|^2$, so that $p_\Psi = p_{\Psi'}$ if $\Psi' = \lambda \cdot \Psi$ for some non-zero $\lambda \in \mathbb{C}^\times$. This motivates the definition of the *projective Hilbert* (or *ray*) *space*

$$\mathcal{P}(\mathcal{H}) \;:=\; (\mathcal{H}\backslash\{0\})\,/\,\mathbb{C}^\times \tag{0.77}$$

as the space of possible *initial states* of the fields[27]. We often use the *Dirac notation* to write elements of the Hilbert space as $|\Psi\rangle \in \mathcal{H}$ and the associated dual vectors as $\langle \Psi| := \langle \Psi, . \rangle \in \mathcal{H}^\vee$. The amplitude of a functional $\Psi \in \mathcal{H}_X$ on a field configuration $\varphi \in \mathcal{F}_X$ may then be written as

$$\Psi[\varphi] \;=\; \langle \varphi | \Psi \rangle \tag{0.78}$$

where $\langle \varphi | \in \mathcal{H}_X^\vee$ represents the *'delta functional'* on $\mathcal{F}_X$ that is peaked at φ.

[25] Here, we denote by $\mathcal{D}\varphi|_X$ the *disintegration* of the measure $\mathcal{D}\varphi$ to $\mathcal{F}_X$, which formally captures the 'restriction' of $\mathcal{D}\varphi$ onto the space of field configurations living on X.

[26] For $z \in \mathbb{C}$, we denote its complex conjugate and modulus squared by z^* and $|z|^2 = z^* z$, respectively.

[27] In the following, we will use the term 'state' rather loosely to refer to non-zero vectors in $\mathcal{H}$ as well as their image under the canonical projection map $\mathcal{H} \to \mathcal{P}(\mathcal{H})$.

2. **Time evolution:** Given a *bordism*[28] between spatial hypersurfaces X and Y (i.e. a d-dimensional spacetime manifold M with an orientation-preserving diffeomorphism $\partial M \cong \overline{X} \sqcup Y$, see Figure 12), we define an associated *time evolution operator*

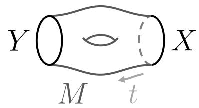


Figure 12

$$\mathcal{U}(M): \mathcal{H}_X \to \mathcal{H}_Y \tag{0.79}$$

to be the linear map that sends an initial state $|\Psi\rangle \in \mathcal{H}_X$ to the final state $\mathcal{U}(M)\,|\Psi\rangle \in \mathcal{H}_Y$ defined via the *Feynman path integral*

$$\langle \phi | \mathcal{U}(M) | \Psi \rangle \;:=\; \int\limits_{\substack{\mathcal{F}(M):\\ \varphi|_Y = \phi}} \mathcal{D}\varphi \cdot e^{iS_M[\varphi]} \cdot \Psi[\varphi|_X]\,. \tag{0.80}$$

Intuitively, what this means is that in order to obtain the amplitude of the final state on a field configuration $\phi \in \mathcal{F}_Y$ on Y, we integrate over all field configurations φ on M that restrict to ϕ on Y, weighted by their amplitudes w.r.t. the initial state $|\Psi\rangle$ on X. Furthermore, we weight any such $\varphi \in \mathcal{F}(M)$ by the exponential of (i times) the action functional evaluated on φ, the effect of which can be described using the so-called *stationary phase approximation*: If we momentarily reintroduce *Planck's constant*[29] $\hbar$ by rescaling the action $S \to \frac{1}{\hbar}S$, then in the limit where $\hbar \to 0$ (which is also called the *classical limit*) we can approximate a generic path integral by

$$\int_{\mathcal{F}} \mathcal{D}\varphi \cdot e^{\frac{i}{\hbar}S[\varphi]} \cdot F[\varphi] \;\;\simeq\;\; \sum_{\varphi_0 \in \mathcal{F}_0} \frac{e^{\frac{i}{\hbar}S[\varphi_0]}}{|\mathrm{H}_{\varphi_0}(S)|^{1/2}} \cdot F[\varphi_0] \;+\; \mathcal{O}(\hbar)\,, \tag{0.81}$$

where $\mathrm{H}_{\varphi_0}(S) := (\delta^2 S/\delta\varphi^2)|_{\varphi_0}$ denotes the *Hessian matrix* of the action S at φ_0 and F is an arbitrary (suitably fast decaying) functional on the space of fields. This shows that when $\hbar \to 0$, the dominant contributions are given by the classical on-shell solutions $\varphi_0 \in \mathcal{F}_0$ of the equations of motion (0.29) as expected. For finite $\hbar$, however, one receives additional contributions from so-called *off-shell* field configurations, which yield quantum corrections to physical observables computed from the path integral.

In order for the above construction of initial states and their time evolution to be well-behaved, we often make the following assumptions about the underlying spaces of fields and associated action functionals [27]:

[28] In order for the gluing of bordisms to be well-defined, one usually equips them with additional data such as collars around their incoming and outgoing boundary components (see e.g. [26] for further details). We will not dwell on this technical issue in what follows. We will frequently use the notation $X \xrightarrow{M} Y$ to denote bordisms from X to Y. Note that, for later convenience, we pictorially represent bordisms with the incoming boundary X on the *right*, e.g. in Figure 12.

[29] For the remainder of this thesis, we choose units in which $\hbar = 1$.

- **Additivity:** Given two spacetimes M and M', the space of fields on their disjoint union is given by the Cartesian product $\mathcal{F}(M \sqcup M') = \mathcal{F}(M) \times \mathcal{F}(M')$ (and similarly for lower-dimensional submanifolds $X \subset M$). Furthermore, given field configurations $\varphi \in \mathcal{F}(M)$ and $\varphi' \in \mathcal{F}(M')$, we have that

$$S_{M\sqcup M'}[(\varphi,\varphi')] \;=\; S_M[\varphi] \,+\, S_{M'}[\varphi'] \,. \tag{0.82}$$

- **Cutting & Gluing:** Given a spacetime M, we can cut it along a codimension-one hypersurface $Y \subset M$ to obtain a new spacetime $M_{||Y}$ with two boundary components diffeomorphic to Y as shown in Figure 13. Every field $\varphi \in \mathcal{F}(M)$ on M then determines a field $\varphi_{||Y} \in \mathcal{F}(M_{||Y})$ on $M_{||Y}$ that has the same boundary values on the two components of $\partial M_{||Y}$ corresponding to Y. We demand that

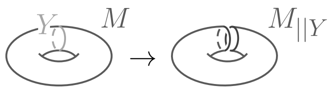


Figure 13

$$S_M[\varphi] \;=\; S_{M_{||Y}}[\varphi_{||Y}] \,. \tag{0.83}$$

- **Orientation:** Given a spacetime M, we have that $\mathcal{F}(M) = \mathcal{F}(\overline{M})$, where $\overline{M}$ denotes the orientation-reversal of M. Furthermore, we have that

$$S_{\overline{M}}[\varphi] \;=\; -\,S_M[\varphi] \,. \tag{0.84}$$

Using these assumptions, one can check that the quantisation procedure that assigns $X \mapsto \mathcal{H}_X$ and $M \mapsto \mathcal{U}(M)$ has the following properties:

- **Multiplicativity:** Given two disjoint spatial hypersurfaces X and Y, the Hilbert space assigned to their union is given by

$$\begin{aligned} \mathcal{H}_{X\sqcup Y} \;&=\; L^2(\mathcal{F}_{X\sqcup Y}, \mathcal{D}\varphi|_{X\sqcup Y}) \\ &=\; L^2(\mathcal{F}_X \times \mathcal{F}_Y, \mathcal{D}\varphi|_X \otimes \mathcal{D}\varphi|_Y) \\ &\cong\; L^2(\mathcal{F}_X, \mathcal{D}\varphi|_X) \otimes L^2(\mathcal{F}_Y, \mathcal{D}\varphi|_Y) \\ &=\; \mathcal{H}_X \otimes \mathcal{H}_Y \,, \end{aligned} \tag{0.85}$$

where we used the additivity axiom as well as the fact that (under suitable assumptions) the L^2-space of the product of two measure spaces is (isometrically) isomorphic to the tensor product of their individual L^2-spaces [28]. Furthermore, we assumed that the restricted measure behaves like $\mathcal{D}\varphi|_{X\sqcup Y} = \mathcal{D}\varphi|_X \otimes \mathcal{D}\varphi|_Y$, where the right hand side denotes the product measure on $\mathcal{F}_X \times \mathcal{F}_Y$. Physically, the relation (0.85) means that states in composite quantum systems consist of (superpositions of) product states for the individual components.

- **Functoriality:** Given spatial hypersurfaces X, Y, and Z and bordisms $X \xrightarrow{M} Y$ and $Y \xrightarrow{M'} Z$, the composition of the associated time evolution operators obeys

$$\begin{aligned}
\langle\phi|\,\mathcal{U}(M')\circ\mathcal{U}(M)\,|\Psi\rangle \;&=\; \int\limits_{\substack{\mathcal{F}(M'):\\ \varphi'|_Z=\phi}} \mathcal{D}\varphi' \cdot e^{iS_{M'}[\varphi']} \cdot \int\limits_{\substack{\mathcal{F}(M):\\ \varphi|_Y=\varphi'|_Y}} \mathcal{D}\varphi \cdot e^{iS_M[\varphi]} \cdot \Psi[\varphi|_X] \\
&=\; \int\limits_{\substack{\mathcal{F}(M'\Box M):\\ \tilde{\varphi}|_Z=\phi}} \mathcal{D}\tilde{\varphi} \,\cdot\, e^{iS_{M'\Box M}[\tilde{\varphi}]} \,\cdot\, \Psi[\tilde{\varphi}|_X] \qquad (0.86)\\
&=\; \langle\phi|\,\mathcal{U}(M'\Box M)\,|\Psi\rangle\ .
\end{aligned}$$

Here, we first used the additivity axiom (0.82) and then identified $M \sqcup M'$ with $(M'\Box M)_{||Y}$, where $M'\Box M$ denotes the result of gluing the bordisms M and M' together along Y as illustrated in Figure 14. Similarly, we identified pairs of field configurations $\varphi \in \mathcal{F}(M)$ and $\varphi' \in \mathcal{F}(M')$ satisfying $\varphi|_Y = \varphi'|_Y$ with $\tilde{\varphi}_{||Y}$ for some $\tilde{\varphi} \in \mathcal{F}(M'\Box M)$, allowing us to apply the cutting and gluing axiom (0.83). Since (0.86) holds for all $\phi \in \mathcal{F}_Z$ and $|\Psi\rangle \in \mathcal{H}_X$, we obtain the operator equation[30]

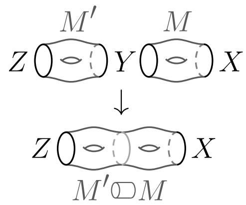


Figure 14

$$\mathcal{U}(M')\circ\mathcal{U}(M) \;=\; \mathcal{U}(M'\Box M)\ . \qquad (0.87)$$

Physically, this means that we can divide the time evolution of states into steps (i.e. evolving an initial state from X to Y and then from Y to Z is equivalent to evolving it from X to Z).

- **Unitarity:** Given a bordism $X \xrightarrow{M} Y$ and states $|\Psi\rangle \in \mathcal{H}_X$ and $|\Phi\rangle \in \mathcal{H}_Y$, we can compute the complex conjugated overlap

$$\begin{aligned}
\langle\Phi|\,\mathcal{U}(M)\,|\Psi\rangle^* \;&=\; \int\limits_{\mathcal{F}(M)} \mathcal{D}\varphi \cdot e^{-iS_M[\varphi]} \cdot \Phi[\varphi|_Y] \cdot \Psi^*[\varphi|_X] \\
&=\; \int\limits_{\mathcal{F}(\overline{M})} \mathcal{D}\varphi \cdot e^{iS_{\overline{M}}[\varphi]} \cdot \Psi^*[\varphi|_{\overline{X}}] \cdot \Phi[\varphi|_{\overline{Y}}] \qquad (0.88)\\
&=\; \langle\Psi|\,\mathcal{U}(M^\dagger)\,|\Phi\rangle\ ,
\end{aligned}$$

where we used the orientation axiom (0.84) and denoted by $Y \xrightarrow{M^\dagger} X$ the bordism that is constructed out of M using the following two canonical operations: Given any bordism $X \xrightarrow{M} Y$ with boundary $\partial M \cong \overline{X} \sqcup Y$, we can assign to it

30 Here, we see that representing bordisms pictorially with the incoming boundary to the right has the advantage that the gluing of bordisms interplays nicely with the composition of the associated time evolution operators as in (0.87).

1. the *dual bordism* $\overline{Y} \xrightarrow{M^\vee} \overline{X}$, where $M^\vee = M$ as an oriented manifold viewed as a bordism with boundary $\overline{\overline{Y}} \sqcup \overline{X}$ (using $\overline{\overline{Y}} \cong Y$),
2. the *opposite bordism* $\overline{X} \xrightarrow{\overline{M}} \overline{Y}$ by flipping the orientation on M and its boundary components X and Y.

We then set $M^\dagger := \overline{M}^\vee$, which corresponds to 'turning the bordism M around' and subsequently reversing its orientation. Since (0.88) holds for all $|\Psi\rangle$ and $|\Phi\rangle$, we obtain the operator equation[31]

$$\mathcal{U}(M^\dagger) \; = \; \mathcal{U}(M)^\dagger \; , \tag{0.89}$$

which states that performing the time evolution along $M^\dagger$ is equivalent to reversing the time evolution along M[32].

Operators

In general, one can consider time evolutions along bordisms $X \xrightarrow{M} Y$ together with so-called *path integral insertions*. Concretely, given a (suitably well-behaved) functional $F : \mathcal{F}(M) \to \mathbb{C}$ on the space of fields on M, we can contruct an associated operator

$$\mathcal{U}(M; F) : \; \mathcal{H}_X \; \to \; \mathcal{H}_Y \tag{0.90}$$

via the path integral that is weighted by F,

$$\langle \phi | \mathcal{U}(M; F) | \Psi \rangle \;\; := \int\limits_{\substack{\mathcal{F}(M): \\ \varphi|_Y = \phi}} \mathcal{D}\varphi \cdot F[\varphi] \cdot e^{iS_M[\varphi]} \cdot \Psi[\varphi|_X] \; , \tag{0.91}$$

where $\phi \in \mathcal{F}_Y$ and $|\Psi\rangle \in \mathcal{H}_X$. We will typically not distinguish between F and $\mathcal{U}(M; F)$ and refer to both simply as 'operators' in what follows. Furthermore, we are often only interested in a small subset of operators that have a well-defined *support* in M. Concretely, given a submanifold $\Sigma \subset M$, we say that a functional F has support in Σ ($\text{supp}(F) = \Sigma$) if for any field $\varphi \in \mathcal{F}(M)$ the amplitude $F[\varphi]$ only depends on the values of φ in an infinitesimal neighbourhood of Σ (see Figure 15). We often write F_Σ for an operator F whose support is given by Σ. It is useful to think of the operator F as the assignment $\Sigma \mapsto F_\Sigma$, which (in favourable circumstances) allows us to obtain

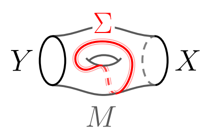


Figure 15

[31] Given a linear map $f : \mathcal{H} \to \mathcal{H}'$ between Hilbert spaces, we denote by $f^\dagger$ the *adjoint* of f, which is the unique linear map $f^\dagger : \mathcal{H}' \to \mathcal{H}$ satisfying $\langle f(\Phi), \Psi \rangle' = \langle \Phi, f^\dagger(\Psi) \rangle$ for all $\Phi \in \mathcal{H}$ and $\Psi \in \mathcal{H}'$.

[32] Note that (0.89) does *not* imply that $\mathcal{U}(M)$ is *unitary* in the sense that $\mathcal{U}^\dagger \circ \mathcal{U} = \mathcal{U} \circ \mathcal{U}^\dagger = \text{id}$. This reflects the fact that M may induce a change in topology from its incoming boundary to its outgoing boundary, so that in general we only expect unitary time evolution to exist for topologically trivial bordisms of the form $M \cong X \times [0, t]$. In other words, unitary time evolution is not a built-in feature of quantum theory but rather a consequence of specific assumptions about the nature of spacetime, as pointed out in [29, 30].

a well-defined path integral insertion for *any* suitable submanifold $\Sigma \subset M$. As a result, we can organise the spectrum of operators according to the dimensionality of their support. In particular, we distinguish between the following two types of operators:

- **Local:** We say that an operator F is *local* if $\dim(\mathrm{supp}(F)) = 0$, meaning that F is supported in an infinitesimal neighbourhood of points in M. We denote by $\mathcal{O}_0$ the space of all local operators, which forms a (generally infinite-dimensional) complex vector space via point-wise addition and scalar multiplication of functionals.

- **Extended:** We say that an operator F is *extended* if $1 \leq \dim(\mathrm{supp}(F)) \leq d-1$. We denote the space of all p-dimensional operators by

$$\mathcal{O}_p \;:=\; \{F \mid \dim(\mathrm{supp}(F)) = p\}\;. \tag{0.92}$$

Apart from *genuine* operators that can be freely inserted on any p-dimensional submanifold, this also includes *non-genuine* operators that sit 'inbetween' non-trivial higher-dimensional operators. Concretely, given two p-dimensional operators $A, B \in \mathcal{O}_p$, we denote by[33] $\mathrm{Hom}_{\mathcal{O}}(A,B) \subset \mathcal{O}_{p-1}$ the space of all $(p-1)$-dimensional operators that can sit at the interface between A and B. Similarly, given $\mu, \nu \in \mathrm{Hom}_{\mathcal{O}}(A,B)$, we denote by $\mathrm{Hom}_{\mathcal{O}}(\mu,\nu) \subset \mathcal{O}_{p-2}$ the space of all $(p-2)$-dimensional operators Θ that can sit at the junction between μ and ν, et cetera. This is illustrated in Figure 16 for $p = 2$. We will provide explicit examples of non-genuine operators further below. Note that for each $F \in \mathcal{O}_p$, there is the 'trivial' interface $\mathrm{id}_F \in \mathrm{Hom}(F,F)$ that corresponds to inserting the trivial functional[34] into the path integral.

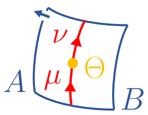


Figure 16

A priori, the space $\mathcal{O}_p$ of p-dimensional operators is again a complex vector space for $p > 0$ via point-wise addition and scalar multiplication of functionals. However, since extended operators often carry a preferred normalisation[35], it is useful to relocate parts of this linear structure to the space $\mathcal{O}_0$ of local operators. Concretely, while we use point-wise addition to define *direct sums* $F \oplus F'$ of p-dimensional operators $F, F' \in \mathcal{O}_p$, we interpret expressions such as $\lambda \cdot F$ (with $\lambda \in \mathbb{C}$) as inserting a multiple of the trivial local operator $\mathrm{id}_F^p \in \mathcal{O}_0$ onto (an arbitrary point on the support of) F as illustrated in Figure 17. This allows us to reduce the linear structure of arbitrary operators to the linear structure of the vector space $\mathcal{O}_0$.

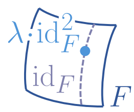


Figure 17

[33] Despite the notation, we do not (yet) think of $\mathcal{O} := \sqcup_{p=0}^{d-1} \mathcal{O}_p$ as a (higher) category due to the lack of appropriate notions of composition. This will change once we restrict ourselves to *topological* operators when discussing generalised global symmetries in Section 0.1.3.

[34] Here, the 'trivial functional' is the one that maps every field configuration in $\mathcal{F}(M)$ to $1 \in \mathbb{C}$.

[35] For example, Wilson lines in gauge theory carry a preferred normalisation corresponding to a choice of character on the gauge group; see further details below.

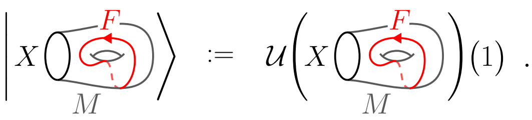
As a special case of the above considerations, we can look at bordisms with partially empty boundary components. For instance, consider a bordism $\varnothing \xrightarrow{M} X$, which corresponds to a spacetime M with only an outgoing boundary component X. Using the canonical identification $\mathcal{H}_\varnothing \cong \mathbb{C}$, this induces a linear map

$$\mathcal{U}(M;F) : \mathbb{C} \to \mathcal{H}_X \tag{0.93}$$

for any operator insertion F on M, which we can canonically identify with a state

$$|M;F\rangle := \mathcal{U}(M;F)(1) \tag{0.94}$$

in the Hilbert space $\mathcal{H}_X$ associated to X. This mapping of operators on spacetime to states on its boundary is called the *operator-state map*[36]. Pictorially, we write it as



(0.95)

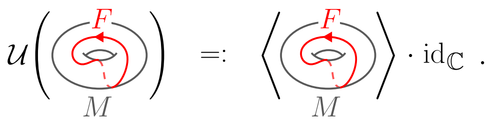
If we furthermore assume that $X = \varnothing$ (so that M is closed), we obtain a linear map

$$\mathcal{U}(M;F) : \mathbb{C} \to \mathbb{C} , \tag{0.96}$$

which we can canonically identify with a complex number

$$\mathcal{U}(M;F) =: \langle M;F\rangle \cdot \mathrm{id}_\mathbb{C} \tag{0.97}$$

called the *correlation function*[37] of F on M. Pictorially, we write this as



(0.98)

The correlation function on a closed spacetime M with no operator insertions is called the *partition function* of M and denoted by

$\mathcal{Z}(M) :=$  .

(0.99)

---

[36] In general, this map is not surjective. However, in special cases such as *conformal field theory* (CFT), one can use the operator-state map to construct a bijection between operators on spacetime and states in the Hilbert space [31]. In this case, one speaks of the operator-state *correspondence*.

[37] Despite $\langle M;F\rangle$ being a complex number, it is referred to as a correlation *function* because it (in principle) depends on the support Σ of the operator F. Studying $\langle M;F\rangle$ as a function of Σ for different operator insertions F typically gives rise to a rich algebraic structure and is one of the main objectives of quantum field theory.

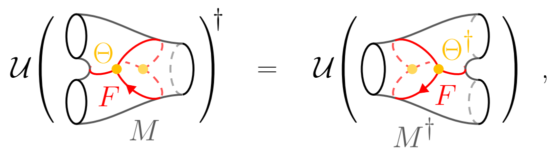
We can derive general properties of correlation functions using the path integral prescription (0.91). For instance, repeating the computation (0.88) with a generic insertion $F : \mathcal{F}(M) \to \mathbb{C}$ leads to the operator equation

$$\mathcal{U}(M;F)^\dagger \;=\; \mathcal{U}(M^\dagger;F^\dagger)\;, \tag{0.100}$$

where $F^\dagger : \mathcal{F}(M^\dagger) \to \mathbb{C}$ is given by the complex conjugated functional $F^\dagger[\varphi] := F[\varphi]^*$. In analogy to the orientation axiom (0.84), we often assume that complex conjugating an *extended* operator F reverses the orientation of its support Σ, i.e.

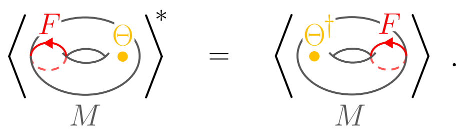
$$F_\Sigma[\varphi]^* \;=\; F_{\overline{\Sigma}}[\varphi] \tag{0.101}$$

when $\dim(\Sigma) > 0$. As a result, we can interpret (0.100) as stating that turning the bordism M and its extended operator insertions F around and subsequently flipping their orientation is equivalent to reversing the 'decorated' time evolution described by $\mathcal{U}(M;F)$. Pictorially, we write this as



(0.102)

where we kept the notation $\Theta^\dagger$ for *local* operators $\Theta \in \mathcal{O}_0$. In particular, by restricting to closed spacetimes M, we see that 'reflecting' the operator content of a correlation function is equivalent to complex conjugation:



(0.103)

As a result, we have that reflection-symmetric correlation functions are necessarily real-valued (and in fact non-negative since they correspond to the norm of a state on a reflection-symmetric hyperplane Π in spacetime). In the context of Euclidean quantum field theory, this is often called the principle of *reflection positivity* [32–35].

Schrödinger Picture

For practical purposes, it is often useful to work infinitesimally in the neighbourhood of a given spatial hypersurface $X \subset M$. Concretely, upon choosing a normal direction to X, we may assume that spacetime locally looks like $M \cong X \times \mathbb{R}$, where $\mathbb{R}$ parameterises the direction of time. If we denote by $\mathcal{H} := \mathcal{H}_X$ the Hilbert space associated to X, we can then describe the time evolution of states using the operator

$$U(t) \;:=\; \mathcal{U}(X \times [0,t])\,, \tag{0.104}$$

which as a consequence of (0.87) and (0.89) obeys

$$U(t) \circ U(t') \;=\; U(t+t') \qquad \text{and} \qquad U(t)^\dagger \;=\; U(-t)\,. \tag{0.105}$$

Furthermore, we assume that $\lim_{t\to 0} U(t) = \mathrm{id}_{\mathcal{H}}$, which together with (0.105) implies that $U(t)$ defines a *unitary* operator on $\mathcal{H}$ for all t. In particular, the time evolution

$$|\Psi(t)\rangle \;:=\; U(t)\,|\Psi\rangle \tag{0.106}$$

of a state $|\Psi\rangle \in \mathcal{H}$ preserves probabilities in the sense that $\|\Psi(t)\|^2 = \|\Psi\|^2$ for all t. Moreover, the rate of change of $|\Psi(t)\rangle$ can be computed to be

$$\begin{aligned} \frac{d}{dt}\,|\Psi(t)\rangle \;&=\; \frac{d}{dt}\,U(t)\,|\Psi\rangle \\ &=\; \lim_{h\to 0}\left(\frac{U(t+h)-U(t)}{h}\right)|\Psi\rangle \\ &=\; \left[\lim_{h\to 0}\left(\frac{U(h)-U(0)}{h}\right)\circ U(t)\right]|\Psi\rangle \\ &=\; (-i)\cdot \widehat{H}\,|\Psi(t)\rangle\ , \end{aligned} \tag{0.107}$$

where we defined the so-called *Hamiltonian operator* $\widehat{H}$ on $\mathcal{H}$ by

$$\widehat{H} \;:=\; i \cdot \lim_{h\to 0}\left(\frac{U(h)-\mathrm{id}_{\mathcal{H}}}{h}\right)\,. \tag{0.108}$$

Rearranging (0.107) then gives the *Schrödinger equation*

$$i\,\frac{d}{dt}\,|\Psi(t)\rangle \;=\; \widehat{H}\,|\Psi(t)\rangle\ , \tag{0.109}$$

which is the quantum analogue of Newton's law of motion describing the time evolution of physical states via a differential equation. The solutions to (0.109) are given by

$$|\Psi(t)\rangle \;=\; e^{-i\widehat{H}t}\,|\Psi\rangle\ , \tag{0.110}$$

which allows us to identify the time evolution operator with $U(t) = e^{-i\widehat{H}t}$. According to Stone's theorem [36], we then have that $U(t)$ is unitary for all t if and only if $\widehat{H}$ is *Hermitian* (or *self-adjoint*) in the sense that $\widehat{H}^\dagger = \widehat{H}$. In this case, $\widehat{H}$ induces an orthogonal decomposition[38]

$$\mathcal{H} \;\cong\; \bigoplus_n \mathcal{H}_n \tag{0.111}$$

[38] More precisely, if $\widehat{H}$ is a compact self-adjoint operator on $\mathcal{H}$, then the *spectral theorem* states that there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $\widehat{H}$ with real eigenvalues [37].

of the Hilbert space into *eigenspaces* $\mathcal{H}_n$ of $\widehat{H}$, where each $|\Psi_n\rangle \in \mathcal{H}_n$ obeys

$$\widehat{H}\,|\Psi_n\rangle \;=\; E_n \cdot |\Psi_n\rangle \tag{0.112}$$

for some (necessarily real) *eigenvalue* $E_n \in \mathbb{R}$ (called the *energy* of $|\Psi_n\rangle$). The set $\{E_n\}_n$ of all eigenvalues is called the *spectrum* of the Hamiltonian, which we assume to be bounded from below. We call the eigenspace $\mathcal{H}_0$ associated to the lowest energy E_0 (if it exists) the space of *ground states* (or *vacua*). We say that an eigenspace $\mathcal{H}_n$ is *degenerate* if $\dim(\mathcal{H}_n) > 1$. Determining the spectrum and its associated degeneracies of a given Hamiltonian is one of the main objectives of quantum field theory.

More generally, we can use the decorated time evolution (0.90) to construct operators on the Hilbert space from generic path integral insertions. Concretely, given a functional F whose support is entirely contained in the spatial hypersurface X, we can define an associated operator $\widehat{F} : \mathcal{H} \to \mathcal{H}$ via

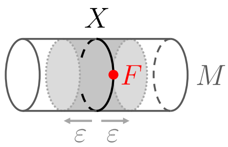


Figure 18

$$\widehat{F} \;:=\; \lim_{\varepsilon\to 0} \mathcal{U}(X_\varepsilon; F)\;, \tag{0.113}$$

where we used the notation $X_\varepsilon := X \times (-\varepsilon, \varepsilon)$. This is illustrated in Figure 18. Intuitively, the operator $\widehat{F}$ allows us to compute 'expectation values'

$$\langle F\rangle_\Psi \;:=\; \frac{\langle\Psi|\widehat{F}|\Psi\rangle}{\|\Psi\|^2} \tag{0.114}$$

of the functional F w.r.t. the probability distribution $p_\Psi = |\Psi|/\|\Psi\|^2$ induced by a non-zero state $|\Psi\rangle \in \mathcal{H}$ on X. Since, from a physical point of view, we are mostly interested in measuring quantities that are real-valued, we often restrict ourselves to functionals F obeying $F^\dagger = F$ (which are also called *observables*). The associated operators $\widehat{F}$ on $\mathcal{H}$ then satisfy

$$\begin{aligned}\widehat{F}^\dagger \;&\equiv\; \lim_{\varepsilon\to 0} \mathcal{U}(X_\varepsilon; F)^\dagger \\ &=\; \lim_{\varepsilon\to 0} \mathcal{U}(X_\varepsilon^\dagger; F^\dagger) \\ &=\; \lim_{\varepsilon\to 0} \mathcal{U}(X_\varepsilon; F) \;\equiv\; \widehat{F}\;,\end{aligned} \tag{0.115}$$

where we used (0.100) as well as the fact that $X_\varepsilon^\dagger \cong X_\varepsilon$. Hence, we see that physical observables are implemented by Hermitian operators on the Hilbert space. In particular, this ensures that their expectation value $\langle F\rangle_\Psi$ is real for any state $|\Psi\rangle$. Moreover, the fact that observables correspond to operators has important consequences for our ability to measure different observables simultaneously. Concretely, if we define the *variance* of an observable F in a state $|\Psi\rangle$ by

$$\sigma(F)_\Psi \;:=\; \sqrt{\langle F^2\rangle_\Psi - \langle F\rangle_\Psi^2}\;, \tag{0.116}$$

then the Cauchy-Schwarz inequality for inner product spaces implies the so-called *Robertson uncertainty relation* [38]

$$\sigma(A)_\Psi\cdot\sigma(B)_\Psi \;\geq\; \frac{1}{2}\left|\big\langle[\widehat{A},\widehat{B}]\big\rangle_\Psi\right| \tag{0.117}$$

for any two observables A and B, where we defined the *commutator* of $\widehat{A}$ and $\widehat{B}$ by $[\widehat{A},\widehat{B}] := \widehat{A}\circ\widehat{B} - \widehat{B}\circ\widehat{A}$. Physically, this means that, given two observables, the precision with which we can measure both of them at the same time in a given state $|\Psi\rangle$ is bounded from below by the expectation value of their commutator.

Examples of quantum field theories and their associated spectra of (local and extended) operators include the following:

- Qubits: A *qubit* is the quantum version of a *bit*, which is a classical system consisting of a single variable s (called *spin*) that can only occupy two possible states (denoted $+$ and $-$ and called 'up' and 'down', respectively). The associated Hilbert space is $\mathcal{H} = \mathbb{C}^2$, which is spanned by the two basis vectors $|+\rangle := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|-\rangle := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. The most general form of the Hamiltonian is given by

$$\widehat{H} \;=\; A\cdot\mathbb{1}_2 + \vec{B}\cdot\vec{\sigma}\;, \tag{0.118}$$

where $A\in\mathbb{R}$ and $\vec{B}\in\mathbb{R}^3$ and we denoted by $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ the vector that contains the three so-called *Pauli-matrices*

$$\sigma_x \;:=\; \begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \sigma_y \;:=\; \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \qquad \sigma_z \;:=\; \begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{0.119}$$

Together with the identity matrix $\mathbb{1}_2$, these form a basis for the space of all Hermitian (2×2)-matrices. Since the parameter A simply shifts the Hamiltonian by a constant, we can without loss of generality assume that $A = 0$ in what follows. The eigenvalues of $\widehat{H}$ are then simply given by

$$E_\pm \;=\; \pm B\;, \tag{0.120}$$

where we denoted by $B := \|\vec{B}\|$ the magnitude of the vector $\vec{B}$. Moreover, the associated time evolution operator can be computed to be

$$U(t) \;\equiv\; e^{-i\widehat{H}t} \;=\; \cos(Bt)\cdot\mathbb{1}_2 \;-\; i\sin(Bt)\cdot(\vec{b}\cdot\vec{\sigma})\;, \tag{0.121}$$

where $\vec{b} := \vec{B}/B$ denotes the unit vector in the direction of $\vec{B}$. Using this, we can compute the expectation value of the *spin operator* $\hat{s} := \sigma_z$ (which measures whether the spin s is up or down) in the time evolved state $U(t)\,|\pm\rangle$ to be

$$\langle s(t)\rangle_{\pm} \;:=\; \langle s\rangle_{U(t)|\pm\rangle} \;=\; \pm\left[1 - 2\cdot b_{\perp}^{2}\cdot \sin^{2}(Bt)\right]\,, \tag{0.122}$$

where we denoted by $b_{\perp} := \|\vec{b}_{\perp}\|$ the magnitude of the transverse component $\vec{b}_{\perp} := \vec{b} - b_z\vec{e}_z$ of the unit vector $\vec{b}$ that is perpendicular to the z-axis in $\mathbb{R}^3$. Physically, this means that if we view the spin s as being aligned along the z-axis and coupled to a constant magnetic field $\vec{B}$, then the latter induces an oscillation of s whose frequency is given by the magnitude B and whose amplitude is determined by the normalised transverse component $\vec{b}_{\perp}$ of $\vec{B}$.

- Quantum mechanics: The quantum theory of probe particles moving on a Riemannian manifold N can be constructed by introducing the so-called *Wiener measure* [39, 40] on the space $\mathcal{F}$ of (continuous) paths $\gamma : [a,b] \to N$. Since the boundaries of time intervals are points and $\mathcal{F}|_{\mathrm{pt}} \cong N$, equation (0.75) then implies that the space of states is given by the Hilbert space

$$\mathcal{H} \;=\; L^2(N) \tag{0.123}$$

of complex functions $\psi : N \to \mathbb{C}$ that are square-integrable w.r.t. the Lebesgue measure induced by the Riemannian metric on N. We usually refer to elements of $\mathcal{H}$ as *wavefunctions* of the probe particle. Physically, given a non-zero $|\psi\rangle \in \mathcal{H}$, we interpret $p_{\psi} := |\psi|^2/\,\|\psi\|^2$ as the probability density to find the probe particle in subregions $S \subset N$ as before. For instance, the state $|n\rangle$ that localises the particle at a fixed position $n \in N$ with probability 1 corresponds to the *Dirac delta function* peaked at n (see Figure 19). The probability to subsequently find the particle at a different position $n' \in N$ after some finite time t is given by

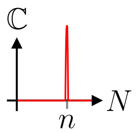


Figure 19

$$\left|\langle n'|U(t)|n\rangle\right|^2\,, \tag{0.124}$$

where, according to (0.80), the time evolution operator $U(t)$ now computes an integral over all paths[39] γ in N that start at n and arrive at n' after time t, weighted by the exponential of the action functional (0.30). This is illustrated in Figure 20.

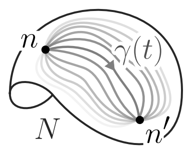


Figure 20

[39] The fact that the general expression (0.80) for the time evolution operator becomes an integral over a space of paths in quantum mechanics is the historical reason for the term 'path integral'.

As an example, consider probe particles moving along trajectories in $N = \mathbb{R}^3$ (or suitable subsets thereof). Typical observables include the position $\vec{r}$ of the particle as well as its momentum $\vec{p} = m \cdot \dot{\vec{r}}$. Using (0.113), their associated Hermitian operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ can be computed to be[40]

$$\hat{r}_i \;=\; r_i \cdot \qquad \text{and} \qquad \hat{p}_i \;=\; -i\partial_i \,, \tag{0.130}$$

where $i = x, y, z$ and $r_i\cdot$ and ∂_i denote multiplication by and differentiation w.r.t. the coordinate r_i, respectively. As a result, the *canonical commutation relations* between the position and momentum operators are given by

$$[\hat{r}_i, \hat{p}_j] \;=\; i \cdot \delta_{ij} \cdot \mathrm{id}_{\mathcal{H}} \,. \tag{0.131}$$

Plugging this into (0.117) yields the *Heisenberg uncertainty relation* [41]

$$\sigma(r) \cdot \sigma(p) \;\geq\; \frac{1}{2} \,, \tag{0.132}$$

where we assumed that we measure the variances $\sigma(r)$ and $\sigma(p)$ of the particle's position and momentum along a common spatial direction in a normalised state.

[40] For example, the expression for the momentum operator $\hat{p}$ can be derived as follows (for simplicity we restrict ourselves to the case of one spatial dimension x): Given a state $|\psi\rangle \in L^2(\mathbb{R})$, its image under the momentum operator is defined to have the position space representation

$$\langle x|\,\hat{p}\,|\psi\rangle \;:=\; \lim_{\varepsilon \to 0} \int\limits_{\substack{y\colon (-\varepsilon,\varepsilon) \to \mathbb{R},\\ y(\varepsilon) = x}} \mathcal{D}y \;\; e^{iS[y]} \cdot \psi(y(-\varepsilon)) \cdot m\dot{y}(0) \,, \tag{0.125}$$

where we used that, classically, the momentum of a particle on a trajectory $y(t)$ is given by $p = m\dot{y}$. For infinitesimal ε, we may approximate any such trajectory by $y(t) = x + \frac{\xi}{2}(\frac{t}{\varepsilon} - 1)$ with $\xi := x - y(-\varepsilon)$, so that $\dot{y} = \frac{\xi}{2\varepsilon}$ and the action functional (0.30) evaluated on y approximates to

$$S[y] \;\equiv\; \int_{-\varepsilon}^{\varepsilon} \left[\frac{1}{2}m\dot{y}^2 - V(y)\right] dt \;\approx\; \frac{m}{4\varepsilon} \cdot \xi^2 \;-\; 2V\big(x - \xi/2\big) \cdot \varepsilon \,. \tag{0.126}$$

Writing $\psi(y(-\varepsilon)) = \psi(x - \xi)$, we can then Taylor expand the integrand of (0.125) about ξ and ε as

$$e^{-\frac{m}{4i\varepsilon} \cdot \xi^2} \cdot \big[1 + \mathcal{O}(\varepsilon)\big] \cdot \big[\psi(x) \,-\, \psi'(x) \cdot \xi \,+\, \mathcal{O}(\xi^2)\big] \cdot \frac{m}{2\varepsilon}\,\xi \tag{0.127}$$

(where $'$ denotes the derivative w.r.t. x), so that switching integration variables from y to ξ and substituting in the Wiener measure $\mathcal{D}y \to \sqrt{\frac{m}{4\pi i\varepsilon}}\, d\xi$ reduces the integral in (0.125) to a series of Gaussian moments for ξ. Evaluating the latter then gives

$$\langle x|\,\hat{p}\,|\psi\rangle \;=\; \lim_{\varepsilon \to 0} \Big(- i\psi'(x) + \mathcal{O}(\varepsilon) \Big) \;\equiv\; -i\psi'(x) \,, \tag{0.128}$$

which shows that the momentum operator can be represented by $\hat{p} = -i\partial_x$ as claimed. Note that the imaginary multiplicative factor i is crucial to ensure that the operator $\hat{p}$ is hermitian:

$$\langle\varphi|\hat{p}|\psi\rangle \;=\; -i \int_{\mathbb{R}} \varphi(x)^* \cdot \psi'(x)\, dx \;=\; i \int_{\mathbb{R}} \varphi'(x)^* \cdot \psi(x)\, dx \;=\; \langle\psi|\hat{p}|\varphi\rangle^* \;, \tag{0.129}$$

where we integrated by parts in the middle and discarded boundary terms due to the fact that φ and ψ (being square-integrable) fall off sufficiently fast at infinity.

Using (0.108), the Hamiltonian can be computed to be

$$\widehat{H} \;=\; -\frac{1}{2m}\Delta + V(\vec{r})\;, \tag{0.133}$$

where $\Delta := \sum_i \partial_i^2$ denotes the *Laplacian* and V is the potential function appearing in the action (0.30). For instance, if $V(\vec{r}) = -\alpha/r$ with $\alpha \in \mathbb{R}_{>0}$, the above Hamiltonian has an infinite series of negative energy eigenvalues[41]

$$E_n \;=\; -\frac{m\alpha^2}{2(n+1)^2} \tag{0.134}$$

indexed by non-negative integers $n = 0, 1, 2, \ldots$ with associated eigenspace degeneracies $\dim(\mathcal{H}_n) = (n+1)^2$. Physically, these correspond to the bound state energies of a quantum particle in a radial $1/r$ potential (such as the electron in the Coulomb potential of the proton, which together form the hydrogen atom).

- Gauge theory: Given a Lie group G, the theory of gauge fields living on a principal G-bundle $\pi : P \to M$ has a canonical class of line operators that can be constructed using the notion of *parallel transport*. Concretely, given a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ and a path $\gamma : [a,b] \to M$ in spacetime, we denote by

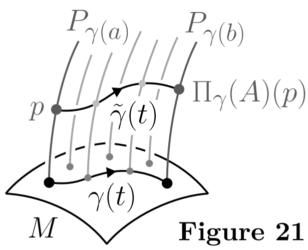


Figure 21

$$\Pi_\gamma(A) : \; P_{\gamma(a)} \;\to\; P_{\gamma(b)} \tag{0.135}$$

the associated parallel transport map[42], which is a diffeomorphism between the fibres $P_{\gamma(a)}$ and $P_{\gamma(b)}$ that is independent of the parameterisation of γ with inverse given by $\Pi_\gamma(A)^{-1} = \Pi_{\bar{\gamma}}(A)$ (where $\bar{\gamma}$ denotes the reverse path of γ) [18]. Under a bundle automorphism $f : P \to P$, it transforms as

$$\Pi_\gamma(f^*(A)) \;=\; f^{-1} \circ \Pi_\gamma(A) \circ f\;, \tag{0.136}$$

which shows that parallel transport is not itself gauge-invariant (as required for a valid path integral insertion). However, we can build gauge-invariant quantities from it by introducing so-called 'matter fields'. Concretely, given a vector space V equipped with a linear (left) G-action $\triangleright : G \times V \to V$, we define the *associated bundle* to be the vector bundle over M with total space

$$P_V \;:=\; (P \times V)/\sim\;, \tag{0.137}$$

[41] The fact that (a part of) the spectrum of a given Hamiltonian is often quantised such as in (0.134) is one of the historical reason for the term 'quantum theory'.
[42] Concretely, the map (0.135) is defined as follows: Given $p \in P_{\gamma(a)}$, we denote by $\tilde{\gamma} : [a,b] \to P$ the unique horizontal lift of γ that satisfies $\pi \circ \tilde{\gamma} = \gamma$ as well as $\dot{\tilde{\gamma}} \in \ker(A)$ and $\tilde{\gamma}(a) = p$. We then set $\Pi_\gamma(A)(p) := \tilde{\gamma}(b)$. This is illustrated in Figure 21.

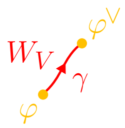
where the equivalence relation identifies $(p \triangleleft g, v) \sim (p, g \triangleright v)$ for $p \in P$, $v \in V$ and $g \in G$. Parallel transport then induces a linear map

$$\Pi^V_\gamma(A):\ (P_V)_{\gamma(a)} \ \to\ (P_V)_{\gamma(b)} \tag{0.138}$$

which sends[43] $[p, v] \mapsto [\Pi_\gamma(A)(p), v]$. If we denote by $V^\vee$ and V^* the dual and complex conjugate space of V equipped with their natural G-actions, respectively, the canonical identifications $(P_V)^\vee \cong P_{V^\vee}$ and $(P_V)^* \cong P_{V^*}$ then imply that

$$[\Pi^V_\gamma(A)]^\vee \ =\ \Pi^{V^\vee}_{\bar{\gamma}}(A) \qquad \text{and} \qquad [\Pi^V_\gamma(A)]^* \ =\ \Pi^{V^*}_\gamma(A)\ . \tag{0.139}$$

If we now assume that our theory contains *matter fields* in the form of sections[44] $\varphi \in \Gamma(P_V)$ and $\varphi^\vee \in \Gamma(P_{V^\vee})$, then the scalar quantity

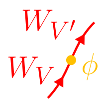


$$:=\ \Big\langle \varphi^\vee_{\gamma(b)}\,,\, \Pi^V_\gamma(A)(\varphi_{\gamma(a)})\Big\rangle \tag{0.140}$$

is invariant under gauge transformations $f : P \to P$ if we simultaneously shift[45]

$$A \ \to\ f^*(A) \qquad \text{and} \qquad \varphi^{(\vee)} \ \to\ f^{-1} \circ \varphi^{(\vee)}\ . \tag{0.141}$$

We call the quantity in (0.140) a *Wilson line* associated to the linear G-space V with endpoints dressed by the matter fields φ and $\varphi^\vee$. This is an example of an extended operator that supports non-genuine local operators at its boundary. More generally, we can consider local junctions



(0.142)

between Wilson lines W_V and $W_{V'}$ associated to (possibly distinct) G-spaces V and V'. By the same reasoning as above, these can be constructed using sections ϕ of the morphism vector bundle

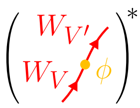
$$P_{V^\vee} \otimes P_{V'} \ \cong\ P_{\mathrm{Hom}(V,V')}\ . \tag{0.143}$$

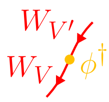
[43] Here, we use the fact that parallel transport obeys $\Pi_\gamma(A) \circ R_g = R_g \circ \Pi_\gamma(A)$ for all $g \in G$ (where $R_g : P \to P$ denotes the right action of g on P), which makes the map (0.138) well-defined.

[44] Given a vector bundle $\pi_E : E \to M$ over M, a *section* of E is a smooth map $s : M \to E$ such that $\pi_E \circ s = \mathrm{id}_M$. We denote the vector space of all smooth sections of E by $\Gamma(E)$.

[45] Here, we use the fact that any bundle automorphism $f : P \to P$ induces an automorphism $P_V \to P_V$ of the associated bundle via $[p, v] \mapsto [f(p), v]$.

If we furthermore assume that V and V' are *unitary* (meaning that $(V^*)^\vee \cong V$ as G-sets and similarly for V'), then (0.139) implies the complex conjugation law



$=$


, (0.144)

where $\phi^\dagger \in \Gamma(P_{\mathrm{Hom}(V',V)})$ is defined by taking pointwise adjoints of linear maps. This is an example of the general conjugation law (0.101) we typically impose on extended operators. Finally, we can consider paths γ whose start and end point coincide (i.e. $\gamma(a) = \gamma(b) =: m$). Parallel transport then induces a linear map $\Pi_\gamma^V(A) : (P_V)_m \to (P_V)_m$, which allows us to define the so-called *Wilson loop*

$$:= \mathrm{Tr}_{(P_V)_m}(\Pi_\gamma^V(A)) \tag{0.145}$$

independently of additional matter fields. One can check that this construction is independent of the choice of base point m of the loop.

As an example, consider a $G = U(1)$ gauge theory with gauge fields given by connection 1-forms[46] $A \in \Omega^1(P, \mathfrak{u}(1))$. Upon choosing a local section $s : U \to P$ that trivialises the principal bundle over an open set $U \subset M$, we can pull back the gauge field to obtain a local one-form $A_s := s^*(A) \in \Omega^1(U, \mathfrak{u}(1))$. Given a path $\gamma : [a, b] \to U$ that is contained in U entirely, the action of the associated parallel transport on $p := s(\gamma(a))$ may then be written as [18]

$$\Pi_\gamma(A)(p) \;=\; q \triangleleft \exp\left(\int_\gamma A_s\right) , \tag{0.146}$$

where $q := s(\gamma(b))$. Similarly, if we denote by V_n the one-dimensional vector space $\mathbb{C}$ on which $z \in U(1)$ acts by multiplication with z^n for some fixed $n \in \mathbb{Z}$, the induced parallel transport $\Pi_\gamma^n(A) := \Pi_\gamma^{V_n}(A)$ is given by

$$\Pi_\gamma^n(A)(v) \;=\; \exp\left(n \int_\gamma A_s\right) \cdot w \, , \tag{0.147}$$

where $v := [p, 1]$ and $w := [q, 1]$. In particular, this shows that for closed γ, the Wilson loop $W_n := W_{V_n}$ associated to V_n is given by the path integral insertion

$$W_n(\gamma) \;=\; \exp\left(n \oint_\gamma A_s\right) \;\equiv\; \exp\left(n \int_S F\right) , \tag{0.148}$$

where the second equality holds *if* the path γ bounds a two-dimensional surface S (i.e. $\partial S = \gamma$). In this case, we can use Stokes' theorem to replace the integral

[46] Here, we denote by $\mathfrak{u}(1) \equiv i\mathbb{R}$ the Lie algebra of $U(1)$.

of A_s over γ by the integral of the pullback $dA_s = s^*(F)$ of the field strength $F = dA \in \Omega^2(P, \mathfrak{u}(1))$ over S. Moreover, since the gauge group $U(1)$ is abelian, equation (0.48) implies that $s^*(F) = (s')^*(F)$ for any other local section s' wherever defined, so that we obtain a globally well-defined 2-form on spacetime M (which by abuse of notation we also call $F \in \Omega^2(M, \mathfrak{u}(1))$ in what follows). This is the 2-form appearing on the right hand side of (0.148). One can check that it satisfies the integral condition[47]

$$\frac{1}{2\pi i}\int_S F \;\in\; \mathbb{Z} \tag{0.151}$$

for any *closed* oriented two-dimensional submanifold $S \subset M$. Physically, equation (0.148) then means that a Wilson loop placed on a closed path measures (the exponential of n times) the magnetic flux through the surface that it bounds.

We can build non-trivial junctions between Wilson lines by assuming that the theory contains an additional complex scalar field ϕ of charge $k \in \mathbb{Z}$ (i.e. ϕ is a section of the line bundle $P_k := P_{V_k}$). In this case, the space of local junctions $\mathrm{Hom}(W_n, W_m)$ between Wilson lines W_n and W_m is non-trivial if and only if $(m-n)$ divides k (i.e. $m - n = k \cdot \ell$ for some $\ell \in \mathbb{Z}$), since the monomial

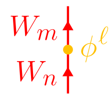


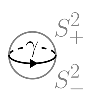
Figure 22

$$\phi^\ell \;:=\; \begin{cases} \phi \otimes \ldots \otimes \phi & \text{if } \ell \geq 0 \\ \phi^* \otimes \ldots \otimes \phi^* & \text{if } \ell < 0 \end{cases} \tag{0.152}$$

(where ϕ^* denotes the complex conjugate of ϕ) defines a non-trivial section of the morphism bundle $P_{\mathrm{Hom}(V_n,V_m)} \equiv P_{m-n}$. This is illustrated in Figure 22.

In addition to Wilson lines, the above example admits yet another type of operator, which instead of being defined in terms of functionals of the gauge field is constructed using certain types of 'boundary conditions' for the latter.

[47] In the case where S is (homotopic to) a 2-sphere S^2, this may be checked as follows: Consider a closed loop γ along the equator of S^2 and denote by $S_\pm$ the north and south hemisphere with boundaries $\partial S_+ = \gamma$ and $\partial S_- = \bar{\gamma}$, respectively. As both S_+ and S_- are contractible, we can find tubular neighbourhoods $U_\pm \subset M$ of $S_\pm$ that trivialise the principal bundle via local sections $s_\pm : U_\pm \to P$. If we denote $A_\pm := s_\pm^*(A) \in \Omega^1(U_\pm, \mathfrak{u}(1))$, we then have



$$\int_{S^2} F \;=\; \int_{S^2_+} F + \int_{S^2_-} F \;=\; \int_\gamma A_+ + \int_{\bar{\gamma}} A_- \tag{0.149}$$

by virtue of Stokes' theorem, which, using (0.146) together with the fact that parallel transport obeys $\Pi_\gamma(A)^{-1} = \Pi_{\bar{\gamma}}(A)$, implies that

$$1 \;=\; \Pi_\gamma(A) \circ \Pi_{\bar{\gamma}}(A) \;=\; \exp\left(\int_\gamma A_+ + \int_{\bar{\gamma}} A_-\right) \;=\; \exp\left(\int_{S^2} F\right). \tag{0.150}$$

Hence, we see that we must have $\int_{S^2} F \in 2\pi i\mathbb{Z}$ as claimed.

Concretely, let $F \in \Omega^2(M, \mathfrak{u}(1))$ denote the globally defined 2-form on M that is induced by the field strength of the gauge field as before. Given a codimension-three submanifold $\Sigma \subset M$ and an integer $n \in \mathbb{Z}$, we can then define an operator T_n with support in Σ by defining the insertion of $T_n(\Sigma)$ into the path integral to mean that we only integrate over those (equivalence classes of) gauge fields on bundles over the modified spacetime $M\backslash\Sigma$ that satisfy

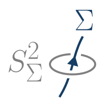


Figure 23

$$\frac{1}{2\pi i}\int_{S^2_\Sigma} F \;=\; n\,, \tag{0.153}$$

where $S^2_\Sigma \subset M$ denotes a 2-sphere that links Σ as illustrated in Figure 23. We will call the operator T_n as defined above a *'t Hooft operator* of charge n in what follows. Physically, it corresponds to the insertion of a magnetic monopole of charge n into the path integral. Note that the dimension $\dim(T_n) = d - 3$ of 't Hooft operators depends on the ambient dimension d of spacetime.

The construction of 't Hooft operators can be generalised to the case of a non-abelian compact simple[48] gauge group G. The result is that while Wilson lines are labelled by linear G-spaces, 't Hooft operators are labelled by linear ${}^LG$-spaces, where ${}^LG$ is the so-called *Langlands dual group*[49] of G [43, 44]. In spacetime dimension $d = 4$ (where the dimensions of Wilson and 't Hooft operators coincide), this can be seen as an instance of *electromagnetic duality* [45, 46], which relates a gauge theory based on G to a gauge theory based on ${}^LG$ and exchanges Wilson and 't Hooft lines in the process.

Symmetries

While the path integral offers an intuitive approach to quantum theory, its usage in the explicit computation of state spaces and correlation functions is often highly non-trivial. In this context, symmetries (again) provide a powerful tool to constrain the dynamics of a quantum field theory. Concretely, suppose that a theory has a symmetry group G with associated group action $\triangleright : G \times \mathcal{F} \to \mathcal{F}$ on the space of fields $\mathcal{F}$ leaving the action functional invariant, i.e. $S[g \triangleright \varphi] = S[\varphi]$ for all $g \in G$ and $\varphi \in \mathcal{F}$. *If* the group action restricts to an action on the space of fields $\mathcal{F}_X$ on some spatial hypersurface $X \subset M$ in spacetime, we obtain an induced action on the Hilbert space $\mathcal{H}_X$ of square-integrable functionals $\Psi : \mathcal{F}_X \to \mathbb{C}$ via

$$(\hat{U}_g \Psi)[\varphi] \;:=\; \Psi\big[g^{-1} \triangleright \varphi\big] \tag{0.154}$$

48 We say that a Lie group is *simple* if its Lie algebra is any of the algebras appearing in the Cartan classification of finite-dimensional simple Lie algebras [42].

49 For instance, one has ${}^L(SU(pq)/\mathbb{Z}_q) = SU(pq)/\mathbb{Z}_p$ while ${}^LSO(2k) = SO(2k)$.

for all $g \in G$ and $\varphi \in \mathcal{F}_X$. In particular, this G-action is *linear* in the sense that

$$\hat{U}_g(\lambda \cdot \Psi + \Phi) \;=\; \lambda \cdot \hat{U}_g(\Psi) \,+\, \hat{U}_g(\Phi) \tag{0.155}$$

for all $\lambda \in \mathbb{C}$ and $\Psi, \Phi \in \mathcal{H}_X$. Furthermore, we have that

$$\hat{U}_g \circ \hat{U}_h \;=\; \hat{U}_{g\cdot h} \tag{0.156}$$

for all $g, h \in G$. Lastly, one can compute that

$$\begin{aligned} \langle\Phi|\hat{U}_g|\Psi\rangle^* \;&=\; \int_{\mathcal{F}_X} \mathcal{D}\varphi|_X \cdot \Psi^*[g^{-1} \triangleright \varphi] \cdot \Phi[\varphi] \\ &=\; \int_{\mathcal{F}_X} \mathcal{D}\widetilde{\varphi}|_X \cdot \Psi^*[\widetilde{\varphi}] \cdot \Phi[g \triangleright \widetilde{\varphi}] \\ &\equiv\; \langle\Psi|\hat{U}_{g^{-1}}|\Phi\rangle \;, \end{aligned} \tag{0.157}$$

where we used the substitution of variables $\widetilde{\varphi} := g^{-1} \triangleright \varphi$ and assumed that the latter leaves the measure invariant, i.e. $\mathcal{D}\widetilde{\varphi} = \mathcal{D}\varphi$. As a result, we obtain that

$$(\hat{U}_g)^\dagger \;=\; \hat{U}_{g^{-1}} \;, \tag{0.158}$$

which, together with (0.156) and $\hat{U}_1 = \mathrm{id}_{\mathcal{H}_X}$, implies that $\hat{U}_g$ is a *unitary* operator on the Hilbert space $\mathcal{H}_X$ for every $g \in G$. The above then motivates the following:

Definition: Given a group G and a complex vector space[50] V, a *(linear) representation* of G on V is a group homomorphism $U : G \to \mathrm{Aut}(V)$. If V is furthermore a Hilbert space, we say that such a representation is *unitary* if U_g is unitary for every $g \in G$. An *intertwiner* between two representations U and U' of G is a linear map $f : V \to V'$ such that $U'_g \circ f = f \circ U_g$ for all $g \in G$. Two representations are said to be *isomorphic* if there exists an invertible intertwiner between them. The *dimension* of a representation U is given by $\dim(U) := \dim(V)$.

Given two representations U and U' of G on vector spaces V and V', their *direct sum* $U \oplus U'$ is the representation on $V \oplus V'$ with diagonal G-action, i.e. $(U \oplus U')_g = U_g \oplus U'_g$. It is often useful to consider representations that cannot be decomposed into direct sums of smaller ones and hence form the ‘building blocks’ for all other representations. Concretely, we introduce the following notions:

- **Indecomposability:** A representation is called *indecomposable* if it is not isomorphic to a direct sum of non-zero other representations.

[50] Although group representations are well-defined on vector spaces over any field $\mathbb{F}$, we will henceforth restrict ourselves to $\mathbb{F} = \mathbb{C}$ and assume that every vector space is complex (unless stated otherwise).

- **Irreducibility:** A representation U of G on a vector space V is called *irreducible* if it does not contain any non-trivial G-invariant subspaces. That is, if $W \subset V$ is such that $U_g(W) \subset W$ for all $g \in G$, then $W = \{0\}$ or $W = V$.

Clearly, irreducibility implies indecomposability but not vice versa. The utility of irreducible representations stems from the fact that the spaces of intertwiners between them are severely constrained. This is the content of *Schur's lemma* [47]:

Lemma: Let U and U' be two irreducible representations of G on vector spaces V and V', respectively. Then, any intertwiner $f : V \to V'$ between U and U' is either zero or an isomorphism, and any other intertwiner is of the form $f' = \lambda \cdot f$ for some $\lambda \in \mathbb{C}$. In particular, since U_z intertwines U for any group element z in the *centre*

$$\mathcal{Z}(G) \;:=\; \{z \in G \mid z \cdot g = g \cdot z \text{ for all } g \in G\} \tag{0.159}$$

of G, we have that $U_z = \lambda(z) \cdot \mathrm{id}_V$ for some group homomorphism $\lambda : \mathcal{Z}(G) \to \mathbb{C}^\times$.

Examples of (unitary) group representations that we will use repeatedly throughout this thesis include the following:

- Given any group G, there exists a one-dimensional representation on $V = \mathbb{C}$ that sends every group element to the identity map $\mathrm{id}_\mathbb{C}$. We will call this the *trivial representation* of G in what follows.

- Representations of the integers $\mathbb{Z}$ on a vector space V can be constructed by picking an invertible $A \in \mathrm{Aut}(V)$ and mapping $n \in \mathbb{Z} \mapsto A^n$. For instance, picking $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in \mathrm{Aut}(\mathbb{C}^2)$ yields a two-dimensional representation that is indecomposable but not irreducible (since $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{C}^2$ spans a non-trivial invariant subspace). Furthermore, this representation is not unitary[51].

- If G is a finite group, a representation of G is irreducible if and only if it is indecomposable. According to *Maschke's theorem*, every representation of G is then isomorphic to a direct sum of irreducible ones [14]. Furthermore, by virtue of *Weyl's unitarity trick*[52], every representation of a finite group is isomorphic to a unitary one. For instance, the dihedral group of order eight

$$D_8 \;=\; \langle\, r, s \mid r^4 = s^2 = 1,\; rsr = s \,\rangle \tag{0.160}$$

[51] Indeed, suppose that there exists an inner product $\langle .\,, . \rangle$ on $\mathbb{C}^2$ such that $\langle A\cdot v, A\cdot w\rangle = \langle v, w\rangle$ for all $v, w \in \mathbb{C}^2$. If we fix $v := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ (so that $A\cdot v = v$), the above implies that $\langle v, A\cdot w - w\rangle = 0$ for all $w \in \mathbb{C}^2$. However, setting $w := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ (so that $A\cdot w = v + w$) yields $0 = \langle v, A\cdot w - w\rangle = \langle v, v\rangle = \|v\|^2$, in contradiction to $v \neq 0$.

[52] Concretely, consider a representation U of a finite group G on a vector space V and fix an arbitrary inner product $\langle .\,, . \rangle$ on V. Define a new inner product via $\langle\!\langle v, w\rangle\!\rangle := \frac{1}{|G|} \cdot \sum_{g\in G} \langle U_g(v), U_g(w)\rangle$. Then, U is a unitary representation w.r.t. $\langle\!\langle .\,, . \rangle\!\rangle$.

has five irreducible representations $1, u, v, uv$ and m that send the generators r and s to the linear automorphisms

$$\begin{array}{c|ccccc} & 1 & u & v & uv & m \\ \hline r & 1 & -1 & 1 & -1 & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ s & 1 & 1 & -1 & -1 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \tag{0.161}$$

and that are unitary w.r.t. the standard inner products on $\mathbb{C}$ and $\mathbb{C}^2$, respectively.

- If A is an abelian group (i.e. $\mathcal{Z}(A) = A$), Schur's lemma implies that every irreducible representation of A is one-dimensional. We denote by $A^\vee := \mathrm{Hom}(A, \mathbb{C}^\times)$ the set of all such one-dimensional representations, which itself forms an abelian group (called the *Pontryagin dual* or *character group* of A) via point-wise multiplication. There exists a canonical pairing

$$\langle ., . \rangle : \; A^\vee \times A \; \to \; \mathbb{C}^\times \; , \tag{0.162}$$

which (if the group A is locally compact) induces a canonical isomorphism $(A^\vee)^\vee \cong A$ between A and its double dual. If A is furthermore finite[53], one has $A^\vee \cong A$, however, this isomorphism is non-canonical in general. For instance, if $A = \mathbb{Z}_n = \langle x \,|\, x^n = 1 \rangle$, its Pontryagin dual is $A^\vee = \langle \widehat{x} \,|\, \widehat{x}^n = 1 \rangle$, where $\widehat{x}$ denotes the character on A defined by $\langle \widehat{x}, x \rangle = e^{2\pi i/n}$.

- If G is a continuous Lie group, we require representations $U : G \to \mathrm{Aut}(V)$ of G to be Lie group homomorphisms (meaning in particular that they are smooth maps between manifolds). It is then often useful to 'linearise' such representations by considering the differential $\rho := D_1 U : \mathfrak{g} \to \mathrm{End}(V)$, which defines a *representation of the Lie algebra* $\mathfrak{g}$ of G in the sense that

$$\rho\big([\varepsilon, \eta]\big) \; = \; \big[\rho(\varepsilon), \rho(\eta)\big] \tag{0.163}$$

for all $\varepsilon, \eta \in \mathfrak{g}$, where the bracket on the right hand side denotes the commutator of linear maps in $\mathrm{End}(V)$. Conversely, if G is a simply connected group[54] and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of its Lie algebra, then there exists a unique representation $U : G \to \mathrm{Aut}(V)$ of G such that $D_1 U = \rho$ [48].

[53] Since for any finite group A we have $a^{|A|} = 1$ for all $a \in A$ by virtue of *Lagrange's theorem*, any character $\chi \in A^\vee$ on A is automatically valued in $U(1) \subset \mathbb{C}^\times$.

[54] We say that G is *simply connected* if it is path-connected (meaning that there exists a continuous path between any two points in G) and has trivial fundamental group $\pi_1(G) = 1$ (meaning that any closed path in G can be shrunk to a point).

As an example, consider $G = SU(2)$, whose Lie algebra $\mathfrak{su}(2)$ is the three-dimensional (real) vector space of traceless skew-hermitian (2×2)-matrices over $\mathbb{C}$ with basis $s_i := -\frac{i}{2}\sigma_i$ (where the σ_i denote the Pauli matrices for $i = x, y, z$ as in (0.119)) and commutation relations[55]

$$[s_i, s_j] \;=\; \varepsilon_{ijk}\, s_k \; . \tag{0.164}$$

Since we are interested in *complex* representations of $\mathfrak{su}(2)$, we can without loss of generality use the *complexification* $\mathbb{C} \otimes \mathfrak{su}(2)$, which is the three-dimensional (complex) vector space of traceless (2×2)-matrices over $\mathbb{C}$ with basis

$$h \;:=\; 2is_z \qquad \text{and} \qquad a_\pm \;:=\; i\,(s_x \pm i\,s_y) \; . \tag{0.165}$$

The Lie brackets of these generators are given by

$$[h, a_\pm] \;=\; \pm 2a_\pm \qquad \text{and} \qquad [a_+, a_-] \;=\; h \; . \tag{0.166}$$

It is then a classic result that for each $n \in \mathbb{N}$ there exists an $(n+1)$-dimensional irreducible complex representation ρ_n of the above algebra on a vector space V_n with basis $\{v_{p,q} \mid 0 \le p, q \le n \text{ s.t. } p + q = n\}$ and generator action

$$\begin{aligned} \rho_n(h)\, v_{p,q} \;&=\; (p-q)\cdot v_{p,q} \; , \\ \rho_n(a_+)\, v_{p,q} \;&=\; q\cdot v_{p+1,\,q-1} \; , \\ \rho_n(a_-)\, v_{p,q} \;&=\; p\cdot v_{p-1,\,q+1} \; , \end{aligned} \tag{0.167}$$

where we set $v_{n+1,-1} = v_{-1,n+1} := 0$ [48]. Since the Lie group $SU(2)$ is simply connected, each ρ_n corresponds to a representation U_n of $SU(2)$, which can be identified with the vector space of homogeneous degree-n polynomials

$$f(\vec{z}) \;=\; f_{n,0}\cdot z_1^n \,+\, \ldots \,+\, f_{p,q}\cdot z_1^p z_2^q \,+\, \ldots \,+\, f_{0,n}\cdot z_2^n \tag{0.168}$$

in two complex variables z_1 and z_2, on which matrices $A \in SU(2)$ act via

$$(U_n(A)f)(\vec{z}) \;:=\; f\,(A^{-1}\!\cdot\vec{z}) \; . \tag{0.169}$$

In particular, this representation is unitary w.r.t. the inner product

$$\langle f, f'\rangle \;:=\; \int_{|\vec{z}|^2=1} d\Omega \;\; f^*(\vec{z})\cdot f'(\vec{z}) \; , \tag{0.170}$$

[55] Here, we denote by ε_{ijk} the totally antisymmetric Levi-Civita tensor defined by $\varepsilon_{xyz} = 1$ and used the *Einstein summation convention* according to which repeated indices are being summed over.

where $d\Omega$ denotes the Lebesgue measure on the unit sphere in $\mathbb{C}^2$. Furthermore, the centre $\mathcal{Z}(SU(2)) = \{\pm\mathbb{1}_2\} \cong \mathbb{Z}_2$ of $SU(2)$ acts via

$$U_n(\pm\mathbb{1}_2) \; = \; (\pm 1)^n \; , \tag{0.171}$$

which shows that U_n induces a representation of $SU(2)/\mathbb{Z}_2 \cong SO(3)$ if and only if n is even. This is an example of a non-simply connected Lie group ($\pi_1(SO(3)) = \mathbb{Z}_2$) whose representations are not in one-to-one correspondence with representations of its Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

Equations (0.155)–(0.158) show that (under favourable circumstances) the Hilbert space associated to a spatial hypersurface $X \subset M$ furnishes a unitary representation $\hat{U}$ of a global symmetry group G in a quantum field theory. Its interplay with the time evolution of physical states along bordisms $X \xrightarrow{M} Y$ is given by

$$\begin{aligned} \langle\phi|\,\mathcal{U}(M)\circ\hat{U}_g\,|\Psi\rangle \; &= \; \int\limits_{\substack{\mathcal{F}(M):\\ \varphi|_Y=\phi}} \mathcal{D}\varphi\cdot e^{iS_M[\varphi]}\cdot\Psi\big[g^{-1}\triangleright\varphi|_X\big] \\ &= \; \int\limits_{\substack{\mathcal{F}(M):\\ g\triangleright\widetilde{\varphi}|_Y=\phi}} \mathcal{D}\widetilde{\varphi}\cdot e^{iS_M[\widetilde{\varphi}]}\cdot\Psi[\widetilde{\varphi}|_X] \\ &\equiv \; \langle\phi|\,\hat{U}_g\circ\mathcal{U}(M)\,|\Psi\rangle \; , \end{aligned} \tag{0.172}$$

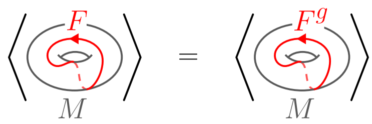
where we used the substitution of variables $\widetilde{\varphi} := g^{-1}\triangleright\varphi$ and assumed that the latter leaves both the measure $\mathcal{D}\varphi$ and the action functional $S_M[\varphi]$ invariant. Since (0.172) holds for any $|\Psi\rangle \in \mathcal{H}_X$ and $\phi \in \mathcal{F}_Y$, we obtain the operator equation

$$\big[\mathcal{U}(M),\hat{U}_g\big] \; = \; 0 \; , \tag{0.173}$$

which means that the unitary action of G on physical states 'commutes' with the time evolution of the latter along any spacetime bordism M. More generally, we can repeat the computation (0.172) for a generic path integral insertion $F : \mathcal{F}(M) \to \mathbb{C}$ and obtain the operator equation

$$\mathcal{U}(M;F)\circ\hat{U}_g \; = \; \hat{U}_g\circ\mathcal{U}(M;F^g) \; , \tag{0.174}$$

where F^g is the functional defined by $(F^g)[\varphi] := F[g\triangleright\varphi]$ for $\varphi \in \mathcal{F}(M)$. This shows that the unitary action of G commutes with decorated time evolution up to transforming possible path integral insertions. In particular, if the bordism $\varnothing \xrightarrow{M} \varnothing$ is closed (so that $\hat{U}$ is the trivial representation on $\mathcal{H}_\varnothing \cong \mathbb{C}$), equation (0.174) implies that correlation functions are invariant under arbitrary group transformations of their operator insertions, i.e.



(0.175)

for all $g \in G$. Physically, this puts strong restrictions on the possible forms of correlation functions and leads to so-called *selection rules*[56] on physical observables.

In the Schrödinger picture, we consider the Hilbert space $\mathcal{H} = \mathcal{H}_X$ associated to a fixed spatial hypersurface $X \subset M$ together with infinitesimal time evolution described by the Hamiltonian operator $\widehat{H}$ on $\mathcal{H}$. The commutation relation (0.173) then becomes

$$[\widehat{H}, \hat{U}_g] \; = \; 0 \tag{0.176}$$

for all $g \in G$, which is the infinitesimal version of the statement that the unitary action of G on $\mathcal{H}$ commutes with time evolution. In particular, if $|\Psi\rangle \in \mathcal{H}$ is an eigenstate of $\widehat{H}$ with eigenvalue $E \in \mathbb{R}$, so is $\hat{U}_g |\Psi\rangle$ for all $g \in G$. Consequently, the decomposition of the Hilbert space $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ into eigenspaces of the Hamiltonian induces a decomposition of $\hat{U} = \bigoplus_n \hat{U}_n$ into unitary subrepresentations

$$\hat{U}_n : \; G \; \to \; \mathrm{Aut}(\mathcal{H}_n) \; , \tag{0.177}$$

which organise states of fixed energy into 'multiplets'[57] of the symmetry group G. For instance, if $\hat{U}_n$ is an irreducible representation of G of dimension greater than one, the associated energy eigenspace is necessarily degenerate. If this happens for $n = 0$ (corresponding to the space $\mathcal{H}_0$ of states with lowest energy), we say that the symmetry is *spontaneously broken*. Physically, this again captures the idea that while the laws that govern the evolution of states may be invariant under a certain symmetry group, a given ground state (or vacuum) need not be. The generalised commutation relation (0.174) in the Schrödinger picture becomes

$$\hat{U}_g \circ \widehat{F} \circ \hat{U}_g^\dagger \; = \; \widehat{{}^gF} \; , \tag{0.178}$$

where ${}^gF := F^{(g^{-1})}$ and $\widehat{F}$ denotes the Hilbert space operator induced by a path integral insertion F as in (0.113). This means that the transformation of an operator $\widehat{F}$ under the symmetry group G can be computed by conjugating $\widehat{F}$ with the unitary operators $\hat{U}_g$ associated to group elements $g \in G$.

[56] Broadly speaking, we use the term *selection rule* to refer to any mechanism that necessitates certain correlation functions (or more generally overlaps) to vanish.

[57] We often use the term *multiplet* to refer to irreducible representations ('irreps') of a given symmetry group G. A one-dimensional irrep is also called a *singlet*.

Examples of symmetries in quantum systems that lead to degeneracies of states or selection rules on correlation functions include the following:

- HYDROGEN ATOM: Consider the quantum mechanics of a probe particle of mass m confined to the potential $V(\vec{r}) = -\alpha/r$ on $\mathbb{R}^3\backslash\{0\}$ with $\alpha \in \mathbb{R}_{>0}$. As before, the Hamiltonian is given by the Hermitian operator

$$\widehat{H} \;=\; \frac{\hat{\vec{p}}^{\,2}}{2m} - \frac{\alpha}{\hat{r}} \;, \tag{0.179}$$

where $\hat{r}_i = r_i\cdot$ and $\hat{p}_i = -i\partial_i$ denote the position and momentum operators, respectively. As in the classical case, there are two sets of conserved quantities, which are now implemented by operators on the Hilbert space. These are

1. the *angular momentum operator*: $\hat{L}_i = [\hat{\vec{r}} \times \hat{\vec{p}}]_i$,
2. the *Runge-Lenz operator*[58]: $\hat{A}_i = \frac{1}{2m}[\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}]_i - \frac{\alpha}{\hat{r}}\hat{r}_i$.

One can check that $[\widehat{H}, \hat{L}_i] = [\widehat{H}, \hat{A}_i] = 0$ for $i = x, y, z$ as required for symmetry transformations. Furthermore, one can check that L and A (where we dropped the $\hat{}$ for better readability) obey the commutation relations

$$\begin{aligned} [L_i, L_j] &= i\varepsilon_{ijk}\, L_k \;, \\ [L_i, A_j] &= i\varepsilon_{ijk}\, A_k \;, \\ [A_i, A_j] &= -\frac{2i}{m}\,\varepsilon_{ijk}\, L_k \circ H \;, \end{aligned} \tag{0.180}$$

together with $\vec{L}\cdot\vec{A} = \vec{A}\cdot\vec{L} = 0$. Lastly, we have that

$$\vec{A}^2 \;=\; \alpha^2 + \frac{2}{m}\, H \circ (\vec{L}^2 + 1) \;. \tag{0.181}$$

Since both the angular momentum and the Runge-Lenz operators commute with the Hamiltonian, we can without loss of generality restrict attention to their actions on some fixed energy eigenspace with corresponding eigenvalue $E \in \mathbb{R}$. We will further restrict to the case of bound states with $E < 0$ in what follows. Upon replacing H by E in (0.180) and rescaling

$$\vec{B} \;:=\; \sqrt{-\frac{m}{2E}}\cdot\vec{A}\,, \tag{0.182}$$

one then obtains the commutation relations

$$[L_i, L_j] = i\varepsilon_{ijk}\, L_k \;, \quad [L_i, B_j] = i\varepsilon_{ijk}\, B_k \;, \quad [B_i, B_j] = i\varepsilon_{ijk}\, L_k \;, \tag{0.183}$$

[58] Here, we (anti)symmetrised the classical expression (0.61) for the Runge-Lenz vector in order to turn $\hat{A}$ into a Hermitian operator.

which can be simplified further by introducing

$$\vec{S} \;:=\; -\frac{i}{2}\,(\vec{L}+\vec{B}) \qquad \text{and} \qquad \vec{S}' \;:=\; -\frac{i}{2}\,(\vec{L}-\vec{B})\;. \tag{0.184}$$

These then satisfy the disentangled commutation relations

$$\begin{aligned} [S_i, S_j] \;&=\; \varepsilon_{ijk}\, S_k\;, \\ [S'_i, S'_j] \;&=\; \varepsilon_{ijk}\, S'_k\;, \\ [S_i, S'_j] \;&=\; 0\;, \end{aligned} \tag{0.185}$$

which, by comparing with (0.164), correspond to a direct sum of two copies of the Lie algebra $\mathfrak{su}(2)$. As discussed before, irreducible representations of the latter are labelled by non-negative integers $n \in \mathbb{N}$ and denoted by $\rho_n : \mathfrak{su}(2) \to \mathrm{End}(V_n)$. The irreducible representations of the direct sum $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are then given by tensor product representations[59] of the form $\rho_n \otimes \rho_m$ [49]. However, in our case, the integers n and m are not unrelated due to the fact that

$$\vec{S}^2 \;=\; \vec{S}'^2 \tag{0.186}$$

(as one can check from the definition (0.184) together with $\vec{L}\cdot\vec{B} = \vec{B}\cdot\vec{L} = 0$), which, using the fact that the representation ρ_n obeys

$$\rho_n(\vec{S}^2) \;=\; -\frac{n}{2}\cdot\left(\frac{n}{2}+1\right)\cdot \mathrm{id}_{V_n}\;, \tag{0.187}$$

implies $n = m$. Upon rewriting (0.181) in terms of $\vec{S}$ and E as

$$4\vec{S}^2 \;=\; \frac{m\alpha^2}{2E} + 1 \tag{0.188}$$

and plugging (0.187) into (0.188), we then see that the possible negative energy eigenvalues are of the form

$$E \;=\; E_n \;=\; -\frac{m\alpha^2}{2(n+1)^2}\;. \tag{0.189}$$

This reproduces the spectrum of bound states in the hydrogen atom from (0.134). Furthermore, we see that the associated eigenspace degeneracies are given by $\dim(\rho_n \otimes \rho_n) \;=\; (n+1)^2$. Note that we again arrived at this result using symmetry considerations. The above derivation of the hydrogen spectrum using the Runge-Lenz operator is originally due to Pauli [50].

[59] Given representations $\rho : \mathfrak{g} \to \mathrm{End}(V)$ and $\rho' : \mathfrak{g}' \to \mathrm{End}(V')$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, their *tensor product* is the representation $\rho\otimes\rho'$ of $\mathfrak{g}\oplus\mathfrak{g}'$ defined by $(\rho\otimes\rho')(\varepsilon+\varepsilon') := \rho(\varepsilon)\otimes \mathrm{id}_{V'} + \mathrm{id}_V \otimes \rho'(\varepsilon')$ for all $\varepsilon \in \mathfrak{g}$ and $\varepsilon' \in \mathfrak{g}'$.

- Conformal field theory: Consider a two-dimensional theory whose spacetime is given by the Riemann sphere $\mathbb{C}^\bullet \equiv \mathbb{C} \cup \{\infty\}$. Further assume that the theory has an internal symmetry group given by $SL(2, \mathbb{C})/\mathbb{Z}_2$, where

$$SL(2,\mathbb{C}) \;=\; \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \;\middle|\; ad - bc = 1 \right\} \tag{0.190}$$

acts on points $z \in \mathbb{C}^\bullet$ via so-called *Möbius transformations*[60]

$$A \triangleright z \;:=\; \frac{az + b}{cz + d} \tag{0.191}$$

with the convention that $A \triangleright \infty = a/c$ and $A \triangleright (-d/c) = \infty$. In particular, the above transformations include

  - *translations* by choosing $A = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbb{C}$, which sends $z \mapsto z + b$,
  - *dilations* by choosing $A = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ with $a \in \mathbb{C}^\times$, which sends $z \mapsto a^2 \cdot z$,
  - *inversions* by choosing $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which sends $z \mapsto -\frac{1}{z}$.

From equation (0.175), we know that correlation functions of any operator F in the theory need to be invariant w.r.t. the above symmetry, i.e.

$$\langle F \rangle \;\stackrel{!}{=}\; \langle F^A \rangle \;, \tag{0.192}$$

where F^A denotes the action of a symmetry transformation $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$ on the operator F. For instance, we can consider so-called *primary operators*, which are local operators F supported on points $z \in \mathbb{C}^\bullet$ with the property that

$$(F^A)(z) \;=\; \frac{F(A \triangleright z)}{(cz + d)^{2k}} \tag{0.193}$$

for some $k \in \mathbb{Z}$ (called the *weight* of F). The condition (0.192) then allows us to constrain the functional form of correlation functions with one or multiple primary operator insertions. For example:

  – Let $f(z) := \langle F(z) \rangle$ be the one-point function of a primary operator of weight k. Translational invariance then implies that $f(z + b) = f(z)$ for all $b \in \mathbb{C}$, so that $f(z) = c$ for some constant $c \in \mathbb{C}$. Furthermore, invariance under dilations implies that $a^{2k} \cdot c = c$ for all $a \in \mathbb{C}^\times$, so that $c = 0$ unless $k = 0$. Thus, we see that one-point functions of primary operators are necessarily of the form $\langle F(z) \rangle = \delta_{k,0} \cdot c$.

[60] The importance of Möbius transformations stems from the fact that they preserve angles between vectors in the two dimensional plane. Concretely, if we identify $(x, y)^T \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$, then the standard Euclidean metric on $\mathbb{R}^2$ can be written as $dx^2 + dy^2 = dz\,dz^*$. Applying a Möbius transformation (0.191) to the latter then shifts $dz\,dz^* \to \Omega(z)^2 \cdot dz\,dz^*$, where $\Omega(z)$ is the real-valued function $\Omega(z) = |cz + d|^{-2}$. This is why we speak of a *conformal* field theory.

– Let $g(z,z') := \langle F(z)F'(z')\rangle$ be the two-point function of primary operators F and F' with weights k and k', respectively, evaluated at distinct points $z, z' \in \mathbb{C}^{\bullet}$. Translational invariance then implies that $g(z+b, z'+b) = g(z,z')$ for all $b \in \mathbb{C}$, so that $g(z,z') = g(z-z')$. Furthermore, invariance under dilations implies $a^{2(k+k')}g(a^2 \cdot (z-z')) = g(z-z')$ for all $a \in \mathbb{C}^{\times}$, so that

$$g(z-z') \;=\; \frac{c}{(z-z')^{k+k'}} \tag{0.194}$$

for some constant $c \in \mathbb{C}$. Lastly, invariance under inversions yields

$$\frac{1}{z^{2k}} \cdot \frac{1}{(z')^{2k'}} \cdot \frac{c}{\left(\frac{1}{z'} - \frac{1}{z}\right)^{k+k'}} \;=\; \frac{c}{(z-z')^{k+k'}} \;, \tag{0.195}$$

which can only hold true if $k = k'$. Thus, we see that two-point functions of primary operators are necessarily of the form

$$\langle F(z)F'(z')\rangle \;=\; \frac{\delta_{k,k'} \cdot c}{(z-z')^{2k}} \;. \tag{0.196}$$

Similarly, one can use the invariance of correlation functions under the transformations (0.193) to compute higher-point functions of any number of primary operators. This is an instance of a theory whose symmetry is powerful enough to determine the functional form of a large class of correlation functions.

To summarise, we have seen from the above discussion that symmetries in quantum theory are useful for (at least) the following two reasons:

1. They organise the spectrum of energy eigenspaces of the Hamiltonian into irreducible representations of global symmetry groups, leading to possible degeneracies of ground and excited states.

2. They constrain the possible forms of correlation functions of arbitrary operator insertions, leading to selection rules on physical observables.

It would be desirable to combine both of the above into a single unified perspective. For this purpose, we ask the following natural question: Given the unitary representation $\hat{U}$ of a global symmetry group G on the Hilbert space $\mathcal{H}$ associated to some spatial hypersurface $X \subset M$, does there exist a path integral insertion U_g for each $g \in G$ whose induced Hilbert space operator is $\hat{U}_g$? In other words, we require

$$\hat{U}_g \;\overset{!}{=}\; \widehat{U_g} \;\equiv\; \lim_{\delta \to 0} \mathcal{U}(X_\delta; U_g) \tag{0.197}$$

for all $g \in G$, where $X_\delta = X \times (-\delta, \delta)$ as before and $\mathcal{U}(X_\delta; U_g)$ denotes the decorated time evolution operator along X_δ as defined in (0.91).

In the case of a continuous Lie group symmetry G, the existence of the path integral insertions U_g can be inferred from a quantum version of Noether's trick. Concretely, consider an infinitesimal symmetry transformation parameterised by an element ε of the Lie algbra $\mathfrak{g}$ of G. Now promote ε to an arbitrary Lie algebra valued function on spacetime M, which we take to be a bordism $X \xrightarrow{M} Y$ between spatial hypersurfaces X and Y. Let $F : \mathcal{F}(M) \to \mathbb{C}$ be an arbitrary path integral insertion on M. If we abbreviate by $\mathcal{U}(F) := \mathcal{U}(M; F)$ the decorated time evolution along M, then

$$\begin{aligned}
\langle \Phi | \, \mathcal{U}(F) \, | \Psi \rangle \;&\equiv\; \int_{\mathcal{F}(M)} \mathcal{D}\widetilde{\varphi} \cdot \Phi^*[\widetilde{\varphi}|_Y] \cdot F[\widetilde{\varphi}] \cdot e^{iS_M[\widetilde{\varphi}]} \cdot \Psi[\widetilde{\varphi}|_X] \\
&=\; \int_{\mathcal{F}(M)} \mathcal{D}\varphi \cdot \Phi^*[e^{-\varepsilon} \triangleright \varphi|_Y] \cdot F[e^{-\varepsilon} \triangleright \varphi] \cdot e^{iS_M[e^{-\varepsilon} \triangleright \varphi]} \cdot \Psi[e^{-\varepsilon} \triangleright \varphi|_X] \\
&=\; \int_{\mathcal{F}(M)} \mathcal{D}\varphi \cdot \big(\hat{U}_{(e^{\varepsilon})}\Phi\big)^*[\varphi|_Y] \cdot \big({}^{(e^{\varepsilon})}F\big)[\varphi] \cdot e^{iS_M[\varphi] \, - \, i\delta_\varepsilon S} \cdot \big(\hat{U}_{(e^{\varepsilon})}\Psi\big)[\varphi|_X] \\
&=\; \big\langle \hat{U}_{(e^{\varepsilon})}(\Phi) \, \big| \, \mathcal{U}\big[{}^{(e^{\varepsilon})}F \cdot e^{-i\delta_\varepsilon S}\big] \, \big| \, \hat{U}_{(e^{\varepsilon})}(\Psi) \big\rangle
\end{aligned} \tag{0.198}$$

for all states $|\Psi\rangle \in \mathcal{H}_X$ and $|\Phi\rangle \in \mathcal{H}_Y$, where we used the substitution of variables $\widetilde{\varphi} =: e^{-\varepsilon} \triangleright \varphi$ and assumed that this leaves the measure invariant, i.e. $\mathcal{D}\widetilde{\varphi} = \mathcal{D}\varphi$. Now let us denote by $\hat{\rho} := D_1\hat{U}$ the representation of the Lie algebra $\mathfrak{g}$ that is induced by the representation $\hat{U}$ of G, which is such that

$$\hat{U}_{(e^{\varepsilon})}|\Psi\rangle \;=\; |\Psi\rangle \,+\, \hat{\rho}_\varepsilon|\Psi\rangle \,+\, \mathcal{O}(\varepsilon^2) \,. \tag{0.199}$$

Furthermore, we write the transformed path integral insertion F as

$${}^{(e^{\varepsilon})}F \;=\; F \,+\, \delta_\varepsilon F \,+\, \mathcal{O}(\varepsilon^2) \,. \tag{0.200}$$

Plugging this into the last line of equation (0.198) and expanding to first order in ε then yields the identity

$$\begin{aligned}
&i \cdot \langle \Phi | \, \mathcal{U}(F \cdot \delta_\varepsilon S) \, | \Psi \rangle \\
&=\; \langle \hat{\rho}_\varepsilon(\Phi) | \, \mathcal{U}(F) \, | \Psi \rangle \,+\, \langle \Phi | \, \mathcal{U}(\delta_\varepsilon F) \, | \Psi \rangle \,+\, \langle \Phi | \, \mathcal{U}(F) \, | \, \hat{\rho}_\varepsilon(\Psi) \rangle \,.
\end{aligned} \tag{0.201}$$

Now, since ε parameterises an infinitesimal symmetry transformation, we have that the first order variation of the action functional is given by[61]

$$\delta_\varepsilon S \;=\; \int_M \langle d\varepsilon \wedge \star j \rangle \;, \tag{0.202}$$

[61] For simplicity, we assume G to be an honest symmetry (as opposed to a quasi-symmetry), so that there are no boundary terms appearing in the variation of the action when ε is constant. This is a special case of the more general transformation behaviour discussed around equation (0.52).

where $j \in \Omega^1(M, \mathfrak{g}^\vee)$ is the $\mathfrak{g}^\vee$-valued Noether current as before. Using integration by parts in combination with Stokes' theorem (as well as the fact that $\partial M = \overline{X} \sqcup Y$), we can rewrite the variation in (0.202) as

$$\delta_\varepsilon S \;=\; Q_\varepsilon(Y) \,-\, Q_\varepsilon(X) \,-\, \int_M \langle \varepsilon, d\star j \rangle \ , \tag{0.203}$$

where we denoted by $Q_\varepsilon(\Sigma) := \int_\Sigma \langle \varepsilon, \star j \rangle$ the Noether charges defined on codimension-one hypersurfaces $\Sigma \subset M$ as before. Plugging this into (0.201) yields

$$\begin{aligned} \langle \Phi | \, \mathcal{U}\big[iQ_\varepsilon(Y) \cdot F \big] \, | \Psi \rangle \,-\, i \int_M \langle \Phi | \, \mathcal{U}\big[\langle \varepsilon, d\star j \rangle \cdot F \big] \, | \Psi \rangle \,-\, \langle \Phi | \, \mathcal{U}\big[F \cdot iQ_\varepsilon(X) \big] \, | \Psi \rangle \\ = \; \langle \hat{\rho}_\varepsilon(\Phi) \, | \, \mathcal{U}(F) \, | \Psi \rangle \,+\, \langle \Phi | \, \mathcal{U}(\delta_\varepsilon F) \, | \Psi \rangle \,+\, \langle \Phi | \, \mathcal{U}(F) \, | \, \hat{\rho}_\varepsilon(\Psi) \rangle \ . \end{aligned} \tag{0.204}$$

Since this equation holds for *any* Lie algebra valued function ε, we can make progress by varying the support of ε in spacetime M:

- If we choose ε to be constant along X and zero everywhere else (and we also assume $F = 1$ for simplicity), equation (0.204) reduces to

$$-\, \langle \Phi | \, \mathcal{U}\big[M; iQ_\varepsilon(X) \big] \, | \Psi \rangle \;=\; \langle \Phi | \, \mathcal{U}(M) \, | \, \hat{\rho}_\varepsilon(\Psi) \rangle \ , \tag{0.205}$$

  where we restored the notational dependence of $\mathcal{U}$ on spacetime M. In particular, by choosing $M = X_\delta$ and taking the limit $\delta \to 0$, we obtain

$$-\, \lim_{\delta \to 0} \langle \Phi | \, \mathcal{U}\big[X_\delta ; iQ_\varepsilon(X) \big] \, | \Psi \rangle \;=\; \langle \Phi | \, \hat{\rho}_\varepsilon(\Psi) \rangle \ . \tag{0.206}$$

  Exponentiating this and using the fact that the states $|\Phi\rangle$ and $|\Psi\rangle$ were arbitrary then yields the operator identity

$$\lim_{\delta \to 0} \mathcal{U}\big[X_\delta ; e^{-iQ_\varepsilon(X)} \big] \;=\; \hat{U}_{(e^\varepsilon)} \ . \tag{0.207}$$

  Similar results follow from choosing ε to be constant on Y and zero elsewhere.

- If we assume ε to be non-zero only in the interior of M and away from the boundary, equation (0.204) yields the operator identity

$$\mathcal{U}(\delta_\varepsilon F) \;=\; -i \int_M \mathcal{U}\big(\langle \varepsilon, d\star j \rangle \cdot F \big) \tag{0.208}$$

  also known as the *Ward-Takahashi identity* [51, 52]. Physically, this is a quantum version of Noether's theorem as it implies $\mathcal{U}\big((d\star j)_m \cdot F \big) = 0$ as long as the insertion point $m \in M$ is away from the support of the operator F. More generally, the insertion of $\langle \varepsilon, d\star j \rangle$ forces any other path integral insertion F to transform infinitesimally wherever the supports of ε and F coincide.

By comparing (0.208) with (0.197), it becomes clear that the path integral insertions

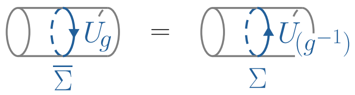
$$U_g(\Sigma) \;:=\; \exp(-iQ_\varepsilon(\Sigma)) \tag{0.209}$$

defined on codimension-one submanifolds $\Sigma \subset M$ and indexed by group elements $g = e^\varepsilon \in G$ with $\varepsilon \in \mathfrak{g}$ have the desired property of inducing the unitary representation $\hat{U}_g$ on the Hilbert space upon being placed along a given spatial hypersurface $X \subset M$. Moreover, they have the following important features (where we take all equalities to hold as insertions into a given time evolution operator):

1. They are compatible with orientation reversal in the sense that[62]

$$U_g(\overline{\Sigma}) \;=\; U_{(g^{-1})}(\Sigma)\;, \tag{0.210}$$

where we used that $Q_\varepsilon(\overline{\Sigma}) = -Q_\varepsilon(\Sigma)$ and $(e^\varepsilon)^{-1} = e^{-\varepsilon}$ for $\varepsilon \in \mathfrak{g}$. Pictorially,

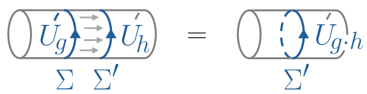


. (0.211)

2. If G is abelian, they 'fuse' according to the group law of G in the sense that

$$\lim_{\Sigma \to \Sigma'} U_g(\Sigma) \cdot U_h(\Sigma') \;=\; U_{g\cdot h}(\Sigma')\;, \tag{0.212}$$

where we used that $e^\varepsilon \cdot e^\eta = e^{\varepsilon+\eta}$ for commuting $\varepsilon, \eta \in \mathfrak{g}$. Pictorially,



. (0.213)

3. They are 'topological' in the sense that

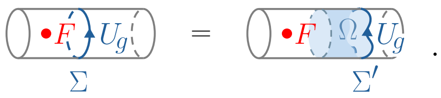
$$U_g(\Sigma) \;=\; U_g(\Sigma') \tag{0.214}$$

as long as Σ can be continuously deformed into Σ' without crossing the support of any other operator insertions F. This follows from the fact that upon choosing $\Omega \subset M$ such that $\partial\Omega = \overline{\Sigma} \sqcup \Sigma'$, we have

$$\begin{aligned} U_g(\Sigma') \cdot U_g(\Sigma)^{-1} \;&=\; \exp\left(iQ_\varepsilon(\Sigma) - iQ_\varepsilon(\Sigma')\right) \\ &=\; \exp\Big(-i\int_\Omega \langle \varepsilon, d{\star}j\rangle\Big) \;=\; 1\;, \end{aligned} \tag{0.215}$$

[62] Furthermore, we have that complex conjugation acts via $U_g^*(\Sigma) = U_g(\overline{\Sigma})$ in line with the general conjugation law (0.101) we impose on extended operators.

where we made use of Stokes' theorem and the fact that, according to the Ward identity, we have $d\star j = 0$ away from the support of the operator F. Pictorially,



(0.216)

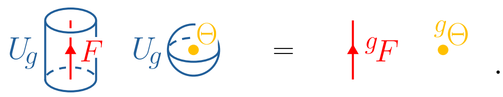
4. They 'act on' other operator insertions F via linking. Concretely, let $S \subset M$ denote the support of F and let N_S be a tubular neighbourhood of S with boundary $\partial N_S =: \Sigma$. Placing Q_ε on Σ then yields

$$-iQ_\varepsilon(\Sigma) \cdot F \;=\; -i\int_{N_S} \langle \varepsilon, d\star j\rangle \cdot F \;=\; \delta_\varepsilon F \tag{0.217}$$

via Stokes' theorem and the Ward identity (0.208), which exponentiates to

$$U_g(\Sigma) \cdot F \;=\; {}^gF \tag{0.218}$$

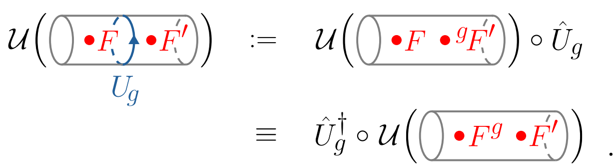
with $g = e^\varepsilon$. Pictorially, this means that 'surrounding' F with U_g is equivalent to inserting the transformed operator ${}^gF$, i.e.



(0.219)

Note that this applies both to local and extended operators.

The above construction of the path integral insertions U_g and their properties relied on the symmetry group G being a continuous Lie group (and moreover abelian). However, we can construct the U_g abstractly for any type of symmetry group G using the properties 1.–4. above together with the defining property (0.197). That is, given a decorated time evolution operator $\mathcal{U}(M;F)$, we simply *define* the insertion of U_g to be the result of topologically moving its support towards the boundary of M and acting on any obstructing operator insertions F with g along the way. Pictorially,



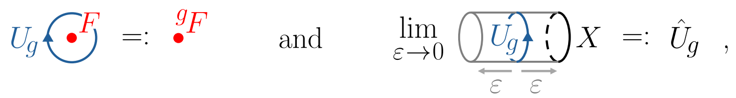
(0.220)

The properties 1.–4. then ensure that this construction is well-defined (i.e. independent of the way we choose to move one or multiple insertions U_g towards the boundary). All in all, the above discussion motivates the following:

Proposition: A quantum theory has a global symmetry group G if and only if there exists a collection of operators U_g indexed by group elements $g \in G$ and supported on codimension-one submanifolds $\Sigma \subset M$ such that

1. they are compatible with orientation reversal, i.e. $U_g(\bar{\Sigma}) = U_{(g^{-1})}(\Sigma)$,
2. they fuse according to the group law of G, i.e. $\lim_{\Sigma \to \Sigma'} U_g(\Sigma) \cdot U_h(\Sigma') = U_{g \cdot h}(\Sigma')$,
3. they are topological, i.e. $U_g(\Sigma) = U_g(\Sigma')$ if Σ can be continuously deformed into Σ' without crossing other operator insertions.

Any such collection then acts on other operator insertions F and the Hilbert spaces associated to spatial hypersurfaces $X \subset M$ via



(0.221)

where we again assume all equalities to hold as insertions into a given time evolution operator. We will often refer to the U_g as *(symmetry) defects* in what follows.

The above characterisation of global symmetries in quantum theories in terms of topological defects has the following advantages:

- It is applicable to both discrete and continuous symmetry groups G that can be abelian as well as non-abelian.
- It makes no reference to the path integral but is defined entirely in terms of decorated time evolution operators and correlation functions.
- It allows for straightforward generalisations, leading to the notion of so-called *generalised (global) symmetries*.

0.1.3 Generalised Symmetries

The description of global symmetries in quantum theories in terms of topological codimension-one defects labelled by group elements $g \in G$ can readily be generalised in two orthogonal directions:

1. We can increase the codimension of symmetry defects, leading to the notion of *higher form symmetries*. Concretely, we say that a theory has a *p-form symmetry* (where $0 \leq p \leq d-1$) if there exists a collection of codimension-$(p+1)$ topological defects that fuse according to the group law of some group G. When $p = 0$, this reduces to the notion of 'ordinary' global symmetries discussed before.
2. We can replace the group G by a more general index set C equipped with a well-defined fusion rule $C \times C \to C$, dropping the assumption that every symmetry defect has an inverse. This leads to the notion of *non-invertible symmetries*.

In the seminal work [13], the authors unified both of the above directions by introducing the following notion of generalised global symmetries:

Definition: A *generalised global symmetry* of a quantum theory is a collection $\mathcal{C}$ of topological defects of codimension between 1 and d.

As before, we implicitly understand adjectives such as 'topological' to refer to properties of the defects in $\mathcal{C}$ that hold upon inserting them into time evolution operators or correlation functions. Two comments are in order:

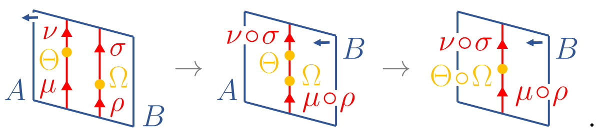
- As in the case of non-topological operators, we can organise the elements of $\mathcal{C}$ according to the (co)dimension of their support. Concretely, for $0 \leq p \leq d-1$, we define the space of *p-form symmetry defects* by

$$\mathcal{C}_p \ := \ \{ U \mid \text{codim}(U) = p+1 \} \ . \tag{0.222}$$

Given two 0-form defects $A, B \in \mathcal{C}_0$, we denote by $\text{Hom}_{\mathcal{C}}(A,B) \subset \mathcal{C}_1$ the space of 1-form defects that can sit at the interface between A and B (which are also called *1-morphisms* between A and B). Similarly, given two 1-morphisms $\mu, \nu \in \text{Hom}_{\mathcal{C}}(A,B)$, we denote by $2\text{Hom}_{\mathcal{C}}(\mu,\nu) \subset \mathcal{C}_2$ the space of so-called *2-morphisms* Θ that can sit at the junction between μ and ν (cf. Figure 16). By iterating this procedure we obtain the space

$$p\text{-Hom}_{\mathcal{C}}(.\,,.) \ \subset \ \mathcal{C}_p \tag{0.223}$$

of *p-morphisms* between two given $(p-1)$-morphisms for $p = 1, ..., d-1$ (where '0-morphisms' are simply elements of $\mathcal{C}_0$, which are also called *objects* of $\mathcal{C}$). Unlike in the case of general operators, the above morphism spaces carry additional structure due to the fact that its element are *topological*. Concretely, given two p-morphisms μ and ρ, we can always bring their supports arbitrarily close to each other to obtain a new p-morphsim $\mu \circ \rho$, which we call the *composition* of μ and ρ. Pictorially, this may be represented as

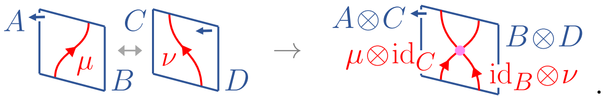


(0.224)

Mathematically, the fact that we can compose morphisms means that the collection $\mathcal{C} = \bigsqcup_{p=0}^{d-1} \mathcal{C}_p$ forms a (higher) category (more precisely, a $(d-1)$-category[63]).

[63] Broadly speaking, an *n-category* for $n \in \mathbb{N}$ is an algebraic structure consisting of a set of objects, a set of morphisms between objects, a set of 2-morphisms between morphisms, and so on and so forth up to n, together with various ways to compose these j-morphisms in a reasonable manner. We refer the reader to [53] for a gentle introduction to n-categories. We note that a 0-category is simply a set, while a 1-category is a category of objects and morphisms in the usual sense.

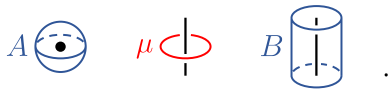
As before, we assume that the top-morphism spaces $(d-1)\text{-Hom}_{\mathcal{C}}(.,.)$ form complex vector spaces, which turns $\mathcal{C}$ into a *linear* category. Furthermore, since we can bring the supports of two objects $A, B \in \mathcal{C}_0$ arbitrarily close to each other, $\mathcal{C}$ is equipped with an additional monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that captures the fusion of topological defects:



(0.225)

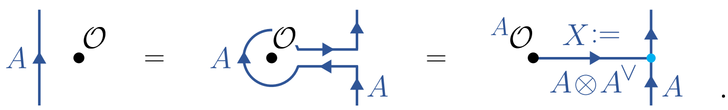
All in all, we see that we should expect generalised symmetries in a d-dimensional quantum theory to form (some version of) a linear monoidal $(d-1)$-category $\mathcal{C}$, which we will call the *symmetry category* in what follows. Understanding the precise nature of $\mathcal{C}$ in simple cases is one of the aims of this thesis.

- As in the case of ordinary global symmetries, generalised symmetries can act on other (non-topological) operators via linking. In order to distinguish topological defects from non-topological operators, we will henceforth use a colour-code and depict the former using chromatic colours and the latter using black or gray-scales. A p-form symmetry defect can then link operators of dimension p and above (which is why it is called a *p-form* defect in the first place), e.g.



(0.226)

However, it is in general not sufficient to only consider the linking action of symmetry defects on *genuine* operators. For instance, consider 'pushing' a 0-form defect A through a local operator $\mathcal{O}$ as depicted here in $d = 2$:



(0.227)

The resulting configuration consists of the linked operator $\mathcal{O}$, which is connected to the symmetry defect A via a small tube formed by A and its orientation reversal $A^\vee$. If A is invertible (i.e. $A = U_g$ for some group element g), we know from (0.210) that $A^\vee$ is given by $U_{(g^{-1})}$, so that the fusion of A and $A^\vee$ yields the trivial defect $U_g \otimes U_{(g^{-1})} = \mathbf{1}$. For generic A, however, we have that $X = A \otimes A^\vee$ is distinct from $\mathbf{1}$, so that the resulting transformed operator ${}^{A}\mathcal{O}$ becomes a so-called *twisted sector* operator for the non-trivial defect X. This shows that the inclusion of twisted sectors is crucial in order to fully capture the action of generalised symmetries. Understanding the precise nature of this action in simple cases is another aim of this thesis.

Examples of (invertible and non-invertible) generalised symmetries that will be used throughout this thesis include the following:

- Maxwell theory: Consider a pure $U(1)$ gauge theory and let $F \in \Omega^2(M, \mathfrak{u}(1))$ denote the global 2-form on M that is induced by the field strength of the gauge field. The equations of motion and the Bianchi identity then imply that

$$d \star F \; = \; 0 \qquad \text{and} \qquad dF \; = \; 0 \, . \tag{0.228}$$

As a result, the exponentiated quantities

$$U_\alpha(\Sigma_{d-2}) \; := \; \exp\left(\alpha \int_\Sigma \star F\right) \qquad \text{and} \qquad V_\beta(\Sigma_2) \; := \; \exp\left(\beta \int_\Sigma F\right) \tag{0.229}$$

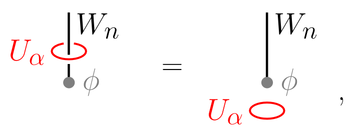
defined on submanifolds $\Sigma_{d-2}, \Sigma_2 \subset M$ and labelled by $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$, respectively, yield two classes of topological defects with fusion rules

$$U_\alpha \otimes U_{\alpha'} \; = \; U_{\alpha+\alpha'} \qquad \text{and} \qquad V_\beta \otimes V_{\beta'} \; = \; V_{\beta+\beta'} \, . \tag{0.230}$$

We call the generalised symmetries generated by U_α and V_β the *electric 1-form* and *magnetic* $(d-3)$*-form symmetry*, respectively. The operators that are charged under these symmetries are given by Wilson lines and 't Hooft operators. Concretely, linking a Wilson line W_n with a symmetry defect U_α and an 't Hooft operator T_m with V_β (where $n, m \in \mathbb{Z}$) yields the multiplicative phases

(0.231)

From this it is easy to see that the inclusion of additional matter fields typically leads to a breaking of higher-form symmetries. For instance, suppose that the theory contains an additional complex scalar ϕ of charge $k \in \mathbb{Z}$, so that the Wilson line W_k can end on a local insertion of ϕ. As a result, any linking of W_k with a topological symmetry defect U_α can be undone by 'sliding' U_α off W_n, i.e.



(0.232)

so that according to (0.231) we must have $e^{2\pi i\alpha k} = 1$, or equivalently $\alpha = i/k$ for $i = 0, ..., k-1$. This is an example of a more general principle: A line operator that can end on a twisted sector local operator cannot be charged under any 1-form symmetries. In the present case, we see that the inclusion of ϕ breaks the electric 1-form symmetry $\mathbb{R}/\mathbb{Z}$ down to the discrete subgroup $\mathbb{Z}_k$.

- GAUGE THEORY: Given a compact connected Lie group G, the theory of gauge fields A living on a principal G-bundle $\pi : P \to M$ possesses analogues of the electric and magnetic symmetries discussed in the previous example (the following discussion is taken from [54]). In order to describe them, consider a closed loop $\gamma : S^1 \to M$ based at $m := \gamma(1) \in M$. Parallel transport along γ yields a map $\Pi_\gamma(A) : P_m \to P_m$, which upon fixing a particular $p \in P_m$ can be identified with a group element $g_p \in G$ such that $\Pi_\gamma(A)(p) = p \triangleleft g_p$. Choosing a different $p' \in P_m$ shifts $g_{p'} = h^{-1} \cdot g_p \cdot h$, where $h \in G$ is such that $p' = p \triangleleft h$. As a result, we obtain a well-defined map Ξ_γ for every γ that sends the gauge field A to the conjugacy class $[g_p]$ of g_p in G. One can check that this map is both gauge-invariant and independent of the choice of basepoint m of the loop γ.

  Now consider a fixed codimension-two submanifold $\Sigma \subset M$. Given a conjugacy class $[g] \in \mathrm{Cl}(G)$ in G, we can construct an associated operator $U_{[g]}$ with support in Σ by defining its insertion into the path integral to mean that we only integrate over those (equivalence classes of) gauge fields A on bundles over the modified spacetime $M \backslash \Sigma$ with

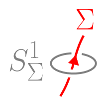
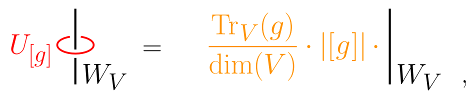
  

  Figure 24

  $$\Xi_{S^1_\Sigma}(A) \;=\; [g]\;, \tag{0.233}$$

  where $S^1_\Sigma \subset M$ denotes an infinitesimal 1-sphere that links Σ as illustrated in Figure 24. We call the operator $U_{[g]}$ as defined above a *Gukov-Witten operator* in what follows [55, 56]. In general, it is not topological. However, we can derive a necessary condition for $U_{[g]}$ to be topological as follows: Consider linking a Wilson line W_V labelled by a linear G-space V with an infinitesimal Gukov-Witten operator $U_{[g]}$. The result is given by the multiplicative phase

  (0.234)

  which generalises the left hand side of (0.231) to the non-abelian case. Since the Wilson line $W_{\mathfrak{g}}$ associated to the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ can always end on the field strength of the gauge field[64], it cannot be charged under any topological 1-form defect. In particular, using (0.234) we see that $U_{[g]}$ can only be topological if

  $$\mathrm{Tr}_{\mathfrak{g}}(\mathrm{Ad}(g)) \;=\; \dim(\mathfrak{g})\;, \tag{0.235}$$

[64] More concretely, let $F \in \Omega^2(P, \mathfrak{g})$ be the field strength of the gauge field and let $s : U \to P$ be a local section of P over $U \subset M$. Upon choosing two vector fields $X, Y \in \mathfrak{X}(M)$, we can define a local section $F_{X,Y} : U \to P_{\mathfrak{g}}$ of the vector bundle associated to the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ by setting $F_{X,Y}(m) := [s(m), s^*(F)_m(X_m, Y_m)]$ for $m \in U$. The transformation law (0.48) for F under changes of the local section s then implies that $F_{X,Y}$ induces a canonical *global* section of $P_{\mathfrak{g}}$, which means that the Wilson line $W_{\mathfrak{g}}$ is able to end as claimed.

which is equivalent to $\mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$. Thus, we see that the topological Gukov-Witten operators are necessarily labelled by elements of

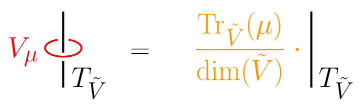
$$\ker(\mathrm{Ad}) \;\equiv\; \mathcal{Z}(G)\,. \tag{0.236}$$

It is known that in a pure gauge theory this condition is also sufficient, so that the topological Gukov-Witten operators form an *electric 1-form symmetry* given by centre of G [54]. If the theory contains additional matter fields in the form of sections $\varphi \in \Gamma(P_V)$ associated to a linear G-space V, then the endability of the Wilson line W_V on φ implies that the electric 1-form symmetry is broken down to the subgroup of those $z \in \mathcal{Z}(G)$ that act trivially on V.

In addition to the electric 1-form symmetry, gauge theories also possess a *magnetic* $(d-3)$*-form symmetry*, which for a simple gauge group G can be shown to be given by the Pontryagin dual $\pi_1(G)^\vee$ of the fundamental group of G. Its action on 't Hooft lines $T_{\tilde{V}}$ labelled by linear ${}^LG$-spaces $\tilde{V}$ (where ${}^LG$ denotes the Langlands dual of G as before) can be described using the fact that

$$\pi_1(G)^\vee \;\cong\; \mathcal{Z}({}^LG) \tag{0.237}$$

which allows us to write down the magnetic analogue of equation (0.234) as



(0.238)

for $\mu \in \pi_1(G)^\vee$. In particular, this is compatible with electromagnetic duality in spacetime dimension $d = 4$, which exchanges Wilson and 't Hooft lines as well as the electric and magnetic 1-form symmetries acting on them.

- Finite gauge theory: Consider a theory $\mathcal{T}$ with an ordinary global symmetry described by some finite group G. From this, we can construct a new theory $\mathcal{T}/G$ by 'gauging' the symmetry G. If G were continuous, this would mean that we introduce a principal G-bundle P over spacetime and path integrate over all (equivalence classes) of gauge fields $A \in \Omega^1(P, \mathfrak{g})$ thereon. However, for finite G, the situation simplifies drastically due to the fact that its Lie algebra vanishes ($\mathfrak{g} = 0$), so that there are no non-trivial gauge fields. As a result, we can gauge G simply by summing over G-bundles P, which can be characterised in the following manner: As before, the total space P comes equipped with a projection $\pi : P \to M$ such that G acts freely and transitively on each fibre $P_m := \pi^{-1}(m)$ for $m \in M$ as illustrated in Figure 25. The principal bundle can be trivialised over contractible open neighbourhoods $U \subset M$ via local

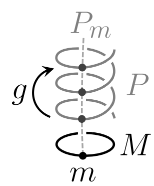


Figure 25

sections $s : U \to P$, which induce isomorphisms $U \times G \cong P_U$ via $(m, g) \mapsto s(m) \triangleleft g$. Now consider an open cover $\bigcup_i U_i = M$ of spacetime by contractible $U_i \subset M$ together with local sections $s_i : U_i \to P$. We denote by $g_{ij} : U_i \cap U_j \to G$ the so-called *transition functions* defined by

$$s_i(m) \ =: \ s_j(m) \triangleleft g_{ij}(m) \tag{0.239}$$

for $m \in U_i \cap U_j$, which satisfy the *cocycle condition*

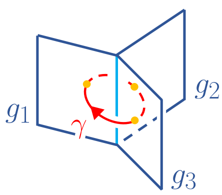
$$g_{ij}(m) \cdot g_{jk}(m) \ = \ g_{ik}(m) \tag{0.240}$$

for all $m \in U_i \cap U_j \cap U_k$. Since G is finite and the g_{ij} are continuous, the latter must necessarily be constant, so that we can label the bundle P by a collection of group elements g_{ij} assigned to double intersections $U_i \cap U_j$ and subject to the condition $g_{ij} \cdot g_{jk} = g_{ik}$. We can represent this pictorially by collapsing the intersections $U_i \cap U_j$ down to codimension-one submanifolds labelled by group elements, which join up at triple intersections $U_i \cap U_j \cap U_k$ in a way such that the product of 'incoming' group elements equals the product of 'outgoing' ones:

. (0.241)

Physically, the above corresponds to inserting a 'network' of topological symmetry defects (also called a *background*) into spacetime, which specifies the bundle P up to equivalence. Summing over all principal bundles P is then equivalent to summing over all possible network configurations (or backgrounds).

In general, the gauged theory $\mathcal{T}/G$ possesses a canonical class of line operators that form the analogue of Wilson lines in a continuous gauge theory. To see this, consider a closed loop $\gamma : S^1 \to M$ based at $m := \gamma(1) \in M$. In the presence of a non-trivial background, this loop will generically pierce through a number of codimension-one symmetry defects that form part of the corresponding network:



. (0.242)

Let $g_1, ..., g_n \in G$ denote the group elements in the order that we pierce through the associated symmetry defects if we follow the oriented curve γ along starting at m. Given a linear G-space V, we then define the corresponding *Wilson loop* to be the path integral insertion

$$W_V(\gamma) \;:=\; \mathrm{Tr}_V(g_1 \cdot \ldots \cdot g_n) \,. \tag{0.243}$$

One can check that this is independent of the choice of basepoint m of the loop. Importantly, in contrast to the case of a continuous gauge group, the line operator W_V is now topological and hence defines a generalised $(d-2)$-form symmetry labelled by linear G-spaces V. Using (0.243), their fusion can be computed to be given by the tensor product

$$W_V \,\otimes\, W_{V'} \;=\; W_{V\otimes V'} \tag{0.244}$$

of G-spaces[65], which shows that for non-abelian G the corresponding symmetry is non-invertible[66]. This provides a simple example of non-invertible symmetries that can be constructed via the discrete gauging of finite invertible symmetries[67]. It was shown in [5, 6] that for $d > 2$, the gauged theory $\mathcal{T}/G$ contains additional higher-dimensional topological defects that are labelled by so-called *higher G-spaces*, which can be non-invertible even for abelian G (see also [57–59]).

0.2 Overview

Generalised global symmetries significantly extend the traditional notion of symmetries in quantum field theory. While the latter can be described using the theory of groups and their representations, the mathematical structures underpinning generalised symmetries and their action on physical observables are less obvious. The aim of this thesis is to provide answers to the following questions:

1. What is the precise mathematical structure that describes generalised symmetries and their properties in arbitrary quantum field theories?
2. How do generalised symmetries act on and thereby organise the spectrum of other physical observables and operators?

Throughout this thesis, we will restrict ourselves to the case of *finite bosonic* symmetries, which allow us to perform explicit calculations and computations. We note that while the motivation for our work is physical, most of its results are mathematical in nature.

---

[65] Given two linear G-spaces V and V', their *tensor product* is defined to be the linear G-space $V \otimes V'$ with associated G-action $g \triangleright (v \otimes v') := (g \triangleright v) \otimes (g \triangleright v')$.

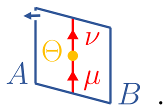
[66] This is due to the fact that any finite non-abelian group G has at least one irreducible representation of dimension greater than one.

[67] Non-invertible symmetries of this type are usually called *non-intrinsic*, since they can be obtained from invertible symmetries via 'topological manipulations' such as discrete gauging. On the other hand, non-invertible symmetries which are not of this type are called *intrinsic*.

0.2.1 Setup

As alluded to in the Motivation 0.1, we expect general global symmetries to be described by (some version of) a higher monoidal category. If the symmetries under consideration are finite and bosonic, it is believed that the appropriate mathematical structure is given by a (higher) fusion category $\mathcal{C}$ (also called the *symmetry category* in what follows) [60]. Without going into technical details, we list some of the most important features of this structure together with their physical significance below:

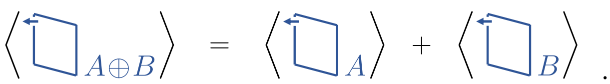
- **Objects & morphisms:** As a higher category, $\mathcal{C}$ contains both objects and morphisms of various degrees, which organise the spectrum of topological defects in $\mathcal{C}$ according to their (co)dimension. Concretely, for each $p = 0, ..., d-1$, there is a space $\mathcal{C}_p$ that contains all topological defects of codimension $p+1$. The elements of $\mathcal{C}_0$ are called the *objects* of $\mathcal{C}$ (and we often simply write $A \in \mathcal{C}$ to denote that A is an object of $\mathcal{C}$). Given two objects $A, B \in \mathcal{C}$, we denote by $\text{Hom}_{\mathcal{C}}(A, B) \subset \mathcal{C}_1$ the space of *morphisms* between A and B, which capture the topological defects at the interface between A and B. Similarly, given $\mu, \nu \in \text{Hom}_{\mathcal{C}}(A, B)$, we denote by $2\text{Hom}_{\mathcal{C}}(\mu, \nu) \subset \mathcal{C}_2$ the space of *2-morphisms* between μ and ν, which capture the topological defects Θ at the junction between μ and ν as illustrated below:



(0.245)

  By iterating this procedure, we arrive at the space[68] $p\text{-Hom}_{\mathcal{C}}(.,.)$ of *p-morphisms* for $p = 1, ...d-1$. Given $\mu \in \mathcal{C}_p$, we denote by $\text{id}_\mu \in (p{+}1)\text{-End}_{\mathcal{C}}(\mu)$ the *identity $(p+1)$-morphism* on μ, which corresponds to the trivial interface between μ and itself. Moreover, we assume that the space $(d{-}1)\text{-Hom}_{\mathcal{C}}(.,.)$ of so-called *top-morphisms* describing topological local defects forms a complex vector space so that compositions of top-morphisms are linear. All in all, the above means that $\mathcal{C}$ is a linear $(d-1)$-category.

- **Finite semisimplicity:** Given two objects $A, B \in \mathcal{C}$, we can define their direct sum $A \oplus B$ to be the symmetry defect whose correlation functions are given by the sum of the correlation functions of A and B, i.e.

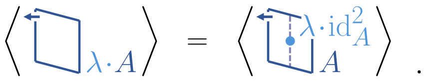


(0.246)

  We assume that any symmetry defect $A \in \mathcal{C}$ can be decomposed into a finite direct sum of *simple* (or *indecomposable*) objects S with $(d{-}1)\text{-End}_{\mathcal{C}}(\text{id}_S^{d-2}) \cong \mathbb{C}$,

[68] More precisely, $p\text{-Hom}_{\mathcal{C}}(.,.)$ is a $(d-1-p)$-category whose objects are p-morphisms in $\mathcal{C}$, morphisms are $(p+1)$-morphisms in $\mathcal{C}$, etc.

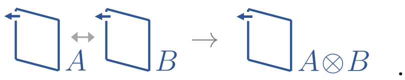
where $\mathrm{id}^p_S := \mathrm{id}_{\mathrm{id}\cdots\mathrm{id}_S}$ denotes the identity p-morphism on S. Furthermore, we assume that there is only a *finite* number of simple objects S (up to a suitable notion of equivalence). Similarly, we assume that every p-morphism can be decomposed into a finite number of simple p-morphisms for $p = 1, ..., d-2$. We define scalar multiples of symmetry defects via the insertion of a multiple of their identity top-morphism into correlation functions, e.g.



(0.247)

for $\lambda \in \mathbb{C}$. All in all, the above means that $\mathcal{C}$ is finite semisimple.

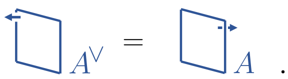
- **Monoidality:** We assume that $\mathcal{C}$ comes equipped with a monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that captures the fusion of topological defects induced by placing the latter parallel to each other and making their supports coincide:



(0.248)

We denote by $\mathbf{1} \in \mathcal{C}$ the *monoidal unit* w.r.t $\otimes$, which corresponds to the trivial codimension-one symmetry defect. We further assume that $\mathbf{1}$ is simple[69].

- **Rigidity:** We assume that every object $A \in \mathcal{C}$ has a *dual*, which is an object $A^\vee \in \mathcal{C}$ that corresponds to the orientation reversal of A, i.e.



(0.249)

Similarly, we assume that every p-morphism has an *adjoint* for $p = 1, ..., d-2$. In addition, we assume that $\mathcal{C}$ is equipped with suitable *duality data*, which allows us to 'bend' topological defects as illustrated below:

(0.250)

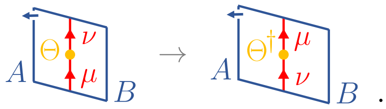
All in all, the above means that $\mathcal{C}$ is a so-called *rigid* category.

- **Unitarity:** We assume that $\mathcal{C}$ is equipped with an involution[70] $\dagger : \mathcal{C} \to \mathcal{C}^{(d-1)\text{-op}}$ that acts trivially on objects and p-morphisms for $p = 1, ..., d-2$ and antilinearly

[69] Physically, this means that there is a unique topological local operator (up to scalar multiplication). Equivalently, the vacuum sector of the theory is indecomposable, i.e. does not split into multiple superselection sectors. Relaxing this assumption leads to the notion of a *multifusion* category.

[70] For $p = 1, ..., d-1$, we denote by $\mathcal{C}^{p\text{-op}}$ the category that has the same objects and q-morphisms as $\mathcal{C}$ for $q = 1, ..., p-1$ but p-morphisms given by $p\text{-Hom}_{(\mathcal{C}^{p\text{-op}})}(\mu, \nu) = p\text{-Hom}_{\mathcal{C}}(\nu, \mu)$ for $\mu, \nu \in \mathcal{C}_{p-1}$.

on top-morphisms. Physically, this captures the behaviour of symmetry defects under spacetime reflections, i.e.



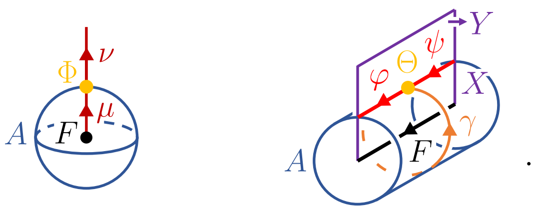
(0.251)

Demanding that † be compatible with the monoidal and rigid structure on $\mathcal{C}$ then turns the latter into a *unitary* category.

While ideas from category theory have played a role in quantum theory since the axiomatisation of *topological* quantum field theories (TQFTs) through Atiyah [61] (see also [62, 63] for the introduction of higher categories for the description of *extended* TQFTs), recent years have demonstrated the utility of higher categorical structures (and in particular higher fusion categories) for the description of symmetries in generic *non-topological* theories. While the mathematical literature on fusion $(d-1)$-categories is well-developed for $d = 2$ (see e.g. [64, 65] for standard references) and an active area of research for $d = 3$ (see e.g. [66–68]), a rigorous treatment for $d > 2$ is more elusive. For this reason, we will only consider the case $d \leq 3$ in detail in this thesis.

0.2.2 Summary

As described in the Motivation 0.1, generalised global symmetries can act on local and extended operators by linking them inside correlation functions. Concretely, given a symmetry category $\mathcal{C}$ and an $(n-1)$-dimensional operator F (where $n = 1, ..., d-1$), we can act on the latter with $\mathcal{C}$ by placing p-form symmetry defects $A, \gamma, ...$ on codimension-$(p+1)$ submanifolds that locally look like $\mathbb{R}^{n-p-1} \times S_F^{d-n}$, where the second factor denotes a sphere that links the n-dimensional support of F. In particular, this is only sensible if $p < n$. Moreover, since we assume F to be a twisted sector operator in general, the p-form defects surrounding F will intersect the $(n-1)$-dimensional symmetry defects $X, \mu, ...$ attached to F along $(n-p-1)$-dimensional loci $\Phi, \varphi, \Theta, ...$. This is illustrated for $d = 3$ and $n = 1, 2$ below:



(0.252)

Note that for a general symmetry category $\mathcal{C}$, the linking action of symmetry defects may change the twisted sectors of local and extended operators. This is one of the hallmarks of non-invertible symmetries – they map untwisted operators to twisted sector operators and vice versa.

We can systematise the above discussion by introducing for each $n = 1, ..., d-1$ two types of n-categories that capture $(n-1)$-dimensional extended operators and their possible linkings with symmetry defects in $\mathcal{C}$, respectively:

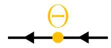


Figure 26

1. ***n*-vector spaces:** We can organise the extended operators of fixed dimension $n-1$ by introducing an $(n-1)$-category whose
   - objects are $(n$–$1)$-dimensional operators (that are generically non-topological and that can be twisted or untwisted),
   - q-morphisms are $(n-q-1)$-dimensional topological interfaces between $(q-1)$-morphisms for $q = 1, ..., n-1$ (see Figure 26).

   For $n = 1$, this simply yields a vector space, representing the fact that local operators carry a linear structure. By analogy, we will call the $(n-1)$-category formed by $(n-1)$-dimensional operators an *n-vector space* in what follows. We denote the n-category formed by all n-vector spaces[71] by $n\mathrm{Vect}$.

2. **Tube *n*-category:** We can organise the possible geometric configurations of symmetry defects in $\mathcal{C}$ with which we can link $(n-1)$-dimensional twisted sector operators by introducing an n-category whose
   - objects are genuine n-dimensional topological symmetry defects[72] in $\mathcal{C}$ that label twisted sectors of $(n-1)$-dimensional operators,
   - q-morphisms are to $(d-q)$-dimensional defects in $\mathcal{C}$ placed on $\mathbb{R}^{n-q} \times S^{d-n}$ together with $(n-q)$-dim. intersection data for $q = 1, ..., n$ (cf. (0.252)).

   We will denote this n-category by n-$\mathrm{T}\mathcal{C}$ and call it the *tube n-category* associated to $\mathcal{C}$ in what follows.

We can use the above constructions to describe the action of symmetry defects in $\mathcal{C}$ on local and extended twisted sector operators. Concretely, we claim that the transformation behaviour of the latter is captured by a suitable notion of representation of the tube category [1, 2]:

Claim: In a d-dimensional quantum field theory with symmetry category $\mathcal{C}$, twisted sector operators of dimension $(n-1)$ transform in so-called *n-representations* of the tube n-category associated to $\mathcal{C}$, by which we mean n-functors

$$\mathcal{F}: \; n\text{-}\mathrm{T}\mathcal{C} \;\to\; n\mathrm{Vect} \tag{0.253}$$

from n-$\mathrm{T}\mathcal{C}$ into the n-category of n-vector spaces (where $n = 1, ..., d-1$).

[71] Concretely, $n\mathrm{Vect}$ is the n-category whose objects are n-vector spaces, morphisms are functors between the corresponding $(n-1)$-categories, 2-morphisms are natural transformations, etc.

[72] More precisely, objects are objects of the *loop space* $\Omega^{d-n-1}(\mathcal{C})$ of $\mathcal{C}$, where for $p = 1, ..., d-1$ we define $\Omega^p(\mathcal{C}) := p\text{-}\mathrm{End}_{\mathcal{C}}(\mathrm{id}_{\mathbf{1}}^{p-1})$ to be the space of p-endomorphisms of the identity $(p$–$1)$-morphism of the monoidal unit $\mathbf{1} \in \mathcal{C}$.

In essence, this generalises the idea that (local) operators in a quantum field theory transform in irreducible representations of ordinary global symmetry groups. The tentative answer to the motivating questions posed at the beginning of this section may hence be written schematically as

	ordinary	generalised
symmetries	group G	symmetry category $\mathcal{C}$
operators	representations of G	representations of T$\mathcal{C}$

.

In this thesis, we will discuss and justify the above in detail in spacetime dimensions $d = 1, 2, 3$. Concretely, the main body of the text is organised as follows:

- In Chapter 1, we review finite global symmetries in one-dimensional quantum systems (a.k.a. quantum mechanics). We discuss finite-dimensional C*-algebras, which generalise the notion of unitary group-like symmetries and which serve as a prototype for non-invertible symmetries in higher dimensions.
- In Chapter 2, we review the construction of the tube category, which captures the action of a fusion category symmetry $\mathcal{C}$ on twisted sector local operators in two dimensions. We describe how its irreducible representations can be classified using the so-called *sandwich construction* (or *Symmetry TFT*) for categorical symmetries. Apart from anomalous group-like symmetries, we provide new examples that include generic Tambara-Yamagami symmetries as well as non-invertible symmetries of Fibonacci and Yang-Lee type.
- In Chapter 3, we construct the tube 1- and 2-categories associated to a fusion 2-category symmetry $\mathcal{C}$ in three dimensions, which capture the action of $\mathcal{C}$ on twisted sector local and line operators, respectively. We classify their irreducible 1- and 2-representations using a higher-dimensional analogue of the sandwich construction and provide explicit examples that include anomalous 2-group symmetries as well as non-invertible 1-form symmetries. Our construction represents a new approach to studying the action of generalised symmetries on local and extended operators in three dimensions in a systematic manner.

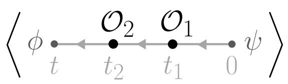
1

ONE DIMENSION

In this chapter, we review the theory of C*-algebras that describe generalised global symmetries in one-dimensional quantum systems (also known as quantum mechanics). We discuss the notion of $*$-representations of C*-algebras and provide examples that include anomalous group symmetries and topological deformations thereof.

1.1 C*-Algebras

Given a quantum mechanical theory $\mathcal{T}$, we are interested in computing correlation functions of observables $\mathcal{O}$ that are inserted into one-dimensional spacetime,



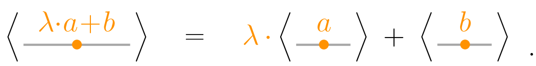
$$= \langle \phi | \, e^{-i(t-t_2)\widehat{H}} \circ \widehat{\mathcal{O}} \circ e^{-i(t_2-t_1)\widehat{H}} \circ \widehat{\mathcal{O}} \circ e^{-it_1\widehat{H}} \, | \psi \rangle \; . \tag{1.1}$$

Here, $\phi, \psi \in \mathcal{H}$ are states in the Hilbert space of the theory, $\widehat{H}$ denotes the Hamiltonian, and the $\mathcal{O}_i = \mathcal{O}(t_i)$ denote different insertions of the observable $\mathcal{O}$, whose associated Hilbert space operator is $\widehat{\mathcal{O}} : \mathcal{H} \to \mathcal{H}$. In the context of symmetries, we are interested in the subspace $\mathcal{A}$ of those operators on $\mathcal{H}$ that commute with the Hamiltonian,

$$\mathcal{A} \;=\; \{ a : \mathcal{H} \to \mathcal{H} \mid [a, \widehat{H}] = 0 \} \; , \tag{1.2}$$

or equivalently those insertions into correlation functions that are topological (as long as no other operator insertions are crossed). We identify $\mathcal{A}$ with the space of generalised symmetry defects in $\mathcal{T}$. Using the structure of linear operators on a Hilbert space, we then expect $\mathcal{A}$ to have the following properties:

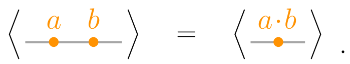
1. *Linearity:* Given $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, their linear combination $\lambda \cdot a + b \in \mathcal{A}$ is given by the linear combination of the respective correlation functions, i.e.



(1.3)

This gives $\mathcal{A}$ the structure of a (complex) vector space.

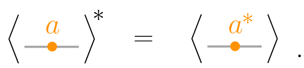
2. *Multiplication:* Given $a, b \in \mathcal{A}$, their product $a \cdot b \in \mathcal{A}$ is obtained by colliding the corresponding topological symmetry defects inside correlation functions, i.e.



(1.4)

This needs to be compatible with the linear structure on $\mathcal{A}$, which turns the latter into an associative algebra. There exists a unit $1 \in \mathcal{A}$ that corresponds to the identity operator on the Hilbert space.

3. *Involution:* Given $a \in \mathcal{A}$, its adjoint $a^* \in \mathcal{A}$ is obtained by complex conjugating (and thereby reflecting) correlation functions with a inserted, i.e.



(1.5)

This equips $\mathcal{A}$ with an antilinear algebra involution that corresponds to taking adjoints of linear operators on the Hilbert space. In particular, the operator norm[1] $\|.\|$ on $\mathcal{A}$ is such that $\|a^* \cdot a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

Physically, the fact that $\mathcal{A}$ forms an algebra means that generic symmetry defects $a \in \mathcal{A}$ are non-invertible, i.e. do not possess an inverse $a^{-1} \in \mathcal{A}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Mathematically, we can capture the algebraic structure on $\mathcal{A}$ as follows:

Definition: A complex unital associative algebra $\mathcal{A}$ is called a $*$-*algebra* if it is equipped with an antilinear map $* : \mathcal{A} \to \mathcal{A}$ that satisfies

$$(a^*)^* \;=\; a \qquad \text{and} \qquad (a \cdot b)^* \;=\; b^* \cdot a^* \tag{1.7}$$

for all $a, b \in \mathcal{A}$. We denote by $\mathcal{A}^\times \subset \mathcal{A}$ the subset of all *invertible* elements of $\mathcal{A}$, i.e. those $a \in \mathcal{A}$ for which there exists an $a^{-1} \in \mathcal{A}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$, where $1 \in \mathcal{A}$ denotes the monoidal unit of $\mathcal{A}$. We say that an element $a \in \mathcal{A}^\times$ is *unitary* if $a^* = a^{-1}$. We say that $a \in \mathcal{A}$ is *self-adjoint* if $a^* = a$. A $*$-algebra $\mathcal{A}$ is called a *C^*-algebra* if it is equipped with a norm $\|.\| : \mathcal{A} \to \mathbb{R}$ that turns $\mathcal{A}$ into a Banach algebra[2] and that satisfies

$$\|a^* \cdot a\| \;=\; \|a\|^2 \tag{1.8}$$

[1] Given a (bounded) linear map $f : \mathcal{H} \to \mathcal{H}'$ between Hilbert spaces, its *operator norm* is defined by

$$\|f\| \;:=\; \sup\{\, \|f(\psi)\|' / \|\psi\| \mid \psi \in \mathcal{H} \backslash \{0\} \} \,. \tag{1.6}$$

[2] An algebra $\mathcal{A}$ is called a *Banach algebra* if it is equipped with a norm $\|.\| : \mathcal{A} \to \mathbb{R}$ such that

$$\text{(i)}\;\; \|\lambda \cdot a\| \;=\; |\lambda| \cdot \|a\| \,, \qquad \text{(ii)}\;\; \|a + b\| \;\leq\; \|a\| + \|b\| \,, \qquad \text{(iii)}\;\; \|a \cdot b\| \;\leq\; \|a\| \cdot \|b\|$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ and $\mathcal{A}$ is complete w.r.t. the norm-induced metric $d(a, b) := \|a - b\|$.

for all $a \in \mathcal{A}$. We note that being C* is a *property* of a $*$-algebra and *not* extra data. Indeed, if $\mathcal{A}$ is a C*-algebra, then the associated norm satisfying (1.8) is uniquely determined by the spectral radius formula [69]

$$\|a\|^2 \;=\; \sup\{\, |\lambda| \mid a^*a - \lambda\cdot 1 \notin \mathcal{A}^\times \} \;. \tag{1.9}$$

An algebra homomorphism $f : \mathcal{A} \to \mathcal{A}'$ between two (C-)$*$-algebras is said to be a $*$-*homomorphism* if $f(a^*) = f(a)^*$ for all $a \in A$. Two (C-)$*$-algebras are said to be *isomorphic* ($\mathcal{A} \cong \mathcal{A}'$) if there exists an invertible $*$-homomorphism between them.

Since for the purposes of this thesis we are only interested in finite symmetries, we will restrict attention to finite-dimensional (C-)$*$-algebras in what follows. In this case, there is a simple criterion that allows us to check whether a given $*$-algebra is C* [70]:

Proposition: Let $\mathcal{A}$ be a finite-dimensional $*$-algebra. Then, $\mathcal{A}$ is C* if and only if the following two equivalent conditions are satisfied:

1. For all $a \in \mathcal{A}$, the equation $a^*a = 0$ implies $a = 0$.
2. There exists a *faithful state* on $\mathcal{A}$, i.e. a linear functional $\Gamma : \mathcal{A} \to \mathbb{C}$ obeying $\Gamma(a^*a) \geq 0$ for all $a \in \mathcal{A}$ with equality if and only if $a = 0$.

As a simple example, consider the algebra $M_k(\mathbb{C})$ of complex square matrices of size $k \in \mathbb{N}$. This is a $*$-algebra if we define the $*$-structure to be the usual conjugate transpose operation. It is furthermore C* since the trace $\mathrm{Tr} : M_k(\mathbb{C}) \to \mathbb{C}$ is a positive linear functional on $M_k(\mathbb{C})$. In fact, this example essentially exhausts all finite-dimensional C*-algebras, as stated by the *Artin-Wedderburn theorem* [71, 72]:

Theorem: Let $\mathcal{A}$ be a finite-dimensional C*-algebra. Then, there exist non-negative integers $n, k_i \in \mathbb{N}$ (with $i = 1, ..., n$) such that

$$\mathcal{A} \;\cong\; \bigoplus_{i=1}^{n} M_{k_i}(\mathbb{C}) \;. \tag{1.10}$$

In general, this isomorphism is non-canonical. We say that $\mathcal{A}$ is *simple* if $n = 1$ (and more generally *semisimple* for $n \geq 1$).

1.2 C*-Representations

Given a C*-algebra $\mathcal{A}$ of symmetry defects, we would like to understand how it acts on the Hilbert space $\mathcal{H}$ of states of a quantum mechanical theory. Mathematically, this is done by introducing an appropriate notion of representations for C*-algebras:

Definition: A $*$-*representation* of a C*-algebra $\mathcal{A}$ is a $*$-homomorphism[3]

$$U : \mathcal{A} \to \mathrm{End}(\mathcal{H}) \tag{1.11}$$

from $\mathcal{A}$ into the C*-algebra of (bounded) endomorphisms of a Hilbert space $\mathcal{H}$ (with $*$-structure given by the adjoint of linear operators). We define the *dimension* of U to be the dimension $\dim(U) := \dim(\mathcal{H})$ of the underlying Hilbert space. Given two $*$-representations U and U', an *intertwiner* between them is a linear map $f : \mathcal{H} \to \mathcal{H}'$ such that $U'(a) \circ f = f \circ U(a)$ for all $a \in \mathcal{A}$. We say that U and U' are *equivalent* if there exists an invertible intertwiner between them. They are said to be *unitarily equivalent* if there exists a unitary intertwiner between them. We denote the category of $*$-representations of $\mathcal{A}$ and intertwiners between them by $\mathrm{Rep}^*(\mathcal{A})$ in what follows.

As in the case of group-like symmetries, we say that a $*$-representation is *irreducible* if the only $\mathcal{A}$-invariant (closed) subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$. We can characterise the irreducible $*$-representations using (a version of) *Schur's lemma*:

Lemma: Let U be a $*$-representation of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Then, U is irreducible if and only if all its intertwiners are of the form $\lambda \cdot \mathrm{id}_{\mathcal{H}}$ for some $\lambda \in \mathbb{C}$. In particular, this implies that two given irreducible $*$-representations U and U' are either unitarily equivalent or have no non-trivial intertwiners between them.

Given two $*$-representations U and U' of $\mathcal{A}$ on Hilbert space $\mathcal{H}$ and $\mathcal{H}'$, respectively, we can define their *direct sum* to be the $*$-representation $U \oplus U'$ of $\mathcal{A}$ that acts on the Hilbert space $\mathcal{H} \oplus \mathcal{H}'$ via

$$(U \oplus U')(a) \cdot (\psi + \psi') \;:=\; U(a) \cdot \psi \,+\, U'(a) \cdot \psi' \tag{1.12}$$

for $\psi \in \mathcal{H}$, $\psi' \in \mathcal{H}'$ and $a \in \mathcal{A}$. Conversely, we can decompose a given $*$-representation $U : \mathcal{A} \to \mathrm{End}(\mathcal{H})$ of $\mathcal{A}$ into direct sums of subrepresentations as follows: Let $\mathcal{V} \subset \mathcal{H}$ be a closed $\mathcal{A}$-invariant subspace of $\mathcal{H}$ and denote by

$$\mathcal{V}^\perp \;=\; \{\, \phi \in \mathcal{H} \mid \langle \phi, \psi \rangle \;=\; 0 \;\text{ for all }\; \psi \in \mathcal{V} \,\} \tag{1.13}$$

its orthogonal complement. For a given $\phi \in \mathcal{V}^\perp$, we then have that

$$\langle U(a) \cdot \phi, \psi \rangle \;=\; \langle \phi, U(a^*) \cdot \psi \rangle \;=\; 0 \tag{1.14}$$

[3] In particular, we require that the unit $1 \in \mathcal{A}$ is mapped to $U(1) = \mathrm{id}_{\mathcal{H}}$, which is equivalent to requiring that U is *non-degenerate* in the sense that $U(a) \cdot \xi = 0$ for all $a \in \mathcal{A}$ implies $\xi = 0$.

for all $\psi \in \mathcal{V}$, which shows that $U(a)\cdot\phi \in \mathcal{V}^\perp$ for all $a \in \mathcal{A}$. As a result, $\mathcal{V}^\perp$ forms another $\mathcal{A}$-invariant subspace of $\mathcal{H}$ and the $*$-representation U decomposes as

$$U \;\cong\; U|_{\mathcal{V}} \oplus U|_{\mathcal{V}^\perp} \;. \tag{1.15}$$

By iterating the above, we can sequentially decompose U into smaller and smaller orthogonal subrepresentations. In particular, if $\mathcal{H}$ is finite-dimensional, this procedure necessarily terminates after a finite number of steps, which yields the following:

Proposition: Every finite-dimensional $*$-representation of a C*-algebra $\mathcal{A}$ is fully reducible, i.e. unitarily equivalent to a finite direct sum of irreducible representations.

For example, up to equivalence, the matrix algebra $M_k(\mathbb{C})$ (where $k \in \mathbb{N}$) has a single irreducible $*$-representation given by the Hilbert space $\mathbb{C}^k$ together with the canonical matrix multiplication action. Similarly, semisimple matrix algebras of the form $\mathcal{A} = \bigoplus_{i=1}^n M_{k_i}(\mathbb{C})$ have n irreducible $*$-representations, in which the action of algebra elements $(a_1, ..., a_n) \in \mathcal{A}$ on vectors $\psi \in \mathbb{C}^{k_i}$ is given by

$$\mathrm{pr}_i(a_1, ..., a_n)\cdot\psi \;:=\; a_i\cdot\psi \;. \tag{1.16}$$

Since the above exhausts all finite-dimensional C*-algebras by virtue of the Artin-Wedderburn theorem, we see that every finite-dimensional C*-algebra $\mathcal{A}$ has a finite number of irreducible $*$-representations U_i. More concretely, upon choosing an isomorphism $\varphi : \mathcal{A} \to \bigoplus_{i=1}^n M_{k_i}(\mathbb{C})$, we set $U_i := \mathrm{pr}_i \circ \varphi$. Furthermore, we define

$$e_i \;:=\; \varphi^{-1}\big((0, ..., \mathbb{1}_{k_i}, ..., 0)\big) \;\in\; \mathcal{A} \tag{1.17}$$

for $i = 1, ..., n$, which yields a distinguished basis of the *centre*

$$\mathcal{Z}(\mathcal{A}) \;:=\; \{z \in \mathcal{A} \mid z\cdot a = a\cdot z \text{ for all } a \in \mathcal{A}\} \tag{1.18}$$

that is independent of the choice of isomorphism φ and that consists of so-called *minimal central idempotents* obeying

$$e_i\cdot e_j \;=\; \delta_{ij}\cdot e_i \;, \qquad e_i^* \;=\; e_i \;, \qquad \sum_{i=1}^n e_i \;=\; 1 \;. \tag{1.19}$$

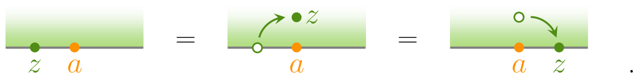
Moreover, their images under the irreducible $*$-representations U_i of $\mathcal{A}$ are given by

$$U_i(e_j) \;=\; \delta_{ij}\cdot \mathrm{id} \;, \tag{1.20}$$

which yields the following useful relation between irreducible $*$-representations and minimal central idempotents in $\mathcal{A}$:

Lemma: In a finite-dimensional C*-algebra, there is a 1:1-correspondence between equivalence classes of irreducible $*$-representations and minimal central idempotents.

Pictorially, we can represent the central elements $z \in \mathcal{Z}(\mathcal{A})$ as those topological operators that can be detached from the one-dimensional line and pushed into an adjacent two-dimensional 'bulk' (and hence have to commute with all other $a \in \mathcal{A}$):



(1.21)

In particular, placing a minimal central idempotent e_i into the bulk will project onto those operators that act on the Hilbert space via the $*$-representation U_i.

1.3 Examples

As discussed in the Motivation 0.1, finite global symmetry groups G act on the Hilbert space $\mathcal{H}$ of a quantum system via unitary representations. More generally, since the physical states are identified with *rays* in $\mathcal{H}$, we can consider so-called *projective representations* of G, which correspond to group homomorphisms

$$U: \; G \; \rightarrow \; \mathrm{Aut}(\mathcal{P}(\mathcal{H})) \tag{1.22}$$

from G into the automorphisms of the projective Hilbert space $\mathcal{P}(\mathcal{H}) = (\mathcal{H} \backslash \{0\}) / \mathbb{C}^\times$. More concretely, we say that an invertible map $U : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H})$ is an *automorphism* of $\mathcal{P}(\mathcal{H})$ if it preserves the so-called *ray product*

$$[\phi] \cdot [\psi] \; := \; \frac{|\langle \phi | \psi \rangle|}{\|\phi\| \cdot \|\psi\|} \tag{1.23}$$

between arbitrary states $[\psi], [\phi] \in \mathcal{P}(\mathcal{H})$ in the sense that $U([\phi]) \cdot U([\psi]) = [\phi] \cdot [\psi]$. It is a famous result due to Wigner that any such map is induced by (anti-)unitary operators on the Hilbert space $\mathcal{H}$ [73]:

Theorem: Let $U : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H})$ be an automorphism of the projective Hilbert space $\mathcal{P}(\mathcal{H})$. Then, there exists a unitary (or antiunitary[4]) operator $\tilde{U} : \mathcal{H} \rightarrow \mathcal{H}$ such that $U([\psi]) = [\tilde{U}(\psi)]$ for all $[\psi] \in \mathcal{P}(\mathcal{H})$. Furthermore, if $\tilde{U}' : \mathcal{H} \rightarrow \mathcal{H}$ is another such operator, then $\tilde{U}' = e^{i\alpha} \cdot \tilde{U}$ for some phase $e^{i\alpha} \in U(1)$.

In the following, we will restrict attention to *unitary* symmetries only. Given a projective representation $U : G \rightarrow \mathrm{Aut}(\mathcal{P}(\mathcal{H}))$, we may then choose for each group

[4] An invertible antilinear map $f : \mathcal{H} \rightarrow \mathcal{H}$ is called *antiunitary* if $\langle f(\phi) | f(\psi) \rangle = \langle \psi, \phi \rangle \equiv \langle \phi | \psi \rangle^*$ for all vectors $\phi, \psi \in \mathcal{H}$.

element $g \in G$ a unitary operator $\tilde{U}_g : \mathcal{H} \to \mathcal{H}$ such that $U_g([\psi]) = [\tilde{U}_g(\psi)]$ for all $[\psi] \in \mathcal{P}(\mathcal{H})$. The composition law $U_g \circ U_h = U_{g \cdot h}$ then implies that

$$\tilde{U}_g \circ \tilde{U}_h \;=\; \mu(g,h) \cdot \tilde{U}_{g \cdot h} \tag{1.24}$$

for some phase $\mu(g,h) \in U(1)$, which, in order to be compatible with associativity of composition of linear maps, needs to obey the *2-cocycle condition*

$$(d\mu)(g,h,k) \;:=\; \frac{\mu(h,k) \cdot \mu(g,hk)}{\mu(gh,k) \cdot \mu(g,h)} \;\overset{!}{=}\; 1 \tag{1.25}$$

for all $g,h,k \in G$. The space of functions $\mu : G^{\times 2} \to U(1)$ obeying (1.25) is called the space of *2-cocycles* on G with coefficients in $U(1)$ and denoted by $Z^2(G, U(1))$. Given another choice of lifts $\tilde{U}'_g$ obeying $U_g([\psi]) = [\tilde{U}'_g(\psi)]$, we have that $\tilde{U}'_g = \nu(g) \cdot \tilde{U}_g$ for some collection of phases $\nu(g) \in U(1)$, which shift the projective 2-cocycle by

$$\mu' \;=\; \mu \cdot d\nu \tag{1.26}$$

with $(d\nu)(g,h) := \nu(g) \cdot \nu(h) / \nu(gh)$. Thus, we see that we can characterise projective representations of G by equivalence classes of 2-cocycles μ, where μ and μ' are considered equivalent if they are related by equation (1.26). The space of all such equivalence classes forms an abelian group, which is called the *second group cohomology* of G with coefficients in $U(1)$ and denoted by $H^2(G, U(1))$. From a physical point of view, elements of this group are often referred to as *('t Hooft) anomalies*, since they capture the possible (controlled) violations of the group law in the action of G on the Hilbert space of the theory.

In the following, we will use the term *unitary μ-projective representation* of G to refer to a collection of unitary operators $\tilde{U}_g$ on a Hilbert space $\mathcal{H}$ that obey the composition rule (1.24) for some fixed 2-cocycle $\mu \in Z^2(G, U(1))$. In order to simplify notation, we will henceforth drop the $\tilde{}$ and write $U(g)$ instead of $\tilde{U}_g$. Without loss of generality, we will always assume $U(1) = \mathrm{id}_{\mathcal{H}}$ and that μ is *normalised*[5] in the sense that

$$\mu(g,1) \;=\; \mu(1,g) \;=\; 1\,. \tag{1.27}$$

A convenient way to construct unitary projective representations of G with a given 2-cocycle μ is using the method of *induction*. Concretely, given a subgroup $H \subset G$

[5] Concretely, given an arbitrary (unnormalised) 2-cocycle $\mu \in Z^2(G, U(1))$, we can set $\nu(g) := \mu(1,g)^*$ and define $\mu' := \mu \cdot d\nu$. Then, μ' is normalised in the sense of (1.27) and $[\mu] = [\mu']$ in $H^2(G, U(1))$.

and a $(\mu|_H)$-projective representation V of H on a Hilbert space $\mathcal{V}$, we can construct a μ-projective representation U of G on $\mathcal{H} := \mathbb{C}[G/H] \otimes \mathcal{V}$ as follows: Let

$$G/H \;=\; \{r_1 H, ..., r_n H\} \; . \tag{1.28}$$

denote the space of *left H-cosets in G* with fixed representatives $r_i \in G$. Using this, we can define for each $i = 1, ..., n$ and $g \in G$ a *little group element*

$$g_i \;:=\; r_i^{-1} \cdot g \cdot r_{g^{-1} \triangleright i} \;\in\; H \; , \tag{1.29}$$

where we set $r_{g \triangleright i} H := g \cdot r_i H$. Upon writing the Hilbert space $\mathcal{H}$ as a direct sum

$$\mathcal{H} \;\cong\; \bigoplus_{i=1}^{n} r_i \mathcal{V} \; , \tag{1.30}$$

the action of group elements $g \in G$ on $\mathcal{H}$ via U may then be decomposed into blocks $U(g)|_i : r_i \mathcal{V} \to r_{g \triangleright i} \mathcal{V}$ defined by

$$U(g)|_i \;:=\; \nu_{g \triangleright i}(g) \cdot V(g_{g \triangleright i}) \; , \tag{1.31}$$

where we defined the multiplicative phases

$$\nu_i(g) \;:=\; \frac{\mu(g, r_{g^{-1} \triangleright i})}{\mu(r_i, g_i)} \; . \tag{1.32}$$

As a result of the 2-cocycle condition (1.25) obeyed by μ, these satisfy

$$\frac{\nu_{g^{-1} \triangleright i}(h) \cdot \nu_i(g)}{\nu_i(gh)} \;=\; \frac{\mu(g, h)}{\mu(g_i, h_{g^{-1} \triangleright i})} \; , \tag{1.33}$$

which ensures that U is μ-projective. Furthermore, one can check that U is unitary if and only if V is. We call U as constructed above the *induction of V from H to G*[6] and denote it by $U = \mathrm{Ind}_H^G(V)$. The induction $\mathrm{Ind}_1^G(\mathbb{1})$ of the trivial representation of the trivial subgroup is called the *regular μ-projective representation* of G.

In the context of the previous subsections, we can recast the above discussion in terms of C*-algebras as follows: Given a finite group G and a 2-cocycle $\mu \in Z^2(G, U(1))$, we can define the *μ-twisted group algebra* $\mathbb{C}^\mu[G]$ to be the $|G|$-dimensional algebra with basis $\{e_g\}_{g \in G}$ and multiplication law

$$e_g \cdot e_h \;:=\; \mu(g, h) \cdot e_{gh} \; . \tag{1.34}$$

[6] One can check that $\mathrm{Ind}_H^G(V)$ depends on the choice of representatives r_i only up to equivalence.

Equivalently, we can define $\mathbb{C}^\mu[G]$ to be the algebra of functions $f : G \to \mathbb{C}$ that multiply according to the twisted convolution

$$(f * f')(g) \;:=\; \frac{1}{|G|} \cdot \sum_{h \cdot k = g} \mu(h,k) \cdot f(h) \cdot f(k) \;. \tag{1.35}$$

The equivalence between the above two descriptions can be established by mapping a function $f : G \to \mathbb{C}$ to the linear combination

$$e_f \;:=\; \frac{1}{|G|} \cdot \sum_g f(g) \cdot e_g \;. \tag{1.36}$$

We can define a $*$-structure on $\mathbb{C}^\mu[G]$ by setting

$$e_g^* \;:=\; e_g^{-1} \;\equiv\; \mu^*(g, g^{-1}) \cdot e_{g^{-1}} \tag{1.37}$$

for all $g \in G$. Since the linear extension of $\Gamma(e_g) := \delta_{g,1}$ defines a faithful state[7] $\Gamma : \mathbb{C}^\mu[G] \to \mathbb{C}$ w.r.t. this $*$-structure, we see that $\mathbb{C}^\mu[G]$ is in fact a C*-algebra. It is easy to convince oneself that its $*$-representations are in 1:1-correspondence with unitary μ-projective representations of G[8]. In particular, the number of *irreducible* $*$-representations of $\mathbb{C}^\mu[G]$ for a given $\mu \in Z^2(G, U(1))$ can be shown to be [74]

$$n \;=\; \frac{1}{|G|} \cdot \sum_{[g,h]=1} \frac{\mu(g,h)}{\mu(h,g)} \;. \tag{1.38}$$

In the special case where $[\mu] = 1 \in H^2(G, U(1))$, this reduces to the number $|\mathrm{Cl}(G)|$ of *conjugacy classes* in the group G.

In general, we know from the previous subsection that the irreducible $*$-representations U_i are in 1:1-correspondence with minimal central idempotents in $\mathbb{C}^\mu[G]$. We can make this correspondence explicit by introducing for each $i = 1, ..., n$ the *character* $\chi_i := \mathrm{Tr}(U_i(.))$ associated to U_i, which has the following properties [75]:

(i) Complex conjugation: $\chi_i^*(g) \;=\; \mu^*(g, g^{-1}) \cdot \chi_i(g^{-1})\,.$

(ii) Twisted class function: $\chi_i(g \cdot h) \;=\; \dfrac{\mu(h,g)}{\mu(g,h)} \cdot \chi_i(h \cdot g)\,.$

(iii) Generalised orthogonality: $\dfrac{1}{|G|} \cdot \sum_g \mu(g,h) \cdot \chi_i^*(g) \cdot \chi_j(g \cdot h) \;=\; \delta_{ij} \cdot \dfrac{\chi_i(h)}{d_i}\,.$

(iv) Generalised regularity: $\dfrac{1}{|G|} \cdot \sum_i d_i \cdot \chi_i(g) \;=\; \delta_{g,1}\,.$

[7] Concretely, we have that $\Gamma(e_f^* \cdot e_f) = \sum_g |f(g)|^2 / \, |G|^2 \geq 0$ for all $f : G \to \mathbb{C}$.

[8] We often denote the category of unitary μ-projective representations by $\mathrm{Rep}^\mu(G) := \mathrm{Rep}^*(\mathbb{C}^\mu[G])$.

Here, we defined $\chi_i(g) := \chi_i(e_g)$ and denoted by $d_i := \dim(U_i)$ the dimension of U_i. Using the above, one can check that the algebra elements

$$e_i \;:=\; \frac{d_i}{|G|} \cdot \sum_g \chi_i^*(g) \cdot e_g \tag{1.39}$$

are central in $\mathbb{C}^\mu[G]$ and satisfy the relations (1.19). Thus, we obtain a direct relation between the minimal central idempotents e_i and the (characters of the) irreducible $*$-representations U_i of $\mathbb{C}^\mu[G]$ as expected.

It is worth noting that two given group algebras can be isomorphic even though the underlying groups are not. For instance, if $\mathbb{Z}_4 = \langle x \rangle$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle u, v \rangle$, then

$$x \;\mapsto\; \frac{1+i}{2} \cdot u \;+\; \frac{1-i}{2} \cdot v \tag{1.40}$$

induces a unitary algebra isomorphism between $\mathbb{C}[\mathbb{Z}_4]$ and $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$. In fact, we have that the group algebras of any two finite abelian groups of the same order are unitarily isomorphic as a consequence of the Artin-Wedderburn theorem. In particular, they share the same irreducible $*$-representations. More generally, we say that two C*-algebras $\mathcal{A}$ and $\mathcal{A}'$ are *Morita equivalent* ($\mathcal{A} \simeq \mathcal{A}'$) if their associated categories of $*$-representations are additively equivalent (i.e. $\text{Rep}^*(\mathcal{A}) \cong \text{Rep}^*(\mathcal{A}')$). For example, we have that $M_k(\mathbb{C}) \simeq \mathbb{C}$ for any positive integer $k \in \mathbb{N}$.

Starting from twisted group algebras, we can obtain new examples of C*-algebras by performing ‘topological deformations’ such as the discrete gauging of certain (sub)symmetries. Concretely, given a quantum mechanical theory $\mathcal{T}$ whose symmetries are given by $\mathbb{C}^\mu[G]$, we can try to gauge the symmetry group G by summing over ‘networks’ of symmetry defects $g \in G$ inserted into one-dimensional spacetime. Pictorially, we have that correlation functions of arbitrary operator insertions $\mathcal{O}_1, \mathcal{O}_2$ in the gauged theory $\mathcal{T}' = \mathcal{T}/G$ are given by

$$\Big\langle \; \mathcal{O}_2 \;\; \mathcal{O}_1 \; \Big\rangle_{\mathcal{T}/G} \;:=\; \Big\langle \; e_G \;\; \mathcal{O}_2 \;\; e_G \;\; \mathcal{O}_1 \;\; e_G \; \Big\rangle_{\mathcal{T}} \;, \tag{1.41}$$

where we defined the ‘generator defect’

$$e_G \;:=\; \frac{1}{|G|} \cdot \sum_{g \in G} \nu(g) \cdot e_g \tag{1.42}$$

for some (yet to be determined) phases $\nu(g) \in U(1)$. In order for (1.41) to be well-defined (i.e. invariant under possible duplicate insertions of the generator e_G), we then require that $(e_G)^2 \overset{!}{=} e_G$, which is equivalent to the condition

$$\mu(g,h) \;\overset{!}{=}\; \frac{\nu(gh)}{\nu(g) \cdot \nu(h)} \;\equiv\; (d\nu^*)(g,h) \;. \tag{1.43}$$

Thus, we see that in order to be able to gauge the entire symmetry group G, we need its associated 't Hooft anomaly to be trivial, $[\mu] = 1 \in H^2(G, U(1))$. In other words, we can view a non-trivial 't Hooft anomaly as an *obstruction* to gauging the global symmetry group G. For a given $\mu \in Z^2(G, U(1))$, we may hence only be able to gauge a subgroup $H \subset G$ on which the anomaly trivialises as $\mu|_H = d\nu^*$ for some $\nu : H \to U(1)$. As before, the resulting gauged theory $\mathcal{T}' = \mathcal{T}/_\nu H$ is then obtained by inserting the generator defect

$$e_H^\nu \;:=\; \frac{1}{|H|} \cdot \sum_{h \in H} \nu(h) \cdot e_h \tag{1.44}$$

into correlation functions. In particular, the Hilbert space of $\mathcal{T}'$ is given by $\mathcal{H}' = e_H^\nu \cdot \mathcal{H}$, whereas the C*-algebra of symmetry defects is $\mathcal{A}' = e_H^\nu \cdot \mathcal{A} \cdot e_H^\nu$ with $\mathcal{A} = \mathbb{C}^\mu[G]$. More concretely, we can identify $\mathcal{A}'$ with the function algebra

$$\mathcal{A}' \;\cong\; \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{matrix} f(g \cdot h) \;=\; \mu(g,h) \cdot \nu(h) \cdot f(g) \\ f(h \cdot g) \;=\; \nu(h) \cdot \mu(h,g) \cdot f(g) \end{matrix} \;\; \text{for all } \; g \in G, \, h \in H \right\}, \tag{1.45}$$

whose algebra product is given by the convolution (1.35) and whose $*$-structure is

$$(f^*)(g) \;:=\; \mu^*(g, g^{-1}) \cdot f(g^{-1})^* \,. \tag{1.46}$$

For instance, if both μ and ν are trivial, we have that $\mathcal{A}' \cong \mathbb{C}[H \backslash G / H]$ is simply given by the double coset ring of H in G. If H is furthermore *normal* in G, we obtain the group algebra $\mathcal{A}' = \mathbb{C}[G/H]$, which captures the 'leftover' symmetry defects in G after gauging H. If $H = G$, we obtain the trivial symmetry algebra $\mathcal{A}' = \mathbb{C}$.

2

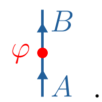
TWO DIMENSIONS

In this chapter, we consider generalised symmetries in two spacetime dimensions that are described by a fusion category $\mathcal{C}$. We review the construction of the tube category associated to $\mathcal{C}$, which captures the action of $\mathcal{C}$ on twisted sector local operators. We describe how its irreducible representations can be classified using the sandwich construction for categorical symmetries. We provide new examples that include generic Tambara-Yamagami symmetries as well as non-invertible symmetries of Fibonacci and Yang-Lee type. The discussion is based on work performed in collaboration with Mathew Bullimore and Andrea Grigoletto [1, 2] as well as the single-author work [4].

2.1 Preliminaries

In this section, we provide some brief mathematical background on the theory of fusion categories, which describe finite bosonic generalised symmetries in two spacetime dimensions (for a more comprehensive review we refer the reader to [64, 65]). Concretely, given a fusion category $\mathcal{C}$, its objects and morphisms correspond to topological line defects and their local junctions on a two-dimensional plane, i.e.

$$A \xrightarrow{\varphi} B \qquad \leftrightarrow \tag{2.1}$$



Moreover, $\mathcal{C}$ comes equipped with a variety of additional structures that capture the topological nature of symmetry defects, the most salient of which we summarise below:

- **Finite semisimplicity:** We assume that $\mathcal{C}$ is enriched over Vect, meaning that the morphism space $\text{Hom}_{\mathcal{C}}(A, B)$ is a finite-dimensional complex vector space for every pair of objects $A, B \in \mathcal{C}$ such that the composition of morphisms is linear. Furthermore, we assume that every object $A \in \mathcal{C}$ can be written as a direct sum of finitely many simple objects $S_i \in \mathcal{C}$ with $\text{Hom}_{\mathcal{C}}(S_i, S_j) = \delta_{ij}\,\mathbb{C}$, i.e.

$$A \;\cong\; \bigoplus_{i=1}^{n} (S_i)^{\oplus A_i} \tag{2.2}$$

  where $A_i = \dim(\text{Hom}_{\mathcal{C}}(A, S_i)) \in \mathbb{N}$. We denote by $\pi_0(\mathcal{C})$ the set of isomorphism classes of simple objects in $\mathcal{C}$, which we assume to be finite. In the following, we will use $\bigoplus_S$ to denote finite direct sums over a fixed set of representatives S of elements in $\pi_0(\mathcal{C})$. We can multiply objects in $\mathcal{C}$ with finite-dimensional vector spaces via the (left) multiplication functor[1]

$$\odot : \text{ Vect} \boxtimes \mathcal{C} \ \to \ \mathcal{C}\,, \quad V \boxtimes A \ \mapsto \ A^{\oplus \dim(V)}\,. \tag{2.3}$$

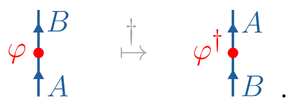
Similarly, one defines the right multiplication $\mathcal{C} \boxtimes \text{Vect} \to \mathcal{C}$, which for compactness we also denote by $\odot$. Using this, we can rewrite the decomposition (2.2) of any object $A \in \mathcal{C}$ into simples as

$$A \ \cong \ \bigoplus_S \text{Hom}_{\mathcal{C}}(S, A) \odot S\,. \tag{2.4}$$

- **Dagger structure:** We assume that $\mathcal{C}$ is compatible with *reflection positivity* in the sense that it allows us to 'reflect' symmetry defects about fixed hyperplanes so that reflection symmetric configurations are positive in an appropriate sense. Mathematically, this is implemented by assuming that $\mathcal{C}$ is a $\dagger$-category:

  Definition: A linear category $\mathcal{C}$ is called a $\dagger$-*category* if it is equipped with a linear functor[2,3] $\dagger : \mathcal{C} \to (\mathcal{C}^{\text{op}})^*$ acting as the identity on objects so that $(\dagger^{\text{op}})^* \circ \dagger = \text{Id}_{\mathcal{C}}$. A morphism $\varphi : A \to B$ in $\mathcal{C}$ is said to be *unitary* if $\varphi^\dagger \circ \varphi = \text{id}_A$ and $\varphi \circ \varphi^\dagger = \text{id}_B$. A linear functor $F : \mathcal{C} \to \mathcal{C}'$ between two $\dagger$-categories is called a $\dagger$-*functor* if $\dagger' \circ F = (F^{\text{op}})^* \circ \dagger$. Given a natural transformation $\eta : F \Rightarrow G$ between two $\dagger$-functors, we define $\eta^\dagger : G \Rightarrow F$ to be the natural transformation with components $(\eta^\dagger)_A := (\eta_A)^{\dagger'}$ for all $A \in \mathcal{C}$. In this way, the category $[\mathcal{C}, \mathcal{C}']^\dagger$ of $\dagger$-functors between $\mathcal{C}$ and $\mathcal{C}'$ naturally becomes a $\dagger$-category itself.

  Pictorially, the action of $\dagger$ can be represented as a reflection of morphisms in $\mathcal{C}$ about a fixed horizontal hyperplane, i.e.



(2.5)

  Positivity is the statement that there exists a faithful state $\Gamma : \text{End}_{\mathcal{C}}(A) \to \mathbb{C}$ on the endomorphism algebra of every object $A \in \mathcal{C}$, which turns $\text{End}_{\mathcal{C}}(A)$ into a C*-algebra for all $A \in \mathcal{C}$.

---

[1] Given two finite abelian categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathcal{C} \boxtimes \mathcal{D}$ their Deligne tensor product [76].

[2] Given a category $\mathcal{C}$, we denote by $\mathcal{C}^{\text{op}}$ its *opposite category*, which has the same objects as $\mathcal{C}$ but reversed morphisms, i.e. $\text{Hom}_{(\mathcal{C}^{\text{op}})}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we denote by $F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ its opposite functor defined in the obvious way.

[3] Given a linear category $\mathcal{C}$, we denote by $\mathcal{C}^*$ its *complex conjugate category*, which has the same objects as $\mathcal{C}$ but complex conjugated morphism spaces, i.e. $\text{Hom}_{(\mathcal{C}^*)}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)^*$. Given a linear functor $F : \mathcal{C} \to \mathcal{D}$, we denote by $F^* : \mathcal{C}^* \to \mathcal{D}^*$ its complex conjugate.

- **Monoidal structure:** We assume that $\mathcal{C}$ encodes the fusion of symmetry defects given by the parallel collision of topological lines in $\mathcal{C}$. Mathematically, this is achieved by endowing $\mathcal{C}$ with a monoidal structure in the following sense:

  Definition: An additive linear category[4] $\mathcal{C}$ is called *monoidal* if it is equipped with an additive linear functor

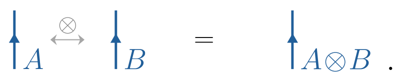
$$\otimes : \ \mathcal{C} \boxtimes \mathcal{C} \ \to \ \mathcal{C} \ , \tag{2.6}$$

  a distinguished object $\mathbf{1} \in \mathcal{C}$ (called the *unit*) and three natural isomorphisms

$$\begin{aligned} \alpha : \ \ \otimes \circ (\otimes \boxtimes \mathrm{Id}) \ &\Rightarrow \ \otimes \circ (\mathrm{Id} \boxtimes \otimes) \ , \\ \lambda : \ \ (\mathbf{1} \otimes \mathrm{Id}) \ &\Rightarrow \ \mathrm{Id} \ , \\ \rho : \ \ (\mathrm{Id} \otimes \mathbf{1}) \ &\Rightarrow \ \mathrm{Id} \end{aligned} \tag{2.7}$$

  (called the *associator*, *left* and *right unitor*, respectively) that satisfy the *pentagon* and *triangle relations* [77]. An additive linear functor $F : \mathcal{C} \to \mathcal{C}'$ between two monoidal categories is called a *monoidal functor* if it is equipped with a natural isomorphism $\nu : \otimes' \circ (F \boxtimes F) \Rightarrow F \circ \otimes$ that satisfies suitable coherence relations and it preserves the monoidal unit, i.e. $F(\mathbf{1}) \cong \mathbf{1}'$.

  Pictorially, we represent the monoidal product of objects $A, B \in \mathcal{C}$ to be the result of bringing the corresponding topological lines together in a parallel fashion:



(2.8)

  The associator then mediates between the two possible ways of bringing three parallel topological lines $A, B, C \in \mathcal{C}$ together:

(2.9)

  For simplicity, we will often omit associators as well as the left and right unitors from graphical representations of symmetry defects in what follows. We depict the monoidal unit $\mathbf{1} \in \mathcal{C}$ by the invisible / transparent line and we assume that it corresponds to a simple object in $\mathcal{C}$, i.e. $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{C}$. In particular, this means that the only genuine topological local operators are given by scalar multiples of the identity morphism on $\mathbf{1}$.

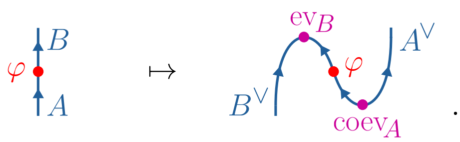
[4] Broadly speaking, a category is *additive* if its morphism spaces are abelian groups such that compositions are bilinear and we can furthermore form direct sums of objects. In particular, there exists a zero object that functions as the neutral element w.r.t. taking direct sums. A functor between additive categories is called *additive* if it preserves direct sums and restricts to group homomorphisms on morphism spaces.

As $\mathcal{C}$ is also a †-category, we require the monoidal structure to be compatible with the dagger structure in the sense that $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ is a †-functor and the associator α and left and right unitors λ and ρ are all unitary. If such a †-structure exists, it is unique up to (unitary monoidal) equivalence [78]. In this case, we say that $\mathcal{C}$ is a *unitary* fusion category.

- **Dual structure:** We assume that $\mathcal{C}$ is equipped with a *dual structure* that allows us to 'bend' topological line defects in the following sense: For each $A \in \mathcal{C}$ there exists a *dual object* $A^\vee \in \mathcal{C}$ (corresponding to the orientation reversal of A) together with evaluation and coevaluation morphisms

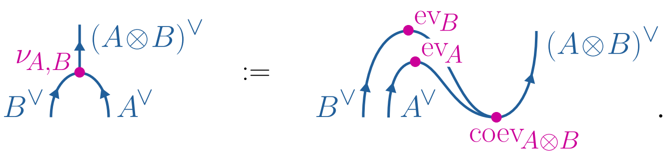
(2.10)

that satisfy suitable *zig-zag relations* [77]. The assignment $A \mapsto A^\vee$ then extends to a functor $\vee : \mathcal{C} \to \mathcal{C}$ that acts on morphisms via



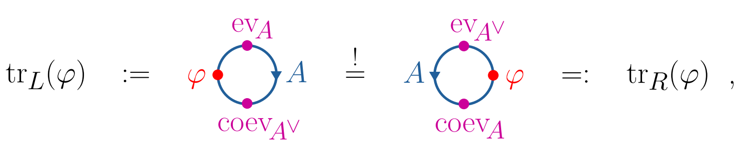
(2.11)

This functor is compatible with the monoidal structure $\otimes$ on $\mathcal{C}$ in the sense that it is op-monoidal with tensorator given by



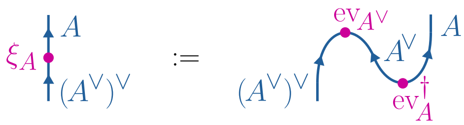
(2.12)

A *pivotal structure* on $\mathcal{C}$ is a choice of natural isomorphism $\xi : \vee^2 \Rightarrow \mathrm{Id}_\mathcal{C}$. We say that the latter is *spherical* if its associated left and right traces agree on all endomorphisms $\varphi \in \mathrm{End}_\mathcal{C}(A)$, i.e.



(2.13)

where we omitted ξ_A from the graphical presentation. In this case, we define $\dim(A) := \mathrm{tr}_{L/R}(\mathrm{id}_A) \in \mathbb{C}$ to be the *quantum dimension* of an object $A \in \mathcal{C}$. We say that the dual structure is *unitary* if it is compatible with the dagger structure on $\mathcal{C}$ in the sense that $\vee : \mathcal{C} \to \mathcal{C}$ is a †-functor and the associated tensorator (2.12) is unitary. In this case, the unitary isomorphisms



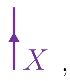
(2.14)

define a canonical pivotal structure on $\mathcal{C}$. Furthermore, there exists a unique (up to suitable equivalence) unitary dual structure on $\mathcal{C}$ such that the above canonical pivotal structure is spherical [79]. We will henceforth assume $\mathcal{C}$ to be equipped with the unique unitary dual structure.

2.2 Tube Category

Given a fusion category $\mathcal{C}$, we can associate to it the so-called *tube category*, which captures the possible linking configurations of twisted sector local operators in two dimensions with symmetry defects in $\mathcal{C}$. Concretely, following [80], we define the tube category $\mathrm{T}\mathcal{C}$ associated to $\mathcal{C}$ to be the additive linear category whose

- objects are given by objects $X \in \mathcal{C}$, i.e.

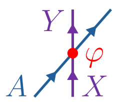


(2.15)

- morphisms between objects $X, Y \in \mathcal{C}$ form the quotient vector space

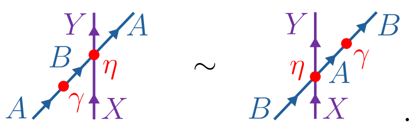
$$\mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y) \;:=\; \bigoplus_{A\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(A\otimes X\,, Y\otimes A) \Big/ \sim \tag{2.16}$$

(where the sum runs over *all* objects $A \in \mathcal{C}$) of local intersection morphisms



(2.17)

in $\mathcal{C}$ subjected to the equivalence relation that is generated by

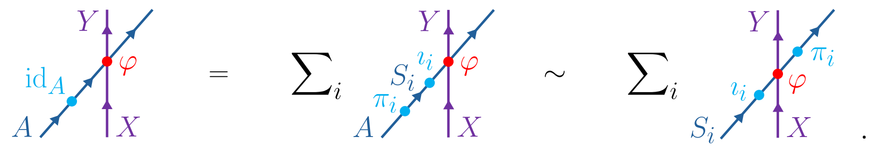


(2.18)

Physically, the equivalence relation (2.18) means that we should think of the symmetry defect A as being placed on a 1-sphere by identifying its two ends. Mathematically, it renders the morphism spaces in (2.16) finite-dimensional. Concretely, let $A \in \mathcal{C}$ be an arbitrary object and consider a decomposition $A \cong \bigoplus_i S_i$ of A into simple objects S_i (possibly with multiplicities). Let us denote by

$$\imath_i : \; S_i \to A \qquad \text{and} \qquad \pi_i : \; A \to S_i \tag{2.19}$$

the associated inclusion and projection morphisms, respectively. Upon inserting the completeness relation $\mathrm{id}_A = \sum_i \imath_i \circ \pi_i$ into (2.17), we then obtain that



(2.20)

If we denote the equivalence class of (2.17) under the relation (2.18) by

$$\langle {}^{Y\,A\,X}_{\quad\varphi} \rangle \;\in\; \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y)\;, \tag{2.21}$$

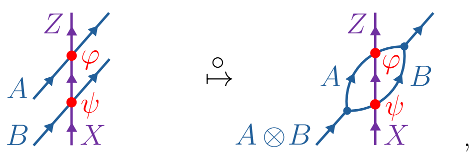
we can rewrite equation (2.20) schematically as the morphism identity

$$\langle {}^{Y\,A\,X}_{\quad\varphi} \rangle \;=\; \sum_i \langle {}^{Y\,S_i\quad X}_{\quad\pi_i\circ\varphi\circ\imath_i} \rangle\;. \tag{2.22}$$

In particular, this shows that, as a vector space, $\mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y)$ is isomorphic to

$$\mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y) \;\cong\; \bigoplus_{[S]\in\pi_0(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(S\otimes X, Y\otimes S)\;, \tag{2.23}$$

where the sum is over a set of fixed representatives S of elements $[S]$ in the set $\pi_0(\mathcal{C})$ of isomorphism classes of simple objects in $\mathcal{C}$. Since the latter is finite by assumption, we see that $\mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y)$ is finite-dimensional as claimed. The composition of morphisms in the tube category is induced by the vertical stacking

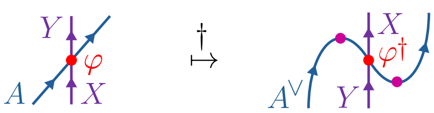


(2.24)

which we denote schematically by

$$\langle {}^{Z\,A\,Y}_{\quad\varphi} \rangle \circ \langle {}^{Y\,B\,X}_{\quad\psi} \rangle \;=\; \langle {}^{Z\;A\otimes B\;X}_{\quad\varphi\circ\psi} \rangle\;. \tag{2.25}$$

Furthermore, the tube category possesses a natural †-structure [81] that is induced by



(2.26)

(where we left the labelling of evaluation and coevaluation morphisms implicit) and which we denote schematically by

$$\langle {}^{Y}\overset{A}{\underset{\varphi}{+}}{}^{X} \rangle^{\dagger} \;=\; \langle {}^{X}\overset{A^{\vee}}{\underset{\varphi^{\dagger}}{+}}{}^{Y} \rangle \,. \tag{2.27}$$

Since $\Gamma\big(\langle {}^{S}\overset{A}{\underset{\varphi}{+}}{}^{S} \rangle\big) := \delta_{A,\mathbf{1}} \cdot \varphi$ defines a faithful state on the endomorphism algebra of each simple object $S \in \mathcal{C}$, we see that $\mathrm{End}_{\mathrm{T}\mathcal{C}}(S)$ is a C*-algebra for all S. In particular, we obtain a C*-structure on the so-called *tube algebra*

$$\mathrm{Tube}(\mathcal{C}) \;:=\; \mathrm{End}_{\mathrm{T}\mathcal{C}}\Big(\bigoplus\nolimits_{[S]\,\in\,\pi_0(\mathcal{C})} S\Big) \tag{2.28}$$

originally introduced by Ocneanu [82], which provides an alternative description of linking configurations of twisted sector local operators in two dimensions.

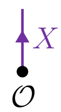
2.3 Tube Representations

Given a two-dimensional quantum field theory with fusion category symmetry $\mathcal{C}$, it was proposed in [83] that twisted sector local operators transform in 'representations' of the tube category associated to $\mathcal{C}$. By this we mean additive linear functors

$$\mathcal{F}:\; \mathrm{T}\mathcal{C} \;\to\; \mathrm{Vect} \tag{2.29}$$

from $\mathrm{T}\mathcal{C}$ into the category of vector spaces, which we will simply call *tube representations* in what follows. Concretely, any such tube representation $\mathcal{F}$ assigns

- to each object $X \in \mathcal{C}$ a vector space $\mathcal{H}_X := \mathcal{F}(X)$ that describes twisted sector local operators $\mathcal{O}$ sitting at the end of the topological line defect X, i.e.

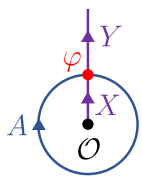


, (2.30)

- to each morphism $\langle {}^{Y}\overset{A}{\underset{\varphi}{+}}{}^{X} \rangle \in \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y)$ a linear map

$$\mathcal{F}\Big(\langle {}^{Y}\overset{A}{\underset{\varphi}{+}}{}^{X} \rangle\Big) \;:\; \mathcal{H}_X \;\to\; \mathcal{H}_Y \tag{2.31}$$

that describes how operators $\mathcal{O}$ in the X-twisted sector get mapped to operators in the Y-twisted sector upon being linked with the symmetry defect A, i.e.



. (2.32)

Given two tube representations, an *intertwiner* between them is a natural transformation between the corresponding functors. We denote the category of all tube representations and intertwiners between them by

$$\mathrm{Rep}(\mathrm{T}\mathcal{C}) \;:=\; [\mathrm{T}\mathcal{C}, \mathrm{Vect}]\;. \tag{2.33}$$

For a given tube representation $\mathcal{F}$, we can use the operator-state map to endow the twisted sectors $\mathcal{H}_X$ with an inner product structure[5], which we assume to be compatible with the action of the tube category in the sense that

$$\mathcal{F}\Big(\big\langle Y \overset{A}{\underset{\varphi}{\longleftarrow}} X \big\rangle^{\dagger}\Big) \;\overset{!}{=}\; \mathcal{F}\Big(\big\langle Y \overset{A}{\underset{\varphi}{\longleftarrow}} X \big\rangle\Big)^{\dagger} \tag{2.34}$$

for all $\langle Y \overset{A}{\underset{\varphi}{\longleftarrow}} X \rangle \in \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X,Y)$ (where the $\dagger$ on the right hand side denotes the adjoint of linear maps). Mathematically, this means that we assume $\mathcal{F}$ to lift to a $\dagger$-functor[6]

$$\mathcal{F}:\; \mathrm{T}\mathcal{C} \xrightarrow{\dagger} \mathrm{Hilb}\;, \tag{2.35}$$

which we will call a *tube $\dagger$-representation* in what follows. We denote the category of all tube $\dagger$-representations and intertwiners between them by

$$\mathrm{Rep}^{\dagger}(\mathrm{T}\mathcal{C}) \;:=\; [\mathrm{T}\mathcal{C}, \mathrm{Hilb}]^{\dagger}\;. \tag{2.36}$$

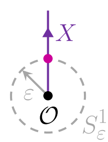
Clearly, every tube $\dagger$-representation reduces to an ordinary tube representation upon forgetting the underlying Hilbert space structure.

A useful way to classify the irreducible tube ($\dagger$-)representations of a fusion category $\mathcal{C}$ is given by the so-called *sandwich construction*. In this picture, we view a two-dimensional theory $\mathcal{T}$ with generalised symmetry $\mathcal{C}$ as being attached to a three-dimensional 'bulk', which hosts topological defects that 'commute' with all other symmetry defects in $\mathcal{C}$. This leads to a categorified notion of the centre:

Definition: Given a fusion category $\mathcal{C}$, its *Drinfeld centre* is the category $\mathcal{Z}(\mathcal{C})$ whose objects are given by pairs $z = (U, \tau_{U,-})$ consisting of

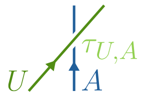
1. an object $U \in \mathcal{C}$ in the fusion category $\mathcal{C}$,
2. a *half-braiding* for U, i.e. a natural isomorphism $\tau_{U,-} : U \otimes - \Rightarrow - \otimes U$ whose components $\tau_{U,A} : U \otimes A \to A \otimes U$ satisfy suitable coherence relations [77].

[5] Concretely, consider a 1-sphere S^1_ε of radius $\varepsilon > 0$ centred around a local operator $\mathcal{O} \in \mathcal{H}_X$ in the X-twisted sector. Using the operator-state map, this induces a state $|\mathcal{O}\rangle_\varepsilon$ in the Hilbert space associated to the (punctured) S^1_ε. Given another X-twisted sector local operator $\mathcal{O}'$, we then define its inner product with $\mathcal{O}$ to be $\langle \mathcal{O}|\mathcal{O}'\rangle_\varepsilon$, which is independent of ε if we assume time evolution along the radial direction to be unitary.



[6] Here, we denote by Hilb the category whose objects are finite-dimensional complex Hilbert spaces and morphisms are linear maps between them.

Pictorially, we represent the component morphisms $\tau_{U,A}$ as 'crossings' that tell us how the object U can be moved across any other topological defect $A \in \mathcal{C}$:



. (2.37)

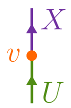
The Drinfeld centre is itself a unitary fusion category with monoidal product given by

$$z \otimes z' \;:=\; (U \otimes U', \tau_{U,-} \circ \tau'_{U',-}) \; . \tag{2.38}$$

It is furthermore *braided*, since $\beta_{z,z'} := \tau_{U,U'} : z \otimes z' \to z' \otimes z$ defines the components of a natural braiding isomorphism β satisfying the *hexagon relations* [77].

The utility of the Drinfeld centre stems from the fact that we can associate to each object $z = (U, \tau_{U,-}) \in \mathcal{Z}(\mathcal{C})$ a tube representation $\mathcal{F}_z \in \mathrm{Rep}(\mathrm{T}\mathcal{C})$ as follows [1, 80]:

- To an object $X \in \mathcal{C}$, the functor $\mathcal{F}_z$ assigns the vector space $\mathcal{H}_X := \mathrm{Hom}_{\mathcal{C}}(U, X)$ of local junction morphisms



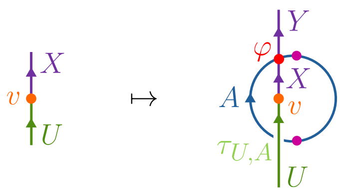
. (2.39)

  We will denote the elements of $\mathcal{H}_X$ by $|\overset{X\;v\;U}{\longrightarrow}\rangle$ in what follows. If X is simple, $\mathcal{H}_X$ is a Hilbert space whose inner product is defined by

$$w \circ v^\dagger \;=:\; \langle v, w \rangle \cdot \mathrm{id}_X \; . \tag{2.40}$$

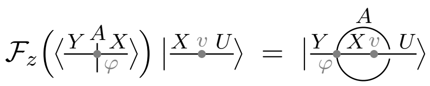
  If X is not simple, we obtain a Hilbert space structure in $\mathcal{H}_X$ by decomposing X into its simple components.

- To a morphism $\langle \overset{Y\,A\,X}{\underset{\varphi}{\longrightarrow}} \rangle \in \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(X, Y)$, the functor $\mathcal{F}_z$ assigns the linear map from $\mathcal{H}_X$ to $\mathcal{H}_Y$ that sends a local junction $v \in \mathrm{Hom}_{\mathcal{C}}(U, X)$ to



(2.41)

  (where we left the labelling of evaluation and coevaluation morphisms implicit). Schematically, we write this as



. (2.42)

As a special case of the above, we can consider the tube representation $\mathbb{1} := \mathcal{F}_{\mathbf{1}}$ associated to the monoidal unit $\mathbf{1} = (\mathbf{1}, \mathrm{id}_-) \in \mathcal{Z}(\mathcal{C})$, which we will call the *trivial* tube representation in what follows. The latter acts on the untwisted sector $\mathcal{H}_{\mathbf{1}} \cong \mathbb{C}$ via the multiplicative factors

$$\mathbb{1}\left(\langle \overset{\mathbf{1}\quad A \quad \mathbf{1}}{\longrightarrow\!\!\!\!\!\!\!\!|_{\varphi}} \rangle\right) \;=\; \mathrm{tr}(\varphi)\,. \tag{2.43}$$

It is known that the assignment $z \mapsto \mathcal{F}_z$ extends to an equivalence

$$\mathcal{Z}(\mathcal{C}) \;\cong\; \mathrm{Rep}(\mathrm{T}\mathcal{C}) \tag{2.44}$$

of linear categories [80, 84, 85]. One can check that the functor $\mathcal{F}_z$ associated to an object $z = (U, \tau_{U,-})$ is a †-functor if and only if the components $\tau_{U,A}$ of the half-braiding are unitary for all $A \in \mathcal{C}$. If we define the collection of such objects to form the so-called *unitary Drinfeld centre* $\mathcal{Z}^{\dagger}(\mathcal{C})$, this yields an equivalence

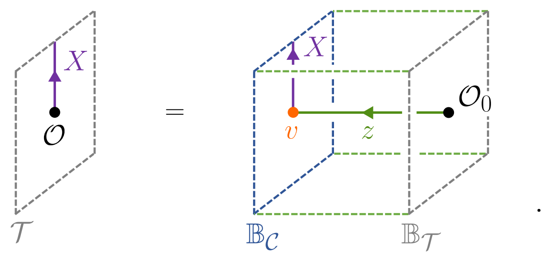
$$\mathcal{Z}^{\dagger}(\mathcal{C}) \;\cong\; \mathrm{Rep}^{\dagger}(\mathrm{T}\mathcal{C}) \tag{2.45}$$

of linear †-categories. In particular, the simple objects of $\mathcal{Z}^{\dagger}(\mathcal{C})$ correspond to the irreducible †-representations of $\mathrm{T}\mathcal{C}$. This may be seen as a two-dimensional analogue of the fact that the irreducible ∗-representations of a C*-algebra $\mathcal{A}$ are in 1:1-correspondence with its minimal central idempotents, which from a canonical basis of the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. Lastly, using the fact that every braiding on a unitary fusion category is automatically unitary [86], one can show that $\mathcal{Z}^{\dagger}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$, which, using (2.44) and (2.45), implies that every tube representation of a given fusion category $\mathcal{C}$ is equivalent to a †-representation. This may be seen as an analogue of the fact that every finite-dimensional representation of a finite group is equivalent to a unitary one.

We can visualise the above construction by viewing the two-dimensional theory $\mathcal{T}$ with generalised symmetry $\mathcal{C}$ as an interval compactification (a.k.a. a sandwich) of a three-dimensional topological theory in the bulk called the *Symmetry Topological Field Theory* (or *Symmetry TFT* for short) [87–89]. In the present case, this is the Turaev-Viro TQFT based on $\mathcal{C}$ (whose category of line defects is indeed given by the Drinfeld centre of $\mathcal{C}$) [90–93]. The latter is equipped with two boundary conditions:

1. A canonical topological boundary condition $\mathbb{B}_{\mathcal{C}}$ on the left that supports the symmetry $\mathcal{C}$ and that is independent of the theory $\mathcal{T}$ under consideration. In particular, the bulk-to-boundary map is given by the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ that sends $z = (U, \tau_{U,-}) \mapsto U$.
2. A physical boundary condition $\mathbb{B}_{\mathcal{T}}$ on the right that depends on the underlying theory $\mathcal{T}$ and that is non-topological in general.

The spectrum of twisted sector local operators $\mathcal{O}$ that transform in a given tube representation $\mathcal{F}_z$ associated to some $z \in \mathcal{Z}(\mathcal{C})$ may then be viewed as the spectrum of topological junctions v between twisted sector lines $X \in \mathcal{C}$ on the left and the bulk line z stretched between the two boundaries of the Symmetry TFT:



(2.46)

Here, $\mathcal{O}_0$ denotes a fixed (and generically non-topological) local operator that terminates z on right boundary. The linking action of the tube category on the operators $\mathcal{O}$ may then be computed using (2.41).

We can offer yet another perspective on the above discussion that uses the notion of the *tube algebra* $\mathrm{Tube}(\mathcal{C})$ associated to $\mathcal{C}$ as defined in (2.28). The latter has the property that it is 'Morita equivalent' to the tube category in the sense that

$$\mathrm{Rep}^*(\mathrm{Tube}(\mathcal{C})) \;\cong\; \mathrm{Rep}^\dagger(\mathrm{T}\mathcal{C})\;, \tag{2.47}$$

which means that there is a 1:1-correspondence between $*$-representations of the tube algebra and $\dagger$-representations of the tube category [80]. One way to see this is by linking twisted sector local operators attached to simple lines $S \in \mathcal{C}$ with symmetry defects $U \in \mathcal{C}$ that can be pushed into the three-dimensional bulk (and hence form part of the defining data of an object $z = (U, \tau_{U,-}) \in \mathcal{Z}(\mathcal{C})$ in the Drinfeld centre). We denote the corresponding tube algebra element by

$$\Big\langle {}_{S}\overset{U}{\underset{\tau_{U,S}}{+}}{}_{S} \Big\rangle \;=:\; \Big\langle {}_{S}\overset{z}{|}{}_{S} \Big\rangle \tag{2.48}$$

in what follows. As a consequence of the hexagon relations obeyed by the half-braiding $\tau_{U,-}$, this then has the property that

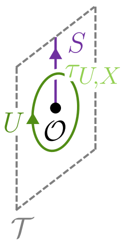
$$\Big\langle {}_{S'}\overset{A}{\underset{\varphi}{+}}{}_{S} \Big\rangle \circ \Big\langle {}_{S}\overset{z}{|}{}_{S} \Big\rangle \;=\; \Big\langle {}_{S'}\overset{z}{|}{}_{S'} \Big\rangle \circ \Big\langle {}_{S'}\overset{A}{\underset{\varphi}{+}}{}_{S} \Big\rangle \tag{2.49}$$

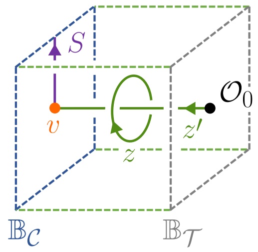
for all $\Big\langle {}_{S'}\overset{A}{\underset{\varphi}{+}}{}_{S} \Big\rangle \in \mathrm{Tube}(\mathcal{C})$. Furthermore, its algebra involution is given by[7]

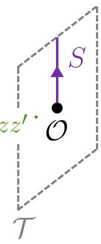
$$\Big\langle {}_{S}\overset{z}{|}{}_{S} \Big\rangle^* \;=\; \Big\langle {}_{S}\overset{z^\vee}{|}{}_{S} \Big\rangle\;, \tag{2.50}$$

[7] To see this, one again uses that any braiding on a unitary fusion category is automatically unitary, which implies that the components $\tau_{U,A}$ of the half-braiding associated to z are all unitary [86].

where $z^\vee$ denotes the dual of z in $\mathcal{Z}(\mathcal{C})$. Using the sandwich picture (2.46), it is then easy to see that the linking action of $\langle \overset{S}{-}\overset{z}{|}\overset{S}{-} \rangle$ on twisted sector local operators $\mathcal{O}$ transforming in a tube representation $\mathcal{F}_{z'}$ associated to some $z' \in \mathcal{Z}(\mathcal{C})$ is given by



$$=$$


$$=: \; d_z \cdot \mathcal{S}_{zz'} \cdot \tag{2.51}$$


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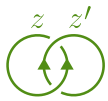
*

where $d_z := \dim(z)$ and we defined the multiplicative factor[8]

$$\mathcal{S}_{zz'} \; := \; \frac{1}{d_z \cdot d_{z'}} \cdot \tag{2.52}$$


.

By letting z and z' run over a fixed set of representatives of isomorphism classes of simple objects in $\mathcal{Z}(\mathcal{C})$, this yields a square matrix

$$\mathcal{S} \, : \, \pi_0(\mathcal{Z}(\mathcal{C})) \, \times \, \pi_0(\mathcal{Z}(\mathcal{C})) \, \to \, \mathbb{C} \tag{2.53}$$

called the *S-matrix* of $\mathcal{Z}(\mathcal{C})$, which has the following properties [65]:

1. It is symmetric, i.e. $\mathcal{S}_{zz'} = \mathcal{S}_{z'z}$.
2. It satisfies $\mathcal{S}_{z^\vee z'} = \mathcal{S}^*_{zz'}$.
3. It is invertible.

In addition, $\mathcal{S}$ obeys the so-called (normalised) *Verlinde formula*

$$\mathcal{S}_{xw} \cdot \mathcal{S}_{yw} \; = \; \sum_z N^z_{xy} \cdot \mathcal{S}_{zw} \; , \tag{2.54}$$

where $\sum_z$ denotes a sum over fixed representatives of elements in $\pi_0(\mathcal{Z}(\mathcal{C}))$ and we defined the (normalised) fusion coefficients

$$N^z_{xy} \; := \; \frac{d_z}{d_x \cdot d_y} \cdot \dim\big(\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(x \otimes y, z)\big) \; . \tag{2.55}$$

The latter allow us to write the product of the tube algebra elements $\langle \overset{S}{-}\overset{z}{|}\overset{S}{-} \rangle$ as

$$\langle \overset{S}{-}\overset{x}{|}\overset{S}{-} \rangle \circ \langle \overset{S}{-}\overset{y}{|}\overset{S}{-} \rangle \; = \; \sum_z \frac{d_x \cdot d_y}{d_z} \cdot N^z_{xy} \cdot \langle \overset{S}{-}\overset{z}{|}\overset{S}{-} \rangle \; . \tag{2.56}$$

[8] Here, we make use of the natural braiding β on the Drinfeld center $\mathcal{Z}(\mathcal{C})$ to make sense of the crossings that appear in (2.52).

Together with the Verlinde formula, this implies that the linear combinations

$$e_z^S \;:=\; \sum_{z'} \frac{1}{d_{z'}} \cdot (\mathcal{S}^{-1})_{zz'} \cdot \langle \underline{S} \overset{z'}{|} \underline{S} \rangle \tag{2.57}$$

define a collection of orthogonal self-adjoint idempotents in Tube($\mathcal{C}$) that are indexed by $[S] \in \pi_0(\mathcal{C})$ and $[z] \in \pi_0(\mathcal{Z}(\mathcal{C}))$ [83], i.e. we have that

$$e_z^S \circ e_{z'}^{S'} \;=\; \delta_{zz'} \cdot \delta^{SS'} \cdot e_z^S \qquad \text{and} \qquad (e_z^S)^* \;=\; e_z^S\,. \tag{2.58}$$

Using this, we can construct the minimal *central* idempotents in Tube($\mathcal{C}$) via

$$e_z \;:=\; \sum_S e_z^S\,, \tag{2.59}$$

where $\sum_S$ denotes a sum over a fixed set of representatives of elements in $\pi_0(\mathcal{C})$. In particular, we see that the minimal central idempotents are labelled by simple objects $z \in \mathcal{Z}(\mathcal{C})$, which, together with (2.47), re-establishes the equivalence (2.45) via their 1:1-correspondence with irreducible $*$-representations of Tube($\mathcal{C}$).

2.4 Examples

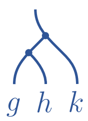
We conclude this section by providing concrete examples of fusion category symmetries and their associated tube categories / algebras. We discuss anomalous group symmetries as well as new examples that include generic Tambara-Yamagami symmetries and non-invertible symmetries of Fibonacci and Yang-Lee type.

2.4.1 Group Symmetry

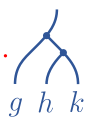
We begin by considering invertible symmetries described by some finite group G. In this case, the simple objects of the fusion category $\mathcal{C}$ correspond to group elements $g \in G$ that fuse according to the group law of G and whose associator is given by

$$(g\cdot h)\cdot k \xrightarrow{\;\alpha(g,h,k)\,\cdot\,\mathrm{id}_{ghk}\;} g\cdot(h\cdot k) \tag{2.60}$$

for some multiplicative phases $\alpha(g,h,k) \in U(1)$. In analogy to the one-dimensional case, we call the collection of the latter an *'t Hooft anomaly*, since they describe the (controlled) violation of associativity in the fusion of symmetry defects:



$$= \quad \alpha(g,h,k) \cdot$$



$$. \tag{2.61}$$

In order to be compatible with the two possible ways to fuse four symmetry defects in different orders, these phases need to satisfy

$$(d\alpha)(g,h,k,\ell) \;:=\; \frac{\alpha(h,k,\ell)\cdot\alpha(g,hk,\ell)\cdot\alpha(g,h,k)}{\alpha(gh,k,\ell)\cdot\alpha(g,h,k\ell)} \;\overset{!}{=}\; 1\,, \tag{2.62}$$

which shows that α defines a 3-cocycle $\alpha \in Z^3(G,U(1))$. The corresponding symmetry category is then given by the fusion category $\mathcal{C} = \mathrm{Hilb}_G^{\alpha}$ of finite-dimensional G-graded Hilbert spaces[9] with associativity twisted by α. The duals and dimensions of simple objects are given by $g^\vee = g^{-1}$ and $\dim(g) = 1$, respectively. This symmetry category is unitary for all G and α.

The tube algebra associated to $\mathcal{C} = \mathrm{Hilb}_G^{\alpha}$ is the $|G|^2$-dimensional algebra with basis vectors $\langle {}^{g}x \overset{g}{\textstyle+} x \rangle$ (where $g, x \in G$ and ${}^{g}x := gxg^{-1}$) and algebra multiplication

$$\langle {}^{g}x \overset{g}{+} x \rangle \circ \langle {}^{h}y \overset{h}{+} y \rangle \;=\; \delta_{x,{}^{h}y}\cdot \tau_y(\alpha)(g,h)\cdot \langle {}^{gh}y \overset{gh}{+} y \rangle\,, \tag{2.63}$$

where we defined the multiplicative phase

$$\tau_x(\alpha)(g,h) \;:=\; \frac{\alpha(g,h,x)\cdot\alpha({}^{gh}x,g,h)}{\alpha(g,{}^{h}x,h)} \tag{2.64}$$

called the *transgression* of the 't Hooft anomaly α (we will henceforth drop the notational dependence of τ on α in order to improve readability). As a result of the cocycle condition (2.62) obeyed by α, it satisfies

$$(d\tau)_x(g,h,k) \;:=\; \frac{\tau_x(h,k)\cdot\tau_x(g,hk)}{\tau_x(gh,k)\cdot\tau_{({}^{k}x)}(g,h)} \;=\; 1\,, \tag{2.65}$$

which ensures that the algebra multiplication in (2.63) is associative. Using (2.26), the $*$-structure on the tube algebra can be computed to be [4]

$$\langle {}^{g}x \overset{g}{+} x \rangle^{*} \;=\; \mu_x(g)\cdot \langle x \overset{g^{-1}}{+} {}^{g}x \rangle\,, \tag{2.66}$$

where we defined the multiplicative phase

$$\mu_x(g) \;:=\; \tau_x^{*}(g^{-1},g)\,. \tag{2.67}$$

As a consequence of (2.65), it satisfies $\mu_x(g^{-1}) = \mu_{(x^g)}(g)$ as well as $d\mu = \widehat{\tau}/\tau$, where we defined the dual transgression 2-cocycle

$$\widehat{\tau}_x(g,h) \;:=\; \tau^{*}_{({}^{gh}x)}(h^{-1},g^{-1})\,. \tag{2.68}$$

9 In particular, the simple objects of Hilb_G^{α} are given by the one-dimensional Hilbert spaces $\mathbb{C}_g$ with G-grading $(\mathbb{C}_g)_h = \delta_{g,h}\mathbb{C}$ for $g,h \in G$.

This ensures that the $*$-structure (2.66) is involutory and compatible with the algebra multiplication (2.63). All in all, the above allows us to identify the tube algebra associated to $\mathcal{C}$ with the so-called *twisted Drinfeld double* of G [94–96], i.e.

$$\mathrm{Tube}(\mathrm{Hilb}_G^\alpha) \;=\; \mathcal{D}^\alpha(G)\,. \tag{2.69}$$

If $\alpha = 1$, the latter may be viewed as the groupoid algebra[10] $\mathbb{C}[G/\!/G]$ associated to the conjugation action of G on itself.

In order to classify the irreducible tube representations of $\mathcal{C} = \mathrm{Hilb}_G^\alpha$ (or equivalently the irreducible representations of $\mathcal{D}^\alpha(G)$), we note that, as a consequence of the delta-function appearing in (2.63), any such $\mathcal{F}$ will decompose into subrepresentations supported on twisted sectors $\mathcal{H}_y$ labelled by elements $y \in [x]$ in the *conjugacy class* of some fixed $x \in G$. If we furthermore restrict to linking with symmetry defects $g \in G$ that lie in the *centraliser* $G_x := \{g \in G \mid {}^g x = x\}$ of x in G, then the linear maps

$$\rho(g) \;:=\; \mathcal{F}\Big(\big\langle \overset{x \quad g \quad x}{\text{———}} \big\rangle\Big) \tag{2.71}$$

define a projective representation of G_x on the Hilbert space $\mathcal{V} := \mathcal{H}_x$ with projective 2-cocycle $\tau_x(\alpha) \in Z^2(G_x, U(1))$. Conversely, given a pair (x, ρ) consisting of

1. a representative $x \in G$ of a conjugacy class $[x] \in \mathrm{Cl}(G)$,
2. an irreducible $\tau_x(\alpha)$-projective representation ρ of G_x,

we can construct an associated tube representation $\mathcal{F}_{(x,\rho)}$ via induction [74]: To this end, fix for each element $y \in [x]$ in the conjugacy class of x a representative $r_y \in G$ such that ${}^{(r_y)}y = x$ (with $r_x := 1$). Using these, we can define

$$g_y \;:=\; r_{({}^g y)} \cdot g \cdot r_y^{-1} \;\in\; G_x \tag{2.72}$$

for all $g \in G$ and $y \in [x]$. If we denote by $\mathcal{V}$ the Hilbert space underlying the projective representation ρ of G_x, then $\mathcal{F}_{(x,\rho)}$ acts on the twisted sectors

$$\mathcal{H}_y \;=\; \begin{cases} \mathcal{V} & \text{if } y \in [x] \\ 0 & \text{otherwise} \end{cases} \tag{2.73}$$

[10] Given a finite groupoid $\mathcal{G}$ (i.e. a category with a finite number of objects x and all morphisms $g : x \to y$ invertible), we define the associated *groupoid algebra* to be the linear span

$$\mathbb{C}[\mathcal{G}] \;:=\; \bigoplus_{x,y\,\in\,\mathcal{G}} \mathbb{C}[\mathrm{Hom}_{\mathcal{G}}(x,y)] \tag{2.70}$$

with algebra multiplication given by composition whenever defined and zero otherwise. We are often interested in the case where $\mathcal{G} = X/\!/G$ is the so-called *action groupoid* associated to the action $\triangleright : G \times X \to X$ of some finite group G on some finite set X. Here, the objects of $X/\!/G$ are given by points $x \in X$ with morphism spaces given by $\mathrm{Hom}_{X/\!/G}(x,y) := \{g \in G \,|\, g \triangleright x = y\}$.

via the non-trivial induced tube actions

$$\mathcal{F}_{(x,\rho)}\Big(\langle\, {}^{g_y} \overset{g}{\text{—}\!\!\!\!+\!\!\!\!\text{—}} {}^{y} \,\rangle\Big) \;:=\; \kappa_y(g) \cdot \rho(g_y) \;, \tag{2.74}$$

where we defined the multiplicative phases

$$\kappa_y(g) \;:=\; \frac{\tau_y(r_{({}^g y)}, g)}{\tau_y(g_y, r_y)} \;. \tag{2.75}$$

As a consequence of (2.65), they satisfy

$$\frac{\kappa_y(h) \cdot \kappa_{({}^h y)}(g)}{\kappa_y(gh)} \;=\; \frac{\tau_y(g,h)}{\tau_x(g_{({}^h y)}, h_y)} \;, \tag{2.76}$$

which ensures that $\mathcal{F}_{(x,\rho)}$ respects the algebra multiplication (2.63). It is straightforward to check that $\mathcal{F}_{(x,\rho)}$ is a $*$-representation of $\mathcal{D}^{\alpha}(G)$ if and only if ρ is a unitary projective representation of G_x. Hence, we see that $*$-representations of the tube algebra form the natural generalisation of unitary representations of group-like symmetries. Furthermore, since every finite-dimensional representation of a finite group is equivalent to a unitary one, this shows that all irreducible representations of $\mathcal{D}^{\alpha}(G)$ are $*$-representations as expected.

All in all, we conclude that the category of $*$-representations of the twisted Drinfeld double $\mathcal{D}^{\alpha}(G)$ admits a direct sum decomposition

$$\mathrm{Rep}^*(\mathcal{D}^{\alpha}(G)) \;\cong\; \boxplus_{[x]\,\in\,\mathrm{Cl}(G)} \mathrm{Rep}^{\tau_x(\alpha)}(G_x) \;. \tag{2.77}$$

In particular, using (2.45), this reproduces the known classification of simple objects in the Drinfeld centre of $\mathcal{C} = \mathrm{Hilb}_G^{\alpha}$ in terms of pairs (x,ρ) as above [97].

2.4.1.1 *Example: $G = \mathbb{Z}_2$*

As a simple example, let us consider $G = \mathbb{Z}_2 =: \langle x \rangle$ with 't Hooft anomaly α given by the non-trivial generator of $H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$ with normalised representative

$$\alpha(x,x,x) \;=\; -1 \;. \tag{2.78}$$

In this case, the $*$-structure on the tube algebra is given by

$$\langle\, {}^{x} \overset{x}{\text{—}\!\!\!\!+\!\!\!\!\text{—}} {}^{x} \,\rangle^* \;=\; (-1) \cdot \langle\, {}^{x} \overset{x}{\text{—}\!\!\!\!+\!\!\!\!\text{—}} {}^{x} \,\rangle \,, \tag{2.79}$$

which admits the following four irreducible tube representations:

- There are two one-dimensional tube representations $\mathcal{F}_1^{\pm}$ acting on the twisted sector $\mathcal{H}_1 = \mathbb{C}$ via the non-trivial tube actions

$$\mathcal{F}_1^{\pm}\Big(\big\langle 1 \overset{x}{\text{—}\!\!\!\!\!\!\!\perp} 1 \big\rangle\Big) \;=\; \pm 1 \;. \tag{2.80}$$

- There are two one-dimensional tube representations $\mathcal{F}_x^{\pm}$ acting on the twisted sector $\mathcal{H}_x = \mathbb{C}$ via the non-trivial tube actions

$$\mathcal{F}_x^{\pm}\Big(\big\langle x \overset{x}{\text{—}\!\!\!\!\!\!\!\perp} x \big\rangle\Big) \;=\; \pm i \;. \tag{2.81}$$

2.4.1.2 Example: $G = D_8$

As another example, let us consider a non-anomalous dihedral group symmetry of order eight, which can be presented as

$$D_8 \;=\; \langle\, r, s \mid r^4 = s^2 = 1,\; srs = r^{-1} \,\rangle \;. \tag{2.82}$$

As described above, its irreducible tube representations can be labelled by pairs (x, ρ) consisting of a representative x of a conjugacy class $[x] \in \mathrm{Cl}(D_8)$ together with an irreducible representation ρ of its centraliser $(D_8)_x$. Concretely, the five conjugacy classes of D_8 together with their centralisers are given by

$$\begin{aligned}
&[1] \;=\; \{1\}\,, && (D_8)_1 \;=\; D_8\,, \\
&[r^2] \;=\; \{r^2\}\,, && (D_8)_{r^2} \;=\; D_8\,, \\
&[r] \;=\; \{r, r^3\}\,, && (D_8)_r \;=\; \langle r\rangle \;\cong\; \mathbb{Z}_4\,, \\
&[s] \;=\; \{s, r^2 s\}\,, && (D_8)_s \;=\; \langle r^2, s\rangle \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_2\,, \\
&[rs] \;=\; \{rs, r^3 s\}\,, && (D_8)_{rs} \;=\; \langle r^2, rs\rangle \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_2\,.
\end{aligned} \tag{2.83}$$

As there are five irreducible representations of D_8, there is a total of 22 irreducible tube representations, which can be described as follows:

- For $x = 1$ and r^2, the centralisers $(D_8)_1$ and $(D_8)_{r^2}$ are equal to the full symmetry group D_8, which has five irreducible representations (four one-dimensional and one two-dimensional one) given by

	1	u	v	uv	m
r	1	-1	1	-1	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
s	1	1	-1	-1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(2.84)

For $x \in \{1, r^2\}$ and $\rho \in \{1, u, v, uv, m\}$, the associated tube representation $\mathcal{F}_{(x,\rho)}$ then acts on the twisted sector $\mathcal{H}_x \cong \mathbb{C}^{\dim(\rho)}$ via the non-trivial tube actions

$$\mathcal{F}_{(x,\rho)}\Big(\langle x \overset{g}{+} x \rangle\Big) \;=\; \rho(g) \tag{2.85}$$

for all group elements $g \in D_8$.

- For $x = r$, the centraliser $(D_8)_r = \langle r \rangle \cong \mathbb{Z}_4$ has four irreducible representations given by its characters $\rho \in (D_8)_r^\vee =: \langle \hat{r} \rangle$ (where $\langle \hat{r}, r \rangle := i$). The corresponding irreducible tube representations act on the twisted sectors $\mathcal{H}_r \cong \mathcal{H}_{r^3} \cong \mathbb{C}$ via the following non-trivial tube actions:

	$\mathcal{F}_{(r,1)}$	$\mathcal{F}_{(r,\hat{r})}$	$\mathcal{F}_{(r,\hat{r}^2)}$	$\mathcal{F}_{(r,\hat{r}^3)}$
$\langle r \overset{r}{+} r \rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}$
$\langle r^3 \overset{r}{+} r^3 \rangle$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$
$\langle r^3 \overset{s}{+} r \rangle$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\langle r \overset{s}{+} r^3 \rangle$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(2.86)

- For $x = s$, the centraliser $(D_8)_s = \langle r^2, s \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ has four irreducible representations $\rho \in (D_8)_s^\vee =: \langle\, \widehat{r^2}, \hat{s}\, \rangle$ (where $\langle \widehat{r^2}, r^2 \rangle = \langle \hat{s}, s \rangle = -1$). The corresponding irreducible tube representations act on the twisted sectors $\mathcal{H}_s \cong \mathcal{H}_{r^2 s} \cong \mathbb{C}$ via the following non-trivial tube actions:

	$\mathcal{F}_{(s,1)}$	$\mathcal{F}_{(s,\,\widehat{r^2})}$	$\mathcal{F}_{(s,\hat{s})}$	$\mathcal{F}_{(s,\,\hat{s}\,\cdot\,\widehat{r^2})}$
$\langle r^2 s \overset{r}{+} s \rangle$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\langle r \overset{r}{+} r^2 s \rangle$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$
$\langle s \overset{s}{+} s \rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$
$\langle r^2 s \overset{s}{+} r^2 s \rangle$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(2.87)

- For $x = rs$, the centraliser $(D_8)_{rs} = \langle r^2, rs \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ has four irreducible representations given by $\rho \in (D_8)_{rs}^\vee =: \langle\, \widehat{r^2}, \widehat{rs}\, \rangle$ (where $\langle \widehat{r^2}, r^2 \rangle = \langle \widehat{rs}, rs \rangle = -1$).

The corresponding irreducible tube representations act on the twisted sectors $\mathcal{H}_{rs} \cong \mathcal{H}_{r^3s} \cong \mathbb{C}$ via the following non-trivial tube actions:

	$\mathcal{F}_{(rs,1)}$	$\mathcal{F}_{(rs,\widehat{r^2})}$	$\mathcal{F}_{(rs,\widehat{rs})}$	$\mathcal{F}_{(rs,\widehat{r^2}\cdot\widehat{rs})}$
$\langle r^3s \overset{r}{+} rs \rangle$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\langle rs \overset{r}{+} r^3s \rangle$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$
$\langle r^3s \overset{s}{+} rs \rangle$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$\langle rs \overset{s}{+} r^3s \rangle$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(2.88)

2.4.2 Tambara-Yamagami Symmetry

As a another example, let us consider a symmetry category $\mathcal{C} = \mathrm{TY}_A^{\chi,s}$ of Tambara-Yamagami type [98], which is specified by the following pieces of data:

1. A finite abelian group A,
2. a non-degenerate symmetric bicharacter $\chi : A \times A \to U(1)$,
3. a square-root s of $1/|A|$.

The simple objects of $\mathcal{C}$ comprise the group elements $a \in A$ that fuse according to the group law of A as well as an additional non-invertible defect m that fuses according to

$$a \otimes m \;=\; m \otimes a = m\,, \qquad m \otimes m \;=\; \bigoplus_{a \in A} a\,. \tag{2.89}$$

The non-trivial components of the associator are given by

$$\begin{aligned}
&\underset{=\,m}{(a\otimes m)\otimes b} \xrightarrow{\;\chi(a,b)\,\cdot\,\mathrm{id}_m\;} \underset{=\,m}{a\otimes(m\otimes b)}\,,\\
&\underset{=\,\bigoplus_{b\in A} b}{(m\otimes a)\otimes m} \xrightarrow{\;\bigoplus_b \chi(a,b)\,\cdot\,\mathrm{id}_b\;} \underset{=\,\bigoplus_{b\in A} b}{m\otimes(a\otimes m)}\,,\\
&\underset{=\,\bigoplus_{a\in A} m}{(m\otimes m)\otimes m} \xrightarrow{\;s\,\cdot\,\bigoplus_{a,b} \chi^*(a,b)\,\cdot\,\mathrm{id}_m\;} \underset{=\,\bigoplus_{b\in A} m}{m\otimes(m\otimes m)}\,.
\end{aligned} \tag{2.90}$$

The dual objects are given by $a^\vee = a^{-1}$ and $m^\vee = m$ with dimensions $\dim(a) = 1$ and $\dim(m) = \sqrt{|A|}$. This symmetry category is unitary for all A, χ and s.

The tube algebra associated to $\mathcal{C} = \mathrm{TY}_A^{\chi,s}$ was computed in [4] and is given by the $2|A|\cdot(|A|+1)$-dimensional algebra that is spanned by the basis vectors

$$\langle x \overset{a}{+} x \rangle\,, \quad \langle y \overset{m}{+} x \rangle\,, \quad \langle m \overset{m}{\underset{a}{+}} m \rangle\,, \quad \langle m \overset{a}{+} m \rangle \tag{2.91}$$

(where $a, x, y \in A$), whose algebra multiplication is given by

$$\begin{aligned}
\langle x \overset{a}{+} x \rangle \circ \langle x \overset{b}{+} x \rangle &= \langle x \overset{ab}{+} x \rangle\,, \\
\langle y \overset{m}{+} x \rangle \circ \langle x \overset{a}{+} x \rangle &= \chi(a,y)\cdot \langle y \overset{m}{+} x \rangle\,, \\
\langle y \overset{a}{+} y \rangle \circ \langle y \overset{m}{+} x \rangle &= \chi(a,x)\cdot \langle y \overset{m}{+} x \rangle\,, \\
\langle z \overset{m}{+} y \rangle \circ \langle y \overset{m}{+} x \rangle &= \delta_{x,z}\cdot \chi^*(x,y)\cdot \textstyle\sum_a \chi^*(a,y)\cdot \langle x \overset{a}{+} x \rangle\,, \\
\langle m \overset{a}{+} m \rangle \circ \langle m \overset{b}{+} m \rangle &= \chi^*(a,b)\cdot \langle m \overset{ab}{+} m \rangle\,, \\
\langle m \overset{a}{+} m \rangle \circ \langle m \overset{m}{\underset{b}{+}} m \rangle &= \langle m \overset{m}{\underset{b}{+}} m \rangle \circ \langle m \overset{a}{+} m \rangle \;=\; \chi(a,ab)\cdot \langle m \overset{m}{\underset{ab}{+}} m \rangle\,, \\
\langle m \overset{m}{\underset{a}{+}} m \rangle \circ \langle m \overset{m}{\underset{b}{+}} m \rangle &= \frac{s}{|A|}\cdot \chi(a,b)\cdot \textstyle\sum_c \chi^*(ab,c)\cdot \langle m \overset{c}{+} m \rangle\,.
\end{aligned} \tag{2.92}$$

Using (2.26), the $*$-structure can be computed to be

$$\begin{aligned}
\langle x \overset{a}{+} x \rangle^* &= \langle x \overset{a^{-1}}{+} x \rangle\,, \\
\langle y \overset{m}{+} x \rangle^* &= \chi(x,y)\cdot \langle x \overset{m}{+} y \rangle\,, \\
\langle m \overset{a}{+} m \rangle^* &= \chi^*(a,a)\cdot \langle m \overset{a^{-1}}{+} m \rangle\,, \\
\langle m \overset{m}{\underset{a}{+}} m \rangle^* &= s\cdot \sum_b \chi^*(a,b)\cdot \langle m \overset{m}{\underset{b}{+}} m \rangle\,.
\end{aligned} \tag{2.93}$$

The associativity of the algebra multiplication as well as its compatibility with the $*$-structure can be checked to hold as a consequence of the character identity

$$\frac{1}{|A|}\cdot \sum_{a\,\in\,A} \chi(a,b) \;=\; \delta_{b,1}\,. \tag{2.94}$$

As described in [2, 4], there is a total of $\frac{1}{2}|A|\cdot(|A|+7)$ irreducible tube representations of $\mathcal{C} = \mathrm{TY}_A^{\chi,s}$, which can be grouped into the following three categories:

- There are $2\cdot|A|$ one-dimensional tube representations $\mathcal{F}_x^\Delta$ labelled by group elements $x \in A$ and a choice of square-root Δ of $\chi^*(x,x) \in U(1)$, which act on the twisted sector $\mathcal{H}_x \cong \mathbb{C}$ via the non-trivial tube actions

$$\begin{aligned}
\mathcal{F}_x^\Delta\big(\langle x \overset{a}{+} x \rangle\big) &= \chi(a,x)\,, \\
\mathcal{F}_x^\Delta\big(\langle x \overset{m}{+} x \rangle\big) &= \frac{1}{s}\cdot \Delta\,.
\end{aligned} \tag{2.95}$$

- There are $2\cdot|A|$ one-dimensional representations $\mathcal{F}_\rho^\Delta$ labelled by antiderivatives[11] ρ of χ and a choice of square-root Δ of $s\cdot\sum_c \rho^*(c)\in U(1)$, which act on the twisted sector $\mathcal{H}_m\cong\mathbb{C}$ via the non-trivial tube actions

$$\begin{aligned} \mathcal{F}_\rho^\Delta\Big(\langle m \overset{a}{\dashv} m\rangle\Big) &= \rho^*(a)\ , \\ \mathcal{F}_\rho^\Delta\Big(\langle m \overset{m}{\underset{a}{\dashv}} m\rangle\Big) &= s\cdot\Delta\cdot\rho(a^{-1})\ . \end{aligned} \tag{2.96}$$

- There are $\frac{1}{2}|A|\cdot(|A|-1)$ two-dimensional representations $\mathcal{F}_{x,y}$ labelled by distinct elements $x,y\in A$, which act on the twisted sectors $\mathcal{H}_x\cong\mathcal{H}_y\cong\mathbb{C}$ via

$$\begin{aligned} \mathcal{F}_{x,y}\Big(\langle x \overset{a}{\dashv} x\rangle\Big) &= \begin{pmatrix}\chi(a,y) & 0\\ 0 & 0\end{pmatrix}\ , \\ \mathcal{F}_{x,y}\Big(\langle y \overset{a}{\dashv} y\rangle\Big) &= \begin{pmatrix}0 & 0\\ 0 & \chi(a,x)\end{pmatrix}\ , \\ \mathcal{F}_{x,y}\Big(\langle y \overset{m}{\dashv} x\rangle\Big) &= \frac{1}{s}\cdot\begin{pmatrix}0 & 0\\ \chi^*(x,y) & 0\end{pmatrix}\ , \\ \mathcal{F}_{x,y}\Big(\langle x \overset{m}{\dashv} y\rangle\Big) &= \frac{1}{s}\cdot\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\ . \end{aligned} \tag{2.97}$$

Using (2.45), the above gives a list of simple objects in the Drinfeld centre of $\mathcal{C}=\mathrm{TY}_A^{\chi,s}$ which reproduces the known classification found in [99].

2.4.2.1 Example: Ising Symmetry

As a simple example, let us consider the case $A=\mathbb{Z}_2=:\langle x\rangle$, which admits a single non-degenerate bicharacter given by $\chi(x,x)=-1$. Hence, there are exactly two corresponding Tambara-Yamagami categories, which are distinguished by the choice of $s=\pm 1/\sqrt{2}$. For $s>0$, one obtains the so-called *Ising category*, which describes the topological defects in the two-dimensional critical Ising CFT, where x corresponds to the invertible spin-flip symmetry and m is non-invertible Kramers-Wannier duality defect [100, 101]. In particular, there are ten conformal primaries (twisted and untwisted), which organise themselves into the nine irreducible tube representations of the Ising category (for a concrete description of the different primaries in the Ising CFT including their conformal weights we refer the reader to [100]):

[11] An antiderivative of χ is a map $\rho: A\to U(1)$ such that $(d\rho)(a,b):=\rho(a)\cdot\rho(b)\cdot\rho^*(ab)\equiv\chi(a,b)$ for all $a,b\in A$. Since χ is symmetric, such a ρ always exists and the set of all antiderivatives forms a torsor over $A^\vee=\mathrm{Hom}(A,U(1))$. This shows that there are $|A^\vee|=|A|$ antiderivatives of χ.

- There are four one-dimensional tube representations $\mathcal{F}_1^{\pm}$ and $\mathcal{F}_x^{\pm}$ that act on the twisted sectors $\mathcal{H}_1 \cong \mathbb{C}$ and $\mathcal{H}_x \cong \mathbb{C}$ via the non-trivial tube actions

$$\begin{aligned} \mathcal{F}_1^{\pm}\Big(\langle 1 \overset{x}{+} 1 \rangle\Big) \;&=\; 1\,, & \mathcal{F}_x^{\pm}\Big(\langle x \overset{x}{+} x \rangle\Big) \;&=\; -1\,, \\ \mathcal{F}_1^{\pm}\Big(\langle 1 \overset{m}{+} 1 \rangle\Big) \;&=\; \pm\sqrt{2}\,, & \mathcal{F}_x^{\pm}\Big(\langle x \overset{m}{+} x \rangle\Big) \;&=\; \pm\sqrt{2}\,i\,. \end{aligned} \tag{2.98}$$

- There are four one-dimensional tube representations $\mathcal{F}_m^{\pm}$ and $\bar{\mathcal{F}}_m^{\pm}$ that act on the twisted sector $\mathcal{H}_m \cong \mathbb{C}$ via the non-trivial tube actions

$$\begin{aligned} \mathcal{F}_m^{\pm}\Big(\langle m \overset{x}{+} m \rangle\Big) \;&=\; i\,, & \bar{\mathcal{F}}_m^{\pm}\Big(\langle m \overset{x}{+} m \rangle\Big) \;&=\; -i\,, \\ \mathcal{F}_m^{\pm}\Big(\langle m \underset{1}{\overset{m}{+}} m \rangle\Big) \;&=\; \pm\,\frac{e^{i\pi/8}}{\sqrt{2}}\,, & \bar{\mathcal{F}}_m^{\pm}\Big(\langle m \underset{1}{\overset{m}{+}} m \rangle\Big) \;&=\; \pm\,\frac{e^{-i\pi/8}}{\sqrt{2}}\,, \\ \mathcal{F}_m^{\pm}\Big(\langle m \underset{x}{\overset{m}{+}} m \rangle\Big) \;&=\; \pm\,\frac{e^{-3\pi i/8}}{\sqrt{2}}\,, & \bar{\mathcal{F}}_m^{\pm}\Big(\langle m \underset{x}{\overset{m}{+}} m \rangle\Big) \;&=\; \pm\,\frac{e^{3\pi i/8}}{\sqrt{2}}\,. \end{aligned} \tag{2.99}$$

- There is one two-dimensional tube representation $\mathcal{F}_{1,x}$ that acts on the twisted sectors $\mathcal{H}_1 \cong \mathcal{H}_x \cong \mathbb{C}$ via the non-trivial tube actions

$$\begin{aligned} \mathcal{F}_{1,x}\Big(\langle 1 \overset{x}{+} 1 \rangle\Big) \;&=\; \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, & \mathcal{F}_{1,x}\Big(\langle x \overset{m}{+} 1 \rangle\Big) \;&=\; \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \\ \mathcal{F}_{1,x}\Big(\langle x \overset{x}{+} x \rangle\Big) \;&=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \mathcal{F}_{1,x}\Big(\langle 1 \overset{m}{+} x \rangle\Big) \;&=\; \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.100}$$

Note that any genuine local operator that is charged under the $\mathbb{Z}_2$ symmetry generated by x necessarily forms part of a two-dimensional multiplet that maps genuine to x-twisted operators under the action of the non-invertible defect m (and vice versa).

2.4.2.2 Example: Rep(D_8)

As another example, let us consider a Tambara-Yamagami category based on the abelian group $A = \mathbb{Z}_2 \times \mathbb{Z}_2 =: \langle u, v \rangle$ with bicharacter χ given by

$$\chi(u,u) \;=\; \chi(v,v) \;=\; 1\,, \qquad \chi(u,v) \;=\; -1\,, \tag{2.101}$$

and square-root $s = +1/2$. It was shown in [98] that this is equivalent to the category Rep(D_8) of finite-dimensional representations of the dihedral group of order eight (see (2.82)), where u and v generate the four one-dimensional irreducible representations and m is the single two-dimensional irreducible representation of D_8 as in (2.84). As before, the irreducible tube representations associated to $\mathcal{C} = \text{Rep}(D_8)$ may then be grouped into three different categories:

- There are eight one-dimensional tube representations $\mathcal{F}_x^{\pm}$ labelled by group elements $x \in A$ that act on the twisted sector $\mathcal{H}_x \cong \mathbb{C}$ via the tube actions

$\mathcal{F}_x^{\pm}$	$\mathcal{F}_1^{\pm}$	$\mathcal{F}_u^{\pm}$	$\mathcal{F}_v^{\pm}$	$\mathcal{F}_{uv}^{\pm}$
$\langle x \overset{a}{+} x \rangle$	$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}_a$	$\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}_a$	$\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}_a$	$\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}_a$
$\langle x \overset{m}{+} x \rangle$	± 2	± 2	± 2	± 2

(2.102)

where the index a runs over $1, u, v$ and uv (in that order).

- There are eight one-dimensional tube representations $\mathcal{F}_\rho^{\pm}$ labelled by the antiderivatives $\rho : A \to U(1)$ of χ, which are given by

	ρ_1	ρ_2	ρ_3	ρ_4
1	1	1	1	1
u	-1	1	1	-1
v	1	-1	1	-1
uv	1	1	-1	-1

(2.103)

These act on the twisted sector $\mathcal{H}_m \cong \mathbb{C}$ via the tube actions

	$\mathcal{F}_{\rho_1}^{\pm}$	$\mathcal{F}_{\rho_2}^{\pm}$	$\mathcal{F}_{\rho_3}^{\pm}$	$\mathcal{F}_{\rho_4}^{\pm}$
$\langle m \overset{a}{+} m \rangle$	$\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}_a$	$\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}_a$	$\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}_a$	$\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}_a$
$\langle m \overset{m}{\underset{a}{+}} m \rangle$	$\pm \frac{1}{2}\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}_a$	$\pm \frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}_a$	$\pm \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}_a$	$\pm \frac{i}{2}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}_a$

(2.104)

where the index a runs over $1, u, v$ and uv (in that order).

- There are six two-dimensional tube representations $\mathcal{F}_{x,y}$ labelled by distinct group elements $x, y \in A$ that act on the twisted sectors $\mathcal{H}_x \cong \mathcal{H}_y \cong \mathbb{C}$ via

$\mathcal{F}_{x,y}$	$\langle x \overset{a}{\perp} x \rangle$	$\langle y \overset{a}{\perp} y \rangle$	$\langle y \overset{m}{\perp} x \rangle$	$\langle x \overset{m}{\perp} y \rangle$
$\mathcal{F}_{1,u}$	$\begin{pmatrix} [1,1,-1,-1]_a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & [1,1,1,1]_a \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$
$\mathcal{F}_{1,v}$	$\begin{pmatrix} [1,1,-1,-1]_a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & [1,1,1,1]_a \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$
$\mathcal{F}_{1,uv}$	$\begin{pmatrix} [1,1,-1,-1]_a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & [1,1,1,1]_a \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$
$\mathcal{F}_{u,v}$	$\begin{pmatrix} [1,-1,1,-1]_a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & [1,1,-1,-1]_a \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$
$\mathcal{F}_{u,uv}$	$\begin{pmatrix} [1,-1,-1,1]_a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & [1,1,-1,-1]_a \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$
$\mathcal{F}_{v,uv}$	$\begin{pmatrix} [1,-1,-1,1]_a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & [1,-1,1,-1]_a \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$,

(2.105)

where the index a runs over $1, u, v$ and uv (in that order).

The fact that the number of irreducible tube representations (twenty-two) for $\text{Rep}(D_8)$ is the same as for Hilb_{D_8} (see subsection 2.4.1.2) is not a coincidence. Rather, it follows from the 'gauge-invariance' of the Drinfeld centre, i.e.

$$\mathcal{Z}(\text{Hilb}_G) \;\cong\; \mathcal{Z}(\text{Rep}(G)) \tag{2.106}$$

for any finite group G, which is an instance of a more general result[12] due to Schauenburg [102]. More concretely, we can motivate the equivalence (2.106) as follows:

[12] Concretely, the result obtained in [102] can be described as follows: Given a theory $\mathcal{T}$ with fusion category symmetry $\mathcal{C}$, we can try to 'gauge' a subsymmetry of $\mathcal{C}$ by choosing an *algebra object* $A \in \mathcal{C}$ and placing it on a fine enough defect network inside correlation functions of $\mathcal{T}$. The symmetries of the resulting theory $\mathcal{T}' = \mathcal{T}/A$ are then described by the category $\mathcal{C}' = {}_A\mathcal{C}_A$ of A-bimodules in $\mathcal{C}$ [103]. In particular, it was shown in [102] that there exists a natural equivalence

$$\mathcal{Z}(\mathcal{C}) \;\cong\; \mathcal{Z}({}_A\mathcal{C}_A) \tag{2.107}$$

between the corresponding Drinfeld centres. Physically, this may be understood from the fact that the process of discrete gauging in reversible, so that the (twisted sector) operator content of $\mathcal{T}'$ must be a (non-trivial) rearrangement of the operator content of $\mathcal{T}$. In particular, any tube representation of $\mathcal{C}$ must give rise to a tube representation of $\mathcal{C}'$ and vice versa, which, using (2.44), yields the equivalence (2.107). In the case where $\mathcal{C} = \text{Hilb}_G$ for some finite group G, choosing $A = \mathbb{C}[G]$ yields $\mathcal{C}' = \text{Rep}(G)$, which, plugged into (2.107), yields the equivalence (2.106). More generally, one may consider anomalous group symmetries G and gauge non-anomalous subgroups $H \subset G$, which leads to the notion of so-called *group-theoretical* fusion categories [64, 104].

Suppose that we are given an object $z \in \mathcal{Z}(\mathrm{Hilb}_G)$. By definition, this means that z corresponds to a pair $z = (U, \tau)$ consisting of

1. a finite-dimensional G-graded Hilbert space $U = \bigoplus_{x \in G} U_x$,
2. a collection τ of unitary linear maps $\tau_g : U \otimes \mathbb{C}_g \to \mathbb{C}_g \otimes U$ labelled by group elements $g \in G$ such that

$$(\mathbb{C}_g \otimes \tau_h) \circ (\tau_g \otimes \mathbb{C}_h) \;=\; \tau_{g \cdot h} \; . \tag{2.108}$$

From this, we would like to construct a corresponding object $z' \in \mathcal{Z}(\mathrm{Rep}(G))$, which is again given by a pair $z' = (R, \varphi)$ consisting of

1. a unitary representation R of G,
2. a collection φ of unitary intertwiners $\varphi_S : R \otimes S \to S \otimes R$ labelled by (arbitrary) representations S of G such that

$$(S \otimes \varphi_T) \circ (\varphi_S \otimes T) \;=\; \varphi_{S \otimes T} \; . \tag{2.109}$$

To do this, we note that upon decomposing U into its graded components U_x and identifying $\mathbb{C}_g \otimes U_x \cong U_{gx}$, we can view the linear maps τ_g as unitary automorphisms of U that shift the grading according to

$$\tau_g : \; U_x \;\to\; U_{x^g} \; , \tag{2.110}$$

where $x^g = g^{-1} x g$. In particular, setting $R_g := \tau_{g^{-1}}$ yields a collection of unitary automorphisms, which as a consequence of (2.108) define a unitary representation R of G on U. Given another representation S of G on a Hilbert space V, we can define an associated linear map $\varphi_S : U \otimes V \to V \otimes U$ via

$$u \otimes v \;\mapsto\; \sum_{x \in G} S_x(v) \otimes u_x \; , \tag{2.111}$$

which can be checked to yield a unitary intertwiner between the tensor product representations $R \otimes S$ and $S \otimes R$. Furthermore, one can check that the collection $\varphi := \{\varphi_S\}_S$ of intertwiners satisfies condition (2.108). If we set $z' := (R, \varphi)$, then the mapping $z \mapsto z'$ establishes the desired equivalence (2.106) between the Drinfeld centres of Hilb_G and $\mathrm{Rep}(G)$.

We can apply the above construction to our case of interest, $G = D_8$, in order to obtain an explicit relation between the irreducible tube representations of Hilb_{D_8} and

$\mathrm{Rep}(D_8)$. Concretely, using the same notation as in subsections 2.4.1.2 and 2.4.2.2, we arrive at the following 1:1-correspondence:

$$
\begin{array}{c|c}
\mathrm{Hilb}_{D_8} & \mathrm{Rep}(D_8) \\
\hline
\mathcal{F}_{(1,1)} & \mathcal{F}_1^+ \\
\mathcal{F}_{(1,u)} & \mathcal{F}_u^+ \\
\mathcal{F}_{(1,v)} & \mathcal{F}_v^+ \\
\mathcal{F}_{(1,uv)} & \mathcal{F}_{uv}^+ \\
\mathcal{F}_{(1,m)} & \mathcal{F}_{\rho 3}^+ \\
\mathcal{F}_{(r^2,1)} & \mathcal{F}_1^-
\end{array}
\qquad
\begin{array}{c|c}
\mathcal{F}_{(r^2,u)} & \mathcal{F}_u^- \\
\mathcal{F}_{(r^2,v)} & \mathcal{F}_v^- \\
\mathcal{F}_{(r^2,uv)} & \mathcal{F}_{uv}^- \\
\mathcal{F}_{(r^2,m)} & \mathcal{F}_{\rho 3}^- \\
\mathcal{F}_{(r,1)} & \mathcal{F}_{1,v} \\
\mathcal{F}_{(r,\hat{r})} & \mathcal{F}_{\rho 4}^- \\
\mathcal{F}_{(r,\hat{r}^2)} & \mathcal{F}_{u,uv} \\
\mathcal{F}_{(r,\hat{r}^3)} & \mathcal{F}_{\rho 4}^+
\end{array}
\qquad
\begin{array}{c|c}
\mathcal{F}_{(s,1)} & \mathcal{F}_{1,u} \\
\mathcal{F}_{(s,\,\widehat{r^2})} & \mathcal{F}_{\rho 2}^+ \\
\mathcal{F}_{(s,\hat{s})} & \mathcal{F}_{v,uv} \\
\mathcal{F}_{(s,\,\hat{s}\cdot\widehat{r^2})} & \mathcal{F}_{\rho 2}^- \\
\mathcal{F}_{(rs,1)} & \mathcal{F}_{1,uv} \\
\mathcal{F}_{(rs,\,\widehat{r^2})} & \mathcal{F}_{\rho 1}^+ \\
\mathcal{F}_{(rs,\,\widehat{rs}\,)} & \mathcal{F}_{u,v} \\
\mathcal{F}_{(rs,\,\widehat{r^2}\cdot\widehat{rs}\,)} & \mathcal{F}_{\rho 1}^-
\end{array}
\;. \tag{2.112}
$$

2.4.3 Fibonacci Symmetry

As a last example, we consider a symmetry category $\mathcal{C}$ with only two simple objects denoted by 1 and τ, whose fusion rules are given by

$$\tau \otimes \tau \;=\; 1 \oplus \tau \,. \tag{2.113}$$

The solution to the pentagon equation for the associator in this case takes the form

$$
\begin{array}{c} (\tau\otimes\tau)\otimes\tau \\ = 1\oplus\tau^{\oplus 2} \end{array}
\xrightarrow{\begin{pmatrix} \mathrm{id}_1 & 0 \\ 0 & A\cdot\mathrm{id}_\tau \end{pmatrix}}
\begin{array}{c} \tau\otimes(\tau\otimes\tau) \\ = 1\oplus\tau^{\oplus 2} \end{array} \;, \tag{2.114}
$$

where the self-inverse (2×2)-matrix A is given by [105]

$$A \;=\; \begin{pmatrix} -a & 1/\lambda \\ -a\lambda & a \end{pmatrix} \,. \tag{2.115}$$

Here, $a \in \mathbb{R}$ is one of the two solutions of the quadratic equation $a^2 = a + 1$ given by

$$a_\pm \;=\; \frac{1}{2}\,(1 \pm \sqrt{5}) \tag{2.116}$$

and $\lambda \in \mathbb{C}^\times$ is a gauge parameter that describes a family of equivalent fusion categories for fixed a. Up to equivalence, there are hence only two distinct fusion categories with fusion rules (2.113), which correspond to choosing $a = a_-$ and $a = a_+$ and which are called the *Fibonacci* (Fib^+) and the *Yang-Lee category* (Fib^-), respectively. The reason for this (perhaps confusing) notation is that the dimension of the non-invertible self-dual object τ in each case is given by

$$\dim(\tau) \;=\; -\frac{1}{a} \;\equiv\; \begin{cases} a_+ > 0 & \text{for } \mathrm{Fib}^+ \\ a_- < 0 & \text{for } \mathrm{Fib}^- \end{cases} . \tag{2.117}$$

Furthermore, since the matrix A is unitary if and only if

$$|\lambda|^2 \;=\; -\frac{1}{a} \,, \tag{2.118}$$

we see that only Fib^+ admits the structure of a unitary fusion category.

The tube algebra associated to $\mathrm{Fib}^\pm$ was computed in [4] and is given by the seven-dimensional algebra that is spanned by the basis vectors

$$\begin{gathered} \langle 1 \overset{1}{+} 1 \rangle , \quad \langle 1 \overset{\tau}{+} 1 \rangle , \quad \langle \tau \overset{\tau}{+} 1 \rangle , \quad \langle 1 \overset{\tau}{+} \tau \rangle , \\ \langle \tau \overset{1}{+} \tau \rangle , \quad \langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle , \quad \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle , \end{gathered} \tag{2.119}$$

whose algebra multiplication is given by

$$\begin{aligned}
\langle 1 \overset{\tau}{+} 1 \rangle \circ \langle 1 \overset{\tau}{+} 1 \rangle &= \langle 1 \overset{1}{+} 1 \rangle + \langle 1 \overset{\tau}{+} 1 \rangle , \\
\langle \tau \overset{\tau}{+} 1 \rangle \circ \langle 1 \overset{\tau}{+} 1 \rangle &= a \cdot \langle \tau \overset{\tau}{+} 1 \rangle , \\
\langle 1 \overset{\tau}{+} \tau \rangle \circ \langle \tau \overset{\tau}{+} 1 \rangle &= \langle 1 \overset{1}{+} 1 \rangle + a \cdot \langle 1 \overset{\tau}{+} 1 \rangle , \\
\langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle \circ \langle \tau \overset{\tau}{+} 1 \rangle &= -a \cdot \langle \tau \overset{\tau}{+} 1 \rangle , \\
\langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle \circ \langle \tau \overset{\tau}{+} 1 \rangle &= a^2 \cdot \langle \tau \overset{\tau}{+} 1 \rangle , \\
\langle 1 \overset{\tau}{+} 1 \rangle \circ \langle 1 \overset{\tau}{+} \tau \rangle &= a \cdot \langle 1 \overset{\tau}{+} \tau \rangle , \\
\langle \tau \overset{\tau}{+} 1 \rangle \circ \langle 1 \overset{\tau}{+} \tau \rangle &= -a \cdot \langle \tau \overset{1}{+} \tau \rangle + \langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle + a^2 \cdot \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle , \\
\langle 1 \overset{\tau}{+} \tau \rangle \circ \langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle &= -a \cdot \langle 1 \overset{\tau}{+} \tau \rangle , \\
\langle 1 \overset{\tau}{+} \tau \rangle \circ \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle &= a^2 \cdot \langle 1 \overset{\tau}{+} \tau \rangle , \\
\langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle \circ \langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle &= -a^3 \cdot \langle \tau \overset{1}{+} \tau \rangle + a^2 \cdot \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle , \\
\langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle \circ \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle &= -a^2 \cdot \langle \tau \overset{1}{+} \tau \rangle + \langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle + a^3 \cdot \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle , \\
\langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle \circ \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle &= \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle \circ \langle \tau \underset{1}{\overset{\tau}{+}} \tau \rangle \;=\; a^2 \cdot \langle \tau \overset{1}{+} \tau \rangle - a^2 \cdot \langle \tau \underset{\tau}{\overset{\tau}{+}} \tau \rangle .
\end{aligned} \tag{2.120}$$

Here, we implicitly understand that $a = a_-$ for Fib^+ and $a = a_+$ for Fib^- as before.

Using (2.26), the $*$-structure[13] can be computed to be

$$
\begin{aligned}
\langle 1\,\overset{\tau}{+}\,1\rangle^* &= \langle 1\,\overset{\tau}{+}\,1\rangle\,, \\
\langle \tau\,\overset{\tau}{+}\,1\rangle^* &= -\frac{1}{a}\cdot\langle 1\,\overset{\tau}{+}\,\tau\rangle\,, \\
\langle 1\,\overset{\tau}{+}\,\tau\rangle^* &= -\,a\cdot\langle \tau\,\overset{\tau}{+}\,1\rangle\,, \\
\langle \tau\,\overset{\tau}{\underset{1}{+}}\,\tau\rangle^* &= -\,a\cdot\langle \tau\,\overset{\tau}{\underset{1}{+}}\,\tau\rangle \;-\; a\cdot\langle \tau\,\overset{\tau}{\underset{\tau}{+}}\,\tau\rangle\,, \\
\langle \tau\,\overset{\tau}{\underset{\tau}{+}}\,\tau\rangle^* &= \langle \tau\,\overset{\tau}{\underset{1}{+}}\,\tau\rangle \;+\; a\cdot\langle \tau\,\overset{\tau}{\underset{\tau}{+}}\,\tau\rangle\,,
\end{aligned}
\tag{2.121}
$$

where again $a = a_-$ for Fib^+ and $a = a_+$ for Fib^-. The associativity of the algebra multiplication as well as its compatibility with the $*$-structure can be checked to hold as a consequence of the identity $a^2 = a + 1$.

Both for Fib^+ and Fib^-, there is a total of four irreducible tube representations, which can be described as follows (here, we again take $a = a_\mp$ for $\mathrm{Fib}^\pm$):

- There is a one-dimensional tube representation $\mathcal{F}_1$ that acts on the twisted sector $\mathcal{H}_1 \cong \mathbb{C}$ via the non-trivial tube action

$$
\mathcal{F}_1\big(\langle 1\,\overset{\tau}{+}\,1\rangle\big) \;=\; -\frac{1}{a}\,. \tag{2.122}
$$

  This is a $*$-representation for both $\mathrm{Fib}^\pm$.

- There are two one-dimensional tube representations $\mathcal{F}^\pm_\tau$ that act on the twisted sector $\mathcal{H}_\tau \cong \mathbb{C}$ via the non-trivial tube actions

$$
\begin{aligned}
\mathcal{F}^\pm_\tau\big(\langle \tau\,\overset{\tau}{\underset{1}{+}}\,\tau\rangle\big) &= x_\pm\,, \\
\mathcal{F}^\pm_\tau\big(\langle \tau\,\overset{\tau}{\underset{\tau}{+}}\,\tau\rangle\big) &= \frac{1}{a}\cdot\Big(1 - \frac{x_\pm}{a}\Big)\,,
\end{aligned}
\tag{2.123}
$$

  where $x_\pm$ are the two solutions of $x^2 + x + a^2 = 0$ given by

$$
x_\pm \;=\; -\frac{1}{2} \pm i\cdot\sqrt{a + \frac{3}{4}}\,. \tag{2.124}
$$

  The latter are related by $x_- = (x_+)^*$ and $x_+ \cdot x_- = a^2$. We have that $\mathcal{F}^\pm_\tau$ are $*$-representations for both $\mathrm{Fib}^\pm$.

[13] Although Fib^- is not unitary, (2.121) still defines an antilinear involution on its tube algebra. However, in contrast to $\mathrm{Tube}(\mathrm{Fib}^+)$, the latter is not a C*-algebra.

- There is one two-dimensional representation $\mathcal{F}_{1,\tau}$ that acts on the twisted sectors $\mathcal{H}_1 \cong \mathcal{H}_\tau \cong \mathbb{C}$ via the non-trivial tube actions

$$
\begin{aligned}
\mathcal{F}_{1,\tau}\Big(\big\langle {}^{1}\!\!-\!\!\overset{\tau}{\bullet}\!\!-\!\!{}^{1} \big\rangle\Big) &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} , \\
\mathcal{F}_{1,\tau}\Big(\big\langle {}^{\tau}\!\!-\!\!\overset{\tau}{\underset{1}{\bullet}}\!\!-\!\!{}^{\tau} \big\rangle\Big) &= \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} , \\
\mathcal{F}_{1,\tau}\Big(\big\langle {}^{\tau}\!\!-\!\!\overset{\tau}{\underset{\tau}{\bullet}}\!\!-\!\!{}^{\tau} \big\rangle\Big) &= \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} , \\
\mathcal{F}_{1,\tau}\Big(\big\langle {}^{\tau}\!\!-\!\!\overset{\tau}{\bullet}\!\!-\!\!{}^{1} \big\rangle\Big) &= \lambda \cdot \begin{pmatrix} 0 & 0 \\ 1+ia & 0 \end{pmatrix} , \\
\mathcal{F}_{1,\tau}\Big(\big\langle {}^{1}\!\!-\!\!\overset{\tau}{\bullet}\!\!-\!\!{}^{\tau} \big\rangle\Big) &= \frac{1}{\lambda} \cdot \begin{pmatrix} 0 & 1-ia \\ 0 & 0 \end{pmatrix} .
\end{aligned} \tag{2.125}
$$

  Here, $\lambda \in \mathbb{C}^\times$ is a gauge parameter that describes equivalent tube representations. One can check that these are $*$-representations if and only if

$$
|\lambda|^2 = -\frac{1}{a} , \tag{2.126}
$$

  which admits a solution only for Fib^+ (where $a = a_-$ so that $-1/a_- = a_+ > 0$). As a result, we see that the non-unitary Yang-Lee category Fib^- admits a tube representation which is not a $*$-representation.

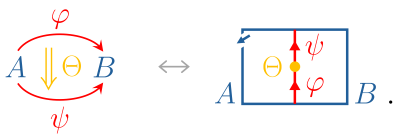
3

THREE DIMENSIONS

In this chapter, we consider generalised symmetries in three spacetime dimensions that are described by a fusion 2-category $\mathcal{C}$. We construct the associated tube 1- and 2-categories, which capture the action of $\mathcal{C}$ on twisted sector local and line operators, respectively. We classify their irreducible 1- and 2-representations using a higher-dimensional analogue of the sandwich construction and we provide explicit examples that include anomalous 2-group symmetries as well as non-invertible 1-form symmetries. The discussion is based on work performed in collaboration with Mathew Bullimore and Andrea Grigoletto [1, 2] as well as the single-author works [3, 4].

3.1 Preliminaries

In this section, we provide some brief mathematical background on the theory of fusion 2-categories, which describe finite bosonic generalised symmetries in three spacetime dimensions (for a more comprehensive account we refer the reader to [66]). Concretely, given a fusion 2-category $\mathcal{C}$, its objects, 1- and 2-morphisms correspond to topological surface defects, their line interfaces and local junctions, respectively, i.e.

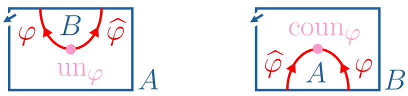


(3.1)

Moreover, $\mathcal{C}$ comes equipped with a variety of additional structures that capture the topological nature of symmetry defects, the most salient of which we summarise below:

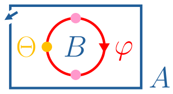
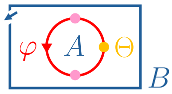
- **Finite semisimplicity:** We assume that $\mathcal{C}$ is enriched over Vect, meaning that the 2-morphism spaces $2\mathrm{Hom}_{\mathcal{C}}(\varphi, \psi)$ are finite-dimensional complex vector space for all 1-morphisms φ and ψ, such that the (vertical and horizontal) composition of 2-morphisms is linear. In particular, this means that for any objects $A, B \in \mathcal{C}$, the 1-morphism space $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a linear 1-category, which we assume to be finite-semisimple. As a result, we can decompose any 1-morphism $\varphi : A \to B$ into a finite direct sum of simple morphisms σ_i, which are such that $2\mathrm{Hom}_{\mathcal{C}}(\sigma_i, \sigma_j) = \delta_{ij}\,\mathbb{C}$. Furthermore, we assume that every 1-morphism

$\varphi : A \to B$ has an *adjoint*, which is a 1-morphism $\widehat{\varphi} : B \to A$ (viewed as the orientation reversal of φ) together with unit and counit 2-morphisms



(3.2)

satisfying suitable *zig-zag relations*, such that $\widehat{\widehat{\varphi}} \cong \varphi$ [66]. Using this, we can define the left and *right traces* of a 2-endomorphism $\Theta \in 2\mathrm{End}_{\mathcal{C}}(\varphi)$ by

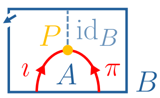
$\mathrm{tr}_L(\Theta) \;:=$  , $\quad \mathrm{tr}_R(\Theta) \;:=$  , (3.3)

where we left the labelling of unit and counit 2-morphisms implicit (also note that $\mathrm{tr}_L(\Theta) \in \mathrm{End}_{\mathcal{C}}(A)$ while $\mathrm{tr}_R(\Theta) \in \mathrm{End}_{\mathcal{C}}(B)$, which means that for generic A and B the left and right traces of 2-morphisms are not comparable).

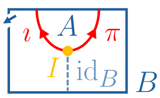
As for 1-morphisms, we assume that we can decompose any object in $\mathcal{C}$ into a direct sum of a finite number of simple objects $S_i \in \mathcal{C}$, i.e.

$$A \;\cong\; \boxplus_{i=1}^{n} (S_i)^{\boxplus A_i} \,, \tag{3.4}$$

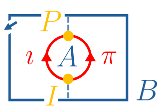
where each S_i is such that $\mathrm{id}_{S_i} \in \mathrm{End}_{\mathcal{C}}(S)$ is simple (or equivalently such that $2\mathrm{End}_{\mathcal{C}}(\mathrm{id}_{S_i}) \cong \mathbb{C}$) and $A_i = \dim(2\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_A, \mathrm{id}_{S_i})) \in \mathbb{N}$. However, unlike in the case of semisimple 1-categories, the 1-morphism space between two non-isomorphic simple objects S_i and S_j need not be trivial. Rather, we say that S_i and S_j are in the same *component* if there exists a non-zero 1-morphism between them. The *set of components* $\pi_0(\mathcal{C})$ is given by the set of simple objects in $\mathcal{C}$ modulo the equivalence relation of being in the same component[1]. Alternatively, we can describe $\pi_0(\mathcal{C})$ as the set of simple objects in $\mathcal{C}$ modulo the notion of 'condensation' [106, 107]: Given two objects A and B, a *condensation* from A onto B (denoted by $A \twoheadrightarrow B$) consists of 1- and 2-morphisms



and



such that



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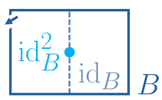
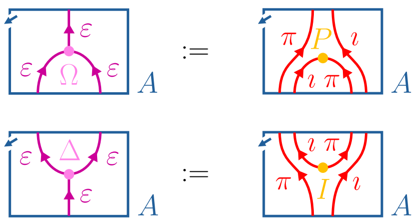


. (3.5)

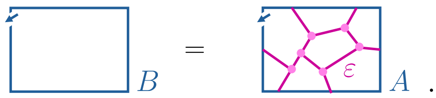
[1] This is indeed an equivalence relation due to the fact that every object has an identity 1-morphism, the composition of two non-zero 1-morphisms between simple objects is non-zero [66], and every 1-morphism has an adjoint by assumption.

Given a condensation $A \twoheadrightarrow B$, the 1-morphism $\varepsilon := \imath \circ \pi$ defines a so-called *condensation monad* in $\mathrm{End}_{\mathcal{C}}(A)$, meaning that it is equipped with 2-morphisms



(3.6)

that satisfy $\Omega \circ \Delta = \mathrm{id}_{\varepsilon}$. Physically, we may then view the surface defect B as a 'condensate' obtained by placing the monad ε on a defect network on A, i.e.



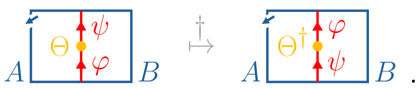
(3.7)

We say that $\mathcal{C}$ is *condensation complete* if every condensation monad ε on an object A is induced by a condensation $A \twoheadrightarrow B$ in the above manner[2]. Physically, this means that all surface defects B that we can construct by 'gauging' a condensation monad ε on a surface A as in (3.7) are already included in $\mathcal{C}$.

- **Dagger structure:** We assume that $\mathcal{C}$ is compatible with *reflection positivity* in the sense that there exists a dagger structure

$$\dagger : \mathrm{Hom}_{\mathcal{C}}(A, B) \to (\mathrm{Hom}_{\mathcal{C}}(A, B)^{\mathrm{op}})^{*} \tag{3.8}$$

on the morphism category between any objects $A, B \in \mathcal{C}$, which allows us to reflect local topological junctions in $\mathcal{C}$ about a fixed hyperplane, i.e.



(3.9)

Positivity is the statement that there exists a faithful state $\Gamma : 2\mathrm{End}_{\mathcal{C}}(\varphi) \to \mathbb{C}$ on the 2-endomorphism algebra of each 1-morphism $\varphi : A \to B$, which turns $2\mathrm{End}_{\mathcal{C}}(\varphi)$ into a C*-algebra for all φ. We assume that the higher coherence data associated to $\mathcal{C}$ as a 2-category (such as 2-associators and unitors for the composition of 1-morphisms) is unitary w.r.t. the above dagger structure. Furthermore, we assume that the latter is compatible with adjoints in $\mathcal{C}$ in the sense that the functor $\wedge : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(B, A)$ that maps $\varphi \mapsto \widehat{\varphi}$ is a $\dagger$-functor

[2] Given any 2-category $\mathcal{C}$, one may construct its so-called *Karoubi envelope*, which is the universal category $\mathrm{Kar}(\mathcal{C})$ that contains $\mathcal{C}$ as well as all possible condensates (and is hence condensation complete) [106]. As a result, we can often omit condensations from our discussion in what follows, since they can be added trivially using the Karoubi envelope.

with unitary coherence data. This turns $\mathcal{C}$ into a so-called †-*2-category* in the sense of [108, 109]. A 2-functor $F : \mathcal{C} \to \mathcal{C}'$ between two †-2-categories is called a †-*2-functor* if it restricts to a †-functor $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}'}(F(A), F(B))$ on the morphism categories between any objects $A, B \in \mathcal{C}$.

- **Monoidal structure:** We assume that $\mathcal{C}$ encodes the fusion of symmetry defects that corresponds to the parallel collision of topological surfaces and their interfaces. Mathematically, this can be achieved by endowing $\mathcal{C}$ with a monoidal structure in the sense of [110] (see also [111] for a review):

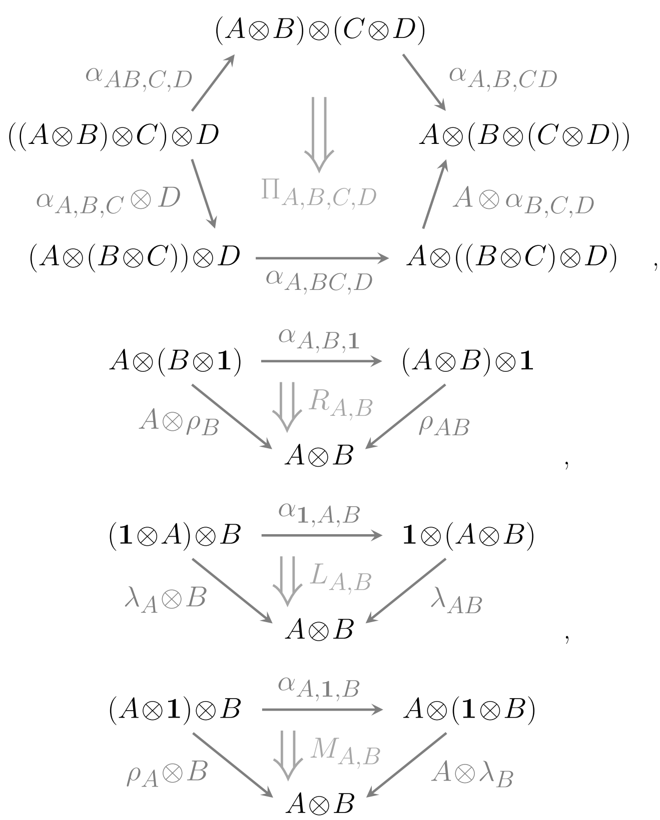
Definition: A *monoidal structure* on $\mathcal{C}$ consists of an additive linear 2-functor

$$\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C} \,, \tag{3.10}$$

a distinguished object $\mathbf{1} \in \mathcal{C}$ (called the *unit*) and three natural isomorphisms

$$\begin{aligned} \alpha :\ & \otimes \circ (\otimes \boxtimes \mathrm{Id}) \Rightarrow \otimes \circ (\mathrm{Id} \boxtimes \otimes) \,, \\ \lambda :\ & (\mathbf{1} \otimes \mathrm{Id}) \Rightarrow \mathrm{Id} \,, \\ \rho :\ & (\mathrm{Id} \otimes \mathbf{1}) \Rightarrow \mathrm{Id} \end{aligned} \tag{3.11}$$

(called the *associator*, *left* and *right unitor*, respectively), whose coherence relations are controlled by modifications



(3.12)

(called the *pentagonator* and *2-unitors*, respectively). In addition, there exist so-called *tensocompositor* 2-isomorphisms

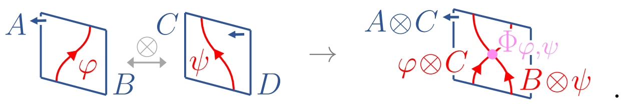
$$A\otimes B \xrightarrow[\mathrm{id}_{A\otimes B}]{\mathrm{id}_A\otimes\mathrm{id}_B} A\otimes B \quad \Downarrow \Phi_{A,B}\,, \qquad A\otimes B \xrightarrow{\varphi\otimes\psi} A'\otimes B' \xrightarrow{\varphi'\otimes\psi'} A''\otimes B''\,,\quad A\otimes B \xrightarrow{(\varphi'\circ\varphi)\otimes(\psi'\circ\psi)} A''\otimes B'' \quad \Downarrow \Phi_{\varphi,\psi;\varphi',\psi'}\,, \tag{3.13}$$

which, together with the above, satisfy suitable coherence relations [110]. An additive linear 2-functor $F:\mathcal{C}\to\mathcal{C}'$ between two monoidal 2-categories is called a *monoidal 2-functor* if it preserves the monoidal unit and is equipped with a natural isomorphism $\nu:\otimes'\circ(F\boxtimes F)\Rightarrow F\circ\otimes$ and a modification

$$\begin{array}{ccccc} & & F(A)\otimes F(B)\otimes F(C) & & \\ & \swarrow^{\nu_{A,B}\otimes C} & & \searrow^{A\otimes\nu_{B,C}} & \\ F(A\otimes B)\otimes F(C) & & \Downarrow V_{A,B,C} & & F(A)\otimes F(B\otimes C) \\ & \searrow_{\nu_{AB,C}} & & \swarrow_{\nu_{AB,C}} & \\ & & F(A\otimes B\otimes C) & & \end{array} \tag{3.14}$$

(where we omitted associators) that satisfies suitable coherence relations [110].

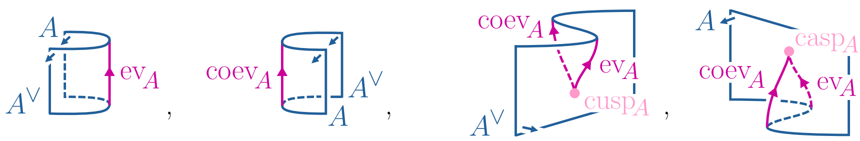
Pictorially, we represent the monoidal product as the result of bringing the corresponding topological symmetry defects together in a parallel fashion:



(3.15)

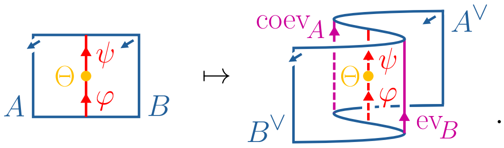
We will often omit any higher coherence data pertinent to associators, unitors, and pentagonators from graphical representations of symmetry defects in what follows. We depict the monoidal unit $\mathbf{1}\in\mathcal{C}$ by the invisible / transparent surface defect and we assume that it is a simple object in $\mathcal{C}$, i.e. $2\mathrm{End}_{\mathcal{C}}(\mathrm{id}_{\mathbf{1}})=\mathbb{C}$. We require the monoidal structure to be compatible with the dagger structure in the sense that $\otimes:\mathcal{C}\boxtimes\mathcal{C}\to\mathcal{C}$ is a †-2-functor and all higher coherence data pertinent to associators, unitors, pentagonars, and tensocompositors is unitary. In this case, we call $\mathcal{C}$ *unitary* fusion 2-category.

- **Dual structure:** We assume that every object $A\in\mathcal{C}$ has a *dual* $A^\vee\in\mathcal{C}$ (viewed as the orientation reversal of A) together with evaluation and coevaluation 1-morphisms as well as cusp and casp 2-morphisms



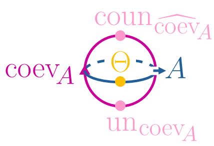
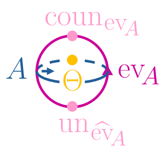
(3.16)

that satisfy suitable *swallowtail relations* [66]. The map $A \mapsto A^\vee$ then extends to an op-monoidal 2-functor $\vee : \mathcal{C} \to \mathcal{C}$ that acts on 1- and 2-morphisms via



(3.17)

A *pivotal structure* on $\mathcal{C}$ is a natural isomorphism $\xi : \vee^2 \Rightarrow \mathrm{Id}_\mathcal{C}$ together with a modification $\Xi_A : (\xi_A)^\vee \overset{\sim}{\Rrightarrow} \xi_{A^\vee}$ satisfying suitable coherence relations. A pivotal structure is called *spherical* if the associated front and back traces of 2-endomorphisms $\Theta \in \mathrm{End}_\mathcal{C}(\mathrm{id}_A)$ agree, i.e.

$\mathrm{Tr}_F(\Theta) \;:=\;$  $\;\overset{!}{=}\;$  $\;=:\; \mathrm{Tr}_B(\Theta)\;,$ (3.18)

where we omitted ξ_A from the graphical notation. In this case, we define the dimension of an object $A \in \mathcal{C}$ by $\dim(A) := \mathrm{Tr}_{F/B}(\mathrm{id}^2_A)$. Similarly, we define the dimension of a 1-morphism φ by[3] $\dim(\varphi) := \mathrm{Tr}_{F/B}(\mathrm{tr}_{L/R}(\mathrm{id}_\varphi))$. We assume the dual structure to be compatible with the dagger structure on $\mathcal{C}$ in the sense that $\vee : \mathcal{C} \to \mathcal{C}$ is a $\dagger$-2-functor with unitary higher coherence data and that the pivotal structure has unitary 2-morphism components.

3.2 1-Twisted Sectors

In this section, we discuss the action of the fusion 2-category symmetry $\mathcal{C}$ on 1-twisted sectors, i.e. local operators attached to topological line defects. We construct the corresponding tube category and show how its irreducible representations can be classified using a higher-dimensional analogue of the sandwich construction. We provide explicit examples that include anomalous 2-groups symmetries as well as non-invertible 1-form symmetries.

3.2.1 Tube Category

The *tube category* associated to a fusion 2-category $\mathcal{C}$ captures the possible linking configurations of twisted sector local operators in three dimensions with symmetry defects in $\mathcal{C}$. Concretely, following [2], we define the tube category $\mathrm{T}\mathcal{C}$ associated to $\mathcal{C}$ to be the additive linear category whose

[3] Here, we use the fact that the front trace obeys $\mathrm{Tr}_F(\mathrm{tr}_L(\Theta)) = \mathrm{Tr}_F(\mathrm{tr}_R(\Theta))$ for any 2-endomorphism Θ [66]. A similar relation holds for the back trace.

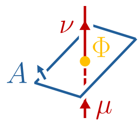
- objects are given by genuine lines μ in the *loop space* $\Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ of $\mathcal{C}$, i.e.

, (3.19)

- morphisms between objects $\mu, \nu \in \Omega\mathcal{C}$ form the quotient vector space

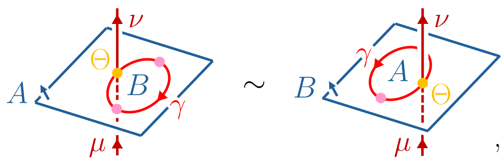
$$\mathrm{Hom}_{\mathrm{T}\mathcal{C}}(\mu,\nu) \;:=\; \bigoplus_{A\in\mathcal{C}} 2\mathrm{Hom}_{\mathcal{C}}(A\otimes\mu, \nu\otimes A) \Big/ \sim \tag{3.20}$$

(where the sum runs over *all* objects $A \in \mathcal{C}$) of local intersection 2-morphisms



(3.21)

in $\mathcal{C}$ subjected to the equivalence relation that is generated by



, (3.22)

where we left the labelling of unit and counit 2-morphisms implicit.

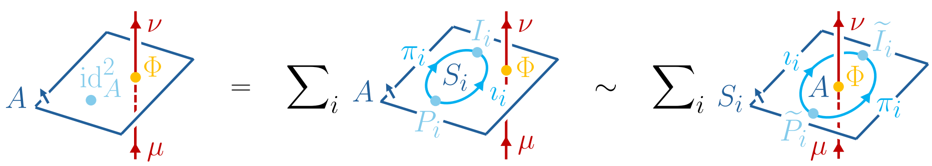
Physically, the equivalence relation (3.22) means that we should think of the symmetry defect A as being wrapped on a 2-sphere. Mathematically, it renders the morphism spaces in (3.20) finite-dimensional. Concretely, let $A \in \mathcal{C}$ be an arbitrary object and consider a decomposition $A \cong \boxplus_i S_i$ of A into a finite number of simple objects S_i (possibly with multiplicities). Let us denote by

$$\imath_i : \; S_i \;\to\; A \qquad \text{and} \qquad \pi_i : \; A \;\to\; S_i \tag{3.23}$$

the associated inclusion and projection 1-morphisms obeying $\bigoplus_i \imath_i \circ \pi_i \cong \mathrm{id}_A$ with associated inclusion and projection 2-morphisms

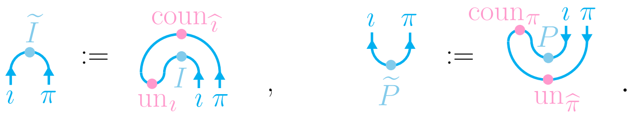
$$I_i : \; \imath_i \circ \pi_i \;\Rightarrow\; \mathrm{id}_A \qquad \text{and} \qquad P_i : \; \mathrm{id}_A \;\Rightarrow\; \imath_i \circ \pi_i \tag{3.24}$$

satisfying the completeness relation $\mathrm{id}^2_A = \sum_i I_i \circ P_i$. Upon inserting the latter into (3.21), we then obtain the relation



, (3.25)

where we defined the modified inclusion and projection 2-morphisms P



(3.26)

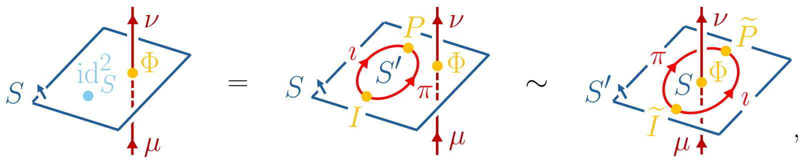
If we denote the equivalence class of (3.21) under the relation (3.22) by

$$\langle {}^{\nu}\!\left[{}^{A}_{\Phi}\right]^{\mu} \rangle \;\in\; \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(\mu,\nu)\,, \tag{3.27}$$

we can rewrite equation (3.25) schematically as the morphism identity

$$\langle {}^{\nu}\!\left[{}^{A}_{\Phi}\right]^{\mu} \rangle \;=\; \sum_i \langle {}^{\nu}\!\left[{}^{S_i}_{\widetilde{I}_i\circ\Phi\circ\widetilde{P}_i}\right]^{\mu} \rangle\,. \tag{3.28}$$

We can decompose the above even further using the notion of condensation. Concretely, given a condensation $S \twoheadrightarrow S'$ between simple objects S and S', we have that

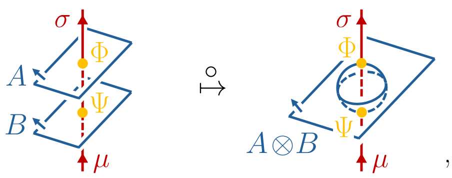


(3.29)

As a result, we see that we can reduce the sum in (3.28) to a sum over a fixed set of representatives S of simple objects modulo condensation. In particular

$$\mathrm{Hom}_{\mathrm{T}\mathcal{C}}(\mu,\nu) \;\cong\; \bigoplus_{[S]\in\pi_0(\mathcal{C})} \left[\,2\mathrm{Hom}_{\mathcal{C}}(S\otimes\mu,\nu\otimes S)\,/\sim\right], \tag{3.30}$$

which shows that the morphism spaces in T$\mathcal{C}$ are finite-dimensional as claimed. The composition of morphisms in T$\mathcal{C}$ is induced by the vertical stacking

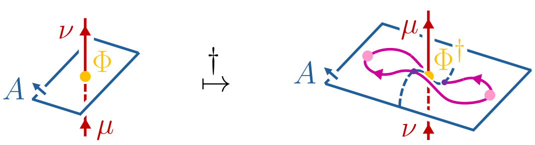


(3.31)

which we denote schematically by

$$\langle {}^{\sigma}\!\left[{}^{A}_{\Phi}\right]^{\nu} \rangle \circ \langle {}^{\nu}\!\left[{}^{B}_{\Psi}\right]^{\mu} \rangle \;=\; \langle {}^{\sigma}\!\left[{}^{A\otimes B}_{\Phi\circ\Psi}\right]^{\mu} \rangle\,. \tag{3.32}$$

Furthermore, the tube category possesses a natural †-structure that is induced by



(3.33)

(where we left the labelling of evaluation and coevaluation 1- and cusp and casp 2-morphisms implicit) and that we denote schematically by

$$\langle \nu \overset{A}{\underset{\Phi}{-\!\!\!-}} \mu \rangle^{\dagger} \;=\; \langle \mu \overset{A^{\vee}}{\underset{\Phi^{\dagger}}{-\!\!\!-}} \nu \rangle\,. \tag{3.34}$$

Since $\Gamma\Big(\langle \sigma \overset{A}{\underset{\Phi}{-\!\!\!-}} \sigma \rangle\Big) := \delta_{A,\mathbf{1}} \cdot \Phi$ defines a faithful state on the endomorphism algebra of each simple line $\sigma \in \Omega\mathcal{C}$, we see that $\mathrm{End}_{\mathrm{T}\mathcal{C}}(\sigma)$ is a C*-algebra for all σ. In particular, we obtain a C*-structure on the *tube algebra*

$$\mathrm{Tube}(\mathcal{C}) \;:=\; \mathrm{End}_{\mathrm{T}\mathcal{C}}\Big(\bigoplus\nolimits_{[\sigma]\in\pi_1(\mathcal{C})} \sigma\Big) \tag{3.35}$$

(where we denoted $\pi_1(\mathcal{C}) := \pi_0(\Omega\mathcal{C})$), which provides an alternative description of linking configurations of twisted sector local operators in three dimensions [2].

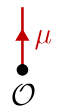
3.2.2 Tube Representations

Given a three-dimensional quantum field theory with fusion 2-category symmetry $\mathcal{C}$, it was proposed in [2] that twisted sector local operators transform in representations of the tube category associated to $\mathcal{C}$, which are additive linear functors

$$\mathcal{F}:\; \mathrm{T}\mathcal{C} \;\to\; \mathrm{Vect} \tag{3.36}$$

from $\mathrm{T}\mathcal{C}$ into the category of vector spaces and which we will again call *tube representations* in what follows. Concretely, any such tube representation $\mathcal{F}$ assigns

- to each object $\mu \in \Omega\mathcal{C}$ a vector space $\mathcal{H}_\mu := \mathcal{F}(\mu)$ that describes twisted sector local operators $\mathcal{O}$ sitting at the end of the topological line defect μ, i.e.

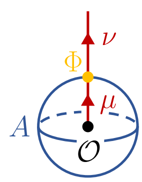


, (3.37)

- to each morphism $\langle \nu \overset{A}{\underset{\Phi}{-\!\!\!-}} \mu \rangle \in \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(\mu,\nu)$ a linear map

$$\mathcal{F}\Big(\langle \nu \overset{A}{\underset{\Phi}{-\!\!\!-}} \mu \rangle\Big):\; \mathcal{H}_\mu \;\to\; \mathcal{H}_\nu \tag{3.38}$$

  that describes how operators $\mathcal{O}$ in the μ-twisted sector get mapped to operators in the ν-twisted sector upon being linked with the symmetry defect A, i.e.



. (3.39)

We denote the category of tube representations and intertwiners between them by

$$\mathrm{Rep}(\mathrm{T}\mathcal{C}) \;:=\; [\mathrm{T}\mathcal{C}, \mathrm{Vect}]\;. \tag{3.40}$$

For a given tube representation $\mathcal{F}$, we can again use the operator-state map to endow the twisted sectors $\mathcal{H}_\mu$ with an inner product structure[4], which we assume to be compatible with the action of the tube category in the sense that

$$\mathcal{F}\Big(\langle \nu \overset{A}{\underset{\Phi}{-}} \mu \rangle^\dagger\Big) \;\overset{!}{=}\; \mathcal{F}\Big(\langle \nu \overset{A}{\underset{\Phi}{-}} \mu \rangle\Big)^\dagger \tag{3.41}$$

for all $\langle \nu \overset{A}{\underset{\Phi}{-}} \mu \rangle \in \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(\mu,\nu)$ (where the $\dagger$ on the right hand side denotes the adjoint of linear maps). Mathematically, this means that we assume $\mathcal{F}$ to lift to a $\dagger$-functor

$$\mathcal{F}:\; \mathrm{T}\mathcal{C} \xrightarrow{\dagger} \mathrm{Hilb}\,, \tag{3.42}$$

which we will call a *tube $\dagger$-representation* in what follows. We denote the category of all tube $\dagger$-representations and intertwiners between them by

$$\mathrm{Rep}^\dagger(\mathrm{T}\mathcal{C}) \;:=\; [\mathrm{T}\mathcal{C}, \mathrm{Hilb}]^\dagger\;. \tag{3.43}$$

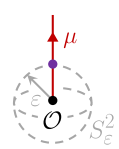
Clearly, every tube $\dagger$-representation reduces to an ordinary tube representation upon forgetting the underlying Hilbert space structure.

As in two dimensions, a useful way to classify the irreducible tube ($\dagger$-)representations of a fusion 2-category $\mathcal{C}$ is given by the *sandwich construction*. In this picture, we view a three-dimensional theory $\mathcal{T}$ with generalised symmetry $\mathcal{C}$ as being attached to a four-dimensional 'bulk', which hosts topological defects that 'commute' with all other symmetry defects in $\mathcal{C}$ and hence form the centre of $\mathcal{C}$:

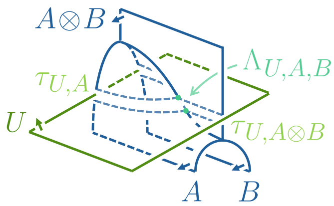
Definition: Given a fusion 2-category $\mathcal{C}$, its *Drinfeld centre* is the 2-category $\mathcal{Z}(\mathcal{C})$ whose objects are given by triples $z = (U, \tau_{U,-}, \Lambda_{U,-,\cdot})$ consisting of

1. an object $U \in \mathcal{C}$ in the fusion 2-category $\mathcal{C}$,
2. a half-braiding for U, i.e. a 2-natural isomorphism $\tau_{U,-} : U \otimes - \Rightarrow - \otimes U$,
3. an invertible modification $\Lambda_{U,-,\cdot} : (-\otimes \tau_{U,\cdot}) \circ (\tau_{U,-} \otimes \cdot) \Rrightarrow \tau_{U,-\otimes\cdot}$ with component 2-isomorphisms $\Lambda_{U,A,B} : (A \otimes \tau_{U,B}) \circ (\tau_{U,A} \otimes B) \Rightarrow \tau_{U,A\otimes B}$ that satisfy suitable coherence relations [112].

[4] Concretely, consider a 2-sphere S^2_ε of radius $\varepsilon > 0$ centred around a local operator $\mathcal{O} \in \mathcal{H}_\mu$ in the μ-twisted sector. Using the operator-state map, this induces a state $|\mathcal{O}\rangle_\varepsilon$ in the Hilbert space associated to the (punctured) S^2_ε. Given another μ-twisted sector local operator $\mathcal{O}'$, we then define its inner product with $\mathcal{O}$ to be $\langle \mathcal{O}|\mathcal{O}'\rangle_\varepsilon$, which is independent of ε if we assume time evolution along the radial direction to be unitary.



Pictorially, we represent the components $\tau_{U,A}$ and $\Lambda_{U,A,B}$ as 'crossings' that tell us how the object U can be moved across any other topological defects $A, B \in \mathcal{C}$:



(3.44)

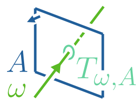
Similarly to the two-dimensional case, the Drinfeld centre inherits the structure of a fusion 2-category that is equipped with a canonical braiding.

For the purpose of describing 1-twisted sector operators using the sandwich construction, we are interested in the genuine topological lines in the four-dimensional bulk, which form the loop space $\Omega\mathcal{Z}(\mathcal{C})$ that can be described as follows [112]:

Proposition: The objects in the loop space $\Omega\mathcal{Z}(\mathcal{C})$ of the Drinfeld centre can be described by pairs $\rho = (\omega, T_{\omega,-})$ consisting of

1. an object $\omega \in \mathcal{C}$ in loop space $\Omega\mathcal{C}$,
2. a half-braiding for ω, i.e. a collection of 2-isomorphisms $T_{\omega,A} : \omega \otimes A \Rightarrow A \otimes \omega$ indexed by objects $A \in \mathcal{C}$ that satisfy suitable coherence relations [112].

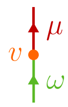
Pictorially, we again represent the components $T_{\omega,A}$ as crossings that tell us how the line defect ω can be moved across any other topological defect $A \in \mathcal{C}$:



(3.45)

We can use the above to associate to each object $\rho = (\omega, T_{\omega,-}) \in \Omega\mathcal{Z}(\mathcal{C})$ in the loop space of the Drinfeld centre a tube representation $\mathcal{F}_\rho \in \mathrm{Rep}(\mathrm{T}\mathcal{C})$ as follows [2]:

- To an object $\mu \in \Omega\mathcal{C}$, the functor $\mathcal{F}_\rho$ assigns the vector space $\mathcal{H}_\mu := 2\mathrm{Hom}_{\mathcal{C}}(\omega, \mu)$ of local junction 2-morphisms



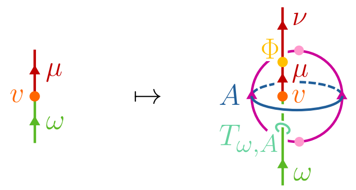
(3.46)

  We will denote the elements of $\mathcal{H}_\mu$ by $|\overset{\mu \;\; v \;\; \omega}{\longleftarrow\!\bullet\!\longleftarrow}\rangle$ in what follows. If μ is simple, $\mathcal{H}_\mu$ is a Hilbert space whose inner product is defined by

$$w \circ v^\dagger \;=:\; \langle v, w \rangle \cdot \mathrm{id}_\mu \,. \tag{3.47}$$

  If μ is not simple, we obtain a Hilbert space structure on $\mathcal{H}_\mu$ by decomposing μ into its simple components.

- To a morphism $\langle {}^{\nu}\overset{A}{\underset{\Phi}{[}}{}^{\mu}\rangle \in \mathrm{Hom}_{\mathrm{T}\mathcal{C}}(\mu,\nu)$, the functor $\mathcal{F}_\rho$ assigns the linear map from $\mathcal{H}_\mu$ to $\mathcal{H}_\nu$ that sends a local junction $v \in 2\mathrm{Hom}_{\mathcal{C}}(\omega,\mu)$ to



(3.48)

(where we left the labelling of evaluation and coevaluation 1- and unit and counit 2-morphisms implicit). Schematically, we write this as

$$\mathcal{F}_\rho\Big(\langle {}^{\nu}\overset{A}{\underset{\Phi}{[}}{}^{\mu}\rangle\Big)\,|{}^{\mu}\,{}^{v}\,{}^{\omega}\rangle \;=\; |{}^{\nu}_{\Phi}\,\overset{A}{(\mu\; v)}\,{}^{\omega}\rangle\;. \tag{3.49}$$

As a special case of the above, we can consider the tube representation $\mathbb{1} := \mathcal{F}_{\mathrm{id}_{\mathbf{1}}}$ associated to the identity 1-morphism $\mathrm{id}_{\mathbf{1}} = (\mathrm{id}_{\mathbf{1}}, \mathrm{id}^2_{-})$ of the monoidal unit $\mathbf{1} \in \mathcal{Z}(\mathcal{C})$, which we will call the *trivial* tube representation in what follows. The latter acts on the untwisted sector $\mathcal{H}_{\mathrm{id}_{\mathbf{1}}} \cong \mathbb{C}$ via the multiplicative factors

$$\mathbb{1}\Big(\langle {}^{\mathrm{id}_{\mathbf{1}}}\overset{A}{\underset{\Phi}{[}}{}^{\mathrm{id}_{\mathbf{1}}}\rangle\Big) \;=\; \mathrm{Tr}(\Phi)\;. \tag{3.50}$$

It was proposed in [2] that the assignment $\rho \mapsto \mathcal{F}_\rho$ extends to an equivalence

$$\Omega\mathcal{Z}(\mathcal{C}) \;\cong\; \mathrm{Rep}(\mathrm{T}\mathcal{C}) \tag{3.51}$$

of linear categories. One can check that the functor $\mathcal{F}_\rho$ associated to $\rho = (\omega, T_{\omega,-})$ is a †-functor if and only if the components $T_{\omega,A}$ of the half-braiding are unitary for all $A \in \mathcal{C}$. If we define the collection of such ρ to form the *unitary loop space* $\Omega\mathcal{Z}^\dagger(\mathcal{C})$ of the Drinfeld centre, this yields an equivalence

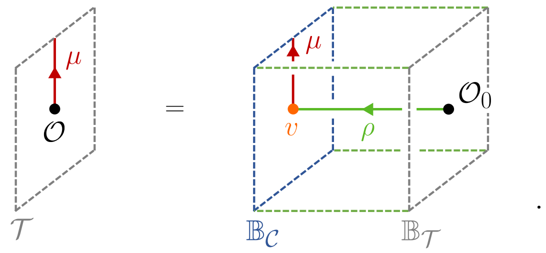
$$\Omega\mathcal{Z}^\dagger(\mathcal{C}) \;\cong\; \mathrm{Rep}^\dagger(\mathrm{T}\mathcal{C}) \tag{3.52}$$

of linear †-categories. In particular, the simple objects of $\Omega\mathcal{Z}^\dagger(\mathcal{C})$ correspond to the irreducible †-representations of $\mathrm{T}\mathcal{C}$. If furthermore $\Omega\mathcal{Z}^\dagger(\mathcal{C}) = \Omega\mathcal{Z}(\mathcal{C})$, then every tube representation of $\mathcal{C}$ is equivalent to a †-representation.

We can again visualise the above construction by viewing the three-dimensional theory $\mathcal{T}$ with generalised symmetry $\mathcal{C}$ as an interval compactification of a four-dimensional Symmetry TFT, which in this case can be identified with the Douglas-Reutter TQFT based on $\mathcal{C}$ (whose 2-category of surface and line defects is given by $\mathcal{Z}(\mathcal{C})$) [66]. The latter is again equipped with two boundary conditions:

1. A canonical topological boundary condition $\mathbb{B}_{\mathcal{C}}$ on the left that supports the symmetry $\mathcal{C}$ and that is independent of the theory $\mathcal{T}$ under consideration. In particular, the bulk-to-boundary map is given by the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ that sends objects $z = (U, \tau_{U,-}, \Lambda_{U,-,\cdot}) \mapsto U$ and lines $\rho = (\omega, T_{\omega,-}) \mapsto \omega$.
2. A physical boundary condition $\mathbb{B}_{\mathcal{T}}$ on the right that depends on the underlying theory $\mathcal{T}$ and that is non-topological in general.

The spectrum of twisted sector local operators $\mathcal{O}$ that transform in a given tube representation $\mathcal{F}_\rho$ associated to some $\rho \in \Omega\mathcal{Z}(\mathcal{C})$ may then be viewed as the spectrum of topological junctions v between twisted line defects $\mu \in \Omega\mathcal{C}$ on the left and the bulk line ρ stretched between the two boundaries of the Symmetry TFT:



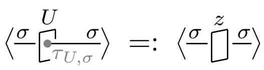
(3.53)

Here, $\mathcal{O}_0$ denotes a fixed (and generically non-topological) local operator that terminates ρ on right boundary. The linking action of the tube category on the operators $\mathcal{O}$ may then be computed using (3.48).

We can offer yet another perspective on the above discussion that uses the notion of the *tube algebra* Tube($\mathcal{C}$) associated to $\mathcal{C}$ as defined in (3.35), whose $*$-representations are in 1:1-correspondence with †-representations of the tube category [2], i.e.

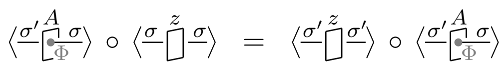
$$\text{Rep}^*(\text{Tube}(\mathcal{C})) \;\cong\; \text{Rep}^\dagger(\text{T}\mathcal{C}) \,. \tag{3.54}$$

One way to see this is by linking twisted sector local operators attached to simple lines $\sigma \in \Omega\mathcal{C}$ with symmetry defects $U \in \mathcal{C}$ that can be pushed into the four-dimensional bulk (and hence form part of the defining data of an object $z = (U, \tau_{U,-}, \Lambda_{U,-,\cdot}) \in \mathcal{Z}(\mathcal{C})$ in the Drinfeld centre). We denote the corresponding tube algebra element by



(3.55)

in what follows. As a consequence of the coherence relations obeyed by the half-braiding $\tau_{U,-}$, this then has the property that

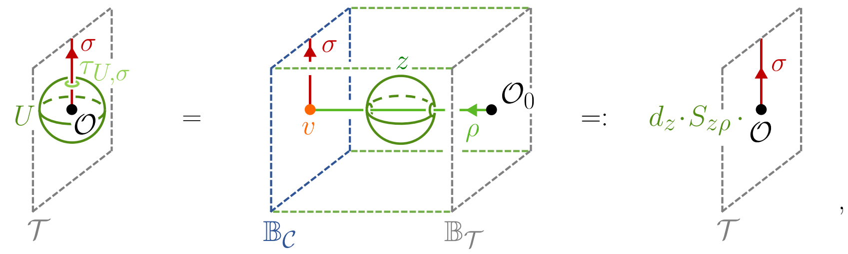


(3.56)

for all $\langle \overset{\nu}{-}\underset{\Phi}{\overset{A}{\square}}\overset{\mu}{-} \rangle \in \mathrm{Tube}(\mathcal{C})$. Furthermore, its algebra involution is given by[5]

$$\langle \overset{\sigma}{-}\overset{z}{\square}\overset{\sigma}{-} \rangle^* \;=\; \langle \overset{\sigma}{-}\overset{z^\vee}{\square}\overset{\sigma}{-} \rangle \,, \tag{3.57}$$

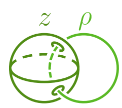
where $z^\vee$ denotes the dual of z in $\mathcal{Z}(\mathcal{C})$. Using the sandwich picture (3.53), it is then easy to see that the linking action of $\langle \overset{\sigma}{-}\overset{z}{\square}\overset{\sigma}{-} \rangle$ on twisted sector local operators $\mathcal{O}$ transforming in a tube representation $\mathcal{F}_\rho$ associated to some $\rho \in \Omega\mathcal{Z}(\mathcal{C})$ is given by



(3.58) ,

where $d_z := \dim(z)$ and we defined the multiplicative factor[6]

$$\mathcal{S}_{z\rho} \;:=\; \frac{1}{d_z \cdot d_\rho} \cdot \tag{3.59}$$



.

One can show that $\mathcal{S}_{z\rho}$ depends on z and ρ only up to condensation and isomorphism, respectively, so that we obtain a well-defined pairing [113, 114]

$$\mathcal{S} \;:\; \pi_0(\mathcal{Z}(\mathcal{C})) \,\times\, \pi_1(\mathcal{Z}(\mathcal{C})) \;\to\; \mathbb{C} \tag{3.60}$$

called the *generalised S-matrix* of $\mathcal{Z}(\mathcal{C})$. The latter has the following properties:

1. It is a square-matrix.
2. It satisfies $\mathcal{S}_{z^\vee\rho} = \mathcal{S}^*_{z\rho}$.
3. It is invertible.

In addition, $\mathcal{S}$ obeys the (generalised) Verlinde formula

$$\mathcal{S}_{x\rho} \cdot \mathcal{S}_{y\rho} \;=\; \sum_z N^z_{xy} \cdot \mathcal{S}_{z\rho} \,, \tag{3.61}$$

where the (generically non-integer) coefficients $N^z_{xy} \in \mathbb{C}$ capture the algebra products

$$\langle \overset{\mu}{-}\overset{x}{\square}\overset{\mu}{-} \rangle \circ \langle \overset{\mu}{-}\overset{y}{\square}\overset{\mu}{-} \rangle \;=\; \sum_z \frac{d_x \cdot d_y}{d_z} \cdot N^z_{xy} \cdot \langle \overset{\mu}{-}\overset{z}{\square}\overset{\mu}{-} \rangle \,. \tag{3.62}$$

---

5 More precisely, equation (3.57) holds as long as $z = (U, \tau_{U,-}, \Lambda_{U,-,\cdot})$ lies in the *unitary Drinfeld centre* $\mathcal{Z}^\dagger(\mathcal{C})$, meaning that the top components associated to the half-braiding $\tau_{U,-}$ and the modification $\Lambda_{U,-,\cdot}$ are all unitary. We assume that $\mathcal{Z}^\dagger(\mathcal{C}) = \mathcal{Z}(\mathcal{C})$ in what follows.

6 Here, we make use of the natural braiding on the Drinfeld centre $\mathcal{Z}(\mathcal{C})$.

Here, $\sum_z$ denotes a sum over fixed representatives of elements in $\pi_0(\mathcal{Z}(\mathcal{C}))$. Together with the Verlinde formula, this implies that the linear combinations

$$e_\rho^\sigma \;:=\; \sum_z \frac{1}{d_z} \cdot (\mathcal{S}^{-1})_{\rho z} \cdot \langle \overset{\sigma}{-}\overset{z}{\square}\overset{\sigma}{-} \rangle \tag{3.63}$$

define a collection of orthogonal self-adjoint idempotents in Tube($\mathcal{C}$) that are indexed by $[\sigma] \in \pi_1(\mathcal{C})$ and $[\rho] \in \pi_1(\mathcal{Z}(\mathcal{C}))$ [4], i.e. we have that

$$e_\rho^\sigma \circ e_{\rho'}^{\sigma'} \;=\; \delta_{\rho\rho'} \cdot \delta^{\sigma\sigma'} \cdot e_\rho^\sigma \qquad \text{and} \qquad (e_\rho^\sigma)^* \;=\; e_\rho^\sigma \,. \tag{3.64}$$

Using this, we can construct the minimal *central* idempotents in Tube($\mathcal{C}$) via

$$e_\rho \;:=\; \sum_\sigma e_\rho^\sigma \,, \tag{3.65}$$

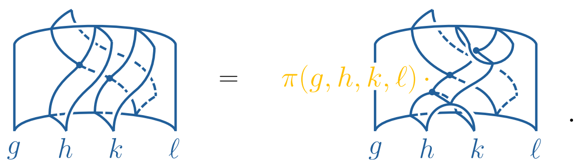
where $\sum_\sigma$ denotes a sum over a fixed set of representatives of elements in $\pi_1(\mathcal{C})$. In particular, we see that the minimal central idempotents are labelled by simple objects $\rho \in \Omega\mathcal{Z}(\mathcal{C})$, which, together with (3.54), re-establishes the equivalence (3.52) via their 1:1-correspondence with irreducible $*$-representations of Tube($\mathcal{C}$).

3.2.3 Examples

We conclude this section with two examples of fusion 2-category symmetries and their associated tube categories / algebras. We discuss anomalous group and 2-group symmetries as well as generic non-invertible 1-form symmetries.

3.2.3.1 Group Symmetry

We begin by considering an invertible symmetry described by some finite group G with 't Hooft anomaly specified by a (normalised) 4-cocycle $\pi \in Z^4(G, U(1))$. Physically, this means that we have simple surface defects labelled by group elements $g \in G$ that fuse according to the group law of G with pentagonator given by



(3.66)

In analogy to two dimensions, we denote the corresponding symmetry 2-category by[7] $\mathcal{C} = 2\mathrm{Hilb}_G^\pi$. The associated tube algebra is simply given by the group algebra $\mathrm{Tube}(2\mathrm{Hilb}_G^\pi) = \mathbb{C}[G]$, whose $*$-representations are unitary representations of G.

[7] We emphasise that this is simply a notation for now. We will discuss the notion of 2-vector and 2-Hilbert spaces in more detail in section 3.3.1.

We now aim to rederive this result using generalised S-matrices. As shown in [112], the Drinfeld centre of $\mathcal{C}$ has connected components

$$\pi_0(\mathcal{Z}(2\mathrm{Hilb}_G^\pi)) \;=\; \mathrm{Cl}(G) \tag{3.67}$$

given by conjugacy classes $[g]$ in G and the loop space given by

$$\Omega\mathcal{Z}(2\mathrm{Hilb}_G^\pi) \;=\; \mathrm{Rep}(G)\ , \tag{3.68}$$

so that $\pi_1(\mathcal{Z}(2\mathrm{Hilb}_G^\pi)) = \mathrm{Irr}(\mathrm{Rep}(G))$ is the set of isomorphism classes of irreducible representations ρ of G. The generalised S-matrix in this case is simply given by the (normalised) character table of G, i.e. corresponds to the canonical pairing

$$\mathcal{S}:\ \mathrm{Cl}(G)\,\times\,\mathrm{Irr}(\mathrm{Rep}(G))\;\to\;\mathbb{C}\,,\quad ([g],[\rho])\;\mapsto\;\frac{\mathrm{Tr}(\rho(g))}{\dim(\rho)}\ . \tag{3.69}$$

In particular, $\mathcal{S}$ is a square-matrix due to the fact that the number of conjugacy classes in G equals the number of irreducible representations, i.e.

$$|\mathrm{Cl}(G)| \;=\; |\mathrm{Irr}(\mathrm{Rep})(G)| \;=:\; n\ . \tag{3.70}$$

In order to simplify notation, we fix for each $i \in \{1, ..., n\}$ a representative $g_i \in G$ of the corresponding conjugacy class $[g_i] \in \mathrm{Cl}(G)$ as well as a representative $\rho_i \in \mathrm{Rep}(G)$ of the corresponding isomorphism class $[\rho_i] \in \mathrm{Irr}(\mathrm{Rep}(G))$ of irreducible representations of G. Furthermore, we denote by $G_i := G_{g_i}$ the centraliser of g_i and by $\chi_i := \mathrm{Tr}(\rho_i(.))$ the character associated to the irreducible representation ρ_i (whose dimension we denote by $d_i := \dim(\rho_i)$). Using this, the S-matrix can be written as the $(n\times n)$-matrix

$$\mathcal{S}_{ij} \;:=\; \mathcal{S}_{[g_i],[\rho_j]} \;=\; \frac{\chi_j(g_i)}{d_j}\ . \tag{3.71}$$

Using the character orthogonality relations

$$\sum_{k=1}^{n} \frac{1}{|G_k|}\cdot \chi_i^*(g_k)\cdot\chi_j(g_k) \;=\; \delta_{ij}\ , \tag{3.72}$$

$$\frac{1}{|G_i|}\cdot\sum_{k=1}^{n} \chi_k^*(g_i)\cdot\chi_k(g_j) \;=\; \delta_{ij}\ , \tag{3.73}$$

one can check that the S-matrix has the following properties:

1. It is invertible with inverse given by $(\mathcal{S}^{-1})_{ij} = \frac{d_i}{|G_j|}\cdot\chi_i^*(g_j)$.
2. It obeys $S_{i^\vee j} = (S_{ij})^*$, where $i^\vee \in \{1, ..., n\}$ is such that $[g_{(i^\vee)}] = [(g_i)^{-1}]$.

3. It satisfies the Verlinde formula

$$S_{i\ell} \cdot S_{j\ell} \;=\; \sum_{k=1}^{n} N_{ij}^{k} \cdot S_{k\ell} \;, \tag{3.74}$$

where the coefficients $N_{ij}^{k} \in \mathbb{C}$ are given by

$$N_{ij}^{k} \;=\; \frac{1}{|G_k|} \cdot \sum_{p=1}^{n} \frac{1}{d_p} \cdot \chi_p(g_i) \cdot \chi_p(g_j) \cdot \chi_p^{*}(g_k) \;. \tag{3.75}$$

By plugging (3.71) into (3.63) and (3.65), we may then compute the minimal central idempotents in the tube algebra to be

$$e_\rho \;=\; \frac{\dim(\rho)}{|G|} \cdot \sum_{g\in G} \chi_\rho^{*}(g) \cdot e_g \;, \tag{3.76}$$

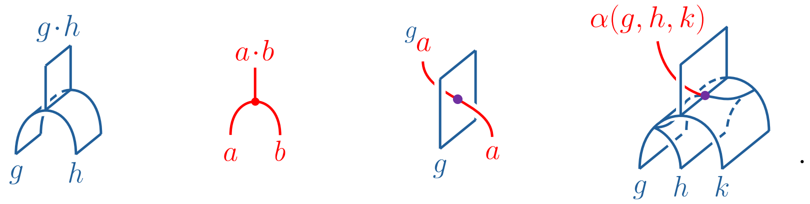
which reproduces the well-known formula for minimal central idempotents in the group algebra $\mathbb{C}[G]$ labelled by irreducible representations ρ of G [115].

3.2.3.2 2-Group Symmetry

As another example, let us consider a finite 2-group symmetry [116–137], by which we mean a quadruple $\mathcal{G} = (G, A, \triangleright, \alpha)$ consisting of the following pieces of data:

1. a finite 0-form symmetry group G,
2. a finite abelian 1-form symmetry group A,
3. a group action[8] $\triangleright : G \to \mathrm{Aut}(A)$,
4. a Postnikov class representative $\alpha \in Z^3_{\triangleright}(G, A)$.

We will often write $\mathcal{G} = A[1] \rtimes_\alpha G$ for a 2-group specified by the above data. Physically, the above means that, in addition to surface defects labelled by group elements $g \in G$, we also have line defects labelled by $a \in A$ which interact with the former via the group action $\triangleright$ and the Postnikov class α:

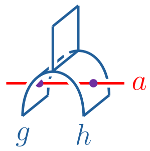


(3.77)

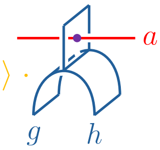
Furthermore, we assume that G and A have a mixed 't Hooft anomaly in the form of a twisted 2-cocycle $\lambda \in Z^2_{\triangleright}(G, A^\vee)$ on G with values in the Pontryagin dual group

[8] We will often abbreviate the action of a group element $g \in G$ on $a \in A$ by $g \triangleright a =: {}^{g}a$.

$A^\vee = \mathrm{Hom}(A, U(1))$ (equipped with the dual G-action), which captures the result of moving a 1-form defect $a \in A$ across the junction of two 0-form defects $g, h \in G$ [120]:



$$= \quad \langle \lambda(g,h), {}^{gh}a \rangle \cdot \tag{3.78}$$



We denote the corresponding symmetry 2-category by $\mathcal{C} = 2\mathrm{Hilb}^{\lambda}_{\mathcal{G}}$ in what follows. The tube algebra associated to $\mathcal{C}$ is the $|G|\cdot|A|$-dimensional algebra with basis vectors $\langle {}^{g}a \overset{g}{\square} a \rangle$ (where $g \in G$, $a \in A$) that multiply according to

$$\langle {}^{g}a \overset{g}{\square} a \rangle \circ \langle {}^{h}b \overset{h}{\square} b \rangle \; = \; \delta_{a,{}^{h}b} \cdot \varepsilon_b(\lambda)(g,h) \cdot \langle {}^{gh}b \overset{gh}{\square} b \rangle \, , \tag{3.79}$$

where we defined the multiplicative phase

$$\varepsilon_a(\lambda)(g,h) \; := \; \langle \lambda(g,h), {}^{gh}a \rangle \; . \tag{3.80}$$

As a result of the cocycle condition obeyed by λ, it satisfies

$$(d\varepsilon)_a(g,h,k) \; := \; \frac{\varepsilon_a(h,k) \cdot \varepsilon_a(g,hk)}{\varepsilon_a(gh,k) \cdot \varepsilon_{({}^{k}a)}(g,h)} \; = \; 1 \tag{3.81}$$

(where we dropped the notational dependence of ε on λ to improve readability), which ensures that the algebra multiplication in (3.79) is associative. Using (3.33), the $*$-structure on the tube algebra can be computed to be [4]

$$\langle {}^{g}a \overset{g}{\square} a \rangle^* \; = \; \mu_a(g) \cdot \langle a \overset{g^{-1}}{\square} {}^{g}a \rangle \, , \tag{3.82}$$

where we defined the multiplicative phase

$$\mu_a(g) \; := \; \varepsilon_a^*(g^{-1}, g) \; . \tag{3.83}$$

As a consequence of (3.81), it satisfies

$$\mu_a(g^{-1}) \; = \; \mu_{(a^g)}(g) \qquad \text{and} \qquad d\mu \; = \; \widehat{\varepsilon}/\varepsilon \, , \tag{3.84}$$

where we defined the dual 2-cocycle

$$\widehat{\varepsilon}_a(g,h) \; := \; \varepsilon^*_{({}^{gh}a)}(h^{-1}, g^{-1}) \; . \tag{3.85}$$

This ensures that the $*$-structure (3.82) is involutory and compatible with the algebra multiplication. If $\lambda = 1$, the above algebra reduces to the groupoid algebra $\mathbb{C}[A//G]$ associated to the group action of G on A.

In order to classify the irreducible tube representations of $\mathcal{C} = 2\mathrm{Hilb}_G^\lambda$, we note that, as a consequence of the delta-function appearing in (3.79), any such $\mathcal{F}$ will decompose into subrepresentations supported on twisted sectors $\mathcal{H}_b$ labelled by elements $b \in [a]$ in the G-orbit of some fixed $a \in A$. If we furthermore restrict to linking with symmetry defects $g \in G$ that lie in the *stabiliser* $G_a := \{g \in G \mid {}^g a = a\}$ of a, then

$$\rho(g) \;:=\; \mathcal{F}\Big(\langle \overset{a}{-}\overset{g}{\boxdot}\overset{a}{-}\rangle\Big) \tag{3.86}$$

defines a projective representation of G_a on the Hilbert space $\mathcal{V} := \mathcal{H}_a$ with projective 2-cocycle $\varepsilon_a(\lambda) \in Z^2(G_a, U(1))$. Conversely, given a pair (a, ρ) consisting of

1. a representative $a \in A$ of a G-orbits $[a] \in A/G$,
2. an irreducible $\varepsilon_a(\lambda)$-projective representation ρ of G_a,

we can construct an associated tube representation $\mathcal{F}_{(a,\rho)}$ via induction [4]: To this end, fix for each $b \in [a]$ in the G-orbit $[a] \subset A$ a representative $r_b \in G$ such that ${}^{(r_b)}b = a$ (with $r_a := 1$). Using these, we can define

$$g_b \;:=\; r_{({}^g b)} \cdot g \cdot r_b^{-1} \;\in\; G_a \tag{3.87}$$

for all $g \in G$ and $b \in [a]$. If we denote by $\mathcal{V}$ the Hilbert space underlying the projective representation ρ of G_a, then $\mathcal{F}_{(a,\rho)}$ acts on the twisted sectors

$$\mathcal{H}_b \;=\; \begin{cases} \mathcal{V} & \text{if } b \in [a] \\ 0 & \text{otherwise} \end{cases} \tag{3.88}$$

via the non-trivial induced tube action

$$\mathcal{F}_{(a,\rho)}\Big(\langle \overset{{}^g b}{-}\overset{g}{\boxdot}\overset{b}{-}\rangle\Big) \;:=\; \kappa_b(g) \cdot \rho(g_b)\;, \tag{3.89}$$

where we defined the multiplicative phases

$$\kappa_b(g) \;:=\; \frac{\varepsilon_b(r_{({}^g b)}, g)}{\varepsilon_b(g_b, r_b)}\;. \tag{3.90}$$

As a consequence of (3.81), they satisfy

$$\frac{\kappa_b(h) \cdot \kappa_{({}^h b)}(g)}{\kappa_b(gh)} \;=\; \frac{\varepsilon_b(g, h)}{\varepsilon_a(g_{({}^h b)}, h_b)}\;, \tag{3.91}$$

which ensures that $\mathcal{F}_{(a,\rho)}$ respects the algebra multiplication (3.79). One can check that $\mathcal{F}_{(a,\rho)}$ is a $*$-representation of the tube algebra if and only if ρ is a unitary projective representation of G_a.

All in all, we conclude that the category of $*$-representations of the tube algebra of $\mathcal{C} = 2\mathrm{Hilb}_{\mathcal{G}}^{\lambda}$ admits a direct sum decomposition

$$\mathrm{Rep}^*(\mathrm{Tube}(2\mathrm{Hilb}_{\mathcal{G}}^{\lambda})) \;\cong\; \boxplus_{[a]\,\in\,A/G} \mathrm{Rep}^{\varepsilon_a(\lambda)}(G_a)\ . \tag{3.92}$$

In particular, using (3.52), this yields a classification of simple objects in the loop space $\Omega\mathcal{Z}(2\mathrm{Hilb}_{\mathcal{G}}^{\lambda})$ of the Drinfeld centre in terms of pairs (a,ρ) as above.

We now aim to rederive this result using generalised S-matrices. To do this, we exploit the fact that the Drinfeld centre is gauge-invariant, i.e. $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{C}')$ if $\mathcal{C}'$ is obtained by gauging a discrete subsymmetry of $\mathcal{C}$ [138]. In particular, if $\mathcal{C} = 2\mathrm{Hilb}_{\mathcal{G}}^{\lambda}$, we can gauge the 1-form symmetry A to obtain a pure 0-form symmetry $\widehat{G}$ that is given by the group extension [5, 6, 57–59, 139]

$$\widehat{G} \;:=\; A^{\vee} \rtimes_{\lambda} G\ , \tag{3.93}$$

which as a set is $A^{\vee} \times G$ with group multiplication defined by

$$(\mu, g)\cdot(\nu, h) \;:=\; \big(\mu \cdot {}^{g}\nu \cdot \lambda(g,h),\, g\cdot h\big)\ . \tag{3.94}$$

This symmetry then has an 't Hooft anomaly that is parameterised by the 4-cocycle $\langle\,.\,,\alpha\rangle \in Z^4(\widehat{G}, U(1))$ defined by

$$\langle\,.\,,\alpha\rangle\,\big[(\mu,g),(\nu,h),(\varphi,k),(\psi,\ell)\big] \;:=\; \big\langle {}^{ghk}\psi,\, \alpha(g,h,k)\big\rangle\ . \tag{3.95}$$

As a result, we can use the discussion from the previous subsection to access the Drinfeld centre $\mathcal{Z}(2\mathrm{Hilb}_{\mathcal{G}}^{\lambda}) \cong \mathcal{Z}\big(2\mathrm{Hilb}_{\widehat{G}}^{\langle\,.\,,\alpha\rangle}\big)$. Concretely, its connected components and loop space can be described as follows:

- The connected components of $\mathcal{Z}(2\mathrm{Hilb}_{\mathcal{G}}^{\lambda})$ are in one-to-one correspondence with the conjugacy classes of $\widehat{G} = A^{\vee} \rtimes_{\lambda} G$. To describe the latter, we note that

  $$^{(\mu,g)}(\chi, x) \;=\; \big({}^{g}\chi \cdot (\mu/{}^{({}^{g}x)}\mu)\cdot \tau_x(g)\,,\, {}^{g}x\big) \tag{3.96}$$

  for any $(\mu,g),(\chi,x) \in \widehat{G}$, where we defined the *transgression* of λ by

  $$\tau_x(\lambda)(g) \;:=\; \frac{\lambda(g,x)}{\lambda({}^{g}x, g)}\ . \tag{3.97}$$

  As a consequence of the twisted 2-cocycle condition obeyed by λ, it satisfies

  $$\frac{{}^{g}[\tau_x(h)]\cdot \tau_{({}^{h}x)}(g)}{\tau_x(gh)} \;=\; \frac{\lambda(g,h)}{^{({}^{gh}x)}\lambda(g,h)}\ , \tag{3.98}$$

where we dropped the notational dependence of τ on λ for better readability. Upon defining $A_x := \{a \in A \,|\, {}^x a = a\}$ and using the canonical identification

$$\frac{A^\vee}{\{\mu/{}^x\mu \mid \mu \in A^\vee\}} \;\cong\; (A_x)^\vee \;, \tag{3.99}$$

this shows that we can label the connected components of $\mathcal{Z}(2\mathrm{Hilb}_G^\lambda)$ by equivalence classes of pairs (x, χ) consisting of a group element $x \in G$ and a character $\chi \in (A_x)^\vee$, where two such pairs (x, χ) and (x', χ') are considered equivalent if there exists a $g \in G$ such that

$$x' \;=\; {}^g x \;, \qquad \chi' \;=\; {}^g\chi \cdot \tau_x(g) \;. \tag{3.100}$$

We will denote the equivalence class of (x, χ) by $[x, \chi]$ in what follows.

- The loop space of $\mathcal{Z}(2\mathrm{Hilb}_G^\lambda)$ corresponds to the set of isomorphism classes of irreducible representations of $\widehat{G} = A^\vee \rtimes_\lambda G$. It is a standard result that the latter can be labelled by pairs (a, ρ) consisting of

  1. a group element $a \in A$, viewed as a character on $A^\vee$,
  2. an irreducible representation ρ of the stabiliser G_a of a with projective 2-cocycle $\langle \lambda, a\rangle \in Z^2(G_a, U(1))$.

  The corresponding irreducible representation $\widehat{\rho}$ of $\widehat{G}$ is given by the induction

$$\widehat{\rho} \;=\; \mathrm{Ind}_{\widehat{G}_a}^{\widehat{G}}(a \otimes \rho) \;, \tag{3.101}$$

  where we set $\widehat{G}_a := A^\vee \rtimes_\lambda G_a$. Two such pairs (a, ρ) and (a', ρ') are considered equivalent if $\widehat{\rho}$ and $\widehat{\rho}'$ are equivalent as representations of $\widehat{G}$. More concretely, (a, ρ) and (a', ρ') are equivalent if there exists a $g \in G$ such that

$$a' \;=\; {}^g a \;, \qquad \rho' \;\cong\; \langle \sigma_g, {}^g a\rangle \otimes {}^g\rho \;, \tag{3.102}$$

  where we defined the multiplicative factor

$$\sigma_g(\lambda)(h) \;:=\; \frac{\lambda(h, g)}{\lambda(g, h^g)} \;. \tag{3.103}$$

  As a result of the twisted 2-cocycle condition for λ, it obeys

$$\frac{{}^h[\sigma_g(k)] \cdot \sigma_g(h)}{\sigma_g(hk)} \;=\; \frac{\lambda(h, k)}{{}^g\lambda(h^g, k^g)} \;, \tag{3.104}$$

  which ensures that (3.102) defines an equivalence relation. We will denote the equivalence class of (a, ρ) by $[a, \rho]$ in what follows.

In terms of the above data, the S-matrix associated to $\mathcal{Z}(2\mathrm{Hilb}_{\mathcal{G}}^{\lambda})$ is the square matrix that is indexed by equivalence classes $[x,\chi]$ and $[a,\rho]$ with entries

$$\mathcal{S}_{[x,\chi],[a,\rho]} \;=\; \frac{\mathrm{Tr}_{\hat{\rho}}(\widehat{x})}{\dim(\hat{\rho})}\;, \tag{3.105}$$

where $\widehat{\rho} = \mathrm{Ind}_{\widehat{G}_a}^{\widehat{G}}(a \otimes \rho) \in \mathrm{Rep}(\widehat{G})$ and $\widehat{x} = (\chi, x) \in \widehat{G}$. Using the character formula for induced representations

$$\mathrm{Tr}_{\hat{\rho}}(\widehat{x}) \;=\; \frac{1}{|\widehat{G}_a|} \cdot \sum_{\substack{\hat{g}\in\widehat{G}:\\ {}^{\hat{g}}\widehat{x}\in\widehat{G}_a}} \mathrm{Tr}_{a\otimes\rho}({}^{\hat{g}}\widehat{x}) \tag{3.106}$$

as well as the character orthogonality relation

$$\frac{1}{|A|} \cdot \sum_{\mu\in A^{\vee}} \mu^{*}(b)\cdot\mu(c) \;=\; \delta_{b,c}\;, \tag{3.107}$$

we can then compute that the S-matrix can be expressed as

$$\mathcal{S}_{[x,\chi],[a,\rho]} \;=\; \frac{1}{\dim(\rho)\cdot|G|} \cdot \sum_{\substack{g\in G:\\ {}^{g}x\in G_a}} \langle \tau_x(g), a\rangle \cdot \mathrm{Tr}_{\rho}({}^{g}x) \cdot \chi(a^{g})\;. \tag{3.108}$$

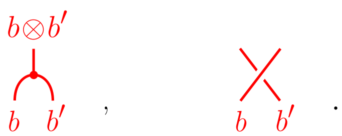
From this, we find the minimal central idempotents in $\mathrm{Tube}(2\mathrm{Hilb}_{\mathcal{G}}^{\lambda})$ to be [4]

$$e_{(a,\rho)} \;=\; \frac{\dim(\rho)}{|G_a|^2} \cdot \sum_{\substack{x,g\in G:\\ {}^{g}x\in G_a}} \langle \tau_x(g), a\rangle^{*} \cdot \mathrm{Tr}_{\rho}^{*}({}^{g}x) \cdot \langle \overset{{}^{g}a}{\text{—}} \overset{x}{\square} \overset{{}^{g}a}{\text{—}} \rangle\;. \tag{3.109}$$

In particular, we see that they are labelled by pairs (a,ρ) as before, which reproduces the decomposition (3.92) of the category of tube representations.

3.2.3.3 Braiding Lines

As a last example, we consider a theory that only has a 1-form symmetry described by some (unitary) braided fusion (1-)category $\mathcal{B}$. Objects $b, b' \in \mathcal{B}$ of the latter correspond to topological line defects that fuse and braid in three-dimensional spacetime:

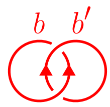


(3.110)

In particular, the fusion of line defects need not be invertible. The corresponding symmetry 2-category is given by Karoubi completion $\mathcal{C} = \Sigma\mathcal{B} := \mathrm{Kar}(B\mathcal{B})$ of the delooping[9] of $\mathcal{B}$ [140]. Physically, this means that the surface defects in $\mathcal{C}$ are all obtained by condensing suitable line defects in $\mathcal{B}$ on the trivial surface. In particular, this means that $\mathcal{C}$ is *connected*[10], i.e. $\pi_0(\mathcal{C}) = 1$. Consequently, using (3.30), the only non-trivial information contained in the tube category $\mathrm{T}\mathcal{C}$ is which of its objects are *zero objects* (i.e. have an endomorphism space isomorphic to the zero vector space). To answer this, we note that for any $b, b' \in \mathcal{B}$ we have

(3.111)

where $\sim$ denotes the equivalence relation defined in (3.22) and

$$\mathcal{S}_{bb'} \;:=\; \frac{1}{d_b \cdot d_{b'}} \cdot$$ 

(3.112)

denotes the (normalised) S-matrix associated to $\mathcal{B}$ with $d_b := \dim(b)$. In particular,

$$(\mathcal{S}_{bb'} - 1) \cdot \mathrm{id}_b \;\sim\; 0\,, \tag{3.113}$$

which implies $\mathrm{id}_b \sim 0$ unless $\mathcal{S}_{bb'} = 1$ for all $b' \in \mathcal{B}$. Thus, we see that the non-zero objects in the tube category are precisely those that form the *Müger centre*

$$\mathcal{Z}_2(\mathcal{B}) \;:=\; \{ b \in \mathcal{B} \mid \mathcal{S}_{bb'} \;=\; 1 \;\text{ for all }\; b' \in \mathcal{B} \} \tag{3.114}$$

of $\mathcal{B}$. Since $\mathcal{Z}_2(\mathcal{B})$ is a finite semisimple category, the Yoneda embedding[11] gives a canonical equivalence [141]

$$\mathcal{Z}_2(\mathcal{B}) \;\cong\; [\mathcal{Z}_2(\mathcal{B}), \mathrm{Vect}] \;\equiv\; \mathrm{Rep}(\mathrm{T}\mathcal{C})\,, \tag{3.116}$$

which shows that the irreducible tube representations of $\mathcal{C} = \Sigma(\mathcal{B})$ are labelled by simple objects in the Müger centre of $\mathcal{B}$. Using (3.51), this reproduces the known characterisation of the loop space of $\mathcal{Z}(\Sigma(\mathcal{B}))$ [140]. Physically, this means that in order for a topological line defect $b \in \mathcal{B}$ to be able to end on a local operator, it has to braid trivially with all other line defects.

---

9 Given a monoidal 1-category $\mathcal{B}$, its *delooping* is the 2-category $B\mathcal{B}$ which has a single object $*$ with endomorphism category given by $\mathrm{End}_{B\mathcal{B}}(*) = \mathcal{B}$. One can show that if $\mathcal{B}$ is a braided fusion 1-category, then $\Sigma\mathcal{B} := \mathrm{Kar}(B\mathcal{B})$ is a fusion 2-category [66].

10 In fact, any connected fusion 2-category $\mathcal{C}$ is of the form $\mathcal{C} = \Sigma(\mathcal{B})$, where $\mathcal{B} \equiv \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ [66].

11 Given a linear category $\mathcal{D}$, the *Yoneda embedding* is the functor $\#\colon \mathcal{D} \to [\mathcal{D}, \mathrm{Vect}]$ that maps

$$D \in \mathcal{D} \;\mapsto\; \mathrm{Hom}_{\mathcal{D}}(-, D)\,. \tag{3.115}$$

It is a well known corollary of the Yoneda Lemma that $\#$ is fully faithful. If $\mathcal{D}$ is finite semisimple, then $\#$ is an equivalence [141].

3.3 2-Twisted Sectors

In this section, we discuss the action of the fusion 2-category symmetry $\mathcal{C}$ on 2-twisted sectors, i.e. line operators attached to topological surface defects. We construct the corresponding tube 2-category and describe how its irreducible 2-representations can be classified using a higher-dimensional analogue of the sandwich construction. We discuss anomalous 2-group symmetries as an example.

3.3.1 2-Vector Spaces

Given that local operators in a quantum field theory generically form a vector space (or a Hilbert space via the operator-state map), it is natural to ask what type of mathematical structure describes collections of extended line operators L. In general, we expect the latter to form (at least) a category $\mathcal{L}$, whose

- objects are distinct line operators L in the theory,
- morphisms between objects L and K are topological local operators v that can sit at the junction between the corresponding line operators as illustrated in Figure 27.

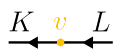
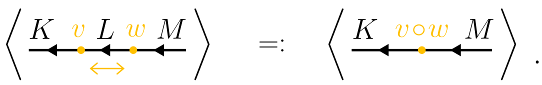


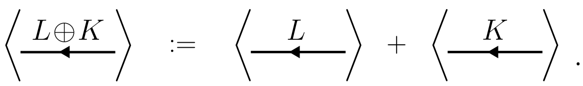
Figure 27

The composition of morphisms in $\mathcal{L}$ is given by the collision of topological junctions inside correlation functions, i.e.



(3.117)

We assume that $\text{Hom}_{\mathcal{L}}(L, K)$ (being a space of local operators) is a complex vector space for all L and K such that the composition operation (3.117) is bilinear. This turns $\mathcal{L}$ into a linear category. Furthermore, we will assume that $\mathcal{L}$ is additive in the sense that we can form direct sums of line operators L and K, which corresponds to adding their respective correlation functions:



(3.118)

Lastly, for the purposes of this thesis, we will also assume that $\mathcal{L}$ is finite semisimple. Concretely, this means that we can decompose any line operator in $\mathcal{L}$ into a direct sum of finitely many simple lines L_i, which are such that $\text{Hom}_{\mathcal{L}}(L_i, L_j) = \delta_{ij}\,\mathbb{C}$. In analogy to the case of local operators, we will call such $\mathcal{L}$ a (finite-dimensional) *2-vector space*, which leads to the following definition [142]:

Definition: The 2-category 2Vect of 2-vector spaces comprises finite semisimple linear 1-categories, additive linear functors and natural transformations between them.

We can obtain a more concrete description of the 2-category 2Vect by noting that, up to equivalence, every $\mathcal{L} \in$ 2Vect is completely determined by its number $n \in \mathbb{N}$ of simple objects L_i (and is hence equivalent to $\text{Vect}^{\boxplus n}$). Any additive linear functor $F : \mathcal{L} \to \mathcal{L}'$ between 2-vector spaces may then written as

$$F(L_j) \;=\; \bigoplus_{i=1}^{n'} V_{ij} \odot L'_i \tag{3.119}$$

for some collection of vector spaces $V_{ij} \in$ Vect, which can be identified with the morphism spaces $V_{ij} \cong \text{Hom}_{\mathcal{L}'}(L'_i, F(L_j))$. Similarly, we can describe any natural transformation $\eta : F \Rightarrow \widetilde{F}$ between two additive linear functors $F, \widetilde{F} : \mathcal{L} \to \mathcal{L}'$ by a collection of linear maps $\varphi_{ij} : V_{ij} \to \widetilde{V}_{ij}$ that determine the component morphisms

$$\eta_{L_j} \;=\; \bigoplus_{i=1}^{n'} \varphi_{ij} \odot L'_i \;. \tag{3.120}$$

As a result, we can think of the 2-category of finite-dimensional 2-vector spaces as capturing 'matrices of vector spaces', which leads to the following model of 2Vect due to Kapranov and Voevodsky [143, 144]:

Proposition: The 2-category 2Vect of finite-dimensional 2-vector spaces can be modelled by the 2-category Mat(Vect) whose

- objects are non-negative integers $n \in \mathbb{N}$,
- 1-morphisms between objects m and n are given by $(n \times m)$-matrices V with vector space entries $V_{ij} \in$ Vect, with composition given by matrix multiplication using tensor products and direct sums of vector spaces,
- 2-morphisms between 1-morphisms V and W are given by $(n \times m)$-matrices φ whose entries are linear maps $\varphi_{ij} : V_{ij} \to W_{ij}$ between the vector space entries of V and W. The vertical composition of 2-morphisms f and g is given by entry-wise composition of linear maps. Their horizontal composition is given by matrix multiplication using tensor products and direct sums of linear maps.

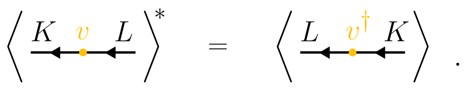
The above shows that we can view 2Vect as a 'coefficient system' for arbitrary fusion 2-categories $\mathcal{C}$ in the sense that there exists a 'multiplication 2-functor'

$$\boxdot : \; \text{2Vect} \boxtimes \mathcal{C} \;\to\; \mathcal{C} \tag{3.121}$$

that acts on objects and 1-morphisms via

$$\begin{aligned} n \boxtimes A \;&\mapsto\; A^{\boxplus n} \;, \\ (m \xrightarrow{V} n) \boxtimes (A \xrightarrow{\gamma} B) \;&\mapsto\; \left(A^{\boxplus m} \xrightarrow{\boxplus_{ij} V_{ij} \odot \gamma} B^{\boxplus n}\right) . \end{aligned} \tag{3.122}$$

Given a category $\mathcal{L}$ of line operators, we often assume the latter to be compatible with reflection positivity in the sense that for each topological junction $v \in \mathrm{Hom}_{\mathcal{L}}(L, K)$ there exists an *adjoint* $v^\dagger \in \mathrm{Hom}_{\mathcal{L}}(K, L)$ that captures the reflection of v associated to complex conjugating correlation functions, i.e.



(3.123)

Using the operator-state map, we can further endow the vector spaces $\mathrm{Hom}_{\mathcal{L}}(L, K)$ with an inner product structure $\langle . | . \rangle$, which we assume to be compatible with adjoints in an appropriate sense. This leads to the notion of a *2-Hilbert space* [145, 146] (for a more recent discussion of higher Hilbert spaces we refer the reader to [147]):

Definition: A *2-Hilbert space* is an abelian †-category $\mathcal{L}$ enriched over Hilb such that for all morphisms u, v, w in $\mathcal{L}$ we have

$$\langle u \circ v \,|\, w \rangle \;=\; \langle v \,|\, u^\dagger \circ w \rangle \;=\; \langle u \,|\, w \circ v^\dagger \rangle \tag{3.124}$$

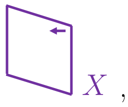
whenever both sides of the equation are defined. We denote by 2Hilb the 2-category of all 2-Hilbert spaces, additive linear †-functors and natural transformations between them. This is itself a †-category upon defining the adjoint of a natural transformation $\eta : F \Rightarrow \widetilde{F}$ between †-functors $F, \widetilde{F} : \mathcal{L} \to \mathcal{L}'$ via $(\eta^\dagger)_L := (\eta_L)^{\dagger'}$ for all $L \in \mathcal{L}$.

The above definition implies that every 2-Hilbert space is in fact semisimple [145], so that we can decompose any $L \in \mathcal{L}$ into a finite direct sum of simple lines L_i with $\mathrm{Hom}_{\mathcal{L}}(L_i, L_j) = \delta_{ij}\, \mathbb{C}_{\lambda_i}$, where $\mathbb{C}_\lambda$ with $\lambda > 0$ denotes $\mathbb{C}$ as a $*$-algebra with inner product given by $\langle a, b \rangle = \lambda \cdot a^* \cdot b$. In particular, if we restrict attention to finite-dimensional 2-Hilbert spaces, we see that any such $\mathcal{L}$ can be characterised by its number $n \in \mathbb{N}$ of simple objects together with a vector $\vec{\lambda} \in \mathbb{R}^n_{>0}$, whose entries we refer to as the *Euler terms* associated to $\mathcal{L}$. Since the latter only ever appear as overall multiplicative factors (and in particular do not interact with symmetry defects), we will henceforth omit them from our discussion entirely. This then allows us to model the 2-category of finite-dimensional 2-Hilbert spaces by 'matrices of Hilbert spaces' that form the 2-category Mat(Hilb).

3.3.2 Tube 2-Category

The *tube 2-category* associated to a fusion 2-category $\mathcal{C}$ captures the possible linking configurations of twisted sector line operators in three dimensions with symmetry defects in $\mathcal{C}$. Concretely, following [2], we define the tube 2-category 2-T$\mathcal{C}$ associated to $\mathcal{C}$ to be the additive linear 2-category whose

- objects are given by objects $X \in \mathcal{C}$, i.e.

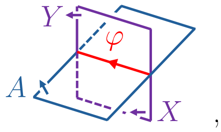


(3.125)

- morphisms between objects $X, Y \in \mathcal{C}$ are given by objects of the 1-category

$$\boxplus_{A \in \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(A \otimes X, Y \otimes A) \tag{3.126}$$

(where the sum runs over *all* objects $A \in \mathcal{C}$) of intersection interfaces

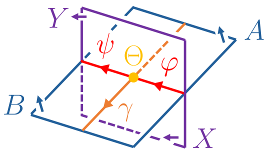


(3.127)

- 2-morphisms between 1-morphisms $\varphi : A \otimes X \to Y \otimes A$ and $\psi : B \otimes X \to Y \otimes B$ form the quotient vector space

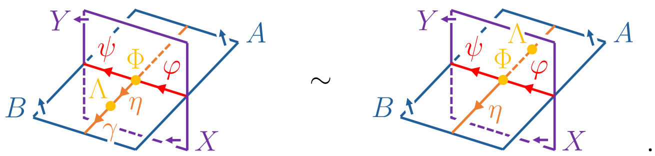
$$2\mathrm{Hom}_{2\text{-}\mathrm{T}\mathcal{C}}(\varphi, \psi) \;:=\; \bigoplus_{\gamma : A \to B} 2\mathrm{Hom}_{\mathcal{C}}(\gamma \circ \varphi, \psi \circ \gamma) \Big/ \sim \tag{3.128}$$

(where the sum runs over *all* $\gamma \in \mathrm{Hom}_{\mathcal{C}}(A, B)$) of intersection 2-morphisms



(3.129)

in $\mathcal{C}$ subjected to the equivalence relation that is generated by

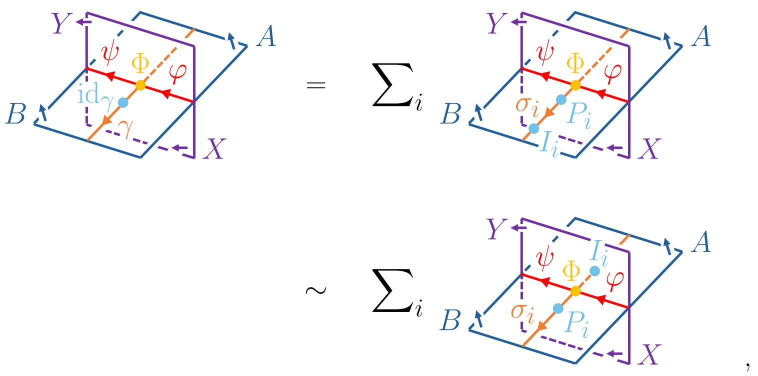


(3.130)

Physically, the equivalence relation (3.130) means that we should think of the symmetry defects A and B as being placed on a cylinder, on which the 1-morphism γ is wrapped around the compact S^1-direction. Mathematically, it renders the 2-morphism spaces in (3.128) finite-dimensional. Concretely, let $\gamma : A \to B$ be an arbitrary 1-morphism in $\mathcal{C}$ and consider a decomposition $\gamma \cong \bigoplus_i \sigma_i$ of γ into a finite number of simples σ_i (possibly with multiplicities). Let us denote by

$$I_i : \; \sigma_i \Rightarrow \gamma \qquad \text{and} \qquad P_i : \; \gamma \Rightarrow \sigma_i \tag{3.131}$$

the associated inclusion and projection 2-morphisms obeying the completeness relation $\sum_i I_i \circ P_i = \mathrm{id}_\gamma$. Upon inserting the latter into (3.129), we then obtain that

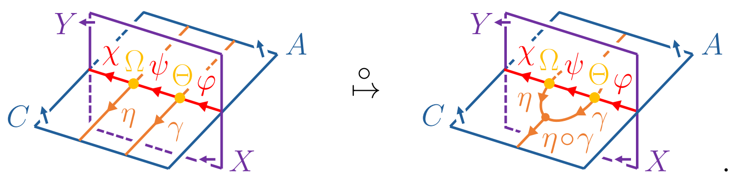


(3.132)

which shows that, as a vector space, the 2-morphism space (3.128) decomposes as

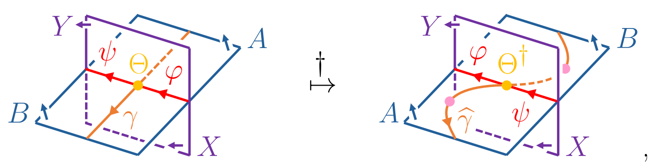
$$2\mathrm{Hom}_{2\text{-}\mathrm{T}\mathcal{C}}(\varphi,\psi) \;\cong\; \bigoplus_\sigma 2\mathrm{Hom}_{\mathcal{C}}(\sigma\circ\varphi,\psi\circ\sigma)\;, \tag{3.133}$$

where σ runs over a set of fixed representatives of isomorphism classes of simple objects in $\mathrm{Hom}_{\mathcal{C}}(A,B)$. Since the latter is finite semisimple by assumption, we see that the 2-morphism space (3.133) is finite-dimensional as claimed. The (vertical) composition of 2-morphisms is induced by



(3.134)

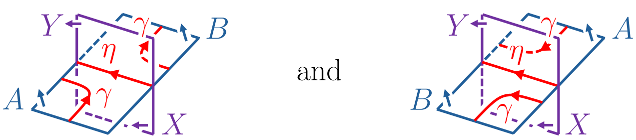
Furthermore, the tube 2-category possesses a natural †-structure that is induced by



(3.135)

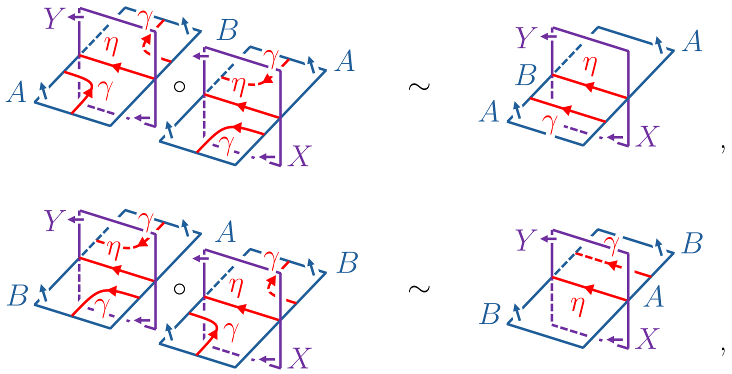
where we left the labelling of unit and conunit 2-morphisms implicit. This turns 2-T$\mathcal{C}$ into a †-2-category in the sense of section 3.1.

We can obtain similar constraints on 1-morphisms in the tube 2-category by noting that, given 1-morphisms $\gamma : A \to B$ and $\eta : B\otimes X \to Y\otimes A$ in $\mathcal{C}$, the 2-morphisms



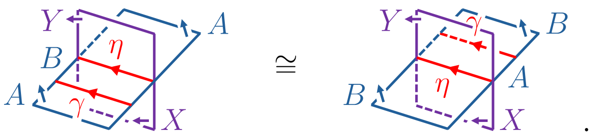
(3.136)

(where we omitted any unit and counit 2-morphisms) obey



(3.137)

and hence establish a 2-isomorphism of 1-morphisms in 2-T$\mathcal{C}$:

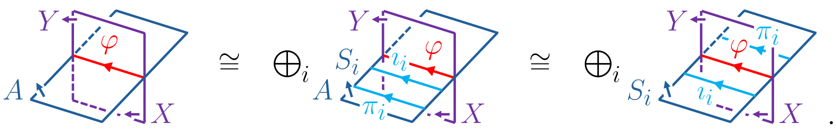


(3.138)

In particular, upon decomposing a generic object $A \in \mathcal{C}$ into its simple components $A \cong \boxplus_i S_i$ (possibly with multiplicities) and denoting by

$$\imath_i : \; S_i \to A \qquad \text{and} \qquad \pi_i : \; A \to S_i \tag{3.139}$$

the associated inclusion and projection 1-morphisms obeying $\bigoplus_i \imath_i \circ \pi_i \cong \mathrm{id}_A$, we have

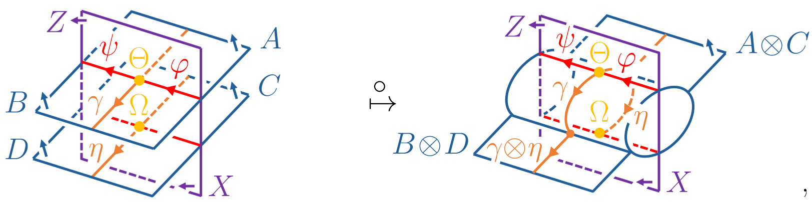


(3.140)

If we denote 1-morphisms of the form (3.127) in the tube 2-category by

$$\langle \overset{A}{Y \sqcap X}_{\varphi} \rangle \in \mathrm{Hom}_{2\text{-T}\mathcal{C}}(X, Y) \,, \tag{3.141}$$

then we can write (3.140) schematically as $\langle \overset{A}{Y \sqcap X}_{\varphi} \rangle \cong \bigoplus_i \langle \overset{S_i}{Y \sqcap X}_{\pi_i \circ \varphi \circ \imath_i} \rangle$. The composition of 1-morphisms (and horizontal composition of 2-morphisms) is given by



(3.142)

which we denote schematically by $\langle \overset{A}{Z \sqcap Y}_{\varphi} \rangle \circ \langle \overset{B}{Y \sqcap X}_{\psi} \rangle = \langle \overset{A \otimes B}{Z \sqcap X}_{\varphi \circ \psi} \rangle$.

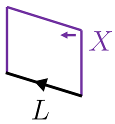
3.3.3 Tube 2-Representations

Given a three-dimensional quantum field theory with fusion 2-category symmetry $\mathcal{C}$, it was proposed in [2] that twisted sector line operators transform in 2-representations of the tube 2-category associated to $\mathcal{C}$, which are additive linear 2-functors

$$\mathcal{F}: \text{2-T}\mathcal{C} \to 2\text{Vect} \tag{3.143}$$

from 2-T$\mathcal{C}$ into the 2-category of 2-vector spaces and which we call *tube 2-representations* in what follows. Concretely, any such tube 2-representation $\mathcal{F}$ assigns

- to each object $X \in \mathcal{C}$ a 2-vector space $\mathcal{L}_X := \mathcal{F}(X)$, which describes the category of line operators L that can sit at the end of the surface defect X, i.e.

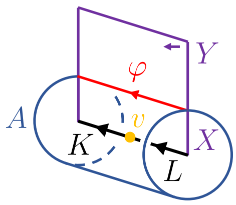


, (3.144)

- to each 1-morphism $\langle Y \xleftarrow[\varphi]{A} X \rangle \in \text{Hom}_{\text{2-T}\mathcal{C}}(X,Y)$ a functor

$$\mathcal{F}\Big(\langle Y \xleftarrow[\varphi]{A} X \rangle\Big): \mathcal{L}_X \to \mathcal{L}_Y \tag{3.145}$$

that describes how lines and junctions in $\mathcal{L}_X$ get mapped to lines and junctions in $\mathcal{L}_Y$ upon being wrapped with the symmetry defect A, i.e.

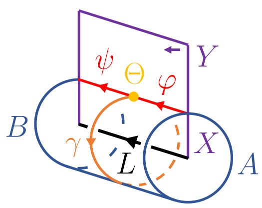


, (3.146)

- to each 2-morphism $\Theta \in 2\text{Hom}_{\text{2-T}\mathcal{C}}\Big(\langle Y \xleftarrow[\varphi]{A} X \rangle, \langle Y \xleftarrow[\psi]{B} X \rangle\Big)$ a natural transformation

$$\mathcal{F}(\Theta): \mathcal{F}\Big(\langle Y \xleftarrow[\varphi]{A} X \rangle\Big) \Rightarrow \mathcal{F}\Big(\langle Y \xleftarrow[\psi]{B} X \rangle\Big), \tag{3.147}$$

whose components capture the topological junction operators that result from shrinking the line defect $\gamma : A \to B$ associated to Θ (cf. (3.129)) to a point, i.e.



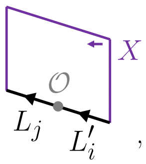
. (3.148)

An *intertwiner* between tube 2-representations $\mathcal{F}$ and $\mathcal{F}'$ is a 2-natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{F}'$ between the corresponding 2-functors. Concretely, any such η assigns

- to each object $X \in \mathcal{C}$ an additive linear functor

$$\eta_X : \mathcal{L}_X \to \mathcal{L}'_X \,, \tag{3.149}$$

  which, using (3.119), can be identified with a collection $\mathcal{H}_X$ of vector spaces $(\mathcal{H}_X)_{ij}$ that describe (possibly non-topological) local operators $\mathcal{O}$ sitting at the junction between the simple lines L'_i of $\mathcal{L}'_X$ and L_j of $\mathcal{L}_X$, i.e.

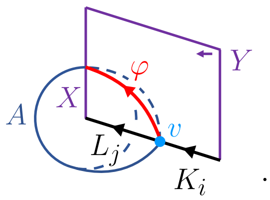


(3.150)

- to each 1-morphism $\langle {}_{Y}\overset{A}{\underset{\varphi}{\sqcap}}{}_{X}\rangle \in \mathrm{Hom}_{2\text{-}\mathrm{T}\mathcal{C}}(X,Y)$ a natural transformation

$$\eta_{\langle {}_{Y}\overset{A}{\underset{\varphi}{\sqcap}}{}_{X}\rangle} : \mathcal{F}'\Big(\langle {}_{Y}\overset{A}{\underset{\varphi}{\sqcap}}{}_{X}\rangle\Big) \circ \eta_X \Rightarrow \eta_Y \circ \mathcal{F}\Big(\langle {}_{Y}\overset{A}{\underset{\varphi}{\sqcap}}{}_{X}\rangle\Big) \tag{3.151}$$

  that can be interpreted as follows: Using (3.119), we can identify the functor $\mathcal{F}\Big(\langle {}_{Y}\overset{A}{\underset{\varphi}{\sqcap}}{}_{X}\rangle\Big)$ with a collection V of vector spaces V_{ij} that capture topological local operators v sitting at the junction between simple lines K_i in $\mathcal{L}_Y$ and simple lines L_j in $\mathcal{L}_X$ wrapped by the symmetry defect A, i.e.

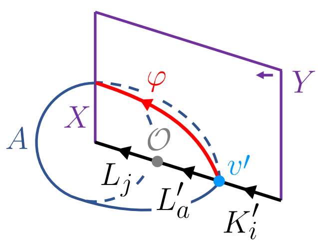


(3.152)

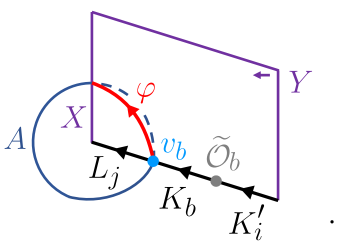
  Using (3.120), we can then identify (3.151) with a collection of linear maps

$$\bigoplus_a V'_{ia} \otimes (\mathcal{H}_X)_{aj} \to \bigoplus_b (\mathcal{H}_Y)_{ib} \otimes V_{bj} \tag{3.153}$$

  that describe how the local operators $\mathcal{O}$ are transformed upon being hit with the topological junction operators v, i.e.



$\mapsto \quad \bigoplus_b$



(3.154)

We denote the 2-category of tube 2-representations and intertwiners between them by

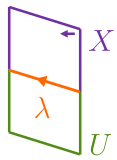
$$\mathrm{2Rep}(\text{2-T}\mathcal{C}) \;:=\; [\text{2-T}\mathcal{C}, \mathrm{2Vect}]\;. \tag{3.155}$$

As in the case of local operators, we can classify the irreducible tube 2-representations using an analogue of the sandwich construction, which yields an equivalence

$$\mathrm{2Rep}(\text{2-T}\mathcal{C}) \;\cong\; \mathcal{Z}(\mathcal{C})\;, \tag{3.156}$$

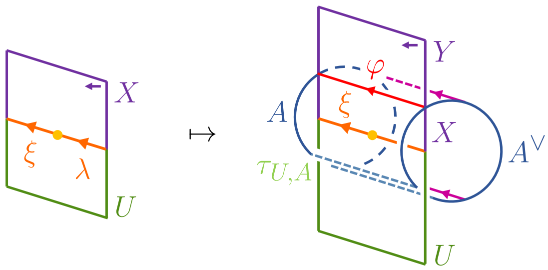
where $\mathcal{Z}(\mathcal{C})$ denotes the Drinfeld centre of $\mathcal{C}$ as defined in section 3.2.2. Concretely, given an object $z \in \mathcal{Z}(\mathcal{C})$ consisting of data $z = (U, \tau_{U,-}, \Lambda_{U,-,\cdot})$ (cf. (3.44)), we can construct an associated tube 2-representation $\mathcal{F}_z$ as follows [2]:

- To an object $X \in \mathcal{C}$, the 2-functor $\mathcal{F}_z$ assigns the 2-vector space (a.k.a. finite semisimple linear category) $\mathcal{L}_X := \mathrm{Hom}_{\mathcal{C}}(U, X)$ of line interfaces



$$\tag{3.157}$$

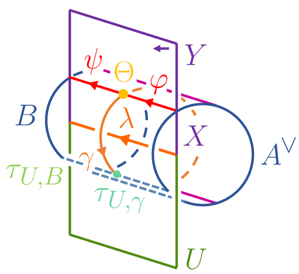
- To a 1-morphism $\langle {}^{Y}\!\overset{A}{\sqcup}{}^{X}_{\varphi} \rangle \in \mathrm{Hom}_{\text{2-T}\mathcal{C}}(X, Y)$, the 2-functor $\mathcal{F}_z$ assigns the functor from $\mathcal{L}_X$ to $\mathcal{L}_Y$ that sends interfaces $\lambda, \xi \in \mathrm{Hom}_{\mathcal{C}}(U, X)$ and their junctions to



$$\tag{3.158}$$

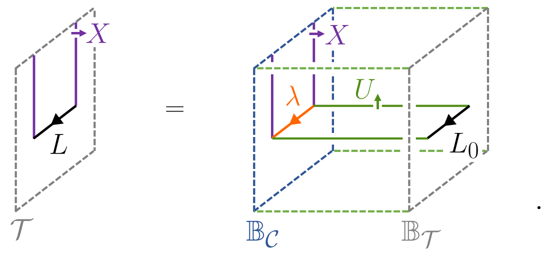
where we left the labelling of evaluation and coevaluation 1-morphisms implicit.

- To a 2-morphism $\Theta \in \mathrm{2Hom}_{\text{2-T}\mathcal{C}}\big(\langle {}^{Y}\!\overset{A}{\sqcup}{}^{X}_{\varphi} \rangle, \langle {}^{Y}\!\overset{B}{\sqcup}{}^{X}_{\psi} \rangle\big)$, the 2-functor $\mathcal{F}_z$ assigns the natural transformation between $\mathcal{F}_z\big(\langle {}^{Y}\!\overset{A}{\sqcup}{}^{X}_{\varphi} \rangle\big)$ and $\mathcal{F}_z\big(\langle {}^{Y}\!\overset{B}{\sqcup}{}^{X}_{\psi} \rangle\big)$ whose components are induced by the line defect $\gamma : A \to B$ associated to Θ (cf. (3.129)) via



$$\tag{3.159}$$

We can again visualise the above construction pictorially by viewing a three-dimensional theory $\mathcal{T}$ with generalised symmetry $\mathcal{C}$ as an interval compactification of the associated four-dimensional Symmetry TFT with topological and physical boundary conditions $\mathbb{B}_{\mathcal{C}}$ and $\mathbb{B}_{\mathcal{T}}$, respectively. The spectrum of twisted sector line operators L that transform in a given tube 2-representation $\mathcal{F}_z$ associated to some $z \in \mathcal{Z}(\mathcal{C})$ may then be viewed as the spectrum of line interfaces λ between surface defects $X \in \mathcal{C}$ on the left and the bulk surface z stretched between the two boundaries of the Symmetry TFT:



(3.160)

Here, L_0 denotes a fixed (and generically non-topological) line operator that terminates z on right boundary. The linking action of the tube 2-category on the operators L can then be computed using (3.159).

As before, we can use the operator-state map to endow spaces of local operators with an inner product structure, which we assume to be compatible with a given tube 2-representation $\mathcal{F}$ in the sense that the latter lifts to a $\dagger$-2-functor

$$\mathcal{F}: \text{ 2-T}\mathcal{C} \xrightarrow{\dagger} \text{2Hilb} , \tag{3.161}$$

called a *tube* $\dagger$*-2-representation* in what follows. We denote the category of all tube $\dagger$-2-representations and intertwiners between them by

$$\text{2Rep}^{\dagger}(\text{2-T}\mathcal{C}) \;:=\; [\text{2-T}\mathcal{C}, \text{2Hilb}]^{\dagger} \,. \tag{3.162}$$

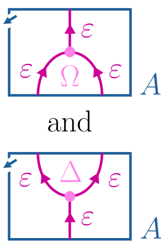
The sandwich construction then yields an equivalence [4]

$$\text{2Rep}^{\dagger}(\text{2-T}\mathcal{C}) \;\cong\; \mathcal{Z}^{\dagger}(\mathcal{C}) \,, \tag{3.163}$$

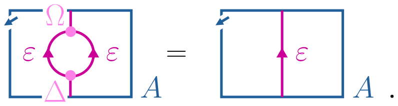
where $\mathcal{Z}^{\dagger}(\mathcal{C})$ denotes the unitary Drinfeld center of $\mathcal{C}$ as before.

A Note on Condensations

When discussing the action of the fusion 2-category $\mathcal{C}$ on twisted sector local operators in section 3.2, we observed that we could restrict ourselves to considering topological surface defects up to condensations (cf. equation (3.30)), since the latter become trivial when being placed on a 2-sphere. It is natural to ask to what extent this holds true when considering the action of surface defects in $\mathcal{C}$ on line operators. To this end, let $A \in \mathcal{C}$ be a surface defect and ε a condensation monad on A, i.e. an endomorphism $\varepsilon \in \mathrm{End}_{\mathcal{C}}(A)$ together with multiplication and comultiplication 2-morphisms



such that

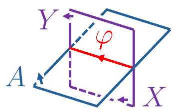
 (3.164)

Given a tube 2-representation $\mathcal{F} : 2\text{-T}\mathcal{C} \to 2\mathrm{Vect}$, we then have that

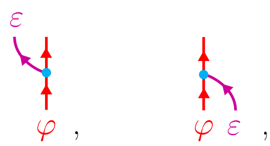
$$E := \mathcal{F}\Big(\langle\ \cdot\ \rangle\Big) \tag{3.165}$$

is a condensation monad on the 2-vector space $\mathcal{L}_A := \mathcal{F}(A)$ with associated multiplication and comultiplication $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Delta)$, respectively. Since the 2-category 2Vect of 2-vector spaces is condensation complete, there exists a (unique up to equivalence) condensation $\mathcal{L}_A \twoheadrightarrow \mathcal{L}_B$, whose associated condensation monad is given by E [66]. In particular, the 2-vector space $\mathcal{L}_B$ can be identified with the image of the surface defect B that is obtained by condensing ε on A (cf. equation (3.7)). Physically, this means that the category of twisted sector lines that $\mathcal{F}$ associates to the condensation defect B is determined by the image of the surface defect A and its condensation monads.

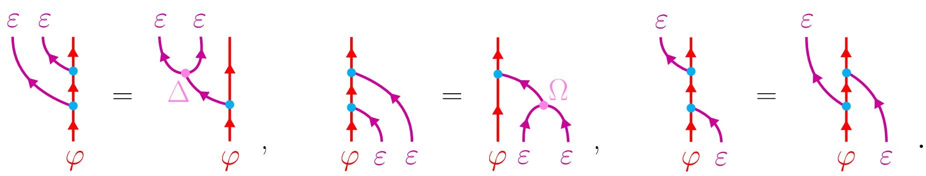
Similarly, we can try to understand the wrapping action of condensation defects using the condensation monad ε. Concretely, consider the 1-morphism

 (3.166)

in the tube 2-category and assume that the intersection interface φ is an ε-bimodule, meaning that it is equipped with left and right action 2-morphisms

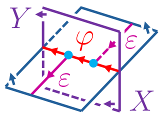
 (3.167)

that satisfy the compatibility conditions

 (3.168)

Physically, this means that φ descends to a well-defined intersection interface $\overline{\varphi}$ for the condensation defect B obtained from A via (3.7). Moreover, (3.168) together with (3.164) implies that the 2-endomorphism

$$e \;:=\; \mathcal{F}\Bigg(\quad \Bigg) \tag{3.169}$$



of $\mathcal{F}\big(\langle \tfrac{Y\,\overset{A}{\sqcap}\,X}{\varphi} \rangle\big) : \mathcal{L}_X \to \mathcal{L}_Y$ is an idempotent, i.e. $e^2 = e$. Since the morphism category $\mathrm{Hom}(\mathcal{L}_X, \mathcal{L}_Y)$ is idempotent complete in 2Vect, the idempotent e *splits*, meaning that there exists a (unique up to isomorphism) $D \in \mathrm{Hom}(\mathcal{L}_X, \mathcal{L}_Y)$ together with inclusion and projection 2-morphisms

$$I:\; D \;\Rightarrow\; \mathcal{F}\big(\langle \tfrac{Y\,\overset{A}{\sqcap}\,X}{\varphi} \rangle\big) \qquad \text{and} \qquad P:\; \mathcal{F}\big(\langle \tfrac{Y\,\overset{A}{\sqcap}\,X}{\varphi} \rangle\big) \;\Rightarrow\; D \tag{3.170}$$

such that $P \circ I = \mathrm{id}_D$ and $I \circ P = e$. In particular, we can identify D with the image of the 1-morphism $\langle \tfrac{Y\,\overset{B}{\sqcap}\,X}{\overline{\varphi}} \rangle$ under $\mathcal{F}$. Physically, this means that the wrapping action of the condensation defect B on twisted sector line operators is again determined by the wrapping action of A and its condensation monads.

3.3.4 Example

As an example, let us consider the case $\mathcal{C} = 2\mathrm{Hilb}^{\pi}_{\mathcal{G}}$ of a finite 2-group symmetry $\mathcal{G} = A[1] \rtimes_{\alpha} G$ with a pure 't Hooft anomaly for the 0-form component G that is specified by a (normalised) 4-cocycle $\pi \in Z^4(G, U(1))$ (cf. (3.66)). The associated tube 2-category can be described as follows [2]:

- Its objects are given by group elements $x \in G$.
- The 1-morphisms between $x, y \in G$ are non-zero if and only if $y = {}^{g}x$ for some $g \in G$, in which case they are given by $\langle \tfrac{{}^{g}x\,\overset{g}{\sqcap}\,x}{a} \rangle$ (where $a \in A$) with composition

$$\langle \tfrac{{}^{gh}y\,\overset{g}{\sqcap}\,{}^{h}y}{a} \rangle \circ \langle \tfrac{{}^{h}y\,\overset{h}{\sqcap}\,y}{b} \rangle \;=\; \langle \tfrac{{}^{gh}y\,\overset{gh}{\sqcap}\,y}{a \cdot {}^{g}b \cdot \tau_y(\alpha)(g,h)} \rangle \,. \tag{3.171}$$

  Here, we denoted by $\tau(\alpha)$ the transgression of the Postnikov class representative $\alpha \in Z^3(G, A)$ of the 2-group, which is defined by

$$\tau_x(\alpha)(g,h) \;:=\; \frac{\alpha(g,h,x) \cdot \alpha({}^{gh}x, g, h)}{\alpha(g, {}^{h}x, h)} \,. \tag{3.172}$$

  As a consequence of the (twisted) cocycle condition obeyed by α, it satisfies

$$(d\tau)_x(\alpha)(g,h,k) \;\equiv\; \frac{{}^{g}[\tau_x(h,k)] \cdot \tau_x(g,hk)}{\tau_x(gh,k) \cdot \tau_{({}^{k}x)}(g,h)} \;=\; \frac{{}^{({}^{ghk}x)}\alpha(g,h,k)}{\alpha(g,h,k)} \,. \tag{3.173}$$

- The vector space of 2-morphisms between two 1-morphisms is given by

$$2\mathrm{Hom}\Big(\langle {}^{g}x \overset{g}{\underset{a}{\leftarrow}} x\rangle, \langle {}^{h}x \overset{h}{\underset{b}{\leftarrow}} x\rangle\Big) \;=\; \delta_{g,h}\cdot \mathbb{C}\big[\, c\in A \,\big|\, {}^{({}^{g}x)}c/c \,=\, b/a \,\big] \,. \tag{3.174}$$

  The vertical composition of 2-morphisms is given by multiplication in A. The horizontal composition of 2-morphisms

$$\begin{aligned} c \;&\in\; 2\mathrm{Hom}\Big(\langle {}^{gh}y \overset{g}{\underset{a}{\leftarrow}} {}^{h}y\rangle, \langle {}^{gh}y \overset{g}{\underset{b}{\leftarrow}} {}^{h}y\rangle\Big) \\ d \;&\in\; 2\mathrm{Hom}\Big(\langle {}^{h}y \overset{h}{\underset{a'}{\leftarrow}} y\rangle, \langle {}^{h}y \overset{h}{\underset{b'}{\leftarrow}} y\rangle\Big) \end{aligned} \tag{3.175}$$

  is given by $c\star d = c\cdot {}^{g}d$. The †-structure acts on 2-morphisms via the (antilinear extension of the) group inversion $c\mapsto c^{-1}$.

- The 2-associator for the composition of 1-morphisms is given by

$$\begin{gathered} \tau_z(\pi)(g,h,k)\cdot\alpha(g,h,k) : \\ \Big[\langle {}^{g}x \overset{g}{\underset{a}{\leftarrow}} x\rangle \circ \langle {}^{h}y \overset{h}{\underset{b}{\leftarrow}} y\rangle\Big] \circ \langle {}^{k}z \overset{k}{\underset{c}{\leftarrow}} z\rangle \;\Rightarrow\; \langle {}^{g}x \overset{g}{\underset{a}{\leftarrow}} x\rangle \circ \Big[\langle {}^{h}y \overset{h}{\underset{b}{\leftarrow}} y\rangle \circ \langle {}^{k}z \overset{k}{\underset{c}{\leftarrow}} z\rangle\Big] \,, \end{gathered} \tag{3.176}$$

  where we defined $y := {}^{k}z$ and $x := {}^{h}y$ and denoted by $\tau(\pi)$ the transgression of the 't Hooft anomaly π given by

$$\tau_x(\pi)(g,h,k) \;:=\; \frac{\pi(g,h,k,x)\cdot\pi(g,{}^{hk}x,h,k)}{\pi(g,h,{}^{k}x,k)\cdot\pi({}^{ghk}x,g,h,k)} \,. \tag{3.177}$$

  As a consequence of the cocycle condition obeyed by π, it satisfies

$$(d\tau)_x(g,h,k,l) \;:=\; \frac{\tau_x(h,k,l)\cdot\tau_x(g,hk,l)\cdot\tau_{({}^{l}x)}(g,h,k)}{\tau_x(gh,k,l)\cdot\tau_x(g,h,kl)} \;=\; 1 \,, \tag{3.178}$$

  which, together with (3.173), ensures that the 2-associator in (3.176) satisfies the analogue of the pentagon relation.

In order to classify the irreducible tube 2-representations of $\mathcal{C} = 2\mathrm{Hilb}_G^{\pi}$, it is useful to 'skeletonise' its associated tube 2-category by identifying isomorphic objects and 1-morphisms. Concretely, this can be done via the following two steps:

1. Given group elements $x, g\in G$, we note that the 1-morphisms

$$\langle {}^{g}x \overset{g}{\underset{1}{\leftarrow}} x\rangle \qquad \text{and} \qquad \langle x \overset{g^{-1}}{\underset{\tau_x(\alpha)(g^{-1},g)^{-1}}{\longleftarrow}} {}^{g}x\rangle \tag{3.179}$$

   admit invertible 2-morphisms

$$\begin{aligned} 1 :\;& \langle x \overset{g^{-1}}{\underset{\tau_x(\alpha)(g^{-1},g)^{-1}}{\longleftarrow}} {}^{g}x\rangle \circ \langle {}^{g}x \overset{g}{\underset{1}{\leftarrow}} x\rangle \;\Rightarrow\; \langle x \overset{1}{\underset{1}{\leftarrow}} x\rangle \,, \\ \alpha(g,g^{-1},g) :\;& \langle {}^{g}x \overset{g}{\underset{1}{\leftarrow}} x\rangle \circ \langle x \overset{g^{-1}}{\underset{\tau_x(\alpha)(g^{-1},g)^{-1}}{\longleftarrow}} {}^{g}x\rangle \;\Rightarrow\; \langle {}^{g}x \overset{1}{\underset{1}{\leftarrow}} {}^{g}x\rangle \,, \end{aligned} \tag{3.180}$$

which establish an isomorphism $x \cong {}^g x$ as objects in the tube 2-category. Since two objects x and y are connected if and only if they are conjugates of one another, this shows that the isomorphism classes of objects in the tube 2-category are disconnected and labelled by conjugacy classes $[x] \in \mathrm{Cl}(G)$.

2. For a given conjugacy class with fixed representative $x \in G$, its 1-endomorphisms are given by $\langle x \xrightarrow[a]{g} x \rangle$ where g lies in the centraliser $G_x \subset G$ of x. From (3.174), two such endomorphisms $\langle x \xrightarrow[a]{g} x \rangle$ and $\langle x \xrightarrow[b]{h} x \rangle$ are connected (and in fact isomorphic) if and only if $g = h$ and $b = a \cdot d$ for some d that lies in the subgroup

$$A^x \;:=\; \{ {}^x c / c \mid c \in A \} \;\subset\; A \,. \tag{3.181}$$

The isomorphism classes of 1-endomorphisms of x can hence be labelled by pairs $([a], g)$ with $[a] \in A/A^x$ and $g \in G_x$, which compose according to

$$([a], g) \circ ([b], h) \;=\; \big(\big[a \cdot {}^g b \cdot \tau_x(\alpha)(g,h) \big] , g \cdot h \big) \,. \tag{3.182}$$

We identify the above as the group law of the group extension of G_x by A/A^x with extension 2-cocycle given by

$$[\tau_x(\alpha)] \;\in\; Z^2(G_x, [A/A^x]) \,. \tag{3.183}$$

To summarise, the skeleton of the tube 2-category associated to $\mathcal{C} = 2\mathrm{Hilb}^{\pi}_{\mathcal{G}}$ decomposes into disconnected components labelled by conjugacy classes $[x] \in \mathrm{Cl}(G)$ with representatives $x \in G$, whose endomorphism categories are given by[12] $\mathrm{Hilb}^{\tau_x(\pi)}[\mathcal{G}_x]$. Here, $\mathcal{G}_x$ denotes the finite 2-group whose

- 0-form component is given by the group extension $[A/A^x] \rtimes_{[\tau_x(\alpha)]} G_x$,
- 1-form component is A, acted upon by the 0-form component via ${}^{([b],g)}a := {}^g a$,
- Postnikov class representative is given by

$$\alpha\big(([a], g), ([b], h), ([c], k) \big) \;:=\; \alpha(g,h,k) \,. \tag{3.184}$$

---

[12] Given any finite 2-group $\mathcal{G} = A[1] \rtimes_\alpha G$ and a 3-cocycle $\omega \in Z^3(G, U(1))$ on its 0-form component, we denote by $\mathrm{Hilb}^\omega[\mathcal{G}]$ the monoidal †-category whose

- objects are G-graded Hilbert spaces (i.e. direct sums of the one-dimensional Hilbert spaces $\mathbb{C}_g$ with G-grading $(\mathbb{C}_g)_h = \delta_{g,h}\,\mathbb{C}$ for $g, h \in G$),
- morphism spaces are $\mathrm{Hom}(\mathbb{C}_g, \mathbb{C}_h) = \delta_{g,h}\,\mathbb{C}[A]$ with composition given by multiplication in A and †-structure given by (the antilinear extension of) $a \mapsto a^{-1}$,
- monoidal structure is given by $\mathbb{C}_g \otimes \mathbb{C}_h = \mathbb{C}_{gh}$ on objects and $a \otimes b = a \cdot {}^g b$ on morphisms $a \in \mathrm{End}(\mathbb{C}_g)$ and $b \in \mathrm{End}(\mathbb{C}_h)$, with associator given by $\omega(g,h,k) \cdot \alpha(g,h,k) \in \mathrm{End}(\mathbb{C}_{ghk})$.

Intuitively, we regard $\mathrm{Hilb}^\omega[\mathcal{G}]$ as a categorification of the notion of the twisted group algebra $\mathbb{C}^\mu[G]$ associated to a finite group G with 2-cocycle $\mu \in Z^2(G, U(1))$.

Similarly, any tube 2-representation $\mathcal{F}$ will decompose into sub-2-representations supported on conjugacy classes in G. Upon fixing a particular $[x] \in \mathrm{Cl}(G)$ together with a representative $x \in G$ and denoting $\mathcal{K} := \mathcal{F}(x) \in 2\mathrm{Vect}$, the remaining data associated to $\mathcal{F}$ then reduces to the data of a monoidal functor

$$R: \ \mathrm{Hilb}^{\tau_x(\pi)}[\mathcal{G}_x] \ \to \ \mathrm{End}(\mathcal{K}) \ , \tag{3.185}$$

which we will call a $\tau_x(\pi)$*-projective 2-representation* of $\mathcal{G}_x$ on the 2-vector space $\mathcal{K}$ in what follows[13]. Conversely, given a pair (x, R) consisting of

1. a representative $x \in G$ of a conjugacy class $[x] \in \mathrm{Cl}(G)$,
2. an irreducible $\tau_x(\pi)$-projective 2-representation R of $\mathcal{G}_x$,

we can construct an associated tube 2-representation $\mathcal{F}_{(x,R)}$ via induction: To this end, we fix for each $y \in [x]$ in the conjugacy class of x a representative $r_y \in G$ such that ${}^{(r_y)}y = x$ (with $r_x := 1$). Using these, we define

$$g_y \ := \ r_{({}^g y)} \cdot g \cdot r_y^{-1} \ \in \ G_x \tag{3.186}$$

for $g \in G$ and $y \in [x]$. If we denote by $\mathcal{K}$ the 2-vector space underlying R, then the 2-functor $\mathcal{F}_{(x,R)} : 2\text{-T}\mathcal{C} \to 2\mathrm{Vect}$ (where $\mathcal{C} = 2\mathrm{Hilb}^{\pi}_{\mathcal{G}}$) can be described as follows:

- To objects $y \in G$, the 2-functor $\mathcal{F}_{(x,R)}$ assigns the twisted sectors

$$\mathcal{L}_y \ \equiv \ \mathcal{F}(y) \ := \ \begin{cases} \mathcal{K} & \text{if } y \in [x] \\ 0 & \text{otherwise} \end{cases} \ . \tag{3.187}$$

- To a 1-morphism $\langle {}^{g}y \xleftarrow[a]{g} y \rangle$, the 2-functor $\mathcal{F}_{(x,R)}$ assigns

$$\mathcal{F}_{(x,R)}\Big(\langle {}^{g}y \xleftarrow[a]{g} y \rangle\Big) \ := \ R\Big(\big[\, {}^{r_{({}^g y)}}a \cdot \kappa_y(\alpha)(g) \big] \, , \, g_y \Big) \ , \tag{3.188}$$

  where we defined the multiplicative factor

$$\kappa_y(\alpha)(g) \ := \ \frac{\tau_y(\alpha)(r_{({}^g y)}, g)}{\tau_y(\alpha)(g_y, r_y)} \ \in \ A \ . \tag{3.189}$$

- To a 2-morphism $c \in \mathrm{Hom}\Big(\langle {}^{g}y \xleftarrow[a]{g} y \rangle , \langle {}^{g}y \xleftarrow[b]{g} y \rangle\Big)$, the 2-functor $\mathcal{F}_{(x,R)}$ assigns

$$\mathcal{F}_{(x,R)}(c) \ := \ R\big({}^{r_{({}^g y)}}c\big) \ . \tag{3.190}$$

[13] Given a finite 2-group $\mathcal{G}$ and a 3-cocycle $\omega \in Z^3(G, U(1))$ on its 0-form component, an ω*-projective 2-representation* of $\mathcal{G}$ on a 2-vector space $\mathcal{L} \in 2\mathrm{Vect}$ is a monoidal functor $R : \mathrm{Hilb}^{\omega}[\mathcal{G}] \to \mathrm{End}(\mathcal{L})$. We denote the 2-category of ω-projective 2-representations of $\mathcal{G}$ by $2\mathrm{Rep}^{\omega}(\mathcal{G})$.

- The compositor 2-isomorphism

$$
\begin{gathered}
\big[\mathcal{F}_{(x,R)}\big]_{\langle {}^{g}y \xleftarrow[a]{g} y\rangle, \langle {}^{h}z \xleftarrow[b]{h} z\rangle} : \\
\mathcal{F}_{(x,R)}\Big(\langle {}^{g}y \xleftarrow[a]{g} y\rangle\Big) \circ \mathcal{F}_{(x,R)}\Big(\langle {}^{h}z \xleftarrow[b]{h} z\rangle\Big) \Rightarrow \mathcal{F}_{(x,R)}\Big(\langle {}^{g}y \xleftarrow[a]{g} y\rangle \circ \langle {}^{h}z \xleftarrow[b]{h} z\rangle\Big)
\end{gathered}
\tag{3.191}
$$

  that controls the composition of 1-morphisms (where we set $y := {}^{h}z$) is given by

$$
\begin{gathered}
\big[\mathcal{F}_{(x,R)}\big]_{\langle {}^{g}y \xleftarrow[a]{g} y\rangle, \langle {}^{h}z \xleftarrow[b]{h} z\rangle} := \\
\phi_z(\pi)(g,h) \cdot R\big(\varphi_z(\alpha)(g,h)\big) \cdot R_{\left(\left[{}^{r({}^{g}y)}a \cdot \kappa_y(g)\right], g_y\right), \left(\left[{}^{r({}^{h}z)}b \cdot \kappa_z(h)\right], h_z\right)} \,,
\end{gathered}
\tag{3.192}
$$

  where we defined the multiplicative factors

$$
\phi_z(\pi)(g,h) := \frac{\tau_z(\pi)\big(g_{({}^{h}z)}, r_{({}^{h}z)}, h\big)}{\tau_z(\pi)\big(r_{({}^{gh}z)}, g, h\big) \cdot \tau_z(\pi)\big(g_{({}^{h}z)}, h_z, r_z\big)} \in U(1) \,, \tag{3.193}
$$

$$
\varphi_z(\alpha)(g,h) := \frac{\alpha\big(g_{({}^{h}z)}, r_{({}^{h}z)}, h\big)}{\alpha\big(r_{({}^{gh}z)}, g, h\big) \cdot \alpha\big(g_{({}^{h}z)}, h_z, r_z\big)} \in A \,. \tag{3.194}
$$

  As a consequence of the cocycle conditions obeyed by $\tau(\pi)$ and α, they satisfy

$$
\frac{\phi_z(h,k) \cdot \phi_z(g,hk)}{\phi_z(gh,k) \cdot \phi_{({}^{k}z)}(g,h)} = \frac{\tau_x(\pi)\big(g_{({}^{hk}z)}, h_{({}^{k}z)}, k_z\big)}{\tau_z(\pi)(g,h,k)} \,, \tag{3.195}
$$

$$
\frac{{}^{g_{({}^{h}z)}}\varphi_z(h,k) \cdot \varphi_z(g,hk)}{\varphi_z(gh,k) \cdot \varphi_{({}^{k}z)}(g,h)} = \frac{\alpha\big(g_{({}^{hk}z)}, h_{({}^{k}z)}, k_z\big)}{{}^{r_{({}^{ghk}z)}}\alpha(g,h,k)} \,, \tag{3.196}
$$

  which ensures that the compositor (3.192) obeys suitable *tetrahedron relations*.

All in all, we conclude that the 2-category of tube 2-representations of $\mathcal{C} = 2\mathrm{Hilb}_{\mathcal{G}}^{\pi}$ admits a direct sum decomposition [2]

$$
2\mathrm{Rep}(2\text{-}\mathrm{T}\mathcal{C}) \cong \boxplus_{[x] \in \mathrm{Cl}(G)} 2\mathrm{Rep}^{\tau_x(\pi)}(\mathcal{G}_x) \,. \tag{3.197}
$$

Using (3.156), this yields a description of the Drinfeld centre of $\mathcal{C}$, which reproduces the known classification of simple objects in terms of pairs (x,R) as above in cases where the 1-form symmetry group A is trivial [112]. One can check that $\mathcal{F}_{(x,R)}$ is a tube †-2-representation if and only if the projective 2-representation R is a monoidal †-functor into the endomorphism category of a 2-Hilbert space $\mathcal{K}$ (in which case we refer to R as a *unitary* 2-representation). We will discuss and classify the (irreducible) unitary 2-representations of finite 2-groups in the following section.

3.4 Unitary 2-Representations

We have seen in the previous section that the wrapping action of an (anomalous) 2-group symmetry $\mathcal{G}$ on twisted sector line operators is completely determined by the (projective) 2-representation theory of $\mathcal{G}$ [1, 148]. In this section, we describe the latter in more detail and provide a classification of the irreducible unitary 2-representations and their intertwiners together with a physical interpretation of the associated data[14]. The discussion is based on [1, 3].

3.4.1 Background

Given a finite 2-group $\mathcal{G} = A[1] \rtimes_\alpha G$, we defined a projective 2-representation of $\mathcal{G}$ with 3-cocycle $\omega \in Z^3(G, U(1))$ to be a monoidal functor

$$R: \ \text{Hilb}^\omega[\mathcal{G}] \ \to \ \text{End}(\mathcal{L}) \tag{3.198}$$

from the 'twisted 2-group algebra' $\text{Hilb}^\omega[\mathcal{G}]$ into the endomorphism category of a fixed 2-vector space $\mathcal{L} \in 2\text{Vect}$. When the projective 3-cocycle ω is trivial, this is equivalent to the data of a monoidal functor

$$R: \ \mathcal{G} \ \to \ \text{End}(\mathcal{L}) \ , \tag{3.199}$$

where by abuse of notation we used $\mathcal{G}$ to denote the monoidal category whose

- objects are given by group elements $g \in G$,
- morphism spaces are given by $\text{Hom}_\mathcal{G}(g,h) = \delta_{g,h} A$ with composition given by group multiplication in A,
- monoidal structure is given by $g \otimes h = g \cdot h$ on objects and $a \otimes b = a \cdot {}^g b$ on morphisms $a \in \text{End}_\mathcal{G}(g)$ and $b \in \text{End}_\mathcal{G}(h)$ with associator $\alpha(g,h,k) \in \text{End}_\mathcal{G}(ghk)$.

Even more abstractly, we can view R as a 2-functor

$$R: \ B\mathcal{G} \ \to \ 2\text{Vect} \tag{3.200}$$

from the delooping of $\mathcal{G}$ into the 2-category of 2-vector spaces such that $R(*) = \mathcal{L}$ (where $*$ denotes the single object of $B\mathcal{G}$). The 2-category of 2-representations of $\mathcal{G}$ is then defined to be the 2-category

$$2\text{Rep}(\mathcal{G}) \ := \ [B\mathcal{G}, 2\text{Vect}] \tag{3.201}$$

[14] We note that the notion of unitary 2-representations of finite groups on 2-Hilbert spaces was already studied extensively in e.g. [149, 150]. A classification of 2-representations of finite 2-groups in the math literature can be found e.g. in [149, 151–153].

of all such 2-functors, their 2-natural transformations and modifications. Similarly, the 2-category of *unitary* 2-representations of $\mathcal{G}$ is the 2-category of †-2-functors

$$2\mathrm{Rep}^\dagger(\mathcal{G}) \;:=\; [B\mathcal{G}, 2\mathrm{Hilb}]^\dagger \;, \tag{3.202}$$

where the †-structure on $B\mathcal{G}$ acts as the inversion $a \mapsto a^{-1}$ of 1-form elements $a \in A$ (which turns $B\mathcal{G}$ into a †-2-category). Clearly, there exists a forgetful 2-functor

$$2\mathrm{Rep}^\dagger(\mathcal{G}) \;\to\; 2\mathrm{Rep}(\mathcal{G}) \;. \tag{3.203}$$

3.4.2 Classification

In order to classify the unitary 2-representations of a finite 2-group $\mathcal{G} = A[1] \rtimes_\alpha G$, it is convenient to model the target 2-category 2Hilb by the 2-category Mat(Hilb) of matrices of Hilbert spaces. The data associated to a †-2-functor $R : B\mathcal{G} \to \mathrm{Mat(Hilb)}$ can then be described as follows:

- To the single object $* \in B\mathcal{G}$, the 2-functor R associates a non-negative integer $n \in \mathbb{N}$, which we call n the *dimension* of the 2-representation R in what follows.
- To the 0-form elements $g \in G$, the 2-functor R assigns invertible $(n \times n)$-matrices $R(g)$ of Hilbert spaces, which up to equivalence need to be of the form

  $$R(g)_{ij} \;=\; \delta_{i,\sigma_g(j)} \cdot \mathbb{C} \tag{3.204}$$

  for some permutation action $\sigma : G \to S_n$ of G on the finite set $[n] := \{1, ..., n\}$. We will abbreviate the action of $g \in G$ on $i \in [n]$ by $g \triangleright i := \sigma_g(i)$ in what follows.
- To 1-form elements $a \in \mathrm{End}_{\mathcal{G}}(g)$, the 2-functor R assigns $(n \times n)$-matrices

  $$R(a) : \; R(g) \;\Rightarrow\; R(g) \tag{3.205}$$

  of unitary linear maps between the Hilbert space entries of $R(g)$. As a consequence of (3.204), these then have to be of the form

  $$R(a)_{ij} \;=\; \delta_{i,g\triangleright j} \cdot \chi_i(a) \tag{3.206}$$

  for some multiplicative phases $\chi_i(a) \in U(1)$ (viewed as a collection of characters $\chi \in (A^\vee)^n$ in the Pontryagin dual group $A^\vee := \mathrm{Hom}(A, U(1))$ of A), which need to be compatible with the group action of G on A in the sense that

  $$\chi_{g\triangleright i}(a) \;=\; \chi_i(a^g) \tag{3.207}$$

  for all $g \in G$, $a \in A$ and $i \in [n]$.

- For each pair of 0-form elements $g, h \in G$, there exists a unitary 2-isomorphism

$$R_{g,h}: \quad R(g) \circ R(h) \;\Rightarrow\; R(g \cdot h)\;, \tag{3.208}$$

which needs to be compatible with the composition of three group elements $g, h, k \in G$ in the sense that the diagram

$$\begin{array}{ccc} & R(g)\circ R(h)\circ R(k) & \\ {\scriptstyle R_{g,h}\star R(k)}\swarrow & & \searrow{\scriptstyle R(g)\star R_{h,k}} \\ R(g\cdot h)\circ R(k) & & R(g)\circ R(h\cdot k) \\ {\scriptstyle R_{gh,k}}\Downarrow & & \Downarrow{\scriptstyle R_{g,hk}} \\ R\big((g\cdot h)\cdot k\big) & \xRightarrow{R(\alpha(g,h,k))} & R\big(g\cdot(h\cdot k)\big) \end{array} \tag{3.209}$$

commutes. Similarly to above, the 2-isomorphisms $R_{g,h}$ can then be identified with invertible $(n \times n)$-matrices of unitary linear maps that are of the form

$$(R_{g,h})_{ij} \;=\; \delta_{i,gh\triangleright j} \cdot c_i(g,h) \tag{3.210}$$

for some multiplicative phases $c_i(g,h) \in U(1)$, which due to (3.209) obey

$$\frac{c_{g^{-1}\triangleright i}(h,k)\cdot c_i(g,hk)}{c_i(gh,k)\cdot c_i(g,h)} \;=\; \chi_i(\alpha(g,h,k))\;. \tag{3.211}$$

The collection of phases $c_i(g,h) \in U(1)$ then defines a twisted group 2-cochain $c \in C^2_\sigma(G, U(1)^n)$ that obeys $d_\sigma c = \langle \chi, \alpha \rangle$, where the abelian group $U(1)^n$ is acted upon by G via the permutation action σ.

To summarise, we can label the unitary 2-representation R of $\mathcal{G} = A[1] \rtimes_\alpha G$ by quadruples $R = (n, \sigma, \chi, c)$ consisting of

1. a non-negative integer $n \in \mathbb{N}$, called the *dimension* of the 2-representation,
2. a permutation action $\sigma : G \to S_n$ of G on $[n] := \{1, ..., n\}$,
3. a collection of n characters $\chi \in (A^\vee)^n$ satisfying $\chi_{g\triangleright i}(a) = \chi_i(a^g)$,
4. a twisted 2-cochain $c \in C^2_\sigma(G, U(1)^n)$ satisfying $d_\sigma c = \langle \chi, \alpha \rangle$.

This reproduces the known classification of (ordinary) 2-representations of $\mathcal{G}$ on Kapranov-Voevodsky 2-vector spaces [151–153]. In particular, we see that every (ordinary) 2-representation of $\mathcal{G}$ is equivalent to a unitary one. The *dual* of a unitary 2-representation $R = (n, \sigma, \chi, c)$ is given by $R^\vee := (n, \sigma, \chi^*, c^*)$. The trivial 2-representation $\mathbb{1}$ of $\mathcal{G}$ is given by $\mathbb{1} = (1, 1, 1, 1, 1)$.

3.4.2.1 Irreducibles

A unitary 2-representation $R = (n, \sigma, \chi, c)$ is irreducible if the associated permutation action $\sigma : G \to S_n$ is transitive. In this case, we can use the orbit-stabiliser theorem to relate the G-orbit $[n] \equiv \{1, ..., n\}$ to the stabiliser subgroup

$$H \;:=\; \mathrm{Stab}_\sigma(1) \;\equiv\; \{h \in G \mid \sigma_h(1) = 1\} \;\subset\; G \tag{3.212}$$

of a fixed element $1 \in [n]$. The remaining data associated to R then gives rise to a one-dimensional unitary 2-representation of $\mathcal{H} := A[1] \rtimes_{(\alpha|_H)} H$ as follows:

- Setting $\lambda := \chi_1$ yields a character $\lambda \in A^\vee$ that is H-invariant in the sense that $\lambda({}^h a) = \lambda(a)$ for all $h \in H$ and $a \in A$.
- Setting $u := c_1|_H$ yields a 2-cochain $u \in C^2(H, U(1))$ obeying $du = \langle \lambda, \alpha|_H \rangle$.

Conversely, given a subgroup $H \subset G$ and a one-dimensional unitary 2-representation (λ, u) of $\mathcal{H}$, we can construct an associated irreducible unitary 2-representation R of $\mathcal{G} = A[1] \rtimes_\alpha G$ via induction:

$$R \;=\; \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\lambda, u)\;. \tag{3.213}$$

To this end, consider a set of fixed representatives $r_i \in G$ of left H-cosets in G, i.e.

$$G/H \;=\; \{r_1 H, ..., r_n H\}\,, \tag{3.214}$$

so that $r_1 := 1$ and $n = |G : H|$. Using this, we can construct the data (n, σ, χ, c) associated to the 2-representation R as follows:

- By multiplying left H-cosets with group element $g \in G$ from the left, we obtain a permutation action $\sigma : G \to S_n$ via

  $$g \cdot r_i H \;=\; r_{\sigma_g(i)} H\,. \tag{3.215}$$

  This allows us to define for each $g \in G$ and $i \in [n]$ a little group element

  $$g_i \;:=\; r_i^{-1} \cdot g \cdot r_{(g^{-1}) \triangleright i} \;\in\; H\;. \tag{3.216}$$

- Given the H-invariant character $\lambda \in A^\vee$, we obtain a collection $\chi \in (A^\vee)^n$ of characters $\chi_i(a) := \lambda(a^{r_i})$ that satisfy $\chi_{g \triangleright i}(a) = \chi_i(a^g)$.
- Given the 2-cochain $u \in C^2(H, U(1))$ obeying $du = \langle \lambda, \alpha|_H \rangle$, we obtain a twisted 2-cochain $c \in C^2_\sigma(G, U(1)^n)$ obeying $d_\sigma c = \langle \chi, \alpha \rangle$ by setting

  $$c_i(g, h) \;:=\; \langle \lambda, \phi_i(\alpha)(g, h) \rangle \,\cdot\, u(g_i, h_{g^{-1} \triangleright i})\;, \tag{3.217}$$

where we defined the multiplicative factor

$$\phi_i(\alpha)(g,h) \;:=\; \frac{\alpha(r_i^{-1},g,h)\,\cdot\,\alpha(g_i,\,h_{g^{-1}\triangleright i},\,r^{-1}_{(gh)^{-1}\triangleright i})}{\alpha(g_i,\,r^{-1}_{g^{-1}\triangleright i},\,h)} \;\in\; A\;. \tag{3.218}$$

To summarise, we can label the irreducible unitary 2-representations R of $\mathcal{G} = A[1]\rtimes_\alpha G$ by triples $R = (H,\lambda,u)$ consisting of

1. a subgroup $H \subset G$,
2. a H-invariant character $\lambda \in A^\vee$,
3. a 2-cochain $u \in C^2(H,U(1))$ satisfying $du = \langle\lambda,\alpha|_H\rangle$.

This reproduces the known classification of ordinary irreducible 2-representations of $\mathcal{G}$ [151–153] (see also [5, 6, 57–59] for a physical interpretation of the latter as Wilson surfaces in three-dimensional discrete gauge theories). The *dual* of $R = (H,\lambda,u)$ is given by $R^\vee = (H,\lambda^*,u^*)$. The trivial 2-representation of $\mathcal{G}$ is $\mathbb{1} = (G,1,1)$.

3.4.2.2 Intertwiners

An intertwiner between two given unitary 2-representations $R = (n,\sigma,\chi,c)$ and $R' = (n',\sigma',\chi',c')$ is a 2-natural transformation $\eta : R \Rightarrow R'$ between the corresponding †-2-functors. The associated data can be described as follows:

- To the single object $* \in B\mathcal{G}$, η assigns a morphism η_* between $R(*) = n$ and $R'(*) = n'$, which is an $(n'\times n)$-matrix V of Hilbert spaces V_{ij}.
- To the 1-form elements $g \in G$, η assigns unitary 2-morphisms

$$\eta_g :\; R'(g)\circ V \;\Rightarrow\; V\circ R(g) \tag{3.219}$$

which need to be compatible with the composition of two 0-form elements $g,h \in G$ in the sense that the diagram

$$\begin{array}{ccccc} & & R'(g)\circ V\circ R(h) & & \\ & \overset{R'(g)\star\eta_h}{\Nearrow} & & \overset{\eta_g\star R(h)}{\Searrow} & \\ R'(g)\circ R'(h)\circ V & & & & V\circ R(g)\circ R(h) \\ {\scriptstyle R'_{g,h}\star V}\Downarrow & & & & \Downarrow{\scriptstyle V\star R_{g,h}} \\ R'(g{\cdot}h)\circ V & & \xRightarrow{\;\eta_{gh}\;} & & V\circ R(g{\cdot}h) \end{array} \tag{3.220}$$

commutes. Upon identifying η_g with an $(n'\times n)$-matrix of unitary linear maps

$$(\eta_g)_{ij} \;=:\; \varphi(g)_{(\sigma'_{g^{-1}})(i),\,j} \tag{3.221}$$

with $\varphi(g)_{ij} : V_{ij} \to V_{g \triangleright (i,j)}$, condition (3.220) becomes equivalent to

$$\varphi(g)_{h \triangleright (i,j)} \circ \varphi(h)_{ij} = \frac{c'_{gh \triangleright i}(g,h)}{c_{gh \triangleright j}(g,h)} \cdot \varphi(g \cdot h)_{ij} \, , \tag{3.222}$$

where we denoted by $g \triangleright (i,j) := (\sigma'_g(i), \sigma_g(j))$ the product action $\sigma' \times \sigma$ of G on $[n'] \times [n]$. Furthermore, in order for η to be compatible with the action of the 1-form symmetry group A, the diagram

$$\begin{array}{ccc} R'(g) \circ V & \overset{\eta_g}{\Longrightarrow} & V \circ R(g) \\ {\scriptstyle R'(a) \star V} \Big\Downarrow & & \Big\Downarrow {\scriptstyle V \star R(a)} \\ R'(g) \circ V & \overset{\eta_g}{\Longrightarrow} & V \circ R(g) \end{array} \tag{3.223}$$

has to commute for all $a \in \mathrm{End}_{\mathcal{G}}(g)$, leading to the condition

$$\chi'_{g \triangleright i}(a) \cdot \varphi(g)_{ij} = \chi_{g \triangleright j}(a) \cdot \varphi(g)_{ij} \, . \tag{3.224}$$

In particular, setting $g = 1$ reveals that $V_{ij} = 0$ unless $\chi'_i = \chi_j \in A^\vee$.

To summarise, intertwiners η between two unitary 2-representations $R = (n, \sigma, \chi, c)$ and $R' = (n', \sigma', \chi', c')$ can be labelled by tuples $\eta = (V, \varphi)$ consisting of

1. an $(n' \times n)$-matrix of Hilbert spaces V_{ij} with $V_{ij} = 0$ unless $\chi'_i = \chi_j$,
2. a collection of unitary linear maps $\varphi(g)_{ij} : V_{ij} \to V_{g \triangleright (i,j)}$ such that

$$\varphi(g)_{h \triangleright (ij)} \circ \varphi(h)_{ij} = \frac{c'_{gh \triangleright i}(g,h)}{c_{gh \triangleright j}(g,h)} \cdot \varphi(g \cdot h)_{ij} \, . \tag{3.225}$$

The identity intertwiner $\mathrm{id}_R : R \Rightarrow R$ of a unitary 2-representation R is specified by the data $\mathrm{id}_\rho = (\mathbb{1}_n, \mathrm{Id})$, where $(\mathbb{1}_n)_{ij} = \delta_{ij} \cdot \mathbb{C}$. The dual and adjoint of an intertwiner $\eta = (V, \varphi) : R \Rightarrow R'$ can be described as follows:

- The *dual* of η is defined to be the intertwiner $\eta^\vee : (R')^\vee \Rightarrow R^\vee$ that has associated data $\eta^\vee = (V^\vee, \varphi^\vee)$ with $(V^\vee)_{ij} = V_{ji}$ and

$$(\varphi^\vee)(g)_{ij} = \varphi(g)_{ji} \, . \tag{3.226}$$

- The *adjoint* of η is defined to be the intertwiner $\widehat{\eta} : R' \Rightarrow R$ that has associated data $\widehat{\eta} = (\widehat{V}, \widehat{\varphi})$ with $(\widehat{V})_{ij} = (V_{ji})^\vee$ and

$$\widehat{\varphi}(g)_{ij} = (\varphi(g)_{ji}^{-1})^T \, , \tag{3.227}$$

where ^{T} denotes the transpose of linear maps between vector spaces.

Simples

In order to classify the simple intertwiners between two irreducible unitary 2-representations $R = (n, \sigma, \chi, c)$ and $R' = (n', \sigma', \chi', c')$ of $\mathcal{G}$, we express the latter as inductions

$$R \;=\; \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\lambda, u, p) \qquad \text{and} \qquad R' \;=\; \mathrm{Ind}_{\mathcal{H}'}^{\mathcal{G}}(\lambda', u', p') \tag{3.228}$$

of the one-dimensional unitary 2-representations (λ, u) and (λ', u') of the sub-2-groups $\mathcal{H}^{(\prime)} = A[1] \rtimes_\alpha H^{(\prime)} \subset \mathcal{G}$ given by

$$H \;=\; \mathrm{Stab}_\sigma(1)\,, \qquad \lambda \;=\; \chi_1|_H\,, \qquad u \;=\; c_1|_H\,, \tag{3.229}$$

and similarly for the $'$-ed variables. We then consider a fixed orbit of the product G-action $\sigma' \times \sigma$ on $[n'] \times [n]$ with fixed representative $(i_0, j_0) \in [n'] \times [n]$. As the G-action σ on $[n]$ is transitive, we may without loss of generality assume that $j_0 = 1$. Similarly, since σ' is transitive on $[n']$, we can fix $x \in G$ such that $x \triangleright 1 = i_0$[15]. Then, the stabiliser of the orbit representative $(i_0, 1) \in [n'] \times [n]$ is given by

$$\begin{aligned} \mathrm{Stab}_{\sigma'\times\sigma}(i_0, 1) \;&=\; \mathrm{Stab}_\sigma(1) \cap \mathrm{Stab}_{\sigma'}(i_0) \\ &=\; \mathrm{Stab}_\sigma(1) \cap {}^x(\mathrm{Stab}_{\sigma'}(1)) \;\equiv\; H \cap {}^xH'\,. \end{aligned} \tag{3.230}$$

Now let $\eta = (V, \varphi)$ be an intertwiner between R and R'. Using the above, we can reduce the data associated to η to the following:

- By defining $W := V_{(i_0,1)}$, we obtain a finite-dimensional Hilbert space that vanishes unless $\chi'_{i_0} = \chi_1$. Since $\chi_1 \equiv \lambda$ and

  $$\chi'_{i_0}(a) \;=\; \chi'_{x\triangleright 1}(a) \;=\; \chi'_1(a^x) \;\equiv\; \lambda'(a^x) \;=:\; ({}^x\lambda')(a) \tag{3.231}$$

  for all $a \in A$, this means that $W = 0$ unless $\lambda = {}^x\lambda'$.

- By defining for each $h \in H \cap {}^xH'$ the unitary linear map

  $$\psi(h) \;:=\; \frac{c'_{i_0}(x, h^x)}{c'_{i_0}(h, x)} \cdot \varphi(h)_{(i_0,1)} \;:\; W \;\to\; W\,, \tag{3.232}$$

  we obtain a unitary representation ψ of $H \cap {}^xH'$ on W with projective 2-cocycle

  $$\frac{{}^xu'}{u} \cdot \langle \lambda, \gamma_x(\alpha) \rangle \;\in\; Z^2(H \cap {}^xH', U(1))\,, \tag{3.233}$$

[15] For fixed i_0, the group element $x \in G$ is unique up to multiplication by elements $h' \in \mathrm{Stab}_{\sigma'}(1) \equiv H'$ from the right. Moreover, multiplying x by elements $h \in \mathrm{Stab}_\sigma(1) \equiv H$ from the left changes the representative $(i_0, 1) \to (h \triangleright i_0, 1)$ of the fixed G-orbit in $[n'] \times [n]$. The element $x \in G$ hence defines a double coset $[x] \in H\backslash G/H'$.

where we defined the multiplicative factor

$$\gamma_x(\alpha)(h,k) \;:=\; \frac{\alpha(h,x,k^x)}{\alpha(h,k,x)\cdot\alpha(x,h^x,k^x)} \;\in\; A\,. \tag{3.234}$$

Conversely, given $x \in G$ such that $\lambda = {}^x\lambda'$ together with a unitary representation ψ of $H \cap {}^xH'$ on a Hilbert space W with projective 2-cocycle (3.233), we obtain an intertwiner $\eta = (V,\varphi)$ between R and R' via induction: To this end, let $\{r_1,...,r_n\}$ and $\{r'_1,...,r'_{n'}\}$ be fixed representatives of left H and H' cosets in G, i.e.

$$\begin{aligned} G/H \;&=\; \{r_1H,...,r_nH\}\,,\\ G/H' \;&=\; \{r'_1H,...,r'_{n'}H\}\,, \end{aligned} \tag{3.235}$$

such that $r_1 = r'_1 = 1$ and $r'_{i_0} = x$. This allows us to define little group elements

$$\begin{aligned} g_j \;&:=\; r_j^{-1}\cdot g\cdot r_{(g^{-1})\triangleright j} \;\in\; H\\ g'_i \;&:=\; (r'_i)^{-1}\cdot g\cdot r'_{(g^{-1})\triangleright i} \;\in\; H' \end{aligned} \tag{3.236}$$

for each $g \in G$ and all $i \in [n']$ and $j \in [n]$. We then define the double index set

$$I_x \;:=\; \big\{(i,j)\in[n']\times[n] \;\big|\; r_j^{-1}r'_i \in HxH'\big\} \;\subset\; [n']\times[n] \tag{3.237}$$

and fix for each $(i,j) \in I_x$ representatives $t_{ij} \in H$ and $t'_{ij} \in H'$ such that

$$r_j^{-1}r'_i \;=\; t_{ij}\cdot x\cdot(t'_{ij})^{-1} \tag{3.238}$$

with $t_{i_0,1} = t'_{i_0,1} = 1$. Using this, we can construct for each $g \in G$ little group elements

$$\begin{aligned} g_{ij} \;&:=\; t^{-1}_{g\triangleright(ij)}\cdot g_{g\triangleright j}\cdot t_{ij}\\ &\equiv\; {}^x\big[(t'_{g\triangleright(ij)})^{-1}\cdot g'_{g\triangleright i}\cdot t'_{ij}\big] \;\in\; H\cap{}^xH'\,, \end{aligned} \tag{3.239}$$

where $(i,j) \in I_x$. The intertwiner $\eta = (V,\varphi)$ is then constructed as follows:

- We define an $(n'\times n)$-matrix V with Hilbert space entries

$$V_{ij} \;:=\; \begin{cases} W & \text{if } (i,j)\in I_x\\ 0 & \text{otherwise}\end{cases}\,. \tag{3.240}$$

- For $(i,j) \in I_x$ and $g \in G$, we construct a unitary map $\varphi(g)_{ij} : V_{ij} \to V_{g\triangleright(ij)}$ by

$$\varphi(g)_{ij} \;:=\; \frac{\nu_{ij}(u)(g)}{\nu'_{ij}(u')(g)}\cdot\Big\langle\lambda,\, \frac{\mu_{ij}(\alpha)(g)}{{}^x[\mu'_{ij}(\alpha)(g)]}\cdot\omega_{x,ij}(\alpha)(g)\Big\rangle\cdot\psi(g_{ij})\,, \tag{3.241}$$

where we defined the multiplicative phases

$$\begin{aligned} \nu_{ij}(u)(g) &:= \frac{u(g_{ij}, t_{ij}^{-1})}{u(t_{g\triangleright(ij)}^{-1}, g_{g\triangleright j})} \in U(1) \ , \\ \nu'_{ij}(u')(g) &:= \frac{u'(g_{ij}^{x}, (t'_{ij})^{-1})}{u'((t'_{g\triangleright(ij)})^{-1}, g'_{g\triangleright i})} \in U(1) \ , \end{aligned} \tag{3.242}$$

as well as the multiplicative factors

$$\begin{aligned} \mu_{ij}(\alpha)(g) &:= \frac{\alpha\Big(t_{g\triangleright(ij)}^{-1}, r_{g\triangleright j}^{-1}, g\Big) \cdot \alpha\Big(g_{ij}, t_{ij}^{-1}, r_j^{-1}\Big)}{\alpha\Big(t_{g\triangleright(ij)}^{-1}, g_{g\triangleright j}, r_j^{-1}\Big)} \in A \ , \\ \mu'_{ij}(\alpha)(g) &:= \frac{\alpha\Big((t'_{g\triangleright(ij)})^{-1}, (r'_{g\triangleright i})^{-1}, g\Big) \cdot \alpha\Big(g_{ij}^{x}, (t'_{ij})^{-1}, (r'_i)^{-1}\Big)}{\alpha\Big((t'_{g\triangleright(ij)})^{-1}, g'_{g\triangleright i}, (r'_i)^{-1}\Big)} \in A \ , \\ \omega_{x,ij}(\alpha)(g) &:= \frac{\alpha\Big(x, g_{ij}^{x}, (r'_i t'_{ij})^{-1}\Big)}{\alpha\Big(x, (r'_{g\triangleright i} t'_{g\triangleright(ij)})^{-1}, g\Big) \cdot \alpha\Big(g_{ij}, x, (r'_i t'_{ij})^{-1}\Big)} \in A \ . \end{aligned} \tag{3.243}$$

The collection of linear maps (3.241) then obeys

$$\varphi(g)_{h\triangleright(ij)} \circ \varphi(h)_{ij} = \frac{c'_{gh\triangleright i}(g,h)}{c_{gh\triangleright j}(g,h)} \cdot \varphi(g\cdot h)_{ij} \ , \tag{3.244}$$

where $c \in C^2_\sigma(G, U(1)^n)$ and $c' \in C^2_{\sigma'}(G, U(1)^{n'})$ are as in (3.217).

To summarise, we can label the simple intertwiners between two irreducible unitary 2-representations $R = (H, \lambda, u)$ and $R' = (H', \lambda', u')$ by tuples $\eta = (x, \psi)$ consisting of

1. a representative $x \in G$ of a double coset $[x] \in H\backslash G/H'$ such that $\lambda = {}^x\lambda'$,
2. an irreducible unitary representation ψ of $H \cap {}^xH'$ with projective 2-cocycle

$$\frac{{}^xu'}{u} \cdot \langle \lambda, \gamma_x(\alpha) \rangle \in Z^2(H \cap {}^xH', U(1)) \ . \tag{3.245}$$

The identity intertwiner $\mathrm{id}_R : R \Rightarrow R$ of $R = (H, \lambda, u)$ is given by $\mathrm{id}_R = (1, \mathbf{1}_H)$, where $\mathbf{1}_H$ denotes the trivial representation of H. The dual and adjoint of a simple intertwiner $\eta = (x, \psi) : R \Rightarrow R'$ can be described as follows:

- The *dual* of η is the intertwiner $\eta^\vee : (R')^\vee \Rightarrow R^\vee$ specified by $\eta^\vee = (x^{-1}, \psi^\vee)$, where $\psi^\vee$ is the representation of $H' \cap H^x$ on W defined by

$$(\psi^\vee)(k) = \frac{\psi({}^xk)}{\langle \lambda', \kappa_x(\alpha)(k) \rangle} \tag{3.246}$$

with $\kappa_x(\alpha)(k) := \beta_{x^{-1},x}(\alpha)(k) \in A$ and $\beta(\alpha)$ as in (3.253) below.

- The *adjoint* of η is the intertwiner $\widehat{\eta} : R' \Rightarrow R$ specified by $\widehat{\eta} = (x^{-1}, \widehat{\psi})$, where $\widehat{\psi}$ is the representation of $H' \cap H^x$ on $W^\vee$ defined by

$$\widehat{\psi}(k) \;:=\; \langle \lambda', \kappa_x(\alpha)(k) \rangle \,\cdot\, \left[\psi({}^x k)^{-1}\right]^T \,. \tag{3.247}$$

Here, ^{T} denotes the transpose of linear maps between vector spaces.

Composition

Given two intertwiners $\eta : R \Rightarrow R'$ and $\eta' : R' \Rightarrow R''$ between unitary 2-representations R, R' and R'', we can compose them to obtain an intertwiner $\eta' \circ \eta : R \Rightarrow R''$. Concretely, if η and η' are specified by data $\eta = (V, \varphi)$ and $\eta' = (V', \varphi')$ as before, their composition has associated data

$$(V', \varphi') \circ (V, \varphi) \;=\; \big(V' \otimes V,\, \varphi' \otimes \varphi\big) \,, \tag{3.248}$$

where defined the matrix of Hilbert spaces and collection of linear maps

$$(V' \otimes V)_{ij} \;=\; \bigoplus_{k=1}^{n'} V'_{ik} \otimes V_{kj} \,, \tag{3.249}$$

$$(\varphi' \otimes \varphi)(g)_{ij} \;=\; \bigoplus_{k=1}^{n'} \varphi'(g)_{ik} \otimes \varphi(g)_{kj} \,. \tag{3.250}$$

Now suppose that R, R' and R'' are all irreducible, so that we can label them by data $R = (H, \lambda, u)$ and similarly for R' and R''. We furthermore assume that η and η' are simple intertwiners, so that we can label them by $\eta = (x, \psi)$ and $\eta' = (x', \psi')$ as before. Then, their composition is the (not necessarily simple) intertwiner labelled by[16]

$$(x, \psi) \circ (x', \psi') \;=\;$$

$$\bigoplus_{[h] \in H^x \backslash H' / {}^{x'}H''} \left(x \cdot h \cdot x',\ \mathrm{Ind}_{H \cap {}^{x}H' \cap {}^{xhx'}H''}^{H \cap {}^{xhx'}H''} \left[\frac{{}^{x}[\varepsilon_h(u')]}{\langle \lambda, \beta_{x,h}(\alpha) \cdot \beta_{xh,x'}(\alpha) \rangle} \cdot (\psi \otimes {}^{xh}\psi') \right] \right) \,, \tag{3.251}$$

where Ind denotes the induction functor for (projective) representations of subgroups and we made use of the 1-cochains

$$\varepsilon_h(u')(k) \;:=\; \frac{u'(h, k^h)}{u'(k, h)} \;\in\; U(1) \,, \tag{3.252}$$

$$\beta_{x,y}(\alpha)(k) \;:=\; \frac{\alpha(k, x, y) \,\cdot\, \alpha(x, y, k^{xy})}{\alpha(x, k^x, y)} \;\in\; A \,. \tag{3.253}$$

[16] For better readability, we temporarily changed the order in which we denote the composition of intertwiners, so that $(x, \psi) \circ (x', \psi')$ denotes the composition of $\eta : R \Rightarrow R'$ and $\eta' : R' \Rightarrow R''$.

The above composition rule simplifies if we restrict attention to endotwiners of an irreducible unitary 2-representation $R = (H, \lambda, u)$ with $H \triangleleft G$ normal. In this case, simple endotwiners $\eta = (x, \psi)$ and $\eta' = (x', \psi')$ are labelled by group elements $[x], [x'] \in G/H$ together with irreducible unitary (projective) representations ψ and ψ' of H, which compose according to

$$(x, \psi) \circ (x', \psi') \;=\; \left(x \cdot x', \; \frac{\psi \otimes {}^{x}\psi'}{\langle \lambda, \beta_{x,x'}(\alpha) \rangle} \right) . \tag{3.254}$$

Equivalences

Having established the notion of intertwiners for unitary 2-representations, we can discuss equivalences between them. Concretely, $R = (n, \sigma, \chi, c)$ and $R' = (n', \sigma', \chi', c')$ are considered equivalent if there exist an invertible intertwiner $\eta : R \Rightarrow R'$ between them. If the latter is specified by data $\eta = (V, \varphi)$ as before, then invertibility of η can be reduced to the following conditions:

- As V is an invertible $(n' \times n)$-matrix of Hilbert spaces, we must have $n = n'$ with V being of the form $V_{ij} = \delta_{i,\tau(j)} \cdot \mathbb{C}$ for some permutation $\tau \in S_n$. Furthermore, since $V_{ij} = 0$ unless $\chi'_i = \chi_j$, we must have $\chi' = {}^{\tau}\chi$, where $({}^{\tau}\chi)_i = \chi_{\tau^{-1}(i)}$.
- As φ provides unitary isomorphisms $\varphi(g)_{ij} : V_{ij} \to V_{\sigma'_g(i), \sigma_g(j)}$ for each $g \in G$, we must have $\sigma'_g = \tau \circ \sigma_g \circ \tau^{-1}$ for all $g \in G$. Furthermore, since the entries of V are one-dimensional, the above linear maps need to be of the form

  $$\varphi(g)_{ij} \;=\; \delta_{i,\tau(j)} \cdot \vartheta_{g \triangleright i}(g) \tag{3.255}$$

  for some multiplicative phases $\vartheta_i(g) \in U(1)$. Plugging this into the composition rule (3.225) then yields the condition

  $$(d\vartheta)_i(g,h) \;\equiv\; \frac{\vartheta_{g^{-1} \triangleright i}(h) \cdot \vartheta_i(g)}{\vartheta_i(gh)} \;=\; \frac{c'_i(g,h)}{c_{\tau^{-1}(i)}(g,h)} \, , \tag{3.256}$$

  which implies that $[c' / {}^{\tau}c] = 1 \in H^2_{\sigma'}(G, U(1)^n)$.

In summary, two given unitary 2-representations $R = (n, \sigma, \chi, c)$ and $R' = (n', \sigma', \chi', c')$ are equivalent if and only if they have the same dimension $n = n'$ and there exists a permutation $\tau \in S_n$ such that

$$\sigma' = {}^{\tau}\sigma \, , \qquad \chi' = {}^{\tau}\chi \, , \qquad [c' / {}^{\tau}c] = 1 \, . \tag{3.257}$$

Now suppose that both R and R' are irreducible, so that we can label them by data $R = (H, \lambda, u)$ and $R' = (H', \lambda', u')$ as before. By repeating the same reasoning as above, one can show that R and R' are equivalent if and only if there exists a group element $x \in G$ such that

$$H' = {}^{x}H \ , \qquad \lambda' = {}^{x}\lambda \ , \qquad \left[\frac{u'}{{}^{x}u} \cdot \langle \lambda, \gamma_x(\alpha) \rangle\right] = 1 \ . \tag{3.258}$$

3.4.2.3 Example

As a simple example, consider the finite 2-groups $\mathcal{G}_\pm = \mathbb{Z}_4[1] \rtimes_\pm \mathbb{Z}_2$, where the 0-form component $\mathbb{Z}_2 =: \langle x \rangle$ acts on the 1-form component $\mathbb{Z}_4 =: \langle a \rangle$ via ${}^{x}a = a^{-1}$ and the $\pm$ indicates the choice of Postnikov class $[\alpha_\pm] \in H^3(\mathbb{Z}_2, \mathbb{Z}_4) \cong \mathbb{Z}_2$ given by

$$\alpha_+(x,x,x) \ = \ 1 \qquad \text{or} \qquad \alpha_-(x,x,x) \ = \ a \ . \tag{3.259}$$

The irreducible 2-representations in each case are labelled by triples (H, u, λ) as before and can be described as follows:

- If $\mathcal{G} = \mathcal{G}_+$, there is no non-trivial choice of 2-cocycle u since $H^2(\mathbb{Z}_2, U(1)) = 1$. Up to equivalence, there are hence five irreducible 2-representations labelled by

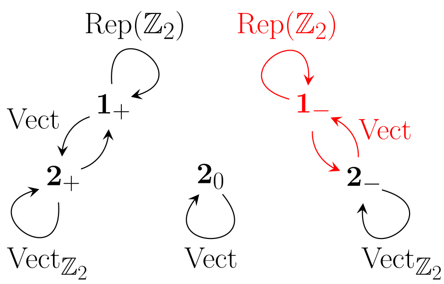
	$\mathbf{1}_+$	$\mathbf{1}_-$	$\mathbf{2}_+$	$\mathbf{2}_0$	$\mathbf{2}_-$
H	$\mathbb{Z}_2$	$\mathbb{Z}_2$	1	1	1
λ	1	$\widehat{a}^2$	1	$\widehat{a}$	$\widehat{a}^2$

(3.260)

  where $\widehat{a}$ denotes the character on the 1-form component defined by $\langle \widehat{a}, a \rangle = i$.

- If $\mathcal{G} = \mathcal{G}_-$, the condition $du = \langle \lambda, \alpha_-|_H \rangle$ admits no solution when $H = \mathbb{Z}_2$ and $\lambda = \widehat{a}^2$. As a result, the corresponding 2-representation $\mathbf{1}_-$ no longer exists.

Pictorially, we represent the irreducible 2-representations of $\mathcal{G}_\pm$ together with their categories of intertwiners by

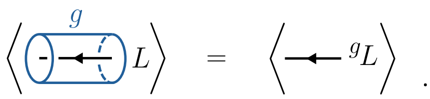


, (3.261)

where the red colouring indicates the absence of the 2-representation $\mathbf{1}_-$ and its intertwiners in the case of $\mathcal{G}_-$.

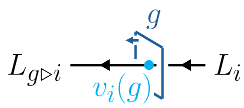
3.4.3 Applications

Having defined and classified unitary 2-representations of finite 2-groups, we can give the associated data a physical interpretation as follows: Consider a quantum field theory with finite 2-group symmetry $\mathcal{G} = A[1] \rtimes_\alpha G$ (i.e. the theory contains topological surface and line defects labelled by 0-form and 1-form group elements $g \in G$ and $a \in A$, respectively, cf. equation (3.77)). As before, the 0-form defects $g \in G$ can act on line operators L in the theory via wrapping, i.e.



(3.262)

Upon fixing a finite set of line operators L_i (where $i = 1, ..., n$) that is closed under the wrapping action of G, we can write ${}^gL_i = L_{\sigma_g(i)}$ for some permutation action $\sigma : G \to S_n$. If we furthermore assume the line operators L_i to be *simple* (meaning that the only topological local operators they host are scalar multiples of the identity), there exist one-dimensional spaces of topological local operators

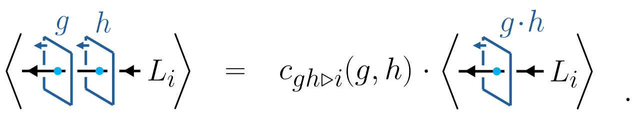


(3.263)

at the intersection of a simple line operator L_i with a symmetry defect $g \in G$. The composition of the junction vectors $v_i(g)$ is then of the form

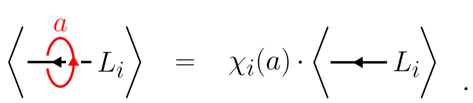
$$v_{h\triangleright i}(g) \circ v_i(h) \;=\; c_{gh\triangleright i}(g,h) \cdot v_i(gh) \tag{3.264}$$

for some multiplicative phases $c_i(g,h) \in U(1)$, which capture the fusion of 0-form defects $g, h \in G$ that intersect the line operators L_i inside correlation functions:



(3.265)

As a result, we can view the collection of the phases $c_i(g,h)$ as an 't Hooft anomaly for the 0-form symmetry G *on* the line operators L_i [1, 148]. In addition, we can link any line operator L_i with a 1-form defect $a \in A$ to produce a phase

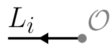


(3.266)

All in all, we see that the data associated to the action of the 2-group symmetry $\mathcal{G}$ on (simple) line operators forms a quadruple (n, σ, χ, c) that labels a unitary 2-

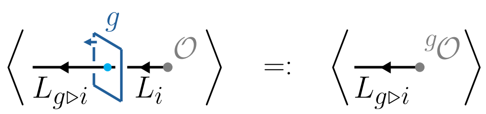
representation R of $\mathcal{G}$. This shows that, in analogy to the fact that local operators transform in unitary representations of symmetry groups, line operators transform in unitary 2-representations of symmetry 2-groups [1, 3, 148]. In the case where the permutation action σ is transitive, we can restrict attention to 0-form defects that lie in the stabiliser subgroup H of a fixed line operator L_1, which reproduces the data of an irreducible 2-unitary 2-representation. Physically, H then corresponds to the 'unbroken' subgroup of G on the line operator L_1.

In addition to the above, one may also consider non-genuine local operators $\mathcal{O}$ that sit at the end of the line operators L_i, i.e.[17]



. (3.267)

Using the operator-state map, the space of all such operators $\mathcal{O}$ then forms a Hilbert space V_i. Since any 1-form defect $a \in A$ can be unlinked from L_i if the latter ends on a local operator $\mathcal{O}$, we see that V_i can be non-zero if and only if $\chi_i(a) = 1$ for all $a \in A$. In this case, intersecting the line operator L_i with a symmetry defect $g \in G$ induces a linear map $\varphi_i(g) : V_i \to V_{g \triangleright i}$ via



, (3.268)

which as a consequence of (3.265) needs to obey the composition rule

$$\varphi_{h \triangleright i}(g) \circ \varphi_i(h) \;=\; c_{gh \triangleright i}(g,h) \cdot \varphi_i(g \cdot h) \,. \tag{3.269}$$

Mathematically, we see that the collection of Hilbert spaces V_i and linear maps φ_i forms the data of an intertwiner $\eta = (V, \varphi)$ between the trivial 2-representation $\mathbb{1}$ and $R = (n, \sigma, \chi, c, s)$. Upon restricting to the case where the latter is irreducible, this then implies that local operators at the end of a fixed line operator L_1 transform in unitary projective representations of the associated unbroken subgroup H.

3.4.3.1 Example: Outer Automorphisms

Consider a pure gauge theory with simply connected gauge group $\mathbb{G}$. This theory has a split[18] 2-group symmetry given by the outer automorphism 2-group

$$\mathcal{G} \;=\; \mathcal{Z}(\mathbb{G})[1] \rtimes \mathrm{Out}(\mathbb{G}) \,. \tag{3.270}$$

[17] More generally, one could consider local junction operators that sit inbetween two non-trivial line operators. We will restrict attention to the above case for simplicity.
[18] We say that a 2-group $\mathcal{G}$ is *split* if its associated Postnikov class is trivial.

Concretely, this is the 2-group whose

- 0-form component is the group $\mathrm{Out}(\mathbb{G}) = \mathrm{Aut}(\mathbb{G})/\mathrm{Inn}(\mathbb{G})$ of outer automorphisms of $\mathbb{G}$, which consists of automorphisms $f : \mathbb{G} \to \mathbb{G}$ modulo precomposition with inner isomorphisms of the form ${}^{g}(.) : \mathbb{G} \to \mathbb{G}$ for some $g \in \mathbb{G}$,
- 1-form component is the centre $\mathcal{Z}(\mathbb{G})$ of $\mathbb{G}$, i.e. consists of group element $z \in \mathbb{G}$ that commute with all other elements of $\mathbb{G}$,
- action of the 0- on the 1-form component is given by $[f] \triangleright z = f(z)$.

A natural class of line operators in the theory is given by Wilson lines W_ρ labelled by irreducible representations ρ of the gauge group $\mathbb{G}$. In particular, they carry an action of the outer automorphism 2-group $\mathcal{G}$ that can be described as follows:

- The wrapping action of 0-form defects $[f] \in \mathrm{Out}(\mathbb{G})$ is given by

$$[f] \triangleright W_\rho \ = \ W_{\rho \circ f^{-1}} \ , \tag{3.271}$$

  where $\rho \circ f^{-1}$ is the irreducible representation of $\mathbb{G}$ obtained by precomposing ρ with the inverse of a representative f of $[f]$.
- The linking action of 1-form defects $z \in \mathcal{Z}(\mathbb{G})$ is given by

$$z \triangleright W_\rho \ = \ \chi_\rho(z) \cdot W_\rho \ , \tag{3.272}$$

  where $\chi_\rho(.) = \mathrm{Tr}[\rho(.)]/\dim(\rho)$ denotes the (normalised) character associated to the irreducible representation ρ.

Below we list a selection of simple examples:

- If $\mathbb{G} = SU(N)$ with $N > 2$, we have $\mathrm{Out}(\mathbb{G}) = \mathbb{Z}_2 =: \langle s \rangle$ and $\mathcal{Z}(\mathbb{G}) = \mathbb{Z}_N =: \langle z \rangle$. If we denote by $\Lambda^k(\mathbf{N})$ the k-th antisymmetric power of the fundamental representation, then the wrapping action of s on the corresponding Wilson lines is

$$s \triangleright W_{\Lambda^k(\mathbf{N})} \ = \ W_{\Lambda^{N-k}(\mathbf{N})} \ . \tag{3.273}$$

  The associated central characters are given by

$$\chi_{\Lambda^k(\mathbf{N})}(z) \ = \ e^{\frac{2\pi i k}{N}} \ . \tag{3.274}$$

- For $\mathbb{G} = \mathrm{Spin}(2N)$ with $N > 4$, we have $\mathrm{Out}(\mathbb{G}) = \mathbb{Z}_2 =: \langle s \rangle$, which exchanges the spinor and conjugate spinor representations $S^{\pm}$ of $\mathrm{Spin}(2N)$. The associated central characters depend on whether N is even or odd:

- ◦ If N is even, we have $\mathcal{Z}(\mathbb{G}) = \mathbb{Z}_2 \times \mathbb{Z}_2 =: \langle z_1, z_2 \rangle$ and the central characters associated to the two spinor representations are

$$\begin{aligned} \chi_{S^+}(z_1) &= 1 \\ \chi_{S^+}(z_2) &= -1 \end{aligned} \quad \text{and} \quad \begin{aligned} \chi_{S^-}(z_1) &= -1 \\ \chi_{S^-}(z_2) &= 1 \end{aligned} \quad . \tag{3.275}$$

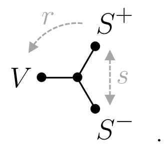
- ◦ If N is odd, we have $\mathcal{Z}(\mathbb{G}) = \mathbb{Z}_4 =: \langle z \rangle$ and the central characters associated to the two spinor representations are

$$\chi_{S^+}(z) = i \qquad \text{and} \qquad \chi_{S^-}(z) = -i \, . \tag{3.276}$$

- If $\mathbb{G} = \text{Spin}(8)$, the group of outer automorphisms enhances to

$$\text{Out}(\mathbb{G}) = S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2 =: \langle r, s \rangle \, , \tag{3.277}$$

which permutes the two spinor and the vector representation as follows:



(3.278)

The associated central characters of $Z(\mathbb{G}) = \mathbb{Z}_2 \times \mathbb{Z}_2 =: \langle z_1, z_2 \rangle$ are given by

$$\begin{aligned} \chi_{S^+}(z_1) &= 1 \\ \chi_{S^+}(z_2) &= -1 \end{aligned} \, , \quad \begin{aligned} \chi_{S^-}(z_1) &= -1 \\ \chi_{S^-}(z_2) &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} \chi_{V}(z_1) &= -1 \\ \chi_{V}(z_2) &= -1 \end{aligned} \quad . \tag{3.279}$$

3.4.3.2 U(1) Gauge Theory

Consider a $\mathbb{G} = U(1)$ gauge theory with two complex scalars ϕ_1 and ϕ_2 of charge $q = 2$. If we neglect magnetic and charge conjugation symmetries, this theory has a split 2-group symmetry that can be described as follows [120]:

- Its 0-form component is the continuous flavour symmetry $SO(3) \cong SU(2)/\mathbb{Z}_2$ that rotates ϕ_1 and ϕ_2, where the $\mathbb{Z}_2$ quotient mods out the diagonal transformation $\phi_i \to -\phi_i$ that can be reabsorbed by a gauge transformation.
- Its 1-form component is the $\mathbb{Z}_2$ subgroup of the center $\mathcal{Z}(\mathbb{G}) = U(1)$ capturing the Gukov-Witten defects that are unbroken in the presence of the scalars ϕ_i.
- Its Postnikov class is given by $\text{Bock}(w_2) \in H^3(SO(3), \mathbb{Z}_2)$, where we denoted by $w_2 \in H^2(SO(3), \mathbb{Z}_2)$ the extension class associated to

$$1 \to \mathbb{Z}_2 \hookrightarrow SU(2) \twoheadrightarrow SO(3) \to 1 \tag{3.280}$$

and Bock : $H^*(-,\mathbb{Z}_2) \to H^{*+1}(-,\mathbb{Z}_2)$ is the Bockstein map associated to

$$1 \;\to\; \mathbb{Z}_2 \;\hookrightarrow \mathbb{Z}_4 \;\twoheadrightarrow\; \mathbb{Z}_2 \;\to\; 1\,. \tag{3.281}$$

Since this 2-group is partly continuous, it falls out of the realm of our current framework. However, one may restrict attention to finite subgroups such as $D_4 \overset{\imath}{\hookrightarrow} SO(3)$, whose associated Postnikov class

$$\alpha \;=\; \imath^*(\text{Bock}(w_2)) \;=\; \text{Bock}(\imath^*(w_2)) \tag{3.282}$$

corresponds to the non-trivial generator of $H^3(D_4,\mathbb{Z}_2) \cong \mathbb{Z}_2$. Since wrapping a Wilson line W_n ($n \in \mathbb{Z}$) with a 0-form defect in D_4 leaves the former invariant, we see that the action of the 2-group $\mathcal{G} = \mathbb{Z}_2[1] \times_\alpha D_4$ on W_n is specified by the following data:

- A character $\chi_n = \chi^n \in \mathbb{Z}_2^\vee$, where χ denotes the generator of $\mathbb{Z}_2^\vee$. This captures the linking action of the topological Gukov-Witten defects generating the $\mathbb{Z}_2$ 1-form symmetry, which is trivial for even n since the associated Wilson line W_n can end on the charged matter fields ϕ_i.
- A 2-cochain $c_n \in C^2(D_4, U(1))$ such that $dc_n = \langle \chi_n, \alpha \rangle$. This captures the consecutive intersection of the Wilson line W_n with two 0-form defects in D_4. In particular, non-closedness of c_n for n odd reflects the fact that the associated Wilson line cannot end on any local operators and necessarily carries an 't Hooft anomaly for the 0-form symmetry D_4.

4

DISCUSSION

In this thesis, we developed systematic tools to describe the action of generalised global symmetries on local and extended operators in a quantum field theory. We generalised the construction of the tube category from two to three spacetime dimensions and provided concrete examples of both invertible and non-invertible symmetries. Below, we list a selection of open questions and possible future research directions.

4.1 Open Questions

While the construction of higher tube categories presented in this thesis marks a first step towards fully capturing the action of generalised global symmetries on physical observables, it leaves open several related questions of both physical and mathematical nature. We mention the following three examples:

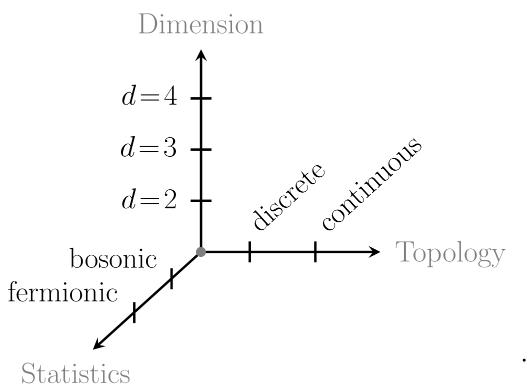
- **Tensor Products:** When considering the action of the tube category on twisted sector operators, we placed the latter on a single isolated spacetime locus (e.g. a point for local operators or a curve for line operators). In general, however, correlation functions will include several operator insertions placed at distinct spacetime loci. In analogy to the group-like case, we then expect the collection of these insertions to transform in an appropriate notion of "tensor product representations" of the tube category. While the latter can be defined explicitly in two dimensions using the so-called Day convolution product [154] (see also [155]), it would be desirable to generalise the construction to higher dimensions and to study the associated implications on both collections of (extended) operators and the structure of the tube category itself.
- **Extended Unitarity:** Throughout this thesis, we realised the principle of unitarity for generalised global symmetries by assuming that the associated symmetry categories are equipped with an appropriate †-structure, which implements spacetime reflections on the top-level of morphisms. In general, however, it was pointed out in [156] that an (∞, n)-category with all adjoints admits a whole variety of possible higher †-structures that are parameterised by different choices of subgroup $\mathfrak{G} \subset \text{Aut}(\text{AdjCat}_{(\infty,n)}) \cong \text{PL}(n)$. In [3], we considered the case $n = 2$ and $\mathfrak{G} = (\mathbb{Z}_2)^2$ for finite 2-group symmetries, while the †-structures

considered in this thesis correspond to choosing $\mathfrak{G} = \mathbb{Z}_2$. It would be desirable to investigate the physical meaning of more general types of †-structures and to study the associated induced structures on the tube category.

- **Physical Applications:** Given the abstract construction of higher tube categories and their representations, it would be desirable to apply the tools and techniques developed in this thesis to derive new dynamical implications of generalised symmetries in concrete physical examples. For instance, recent advances include the development of a categorical Landau paradigm that classifies gapped phases of matter via their generalised symmetries [157–160] as well as the construction of quantum lattice models that realise a given (higher) fusion category as their symmetry [161–163].

4.2 Outlook

Throughout this thesis, we considered generalised global symmetries in spacetime dimension $d \leq 3$ that were of finite bosonic type. In general, however, quantum field theories in any number of dimensions will typically exhibit symmetries that are both bosonic and fermionic as well as discrete and continuous. It is hence natural to try to extend the categorical description of generalised symmetries in three orthogonal directions as illustrated below:



While finite fermionic symmetries are believed to be captured by (higher) superfusion categories (see e.g. [164–166]), a rigorous mathematical treatment of continuous non-invertible symmetries seems more elusive. We expect that addressing these challenges will lead to novel insights and further progress in our physical and mathematical understanding of generalised symmetries.

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