On Cosmological Correlators In The α-Vacua

CHOPPING, ALISTAIR,JAMES

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Alistair James Chopping

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Abstract: This thesis, based largely on [1], deals with new techniques for computing cosmological correlation functions of scalar fields with respect to a class of vacuum states known as the α-vacua. By extending the Mellin space formalism for computing such correlators in the Bunch-Davies vacuum to the case of non-zero α, we show that to all orders in perturbation theory, late-time de Sitter boundary correlators can be written as a linear combination of their counterparts in the Bunch-Davies vacuum. The constituent perturbative Bunch-Davies contributions feature points in both the expanding and the contracting Poincaré patches. In turn, this reformulation allows us to relate these perturbative de Sitter correlators to Witten diagrams in Euclidean Anti-de Sitter space. In particular, we show that any de Sitter diagram can be written as a linear combination of EAdS Witten diagrams with analytically continued momenta, each dressed with α-dependent coefficients. In addition, we use our de Sitter results to compute the inflationary two- and three-point functions of inflaton perturbations at leading order in slow-roll, for arbitrary spacetime dimension and arbitrary choice of α-vacuum.

Declaration

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To my parents.

That’s all physics, you know.

— Jayne Elizabeth Chopping

You can do anything if you want it badly enough.

— Wayne Edward Chopping

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Chapter 1

Introduction

Millennia of scientific research have culminated in two physical theories that describe our universe at the largest and the smallest scales accessible to human beings. This thesis explores a small part of the story that unfolds when we try to combine them.

Quantum Field Theory (QFT) is a mathematical framework which at its most basic level describes the dynamics of elementary particles. The particular QFT that describes three of the four fundamental forces in our universe, known as the Standard Model of Particle Physics, has been tested to extraordinary accuracy. The part of the standard model describing electromagnetism, Quantum Electrodynamics, can be used to derive a property of the electron called its *anomalous magnetic moment*, whose theoretical prediction matches the experimental measurement to more than 10 significant figures [2]. To put this into perspective, this level of accuracy is analogous to predicting the distance between London and Athens, and this prediction agreeing with the measured value to within the width of a grain of salt[1].

At the other end of the universe's ruler, we have Einstein's theory of General Relativity (GR). GR has stood fast as our best description of gravity for over one

[1]The distance between London (England) and Athens is roughly $3,100$km, and a grain of salt is roughly half a millimetre in width. Thus, measuring the distance between London and Athens to within the width of a grain of salt is a precision of roughly $\frac{0.5}{3.1\times10^9} \approx 1.6\times10^{-10}$; around one part in 10^{10}.

hundred years, describing the universe on macroscopic scales - from planets all the way up to enormous filaments of galaxies billions of light-years in length. Amongst General Relativity's predictions are the existence of black holes and gravitational waves, the direct detection of which [3, 4] have been major recent achievements in modern experimental physics.

However, despite the overwhelming successes of QFT and GR, the two theories seem to conspire against unification. Treating General Relativity as a Quantum Field Theory leads to issues of *non-renormalisability*, whereby amplitudes for high-energy gravitational processes yield infinities that cannot be cured. Thought experiments concerning quantum aspects of black hole event horizons lead to issues like the so-called "firewall paradox" [5], which suggests that for GR and Quantum Mechanics to be compatible one may have to let go of some sacred tenet of one of the two. Experiment can't shed any light on the matter either - experimental detection of the would-be particle mediating gravitational interactions, the graviton, seems completely impractical. For instance, it was estimated in [6] that a detector with a mass comparable to that of Jupiter, in close orbit around a neutron star, would expect to detect one graviton every decade[2]. The revelation that GR and QFT in their current state simply do not mesh together[3] is a rather startling one - how can two theories that work so well in their own regimes, fail so spectacularly when we try to combine them? Ultimately, we live in one universe, so it is philosophically nonsensical to have two theories describing it that are incompatible with each other.

The last ~ 50 years or so have seen surprising connections between these two seemingly disparate paradigms, hinting at some underlying unified description. The Anti-de Sitter/Conformal Field Theory correspondence [8–10] is a conjecture positing that certain gravitational theories, in spacetimes with negative cosmological constant, are secretly equivalent to certain quantum field theories without gravity in one dimension lower. These non-gravitational QFTs are often easier to study

[2] We note that while direct graviton detection is highly unlikely, cunning proposals for measuring quantum gravitational effects in the laboratory do appear from time to time. See [7] for an overview.

[3] At least, not at high energies.

than their gravitational duals, and the correspondence is conjectured to hold even at the quantum level on the gravitational side, providing a handle on previously intractable problems in Quantum Gravity. Perhaps the most famous example of the correspondence is the conjecture that (on the gravitational side) type IIB string theory on $\text{AdS}_5 \times S^5$ is dual to (on the field theory side) $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $\mathbb{R}^{3,1}$.

In the most well-understood examples of AdS/CFT, the gravitational side of the correspondence is a ten-dimensional string theory, which at low energies is described by a theory of supergravity. The corresponding QFT in the conjecture is most often supersymmetric, conformally invariant, and non-confining. By contrast, we clearly live in four spacetime dimensions (not ten) and our universe is described by General Relativity (not supergravity) at low energies. In addition, the kinds of QFTs that describe the particle physics we have observed are non-supersymmetric, non-conformal, and (in the case of the strong force) confining. Thus, despite the enormous success of AdS/CFT, throughout its ~ 30 year tenure one particularly persistent hope has remained. Can we extend our understanding from AdS/CFT to gain insight into theories closer to those that describe our universe?

It is widely believed that our universe began with a period of extremely rapid accelerated expansion known as *inflation*, ending approximately 300,000 years before the beginning of the hot big bang. The inflationary paradigm provides an explanation for a number of puzzles that arose from our pre-inflationary understanding of early-universe cosmology. One such puzzle is the so-called *horizon problem*, in which the observed homogeneity of the Cosmic Microwave Background (CMB) implies that seemingly causally disconnected regions of the universe must have been in thermal equilibrium at some point during the universe's evolution, despite the absence of a mechanism for being so. The widely accepted solution to the horizon problem is that the period before the CMB was produced must have been longer than once thought, and that the offending regions of spacetime must have been in causal contact during this "inflationary" period.

Correlations between quantum fluctuations in the inflationary phase give rise to correlations in the CMB, which in turn give rise to correlations in the positions of the large-scale structures we see in the universe today. Thus, observations of large-scale structure can in principle be traced back through the evolution of the universe and into the inflationary phase, giving us a window into the highest-energy observable process in the history of our universe. The geometry of the inflationary phase is considered to be approximately that of de Sitter space (dS), a maximally symmetric vacuum solution of Einstein's equations with positive scalar curvature. Thus, understanding QFT in de Sitter space is paramount to understanding our universe as a whole.

It is important to note that an active area of research in modern experimental physics is dedicated to looking through that window into the dynamics of the inflationary phase. For example, the *temperature power spectrum* is a classic observable measured by *Planck* [11] (amongst others) that describes temperature inhomogeneities in the CMB. This power spectrum is "seeded" by the inflationary two-point function of primordial fluctuations on the reheating surface at the end of inflation, and so by measuring the temperature power spectrum we are peeking behind the curtain that hides the dynamics of inflation. Future experiments hope to measure so-called *non-Gaussianities* - higher-point correlations that could shed light on the spectrum of particles and their interactions during inflation. For instance, observables known as *bispectra* are related to inflationary three-point functions, and their measurement could in principle allow us to infer the masses of the fields present in the inflationary phase [12].

dS can be thought of as a closely related cousin of AdS, differing "only" by a change in sign of the scalar curvature. Despite the seemingly small difference between the two spaces, our understanding of quantum field theory in de Sitter space pales in comparison to that of AdS. Through AdS/CFT one can translate poorly-understood questions about quantum gravity in AdS into sharply-defined questions about the dual conformal field theory, which can then be answered through such techniques as integrability and the conformal bootstrap[4]. The situation with de Sitter however is far murkier - there are very few robust examples of dS/CFT, and in de Sitter holography as a whole even fundamental questions like where exactly the putative dual field theory should live[5] are poorly understood. Even more pedestrian issues such as the existence and definition of a de Sitter S-matrix[6] and, most pertinent to this thesis, the non-existence of a unique vacuum state further muddy the waters.

Scalar field theory in de Sitter space in fact admits an infinite one-parameter family of vacuum states that are all invariant under the de Sitter isometries [25, 26], which are often referred to as the *α-vacua*, with α a superselection parameter. The Bunch-Davies vacuum, corresponding to $\alpha = 0$, is the unique state that extrapolates to the standard Minkowski vacuum at early times. Two-point functions in the Bunch-Davies vacuum therefore have the usual short-distance singularity where the two points are null separated, while the other dS invariant vacua (also) have a singularity when the points are antipodal to one another. The latter is rather unconventional since in dS antipodal points are separated by a cosmological horizon, and whether or not interacting QFTs in vacua with $\alpha \neq 0$ are consistent has been the subject of debate [27–34]. For these reasons, late-time correlators with $\alpha \neq 0$ are less studied than their Bunch-Davies counterparts – though their phenomenological significance has been discussed e.g. in [35–42]. From a holographic perspective, it has been argued that in dS/CFT the bulk α-vacua correspond to a one-parameter family of marginal deformations of the putative dual CFT on the boundary [43, 44].

The standard approach for computing inflationary and de Sitter correlation functions is to use the *in-in formalism* [45, 46], which has given rise to the first perturbative

[4]The literature on these lines of research is vast; see for instance [13–15] for examples of the conformal bootstrap approach, and [16] for a review of the utility of integrability in the context of AdS/CFT.

[5]In the traditional dS/CFT approach [17] the CFT lives on the future boundary of de Sitter space, but other approaches such as static patch holography [18] and de Sitter world line holography [19], in which the dual theory lives elsewhere, have been considered.

[6]The asymptotic past and future of de Sitter space are separated by a cosmological horizon, and all observers are forbidden by causality from accessing both $\mathcal{I}^-$ and $\mathcal{I}^+$, making a traditional S-matrix difficult to define - see [20, 21]. For recent progress regarding the definition of a de Sitter S-matrix, see [22–24].

results [47–53] for late-time correlators and a rich spectrum of inflationary phenomenology. In recent years the *Cosmological Bootstrap*[7] has emerged as a complementary approach which aims to construct such correlators directly on the future boundary (making no reference to bulk time evolution) using physical principles such as unitarity, locality and symmetry as consistency requirements [55–60]. Invariance under the full set of dS isometries in particular implies that correlation functions on the late-time boundary of de Sitter space are constrained by conformal Ward identities [12, 55, 61–73]. The latter, combined with consistent on-shell factorisation and choice of initial state, has led to the first complete analytic understanding of the four-point function for massless scalars mediated by the exchange of a massive scalar in the Bunch-Davies vacuum [55]. In the former approach, the in-in formalism, one often encounters complicated bulk time integrals involving Bessel functions that are notoriously difficult to compute. In recent years, the *Mellin space* formalism [74–77] for cosmological correlators has proven to be extremely useful not only for tackling this issue, but also for providing new insights into the analytic structure of both de Sitter and inflationary correlation functions[8].

In particular, one approach to cosmological correlators has been to try to draw lessons from the relatively well understood Anti-de Sitter (AdS) case, where boundary correlation functions are, through the AdS/CFT correspondence, constrained non-perturbatively by the axioms of the Conformal Bootstrap [78, 79]. Using the structural similarities between dS and AdS, perturbative late-time correlators in the in-in formalism can be reformulated in the more familiar language of Witten diagrams in Euclidean Anti-de Sitter (EAdS) space. It is well-known that dS and EAdS are related by analytic continuation, and it turns out that this fact filters through to the propagators for quantum field theories on the two spaces. In particular, dS

[7]For an overview of the Cosmological Bootstrap programme see e.g. [54].

[8]We should stress that the relationship between de Sitter space and the inflationary geometry is *approximate* - inflationary correlation functions need not respect all of the isometries of de Sitter space, and the departure from an exact de Sitter background is controlled by a dependence on the *slow-roll parameter* ε that does not appear in the analogous de Sitter correlators. However, de Sitter space provides a somewhat simpler toy model of inflation, and in this sense is a “first step” towards an understanding of a more realistic setting.

propagators are a linear combination of appropriately analytically continued solutions to the propagator equation for a field of the same quantum numbers in EAdS. These relations have been understood for late-time correlators in the Bunch-Davies vacuum [75–77], and from them it follows that any given perturbative contribution can be expressed in terms of corresponding Witten diagrams in EAdS. Late-time correlators in the Bunch-Davies vacuum therefore have a similar analytic structure to their Euclidean AdS counterparts, and this has led to various new insights [76, 77, 80–86] into their properties - including at the non-perturbative level. These relations between dS and AdS open up the possibility to import techniques, results and understanding from the relatively well-understood AdS setting. One central question that this thesis attempts to answer is: *How do these relations manifest themselves for de Sitter vacua away from the Bunch-Davies vacuum?*

1.1 This Thesis

In this thesis, based in large part on [1], we develop new techniques for the computation of late-time boundary correlation functions in de Sitter space for arbitrary choice of α-vacuum. We also show how our results for exact de Sitter correlators can be used to obtain two- and three-point functions in an inflationary context. The main results of the thesis are:

- **Correlators in the α-vacua from Bunch-Davies** (sections 3.2 & 4). Perturbative contributions to late-time correlators for a generic α-vacuum in dS_{d+1} can be re-written in terms of their counterparts in the Bunch-Davies vacuum with points antipodally transformed to the boundary at past infinity. In momentum space this corresponds to flipping the magnitudes of the boundary momenta by phases $e^{\pm \pi i}$. This is proven in section 3.2 for massive scalar fields (i.e. principal series representations) by rewriting the propagators for generic α-vacuum in terms of their Bunch-Davies counterparts. It is possible to obtain expressions for other representations by analytic continuation and, where appropriate, renormalisation.

- **Correlators in the α-vacua from EAdS** (sections 3.3 & 4).

  Perturbative late-time correlators for a generic α-vacuum in dS_{d+1} can be reformulated in terms of corresponding Witten diagrams in EAdS_{d+1} with points antipodally transformed from the boundary on the upper to the lower sheet of the EAdS hyperboloid. As above, in momentum space this corresponds to flipping the magnitudes of the boundary momenta by phases $e^{\pm\pi i}$. This is achieved in section 3.3 by rewriting the Bunch-Davies propagators involved in the results of section 3.2 in terms of their EAdS counterparts. This extends the results of [75–77] for late-time correlators in the Bunch-Davies vacuum to initial states related to the latter via Bogoliubov transformation. Various examples for contact diagrams in the α-vacua are given in chapter 4, and we show how to construct the four-point exchange diagram of general scalars. A full expression for the scalar four-point de Sitter exchange written as a linear combination of EAdS exchanges is available at [87].

- **Mellin space for generic α-vacuum**. We extend the Mellin space representation of momentum space conformal correlators [74–77] to include late-time correlators in a generic α-vacuum. These differ from their Bunch-Davies ($\alpha = 0$) counterparts by phases in the Mellin variables. We review the solution of the momentum space conformal Ward identities using the Mellin representation (appendix C), where dilatation symmetry is manifest and the Ward identity for special conformal transformations is reduced to a recursion relation. For this reason, various propagator and correlator identities appearing in this work are manifest in Mellin space.

- **Inflationary correlators for arbitrary choice of α-vacuum** (chapter 5).

  The final chapter of the thesis is based on unpublished work. We use our results for three-point contact and four-point exchange diagrams in the α-vacua to compute corrections to the inflationary two- and three-point (respectively) correlation functions, to leading order in slow-roll. We find that our result for the inflationary three-point function (which we calculate in the squeezed limit) is a product of two inflationary two-point functions.

Notation and Conventions. We work in $(d+1)$-dimensional (EA)dS with the "mostly plus" metric signature. Momentum vectors are denoted $\vec{k}$, and have magnitude $k \equiv |\vec{k}|$. The momentum of the i-th external leg in a correlation function is denoted $\vec{k}_i$, while $\vec{k}$ is reserved for the momentum of an exchanged particle. Ambient space indices are denoted by capital Latin letters $A, B \in \{0, 1, ..., d+1\}$. Bulk scalar fields are denoted by ϕ, and have scaling dimension $\Delta = \frac{d}{2} + i\nu$ unless otherwise stated. s_i and u_i are reserved for Mellin variables associated with external and internal particles, respectively. Unless otherwise stated we set the curvature radius L of (EA)dS to $L = 1$.

Chapter 2

Background

In this chapter we review the salient aspects of quantum field theory in de Sitter and Anti-de Sitter spacetime. We begin with a description of the classical geometry of these manifolds, before reviewing scalar field theory on both dS and AdS. We discuss the classification of particles in de Sitter according to Unitary Irreducible Representations of the de Sitter isometry group. We then detail the issues pertinent to vacuum states in de Sitter space, explaining how the α-vacua arise.

2.1 The Geometry of (Anti-) de Sitter Spacetime

2.1.1 Anti-de Sitter Spacetime

Anti-de Sitter (AdS) spacetime is a maximally symmetric vacuum solution to Einstein's equations with negative cosmological constant. AdS_{d+1} can be embedded in Minkowski $\mathbb{R}^{d,2}$ with *embedding coordinates* $X_i,\ i \in \{0, ..., d+1\}$ and metric

$$ds^2_{\mathbb{R}^{d,2}} = -dX_0^2 - dX_{d+1}^2 + dX_1^2 + ... + dX_d, \tag{2.1.1}$$

subject to the hyperbolic embedding equation

$$-X_0^2 - X_{d+1}^2 + X_1^2 + ... + X_d^2 = -L^2_{\mathrm{AdS}}, \tag{2.1.2}$$

where L_{AdS} is a length scale called the AdS radius, often set to 1 in this thesis. From the embedding equation one can immediately see that the isometry group of AdS_{d+1} is $O(d,2)$. Different solutions to this equation then parameterise different charts on the AdS hyperboloid. In this thesis we will be most concerned not with Anti-de Sitter, but with *Euclidean* Anti-de Sitter (EAdS)[1]. From now on we will use "AdS" and "EAdS" interchangeably to mean EAdS. EAdS_{d+1} can be viewed as a codimension-1 surface embedded in $\mathbb{R}^{d+1,1}$ subject to the hyperbolic embedding equation

$$-X_0^2 + X_1^2 + ... + X_d^2 + X_{d+1}^2 = -L_{\text{EAdS}}^2, \tag{2.1.3}$$

whose different solutions then result in different charts on EAdS. Note that EAdS is a two-sheeted hyperboloid; rearranging for X_0 we find

$$X_0^2 = \sum_{j=1}^{d+1} X_j^2 + L_{\text{EAdS}}^2. \tag{2.1.4}$$

Since $X_0^2 = 0$ has no solutions (for real X_j), the regions of the hyperboloid with $\text{sgn}(X_0) = \pm 1$ are disconnected from one another, and EAdS is a two-sheeted hyperboloid. The chart we will be most concerned with throughout is the *Poincaré patch*, obtained via the solution (setting $L_{\text{AdS}} = 1$)

$$X_0 = \frac{1+x^2+z^2}{2z} \ , \ \ X_i = \frac{x_i}{z} \ , \ \ X_{d+1} = \frac{1-x^2-z^2}{2z}, \qquad z > 0 \tag{2.1.5}$$

where $i \in \{1, 2, ..., d\}$. This is a chart on the upper sheet of the AdS hyperboloid, since $X_0 > 0$. We can multiply each coordinate by -1 to obtain the analogous chart on the lower sheet;

$$X_0 = -\frac{1+x^2+z^2}{2z} \ , \ \ X_i = -\frac{x_i}{z} \ , \ \ X_{d+1} = -\frac{1-x^2-z^2}{2z}, \qquad z > 0 \tag{2.1.6}$$

is also a solution to the embedding equation, this time with $X_0 < 0$. We can therefore summarise EAdS in Poincaré coordinates as the solution

$$X_{\text{AdS}^\pm} \equiv (X^0_{\text{AdS}^\pm}, X^i_{\text{AdS}^\pm}, X^{d+1}_{\text{AdS}^\pm})$$

[1] Or simply *hyperbolic space* in mathematical literature.

$$= \pm\frac{1}{z}\left(\frac{1+z^2+x^2}{2}, x^i, \frac{1-z^2-x^2}{2}\right), \qquad z>0, \tag{2.1.7}$$

where the upper sheet is parameterised by X_{AdS^+} with $X^0_{\mathrm{AdS}^+} > 0$ and the lower sheet by X_{AdS^-} with $X^0_{\mathrm{AdS}^-} < 0$.

Plugging the above into the metric for $\mathbb{R}^{d+1,1}$ we obtain the metric for Euclidean AdS_{d+1} in Poincaré coorindates,

$$ds^2_{\mathrm{EAdS}_{d+1}} = \frac{L^2_{\mathrm{AdS}}}{z^2}(dz^2 + \delta_{ij}dx^i dx^j), \tag{2.1.8}$$

where $z \in [0,\infty)$. One important feature of Anti-de Sitter space is the presence of a *conformal boundary* at $z=0$. Notice that a Weyl rescaling of the metric and subsequently sending $z \to 0$ results in the metric for Euclidean $\mathbb{R}^d$. I.e,

$$\lim_{z\to 0}\left(\frac{z^2}{L^2_{\mathrm{EAdS}}} ds^2_{\mathrm{EAdS}_{d+1}}\right) = ds^2_{\mathbb{R}^d} := \delta_{ij}dx^i dx^j. \tag{2.1.9}$$

The EAdS boundary[2] is therefore a copy of $\mathbb{R}^d$. We can also see this by considering coordinates Q defining an embedding of $\mathbb{R}^d$ in $\mathbb{R}^{d+1,1}$, defined by

$$Q^0 = \frac{1+x^2}{2} \ , \ Q^i = x^i \ , \ Q^{d+1} = \frac{1-x^2}{2}, \tag{2.1.10}$$

where again $i \in \{1, ..., d\}$. Notice that in the $z \longrightarrow 0$ limit, we have

$$zX \longrightarrow Q, \tag{2.1.11}$$

namely the coordinates defining an embedding of AdS_{d+1} in $\mathbb{R}^{d+1,1}$ (ie, the X) coincide with the coordinates defining an embedding of $\mathbb{R}^d$ in $\mathbb{R}^{d+1,1}$ (ie, the Q) up to some overall factor.

From (2.1.3) we see that the isometry group of EAdS is $O(d+1,1)$. This happens to also be the isometry group of de Sitter space, and the two manifolds turn out to be related by a Wick rotation. We will exploit this fact heavily in this work.

[2]Note that AdS is maximally symmetric with constant Ricci scalar everywhere, and therefore does not have a boundary in the sense of a "manifold with boundary". In all cases in this thesis when we refer to the AdS or dS "boundary", we mean the *conformal* boundary as defined above.

2.1.2 de Sitter Spacetime

de Sitter (dS) spacetime is a maximally symmetric vacuum solution to Einstein's equations with positive cosmological constant. dS_{d+1} can be embedded in Minkowski $\mathbb{R}^{d+1,1}$ with coordinates $X_i,\ i \in \{0, ..., d+1\}$ and metric

$$ds^2_{\mathbb{R}^{d+1,1}} = -dX_0^2 + dX_1^2 + ... + dX_d + dX_{d+1}^2, \tag{2.1.12}$$

subject to the hyperbolic embedding equation

$$-X_0^2 + X_1^2 + ... + X_d^2 + X_{d+1}^2 = L_{\mathrm{dS}}^2, \tag{2.1.13}$$

where analogously to the AdS case L_{dS} is a length scale called the de Sitter radius[3]. Unlike EAdS, the equation $X_0^2 = 0$ has solutions (of the form $\sum_{i=1}^{d+1} X_i^2 = L_{\mathrm{dS}}^2$), and so the regions for $X_0 < 0$ and $X_0 > 0$ are connected to one another. de Sitter space is therefore a one-sheeted hyperboloid. One can see from (2.1.13) that the isometry group of de Sitter space is $O(d+1,1)$, which has four connected components[4]. The component connected to the identity is denoted $SO^+(d+1,1)$, whose action preserves the time direction and the space orientation of de Sitter space. Discrete elements of $O(d+1,1)$ include time reversal and the parity transformation, given in embedding space by[5]

$$T := \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{d+1}) \tag{2.1.14}$$

$$P = \mathrm{diag}(1, -1, \underbrace{1, \ldots, 1}_{d}). \tag{2.1.15}$$

[3]Again, often set to 1.

[4]An $O(d+1,1)$ transformation acts on the coordinates as $X^\mu \to \Lambda^\mu{}_\nu X^\nu$. The four connected components are then characterised by the sign of $\Lambda^0{}_0$ and $\det(\Lambda) = \pm 1$. The component with $\det(\Lambda) = +1$ and $\Lambda^0{}_0 > 0$ is denoted $SO^+(d+1,1)$.

[5]In general dimensions, a parity transformation is defined to be any transformation that flips the sign of *one* spatial coordinate; on $\mathbb{R}^3$ it turns out that flipping the sign of *all* coordinates is also a valid parity transformation. We choose to flip the X_1 coordinate by convention.

respectively. An additional discrete element which will prove essential to this work is the antipodal transformation, defined

$$A := \operatorname{diag}\underbrace{(-1, ..., -1)}_{d+2}, \tag{2.1.16}$$

which acts in embedding space on a point $x \in \mathrm{dS}_{d+1}$ as

$$A : X(x) \longmapsto X(\bar{x}) := -X(x). \tag{2.1.17}$$

Solutions to (2.1.13) correspond to various coordinate charts on dS_{d+1}. One solution covers the *Expanding Poincaré Patch* (EPP), and is parameterised by the coordinates (t_+, x_+^i);

$$\begin{aligned} X_{\mathrm{dS}^+} &\equiv (X^0_{\mathrm{dS}^+}, X^i_{\mathrm{dS}^+}, X^{d+1}_{\mathrm{dS}^+}) \\ &= \left(L \sinh\left(\frac{t_+}{L}\right) + \frac{\vec{x}_+^2}{2L}\, e^{\frac{t_+}{L}}\ ,\ \frac{x^i_+}{L}\, e^{\frac{t_+}{L}}\ ,\ -L\cosh\left(\frac{t_+}{L}\right) + \frac{\vec{x}_+^2}{2L}\, e^{\frac{t_+}{L}}\right), \end{aligned} \tag{2.1.18}$$

giving the metric

$$ds^2_{\mathrm{dS}^+} = -dt_+^2 + e^{+\frac{2t_+}{L}} d\vec{x}_+^2. \tag{2.1.19}$$

The EPP covers half of the de Sitter manifold, with the above parameterisation satisfying $X^0 \geq X^{d+1}$. The other half of the manifold is covered by the *Contracting Poincaré Patch* (CPP), which corresponds to the solution

$$\begin{aligned} X_{\mathrm{dS}^-} &\equiv (X^0_{\mathrm{dS}^-}, X^i_{\mathrm{dS}^-}, X^{d+1}_{\mathrm{dS}^-}) \\ &= \left(L \sinh\left(\frac{t_-}{L}\right) - \frac{\vec{x}_-^2}{2L}\, e^{-\frac{t_-}{L}}\ ,\ \frac{x^i_-}{L}\, e^{-\frac{t_-}{L}}\ ,\ L\cosh\left(\frac{t_-}{L}\right) - \frac{\vec{x}_-^2}{2L}\, e^{-\frac{t_-}{L}}\right), \end{aligned} \tag{2.1.20}$$

giving the metric

$$ds^2_{\mathrm{dS}^-} = -dt_-^2 + e^{-\frac{2t_-}{L}} d\vec{x}_-^2. \tag{2.1.21}$$

In the context of inflationary cosmology, the EPP is most commonly used after an additional transformation to conformal time. In fact, defining

$$\eta_\pm = L e^{\mp\frac{t_\pm}{L}}, \tag{2.1.22}$$

the solution to the embedding equation becomes

$$X_{\mathrm{dS}^+} = \left(\frac{L^2 + \vec{x}_+^2 - \eta_+^2}{2\eta_+} \, , \, \frac{x_+^i}{\eta_+} \, , \, \frac{-L^2 + \vec{x}_+^2 - \eta_+^2}{2\eta_+} \right) \tag{2.1.23}$$

for the EPP, and

$$X_{\mathrm{dS}^-} = \left(\frac{-L^2 - \vec{x}_-^2 + \eta_-^2}{2\eta_-} \, , \, \frac{x_-^i}{\eta_-} \, , \, \frac{L^2 - \vec{x}_-^2 + \eta_-^2}{2\eta_-} \right) \tag{2.1.24}$$

for the CPP. The metric for each patch then takes the same form;

$$ds^2 = \frac{L^2}{\eta_\pm^2}(-d\eta_\pm^2 + \delta_{ij} dx_\pm^i dx_\pm^j). \tag{2.1.25}$$

The difference between these is that in the EPP, the conformal time is related to t_+ as

$$\eta_+ \in (\infty, 0) \iff t_+ \in (-\infty, +\infty), \tag{2.1.26}$$

whereas in the CPP we have

$$\eta_- \in (0, \infty) \iff t_- \in (-\infty, +\infty). \tag{2.1.27}$$

As such, the conformal boundary of the EPP is at future infinity where $\eta_+ = 0$, and that of the CPP is at past infinity where $\eta_- = 0$. One can also see the presence of these boundaries in the embedding space formalism. One can define an embedding of $\mathbb{R}^d$ in $\mathbb{R}^{d+1,1}$ by coordinates $Q_\pm$, where

$$Q_\pm = \pm \left(\frac{L^2 + \vec{x}^2}{2}, \, x^i \, , \frac{-L^2 + \vec{x}^2}{2} \right). \tag{2.1.28}$$

In the $\eta_\pm \to 0$ limit, we see that

$$\lim_{\eta_\pm \to 0} \eta_\pm X_{\mathrm{dS}^\pm} = Q_\pm, \tag{2.1.29}$$

if we identify $x_+^i = -x_-^i$. In this way, we see that a point described by Q_+ on the future boundary of dS is related to a point described by Q_- on the past boundary via the antipodal map;

$$A : Q_\pm \longmapsto Q_\mp. \tag{2.1.30}$$

The Penrose diagram in figure 2.1 shows the regions of de Sitter space that the EPP and CPP cover, and the locations of the conformal boundaries.

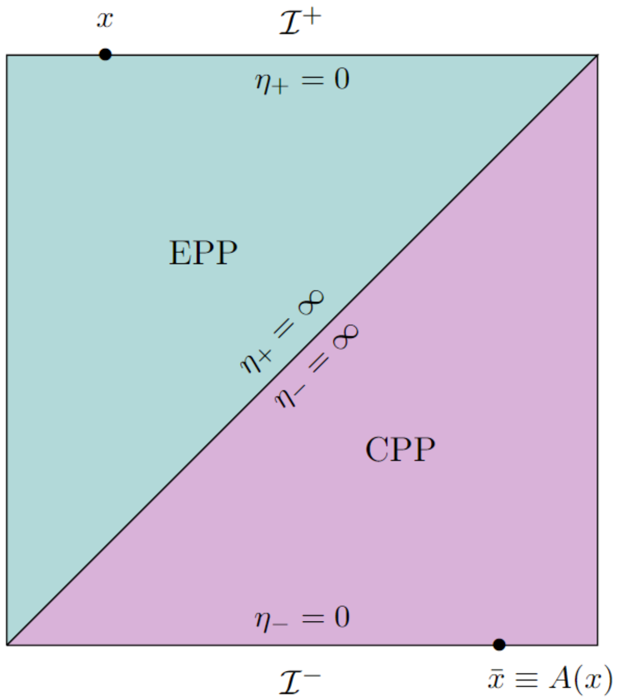


Figure 2.1: The Penrose diagram for de Sitter space, showing the Expanding and Contracting Poincaré patches and the locations of the conformal boundaries at past and future infinity. The antipodal map takes a point on the boundary at future infinity $\mathcal{I}^+$ to its opposite point on the boundary at past infinity $\mathcal{I}^-$.

We can cover both the CPP and the EPP simultaneously by simply "gluing" the patches together; if we make the replacements $x_\pm \to x$ and $\eta_\pm \to \eta \in (-\infty, \infty)$, then the EPP is covered by $\eta = -\eta_+ \leq 0$ and the CPP is covered by $\eta = \eta_- \geq 0$. Then, the metric is given simply by

$$ds^2 = \frac{L_{\text{dS}}^2}{\eta^2}(-d\eta^2 + \delta_{ij}dx^i dx^j), \quad \eta \in (-\infty, \infty). \tag{2.1.31}$$

The antipodal transformation then acts by flipping the sign of the conformal time;

$$A : x \longmapsto \bar{x} = (-\eta, \vec{x}), \quad \text{where} \quad x = (\eta, \vec{x}). \tag{2.1.32}$$

In these coordinates, the past and future conformal boundaries are reached by sending η to zero from above and below, respectively. Namely, in sending $\eta \to 0^-$

we are reaching 0 from the EPP, probing the boundary at future infinity. Similarly, in sending $\eta \to 0^+$ we are reaching 0 from the CPP, probing the boundary at past infinity. Figure 2.2 shows a rearrangement of the Penrose diagram for de Sitter, illustrating how the η coordinate covers both patches.

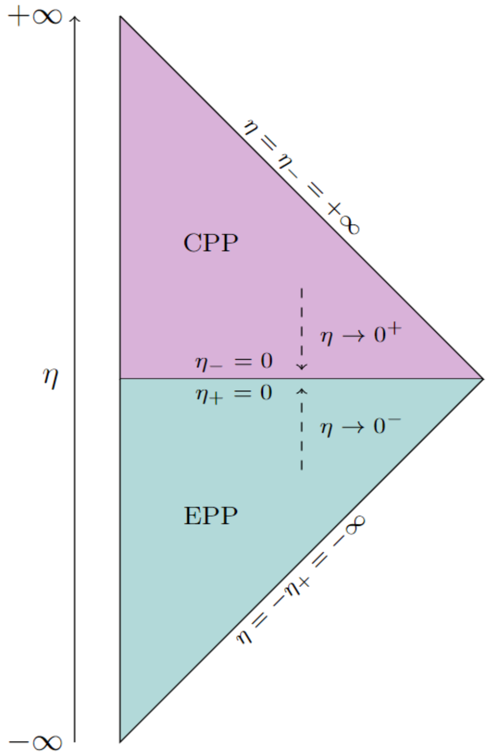


Figure 2.2: The EPP and CPP can simultaneously be covered by the time coordinate $\eta \in (-\infty, +\infty)$. The boundary at future infinity is reached from the EPP by sending $\eta \to 0^-$, and the boundary at past infinity is reached from the CPP by sending $\eta \to 0^+$.

2.2 Quantum Field Theory in de Sitter Space

In this section we review the salient aspects of scalar Quantum Field Theory in de Sitter space.

2.2.1 Late-Time Behaviour of the Scalar Field

Consider a massive scalar field on a fixed de Sitter background, propagating in the expanding Poincaré patch. The dynamics of the scalar are given by the canonical wave equation,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi(\eta,x)) - m^2\phi(\eta,x) = 0, \tag{2.2.1}$$

with the time coordinate restricted to $\eta \in (-\infty, 0)$ for the EPP. Using the metric (2.1.31), we find

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi(\eta,x)) - m^2\phi(\eta,x) \\
&= (-\eta)^{d+1}\partial_\mu\left[\left(-\frac{1}{\eta}\right)^{d+1} g^{\mu\nu}\partial_\nu\phi(\eta,x)\right] - m^2\phi(\eta,x) \\
&= (-\eta)^{d+1}\left\{\partial_\eta\left[\left(-\frac{1}{\eta}\right)^{d+1} \underbrace{g^{\eta\eta}}_{-\frac{\eta^2}{L^2}}\partial_\eta\phi(\eta,x)\right] + \partial_i\left[\left(-\frac{1}{\eta}\right)^{d+1} \underbrace{g^{ij}}_{\frac{\eta^2}{L^2}\delta^{ij}}\partial_j\phi(\eta,x)\right]\right\} - m^2\phi(\eta,x) \\
&= \frac{(-\eta)^{d+1}}{L^2}\left\{-\partial_\eta\left[(-\eta)^{-d+1}\partial_\eta\phi(\eta,x)\right] + \left(-\frac{1}{\eta}\right)^{d+1}\partial_i\partial^i\phi(\eta,x)\right\} - m^2\phi(\eta,x) \\
&= \ldots \\
&= \left((-\eta)^2\frac{\partial^2}{\partial(-\eta)^2} - (d-1)(-\eta)\frac{\partial}{\partial(-\eta)} - (-\eta)^2\partial_i\partial^i + m^2L^2\right)\phi(\eta,x) = 0.
\end{aligned} \tag{2.2.2}$$

Passing to momentum space in the spatial directions, this becomes

$$\left((-\eta)^2\frac{\partial^2}{\partial(-\eta)^2} - (d-1)(-\eta)\frac{\partial}{\partial(-\eta)} + (-\eta)^2k^2 + m^2L^2\right)\phi_{\vec{k}}(\eta) = 0. \tag{2.2.3}$$

Let us now consider an ansatz $\phi_{\vec{k}}(\eta) = \mathcal{O}(k)(-\eta)^\Delta$, with $\mathcal{O}(k)$ some arbitrary function of the momentum. The wave equation then implies

$$\Delta(\Delta-1) + \Delta(1-d) + (-\eta)^2k^2 + m^2L^2 = 0. \tag{2.2.4}$$

Close to the late-time boundary, namely $\eta \to 0$, we have

$$\begin{aligned} 0 &\approx \Delta(\Delta-1) + \Delta(1-d) + m^2L^2 \\ &= \Delta^2 - d\Delta + m^2L^2, \end{aligned} \tag{2.2.5}$$

which has solutions

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - m^2L^2}. \tag{2.2.6}$$

Thus, we see that the behaviour of the field close to the future boundary takes a particularly simple form given by

$$\phi_{\vec{k}}(\eta \to 0^-) \sim \mathcal{O}_+(k)(-\eta)^{\Delta_+} + \mathcal{O}_-(k)(-\eta)^{\Delta_-}, \tag{2.2.7}$$

where it will prove convenient to parameterise $\Delta_\pm$ as

$$\Delta_\pm = \frac{d}{2} \pm i\nu. \tag{2.2.8}$$

2.2.2 Classification of Particles in de Sitter Space

A basic consistency requirement for sensible physics is that the outcomes of experiments should be unchanged when the system is acted upon by a symmetry transformation. For instance, performing some particular experiment should yield identical results regardless of whether we perform it in Durham, or in London - the results should be the same if we translate the experiment by a few hundred miles.

It turns out that this idea (admittedly, pushed rather far) leads to the classification of particle states in quantum field theory, telling us the kinds of particles that are "allowed" if we want consistent physics. As a simple illustrative example, let $\mathcal{M}$ and $\mathcal{S}$ be two operators on a vector space V corresponding to the mass and spin of some particle, and let

$$\begin{aligned} \rho : G &\longrightarrow GL(V) \\ g &\longmapsto \rho(g) \end{aligned} \tag{2.2.9}$$

be some representation of a Lie group G that we want our system to be invariant under. In particular, since the mass and spin of the particle are observables we want the operators $\mathcal{M}$ and $\mathcal{S}$ to be invariant under the action of G, namely

$$\rho(g)\mathcal{M}\rho(g)^{-1} \stackrel{!}{=} \mathcal{M} \iff [\rho(g), \mathcal{M}] \stackrel{!}{=} 0, \tag{2.2.10}$$

$$\rho(g)\mathcal{S}\rho(g)^{-1} \stackrel{!}{=} \mathcal{S} \iff [\rho(g), \mathcal{S}] \stackrel{!}{=} 0. \tag{2.2.11}$$

We also want these operators to assume fixed values on each representation; namely we want that the action of $\rho(g)$ on a state $|\psi\rangle$ does not change the mass or spin of $|\psi\rangle$. If $\rho(g)$ is an *irreducible* representation (irrep) of G, then by Schur's lemma

$$[\rho(g), \mathcal{M}] = 0 \implies \mathcal{M} \propto \mathbb{I}, \tag{2.2.12}$$

namely $\mathcal{M} = m\mathbb{I}$ with m some proportionality constant. Thus, ρ being irreducible is a sufficient condition for $\mathcal{M}$ to assume a fixed value m. Note that different representations ρ will correspond to different m.

We also want to insist that inner products of the states remain invariant under the action of G. In particular, if we transform

$$|m\rangle \longrightarrow |\tilde{m}\rangle := \rho(g)\,|m\rangle\,, \tag{2.2.13}$$

we should insist that

$$\langle \tilde{m}|\tilde{m}\rangle \stackrel{!}{=} \langle m|m\rangle \implies \rho(g)^\dagger \rho(g) \stackrel{!}{=} \mathbb{I}. \tag{2.2.14}$$

Thus, we see that some basic consistency requirements imply that states should transform in *unitary, irreducible* representations (UIRs) of the symmetry group.

How should we classify these representations? One convenient method is provided by *Casimir invariants* - operators that can be constructed from generators of the Lie algebra $\mathfrak{g} \equiv \mathrm{Lie}(G)$. Let

$$\tilde{\rho} : \mathfrak{g} \longrightarrow \mathrm{End}(V) \tag{2.2.15}$$

be a representation of the Lie algebra $\mathfrak{g}$ on a vector space V, and $\{X_i\}_{i=1,\ldots,\dim(\mathfrak{g})}$ be any basis of $\mathfrak{g}$. Define the $\tilde{\rho}-$*Killing form* to be the map[6]

$$\begin{aligned} \kappa_{\tilde{\rho}} : \mathfrak{g} \times \mathfrak{g} &\longrightarrow F \\ (x, y) &\longmapsto \kappa_{\tilde{\rho}}(x, y) := \mathrm{Tr}(\tilde{\rho}(x) \circ \tilde{\rho}(y)), \end{aligned} \tag{2.2.16}$$

with F the field over which $\mathfrak{g}$ is defined (usually $\mathbb{R}$ or $\mathbb{C}$). Let $\{\xi_i\}$ be the dual basis with respect to $\{X_i\}$; the basis for which $\kappa_{\tilde{\rho}}(X_i, \xi_j) = \delta_{ij}$. The Casimir invariant associated with the representation $\tilde{\rho}$ is then defined

$$\mathcal{C}_{\tilde{\rho}} := \sum_{i=1}^{\dim(\mathfrak{g})} \tilde{\rho}(X_i) \circ \tilde{\rho}(\xi_i). \tag{2.2.17}$$

It can be shown that the Casimir invariant commutes with all elements of the Lie algebra, namely

$$[\mathcal{C}_{\tilde{\rho}}, \tilde{\rho}(a)] = 0, \ \forall a \in \mathfrak{g}. \tag{2.2.18}$$

Therefore, if $\tilde{\rho}$ is irreducible, then by Schur's lemma,

$$\mathcal{C}_{\tilde{\rho}} = c_{\tilde{\rho}} \cdot \mathrm{id}_V, \tag{2.2.19}$$

with $c_{\tilde{\rho}}$ a constant. Different irreps will result in different Casimir operators and therefore different $c_{\tilde{\rho}}$, and since $c_{\tilde{\rho}}$ are constant on each irrep, Casimir operators provide a convenient classification of irreps.

Let us now apply these ideas to the case at hand; particles in de Sitter space[7]. The generators of the d-dimensional Euclidean conformal group $\mathrm{Conf}(\mathbb{R}^d)$ are[8]

$$\underbrace{M_{ij}}_{\text{Spatial Rotations}} \quad , \quad \underbrace{D}_{\text{Dilatations}} \quad , \quad \underbrace{P_i}_{\text{Translations}} \quad , \quad \underbrace{K_i}_{\text{SCTs}}, \tag{2.2.20}$$

with $M_{ij} = -M_{ji}$. The isomorphism between $\mathrm{Conf}(\mathbb{R}^d)$ and $SO(d+1,1)$ is made manifest by defining

$$L_{ij} := M_{ij} \ , \ L_{0,d+1} := \mathcal{D} \ , \ L_{d+1,i} := \frac{1}{2}(P_i + K_i) \ , \ L_{0,i} := \frac{1}{2}(P_i - K_i), \tag{2.2.21}$$

[6]Once a basis for V is chosen, the map Tr is the elementary trace of a matrix.
[7]See [88] for an excellent review of the representation theory of $SO(d+1,1)$.
[8]With *SCTs* being short for Special Conformal Transformations.

with $L_{AB} = -L_{BA}$, which indeed generate the algebra $\mathfrak{so}(d+1,1)$[9],

$$[L_{AB}, L_{CD}] = \eta_{BC}L_{AD} - \eta_{AC}L_{BD} + \eta_{AD}L_{BC} - \eta_{BD}L_{AC}, \tag{2.2.22}$$

where

$$\eta_{AB} = \mathrm{diag}(-1, \underbrace{1, ..., 1}_{d+1}), \tag{2.2.23}$$

is used to raise and lower indices of the L_{AB}. To get a *unitary* representation of $SO(d+1,1)$ we also require the reality condition

$$L^{\dagger}_{AB} := (L_{BA})^* = (-L_{AB})^* \overset{!}{=} -L_{AB}. \tag{2.2.24}$$

The (quadratic) Casimir invariant is chosen to be

$$\begin{aligned}
\mathcal{C}_2 &:= \frac{1}{2}L_{AB}L^{AB} \\
&= \frac{1}{2}(2L_{0i}L^{0i} + 2L_{0,d+1}L^{0,d+1} + L_{ij}L^{ij} + 2L_{i,d+1}L^{i,d+1}) \\
&= \frac{1}{2}(-2L_{0i}L_{0i} - 2L_{0,d+1}L_{0,d+1} + L_{ij}L^{ij} + 2L_{i,d+1}L_{i,d+1}) \\
&= \frac{1}{2}(P_iK^i + K_iP^i - 2\mathcal{D}^2 + M_{ij}M^{ij}).
\end{aligned} \tag{2.2.25}$$

Now, using $[K_i, P^j] = 2\delta_i^{\ j}\mathcal{D} - 2M_i^{\ j}$ and $M_i^{\ i} = 0$ (since M_{ij} is antisymmetric), we have

$$\mathcal{C}_2 = \mathcal{D}(d-\mathcal{D}) + P_iK^i + \frac{1}{2}M_{ij}M^{ij}. \tag{2.2.26}$$

To find the eigenvalue of this Casimir operator, c_2, we allow $\mathcal{C}_2$ to act on a state that transforms in some arbitrary irreducible representation of $\mathrm{Conf}(\mathbb{R}^d)$. Since the Casimir operator is constant on each representation, the value of c_2 doesn't depend on the specific state we choose (only the representation the state transforms in), and so we can make things easier by choosing the state to be a conformal primary $|\Delta, s\rangle$, satisfying (by definition)

$$\mathcal{D}\,|\Delta, s\rangle = \Delta\,|\Delta, s\rangle \tag{2.2.27}$$

$$K^i\,|\Delta, s\rangle = 0 \tag{2.2.28}$$

[9]This can be seen by plugging in the commutation relations for the conformal algebra.

$$\frac{1}{2}M_{ij}M^{ij}\left|\Delta,s\right\rangle = -s(s+d-2)\left|\Delta,s\right\rangle. \tag{2.2.29}$$

Acting on this state with the Casimir operator, we find

$$\mathcal{C}_2\left|\Delta,s\right\rangle = (\Delta(d-\Delta)-s(s+d-2))\left|\Delta,s\right\rangle, \tag{2.2.30}$$

and so we read off the eigenvalue

$$c_2 = \Delta(d-\Delta)-s(s+d-2). \tag{2.2.31}$$

It turns out that these two parameters Δ and s (the conformal dimension and the spin, respectively) are enough to classify (or "label") all unitary irreducible representations of $SO(d+1,1)$ that describe traceless - symmetric fields in de Sitter space.

Furthermore, for scalar fields $s=0$ the UIRs of the de Sitter isometry group fall into three categories known as the *Principal Series*, the *Complementary Series* and the *Discrete Series*. Recall that the boundary behaviour of a massive scalar in de Sitter space was given by

$$\phi(\eta\to 0^-,k)\sim\mathcal{O}_+(k)(-\eta)^{\Delta_+}+\mathcal{O}_-(k)(-\eta)^{\Delta_-}, \tag{2.2.32}$$

with $\mathcal{O}_\pm$ transforming under Conf($\mathbb{R}^d$) like a scalar primary operator in a Euclidean conformal field theory, with conformal dimension $\Delta_\pm$. It is convenient to use the parameterisation $\Delta_\pm=\frac{d}{2}\pm i\nu$, and describe the three series of UIRs by specifying the value of ν. These three series are as follows:

1. *The Principal Series*

   - For the principal series, $\nu \in \mathbb{R}$ ($i\nu := \sqrt{\frac{d^2}{4} - m^2L^2}$ is imaginary) and so $m^2L^2 \geq \frac{d^2}{4}$. This class therefore corresponds to "heavy" particles.

2. *The Complementary Series*

   - For the complementary series, we define $\nu =: -i\mu$ with $\mu \in \mathbb{R}$ ($i\nu := \sqrt{\frac{d^2}{4} - m^2L^2}$ is real), and so $\Delta_\pm = \frac{d}{2} \pm \mu$. In this series we have $0 < m^2L^2 < \frac{d^2}{4}$, and so $\mu \in (0, \frac{d}{2})$. This class corresponds to "light" particles. Massless particles correspond to $\mu = \frac{d}{2}$ and lie on the boundary of the complementary series - sometimes referred to as the *Exceptional Series*.

3. *The Discrete Series*

   - The discrete series is subtle, and we will not consider it in this work. The discrete series poses a challenge on account of its containing tachyons, but recent work on these UIRs can be found in [89–91].

We should note that there are also spinning versions of the principal, complementary and discrete series. In particular, for traceless-symmetric spin-s fields (with s strictly greater than 0) in de Sitter space we have

$$\Delta \equiv \Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} - m^2L^2 + s} \equiv \frac{d}{2} + i\nu_s. \tag{2.2.33}$$

For spinning fields the representation theory of $SO(d+1,1)$ is more complicated, and depends on the dimension d. The spinning Principal, Complementary and Exceptional Series are:

- The Principal Series : $\nu_s \in \mathbb{R}$, so $m^2L^2 \geq \frac{d^2}{4} + s$.

- The Complementary Series : $\nu_s := -i\mu_s \implies \Delta = \frac{d}{2} + \mu_s$, with $\mu_s \in \mathbb{R}$ such that $0 < \mu_s < \frac{d}{2} - 1$. This corresponds to $d - 1 < m^2 L < \frac{d^2}{4} + s$.

- The Exceptional Series : Describes the boundary points of the complementary series.

We also have the spinning Discrete Series, which exists only for odd d (ie, even spacetime dimension $d+1$) and describes non-traceless-symmetric fields. For the Discrete Series, the spin vector $\vec{s} = (s_1, \ldots, s_r)$ has all entries non-vanishing, and the label Δ is given by $\Delta = \frac{d}{2} + k$ with $k \in \frac{1}{2}\mathbb{N}$ and $0 < k \leq s_r$.

de Sitter space also admits so-called "partially massless" fields, with the parameter ν given by $\nu = -\frac{i}{2}(d - 2 + 2(s - \zeta) - 2)$ for totally symmetric representations, where $\zeta \in \{0, 1, \ldots, s-1\}$. An excellent review of all of the above can be found in [92], along with a thorough discussion of non-traceless-symmetric representations.

2.2.3 de Sitter Vacua I - The Bunch-Davies Vacuum

As is standard procedure in quantum field theory, in Fourier space one can expand a scalar field[10] $\phi_{\vec{k}}(t)$ in a basis of mode functions[11] $\{f_{\vec{k}}(t), \bar{f}_{\vec{k}}(t)\}$,

$$\phi_{\vec{k}}(t) = f_k(t)\, a^\dagger_{\vec{k}} + \bar{f}_k(t)\, a_{-\vec{k}}, \tag{2.2.35}$$

which are normalised with respect to the Klein-Gordon inner product, defined

$$(f, g) := -i \int_\Sigma \mathrm{d}^d x \, \sqrt{\gamma} n^\mu \left(f^* \partial_\mu g - g \partial_\mu f^*\right), \tag{2.2.36}$$

where Σ is a spatial slice with induced metric γ_{ij} and n^μ is a timelike unit vector normal to Σ. Defining $\phi_1(t, \vec{x}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{+i\vec{k}\cdot\vec{x}} \bar{f}_k(t)$ and $\phi_2(t, \vec{x}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{+i\vec{k}\cdot\vec{x}} f_k(t)$,

[10] In position space we have

$$\phi(t, x) = \int \frac{d^d k}{(2\pi)^d} \phi_{\vec{k}}(t)\, e^{+i\vec{k}\cdot\vec{x}}. \tag{2.2.34}$$

[11] We follow the conventions of [12].

we require

$$(\phi_1, \phi_1) = 1, \quad (\phi_2, \phi_2) = -1, \quad (\phi_1, \phi_2) = 0. \tag{2.2.37}$$

$a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ are the standard creation and annihilation operators (respectively), satisfying the canonical commutation relations;

$$[a_{\vec{k}_1}, a_{\vec{k}_2}^\dagger] = (2\pi)^d \, \delta(\vec{k}_1 - \vec{k}_2), \qquad [a_{\vec{k}_1}, a_{\vec{k}_2}] = [a_{\vec{k}_1}^\dagger, a_{\vec{k}_2}^\dagger] = 0\,. \tag{2.2.38}$$

One then defines a vacuum state $|\Omega\rangle$ by demanding

$$a_{\vec{k}} \, |\Omega\rangle := 0, \tag{2.2.39}$$

thereby marrying the vacuum to a particular choice of the basis mode functions $f_{\vec{k}}(\eta)$.

We can use the Wightman function to characterise a vacuum state via

$$G_W^{(\Omega)}(x_1, x_2) := \langle\Omega| \, \phi(x_1)\phi(x_2) \, |\Omega\rangle\,, \tag{2.2.40}$$

whose form will depend on the vacuum $|\Omega\rangle$. The Wightman function solves the source-free wave equation; we see that since

$$\phi(x,t) = \int \frac{d^d k}{(2\pi)^d} (f_k(t) \, a_{\vec{k}}^\dagger \; + \; \bar{f}_k(t) \, a_{-\vec{k}}) e^{i\vec{k}\cdot\vec{x}}, \tag{2.2.41}$$

we have

$$\begin{aligned} G_W^{(\Omega)}(x_1, x_2) :=& \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \, \langle\Omega| \, (f_{k_1}(t_1) \, a_{\vec{k}_1}^\dagger \; + \; \bar{f}_{k_1}(t_1) \, a_{-\vec{k}_1}) \\ & \qquad \times (f_{k_2}(t_2) \, a_{\vec{k}_2}^\dagger \; + \; \bar{f}_{k_2}(t_2) \, a_{-\vec{k}_2}) \, |\Omega\rangle \, e^{i\vec{k}_1\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2} \\ =& \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \bar{f}_{k_1}(t_1) f_{k_2}(t_2) \underbrace{\langle\Omega| \, a_{-\vec{k}_1} a_{\vec{k}_2}^\dagger \, |\Omega\rangle}_{\langle\Omega| \, [a_{-\vec{k}_1}, a_{\vec{k}_2}^\dagger] \, |\Omega\rangle} e^{i\vec{k}_1\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2}. \end{aligned} \tag{2.2.42}$$

Sending $\vec{k}_1 \to -\vec{k}_1$ and using the canonical commutation relation (2.2.38), we obtain

$$\begin{aligned} G_W^{(\Omega)}(x_1, x_2) &= \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \bar{f}_k(t_1) f_k(t_2) e^{-i\vec{k}_1\cdot\vec{x}_1} e^{i\vec{k}_2\cdot\vec{x}_2} \; (2\pi)^d \, \delta^{(d)}(\vec{k}_1 - \vec{k}_2) \\ &= \int \frac{d^d k}{(2\pi)^d} \; \bar{f}_k(t_1) f_k(t_2) \; e^{i\vec{k}\cdot(\vec{x}_2 - \vec{x}_1)}. \end{aligned} \tag{2.2.43}$$

Since the mode functions by definition solve the homogeneous wave equation

$$(\nabla^2 - m^2)f(t,\vec{x}) = 0 \implies \int \frac{d^dk}{(2\pi)^d}(\nabla^2 - m^2)f_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} = 0, \tag{2.2.44}$$

the Wightman function also solves the homogeneous wave equation;

$$(\nabla^2_{x_1} - m^2)G_W^{(\Omega)}(x_1,x_2) = 0. \tag{2.2.45}$$

We also see that in Fourier space, the Wightman function is given simply by a product of the mode functions corresponding to $|\Omega\rangle$;

$$G_W^{(\Omega)}(t_1,t_2;\vec{k}) = \bar{f}_{\vec{k}}(t_1)f_{\vec{k}}(t_2), \tag{2.2.46}$$

a crucial fact that will prove extremely useful later on.

The Wightman function is required to be invariant under the isometry group of the background on which it is defined. In the case of de Sitter space, the Wightman function should therefore be invariant under the de Sitter isometry group $O(d+1,1)$. As such, $G_W^{(\Omega)}(x_1,x_2)$ must depend only on an $O(d+1,1)$-invariant combination of x_1 and x_2, which we define to be

$$\begin{aligned} s(x_1,x_2) &:= \eta_{AB}X_1(x_1)^A X_2(x_2)^B \equiv X_1(x_1)\cdot X_2(x_2) \\ &= 1 + \frac{(\eta_1-\eta_2)^2-(x_1-x_2)^2}{2\eta_1\eta_2}, \end{aligned} \tag{2.2.47}$$

where we have given its specific form in the Poincaré patch for convenience later on. This turns out to be more convenient to express via the variable

$$\sigma(x_1,x_2) := \frac{1+s(x_1,x_2)}{2} = 1 + \frac{(\eta_1-\eta_2)^2-(x_1-x_2)^2}{4\eta_1\eta_2}, \tag{2.2.48}$$

where we have

$$\sigma(x,x) = 1 \ , \ \sigma(x,\bar{x}) = 0. \tag{2.2.49}$$

In terms of the coordinate σ, it is well-known[12] that the wave equation becomes

[12] See, for instance, [93].

Euler's Hypergeometric equation,

$$\left[\sigma\left(1-\sigma\right)\partial_\sigma^2 G^{(\Omega)}\left(\sigma\right)-\left(\tfrac{d+1}{2}\right)\left(2\sigma-1\right)\partial_\sigma G^{(\Omega)}\left(\sigma\right)\right]-m^2 G^{(\Omega)}\left(\sigma\right)=0, \qquad (2.2.50)$$

whose solution is a linear combination of two Gauss Hypergeometric functions. In particular, we have

$$G^{(\Omega)}\left(\sigma\right)=A\,{}_2F_1\!\left(\tfrac{d}{2}+i\nu,\tfrac{d}{2}-i\nu;\tfrac{d+1}{2};\sigma\right)+B\,{}_2F_1\!\left(\tfrac{d}{2}+i\nu,\tfrac{d}{2}-i\nu;\tfrac{d+1}{2};1-\sigma\right), \quad (2.2.51)$$

with $A, B \in \mathbb{C}$. The Gauss Hypergeometric function is defined for $|z|<1$ as

$${}_2F_1(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad (2.2.52)$$

with $(a)_n$ the rising factorial[13], and can be defined elsewhere by analytic continuation. This has a branch point at $z=1$, with the corresponding branch cut chosen to be along $[1,\infty)$. This implies that (2.2.51) in turn is singular for $\sigma=1$ when $A\neq 0, B=0$, and for $\sigma=0$ when $A=0, B\neq 0$. For the former, we have

$$\sigma(x_1,x_1)=\frac{1+X_1(x_1)\cdot X_2(x_2)}{2}=1 \implies X_1(x_1)\cdot X_2(x_2)=1, \qquad (2.2.54)$$

which occurs when x_1 and x_2 are null separated, or lie on the lightcone of each-other - we will henceforth call this the *lightcone singularity*. For the $A=0$ singularity however,

$$\begin{aligned}\sigma(x_1,x_1)=\frac{1+X_1(x_1)\cdot X_2(x_2)}{2}&=0 \implies X_1(x_1)\cdot X_2(x_2)=-1\\ \implies X_1(x_1)\cdot X_2(\bar{x}_2)&=1.\end{aligned} \qquad (2.2.55)$$

Namely, the Wightman function is singular when x_1 is null separated from the point antipodal to x_2 (and vice-versa). This latter singularity is at first sight a rather unphysical one, though as can be seen from figure 2.1, antipodal points in de Sitter space are separated by a cosmological horizon. Therefore a single observer,

[13] The rising factorial, or *Pochhammer symbol*, is given by

$$(a)_n:=\frac{\Gamma(a+n)}{\Gamma(a)}. \qquad (2.2.53)$$

who is causally disconnected from half of de Sitter space and cannot access the point antipodal to them, could never detect such a singularity. We see that (2.2.51) therefore implies the existence not of a single vacuum state, but rather an entire family of vacua corresponding to linear combinations of the two solutions, known as the α-vacua. We will shortly see that in the Poincaré patch, this family can be characterised by two parameters α and β.

The solution with $B = 0$ defines the so-called *Bunch-Davies vacuum*, whose only singularity is the lightcone singularity. The Bunch-Davies vacuum, denoted $|0\rangle$, is referred to in the literature by a number of different names - the *Hartle-Hawking* vacuum, the *Euclidean* vacuum[14], and the *Thermal* vacuum are all used. The lightcone singularity also occurs in Minkowksi space, and the coefficient A can be fixed by insisting that the lightcone singularity in de Sitter and in Minkowski space have the same strength[15], resulting in

$$G^{(0)}(\sigma) = \frac{\Gamma\left(\frac{d}{2}+i\nu\right)\Gamma\left(\frac{d}{2}-i\nu\right)}{(4\pi)^{\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)} {}_2F_1\left(\tfrac{d}{2}+i\nu, \tfrac{d}{2}-i\nu; \tfrac{d+1}{2}; \sigma\right), \tag{2.2.56}$$

where the superscript (0) signifies the Bunch-Davies vacuum. Note that we still need to be able to access the region for spacelike separation $\sigma > 1$, so we need to specify how to deal with the branch cut along $[0, \infty)$. The different $i\epsilon$ prescriptions for dealing with this branch cut will then give us the time-ordered, anti-time-ordered and Wightman functions. We note that in the short-distance limit in the Bunch-Davies vacuum, the $i\epsilon$ prescription should coincide with that for the Minkowski space Green's functions[16]. The solution (2.2.56) is a function of σ, which in the short-distance limit $\eta_1 \approx \eta_2$, $\vec{x}_1 \approx \vec{x}_2$ is given by

$$\sigma = 1 + \frac{(\eta_1-\eta_2)^2 - (\vec{x}_1-\vec{x}_2)^2}{4\eta_1\eta_2} \sim 1 + (\eta_1-\eta_2)^2 - (\vec{x}_1-\vec{x}_2)^2. \tag{2.2.57}$$

[14] Since it is the unique vacuum reached by analytic continuation from the Euclidean sphere.

[15] This is a sensible consistency criterion; if we bring x_1 and x_2 close enough together they cannot "feel" the curvature of the background and so the singularity should be of the same strength as its Minkowski counterpart.

[16] I thank Joe Marshall for making me aware of footnote 12 of [81], which contains the following simple argument.

The Feynman propagator in Minkowski space depends on the combination of coordinates (with the $i\epsilon$ prescription included) $(t_1 - t_2)^2 - (\vec{x}_1 - \vec{x}_2)^2 - i\epsilon$, and so we should choose $\sigma \to \sigma - i\epsilon$ for the prescription to obtain the time-ordered Green's function. A similar argument can be made for the anti-time-ordered Green's function, obtaining

$$G^{(0)}_{T}(\sigma) = G^{(0)}(\sigma - i\epsilon)\,, \tag{2.2.58a}$$

$$G^{(0)}_{\bar{T}}(\sigma) = G^{(0)}(\sigma + i\epsilon)\,. \tag{2.2.58b}$$

These can be equivalently written as

$$G^{(0)}_{T}(x_1, x_2) = \theta(\eta_1 - \eta_2) G^{(0)}_{W}(x_1, x_2) + \theta(\eta_2 - \eta_1) G^{(0)}_{W}(x_2, x_1)\,, \tag{2.2.59a}$$

$$G^{(0)}_{\bar{T}}(x_1, x_2) = \theta(\eta_1 - \eta_2) G^{(0)}_{W}(x_2, x_1) + \theta(\eta_2 - \eta_1) G^{(0)}_{W}(x_1, x_2)\,, \tag{2.2.59b}$$

in terms of the Heaviside step function θ, with the $i\epsilon$ prescription for the Wightman function given by

$$G^{(0)}_{W}(x_1, x_2) = G^{(0)}(\sigma - i\epsilon\,\mathrm{sgn}(\eta_1 - \eta_2))\,. \tag{2.2.60}$$

It is often useful to decompose dS two-point functions in the Bunch-Davies vacuum according to

$$G^{(0)}(\sigma) = G^{(0)}_{\Delta_+}(\sigma) + G^{(0)}_{\Delta_-}(\sigma)\,, \tag{2.2.61}$$

where

$$G^{(0)}_{\Delta}(\sigma) = C^{\mathrm{dS}}_{\Delta}(-4\sigma)^{-\Delta}\,{}_2F_1\left(\begin{matrix}\Delta, \Delta - \frac{d}{2} + \frac{1}{2} \\ 2\Delta - d + 1\end{matrix}; \frac{1}{\sigma}\right), \tag{2.2.62}$$

with C^{dS}_{Δ} a constant we call the *two-point coefficient*. In the case that one of the two points lies on the future or past de Sitter boundary one can then expand (2.2.58) for large s, giving:

$$G^{(0)}_{T(\bar{T})}(s \to \infty) = K^{(0)}_{\Delta_+, T(\bar{T})}(s) + K^{(0)}_{\Delta_-, T(\bar{T})}(s)\,, \tag{2.2.63}$$

which naturally identifies time-ordered and anti-time-ordered *bulk-to-boundary propag-*

ators,

$$K^{(0)}_{\Delta,T}\,(s) = \frac{C^{\rm dS}_{\Delta}}{(-2s+i\epsilon)^{\Delta}}, \tag{2.2.64a}$$

$$K^{(0)}_{\Delta,\bar{T}}\,(s) = \frac{C^{\rm dS}_{\Delta}}{(-2s-i\epsilon)^{\Delta}}, \tag{2.2.64b}$$

with scaling dimension Δ and two-point coefficient

$$C^{\rm dS}_{\Delta} = \frac{1}{4\pi^{\frac{d+2}{2}}}\Gamma\left(\Delta\right)\Gamma(\tfrac{d}{2}-\Delta). \tag{2.2.65}$$

In Poincaré coordinates these read

$$K^{(0)}_{\Delta,T}\,(\eta,\vec{x};\vec{y}) = C^{\rm dS}_{\Delta}\left(\frac{\eta\eta_0}{-\eta^2+(\vec{x}-\vec{y})^2+i\epsilon}\right)^{\Delta}, \tag{2.2.66a}$$

$$K^{(0)}_{\Delta,\bar{T}}\,(\eta,\vec{x};\vec{y}) = C^{\rm dS}_{\Delta}\left(\frac{\eta\eta_0}{-\eta^2+(\vec{x}-\vec{y})^2-i\epsilon}\right)^{\Delta}, \tag{2.2.66b}$$

at late times $\eta_0 \sim 0$.

As we showed in section 2.2.1, in the expanding Poincaré patch, rather than becoming Euler's hypergeometric equation the wave equation (2.2.45) becomes (after going to momentum space)

$$\left((-\eta)^2\frac{d^2}{d(-\eta)^2} - (d-1)(-\eta)\frac{d}{d(-\eta)} + (-\eta)^2k^2 + m^2\right)g_{\vec{k}}(\eta) = 0. \tag{2.2.67}$$

The solution to this equation can be expressed as a linear combination of Hankel functions of the first and second kind;

$$g_{\vec{k}}(\eta) = \mathcal{C}_1\frac{1}{2}(-\eta)^{\frac{d}{2}}H^{(1)}_{i\nu}(-k\eta) + \mathcal{C}_2\frac{1}{2}(-\eta)^{\frac{d}{2}}H^{(2)}_{i\nu}(-k\eta), \tag{2.2.68}$$

with $\mathcal{C}_i$ constants. By insisting that these solutions reduce to those for the wave equation in Minkowski space in the $\eta \to -\infty$ limit, we can fix the normalisation constants and obtain the Bunch-Davies mode functions. In particular, insisting that

$$\lim_{\eta\to 0}\left(\mathcal{C}_1\frac{1}{2}(-\eta)^{\frac{d}{2}}H^{(1)}_{i\nu}(-k\eta)\right) = \lim_{\eta\to 0}\left(\mathcal{C}_2\frac{1}{2}(-\eta)^{\frac{d}{2}}H^{(2)}_{i\nu}(-k\eta)\right) \stackrel{!}{=} \frac{1}{\sqrt{2k}}e^{-ik\eta}, \tag{2.2.69}$$

leads to the mode functions in the Bunch-Davies vacuum $f_{\vec{k}}^{(0)}$ and $\bar{f}_{\vec{k}}^{(0)}$, given by

$$f_{\vec{k}}^{(0)}(\eta) = (-\eta)^{\frac{d}{2}} \frac{\sqrt{\pi}}{2} e^{\frac{\pi\nu}{2}} H_{i\nu}^{(2)}(-k\eta), \tag{2.2.70a}$$

$$\bar{f}_{\vec{k}}^{(0)}(\eta) = (-\eta)^{\frac{d}{2}} \frac{\sqrt{\pi}}{2} e^{-\frac{\pi\nu}{2}} H_{i\nu}^{(1)}(-k\eta). \tag{2.2.70b}$$

These are simply related under anti-podal transformation:

$$\bar{f}_{\vec{k}}^{(0)}(\eta) = -\, e^{\frac{i\pi d}{2}} f_{\vec{k}}^{(0)}(\eta\, e^{-i\pi}). \tag{2.2.71}$$

2.2.4 de Sitter Vacua II - The α-Vacua

de Sitter space admits a family of inequivalent vacuum states known as the α-vacua, which can be defined by a Bogoliubov transformation from the Bunch-Davies vacuum[17]. The mode functions corresponding to the α-vacua are thus a linear combination of the Bunch-Davies mode functions defined above, namely

$$f_{\vec{k}}^{(\alpha)}(\eta) := A\, f_{\vec{k}}^{(0)}(\eta) + B\, \bar{f}_{\vec{k}}^{(0)}(\eta), \tag{2.2.72}$$

$$\bar{f}_{\vec{k}}^{(\alpha)}(\eta) := A^*\, \bar{f}_{\vec{k}}^{(0)}(\eta) + B^*\, f_{\vec{k}}^{(0)}(\eta). \tag{2.2.73}$$

The set $\{f_{\vec{k}}^{(\alpha)}(\eta), \bar{f}_{\vec{k}}^{(\alpha)}(\eta)\}$ then forms a basis of solutions to the Klein-Gordon equation, and the field ϕ can be expressed as

$$\phi_{\vec{k}}(\eta) = f_{\vec{k}}^{(\alpha)}(\eta)\, b_{\vec{k}}^{\dagger} + \bar{f}_{\vec{k}}^{(\alpha)}(\eta)\, b_{\vec{k}}, \tag{2.2.74}$$

with $b_{\vec{k}}^{\dagger}$, $b_{\vec{k}}$ the creation and annihilation operators associated with the α-vacuum modes. Plugging in (2.2.72), we see that

$$\begin{aligned}\phi_{\vec{k}}(\eta) &= f_{\vec{k}}^{(\alpha)}(\eta)\, b_{\vec{k}}^{\dagger} + \bar{f}_{\vec{k}}^{(\alpha)}(\eta)\, b_{\vec{k}} \\ &= (A\, f_{\vec{k}}^{(0)}(\eta) + B\, \bar{f}_{\vec{k}}^{(0)}(\eta))\, b_{\vec{k}}^{\dagger} + (A^*\, \bar{f}_{\vec{k}}^{(0)}(\eta) + B^*\, f_{\vec{k}}^{(0)}(\eta))\, b_{\vec{k}} \\ &= (A\, b_{\vec{k}}^{\dagger} + B^*\, b_{\vec{k}}) f_{\vec{k}}^{(0)}(\eta) + (B\, b_{\vec{k}}^{\dagger} + A^*\, b_{\vec{k}}) \bar{f}_{\vec{k}}^{(0)}(\eta)\end{aligned}$$

[17]This is analogous to the Unruh effect in which the accelerated observer's "vacuum" is related by a Bogoliubov transformation to that of the inertial observer. However, in that case the accelerated vacuum breaks translation invariance and thus is not a vacuum in the sense of this thesis, where "vacua" are states which are invariant under all the isometries of the background spacetime.

$$\stackrel{!}{=} f^{(0)}_{\vec{k}}(\eta)\, a^\dagger_{\vec{k}} + \bar{f}^{(0)}_{\vec{k}}(\eta)\, a_{\vec{k}}, \tag{2.2.75}$$

and so we identify the Bogoliubov transformation between the Bunch-Davies creation and annihilation operators $a^\dagger_{\vec{k}}$, $a_{\vec{k}}$, and those for the α-vacua $b^\dagger_{\vec{k}}$, $b_{\vec{k}}$,

$$a^\dagger_{\vec{k}} \stackrel{!}{=} A\, b^\dagger_{\vec{k}} + B^*\, b_{\vec{k}}, \quad a_{\vec{k}} \stackrel{!}{=} B\, b^\dagger_{\vec{k}} + A^*\, b_{\vec{k}}. \tag{2.2.76}$$

If we now insist that the new creation & annihilation operators satisfy the same commutation relations as those for the Bunch-Davies vacuum, we find

$$\begin{aligned}
[a_{\vec{k}_1}, a^\dagger_{\vec{k}_2}] &= (B\, b^\dagger_{\vec{k}_1} + A^*\, b_{\vec{k}_1})(A\, b^\dagger_{\vec{k}_2} + B^*\, b_{\vec{k}_2}) - (A\, b^\dagger_{\vec{k}_2} + B^*\, b_{\vec{k}_2})(B\, b^\dagger_{\vec{k}_1} + A^*\, b_{\vec{k}_1}) \\
&= |B|^2\, b^\dagger_{\vec{k}_1} b_{\vec{k}_2} + |A|^2\, b_{\vec{k}_1} b^\dagger_{\vec{k}_2} - |A|^2\, b^\dagger_{\vec{k}_2} b_{\vec{k}_1} - |B|^2\, b_{\vec{k}_2} b^\dagger_{\vec{k}_1} \\
&= (|A|^2 - |B|^2) b_{\vec{k}_1} b^\dagger_{\vec{k}_2} - (|A|^2 - |B|^2) b^\dagger_{\vec{k}_2} b_{\vec{k}_1} \\
&= (|A|^2 - |B|^2)[b_{\vec{k}_1}, b^\dagger_{\vec{k}_2}],
\end{aligned} \tag{2.2.77}$$

and so insisting that $[a_{\vec{k}_1}, a^\dagger_{\vec{k}_2}] \stackrel{!}{=} [b_{\vec{k}_1}, b^\dagger_{\vec{k}_2}]$ requires

$$|A|^2 - |B|^2 \stackrel{!}{=} 1. \tag{2.2.78}$$

This equation is solved by

$$A = \cosh\alpha, \qquad B = e^{i\beta}\sinh\alpha, \tag{2.2.79}$$

with $\alpha \in \mathbb{R}$ and $\beta \in [0, 2\pi)$, where we have neglected a possible overall phase that does not contribute to expectation values. It can be shown that the Wightman function is invariant under the component of the isometry group $O(d+1, 1)$ connected to the identity only if α and β are momentum-independent, and under the full disconnected isometry group only if $\beta = 0$ [25]. Note that one reaches the Bunch-Davies vacuum simply by setting $\alpha = \beta = 0$; one can view the Bunch-Davies vacuum as a distinguished point in the space of α-vacua.

An important remark to make is that since the creation and annihilation operators for the Bunch-Davies vacuum are each a linear combination of creation and annihilation

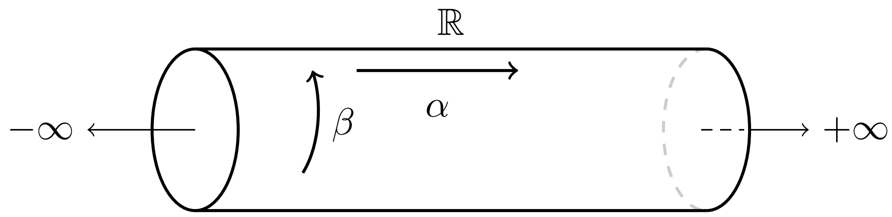


Figure 2.3: Each momentum-independent pair $(\alpha, \beta) \in \mathbb{R} \times [0, 2\pi)$ defines the unique vacuum state of a different Fock space. The vacuum manifold parameterised by this pair is a cylinder.

operators for the α-vacua, an α-vacuum $|\alpha, \beta\rangle$ defined by

$$b_{\vec{k}} \, |\alpha, \beta\rangle := 0, \tag{2.2.80}$$

will *not* be a vacuum state with respect to the Bunch-Davies annihilation operator;

$$a_{\vec{k}} \, |\alpha, \beta\rangle \neq 0. \tag{2.2.81}$$

From this perspective one can view the α-vacua as being states containing Bunch-Davies excitations, analogous to the view that one takes in the context of the Unruh effect where the accelerated observer finds particles in the state which the inertial observer considers to be a vacuum. In addition, one should note that each momentum-independent α-vacuum defines a unique de Sitter invariant ground state of a *different* Fock space, with α acting as a superselection parameter. Alternatively if we allow α, β to be momentum-dependent, it turns out that the dependence can be chosen such that $|\alpha_k\rangle$ and $|0\rangle$ lie in the same Hilbert space, in which case the α_k-vacua are literally excited states over the Bunch-Davies vacuum [32, 94]. Such general states defined by momentum-dependent parameters α_k, β_k have been called *Bogoliubov Initial States* [1, 95, 96] or simply *Alpha States* [94] in the literature.

Henceforth, the goal is to compute correlation functions for scalar fields in the α-vacua in de Sitter space, which can be extended to the more general family of Bogoliubov initial states simply by allowing α and β to be appropriate functions of the momenta; α_k and β_k for each mode of momentum k. In particular, we will use the notation

$$\langle \phi_1(\vec{k}_1)...\phi_n(\vec{k}_n)\rangle^{(\alpha)} \equiv \langle\alpha|\, \phi_1(\vec{k}_1)...\phi_n(\vec{k}_n)\, |\alpha\rangle\,, \tag{2.2.82}$$

with $|\alpha\rangle \equiv |\alpha, \beta\rangle$ assumed to indicate non-zero β unless otherwise specified, and

$$\langle \phi_1(\vec{k}_1)...\phi_n(\vec{k}_n)\rangle^{(0)} \tag{2.2.83}$$

denoting a correlator in the Bunch-Davies vacuum. In the next section, we introduce the general method we will use to compute de Sitter correlators, known as the *in-in formalism*.

2.2.5 dS Scattering: The In-In Formalism

In this section, we argue that scattering in de Sitter is more subtle than in flat space (and indeed, in AdS), and we introduce the *in-in* or *Schwinger-Keldysh* formalism for computing boundary correlators in de Sitter space.

Scattering in Flat Space. We begin by reviewing the procedure for calculating scattering amplitudes in flat space in perturbation theory, following the review in [97]. We define the two-point Green's function of a (generally complex) field $\mathcal{O}$ in the Heisenberg picture (where states are fixed and operators are time-dependent) between two spacetime points x and x' to be

$$G(x, x') := \langle\Omega|\, T(\mathcal{O}(x)\mathcal{O}^\dagger(x'))\, |\Omega\rangle\,, \tag{2.2.84}$$

where $|\Omega\rangle$ is defined to be the ground state of the full interacting theory[18]. To compute scattering amplitudes in flat space, one proceeds perturbatively by splitting the Hamiltonian of the interacting theory into a free part and a perturbation characterising the interaction, namely

$$H = H_0 + H_{\text{int}}. \tag{2.2.85}$$

[18] Namely, $|\Omega\rangle$ is the lowest-eigenvalue eigenstate of the full Hamiltonian H.

We can then switch to the interaction picture (where operators evolve with the free Hamiltonian H_0 and states evolve with $H_{\rm int}$) and define the time evolution operator to be the time-ordered exponential of the interaction Hamiltonian. Precisely, the evolution of a state $|\psi\rangle_I$ from some initial t_0 to a final t in the interaction picture is given by

$$|\psi(t)\rangle_I = U(t,t_0)\,|\psi(t_0)\rangle_I\,, \tag{2.2.86}$$

where the time evolution operator is given by

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt' H_{\rm int}(t')\right). \tag{2.2.87}$$

Equation (2.2.84) is in general difficult to work with, since we do not necessarily know what the state $|\Omega\rangle$ is. Hence, we instead assume that the interactions switch off in the asymptotic past and future, and work with the ground state of the free theory $|0\rangle$[19]. We derive

$$\begin{aligned}\langle\mathcal{O}(x)\mathcal{O}^\dagger(x')\rangle \equiv G(x,x') &= \frac{1}{\langle 0|\,S\,|0\rangle}\,\langle 0|\,T(S\,\mathcal{O}_I(x)\mathcal{O}_I^\dagger(x'))\,|0\rangle \\ &= \langle 0|\;S^\dagger\,T(\mathcal{O}_I(x)\mathcal{O}_I^\dagger(x'))\,S\;|0\rangle\,,\end{aligned} \tag{2.2.88}$$

where $\mathcal{O}_I$ denote the fields in the interaction picture, and the S-matrix is defined by

$$S := U(-\infty,\infty). \tag{2.2.89}$$

A perturbative expansion of this object yields Wick contractions between fields from the interaction Hamiltonian in the time-ordered exponentials, thus giving contributions from the two-point Green's function $G(x,x')$.

In the above formalism, we have implicitly assumed that the vacuum of the free theory $|0\rangle$ is unchanged throughout the evolution of the system. Namely, we have one vacuum state that the interacting theory asymptotes to in the far past and future. In other words, we have general in- and out- states schematically defined

[19]Namely, $H_0\,|0\rangle = 0$.

through a single vacuum state,

$$|\text{in}\rangle := a^\dagger ... a^\dagger |0\rangle \quad , \quad |\text{out}\rangle := b^\dagger ... b^\dagger |0\rangle \, . \tag{2.2.90}$$

However, this assumption is not justified in the expanding Poincaré patch of de Sitter space. In expanding spacetimes particle production can occur, and thus the state we define as the vacuum on one fixed time slice will in general contain particles at some later time. In addition, the asymptotic regions of de Sitter space are separated by a cosmological horizon, so it is difficult for an observer to set up an "in" state on $\mathcal{I}^-$, and subsequently observe an "out" state on $\mathcal{I}^+$ analogously to the Minkowski case. We therefore seek a new formalism for dealing with scattering in de Sitter.

The In-In Formalism. The *in-in formalism* deals with the above issues by only ever making reference to the initial state. Here, we take the initial state to be the vacuum state of the free Hamiltonian on a time slice in the infinite past.

We want to evolve the system in time such that we can consider non-trivial dynamics, and yet we also want to avoid making reference to any kind of "out" state (since as we described above, such amplitudes are difficult to make sense of). We do this by evolving the system forward in time, allowing interactions to influence the system, and then evolving the system backwards in time to the initial time slice. Namely, we are interested in so-called "in-in" correlators, heuristically

$$\langle \text{in} | \, S \, | \text{in} \rangle \, . \tag{2.2.91}$$

We can achieve this kind of evolution by complexifying the time coordinate and considering a contour $\mathcal{C}$ in the complex plane. Such a contour is schematically shown in figure 2.4. We implement the time evolution by integrating from $-\infty$ to some time t at which we have an operator insertion[20] in the correlator, and then from t back to $-\infty$. Concretely, we have that the correlation function of some product of

[20] Since in this thesis we are interested in boundary correlators in de Sitter space, the operators will be inserted on the late-time boundary, at time $t \equiv \eta \sim 0$.

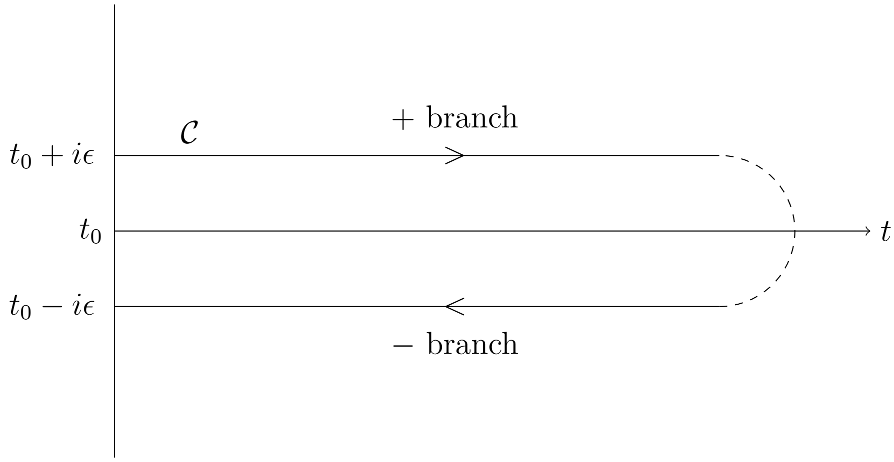


Figure 2.4: The complex time contour $\mathcal{C}$ along which we evolve the system (often referred to as the in-in contour). The upper leg of the contour is referred to as the "+ branch" and the lower leg as the "− branch". The contour is closed at some time t and runs clockwise - operators inserted on the + branch are therefore time-ordered, and those inserted on the − branch are anti-time ordered.

operators $Q(t)$ inserted at a time t is given by [46]

$$\langle Q(t)\rangle = \langle 0| \left[\bar{T}\exp\left(-i\int_{-\infty}^{t} dt' H\right)\right] Q(t) \left[T\exp\left(+i\int_{-\infty}^{t} dt' H\right)\right] |0\rangle\,. \tag{2.2.92}$$

A perturbative expansion of this object will yield not only Wick contractions arising from within the time-ordered part (giving the usual time-ordered Feynman propagator) and those from within the anti-time-ordered part (giving the anti-time-ordered Feynman propagator), but also Wick contractions *between the two.* These result in contributions from two new Green's functions that we will denote G_{+-} and G_{-+}.

Note that we do not need to evolve the system into the infinite future before we turn the contour around and evolve back to the initial time slice - we only need to consider the part of the contour from the initial time $t = -\infty$ to the future-most operator insertion. In other words, we do not concern ourselves with how the system evolves to the future of all operator insertions.

Armed with the above, one can define the two-point in-in-Green's function via

$$G_{\mathcal{C}}^{(\Omega)}(x_1, x_2) := \langle \Omega|\, T_{\mathcal{C}}(\mathcal{O}(x_1)\mathcal{O}^{\dagger}(x_2))\, |\Omega\rangle\,, \tag{2.2.93}$$

with $T_{\mathcal{C}}$ the "contour-ordering" operator, which orders operator insertions $\mathcal{O}$ according to where on the in-in contour they are inserted. From the above discussion regarding the in-in contour time-ordering, the Green's function decomposes into four "component" Green's functions, corresponding to the four possible distributions of the operators along the in-in contour. The first two are

$$G^{(\Omega)}_{++}(x_1, x_2) \equiv G^{(\Omega)}_{T}(x_1, x_2) = \langle\Omega|\, T(\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2))\,|\Omega\rangle \tag{2.2.94}$$

corresponding to both operators being inserted on the + branch, and

$$G^{(\Omega)}_{--}(x_1, x_2) \equiv G^{(\Omega)}_{\bar{T}}(x_1, x_2) = \langle\Omega|\, \bar{T}(\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2))\,|\Omega\rangle \tag{2.2.95}$$

corresponding to both operators being inserted on the − branch. Note that (2.2.94) corresponds to the usual time-ordered Feynman propagator, and (2.2.95) corresponds to the anti-time-ordered Feynman propagator. We also have two new non-trivial Green's functions arising from one operator being on the + branch and the other being on the − branch, namely

$$G^{(\Omega)}_{+-}(x_1, x_2) \equiv G^{(\Omega)}_{W}(x_2, x_1) = \langle\Omega|\, \mathcal{O}^\dagger(x_2)\mathcal{O}(x_1)\,|\Omega\rangle \tag{2.2.96}$$

and

$$G^{(\Omega)}_{-+}(x_1, x_2) \equiv G^{(\Omega)}_{W}(x_1, x_2) = \langle\Omega|\, \mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\,|\Omega\rangle\,. \tag{2.2.97}$$

Here, (2.2.96) corresponds to the future-most field insertion being on the + branch, and (2.2.97) corresponds to the future-most field insertion being on the − branch.

We now apply the above formalism to the case at hand; scattering in de Sitter space.

Scattering in de Sitter Space. In this thesis we are interested in correlation functions of operators inserted on the future boundary of the expanding Poincaré patch of de Sitter space. Following the above, the in-in correlation function of some set of scalar fields on a given time slice at time η is given by

$$\langle\phi(\eta, \vec{x}_1)\ldots\phi(\eta, \vec{x}_n)\rangle \tag{2.2.98}$$

$$= \langle\Omega(\eta_i)| \left[\bar{T}\exp\left(-i\int_{\eta_i}^{\eta} d\eta' H\right)\right] \phi(\eta,\vec{x}_1)\dots\phi(\eta,\vec{x}_n) \left[T\exp\left(+i\int_{\eta_i}^{\eta} d\eta' H\right)\right] |\Omega(\eta_i)\rangle \,.$$

One starts at initial time[21] $\eta_i = -\infty$ with vacuum $|\Omega(\eta_i)\rangle$ in the interacting theory, evolves in a time-ordered (T) fashion with respect to Hamiltonian H to the time η and then evolves back in an anti-time-ordered fashion ($\bar{T}$). To compute such correlators perturbatively, as usual we work in the interaction picture, again splitting the Hamiltonian into a free and interacting part $H = H_0 + H_{\text{int}}$. The correlator (2.2.98) can then be re-written as:

$$\langle\phi(\eta,\vec{x}_1)\dots\phi(\eta,\vec{x}_n)\rangle \tag{2.2.99}$$
$$= \frac{{}_0\langle 0|\,\bar{T}\left(\exp[-i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{\text{int}}(\phi_0)]\right)\phi_0(\eta,\vec{x}_1)\dots\phi_0(\eta,\vec{x}_n)T\left(\exp[+i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{\text{int}}(\phi_0)]\right)|0\rangle_0}{{}_0\langle 0|\,\bar{T}\left(\exp[i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{\text{int}}(\phi_0)]\right)T\left(\exp[-i\int_{-\infty}^{\eta}\mathrm{d}\eta' H_{\text{int}}(\phi_0)]\right)|0\rangle_0},$$

where $|0\rangle_0$ is the free theory vacuum and and the ϕ_0 evolve according to the free theory Hamiltonian H_0. At late times we have $\eta \sim 0$. In this thesis we take $|0\rangle_0$ to be an α-vacuum defined by (2.2.80). The perturbative expansion of this object is then generated by expanding the exponentials so that the correlator can then be computed perturbatively in the usual way using Wick contractions. As described earlier, this gives rise to the four bulk-to-bulk propagators:

$$G_{++}(x_1;x_2) = G_T(x_1;x_2) = {}_0\langle 0|T(\phi_0(x_1)\phi_0(x_2))\,|0\rangle_0\,, \tag{2.2.100a}$$

$$G_{+-}(x_1;x_2) = G_W(x_2;x_1) = {}_0\langle 0|\phi_0(x_2)\phi_0(x_1)\,|0\rangle_0\,, \tag{2.2.100b}$$

$$G_{-+}(x_1;x_2) = G_W(x_1;x_2) = {}_0\langle 0|\phi_0(x_1)\phi_0(x_2)\,|0\rangle_0\,, \tag{2.2.100c}$$

$$G_{--}(x_1;x_2) = G_{\bar{T}}(x_1;x_2) = {}_0\langle 0|\bar{T}(\phi_0(x_1)\phi_0(x_2))\,|0\rangle_0\,, \tag{2.2.100d}$$

[21] We should note that an $i\epsilon$ prescription is commonly imposed at early times, $\eta_i = -\infty(1 \pm i\epsilon)$, that projects onto the Bunch-Davies vacuum and ensures that the mode functions are suppressed as $\eta \to -\infty$. Away from the Bunch-Davies vacuum however, the mode functions are an admixture of positive and negative frequency modes, and one could worry about divergences coming from the early-time behaviour of the scalar field. However, imposing such an $i\epsilon$ prescription appears to be unimportant for the convergence of the η integrals we encounter in this work, and in fact, the Mellin space approach that we employ later in the thesis seems to take care of this issue automatically. See chapter 4.2 of [73] for a more detailed discussion of this issue.

and two bulk-to-boundary propagators:

$$K_{\Delta,+}(x,\vec{y}) \equiv K_{\Delta,T}(x,\vec{y}), \tag{2.2.101a}$$

$$K_{\Delta,-}(x,\vec{y}) \equiv K_{\Delta,\bar{T}}(x,\vec{y}), \tag{2.2.101b}$$

where x, x_n are bulk points and $\vec{y}$ is a boundary point. The bulk-to-boundary propagators can be computed simply by taking one of the points in a given bulk-to-bulk propagator to the boundary.

We will almost always go to Fourier space in the spatial directions, and we will give the functional form of all of these propagators in the expanding Poincaré patch in chapter 3.

2.3 Quantum Field Theory in Anti-de Sitter Space

In later chapters we will be interested in the relationship between correlation functions in Anti-de Sitter space and those in de Sitter. In this section we review the salient aspects of scalar Quantum Field Theory in Anti-de Sitter space.

2.3.1 Boundary Behaviour of the Scalar Field

Throughout this thesis we will consider scalar field theories in (EA)dS whose actions in the most general case take the form

$$S = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{|g|}\left(\mathcal{R} - 2\Lambda + \mathcal{L}_{\text{matter}}\right), \tag{2.3.1}$$

where G_{d+1} is Newton's constant in $d+1$ dimensions and

$$\mathcal{L}_{\text{matter}} = \frac{1}{2}\sum_i \nabla_\mu\phi_i\nabla^\mu\phi_i + \frac{1}{2}\sum_i m_i^2\phi_i^2 + \sum\frac{g}{3!}\sum_{i,j,k}\phi_i\phi_j\phi_k + \frac{\lambda}{4!}\sum_{i,j,k,l}\phi_i\phi_j\phi_k\phi_l + \dots \ . \tag{2.3.2}$$

In particular, when we consider ϕ^3 or ϕ^4 theory, we will endeavour to keep the interaction as general as possible and allow the cubic- or quartic interactions to be between *different* scalars with different masses. For simplicity, in this section we will just consider a single scalar with mass m.

Consider the wave equation (2.2.1) for a massive scalar on a general background g,

$$\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\partial^\mu\phi(z,x)) - m^2\phi(z,x) = 0. \tag{2.3.3}$$

For EAdS$_{d+1}$ in Poincaré coordinates,

$$ds^2 = \frac{L_{\text{AdS}}^2}{z^2}(dz^2 + \delta_{ij}dx^i dx^j), \tag{2.3.4}$$

and so the wave equation becomes

$$\begin{aligned}
0 &= \frac{z^{d+1}}{L_{\text{AdS}}^{d+1}}\partial_\mu\left(\frac{L_{\text{AdS}}^{d+1}}{z^{d+1}}\partial^\mu\phi(z,x)\right) - m^2\phi(z,x) \\
&= z^{d+1}\partial_\mu\left(\frac{1}{z^{d+1}}g^{\mu\nu}\partial_\nu\phi(z,x)\right) - m^2\phi(z,x) \\
&= z^{d+1}\partial_z\left(\frac{1}{z^{d+1}}\frac{z^2}{L_{\text{AdS}}^2}\partial_z\phi(z,x)\right) + z^{d+1}\partial_\alpha\left(\frac{1}{z^{d+1}}\frac{z^2}{L_{\text{AdS}}^2}\delta^{\alpha\beta}\partial_\beta\phi(z,x)\right) - m^2\phi(z,x) \\
&= z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi(z,x)\right) + z^2\partial_\alpha\partial^\alpha\phi(z,x) - m^2L_{\text{AdS}}^2\phi(z,x).
\end{aligned} \tag{2.3.5}$$

Passing to Fourier space in the directions parallel to the boundary, we find

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi(z,k)) - k^2z^2\phi(z,k) - m^2L_{\text{AdS}}^2\phi(z,k) = 0. \tag{2.3.6}$$

Close to the boundary at $z \to 0$, we can drop the z^2 term and write

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi) - m^2L_{\text{AdS}}^2\phi = 0. \tag{2.3.7}$$

Making the ansatz $\phi(z,k) = C(k)z^\Delta$, we find

$$\begin{aligned}
0 &= z^{d+1}\partial_z(z^{1-d}\partial_z(C(k)z^\Delta)) - m^2L_{\text{AdS}}^2C(k)z^\Delta \\
&= z^{d+1}\Delta\partial_z(z^{\Delta-d}) - m^2L_{\text{AdS}}^2z^\Delta \\
&= \Delta(\Delta-d) - m^2L_{\text{AdS}}^2
\end{aligned} \tag{2.3.8}$$

which is satisfied for

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L_{\text{AdS}}^2}. \tag{2.3.9}$$

Defining $\nu := \sqrt{\frac{d^2}{4} + m^2 L_{\text{AdS}}^2}$, we therefore have two independent solutions;

$$\Delta_+ \equiv \frac{d}{2} + \nu, \tag{2.3.10}$$

and

$$\Delta_- \equiv \frac{d}{2} - \nu = d - \Delta_+, \tag{2.3.11}$$

with

$$m^2 L_{\text{AdS}}^2 = -\Delta_+ \Delta_-. \tag{2.3.12}$$

We can therefore expand $\phi(z,k)$ near the boundary in terms of coefficients $A(k)$ and $B(k)$ as

$$\lim_{z \to 0} \phi(z,k) = A(k) z^{\Delta_-} + B(k) z^{\Delta_+}. \tag{2.3.13}$$

Note that the exponents are real if

$$m^2 L_{\text{AdS}}^2 \geq -\frac{d^2}{4} \tag{2.3.14}$$

is satisfied. This is the so-called *Breitenlohner-Freedman bound*, and its violation can lead to instabilities in the bulk[22]. Notice that if $\nu < 0$, we have

$$\begin{aligned} \sqrt{\frac{d^2}{4} + m^2 L_{\text{AdS}}^2} &< 0 \\ \implies m^2 L_{\text{AdS}}^2 &< -\frac{d^2}{4}, \end{aligned} \tag{2.3.15}$$

in violation of the BF bound. Thus, we require that $\nu \overset{!}{\geq} 0$.

Let us investigate further these modes A and B. Consider the Klein-Gordon inner

[22] In particular, when the BF bound is violated in Lorentzian AdS there exist modes in the bulk which grow exponentially with increasing time. See [98, 99] for the original papers, and appendix B of [100] for a review.

product (2.2.36)[23]

$$(\phi_1, \phi_2) := -i \int\limits_{\Sigma_t} dz d\vec{x} \sqrt{-g} g^{tt} (\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*), \tag{2.3.16}$$

where Σ_t is a constant time slice and the integral is over the spatial directions. In EAdS we have no time coordinate, so we can choose any of the $\vec{x}$ coordinates as our "time" direction. Using $g_{tt} = \frac{1}{z^2} \implies g^{tt} = z^2$ and $\sqrt{-g_{\text{AdS}}} = \frac{1}{z^{d+1}}$, we can plug in the A mode $\phi_A(z, x) := A(x) z^{d-\Delta_+}$ and find

$$\begin{aligned}(\phi_A, \phi_A) &= -i \int\limits_{\Sigma_t} dz d\vec{x} \frac{|z^{d-\Delta_+}|^2}{z^{d-1}} (A^* \partial_t A - A \partial_t A*) \\ &\sim \int dz z^{-2\Delta_+ + d + 1} \\ &\sim \frac{z^{-2\Delta_+ + d + 2}}{-2\Delta_+ + d + 2}.\end{aligned} \tag{2.3.17}$$

The integral therefore is only convergent near the boundary $z \longrightarrow 0$ if

$$\begin{aligned} &-2\Delta_+ + d + 2 > 0 \\ \iff &-2\left(\frac{d}{2} + \nu\right) + d + 2 > 0 \\ &\implies \nu < 1, \end{aligned} \tag{2.3.18}$$

where we have used $\Delta_+ = \frac{d}{2} + \nu$. Thus, for the A-modes to be normalisable we have the condition $0 \leq \nu < 1$.

Now let's look at the "B" modes. Defining $\phi_B := B(x) z^{\Delta_+}$, we have

$$\begin{aligned}(\phi_B, \phi_B) &\sim \int dz \frac{z^{2\Delta_+}}{z^{d-1}} \\ &\sim \frac{z^{2\Delta_+ - d + 2}}{2\Delta_+ - d + 2},\end{aligned} \tag{2.3.19}$$

and so the inner product is convergent only for $2\Delta_+ - d + 2 > 0$, which corresponds to $\nu > -1$. Since we already have the condition that $\nu \geq 0$ from the BF bound, this constraint is always satisfied and so the B modes are always normalisable.

[23] The unit normal vector points in the t-direction and is chosen to be $n^t = 1, n^j = 0\ \forall j \neq t$, so $n^\mu = g^{\mu\nu} n_\nu = g^{tt}$. We also have $\gamma_{ij} = g_{ij}$.

From the above, we see that we need to impose certain conditions on these modes when we quantise the theory in order for us to avoid having non-physical states in our Hilbert space. For $\nu \geq 1$, we get rid of the non-normalisable A mode by simply imposing the boundary condition $A = 0$. For $0 \leq \nu < 1$ both modes are normalisable, so we have a free choice between *Neumann* boundary conditions[24] $A = 0$, *Dirichlet* boundary conditions[25] $B = 0$, or neither, in which case we have "mixed" boundary conditions.

The normalisable modes are the ones we quantise and use to build the Hilbert space of the bulk theory. But what role do the non-normalisable modes play? Let's consider standard quantisation, where the non-normalisable mode is A. Before we impose the $A = 0$ condition, we have

$$\begin{aligned} \phi(z,x) &= A(x) z^{d-\Delta_+} + B(x) z^{\Delta_+} + \mathcal{O}(z^2) \\ &= A(x) z^{\frac{d}{2}-\nu} + B(x) z^{\frac{d}{2}+\nu} + \mathcal{O}(z^2). \end{aligned} \tag{2.3.20}$$

Therefore, near the boundary at $z \longrightarrow 0$ the A term dominates, assuming $\nu > 0$. Therefore, we interpret $A(x)$ as setting the boundary value of the bulk field ϕ, namely $A = \phi|_{\partial \text{AdS}}$.

2.3.2 The AdS/CFT Correspondence

The AdS/CFT correspondence represents a concrete realisation of the holographic principle, and posits that a quantum theory of gravity in asymptotically AdS_{d+1} spacetimes can be described by a d-dimensional Conformal Field Theory (CFT_d) living on the AdS boundary. AdS/CFT provides us with a non-perturbative definition of quantum gravity in such spacetimes, and allows us to answer hard-to-attack questions about quantum gravity by instead answering sharp questions about the dual CFT. There are a plethora of concrete examples of AdS/CFT in a variety of dimensions, the most well-understood being the conjecture that Type IIB string

[24] Known as "standard quantisation".
[25] Known as "alternative quantisation".

theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ is dual to $\mathcal{N}=4$ Supersymmetric Yang-Mills theory on $\mathbb{R}^{3,1}$. We will not be interested in concrete examples in this thesis, but it will be useful to give a short review of AdS/CFT in generality in order to appreciate the significance of Witten diagrams. As always we use Poincaré coordinates, where the metric for Euclidean AdS is given by (2.1.8).

The GKPW Formula

The basic statement of the duality is that the generating functional of correlators in certain[26] d-dimensional Conformal Field Theories coincides with the partition function of some gravitational theory on AdS_{d+1}, if one imposes a certain set of conditions which we elucidate here. Considering a single bulk scalar ϕ and ignoring all other fields for simplicity, the so-called *Gubser-Klebanov-Polyakov/Witten* (GKPW) formula [9, 10] provides a concrete statement of the AdS/CFT correspondence,

$$\left\langle \exp\left(-\int d^d x \phi_0(x)\mathcal{O}(x)\right)\right\rangle_{\mathrm{CFT}} = \int_{\phi|_{\partial AdS}=\phi_0(x)} \mathcal{D}\phi \exp\left[-\frac{1}{G} S_{\mathrm{AdS}_{d+1}}[\phi]\right]. \quad (2.3.21)$$

This formula is a central entry in the AdS/CFT dictionary, and analogous formulae exist for bulk fields of non-zero spin. On the RHS, we are evaluating the partition function on the gravity side with the condition that as we approach the AdS boundary, the bulk field ϕ approaches some given function ϕ_0[27]. The AdS/CFT dictionary then tells us that we should interpret this given function ϕ_0 as the source for a CFT operator $\mathcal{O}$, which we take to be the operator "dual" to ϕ. In Poincaré coordinates (z,x) (where the AdS boundary is at $z\to 0$), a more precise statement of the boundary condition is

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \phi_0(x), \quad (2.3.22)$$

where Δ is the conformal dimension of the CFT operator $\mathcal{O}$ dual to the AdS bulk field ϕ. In particular, this means that one can take two perspectives on the source

[26] It was argued in [101] that any CFT satisfying a certain list of criteria should have an AdS dual.

[27] Namely, we integrate only over bulk field configurations that satisfy $\phi|_{\partial AdS}=\phi_0(x)$

- ϕ_0 can either be viewed as a source for the boundary CFT operator $\mathcal{O}$, or as a source for its dual bulk field ϕ.

The above implies that one can obtain n-point correlators on the CFT side of the duality by taking functional derivatives of the AdS partition function, namely[28]

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \mathcal{N}\frac{\delta^n}{\delta\phi_0(x_1)...\delta\phi_0(x_n)}\int_{\phi|_{\partial AdS}=\phi_0(x)} \mathcal{D}\phi \exp\left[-\frac{1}{G}S_{\text{AdS}}[\phi]\right]\Bigg|_{\phi_0=0}. \tag{2.3.23}$$

The AdS Feynman diagrams that contribute to these correlators are known as *Witten diagrams* [9], and are an invaluable tool in AdS/CFT calculations. Since the duality is strong/weak, the possibility to study strongly-coupled conformal field theories via bulk perturbation theory opens up.

As an aside, note that one can imagine an alternative method for holographically computing CFT correlators - by computing bulk correlators before extrapolating them to the boundary. This idea (for identical scalar operators/bulk fields) gives the *extrapolate* dictionary

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle_{\text{CFT}} = \lim_{z\to 0} z^{-n\Delta}\langle\phi(z,x_1)...\phi(z,x_n)\rangle_{\text{bulk}}. \tag{2.3.24}$$

It has been shown that this method is equivalent to the differentiate dictionary in AdS, namely that the resulting correlators for each method coincide [102].

2.3.3 Witten Diagrams

As explained above, boundary correlators in EAdS_{d+1} can be computed perturbatively as a sum of *Witten diagrams*; the AdS analogue of Feynman diagrams where each external leg is anchored to the boundary. Internal legs of said diagrams are associated with bulk-to-bulk propagators, which solve the wave equation with a Dirac delta function source;

$$(\nabla^2 - m^2)G_\Delta^{\text{AdS}}(x,x') = -\frac{1}{\sqrt{|g|}}\delta^{d+1}(x-x'). \tag{2.3.25}$$

[28] This is known as the *differentiate* dictionary.

It turns out[29] [103] that one can express the bulk-to-bulk propagator in terms of a basis of functions $\Omega_\nu^{\text{AdS}}(x, x')$ that are harmonic with respect to the AdS Laplacian,

$$(\nabla^2 - m^2)\Omega_\nu^{\text{AdS}}(x, x') = 0, \tag{2.3.26}$$

and admit a *split representation* as a product of bulk-to-boundary propagators. Namely, going to Fourier space in the directions parallel to the boundary the AdS harmonic function is given by

$$\Omega_{\nu,\vec{k}}^{\text{AdS}}(z, z') = \frac{\nu^2}{\pi} K_{\Delta_+}^{\text{AdS}}(z, \vec{k}) K_{\Delta_-}^{\text{AdS}}(z', -\vec{k}), \tag{2.3.27}$$

with the bulk-to-boundary propagator $K_\Delta^{\text{AdS}}(z, \vec{k})$ given by a modified Bessel function of the second kind[30] [10]. The bulk-to-boundary propagator can also be understood as an integral kernel used to solve the homogeneous wave equation for a bulk field $\phi(z, x)$,

$$(\nabla^2 - m^2)\phi = 0, \tag{2.3.28}$$

subject to the boundary condition $\lim_{z\to 0} z^{\Delta-d}\phi(z, x) = \phi_0(x)$. In particular, we seek a solution to (2.3.28) of the form

$$\phi(z, x) = \int d^d x' K_\Delta(z, x; x')\phi_0(x'), \tag{2.3.29}$$

where we impose

$$(\nabla^2 - m^2)K_\Delta^{\text{AdS}}(z, x; x') = 0, \quad \lim_{z\to 0} z^{\Delta-d} K_\Delta^{\text{AdS}}(z, x; x') = \delta^{(d)}(x - x'). \tag{2.3.30}$$

It can be easily verified that (2.3.29) is indeed a formal solution of the homogeneous wave equation, satisfying the required boundary condition. It is well-known that in position space, the bulk-to-boundary propagator satisfying the above is then given by

$$K_\Delta^{\text{AdS}}(z, x, x') = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}} L_{\text{AdS}}^{\frac{d-2}{2}} \Gamma\left(\Delta - \frac{d}{2} + 1\right)} \left(\frac{z}{z^2 + (x - x')^2}\right)^\Delta. \tag{2.3.31}$$

[29]See section 3.2.1 for more details.

[30]In later sections we will give the functional form of the bulk-to-bulk and bulk-to-boundary propagators in the Mellin space representation.

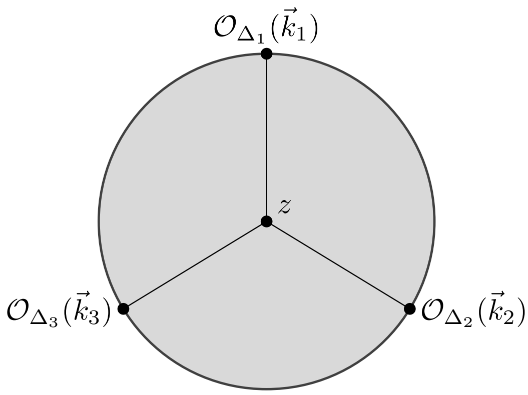


Figure 2.5: The Witten diagram corresponding to a three-point contact interaction between scalar fields of mass m_i at a point z in the bulk. This is interpreted in the AdS/CFT correspondence as a tree-level perturbative contribution to a three-point correlator of operators $\mathcal{O}$ with conformal dimension Δ_i in the dual CFT on the boundary.

Witten diagrams can then be constructed from the above defined propagators by integrating them over the bulk points involved. For instance, the contribution to a boundary three-point function from a bulk three-point contact diagram is given by

$$\langle \mathcal{O}_{\Delta_1}(\vec{k}_1)\mathcal{O}_{\Delta_2}(\vec{k}_2)\mathcal{O}_{\Delta_3}(\vec{k}_3)\rangle = \int_0^\infty \frac{dz}{z^{d+1}}\, K^{\rm AdS}_{\Delta_1}(z,\vec{k}_1)K^{\rm AdS}_{\Delta_2}(z,\vec{k}_2)K^{\rm AdS}_{\Delta_3}(z,\vec{k}_3), \quad (2.3.32)$$

with the integral over the bulk point z. This is represented diagrammatically in figure 2.5.

2.3.4 A (Not-So) Trivial Observation

The keen reader will note that while there are clearly fundamental geometric differences between dS and AdS, the line elements for the two spaces are very similar, and are in fact related by a Wick rotation. In particular, one can analytically continue to the flat slicing of EAdS_{d+1} from the flat slicing of dS_{d+1} by starting with

$$ds^2_{\mathrm{dS}_{d+1}} = \frac{L^2_{\rm dS}}{\eta^2}(-d\eta^2 + \delta_{ij}dx^i dx^j), \quad (2.3.33)$$

and Wick rotating via the identification

$$z = -\eta e^{\pm i\frac{\pi}{2}} \;\;,\;\; L_{\rm AdS} = -L_{\rm dS}e^{\pm i\frac{\pi}{2}}, \quad (2.3.34)$$

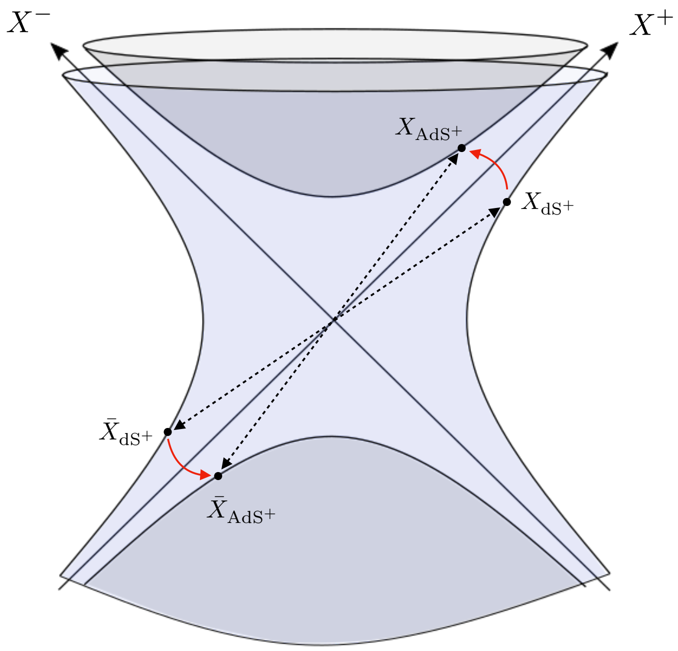


Figure 2.6: dS and EAdS in embedding space. EAdS is a two-sheeted hyperboloid. The Expanding Poincaré patch of de Sitter space analytically continues to the upper sheet of the EAdS hyperboloid under $\eta = -ze^{\pm i\frac{\pi}{2}}$, while the Contracting Poincaré patch analytically continues to the lower sheet.

resulting in the metric for EAdS_{d+1}

$$ds^2_{\mathrm{EAdS}_{d+1}} = \frac{L^2_{\mathrm{EAdS}}}{z^2}(dz^2 + \delta_{ij}dx^i dx^j). \tag{2.3.35}$$

We notice also that by comparing (2.1.23) and (2.1.24) with (2.1.7), the Expanding Poincaré Patch analytically continues to the upper sheet of the EAdS hyperboloid, and the Contracting Poincaré patch analytically continues to the lower sheet.

This is an essential observation which underpins much of what follows. In fact, it turns out that this relation between the line elements is not just a happy-but-useless coincidence, but it persists through to the propagators in quantum field theories on each space. In particular, we will see in the next chapter that the bulk-to-bulk and bulk-to-boundary propagators for scalar fields in dS and EAdS are related via certain analytic continuations of the radial coordinate, which can be traded for analytic continuations of the boundary momenta. This implies that to all orders in perturbation theory, late-time boundary correlation functions in de Sitter space can be recast as a linear combination of Witten diagrams in EAdS, with analytically continued boundary momenta. The corresponding position space statement is that correlation functions in de Sitter, when viewed from an AdS perspective, feature points on both the upper and the lower sheets of the AdS hyperboloid.

Having explored the necessary background material, we now endeavour to derive the propagators for scalar fields in the α-vacua in de Sitter space.

Chapter 3

Propagators from Bunch-Davies & EAdS

This chapter is based on sections 3 and 4 of [1]. In this chapter we derive relations between the de Sitter bulk-to-bulk and bulk-to-boundary propagators in the α-vacua and the corresponding propagators in the Bunch-Davies vacuum, and in turn between the α-vacua and the corresponding propagators in EAdS. This extends the work of [74–77], in which it was shown that Bunch-Davies correlators can be rewritten in terms of EAdS Witten diagrams, to arbitrary choice of α-vacuum.

The main results of this chapter are:

- **α-Vacua From Bunch-Davies.** Bulk-to-bulk and bulk-to-boundary propagators in the in-in formalism for arbitrary choice of α-vacuum in the Expanding Poincaré patch are written in terms of their counterparts in the Bunch-Davies vacuum. The Bunch-Davies propagators involved feature points in both the Expanding and the Contracting Poincaré patches.

- **α-Vacua From EAdS.** In turn, we re-write the α-vacuum bulk-to-bulk and bulk-to-boundary propagators in the in-in formalism in terms of the corresponding EAdS propagators. As a consequence of the α-vacuum propagators involving points in both the EPP and the CPP, the constituent EAdS propagators appearing in the α-vacuum expressions involve points on both the upper and the lower sheet of the EAdS hyperboloid. We will see that the consequence of this in momentum space is that the EAdS propagators feature momenta that are analytically continued as $k \to e^{\pm i\pi}k$. As such, unlike those for the Bunch-Davies vacuum, late-time correlators in the α-vacua do not share the same analytic structure as the EAdS boundary correlators appearing in the context of the AdS/CFT correspondence, where all points are on the upper sheet of the EAdS hyperboloid.

The results in this chapter extend to the Bogoliubov initial states by allowing α and β to depend appropriately[1] on the momentum, $(\alpha, \beta) \to (\alpha_k, \beta_k)$ for a mode of momentum k.

Throughout this and subsequent chapters, we will make heavy use of the *Mellin space formalism* [74–77], which has proven a useful tool for studying boundary correlation functions in both dS and EAdS. We begin this chapter with a review.

3.1 Mellin Space

The computation of boundary correlation functions in dS and EAdS involves complicated integrals over the radial coordinates η and z, respectively. In this section, we demonstrate that in the presence of dilatation symmetry it is possible to go to a basis of the Hilbert space in which these integrals are trivialised, becoming Dirac delta functions. This basis can be identified with the so-called *Mellin transform*, and the Dirac delta function that appears involves the *Mellin variables*, both of which we define. We draw analogy with the setting of momentum space in the presence of translation symmetry, following section 2.2 of [77]. Further details regarding the

[1] We note that Bogoliubov initial states do not promote α and β to *arbitrary* functions of the momentum, but rather to functions with a certain fall-off. See [94] for details.

construction of the Mellin transform and its relationship with the dilatation group can be found in [104].

In the presence of translation symmetry it is often convenient to work in momentum space, i.e. in a basis of plane waves $e^{\pm ikx}$ that diagonalise the translation generator[2] and thus make the symmetry manifest. Namely, we can decompose a function $f(x) \in L^2\left(\mathbb{R}, \mathrm{d}x\right)$ as

$$f\left(x\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} f(k)\, e^{ikx}, \tag{3.1.1}$$

where

$$f\left(k\right) = \int_{-\infty}^{\infty} \mathrm{d}x f(x)\, e^{-ikx}. \tag{3.1.2}$$

For this reason, in de Sitter space (and indeed, in any spacetime with translation symmetry) one often considers late-time correlators in momentum space due to the presence of translation symmetry in the spatial (boundary) directions. The remaining direction is the bulk time direction, in which there is no translation symmetry, owing to the time dependence of the de Sitter background. There is therefore less benefit from transforming to Fourier space in the time direction. The situation is similar for EAdS, where the presence of translation symmetry along the directions parallel to the boundary make Fourier space a useful setting for studying boundary correlation functions.

In dS and EAdS we not only have translation symmetry, but also symmetry under dilatations; scale transformations of the form (for EAdS) $(z, \vec{x}) \to (\lambda z, \lambda \vec{x})$. In the presence of dilatation symmetry, it is natural to work in Mellin space [74–77]. We can decompose functions $f(z) \in L^2\left(\mathbb{R}^+, \frac{\mathrm{d}z}{z^{d+1}}\right)$ according to

$$f(z) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}s}{2\pi i} 2 f(s) z^{-\left(2s-\frac{d}{2}\right)}, \tag{3.1.3a}$$

$$f(s) = \int_{0}^{\infty} \frac{\mathrm{d}z}{z} f(z) z^{2s-\frac{d}{2}}, \tag{3.1.3b}$$

where the monomials $z^{\mp\left(2s-\frac{d}{2}\right)}$ are analogous to the plane waves in momentum space,

[2] Namely, the plane waves $e^{\pm ikx}$ are eigenfunctions of the translation operator.

and in this case diagonalise the dilatation generator. We will illustrate this point (following section 2.2 of [77]) in the context of EAdS, but the same holds for de Sitter space with respect to the time coordinate η.

For theories with dilatation symmetry, namely those invariant under

$$z \to \lambda z, \quad x \to \lambda x, \tag{3.1.4}$$

Mellin space is analogous in a number of ways to Fourier space in the presence of translation symmetry. To illustrate this, consider a function

$$\begin{aligned} f : (0, \infty) &\longrightarrow \mathbb{R} \\ z &\longmapsto f(z). \end{aligned} \tag{3.1.5}$$

We want to study dilatations acting on such a function, so we define a representation of the dilatation group $G_{\mathcal{D}}$ on the space of smooth functions on $\mathbb{R}$;

$$\begin{aligned} \mathcal{T} : G_{\mathcal{D}} &\longrightarrow \mathrm{End}(C^\infty(\mathbb{R})) \\ \lambda &\longmapsto \mathcal{T}_\lambda, \end{aligned} \tag{3.1.6}$$

where the *dilatation generator* is formally the map

$$\begin{aligned} \mathcal{T}_\lambda : C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto \mathcal{T}_\lambda[f], \end{aligned} \tag{3.1.7}$$

where we define

$$\mathcal{T}_\lambda[f](z) \equiv \mathcal{T}_\lambda[f(z)] := \lambda^{-\frac{d}{2}} f(\lambda z). \tag{3.1.8}$$

We claim that the dilatation generator acting on a function $f(z)$ can be explicitly written

$$\mathcal{T}_\lambda = e^{\lambda \mathcal{D}_z}, \quad \mathcal{D}_z := z\partial_z - \frac{d}{2}. \tag{3.1.9}$$

We can see this by first looking at translations. The translation generator acts as

$$e^{a\partial_x} f(x) = f(x+a), \tag{3.1.10}$$

which can easily be seen by inspection of the Taylor expansion of the right-hand-side. We can use this to deduce the action of the operator $e^{\lambda x\partial_x}$ on a function; we have

$$e^{\lambda x\partial_x} f(x) = e^{\lambda\partial_{\ln(x)}} f(x) = e^{\lambda\partial_y} f(e^y) = f(e^{y+\lambda}), \tag{3.1.11}$$

where we have re-written the operator as a translation generator and used (3.1.10) in the last equality. We then have

$$e^{\lambda x\partial_x} f(x) = f(e^{y+\lambda}) = f(e^\lambda x) \equiv f(\tilde{\lambda} x). \tag{3.1.12}$$

We will abuse notation somewhat by dropping the tilde, resulting in

$$e^{\lambda x\partial_x} f(x) = f(\lambda x). \tag{3.1.13}$$

We therefore see that this operator generates scale transformations on the coordinates. Using this, the action of $e^{\lambda \mathcal{D}_z}$ on a function $f(z)$ is given by

$$\begin{aligned} e^{\lambda \mathcal{D}_z} f(z) &= e^{\lambda z\partial_z} f(z) e^{-\lambda \frac{d}{2}} \\ &= f(e^\lambda z)(e^\lambda)^{-\frac{d}{2}} \\ &\equiv \lambda^{-\frac{d}{2}} f(\lambda z), \end{aligned} \tag{3.1.14}$$

by our abuse of notation, and so indeed the action of this operator coincides with our definition of $\mathcal{T}_\lambda$ above.

$\mathcal{T}_\lambda$ generates a symmetry transformation by preserving the sesquilinear inner product defined by

$$\langle f|g\rangle := \int_0^\infty \frac{dz}{z^{d+1}} f(z) g^*(z). \tag{3.1.15}$$

Indeed, we see that

$$\begin{aligned} \langle \mathcal{T}[f]|\mathcal{T}[g]\rangle &= \int_0^\infty \frac{dz}{z^{d+1}} \mathcal{T}[f(z)]\mathcal{T}[g(z)]^* \\ &= \int_0^\infty \frac{dz}{z^{d+1}} \lambda^{-d} f(\lambda z) g^*(\lambda z) \\ &= \int_0^\infty \frac{dw}{\lambda^{-d} w^{d+1}} \lambda^{-d} f(w) g^*(w) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{dz}{z^{d+1}} f(z) g^*(z) \\
&= \langle f|g \rangle \, , \qquad (3.1.16)
\end{aligned}
$$

where we performed the simple change of variables $w := \lambda z$. Defining a norm on $C^\infty(\mathbb{R}^+)$ by

$$
|| \cdot || := \sqrt{\langle \cdot \, | \, \cdot \rangle}, \qquad (3.1.17)
$$

the pair $(C^\infty(\mathbb{R}^+), || \cdot ||)$ naturally forms a Hilbert space, $L^2\left(\mathbb{R}^+, \frac{dz}{z^{d+1}}\right)$. A theory defined by this Hilbert space will then have dilatation symmetry, namely its inner products will be preserved under the action of $\mathcal{T}_\lambda$.

One can easily see that the eigenfunctions of the dilatation generator are of the form

$$
f_\alpha(z) = z^{\frac{d}{2} - i\alpha}, \quad \mathcal{T}_\lambda(f_\alpha(z)) = \lambda^{-i\alpha} f_\alpha(z). \qquad (3.1.18)
$$

These eigenfunctions are orthogonal;

$$
\begin{aligned}
\langle f_\alpha | f_\beta \rangle &= \int_0^\infty \frac{dz}{z^{d+1}} z^{\frac{d}{2} - i\alpha} z^{\frac{d}{2} + i\beta} \\
&= \int_0^\infty \frac{dz}{z} z^{i(\beta - \alpha)} \\
&= \int_{-\infty}^\infty dx \, e^{i(\beta - \alpha)x} \\
&= 2\pi \delta(\beta - \alpha), \qquad (3.1.19)
\end{aligned}
$$

and satisfy a completeness relation;

$$
\begin{aligned}
\int_{-\infty}^\infty \frac{d\alpha}{2\pi} f_\alpha^*(z_1) f_\alpha(z_2) &= \int_{-\infty}^\infty \frac{d\alpha}{2\pi} z_1^{i\alpha + \frac{d}{2}} z_2^{-i\alpha + \frac{d}{2}} \\
&= (z_1 z_2)^{\frac{d}{2}} \int_{-\infty}^\infty \frac{d\alpha}{2\pi} \left(\frac{z_1}{z_2} \right)^{i\alpha} \\
&= (z_1 z_2)^{\frac{d}{2}} \int_{-\infty}^\infty \frac{d\alpha}{2\pi} \left(e^{\ln\left(\frac{z_1}{z_2}\right)} \right)^{i\alpha} \\
&= (z_1 z_2)^{\frac{d}{2}} \, \delta\left(\ln\left(\frac{z_1}{z_2} \right) \right) \\
&= z_1^{d+1} \, \delta(z_1 - z_2), \qquad (3.1.20)
\end{aligned}
$$

where we used the well-known identity (treating z_2 as constant)

$$\delta(f(x)) = \frac{\delta(x-x_0)}{|f'(x_0)|}, \tag{3.1.21}$$

with x_0 the root of $f(x)$, and that $\delta(z_1 - z_2)$ is nonzero only for $z_1 = z_2$. Therefore, the set of eigenfunctions $\{f_\alpha(z)\} \equiv \{z^{\frac{d}{2}-i\alpha}\}$ of the dilatation operator forms a basis of the Hilbert space $L^2(\mathbb{R}^+, \frac{dz}{z^{d+1}})$. We can therefore decompose an element of this Hilbert space into an integral against these eigenfunctions, analogously to the case for an infinite discrete basis $\{\vec{e_i}\}$ and the decomposition

$$\vec{v} = \sum_{i=1}^{\infty} (\vec{v}\cdot\vec{e_i})\vec{e_i}. \tag{3.1.22}$$

In particular, we can decompose an element g of $L^2(\mathbb{R}^+, \frac{dz}{z^{d+1}})$ in terms of the eigenfunctions f_α as

$$g(z) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \langle g|f_\alpha\rangle\, f_\alpha(z), \tag{3.1.23}$$

with

$$\begin{aligned} \langle g|f_\alpha\rangle &= \int_0^\infty \frac{dz}{z^{d+1}} g(z) f_\alpha^*(z) \\ &= \int_0^\infty \frac{dz}{z^{d+1}} g(z) z^{i\alpha+\frac{d}{2}} \\ &= \int_0^\infty \frac{dz}{z} g(z) z^{i\alpha-\frac{d}{2}}. \end{aligned} \tag{3.1.24}$$

Notice that by identifying $i\alpha = 2s$, this is nothing but the *Mellin transform* of g;

$$g(s) = \int_0^\infty \frac{dz}{z} g(z) z^{2s-\frac{d}{2}}. \tag{3.1.25}$$

In summary, the decomposition of a function $g \in L^2(\mathbb{R}^+, \frac{dz}{z^{d+1}})$ into eigenfunctions of the dilatation operator $\mathcal{T}_\lambda$ is therefore naturally identified with the Mellin transform of g. We can also easily identify (3.1.23) with the *inverse Mellin transform*, by

$$g(z) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \underbrace{\langle g|f_\alpha\rangle}_{\equiv g(i\alpha)} \underbrace{f_\alpha(z)}_{z^{-\left(i\alpha-\frac{d}{2}\right)}}$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} 2g(s) z^{-\left(2s-\frac{d}{2}\right)}, \tag{3.1.26}$$

where again we replaced $i\alpha = 2s$. Using this replacement and the completeness relation it is straightforward to show that (3.1.25) and (3.1.26) are consistent.

We have demonstrated that in the presence of dilatation symmetry, it is possible to construct a basis of the Hilbert space whose elements are eigenfunctions of the dilatation operator; $z^{\pm(2s-\frac{d}{2})}$. We have also shown that the decomposition of elements of that Hilbert space with respect to said basis can be identified with the Mellin transform. It is useful to go to this particular basis, and therefore useful to go to Mellin space in the presence of dilatation symmetry[3], because the integrals over the radial coordinate (z for EAdS, η for dS) that appear in computations of boundary correlators in dS and EAdS then trivialise to Dirac delta functions in the Mellin variables. In turn, these delta functions enforce a constraint on the Mellin variables analogous to the momentum-conserving delta function that appears when we study correlators in Fourier space. We will now show how this delta function appears, both for the case of Fourier space and translation symmetry as well as for Mellin space and dilatation symmetry.

First, let's consider the case for translation symmetry and the appearance of a delta function in momentum space. Consider a three-point function $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$ for a CFT on $\mathbb{R}^d$, and impose translation invariance,

$$\begin{aligned} &\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle \overset{!}{=} \langle \mathcal{O}_1(x_1+a)\mathcal{O}_2(x_2+a)\mathcal{O}_3(x_3+a)\rangle \\ \implies &\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \langle \mathcal{O}_1(x_1-x_3)\mathcal{O}_2(x_2-x_3)\mathcal{O}_3(0)\rangle, \end{aligned} \tag{3.1.27}$$

where we chose $a = -x_3$. In momentum space, we have

$$\langle \mathcal{O}_1(\vec{k}_1)\mathcal{O}_2(\vec{k}_2)\mathcal{O}_3(\vec{k}_3)\rangle = (2\pi)^d \int d^d x_1 d^d x_2 d^d x_3 \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle e^{i(\vec{k}_1\cdot x_1+\vec{k}_2\cdot x_2+\vec{k}_3\cdot x_3)}$$

[3]Note that since we are just performing an integral transform, we could in principle go to Mellin space in the absence of dilatation symmetry, namely, by equipping the Hilbert space with some other inner product that isn't preserved under dilatations. However, in that case the Mellin transform $g(s)$ (which we've just shown is the same thing as the inner product) will not be preserved under dilatations, and therefore the physics of dilatations will not be present in the resulting object.

$$= (2\pi)^d \int d^d x_1 d^d x_2 d^d x_3 \langle \mathcal{O}_1(x_1 - x_3)\mathcal{O}_2(x_2 - x_3)\mathcal{O}_3(0)\rangle e^{i(\vec{k}_1\cdot x_1 + \vec{k}_2\cdot x_2 + \vec{k}_3\cdot x_3)}.$$

Let $u = x_1 - x_3$ and $v = x_2 - x_3$. Using $d^d x_1 d^d x_2 = |J| d^d u d^d v$ (where J is the Jacobian) we find

$$\begin{aligned}
\langle \mathcal{O}_1(\vec{k}_1)\mathcal{O}_2(\vec{k}_2)\mathcal{O}_3(\vec{k}_3)\rangle &= (2\pi)^d \int d^d u d^d v d^d x_3 \langle \mathcal{O}_1(u)\mathcal{O}_2(v)\mathcal{O}_3(0)\rangle e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot x_3} e^{i(\vec{k}_1\cdot u + \vec{k}_2\cdot v)} \\
&= (2\pi)^d \delta^{(d)}(\vec{k}_1+\vec{k}_2+\vec{k}_3) \int d^d u d^d v \langle \mathcal{O}_1(u)\mathcal{O}_2(v)\mathcal{O}_3(0)\rangle e^{i(\vec{k}_1\cdot u + \vec{k}_2\cdot v)} \\
&\equiv (2\pi)^d \delta^{(d)}(\vec{k}_1+\vec{k}_2+\vec{k}_3) \langle \mathcal{O}_1(\vec{k}_1)\mathcal{O}_2(\vec{k}_2)\mathcal{O}_3(\vec{k}_3)\rangle'. \qquad (3.1.28)
\end{aligned}$$

Due to rotation invariance together with momentum conservation, the three-point function can only depend on the *magnitudes* of the momenta. In particular, rotation invariance implies that the three-point function must be a function of dot products of the momenta of the form

$$\vec{k}_a \cdot \vec{k}_b, \quad a, b \in \{1, 2, 3\}. \qquad (3.1.29)$$

Combining this with momentum conservation, we see that any dot product for which $a \neq b$ can be re-written in terms of those for which $a = b$; for instance,

$$\begin{aligned}
\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \implies \vec{k}_3 &= -(\vec{k}_1 + \vec{k}_2) \\
\implies |\vec{k}_3|^2 &= (\vec{k}_1 + \vec{k}_2)\cdot(\vec{k}_1 + \vec{k}_2) \\
&= |\vec{k}_1|^2 + |\vec{k}_2|^2 + 2\vec{k}_1\cdot\vec{k}_2 \\
\iff \vec{k}_1\cdot\vec{k}_2 &= \frac{1}{2}\left(|\vec{k}_3|^2 - |\vec{k}_2|^2 - |\vec{k}_1|^2\right).
\end{aligned}$$

Therefore, we have

$$\langle \mathcal{O}_1(\vec{k}_1)\mathcal{O}_2(\vec{k}_2)\mathcal{O}_3(\vec{k}_3)\rangle = (2\pi)^d \delta^{(d)}(\vec{k}_1+\vec{k}_2+\vec{k}_3) \langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle', \qquad (3.1.30)$$

with $|\vec{k}_i| \equiv k_i$. We see that in momentum space, the relic of the imposition of translation symmetry in position space is the appearance of a Dirac delta function that imposes momentum conservation. This is to be expected - the Noether charge of spatial translations is spatial momentum, and so we would expect that requiring translation invariance in position space corresponds to some kind of constraint imposing momentum conservation in momentum space.

Let us now go to Mellin space, and investigate what happens when we impose invariance (or more precisely covariance) under dilatations. In the Mellin-Barnes representation, (3.1.30) becomes[4]

$$\begin{aligned}\langle\mathcal{O}_1(\vec{k}_1)\mathcal{O}_2(\vec{k}_2)\mathcal{O}_3(\vec{k}_3)\rangle &= (2\pi)^d\delta^{(d)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\\ &\times\int_{-i\infty}^{i\infty}[ds_j]_3\langle\mathcal{O}_1(s_1)\mathcal{O}_2(s_2)\mathcal{O}_3(s_3)\rangle\prod_{j=1}^{3}\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j},\end{aligned} \tag{3.1.31}$$

where

$$[ds_j]_n := \frac{ds_1}{2\pi i}\cdots\frac{ds_n}{2\pi i}. \tag{3.1.32}$$

Imposing the dilatation Ward identity (a derivation of which is given in appendix C) on the three-point function gives

$$\left(-d+\sum_{j=1}^{3}\mathcal{D}_j\right)\langle\mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' \stackrel{!}{=} 0, \tag{3.1.33}$$

where $\mathcal{D}_j := -(\Delta_j - d) + k_j\partial_{k_j}$ is the generator of dilatations in momentum space. Plugging in the Mellin-Barnes representation of $\langle\mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle'$, this implies

$$\int_{-i\infty}^{i\infty}[ds_j]_3\left(\frac{d}{2}-2(s_1+s_2+s_3)\right)\langle\mathcal{O}_1(s_1)\mathcal{O}_2(s_2)\mathcal{O}_3(s_3)\rangle\prod_{j=1}^{3}\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j} \stackrel{!}{=} 0. \tag{3.1.34}$$

Therefore, imposing invariance under dilatations enforces a constraint on the Mellin variables,

$$\sum_{j=1}^{3}2s_j-\frac{d}{2}=0. \tag{3.1.35}$$

However, it turns out that this constraint can also be shown to appear in a different way. By going to Mellin space in the momenta, we will see in later chapters that integrals of the form

$$\int_{-\infty}^{0}\frac{\mathrm{d}\,(-\eta)}{(-\eta)^{d+1}}\,(-\eta)^{-\sum_i\left(2s_i-\frac{d}{2}\right)} = 2\pi i\,\delta\left(d+\sum_i\left(2s_i-\tfrac{d}{2}\right)\right), \tag{3.1.36}$$

[4]See section 3.1 and appendix C.2 of [75] for further details on the Mellin representation of three-point conformal structures.

appear, which enforces the same constraint we obtained from the dilatation Ward identity, namely (3.1.35). Therefore, we can view the appearance of a delta function in the Mellin variables as the relic of dilatation symmetry in Mellin space; analogous to the appearance of a momentum-conserving delta function as the relic of translation invariance in momentum space. A proof of (3.1.36) can be found in appendix B.

The structure of the Mellin transform makes manifest certain identities satisfied by propagators and their corresponding boundary correlators, both in dS and EAdS - as well as the relationship between the two under analytic continuation. In the following sections we derive these various identities, using the Mellin space formalism described above.

3.2 α-vacua from Bunch-Davies

In this section we derive the bulk-to-bulk and bulk-to-boundary propagators for scalar fields in an arbitrary choice of α-vacuum, expressing them in terms of their corresponding propagators in the Bunch-Davies vacuum. These results extend to the Bogoliubov initial states by allowing α and β to depend appropriately on the momentum of the relevant mode. We begin with the bulk-to-bulk propagators, computing the Wightman function and the (anti-) time-ordered Green's functions.

3.2.1 Bulk-to-Bulk Propagators

Wightman Function. Recall that the Wightman function in momentum space is given simply by the product of mode functions, as shown by equation (2.2.46). In particular, for a vacuum state $|\Omega\rangle$ and corresponding mode functions $f_{\vec{k}}(t)$, we have the general formula

$$G_W^{(\Omega)}(t_1, t_2; \vec{k}) = \bar{f}_{\vec{k}}(t_1) f_{\vec{k}}(t_2). \tag{3.2.1}$$

In particular, for a scalar field in a generic α-vacuum in de Sitter space, we have

$$G_W^{(\alpha)}(\eta_1, \eta_2; \vec{k}) = \bar{f}_{\vec{k}}^{(\alpha)}(\eta_1) f_{\vec{k}}^{(\alpha)}(\eta_2), \tag{3.2.2}$$

where

$$f_{\vec{k}}^{(\alpha)}(\eta) = A\, f_{\vec{k}}^{(0)}(\eta) + B\, \bar{f}_{\vec{k}}^{(0)}(\eta), \tag{3.2.3}$$

$$\bar{f}_{\vec{k}}^{(\alpha)}(\eta) = A^*\, \bar{f}_{\vec{k}}^{(0)}(\eta) + B^*\, f_{\vec{k}}^{(0)}(\eta), \tag{3.2.4}$$

with $f_{\vec{k}}^{(0)}(\eta)$ the mode functions for the Bunch-Davies vacuum and we recall the parameterisation

$$A = \cosh(\alpha), \quad B = e^{i\beta} \sinh(\alpha). \tag{3.2.5}$$

To compute the Wightman function, we use the Mellin space representation of the Bunch-Davies mode functions, writing

$$f_{\vec{k}}^{(0)}(\eta) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}u}{2\pi i} f_{\vec{k}}^{(0)}(u)(-\eta)^{-\left(2u-\frac{d}{2}\right)}, \tag{3.2.6}$$

$$\bar{f}_{\vec{k}}^{(0)}(\eta) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}u}{2\pi i} \bar{f}_{\vec{k}}^{(0)}(u)(-\eta)^{-\left(2u-\frac{d}{2}\right)}, \tag{3.2.7}$$

where we recall that these functions were given by Hankel functions as in (2.2.70). The integrands in the above are therefore given simply by the Mellin-Barnes representation of Hankel functions of the second and first kind (respectively), namely

$$f_{\vec{k}}^{(0)}(u) = +\frac{i}{2\sqrt{\pi}}\Gamma\left(u + \frac{i\nu}{2}\right)\Gamma\left(u - \frac{i\nu}{2}\right)\left(\frac{1}{2}\, k\, e^{+\frac{i\pi}{2}}\right)^{-2u}, \tag{3.2.8}$$

$$\bar{f}_{\vec{k}}^{(0)}(u) = -\frac{i}{2\sqrt{\pi}}\Gamma\left(u + \frac{i\nu}{2}\right)\Gamma\left(u - \frac{i\nu}{2}\right)\left(\frac{1}{2}\, k\, e^{-\frac{i\pi}{2}}\right)^{-2u}. \tag{3.2.9}$$

It is then straightforward to see the useful relation

$$f_{\vec{k}}^{(0)}(u) = -e^{-2i\pi u} \bar{f}_{\vec{k}}^{(0)}(u), \tag{3.2.10}$$

which we will use shortly. We will start by defining the Mellin-Barnes representation of the Wightman function,

$$G_W^{(\alpha)}(\eta_1; \eta_2) = \int\limits_{-i\infty}^{i\infty} [ds]_2\, G_W^{(\alpha)}(u_1; u_2)(-\eta_1)^{-2s_1+\frac{d}{2}}(-\eta_2)^{-2s_2+\frac{d}{2}}, \tag{3.2.11}$$

with

$$G_W^{(\alpha)}(u_1; u_2) := \bar{f}_{\vec{k}}^{(\alpha)}(u_1) f_{\vec{k}}^{(\alpha)}(u_2). \tag{3.2.12}$$

Now, using that the α-vacuum mode functions are a linear combination of their Bunch-Davies counterparts as in (3.2.3), we have

$$\begin{aligned} G_W^{(\alpha)}(u_1;u_2) &= \bar{f}_{\vec{k}}^{(\alpha)}(u_1) f_{\vec{k}}^{(\alpha)}(u_2) \\ &= \left(A^* \bar{f}_{\vec{k}}^{(0)}(u_1) + B^* f_{\vec{k}}^{(0)}(u_1)\right)\left(A f_{\vec{k}}^{(0)}(u_2) + B \bar{f}_{\vec{k}}^{(0)}(u_2)\right) \\ &= |A|^2 \bar{f}_{\vec{k}}^{(0)}(u_1) f_{\vec{k}}^{(0)}(u_2) + |B|^2 f_{\vec{k}}^{(0)}(u_1)\bar{f}_{\vec{k}}^{(0)}(u_2) + B^* A f_{\vec{k}}^{(0)}(u_1) f_{\vec{k}}^{(0)}(u_2) \\ &\quad + A^* B \bar{f}_{\vec{k}}^{(0)}(u_1)\bar{f}_{\vec{k}}^{(0)}(u_2). \end{aligned} \tag{3.2.13}$$

Using (3.2.10) and (3.2.12) we can express this in terms of the Bunch-Davies Wightman function,

$$\begin{aligned} G_W^{(\alpha)}(u_1;u_2) = |A|^2 G_W^{(0)}(u_1;u_2) + |B|^2 e^{-2\pi i(u_1-u_2)} G_W^{(0)}(u_1;u_2) - B^* A e^{-2i\pi u_1} G_W^{(0)}(u_1;u_2) \\ - A^* B e^{2i\pi u_2} G_W^{(0)}(u_1;u_2). \end{aligned} \tag{3.2.14}$$

Using the Mellin-Barnes representation (3.2.11) one can then show that

$$e^{-2i\pi u_1} G_W^{(0)}(u_1;u_2) \iff e^{-\frac{i\pi d}{2}} G_W^{(0)}(e^{i\pi}\eta_1;\eta_2) \tag{3.2.15}$$

$$e^{2i\pi u_2} G_W^{(0)}(u_1;u_2) \iff e^{+\frac{i\pi d}{2}} G_W^{(0)}(\eta_1;e^{-i\pi}\eta_2) \tag{3.2.16}$$

$$e^{-2i\pi(u_1-u_2)} G_W^{(0)}(u_1;u_2) \iff G_W^{(0)}(e^{i\pi}\eta_1;e^{-i\pi}\eta_2). \tag{3.2.17}$$

Thus, taking the inverse Mellin transform of both sides of (3.2.14) results in

$$\begin{aligned} G_W^{(\alpha)}(\eta_1;\eta_2) = |A|^2 G_W^{(0)}(\eta_1;\eta_2) + |B|^2 G_W^{(0)}(\bar{\eta}_1^-;\bar{\eta}_2^+) - B^* A e^{-\frac{i\pi d}{2}} G_W^{(0)}(\bar{\eta}_1^-;\eta_2) \\ - A^* B e^{+\frac{i\pi d}{2}} G_W^{(0)}(\eta_1;\bar{\eta}_2^+), \end{aligned} \tag{3.2.18}$$

where we have introduced the notation

$$\bar{\eta}_i^{\pm} := e^{\mp i\pi}\eta_i. \tag{3.2.19}$$

Finally, using our parameterisation of the coefficients A and B (3.2.5) we obtain the Wightman function for a scalar field in a generic α-vacuum,

$$\boxed{\begin{aligned} G_W^{(\alpha)}(\eta_1;\eta_2) &= \cosh^2\alpha\, G_W^{(0)}(\eta_1;\eta_2) + \sinh^2\alpha\, G_W^{(0)}(\bar{\eta}_1^-;\bar{\eta}_2^+) \\ &\quad - \frac{1}{2}\sinh 2\alpha [e^{i\left(\beta+\frac{\pi d}{2}\right)}\, G_W^{(0)}(\eta_1;\bar{\eta}_2^+) + e^{-i\left(\beta+\frac{\pi d}{2}\right)}\, G_W^{(0)}(\bar{\eta}_1^-;\eta_2)]. \end{aligned}} \tag{3.2.20}$$

Note that the Wightman function for generic α-vacuum in the *expanding* Poincaré patch involves Bunch-Davies propagators with points anti-podally transformed to the *contracting* Poincaré patch, $\eta \to \bar{\eta}^{\pm}$. This will also be true for the (anti-) time-ordered propagators, and will ultimately have interesting consequences for the analytic structure of perturbative correlators in the α-vacua[5].

(Anti-) Time-Ordered Propagators. We are also interested in the time-ordered and anti-time-ordered propagators in terms of their Bunch-Davies counterparts. We begin with the definition

$$G_T^{(\alpha)}(\eta_1;\eta_2) = \theta(\eta_1-\eta_2)\, G_W^{(\alpha)}(\eta_1;\eta_2) + \theta(\eta_2-\eta_1)\, G_W^{(\alpha)}(\eta_2;\eta_1)\,, \tag{3.2.21a}$$

$$G_{\bar{T}}^{(\alpha)}(\eta_1;\eta_2) = \theta(\eta_1-\eta_2)\, G_W^{(\alpha)}(\eta_2;\eta_1) + \theta(\eta_2-\eta_1)\, G_W^{(\alpha)}(\eta_1;\eta_2)\,, \tag{3.2.21b}$$

which we combine with the above expression (3.2.20) for the Wightman function $G_W^{(\alpha)}$ in terms of its Bunch-Davies counterpart $G_W^{(0)}$. Using the Mellin-Barnes representation (3.2.11) it is straightforward to show that

$$\begin{aligned} G_W^{(0)}(\bar{\eta}_1^-;\bar{\eta}_2^+) &= \int_{-i\infty}^{i\infty} [\mathrm{d}u]_2\, G_W^{(0)}(u_1,k_1;u_2,k_2)\left(-\eta_1 e^{+\pi i}\right)^{-\left(2u_1-\frac{d}{2}\right)}\left(-\eta_2 e^{-\pi i}\right)^{-\left(2u_2-\frac{d}{2}\right)}, \\ &= \int_{-i\infty}^{i\infty} [\mathrm{d}u]_2\, e^{-2\pi i(u_1-u_2)}\,\bar{f}_{\vec{k}_1}^{(0)}(u_1) f_{\vec{k}_2}^{(0)}(u_2)\,(-\eta_1)^{-\left(2u_1-\frac{d}{2}\right)}(-\eta_2)^{-\left(2u_2-\frac{d}{2}\right)}, \\ &= \int_{-i\infty}^{i\infty} [\mathrm{d}u]_2\, \bar{f}_{\vec{k}_1}^{(0)}(u_2) f_{\vec{k}_2}^{(0)}(u_1)\,(-\eta_1)^{-\left(2u_1-\frac{d}{2}\right)}(-\eta_2)^{-\left(2u_2-\frac{d}{2}\right)}, \\ &= \int_{-i\infty}^{i\infty} [\mathrm{d}u]_2\, G_W^{(0)}(u_1,k_1;u_2,k_2)\,(-\eta_2)^{-\left(2u_1-\frac{d}{2}\right)}(-\eta_1)^{-\left(2u_2-\frac{d}{2}\right)}, \end{aligned}$$

[5]Note that since the Hankel functions in (2.2.70) have a branch cut along the negative real axis, one could worry about a rotation from the positive real axis by $e^{\pm i\pi}$. However, whenever we perform such a rotation there is an implicit $i\epsilon$ prescription that prevents us from rotating exactly onto said branch cut. We can either view this as an infinitesimal movement of the branch cut, or as a rotation by slightly less than π of the form $e^{\pm i(\pi-\epsilon)}$.

$$= G_W^{(0)}(\eta_2;\eta_1), \tag{3.2.22}$$

where in the penultimate line we performed the trivial change of variables $u_1 \leftrightarrow u_2$ in the integral. Thus, we can express the time-ordered Green's function as

$$\begin{aligned} G_T^{(\alpha)}(\eta_1;\eta_2) &= \theta(\eta_1-\eta_2)\, G_W^{(\alpha)}(\eta_1;\eta_2) + \theta(\eta_2-\eta_1)\, G_W^{(\alpha)}(\eta_2;\eta_1) \\ &= \cosh^2(\alpha)[\theta(\eta_1-\eta_2)G_W^{(0)}(\eta_1;\eta_2) + \theta(\eta_2-\eta_1)G_W^{(0)}(\eta_2;\eta_1)] \\ &\quad + \sinh^2(\alpha)[\theta(\eta_1-\eta_2)G_W^{(0)}(\bar{\eta}_1^-;\bar{\eta}_2^+) + \theta(\eta_2-\eta_1)G_W^{(0)}(\bar{\eta}_2^-;\bar{\eta}_1^+)] \\ &\quad - \frac{1}{2}\sinh(2\alpha)\Big\{\theta(\eta_1-\eta_2)[e^{i\beta+\frac{i\pi d}{2}}G_W^{(0)}(\eta_1;\bar{\eta}_2^+) + e^{-i\beta-\frac{i\pi d}{2}}G_W^{(0)}(\bar{\eta}_1^-;\eta_2)] \\ &\qquad \theta(\eta_2-\eta_1)[e^{i\beta+\frac{i\pi d}{2}}G_W^{(0)}(\eta_2;\bar{\eta}_1^+) + e^{-i\beta-\frac{i\pi d}{2}}G_W^{(0)}(\bar{\eta}_2^-;\eta_1)]\Big\} \\ &= \cosh^2(\alpha)G_T^{(0)}(\eta_1;\eta_2) + \sinh^2(\alpha)G_{\bar{T}}^{(0)}(\eta_1;\eta_2) \\ &\quad - \frac{1}{2}\sinh(2\alpha)\Big\{\theta(\eta_1-\eta_2)[e^{i\beta+\frac{i\pi d}{2}}G_W^{(0)}(\eta_1;\bar{\eta}_2^+) + e^{-i\beta-\frac{i\pi d}{2}}G_W^{(0)}(\bar{\eta}_1^-;\eta_2)] \\ &\qquad \theta(\eta_2-\eta_1)[e^{i\beta+\frac{i\pi d}{2}}G_W^{(0)}(\eta_2;\bar{\eta}_1^+) + e^{-i\beta-\frac{i\pi d}{2}}G_W^{(0)}(\bar{\eta}_2^-;\eta_1)]\Big\}. \end{aligned} \tag{3.2.23}$$

One can show using the Mellin representation that (recalling the notation (3.2.19))

$$G_W^{(0)}(\bar{\eta}_2^-;\eta_1) = G_W^{(0)}(\bar{\eta}_1^-;\eta_2), \tag{3.2.24a}$$

$$G_W^{(0)}(\eta_2;\bar{\eta}_1^+) = G_W^{(0)}(\eta_1;\bar{\eta}_2^+), \tag{3.2.24b}$$

and so this further simplifies to

$$\begin{aligned} G_T^{(\alpha)}(\eta_1;\eta_2) &= \cosh^2(\alpha)G_T^{(0)}(\eta_1;\eta_2) + \sinh^2(\alpha)G_{\bar{T}}^{(0)}(\eta_1;\eta_2) \\ &\quad - \frac{1}{2}\sinh(2\alpha)\Big\{e^{i\beta+\frac{i\pi d}{2}}G_W^{(0)}(\eta_1;\bar{\eta}_2^+) + e^{-i\beta-\frac{i\pi d}{2}}G_W^{(0)}(\bar{\eta}_1^-;\eta_2)\Big\}. \end{aligned} \tag{3.2.25}$$

We now aim to express the remaining Wightman functions in this expression in terms of (anti) time-ordered Green's functions.

First, a small detour. It turns out that the in-in bulk-to-bulk propagators can be expressed in terms of homogeneous solutions to the Klein-Gordon equation, called *harmonic functions*. Originally introduced in the context of AdS [103, 105], these harmonic functions are a difference of AdS bulk-to-bulk propagators[6];

$$\Omega_{\nu}^{\mathrm{AdS}}(x_1,x_2)=\frac{i\nu}{2\pi}\left(G^{\mathrm{AdS}}_{\frac{d}{2}+i\nu}(x_1,x_2)-G^{\mathrm{AdS}}_{\frac{d}{2}-i\nu}(x_1,x_2)\right). \tag{3.2.26}$$

One can define analogous harmonic functions for each of the de Sitter in-in propagators, and express the propagators in Mellin space as [76]

$$\begin{aligned} G^{(0)}_{\pm\hat{\pm}}(u_1,u_2) &= \csc(\pi(u_1+u_2))(\alpha^{\pm\hat{\pm}}\omega_{\Delta_+}(u_1,u_2)+\beta^{\pm}\omega_{\Delta_-}(u_1,u_2)) \\ &\qquad\qquad \times\Gamma(i\nu)\Gamma(-i\nu)\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2), \qquad (3.2.27)\\ &\equiv G^{(0)}_{\Delta_+,\pm\hat{\pm}}(u_1,u_2)+G^{(0)}_{\Delta_-,\pm\hat{\pm}}(u_1,u_2), \qquad (3.2.28)\end{aligned}$$

where the dS Harmonic functions are defined in terms of the AdS harmonic function[7];

$$\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2)=c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}e^{\mp\pi i\left(u_1+\frac{i\nu}{2}\right)}e^{\hat{\mp}\pi i\left(u_2-\frac{i\nu}{2}\right)}\Omega^{\mathrm{AdS}}_{\nu,\vec{k}}(u_1,u_2), \tag{3.2.29}$$

with

$$c^{\text{dS-AdS}}_{\Delta}:=\frac{1}{2}\csc\left(\pi\left(\frac{d}{2}-\Delta\right)\right), \tag{3.2.30}$$

accounting for the change in two-point coefficient in going from dS to EAdS. We will give the functional form of the EAdS harmonic function when we discuss bulk-to-boundary propagators in the following section - the above is sufficient for now. We also define the functions

$$\omega_{\Delta_\pm}(u_1,u_2):=2\sin\left(\pi\left(u_1\mp\frac{i\nu}{2}\right)\right)\sin\left(\pi\left(u_2\mp\frac{i\nu}{2}\right)\right), \tag{3.2.31}$$

which serve to project onto the contributions from each of $\Delta_\pm$. The coefficients of these functions in (3.2.27) are defined

$$\alpha^{\pm\pm}:=\frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}}e^{\pm\pi\nu},\quad \beta^{\pm\pm}:=\frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}}e^{\mp\pi\nu}, \tag{3.2.32a}$$

[6]Intuitively, the difference of bulk-to-bulk propagators for different boundary conditions cannot be zero, but will solve the homogeneous Klein-Gordon equation since the delta functions arising from acting with the Klein-Gordon operator on the bulk-to-bulk propagators will cancel. This difference will thus be a "harmonic function" in the $\nabla^2 f=0$ sense.

[7]Note that in position space this expression corresponds to the de Sitter harmonic functions being an analytic continuation of their AdS counterparts.

$$\alpha^{\pm\mp} := \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}} e^{\mp\pi\nu}, \quad \beta^{\pm\mp} := \frac{1}{c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}} e^{\mp\pi\nu}, \tag{3.2.32b}$$

which can be written in the compact form

$$\alpha^{\hat{\pm}} := \alpha^{\pm\hat{\pm}}, \quad \beta^{\pm} := \beta^{\pm\hat{\pm}}, \tag{3.2.33}$$

and satisfy the identities

$$\alpha^{\pm} = e^{\pm 2\pi\nu}\alpha^{\mp}, \quad \beta^{\pm} = e^{\mp 2\pi\nu}\beta^{\mp}, \tag{3.2.34}$$

which we use extensively. We have also defined propagators for each boundary condition[8] $\Delta_\pm$,

$$G^{(0)}_{\Delta_+,\pm\hat{\pm}}(u_1,u_2) \equiv \csc(\pi(u_1+u_2))\alpha^{\hat{\pm}}\omega_{\Delta_+}(u_1,u_2)\Gamma(i\nu)\Gamma(-i\nu)\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2), \tag{3.2.35a}$$

$$G^{(0)}_{\Delta_-,\pm\hat{\pm}}(u_1,u_2) \equiv \csc(\pi(u_1+u_2))\beta^{\pm}\omega_{\Delta_-}(u_1,u_2)\Gamma(i\nu)\Gamma(-i\nu)\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2), \tag{3.2.35b}$$

and it is straightforward to show from (3.2.29) the useful identities

$$e^{-2i\pi u_1}\Omega^{-+}_{\nu,\vec{k}}(u_1,u_2) = e^{-\pi\nu}\Omega^{++}_{\nu,\vec{k}}(u_1,u_2), \tag{3.2.36a}$$

$$e^{2i\pi u_2}\Omega^{-+}_{\nu,\vec{k}}(u_1,u_2) = e^{-\pi\nu}\Omega^{--}_{\nu,\vec{k}}(u_1,u_2), \tag{3.2.36b}$$

$$e^{2i\pi(u_1+u_2)}\Omega^{++}_{\nu,\vec{k}}(u_1,u_2) = \Omega^{--}_{\nu,\vec{k}}(u_1,u_2). \tag{3.2.36c}$$

We now aim to express the Wightman functions in (3.2.25) in terms of (anti-) time-ordered Green's functions. We first go to Mellin space, where it can be shown that[9]

$$G^{(0)}_W(\eta_1,\bar{\eta}_2^+) = e^{-i\pi\Delta_+}G^{(0)}_{\Delta_+,\bar{T}}(\eta_1,\eta_2) + e^{-i\pi\Delta_-}G^{(0)}_{\Delta_-,\bar{T}}(\eta_1,\eta_2). \tag{3.2.37}$$

Following the same procedure, one can also show the following identities;

$$G^{(0)}_W(\bar{\eta}_1^-,\eta_2) = e^{+\pi i\Delta_+}G^{(0)}_{\Delta_+,T}(\eta_1,\eta_2) + e^{+\pi i\Delta_-}G^{(0)}_{\Delta_-,T}(\eta_1,\eta_2), \tag{3.2.38a}$$

[8]Note that "boundary condition" is something of an abuse of terminology in this context - there are two solutions to the wave equation with the most generic behaviour described by a linear combination of the two; see (2.2.7). "Boundary condition" here simply refers to which of these solutions we want to consider. In the context of AdS/CFT, these correspond to Dirichlet, Neumann, or mixed boundary conditions - hence the terminology.

[9]See appendix A for details.

$$G^{(0)}_{\Delta_\pm,T}(\eta_1,\eta_2) = e^{-2\pi i\Delta_\pm} G^{(0)}_{\Delta_\pm,\bar{T}}(\bar{\eta}_1^-,\bar{\eta}_2^-), \tag{3.2.38b}$$

$$G^{(0)}_{\Delta_\pm,\bar{T}}(\eta_1,\eta_2) = e^{2\pi i\Delta_\pm} G^{(0)}_{\Delta_\pm,T}(\bar{\eta}_1^+,\bar{\eta}_2^+). \tag{3.2.38c}$$

We can then plug these into (3.2.25), and obtain our final result for the time-ordered propagator,

$$\boxed{\begin{aligned} G^{(\alpha)}_T(\eta_1,\eta_2) = P^+_{\Delta_+} G^{(0)}_{\Delta_+,T}(\eta_1,\eta_2) + e^{2\pi i\Delta_+} P^-_{\Delta_+} G^{(0)}_{\Delta_+,T}(\bar{\eta}_1^+,\bar{\eta}_2^+) \\ + (\Delta_+ \longrightarrow \Delta_-), \end{aligned}} \tag{3.2.39}$$

where we have defined

$$P^+_\Delta := \left(\cosh^2(\alpha) - \frac{1}{2}\sinh(2\alpha)e^{-i\beta-\pi\nu}\right), \tag{3.2.40a}$$

$$P^-_\Delta := \left(\sinh^2(\alpha) - \frac{1}{2}\sinh(2\alpha)e^{+i\beta+\pi\nu}\right). \tag{3.2.40b}$$

In (3.2.40) we have parameterised $\Delta = \frac{d}{2} + i\nu$, and in (3.2.39) the $(\Delta_+ \to \Delta_-)$ includes the subscripts on the $P^\pm_{\Delta_\pm}$.

For the anti-time-ordered propagator we follow the same procedure. Beginning with the definition (3.2.21), we have

$$\begin{aligned} G^{(\alpha)}_{\bar{T}}(\eta_1;\eta_2) &= \theta(\eta_1-\eta_2)\,G^{(\alpha)}_W(\eta_2;\eta_1) + \theta(\eta_2-\eta_1)\,G^{(\alpha)}_W(\eta_1;\eta_2) \\ &= \cosh^2(\alpha)[\theta(\eta_1-\eta_2)G^{(0)}_W(\eta_2,\eta_1) + \theta(\eta_2-\eta_1)G^{(0)}_W(\eta_1,\eta_2)] \\ &\quad + \sinh^2(\alpha)[\theta(\eta_1-\eta_2)G^{(0)}_W(\eta_1,\eta_2) + \theta(\eta_2-\eta_1)G^{(0)}_W(\eta_2,\eta_1)] \\ &\quad - \frac{1}{2}\sinh(2\alpha)[e^{i\left(\beta+\frac{\pi d}{2}\right)}G^{(0)}_W(\eta_1,\bar{\eta}_2^+) + e^{-i\left(\beta+\frac{\pi d}{2}\right)}G^{(0)}_W(\bar{\eta}_1^-,\eta_2)], \end{aligned} \tag{3.2.41}$$

where in going from the first to the second line we used identities (3.2.22), (3.2.24) and

$$\theta(\eta_1-\eta_2) + \theta(\eta_2-\eta_1) = 1. \tag{3.2.42}$$

From here we see that

$$G^{(\alpha)}_{\bar{T}}(\eta_1;\eta_2) = \cosh^2(\alpha)G^{(0)}_{\bar{T}}(\eta_1,\eta_2) + \sinh^2(\alpha)G^{(0)}_T(\eta_1,\eta_2)$$

$$- \frac{1}{2}\sinh(2\alpha)[e^{i\left(\beta+\frac{\pi d}{2}\right)}G_W^{(0)}(\eta_1,\bar{\eta}_2^+) + e^{-i\left(\beta+\frac{\pi d}{2}\right)}G_W^{(0)}(\bar{\eta}_1^-,\eta_2)], \tag{3.2.43}$$

which is identical to (3.2.25) after swapping[10] $\sinh(\alpha) \leftrightarrow \cosh(\alpha)$. Performing this swapping on (3.2.39) we can therefore immediately write down

$$G_{\bar{T}}^{(\alpha)}(\eta_1,\eta_2) = M_{\Delta_+}^+ G_{\Delta_+,T}^{(0)}(\eta_1,\eta_2) + e^{2\pi i\Delta_+} M_{\Delta_+}^- G_{\Delta_+,T}^{(0)}(\bar{\eta}_1^+,\bar{\eta}_2^+) + (\Delta_+ \longrightarrow \Delta_-), \tag{3.2.44}$$

where we have defined

$$M_\Delta^+ := \left(\sinh^2(\alpha) - \frac{1}{2}\sinh(2\alpha)e^{-i\beta-\pi\nu}\right), \tag{3.2.45a}$$

$$M_\Delta^- := \left(\cosh^2(\alpha) - \frac{1}{2}\sinh(2\alpha)e^{+i\beta+\pi\nu}\right). \tag{3.2.45b}$$

Using (3.2.38) we reach our final result for the anti-time-ordered propagator,

$$\boxed{G_{\bar{T}}^{(\alpha)}(\eta_1,\eta_2) = M_{\Delta_+}^- G_{\Delta_+,\bar{T}}^{(0)}(\eta_1,\eta_2) + e^{-2\pi i\Delta_+} M_{\Delta_+}^+ G_{\Delta_+,\bar{T}}^{(0)}(\bar{\eta}_1^-,\bar{\eta}_2^-) + (\Delta_+ \longrightarrow \Delta_-).} \tag{3.2.46}$$

Having derived the Wightman (equation (3.2.20)), time-ordered (equation (3.2.39)) and anti-time-ordered (equation (3.2.46)) bulk-to-bulk propagators in terms of their Bunch-Davies counterparts, we now derive analogous expressions for the bulk-to-boundary propagators.

3.2.2 Bulk-to-Boundary Propagators

Bulk-to-boundary propagators are obtained by taking the late-time limit of one of the bulk points in the free theory two-point functions presented above, which is straightforward to perform in the Mellin-Barnes representation. We'll choose η_2, and so the calculation boils down to simply computing a Mellin-Barnes integral via the residue theorem, and taking the leading residue in the $\eta_2 \to 0$ limit. The expansion

[10] This is clearer if we note the identity $\frac{1}{2}\sinh(2\alpha) = \cosh(\alpha)\sinh(\alpha)$.

for small η_2 is generated by the residues of the poles in the corresponding Mellin variable u_2:

$$u_2 = \pm\tfrac{i\nu}{2} - n, \qquad n = 0,\, 1,\, 2,\, \dots\,, \tag{3.2.47}$$

where the leading terms are encoded the $n = 0$ poles. In particular,

$$\lim_{\eta_2 \to 0} G^{(\alpha)}_{\pm\mp}(\eta_1;\eta_2) = K^{(\alpha)}_{\Delta_+,\pm}(\eta_1,k) + K^{(\alpha)}_{\Delta_-,\pm}(\eta_1,k), \tag{3.2.48}$$

which identifies the bulk-to-boundary propagators,

$$K^{(\alpha)}_{\Delta_\pm,\hat{\pm}}(\eta_1,k) = \int_{-i\infty}^{+i\infty} \frac{du_1}{2\pi i}\, \text{Res}_{u_2=\mp\frac{i\nu}{2}} \left[G^{(\alpha)}_{\hat{\pm}\bullet}(u_1,u_2)\,(-\eta_2)^{-\left(2u_2-\frac{d}{2}\right)} \right] \times (-\eta_1)^{-\left(2u_1-\frac{d}{2}\right)}, \tag{3.2.49}$$

where $\bullet$ indicates that the branch of the in-in contour the second bulk point is on makes no difference to the result[11].

For generic α, combining the bulk-bulk propagators derived earlier (and summarised later in equation (3.2.64), where we use the notation of the in-in formalism) with (3.2.49) above one obtains

$$\boxed{\begin{aligned} K^{(\alpha)}_{\Delta,+}(\eta,k) &= P^+_\Delta K^{(0)}_{\Delta,+}(\eta,k) + P^-_\Delta e^{\Delta\pi i} K^{(0)}_{\Delta,+}(\bar{\eta}^+,k), && \text{(3.2.50a)} \\ K^{(\alpha)}_{\Delta,-}(\eta,k) &= M^+_\Delta e^{-\Delta\pi i} K^{(0)}_{\Delta,-}(\bar{\eta}^-,k) + M^-_\Delta K^{(0)}_{\Delta,-}(\eta,k), && \text{(3.2.50b)} \end{aligned}}$$

where in the Mellin space representation the bulk-to-boundary propagators for the Bunch-Davies vacuum read

$$K^{(0)}_{\Delta,\pm}(\eta,k) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} K^{(0)}_{\Delta,\pm}(s,k)\,(-\eta)^{-\left(2s-\frac{d}{2}\right)}, \tag{3.2.51}$$

where

$$K^{(0)}_{\Delta,\pm}(s,k) = (-\eta_0)^{\Delta}\, \frac{\Gamma(-i\nu)}{4\pi} e^{\mp\left(s+\frac{i\nu}{2}\right)\pi i}\, \Gamma(s+\tfrac{i\nu}{2})\Gamma(s-\tfrac{i\nu}{2}) \left(\frac{k}{2}\right)^{-2s+i\nu}, \tag{3.2.52}$$

with the parameterisation $\Delta = \frac{d}{2} + i\nu$.

[11] Which of $K^{(\alpha)}_{\Delta,\hat{\pm}}(\eta_1,k)$ we obtain depends only on the in-in branch for the first bulk point, since we are leaving this point in the bulk and taking the other to the boundary.

It should be noted that it is possible to trade the analytic continuations of the time coordinate η in the above propagators for analytic continuations of the boundary momentum k, a crucial fact that we will make heavy use of. To see this, we consider the Mellin representation of an analytically continued bulk-to-boundary propagator and simply shuffle the analytic continuation from the η to the k;

$$\begin{aligned} K^{(0)}_{\Delta,\hat{\pm}}(\bar{\eta}^{\pm},k) &= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} K^{(0)}_{\Delta,\hat{\pm}}(s,k)\,(-\bar{\eta}^{\pm})^{-2s+\frac{d}{2}} \\ &= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\,(-\eta_0)^{\Delta}\,\frac{\Gamma(-i\nu)}{4\pi} e^{\hat{\mp}\left(s+\frac{i\nu}{2}\right)\pi i}\Gamma(s+\tfrac{i\nu}{2})\Gamma(s-\tfrac{i\nu}{2})\left(\frac{k}{2}\right)^{-2s+i\nu} \\ &\qquad \times \left(e^{\mp\pi i}\right)^{\left(-2s+\frac{d}{2}\right)}(-\eta)^{-2s+\frac{d}{2}} \\ &= e^{\pm\pi i(\Delta-d)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\,(-\eta_0)^{\Delta}\,\frac{\Gamma(-i\nu)}{4\pi} e^{\hat{\mp}\left(s+\frac{i\nu}{2}\right)\pi i}\Gamma(s+\tfrac{i\nu}{2})\Gamma(s-\tfrac{i\nu}{2}) \\ &\qquad \times \left(\frac{e^{\mp\pi i}k}{2}\right)^{-2s+i\nu}(-\eta)^{-2s+\frac{d}{2}} \\ &= e^{\pm\pi i(\Delta-d)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} K^{(0)}_{\Delta,\hat{\pm}}\left(s,\bar{k}^{\pm}\right)(-\eta)^{-2s+\frac{d}{2}}, \end{aligned} \tag{3.2.53}$$

giving us the identity

$$K^{(0)}_{\Delta,\hat{\pm}}(\bar{\eta}^{\pm},k) = e^{\pm\pi i(\Delta-d)}K^{(0)}_{\Delta,\hat{\pm}}(\eta,\bar{k}^{\pm}), \tag{3.2.54}$$

where we parameterised $\Delta = \frac{d}{2} + i\nu$ and introduced the notation

$$\bar{k}^{\pm} = e^{\mp\pi i}k, \tag{3.2.55}$$

analogous to the notation in (3.2.19). The equivalent identity at the level of the Mellin-Barnes representation is

$$e^{\mp\pi i\left(-2s+\frac{d}{2}\right)}K^{(0)}_{\Delta,\hat{\pm}}(s,k) = e^{\pm\pi i(\Delta-d)}K^{(0)}_{\Delta,\hat{\pm}}\left(s,\bar{k}^{\pm}\right), \tag{3.2.56}$$

which can be seen from the intermediate steps in the derivation of (3.2.54).

The same trade can be made in the bulk-to-bulk propagators. Recall the definition of the dS harmonic function in Mellin space from equation (3.2.29),

$$\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2) = c^{\text{dS-AdS}}_{\frac{d}{2}+i\nu}c^{\text{dS-AdS}}_{\frac{d}{2}-i\nu}e^{\mp\pi i\left(u_1+\frac{i\nu}{2}\right)}e^{\hat{\mp}\pi i\left(u_2-\frac{i\nu}{2}\right)}\Omega^{\text{AdS}}_{\nu,\vec{k}}(u_1,u_2). \tag{3.2.57}$$

This has a representation in terms of dS bulk-to-boundary propagators;

$$\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2) = \frac{\nu^2}{\pi}K^{(0)}_{\Delta_+,\pm}\left(u_1,k_1\right)K^{(0)}_{\Delta_-,\hat{\pm}}\left(u_2,k_2\right), \tag{3.2.58}$$

which can be seen by noting that the EAdS harmonic function is a product of EAdS bulk-boundary propagators [103, 105], which in Mellin space reads [76]

$$\Omega^{\text{AdS}}_{\nu,\vec{k}}(u_1,u_2) = \frac{\nu^2}{\pi}K^{\text{AdS}}_{\Delta_+}\left(u_1,k_1\right)K^{\text{AdS}}_{\Delta_-}\left(u_2,k_2\right), \tag{3.2.59}$$

with the EAdS bulk-to-boundary propagator given by

$$K^{\text{AdS}}_{\Delta}\left(s,k\right) = \frac{1}{2\Gamma\left(i\nu+1\right)}\Gamma(s+\tfrac{i\nu}{2})\Gamma(s-\tfrac{i\nu}{2})\left(\frac{k}{2}\right)^{-2s+i\nu}. \tag{3.2.60}$$

From this representation of the dS harmonic functions, the bulk-to-bulk propagators inherit identities analogous to (3.2.54). In particular, we see that

$$\begin{aligned}
G^{(0)}_{\pm\hat{\pm}}(\bar{\eta}^{\pm}_1,k_1;\bar{\eta}^{\hat{\pm}}_2,k_2) &= \int_{-i\infty}^{i\infty}[ds]_2G^{(0)}_{\pm\hat{\pm}}(s_1,k_1;s_2,k_2)(-\bar{\eta}^{\pm}_1)^{-2s_1+\frac{d}{2}}(-\bar{\eta}^{\hat{\pm}}_2)^{-2s_2+\frac{d}{2}}\\
&= \int_{-i\infty}^{i\infty}[ds]_2e^{\mp\pi i(-2s_1+\frac{d}{2})}e^{\hat{\mp}\pi i(-2s_2+\frac{d}{2})}\csc(\pi(s_1+s_2))\\
&\qquad\times(\alpha^{\hat{\pm}}\omega_{\Delta_+}(s_1,s_2)+\beta^{\pm}\omega_{\Delta_-}(s_1,s_2))\Gamma(i\nu)\Gamma(-i\nu)\\
&\qquad\qquad\times\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2)(-\eta_1)^{-2s_1+\frac{d}{2}}(-\eta_2)^{-2s_2+\frac{d}{2}}\\
&= \int_{-i\infty}^{i\infty}[ds]_2e^{\mp\pi i(-2s_1+\frac{d}{2})}e^{\hat{\mp}\pi i(-2s_2+\frac{d}{2})}\csc(\pi(s_1+s_2))\\
&\qquad\times(\alpha^{\hat{\pm}}\omega_{\Delta_+}(s_1,s_2)+\beta^{\pm}\omega_{\Delta_-}(s_1,s_2))\Gamma(i\nu)\Gamma(-i\nu)\\
&\qquad\times\frac{\nu^2}{\pi}K^{(0)}_{\Delta_+,\pm}\left(s_1,k_1\right)K^{(0)}_{\Delta_-,\hat{\pm}}\left(s_2,k_2\right)(-\eta_1)^{-2s_1+\frac{d}{2}}(-\eta_2)^{-2s_2+\frac{d}{2}}\\
&= e^{\pm\pi i(\Delta_+-d)}e^{\hat{\pm}\pi i(\Delta_--d)}\int_{-i\infty}^{i\infty}[ds]_2\csc(\pi(s_1+s_2))\\
&\qquad\times(\alpha^{\hat{\pm}}\omega_{\Delta_+}(s_1,s_2)+\beta^{\pm}\omega_{\Delta_-}(s_1,s_2))\Gamma(i\nu)\Gamma(-i\nu)
\end{aligned}$$

$$\times \frac{\nu^2}{\pi} K^{(0)}_{\Delta_+,\pm}\left(s_1,\bar{k}_1^{\pm}\right) K^{(0)}_{\Delta_-,\hat{\pm}}\left(s_2,\bar{k}_2^{\hat{\pm}}\right)(-\eta_1)^{-2s_1+\frac{d}{2}}(-\eta_2)^{-2s_2+\frac{d}{2}}$$

$$= e^{\pm\pi i(\Delta_+-d)} e^{\hat{\pm}\pi i(\Delta_--d)} \int_{-i\infty}^{i\infty} [ds]_2 G^{(0)}_{\pm\hat{\pm}}(s_1,\bar{k}_1^{\pm};s_2,\bar{k}_2^{\hat{\pm}}) \times (-\eta_1)^{-2s_1+\frac{d}{2}}(-\eta_2)^{-2s_2+\frac{d}{2}}, \tag{3.2.61}$$

and we obtain the identity

$$G^{(0)}_{\pm\hat{\pm}}(\bar{\eta}_1^{\pm},k_1;\bar{\eta}_2^{\hat{\pm}},k_2) = e^{\pm\pi i(\Delta_+-d)} e^{\hat{\pm}\pi i(\Delta_--d)} G^{(0)}_{\pm\hat{\pm}}(\eta_1,\bar{k}_1^{\pm};\eta_2,\bar{k}_2^{\hat{\pm}}). \tag{3.2.62}$$

Analogous identities for a single analytic continuation that can be obtained in the same way are

$$G^{(0)}_{\pm\mp}(\bar{\eta}_1^{\pm},k_1;\eta_2,k_2) = (e^{\pm\pi i})^{\Delta_+-d} G^{(0)}_{\pm\hat{\pm}}(\eta_1,\bar{k}_1^{\pm};\eta_2,k_2), \tag{3.2.63a}$$

$$G^{(0)}_{\pm\mp}(\eta_1,k_1;\bar{\eta}_2^{\mp},k_2) = (e^{\mp\pi i})^{\Delta_--d} G^{(0)}_{\pm\mp}(\eta_1,k_1;\eta_2,\bar{k}_2^{\mp}). \tag{3.2.63b}$$

In this section we reached some of the main results of the thesis. The Wightman and (anti-)time-ordered propagators for arbitrary choice of α-vacuum in the Expanding Poincaré patch were written in terms of their counterparts in the Bunch-Davies vacuum, and similar expressions relating the α-vacuum bulk-to-boundary propagators to those for the Bunch-Davies vacuum were obtained. The Bunch-Davies propagators involved feature points that are analytically continued to the Contracting Poincaré patch. We summarise these results below.

α-Vacuum Propagators from Bunch-Davies. Summarising the above in the notation of the in-in formalism, for the bulk-to-bulk propagators we have

$$G^{(\alpha)}_T(\eta_1;\eta_2) \equiv G^{(\alpha)}_{++}(\eta_1;\eta_2) = P^+_{\Delta_+} G^{(0)}_{\Delta_+,++}(\eta_1;\eta_2) + e^{2\Delta_+\pi i} P^-_{\Delta_+} G^{(0)}_{\Delta_+,++}(\bar{\eta}_1^+;\bar{\eta}_2^+) + (\Delta_+ \to \Delta_-), \tag{3.2.64a}$$

$$G^{(\alpha)}_{\bar{T}}(\eta_1;\eta_2) \equiv G^{(\alpha)}_{--}(\eta_1;\eta_2) = M^-_{\Delta_+} G^{(0)}_{\Delta_+,--}(\eta_1;\eta_2) + e^{-2\Delta_+\pi i} M^+_{\Delta_+} G^{(0)}_{\Delta_+,--}(\bar{\eta}_1^-;\bar{\eta}_2^-) + (\Delta_+ \to \Delta_-), \tag{3.2.64b}$$

$$
\begin{aligned}
G_W^{(\alpha)}(\eta_1;\eta_2) \equiv G_{-+}^{(\alpha)}(\eta_1;\eta_2) &= \cosh^2\alpha\, G_{-+}^{(0)}(\eta_1;\eta_2) + \sinh^2\alpha\, G_{-+}^{(0)}(\bar{\eta}_1^-;\bar{\eta}_2^+) \qquad (3.2.64c)\\
&- \frac{1}{2}\sinh 2\alpha\big[e^{i\left(\beta+\frac{\pi d}{2}\right)}\, G_{-+}^{(0)}(\eta_1;\bar{\eta}_2^+) + e^{-i\left(\beta+\frac{\pi d}{2}\right)}\, G_{-+}^{(0)}(\bar{\eta}_1^-;\eta_2)\big],\\
G_W^{(\alpha)}(\eta_2;\eta_1) \equiv G_{+-}^{(\alpha)}(\eta_1;\eta_2) &= \cosh^2\alpha\, G_{+-}^{(0)}(\eta_1;\eta_2) + \sinh^2\alpha\, G_{+-}^{(0)}(\bar{\eta}_1^+;\bar{\eta}_2^-) \qquad (3.2.64d)\\
&- \frac{1}{2}\sinh 2\alpha\big[e^{i\left(\beta+\frac{\pi d}{2}\right)}\, G_{+-}^{(0)}(\bar{\eta}_1^+;\eta_2) + e^{-i\left(\beta+\frac{\pi d}{2}\right)}\, G_{+-}^{(0)}(\eta_1;\bar{\eta}_2^-)\big]
\end{aligned}
$$

where we used (3.2.22) and $G_{+-}^{(0)}(\eta_1,\eta_2) = G_{-+}^{(0)}(\eta_2,\eta_1)$. For the bulk-to-boundary propagators, we found

$$K_{\Delta,+}^{(\alpha)}(\eta,k) = P_\Delta^+ K_{\Delta,+}^{(0)}(\eta,k) + P_\Delta^- e^{\Delta\pi i} K_{\Delta,+}^{(0)}(\bar{\eta}^+,k), \qquad (3.2.65a)$$

$$K_{\Delta,-}^{(\alpha)}(\eta,k) = M_\Delta^+ e^{-\Delta\pi i} K_{\Delta,-}^{(0)}(\bar{\eta}^-,k) + M_\Delta^- K_{\Delta,-}^{(0)}(\eta,k), \qquad (3.2.65b)$$

where the coefficients are given by

$$P_\Delta^+ = \left(\cosh^2\alpha - \frac{1}{2}e^{-i\beta}\sinh 2\alpha e^{-\pi\nu}\right), \qquad (3.2.66a)$$

$$P_\Delta^- = \left(\sinh^2\alpha - \frac{1}{2}e^{+i\beta}\sinh 2\alpha e^{+\pi\nu}\right), \qquad (3.2.66b)$$

$$M_\Delta^+ = \left(\sinh^2\alpha - \frac{1}{2}e^{-i\beta}\sinh 2\alpha e^{-\pi\nu}\right), \qquad (3.2.66c)$$

$$M_\Delta^- = \left(\cosh^2\alpha - \frac{1}{2}e^{+i\beta}\sinh 2\alpha e^{+\pi\nu}\right). \qquad (3.2.66d)$$

These results were originally obtained in [1].

The central point of this section was to show that the in-in propagators in a generic α-vacuum in the Expanding Poincaré Patch can be expressed in terms of their Bunch-Davies counterparts, with points antipodally transformed to the Contracting Poincaré Patch. We will see in the next section that the above can be expressed in terms of EAdS propagators, with the analytic continuations on the dS time coordinate leading to expressions involving EAdS propagators with points on both the upper and the lower sheet of the EAdS hyperboloid.

3.3 α-vacua from EAdS

Recent years have seen substantial progress in our understanding of the relationship between in-in propagators in the Bunch-Davies vacuum and bulk-to-bulk and bulk-to-boundary propagators in Euclidean AdS, and in turn the relationship between perturbative correlators in the two spacetimes [74–77]. By computing the α-vacuum bulk-to-bulk and bulk-to-boundary propagators in terms of their Bunch-Davies counterparts, we can therefore straightforwardly extend existing results to express α-vacuum propagators in terms of their counterparts in EAdS.

3.3.1 Bunch-Davies and EAdS

We begin with a review of the relations between in-in propagators in the Bunch-Davies vacuum and the corresponding EAdS propagators; see [74–77] for details. To express the bulk-to-bulk and bulk-to-boundary propagators in terms of their EAdS counterparts, we first note that the bulk-to-bulk propagator in EAdS can be expressed in the form [76, 77]

$$G_{\Delta}^{\text{AdS}}(z_1, k_1; z_2, k_2) = \int_{-i\infty}^{i\infty} [\mathrm{d}u]_2 \, G_{\Delta}^{\text{AdS}}(u_1, k_1; u_2, k_2) z_1^{-\left(2u_1 - \frac{d}{2}\right)} z_2^{-\left(2u_2 - \frac{d}{2}\right)}, \quad (3.3.1)$$

where

$$G_{\Delta}^{\text{AdS}}(u_1, k_1; u_2, k_2) = \csc\left(\pi(u_1 + u_2)\right) \omega_{\Delta}(u_1, u_2) \Gamma(i\nu)\Gamma(-i\nu) \, \Omega_{\nu,\vec{k}}^{\text{AdS}}(u_1, u_2), \quad (3.3.2)$$

with the AdS harmonic function defined as a product of bulk-to-boundary propagators in (3.2.59). Repeating for convenience, the Mellin representation of the AdS bulk-to-boundary propagator is given by

$$K_{\Delta}^{\text{AdS}}(s, k) = \frac{1}{2\Gamma\left(i\nu + 1\right)} \Gamma(s + \tfrac{i\nu}{2}) \Gamma(s - \tfrac{i\nu}{2}) \left(\frac{k}{2}\right)^{-2s + i\nu}. \quad (3.3.3)$$

By comparing with (3.2.52), it is straightforward to see that in the Mellin representation (after stripping off the $(-\eta_0)^{\Delta}$ factor), the Bunch-Davies bulk-to-boundary propagators in de Sitter are related to the corresponding EAdS propagator by

$$K^{(0)}_{\Delta,\pm}\left(s,k\right)=c^{\text{dS-AdS}}_{\Delta}e^{\mp\left(s+\frac{1}{2}\left(\Delta-\frac{d}{2}\right)\right)\pi i}K^{\text{AdS}}_{\Delta}\left(s,k\right). \tag{3.3.4}$$

Taking the Mellin transform of either side yields

$$K^{(0)}_{\Delta,\pm}(\eta,k)=c^{\text{dS-AdS}}_{\Delta}e^{\mp\frac{i\pi}{2}\Delta}K^{\text{AdS}}_{\Delta}(-e^{\pm\frac{i\pi}{2}}\eta,k). \tag{3.3.5}$$

The identity (3.3.4) then implies the relation (3.2.29) between the EAdS and dS harmonic functions, via (3.2.58) and (3.2.59). We see that

$$\begin{aligned}\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2)&=\frac{\nu^2}{\pi}K^{(0)}_{\Delta_+,\pm}\left(u_1,k_1\right)K^{(0)}_{\Delta_-,\hat{\pm}}\left(u_2,k_2\right)\\&=c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}\frac{\nu^2}{\pi}K^{\text{AdS}}_{\Delta_+}\left(u_1,k_1\right)K^{\text{AdS}}_{\Delta_-}\left(u_2,k_2\right)\\&=c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}\Omega^{\text{AdS}}_{\nu,\vec{k}}(u_1,u_2),\end{aligned} \tag{3.3.6}$$

which in turn implies relations between the EAdS and dS bulk-to-bulk and bulk-to-boundary propagators. Starting from (3.2.27) and using the above identity between the dS and AdS harmonic functions we find

$$\begin{aligned}G^{(0)}_{\pm\hat{\pm}}(u_1,u_2)&=\csc(\pi(u_1+u_2))(\alpha^{\hat{\pm}}\omega_{\Delta_+}(u_1,u_2)+\beta^{\pm}\omega_{\Delta_-}(u_1,u_2))\\&\qquad\times\Gamma(i\nu)\Gamma(-i\nu)\Omega^{\pm\hat{\pm}}_{\nu,\vec{k}}(u_1,u_2)\\&=\csc(\pi(u_1+u_2))(\alpha^{\hat{\pm}}\omega_{\Delta_+}(u_1,u_2)+\beta^{\pm}\omega_{\Delta_-}(u_1,u_2))\Gamma(i\nu)\Gamma(-i\nu)\\&\quad\times c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}\Omega^{\text{AdS}}_{\nu,\vec{k}}(u_1,u_2)\\&=\alpha^{\hat{\pm}}c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}\\&\quad\times\csc(\pi(u_1+u_2))\omega_{\Delta_+}(u_1,u_2)\Gamma(i\nu)\Gamma(-i\nu)\Omega^{\text{AdS}}_{\nu,\vec{k}}(u_1,u_2)\\&\quad+\beta^{\pm}c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}\\&\quad\times\csc(\pi(u_1+u_2))\omega_{\Delta_-}(u_1,u_2)\Gamma(i\nu)\Gamma(-i\nu)\Omega^{\text{AdS}}_{\nu,\vec{k}}(u_1,u_2)\\&=\alpha^{\hat{\pm}}c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}G^{\text{AdS}}_{\Delta_+}(u_1,u_2)\\&\quad+\beta^{\pm}c^{\text{dS-AdS}}_{\Delta_+}c^{\text{dS-AdS}}_{\Delta_-}e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i}e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}G^{\text{AdS}}_{\Delta_-}(u_1,u_2),\end{aligned} \tag{3.3.7}$$

leading to the identity

$$G^{(0)}_{\pm\hat{\pm}}(u_1,u_2) = c^{\text{dS-AdS}}_{\Delta_+} c^{\text{dS-AdS}}_{\Delta_-} e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i} e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i}\left(\alpha^{\hat{\pm}} G^{\text{AdS}}_{\Delta_+}(u_1,u_2) + \beta^{\pm} G^{\text{AdS}}_{\Delta_-}(u_1,u_2)\right). \tag{3.3.8}$$

By noting that

$$\alpha^{\hat{\pm}} c^{\text{dS-AdS}}_{\Delta_+} c^{\text{dS-AdS}}_{\Delta_-} = c^{\text{dS-AdS}}_{\Delta_+} e^{\hat{\pm}\pi\nu}, \quad \beta^{\pm} c^{\text{dS-AdS}}_{\Delta_+} c^{\text{dS-AdS}}_{\Delta_-} = c^{\text{dS-AdS}}_{\Delta_-} e^{\pm\pi\nu}, \tag{3.3.9}$$

we can write this as

$$\begin{aligned} G^{(0)}_{\pm\hat{\pm}}(u_1,u_2) =& c^{\text{dS-AdS}}_{\Delta_+} e^{\hat{\pm}\pi\nu} e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i} e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i} G^{\text{AdS}}_{\Delta_+}(u_1,u_2) \\ &+ c^{\text{dS-AdS}}_{\Delta_-} e^{\pm\pi\nu} e^{\mp\left(u_1+\frac{i\nu}{2}\right)\pi i} e^{\hat{\mp}\left(u_2-\frac{i\nu}{2}\right)\pi i} G^{\text{AdS}}_{\Delta_-}(u_1,u_2), \end{aligned} \tag{3.3.10}$$

and taking the inverse Mellin transform of either side results in

$$\begin{aligned} G^{(0)}_{\pm\hat{\pm}}(\eta_1,\eta_2) =& c^{\text{dS-AdS}}_{\Delta_+} e^{\mp\frac{i\pi}{2}\Delta_+} e^{\hat{\mp}\frac{i\pi}{2}\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\pm\frac{i\pi}{2}}\eta_1, -e^{\hat{\pm}\frac{i\pi}{2}}\eta_2) \\ &+ (\Delta_+ \longrightarrow \Delta_-), \end{aligned} \tag{3.3.11}$$

as was first derived in [77].

We see that bulk-to-bulk and bulk-to-boundary propagators in the Bunch-Davies vacuum are related by analytic continuation to their counterparts in Euclidean Anti-de Sitter. Via the relations (3.2.64) and (3.2.65), it should then be straightforward to express the propagators for arbitrary choice of α-vacuum in terms of corresponding propagators in EAdS. Writing down these relations is the goal of the next section.

3.3.2 α-vacua and EAdS

To express the de Sitter bulk-to-bulk and bulk-to-boundary propagators for the α-vacua in terms of EAdS objects, we can simply plug the relations for the Bunch-Davies propagators in terms of their EAdS counterparts into the relations for α-vacuum propagators in terms of Bunch-Davies. For example, plugging (3.3.11) into (3.2.64) will result in expressions for the in-in bulk-to-bulk propagators for the α-vacua in terms of corresponding EAdS propagators, with the momenta analytically continued according to (3.2.62) and (3.2.63).

We start with the analogue of (3.3.11) for each of the boundary conditions $\Delta_\pm$, namely

$$G^{(0)}_{\Delta_+,\pm\hat{\pm}}(\eta_1,\eta_2) = c^{\text{dS-AdS}}_{\Delta_+} e^{\mp\frac{i\pi}{2}\Delta_+} e^{\hat{\mp}\frac{i\pi}{2}\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\pm\frac{i\pi}{2}}\eta_1, -e^{\hat{\pm}\frac{i\pi}{2}}\eta_2), \tag{3.3.12}$$

and similar for Δ_-. For the time-ordered propagator, this implies

$$G^{(0)}_{\Delta_+,++}(\eta_1,\eta_2) = c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, -e^{\frac{i\pi}{2}}\eta_2), \tag{3.3.13}$$

$$G^{(0)}_{\Delta_+,++}(\bar{\eta}_1^+,\bar{\eta}_2^+) = c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\bar{\eta}_1^+, -e^{\frac{i\pi}{2}}\bar{\eta}_2^+). \tag{3.3.14}$$

As explained in the previous section, the analytic continuations $\bar{\eta}^\pm \equiv e^{\mp i\pi}\eta$ can be traded for analytic continuations of the momenta. In particular, using the Mellin representation of the AdS bulk-to-bulk propagator (3.3.1) and the split representation of the AdS harmonic function (3.2.59) it can be shown that

$$G^{\text{AdS}}_{\Delta_+}(\bar{z}_1^\pm, k; \bar{z}_2^\pm, k) = e^{\mp i\pi d} G^{\text{AdS}}_{\Delta_+}(z_1, \bar{k}^\pm; z_2, \bar{k}^\pm), \tag{3.3.15}$$

and so

$$\begin{aligned} G^{(0)}_{\Delta_+,++}(\bar{\eta}_1^+,\bar{\eta}_2^+) &= c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\bar{\eta}_1^+, e^{\frac{i\pi}{2}}\bar{\eta}_2^+) \\ &= c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\Delta_+} e^{-i\pi d} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, \bar{k}^+; -e^{\frac{i\pi}{2}}\eta_2, \bar{k}^+). \end{aligned} \tag{3.3.16}$$

Plugging these into (3.2.64a), we then find the time-ordered propagator for arbitrary choice of α-vacuum in terms of the AdS bulk-to-bulk propagator,

$$\boxed{\begin{aligned} G^{(\alpha)}_{++}(\eta_1,k;\eta_2,k) = c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\Delta_+}\Big(& P^+_{\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, k; -e^{\frac{i\pi}{2}}\eta_2, k) \\ & + P^-_{\Delta_+} e^{-2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, \bar{k}^+; -e^{\frac{i\pi}{2}}\eta_2, \bar{k}^+)\Big) \\ & + (\Delta_+ \to \Delta_-). \end{aligned}} \tag{3.3.17}$$

The same steps can be followed for the anti-time-ordered propagator and plugging into (3.2.64b), resulting in

$$\boxed{\begin{aligned} G^{(\alpha)}_{--}(\eta_1, k; \eta_2, k) = c^{\text{dS-AdS}}_{\Delta_+} e^{i\pi\Delta_+}\Big(& M^-_{\Delta_+}\, G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, k; -e^{-\frac{i\pi}{2}}\eta_2, k) \\ & + M^+_{\Delta_+} e^{2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, \bar{k}^-; -e^{-\frac{i\pi}{2}}\eta_2, \bar{k}^-)\Big) \\ & + (\Delta_+ \to \Delta_-). \end{aligned}} \tag{3.3.18}$$

The derivations for the other propagators are similar. For $G^{(\alpha)}_{-+}$ and $G^{(\alpha)}_{+-}$ we need the identities

$$G^{\text{AdS}}_{\Delta}(\bar{z}_1^{\pm}, k_1; z_2, k_2) = e^{\mp i\pi(d-\Delta_+)} G^{\text{AdS}}_{\Delta}(z_1, \bar{k}_1^{\pm}; z_2, k_2) \tag{3.3.19a}$$

$$G^{\text{AdS}}_{\Delta}(z_1, k_1; \bar{z}_2^{\pm}, k_2) = e^{\mp i\pi\Delta_+} G^{\text{AdS}}_{\Delta}(z_1, k_1; z_2, \bar{k}_2^{\pm}) \tag{3.3.19b}$$

$$G^{\text{AdS}}_{\Delta}(\bar{z}_1^{\pm}, k_1; \bar{z}_2^{\hat{\pm}}, k_2) = e^{\mp i\pi(d-\Delta_+)} e^{\hat{\mp} i\pi\Delta_+} G^{\text{AdS}}_{\Delta}(z_1, \bar{k}_1^{\pm}; z_2, \bar{k}_2^{\hat{\pm}}), \tag{3.3.19c}$$

which can again be obtained with the Mellin representation of the AdS bulk-to-bulk propagator (3.3.1). Using (3.3.11) these lead to

$$\begin{aligned} G^{(0)}_{-+}(\eta_1, \bar{\eta}_2^{+}) = c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\left(\frac{d}{2}+i\nu\right)} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, k_1; -e^{+\frac{i\pi}{2}}\eta_2, \bar{k}_2^{+}) \\ + (\Delta_+ \to \Delta_-) \end{aligned} \tag{3.3.20a}$$

$$\begin{aligned} G^{(0)}_{-+}(\bar{\eta}_1^{-}, \eta_2) = c^{\text{dS-AdS}}_{\Delta_+} e^{i\pi\left(\frac{d}{2}-i\nu\right)} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, \bar{k}_1^{-}; -e^{+\frac{i\pi}{2}}\eta_2, k_2) \\ + (\Delta_+ \to \Delta_-) \end{aligned} \tag{3.3.20b}$$

$$\begin{aligned} G^{(0)}_{-+}(\bar{\eta}_1^{-}, \bar{\eta}_2^{+}) = c^{\text{dS-AdS}}_{\Delta_+} e^{2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, \bar{k}_1^{-}; -e^{+\frac{i\pi}{2}}\eta_2, \bar{k}_2^{+}) \\ + (\Delta_+ \to \Delta_-), \end{aligned} \tag{3.3.20c}$$

$$\begin{aligned} G^{(0)}_{+-}(\eta_1, \bar{\eta}_2^{-}) = c^{\text{dS-AdS}}_{\Delta_+} e^{i\pi(\frac{d}{2}+i\nu)} G^{\text{AdS}}_{\Delta_+}(-e^{+\frac{i\pi}{2}}\eta_1, k_1; -e^{-\frac{i\pi}{2}}\eta_2, \bar{k}_2^{-}) \\ + (\Delta_+ \to \Delta_-) \end{aligned} \tag{3.3.20d}$$

$$\begin{aligned} G^{(0)}_{+-}(\bar{\eta}_1^{+}, \eta_2) = c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi(\frac{d}{2}-i\nu)} G^{\text{AdS}}_{\Delta_+}(-e^{+\frac{i\pi}{2}}\eta_1, \bar{k}_1^{+}; -e^{-\frac{i\pi}{2}}\eta_2, k_2) \\ + (\Delta_+ \to \Delta_-) \end{aligned} \tag{3.3.20e}$$

$$\begin{aligned} G^{(0)}_{+-}(\bar{\eta}_1^{+}, \bar{\eta}_2^{-}) = c^{\text{dS-AdS}}_{\Delta_+} e^{-2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{+\frac{i\pi}{2}}\eta_1, \bar{k}_1^{+}; -e^{-\frac{i\pi}{2}}\eta_2, \bar{k}_2^{-}) \\ + (\Delta_+ \to \Delta_-). \end{aligned} \tag{3.3.20f}$$

Plugging (3.3.20a), (3.3.20b) and (3.3.20c) into (3.2.64c) then gives

$$\boxed{\begin{aligned} G^{(\alpha)}_{-+}(\eta_1,\eta_2) &= \cosh^2(\alpha) c^{\text{dS-AdS}}_{\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, k; -e^{\frac{i\pi}{2}}\eta_2, k) \\ &\quad + \sinh^2(\alpha) c^{\text{dS-AdS}}_{\Delta_+} e^{2\pi\nu}\, G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, \bar{k}^-; -e^{\frac{i\pi}{2}}\eta_2, \bar{k}^+) \\ &\quad - \frac{1}{2}\sinh(2\alpha) c^{\text{dS-AdS}}_{\Delta_+} e^{\pi\nu}\Big[e^{i\beta}\, G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, k; -e^{\frac{i\pi}{2}}\eta_2, \bar{k}^+) \\ &\qquad + e^{-i\beta}\, G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1, \bar{k}^-; -e^{\frac{i\pi}{2}}\eta_2, k)\Big] + (\Delta_+ \to \Delta_-). \end{aligned}} \tag{3.3.21}$$

In the same way, plugging (3.3.20d), (3.3.20e) and (3.3.20f) into (3.2.64d) gives $G^{(\alpha)}_{+-}$,

$$\boxed{\begin{aligned} G^{(\alpha)}_{+-}(\eta_1,\eta_2) &= \cosh^2(\alpha) c^{\text{dS-AdS}}_{\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, k; -e^{-\frac{i\pi}{2}}\eta_2, k) \\ &\quad + \sinh^2(\alpha) c^{\text{dS-AdS}}_{\Delta_+} e^{-2\pi\nu}\, G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, \bar{k}^+; -e^{-\frac{i\pi}{2}}\eta_2, \bar{k}^-) \\ &\quad - \frac{1}{2}\sinh(2\alpha) c^{\text{dS-AdS}}_{\Delta_+} e^{-\pi\nu}\Big[e^{i\beta}\, G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, \bar{k}^+; -e^{-\frac{i\pi}{2}}\eta_2, k) \\ &\qquad + e^{-i\beta}\, G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1, k; -e^{-\frac{i\pi}{2}}\eta_2, \bar{k}^-)\Big] + (\Delta_+ \to \Delta_-). \end{aligned}} \tag{3.3.22}$$

We can similarly rewrite the bulk-to-boundary propagators in terms of their EAdS counterparts, by plugging (3.3.5) into (3.2.65). For the time-ordered bulk-to-boundary propagator, we have

$$\begin{aligned} K^{(\alpha)}_{\Delta,+}(\eta,k) &= P^+_\Delta K^{(0)}_{\Delta,+}(\eta,k) + P^-_\Delta e^{i\pi\Delta} K^{(0)}_{\Delta,+}(\bar{\eta}^+,k) \\ &= P^+_\Delta c^{\text{dS-AdS}}_\Delta e^{-\frac{i\pi}{2}\Delta} K^{\text{AdS}}_\Delta(-e^{+\frac{i\pi}{2}}\eta,k) + P^-_\Delta c^{\text{dS-AdS}}_\Delta e^{+\frac{i\pi}{2}\Delta} K^{(0)}_{\Delta,+}(-e^{+\frac{i\pi}{2}}\bar{\eta}^+,k). \end{aligned} \tag{3.3.23}$$

Trading the analytic continuation of the time coordinate for an analytic continuation of the momentum, ie using

$$K^{\text{AdS}}_\Delta(-e^{\frac{i\pi}{2}}\bar{\eta}^+,k) = e^{-i\pi(d-\Delta)} K^{\text{AdS}}_\Delta(-e^{\frac{i\pi}{2}}\eta,\bar{k}^+), \tag{3.3.24}$$

this gives

$$\boxed{\begin{aligned} K^{(\alpha)}_{\Delta,+}(\eta,k) = c^{\text{dS-AdS}}_\Delta e^{-\frac{i\pi}{2}\Delta}\Big[&P^+_\Delta K^{\text{AdS}}_\Delta(-e^{\frac{i\pi}{2}}\eta,k) \\ &+ P^-_\Delta e^{i\pi(2\Delta-d)} K^{\text{AdS}}_\Delta(-e^{\frac{i\pi}{2}}\eta,\bar{k}^+)\Big]. \end{aligned}} \tag{3.3.25}$$

The derivation for the anti-time-ordered bulk-to-boundary propagator is exactly the same. We find

$$\begin{aligned} K^{(\alpha)}_{\Delta,-}(\eta,k) &= M^{+}_{\Delta} e^{-i\pi\Delta} K^{(0)}_{\Delta,-}(\bar{\eta}^{-},k) + M^{-}_{\Delta} K^{(0)}_{\Delta,-}(\eta,k) \\ &= M^{+}_{\Delta} c^{\text{dS-AdS}}_{\Delta} e^{-\frac{i\pi}{2}\Delta} K^{\text{AdS}}_{\Delta}(-e^{-\frac{i\pi}{2}}\bar{\eta}^{-},k) + M^{-}_{\Delta} c^{\text{dS-AdS}}_{\Delta} e^{+\frac{i\pi}{2}\Delta} K^{\text{AdS}}_{\Delta}(-e^{-\frac{i\pi}{2}}\eta,k), \end{aligned} \tag{3.3.26}$$

which after again trading the analytic continuation on the time coordinate for one on the momentum, gives

$$\boxed{\begin{aligned} K^{(\alpha)}_{\Delta,-}(\eta,k) = c^{\text{dS-AdS}}_{\Delta} e^{\frac{i\pi}{2}\Delta} \Big[& M^{-}_{\Delta} K^{\text{AdS}}_{\Delta}(-e^{-\frac{i\pi}{2}}\eta,k) \\ & + M^{+}_{\Delta} e^{-i\pi(2\Delta-d)} K^{\text{AdS}}_{\Delta}(-e^{-\frac{i\pi}{2}}\eta,\bar{k}^{-}) \Big]. \end{aligned}} \tag{3.3.27}$$

In this section we found that the in-in propagators in de Sitter space for generic α-vacuum can be re-written in terms of their counterparts in Euclidean AdS, with some of the momenta analytically continued as $k \to \bar{k}^{\pm} := e^{\mp i\pi}k$. We summarise the results below.

α-Vacuum Propagators from EAdS Summarising the above, we have

$$\begin{aligned} G^{(\alpha)}_{++}(\eta_1,k;\eta_2,k) = c^{\text{dS-AdS}}_{\Delta_+} e^{-i\pi\Delta_+} \Big(& P^{+}_{\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1,k;-e^{\frac{i\pi}{2}}\eta_2,k) \\ & + P^{-}_{\Delta_+} e^{-2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1,\bar{k}^{+};-e^{\frac{i\pi}{2}}\eta_2,\bar{k}^{+}) \Big) + (\Delta_+ \to \Delta_-), \end{aligned} \tag{3.3.28a}$$

$$\begin{aligned} G^{(\alpha)}_{--}(\eta_1,k;\eta_2,k) = c^{\text{dS-AdS}}_{\Delta_+} e^{i\pi\Delta_+} \Big(& M^{-}_{\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1,k;-e^{-\frac{i\pi}{2}}\eta_2,k) \\ & + M^{+}_{\Delta_+} e^{2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1,\bar{k}^{-};-e^{-\frac{i\pi}{2}}\eta_2,\bar{k}^{-}) \Big) + (\Delta_+ \to \Delta_-), \end{aligned} \tag{3.3.28b}$$

$$\begin{aligned} G^{(\alpha)}_{-+}(\eta_1,k;\eta_2,k) = & \cosh^2(\alpha) c^{\text{dS-AdS}}_{\Delta_+} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1,k;-e^{\frac{i\pi}{2}}\eta_2,k) \\ & + \sinh^2(\alpha) c^{\text{dS-AdS}}_{\Delta_+} e^{2\pi\nu} G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1,\bar{k}^{-};-e^{\frac{i\pi}{2}}\eta_2,\bar{k}^{+}) \end{aligned} \tag{3.3.28c}$$

$$- \frac{1}{2}\sinh(2\alpha)c^{\text{dS-AdS}}_{\Delta_+}e^{\pi\nu}\Big[e^{i\beta}\,G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1,k;-e^{\frac{i\pi}{2}}\eta_2,\bar{k}^+)$$
$$+ e^{-i\beta}\,G^{\text{AdS}}_{\Delta_+}(-e^{-\frac{i\pi}{2}}\eta_1,\bar{k}^-;-e^{\frac{i\pi}{2}}\eta_2,k)\Big] + (\Delta_+ \to \Delta_-)$$

$$G^{(\alpha)}_{+-}(\eta_1,k;\eta_2,k) = \cosh^2(\alpha)c^{\text{dS-AdS}}_{\Delta_+}G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1,k;-e^{-\frac{i\pi}{2}}\eta_2,k) \tag{3.3.28d}$$
$$+ \sinh^2(\alpha)c^{\text{dS-AdS}}_{\Delta_+}e^{-2\pi\nu}\,G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1,\bar{k}^+;-e^{-\frac{i\pi}{2}}\eta_2,\bar{k}^-)$$
$$- \frac{1}{2}\sinh(2\alpha)c^{\text{dS-AdS}}_{\Delta_+}e^{-\pi\nu}\Big[e^{i\beta}\,G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1,\bar{k}^+;-e^{-\frac{i\pi}{2}}\eta_2,k)$$
$$+ e^{-i\beta}\,G^{\text{AdS}}_{\Delta_+}(-e^{\frac{i\pi}{2}}\eta_1,k;-e^{-\frac{i\pi}{2}}\eta_2,\bar{k}^-)\Big] + (\Delta_+ \to \Delta_-),$$

and

$$K^{(\alpha)}_{\Delta,+}(\eta,k) = c^{\text{dS-AdS}}_{\Delta}e^{-\frac{i\pi}{2}\Delta}(-\eta_0)^{\Delta}\Big[P^+_{\Delta}K^{\text{AdS}}_{\Delta}(-e^{+\frac{i\pi}{2}}\eta,k) \tag{3.3.29a}$$
$$+ P^-_{\Delta}e^{+i\pi(2\Delta-d)}K^{\text{AdS}}_{\Delta}(-e^{+\frac{i\pi}{2}}\eta,\bar{k}^+)\Big],$$
$$K^{(\alpha)}_{\Delta,-}(\eta,k) = c^{\text{dS-AdS}}_{\Delta}e^{+\frac{i\pi}{2}\Delta}(-\eta_0)^{\Delta}\Big[M^-_{\Delta}K^{\text{AdS}}_{\Delta}(-e^{-\frac{i\pi}{2}}\eta,k) \tag{3.3.29b}$$
$$+ M^+_{\Delta}e^{-i\pi(2\Delta-d)}K^{\text{AdS}}_{\Delta}(-e^{-\frac{i\pi}{2}}\eta,\bar{k}^-)\Big],$$

where we have restored the factors of $(-\eta_0)^{\Delta}$ for completeness and we recall the definitions of $P^{\pm}_{\Delta}$ and $M^{\pm}_{\Delta}$ from (3.2.66). These relations extend to a generic Bogoliubov initial state by allowing α and β to depend appropriately on the momentum associated with the propagators.

Summary. In this chapter we have developed some of the main results critical to the latter half of the thesis. The identities (3.3.28) and (3.3.29) provide a proof, to all orders in perturbation theory, that de Sitter boundary correlation functions in the α-vacua can be expressed as a linear combination of EAdS Witten diagrams with analytically continued momenta, each dressed with α- and β-dependent coefficients.

Clearly, unlike for the Bunch-Davies vacuum, late-time correlators in a generic α-vacuum do not share the same analytic structure as EAdS boundary correlators encountered in the context of the AdS/CFT correspondence – where all points are on the upper sheet of the EAdS hyperboloid. Late-time correlators in a generic

α-vacuum are well-known to exhibit singularities for collinear momentum configurations, which can be viewed from an AdS perspective through (3.3.28) and (3.3.29) as a consequence of the correlators featuring points on both the upper and the lower sheet of the EAdS hyperboloid. The corresponding analytic continuations of the external momenta convert the "total energy" singularity of cosmological correlators in the Bunch-Davies vacuum [62, 106, 107], into the additional collinear singularities characteristic of the α-vacua.

In the following chapter, we will apply the above identities to some examples, computing perturbative late-time correlation functions in momentum space by re-writing them in terms of corresponding EAdS Witten diagrams.

Chapter 4

Perturbative Correlators in de Sitter

In this chapter, based on section 5 of [1], we study some examples of late-time de Sitter correlation functions in the α-vacua, utilising the tools described in the previous chapter. Our results extend to Bogoliubov initial states by allowing α and β to be appropriate functions of the momentum, sending $(\alpha, \beta) \to (\alpha_k, \beta_k)$ for each mode of momentum k. The main results of this chapter are, for arbitrary choice of α-vacuum and general spacetime dimension d:

- The late-time boundary two-point function of a general scalar field.
- Expressions for n-point late-time contact diagrams of general scalars in terms of their Bunch-Davies counterparts, through which they are also expressed in terms of corresponding Witten diagrams in EAdS.
- Examples for the $n = d = 3$ case - three massless scalars, as well as two conformally coupled scalars and a massless scalar.
- We also show how to construct the late-time boundary four-point exchange diagram of general scalars in the in-in formalism. The full de Sitter exchange written as a linear combination of EAdS Witten diagrams is available at [87].

4.1 Two-point Function

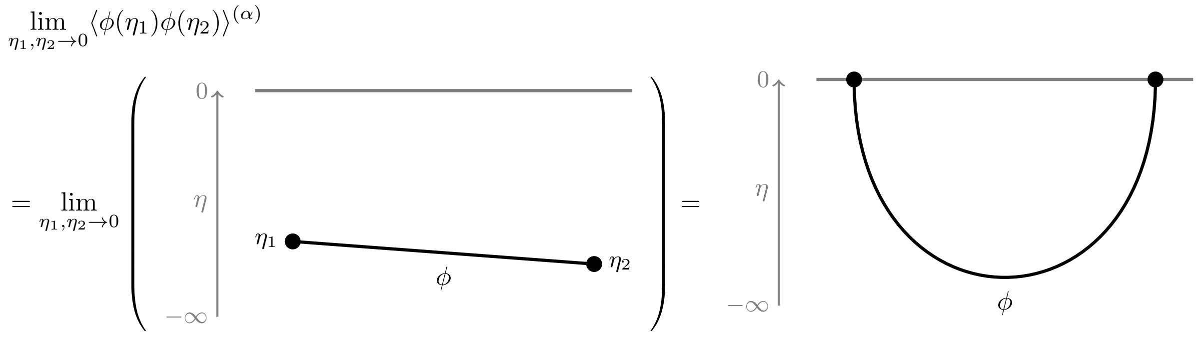


Figure 4.1: The late-time boundary two-point function in de Sitter space is obtained by taking the $\eta_i \to 0$ limit of both points in a bulk-to-bulk propagator.

We begin with the late-time two-point function in de Sitter space, which is easily obtained simply by taking both points of a bulk-to-bulk propagator to the boundary. This is shown diagrammatically in figure 4.1. We choose the Wightman function in this case, with the late-time two-point function defined simply by (3.2.20),

$$
\begin{aligned}
\lim_{\eta_1,\eta_2\to 0}\langle\phi(\eta_1)\phi(\eta_2)\rangle^{(\alpha)} &= \lim_{\eta_1,\eta_2\to 0} G_W^{(\alpha)}(\eta_1,\eta_2), \qquad (4.1.1)\\
&= \lim_{\eta_1,\eta_2\to 0}\Big[\cosh^2\alpha\,\langle\phi(\eta_1)\phi(\eta_2)\rangle^{(0)} + \sinh^2\alpha\,\langle\phi(\bar\eta_1^-)\phi(\bar\eta_2^+)\rangle^{(0)} \\
&\quad -\frac{1}{2}\sinh 2\alpha\left(e^{i\left(\beta+\frac{\pi d}{2}\right)}\langle\phi(\eta_1)\phi(\bar\eta_2^+)\rangle^{(0)} + e^{-i\left(\beta+\frac{\pi d}{2}\right)}\langle\phi(\bar\eta_1^-)\phi(\eta_2)\rangle^{(0)}\right)\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\lim_{\eta_1,\eta_2\to 0}\langle\phi(\eta_1)\phi(\eta_2)\rangle^{(0)} &= \lim_{\eta_1,\eta_2\to 0} G_W^{(0)}(\eta_1;\eta_2),\\
&\equiv \lim_{\eta_1,\eta_2\to 0} G_{-+}^{(0)}(\eta_1;\eta_2)\\
&= \lim_{\eta_1,\eta_2\to 0}\left[G_{\Delta_+,-+}^{(0)}(\eta_1;\eta_2) + G_{\Delta_-,-+}^{(0)}(\eta_1;\eta_2)\right]. \qquad (4.1.2)
\end{aligned}
$$

To compute the late-time limit of these Bunch-Davies two-point functions, we note that by taking *one* bulk point to the boundary we obtain a bulk-to-boundary propagator. In particular,

$$
\lim_{\eta_1,\eta_2\to 0} G_{\Delta_\pm,-+}^{(0)}(\eta_1;\eta_2) = \lim_{\eta_1\to 0} K_{\Delta_\pm,-}^{(0)}(\eta_1,k). \qquad (4.1.3)
$$

To compute the right-hand-side, we can use the Mellin representation of the bulk-to-boundary propagator, writing

$$\lim_{\eta_1,\eta_2\to 0} G^{(0)}_{\Delta_\pm,-+}(\eta_1;\eta_2) = \lim_{\eta_1\to 0}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}K^{(0)}_{\Delta_\pm,-}(s,k)(-\eta_1)^{-2s+\frac{d}{2}}$$
$$= (-\eta_2)^{\Delta_\pm}\frac{\Gamma(-i\nu)}{4\pi}e^{-\frac{\pi\nu}{2}}\left(\frac{k}{2}\right)^{i\nu}\lim_{\eta_1\to 0}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}e^{i\pi s}\Gamma\left(s+\frac{i\nu}{2}\right)\Gamma\left(s-\frac{i\nu}{2}\right)\left(\frac{k}{2}\right)^{-2s}(-\eta_1)^{-2s+\frac{d}{2}}$$
$$\equiv (-\eta_2)^{\Delta_\pm}\frac{\Gamma(-i\nu)}{4\pi}e^{-\frac{\pi\nu}{2}}\left(\frac{k}{2}\right)^{i\nu}\lim_{\eta_1\to 0}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}f(s), \tag{4.1.4}$$

where we used (3.2.52) and called the integrand $f(s)$ for compactness. $f(s)$ has poles on the left of the imaginary axis at

$$s = \pm\frac{i\nu}{2} - n, \quad n\in\mathbb{Z}_{\geq 0}, \tag{4.1.5}$$

and we compute the integral with the residue theorem,

$$\lim_{\eta_1\to 0}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}f(s) = \lim_{\eta_1\to 0}\sum_\pm\sum_n \text{Res}_{s=\pm\frac{i\nu}{2}-n}(f(s)). \tag{4.1.6}$$

From the $(-\eta_1)^{-2s+\frac{d}{2}}$ term in the integrand we can see that the terms for $n\neq 0$ will be subleading in the $\eta_1\to 0$ limit. We therefore have

$$\lim_{\eta_1\to 0}\sum_\pm\sum_n \text{Res}_{s=\pm\frac{i\nu}{2}-n}(f(s)) = \sum_\pm \text{Res}_{s=\pm\frac{i\nu}{2}}(f(s)), \tag{4.1.7}$$

and so

$$\lim_{\eta_1\to 0}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}f(s) = \lim_{s\to\frac{i\nu}{2}}\left(s-\frac{i\nu}{2}\right)\Gamma\left(s-\frac{i\nu}{2}\right)\Gamma\left(s+\frac{i\nu}{2}\right)e^{i\pi s}\left(\frac{k}{2}\right)^{-2s}(-\eta_1)^{-2s+\frac{d}{2}}$$
$$+\lim_{s\to-\frac{i\nu}{2}}\left(s+\frac{i\nu}{2}\right)\Gamma\left(s+\frac{i\nu}{2}\right)\Gamma\left(s-\frac{i\nu}{2}\right)e^{i\pi s}\left(\frac{k}{2}\right)^{-2s}(-\eta_1)^{-2s+\frac{d}{2}}$$
$$=\Gamma(i\nu)e^{-\frac{\pi\nu}{2}}\left(\frac{k}{2}\right)^{-i\nu}(-\eta_1)^{\frac{d}{2}-i\nu}+\Gamma(-i\nu)e^{\frac{\pi\nu}{2}}\left(\frac{k}{2}\right)^{i\nu}(-\eta_1)^{\frac{d}{2}+i\nu}, \tag{4.1.8}$$

where we used the identity

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\frac{1}{1+\frac{z}{n}}\left(1+\frac{1}{n}\right)^z\right]. \tag{4.1.9}$$

Plugging this into equation (4.1.4), we have

$$\lim_{\eta_1,\eta_2\to 0} G^{(0)}_{\Delta_\pm,-+}(\eta_1;\eta_2) = \left(\frac{k}{2}\right)^{\pm i\nu}\left(\frac{k}{2}\right)^{-i\nu}(-\eta_2)^{\Delta_\pm}(-\eta_1)^{\Delta_-}\frac{\Gamma(\mp i\nu)\Gamma(i\nu)}{4\pi}e^{\mp\frac{\pi\nu}{2}}e^{\frac{\pi\nu}{2}}$$
$$+\left(\frac{k}{2}\right)^{\pm i\nu}\left(\frac{k}{2}\right)^{i\nu}(-\eta_2)^{\Delta_\pm}(-\eta_1)^{\Delta_+}\frac{\Gamma(\mp i\nu)\Gamma(-i\nu)}{4\pi}e^{\mp\frac{\pi\nu}{2}}e^{-\frac{\pi\nu}{2}}, \qquad (4.1.10)$$

which can be expressed in the compact form

$$\lim_{\eta_1,\eta_2\to 0} G^{(0)}_{\Delta_\pm,-+}(\eta_1;\eta_2) = \frac{(\eta_1\eta_2)^{\frac{d}{2}}}{4\pi}\Gamma\left(\frac{d}{2}-\Delta_\pm\right)^2\left(\frac{\eta_1\eta_2 k^2}{4}\right)^{\Delta_\pm-\frac{d}{2}} + \ldots, \qquad (4.1.11)$$

where ... denotes a local term constant in k. Combining this with (4.1.2) and plugging into (4.1.1) we obtain the late-time free theory two-point function for arbitrary choice of α-vacuum,

$$\lim_{\eta_1,\eta_2\to 0}\langle\phi(\eta_1)\phi(\eta_2)\rangle^{(\alpha)} = \left[\cosh 2\alpha - \sinh 2\alpha\cos\left(\pi\left(\tfrac{\beta}{\pi}+\tfrac{d}{2}-\Delta_+\right)\right)\right]\lim_{\eta_1,\eta_2\to 0} G^{(0)}_{\Delta_+,W}(\eta_1;\eta_2)$$
$$+\left(\Delta_+\to\Delta_-\right). \quad (4.1.12)$$

For a massless scalar (i.e. $\nu = \frac{id}{2}$, corresponding to $\Delta_+ = 0$ and $\Delta_- = d$) in $d = 3$ this reduces to

$$\lim_{\eta_1,\eta_2\to 0}\langle\phi_{\vec{k}}(\eta_1)\phi_{-\vec{k}}(\eta_2)\rangle^{(\alpha)} = \frac{\cosh 2\alpha - \sin\beta\sinh 2\alpha}{2k^3} + \ldots, \qquad (4.1.13)$$

where ... is a local term proportional to η_1 and η_2.

At this point, there is a natural question to ask in the context of cosmology - which value of α does our universe correspond to? At the level of the two-point function, measuring our universe's value of α would be a challenge, since (treating α as momentum-independent) (4.1.13) differs from the Bunch-Davies result simply by an overall multiplicative factor. However, at the level of non-Gaussianities, correlation functions in the α-vacua feature poles for collinear momentum configurations that are not present in the Bunch-Davies vacuum. Such *folded singularities* are a hallmark of the α-vacua, and their detection via future cosmological surveys could provide a route to measuring α as a cosmological observable[1].

[1]See also [37] for a discussion of the effects of non-zero α on the CMB at the level of the

We now move on to perturbative contributions to higher-point correlators, first focusing on contact diagrams.

4.2 Contact Diagrams

Consider the n-point late-time contact diagram generated by the non-derivative interaction

$$\mathcal{V}_{12\ldots n} = -g\phi_1\phi_2\ldots\phi_n, \tag{4.2.1}$$

of scalar fields ϕ_i, $i = 1, 2, \ldots, n$ with dS mass $m_i^2 = \Delta_i(d - \Delta_i)$. In the in-in formalism, for a generic α-vacuum this is given by

$$^{(\alpha)}\mathcal{A}^{\mathcal{V}_{12\ldots n}}_{\Delta_1\Delta_2\ldots\Delta_n} = {}^{(\alpha,+)}\mathcal{A}^{\mathcal{V}_{12\ldots n}}_{\Delta_1\Delta_2\ldots\Delta_n} + {}^{(\alpha,-)}\mathcal{A}^{\mathcal{V}_{12\ldots n}}_{\Delta_1\Delta_2\ldots\Delta_n}, \tag{4.2.2}$$

with contributions from the time-ordered (+) and anti-time-ordered (−) branches of the in-in contour,

$$^{(\alpha,\pm)}\mathcal{A}^{\mathcal{V}_{1\ldots n}}_{\Delta_1\ldots\Delta_n}(k_1, k_2, \ldots, k_n) = \pm ig\int_{-\infty}^{\eta_0}\frac{\mathrm{d}\eta}{(-\eta)^{d+1}}\,K^{(\alpha)}_{\Delta_1,\pm}(\eta, k_1)\ldots K^{(\alpha)}_{\Delta_n,\pm}(\eta, k_n)\,, \tag{4.2.3}$$

where $\eta_0 \sim 0$. Equation (4.2.2) comes directly from a perturbative expansion of the definition of an in-in correlation function. In particular (schematically), for a cubic interaction we define $H_{\text{int}}(\eta') := \phi^3_\pm(\eta')$ and for the in-in three-point function we find that the numerator of equation (2.2.99) becomes

$$\begin{aligned}
&\langle\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3)\rangle \\
&\quad\sim \langle\bar{T}\left(\exp[-ig\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta)^{d+1}}\phi^3_-(\eta')]\right)\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3) \\
&\qquad\times T\left(\exp[+ig\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta)^{d+1}}\phi^3_+(\eta')]\right)\rangle,
\end{aligned} \tag{4.2.4}$$

where the notation $\phi_\pm$ is simply to denote which of the time-ordered or anti-time-ordered branches of the in-in contour the field belongs to. Expanding the exponentials

two-point function, albeit for a more intricate setup involving a momentum scale above which only Bunch-Davies modes are excited.

and using Wick's theorem, at leading order we find

$$
\begin{aligned}
&\langle \phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3)\rangle \\
&\quad\sim \langle \bar{T}\left(\exp[-\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta')^{d+1}}\phi_-^3(\eta')]\right)\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3) T\left(\exp[+i\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta')^{d+1}}\phi_+^3(\eta')]\right)\rangle, \\
&\quad\sim +ig\,\langle T\Big(\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta')^{d+1}}\phi_+^3(\eta')\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3)\Big)\rangle \\
&\qquad\qquad - ig\,\langle \bar{T}\Big(\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta')^{d+1}}\phi_-^3(\eta')\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3)\Big)\rangle \\
&\quad\sim +ig\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta')^{d+1}}\phi_+(\eta')\phi_+(\eta')\phi_+(\eta')\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3) \\
&\qquad\qquad - ig\int_{-\infty}^{\eta_0}\frac{d\eta'}{(-\eta')^{d+1}}\phi_-(\eta')\phi_-(\eta')\phi_-(\eta')\phi(\eta,\vec{x}_1)\phi(\eta,\vec{x}_2)\phi(\eta,\vec{x}_3),
\end{aligned}
\tag{4.2.5}
$$

which is exactly (4.2.2) for $n = 3$. For general n and $\eta_0 \sim 0$ this is represented diagrammatically in figure 4.2.

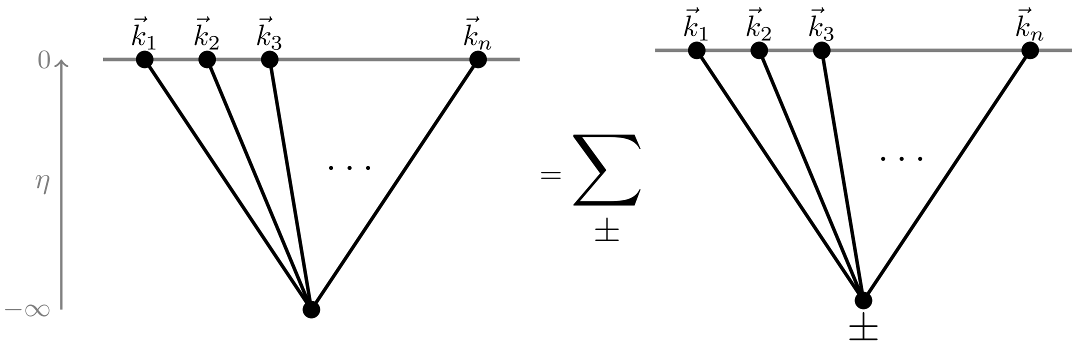


Figure 4.2: The n-point late-time contact diagram in de Sitter space is given by the sum of each contribution from the $\pm$ branches of the in-in contour. The solid grey line at the top of each diagram represents the late-time boundary, with the time coordinate running from $\eta = -\infty$ deep in the bulk to $\eta = 0$ on the boundary.

Using the expressions (3.2.65) for bulk-to-boundary propagators for generic α in terms of those in the Bunch-Davies vacuum ($\alpha = 0$), each contribution can be recast in terms of the corresponding contribution in the Bunch-Davies vacuum with the momenta appropriately rotated,

$$
{}^{(\alpha,+)}\mathcal{A}^{\mathcal{V}_{1\ldots n}}_{\Delta_1\ldots\Delta_n}(k_1,k_2,\ldots,k_n) = P^+_{\Delta_1}\ldots P^+_{\Delta_n}\,{}^{(0,+)}\mathcal{A}^{\mathcal{V}_{1\ldots n}}_{\Delta_1\ldots\Delta_n}(k_1,k_2,\ldots,k_n) \tag{4.2.6}
$$

$$
\begin{aligned}
&+\sum_i e^{(2\Delta_i-d)\pi i} P^+_{\Delta_1}\dots P^-_{\Delta_i}\dots P^+_{\Delta_n}\, {}^{(0,+)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}\left(k_1,\dots,\bar{k}^+_i,\dots,k_n\right)\\
&+\sum_{i<j}\Big(e^{(2\Delta_i-d)\pi i}e^{(2\Delta_j-d)\pi i}P^+_{\Delta_1}\dots P^-_{\Delta_i}\dots P^-_{\Delta_j}\dots P^+_{\Delta_n}\\
&\qquad\qquad\qquad\times {}^{(0,+)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}\left(k_1,\dots,\bar{k}^+_i,\dots,\bar{k}^+_j,\dots,k_n\right)\Big)\\
&\vdots\\
&+e^{(2\Delta_1-d)\pi i}\dots e^{(2\Delta_n-d)\pi i}P^-_{\Delta_1}\dots P^-_{\Delta_n}\, {}^{(0,+)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}\left(\bar{k}^+_1,\dots,\bar{k}^+_n\right),
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(\alpha,-)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}(k_1,k_2,\dots,k_n) &= M^-_{\Delta_1}\dots M^-_{\Delta_n}\, {}^{(0,-)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}(k_1,k_2,\dots,k_n)\\
&+\sum_i e^{-(2\Delta_i-d)\pi i}M^-_{\Delta_1}\dots M^+_{\Delta_i}\dots M^-_{\Delta_n}\, {}^{(0,-)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}\left(k_1,\dots,\bar{k}^-_i,\dots,k_n\right)\\
&+\sum_{i<j}\Big(e^{-(2\Delta_i-d)\pi i}e^{-(2\Delta_j-d)\pi i}M^-_{\Delta_1}\dots M^+_{\Delta_i}\dots M^+_{\Delta_j}\dots M^-_{\Delta_n}\, {}^{(0,-)}\\
&\qquad\qquad\qquad\times \mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}\left(k_1,\dots,\bar{k}^-_i,\dots,\bar{k}^-_j,\dots,k_n\right)\Big)\\
&\vdots\\
&+e^{-(2\Delta_1-d)\pi i}\dots e^{-(2\Delta_n-d)\pi i}M^+_{\Delta_1}\dots M^+_{\Delta_n}\, {}^{(0,-)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}\left(\bar{k}^-_1,\dots,\bar{k}^-_n\right),
\end{aligned}
\tag{4.2.7}
$$

where the $P^\pm_{\Delta_i}$ and $M^\pm_{\Delta_i}$ were summarised in (3.2.66). These, in turn, can be expressed in terms of corresponding Witten diagrams in EAdS using the analytic continuations (3.3.29), where [74]:

$$
\begin{aligned}
&{}^{(0,\pm)}\mathcal{A}^{\mathcal{V}_{1\dots n}}_{\Delta_1\dots\Delta_n}(k_1,k_2,\dots,k_n)\\
&\quad=\pm i\, e^{\mp\frac{i\pi}{2}\left(\frac{(n-2)d}{2}+i(\nu_1+\dots+\nu_n)\right)}\left(\prod_{i=1}^n c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right)\mathcal{A}^{\text{AdS}}_{\Delta_1\dots\Delta_n}(k_1,\dots,k_n),
\end{aligned}
\tag{4.2.8}
$$

with

$$
\mathcal{A}^{\text{AdS}}_{\Delta_1\dots\Delta_n}(k_1,\dots,k_n) = -g\int_0^\infty \frac{\mathrm{d}z}{z^{d+1}}\,K^{\text{AdS}}_{\Delta_1}(z,k_1)\dots K^{\text{AdS}}_{\Delta_n}(z,k_n)\,. \tag{4.2.9}
$$

Using the Mellin representation of the bulk-to-boundary propagators

$$
K^{\text{AdS}}_{\Delta}(z,k) = \int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}K^{\text{AdS}}_{\Delta}(s,k)\,z^{-2s+\frac{d}{2}}, \tag{4.2.10}
$$

and (3.2.60), this can be expressed in the Mellin representation

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\ldots\Delta_n}\left(k_1,\ldots,k_n\right) \quad = \quad \int_{-i\infty}^{+i\infty}[\mathrm{d}s]_n\,\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\ldots\Delta_n}\left(s_1,k_1;s_2,k_2;\ldots;s_n,k_n\right), \quad (4.2.11)$$

where

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\ldots\Delta_n}\left(s_1,k_1;s_2,k_2;\ldots;s_n,k_n\right) = -g\int_0^\infty \frac{\mathrm{d}z}{z^{d+1}}\,z^{-\sum\limits_{i=1}^{n}\left(2s_i-\frac{d}{2}\right)} \quad (4.2.12)$$
$$\times\prod_{i=1}^{n}\frac{\Gamma\left(s_i+\frac{1}{2}\left(\frac{d}{2}-\Delta_i\right)\right)\Gamma\left(s_i-\frac{1}{2}\left(\frac{d}{2}-\Delta_i\right)\right)}{2\Gamma\left(\Delta_i-\frac{d}{2}+1\right)}\left(\frac{k_i}{2}\right)^{-2s_i+\Delta_i-\frac{d}{2}}.$$

The integral over z then trivialises to a Dirac delta function in the Mellin variables according to (3.1.36) (and explained in appendix B), resulting in

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\ldots\Delta_n}\left(s_1,k_1;s_2,k_2;\ldots;s_n,k_n\right) = -g\,2\pi i\,\delta\left(d+\sum_{i=1}^{n}\left(2s_i-\frac{d}{2}\right)\right) \quad (4.2.13)$$
$$\times\prod_{i=1}^{n}\frac{\Gamma\left(s_i+\frac{1}{2}\left(\frac{d}{2}-\Delta_i\right)\right)\Gamma\left(s_i-\frac{1}{2}\left(\frac{d}{2}-\Delta_i\right)\right)}{2\Gamma\left(\Delta_i-\frac{d}{2}+1\right)}\left(\frac{k_i}{2}\right)^{-2s_i+\Delta_i-\frac{d}{2}}.$$

Note that in the above we have used that rotations in the momenta are equivalent to the antipodal transformation of the corresponding bulk point, as explained in section 3.2.2. In particular,

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\ldots\Delta_n}\left(k_1,\ldots,\bar{k}_i^{\pm},\ldots,k_n\right)$$
$$= -g(e^{\mp\pi i})^{\Delta_i-d}\int_0^\infty\frac{\mathrm{d}z}{z^{d+1}}\,K^{\text{AdS}}_{\Delta_1}\left(z,k_1\right)\ldots K^{\text{AdS}}_{\Delta_i}\left(\bar{z}^{\pm},k_1\right)\ldots K^{\text{AdS}}_{\Delta_n}\left(z,k_n\right). \quad (4.2.14)$$

4.2.1 Three-point Function

Let us examine the $n=3$ case in more detail. It turns out that one can trade an analytic continuation of two momenta for an analytic continuation of just one. We can see this in Mellin space with a straightforward calculation; we have

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(k_1e^{\mp i\pi},k_2e^{\mp i\pi},k_3\right) = \int_0^\infty\frac{dz}{z^{d+1}}K^{\text{AdS}}_{\Delta_1}(z,e^{\mp i\pi}k_1)K^{\text{AdS}}_{\Delta_2}(z,e^{\mp i\pi}k_2)K^{\text{AdS}}_{\Delta_3}(z,k_3)$$
$$= \int_{-i\infty}^{i\infty}[ds]_3K^{\text{AdS}}_{\Delta_1}(s_1,e^{\mp i\pi}k_1)K^{\text{AdS}}_{\Delta_2}(s_2,e^{\mp i\pi}k_2)K^{\text{AdS}}_{\Delta_3}(s_3,k_3)\int_0^\infty\frac{dz}{z^{d+1}}z^{-2(s_1+s_2+s_3)+\frac{3d}{2}}$$

$$= e^{\mp i\pi(i\nu_1+i\nu_2)} \int_{-i\infty}^{i\infty} [ds]_3 \prod_{j=1}^{3} \left\{ \frac{\Gamma\left(s_j + \frac{i\nu_j}{2}\right)\Gamma\left(s_j - \frac{i\nu_j}{2}\right)}{2\Gamma(1+i\nu_j)} \left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j} \right\} e^{\pm 2i\pi(s_1+s_2)}$$
$$\times 2i\pi\delta\left(s_1+s_2+s_3-\frac{d}{4}\right), \tag{4.2.15}$$

where we used (3.1.36) to perform the z integral and (3.2.60). Using the delta function to impose the constraint $s_1 + s_2 = \frac{d}{4} - s_3$, we can write

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(k_1 e^{\mp i\pi}, k_2 e^{\mp i\pi}, k_3\right)$$
$$= e^{\mp i\pi(i\nu_1+i\nu_2)} \int_{-i\infty}^{i\infty} [ds]_3 \prod_{j=1}^{3} \left\{ \frac{\Gamma\left(s_j + \frac{i\nu_j}{2}\right)\Gamma\left(s_j - \frac{i\nu_j}{2}\right)}{2\Gamma(1+i\nu_j)} \left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j} \right\} e^{\pm 2i\pi\left(\frac{d}{4}-s_3\right)}$$
$$\times 2\pi i\delta\left(s_1+s_2+s_3-\frac{d}{4}\right)$$
$$= e^{\mp i\pi(-\frac{d}{2}+i\nu_1+i\nu_2+i\nu_3)} \int_{-i\infty}^{i\infty} [ds]_3 \prod_{j=1}^{3} \left\{ \frac{\Gamma\left(s_j + \frac{i\nu_j}{2}\right)\Gamma\left(s_j - \frac{i\nu_j}{2}\right)}{2\Gamma(1+i\nu_j)} \left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j} \right\} e^{\pm i\pi(-2s_3+i\nu_3)}$$
$$\times 2\pi i\delta\left(s_1+s_2+s_3-\frac{d}{4}\right), \tag{4.2.16}$$

which leads to the identity

$$\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(k_1 e^{\mp i\pi}, k_2 e^{\mp i\pi}, k_3\right) = \left(e^{\mp\pi i}\right)^{\Delta_1+\Delta_2+\Delta_3-2d} \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, k_3 e^{\pm i\pi}\right). \tag{4.2.17}$$

This in turn gives rise to the following compact expression for the α-vacuum three-point contact diagram:

$$^{(\alpha)}\mathcal{A}^{\mathcal{V}_{123}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,k_3) = i\left(\prod_{i=1}^{3} c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right) \tag{4.2.18}$$
$$\times \sum_{\pm} e^{\mp\frac{i\pi}{2}\left(\frac{d}{2}+i(\nu_1+\nu_2+\nu_3)\right)} \Big\{ \left(P^{\pm}_{\Delta_1}P^{\pm}_{\Delta_2}P^{\pm}_{\Delta_3} - M^{\pm}_{\Delta_1}M^{\pm}_{\Delta_2}M^{\pm}_{\Delta_3}\right)\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,k_3)$$
$$+ \left(P^{\mp}_{\Delta_1}P^{\pm}_{\Delta_2}P^{\pm}_{\Delta_3} - M^{\mp}_{\Delta_1}M^{\pm}_{\Delta_2}M^{\pm}_{\Delta_3}\right) e^{\mp 2\pi\nu_1}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(e^{\mp i\pi}k_1,k_2,k_3)$$
$$+ \left(P^{\pm}_{\Delta_1}P^{\mp}_{\Delta_2}P^{\pm}_{\Delta_3} - M^{\pm}_{\Delta_1}M^{\mp}_{\Delta_2}M^{\pm}_{\Delta_3}\right) e^{\mp 2\pi\nu_2}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(k_1,e^{\mp i\pi}k_2,k_3)$$
$$+ \left(P^{\pm}_{\Delta_1}P^{\pm}_{\Delta_2}P^{\mp}_{\Delta_3} - M^{\pm}_{\Delta_1}M^{\pm}_{\Delta_2}M^{\mp}_{\Delta_3}\right) e^{\mp 2\pi\nu_3}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,e^{\mp i\pi}k_3)\Big\},$$

with the $P^{\pm}_{\Delta_i}$ and $M^{\pm}_{\Delta_i}$ summarised in (3.2.66). It is also convenient to express the contact diagram in the following form, which makes manifest that one recovers the known Bunch-Davies result upon setting $\alpha = 0$:

$$
\begin{aligned}
{}^{(\alpha)}\mathcal{A}^{\mathcal{V}_{123}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,k_3) &= i\left(\prod_{i=1}^{3} c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right)\sum_{\pm} e^{\mp\frac{i\pi}{4}(d+2i(\nu_1+\nu_2+\nu_3))} \\
\times\Big\{ {}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,k_3) &+ \sinh(2\alpha)\,{}^{(\alpha)}C^{\pm}_{\Delta_1\Delta_2\Delta_3}\Big(e^{\mp\pi\nu_1}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(e^{\mp i\pi}k_1,k_2,k_3\right) \\
+e^{\mp\pi\nu_2}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(k_1,e^{\mp i\pi}k_2,k_3\right) &+ e^{\mp\pi\nu_3}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}\left(k_1,k_2,e^{\mp i\pi}k_3\right)\Big)\Big\}, \quad (4.2.19)
\end{aligned}
$$

where the coefficients are defined

$$
\begin{aligned}
{}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3} := \frac{1}{8}\Big[&\pm(3\cosh(4\alpha)+5)\mp 2e^{\mp i\beta}\sinh(4\alpha)\left(e^{\mp\pi\nu_1}+e^{\mp\pi\nu_2}+e^{\mp\pi\nu_3}\right) \\
&\pm 2e^{\mp 2i\beta}\sinh^2(2\alpha)\left(e^{\mp\pi(\nu_1+\nu_2)}+e^{\mp\pi(\nu_1+\nu_3)}+e^{\mp\pi(\nu_2+\nu_3)}\right)\Big], \quad (4.2.20a)
\end{aligned}
$$

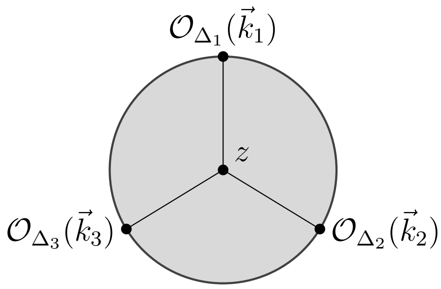
$$
\begin{aligned}
{}^{(\alpha)}C^{\pm}_{\Delta_1\Delta_2\Delta_3} := \frac{1}{4}\Big[&\mp 2e^{\pm i\beta}\cosh(2\alpha)e^{\pm\pi(\nu_1+\nu_2+\nu_3)} \\
&\pm\sinh(2\alpha)\left(-e^{\mp 2i\beta}+e^{\pm\pi(\nu_1+\nu_2)}+e^{\pm\pi(\nu_1+\nu_3)}+e^{\pm\pi(\nu_2+\nu_3)}\right)\Big]. \quad (4.2.20b)
\end{aligned}
$$

Indeed, for $\alpha = 0$ the terms proportional to $\sinh(2\alpha)$ vanish and ${}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3} = \pm 1$, so the above yields

$$
\begin{aligned}
{}^{(0)}\mathcal{A}^{\mathcal{V}_{123}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,k_3) = 2\left(\prod_{i=1}^{3} c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right)&\sin\left(\frac{\pi}{2}\left(\frac{d}{2}+i(\nu_1+\nu_2+\nu_3)\right)\right) \\
&\times\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(k_1,k_2,k_3), \quad (4.2.21)
\end{aligned}
$$

recovering the result of [74] for contact diagrams in the Bunch-Davies vacuum.

For generic scaling dimensions Δ_i on the Principal Series, conformal three-point functions in Fourier space are given by Appell's F_4 function [70]; a generalised hypergeometric function of two variables which admits a "triple-K" integral representation as a product of three modified Bessel functions of the second kind ("Bessel - K"s). As expected from AdS/CFT this representation coincides with the three-point contact diagram in AdS,



$$= \int_0^\infty \frac{dz}{z^{d+1}} K^{\text{AdS}}_{\Delta_1}(z,k_1) K^{\text{AdS}}_{\Delta_2}(z,k_2) K^{\text{AdS}}_{\Delta_3}(z,k_3), \tag{4.2.22}$$

with each bulk field dual to the CFT operator $\mathcal{O}_{\Delta_i}(\vec{k}_i)$ having a corresponding bulk-to-boundary propagator containing a Bessel-K. In Mellin space, n-point contact diagrams take the form (4.2.13). Away from the Principal Series, for certain special values of the scaling dimensions this expression simplifies and in some cases there are IR divergences. We will see examples of these in the following.

In the following examples we will set the parameter $\beta = 0$.

Three massless scalars in $d = 3$. In this case we have $\nu_i = \frac{3i}{2}$ i.e. $\Delta_i = 0$:

$$\frac{{}^{(\alpha)}\mathcal{A}^{\mathcal{V}_{123}}_{000}\left(k_1,k_2,k_3\right)}{\left(c_0^{\text{dS-AdS}}\right)^3} = 2\mathcal{A}^{\text{AdS}}_{000}\left(k_1,k_2,k_3\right) - \sinh\left(2\alpha\right)\sum_{\pm}\left(\sinh\left(2\alpha\right) \pm \frac{i}{2}\cosh\left(2\alpha\right)\right)$$
$$\times\left(\mathcal{A}^{\text{AdS}}_{000}\left(e^{\mp i\pi}k_1,k_2,k_3\right) + \mathcal{A}^{\text{AdS}}_{000}\left(k_1,e^{\mp i\pi}k_2,k_3\right) + \mathcal{A}^{\text{AdS}}_{000}\left(k_1,k_2,e^{\mp i\pi}k_3\right)\right). \tag{4.2.23}$$

The corresponding EAdS contact diagram is given by[2]

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{000}\left(k_1,k_2,k_3\right) = \frac{1}{3k_1^3k_2^3k_3^3}\Big[&2k_1^2(k_2+k_3) + 2k_1\left(k_2^2 - k_2k_3 + k_3^2\right) + 2k_2k_3(k_2+k_3)\\ &+\frac{4\left(k_1^3+k_2^3+k_3^3\right)}{d-3} - \frac{1}{3}(6\gamma - 11)\left(k_1^3+k_2^3+k_3^3\right)\\ &-\left(k_1^3+k_2^3+k_3^3\right)\left(2\log(k_1+k_2+k_3)+1\right)\Big],\end{aligned} \tag{4.2.24}$$

which, for example, can be obtained from the Mellin-Barnes representation (4.2.13) of three-point contact diagrams in dimensional regularisation (see appendix A). Note

[2]Note the presence of a *total energy singularity* $k_T := k_1 + k_2 + k_3 \to 0$. This is a characteristic singularity of cosmological correlators, and is related to the high-energy limit of the corresponding flat-space amplitude [12, 55, 62].

the presence of a local IR divergence for $d = 3$; in an inflationary context this is regularised by a natural IR cut-off $\eta_0 \approx 0$ (ie, the end of inflation) near the late-time boundary, where this term is of the form[3] $\left(\sum_{i=1}^{3} k_i^3\right) \log(-\eta_0 k_T)$ [63]. Such local IR divergences can be cancelled in the in-in formalism by adding local counterterms at the future boundary of dS [109].

The expression (4.2.23) agrees with the result originally obtained in [73], up to local terms. Local terms (or *contact terms*) are those that in position space (namely, after taking a Fourier transform) correspond to a Dirac delta function (or some derivative acting on one), and so only contribute for coincident points. Examples of local terms are constants and polynomials in one or more momenta; terms of the form $k_1 k_2$, $k_1^2 k_2 k_3$, $\frac{k_2^2 k_3}{k_1}$ etc[4]. Terms like these produce a (derivative of) a delta function when taking a Fourier transform back to position space, for instance

$$\begin{aligned}
(k_1^3 + k_2^3 + k_3^3)\,\delta(k_1 + k_2 + k_3) \xrightarrow{\text{F.T.}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\,(k_1^3 + k_2^3 + k_3^3)\, e^{ik_1 x} e^{ik_2 y} e^{ik_3 z} \\
\times\, \delta(k_1 + k_2 + k_3) \\
= \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\,(k_1^3 + k_2^3 - (k_1 + k_2)^3) e^{ik_1(x-z)} e^{ik_2(y-z)} \\
= i\Big(\int_{-\infty}^{\infty} \frac{dk_2}{2\pi}\, e^{ik_2(y-z)}\Big)\frac{d^3}{d(x-z)^3}\Big(\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\, e^{ik_1(x-z)}\Big) \\
+ i\Big(\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\, e^{ik_1(x-z)}\Big)\frac{d^3}{d(y-z)^3}\Big(\int_{-\infty}^{\infty} \frac{dk_2}{2\pi}\, e^{ik_2(y-z)}\Big) + \ldots \\
= i\delta(y-z)\frac{d^3}{d(x-z)^3}\delta(x-z) + i\delta(x-z)\frac{d^3}{d(y-z)^3}\delta(y-z) \\
+ \ldots, \qquad (4.2.25)
\end{aligned}$$

where ... denotes similar terms that come from expanding the $(k_1+k_2)^3$ term. Clearly,

[3]Setting $d = 3 + \delta$ in (4.2.24) results in a $\frac{1}{\delta}$ pole. Such poles in dimensional regularisation are analogous to $\log(\Lambda)$ terms in cut-off regularisation (with Λ an IR cutoff), which connects the divergence in (4.2.24) to the $\log(-\eta_0 k_T)$ divergence for $d = 3$. See chapter 3 and appendix A of [108] for a discussion in the context of the contact diagram of two conformally coupled and one massless scalar.

[4]While the latter is clearly not analytic in k_1, it will still produce a derivative of a delta function involving y and z if we take a Fourier transform.

such terms only contribute in the limit that the points are coincident, and are hence ignored under the assumption that we are interested in correlation functions with separated points. By contrast, terms like those in (4.2.23) of the form $\frac{1}{k^p}$ for some power $p > 0$ do not correspond to delta functions in position space, and so are said to be non-local.

Two conformally coupled and one general scalar. In this case we have $\nu_1 = \nu_2 = \frac{i}{2}$ and generic $\Delta_3 \equiv \Delta$. We find

$$\frac{{}^{(\alpha)}\mathcal{A}^{\nu_{123}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}(k_1,k_2,k_3)}{\left(\prod_{i=1}^{3} c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right)} = \sum_{\pm} {}^{(0)}C^{\pm}_{\frac{d-1}{2}\frac{d-1}{2}\Delta} e^{\mp\frac{i\pi}{4}(d-2i\nu+2)}\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}(k_1,k_2,k_3) \tag{4.2.26}$$

$$+\sinh(2\alpha)\sum_{\pm} {}^{(\alpha)}C^{\pm}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}\Big[e^{\mp\frac{i\pi}{4}(d-2i\nu+4)}\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}\left(e^{\mp i\pi}k_1,k_2,k_3\right)$$

$$+e^{\mp\frac{i\pi}{4}(d-2i\nu+4)}\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}\left(k_1,e^{\mp i\pi}k_2,k_3\right)$$

$$+e^{\mp\frac{i\pi}{4}(d-6i\nu+2)}\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}\left(k_1,k_2,e^{\mp i\pi}k_3\right)\Big],$$

with

$$\begin{aligned} {}^{(0)}C^{\pm}_{\frac{d-1}{2}\frac{d-1}{2}\Delta} &= \frac{1}{8}\left[\mp i(3\cosh(4\alpha)+5)\left(\mp 4e^{\mp\pi\nu}+2i\right)\sinh^2(2\alpha)+\left(4\pm 2ie^{\mp\pi\nu}\right)\sinh(4\alpha)\right], \\ {}^{(\alpha)}C^{\pm}_{\frac{d-1}{2}\frac{d-1}{2}\Delta} &= \pm\frac{i}{4}\left[2\cosh(2\alpha)e^{\pm\pi(i+\nu)}+2\sinh(2\alpha)\left(1\mp ie^{\pm\pi\nu}\right)\right]. \end{aligned} \tag{4.2.27}$$

In this case the three-point conformal structure reduces to a Gauss hypergeometric function[5] of the form

$$\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}(k_1,k_2,k_3) = \pi^{3/2}\frac{2^{4-d}}{k_1k_2}\left(\frac{k_3}{2}\right)^{i\nu-\frac{d}{2}+1}\frac{\Gamma\left(\frac{d}{2}-i\nu-1\right)\Gamma\left(\frac{d}{2}+i\nu-1\right)}{\Gamma\left(\frac{d-1}{2}\right)}$$

$$\times\, {}_2F_1\left(\begin{matrix}\frac{d}{2}-1-i\nu,\frac{d}{2}-1+i\nu \\ \frac{d-1}{2}\end{matrix};\frac{k_3-k_1-k_2}{2k_3}\right). \tag{4.2.28}$$

OPE limit. As we shall see in the next section, exchange diagrams factorise at the level of each individual Witten diagram that appears in the expression. The OPE limit where the exchanged particle goes on-shell, $k \to 0$, can therefore be accessed by

[5] See for example section 3.3 of [74].

considering the soft limit of one leg of the constituent three-point contact diagrams. We will choose the third leg $k_3 \to 0$ to study the OPE limit of correlators for generic choice of α vacuum.

Each term in the three-point function (4.2.26) can be characterised by the notation $\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}\left(e^{\pm i\pi a}k_1, e^{\hat{\pm} i\pi b}k_2, e^{\tilde{\pm} i\pi c}k_3\right)$, with $a, b, c \in \{0, 1\}$ parameterising the analytic continuations. For a given term in said three-point function, the argument of the Gauss hypergeometric function (4.2.28) then takes the form[6]

$$\begin{aligned}
z &= \frac{k_3 - e^{i\pi(\pm a \tilde{\mp} c)}k_1 + e^{i\pi(\hat{\pm} b \tilde{\mp} c)}k_2}{2k_3} \\
&= \frac{1}{2} - \frac{1}{2}e^{i\pi(\pm a \tilde{\mp} c)}\frac{k_1}{k_3} - \frac{1}{2}e^{i\pi(\hat{\pm} b \tilde{\mp} c)}\frac{1}{k_3}\sqrt{k_1^2 + k_3^2 + 2k_1 k_3 \cos\theta_{k_1,k_3}} \\
&\overset{k_3 \to 0}{=} \frac{1}{2} - \frac{1}{2}\left(e^{i\pi(\pm a \tilde{\mp} c)} + e^{i\pi(\hat{\pm} b \tilde{\mp} c)}\right)\frac{k_1}{k_3} - \frac{1}{2}e^{i\pi(\hat{\pm} b \tilde{\mp} c)}\cos\theta_{k_1,k_3} + \mathcal{O}\left(\frac{k_3}{k_1}\right),
\end{aligned} \tag{4.2.29}$$

where θ_{k_1,k_3} is the angle between vectors $\vec{k}_1$ and $\vec{k}_3$. In the Bunch-Davies vacuum (i.e. $a = b = c = 0$) the k_1/k_3 term is leading, so that

$$ {}_2F_1\left(\begin{matrix}\frac{d}{2} - 1 - i\nu, \frac{d}{2} - 1 + i\nu \\ \frac{d-1}{2}\end{matrix}; \frac{k_3 - k_1 - k_2}{2k_3}\right) \sim \left(\frac{k_1}{k_3}\right)^{-\frac{d}{2} - i\nu + 1}, \tag{4.2.30}$$

and

$$ {}^{(0)}\mathcal{A}^{\mathcal{V}_{123}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}(k_1, k_2, k_3 \to 0) \sim \left(\frac{k_3}{k_1}\right)^{2i\nu}, \tag{4.2.31}$$

where we stripped off the overall power of k_1 carrying the dimension of the three-point function. In a generic α-vacuum instead, for certain values of a, b, c the coefficient of this term vanishes[7]. For such values of a, b, c we have

$$\begin{aligned}
&\mathcal{A}^{\text{AdS}}_{\frac{d-1}{2}\frac{d-1}{2}\Delta}\left(e^{a\pi i}k_1, e^{b\pi i}k_2, e^{c\pi i}k_3\right) \\
&\quad \sim \left(\frac{k_3}{k_1}\right)^{i\nu - \frac{d}{2} + 1} {}_2F_1\left(\begin{matrix}\frac{d}{2} - 1 - i\nu, \frac{d}{2} - 1 + i\nu \\ \frac{d-1}{2}\end{matrix}; \frac{1 - e^{i\pi(\hat{\pm} b \tilde{\mp} c)}\cos\theta_{k_1,k_3}}{2}\right),
\end{aligned} \tag{4.2.32}$$

which, for massive particles (i.e. $\nu \in \mathbb{R}$), dominates over the Bunch-Davies contribution (4.2.31) for $d > 2$. Note that this develops a branch cut for $\cos\theta_{k_1,k_3} = \pm 1$ i.e.

[6] Note that by momentum conservation we have $k_2^2 = k_1^2 + k_3^2 + 2k_1 k_3 \cos\theta_{k_1,k_3}$.

[7] For instance, $(\pm a, \hat{\pm} b, \tilde{\pm} c) = (+1, 0, -1)$ or $(0, -1, 0)$.

when $\vec{k_1}$ and $\vec{k_3}$ are collinear, as expected.

Let us note that due to their collinear singularities, three-point correlators for generic α-vacuum have been interpreted in the literature as being "inconsistent" with the OPE limit by the following general argument, which can be also be found in [62, 110, 111]. For conformally coupled scalars $\Delta = 2$ in $d = 3$, the dilatation Ward identity[8] simplifies to

$$\left(-d + \sum_{i=1}^{3} \mathcal{D}_i\right) \langle \mathcal{O}(\vec{k}_1)\mathcal{O}(\vec{k}_2)\mathcal{O}(\vec{k}_3)\rangle = \left(\sum_{i=1}^{3} k_i \partial_{k_i}\right) \langle \mathcal{O}(\vec{k}_1)\mathcal{O}(\vec{k}_2)\mathcal{O}(\vec{k}_3)\rangle = c, \quad (4.2.33)$$

with c some constant[9]. It is straightforward to show that a combination of logarithms solves this equation;

$$\begin{aligned} \langle \mathcal{O}(\vec{k}_1)\mathcal{O}(\vec{k}_2)\mathcal{O}(\vec{k}_3)\rangle =& A\ln(k_1+k_2+k_3) + B\ln(-k_1+k_2+k_3) \\ &+ C\ln(k_1-k_2+k_3) + D\ln(k_1+k_2-k_3), \end{aligned} \quad (4.2.34)$$

with symmetry under permutation of the momenta requiring $B = C = D$. In position space, the OPE limit corresponds to the limit in which two points become coincident. We assume that the unique solution to the conformal Ward identities in position space is given by

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{\mathcal{C}_{\Delta_1\Delta_2\Delta_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_1+\Delta_3-\Delta_2}}, \quad (4.2.35)$$

with $x_{ij} := |x_i - x_j|$ and $\mathcal{C}_{\Delta_1\Delta_2\Delta_3}$ a constant. If we take x_1 and x_2 coincident, ie we consider the $x_{12} \to 0$ limit, this becomes

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle \sim \frac{1}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{2\Delta_3}}. \quad (4.2.36)$$

One can show that the Fourier transform of this object is proportional to

$$\langle \mathcal{O}_{\vec{k}_1}\mathcal{O}_{\vec{k}_2}\mathcal{O}_{\vec{k}_3}\rangle \propto \frac{1}{k_1^{d-\Delta_1-\Delta_2+\Delta_3} k_3^{d-2\Delta_3}} = \frac{k_3}{k_1}, \quad (4.2.37)$$

[8] See appendix C for a derivation of the dilatation Ward identity.

[9] Note that we are allowing solutions that satisfy the dilatation Ward identity up to contact terms, namely the constant on the right-hand-side is allowed to be non-zero (following section 4.3 of [62]). A constant in momentum space becomes a delta function in position space, which vanishes for separated points. See the discussion around equation (4.2.25).

where in the last equality we set $d = 3, \Delta_j = 2 \, \forall j \in \{1, 2, 3\}$. Note that this is exactly the behaviour of the Bunch-Davies limit of the three-point function (4.2.31)[10] if we set the general scalar conformally coupled, $\nu = \frac{i}{2}$. This establishes the result of going to momentum space after taking the position-space OPE limit $x_{12} \to 0$ of the position space solution to the conformal Ward identities (4.2.35), assuming that (4.2.35) is the unique solution in position space.

Now we want to compare this behaviour to that of the solution to the momentum-space dilatation Ward identity (4.2.34), taking the corresponding OPE limit directly in momentum space. In the $\vec{k}_3 \to 0$ limit (whose position space equivalent is the above OPE limit $x_{12} \to 0$), momentum conservation $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$ implies $k_1 \approx k_2$. The first term of (4.2.34) therefore scales as

$$\begin{aligned}
\ln(k_1 + k_2 + k_3) &\sim \ln(2k_1 + k_3) \\
&= \ln\left(k_3\left(1 + \frac{2k_1}{k_3}\right)\right) \\
&= \ln(k_3) + \ln\left(1 + \frac{2k_1}{k_3}\right), \\
&= \ln(k_3) + \log\left(\frac{2k_1}{k_3}\right) + \frac{k_3}{2k_1} + \mathcal{O}(k_3^2)
\end{aligned} \tag{4.2.38}$$

which in the $k_3 \to 0$ limit scales as

$$\ln(k_1 + k_2 + k_3) \sim \frac{k_3}{k_1}. \tag{4.2.39}$$

It can be shown that the other terms, those involving collinear singularities, do not behave this way, for instance

$$\ln(-k_1 + k_2 + k_3) \sim \ln(k_3), \tag{4.2.40}$$

for $k_1 \approx k_2$. Therefore, the position-space OPE limit of (4.2.35), given by (4.2.37), is only consistent with the corresponding momentum-space OPE limit of (4.2.34) if $B = C = D = 0$. As such, solutions to the conformal Ward identities with

[10] We note that [62, 110, 111] parameterise $\Delta = \frac{d}{2} - i\nu$, giving $\Delta = 2$ for conformally coupled scalars $\nu = \frac{i}{2}$ in $d = 3$, while we parameterise $\Delta = \frac{d}{2} + i\nu$. As such, to compare with the results of this section we should send $\nu \to -\nu$ before plugging $\nu = \frac{i}{2}$ into (4.2.26).

singularities for collinear momentum configurations have been previously considered "inconsistent" with the OPE limit.

However, the above argument assumes that the solution to the position space conformal Ward identity is uniquely given by (4.2.35). In fact, as was shown in [1], the space of solutions to the position space conformal Ward identities must be enlarged to account for those with collinear singularities in momentum space, and a correct OPE limit in position space should take the presence of these additional solutions into account.

4.2.2 Four-point function

One can obtain similar expressions for higher-point contact diagrams. To demonstrate, for $n=4$ we have

$$\begin{aligned}
&\frac{{}^{(\alpha)}\mathcal{A}^{\mathcal{V}_{1234}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(k_1,k_2,k_3,k_4)}{i\left(\prod\limits_{i=1}^{4} c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right)} \\
&\qquad= \sum_{\pm} e^{\mp\frac{i\pi}{2}(d+i(\nu_1+\nu_2+\nu_3+\nu_4))}\Big\{{}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3\Delta_4}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(k_1,k_2,k_3,k_4) \\
&\qquad+\sinh(2\alpha)^{(\alpha,1)}C^{\pm}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Big[e^{\mp\pi\nu_1}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Big(e^{\mp\pi i}k_1,k_2,k_3,k_4\Big)+\ldots\Big] \\
&+\sinh^2(2\alpha)^{(\alpha,2)}C^{\pm}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Big[e^{\mp\pi(\nu_1+\nu_2)}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Big(e^{\mp\pi i}k_1,e^{\mp\pi i}k_2,k_3,k_4\Big) \\
&+e^{\mp\pi(\nu_1+\nu_3)}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Big(e^{\mp\pi i}k_1,k_2,e^{\mp\pi i}k_3,k_4\Big)+e^{\mp\pi(\nu_1+\nu_4)}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Big(e^{\mp\pi i}k_1,k_2,k_3,e^{\mp\pi i}k_4\Big)\Big]\Big\},
\end{aligned} \tag{4.2.41}$$

where

$$\begin{aligned}
{}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3\Delta_4} &= \pm\frac{1}{32}\left[4(7\cosh(2\alpha)+\cosh(6\alpha))\right. \\
&-4e^{\mp 3i\beta}\sinh^3(2\alpha)\Big(e^{\mp\pi(\nu_1+\nu_2+\nu_3)}+e^{\mp\pi(\nu_1+\nu_2+\nu_4)}+e^{\mp\pi(\nu_1+\nu_3+\nu_4)}+e^{\mp\pi(\nu_2+\nu_3+\nu_4)}\Big) \\
&+4e^{\mp 2i\beta}\sinh(4\alpha)\sinh(2\alpha)\Big(e^{\mp\pi(\nu_1+\nu_2)}+e^{\mp\pi(\nu_1+\nu_3)}+e^{\mp\pi(\nu_1+\nu_4)}+e^{-\pi(\nu_2+\nu_3)}+e^{\mp\pi(\nu_2+\nu_4)} \\
&+e^{\mp\pi(\nu_3+\nu_4)}\Big)-e^{\mp i\beta}(7\sinh(2\alpha)+3\sinh(6\alpha))\Big(e^{\mp\pi\nu_1}+e^{\mp\pi\nu_2}+e^{\mp\pi\nu_3}+e^{\mp\pi\nu_4}\Big)\Big],
\end{aligned} \tag{4.2.42}$$

$$
{}^{(\alpha,1)}C^{\pm}_{\Delta_1\Delta_2\Delta_3\Delta_4} = \pm\frac{1}{16}\Big[2e^{\mp 3i\beta}e^{\mp\pi(\nu_1+\nu_2+\nu_3+\nu_4)}\sinh^2(2\alpha) - e^{\pm i\beta}(3\cosh(4\alpha)+5)
$$
$$
-2e^{\mp i\beta}\sinh^2(2\alpha)\left(e^{\mp\pi(\nu_1+\nu_2)}+e^{\mp\pi(\nu_1+\nu_3)}+e^{\mp\pi(\nu_1+\nu_4)}+e^{\mp\pi(\nu_2+\nu_3)}+e^{\mp\pi(\nu_2+\nu_4)}+e^{\mp\pi(\nu_3+\nu_4)}\right)
$$
$$
+2\sinh(4\alpha)\left(e^{\mp\pi\nu_1}+e^{\mp\pi\nu_2}+e^{\mp\pi\nu_3}+e^{\mp\pi\nu_4}\right)\Big], \quad (4.2.43)
$$

$$
{}^{(\alpha,2)}C^{\pm}_{\Delta_1\Delta_2\Delta_3\Delta_4} = \pm\frac{1}{8}\Big[-e^{\pm i\beta}\sinh(2\alpha)\left(e^{\mp\pi\nu_1}+e^{\mp\pi\nu_2}+e^{\mp\pi\nu_3}+e^{\mp\pi\nu_4}\right)
$$
$$
+e^{\mp i\beta}\sinh(2\alpha)\left(e^{\mp\pi(\nu_1+\nu_2+\nu_3)}+e^{\mp\pi(\nu_1+\nu_2+\nu_4)}+e^{\mp\pi(\nu_1+\nu_3+\nu_4)}+e^{\mp\pi(\nu_2+\nu_3+\nu_4)}\right)
$$
$$
-2e^{\mp 2i\beta}\cosh(2\alpha)e^{\mp\pi(\nu_1+\nu_2+\nu_3+\nu_4)}+2e^{\pm 2i\beta}\cosh(2\alpha)\Big], \quad (4.2.44)
$$

extending the coefficients (4.2.20) for generic scalars to $n = 4$.

4.3 Four-point Exchange

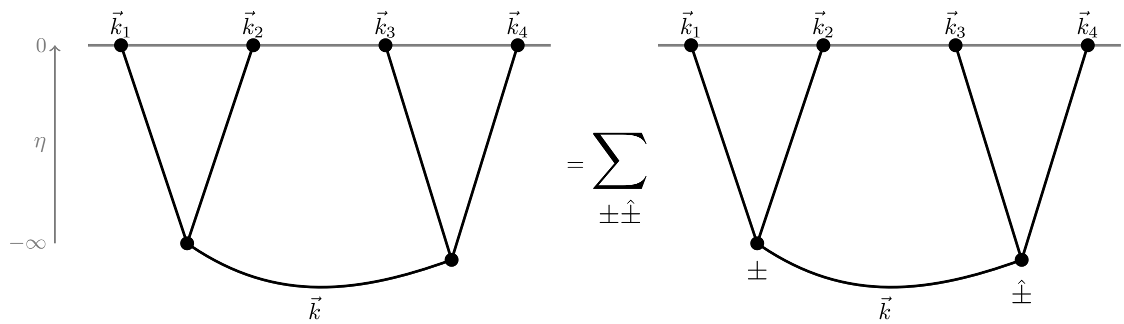


Figure 4.3: The four-point exchange diagram in de Sitter space is given by a sum of each contribution from the $\pm$ branches of the in-in contour, of which there are four - two for each bulk point. The solid grey line at the top of each diagram represents the late-time boundary, with the time coordinate running from $\eta = -\infty$ deep in the bulk to $\eta = 0$ on the boundary.

In the same way, one can express four-point exchange diagrams in a generic α-vacuum in terms of corresponding exchanges in Euclidean AdS with the momenta appropriately rotated. In this case one also makes use of the identities (3.2.64) for the bulk-to-bulk propagators in the Schwinger-Keldysh formalism which, under the analytic continuations in section 3.3, are expressed in terms of corresponding bulk-to-bulk propagators in EAdS with rotations of the exchanged momentum.

In fact, it is possible to trade such rotations of the exchanged momenta for rotations of the momentum of the external fields. Consider the exchange diagram mediated by the cubic vertices

$$\mathcal{V}_{12\phi} = g_{12}\phi_1\phi_2\phi, \qquad \mathcal{V}_{34\phi} = g_{34}\phi_3\phi_4\phi. \tag{4.3.1}$$

Using the representation (3.3.2) of bulk-to-bulk propagators together with (3.2.59), the s-channel exchange diagram generated by the above cubic vertices in EAdS takes the form, in the Mellin space representation,

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\left(k_1,k_2;p_1;p_2;k_3,k_4\right) = \int_{-i\infty}^{+i\infty}[\mathrm{d}s]_4[\mathrm{d}u]_2 \\ \times \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\left(k_1,s_1;k_2,s_2;p_1,u_1;p_2,u_2;k_3,s_3;k_4,s_4\right),\end{aligned} \tag{4.3.2}$$

with

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}&\left(k_1,s_1;k_2,s_2;p_1,u_1;p_2,u_2;k_3,s_3;k_4,s_4\right) \\ &= \frac{\Gamma\left(1+i\nu\right)\Gamma\left(1-i\nu\right)}{\pi}\csc\left(\pi\left(u_1+u_2\right)\right)\omega_\Delta\left(u_1,u_2\right) \\ &\times \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta}\left(s_1,k_1;s_2,k_2;u_1,p_1\right)\mathcal{A}^{\text{AdS}}_{\Delta_3\Delta_4 d-\Delta}\left(s_3,k_3;s_4,k_4;u_2,p_2\right),\end{aligned} \tag{4.3.3}$$

in terms of the constituent three-point contact diagrams (4.2.13) in Mellin space. With the same analysis used to derive the property (4.2.17) of the constituent three-point functions, it can then be shown that rotations of the internal momentum can be traded for rotations of the external momenta,

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\left(k_1,k_2;\bar{p}_1^{\pm};\bar{p}_2^{\hat{\pm}};k_3,k_4\right) = \left(e^{\pm i\pi}\right)^{\frac{d}{2}-i\nu-i\nu_1-i\nu_2}\left(e^{\hat{\pm} i\pi}\right)^{\frac{d}{2}+i\nu-i\nu_3-i\nu_4} \\ \times \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta|\Delta_3\Delta_4}\left(\bar{k}_1^{\mp},\bar{k}_2^{\mp};p_1;p_2;\bar{k}_3^{\hat{\mp}},\bar{k}_4^{\hat{\mp}}\right).\end{aligned} \tag{4.3.4}$$

In de Sitter space, the contribution to the s-channel exchange from the $++$ branch of the in-in contour (i.e. both bulk points time-ordered) is

$$\begin{aligned}{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\left(k_1,k_2,k_3,k_4\right) = \left(+ig_{12}\right)\left(+ig_{34}\right)\int_{-\infty}^{0}\frac{\mathrm{d}\eta_1}{\left(-\eta_1\right)^{d+1}}\int_{-\infty}^{0}\frac{\mathrm{d}\eta_2}{\left(-\eta_2\right)^{d+1}} \\ \times K^{(\alpha)}_{\Delta_1,+}\left(\eta_1,k_1\right)K^{(\alpha)}_{\Delta_2,+}\left(\eta_1,k_2\right)K^{(\alpha)}_{\Delta_3,+}\left(\eta_2,k_3\right)K^{(\alpha)}_{\Delta_4,+}\left(\eta_2,k_4\right)\end{aligned}$$

$$\times\left[P^{+}_{\Delta_+}G^{(0)}_{\Delta_+,++}(\eta_1,k;\eta_2,k)+e^{-2\nu\pi}P^{-}_{\Delta_+}G^{(0)}_{\Delta_+,++}(\eta_1,\bar{k}^{+};\eta_2,\bar{k}^{+})+(\Delta_+\to\Delta_-)\right], \tag{4.3.5}$$

where we inserted the expression (3.2.64) for the α-vacuum bulk-to-bulk propagator in terms of its counterpart in the Bunch-Davies vacuum. By exchanging analytic continuations of internal momenta for analytic continuations of external momenta, this can be written in the form

$$\begin{aligned}{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(k_1,k_2,k_3,k_4)&=P^{+}_{\Delta_+}\,{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(k_1,k_2,k_3,k_4)\\&+P^{-}_{\Delta_+}\left(e^{+i\pi}\right)^{2\Delta_+-i(\nu_1+i\nu_2+i\nu_3+i\nu_4)}\,{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(\bar{k}^{-}_1,\bar{k}^{-}_2,\bar{k}^{-}_3,\bar{k}^{-}_4\right)\\&+(\Delta_+\to\Delta_-)\,,\end{aligned} \tag{4.3.6}$$

with only the external momenta k_i rotated, where

$$\begin{aligned}{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(k_1,k_2,k_3,k_4)&=(+ig_{12})(+ig_{34})\int_{-\infty}^{0}\frac{\mathrm{d}\eta_1}{(-\eta_1)^{d+1}}\int_{-\infty}^{0}\frac{\mathrm{d}\eta_2}{(-\eta_2)^{d+1}}\\&\times K^{(\alpha)}_{\Delta_1,+}(\eta_1,k_1)\,K^{(\alpha)}_{\Delta_2,+}(\eta_1,k_2)\,G^{(0)}_{\Delta_+,++}(\eta_1,k;\eta_2,k)K^{(\alpha)}_{\Delta_3,+}(\eta_2,k_3)\,K^{(\alpha)}_{\Delta_4,+}(\eta_2,k_4)\,.\end{aligned} \tag{4.3.7}$$

Applying the identities (3.2.65) for the bulk-to-boundary propagators as well, this can be expressed in terms of exchanges in the Bunch-Davies vacuum. In particular, we find

$$\begin{aligned}{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(k_1,k_2,k_3,k_4)&=P^{+}_{\Delta_1}P^{+}_{\Delta_2}P^{+}_{\Delta_3}P^{+}_{\Delta_4}\,{}^{(0,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(k_1,k_2,k_3,k_4)\\&+\sum_i e^{(2\Delta_i-d)\pi i}P^{+}_{\Delta_1}\dots P^{-}_{\Delta_i}\dots P^{+}_{\Delta_4}\,{}^{(0,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(k_1,\dots,\bar{k}^{+}_i,\dots,k_4\right)\\&+\sum_{i<j}\Big(e^{(2\Delta_i-d)\pi i}e^{(2\Delta_j-d)\pi i}P^{+}_{\Delta_1}\dots P^{-}_{\Delta_i}\dots P^{-}_{\Delta_j}\dots P^{+}_{\Delta_4}\\&\qquad\times\,{}^{(0,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(k_1,\dots,\bar{k}^{+}_i,\dots,\bar{k}^{+}_j,\dots,k_4\right)\Big)\\&+e^{\sum_i(2\Delta_i-d)\pi i}\sum_i e^{-(2\Delta_i-d)\pi i}P^{-}_{\Delta_1}\dots P^{+}_{\Delta_i}\dots P^{-}_{\Delta_4}\,{}^{(0,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(\bar{k}^{+}_1,\dots,k_i,\dots,\bar{k}^{+}_4\right)\\&+\left(\prod_{i=1}^{4}e^{(2\Delta_i-d)\pi i}P^{-}_{\Delta_i}\right)\,{}^{(0,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(\bar{k}^{+}_1,\bar{k}^{+}_2,\bar{k}^{+}_3,\bar{k}^{+}_4\right),\end{aligned} \tag{4.3.8}$$

which in turn, using the analytic continuations (3.3.5) and (3.3.11), can be written in terms of the corresponding exchange diagram in EAdS [76, 77]:

$$\begin{aligned}{}^{(0,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(k_1,k_2,k_3,k_4\right)\\ =\left(\prod_{i=1}^{4}c^{\text{dS-AdS}}_{\Delta_i}\left(-\eta_0\right)^{\Delta_i}\right)c^{\text{dS-AdS}}_{\Delta_+}e^{-\left(\frac{-d+\Delta_++\Delta_1+\Delta_2}{2}\right)\pi i}e^{-\left(\frac{-d+\Delta_++\Delta_3+\Delta_4}{2}\right)\pi i}\\ \times\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(k_1,k_2,k_3,k_4\right),\end{aligned}\tag{4.3.9}$$

where

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}\left(k_1,k_2,k_3,k_4\right)=g_{12}g_{34}\int_0^{\infty}\frac{\mathrm{d}z_1}{z_1^{d+1}}\int_0^{+\infty}\frac{\mathrm{d}z_2}{z_2^{d+1}}\\ \times K^{\text{AdS}}_{\Delta_1}\left(z_1,k_1\right)K^{\text{AdS}}_{\Delta_2}\left(z_1,k_2\right)G^{\text{AdS}}_{\Delta_+}(z_1,k;z_2,k)K^{\text{AdS}}_{\Delta_3}\left(z_2,k_3\right)K^{\text{AdS}}_{\Delta_4}\left(z_2,k_4\right).\end{aligned}\tag{4.3.10}$$

We have seen that by the split representation of the EAdS harmonic function (3.2.59), the EAdS exchange diagrams "factorise", or can be expressed as a product of three-point contact diagrams. The OPE limit $k\to 0$ where the exchanged particle goes on-shell can therefore be equivalently probed by considering the soft limit of one leg of the constituent three-point contact diagrams, which was studied in the previous section.

The above procedure can be repeated for all other contributions (i.e. $+-$, $-+$, $--$) to the exchange in the in-in formalism, expressing the result in terms of corresponding exchange diagrams in EAdS with appropriate rotations of the external momenta. This expression for the full four-point s-channel de Sitter exchange can be found in the *Mathematica* file in [87].

The t- and u-channel exchanges are obtained simply by permuting the external legs in the s-channel exchange; see figure 4.4. In particular, the $(++)$ contribution to the s-channel exchange is given by

$$\begin{aligned}{}^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\left(k_1,k_2,k_3,k_4\right)=\left(+ig_{12}\right)\left(+ig_{34}\right)\int_{-\infty}^{0}\frac{\mathrm{d}\eta_1}{\left(-\eta_1\right)^{d+1}}\int_{-\infty}^{0}\frac{\mathrm{d}\eta_2}{\left(-\eta_2\right)^{d+1}}\\ \times K^{(\alpha)}_{\Delta_1,+}\left(\eta_1,k_1\right)K^{(\alpha)}_{\Delta_2,+}\left(\eta_1,k_2\right)G^{(\alpha)}_{\Delta_+,++}(\eta_1,k;\eta_2,k)K^{(\alpha)}_{\Delta_3,+}\left(\eta_2,k_3\right)K^{(\alpha)}_{\Delta_4,+}\left(\eta_2,k_4\right),\\ =\left(+ig_{12}\right)\left(+ig_{34}\right)\int[ds]_4[du]_2K^{(\alpha)}_{\Delta_1,+}\left(s_1k_1\right)K^{(\alpha)}_{\Delta_2,+}\left(s_2,k_2\right)G^{(\alpha)}_{\Delta_+,++}(u_1,k;u_2,k)\end{aligned}$$

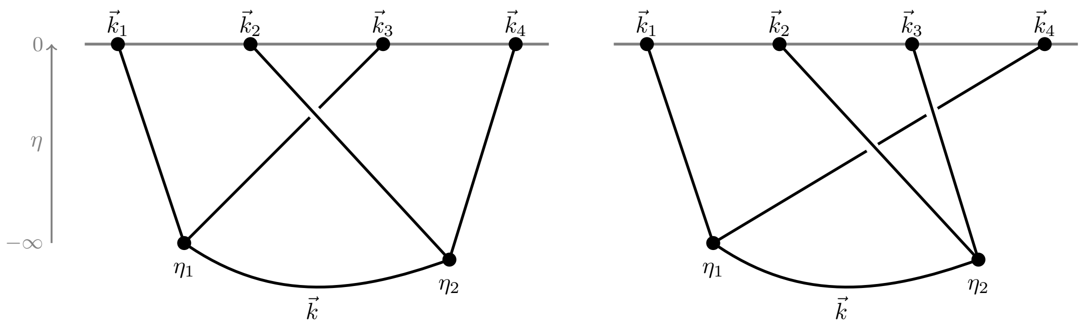


Figure 4.4: The t-channel (left) and u-channel contributions to the de Sitter exchange are obtained from the s-channel exchange by a permutation of the external legs. At the level of the Mellin-Barnes representation, this amounts simply to a swapping of the external Mellin variables in the Dirac delta functions arising from the bulk time integrals.

$$\times K^{(\alpha)}_{\Delta_3,+}(s_3,k_3)\, K^{(\alpha)}_{\Delta_4,+}(s_4,k_4)$$
$$\times \int_{-\infty}^{0} \frac{\mathrm{d}\eta_1}{(-\eta_1)^{d+1}}(-\eta_1)^{-2(s_1+s_2+u_1)+\frac{3d}{2}} \int_{-\infty}^{0} \frac{\mathrm{d}\eta_2}{(-\eta_2)^{d+1}}(-\eta_2)^{-2(s_3+s_4+u_2)+\frac{3d}{2}},$$
$$= (+ig_{12})(+ig_{34}) \int [ds]_4[du]_2 K^{(\alpha)}_{\Delta_1,+}(s_1k_1)\, K^{(\alpha)}_{\Delta_2,+}(s_2,k_2)\, G^{(\alpha)}_{\Delta_+,++}(u_1,k;u_2,k)$$
$$\times K^{(\alpha)}_{\Delta_3,+}(s_3,k_3)\, K^{(\alpha)}_{\Delta_4,+}(s_4,k_4)$$
$$\times (2\pi i)\delta\left(\frac{d}{4}-s_1-s_2-u_1\right)(2\pi i)\delta\left(\frac{d}{4}-s_3-s_4-u_2\right), \tag{4.3.11}$$

while the t-channel contribution is given by

$$^{(\alpha,++)}\mathcal{A}^{\mathcal{V}_{12\phi}\mathcal{V}_{34\phi}}_{\Delta_1\Delta_3\Delta_2\Delta_4}(k_1,k_2,k_3,k_4) = (+ig_{12})(+ig_{34}) \int_{-\infty}^{0} \frac{\mathrm{d}\eta_1}{(-\eta_1)^{d+1}} \int_{-\infty}^{0} \frac{\mathrm{d}\eta_2}{(-\eta_2)^{d+1}}$$
$$\times K^{(\alpha)}_{\Delta_1,+}(\eta_1,k_1)\, K^{(\alpha)}_{\Delta_3,+}(\eta_1,k_3)\, G^{(\alpha)}_{\Delta_+,++}(\eta_1,k;\eta_2,k) K^{(\alpha)}_{\Delta_2,+}(\eta_2,k_2)\, K^{(\alpha)}_{\Delta_4,+}(\eta_2,k_4),$$
$$= (+ig_{12})(+ig_{34}) \int [ds]_4[du]_2 K^{(\alpha)}_{\Delta_1,+}(s_1k_1)\, K^{(\alpha)}_{\Delta_2,+}(s_2,k_2)\, G^{(\alpha)}_{\Delta_+,++}(u_1,k;u_2,k)$$
$$\times K^{(\alpha)}_{\Delta_3,+}(s_3,k_3)\, K^{(\alpha)}_{\Delta_4,+}(s_4,k_4)$$
$$\times (2\pi i)\delta\left(\frac{d}{4}-s_1-s_3-u_1\right)(2\pi i)\delta\left(\frac{d}{4}-s_2-s_4-u_2\right). \tag{4.3.12}$$

At the level of the Mellin-Barnes representation, we see that the difference is ultimately just a swapping of $s_2 \leftrightarrow s_3$ in the delta functions.

Summary. In this chapter we applied the relations derived in chapter 3 to study perturbative correlators in de Sitter space. In particular, we computed the late-time boundary two-point function and derived expressions for n-point contact diagrams for generic choice of α-vacuum. We also focussed on some examples in the $n = d = 3$ case - the contact diagram of three massless scalars and that of two conformally coupled and one general scalar. We showed that both contact and exchange diagrams can be expressed in terms of their Bunch-Davies counterparts, as well as in terms of EAdS Witten diagrams, with appropriately rotated external momenta. This opens the door to a deeper understanding of de Sitter boundary correlators in the α-vacua - by reformulating them in terms of the comparatively better-understood EAdS Witten diagrams, one can then import techniques and understanding from the EAdS case to the dS case.

In the following chapter, we extend the tools described above to the context of inflation.

Chapter 5

Inflationary Correlators from EAdS

This chapter is based on unpublished work. We extend the tools described above to the context of inflationary correlation functions. In particular, we derive objects that we refer to as *master formulae* for "inflationary" three-point contact and four-point scalar exchange diagrams in EAdS, with legs analytically continued. In plugging these master formulae into the three-point contact and four-point de Sitter exchange diagrams for arbitrary choice of α-vacuum, we obtain corrections to the inflationary two- and three-point functions of inflaton perturbations $\delta\phi$ at leading order in slow-roll.

The main results of this chapter are:

- **Inflationary EAdS Master Formulae.** We derive general formulae for EAdS three-point contact and four-point exchange diagrams with external legs analytically continued (as $k \to e^{\pm i\pi}k$) in every way that appears in the dS three-point contact diagram (4.2.18), and the four-point exchange. We use the Mellin space formalism to compute these formulae in certain soft limits, with each leg having a small mass ϵ related to the slow-roll parameters.
- **Inflationary Correlators from EAdS.** By combining these "inflationary

EAdS" master formulae with the formulae for dS correlators from EAdS Witten diagrams described in chapter 4, we compute the leading correction to the inflationary two- and three-point functions of inflaton perturbations.

5.1 Slow-Roll Inflation

We begin with a short overview of single-field slow-roll inflation. An excellent review can be found in [112], which we loosely follow here.

5.1.1 Motivation

Correlations in the large-scale structure we see in the sky today are ultimately seeded by quantum fluctuations in a phase of the universe's evolution called *inflation*, a period of extremely rapid accelerated expansion before the hot big bang. The inflationary paradigm solves a number of issues with our pre-inflationary understanding of early-universe cosmology, one being the *horizon problem*. This is the statement that the observed homogeneity of the Cosmic Microwave Background (CMB) implies that seemingly causally disconnected regions of the universe must have been in thermal equilibrium at some point during the universe's evolution, despite the absence of a mechanism for being so. Inflation solves this issue by effectively including more time before the CMB was produced, and allowing these regions to be in causal contact during the inflationary phase.

The idea for how correlations in large-scale structure arose in the inflationary model is as follows. Fluctuations in the metric and a scalar field called the *inflaton* get stretched by the extremely rapid accelerated expansion of an approximately de Sitter spacetime, ultimately being "caught" on the future boundary when the inflationary period ends, depositing some energy onto this late-time surface. These energy deposits then manifest themselves as temperature inhomogeneities in the Cosmic Microwave Background, and create potential wells and hills for matter to fall into and out of further down the line in the universe's evolution. In this way, correlations between fluctuations in the inflationary phase ultimately lead to correlations in the positions of large-scale structures in the sky today.

A central goal of modern experimental cosmology is to uncover the physics governing the inflationary phase. An approach known as *Cosmological Collider Physics* [12] puts forward the idea of viewing inflation as an extremely high-energy particle collider, whereby observing large-scale structure could allow us to infer not only the spectrum of particles present during the inflationary phase but also their dynamics. For instance, measurements of *power spectra*, quantities related to the two-point function of primordial fluctuations, by *Planck* [11] and others have found that inflationary physics must have been approximately scale invariant. However, the power spectrum alone is insufficient to allow us to infer the spectrum of particles and their dynamics - to glean information about interactions for instance, one must measure higher-point correlations. Probing such *non-Gaussianities* is a central goal of the *Euclid* mission [113], which aims to constrain primordial non-Gaussianities by detecting their imprint on the large-scale distribution of galaxies. The eventual hope is that such cosmological surveys, in combination with future generations of CMB experiments like *CMB-S4*, could improve existing constraints on non-Gaussianities and uncover subtle features of inflationary dynamics, such as the kinds of interactions between fields and initial conditions [114, 115]. As such, theoretical predictions for the kinds of measurements these experiments hope to obtain are of paramount importance.

In this thesis we are concerned only with the simplest model of inflationary dynamics, known as *single-field slow-roll inflation.* The next section provides a lightning fast review of the aspects of this model most pertinent to the thesis.

5.1.2 Single-Field Slow-Roll Inflation

Accelerated expansion requires a source of negative pressure, which in the simplest models of inflation is provided by the presence of a single scalar field ϕ called the *inflaton*. The action for a scalar field on a general fixed background $g_{\mu\nu}$ is given by[1]

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big[\frac{1}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\Big], \tag{5.1.1}$$

where the *inflaton potential* $V(\phi)$ captures the mass term and self-interactions of the scalar. From this action one can derive the energy-momentum tensor

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\nabla^\sigma\phi\nabla_\sigma\phi + V(\phi)\right), \tag{5.1.2}$$

with the equation of motion for the scalar being given by

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{dV(\phi)}{d\phi} = 0. \tag{5.1.3}$$

Now, as a general model of an expanding spacetime we assume that $g_{\mu\nu}$ is of Friedmann-Lemaître-Robertson-Walker (FLRW) form, whose line element is given in polar coordinates by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right) \tag{5.1.4}$$

with the function $a(t)$ being the *scale factor* describing how spatial sections grow in time. The parameter k describes the geometry of the spatial slices; $k = 0$ corresponds to flat space, $k = +1$ to a sphere, and $k = -1$ to hyperbolic spatial slices. We will be mostly concerned with flat spatial slices, and work in Cartesian coordinates where the metric is given by

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{5.1.5}$$

Making the simplifying assumption that the scalar is homogeneous in space, namely $\phi(t,\vec{x}) = \bar{\phi}(t)$, we find that the energy-momentum tensor takes the form of that of a

[1]We specialise to $d + 1 = 4$ in this short review for convenience.

perfect fluid. Namely, we find

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho_\phi & 0 & 0 & 0 \\ 0 & -p_\phi & 0 & 0 \\ 0 & 0 & -p_\phi & 0 \\ 0 & 0 & 0 & -p_\phi \end{pmatrix}, \tag{5.1.6}$$

where the energy density is given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{5.1.7}$$

and the pressure is

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{5.1.8}$$

Indeed, we see that large negative pressure can be achieved if the scalar field satisfies $\frac{1}{2}\dot{\phi}^2 << V(\phi)$.

The Einstein equations then imply two coupled differential equations; the first known as the *Friedmann Equation*,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{5.1.9}$$

and the second (coming from the trace of the Einstein equations) known as the *acceleration equation*,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{5.1.10}$$

These two equations together then imply the *continuity equation*

$$\dot{\rho} = -3H(\rho + p), \tag{5.1.11}$$

where we have defined the *Hubble parameter*

$$H := \frac{\dot{a}}{a}. \tag{5.1.12}$$

Thus, the presence of the scalar field affects the geometry; through the Friedmann equation we have

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_\phi = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})\right), \tag{5.1.13}$$

and so starting from the FLRW metric we find

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \\ &= -dt^2 + \frac{3}{8\pi G}\frac{\dot{a}^2}{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})}(dx^2 + dy^2 + dz^2). \end{aligned} \tag{5.1.14}$$

The equation of motion for the homogeneous scalar field is given by

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \frac{\partial V}{\partial \bar{\phi}} = 0. \tag{5.1.15}$$

Note that this is the equation for a particle rolling down its potential, with $3H\dot{\bar{\phi}}$ playing the role of the friction term[2]. If the friction term is sufficiently large, the acceleration of this particle will be small, the particle will "roll slowly" down its potential, and we will have $\dot{\bar{\phi}} \sim \frac{1}{3H}V'(\phi)$. This is the origin of the "slow-roll" in slow-roll inflation. This "slow-roll condition" $|\ddot{\bar{\phi}}| << |3H\dot{\bar{\phi}}|, |V'(\bar{\phi})|$ is achieved if

$$\eta_H := -\frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} << 1, \tag{5.1.16}$$

is satisfied, where we have defined the *slow-roll parameter* η_H.

The equation of state is characterised by the variable

$$W_\phi := \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\bar{\phi}}^2 - V(\bar{\phi})}{\frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi})}. \tag{5.1.17}$$

If we insist on accelerated expansion, namely $\ddot{a} \overset{!}{>} 0$, we find from the acceleration equation (with $a > 0$) that

$$\begin{aligned} \ddot{a} &= -\frac{4\pi G a}{3}(\rho_\phi + 3p_\phi) \\ &= -\frac{4\pi G a \rho_\phi}{3}(1 + 3W_\phi) \overset{!}{>} 0, \end{aligned} \tag{5.1.18}$$

[2]A particle rolling down a hill modelled by a potential $V(x)$ experiences forces $F = m\ddot{x}$ and $F_{\text{hill}} = -V'(x) - \gamma\dot{x}$ with γ the friction coefficient. Equating the forces we get the EOM $\ddot{x} + \frac{\gamma}{m}\dot{x} + V'(x) = 0$.

and so accelerated expansion requires $W_\phi \overset{!}{<} -\frac{1}{3}$. In addition, assuming a positive scale factor and energy density the requirement for negative pressure is $W_\phi \overset{!}{<} 0$.

Beginning with the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{5.1.19}$$

we can write

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G\rho}{3}\left(1 + 3W_\phi\right) \\ &= -\frac{H^2}{2}(1 + 3W_\phi) \\ &= -H^2(\frac{1}{2} + \frac{3}{2}W_\phi) \\ &= H^2(1 - \varepsilon),\end{aligned} \tag{5.1.20}$$

where in going from the first to the second line we used the Friedmann equation and assumed flat spatial sections $k = 0$. We have also defined another slow-roll parameter[3]

$$\varepsilon := \frac{3}{2}(1 + W_\phi) = \frac{3}{2}\frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{8\pi G}{2}\frac{\dot{\phi}^2}{H^2}, \tag{5.1.21}$$

where we used equation (5.1.13). The condition for accelerated expansion $\ddot{a} > 0$ then translates into a condition on the slow-roll parameter; $\varepsilon < 1$. The slow-roll parameter can also be related to the Hubble parameter, which can be seen first by taking the time derivative,

$$H = \frac{\dot{a}}{a} \implies \dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2, \tag{5.1.22}$$

and then using (5.1.20), giving

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln(H)}{dN}. \tag{5.1.23}$$

We have introduced the *number of e-folds* N, defined by $dN := H(t)dt = d\ln(a)$, namely $e^N := a$. It turns out that to solve the horizon problem, namely for the light-cones of two apparently causally disconnected regions of the CMB to intersect

[3]The $8\pi G$ is often dropped, leaving just $\varepsilon = \frac{\dot{\phi}^2}{2H^2}$.

in inflation, we require the inflationary phase to be maintained for at least ~ 60 e-folds, namely $N \overset{!}{\gtrsim} 60$. In slow-roll inflation we take[4] $\varepsilon << 1$, and so the acceleration equation becomes

$$\frac{\ddot{a}}{a} = H^2(1-\varepsilon) \approx H^2. \tag{5.1.24}$$

Taking H to vary slowly enough to be considered approximately constant, we then solve this differential equation to find

$$a(t) \approx e^{Ht}. \tag{5.1.25}$$

Plugging this back into the FLRW metric we find

$$\begin{aligned} ds^2 &= -dt^2 + a(t)(dx^2 + dy^2 + dz^2) \\ &\approx \frac{1}{H^2\eta^2}\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right), \end{aligned} \tag{5.1.26}$$

after transforming to conformal time. Thus, we see that the geometry of the universe during slow-roll inflation was approximately that of de Sitter space.

In the slow-roll approximation, the slow-roll parameters can be expressed as conditions on the shape of the potential as follows. In slow-roll where the potential energy of the scalar dominates over the kinetic energy $\frac{1}{2}\dot{\bar{\phi}} << V(\bar{\phi})$, the Friedmann equation (5.1.13) becomes

$$\begin{aligned} H^2 &\approx \frac{8\pi G}{3} V(\bar{\phi}) \\ \implies \dot{H} &\approx \frac{8\pi G}{3}\frac{1}{6H^2} V'(\bar{\phi}), \end{aligned} \tag{5.1.27}$$

where we differentiated both sides and used that $\dot{\bar{\phi}} \sim \frac{1}{3H} V'(\bar{\phi})$ in slow-roll. After some algebra and using the Friedmann equation again, we then have

$$\varepsilon := -\frac{\dot{H}}{H^2} \approx -\frac{1}{16\pi G}\left(\frac{V'(\bar{\phi})}{V(\bar{\phi})}\right)^2, \tag{5.1.28}$$

in the slow-roll approximation. For the other slow-roll parameter, we start with

[4]See appendix D of [112] for details on the smallness of the slow-roll parameters in inflation.

$\dot{\bar{\phi}} \sim \frac{1}{3H} V'(\bar{\phi})$ and differentiate both sides. We find

$$\ddot{\bar{\phi}} \approx \frac{1}{3}\left(-\frac{\dot{H}}{H^2}V'(\bar{\phi}) + \frac{\dot{\bar{\phi}}}{H}V''(\bar{\phi})\right), \tag{5.1.29}$$

which using $\dot{\bar{\phi}} \sim \frac{1}{3H} V'(\bar{\phi})$ again leads to

$$\eta_H = -\frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} \approx \varepsilon - \frac{1}{8\pi G}\frac{V''(\bar{\phi})}{V(\bar{\phi})}. \tag{5.1.30}$$

For completeness, these ratios of (derivatives of) the potential are sometimes defined to be two further *potential* slow-roll parameters (to distinguish them from the *Hubble* slow-roll parameters above), leading to

$$\varepsilon \approx \varepsilon_v, \quad \eta_H \approx \varepsilon - \eta_v. \tag{5.1.31}$$

In order to obtain large-scale structure in the universe today, there must have been deviations from the homogeneous cosmology described above. To take these deviations into account, we consider small perturbations around the homogeneous solution for the inflaton,

$$\phi(t, \vec{x}) = \bar{\phi}(t) + \delta\phi(t, \vec{x}), \tag{5.1.32}$$

where we call $\delta\phi(t, \vec{x})$ the *inflaton perturbation*. In cosmological perturbation theory we also often consider perturbations in the metric,

$$g_{\mu\nu}(t, \vec{x}) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(t, \vec{x}), \tag{5.1.33}$$

with $h_{\mu\nu}$ called the *curvature-* or *metric perturbation*. It is the correlations between these fluctuations in the inflationary phase that ultimately lead to the correlations we observe in large-scale structure in the universe today. In this chapter we will only consider correlators involving the inflaton perturbations, and we will ignore the metric perturbations[5].

[5]For realistic inflationary dynamics, the inflaton perturbations induce perturbations in the stress tensor, which in turn source fluctuations in the metric. As such, for a full and consistent treatment we should include the metric fluctuations. However, focussing only on the inflaton perturbations will allow us to demonstrate the power of the new techniques presented in this chapter, and we leave inflationary correlators involving gravitons and the effects of metric perturbations to future work.

One final point to make is that the slow-roll parameters ε and η_H can be associated with the mass of the inflaton in the following way. Consider the action (5.1.1), and consider expanding the scalar field in a homogeneous part and a fluctuation as above, $\phi(t,\vec{x}) = \bar{\phi}(t) + \delta\phi(t,\vec{x})$. We find that the matter part of the action becomes

$$\begin{aligned}\mathcal{S}_{\text{matter}} &= \int d^d x\sqrt{-g}\left[\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right] \\ &= \int d^d x\sqrt{-g}\Big[\frac{1}{2}\nabla_\mu\delta\phi\nabla^\mu\delta\phi - V(\bar{\phi}) - V'(\bar{\phi})\delta\phi - \frac{1}{2}V''(\bar{\phi})\delta\phi^2 \\ &\qquad\qquad - \frac{1}{3!}V^{(3)}(\bar{\phi})\delta\phi^3 + \dots\Big] \\ &= \mathcal{S}^{(2)} + \mathcal{S}^{(3)} + \dots,\end{aligned} \tag{5.1.34}$$

where the part of the action containing quadratic terms $\mathcal{S}^{(2)}$ encodes the free dynamics of the perturbation, with $\mathcal{S}^{(3)}$ and higher-order terms in the expansion encoding interactions. We see that the quadratic action contains a mass term for the perturbation, with the mass given by

$$m^2 = V''(\bar{\phi}). \tag{5.1.35}$$

Recalling the potential slow-roll parameter η_v from (5.1.30), the mass of the inflaton perturbation can be then expressed

$$\begin{aligned}m^2 &\approx 8\pi G\eta_v V(\bar{\phi}) \\ &\approx 3H^2\eta_v \\ &\approx 3H^2(\varepsilon - \eta_H),\end{aligned} \tag{5.1.36}$$

where we used the Friedmann equation $H^2 \approx \frac{8\pi G}{3}V(\bar{\phi})$ in slow-roll. We see that in slow-roll inflation, the mass of the inflaton perturbation is related to the slow-roll parameters. This provides important intuition for the recipe in the next section.

5.1.3 Inflationary Correlators from de Sitter

Since we have good control over the computation of de Sitter correlation functions via the in-in formalism, a natural question to ask is whether or not it is possible to obtain correlation functions involving inflaton perturbations in the inflationary background from knowledge of analogous correlators in de Sitter.

In slow-roll inflation we have two small parameters intrinsic to the theory; $\varepsilon, \eta_H << 1$. Thus, a natural approach to inflationary correlators could be to compute a de Sitter correlation function, perturb it using some combination ϵ of the slow-roll parameters, and perform a series expansion using that combination as a perturbative parameter. This way, we could interpret terms in the expansion proportional to powers of ϵ as corrections to the de Sitter result, which arise due to the breaking of the de Sitter isometries in an inflationary context. Indeed, this is exactly the approach we take.

We will follow a by-now-standard recipe [12, 116] inspired by the above logic for computing inflationary two-point functions from de Sitter three-point functions, and inflationary three-point functions from de Sitter exchange four-point functions of general scalars. The recipe is as follows.

dS Correlators $\longrightarrow$ Inflationary Correlators:

1. Give each leg in the dS diagram a small mass, setting $\nu_j = i\left(\frac{d}{2} - \epsilon\right)$, with ϵ related to the slow-roll parameters.
2. Send one of the legs soft and expand the result in powers of ϵ.

The intuition for this recipe is as follows. Regarding step 1, we want to produce a correlation function of inflaton perturbations $\delta\phi$ in inflation from a correlation function of a general scalar field in exact de Sitter. Since the inflaton perturbation $\delta\phi$ is not perfectly massless, but rather has a small mass related to the slow-roll parameters through (5.1.36), we give each leg in the de Sitter correlator a small mass by setting $\nu_j = i(\frac{d}{2} - \epsilon)$, with ϵ related to the slow-roll parameters. In step 2, we then take the soft limit of one leg of the correlator. The limit $k \to 0$ (soft spatial momentum, or small wavenumber) while keeping the other legs hard corresponds

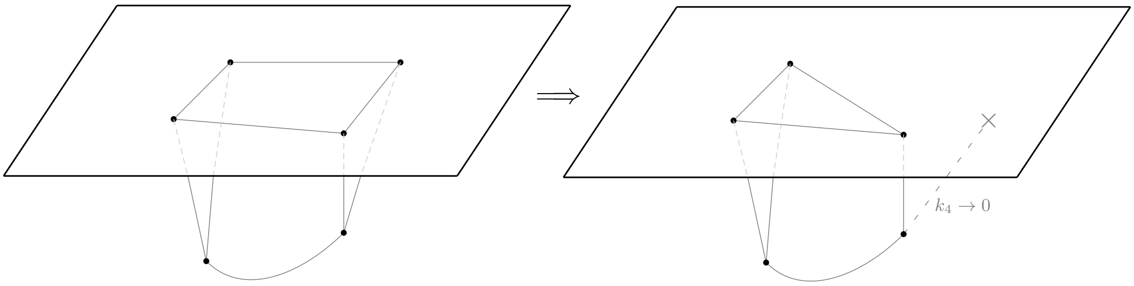


Figure 5.1: Inflationary three-point functions can be obtained from exact de Sitter four-point functions by taking one of the external legs soft. The soft limit, or the long-wavelength limit, results in a correlator with three hard perturbations on top of an approximately spatially static background field. This is exactly the situation in slow-roll inflation, where we expand the inflaton as $\phi(t,\vec{x}) = \bar{\phi}(t) + \delta\phi(t,\vec{x})$.

to the leg in question having very long wavelength compared to the other legs, and as such we treat the soft leg as an approximately spatially static background field. The result is a lower-point correlation function, whose hard legs are perturbations on top of a classical, spatially constant background field. This is exactly a slow-roll inflationary scenario, with the perturbations $\delta\phi(t,\vec{x})$ on top of the homogeneous classical background field $\bar{\phi}(t)$. By expanding in ϵ, we obtain inflationary corrections to the corresponding de Sitter correlators. Note that while we could in principle compute inflationary three-point functions from de Sitter four-point contact diagrams, it is particularly interesting to compute them from de Sitter four-point exchanges because the resulting inflationary three-point function will then retain features of the exchanged particle. In particular, four-point exchanges contain non-analytic (ie, oscillatory) terms signalling particle production[6] in the OPE limit, where the exchanged particle goes on-shell $k \to 0$. Inflationary three-point functions computed from such de Sitter diagrams then retain this oscillatory signal in the *squeezed limit*, where (if we choose k_4 to be the soft leg in the above recipe) $k_3 \to 0$[7]. In particular,

[6]Namely, these terms are interpreted as signalling the production of a physical, on-shell particle by the time-dependent background, which then decays into two pairs forming the external legs of the exchange diagram. This is in contrast to the usual interpretation where the exchanged particle is virtual.

[7]For an exchange diagram momentum conservation for one of the subdiagrams is given by $\vec{k} + \vec{k}_3 + \vec{k}_4 = 0$. Sending $\vec{k}_4 \to 0$ then implies $|k| = |k_3|$, and so the $k \to 0$ limit for the exchanged

such correlators contain terms of the form $\left(\frac{k_3}{k_1}\right)^{\frac{d}{2}+i\nu}$, from which we see that the frequency of the oscillations is set by the mass of the exchanged particle. This motivates the notion of *Cosmological Collider Physics* [12] - by experimentally measuring the inflationary three-point function, one could in principle glean information about the spectrum of particles during the inflationary phase.

In relating dS correlators for the α-vacua to a combination of Witten diagrams in EAdS in chapter 4, we now have a clear path towards a systematic understanding of such correlators (via EAdS) in an inflationary context. If we could compute a "master formula" for an "inflationary" EAdS diagram[8] with the momenta of the external legs analytically continued, we could simply combine these in the way described in chapter 4 to obtain inflationary correlators for arbitrary choice of α-vacuum. This is the goal of the remainder of this chapter.

5.2 EAdS Inflationary Master Formulae

In this section we derive three- and four-point master formulae for analytically continued, inflationary EAdS Witten diagrams.

5.2.1 Three-point Contact Diagram

The EAdS three-point contact diagram can be expressed in Mellin space as[9]

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}&(e^{\pm ai\pi}k_1, e^{\mp bi\pi}k_2, e^{\mp ci\pi}k_3)\\ &= \int_{-i\infty}^{i\infty}[\mathrm{d}s]_3 K^{\text{AdS}}_{\Delta_1}(s_1, e^{\pm ai\pi}k_1)K^{\text{AdS}}_{\Delta_2}(s_2, e^{\mp bi\pi}k_2)K^{\text{AdS}}_{\Delta_3}(s_3, e^{\mp ci\pi}k_3)\\ &\qquad \times (2i\pi)\delta\left(\frac{d}{4}-s_1-s_2-s_3\right)\end{aligned} \tag{5.2.1}$$

particle is equivalent to $k_3 \to 0$, which is known as the *squeezed limit*.

[8] We define "inflationary" in the context of EAdS to simply be an EAdS Witten diagram for which we follow the above recipe.

[9] Note that we have implicitly already performed the integral over the bulk coordinate z, which gives the delta function. See equation (3.1.36).

with $K^{\rm AdS}_{\Delta}(s,k)$ given by (3.3.3), and $a,b,c\in\{0,1\}$ characterising the various analytic continuations of the momenta. This is a "master formula" in the sense that by choosing $\pm a, \hat{\pm} b$ and $\tilde{\pm} c$ appropriately, the formula describes all possible analytic continuations of the legs that appear in the corresponding de Sitter diagram (4.2.18). Using (3.3.3) and eliminating the s_1 integral with the delta function, we are left with

$$\begin{aligned}
&\mathcal{A}^{\rm AdS}_{\Delta_1\Delta_2\Delta_3}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3)\\
&= \frac{2^{\frac{d}{2}-\sum\limits_{j=1}^{3} i\nu_j}}{\prod\limits_{j=1}^{3} 2\Gamma(1+i\nu_j)}\int\limits_{-i\infty}^{i\infty}\frac{{\rm d}s_2}{2\pi i}\frac{{\rm d}s_3}{2\pi i}\,\Gamma\left(s_2+\frac{i\nu_2}{2}\right)\Gamma\left(s_2-\frac{i\nu_2}{2}\right)\Gamma\left(s_3+\frac{i\nu_3}{2}\right)\Gamma\left(s_3-\frac{i\nu_3}{2}\right)\\
&\qquad\times\Gamma\left(\frac{d}{4}-\frac{i\nu_1}{2}-s_2-s_3\right)\Gamma\left(\frac{d}{4}+\frac{i\nu_1}{2}-s_2-s_3\right)\\
&\qquad\times(e^{\pm i\pi a}k_1)^{-\frac{d}{2}+i\nu_1+2s_2+2s_3}(e^{\hat{\pm} i\pi b}k_2)^{-2s_2+i\nu_2}(e^{\tilde{\pm} i\pi c}k_3)^{-2s_3+i\nu_3}.
\end{aligned}\tag{5.2.2}$$

We choose to perform the s_2 integral first. Isolating it, we have

$$\begin{aligned}
\mathcal{I}_{s_2} &:= \int\limits_{-i\infty}^{i\infty}\frac{{\rm d}s_2}{2\pi i}\Gamma\left(s_2+\frac{i\nu_2}{2}\right)\Gamma\left(s_2-\frac{i\nu_2}{2}\right)\Gamma\left(\frac{d}{4}-\frac{i\nu_1}{2}-s_2-s_3\right)\Gamma\left(\frac{d}{4}+\frac{i\nu_1}{2}-s_2-s_3\right)\\
&\qquad\times(e^{\pm i\pi a}k_1)^{-\frac{d}{2}+i\nu_1+2s_2+2s_3}(e^{\hat{\pm} i\pi b}k_2)^{-2s_2+i\nu_2},
\end{aligned}\tag{5.2.3}$$

which has poles at

$$s_2 = \pm\frac{i\nu_2}{2}-n,\quad n\in\mathbb{Z}_{\geq 0},\tag{5.2.4}$$

$$s_2 = \frac{d}{4}-s_3\pm\frac{i\nu_1}{2}+m,\quad m\in\mathbb{Z}_{\geq 0}.\tag{5.2.5}$$

We choose[10] to close the contour to the left of the imaginary axis, picking up the first set of poles; those from the $\Gamma\left(s_2\pm\frac{i\nu_2}{2}\right)$ factors. Given that we are interested in eventually computing inflationary correlators, we need only consider the leading s_2 pole in the soft limit $k_2\to 0$. This turns out to be the pole at $s_2=-\frac{i\nu_2}{2}$. After

[10]Note that this is not in general a free choice; one must be careful to choose the contour such that the integral converges. Details can be found in appendix A.

invoking the residue theorem, we find

$$\lim_{k_2\to 0}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\hat{\pm}ci\pi}k_3) = \frac{2^{\frac{d}{2}-i(\nu_1+\nu_2+\nu_3)}\Gamma(-i\nu_2)(e^{\hat{\pm}i\pi b}k_2)^{2i\nu_2}}{\prod_{j=1}^{3}2\Gamma(1+i\nu_j)}$$
$$\times\int_{-i\infty}^{i\infty}\frac{\mathrm{d}s_3}{2\pi i}\Gamma\left(s_3+\frac{i\nu_3}{2}\right)\Gamma\left(s_3-\frac{i\nu_3}{2}\right)\Gamma\left(\frac{d}{4}-s_3-\frac{i}{2}(\nu_1-\nu_2)\right) \tag{5.2.6}$$
$$\times\,\Gamma\left(\frac{d}{4}-s_3+\frac{i}{2}(\nu_1+\nu_2)\right)(e^{\pm i\pi a}k_3)^{-\frac{d}{2}+i(\nu_1-\nu_2)+2s_3}(e^{\hat{\pm}i\pi c}k_3)^{-2s_3+i\nu_3},$$

where we used that for $k_2\to 0$, we have $k_1\sim k_3$.

Finally, we deal with the s_3 integral. For this we close the contour to the right of the imaginary axis, picking up the poles at

$$s_3 = \frac{d}{4}+\frac{i}{2}(\nu_1+\nu_2)+n,\quad n\in\mathbb{Z}_{\geq 0} \tag{5.2.7}$$
$$s_3 = \frac{d}{4}-\frac{i}{2}(\nu_1-\nu_2)+m,\quad m\in\mathbb{Z}_{\geq 0}. \tag{5.2.8}$$

Once again invoking the residue theorem, we find the soft EAdS three-point function

$$\lim_{k_2\to 0}\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\hat{\pm}ci\pi}k_3)$$
$$= 2^{\frac{d}{2}-i(\nu_1+\nu_2+\nu_3)}(e^{\hat{\pm}i\pi b}k_2)^{2i\nu_2}\frac{\Gamma(-i\nu_1)\Gamma(-i\nu_2)\Gamma(1-\frac{d}{2}-i\nu_2)}{2^3\Gamma(1+i\nu_2)\Gamma(1+i\nu_3)} \tag{5.2.9}$$
$$\times\left\{\frac{(e^{\pm i\pi a}k_3)^{2i\nu_1}(e^{\hat{\pm}i\pi c}k_3)^{-\frac{d}{2}-i(\nu_1+\nu_2-\nu_3)}\Gamma(\frac{d}{4}+\frac{i}{2}(\nu_1+\nu_2-\nu_3))\Gamma(\frac{d}{4}+\frac{i}{2}(\nu_1+\nu_2+\nu_3))}{\Gamma(1-\frac{d}{4}-\frac{i}{2}(-\nu_1+\nu_2+\nu_3))\Gamma(1-\frac{d}{4}+\frac{i}{2}(\nu_1-\nu_2+\nu_3))}\right.$$
$$\left.-\frac{(e^{\hat{\pm}i\pi c}k_3)^{-\frac{d}{2}+i(\nu_1-\nu_2+\nu_3)}\Gamma(\frac{d}{4}-\frac{i}{2}(\nu_1-\nu_2+\nu_3))\Gamma(\frac{d}{4}-\frac{i}{2}(\nu_1-\nu_2-\nu_3))}{\Gamma(1-\frac{d}{4}-\frac{i}{2}(\nu_1+\nu_2+\nu_3))\Gamma(1-\frac{d}{4}-\frac{i}{2}(\nu_1+\nu_2-\nu_3))}\right\}.$$

To obtain the building block for a massless inflationary two-point function, we first multiply by a factor of $\prod_{j=1}^{3}c^{\text{dS-AdS}}_{\Delta_j}$ before setting $\nu_1=\nu_2=\nu_3=i\left(\frac{d}{2}-\epsilon\right)$, and expanding around $\epsilon=0$. The leading term in ϵ then gives us our final result for the "inflationary" EAdS two-point function,

$$\boxed{\begin{aligned}&\prod_{j=1}^{3}\left(c^{\text{dS-AdS}}_{\Delta_j}\right)\lim_{k_2\to 0}\mathcal{A}^{\text{AdS}}_{\epsilon\epsilon\epsilon}(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\hat{\pm}ci\pi}k_3)\\ &\quad\sim-\frac{4^d\csc\left(\frac{\pi d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)^4}{\epsilon\, d^5\,\pi^2(e^{\pm i\pi a}k_3)^d(e^{\hat{\pm}i\pi b}k_2)^d}-\frac{4^d\csc\left(\frac{\pi d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)^4}{\epsilon\, d^5\,\pi^2(e^{\hat{\pm}i\pi c}k_3)^d(e^{\hat{\pm}i\pi b}k_2)^d}.\end{aligned}} \tag{5.2.10}$$

We will later plug this object into the de Sitter three-point contact diagram (4.2.18) to find the leading correction to the inflationary two-point function for a generic choice of α-vacuum.

5.2.2 Four-point Exchange Diagram

Here we follow a similar procedure to the above, but for EAdS four-point exchange diagrams. In particular, we aim to compute an inflationary master formula for an EAdS exchange with one soft leg (which we choose to be $k_4 \to 0$), in the squeezed limit $k_3 \to 0$. By plugging this object into the de Sitter exchange described in section 4.3 (and available in the GitHub file at [87]), we then obtain the inflationary three-point function of inflaton perturbations $\delta\phi$ in the squeezed limit. We begin by computing a master formula for the s-channel exchange.

s-channel In particular, we are interested in computing the master formula

$$\begin{aligned}
&\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) \\
&= \int [\mathrm{d}s]_4 [\mathrm{d}u]_2\, \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(u_1, u_2, s_i | e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4), \quad (5.2.11)
\end{aligned}$$

where in the s-channel we define the Mellin representation of the four-point exchange as[11]

$$\begin{aligned}
\mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}&(u_1, u_2, s_i | e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) := G^{\text{AdS}}_{\Delta_+}(u_1, k; u_2, k) \\
&\times K^{\text{AdS}}_{\Delta_1}(s_1, e^{\pm ai\pi}k_1) K^{\text{AdS}}_{\Delta_2}(s_2, e^{\hat{\pm} bi\pi}k_2) K^{\text{AdS}}_{\Delta_3}(s_3, e^{\tilde{\pm} ci\pi}k_3) K^{\text{AdS}}_{\Delta_4}(s_4, e^{\bar{\pm} fi\pi}k_4) \\
&\times (2\pi i)\delta\left(\frac{d}{4} - s_1 - s_2 - u_1\right)(2\pi i)\delta\left(\frac{d}{4} - s_3 - s_4 - u_2\right). \quad (5.2.12)
\end{aligned}$$

Similarly to the three-point contact Witten diagram (5.2.1), the parameters $a, b, c, f \in \{0, 1\}$ describe all possible analytic continuations of the external legs. It is in this sense that (5.2.11) is a "master formula", and we will see that the Mellin integrals are defined for all analytic continuations appearing in the four-point de Sitter exchange.

[11] Note that again we have implicitly performed the integrals over the bulk points z_1 and z_2, resulting in the delta functions.

Plugging in the Mellin representations of the EAdS bulk-to-bulk and bulk-to-boundary propagators (3.3.2) and (3.3.3), one can write this as

$$\mathcal{A}^{\mathrm{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) = \frac{1}{4\pi}e^{\mp a\pi\nu_1}e^{\hat{\mp} b\pi\nu_2}e^{\tilde{\mp} c\pi\nu_3}e^{\bar{\mp} f\pi\nu_4}$$
$$\times \prod_{j=1}^{4}\left(\frac{1}{2\Gamma(1+i\nu_j)}\right)\int_{-i\infty}^{i\infty}[\mathrm{d}s]_3[\mathrm{d}u]_2 \csc(\pi(u_1+u_2))\omega_{\Delta_+}(u_1,u_2)\Gamma\left(u_1+\frac{i\nu}{2}\right)\Gamma\left(u_1-\frac{i\nu}{2}\right)$$
$$\times \Gamma\left(u_2+\frac{i\nu}{2}\right)\Gamma\left(u_2-\frac{i\nu}{2}\right)\left(\frac{k}{2}\right)^{-2(u_1+u_2)} e^{\mp 2ai\pi s_1}e^{\hat{\mp}2bi\pi s_2}e^{\tilde{\mp}2ci\pi s_3} \tag{5.2.13}$$
$$\times \prod_{j=1}^{3}\left\{\Gamma\left(s_j+\frac{i\nu_j}{2}\right)\Gamma\left(s_j-\frac{i\nu_j}{2}\right)\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j}\right\}(2\pi i)\delta\left(u_1-\left(\frac{d}{4}-s_1-s_2\right)\right)$$
$$\times \int_{-i\infty}^{i\infty}\frac{\mathrm{d}s_4}{2\pi i}\Gamma\left(s_4+\frac{i\nu_4}{2}\right)\Gamma\left(s_4-\frac{i\nu_4}{2}\right)e^{\bar{\mp}2fi\pi s_4}\left(\frac{k_4}{2}\right)^{-2s_4+i\nu_4}(2\pi i)\delta\left(u_2-\left(\frac{d}{4}-s_3-s_4\right)\right).$$

Recall that when we compute inflationary correlators from correlators in de Sitter, we are interested in the soft limit with respect to one of the external legs. We will choose to consider the $k_4 \to 0$ limit, for which we need to perform the s_4 integral, which has poles at

$$s_4 = \pm\frac{i\nu_4}{2} - n. \tag{5.2.14}$$

It turns out that the leading pole in the soft k_4 limit is the $s_4 = -\frac{i\nu_4}{2}$ pole, with all others subleading. The corresponding residue is given by

$$\Gamma(-i\nu_4)\, e^{\bar{\mp} f\pi\nu_4}\left(\frac{k_4}{2}\right)^{2i\nu_4}\delta\left(u_2-\left(\frac{d}{4}-s_3+\frac{i\nu_4}{2}\right)\right). \tag{5.2.15}$$

Invoking the residue theorem, we can therefore write

$$\lim_{k_4\to 0}\mathcal{A}^{\mathrm{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) = \frac{\Gamma(-i\nu_4)}{4\pi}e^{\mp a\pi\nu_1}e^{\hat{\mp} b\pi\nu_2}e^{\tilde{\mp} c\pi\nu_3}e^{\bar{\mp}2f\pi\nu_4}$$
$$\times \left(\frac{k_4}{2}\right)^{2i\nu_4}\prod_{j=1}^{4}\left(\frac{1}{2\Gamma(1+i\nu_j)}\right)\int_{-i\infty}^{i\infty}[\mathrm{d}s]_3[\mathrm{d}u]_2\csc(\pi(u_1+u_2))\omega_{\Delta_+}(u_1,u_2)\Gamma\left(u_1+\frac{i\nu}{2}\right)$$
$$\times \Gamma\left(u_1-\frac{i\nu}{2}\right)\Gamma\left(u_2+\frac{i\nu}{2}\right)\Gamma\left(u_2-\frac{i\nu}{2}\right)\left(\frac{k}{2}\right)^{-2(u_1+u_2)} e^{\mp 2ai\pi s_1}e^{\hat{\mp}2bi\pi s_2}e^{\tilde{\mp}2ci\pi s_3}$$
$$\times \prod_{j=1}^{3}\left\{\Gamma\left(s_j+\frac{i\nu_j}{2}\right)\Gamma\left(s_j-\frac{i\nu_j}{2}\right)\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j}\right\}(2\pi i)\delta\left(u_1-\left(\frac{d}{4}-s_1-s_2\right)\right)$$
$$\times (2\pi i)\delta\left(u_2-\left(\frac{d}{4}+\frac{i\nu_4}{2}-s_3\right)\right). \tag{5.2.16}$$

We now use the delta functions to eliminate the u_1 and u_2 integrals, obtaining

$$
\begin{aligned}
&\lim_{k_4\to 0} \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) = e^{\mp a\pi\nu_1}e^{\hat{\mp} b\pi\nu_2}e^{\tilde{\mp} c\pi\nu_3}e^{\bar{\mp}2f\pi\nu_4} \\
&\times \frac{\Gamma(-i\nu_4)}{4\pi}\left(\frac{k_4}{2}\right)^{2i\nu_4}\prod_{j=1}^{4}\left(\frac{1}{2\Gamma(1+i\nu_j)}\right)\int_{-i\infty}^{i\infty}[\mathrm{d}s]_3 \csc\left(\pi\left(\frac{d}{2}+\frac{i\nu_4}{2}-s_1-s_2-s_3\right)\right) \\
&\times \omega_{\Delta_+}\left(\frac{d}{4}-s_1-s_2, \frac{d}{4}+\frac{i\nu_4}{2}-s_3\right)\Gamma\left(\frac{d}{4}+\frac{i\nu}{2}-s_1-s_2\right)\Gamma\left(\frac{d}{4}-\frac{i\nu}{2}-s_1-s_2\right) \\
&\times \Gamma\left(\frac{d}{4}+\frac{i\nu_4}{2}+\frac{i\nu}{2}-s_3\right)\Gamma\left(\frac{d}{4}+\frac{i\nu_4}{2}-\frac{i\nu}{2}-s_3\right)\left(\frac{k_3}{2}\right)^{-2(\frac{d}{2}+\frac{i\nu_4}{2}-s_1-s_2-s_3)} \\
&\times e^{\mp 2ai\pi s_1}e^{\hat{\mp} 2bi\pi s_2}e^{\tilde{\mp} 2ci\pi s_3}\prod_{j=1}^{3}\left\{\Gamma\left(s_j+\frac{i\nu_j}{2}\right)\Gamma\left(s_j-\frac{i\nu_j}{2}\right)\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j}\right\}, \qquad (5.2.17)
\end{aligned}
$$

where we used that in the soft $k_4 \to 0$ limit, $k \sim k_3$.

From here, we note that we are particularly interested in the *squeezed* limit of the inflationary three-point function, which we obtain from the $k_3 \to 0$ limit of the above. In said limit, $k_2 \sim k_1$, and it will also prove useful to redefine $s_1 \to s_1 - s_2$. After some algebra and plugging in the projector (3.2.31), this results in

$$
\begin{aligned}
&\lim_{k_3,k_4\to 0} \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2|\Delta_+|\Delta_3\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\tilde{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) = \frac{\Gamma(-i\nu_4)}{2\pi}\prod_{j=1}^{4}\left(\frac{1}{2\Gamma(1+i\nu_j)}\right) \\
&\times \left(\frac{k_4}{2}\right)^{2i\nu_4}\left(\frac{k_1}{2}\right)^{i(\nu_1+\nu_2)}\left(\frac{k_3}{2}\right)^{-d+i(\nu_3-\nu_4)} e^{\mp a\pi\nu_1}e^{\hat{\mp} b\pi\nu_2}e^{\tilde{\mp} c\pi\nu_3}e^{\bar{\mp}2f\pi\nu_4}\,\mathcal{I}_{s_1}\,\mathcal{I}_{s_3}\,\mathcal{I}_{s_2}, \qquad (5.2.18)
\end{aligned}
$$

where we define

$$
\begin{aligned}
\mathcal{I}_{s_2} := \int_{-i\infty}^{i\infty}\frac{\mathrm{d}s_2}{2\pi i}\,&\Gamma\left(s_2+\frac{i\nu_2}{2}\right)\Gamma\left(s_2-\frac{i\nu_2}{2}\right)\Gamma\left(\frac{i\nu_1}{2}+s_1-s_2\right)\Gamma\left(-\frac{i\nu_1}{2}+s_1-s_2\right) \\
&\times e^{2i\pi s_2(\pm a\hat{\mp} b)}, \qquad (5.2.19)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{s_3} := \int_{-i\infty}^{i\infty}\frac{\mathrm{d}s_3}{2\pi i}\,&\sin\left(\pi\left(\frac{d}{4}+\frac{i\nu_4}{2}-\frac{i\nu}{2}-s_3\right)\right)\csc\left(\pi\left(\frac{d+i\nu_4}{2}-s_3-s_1\right)\right)e^{\tilde{\mp}2ci\pi s_3} \\
&\times \prod_{\pm}\Gamma\left(s_3\pm\frac{i\nu_3}{2}\right)\Gamma\left(\frac{d}{4}+\frac{i\nu_4}{2}\pm\frac{i\nu}{2}-s_3\right), \qquad (5.2.20)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{s_1} := \int_{-i\infty}^{i\infty}\frac{\mathrm{d}s_1}{2\pi i}\,&\sin\left(\pi\left(\frac{d}{4}-\frac{i\nu}{2}-s_1\right)\right)\Gamma\left(\frac{d}{4}+\frac{i\nu}{2}-s_1\right)\Gamma\left(\frac{d}{4}-\frac{i\nu}{2}-s_1\right) \\
&\times \left(\frac{k_3}{k_1}\right)^{2s_1} e^{\mp 2ai\pi s_1}. \qquad (5.2.21)
\end{aligned}
$$

We then compute these integrals with the residue theorem, being careful in our choices of contour such that each integral converges. The easiest integral is that over s_2, which is a variation of Barnes' lemma. This has poles at

$$s_2 = \pm\frac{i\nu_2}{2} - n, \quad n \in \mathbb{Z}_{\geq 0}, \tag{5.2.22}$$

$$s_2 = \pm\frac{i\nu_1}{2} - s_1 + m, \quad m \in \mathbb{Z}_{\geq 0}, \tag{5.2.23}$$

on the left and the right of the imaginary axis, respectively. Using the techniques in appendix A, it turns out that we can choose to close the contour in either direction for the combinations of $\pm a$ and $\hat{\pm} b$ that appear in the expression for the de Sitter exchange diagram[12]. Closing to the left, resummation of the corresponding residues results in

$$\mathcal{I}_{s_2} = e^{\pm\pi a\nu_2 \hat{\mp}\pi b\nu_2} \frac{\Gamma(1+i\nu_2)\Gamma(-i\nu_2)\Gamma(1-2s_1)\Gamma\left(s_1 - \frac{i(\nu_1-\nu_2)}{2}\right)\Gamma\left(s_1 + \frac{i(\nu_1+\nu_2)}{2}\right)}{\Gamma\left(1 - s_1 + \frac{i(-\nu_1+\nu_2)}{2}\right)\Gamma\left(1 - s_1 + \frac{i(\nu_1+\nu_2)}{2}\right)} + (\nu_2 \to -\nu_2) \tag{5.2.24}$$

We then compute the s_3 integral (5.2.20). Setting $s_3 = Re^{i\theta}$ and again using the techniques in appendix A, the integrand has behaviour

$$R^{\mathrm{Re}(\Delta_4)-2} e^{-2\pi R(|\sin\theta| \tilde{\mp} c\sin\theta)}, \tag{5.2.25}$$

in the $R \to \infty$ limit. Thus, for decay of the integrand we require

$$|\sin\theta| \tilde{\mp} c\sin\theta \overset{!}{\geq} 0, \tag{5.2.26}$$

where in the case that the bound is saturated we also require

$$\mathrm{Re}(\Delta_4) \overset{!}{<} 1. \tag{5.2.27}$$

Note that condition (5.2.27) is satisfied in the case we are interested in, namely for massless particles where $\nu_4 = \frac{id}{2}$. Note also that (5.2.26) is always satisfied for

[12] These are $\left(\pm a, \hat{\pm} b\right) = (0,0)$, $(\pm 1, \pm 1)$, $(\pm 1, 0)$, $(0, \pm 1)$. Note in particular that the combinations $(1,-1)$ and $(-1,1)$ do not appear.

$c \in \{0, 1\}$. We choose to close the contour to the right, enclosing the poles at

$$s_3 = \frac{d + 2i\nu_4 + 2i\nu}{4} + n, \quad n \in \mathbb{Z}_{\geq 0}, \tag{5.2.28}$$

$$s_3 = \frac{d + i\nu_4 - 2s_1}{2} + m, \quad m \in \mathbb{Z}_{\geq 0}, \tag{5.2.29}$$

which gives

$$\begin{aligned}
\mathcal{I}_{s_3} = &\Gamma\left(1 - \frac{d}{4} + \frac{i\nu}{2} + s_1\right)\Gamma\left(\frac{d}{4} - \frac{i\nu}{2} - s_1\right) \\
&\times \frac{e^{\tilde{\mp} i\pi c\left(\frac{d}{2} + i(\nu+\nu_4)\right)}\Gamma\left(\frac{d}{4} + \frac{i(\nu-\nu_3+\nu_4)}{2}\right)\Gamma\left(\frac{d}{4} + \frac{i(\nu+\nu_3+\nu_4)}{2}\right)\Gamma\left(1 - \frac{d}{2} - i\nu_4\right)}{\Gamma\left(1 - \frac{d}{4} + i\nu - \frac{i(\nu-\nu_3+\nu_4)}{2}\right)\Gamma\left(1 - \frac{d}{4} + i\nu - \frac{i(\nu+\nu_3+\nu_4)}{2}\right)} \\
&+ e^{\tilde{\mp} i\pi c(d + i\nu_4 - 2s_1)}\Gamma\left(\frac{(d - i\nu_3 + i\nu_4)}{2} - s_1\right)\Gamma\left(\frac{(d + i\nu_3 + i\nu_4)}{2} - s_1\right)\frac{\Gamma\left(-\frac{d}{4} + \frac{i\nu}{2} + s_1\right)}{\Gamma\left(1 + \frac{d}{4} + \frac{i\nu}{2} - s_1\right)} \\
&\times {}_3F_2\left(1, \tfrac{d}{2} - \tfrac{i(\nu_3-\nu_4)}{2} - s_1, \tfrac{d}{2} + \tfrac{i(\nu_3+\nu_4)}{2} - s_1; \tfrac{d}{4} - \tfrac{i\nu}{2} - s_1 + 1, \tfrac{d}{4} + \tfrac{i\nu}{2} - s_1 + 1; 1\right).
\end{aligned} \tag{5.2.30}$$

To evaluate the remaining integral in s_1, setting $s_1 = Re^{i\theta}$ we need to determine the behaviour of the ${}_3F_2$ hypergeometric function in (5.2.30) as $R \to \infty$. To that end, using a transformation formula[13] for ${}_3F_2$ we have

$$\begin{aligned}
&{}_3F_2\left(1, \tfrac{d}{2} - \tfrac{i(\nu_3-\nu_4)}{2} - s_1, \tfrac{d}{2} + \tfrac{i(\nu_3+\nu_4)}{2} - s_1; \tfrac{d}{4} - \tfrac{i\nu}{2} - s_1 + 1, \tfrac{d}{4} + \tfrac{i\nu}{2} - s_1 + 1; 1\right) \\
&\qquad = \frac{\Gamma\left(-\frac{d}{2} - i\nu_4 + 1\right)\Gamma\left(\frac{d}{4} - s_1 - \frac{i\nu}{2} + 1\right)}{\Gamma\left(-\frac{d}{2} - i\nu_4 + 2\right)\Gamma\left(\frac{d}{4} - s_1 - \frac{i\nu}{2}\right)} \\
&\times \underbrace{{}_3F_2\left(1, -\tfrac{d}{4} + \tfrac{i(\nu-\nu_3-\nu_4)}{2} + 1, -\tfrac{d}{4} + \tfrac{i(\nu+\nu_3-\nu_4)}{2} + 1; \tfrac{d}{4} + \tfrac{i\nu}{2} - s_1 + 1, -\tfrac{d}{2} - i\nu_4 + 2; 1\right)}_{\sim 1} \\
&\qquad \sim \log R,
\end{aligned} \tag{5.2.32}$$

where in the final equality we used the techniques of appendix A to give the behaviour of the ratio of Γ-functions as $R \to \infty$. Using the same techniques one can determine

---

[13] In particular,

$$\begin{aligned}
{}_3F_2(a_1, a_2, a_3; b_1, b_2; 1) = &\frac{\Gamma(b_1)\Gamma(-a_1 - a_2 - a_3 + b_1 + b_2)}{\Gamma(b_1 - a_1)\Gamma(-a_2 - a_3 + b_1 + b_2)} \\
&\times {}_3F_2(a_1, b_2 - a_2, b_2 - a_3; b_2, -a_2 - a_3 + b_1 + b_2; 1).
\end{aligned} \tag{5.2.31}$$

that exponential decay of the s_1 integrand requires

$$|\sin\theta| \mp a\sin\theta \overset{!}{\geq} 0, \tag{5.2.33}$$

$$2|\sin\theta| + (\tilde{\pm}c \pm a)\sin\theta \overset{!}{>} 0, \tag{5.2.34}$$

where in the case that these bounds are saturated, decay of the integrand requires that

$$d \overset{!}{\leq} 3, \tag{5.2.35}$$

and

$$d - \mathrm{Im}(\nu_4) \overset{!}{\leq} \frac{5}{2}, \tag{5.2.36}$$

respectively. In our case of interest when the fourth leg is massless and $\nu_4 = \frac{id}{2}$, the latter condition becomes $d \overset{!}{\leq} 5$. After computing the integral we can then relax this condition by analytic continuation. Note that the left-hand-side of the first condition is at its smallest when $\mp a = -1$, and that of the second condition is at its smallest when $\tilde{\pm}c \pm a = -2$. Both conditions are therefore always satisfied. We choose to close the integral over s_1 to the right of the imaginary axis, where the integrand contains three families of poles;

$$s_1 = \frac{d}{4} \pm \frac{i\nu}{2} + n, \quad n \in \mathbb{Z}_{\geq 0}, \tag{5.2.37}$$

$$s_1 = \frac{1+n}{2}, \quad n \in \mathbb{Z}_{\geq 0}, \tag{5.2.38}$$

$$s_1 = \frac{d \pm i\nu_3 + i\nu_4}{2} + n, \quad n \in \mathbb{Z}_{\geq 0}. \tag{5.2.39}$$

Performing the integral with the residue theorem results in a series expansion in $\frac{k_3}{k_1}$, whose leading terms in the $k_3 \to 0$ limit are encoded in the above poles for which $n = 0$. Our final task to obtain the master formula for the s-channel exchange is to compute the contribution to the s_1 integral from each of the above poles in s_1.

We first compute the contribution from the (5.2.37) poles. After multiplying the result by the $\prod_{i=1}^{4}\left(c^{\text{dS-AdS}}_{\Delta_i}\right) c^{\text{dS-AdS}}_{\Delta_+}$ factor, we then take the inflationary limit, setting $\nu_j = \nu = i\left(\frac{d}{2} - \epsilon\right)$ $\forall j \in \{1,2,3,4\}$ and expanding around $\epsilon = 0$. The leading term in the $\epsilon \to 0$ limit is given by

$$\prod_{i=1}^{4}\left(c_{\Delta_i}^{\text{dS-AdS}}\right) c_{\Delta_+}^{\text{dS-AdS}}\, \mathcal{A}^{\text{AdS}}_{\epsilon\epsilon\epsilon\epsilon}\left(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\tilde{\pm}ci\pi}k_3, e^{\bar{\pm}fi\pi}k_4\right)\Big|_{s_1=\frac{d}{4}\pm\frac{i\nu}{2}+n}$$
$$\sim \frac{2^{3d-4}\csc\left(\frac{\pi d}{2}\right)^4\Gamma\left(\frac{d}{2}\right)^4 e^{-i\pi d(\pm a\hat{\pm}b\tilde{\pm}c\bar{\pm}f)}(e^{\pm i\pi ad}+e^{\hat{\pm}i\pi bd})(1+e^{\tilde{\pm}i\pi cd})}{d^4\pi\epsilon^2\Gamma\left(-\frac{d}{2}\right)^2(k_1k_3k_4)^d}. \quad (5.2.40)$$

Computing the contribution from the (5.2.39) poles and performing the above procedure results in

$$\prod_{i=1}^{4}\left(c_{\Delta_i}^{\text{dS-AdS}}\right) c_{\Delta_+}^{\text{dS-AdS}}\, \mathcal{A}^{\text{AdS}}_{\epsilon\epsilon\epsilon\epsilon}\left(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\tilde{\pm}ci\pi}k_3, e^{\bar{\pm}fi\pi}k_4\right)\Big|_{s_1=\frac{d\pm i\nu_3+i\nu_4}{2}+n}$$
$$\sim -\frac{2^{3d-4}\csc\left(\frac{\pi d}{2}\right)^4\Gamma\left(\frac{d}{2}\right)^4 e^{-i\pi d(\pm a\hat{\pm}b\tilde{\pm}c\bar{\pm}f)}(e^{\pm i\pi ad}+e^{\hat{\pm}i\pi bd})}{d^4\pi\epsilon^2\Gamma\left(-\frac{d}{2}\right)^2(k_1k_3k_4)^d}. \quad (5.2.41)$$

Finally, computing the contribution from the leading (5.2.38) pole and performing the above procedure results in

$$\prod_{i=1}^{4}\left(c_{\Delta_i}^{\text{dS-AdS}}\right) c_{\Delta_+}^{\text{dS-AdS}}\, \mathcal{A}^{\text{AdS}}_{\epsilon\epsilon\epsilon\epsilon}\left(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\tilde{\pm}ci\pi}k_3, e^{\bar{\pm}fi\pi}k_4\right)\Big|_{s_1=\frac{1+n}{2}}$$
$$\sim -\frac{4^{d-3}\csc\left(\frac{\pi d}{2}\right)^2\Gamma\left(\frac{d}{2}\right)^4\Gamma(d-1)e^{\mp i\pi a-i\pi d(\pm a\hat{\pm}b\bar{\pm}f)}(e^{\pm i\pi ad}-e^{\hat{\pm}i\pi bd})}{d\pi^2\,\epsilon\, k_1^{d+1}k_3^{d-1}k_4^d}. \quad (5.2.42)$$

We see that (5.2.42) is subleading in both ϵ and in $\frac{k_3}{k_1}$, and so can be ignored. Summing (5.2.40) and (5.2.41) gives us our final result for the "inflationary" EAdS exchange in the s-channel,

$$\boxed{\begin{aligned}&\prod_{i=1}^{4}\left(c_{\Delta_i}^{\text{dS-AdS}}\right) c_{\Delta_+}^{\text{dS-AdS}}\, \mathcal{A}^{\text{AdS}}_{\epsilon\epsilon\epsilon\epsilon}\left(e^{\pm ai\pi}k_1, e^{\hat{\pm}bi\pi}k_2, e^{\tilde{\pm}ci\pi}k_3, e^{\bar{\pm}fi\pi}k_4\right)\\ &\qquad\sim \frac{2^{3d-4}\csc\left(\frac{\pi d}{2}\right)^4\Gamma\left(\frac{d}{2}\right)^4 e^{-i\pi d(\pm a\hat{\pm}b\bar{\pm}f)}(e^{\pm i\pi ad}+e^{\hat{\pm}i\pi bd})}{d^4\pi\epsilon^2\Gamma\left(-\frac{d}{2}\right)^2(k_1k_3k_4)^d}.\end{aligned}} \quad (5.2.43)$$

This result is useful because it will later be plugged into the de Sitter exchange described in section 4.3 (and in the GitHub file at [87]) to obtain the squeezed limit of the inflationary three-point function of inflaton perturbations $\delta\phi$.

***t*- and *u*-channel.** It is also important to consider the contributions from the t- and u-channel de Sitter exchanges (see figure 4.4) to the squeezed limit of the inflationary three-point function. As we shall see in the following, these contribute at higher orders in the slow-roll approximation, and so can be neglected if we are only interested in the leading-order correction to the inflationary three-point function. We focus without loss of generality on the t-channel exchange diagram, where the contribution from the u-channel follows by exchanging $k_1 \leftrightarrow k_2$.

As before we employ the decomposition of the dS exchange diagram in a generic α-vacuum in terms of corresponding exchange diagrams in EAdS. For the latter, the soft limit[14] $k_4 \to 0$ is given by:

$$\lim_{k_4\to 0} \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_3|\Delta_+|\Delta_2\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\bar{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4) = \frac{\Gamma(-i\nu_4)}{4\pi} e^{\mp a\pi\nu_1} e^{\hat{\mp} b\pi\nu_2} e^{\bar{\mp} c\pi\nu_3} e^{\bar{\mp} 2f\pi\nu_4}$$
$$\times \left(\frac{k_4}{2}\right)^{2i\nu_4} \prod_{j=1}^{4} \left(\frac{1}{2\Gamma(1+i\nu_j)}\right) \int_{-i\infty}^{i\infty} [\mathrm{d}s]_3 [\mathrm{d}u]_2 \csc(\pi(u_1+u_2))\omega_{\Delta_+}(u_1,u_2)\Gamma\left(u_1+\frac{i\nu}{2}\right)$$
$$\times \Gamma\left(u_1-\frac{i\nu}{2}\right)\Gamma\left(u_2+\frac{i\nu}{2}\right)\Gamma\left(u_2-\frac{i\nu}{2}\right)\left(\frac{k}{2}\right)^{-2(u_1+u_2)} e^{\mp 2ai\pi s_1} e^{\hat{\mp} 2bi\pi s_2} e^{\bar{\mp} 2ci\pi s_3}$$
$$\times \prod_{j=1}^{3}\left\{\Gamma\left(s_j+\frac{i\nu_j}{2}\right)\Gamma\left(s_j-\frac{i\nu_j}{2}\right)\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j}\right\}(2\pi i)\delta\left(u_1-\left(\frac{d}{4}-s_1-s_3\right)\right)$$
$$\times (2\pi i)\delta\left(u_2-\left(\frac{d}{4}+\frac{i\nu_4}{2}-s_2\right)\right). \tag{5.2.44}$$

It is then straightforward to compute the u_1 and u_2 integrals with the delta functions. Computing these and the s_3 integral, in the squeezed limit $k_3 \to 0$ we have

$$\lim_{k_3,k_4\to 0} \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_3|\Delta_+|\Delta_2\Delta_4}(e^{\pm ai\pi}k_1, e^{\hat{\pm} bi\pi}k_2, e^{\bar{\pm} ci\pi}k_3, e^{\bar{\pm} fi\pi}k_4)$$
$$= \pi\frac{\Gamma(-i\nu_4)\Gamma(-i\nu_3)}{2} e^{\mp a\pi\nu_1} e^{\hat{\mp} b\pi\nu_2} e^{\bar{\mp} 2c\pi\nu_3} e^{\bar{\mp} 2f\pi\nu_4}\left(\frac{k_3}{2}\right)^{2i\nu_3}\left(\frac{k_4}{2}\right)^{2i\nu_4}$$
$$\times \left(\frac{k}{2}\right)^{-d+i(\nu_1-\nu_3)+i(\nu_2-\nu_4)} \prod_{j=1}^{4}\left(\frac{1}{2\Gamma(1+i\nu_j)}\right) I^{(t)}_{s_1,s_2}, \tag{5.2.45}$$

with remaining Mellin-Barnes integrals in s_1 and s_2 given by

$$I^{(t)}_{s_1,s_2} = \int_{-i\infty}^{i\infty} [\mathrm{d}s]_2\, e^{\mp 2ai\pi s_1} e^{\hat{\mp} 2bi\pi s_2} \csc(\pi(\tfrac{d}{2} + \tfrac{i(\nu_1+\nu_2)}{2} - s_1 - s_2))$$

[14] Namely, after computing the s_4 integral and taking the leading term in $k_4 \to 0$.

$$\times \frac{\Gamma\left(\frac{d}{4}+\frac{i(\nu+\nu_3)}{2}-s_1\right)\Gamma\left(\frac{d}{4}+\frac{i(\nu+\nu_4)}{2}-s_2\right)}{\Gamma\left(s_1+1-\frac{d}{4}+\frac{i(\nu-\nu_3)}{2}\right)\Gamma\left(s_2+1-\frac{d}{4}+\frac{i(\nu-\nu_4)}{2}\right)}$$
$$\times\Gamma\left(s_1+\frac{i\nu_1}{2}\right)\Gamma\left(s_1-\frac{i\nu_1}{2}\right)\Gamma\left(s_2+\frac{i\nu_2}{2}\right)\Gamma\left(s_2-\frac{i\nu_2}{2}\right), \quad (5.2.46)$$

where we used that $k_1, k_2 \to k$ for $k_3, k_4 \to 0$. Without loss of generality we can evaluate the integral in s_2 first, where the integrand has the following poles for $n \in \mathbb{Z}_{\geq 0}$,

$$s_2 = \pm\frac{i\nu_2}{2} - n, \quad (5.2.47)$$
$$s_2 = \frac{d-2}{2} + \frac{i(\nu_1+\nu_2)}{2} - s_1 - n, \quad (5.2.48)$$
$$s_2 = \frac{d}{2} + \frac{i(\nu_1+\nu_2)}{2} - s_1 + n, \quad (5.2.49)$$
$$s_2 = \frac{d}{2} + \frac{i(\nu+\nu_4)}{2} + n. \quad (5.2.50)$$

For all $a, b \in \{0, 1\}$ the contour can be closed either side assuming that $2\mathrm{Re}[\Delta_3] < 3$. Closing the contour to the right of the imaginary axis, this gives

$$I^{(t)}_{s_1,s_2} = \frac{e^{\mp 2i\pi a s_1}\Gamma\left(s_1-\frac{i\nu_1}{2}\right)\Gamma\left(s_1+\frac{i\nu_1}{2}\right)\Gamma\left(\frac{d}{4}-s_1+\frac{i(\nu+\nu_3)}{2}\right)}{\Gamma\left(-\frac{d}{4}+s_1+\frac{1}{2}i(\nu-\nu_3)+1\right)}$$
$$\times\csc\left(\pi\left(\frac{d}{4}-\frac{i(\nu-\nu_1-\nu_2+\nu_4)}{2}-s_1\right)\right)$$
$$\times\left[e^{\hat{\mp}\frac{1}{2}i\pi b(d+2i(\nu+\nu_4))}\frac{\Gamma\left(-\frac{d}{2}-i\nu_4+1\right)\Gamma\left(\frac{d}{4}+\frac{i(\nu-\nu_2+\nu_4)}{2}\right)\Gamma\left(\frac{d}{4}+\frac{i(\nu+\nu_2+\nu_4)}{2}\right)}{\Gamma\left(1+\frac{i(\nu-\nu_2-\nu_4)}{2}-\frac{d}{4}\right)\Gamma\left(1+\frac{i(\nu+\nu_2-\nu_4)}{2}-\frac{d}{4}\right)}\right.$$
$$-e^{\hat{\mp}i\pi b(d+i(\nu_1+\nu_2)-2s_1)}\Gamma\left(\frac{d+i\nu_1}{2}-s_1\right)\Gamma\left(\frac{d+i(\nu_1+2\nu_2)}{2}-s_1\right)$$
$$\left.\times\,{}_3F_2\left(\begin{matrix}1,\frac{d+i\nu_1}{2}-s_1,\frac{d+i(\nu_1+2\nu_2)}{2}-s_1\\ \frac{d+4-2i(\nu-\nu_1-\nu_2+\nu_4)}{4}-s_1,\frac{d+4+2i(\nu+\nu_1+\nu_2-\nu_4)}{4}-s_1\end{matrix};1\right)\right]. \quad (5.2.51)$$

Since we are interested in the three-point functions of scalar perturbations at leading order in the slow-roll approximation, at this point is it convenient to set $\nu_i = \nu = i\left(\frac{d}{2}-\epsilon\right)$ and consider the leading contribution in ϵ,

$$I^{(t)}_{s_1,s_2} = \frac{2}{\epsilon}e^{\hat{\pm}\frac{i\pi}{2}bd}e^{\mp 2i\pi a s_1}\frac{\Gamma\left(-\frac{d}{2}\right)\csc\left(\pi\left(\frac{d}{4}-s_1\right)\right)\Gamma\left(-\frac{d}{4}-s_1\right)\Gamma\left(s_1-\frac{d}{4}\right)\Gamma\left(s_1+\frac{d}{4}\right)}{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(s_1+1-\frac{d}{4}\right)}$$
$$+O(1). \quad (5.2.52)$$

The poles in s_1 are

$$s_1 = \pm\frac{d}{4} + n, \tag{5.2.53}$$

$$s_1 = \pm\frac{d}{4} - n, \tag{5.2.54}$$

$$s_1 = \frac{d-4}{4} - n. \tag{5.2.55}$$

One can then show that for each possible $(\pm a, \hat{\pm} b)$ the result is given by a simple expression involving harmonic numbers H_m. For instance,

$$I^{(t)}_{s_1,s_2}\Big|_{(\pm a,\hat{\pm} b)=(1,1)} = \frac{8\csc\left(\frac{\pi d}{2}\right)}{d^2\epsilon}\left(e^{i\pi d}H_{\frac{d}{2}} + H_{-\frac{d}{2}}\right), \tag{5.2.56}$$

and similar for all other $(\pm a, \hat{\pm} b) = (\pm 1, \hat{\pm} 1)$.

Notice that this is subleading compared to the s-channel contributions[15] (5.2.43), which are $O\left(\epsilon^{-2}\right)$. Since the contributions from the t-channel, and therefore u-channel, are subleading to the s-channel contributions, and we are only interested in the leading correction to the inflationary three-point function, we ignore them in the following.

5.3 Inflationary Correlators in the α-vacua

Inflationary Two-Point Function. After dividing (4.2.18) by the dS two-point function (4.1.12) for momentum k_2 and plugging in (5.2.10), we obtain the leading correction to the two-point function of the inflaton perturbation $\delta\phi$ for general spacetime dimension $d+1$. We find that

$$\boxed{\langle \delta\phi_{\vec{k}}\, \delta\phi_{-\vec{k}}\rangle^{(\alpha)}_{(\text{Infl.})} \sim \frac{2^d\,\Gamma\left(\frac{d}{2}\right)^2\left(\cosh 2\alpha - \cos\left(\frac{\pi d}{2}+\beta\right)\sinh 2\alpha\right)}{k^d\, d\pi\epsilon},} \tag{5.3.1}$$

[15] One could be concerned that this result is inconsistent with permutation symmetry, given that the de Sitter four-point exchange in the s-, t- and u-channels are related to each other by permutations of the external legs. However, the squeezed limit $k_3 \to 0$ breaks this permutation symmetry and renders only the resulting t- and u-channel diagrams permutations of each other. Therefore, the fact that the t- and u-channel diagrams have a different dependence on ϵ than the s-channel diagram is not inconsistent with permutation symmetry.

where we relabelled $k_3 \to k$. In the Bunch-Davies limit $\alpha = \beta = 0$, this agrees with (3.55) of [75][16]. For $d = 3$ this reduces to the simple form

$$\langle \delta\phi_{\vec{k}}\, \delta\phi_{-\vec{k}} \rangle^{(\alpha)}_{(\text{Infl.})}\Big|_{d=3} \sim \frac{2\left(\cosh 2\alpha - \sin\beta \sinh 2\alpha\right)}{3\, k^3 \epsilon}. \tag{5.3.2}$$

Inflationary Three-Point Function. After plugging (5.2.43) into the de Sitter exchange described in section 4.3, we obtain the leading correction to the squeezed limit of the three-point function of inflaton perturbations $\delta\phi$ for general spacetime dimension $d+1$. We find

$$\boxed{\lim_{k_3 \to 0} \langle \delta\phi_{\vec{k}_1} \delta\phi_{\vec{k}_2} \delta\phi_{\vec{k}_3} \rangle^{(\alpha)}_{(\text{Infl.})} \sim -\frac{2^{2d-3}\Gamma\left(\frac{d}{2}\right)^4 \left(\cosh 2\alpha - \cos\left(\frac{\pi d}{2} + \beta\right) \sinh 2\alpha\right)^2}{(k_1 k_3)^d\, d\pi^2 \epsilon},} \tag{5.3.3}$$

where we have divided by the inflationary two-point function (5.3.1) of the soft leg, k_4. Note that the above inflationary three-point function is proportional to a product of inflationary two-point functions. Namely, we see that

$$\lim_{k_3 \to 0} \langle \delta\phi_{\vec{k}_1} \delta\phi_{\vec{k}_2} \delta\phi_{\vec{k}_3} \rangle^{(\alpha)}_{(\text{Infl.})} = -\frac{d\epsilon}{8} \langle \delta\phi_{\vec{k}_1} \delta\phi_{-\vec{k}_1} \rangle^{(\alpha)}_{(\text{Infl.})} \langle \delta\phi_{\vec{k}_3} \delta\phi_{-\vec{k}_3} \rangle^{(\alpha)}_{(\text{Infl.})}. \tag{5.3.4}$$

A few comments are in order:

- We note that (5.3.3) is not the full inflationary three-point function, but rather the contribution to it from the exchange of an inflaton. To obtain the full three-point function one should also take into account the contribution from the exchange of a graviton.
- We also note the $\frac{1}{\epsilon}$ pole in the two- and three-point functions of $\delta\phi$. While this would be expected of the scalar part of the curvature perturbation ζ [117], we are yet to understand its presence in the correlators of $\delta\phi$.

[16] Or more precisely, it agrees upon setting the spin $\ell = 0$, setting $\nu_1 = \nu_3 = i(\frac{d}{2} - \epsilon)$ and expanding in ϵ.

- The *Maldacena consistency condition* [45] is a relation between the squeezed limit of the inflationary three-point function of the scalar part of the curvature perturbations ζ and a product of inflationary two-point functions, of the form

$$\lim_{k_3\to 0}\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle^{(0)}_{(\text{Infl.})} = -n_s\langle\zeta_{\vec{k}_1}\zeta_{-\vec{k}_1}\rangle^{(0)}_{(\text{Infl.})}\langle\zeta_{\vec{k}_3}\zeta_{-\vec{k}_3}\rangle^{(0)}_{(\text{Infl.})}, \tag{5.3.5}$$

  where the proportionality constant n_s is called the *scalar spectral tilt.* In [73], it was found that the Maldacena consistency condition is not satisfied for vacua away from the Bunch-Davies vacuum. The consistency condition has been shown to arise from the conformal Ward identities [116], so its violation would imply that the α-vacua are inconsistent with conformal symmetry, at odds with recent progress at the four-point level [118]. We note that while our result (5.3.4) is reminiscent of the Maldacena consistency condition, it involves correlators of the inflaton perturbation $\delta\phi$ rather than those of the curvature perturbation ζ. However, the two are in principle related simply by a change of gauge, giving hope that the Maldacena consistency condition could in fact be satisfied for vacua with $\alpha \neq 0$.

Chapter 6

Conclusion

6.1 Summary

In this thesis, based in large part on [1], we developed new techniques for the computation of late-time boundary correlators in de Sitter space for generic choice of α-vacuum. Many of our results extend to the more general superset of states known as the *Bogoliubov initial states*, by allowing the parameters (α, β) to be appropriate functions of the momentum as $(\alpha, \beta) \to (\alpha_k, \beta_k)$, for each mode of momentum k.

In particular, we showed in chapter 3 that the in-in bulk-to-bulk and bulk-to-boundary propagators for arbitrary choice of α-vacuum can be re-expressed in terms of their Bunch-Davies counterparts, via analytic continuation of the bulk time coordinate. We showed that this analytic continuation of the time coordinate can be traded for an analytic continuation of the boundary momentum, which at the level of the correlation functions converts the standard total-energy singularity present in cosmological correlators to the folded singularities characteristic of the α-vacua. By leveraging recent work on the reformulation of Bunch-Davies correlators in terms of Witten diagrams in EAdS, we then showed that α-vacuum propagators can be expressed as combinations of EAdS propagators with analytically continued momenta.

In chapter 4 we studied a number of examples of perturbative correlation functions in the α-vacua, utilising the tools we developed in chapter 3 to express these correlators as EAdS Witten diagrams with analytically continued momenta. We computed the late-time boundary two-point function, as well as a general formula for contact diagrams of scalar fields with any mass in arbitrary spacetime dimension $d+1$, and studied the $d=3$ case in more detail. In $d=3$ we computed the three-point contact diagram of massless scalars, as well as that of one general scalar and two conformally coupled scalars. We also computed the OPE limit of the latter, and commented on the consistency of the α-vacua with this limit. We gave an algorithm for constructing the four-point exchange diagram of scalars with generic mass, for generic spacetime dimension, in terms of EAdS exchanges. The full expression is available in the GitHub file at [87].

Finally, in chapter 5 (based on unpublished work) we extended our results to an inflationary context. In particular, we computed what we call inflationary EAdS *master formulae* - EAdS Witten diagrams with a soft leg and masses related to the slow-roll parameter, with analytically continued boundary momenta. These were then inserted into our results for de Sitter diagrams in terms of their EAdS counterparts to obtain the leading correction to the inflationary two-point function, and the leading correction to the squeezed limit of the inflationary three-point function for any choice of α-vacuum and spacetime dimension.

In particular we showed how the three-point de Sitter contact diagram can be used to derive the correction to the inflationary two-point function of inflaton perturbations $\delta\phi$ at leading order in slow-roll using the above procedure. We also computed the inflationary three-point function in the same way, and found that it is proportional to a product of inflationary two-point functions.

6.2 Directions for Future Work

There are a number of interesting possible directions for future work that we neglected to mention in this thesis.

- **Fields with spin.** While we focused on scalar field theories, the extension to spinning fields should be straightforward. Indeed, the perturbative reformulation of late-time correlators in the Bunch-Davies vacuum in terms of their EAdS counterparts has also been derived for fields of arbitrary integer spin [75–77] and fermions [82].

- **More general initial states.** It would be interesting to use the techniques presented in this work to examine the properties of late-time correlators in Bogoliubov initial states in more detail and to explore whether they might be extended to more general states that are not the Bogoliubov transform of the Bunch-Davies vacuum.

- **Celestial correlators.** Holographic correlation functions on the celestial sphere of Minkowski space have recently been defined by considering the Mellin transform of time-ordered Minkowski correlation functions with respect to the radial direction in the hyperbolic slicing [119, 120]. The effective reduced vacuum on the de Sitter slicing is the Bunch-Davies vacuum and accordingly such celestial correlation functions can be re-written in terms of Witten diagrams in EAdS, recycling the results [76, 77] for the Bunch-Davies vacuum. If one instead considers Lorentz (not Poincaré) invariant vacua, the effective reduced vacua on the de Sitter slicing are α-vacua [121]. It would therefore be interesting to understand whether celestial correlators in Lorentz (not Poincaré) invariant vacua can be studied along similar lines using the results of this work.

- **Holography.** In the context of dS/CFT, the bulk α-vacua correspond to a family of marginal deformations of the putative dual boundary CFT [43, 44]. Additionally, in [44] a definition of holographic CFT correlators was proposed that corresponds to bulk particle scattering from the de Sitter boundary at past infinity, $\mathcal{I}^-$, to the boundary at future infinity, $\mathcal{I}^+$. Late-time de Sitter correlators in the α-vacua contain singularities for collinear momentum configurations, which we showed can be understood as arising from antipodal transformations of the points from the future to the past boundary via equation (3.2.54). In light of our results for late-time de Sitter correlators in the α-vacua featuring points on both boundaries[1], it would be interesting to explore connections to holography along these lines.

[1]See equations (3.2.64) and (3.2.65) for the bulk-to-bulk and bulk-to-boundary propagators, explicitly showing the analytic continuations of the time coordinate $\eta \to \bar{\eta}^{\pm}$.

Appendix A

Mellin-Barnes Integrals

In this work we regularly encounter *Mellin-Barnes integrals*; integrals of the general form

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} g(s), \tag{A.0.1}$$

where the integrand is in general a ratio of products of Gamma functions,

$$g(s) = \frac{\Gamma(a_1 + A_1 s)...\Gamma(a_n + A_n s)\Gamma(b_1 - B_1 s)...\Gamma(b_n - B_n s)}{\Gamma(c_1 + C_1 s)...\Gamma(c_n + C_n s)\Gamma(d_1 - D_1 s)...\Gamma(d_n - D_n s)}, \tag{A.0.2}$$

with $A_i, B_i, C_i, D_i > 0$. As in the standard treatment of Mellin-Barnes integrals, the contour is from $-i\infty$ to $i\infty$ with suitable indentations to separate sequences of poles of type $\Gamma(s + a_i)$ from those of type $\Gamma(-s + b_i)$. This contour prescription is well-defined for parameters a_i and b_i such that poles from Γ-functions of the former type do not collide with poles from those of the latter type. In the context of QFT in de Sitter space, this is always the case for Principal Series representations where $\Delta_i = \frac{d}{2} + i\nu_i$, $\nu_i \in \mathbb{R}$, and such poles are therefore always separated[1]. This is not always the case for the other unitary representations (i.e. the complementary series representations, see section 2.2.2), where ν_i is imaginary, which can lead to pinching of the integration contour. The latter gives rise to singularities which require careful

[1]A simple illustration is an integrand of the form $\Gamma(s + \Delta_i)\Gamma(-s + \Delta_i)$. The two sets of poles overlap if $\Delta_i = -n$, which has no solutions for Principal series representations with $\nu_i \in \mathbb{R}$.

regularisation and in some cases also renormalisation[2].

An example of pole collision that we encounter in this work is the three-point contact diagram (4.2.18) with $\nu_i = \frac{3i}{2}$, which in $d=3$ corresponds to massless scalar fields. Taking for the moment d arbitrary, from the Mellin space expression (4.2.13) we have (following section 3.4 of [74])

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{000}(k_1,k_2,k_3) = &-\frac{2}{k_2^3k_3^3}\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}s_1}{2\pi i}\left(d-4(s_1+1)\right)\left(2s_1-\tfrac{1}{2}\right)\Gamma\left(2s_1-\tfrac{3}{2}\right)\Gamma\left(\tfrac{d}{2}-2s_1-3\right)\\ &\times\left(\frac{1}{2}k_2k_3(d-4s_1-6)+(k_2+k_3)^2\right)(k_2+k_3)^{-\frac{d}{2}+2s_1+1}\,k_1^{-2s_1-\frac{3}{2}}, \qquad \text{(A.0.3)}\end{aligned}$$

where we eliminated the s_3 integral using the Dirac delta function arising from (4.2.12) and the s_2 integral by applying Cauchy's residue theorem. For the remaining s_1 integral, the Γ-function poles are

$$s_1 = \frac{3}{4}-\frac{n}{2}, \qquad n\in\mathbb{Z}_{\geq 0} \qquad \text{(A.0.4)}$$

$$s_1 = \frac{d-6}{4}+\frac{m}{2}, \qquad m\in\mathbb{Z}_{\geq 0}, \qquad \text{(A.0.5)}$$

which are overlapping for $d\leq 9$. Keeping d arbitrary, evaluating the s_1 integral by closing the contour to the right one obtains

$$\begin{aligned}\mathcal{A}^{\text{AdS}}_{000}(k_1,k_2,k_3) = \frac{1}{3k_1^3k_2^3k_3^3}\Big[&2k_1^2(k_2+k_3)+2k_1\left(k_2^2-k_2k_3+k_3^2\right)+2k_2k_3(k_2+k_3)\\ &+\frac{4\left(k_1^3+k_2^3+k_3^3\right)}{d-3}-\frac{1}{3}(6\gamma-11)\left(k_1^3+k_2^3+k_3^3\right)\\ &-\left(k_1^3+k_2^3+k_3^3\right)\left(2\log(k_1+k_2+k_3)+1\right)\Big]. \qquad \text{(A.0.6)}\end{aligned}$$

Note that there is a simple pole for $d=3$ which requires renormalisation. This divergence is local and therefore, in the corresponding dS contact diagram (4.2.23), can be cancelled in the in-in formalism by adding local counterterms at the future boundary of dS [109].

Mellin space is also highly useful for deriving various propagator and correlator identities. For example, equation (3.2.37) is straightforward to derive using the

[2]See [70, 109, 122–125] for authoritative works on regularisation and renormalisation of momentum space correlation functions in CFT.

Mellin space formalism;

$$
\begin{aligned}
G_W^{(0)}(\eta_1,\bar{\eta}_2^+) &\equiv G_{-+}^{(0)}(\eta_1,\bar{\eta}_2^+) \\
&= \int_{-i\infty}^{+i\infty}[du]_2 G_{-+}^{(0)}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-e^{-\pi i}\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-\frac{i\pi d}{2}}\int_{-i\infty}^{+i\infty}[du]_2 e^{2i\pi u_2}G_{-+}^{(0)}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-\frac{i\pi d}{2}}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))(\alpha^+\omega_{\Delta_+}(u_1,u_2)+\beta^-\omega_{\Delta_-}(u_1,u_2))\Gamma(i\nu)\Gamma(-i\nu) \\
&\qquad\qquad \times e^{2i\pi u_2}\Omega_{\nu,\vec{k}}^{-+}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-\frac{i\pi d}{2}}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))(\alpha^+\omega_{\Delta_+}(u_1,u_2)+\beta^-\omega_{\Delta_-}(u_1,u_2))\Gamma(i\nu)\Gamma(-i\nu) \\
&\qquad\qquad \times e^{-\pi\nu}\Omega_{\nu,\vec{k}}^{--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-i\pi\left(\frac{d}{2}-i\nu\right)}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))(\alpha^+\omega_{\Delta_+}(u_1,u_2)+\beta^-\omega_{\Delta_-}(u_1,u_2)) \\
&\qquad\qquad \times \Gamma(i\nu)\Gamma(-i\nu)\Omega_{\nu,\vec{k}}^{--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-i\pi\left(\frac{d}{2}-i\nu\right)}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))\alpha^+\omega_{\Delta_+}(u_1,u_2)\Gamma(i\nu)\Gamma(-i\nu) \\
&\qquad\qquad\qquad \times \Omega_{\nu,\vec{k}}^{--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&\quad + e^{-i\pi\left(\frac{d}{2}-i\nu\right)}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))\beta^-\omega_{\Delta_-}(u_1,u_2))\Gamma(i\nu)\Gamma(-i\nu) \\
&\qquad\qquad\qquad \times \Omega_{\nu,\vec{k}}^{--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-i\pi\left(\frac{d}{2}-i\nu\right)}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))e^{2\pi\nu}\alpha^-\omega_{\Delta_+}(u_1,u_2)\Gamma(i\nu)\Gamma(-i\nu) \\
&\qquad\qquad\qquad \times \Omega_{\nu,\vec{k}}^{--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&\quad + e^{-i\pi\left(\frac{d}{2}-i\nu\right)}\int_{-i\infty}^{+i\infty}[du]_2 \csc(\pi(u_1+u_2))\beta^-\omega_{\Delta_-}(u_1,u_2))\Gamma(i\nu)\Gamma(-i\nu) \\
&\qquad\qquad\qquad \times \Omega_{\nu,\vec{k}}^{--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}} \\
&= e^{-i\pi\left(\frac{d}{2}+i\nu\right)}\int_{-i\infty}^{+i\infty}[du]_2 G_{\Delta_+,--}^{(0)}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}}
\end{aligned}
$$

$$+ e^{-i\pi\left(\frac{d}{2}-i\nu\right)} \int_{-i\infty}^{+i\infty} [du]_2 G^{(0)}_{\Delta_-,--}(u_1,u_2)(-\eta_1)^{-2u_1+\frac{d}{2}}(-\eta_2)^{-2u_2+\frac{d}{2}}, \tag{A.0.7}$$

and so we obtain equation (3.2.37),

$$G^{(0)}_W(\eta_1,\bar{\eta}_2^+) = e^{-i\pi\Delta_+}G^{(0)}_{\Delta_+,\bar{T}}(\eta_1,\eta_2) + e^{-i\pi\Delta_-}G^{(0)}_{\Delta_-,\bar{T}}(\eta_1,\eta_2). \tag{A.0.8}$$

A.1 Convergence of Mellin-Barnes Integrals

Here we review the convergence of Mellin-Barnes integrals, largely following chapter 2.4 of [126]. We then prove a modified version of Barnes' first lemma featuring a phase dependent on the Mellin variable.

We are interested in the behaviour of the integrand of (A.0.1) as $|s| \longrightarrow \infty$. If the integrand exponentially decays in this limit, then the contour can be closed at infinity and the unwanted part of the contour will vanish. We begin by parameterising

$$s = Re^{i\theta}, \tag{A.1.1}$$

and in this parameterisation we are interested in the $|s| = R \longrightarrow \infty$ limit. Using Stirling's approximation,

$$\log(\Gamma(z)) \sim \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi), \quad -\pi < \arg(z) < \pi, \tag{A.1.2}$$

we can look at Gamma functions of the type $\Gamma(\alpha \pm \beta s)$ by writing

$$\log(\Gamma(\alpha \pm \beta Re^{i\theta})) \sim \left(\alpha \pm \beta Re^{i\theta} - \frac{1}{2}\right)\log(\alpha \pm \beta Re^{i\theta}) \mp \beta Re^{i\theta} + (s\text{-independent terms}). \tag{A.1.3}$$

We see that the approximation of $\log(\Gamma(\alpha - \beta Re^{i\theta}))$ contains a term involving $\log(\alpha - \beta Re^{i\theta})$. Using the Euler reflection formula

$$\Gamma(1-z) = \frac{\pi}{\sin(\pi z)\Gamma(z)}, \tag{A.1.4}$$

and Stirling's approximation, we can obtain

$$\begin{aligned}\log(\sin(\pi z)) &= \log(\pi) - \log(\Gamma(z)) - \log(\Gamma(1-z)) - \log(\Gamma(1-z)) \\ &\sim -\left(z - \frac{1}{2}\right)\log(z) + \left(z - \frac{1}{2}\right)\log(1-z) + \mathcal{O}(1) \\ \implies \log(z) &\sim \log(1-z) - \frac{1}{z-\frac{1}{2}}\log(\sin(\pi z)),\end{aligned} \tag{A.1.5}$$

and so

$$\log(\alpha - \beta R e^{i\theta}) \sim \log(1-\alpha+\beta R e^{i\theta}) - \frac{1}{\alpha - \beta R e^{i\theta} - \frac{1}{2}}\log(\sin(\pi(\alpha - \beta R e^{i\theta}))). \tag{A.1.6}$$

Plugging this into the above, we then have

$$\begin{aligned}\log(\Gamma(\alpha - \beta R e^{i\theta})) \sim \left(\alpha - \beta R e^{i\theta} - \frac{1}{2}\right)&\log(1-\alpha+\beta R e^{i\theta}) \\ &- \log(\sin(\pi(\alpha - \beta R e^{i\theta}))) + \beta R e^{i\theta},\end{aligned} \tag{A.1.7}$$

where we have dropped the $\mathcal{O}(1)$ terms. The series expansion of $\log(1-\alpha+\beta R e^{i\theta})$ at $R = \infty$ is given by

$$\log(1-\alpha+\beta R e^{i\theta}) = \log(\beta R e^{i\theta}) + \mathcal{O}\left(\frac{1}{R}\right), \tag{A.1.8}$$

and so we can simplify the above to

$$\begin{aligned}\log(\Gamma(\alpha - \beta R e^{i\theta})) \sim &-\log(\sin(\pi(\alpha - \beta R e^{i\theta}))) \\ &+ \left(\alpha - \beta R e^{i\theta} - \frac{1}{2}\right)\log(\beta R e^{i\theta}) + \beta R e^{i\theta}.\end{aligned} \tag{A.1.9}$$

Recall also that we had

$$\log(\Gamma(\alpha + \beta R e^{i\theta})) \sim \left(\alpha + \beta R e^{i\theta} - \frac{1}{2}\right)\log(\alpha + \beta R e^{i\theta}) - \beta R e^{i\theta}. \tag{A.1.10}$$

We are particularly interested in the behaviour of the *modulus* of the integrand. We therefore have

$$\begin{aligned}\log|\Gamma(\alpha + \beta R e^{i\theta})| &= \mathrm{Re}(\log(\Gamma(\alpha + \beta R e^{i\theta}))) \\ &= \mathrm{Re}\left[\left(\alpha + \beta R e^{i\theta} - \frac{1}{2}\right)\log(\alpha + \beta R e^{i\theta}) - \beta R e^{i\theta}\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Re}\left[\left(\alpha + \beta R e^{i\theta} - \frac{1}{2}\right)\log(\alpha + \beta R e^{i\theta})\right] - \beta R\cos(\theta) \\
&= \mathrm{Re}\left[\alpha\log(\alpha + \beta R e^{i\theta})\right] + \beta R \mathrm{Re}\left[e^{i\theta}\log(\alpha + \beta R e^{i\theta})\right] - \frac{1}{2}\mathrm{Re}\left[\log(\alpha + \beta R e^{i\theta})\right] \\
&\quad - \beta R\cos(\theta) \\
&\sim \mathrm{Re}\left[\alpha\log(\beta R e^{i\theta})\right] + \beta R \mathrm{Re}\left[e^{i\theta}\log(\beta R e^{i\theta})\right] - \frac{1}{2}\mathrm{Re}\left[\log(\beta R e^{i\theta})\right] - \beta R\cos(\theta).
\end{aligned}
$$

From here we note

$$
\begin{aligned}
\mathrm{Re}\left[\alpha\log(\beta R e^{i\theta})\right] &= \mathrm{Re}\left[\alpha\log|\beta R e^{i\theta}| + i\arg(\beta R e^{i\theta})\right] \\
&= \mathrm{Re}\left[\alpha\log|\beta R e^{i\theta}|\right] \\
&= \mathrm{Re}\left[\alpha\log(\beta R)\right] \\
&= \mathrm{Re}\left[\alpha\right]\mathrm{Re}\left[\log(\beta R)\right] - \mathrm{Im}\left[\alpha\right]\underbrace{\mathrm{Im}\left[\log(\beta R)\right]}_{0}, \qquad \text{(A.1.11)}
\end{aligned}
$$

and a similar procedure for the other terms. We then find

$$
\begin{aligned}
\log|\Gamma(\alpha + \beta R e^{i\theta})| \sim \left(\mathrm{Re}(\alpha) - \frac{1}{2}\right)\log(\beta R) - \beta R(\theta\sin(\theta) + \cos(\theta)) \\
+ \beta R\cos(\theta)\log(\beta R), \qquad \text{(A.1.12)}
\end{aligned}
$$

where again we have dropped all $\mathcal{O}(1)$ terms. Following exactly the same procedure we can also obtain

$$
\begin{aligned}
\log|\Gamma(\alpha - \beta R e^{i\theta})| \sim &- \beta R\cos(\theta)\log(\beta R) + \beta R(\theta\sin(\theta) + \cos(\theta)) \\
&+ \left(\mathrm{Re}(\alpha) - \frac{1}{2}\right)\log(\beta R) - \log|\sin(\pi(\alpha - \beta R e^{i\theta}))|.
\end{aligned}
$$

The $\log|\sin(\pi(\alpha - \beta R e^{i\theta}))|$ can be manipulated further for large R - in particular

$$
\begin{aligned}
\log|\sin(\pi(\alpha - \beta R e^{i\theta}))| &=: \mathrm{Re}\left[\log\sin(\pi(\alpha - \beta R e^{i\theta}))\right] \\
&= \mathrm{Re}\left[\log\left(\frac{1}{2i}\left(e^{i\pi(\alpha - \beta R e^{i\theta})} - e^{-i\pi(\alpha - \beta R e^{i\theta})}\right)\right)\right] \\
&= \mathrm{Re}\left[\log\left(\frac{1}{2i}\left(e^{i\pi(\alpha - \beta R(\cos\theta + i\sin\theta))} - e^{-i\pi(\alpha - \beta R(\cos\theta + i\sin\theta))}\right)\right)\right].
\end{aligned}
\qquad \text{(A.1.13)}
$$

The oscillatory pieces contribute nothing to convergence in the $R \longrightarrow \infty$ limit, and so we see that

$$\log|\sin(\pi(\alpha-\beta Re^{i\theta}))| \sim \begin{cases} \mathrm{Re}\left[\log\left(\frac{1}{2i}e^{\pi\beta R\sin(\theta)}\right)\right] & \text{for } \sin(\theta)>0 \\ \mathrm{Re}\left[\log\left(\frac{1}{2i}e^{\pi\beta R(-\sin(\theta))}\right)\right] & \text{for } \sin(\theta)<0. \end{cases} \tag{A.1.14}$$

Namely,

$$\log|\sin(\pi(\alpha-\beta Re^{i\theta}))| \sim \mathrm{Re}\left[\log\left(\frac{1}{2i}e^{\pi\beta R|\sin(\theta)|}\right)\right]. \tag{A.1.15}$$

Continuing, we therefore have

$$\begin{aligned} \log|\sin(\pi(\alpha-\beta Re^{i\theta}))| &\sim \log\left|\frac{1}{2i}e^{\pi\beta R|\sin(\theta)|}\right| \\ &= \log\left(\sqrt{\frac{1}{4}e^{2\pi\beta R|\sin(\theta)|}}\right) \\ &= \log\left|\frac{1}{2}e^{\pi\beta R|\sin(\theta)|}\right| \\ &\sim \log\left|e^{\pi\beta R|\sin(\theta)|}\right| \\ &\sim \pi\beta R|\sin(\theta)|, \end{aligned} \tag{A.1.16}$$

remembering at every step that $R \longrightarrow \infty$. So finally, we have

$$\begin{aligned} \log|\Gamma(\alpha-\beta Re^{i\theta})| \sim &-\beta R\cos(\theta)\log(\beta R)+\beta R(\theta\sin(\theta)+\cos(\theta)) \\ &+\left(\mathrm{Re}(\alpha)-\frac{1}{2}\right)\log(\beta R)-\pi\beta R|\sin(\theta)||. \end{aligned} \tag{A.1.17}$$

Barnes' First Lemma with a Phase. Barnes' first lemma is given by

$$\begin{aligned} &\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) \\ &\qquad = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}. \end{aligned} \tag{A.1.18}$$

This can be easily verified in *Mathematica*, and it can also be checked that it holds regardless of the direction in which we close the contour. This can be seen from the above formulae, where we find (defining $g(s)$ to be the integrand)

$$\begin{aligned} \log|g(s)| &= \mathrm{Re}(\log(g(s))) \\ &= \log|\Gamma(a+s)|+\log|\Gamma(b+s))|+\log|\Gamma(c-s))|+\log|\Gamma(c-s))|\,. \end{aligned} \tag{A.1.19}$$

Using the formulae (A.1.12) and (A.1.17), we see that

$$\log|g(s)| = -2\pi R|\sin(\theta)| + \log(R)[\mathrm{Re}(a+b+c+d)-2], \tag{A.1.20}$$

as $R \longrightarrow \infty$ where $s = Re^{i\theta}$. Thus, the integrand exponentially decays regardless of the direction we close the contour;

$$\begin{aligned}|g(s)| &\sim e^{-2\pi R|\sin(\theta)|}e^{\log(R)[\mathrm{Re}(a+b+c+d)-2]} \\ &= e^{-2\pi R|\sin(\theta)|}R^{\mathrm{Re}(a+b+c+d)-2}.\end{aligned} \tag{A.1.21}$$

In this work, a modified version of the above integral appears in the computation of the inflationary three-point function, which is similar to that which appears in Barnes' lemma but with an additional phase factor. Namely, (5.2.19) is an integral of the form

$$I_2^{\pm} = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)e^{\pm 2i\pi s}, \tag{A.1.22}$$

where we will define

$$f_{\pm}^{(2)}(s) := \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)e^{\pm 2i\pi s}. \tag{A.1.23}$$

We now follow the above procedure to determine in which direction we should close the contour. We have

$$\begin{aligned}\log|f_{\pm}^{(2)}(s)| &= \mathrm{Re}(\log(f_{\pm}(s))) \qquad \text{(A.1.24)} \\ &= \log|\Gamma(a+s)| + \log|\Gamma(b+s))| + \log|\Gamma(c-s))| + \log|\Gamma(c-s))| + \log|e^{\pm 2i\pi s}|.\end{aligned}$$

The $\log|e^{\pm 2\pi i s_2}|$ is given by

$$\begin{aligned}\log\left|e^{\pm 2\pi i s_2}\right| &= \log\sqrt{e^{\pm 2\pi i s_2}e^{\mp 2\pi i s_2^*}} \\ &= \log\left(e^{\pm i\pi(s_2 - s_2^*)}\right) \\ &= \pm i\pi(s_2 - s_2^*) \\ &= \mp 2\pi R\sin(\theta),\end{aligned} \tag{A.1.25}$$

where in the last line we set $s_2 = Re^{i\theta}$. This then gives

$$\begin{aligned}\log|f_{\pm}^{(2)}(s)| &= \mathrm{Re}(\log(f_{\pm}(s))) \\ &= \log|\Gamma(a+s)| + \log|\Gamma(b+s))| + \log|\Gamma(c-s))| \\ &\quad + \log|\Gamma(c-s))| \mp 2\pi R\sin(\theta). \qquad \text{(A.1.26)}\end{aligned}$$

Plugging in (A.1.12) & (A.1.17) we find

$$\log|f_{\pm}^{(2)}(s)| \sim -2\pi R|\sin(\theta)| + \log(R)[\mathrm{Re}(a+b+c+d)-2] \mp 2\pi R\sin(\theta). \quad \text{(A.1.27)}$$

Therefore, for each of $f_{\pm}^{(2)}$ the modulus of the integrand is given by

$$|f_{+}^{(2)}(s)| \sim e^{-2\pi R|\sin(\theta)|} R^{\mathrm{Re}(a+b+c+d)-2} e^{-2\pi R\sin(\theta)}, \qquad \text{(A.1.28)}$$

and

$$|f_{-}^{(2)}(s)| \sim e^{-2\pi R|\sin(\theta)|} R^{\mathrm{Re}(a+b+c+d)-2} e^{+2\pi R\sin(\theta)}. \qquad \text{(A.1.29)}$$

We now consider the behaviour of each integrand separately as we close the contour to the right and to the left.

First consider $f_{+}^{(2)}(s)$. For $\sin(\theta) > 0$, we see that $|f_{+}^{(2)}(s)|$ exponentially decays and the unwanted part of the contour will vanish for $R \longrightarrow \infty$. For $\sin(\theta) < 0$, the exponential factors will cancel and the asymptotic behaviour is given by

$$|f_{-}^{(2)}(s)| \sim R^{\mathrm{Re}(a+b+c+d)-2}. \qquad \text{(A.1.30)}$$

This decays as $R \longrightarrow \infty$ if we insist on the restriction $\mathrm{Re}(a+b+c+d) \leq 1$. This condition can then be relaxed by analytic continuation. We therefore find that I_2^{+} can be computed by closing the contour to either side.

Now consider $f_{-}^{(2)}(s)$. For $\sin(\theta) > 0$, we see that the exponential factors cancel and we have the same situation as for $\sin(\theta) < 0$ in the $|f_{+}^{(2)}(s)|$ case. I.e, for the integrand to decay and the unwanted part of the contour to vanish we again require $\mathrm{Re}(a+b+c+d) \overset{!}{\leq} 1$. For $\sin(\theta) < 0$, the modulus of the exponentially decays and the unwanted part of the contour will vanish for $R \longrightarrow \infty$. We therefore find that

I_2^- can be computed by closing the contour to either side, provided we insist on the restriction $\text{Re}(a+b+c+d) \leq 1$.

With the above contour prescription these two integrals are then easily computed in *Mathematica*, and are indeed seen to be a modification of Barnes' First Lemma,

$$\begin{aligned} I_2^{\pm} = \pm\frac{1}{2i}&\left(e^{\pm i\pi(-a+b+c+d)} + e^{\pm i\pi(a-b+c+d)} - e^{\mp i\pi(a+b-c+d)} - e^{\mp i\pi(a+b+c-d)}\right) \\ &\times \csc(\pi(a+b+c+d))\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}. \end{aligned} \tag{A.1.31}$$

Appendix B

Trivialisation of η integrals

Here we prove[1] the simple identity;

$$\int_{-\infty}^{0} d\eta(-\eta)^{s-1} = 2\pi i\delta(s), \tag{B.0.1}$$

for $s \in \mathbb{C}$ which can be trivially extended to the cases that appear in this work. This integral appears in the calculation of correlation functions in both dS and EAdS when we go to Mellin space in the boundary directions, trivialising the integrals over the radial coordinate and producing a Dirac delta function that constrains the Mellin variables. Here we give the proof in the setting of dS and an integral over the time coordinate η, but the analogous identity can be proven for EAdS and the radial coordinate z.

We begin by defining

$$\phi(s) := \int_{-\infty}^{0} d\eta(-\eta)^{s-1}, \tag{B.0.2}$$

and integrating this against a test function $f : \mathbb{C} \to \mathbb{C}$, which we take to be holomorphic (ie, without poles) everywhere. Namely, we aim to evaluate

$$I[f] := \int_{\mathcal{C}} \frac{ds}{2\pi i} f(s)\phi(s), \tag{B.0.3}$$

where $\mathcal{C}$ is some arbitrary contour that we will find to be restricted. We first send

[1] I am grateful to Rudolfs Treilis for bringing this elementary proof to my attention.

$\eta \to -\eta$, and then split up the integral $\phi(s)$ using a regulator a;

$$\begin{aligned} I[f] &= \int_{\mathcal{C}} \frac{ds}{2\pi i} f(s) \int_{-\infty}^{0} d\eta (-\eta)^{s-1} \\ &= \int_{\mathcal{C}} \frac{ds}{2\pi i} f(s) \int_{0}^{\infty} d\eta\, \eta^{s-1} \\ &= \int_{\mathcal{C}} \frac{ds}{2\pi i} f(s) \left(\int_{0}^{a} d\eta\, \eta^{s-1} + \int_{a}^{\infty} d\eta\, \eta^{s-1} \right). \end{aligned} \tag{B.0.4}$$

Considering the first η integral, we have

$$\int_{0}^{a} d\eta\, \eta^{s-1} = \frac{a^s}{s}, \; \mathrm{Re}(s) \overset{!}{>} 0, \tag{B.0.5}$$

where the requirement on s comes from insisting on convergence of the integral. For the second η integral, we have

$$\int_{a}^{\infty} d\eta\, \eta^{s-1} = -\frac{a^s}{s}, \; \mathrm{Re}(s) \overset{!}{<} 0, \tag{B.0.6}$$

where again the requirement on s comes from insisting on convergence of the integral. These requirements on s then restrict the contour $\mathcal{C}$ for each s integral;

$$I[f] = \int_{\mathcal{C}_1, \mathrm{Re}(s)<0} \frac{ds}{2\pi i} f(s) \frac{a^s}{s} - \int_{\mathcal{C}_2, \mathrm{Re}(s)>0} \frac{ds}{2\pi i} f(s) \frac{a^s}{s}, \tag{B.0.7}$$

where both integrands have a pole at $s = 0$ and $\mathcal{C}_1$ and $\mathcal{C}_2$ are indicated in figure B.1. These contours can then be deformed into a single contour that encloses the pole at $s = 0$; see figure B.1.

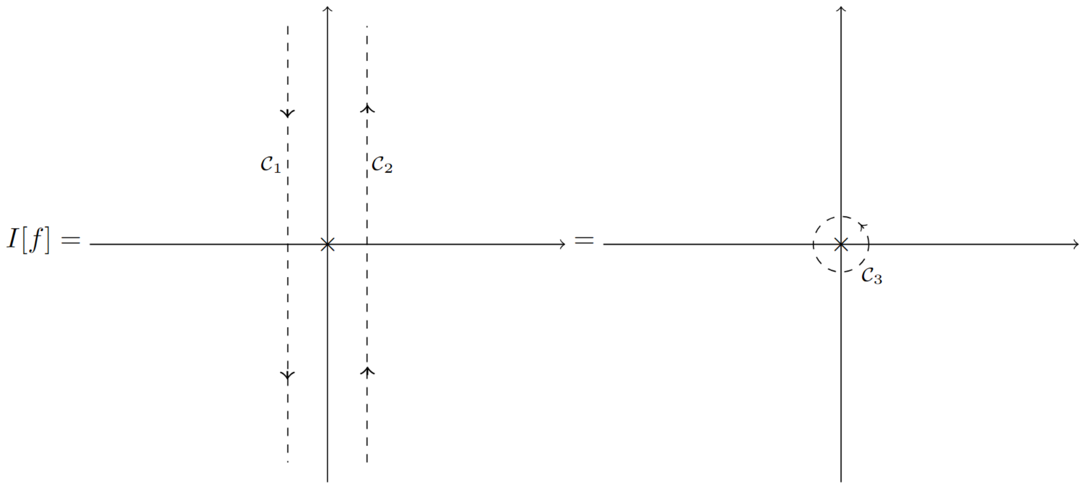


Figure B.1: Contours $\mathcal{C}_1$ and $\mathcal{C}_2$ for the integrals contributing to $I[f]$, which can be deformed into a single contour $\mathcal{C}_3$ that encloses the pole at $s = 0$.

Deforming the contours into $\mathcal{C}_3$ we can then write

$$
\begin{aligned}
I[f] &= \int\limits_{\mathcal{C}_1, \mathrm{Re}(s)<0} \frac{ds}{2\pi i} f(s) \frac{a^s}{s} - \int\limits_{\mathcal{C}_2, \mathrm{Re}(s)>0} \frac{ds}{2\pi i} f(s) \frac{a^s}{s} \\
&= \int\limits_{\mathcal{C}_3} \frac{ds}{2\pi i} f(s) \frac{a^s}{s} \\
&= \mathrm{Res}\left(f(s) \frac{a^s}{s}, s = 0 \right) \\
&= f(0).
\end{aligned} \tag{B.0.8}
$$

Therefore, we find that

$$
\int\limits_{\mathcal{C}} \frac{ds}{2\pi i} f(s) \phi(s) = f(0), \tag{B.0.9}
$$

and so we identify

$$
\phi(s) := \int\limits_{-\infty}^{0} d\eta (-\eta)^{s-1} = 2\pi i \delta(s). \tag{B.0.10}
$$

□

Appendix C

Ward Identities & Boundary Correlators

Here we give an overview of Ward identities for scalar operators. Let $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$ be a classical global symmetry of a scalar field theory. To investigate this symmetry at the quantum level, we transform the path integral as

$$\int \mathcal{D}\phi e^{iS[\phi]} \longrightarrow \int \mathcal{D}\phi' e^{iS[\phi'] + \int d^d x \partial_\mu j^\mu(x)}, \tag{C.0.1}$$

where $j^\mu(x)$ is the Noether current associated with the symmetry (the action is allowed to transform up to a total derivative term under the symmetry transformation). Using $S[\phi'] = S[\phi]$ for a classical symmetry, we therefore have

$$\begin{aligned}
\int \mathcal{D}\phi e^{iS[\phi]} \longrightarrow &\int \mathcal{D}\phi' e^{iS[\phi] + \int d^d x \partial_\mu j^\mu(x)} \\
&= \int \mathcal{D}\phi' e^{iS[\phi]} \left(1 + \int d^d x \partial_\mu j^\mu(x) + ...\right) \\
&= \int \mathcal{D}\phi' e^{iS[\phi]} + \int \mathcal{D}\phi' \left(\int d^d x \partial_\mu j^\mu(x)\right) e^{iS[\phi]} + ... \\
&= \int \mathcal{D}\phi' e^{iS[\phi]} + \left\langle \int d^d x \partial_\mu j^\mu(x) \right\rangle + ...
\end{aligned} \tag{C.0.2}$$

Therefore, for the symmetry to hold at the quantum level (to leading order in perturbation theory at least), we need the path integral measure to be invariant and we also require that

$$\langle \partial_\mu j^\mu(x) \rangle \stackrel{!}{=} 0. \tag{C.0.3}$$

Equation (C.0.3) is an example of a *Ward identity* - a condition that a correlation function must satisfy in order for a transformation (in this case $\phi \to \phi + \delta\phi$) to be a symmetry at the quantum level.

C.1 Dilatation Ward Identity

In this thesis we are most concerned with conformal symmetry. One relevant Ward identity is therefore that for dilatations $x_i \to \lambda x_i$ acting on three-point correlators of primary operators in momentum space. To derive the dilatation Ward identity, we begin by acting on the coordinates in the correlator with a dilatation and taking the Fourier transform, namely

$$\begin{aligned}
\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\mathcal{O}_3(\lambda x_3)\rangle &= \int \frac{d^d\mathbf{k}_1}{(2\pi)^d}\frac{d^d\mathbf{k}_2}{(2\pi)^d}\frac{d^d\mathbf{k}_3}{(2\pi)^d}\langle \mathcal{O}_1(\mathbf{k}_1)\mathcal{O}_2(\mathbf{k}_2)\mathcal{O}_3(\mathbf{k}_3)\rangle e^{i\lambda(\mathbf{k}_1\cdot x_1+\mathbf{k}_2\cdot x_2+\mathbf{k}_3\cdot x_3)} \\
&= \int \frac{d^d\mathbf{k}_1}{(2\pi)^d}\frac{d^d\mathbf{k}_2}{(2\pi)^d}\frac{d^d\mathbf{k}_3}{(2\pi)^d}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle e^{i\lambda(\mathbf{k}_1\cdot x_1+\mathbf{k}_2\cdot x_2+\mathbf{k}_3\cdot x_3)} \\
&\qquad \times \delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \\
&= \int \frac{d^d\mathbf{k}_1}{(2\pi)^d}\frac{d^d\mathbf{k}_2}{(2\pi)^d}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' e^{i\lambda(\mathbf{k}_1\cdot x_1+\mathbf{k}_2\cdot x_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot x_3)},
\end{aligned} \tag{C.1.1}$$

where we have used that imposing translation and rotation invariance of the correlator implies that it is a function of the magnitudes of the momenta $k_i \equiv |\mathbf{k}_i|$ and that it is proportional to a momentum-conserving delta function. We have also defined

$$\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' \equiv \langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(-(k_1+k_2))\rangle. \tag{C.1.2}$$

Continuing, we have

$$\begin{aligned}
\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\mathcal{O}_3(\lambda x_3)\rangle &= \int \frac{d^d\mathbf{k}_1}{(2\pi)^d}\frac{d^d\mathbf{k}_2}{(2\pi)^d}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' e^{i\lambda(\mathbf{k}_1\cdot x_1+\mathbf{k}_2\cdot x_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot x_3)} \\
&= \frac{1}{\lambda^{2d}}\int \frac{d^d\mathbf{k}'_1}{(2\pi)^d}\frac{d^d\mathbf{k}'_2}{(2\pi)^d}\left\langle \mathcal{O}_1\left(\frac{k'_1}{\lambda}\right)\mathcal{O}_2\left(\frac{k'_2}{\lambda}\right)\mathcal{O}_3\left(\frac{k'_3}{\lambda}\right)\right\rangle'
\end{aligned}$$

$$\times e^{i(\mathbf{k}_1'\cdot x_1+\mathbf{k}_2'\cdot x_2-(\mathbf{k}_1'+\mathbf{k}_2')\cdot x_3)}, \tag{C.1.3}$$

where we have changed variables to $\mathbf{k}_i' \equiv \lambda \mathbf{k}_i$. Using that $\mathcal{O}_j(x_j) \to \lambda^{-\Delta_j}\mathcal{O}_j(x_j)$ under a dilatation, we want the above to be equal to

$$\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\mathcal{O}_3(\lambda x_3)\rangle = \lambda^{-\Delta}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle, \tag{C.1.4}$$

where we have defined $\Delta := \sum_{i=1}^{3} \Delta_i$. We therefore have

$$\begin{aligned}\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\mathcal{O}_3(\lambda x_3)\rangle &= \lambda^{-\Delta}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle \\ &= \lambda^{-\Delta}\int \frac{d^d\mathbf{k}_1}{(2\pi)^d}\frac{d^d\mathbf{k}_2}{(2\pi)^d}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' \\ &\qquad \times e^{i\lambda(\mathbf{k}_1\cdot x_1+\mathbf{k}_2\cdot x_2-(\mathbf{k}_1+\mathbf{k}_2)\cdot x_3)}.\end{aligned}$$

Comparing (C.1.3) to (C.1.5), we see that

$$\begin{aligned}\lambda^{-\Delta}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' &= \frac{1}{\lambda^{2d}}\left\langle \mathcal{O}_1\left(\frac{k_1}{\lambda}\right)\mathcal{O}_2\left(\frac{k_2}{\lambda}\right)\mathcal{O}_3\left(\frac{k_3}{\lambda}\right)\right\rangle' \\ \iff \lambda^{2d-\Delta}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' &= \left\langle \mathcal{O}_1\left(\frac{k_1}{\lambda}\right)\mathcal{O}_2\left(\frac{k_2}{\lambda}\right)\mathcal{O}_3\left(\frac{k_3}{\lambda}\right)\right\rangle', \end{aligned} \tag{C.1.5}$$

where we have relabelled the integration variable $\mathbf{k}_i' \to \mathbf{k}_i$. Setting $\frac{1}{\lambda} = (1+\epsilon)$ for ϵ small, we obtain

$$\begin{aligned}(1+\epsilon)^{-(2d-\Delta)}\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' &= \langle \mathcal{O}_1\left(k_1+\epsilon k_1\right)\mathcal{O}_2\left(k_2+\epsilon k_2\right)\mathcal{O}_3\left(k_3+\epsilon k_3\right)\rangle' \\ &= \left(1+\epsilon\sum_{i=1}^{3}k_i\frac{\partial}{\partial k_i}\right)\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle'\end{aligned} \tag{C.1.6}$$

where we have performed a Taylor expansion to $\mathcal{O}(\epsilon)$ on the right-hand-side to get from the first to the second line. Using that

$$\frac{1}{(1+\epsilon)^{2d-\Delta}} = 1-(2d-\Delta)\epsilon+\mathcal{O}(\epsilon^2), \tag{C.1.7}$$

we can obtain

$$\left(2d-\Delta+\sum_{i=1}^{3}k_i\frac{\partial}{\partial k_i}\right)\langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' = 0. \tag{C.1.8}$$

Defining

$$\mathcal{D}_i := -(\Delta_i - d) + k_i \frac{\partial}{\partial k_i}, \tag{C.1.9}$$

we obtain the dilatation ward identity,

$$\left(-d + \sum_{i=1}^{3} \mathcal{D}_i\right) \langle \mathcal{O}_1(k_1)\mathcal{O}_2(k_2)\mathcal{O}_3(k_3)\rangle' = 0. \tag{C.1.10}$$

C.2 Boundary Correlators & Mellin Space

C.2.1 Bulk Perspective

In section 3.2 we saw that the propagators for late-time correlators in a generic α vacuum could be re-written in terms of the corresponding propagators in the Bunch-Davies vacuum ($\alpha = 0$) with appropriate rotations of the time coordinate. These rotations of the time coordinate could be traded for corresponding rotations of the modulus of the boundary momenta. Since in Mellin space the dependence on the modulus of the boundary momentum enters as a power, with the exponent given by the corresponding Mellin variable (see equation (3.2.52)), the Mellin space representation of propagators for generic α-vacuum are given by those in the Bunch-Davies vacuum appropriately dressed by phases in the Mellin variables.

In particular, from the identities (3.2.64) and (3.2.65) it follows that propagators in a generic α vacuum take the following form in Mellin space:

$$\begin{aligned}
G^{(\alpha)}_{++}(u_1,k;u_2,k) &= \left[P^{+}_{\Delta_+} + e^{-2\nu\pi} P^{-}_{\Delta_+}\, e^{2(u_1+u_2)\pi i}\right] G^{(0)}_{\Delta_+,++}(u_1,k;u_2,k) + (\Delta_+ \to \Delta_-),\\
G^{(\alpha)}_{--}(u_1,k;u_2,k) &= \left[M^{-}_{\Delta_+} + e^{2\nu\pi} M^{+}_{\Delta_+}\, e^{-2(u_1+u_2)\pi i}\right] G^{(0)}_{\Delta_+,--}(u_1,k;u_2,k) + (\Delta_+ \to \Delta_-),\\
G^{(\alpha)}_{-+}(u_1,k;u_2,k) &= \Big[\cosh^2\alpha + \sinh^2\alpha\, e^{-2(u_1-u_2)\pi i}\\
&\qquad -\frac{1}{2}\sinh 2\alpha\, e^{+\nu\pi}\left(e^{i\beta}\, e^{2u_2\pi i} + e^{-i\beta}\, e^{-2u_1\pi i}\right)\Big] G^{(0)}_{\Delta_+,-+}(u_1,k;u_2,k)\\
&\quad + (\Delta_+ \to \Delta_-),\\
G^{(\alpha)}_{+-}(u_1,k;u_2,k) &= \Big[\cosh^2\alpha + \sinh^2\alpha\, e^{2(u_1-u_2)\pi i}\\
&\qquad -\frac{1}{2}\sinh 2\alpha\, e^{-\nu\pi}(e^{i\beta}\, e^{-2u_2\pi i} + e^{-i\beta}\, e^{2u_1\pi i})\Big] G^{(0)}_{\Delta_+,+-}(u_1,k;u_2,k)
\end{aligned}$$

$$+ (\Delta_+ \to \Delta_-), \tag{C.2.1}$$

and

$$K^{(\alpha)}_{\Delta,+}(s,k) = \left[P^+_\Delta + P^-_\Delta e^{-\nu\pi} e^{2s\pi i}\right] K^{(0)}_{\Delta,+}(s,k)\,, \tag{C.2.2a}$$

$$K^{(\alpha)}_{\Delta,-}(s,k) = \left[M^-_\Delta + M^+_\Delta e^{\nu\pi} e^{-2s\pi i}\right] K^{(0)}_{\Delta,-}(s,k)\,. \tag{C.2.2b}$$

The upshot is that in Mellin space, perturbative late-time correlators in a generic α vacuum are given by the Mellin space representation of the corresponding process in the Bunch-Davies vacuum, dressed by phases in the Mellin variables. Owing to the Dirac delta function (3.1.36) enforcing invariance under dilatations, such phases moreover can be expressed purely in terms of Mellin variables associated to external legs.

For example, in Mellin space the three-point contact diagram (4.2.19) in a generic α-vacuum is given by

$$\begin{aligned}{}^{(\alpha)}\mathcal{A}^{\nu_{123}}_{\Delta_1\Delta_2\Delta_3}(s_1,k_1;s_2,k_2;s_3,k_3) &= i\left(\prod_{i=1}^{3} c^{\text{dS-AdS}}_{\Delta_i}(-\eta_0)^{\Delta_i}\right)\sum_{\pm} e^{\mp\frac{i\pi}{4}(d+2i(\nu_1+\nu_2+\nu_3))}\\ &\times\left\{{}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3} + \sinh(2\alpha)\,{}^{(\alpha)}C^{\pm}_{\Delta_1\Delta_2\Delta_3}\left(e^{\pm 2s_1\pi i} + e^{\pm 2s_2\pi i} + e^{\pm 2s_3\pi i}\right)\right\}\\ &\times \mathcal{A}^{\text{AdS}}_{\Delta_1\Delta_2\Delta_3}(s_1,k_1;s_2,k_2;s_3,k_3)\,, \end{aligned} \tag{C.2.3}$$

where ${}^{(0)}C^{\pm}_{\Delta_1\Delta_2\Delta_3}$ and ${}^{(\alpha)}C^{\pm}_{\Delta_1\Delta_2\Delta_3}$ were defined in (4.2.20).

C.2.2 Boundary Perspective

The structure of late-time correlators in a generic α-vacuum in Mellin space can also be understood from a boundary perspective, following (and building on) section 3.1 of [77]. The (EA)dS isometry group acts on the boundary like the Euclidean conformal group, and boundary correlators in these spaces are therefore constrained by conformal Ward identities. Here, we review the implications of such conformal Ward identities in Mellin space, focusing for simplicity on three-point functions.

In addition to translations and rotations, the conformal group is generated by dilatations and special conformal transformations. In momentum space, the dilatation generator is given by (C.1.9), while the special conformal generator reads

$$\mathcal{K}_{k_i} = 2\left(\Delta - d\right)\partial_{k_i} - 2k^j \partial_{k_j}\partial_{k_i} + k^i \partial^2_{k_j}. \tag{C.2.4}$$

As we saw above, translation and rotation symmetry enforce that three-point functions are proportional to a momentum-conserving delta function, whose coefficient is given by a function of the magnitudes $k_i = |\vec{k}_i|$ of the boundary momenta,

$$F_{\Delta_1\Delta_2\Delta_3}\left(\vec{k}_1, \vec{k}_2, \vec{k}_3\right) = (2\pi)^d\, \delta^{(d)}\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right) F'_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, k_3\right), \tag{C.2.5}$$

where $F_{\Delta_1\Delta_2\Delta_3}$ is a momentum space three-point function of operators with scaling dimension Δ_i.

To solve the Ward identities associated to dilatations and special conformal transformations we can go to Mellin space, where the three-point conformal structure becomes [75]

$$F'_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, k_3\right) = \int_{-i\infty}^{+i\infty} [\mathrm{d}s]_3\, F_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right) \prod_{i=1}^{3} \left(\frac{k_i}{2}\right)^{-2s_i + i\nu_i}, \tag{C.2.6}$$

and $F_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right)$ is the Mellin transform of $F'_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, k_3\right)$. As we derived in the previous section, the dilatation Ward identity is given by

$$0 = \left(-d + \sum_{i=1}^{3} \mathcal{D}_j\right) F'_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, k_3\right), \tag{C.2.7}$$

which in Mellin space translates into

$$0 = \int_{-i\infty}^{+i\infty} [\mathrm{d}s]_3 \left(\tfrac{d}{2} - 2\left(s_1 + s_2 + s_3\right)\right) F_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right) \prod_{i=1}^{3} \left(\frac{k_i}{2}\right)^{-2s_i + i\nu_i}. \tag{C.2.8}$$

This implies that $F_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right)$ takes the form:

$$F_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right) = 2\pi i\, \delta\left(\tfrac{d}{4} - s_1 - s_2 - s_3\right) F'_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right), \tag{C.2.9}$$

which is the Mellin space analogue of momentum conservation (C.2.5) implied by translation invariance.

The Ward identity associated to special conformal transformations is given by

$$\left(\prod_{i=1}^{3} \mathcal{K}^j_{k_i}\right) F'_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, k_3\right) = 0. \tag{C.2.10}$$

In [70] this was solved by reducing the Ward identity to two independent scalar equations;

$$0 = \left[\left(\frac{\partial^2}{\partial k_1^2} + \frac{d+1-2\Delta_1}{k_1}\frac{\partial}{\partial k_1}\right) - \left(\frac{\partial^2}{\partial k_3^2} + \frac{d+1-2\Delta_3}{k_3}\frac{\partial}{\partial k_3}\right)\right] F'_{\Delta_1\,\Delta_2\,\Delta_3}\left(k_1, k_2, k_3\right), \tag{C.2.11a}$$

$$0 = \left[\left(\frac{\partial^2}{\partial k_2^2} + \frac{d+1-2\Delta_2}{k_2}\frac{\partial}{\partial k_2}\right) - \left(\frac{\partial^2}{\partial k_3^2} + \frac{d+1-2\Delta_3}{k_3}\frac{\partial}{\partial k_3}\right)\right] F'_{\Delta_1\,\Delta_2\,\Delta_3}\left(k_1, k_2, k_3\right), \tag{C.2.11b}$$

which is achieved by taking $\vec{k}_1$ and $\vec{k}_2$ to be independent momenta. Going to Mellin space by plugging in (C.2.6), these translate into a system of difference equations,

$$\begin{aligned}&\left(s_1 - 1 + \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_1\right)\right)\left(s_1 - 1 - \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_1\right)\right) F'_{\Delta_1\Delta_2\Delta_3}\left(s_1 - 1, s_2, s_3\right)\\ &= \left(s_3 - 1 + \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_3\right)\right)\left(s_3 - 1 - \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_3\right)\right) F'_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3 - 1\right),\end{aligned} \tag{C.2.12}$$

$$\begin{aligned}&\left(s_2 - 1 + \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_2\right)\right)\left(s_2 - 1 - \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_2\right)\right) F'_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2 - 1, s_3\right)\\ &= \left(s_3 - 1 + \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_3\right)\right)\left(s_3 - 1 - \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_3\right)\right) F'_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3 - 1\right),\end{aligned} \tag{C.2.13}$$

which are solved by [77]

$$\begin{aligned}F'_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right) &= p_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right)\\ &\times \prod_{i=1}^{3} \Gamma\left(s_i + \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_i\right)\right)\Gamma\left(s_i - \tfrac{1}{2}\left(\tfrac{d}{2} - \Delta_i\right)\right),\end{aligned} \tag{C.2.14}$$

with $p_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right)$ a periodic function of unit period in the Mellin variables[1]. This a priori arbitrary function is further constrained by the requirement that the Mellin integrals converge. Different choices for $p_{\Delta_1\Delta_2\Delta_3}\left(s_1, s_2, s_3\right)$ yield different solutions to the system of difference equations, of which there are four in total, since

[1] See chapter 4.4 of [126] for a detailed discussion of the application of the Mellin transform to the solution of differential equations.

we ultimately seek solutions to two second order differential equations (C.2.11).

The solution with no folded singularities is $p_{\Delta_1\Delta_2\Delta_3}(s_1, s_2, s_3) =$ constant, which corresponds to the conformal structure accounting for boundary correlators in EAdS and in the Bunch-Davies vacuum of dS. This is the Mellin space representation of the "triple-K" integral solution [70] to the conformal Ward identities, given by (4.2.9) with $n = 3$. It is therefore manifest that in Mellin space the other solutions to the conformal Ward identities are given by dressings of this solution by phases of unit period in the Mellin variables, as in the expression (C.2.3) for late-time three-point functions in the α-vacua. In Mellin space, multiplication of the Bunch-Davies solution by phases in the Mellin variables corresponds, in momentum space, to a flipping of the boundary momenta $k \to e^{\pm i\pi}k$. A dressing of the Bunch-Davies solution by phases, namely the function $p_{\Delta_1\Delta_2\Delta_3}(s_1, s_2, s_3)$ being given by a phase in the Mellin variables of the form $e^{\pm 2i\pi s_j}$, therefore indeed results in folded singularities. For instance,

$$\begin{aligned} F'_{\Delta_1\Delta_2\Delta_3}(k_1, k_2, k_3) &= \int_{-i\infty}^{+i\infty} [\mathrm{d}s]_3 \, F'_{\Delta_1\Delta_2\Delta_3}(s_1, s_2, s_3) \prod_{i=1}^{3} \left(\frac{k_i}{2}\right)^{-2s_i+i\nu_i} \\ &\longrightarrow \int_{-i\infty}^{+i\infty} [\mathrm{d}s]_3 e^{\pm 2i\pi s_3} F'_{\Delta_1\Delta_2\Delta_3}(s_1, s_2, s_3) \prod_{i=1}^{3} \left(\frac{k_i}{2}\right)^{-2s_i+i\nu_i} \\ &= e^{\mp\pi\nu_3} \int_{-i\infty}^{+i\infty} [\mathrm{d}s]_3 F'_{\Delta_1\Delta_2\Delta_3}(s_1, s_2, s_3) \left(\frac{e^{\mp i\pi}k_3}{2}\right)^{-2s_3+i\nu_3} \prod_{i=1}^{2} \left(\frac{k_i}{2}\right)^{-2s_i+i\nu_i} \\ &= e^{\mp\pi\nu_3} F'_{\Delta_1\Delta_2\Delta_3}\left(k_1, k_2, e^{\mp i\pi}k_3\right). \end{aligned} \tag{C.2.15}$$

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