Generalized Symmetries in the Strong Coupling Limit

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Abstract

This thesis examines applications of generalized symmetries to strongly coupled Yang-Mills theories in four dimensions. We begin by providing a brief overview of recent developments in generalized symmetries, namely higher-form symmetries, non-invertible symmetries, higher-group symmetries, and generalized 't Hooft anomalies. We then proceed to study several examples of their applications in understanding the infrared (IR) behaviour of Yang-Mills theories as they become strongly coupled. Firstly, we study a family of 2-index chiral gauge theories, which exhibit generalized anomalies arising from the presence of fractionally charged backgrounds, known as 't Hooft fluxes (or twists). We leverage the 't Hooft anomalies to constrain their IR phases. In some cases, the generalized anomalies allow us to eliminate the possibility of composite fermions, which was not previously possible with ordinary 't Hooft anomalies. After studying higher-form symmetries, we then proceed to analyze the non-invertible symmetry in Yang-Mills theories arising from 't Hooft twists, and we provide an explicit method to construct such symmetries in the Hamiltonian formalism. Finally, we turn to axion physics and argue that a three-form gauge theory is a good effective field theory description for axion-Yang-Mills in the IR, incorporating both higher-form symmetries and higher-group symmetries.

Supervisors: Mohamed Anber

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Contents

Declaration

The work in this thesis is based on research carried out at the Centre of Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification, and it is the sole work of the author unless referenced to the contrary in the text.

Some of the work presented in this thesis has been published in journals - the relevant publications are [1, 2, 3]. All three publications were written in collaboration with Mohamed Anber.

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Chapter 1

Introduction

Since the advent of quantum field theory, our best explanations of particle interactions can be formulated using gauge theories. Yet many questions still remain unanswered, especially in the case of Yang-Mills theory, where the gauge group is non-abelian. One of the biggest mysteries is confinement - despite observing quarks combining to form hadrons, we are still missing an analytic explanation of how quantum chromodynamics (QCD) predicts confinement.

One reason why studying Yang-Mills theory is so difficult is the lack of analytic methods to study the theory at relevant energies. Famously, the Yang-Mills action is given by:

$$S_{YM}[a_\mu^c] = \frac{1}{2g^2} \int dx^4 \, \mathrm{tr} f_{\mu\nu}^c f^{c,\mu\nu} \tag{1.1}$$

where g is the Yang-Mills coupling constant, and $f_{\mu\nu}^c = \partial_\mu a_\nu^c - \partial_\nu a_\mu^c + i[a_\mu, a_\nu]$ is the non-abelian field strength, given in terms of the gauge field a_μ^c. The path integral in Lorentzian signature is given by

$$Z = \int \mathcal{D}a_\mu^c e^{iS_{YM}[a_\mu^c]} \tag{1.2}$$

If the coupling g^2 is small, the path integral is well-approximated by the stationary phase approximation, given by the minimum of the action. In this case we can study fluctuations about the classical equations of motion, which gives us a good foothold to understand the physics. These are *perturbative* approaches to understanding gauge theories. But when the coupling is large, also known as *strong* coupling, the weight of field configurations away from the minimum become significant. In this regime we can no longer use perturbative methods to understand the physics - we must find other methods to understand the physical behaviour of these systems.

One might hope that the coupling constant of Yang-Mills theory is a small constant, and we are able to extract many physical predictions by studying the perturbative aspects of quantum field theory. Unfortunately, developments in the twentieth century on the renormalization group flow of quantum field theories tells us that life is not so simple. The value of the coupling constant depends on the energy scale we are considering. In particular, the Yang-Mills coupling is governed by the *beta function*:

$$\mu \frac{dg}{d\mu} = \beta_0 g^3, \tag{1.3}$$

where μ is the energy scale and β_0 is a negative constant depending on group theoretic factors of the gauge group. From this equation we can deduce that the theory is *asymptotically free* - the coupling is large at low energies and small at high energies.

The upshot is that we can no longer apply perturbative techniques to understand non-abelian gauge theories at longer length scales. We must therefore turn to other methods to try and probe for infra-red (IR) phenomena. One handle we have on the low energy behaviour of gauge theories is the theory's global symmetries and their anomalies. The presence of symmetry in quantum field theory can therefore give us new insight into the IR behaviour of theories. New symmetries could perhaps lead to new insights. In the past decade, there has been rapid developments in the study of symmetries in quantum field theory, falling under the umbrella of *generalized* symmetries [4]. This has led to new insights to the IR behaviour of non-abelian gauge theories. This thesis will focus on applications of generalized symmetries to study strongly coupled gauge theories.

1.1 Outline of Thesis

Chapter two will be an introduction to generalized symmetries. We begin by introducing anomalies and deriving the ABJ anomaly via the Fujikawa method. We also discuss 't Hooft anomaly matching. We then turn to higher-form symmetries and discuss how to explicitly construct 1-form symmetry backgrounds on the four-torus. We will also introduce non-invertible symmetries and higher-group symmetries, providing examples that will be relevant for the later chapters.

Chapter three, four and five are based on my publications [1], [2], and [3] respectively. The thesis will be arranged as follows:

In chapter three, we will investigate a family of 4-dimensional $SU(N)$ chiral gauge theories and investigate their faithful global symmetries and dynamics. Despite their prevalence, chiral gauge theories are still poorly understood, partly due to our inability to simulate them on the lattice thanks to the Nielsen-Ninomiya theorem [5, 6]. We will study a finite set of theories with fermions in the 2-index symmetric and anti-symmetric representations, with no fundamentals, and they do not admit a large-N limit. We employ a combination of perturbative and nonperturbative methods, enabling us to constrain their infrared (IR) phases. Specifically, we leverage the 't Hooft anomalies associated with continuous and discrete groups to eliminate a few scenarios. In some cases, the 1-form anomalies arising from turning on 't Hooft fluxes allow us to rule out the possibility of fermion composites. In other cases, the interplay between the continuous and discrete anomalies leads to multiple higher-order condensates, which inevitably form to match the anomalies. Further, we pinpoint the most probable symmetry-breaking patterns by searching for condensates that match the full set of anomalies resulting in the smallest number of IR degrees of freedom. Higher-loop β-function analysis suggests that a few theories may flow to a conformal fixed point. We hope that by studying this family of theories we can uncover general lessons to better understand quantum field theories.

In chapter four, we will turn to non-invertible symmetries. Non-invertible symmetries have cropped up in the past few years, and it is desirable to study their implications for non-abelian gauge theories. We devise a general method for obtaining 0-form noninvertible discrete chiral symmetries in 4-dimensional $SU(N)/\mathbb{Z}_p$ and $SU(N) \times U(1)/\mathbb{Z}_p$ gauge theories with matter in arbitrary representations, where $\mathbb{Z}_p$ is a subgroup of the electric 1-form center symmetry. Our approach involves placing the theory on a three-torus and utilizing the Hamiltonian formalism to construct noninvertible operators by introducing twists compatible with the gauging of $\mathbb{Z}_p$. These theories exhibit electric 1-form and magnetic 1-form global symmetries, and their generators play a crucial role in constructing the corresponding Hilbert space. The noninvertible operators are demonstrated to project onto specific Hilbert space sectors characterized by particular magnetic fluxes. Furthermore, when subjected to twists by the electric 1-form global symmetry, these surviving sectors reveal an anomaly between the noninvertible and the 1-form symmetries. We argue that an anomaly implies that certain sectors, characterized by the eigenvalues of the electric symmetry generators, exhibit multi-fold degeneracies. When we couple these theories to axions, infrared axionic noninvertible operators inherit the ultraviolet structure of the theory, including the projective nature of the operators and their anomalies. We discuss various examples of vector and chiral gauge theories that showcase the versatility of our approach.

The axion is a popular candidate for dark matter [7], and is also an appealing solution to the Strong CP problem. Simultaneously, with the introduction of higher-form symmetries, we can also describe axion physics in terms of higher-form gauge theories. In chapter five, we study the proposition that axion-Yang-Mills systems are characterized by a 3-form gauge theory in the deep infrared regime. This hypothesis is rigorously examined by initially developing a systematic framework for analyzing 3-form gauge theory coupled to an axion, specifically focusing on its global properties. The theory consists of a BF term deformed by marginal and irrelevant operators and describes a network of vacua separated by domain walls converging at the junction of an axion string. It encompasses 0- and 3-form spontaneously broken global symmetries. Utilizing this framework, in conjunction with effective field theory techniques and 't Hooft anomaly-matching conditions, we argue that the 3-form gauge theory faithfully captures the infrared physics of the axion-Yang-Mills system. The ultraviolet theory is an $SU(N)$ Yang-Mills theory endowed with a massless Dirac fermion coupled to a complex scalar and is characterized by chiral and genuine $\mathbb{Z}_m^{(1)}$ 1-form center symmetries, with a mixed anomaly between them. It features two scales: the vev of the complex scalar, v, and the strong-coupling scale, Λ, with $\Lambda \ll v$. Below v, the fermion decouples and a $U(1)^{(2)}$ 2-form winding symmetry emerge, while the 1-form symmetry is enhanced to $\mathbb{Z}_N^{(1)}$. As we flow below Λ, matching the mixed anomaly necessitates introducing a dynamical 3-form gauge field of $U(1)^{(2)}$, which appears as the incarnation of a long-range tail of the color field. The infrared theory possesses spontaneously broken chiral and emergent 3-form global symmetries. It passes several checks, among which: it displays the expected restructuring in the hadronic sector upon transition between the vacua, and it is consistent under the gauging of the genuine $\mathbb{Z}_m^{(1)} \subset \mathbb{Z}_N^{(1)}$ symmetry.

1.2 Notation

A brief word on the notation used throughout this thesis: When discussing differential forms, the subscript denotes the degree. For example, j_1 denotes that j is a 1-form. A gauge field is written in lower case letters, e.g. a_1, is a dynamical gauge field. A gauge field written in capital letters, e.g. B_2, is a background gauge field. We will also use $\sim$ above groups and operators to denote non-invertible symmetries, for example U denotes an invertible symmetry, and $\tilde{U}$ represents a non-invertible symmetry.

Chapter **2**

Introduction to anomalies and generalized symmetries

In rudimentary quantum field theory in 3+1 spacetime dimensions, we learn that global continuous abelian symmetries in quantum field theory come with a Noether current j_1 and Noether charge $Q = \int dx^4\, j^0$. For a local field ψ transforming under this symmetry $\psi + \alpha\Delta\psi$, where α is an infinitesimal parameter, the transformation is generated by the charge:

$$\Delta\psi = i[Q, \psi]. \tag{2.1}$$

The Noether current satisfies the Ward identity:

$$\partial_\mu j^\mu(x)\psi(y) = \delta^4(x-y)\Delta\psi(x), \tag{2.2}$$

where the above expression is understood to hold inside correlation functions [6, 8].

We can reframe a continuous, global symmetry in terms of topological operators. Consider the operator (using differential form notation) defined on the closed 3-dimensional sub-manifold $\mathcal{M}_3$ [4, 9, 10]:

$$U_\alpha(\mathcal{M}_3) = e^{i\alpha \int_{\mathcal{M}_3} \star j_1}. \tag{2.3}$$

The operator is labelled by the transformation parameter α. For example, if we have a $U(1)$ symmetry, we have $\alpha \in [0, 2\pi)$. This operator is topological - under deformations of the manifold $\mathcal{M}_3 \to \mathcal{M}_3'$:

$$U_\alpha(\mathcal{M}_3)U_\alpha^{-1}(\mathcal{M}_3') = e^{i\alpha \int_{\mathcal{M}_3 - \mathcal{M}_3'} \star j_1} = e^{i\alpha \int_{\mathcal{M}_4'} d\star j_1} = 1, \tag{2.4}$$

where $\partial\mathcal{M}_4' = \mathcal{M}_3 - \mathcal{M}_3'$ and we obtain the last equality via the Ward identity. If we have a local operator $\phi(x)$ charged under the symmetry with charge q, when deforming $U_\alpha(\mathcal{M}_3)$ past the point x we obtain a phase:

$$U_\alpha(\mathcal{M}_3)\phi(x) = e^{iq\alpha}\phi(x)U_\alpha(\mathcal{M}_3'). \tag{2.5}$$

For grouplike symmetries, the operators also obey a fusion rule when we deform the support of one symmetry operator onto another. For example, for an abelian symmetry the topological operators obey:

$$\lim_{\mathcal{M}_3' \to \mathcal{M}_3} U_\alpha(\mathcal{M}_3) U_\beta(\mathcal{M}_3') = U_{\alpha+\beta}(\mathcal{M}_3). \tag{2.6}$$

Before moving to generalized symmetries, we note that we can describe symmetries in terms of topological operators without needing to refer to a conserved current. One advantage of describing symmetries this way is that they allow us to describe discrete symmetries, which do not have a conserved current, on an equal footing as continuous symmetries.

There are several ways to generalize this construction. We could change the dimensions of the topological operators, letting them act on operators of higher dimensions. These are known as *higher form* symmetries. We could also modify the fusion rules satisfied by the operators. Instead of a group like structure, where every transformation has a corresponding inverse, we could consider operators that do not have inverses - these are labelled as *noninvertible* symmetries. There could also be non-trivial interactions between symmetries of different dimensions - the structure of the symmetry generalizes to a *higher group*. In the rest of this chapter we will introduce these new symmetries and highlight possible implications they have for IR physics.

Much has been said about generalized symmetries; see [4, 9, 10, 11, 12, 13] for reviews on the topic. We will not attempt to cover as much as possible - instead we will focus on the aspects that will allow us to understand the subsequent chapters.

2.1 Higher-Form Symmetries

A higher-form symmetry is a symmetry acting on higher-dimensional operators. A symmetry acting on a p-form operator in $d+1$ spacetime dimensions can be described by topological operators defined on submanifolds of co-dimension $d-p$. For example, symmetry operators in 3+1 dimensions acting on 0-form, or local, operators has co-dimension three, as described previously.

We could also consider symmetries acting on line operators, which are topological operators with co-dimension $p+1$ - under a topological deformation $\mathcal{M}_{d-p} \to \mathcal{M}'_{d-p}$

$$U^{(p)}(\mathcal{M}_{d-p}) = U^{(p)}(\mathcal{M}'_{d-p}). \tag{2.7}$$

Here we use the superscript (p) to denote that $U^{(p)}(\mathcal{M}_{d-p})$ is p-form symmetry operator. For simplicity all symmetries discussed in this section are assumed to be invertible - for every symmetry operator $U^{(p)}(\mathcal{M}_{d-p})$ there is a corresponding operator $U^{-1(p)}(\mathcal{M}_{d-p})$ such that

$$\lim_{\mathcal{M}'_{d-p} \to \mathcal{M}_{d-p}} U^{(p)}(\mathcal{M}_{d-p}) U^{-1(p)}(\mathcal{M}'_{d-p}) = 1, \tag{2.8}$$

where 1 denotes the identity operator.

In the presence of a p-form operator $\mathcal{O}^{(p)}(\Sigma_p)$ charged under this symmetry, when deforming $\mathcal{M}_{d-p}$ past Σ_p to $\mathcal{M}'_{d-p}$, the operator picks up a phase:

$$U^{(p)}(\mathcal{M}_{d-p})\mathcal{O}^{(p)}(\Sigma_p) = e^{i\theta}\mathcal{O}^{(p)}(\Sigma_p)U^{(p)}(\mathcal{M}_{d-p}). \tag{2.9}$$

For operators with dimension less than p, we can deform the symmetry past the operator without any intersection. In general, p-form symmetries can only act on operators of dimension greater than or equal to p [9]. We also observe that two topological operators with co-dimension greater than one can be topologically deformed past each other without crossing. The symmetry group for a higher-form symmetry must therefore be abelian. They satisfy the fusion rule

$$\lim_{\mathcal{M}'_3\to\mathcal{M}_3} U_\alpha^{(p)}(\mathcal{M}_3)U_\beta^{(p)}(\mathcal{M}'_3) = U_{\alpha+\beta}(\mathcal{M}_3), \tag{2.10}$$

where α, β are transformation parameters for an abelian symmetry.

Continuous higher-form symmetries are also equipped with conserved currents. A continuous p-form symmetry has a $p+1$-form current j_{p+1}, which satisfies the conservation law

$$d\star j_{p+1} = 0. \tag{2.11}$$

We can construct the corresponding symmetry operators in a similar manner to 0-form symmetries. We can integrate the hodge dual of the current over a co-dimension $p+1$ manifold:

$$U_\alpha^{(p+1)}(\mathcal{M}_{d-p}) = e^{i\alpha\int_{\mathcal{M}_{d-p}}\star j_{p+1}}. \tag{2.12}$$

Let us consider some concrete examples of higher-form symmetries - the following examples are all in 3+1 spacetime dimensions:

In pure Maxwell theory, the Lagrangian is

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{2e^2}f_2\wedge\star f_2, \tag{2.13}$$

where f_2 is the 2-form $U(1)$ field strength and e is the Maxwell coupling. The equation of motion is

$$d\star f_2 = 0. \tag{2.14}$$

The field strength also satisfies the Bianchi identity

$$df_2 = 0. \tag{2.15}$$

$\star f_2$ and f_2 satisfy current conservation equations - we can interpret them as conserved currents, with the corresponding symmetry operators

$$U_{e,\alpha}^{(1)}(\mathcal{M}_2) = e^{i\alpha\int_{\mathcal{M}_2}\star f_2}, \tag{2.16}$$

$$U_{m,\beta}^{(1)}(\mathcal{M}_2) = e^{i\beta\int_{\mathcal{M}_2} f_2}, \tag{2.17}$$

where $\alpha, \beta \in [0, 2\pi)$. These are $U(1)^{(1)}_e$ electric and $U(1)^{(1)}_m$ magnetic symmetries, acting on Wilson lines and 't Hooft lines respectively. Given a Wilson line $W_q[C]$, which we can interpret as a probe particle of infinite mass with electric charge q, the electric 1-form symmetry operator $U^{(1)}_{e,\alpha}(\mathcal{M}_2)$ defined on a manifold $\mathcal{M}_2$ acts on the Wilson line via:

$$U^{(1)}_{e,\alpha}(\mathcal{M}_2)W_q[C] = e^{iq\alpha \text{Link}(\mathcal{M}_2,C)}W_q[C]U^{(1)}_{e,\alpha}(\mathcal{M}'_2), \tag{2.18}$$

where $\text{Link}(\mathcal{M}_2, C)$ is the linking number of $\mathcal{M}_2$ and C, and $\mathcal{M}'_2$ is a manifold with no non-trivial linking with the curve C. $U(1)^{(1)}_m$ acts similarly on 't Hooft lines.

If we couple the theory to an electrically charged particle ϕ with charge k, a Wilson line with charge k can terminate on ϕ as a gauge-invariant configuration. The action of the symmetry operator must be invariant under any topological deformation, so the action of the 1-form symmetry $U^{(1)}_\alpha$ on the Wilson line must be trivial (see figure 2.1).

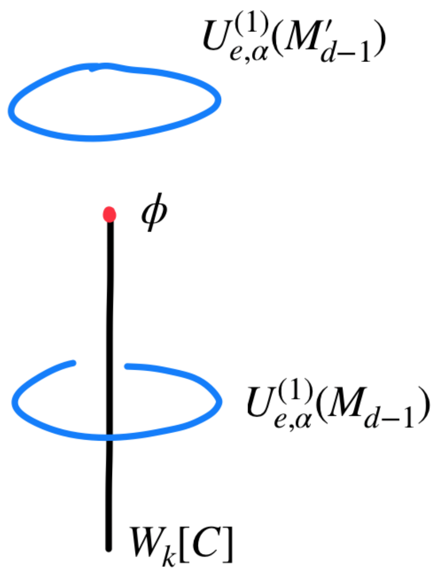

Figure 2.1: The action of the symmetry operator acting on the line must be the same as the action of the symmetry operator acting on the trivial line as it must be invariant under topological deformations.

In particular, the phase obtained by the Wilson line under the action of the 1-form symmetry $U^{(1)}_{e,\alpha}$ must be trivial:

$$e^{2\pi ik\alpha} = 1. \tag{2.19}$$

This restricts $\alpha \in \frac{1}{k}\mathbb{Z}$. We see that in the presence of electrically charged matter, the electric 1-form symmetry breaks from $U(1)^{(1)}_e$ to $\mathbb{Z}^{(1)}_k$.

Pure Yang-Mills theory with gauge group $SU(N)$ has a $\mathbb{Z}_N^{(1)}$ electric symmetry. Given a Wilson line characterized by the irreducible representation $\mathcal{R}$ of $SU(N)$,

$$W_{\mathcal{R}}[C] = \mathrm{tr}_{\mathcal{R}} \mathcal{P} e^{2\pi i \int_C a^c}, \tag{2.20}$$

the $\mathbb{Z}_N^{(1)}$ symmetry operator $U_{k=1}^{(1)}(\mathcal{M}_2)$ acts on $W_{\mathcal{R}}[C]$ via

$$U_{k=1}^{(1)}(\mathcal{M}_2) W_{\mathcal{R}}[C] = e^{\frac{2\pi i}{N} n_{\mathcal{R}} \mathrm{Link}(\mathcal{M}_2, C)} W_{\mathcal{R}}[C] U_{k=1}^{(1)}(\mathcal{M}_2'), \tag{2.21}$$

where $n_{\mathcal{R}}$ is the *n-ality* of the representation $\mathcal{R}$, given by the number of boxes in the Young tableau representation of $\mathcal{R}$.

If the $SU(N)$ gauge theory is coupled to matter ψ transforming in the representation $\mathcal{R}'$, Wilson lines in the appropriate representation can end on ψ. Similar to the case of $U(1)$ gauge theory, the action of the 1-form symmetry $U_k^{(1)}$ on the Wilson line attached to ψ must be trivial as we can topologically deform the operator past the Wilson line without crossing. In other words, the phase obtained by the Wilson line must be trivial:

$$e^{2\pi i n_{\mathcal{R}'} k/N} = 1. \tag{2.22}$$

This restricts k to be a multiple of $N/\gcd(N, n_{\mathcal{R}'})$. Therefore the 1-form symmetry is broken to $\mathbb{Z}^{(1)}_{\gcd(N, n_{\mathcal{R}'})}$.

2.1.1 1-form symmetry and Confinement

Given a symmetry, we would like to apply the Landau prescription to classify phases of a gauge theory. In gauge theories, the order parameter for confinement is the large loop limit of the Wilson loop expectation value $\lim_{C\to\infty} < W[C] >$ * If the Wilson loop obeys area law, i.e.

$$\lim_{C\to\infty} < W[C] > \sim e^{-\mu A[C]}, \tag{2.23}$$

for some constant μ, the theory *confines*. Otherwise the theory *deconfines*. If the theory has a one-form electric symmetry, the charged operator would be the Wilson line. The perimeter law, in other words deconfinement, indicates spontaneously broken one-form symmetry.

For example, pure Maxwell theory is understood to be in the Coulomb phase. Static electric probe charges with separation R obey the Coulomb law, with potential $V(r) \sim \frac{1}{R}$. The theory is therefore deconfining, as it is energetically favourable to increase the separation of the charges. To reframe this in the language of 1-form symmetries, consider the rectangular Wilson loop for two static charges evolving in time. It has expectation value:

$$< W[C] > \sim e^{-V(R)T} = e^{-\frac{T}{R}}. \tag{2.24}$$

*The notation $C \to \infty$ denotes taking the perimeter of the loop to infinity.

As we take $R, T \to \infty$, $< W[C] >$ tends to a non-zero constant, so the 1-form $U(1)_e^{(1)}$ symmetry is spontaneously broken. As the spontaneously broken symmetry is continuous, the IR spectrum has a Goldstone mode: the photon [4].

2.1.2 Coupling Gauge Backgrounds for Higher-Form Symmetries

As mentioned in the introduction, one way global symmetries can help determine the IR spectrum of a theory is requiring any 't Hooft anomalies to be matched. We will discuss this in a later section. First we must introduce background gauge fields for higher-form symmetries. For a continuous p-form symmetry (which must have symmetry group $U(1)$ - recall that higher-form symmetries must be abelian), we can introduce a $U(1)$-valued $p+1$-form gauge field B_{p+1} with field strength $F_{p+2} = dB_{p+1}$ by coupling it to the $p+1$-form current in the action:

$$\star J_{p+1} \wedge B_{p+1}. \tag{2.25}$$

This term is invariant under a background gauge transformation

$$B_{p+1} \to B_{p+1} + d\Lambda_p, \tag{2.26}$$

as the current is conserved. We can gauge the p-form symmetry by summing over all such gauge configurations in the path integral. In chapter 5, we will consider an example of a 3-form $U(1)$ gauge theory.

Turning on a background gauge field is less straightforward for a discrete p-form symmetry. Let us focus on the case of the $\mathbb{Z}_N^{(1)}$ 1-form symmetry in $SU(N)$ gauge theory [14, 15, 16, 17]. In order to implement discrete background gauge transformations, we will first promote the $SU(N)$ gauge theory to a $U(N)$ gauge theory. We can do this by adding a 1-form $U(1)$ background gauge field B_1 to the $SU(N)$ gauge field a_1^c:

$$\hat{a}_1^c = a_1^c + \frac{1}{N} B_1, \tag{2.27}$$

with field strength

$$\hat{f}_2^c = d\hat{a}_1^c + \hat{a}_1^c \wedge \hat{a}_1^c. \tag{2.28}$$

We now have an extra background gauge transformation - the field strength is not invariant under a $U(1)$ 1-form valued-transformation $B_1 \to B_1 + N\Lambda_1$. It transforms as $\hat{f}_2^c \to \hat{f}_2^c + d\Lambda_1$. To remedy this, we introduce an extra $U(1)$ 2-form background gauge field B_2 which is constrained by the relation $dB_1 = NB_2$, killing the extra degree of freedom. Then the combination $f_2^c - B_2$ is invariant under $U(1)$ background gauge transformations. We thus obtain an $SU(N)$ gauge theory coupled to $\mathbb{Z}_N^{(1)}$ background gauge fields:

$$S_{SU(N)/\mathbb{Z}_N} = \int \mathrm{tr}\,(f_2^c - B_2) \wedge \star(f_2^c - B_2) + \theta Q, \tag{2.29}$$

where B_2 is the gauge field for the $\mathbb{Z}_N^{(1)}$ symmetry, and we included the theta term with the topological charge Q, given by

$$Q = \frac{1}{8\pi^2}\int \mathrm{tr}\,(\hat{f}_2^c - B_2)\wedge(\hat{f}_2^c - B_2) = \frac{1}{8\pi^2}\int \mathrm{tr}\,\hat{f}_2^c\wedge\hat{f}_2^c + \frac{N}{8\pi^2}\int B_2\wedge B_2. \quad (2.30)$$

The relation $dB_1 = NB_2$ tells us that the $\mathbb{Z}_N^{(1)}$ background gauge field is flat:

$$dB_2 = 0, \quad (2.31)$$

When integrating over a closed manifold $\mathcal{M}_2$, we also have the quantisation condition

$$\int_{\mathcal{M}_2} B_2 \in \frac{2\pi}{N}\mathbb{Z}. \quad (2.32)$$

It follows that the topological charge of $SU(N)/Z_N$ background are also fractional, i.e. $Q \in \frac{2\pi}{N}\mathbb{Z}$. Coupling a $\mathbb{Z}_N^{(1)}$ gauge background leads to the interpretation that we are coupling to a gauge background with fractional topological charge. Gauging the $\mathbb{Z}_N^{(1)}$ symmetry amounts to summing over these fractionally charged backgrounds in the path integral.

2.1.3 $\mathbb{Z}_N^{(1)}$ 1-form background on $\mathbb{T}^4$: 't Hooft Fluxes

In the absence of matter, in order to define an $SU(N)$ gauge background on a manifold $\mathcal{M}_4$ divided into patches U_i, the transition functions g_{ij} on the intersection $U_i \cap U_j$ must satisfy the cocycle condition

$$g_{ij}g_{jk}g_{ki} = 1. \quad (2.33)$$

A gauge background for $SU(N)/\mathbb{Z}_N$ must satisfy relaxed cocycle conditions

$$g_{ij}g_{jk}g_{ki} = e^{\frac{2\pi i}{N}k},\ k \in \mathbb{Z}. \quad (2.34)$$

In other words, in order to construct 1-form symmetry backgrounds we must construct gauge backgrounds that satisfy the relaxed cocycle conditions.

We are able to construct these backgrounds explicitly on the 4-torus $\mathbb{T}^4$. For simplicity, suppose all cycles on the torus have length L. One cycle of the torus, x_i, can be covered by one coordinate patch $[0, L]$, and the transition function at $x_i = L$ is equivalent to defining a boundary condition. When the boundary condition is twisted, i.e. the transition function is valued in $\mathbb{Z}_N$, we obtain background gauge configurations for 1-form symmetries, known as *'t Hooft fluxes* [18, 19, 20].

In general, the $SU(N)$ gauge fields a_μ^c are taken to obey the boundary conditions

$$a_\nu^c(x + L_\mu\hat{e}_\mu) = \Omega_\mu \circ a_\nu^c(x) \equiv \Omega_\mu(x)a_\nu^c(x)\Omega_\mu^{-1}(x) - i\Omega_\mu(x)\partial_\nu\Omega_\mu^{-1}(x)\,, \quad (2.35)$$

upon traversing $\mathbb{T}^4$ in each direction. The transition functions Ω_μ are $N \times N$ unitary matrices in the defining representation of $SU(N)$, and $\hat{e}_\nu$ are unit vectors in the x_ν direction. The subscript μ in Ω_μ means that the function Ω_μ does not depend on the coordinate x_μ. Then, the compatibility of (2.35) at the corners of the $x_\mu - x_\nu$ plane of $\mathbb{T}^4$ gives the cocycle condition

$$\Omega_\mu(x + \hat{e}_\nu L_\nu)\, \Omega_\nu(x) = e^{i\frac{2\pi n_{\mu\nu}}{N}}\, \Omega_\nu(x + \hat{e}_\mu L_\mu)\, \Omega_\mu(x)\,. \tag{2.36}$$

The exponent $e^{i\frac{2\pi n_{\mu\nu}}{N}}$, with anti-symmetric integers $n_{\mu\nu} = -n_{\nu\mu}$, is the $\mathbb{Z}_N$ center of $SU(N)$. The freedom to twist by elements of the center stems from the fact that both the transition function and its inverse appear in (2.35). This is also equivalent to the fact that the Wilson lines in pure $SU(N)$ gauge theory are charged under the electric $\mathbb{Z}_N^{(1)}$ 1-form center symmetry. The fundamental (defining representation) Wilson lines wind around the 4 cycles and are given by

$$W_\mu = \mathrm{tr}_\square \left[P e^{i\int_{x_\mu=0}^{x_\mu=L_\mu} a_\mu^c} \Omega_\mu \right], \tag{2.37}$$

where $\square$ denotes the defining representation of $SU(N)$ and the insertion of the transition function Ω_μ ensures the gauge invariance of the lines.

It will be useful for the rest of this thesis to have an explicit construction of 't Hooft fluxes. Consider a particular gauge configuration

$$a_1^c = \frac{2\pi m_1}{L^2}\mathbf{H_c}\cdot\boldsymbol{\nu_c} x_2, \qquad a_2^c = 0, \qquad a_3^c = \frac{2\pi m_2}{L^2}\mathbf{H_c}\cdot\boldsymbol{\nu_c} x_4, \qquad a_4^c = 0, \tag{2.38}$$

where $m_1, m_2 \in \mathbb{Z}_N$, ν_c is a weight of the fundamental representation * of $SU(N)$, and $\mathbf{H_c} = (H_1, \ldots, H_{N-1})$ are the generators of its Cartan subalgebra. Note that the gauge fields as written are not single-valued on the torus. However $a_1^c(x_2 = L)$ and $a_1^c(x_2 = 0)$ are related by gauge transformations Ω_μ †:

$$a_1^c(x_2 = L) = a_1^c(x_2 = 0) + \frac{2\pi m_1}{L}\mathbf{H_c}\cdot\boldsymbol{\nu_c} = \Omega_1^\dagger(x_1)(a_1^c(x_2 = 0) + i\partial_1)\Omega_1(x_1), \tag{2.39}$$

where we use the transition functions

$$\Omega_1(x_1) = e^{\frac{2\pi i m_1}{L}\mathbf{H_c}\cdot\boldsymbol{\nu_c} x_1}. \tag{2.40}$$

$\Omega_1(x_1)$ is not periodic in x_1. At $x_1 = L$, the gauge transformation transforms as ‡

$$\Omega_1(L) = e^{2\pi i m_1 \mathbf{H_c}\cdot\boldsymbol{\nu_c}}\Omega_1(0) = e^{-2\pi i m_1 \frac{1}{N}}\Omega_1(0), \tag{2.41}$$

where we use the normalization $\boldsymbol{\nu_{c,A}} \cdot \boldsymbol{\nu_{c,B}} = \delta_{AB} - \frac{1}{N}$. The transition function at $x_1 = L$ is given by $e^{-2\pi i m_1 \frac{1}{N}}$. Similarly the transition function at $x_3 = L$ is also given by $e^{-2\pi i m_1 \frac{1}{N}}$. It is clear that they satisfy the relaxed cocycle conditions 2.34.

*Also referred to as the *defining* representation in this thesis.

†We use the notation Ω_μ instead of g_{ij} for gauge transformations on the torus $\mathbb{T}^4$.

‡See Appendix A for further details and explicit computations for the gauge group $SU(2)$ and $SU(3)$.

The field strength of the 't Hooft flux is:

$$f_{12}^c = -f_{21}^c = -\frac{2\pi m_1}{L^2}\mathbf{H_c}\cdot\boldsymbol{\nu_c}, \qquad f_{34}^c = -f_{43}^c = -\frac{2\pi m_2}{L^2}\mathbf{H_c}\cdot\boldsymbol{\nu_c}, \tag{2.42}$$

The topological charge is therefore given by:

$$Q = \frac{1}{8\pi^2}\int_{\mathbb{T}^4} \operatorname{tr} f^c \wedge f^c = m_1 m_2\left(1-\frac{1}{N}\right). \tag{2.43}$$

We see that the 't Hooft flux does indeed have fractional topological charge.

In chapter three we will study 't Hooft anomalies in the presence of generalized 't Hooft fluxes, arising from the presence of a discrete quotient in the global symmetry group. In chapter four we will study a non-invertible symmetry in $SU(N)/\mathbb{Z}_N$ gauge theory by implementing 't Hooft twists in the Hamiltonian formalism.

2.2 Introduction to Anomalies

An *anomaly* occurs when a symmetry of the classical action fails to be a symmetry in the quantum theory. We say such a symmetry is *anomalous*. How does an anomaly arise in quantum field theory? When quantizing the classical action we place it inside the path integral:

$$Z = \int \mathcal{D}\psi\, e^{S[\psi]}. \tag{2.44}$$

Under a classical symmetry $\psi \to \psi'$, the path integral transforms as:

$$Z = \int \mathcal{D}\psi\, e^{S[\psi]} \to \int \mathcal{D}\psi'\, e^{S[\psi']} = \int \mathcal{D}\psi'\, e^{S[\psi]}. \tag{2.45}$$

So the obstruction for the transformation to be a symmetry in the quantum theory must arise from the measure of our matter fields. In fact, as we will see later, for a transformation $\psi' = e^{i\epsilon(x)t}\psi$, $e^{i\epsilon(x)t} \in G$, where t is the symmetry generator with parameter $\epsilon(x)$, the anomaly arises as a phase in the path integral:

$$\mathcal{D}\psi' = \mathcal{D}\psi e^{i\int d^4x \epsilon(x)\mathcal{A}_\alpha(x)}. \tag{2.46}$$

where $\mathcal{A}_\alpha(x)$ is the anomaly. Let us compute the anomaly explicitly for the chiral symmetry of the Dirac fermion coupled to a $U(1)$ gauge theory, known as the ABJ (Adler-Bell-Jackiw) anomaly. We will follow the derivation in [21].

2.2.1 The ABJ anomaly

An anomaly arises from the matter measure. To define the matter measure, we need to first define the Lagrangian for our matter fields. For the Dirac fermion ψ, i.e.

$$\mathcal{L}_{\text{matter}}\left[\psi, \overline{\psi}, D_\mu\psi, D_\mu\overline{\psi}\right] = \overline{\psi}\left(\not{D} + m\right)\psi. \tag{2.47}$$

Let ψ_n be a basis of eigenfunctions of the operator $\not{D}+m$, i.e.

$$(\not{D}+m)\,\psi_n = \lambda_n \psi_n. \tag{2.48}$$

We can expand any field ϕ in this basis of eigenfunctions:

$$\psi = \sum_n c_n \psi_n. \tag{2.49}$$

The matter measure is then defined as:

$$\mathcal{D}\psi = \prod_n c_n. \tag{2.50}$$

We want to see how the matter measure transforms under the chiral symmetry. Under a general local transformation

$$\psi \to \widetilde{\psi} = U(x)\psi, \tag{2.51}$$

the Dirac conjugate transforms as:

$$\overline{\psi} = \psi^\dagger i\gamma^0 \to (U(x)\psi)^\dagger\, i\gamma^0 = \psi^\dagger i\gamma^0 \left(i\gamma^0 U(x)^\dagger i\gamma^0\right) = \overline{\psi}\,\overline{U}(x) = \widetilde{\overline{\psi}}. \tag{2.52}$$

We can write the local transformation as

$$\widetilde{\psi}(x) = \int d^4y\, \mathcal{U}(x,y)\psi(y), \tag{2.53}$$

$$\widetilde{\overline{\psi}}(x) = \int d^4y\, \overline{\mathcal{U}}(x,y)\psi(y), \tag{2.54}$$

where the operators $\mathcal{U}$ and $\overline{\mathcal{U}}$ are given by:

$$\mathcal{U}(x,y) = \langle x|\,\mathcal{U}\,|y\rangle = U(x)\delta^{(4)}(x-y), \tag{2.55}$$

$$\overline{\mathcal{U}}(x,y) = \langle x|\,\overline{\mathcal{U}}\,|y\rangle = \overline{U}(x)\delta^{(4)}(x-y). \tag{2.56}$$

Dirac fermions anticommute, so the fermion measure transforms as:

$$\mathcal{D}\widetilde{\psi} = (\mathrm{Det}\,\mathcal{U})^{-1}\,\mathcal{D}\psi, \qquad \mathcal{D}\widetilde{\overline{\psi}} = \left(\mathrm{Det}\,\overline{\mathcal{U}}\right)^{-1}\mathcal{D}\overline{\psi}, \tag{2.57}$$

$$\implies \mathcal{D}\psi\mathcal{D}\overline{\psi} \to \mathcal{D}\widetilde{\psi}\mathcal{D}\widetilde{\overline{\psi}} = \left(\mathrm{Det}\,\mathcal{U}\,\mathrm{Det}\,\overline{\mathcal{U}}\right)^{-1}\mathcal{D}\psi\mathcal{D}\overline{\psi} \tag{2.58}$$

For a symmetry transformation to be non-anomalous, we require $\mathrm{Det}\,\mathcal{U}\mathrm{Det}\,\overline{\mathcal{U}} = |\mathrm{Det}\,\mathcal{U}| = 1$, or in other words we require the transformation to be unitary.

The chiral transformation acts on the fermion as *:

$$U(x) = e^{i\epsilon(x)t\gamma_5}, \qquad [t,\gamma^\mu] = 0, \tag{2.59}$$

*We pick the convention $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

where t is the generator for the $U(1)$ symmetry. Note that the requirement $[t, \gamma^\mu] = 0$ tells us that the transformation is a symmetry of the matter Lagrangian. Under a chiral transformation, the Dirac conjugate matrix $\overline{U}$ transforms as:

$$\overline{U}(x) = i\gamma^0 U^\dagger(x) i\gamma^0 = i\gamma^0 e^{-i\epsilon(x)t\gamma_5} i\gamma^0 = e^{i\epsilon(x)t\gamma_5} = U(x), \tag{2.60}$$

where we used $\left(\sum_n (\gamma_5)^n\right) i\gamma^0 = i\gamma^0 \sum_n (-\gamma_5)^n$. Therefore $\mathcal{U} = \overline{\mathcal{U}}$ and:

$$\mathrm{Det}\,\mathcal{U}\,\mathrm{Det}\,\overline{\mathcal{U}} = e^{2\int d^4x \mathrm{tr}\,(i\epsilon(x)t\gamma_5)\Lambda^4} \neq 1, \tag{2.61}$$

where the trace is taken over Dirac indices. So a chiral transformation leads to a non-trivial transformation of the fermion measure.

2.2.2 The Fujikawa Method

Given that the fermion measure transforms non-trivially under a chiral transformation, we can write

$$\left(\mathrm{Det}\,\mathcal{U}\,\mathrm{Det}\,\overline{\mathcal{U}}\right)^{-1} = (\mathrm{Det}\,\mathcal{U})^{-2} = e^{i\int d^4x \epsilon(x)\mathcal{A}(x)}, \tag{2.62}$$

where $\mathcal{A}(x) = -2\Lambda^4 \mathrm{tr}\,(t\gamma_5)$ is the anomaly. However we need to be careful, as the Dirac trace of γ_5 is zero, and taking the limit of $\Lambda \to \infty$ gives

$$\mathcal{A}(x) = -2(0 \times \infty). \tag{2.63}$$

To obtained a well-defined expression for the anomaly, we will have to regularize it. What follows is a calculation for the abelian anomaly from the transformation of the path integral, known as the *Fujikawa method*.

Let us introduce a cutoff function, g, satisfying the properties $g(0) = 1, g(\infty) = 0, sg'(s) = 0$ at $s = 0, \infty$ *. The regulated exponent is:

$$\begin{aligned}
\mathrm{Tr}\,\log\mathcal{U}\Big|_{\mathrm{reg}} &= i\mathrm{Tr}\left(g\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)\epsilon(x)t\gamma_5\right) \\
&= i\int d^4x \mathrm{tr}\,\langle x|\,g\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)\epsilon(\hat{x})t\gamma_5\,|x\rangle \\
&= i\int d^4x\,\epsilon(x)\int d^4p\,\langle x|p\rangle\,\langle p|\,\mathrm{tr}\,g\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)t\gamma_5\,|x\rangle \\
&= i\int d^4x\,\epsilon(x)\int d^4p\frac{e^{ip\cdot x}}{(2\pi)^2}\mathrm{tr}\,g\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)t\gamma_5\frac{e^{-ip\cdot x}}{(2\pi)^2},
\end{aligned}$$

*An example of such a cutoff function would be $g(s) = e^{-s}$.

where we used $\langle x|p\rangle = \frac{e^{ip\cdot x}}{(2\pi)^2}$. To proceed, it is helpful to compute:

$$\begin{aligned}(i\not{D})^2 = -\gamma^\mu\gamma^\nu D_\mu D_\nu &= -\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right) D_\mu D_\nu \\ &= -\frac{1}{2}2\eta^{\mu\nu}D_\mu D_\nu - \gamma^{\mu\nu}D_\mu D_\nu \qquad\qquad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \\ &= -D_\mu D_\nu + \frac{i}{2}\gamma^{\mu\nu}f_{\mu\nu},\end{aligned}$$

where f is the $U(1)$ field strength. We therefore have:

$$\implies g\left(\left(\frac{i\not{D}}{\Lambda}\right)^2\right)e^{-ip\cdot x} = e^{-ip\cdot x} g\left(\frac{-(D_\mu - ip_\mu)(D^\mu - ip^\mu) + \frac{i}{2}\gamma^{\mu\nu}f_{\mu\nu}}{\Lambda^2}\right). \tag{2.64}$$

We thus have:

$$\begin{aligned}\text{Tr }\log\mathcal{U}\Big|_{\text{reg}} &= i\int d^4x\,\epsilon(x)\int\frac{d^4p}{(2\pi)^4}\text{tr}\,g\left(\frac{-(D_\mu - ip_\mu)(D^\mu - ip^\mu) + \frac{i}{2}\gamma^{\mu\nu}f_{\mu\nu}}{\Lambda^2}\right)t\gamma_5 \\ &= i\int d^4x\,\epsilon(x)\int\frac{d^4q}{(2\pi)^4}\Lambda^4\text{tr}\,g\left(\frac{-(D_\mu - i\Lambda q_\mu)(D^\mu - i\Lambda q^\mu) + \frac{i}{2}\gamma^{\mu\nu}f_{\mu\nu}}{\Lambda^2}\right)t\gamma_5 \\ &= i\int d^4x\,\epsilon(x)\int\frac{d^4q}{(2\pi)^4}\Lambda^4\text{tr}\,g\left(q^2 + \frac{2iq\cdot D}{\Lambda^2} - \frac{D^2}{\Lambda^2} + \frac{i}{2\Lambda^2}\gamma^{\mu\nu}f_{\mu\nu}\right)t\gamma_5,\end{aligned}$$

where to go from the first line to the second line we substituted $p^\mu = \Lambda q^\mu$. We wish to take the limit $\Lambda \to \infty$, so we Taylor expand about q^2. Taking the Dirac trace of gamma matrices, we discover that only quadratic or higher powers of $\frac{i}{2\Lambda^2}\gamma^{\mu\nu}f_{\mu\nu}$ are non-vanishing, as we have the gamma matrix identity

$$\text{tr}\,\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma_5 \neq 0 \qquad \text{iff } n \geq 4. \tag{2.65}$$

On the other hand, only the quadratic term is non-vanishing when we take the limit $\Lambda \to \infty$ as there is a factor of Λ^4 in the integrand. Therefore as we take $\Lambda \to \infty$, we extract only the quadratic term and obtain:

$$\text{Tr }\log\mathcal{U}\Big|_{\text{reg}} = i\int d^4x\,\epsilon(x)\int\frac{d^4q}{(2\pi)^4}\frac{1}{2}g''\left(q^2\right)\text{tr}\left(\frac{i}{2}\gamma^{\mu\nu}f_{\mu\nu}\right)t\gamma_5, \tag{2.66}$$

$$\begin{aligned}\text{tr}\left(\frac{i}{2}\gamma^{\mu\nu}f_{\mu\nu}\right)^2 t\gamma_5 &= -\frac{1}{4}(f_{\mu\nu}f_{\rho\sigma}t)\text{tr}\,(\gamma^{\mu\nu}\gamma^{\rho\sigma}\gamma_5) \\ &= -\frac{1}{4}(f_{\mu\nu}f_{\rho\sigma}t)\text{tr}\left(\frac{1}{4}[\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\gamma_5\right) \\ &= -\frac{1}{4}(f_{\mu\nu}f_{\rho\sigma}t)4i\epsilon^{\mu\nu\rho\sigma} \\ &= -i\epsilon^{\mu\nu\rho\sigma}f_{\mu\nu}f_{\rho\sigma}t,\end{aligned}$$

where we used $\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 = 4i\epsilon^{\mu\nu\rho\sigma}$. We also have:

$$\begin{aligned}
\int \frac{d^4 q}{(2\pi)^4} \frac{1}{2} g''\left(q^2\right) &= \frac{i}{(2\pi)^4} \int d^4 q_E \frac{1}{2} g''\left(q_E^2\right) && \text{by Wick rotation} \\
&= \frac{i}{16\pi^4} \int dq_E\, q_E^3 \operatorname{vol}\left(S^3\right) \frac{1}{2} g''\left(q_E^2\right) \\
&= \frac{i}{16\pi^2} \int dq_E\, q_E^3\, g''\left(q_E^2\right) && \text{as } \operatorname{vol}\left(S^3\right) = 2\pi^2 \\
&= \frac{i}{16\pi^2} \int du\, \frac{1}{2} u g''(u) && u = q_E^2 \\
&= \frac{i}{32\pi^2} \left(\left[u g'(u)\right]_0^\infty - \int_0^\infty g'(u) du \right) \\
&= \frac{i}{32\pi^2},
\end{aligned}$$

where in the first line we Wick rotated to Euclidean coordinates in order to use spherical coordinates in $\mathbb{R}^4$, and we computed the final integral using the properties $sg'(s) = 0$ at $s = 0, \infty$, and $g(0) = 1, g(\infty) = 0$. Combining both expressions, we obtain:

$$\operatorname{Tr} \log \mathcal{U}\Big|_{\text{reg}} = i \int d^4 x\, \epsilon(x) \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} f_{\mu\nu}(x) f_{\rho\sigma}(x) t. \tag{2.67}$$

Comparing with out previous expressions,

$$e^{i \int d^4 x \epsilon(x) \mathcal{A}(x)} = (\operatorname{Det} \mathcal{U})^{-2} = e^{-2 \operatorname{Tr} \log \mathcal{U}}. \tag{2.68}$$

The ABJ, or axial, anomaly is therefore given by:

$$\mathcal{A}(x) = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu}(x) f_{\rho\sigma}(x) t. \tag{2.69}$$

If the fermion transforms with charge q under the chiral symmetry, the anomaly obtains the coefficient q.

The Fujikawa method readily generalizes to non-abelian gauge theories, and we omit the derivation for brevity. For N_f Dirac fermions transforming in the representation $\mathcal{R}$ under $SU(N)$ gauge transformation, the ABJ anomaly is given by a similar expression:

$$\mathcal{A}(x) = -\frac{N_f T_{\mathcal{R}}}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} f^c_{\mu\nu}(x) f^c_{\rho\sigma}(x) t, \tag{2.70}$$

where now the trace is taken over Lie algebra indices. In differential form notation, we have

$$\mathcal{A} = -\frac{N_f T_{\mathcal{R}}}{8\pi^2} f_2^c \wedge f_2^c. \tag{2.71}$$

2.2.3 The Anomalous Ward Identity

Classically, Noether's theorem tells us that a global symmetry gives rise to a conserved Noether current j^μ, i.e. $\partial_\mu j^\mu = 0$ if the equations of motion are satisfied.

If the symmetry is non-anomalous, we would also have Ward identities, where the current conservation equation is understood to hold inside correlation functions:

$$\langle \partial_\mu j^\mu \rangle = 0. \tag{2.72}$$

In the presence of an anomaly, the Ward identity does not hold and the non-conservation of the current is given by the anomaly itself. The global axial symmetry of the massless fermion gives rise to a Noether current

$$j_A^\mu = i\overline{\psi}\gamma^\mu\gamma_5\psi, \tag{2.73}$$

which is conserved on the classical level. But the presence of the abelian anomaly means that the Ward identities are no longer trivially true. Let us try to compute $\langle \partial_\mu j_A^\mu \rangle$:

$$\begin{aligned}\int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_{\text{matter}}[\psi,\overline{\psi},A]} &= \int \mathcal{D}\psi'\mathcal{D}\overline{\psi}'e^{iS_{\text{matter}}[\psi',\overline{\psi}',A]} \\ &= \int \mathcal{D}\psi\mathcal{D}e^{i\int \epsilon(x)a(x)d^4x}e^{iS_{\text{matter}}[\psi,\overline{\psi},A]+\int \partial_\mu j_A^\mu(x)\epsilon(x)d^4x} \\ &\approx \int \mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_{\text{matter}}}\left(1+i\int d^4x\,\epsilon(x)\left(\mathcal{A}(x)+\partial_\mu j_A^\mu(x)\right)\right).\end{aligned}$$

So instead of the usual Ward identity, we have:

$$\partial_\mu \langle j_A^\mu \rangle = -\mathcal{A}(x), \tag{2.74}$$

where the correlation function is understood to be taken with a fixed background gauge field.

2.2.4 't Hooft Anomalies

From our previous discussion, we can interpret anomalies as arising from transformations in the presence of gauge backgrounds. Given a non-anomalous global symmetry G_{global}, it is possible that coupling backgrounds for G_{global} can give rise to new anomalies. This is known as a *'t Hooft anomaly.* The presence of a 't Hooft anomaly can give us some information about the IR phase of the theory via *'t Hooft anomaly matching.*

't Hooft's original argument is as follows. Suppose we have a theory equipped with a global symmetry G_{global} in the UV, which would have a gauge anomaly if we attempt to gauge it. This is known as a *'t Hooft anomaly.* If we attempt to gauge the global symmetry, the theory will be inconsistent due to the anomaly. To remedy this we can introduce "spectator" massless Weyl fermions that couple only to the gauge field of G_{global} such that net 't Hooft anomaly vanishes. We can choose to have the confining scale of the G_{global} gauge field to be smaller than every other energy scale of the original theory such that the original IR spectrum will not be affected by introducing the G_{global} gauge field. Since anomalies are invariant under renormalization group (RG) flow, the net G_{global} anomaly in the IR remains zero, but the spectator fermions still contribute a non-vanishing 't Hooft anomaly. Therefore the spectrum of the original theory must contain states that cancel the anomaly arising from the spectator fermions. For example, G_{global} could be spontaneously broken leading to massless Goldstone bosons. Or the theory could confine, preserving G_{global} and containing massless composite fermions in the IR. In essence, a theory with a 't Hooft anomaly cannot be trivially gapped in the IR.

In our discussion of higher-form symmetries, we have constructed 't Hooft fluxes for 1-form symmetries on the 4-torus. As there are new gauge backgrounds thanks to the presence of higher-form symmetries, we might obtain new anomalies. As an example, consider an $SU(N)$ gauge theory with a single Dirac fermion ψ transforming in the representation $\mathcal{R}$ of the gauge group. Thanks to the ABJ anomaly, the path integral transforms by a phase under an axial $U(1)_A$ rotation:

$$\mathcal{Z}[\psi] \longrightarrow \mathcal{Z}[\psi] \exp\left(\frac{2T_{\mathcal{R}}}{8\pi^2}\int f_2^c \wedge f_2^c\right). \tag{2.75}$$

For the path integral to remain invariant, the axial $U(1)_A$ symmetry is broken to $\mathbb{Z}_{2T_{\mathcal{R}}}$, where $T_{\mathcal{R}}$ is the Dynkin index of the representation $\mathcal{R}$, normalized such that in the fundamental representation, $\operatorname{tr} T^a T^b = 1$. As explained in the discussion around equation 2.22, in the presence of the fermion ψ the 1-form symmetry is $\mathbb{Z}_p, p = \gcd(N, n_{\mathcal{R}})$. When we gauge this 1-form symmetry, the theory permits gauge backgrounds with topological charge $Q \in \frac{1}{p}\mathbb{Z}$. Since the ABJ anomaly depends on the quantization of the topological charge of gauge configurations, there is a mixed anomaly between the chiral symmetry and the 1-form symmetry. In chapter four we will leverage this mixed anomaly to define a non-invertible symmetry.

2.3 Non-Invertible Symmetries

It is possible that given a symmetry $U^{(p)}(\mathcal{M}_{d-p})$, there is no symmetry operator $U^{-1(p)}(\mathcal{M}_{d-p})$ which satisfies

$$U^{(p)}(\mathcal{M}_{d-p})\, U^{-1(p)}(\mathcal{M}_{d-p}) = 1. \tag{2.76}$$

We say that $U^{(p)}(\mathcal{M}_{d-p})$ is a *non-invertible* symmetry.

A relevant example for our this thesis is the non-invertible $U(1)_A$ axial symmetry in QED [22, 23, 24]. For concreteness, consider QED with a single Dirac fermion in Euclidean spacetime. The ABJ anomaly in this case is given by:

$$d \star j_A = \frac{2}{8\pi^2} f_2 \wedge f_2. \tag{2.77}$$

Thanks to the anomaly, the naive symmetry operator for the axial symmetry

$$U_\alpha(\mathcal{M}_3) = \exp\left(i\alpha \int_{\mathcal{M}_3} \star j_A\right) \tag{2.78}$$

is not topological as the current is not conserved. However we can define the new current

$$\hat{j}_A = j_A - \frac{1}{4\pi^2} a \wedge da, \tag{2.79}$$

which is conserved thanks to the ABJ anomaly. One might therefore be tempted to write the topological operator

$$\exp\left(i\alpha \int_{\mathcal{M}_3} \star j_A - \frac{2}{8\pi^2} a \wedge da\right). \tag{2.80}$$

For manifolds with non-trivial $H_2(\mathcal{M}_3, \mathbb{Z})$, this operator is not invariant under large gauge transformations for arbitrary $\alpha \in [0, \pi)$ * as the Chern-Simons current, $a \wedge da$, is improperly quantized. On a spin manifold, one would require the level of the Chern-Simons current to be an integer, i.e. $\alpha \in 2\pi\mathbb{Z}$. There would be no non-trivial symmetries as a consequence.

If we take $\alpha = \frac{\pi}{N}$, we would nonetheless obtain an interesting symmetry operator. The non-gauge invariant term is

$$-\frac{i}{4\pi N} \int_{\mathcal{M}_3} a \wedge da. \tag{2.81}$$

There is a gauge invariant version of this term - it is the action for the fractional quantum hall state in 2+1D with filling fraction $\frac{1}{N}$ [25, 22, 23, 10]

$$\int_{\mathcal{M}_3} \frac{iN}{4\pi} b \wedge db + \frac{i}{2\pi} b \wedge da, \tag{2.82}$$

where b is an auxiliary dynamical $U(1)$ gauge field living on $\mathcal{M}_3$. We could try to integrate out b to obtain the term 2.81, but the field $b = -a/N$ would be an improperly quantized $U(1)$ gauge field. Using this action we can write down a gauge-invariant, topological operator:

$$\tilde{U}_{\alpha=\frac{\pi}{N}}(\mathcal{M}_3) = \int [\mathcal{D}b] \exp\left(i \int_{\mathcal{M}_3} \frac{\pi}{N} \star j_A + \frac{N}{4\pi} b \wedge db + \frac{1}{2\pi} b \wedge da\right). \tag{2.83}$$

In general we can define a symmetry operator for any $\alpha = p\pi/N, \gcd(p, N) = 1$:

$$\tilde{U}_{\alpha=\frac{p\pi}{N}}(\mathcal{M}_3) = \int [\mathcal{D}b] \exp\left(ip \int_{\mathcal{M}_3} \frac{\pi}{N} \star j_A + \frac{N}{4\pi} b \wedge db + \frac{1}{2\pi} b \wedge da\right). \tag{2.84}$$

This operator acts on local operators via a chiral rotation - the fermion $\Psi(x)$ located inside a contractible $\mathcal{M}_3$ picks up a phase when we deform the symmetry operator to a point:

$$\tilde{U}_{\alpha=\frac{p\pi}{N}}(\mathcal{M}_3)\Psi(x) = e^{i\alpha\gamma_5}\Psi(x). \tag{2.85}$$

*In this normalization $a \in [0, \pi)$ as $\alpha = \pi$ acts as the fermion number transformation, $\Psi \to (-1)\Psi$, which is redundant with the $U(1)$ gauge symmetry.

To see its non-invertible nature, we can look at its action on 't Hooft lines. When acting on a 't Hooft line, the 't Hooft line becomes attached to a topological surface operator $\exp\left(\frac{ip}{N}\int F\right)$ stretching from the 't Hooft line to the symmetry operator. We can also consider the condition arising from integrating out b on the manifold $\mathcal{M}_3$, $b = -a/N$. b is only a properly quantized $U(1)$ gauge field if a has flux N. In other words, $\tilde{U}_{\alpha=\frac{p\pi}{N}}(\mathcal{M}_3)$ projects out all gauge configurations in the path integral except for those with topological charge $N\mathbb{Z}$.

In chapter four, we will study an analogous non-invertible symmetry in non-abelian gauge theories. We will see that the chiral symmetry in $SU(N)$ gauge theory becomes non-invertible after gauging a $\mathbb{Z}_q$ subgroup of the Z_N center, and has a similar effect of projecting out certain 't Hooft flux backgrounds in the path integral. We will investigate the associated anomalies for this symmetry and argue that the ground state spectrum have multi-fold degeneracies.

2.4 Higher Group Symmetries

Consider a theory with a p-form and q-form symmetry, $p > q$, coupled to background gauge fields with A_{p+1} and A_{q+1} respectively. If the background field A_{p+1} transforms non-trivially under a background gauge transformation for the q-form symmetry, we say the theory has a $p+1$-*group symmetry.*

The higher group structure of an emergent symmetry contains information about the order in which the associated symmetries emerge along RG flows [26, 27, 10]. Suppose a p-form symmetry emerges at the energy scale E_p, and the q-form symmetry emerges at the scale E_q. In the scenario where the symmetries combine to form a $p+1$-group symmetry ($p > q$), if we couple A_{q+1}, A_{p+1} must be present in order for the theory to remain invariant under background q-form symmetry transformations. Therefore the energy scales of emergence obey the inequality

$$E_p \geq E_q. \tag{2.86}$$

In chapter five, we will leverage higher group symmetries in Axion Yang-Mills systems to study the IR behaviour of the theory. One of the higher-group symmetries we discuss was first introduced in [27]. Consider an axion a coupled to an $SU(N)$ gauge theory, with Lagrangian

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{1}{2g^2} \mathrm{tr}\, f_2^c \wedge \star f_2^c - \frac{K}{8\pi^2} a \mathrm{tr}\, f_2^c \wedge f_2^c \tag{2.87}$$

for some integer K determined by the UV completion of the theory. The two symmetries involved in a 3-group structure is the 1-form $\mathbb{Z}_N^{(1)}$ symmetry and the 2-form winding symmetry has the three-form current

$$j_3 = \star da. \tag{2.88}$$

The charged object of the 2-form symmetry is the axion string, which we assume to transform with charge 1.

If we simultaneously couple the background field for the winding symmetry, C_3, and the background field for the $\mathbb{Z}_N^{(1)}$ form symmetry, $B_2^{(N)}$, the Lagrangian becomes:

$$\begin{aligned}\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{1}{2g^2} \mathrm{tr}\,(f_2^c - B_2^{(N)}) \wedge \star (f_2^c - B_2^{(N)}) \\ - \frac{K}{8\pi^2} a \mathrm{tr}\,(f_2^c - B_2^{(N)}) \wedge (f_2^c - B_2^{(N)}) - \frac{1}{2\pi} da \wedge C_3. \end{aligned} \tag{2.89}$$

The axion couples to the background fields in the following manner:

$$\frac{1}{2\pi} a G_4 = \frac{1}{2\pi} a \left(dC_3 - \frac{KN}{4\pi} B_2^{(N)} \wedge B_2^{(N)} \right). \tag{2.90}$$

The field strength G_4 combines both background fields. It is invariant under a $\mathbb{Z}_N^{(1)}$ background gauge transformation, $B_2^{(N)} \to B_2^{(N)} + d\lambda_1^{(N)}$, only if C_3 also transforms as:

$$C_3 \to C_3 + \frac{KN}{2\pi} \lambda_1^{(N)} \wedge B_2^{(N)} + \frac{KN}{4\pi} \lambda_1^{(N)} \wedge d\lambda_1^{(N)}. \tag{2.91}$$

We see that the $U(1)^{(2)}$ and $\mathbb{Z}_N^{(1)}$ symmetries combine to form a 3-group structure.

Chapter 3

2 Index chiral gauge theories

3.1 Introduction

We begin by studying the generalized symmetries and anomalies in chiral gauge theories. Chiral gauge theories form the fundamental framework of the Standard Model (SM) of particle physics. Within the SM, the electroweak sector undergoes Higgsing at weak coupling, allowing us to apply perturbative techniques. However, without a Higgs field, gauge theories generally flow towards a strongly-coupled regime, rendering their study considerably more challenging. A non-comprehensive list of some of the recent papers that studied chiral gauge theories is [28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

This chapter focuses on a class of $SU(N)$ chiral gauge theories that accommodate fermions in the 2-index symmetric and anti-symmetric representations. These theories, referred to as 2-index chiral gauge theories, can be characterized by the pair (N, k), where N represents the color and k serves as a common divisor of $N+4$ and $N-4$. Moreover, k is directly associated with the number of flavors in the 2-index symmetric and anti-symmetric representations. What makes these theories particularly intriguing is the absence of a requirement to introduce fundamental fermions to cancel the gauge anomaly. Additionally, they are non-renormalizable for $N > 44$. This class encompasses a collection of 14 distinct theories, occupying a distinct region within asymptotically-free chiral gauge theories. Consequently, a systematic approach to studying this class is justified. It is divided into two subclasses: bosonic and fermion theories. The latter can accommodate gauge-invariant massless fermions. In comparison, the gauge-invariant operators in bosonic theories cannot have a spinor index, as the fermion number is gauged.

This study was first initiated in [35], utilizing 't Hooft anomalies to constrain the infrared dynamics of two theories. Namely, these are $(N=8, k=4)$ and $(N=8, k=2)$ theories. One important development was the identification of the faithful global symmetry acting on fermions. This led to turning on the most general discrete fluxes, the color-flavor-$U(1)$ (CFU) fluxes compatible with the theory and, thus, utilize the full power of 't Hooft anomaly matching conditions. These anomalies are dubbed CFU anomalies. A theory with 't Hooft anomalies cannot be trivially gapped; the infrared (IR) spectrum must contain massless particles or multi-vacua. In the case $(N = 8, k = 4)$, it was found that the condensation of two operators can saturate the anomalies.

Continuing the explorations within this comprehensive framework, our investigation exhausts all the 14 theories and introduces a few novel aspects.

1. We incorporate anomalies stemming from discrete symmetries, thereby imposing additional constraints on the infrared spectra. In the context of the 2-index chiral gauge theories we are examining, in addition to continuous non-abelian flavor symmetries, an axial $U(1)_A$ symmetry comes into play. When a bosonic operator condenses, it generally breaks the $U(1)_A$ symmetry down to a discrete subgroup, which typically is anomalous. Consequently, we face the challenge of identifying a set of condensates that not only matches the anomalies associated with non-abelian symmetries but also avoids the presence of any anomalous unbroken discrete subgroups. For example, the two candidate condensates we previously considered in the case $(N = 8, k = 4)$, [35], fail to match the anomaly of an unbroken discrete group. We revise the situation in light of the new understanding and propose that the set of anomalies can be matched by other condensates.

   Interestingly, in a few cases, matching the full set of anomalies, particularly anomalies of discrete symmetries, can be achieved only via the condensation of multiple higher-order operators. Given the strong dynamics, such formation is not a surprise. However, anomalies explain the kinematical reasons why such condensates have to form. *

2. Another significant aspect of this analysis lies in our pursuit of the minimal scenario that satisfies the entire set of anomalies and yields the smallest number of massless particles in the infrared spectrum. Such a scenario holds particular appeal as it minimizes the free energy associated with the theory.

3. We adhered to a systematic algorithm during our quest to identify composite massless fermions capable of satisfying the anomalies within the fermionic class. Regrettably, we were unable to find such composites. Notably, in the case of $(N = 6, k = 2)$, we demonstrate that these composites cannot solely match the CFU anomaly. Consequently, we are left with two plausible explanations: either these composites do not exist altogether, or the formation of condensates alongside the composites is necessary to match the anomaly. However, the latter scenario is rather contrived in that it requires the formation of special condensates (that do not alter the anomalies matched by

*The CFU anomalies also allow us to eliminate the possibility of anomalous gapped sectors. We discuss this possibility briefly in section 3.3.

the composites) in addition to the composite fermions, prompting us to lean toward the likelihood that composites cannot form in this case.

4. To complete our study, we also examined the possibility that a theory flows to a conformal fixed point in the IR. Generally speaking, a theory can form a strongly-coupled IR fixed point, which is beyond the scope of any perturbative analysis. Such a fixed point automatically satisfies the full set of anomalies, albeit it remains an open question how this can be seen. Nevertheless, by scrutinizing the higher-order β-function, we successfully identified several instances where the analysis of the β-function offers indications suggestive of a perturbative nature inherent to a fixed point.

This chapter is organized as follows. In Section 3.2, we review the global symmetries and anomalies of the 2-index chiral theories. This includes the anomalies of continuous symmetries, the CFU anomalies, as well as anomalies of discrete symmetries. In Section 3.3, we revise the matching of the CFU anomalies in the IR and introduce the novelty of matching the discrete subgroups of the axial $U(1)_A$ that can be left unbroken by a condensate. Sections 3.4 and 3.5 are devoted to applying these ideas to both the fermionic and bosonic field theories, respectively. We summarize our findings in Section 3.6, and in particular, the reader is referred to Table 3.10, which, for all theories, it gives the global symmetries, the proposed IR condensates that yield the smallest number of Goldstones, and the fate of the symmetries in the IR.

3.2 Theory: symmetry structure and anomalies

We consider $SU(N)$ gauge theory with n_ψ flavors of left-handed Weyl fermions ψ transforming in the 2-index symmetric representation along with n_χ flavors of left-handed Weyl fermions χ transforming in the complex conjugate 2-index anti-symmetric representation:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}[f^c \wedge \star f^c] - i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi - i\bar{\chi}\bar{\sigma}^\mu D_\mu \chi \,, \tag{3.1}$$

where $D_\mu \equiv \partial_\mu - i a^c_\mu$ is the covariant derivative, a^c is the color gauge field, and f^c is its field strength. In this work, we use the lower-case letters, a^c and $f^c = da^c$, to denote the dynamical (color) 1-form gauge field and its field strength, while we use upper-case letters, A and $F = dA$, for background fields. To keep track of the color indices, we choose $\psi_{(a_1 a_2)}$ to carry two down indices, while $\chi_{[a_1 a_2 \ldots a_{N-2}]}$ carries $N-2$ down indices. A round (square) bracket indicates symmetrizing (anti-symmetrizing) over the indices. The cubic anomaly coefficients of the 2-index symmetric and the conjugate of the 2-index anti-symmetric representations are $\mathcal{A}_\psi = N+4$, $\mathcal{A}_\chi = -(N-4)$, respectively. Cancellation of the gauge anomaly

N	5	6	8	10	12	16	20	28	36	44
k	1	1,2	2,4	2	4,8	4	4,8	8	8	8

.

Table 3.1: A list of the 2-index chiral gauge theories.

demands that n_ψ and n_χ are fixed as

$$n_\psi = \frac{N-4}{k}\,, \quad n_\chi = \frac{N+4}{k}\,, \tag{3.2}$$

where k is a common divisor of $N-4$ and $N+4$. The theory is asymptotically free provided that $11N - \frac{2(N^2-8)}{k} > 0$. This leaves us with the finite set of theories in Table 3.1. These theories do not possess a large-N limit, as they become infrared-free for $N > 44$. Also, except for $N = 5, 6, 10$, the other allowed colors are multiples of 4. These are bosonic theories because all their gauge invariant operators cannot carry a spinor index. In other words, the $(-1)^F$ fermion number in bosonic theories is gauged, and thus, they cannot have gauge-singlet fermionic operators.

One important aspect of this work is to systematically analyze these theories, paying particular attention to the faithful global symmetries, and exhausting the class of generalized 't Hooft anomalies that enable us to constrain the infrared phases.

3.2.1 Symmetries

The theory enjoys two global flavor groups $SU(n_\psi) = SU((N-4)/k)$ and $SU(n_\chi) = SU((N+4)/k)$ acting on ψ and χ, respectively. In addition, the theory is endowed with two $U(1)$ global classical symmetries, $U(1)_1 \times U(1)_2$. Their action on ψ and χ is chosen as

$$\begin{aligned} U(1)_1:& \quad \psi \longrightarrow e^{i\alpha_1}\psi\,, \quad \chi \longrightarrow e^{i\beta_1}\chi\,,\\ U(1)_2:& \quad \psi \longrightarrow e^{i\alpha_2}\psi\,, \quad \chi \longrightarrow e^{i\beta_2}\chi\,. \end{aligned} \tag{3.3}$$

The two transformations $U(1)_1$ and $U(1)_2$ come naturally with two parameters. Here, however, we introduce the 4 parameters $\alpha_{1,2}$ and $\beta_{1,2}$ to account for the fermions charges, in addition to the transformation parameters. The gauge sector instantons break most of the classical $U(1)$ symmetries. The effective action in the instanton background acquires the terms

$$\begin{aligned} \Delta S = i\left(n_\psi \alpha_1 T_\psi + n_\chi \beta_1 T_\chi\right) \int_{\mathcal{M}^4} \lambda_0^{u_1} \frac{\text{tr}\left[f^c \wedge f^c\right]}{8\pi^2} \\ + i\left(n_\psi \alpha_2 T_\psi + n_\chi \beta_2 T_\chi\right) \int_{\mathcal{M}^4} \lambda_0^{u_2} \frac{\text{tr}\left[f^c \wedge f^c\right]}{8\pi^2} \end{aligned} \tag{3.4}$$

upon performing simultaneous transformations of $U(1)_1 \times U(1)_2$, where

$$T_\psi = N+2\,, \quad T_\chi = N-2 \tag{3.5}$$

are the Dynkin indices of the representations. Here, f^c is the 2-form field strength of the color group, while λ^{u_1} and λ^{u_2} are the gauge parameters of $U(1)_1$ and $U(2)_2$, respectively, i.e., $A^{u_{1,2}} \longrightarrow A^{u_{1,2}} + d\lambda^{u_{1,2}}$. We can find a combination of the parameters α_1 and β_1 that kills the first term in ΔS, leaving behind a genuine symmetry. We call this symmetry the axial $U(1)_A$. It acts on ψ and χ with transformation parameter α:

$$U(1)_A : \quad \psi \longrightarrow e^{i2\pi\alpha q_\psi}\psi\,, \quad \chi \longrightarrow e^{i2\pi\alpha q_\chi}\chi\,, \tag{3.6}$$

and we have defined the $U(1)_A$ charges of ψ and χ as

$$q_\psi \equiv -\frac{N_\chi}{r}\,, \quad q_\chi \equiv \frac{N_\psi}{r}\,, \tag{3.7}$$

where $r = \gcd(n_\chi T_\chi, n_\psi T_\psi)$, and

$$N_\chi \equiv n_\chi T_\chi\,, \quad N_\psi \equiv n_\psi T_\psi\,. \tag{3.8}$$

Yet, we can find values of α_2 and β_2 that leave the discrete subgroup $\mathbb{Z}_{p_\psi N_\psi + p_\chi N_\chi} \subset U(1)_2$ invariant in the color background, where p_ψ and p_χ are arbitrary integers. In Appendix B, we show that most of the $\mathbb{Z}_{p_\psi N_\psi + p_\chi N_\chi}$ elements belong to $U(1)_A$ and that only a subgroup $\mathbb{Z}_r \subset \mathbb{Z}_{p_\psi N_\psi + p_\chi N_\chi}$, which is p_χ and p_ψ-independent, can *potentially* act as a *genuine* symmetry on the fermions. Also, we can always choose $\mathbb{Z}_r$ to act solely on χ:

$$\mathbb{Z}_r : \quad (\psi, \chi) \longrightarrow \left(\psi, e^{i\frac{2\pi\ell}{r}}\chi\right)\,, \quad \ell = 0, 1, 2, ..., r-1\,. \tag{3.9}$$

Yet, one must check that $\mathbb{Z}_r$ or a subgroup of it cannot be absorbed in the centers of the color or flavor groups, which leaves a proper subgroup of $\mathbb{Z}_r$ as the genuine discrete symmetry. This will be checked on a case-by-case basis. In the following, we will use $\mathbb{Z}_p^{d\chi} \subseteq \mathbb{Z}_r$ to denote the genuine discrete chiral symmetry. For completeness, we remind the reader that the fermion number symmetry $\mathbb{Z}_2^F = (-1)^F$ operates on ψ and χ as $(\psi, \chi) \longrightarrow -(\psi, \chi)$.

Finally, we also note that when N is even, the theory is endowed with a $\mathbb{Z}_2^{(1)}$ 1-form center symmetry acting on the fundamental Wilson loops. In summary, the good global symmetry of the theory is

$$G_{\text{global}} \sim SU(n_\psi) \times SU(n_\chi) \times U(1)_A \times \mathbb{Z}_p^{d\chi} \times \mathbb{Z}^{(1)}_{\gcd(N,2)}\,, \tag{3.10}$$

where the tilde indicates that this is the correct group modulo a discrete group needed to fix the faithful global symmetry. Thus, the faithful global symmetry is a quotient group.

To determine the correct quotient group, we follow [20, 35]. Here, we keep our treatment short as the details can be found in [35]. We put the theory on a general compact 4-D spin manifold $\mathcal{M}^4$, define a principal bundle of the continuous part of the global symmetry G_{global} on $\mathcal{M}^4$, and take the transition functions of G_{global} to act on fibers by left multiplication. Spinors are sections of the bundle, and we use the notations ψ_i and χ_i for their values on a local patch $U_i \subset \mathcal{M}^4$. We denote the transition functions of the color $SU(N)$, (non-abelian) flavor, and $U(1)_A$ group as g, f, and u, respectively, along with the proper superscript to distinguish those of ψ and χ. On the overlap $U_i \cap U_j$ we have

$$\psi_i = (g^\psi, f^\psi, u^\psi)_{ij}\, \psi_j \ , \quad \chi_i = (g^\chi, f^\chi, u^\chi)_{ij}\, \chi_j \ . \tag{3.11}$$

The fermions are well defined on $\mathcal{M}^4$ provided they satisfy the cocycle condition (a necessary consistency condition) on the triple overlap $U_i \cap U_j \cap U_k$. Now, we turn on background gauge fields for centers of the gauge, flavor, and $U(1)_A$ groups and determine the most general combination compatible with the cocycle condition *, which reads

$$\left(g^\psi, f^\psi, u^\psi\right)_{ij} \circ \left(g^\psi, f^\psi, u^\psi\right)_{jk} \circ \left(g^\psi, f^\psi, u^\psi\right)_{ki} = (z_c, z_f, z_u) \qquad \text{with} \quad z_c z_f z_u = 1 \tag{3.12}$$

where z's refer to the center elements: $z_c \in \mathbb{Z}_{N/\gcd(N,2)}$, $z_f \in \mathbb{Z}_{n_\psi}$, and $z_u \in U(1)_A$. The condition $z_c z_f z_u = 1$ is required for the equivalence relation

$$(z_c, z_f, z_u) \sim (1,1,1) \ , \tag{3.13}$$

which is needed to obtain the correct compatibility condition. Similar expressions hold for the cocycle condition of χ. The following two equations give the consistency (compatibility) conditions

$$\begin{aligned} \psi \quad &: \quad \underbrace{e^{i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{i2\pi\frac{pk}{N-4}}}_{z_\psi}\underbrace{e^{-i2\pi s\frac{(N+4)(N-2)}{kr}}}_{z_u} = 1\,, \\ \chi \quad &: \quad \underbrace{e^{-i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{-i2\pi\frac{p'k}{N+4}}}_{z_\chi}\underbrace{e^{i2\pi s\frac{(N-4)(N+2)}{kr}}}_{z_u} = 1\,. \end{aligned} \tag{3.14}$$

Here, $m \in \mathbb{Z}_{N/\gcd(N,2)}$, $p \in \mathbb{Z}_{n_\psi}$, $p' \in \mathbb{Z}_{n_\chi}$ and s is a $U(1)_A$ parameter. The factor of 2 that appears in z_c accounts for the N-ality of ψ and χ, and the negative sign that appears in the z_c factor in the second line accounts for the fact that χ transforms in the complex conjugate of the 2-index anti-symmetric representation. We also take ψ to transform in the fundamental representation of $SU(n_\psi)$ and χ to transform in the anti-fundamental representation of $SU(n_\chi)$. Following [35], we shall dub the discrete color-flavor-$U(1)_A$ fluxes as the CFU fluxes. The full set of solutions of (3.14) determines the quotient group in (3.10). These solutions will be found on a case-by-case basis. In general, we divide (3.10) by $\mathbb{Z}_{N/\gcd(N,2)} \times \mathbb{Z}_{(N-4)/k} \times \mathbb{Z}_{(N+4)/k}$ or a subgroup of it.

Once a non-trivial solution of (3.14) is found, we can calculate the topological charges associated with the center fluxes, which are fractional charges in general.

*See [38, 39, 17, 40] for applications of the anomalies resulting from turning on these fluxes.

Let $\mathcal{M}^4$ admit two independent 2-cycles and let two integers, e.g., m_1 and m_2, account for the number of quanta piercing through them. For example, we can take $\mathcal{M}^4 = \mathbb{T}^4$, a 4-torus with a period length L, and turn on the gauge fields that are compatible with the cocycle condition:

$$\begin{aligned}
a_1^c &= \frac{2\pi m_1}{L^2}\boldsymbol{H}_c\cdot\boldsymbol{\nu}_c\,, \quad a_2^c = 0\,, \quad a_3^c = \frac{2\pi m_2}{L^2}\boldsymbol{H}_c\cdot\boldsymbol{\nu}_c\,, \quad a_4^c = 0\,,\\
A_1^\psi &= \frac{2\pi p_1}{L^2}\boldsymbol{H}_\psi\cdot\boldsymbol{\nu}_\psi\,, \quad A_2^\psi = 0\,, \quad A_3^\psi = \frac{2\pi p_2}{L^2}\boldsymbol{H}_\psi\cdot\boldsymbol{\nu}_\psi\,, \quad A_4^\psi = 0\,,\\
A_1^\chi &= \frac{2\pi p_1'}{L^2}\boldsymbol{H}_\chi\cdot\boldsymbol{\nu}_\chi\,, \quad A_2^\chi = 0\,, \quad A_3^\chi = \frac{2\pi p_2'}{L^2}\boldsymbol{H}_\chi\cdot\boldsymbol{\nu}_\chi\,, \quad A_4^\chi = 0\,,\\
A_1^u &= \frac{2\pi s_1}{L^2}\,, \quad A_2^u = 0\,, \quad A_3^u = \frac{2\pi s_2}{L^2}\,, \quad A_4^u = 0\,.
\end{aligned} \tag{3.15}$$

a_μ^c, A_μ^ψ, A_μ^χ, and A_μ^u are the background gauge fields of the center of the color, $SU(n_\psi)$ flavor, $SU(n_\chi)$, flavor, and $U(1)_A$, respectively. The bold-face letters $\boldsymbol{H} \equiv (H_1, ..., H_{N-1})$ are the Cartan generators of $SU(N)$ group, while $\boldsymbol{\nu} \equiv (\nu_1, ..., \nu_{N-1})$ is a weight in the defining representation of the group. Notice that the integers $m_{1,2}, p_{1,2}, p'_{1,2}, s_{1,2}$ are the same integers that solve the consistency conditions (3.14). Given the set of the background fields (3.15), one immediately obtains the topological charges defined as

$$\begin{aligned}
Q_c &= \int_{\mathbb{T}^4} \frac{\mathrm{tr}\,[f^c\wedge f^c]}{8\pi^2}\,, &\qquad Q_\psi &= \int_{\mathbb{T}^4} \frac{\mathrm{tr}\left[F^\psi\wedge F^\psi\right]}{8\pi^2}\,,\\
Q_\chi &= \int_{\mathbb{T}^4} \frac{\mathrm{tr}\,[F^\chi\wedge F^\chi]}{8\pi^2}\,, &\qquad Q_u &= \int_{\mathbb{T}^4} \frac{F^u\wedge F^u}{8\pi^2}
\end{aligned} \tag{3.16}$$

and f^c, $F^{\psi,\chi,u}$ are the field strengths of the corresponding background. Substituting (3.15) into (3.16), we obtain

$$\begin{aligned}
Q_c &= k_c - \frac{m_1 m_2}{N}\,, \quad Q_\psi = k_\psi - \frac{p_1 p_2 k}{N-4}\,,\\
Q_\chi &= k_\chi - \frac{p_1' p_2' k}{N+4}\,, \quad Q_u = (s_1 - k_1)(s_2 - k_2)\,,
\end{aligned} \tag{3.17}$$

and $k_c, k_\psi, k_\chi, k_1, k_2 \in \mathbb{Z}$ are arbitrary integers that are always allowed. These fluxes will support fermion zero modes, and the Dirac indices give their number:

$$\begin{aligned}
\mathcal{I}_\psi &= n_\psi T_\psi Q_c + \mathrm{dim}_\psi Q_\psi + \mathrm{dim}_\psi n_\psi q_\psi^2 Q_u\,,\\
\mathcal{I}_\chi &= n_\chi T_\chi Q_c + \mathrm{dim}_\chi Q_\chi + \mathrm{dim}_\chi n_\chi q_\chi^2 Q_u\,,
\end{aligned} \tag{3.18}$$

and $\mathrm{dim}_\psi = \frac{N(N+1)}{2}$, $\mathrm{dim}_\chi = \frac{N(N-1)}{2}$ are the dimensions of the representations. Dirac indices count the number of the Weyl zero modes in the background of center fluxes. The integrality of the indices can work as a check on the consistency of the fluxes on $\mathcal{M}^4$.

One may also turn on the CFU fluxes on nonspin $\mathcal{M}^4$. A nonspin manifold does not admit fermions in the sense that there is an obstruction in lifting the $SO(4)$ rotation group bundle to a Spin(4) bundle on $\mathcal{M}^4$. A diagnosis of a non-spin manifold is that the Dirac index of a Weyl fermion, $\mathcal{I} = \frac{1}{196\pi^2}\int_{\mathcal{M}^4} \text{tr}R \wedge R$, where R is the curvature 2-form, is non-integer. An example of a nonspin manifold is $\mathbb{CP}^2$, which has $\frac{1}{196\pi^2}\int_{\mathbb{CP}^2} \text{tr}R \wedge R = -\frac{1}{8}$. To put a Weyl fermion on $\mathbb{CP}^2$, we need to excite a $U(1)$ flux F through its 2-cycle $\mathbb{CP}^1 \subset \mathbb{CP}^2$ and demand that $\int_{\mathbb{CP}^1} F \in \pi(2\mathbb{Z}+1)$, which implies $\frac{1}{8\pi^2}\int_{\mathbb{CP}^2} F \wedge F \in \frac{\mathbb{Z}}{8}$. Now, one can easily check the integrality of the Dirac index $\frac{1}{196\pi^2}\int_{\mathbb{CP}^2} \text{tr}R \wedge R + \frac{1}{8\pi^2}\int_{\mathbb{CP}^2} F \wedge F \in \mathbb{Z}$, and thus, the fermions are well-defined on $\mathbb{CP}^2$ in the presence of such $U(1)$ fluxes. Here, although one cannot define a Spin(4) bundle on pure $\mathbb{CP}^2$, in the sense that the corresponding cocycle condition fails on a triple overlap, nonetheless, we can define the Spin$_c$(4) structure Spin(4) $\times\, U(1)/\mathbb{Z}_2$ in the presence of the $U(1)$ background.

This idea can be generalized in the presence of the CFU fluxes; see [41] for details. One just needs to replace the consistency conditions (3.14) with

$$\begin{aligned} \psi \quad &: \quad \underbrace{e^{i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{i2\pi\frac{pk}{N-4}}}_{z_\psi}\underbrace{e^{-i2\pi s\frac{(N+4)(N-2)}{kr}}}_{z_u} = -1\,, \\ \chi \quad &: \quad \underbrace{e^{-i2\pi\frac{2m}{N}}}_{z_c}\underbrace{e^{-i2\pi\frac{p'k}{N+4}}}_{z_\chi}\underbrace{e^{i2\pi s\frac{(N-4)(N+2)}{kr}}}_{z_u} = -1\,. \end{aligned} \tag{3.19}$$

The minus sign on the right-hand side compensates for the minus sign arising from parallel transporting the spinor fields around appropriate closed paths in $\mathbb{CP}^2$; see the detailed discussion in [41]. Given that a solution, $m \in \mathbb{Z}_{N/\gcd(N,2)}$, $p \in \mathbb{Z}_{n_\psi}$, $p' \in \mathbb{Z}_{n_\chi}$ and s, to (3.19) can be found, the topological charges corresponding to the CFU fluxes and gravity are given by (see [41])

$$\begin{aligned} Q_c \quad &= \frac{m^2}{2}\left(1-\frac{1}{N}\right)\,, \quad Q_\psi = \frac{p^2}{2}\left(1-\frac{k}{N-4}\right)\,, \\ Q_\chi \quad &= \frac{p'^2}{2}\left(1-\frac{k}{N+4}\right)\,, \quad Q_u = \frac{1}{2}s^2\,, \quad Q_g = -\frac{1}{8}\,. \end{aligned} \tag{3.20}$$

The Dirac-indices of ψ and χ are

$$\begin{aligned} \mathcal{I}_\psi^{\mathbb{CP}^2} &= n_\psi T_\psi Q_c + \dim_\psi Q_\psi + \dim_\psi n_\psi\left(q_\psi^2 Q_u + Q_g\right)\,, \\ \mathcal{I}_\chi^{\mathbb{CP}^2} &= n_\chi T_\chi Q_c + \dim_\chi Q_\chi + \dim_\chi n_\chi\left(q_\chi^2 Q_u + Q_g\right)\,, \end{aligned} \tag{3.21}$$

which are always integers. Except for $(N=6, k=2)$ and $(N=10, k=2)$ in Table 3.1, we can always find solutions to (3.19), and thus, we can put these theories on $\mathbb{CP}^2$.

3.2.2 Anomalies

The theory has a set of 't Hooft anomalies that can help constrain the possible IR phases. In the following, we list the 't Hooft anomalies we shall encounter in our study.

(I) $[SU(n_\psi)]^3$ and $[SU(n_\chi)]^3$ anomalies

These are perturbative (triangle) anomalies and their inflow from 5-D to 4-D is captured via 5-D Chern-Simons theories:

$$\begin{aligned}
{[SU(n_\psi)]^3} \quad &: \quad \exp\left[i\,\dim_\psi \int_{\mathcal{M}^5} \omega_5(A^\psi)\right] , \\
{[SU(n_\chi)]^3} \quad &: \quad \exp\left[i\,\dim_\chi \int_{\mathcal{M}^5} \omega_5(A^\chi)\right] ,
\end{aligned} \tag{3.22}$$

where A^ψ and A^χ are the $SU(n_\psi)$ and $SU(n_\chi)$ 1-form background gauge fields, extended from 4-D to 5-D, and $\omega_5(A)$ is the 5-D Chern-Simons form defined via the descent equation:

$$d\omega_5(A) = \frac{1}{3!(2\pi)^2} \mathrm{tr}_\Box F^3 \,, \tag{3.23}$$

and F is the 2-form field strength of A. We review how to obtain the anomaly from the anomaly descent mechanism in appendix C.

(II) $U(1)_A$- and $\mathbb{Z}_p^{d\chi}$-gravitational anomalies

These anomalies are captured via the 5-D anomaly inflow actions:

$$\begin{aligned}
U(1)_A[\text{grav}] \quad &: \quad \exp\left[i\left(q_\psi n_\psi \mathrm{dim}_\psi + q_\chi n_\chi \mathrm{dim}_\chi\right)\int_{\mathcal{M}^5} A^u \wedge \frac{p_1(\mathcal{M}^5)}{24}\right] , \\
\mathbb{Z}_p^{d\chi}[\text{grav}] \quad &: \quad \exp\left[i\left(n_\chi \mathrm{dim}_\chi\right)\int_{\mathcal{M}^5} A^{d\chi} \wedge \frac{p_1(\mathcal{M}^5)}{24}\right] .
\end{aligned} \tag{3.24}$$

The 1-form gauge fields A^u and $A^{d\chi}$ are the backgrounds of $U(1)_A$ and $\mathbb{Z}_p^{d\chi}$, respectively. $p_1(\mathcal{M}^5) = -\frac{1}{8\pi^2} R \wedge R$ is the first Pontryagin number and R is the curvature 2-form. On a spin manifold, we have $\int_{\mathcal{M}^4} p_1(\mathcal{M}^4) \in 48\,\mathbb{Z}$, and thus, there are 2 zero modes per Weyl fermion in a gravitational background. Under $U(1)_A$ and $\mathbb{Z}_p^{d\chi}$ transformations we have $A^u \longrightarrow A^u + d\lambda^u$ with $\oint d\lambda^u = 2\pi\mathbb{Z}$ and $A^{d\chi} \longrightarrow A^{d\chi} + d\lambda^{d\chi}$, with $\oint d\lambda^{d\chi} = \frac{2\pi\mathbb{Z}}{p}$, and the anomaly inflow actions produce the 4-D anomalies.

The result (3.24) is "perturbative" as it can be seen from a triangle diagram with two vertices that couple the fermions to a gravitational background via the energy-momentum tensor, while the third vertex couples the fermions to an external $U(1)_A$ or $\mathbb{Z}_p^{d\chi}$ sources.

(III) CFU anomalies

These anomalies were identified in [20]; however, see [42, 43] for earlier encounters. They are anomalies of $U(1)_A$ and $\mathbb{Z}_p^{d\chi}$ symmetries in the background of the CFU fluxes that are supported on a general spin manifold. As was shown in [35], 5-D anomaly inflow actions can also capture them. However, we find it more convenient to express such anomalies in terms of the non-trivial phases that are acquired by the partition function $\mathcal{Z}$ under the action of $U(1)_A$ and $\mathbb{Z}_p^{d\chi}$ symmetries in the background of the CFU fluxes:

$$\begin{aligned} U(1)_A[\text{CFU}] &: \mathcal{Z} \longrightarrow e^{i2\pi\alpha\left(q_\psi \mathcal{I}_\psi + q_\chi \mathcal{I}_\chi\right)} \mathcal{Z}\,, \\ \mathbb{Z}_p^{d\chi}[\text{CFU}] &: \mathcal{Z} \longrightarrow e^{i\frac{2\pi}{p}\mathcal{I}_\chi} \mathcal{Z}\,, \end{aligned} \tag{3.25}$$

and the Dirac indices $\mathcal{I}_\psi$ and $\mathcal{I}_\chi$ are given in (3.18). The contribution from the color topological charge Q_c drops out in the computation of the $U(1)_A[\text{CFU}]$ anomaly, as can be easily checked, since $U(1)_A$ is a good symmetry in the background of the color flux. This is not the case with $\mathbb{Z}_p^{d\chi}[\text{CFU}]$ anomaly, where Q_c contributes to the anomaly. As we shall discuss, this observation has important consequences for anomaly matching in the IR.

It is also important to notice that the perturbative anomalies $U(1)_A[SU(n_\psi)]^2$, $U(1)_A[SU(n_\chi)]^2$, $\mathbb{Z}_p^{d\chi}[SU(n_\chi)]^2$, and $[U(1)_A]^3$ are a subset of the CFU anomalies, obtained by turning off the center fluxes and keeping only the integer topological charges in (3.17). Again, one can express them using anomaly inflow actions as

$$\begin{aligned} U(1)_A[SU(n_\psi)]^2 &: \exp\left[i q_\psi n_\psi \text{dim}_\psi \int_{\mathcal{M}^5} A^u \wedge \frac{\text{tr}\left[F^\psi \wedge F^\psi\right]}{8\pi^2}\right] \\ U(1)_A[SU(n_\chi)]^2 &: \exp\left[i q_\chi n_\chi \text{dim}_\chi \int_{\mathcal{M}^5} A^u \wedge \frac{\text{tr}\left[F^\chi \wedge F^\chi\right]}{8\pi^2}\right] \\ \mathbb{Z}_p^{d\chi}[SU(n_\chi)]^2 &: \exp\left[i n_\chi \text{dim}_\chi \int_{\mathcal{M}^5} A^{d\chi} \wedge \frac{\text{tr}\left[F^\chi \wedge F^\chi\right]}{8\pi^2}\right] \\ [U(1)_A]^3 &: \exp\left[i\left(q_\psi^3 n_\psi \text{dim}_\psi + q_\chi^3 n_\chi \text{dim}_\chi\right) \int_{\mathcal{M}^5} A^u \wedge \frac{F^u \wedge F^u}{24\pi^2}\right] \end{aligned} \tag{3.26}$$

In addition, for N even, the CFU anomalies encompass the $U(1)_A$ 0-form/ $\mathbb{Z}_2^{[1]}$ 1-form as well as the $\mathbb{Z}_r^{d\chi}$ 0-form/ $\mathbb{Z}_2^{[1]}$ 1-form mixed anomalies. These can be easily found by turning off the flavor and the $U(1)_A$ fluxes. In practice, one uses (3.25), (3.17), and (3.18), after setting $p_{1,2} = p'_{1,2} = s_{1,2} = 0$ and $m_1 = m_2 = N/2$. This choice enforces the consistency conditions (3.14) and gives $Q_\psi = Q_\chi = Q_u = 0$ and $Q_c = \frac{N}{4}$.

One may also use the CFU fluxes on $\mathbb{CP}^2$ to calculate the $U(1)_A\,[\text{CFU}]$ and $\mathbb{Z}_p\,[\text{CFU}]$ anomalies, which sometimes are more restrictive than the corresponding anomalies on a spin manifold. We use the Dirac indices on $\mathbb{CP}^2$, as given by (3.21), to find

$$\begin{aligned} U(1)_A\,[\text{CFU}]_{\mathbb{CP}^2} &: \mathcal{Z} \longrightarrow e^{i2\pi\alpha\left[q_\psi \mathcal{I}_\psi^{\mathbb{CP}^2} + q_\chi \mathcal{I}_\chi^{\mathbb{CP}^2}\right]} \mathcal{Z}\,, \\ \mathbb{Z}_p^{d\chi}\,[\text{CFU}]_{\mathbb{CP}^2} &: \mathcal{Z} \longrightarrow e^{i\frac{2\pi}{p}\mathcal{I}_\chi^{\mathbb{CP}^2}} \mathcal{Z}\,. \end{aligned} \tag{3.27}$$

From here on, we shall write all anomalies in terms of their phases to reduce clutter. For example, instead of (3.25), we write:

$$U(1)_A[\text{CFU}] : q_\psi \mathcal{I}_\psi + q_\chi \mathcal{I}_\chi\,, \quad \mathbb{Z}_p^{d_\chi}[\text{CFU}] : \mathcal{I}_\chi\,. \tag{3.28}$$

(IV) Anomalies of discrete groups

Here, we consider anomalies of a discrete symmetry $\mathbb{Z}_n$, where n is a general positive integer. An example of the discrete symmetry is the $\mathbb{Z}_p^{d_\chi}$ chiral symmetry or a discrete subgroup of $U(1)_A$ left unbroken in the IR. The proper way to detect anomalies of discrete symmetries is to use the Dai-Freed prescription [44, 45]. The idea stems from the fact that a chiral massless fermion defined on $\mathcal{M}^4$ can be realized as the chiral zero mode residing on the boundary $\mathcal{M}^4$ of a 5-dimensional manifold $\mathcal{M}^5$ that is endowed with massive fermions, with $\mathbb{Z}_n$ turned on in the 5-dimensional manifold. One can also consider a different 5-dimensional manifold $\mathcal{M}'^5$ with the same boundary $\mathcal{M}^4$. If the partition functions defined on $\mathcal{M}^5$ and $\mathcal{M}'^5$ have the same phase, then the theory on $\mathcal{M}^4$ is uniquely defined and anomaly-free; otherwise, it is anomalous. Applying the Dai-Freed prescription to study the IR phases of strongly-coupled theories is innovative. However, see [46, 39] for previous applications*.

If $\mathcal{M}^4$ is a spin manifold, the geometrical obstruction of uniquely extending a 4-D theory to a 5-D bulk can be inferred by computing the bordism group $\Omega_5^{\text{Spin}}(B\mathbb{Z}_n)$, where $B\mathbb{Z}_n$ is the classifying space of $\mathbb{Z}_n$†. If $\Omega_5^{\text{Spin}}(B\mathbb{Z}_n)$ is non-trivial, the theory might have a nonperturbative anomaly. To find the anomaly, one computes the η-invariant, a resolvent of the spectral asymmetry of the Dirac operator, on specific closed 5-dimensional spin manifolds that can detect the anomaly. For example, one puts the theory on Lens spaces to gauge $\mathbb{Z}_n$ and discover whether the theory exhibits a nonperturbative anomaly. For n even, $n = 2m$, one can take $\mathcal{M}^4$ to be nonspin by employing the twisted symmetry group $\text{Spin}^{\mathbb{Z}_{2m}}(4) = (\text{Spin}(4) \times \mathbb{Z}_{2m})/\mathbb{Z}_2$ instead of Spin(4). Here, one needs to compute the η-invariant that detects the bordism group $\Omega_5^{\text{Spin}^{\mathbb{Z}_{2m}}}$.

The computations of the relevant η-invariants were carried out in [49] (see also [50] for an alternative perspective). For a theory of a left-handed Weyl fermion with a charge s under $\mathbb{Z}_n$ defined on a spin manifold, the anomaly is given by the pair of phases:

$$\left\{(n^2 + 3n + 2)s^3 \bmod 6n\,, 2s \bmod n\right\}\,. \tag{3.29}$$

*Also, see [47, 48] for applications of Dai-Freed anomalies in particle physics.

†A classifying space of symmetry G is an infinite dimensional space with the property that any principal G-bundle on a manifold $\mathcal{M}$ is the pullback via some map $f : \mathcal{M} \longrightarrow BG$. Then, the set of topologically distinct principal G-bundles over $\mathcal{M}$ is equivalent to the set of the homotopy classes of maps from $\mathcal{M}$ to BG.

This pair can be thought of as contributions from $[\mathbb{Z}_n]^3$ and mixed $\mathbb{Z}_n$ [grav] anomalies. Indeed, the second entry in (3.29) is precisely the anomaly we computed in the second line of (3.24). The first entry can be obtained from pure $[U(1)]^3$ anomaly by restricting $U(1)$ to a $\mathbb{Z}_n$ discrete group; this is the Ibanez-Ross anomaly we comment on below. Also, a Weyl fermion defined on a twisted background and carrying a charge s under $\mathbb{Z}_{2m}$ has an anomaly given by the pair*

$$\left\{\left((2m^2+m+1)s^3-(m+3)s\right) \bmod 48m\,, \left(ms^3+s\right) \bmod 2m\right\}\,. \tag{3.30}$$

The charge s is assumed to be odd such that the fermion transforms under the $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_{2m}$. Generally, the anomaly (3.30) is more restrictive than (3.29). We shall use both (3.30) and (3.29) to constrain our theories. It is important to note that $\mathbb{Z}_2$ symmetry is anomaly-free, as can be easily seen from (3.29) and (3.30). This observation will play an essential role in the IR anomaly matching by condensates, as many of them break the global symmetries to $\mathbb{Z}_2$, as we discuss below.

We also comment on the Ibanez-Ross anomaly-matching conditions [51]. These are obtained from $[U(1)]^3$ and $U(1)$[grav] anomalies by restricting $U(1)$ to a $\mathbb{Z}_n$ subgroup. The Ibanez-Ross anomaly-cancellation conditions read:

$$s^3=p'n+\frac{r'n^3}{8}\,,\quad s=p''n+r''n\,, \tag{3.31}$$

where $p',r',p'',r''\in\mathbb{Z}$, $p'\in 3\mathbb{Z}$ if $n\in 3\mathbb{Z}$, and $r',r''=0$ if n is odd. It can be shown that (3.31) and (3.29) are equivalent [49]. Hence, in what follows, we use either (3.29) or (3.30) to calculate discrete anomalies.

Finally, we also may have a discrete anomaly of the form $\mathbb{Z}_m[\mathbb{Z}_n]^2$. Such an anomaly can descend from $U(1)_A$[CFU] anomaly after a given condensate breaks $U(1)_A$ down to a discrete subgroup. Let s and s' be the charges of a left-handed Weyl fermion under $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. Then, the anomaly cancelation condition, which follows from the Ibanez-Ross conditions, is given by [52]

$$s^2s'=p'\,\gcd(m,n)+\frac{p''}{8}mn^2\,, \tag{3.32}$$

where $p',p''\in\mathbb{Z}$ and p'' can be non-vanishing only if n and m are even. Notice that this anomaly is trivial when $\mathbb{Z}_m=\mathbb{Z}_2$.

*More generally, the bordism group $\Omega_5^{\mathrm{Spin}^{\mathbb{Z}_{2m}}}\cong\mathbb{Z}_a\times\mathbb{Z}_b$, where a,b, and the associated anomalies are given by Eqs. (2.11)- (2.13) in [39]. In the present work, Eq. (3.30) suffices to tackle the theories at hand.

3.3 Anomaly matching and the IR phase

3.3.1 IR anomaly matching

’t Hooft anomalies preclude a trivially gapped IR phase: a theory with ’t Hooft anomaly must have gapless excitations, degenerate vacua, or symmetry-preserving topological quantum field theory (TQFT). In this work, we will assume that the gauge group does not break spontaneously under its strong dynamics. The breaking of a gauge group is under control in the presence of scalars at weak coupling, an ingredient absent from our theory from the get-go*. This leaves us with three possible IR scenarios.

(I) Conformal fixed point

In the first scenario, the theory flows to a conformal field theory (CFT). When the CFT is weakly coupled, the renormalization group flow from the UV to the IR is very slow. The UV matter content (fermions) can be considered the IR gapless excitations, and the anomalies are automatically matched. Anomaly matching by strongly interacting CFT is still an open problem; we have nothing to say here. The existence of a well-controlled Banks-Zaks fixed point implies a strictly weakly coupled CFT. However, such a reliable fixed point can only be obtained in the large-N limit. The 3-loop β-function reads

$$\beta(g) = \quad -\beta_0 \frac{g^3}{(4\pi)^2} - \beta_1 \frac{g^5}{(4\pi)^4} - \beta_2 \frac{g^7}{(4\pi)^6}\,. \tag{3.33}$$

If $\beta_0 > 0$ and $\beta_1 < 0$, the theory flows to an IR fixed point, $g_*^2 = -\frac{(4\pi)^2\beta_0}{\beta_1}$, up to corrections from β_2. In the large-N limit and close to the boundary of the asymptotic-freedom region, the contribution from the third term is suppressed compared to the second term, and thus, this term and higher-order terms can be safely neglected. Our theories, however, do not admit large-N analysis. Yet, as we shall discuss, in a few cases, the numerical value of the third term is extremely small compared to the first two terms, so one can conclude that such a weakly-coupled CFT exists. In Appendix D, we work out the fixed points using the 2 and 3-loop β-functions. We consider values of $\frac{g_*^2}{4\pi} < 0.1$, using both the 2-loop and 3-loop calculations, small enough to conclude that the theory has an IR fixed point. Also, the perturbative nature of the β-function calculations will be trusted when the third term in (3.33) is small compared to the first two terms. More stringent coupling constant values at the fixed point could also be assumed. This, however, will only mean doing more work to find out the fate of the IR phases of such theories.

*Tumbling, [53], is a mechanism by which the breakdown of a gauge group occurs without the aid of fundamental scalar fields. We do not discuss tumbling in this work.

(II) Composite massless fermions

In the second scenario, the theory becomes strongly coupled; it confines (for N even), preserves the global symmetries, and flows to a phase of composite massless fermions. This can happen in the non-bosonic theories $N = 5, 6, 10$. We sketch how one can systematically search for such composites. Let $\mathcal{F}_i$ be a gauge-invariant fermionic operator (a composite that transforms as a left-handed Weyl fermion under the Lorentz group) built of ψ and χ:

$$\mathcal{F}_i = \psi^{\kappa_i} \chi^{\rho_i} \,, \tag{3.34}$$

where $\kappa_i, \rho_i \in \{0\} \cup \mathbb{Z}^+$, and we suppressed the color and spinor indices to reduce notational clutter. Insertion of gluon fields can be used whenever fermi statistics cause $\mathcal{F}_i$ to vanish. Using the convention that ψ and χ carry 2 and $N-2$ indices, respectively, and demanding that $\mathcal{F}_i$ be a gauge invariant fermion yields the two conditions:

$$2\kappa_i + (N-2)\rho_i \in N\mathbb{Z}^+ \,, \quad \kappa_i + \rho_i \in (2\mathbb{Z}^+ - 1) \,. \tag{3.35}$$

The $U(1)_A$ charge of $\mathcal{F}_i$ is

$$q_{\mathcal{F}_i} = \frac{-\kappa_i N_\chi + \rho_i N_\psi}{r} \,. \tag{3.36}$$

Generally, the composites $\mathcal{F}_i$ transform in higher representations of $SU(n_\psi)$ and $SU(n_\chi)$, making the process of matching anomalies containing flavor groups a daunting task. Thus, it is more convenient to start with matching $[U(1)_A]^3$, $U(1)_A[\text{grav}]$, and nonperturbative $\mathbb{Z}_p^{d\chi}$ anomalies. For generality, we assume there are $\mathcal{N}_i$ copies of composites $\mathcal{F}_i$. Then, matching these anomalies gives the conditions

$$\begin{aligned}
\sum_i \mathcal{N}_i q_{\mathcal{F}_i}^3 &= q_\psi^3 n_\psi \text{dim}_\psi + q_\chi^3 n_\chi \text{dim}_\chi \\
\sum_i \mathcal{N}_i q_{\mathcal{F}_i} &= q_\psi n_\psi \text{dim}_\psi + q_\chi n_\chi \text{dim}_\chi \\
2\sum_i \mathcal{N}_i &= 2 n_\chi \text{dim}_\chi \, (\text{mod}\, p) \\
(p^2+3p+2)\sum_i \mathcal{N}_i &= (p^2+3p+2) n_\chi \text{dim}_\chi \, (\text{mod}\, 6p)
\end{aligned} \tag{3.37}$$

The number of the IR fermionic species $\mathcal{N} = \sum_i \mathcal{N}_i$ is bounded from above by the a-theorem:

$$\underbrace{\underbrace{2(N^2-1)}_{\text{gluons}} + \frac{7}{4}\left(n_\psi \text{dim}_\psi + n_\chi \text{dim}_\chi\right)}_{\text{UV degrees of freedom}} \geq \underbrace{\frac{7\mathcal{N}}{4}}_{\text{IR degrees of freedom}} \,. \tag{3.38}$$

In principle, one could systematically search for copies of composites $\{\mathcal{N}_1, \mathcal{N}_2, ...\}$ that satisfy (3.37). However, this would require finding the partitions of $\mathcal{N}$ (all integers that their sums give $\mathcal{N}$), a number that grows exponentially with $\sqrt{\mathcal{N}}$. The composites that satisfy (3.37) must also match the rest of the anomalies that involve the flavor groups. In all non-bosonic theories, $N = 5, 6, 10$, we could not find a set of composites that matched the full set of anomalies using the systematic approach sketched above. Simply, the algorithm takes an extremely long time, which makes such a systematic search impractical.

In fact, we can utilize the $\mathbb{Z}_p^{d\chi}$[CFU] anomaly to show that in some cases, such candidates, if they exist, cannot solely match this anomaly. This approach was used in [20] in the case of vector-like theories, and we repeat it here for chiral theories. To this end, we assume that there exists a set of gauge-invariant composite fermions that match $[SU(n_\psi)]^3$, $[SU(n_\chi)]^3$, $[U(1)_A]^3$, $\mathbb{Z}_p^{d\chi}[U(1)_A]^2$, $\mathbb{Z}_p^{d\chi}[SU(n_\psi)]^2$, $\mathbb{Z}_p^{d\chi}[SU(n_\chi)]^2$, $U(1)_A$[grav], and $\mathbb{Z}_p^{d\chi}$[grav] anomalies. Then, we turn the CFU fluxes on $\mathcal{M}^4$ and perform a $\mathbb{Z}_p^{d\chi}$ rotation. We denote the UV coefficients that multiply Q_c, Q_χ, and Q_u in (3.25, 3.18) by $D_c^{\rm UV} \equiv n_\chi T_\chi$, $D_\chi^{\rm UV} \equiv \dim_\chi$, and $D_u^{\rm UV} \equiv q_\chi^2 n_\chi \dim_\chi$. Under a discrete chiral rotation, the UV partition function transforms as

$$\mathcal{Z}_{\rm UV} \longrightarrow e^{i\frac{2\pi}{p}\left(D_c^{\rm UV}Q_c + D_\chi^{\rm UV}Q_\chi + D_u^{\rm UV}Q_u\right)}\mathcal{Z}_{\rm UV}\,, \tag{3.39}$$

while the IR partition function transforms as*

$$\mathcal{Z}_{\rm IR} \longrightarrow e^{i\frac{2\pi}{p}\left(D_\chi^{\rm IR}Q_\chi + D_u^{\rm IR}Q_u\right)}\mathcal{Z}_{\rm IR}\,, \tag{3.40}$$

where $D_\chi^{\rm IR}, D_u^{\rm IR} \in \mathbb{Z}$ are group-theoretical coefficients that are chosen to match $\mathbb{Z}_p^{d\chi}[U(1)_A]^2$ and $\mathbb{Z}_p^{d\chi}[SU(n_\chi)]^2$ anomalies. Since the UV-IR anomaly matching is mod p, we must have†

$$D_c^{\rm UV} = p\ell_c\,, \quad D_\chi^{\rm UV} - D_\chi^{\rm IR} = p\ell_\chi\,, \quad D_u^{\rm UV} - D_u^{\rm IR} = p\ell_u\,, \tag{3.41}$$

for some $\ell_{c,\chi,u} \in \mathbb{Z}$. Thus, the ratio between the UV and IR partition functions reads

$$\frac{\mathcal{Z}_{\rm UV}}{\mathcal{Z}_{\rm IR}} = e^{i2\pi(\ell_c Q_c + \ell_\chi Q_\chi + \ell_u Q_u)}\,, \tag{3.42}$$

and the matching of the $\mathbb{Z}_p^{d\chi}$[CFU] anomaly requires

$$\ell_c Q_c + \ell_\psi Q_\chi + \ell_u Q_u \in \mathbb{Z}\,, \tag{3.43}$$

for all allowed topological charges. Suppose no integers $\ell_{c,\chi,u}$ exist that satisfy this condition for a given allowed fractional topological charges. In that case, the composites cannot solely match the $\mathbb{Z}_p^{d\chi}$[CFU] anomaly.

*Q_c does not contribute to the IR phase since the composites are color singlets.

†$\mathbb{Z}_p^{d\chi}$ is a good symmetry in the color background, and thus we must have $D_c^{\rm UV} = p\ell_c$.

A minimal way out would be breaking $\mathbb{Z}_p^{d\chi} \longrightarrow \mathbb{Z}_{q<p}$ via condensate formation provided that the anomaly $\mathbb{Z}_q$[CFU] vanishes. Usually, a condensate would ordinarily break $SU(n_\psi)$, $SU(n_\chi)$, and $U(1)_A$. Thus, one must postulate that all gauge-invariant condensates charged under these symmetries have zero vacuum expectation values. Otherwise, the condensation of such operators would oversaturate these anomalies, which are assumed to be matched by composites. In addition, one needs to build a neutral operator under the continuous symmetries, charged under $\mathbb{Z}_p^{d\chi}$, and has a non-zero vacuum expectation value. If it exists, such an operator would have a scaling dimension larger than the vanishing lower-order condensates. Although this scenario cannot be ruled out, we find it contrived in the examples of the 2-index chiral theories we discuss here.

This leaves us with the possibility that if condition (3.43) is violated, the $\mathbb{Z}_p^{d\chi}$[CFU] anomaly can be matched by a symmetry-preserving topological quantum field theory (TQFT). In [54, 55], it was shown that the matching of $\mathbb{Z}_p^{d\chi}$-gravitational anomalies by a unitary and symmetry-preserving TQFT is obstructed on a spin manifold. This obstruction can also be shown to hold in the case of $\mathbb{Z}_p^{d\chi}$[CFU] anomaly [35].

We conclude that if condition (3.43) is violated, the theory probably cannot flow to a phase with massless composites.

(III) Spontaneous symmetry-breaking

In this scenario, the theory becomes strongly coupled; it confines (for N even) and breaks its global symmetries spontaneously. We say that the theory flows to a spontaneous symmetry-breaking (SSB) phase. An important aspect of this work involves identifying the minimal set of condensates that break global symmetries while matching the anomalies. These condensates break $G^{\rm g}$ down to $H \subset G^{\rm g}$, with the requirement that H remains anomaly-free. Without satisfying this condition, the symmetry breaking alone would not sufficiently match the UV anomaly. It is possible for composite fermions to match a non-vanishing anomaly in H, but it is crucial that these fermions do not undermine the matching of the $G^{\rm g}$ anomalies achieved by the condensates. Our focus did not involve searching for composites that could match the anomalous unbroken subgroups.

Generally, H can be expressed as $H = H^c \times \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$, where H^c represents the continuous part of H, $\mathbb{Z}_{q_1}$ "collectively" denotes the unbroken discrete subgroups of $SU(n_\psi) \times SU(n_\chi) \times U(1)_A$, and $\mathbb{Z}_{q_2}$ represents the unbroken subgroup of $\mathbb{Z}_p^{d\chi}$. If the condensates leave a discrete subgroup unbroken, we must examine its anomalies. In addition, if the theory possesses a 1-form/0-form mixed anomaly, there can be fractionalization classes, and hence, an ambiguity in calculating the cubic discrete anomalies [56]. One must ensure the condensates do not leave any discrete anomaly in any fractionalization class. In a few examples, we observe that lower-order condensates (such as the 2-fermion condensates) lead to anomalous unbroken discrete subgroups. Consequently, the formation of other (higher-order) condensates becomes necessary to break the symmetries into non-anomalous subgroups.

In strongly-coupled theories, it is generally believed that higher-order bosonic operators undergo condensation. In this work, through anomaly matching conditions, we provide kinematical reasons behind this condensation.

Now we turn our attention to the matching of CFU anomalies. Given the better understanding of the nature of the unbroken discrete subgroups of $U(1)_A$ and their anomalies, here, we provide a more in-depth discussion of the CFU anomalies than the earlier work [35]. As mentioned above, we encounter two types of such anomalies: $U(1)_A$[CFU] and $\mathbb{Z}_p^{d\chi}$[CFU] anomalies. In all the examples we have examined, we consistently observe the trivialization of $\mathbb{Z}_p^{d\chi}$[CFU] by the condensates, which break the $U(1)_A$ symmetry. On the other hand, the matching of the $U(1)_A$[CFU] anomaly through condensates is a more intricate process that required closer examination.

As emphasized earlier, the color topological charge does not play a role in this particular anomaly. Consequently, we can view it as an anomaly of $U(1)_A$ in the presence of the flavor center and $U(1)_A$ fluxes. In the examples we have studied, the condensates break the flavor center, rendering this anomaly irrelevant. In simpler terms, the full breaking of the flavor center automatically matches the $U(1)_A$[CFU] anomaly. This can be understood through the following principle: if a triangle (anomaly) diagram involves three abelian symmetries, namely G_1, G_2, and G_3 (in this case, G_1 through G_3 are the abelian discrete groups corresponding to turning on the CFU fluxes), the complete breaking of at least one of these symmetry groups will resolve the anomaly.

To extract more valuable insights from this anomaly, we can focus our attention solely on the color-$U(1)_A$ fluxes by deactivating the flavor background. In doing so, the $U(1)_A$[CU] anomaly becomes a mixed anomaly of $U(1)_A$ in the presence of fractional $U(1)_A$ flux (keeping in mind that the $U(1)_A$ flux still needs to combine with the color flux to satisfy the cocycle conditions. Yet, the color topological charge remains uninvolved in the anomaly). Superficially, one might consider this to be equivalent to the $[U(1)_A]^3$ anomaly. However, this is not the case since the latter anomaly only encompasses integer fluxes of $U(1)_A$, whereas the $U(1)_A$[CU] anomaly incorporates the minimal flux of $U(1)_A$. Consequently, the latter is more restrictive in nature compared to the $[U(1)_A]^3$ anomaly. Let the discrete flux of $U(1)_A$ be a $\mathbb{Z}_n$ flux, and let a particular condensate break $U(1)_A$ to $\mathbb{Z}_m \supseteq \mathbb{Z}_n$. Then the $U(1)$[CU] anomaly can be thought of as a $\mathbb{Z}_m[\mathbb{Z}_n]^2$ anomaly, which can be checked via (3.32). If $\mathbb{Z}_m \subset \mathbb{Z}_n$ and the anomaly $[\mathbb{Z}_m]^3$ vanishes, the breaking of $U(1)_A$ to $\mathbb{Z}_m$ automatically matches the $U(1)$[CU] anomaly as the symmetry corresponding to the discrete flux of $U(1)_A$ is broken.

Next, we discuss the condensates that cause the symmetries to break. A gauge-invariant condensate is a bosonic operator

$$\mathcal{C} = \psi^{\alpha_\psi} \chi^{\alpha_\chi} \,, \tag{3.44}$$

and α_ψ and α_χ satisfy the conditions

$$2\alpha_\psi + (N-2)\alpha_\chi \in N\mathbb{Z}^+ \,, \quad \alpha_\psi + \alpha_\chi \in 2\mathbb{Z}^+ \,. \tag{3.45}$$

One needs as many condensates as necessary to break G^{g} to an anomaly-free subgroup. Distinct condensates will break G^{g} down to $H_1 \subset G^{\mathrm{g}}$, $H_2 \subset G^{\mathrm{g}}$, etc. If the subgroups $\{H_1, H_2, ..\}$ do not share common generators, G^{g} will break to unity. Finding the breaking patterns of a group G^{g} because of the condensation of single or many operators transforming in the defining or higher-dimensional representations of G^{g} is, in general, a complicated problem. Only a few cases have been discussed in the literature; see, e.g., [57, 58, 59, 60, 61] and references therein. The question then is, how many condensates does the theory develop in the IR? There is no known answer to this question. However, there must be at least as many condensates as needed to match all anomalies.

3.3.2 Minimizing the IR degrees of freedom

Beyond 't Hooft anomalies, are there additional sources of information that can be harnessed to make conjectures about the infrared (IR) phase of a strongly coupled theory? In [62, 63, 64], a constraint on the structure of strongly coupled asymptotically-free field theories was proposed. The constraint is an inequality favoring an IR phase with fewer degrees of freedom (DOF). It was also proposed to use the free energy to characterize DOF. The effective degrees of freedom $\mathcal{A}$ of free n_B massless real scalars and free n_f massless Weyl fermions are given in terms of the free energy density F as (T is an infinitesmal temperature)

$$\mathcal{A} \equiv \frac{90F}{\pi^2 T^4} = n_B + \frac{7}{4} n_f \,. \tag{3.46}$$

First, we may use (3.46) to favor between a phase of composite fermions or a phase with spontaneous symmetry breaking (SSB). As we pointed out above, we could not find composite fermions that matched the anomalies. Yet, one may be tempted to use (3.46) to predict whether the theory flows to an IR CFT. In a weakly-coupled CFT, the IR DOF are the same UV DOF. On the other hand, the DOF in a spontaneously broken phase, assuming the global symmetry $SU(N_f)$ entirely breaks, are $N_f^2 - 1$ Goldstones*. Let us define $\Delta\mathcal{A}$ as the difference between the DOF in the two scenarios. Then, we have

$$\Delta\mathcal{A} = \underbrace{n_\psi^2 + n_\chi^2 - 2}_{\text{Goldstones}} - \left\{ \underbrace{2(N^2-1)}_{\text{gluons DOF}} + \frac{7}{4}\left(n_\psi \mathrm{dim}_\psi + n_\chi \mathrm{dim}_\chi\right) \right\} . \tag{3.47}$$

*Notice that a theory that fully breaks its global symmetries will match its 't Hooft anomalies in the IR. We assume that enough condensates form to obey the matching conditions.

According to the conjecture, a theory with $\Delta\mathcal{A} > 0$ disfavors an SSB phase. It can be easily checked that all the theories in Table 3.1 yield $\Delta\mathcal{A} < 0$, favoring a phase with broken symmetries. This is to be expected since $n_\psi, n_\chi \sim N$ and $\dim_\psi, \dim_\chi \sim N^2$. Thus, while the SSB phase has $\sim N^2$ DOF, a phase with CFT has $\sim N^3$ DOF. Then, one may naively conclude that all theories in Table 3.1 will break their symmetries and flow to a Goldstone phase. This conclusion, however, totally ignores the dynamics of the theory on the way from UV to IR. A theory must enter a strongly-coupled regime to form condensates and break its continuous symmetries, i.e., breaking the symmetries has to happen dynamically since no elementary scalars exist. As we argued above, some of our theories have robust IR fixed points at weak coupling, indicating that it is most unlikely they can form condensates. Consequently, in the subsequent analysis, we avoid employing the aforementioned hypothesis to favor between an SSB or CFT phase. Instead, we use the β-function analysis to check whether a theory flows to an IR CFT*.

However, assuming the existence of multiple sets of condensates, each capable of accounting for all observed anomalies via SSB, we can employ the aforementioned line of reasoning to make a prediction. Presumably, the set of condensates that causes the flavor group to break into the largest subgroup will be preferred due to its associated reduction in the number of infrared degrees of freedom.

The following sections are devoted to systematically applying the above ideas to the concrete theories in Table 3.1. We start our discussion by working out all the details. As we progress through the list of theories, we build on the previous experience and shorten our discussion.

3.4 Fermionic theories

This section systematically studies theories that admit fermionic operators in their spectrum. These are $(N = 5, k = 1)$, $(N = 6, k = 2)$, $(N = 6, k = 1)$, and $(N = 10, k = 2)$. Our analysis indicates that the first two theories form condensates and break their global symmetries, while the last two flow to a CFT.

3.4.1 $\boxed{SU(5), k = 1}$

This theory admits a single Weyl fermion ψ and $n_\chi = 9$ flavors of χ Weyl fermions. In addition, we have $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = \gcd(7, 27) = 1$, indicating that the theory does not possess a discrete chiral symmetry. The solutions to the cocycle

*This method was used in [39] to predict the IR phase of a theory with a single adjoint and N_f fundamental flavors of Weyl fermions. It was found that the $\Delta\mathcal{A}$ calculations are consistent with the prediction of perturbative β-function. The fact that this analysis does not hold for the 2-index chiral theories is attributed to the large number of degrees of freedom of a CFT, which always exceeds the number of degrees of freedom of an SSB phase.

conditions (3.14) give $\mathbb{Z}_5 \times \mathbb{Z}_9$ as the discrete division group. Thus, the faithful global symmetry is

$$G_{\text{global}} = \frac{SU(9)_\chi \times U(1)_A}{\mathbb{Z}_5 \times \mathbb{Z}_9}\,, \tag{3.48}$$

and the $U(1)_A$ charges of ψ and χ are

$$q_\psi = -27\,, \quad q_\chi = 7\,. \tag{3.49}$$

Since both q_ψ and q_χ are odd, $(-1)^F \equiv \mathbb{Z}_2^F$ fermion-number symmetry, which acts on (ψ, χ) as $(\psi, \chi) \longrightarrow -(\psi, \chi)$, is a subgroup of $U(1)_A$.

The topological charges of the CFU fluxes are given by:

$$Q_c = \frac{4m^2}{5}\,, m \in \mathbb{Z}_5\,, \quad Q_\chi = \frac{8p'^2}{9}\,, p' \in \mathbb{Z}_9\,, \quad Q_u = s^2\,, s \in \mathbb{Z}_{45}\,, \tag{3.50}$$

and (m, p', s) are chosen to satisfy (3.14). The theory admits a set of anomalies listed in Table 3.2 (from here on, we give the phase of the corresponding anomaly).

Anomaly	Equation	Value
$[U(1)_A]^3$	$\kappa_u^3 = n_\psi q_\psi^3 \dim\psi + n_\chi q_\chi^3 \dim\chi$	-264375
$U(1)_A[SU(9)_\chi]^2$	$q_\chi \dim\chi$	70
$[SU(9)_\chi]^3$	$\dim\chi$	10
$U(1)_A[\text{grav}]$	$2(n_\psi q_\psi \dim\psi + n_\chi q_\chi \dim\chi)$	450
$U(1)_A[\text{CFU}]$	$q_\chi \dim\chi\, Q_\chi + \kappa_{u^3} Q_u$	$\frac{560}{9}p'^2 - 264375s^2$

Table 3.2: Anomalies of $SU(5), k = 1$.

Notice that, as pointed out above, the $U(1)_A[\text{CFU}]$ anomaly does not depend on the color topological charge. We can also put the theory on $\mathbb{CP}^2$ by employing fluxes in the centers of $SU(5)$ and $SU(9)_\chi$ accompanied by a $U(1)_A$ flux, as can be easily checked from (3.19).

The 2-loop and 3-loop β-function analysis show that the theory has an IR fixed point at *somewhat large* coupling-constant: $\frac{g_*^2}{4\pi} \approx 0.64$ and $\frac{g_*^2}{4\pi} \approx 0.34$, respectively. Therefore, such a fixed point is not robust. We conclude that either the theory forms composite fermions or flows to an SSB phase.

Matching by composites

We used the systematic approach discussed in Section 3.3 to search for a set of composite fermions. We found a pair of operators

$$\mathcal{F}_1 = \psi\chi^6\,, \quad \mathcal{F}_2 = \psi^7\chi^{22}\,, \tag{3.51}$$

with $\mathcal{N}_1 = 36$ and $\mathcal{N}_2 = 9$ copies that matched the $[U(1)_A]^3$ and $U(1)_A[\text{grav}]$ anomalies. Yet, this pair failed to match the $U(1)_A\,[SU(9)_\psi]^2$ anomaly. The upper bound on the number of the IR fermion species is $\mathcal{N} \sim 132$. The large number of partitions of $\mathcal{N}$ is $\mathcal{O}(10^7)$, which hindered the abilities of our search algorithm. We failed to find a set of composites that matches the full set of anomalies.

Matching by condensates

We now turn to the formation of condensates. The lowest-order condensate is

$$\mathcal{C}_1^i = \epsilon^{a_1a_2a_3a_4a_5}\epsilon_{\alpha_1\alpha_2}\psi^{\alpha_1}_{(a_1a_2)}\chi^{\alpha_2,\,i}_{[a_3a_4a_5]}\,, \tag{3.52}$$

where $a_1, .., a_5$ are color, α_1, α_2 are spinor, and i is a $SU(9)_\chi$ flavor indices. This condensate vanishes identically owing to the symmetrizing over a_1, a_2. Yet, one can evade this problem by inserting gauge-covariant gluonic fields $(f^c_{\mu\nu})^{a_j}_{a_i}\sigma^{\mu\nu}$:

$$\mathcal{C}_1^i \longrightarrow \tilde{\mathcal{C}}_1^i = \epsilon^{a_1a_2a_3a_4a_5}\epsilon_{\alpha_1\alpha_2}(f^c_{\mu\nu})^{a_6}_{a_2}\sigma^{\mu\nu}\psi^{\alpha_1}_{(a_1a_6)}\chi^{\alpha_2,\,i}_{[a_3a_4a_5]}\,. \tag{3.53}$$

This trick will always be followed whenever the statistics of indices cause some operator to vanish. $\mathcal{C}_1^i$ transforms in the defining representation of $SU(9)_\chi$, and thus, it breaks it down to $SU(8)$. However, the condensation of $\mathcal{C}_1^i$ leaves a $U(1)$ generator of $SU(9)\times U(1)_A$ unbroken. To see that, we go to a basis where $\mathcal{C}_1^i \propto \delta_{i,9}$. In this basis, the unbroken $SU(8)$ group acts on the 8×8 upper block matrices of the original 9×9 unitary matrices of $SU(9)$. Now, it is easy to see that the $SU(9)$ Cartan generator $H_8 = \text{diag}\,(1, 1, ..., -8)$ combines with the $U(1)_A$ generator to leave the vacuum $\delta_{i,9}$ invariant:

$$e^{i2\pi(-20\beta)}\begin{bmatrix} e^{i2\pi\alpha} & 0... & 0 \\ ... & ... & ... \\ 0 & ... & e^{i2\pi(-8\alpha)} \end{bmatrix}\begin{bmatrix} 0 \\ ... \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ ... \\ 1 \end{bmatrix}, \tag{3.54}$$

where α and β are the Cartan and $U(1)_A$ phases, respectively. The direction $2\alpha = -5\beta$ is the unbroken $U(1)$ direction. The unbroken $SU(8)$ has a non-vanishing cubic anomaly. In addition, the unbroken $U(1)$ symmetry inherits the $U(1)_A[\text{grav}]$ anomaly, signaling that such breaking is incomplete or inconsistent with the anomaly-matching conditions.

Another condensate is (we suppress color and spinor indices to reduce clutter)

$$\mathcal{C}_2^{(ij)} = \psi^2\chi^{(i}\chi^{j)}\,, \tag{3.55}$$

which transforms in the 2-index symmetric representation of the flavor group*. The general form of the "Higgs" potential of the condensate is

$$V(\mathcal{C}_2) = -\frac{1}{2}\mu^2\mathcal{C}_2^{(ij)}\mathcal{C}_{2(ij)} + \frac{1}{4}\lambda_1(\mathcal{C}_2^{(ij)}\mathcal{C}_{2(ij)})^2 + \frac{1}{4}\lambda_2(\mathcal{C}_{2(ij)}\mathcal{C}_2^{(jk)}\mathcal{C}_{2(kl)}\mathcal{C}_2^{(li)})\,, \tag{3.56}$$

*We also insert gluons if the statistics cause the condensate to vanish.

for some real parameters $\mu^2 > 0$, λ_1, and λ_2. In the case $\lambda_2 > 0$, the condensate has a non-zero vacuum expectation value and we can pick the form of the condensate to be $\mathcal{C}_2 \propto I_9$ [57]. This breaks $SU(9)$ to the anomaly-free subgroup $SO(9)$.

Is there a combination of $SU(9)_\chi \times U(1)_A$ that breaks to a remaining $U(1)$ symmetry? As $U(1)_A$ is abelian, we need to consider the subgroup generated by the Cartan subalgebra of $SU(9)$. The “unnormalized" generators of the Cartan subalgebra of $SU(9)$ are:

$$[H_m]_{ij} = \sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1}\,, \quad m = 1, 2, ..., 8\,. \tag{3.57}$$

A general $SU(9)$ element generated by the Cartan subalgebra has the form $\exp(2\pi i \alpha_m H_m)$, $m = 1, 2, .., 8$ and $\alpha_m \in [0, 1)$. A combined $SU(9) \times U(1)_A$ transformation acts on $\mathcal{C}^{ij}$ via:

$$\mathcal{C}_2'^{(ij)} = e^{2\pi i(-40\beta)} \left(e^{2\pi i \alpha_m H_m}\right)^{ik} \left(e^{2\pi i \alpha_m H_m}\right)^{jl} \mathcal{C}_{2(kl)}\,, \tag{3.58}$$

and should leave the vacuum expectation value invariant. Thus, we need

$$e^{2\pi i(-40\beta)} \left(e^{2\pi i \alpha_m H_m}\right)^{ik} \left(e^{2\pi i \alpha_m H_m}\right)^{jl} I_9 = I_9\,. \tag{3.59}$$

It can be easily checked that there are no nontrivial solutions to the above equation, indicating that no $U(1)$ direction is left unbroken.

Under the action of $U(1)_A$, the condensate transforms as $\mathcal{C}_2^{(ij)} = \psi^2\chi^{(i}\chi^{j)} \longrightarrow \mathcal{C}_2'^{(ij)} = e^{i2\pi(-40\beta)}\psi^2\chi^{(i}\chi^{j)}$, where $\beta \in [0, 1)$ is the $U(1)_A$ parameter. So it appears that the condensate is invariant under a discrete $\mathbb{Z}_{40}$ subgroup of $U(1)_A$. But recall that the global symmetry group includes a division by the $\mathbb{Z}_5$ center of the color group. $\mathbb{Z}_5$ is not a subgroup of $SU(9)$, therefore it can only quotient $U(1)_A$, so the parameter β is in fact a $U(1)_A/\mathbb{Z}_5$ parameter and $\beta \in [0, 1/5)$. Therefore the condensate only exhibits an unbroken $\mathbb{Z}_8$ symmetry.

The discrete symmetry $\mathbb{Z}_8$ has non-perturbative anomalies, as can easily be checked using (3.29) and (3.30), meaning that the condensation of $\mathcal{C}_2^{(ij)}$ is insufficient to match the full set of anomalies. Notice that since both ψ and χ have odd charges under $U(1)_A$, any unbroken discrete subgroup of $U(1)_A$ necessarily contains $(-1)^F$, and thus, we can use the twisted group $\mathrm{Spin}^{\mathbb{Z}_{2m}}$ to detect the nonperturbative anomaly as given from (3.30). Moreover, since the theory does not possess a 1-form symmetry, there is no ambiguity in calculating the discrete-symmetry anomaly [56].

In searching for a condensate that does not leave behind a non-anomalous $U(1)$ or discrete subgroup, we consider the most general bosonic operator:

$$\mathcal{C} = \psi^{\alpha_\psi}\chi^{\alpha_\chi}\,, \quad 2\alpha_\psi + 3\alpha_\chi \in 5\mathbb{Z}^+\,, \alpha_\psi + \alpha_\chi \in 2\mathbb{Z}^+\,. \tag{3.60}$$

This condensate carries a charge of $-27\alpha_\psi + 7\alpha_\chi$ under $U(1)_A$, and thus, breaks $U(1)_A$ down to $\mathbb{Z}_{(-27\alpha_\psi+7\alpha_\chi)/5}$. We used both (3.29) and (3.30) to check the non-perturbative anomalies of $\mathbb{Z}_{(-27\alpha_\psi+7\alpha_\chi)/5}$ and found that both untwisted and twisted backgrounds yield the same results. The lowest-dimensional condensate that breaks $U(1)_A$ to a non-anomalous subgroup has $\alpha_\psi = 1$ and $\alpha_\chi = 11$. In this case, $\mathbb{Z}_{10}$ is the anomaly-free subgroup.

The condensate

$$\mathcal{C}_3 = \psi\chi^{11} \tag{3.61}$$

transforms in a higher representation of $SU(9)_\chi$. One can contract 9 out of the 11 flavor indices of $\mathcal{C}_3$ with the Levi-Civita tensor, leaving 2 free indices. Then, we can rearrange the free indices (possibly with insertions of gluons in case the statistics cause the condensate to vanish) such that $\mathcal{C}_3$ transforms in the 2-index symmetric representation of $SU(9)$:

$$\mathcal{C}_3^{(ij)} = \psi\chi^9\chi^{(i}\chi^{j)}\,. \tag{3.62}$$

The condensing of $\mathcal{C}_3^{(ij)}$ breaks $\frac{SU(9)_\chi\times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_9}$ down to the anomaly-free subgroup $SO(9)\times\mathbb{Z}_{10}$.

Alternatively, one can search for a companion condensate to $\mathcal{C}_2^{(ij)}$ that breaks $U(1)_A$ to a discrete subgroup $\mathbb{Z}_q$, such that $\gcd(q,8)=2$. This ensures that the formation of these two condensates breaks $U(1)_A$ down to the anomaly-free subgroup $\mathbb{Z}_2^F$, which is the fermion number. The companion condensate with the lowest dimension is $\mathcal{C}_4 = \chi^{10}$, which, superficially, breaks $U(1)_A$ down to $\mathbb{Z}_q = \mathbb{Z}_{14}$. This, however, is an immature conclusion. One can contract 9 flavor indices of $\mathcal{C}_4$ with a Levi-Civita tensor leaving one free index. Then, $\mathcal{C}_4$ transforms in the fundamental representation of $SU(9)$, and according to the discussion preceding (3.54), it breaks it down to $SU(8)\times U(1)$. Because of the unbroken $U(1)$ generator, the condensation of $\mathcal{C}_2$ along with $\mathcal{C}_4$ break $\frac{SU(9)_\chi\times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_9}$ down to a subgroup that contains the anomalous $\mathbb{Z}_8$. More than this is needed to match the full set of anomalies.

We might continue searching for suitable condensates that break G^{g} to an anomaly-free subgroup. However, the lesson from the above discussion is that it is generically a complex exercise.

Since $SO(9)$ is the largest anomaly-free subgroup of $SU(9)$, the condensation of $\mathcal{C}_3^{(ij)}$ leads to the smallest number of the IR Goldstones, and hence, we predict that the theory will flow to a phase with the global symmetry broken down to the anomaly-free $SO(9)\times\mathbb{Z}_{10}$. This is the minimal scenario. However, because of strong dynamics, nothing forbids the theory from forming all kinds of condensates, breaking G^{g} down to the anomaly-free $\mathbb{Z}_2^F$ fermion number symmetry.

In an equally alternative scenario, $SU(9)$ could be broken down to the anomaly-free $Sp(8)$ by a condensate transforming in the 2-index anti-symmetric representation.

However, since the dimensions of $Sp(8)$ and $SO(9)$ are identical, anomalies and the argument of the number of Godstones cannot distinguish between the two possible symmetry-breaking scenarios. In general, a condensate transforming in the 2-index symmetric representation of $SU(2N+1)$ breaks this group down to $SO(2N+1)$, while a condensate in the 2-index anti-symmetric representation breaks it down to $Sp(2N)$. Both $SO(2N+1)$ and $Sp(2N)$ have dimension $N(2N+1)$.

Interestingly, the operator $\mathcal{C}_3^{(ij)}$ has a scaling dimension of at least 15 (it could have a higher dimension if gluon fields are needed to avoid the vanishing of the condensate because of fermi-statistics). That such condensate with a large-scaling dimension must condense in the IR to match the complete set of anomalies is remarkable. Generally, it is natural to expect that a strongly coupled theory forms higher-order condensates. In this example, however, this formation is not a question about the dynamics; rather, it is a necessary condition for the theory to obey the kinematical constraints imposed by anomalies.

Does our proposed condensate $\mathcal{C}_3^{(ij)}$ match the $U(1)_A$[CFU] anomaly? The answer is affirmative. $\mathcal{C}_3^{(ij)}$ breaks $SU(9)_\chi$ flavor group down to $SO(9)$. The latter does not have a center symmetry, while the former group has a $\mathbb{Z}_9$ center. Thus, we conclude that the condensate breaks $\mathbb{Z}_9$ maximally, matching the $U(1)_A$[CFU] anomaly. Next, we may turn off the flavor background, restricting ourselves to the color center and $U(1)_A$ (CU) fluxes. In this case, we have $(m, p', s) = (1, 0, \frac{1}{5})$, and keeping in mind that the CU anomaly does not depend on the color topological charge, we find that this is an anomaly of the axial current in the background of a $\mathbb{Z}_5$ flux. The condensation of $\mathcal{C}_3$ breaks $U(1)_A$ to $\mathbb{Z}_{10}$. Thus, the $U(1)_A$[CU] anomaly becomes the $\mathbb{Z}_{10}[\mathbb{Z}_5]^2$ anomaly discussed around Eq. (3.32). However, from the last line in Table 3.2, the anomaly coefficient becomes, $\frac{-264375}{5^2} = -10575$, which is 0 modulo 5. Therefore, in this case, the anomaly becomes trivial, and the $U(1)_A$[CU] anomaly is automatically matched.

3.4.2 $\boxed{SU(6), k=2}$

This theory has a single ψ Weyl fermion along with 5 flavors of χ fermions. Thus, the continuous global symmetry is $SU(5)_\chi \times U(1)_A$. The charges of ψ and χ under $U(1)_A$ are

$$q_\psi = -5\,, \quad q_\chi = 2\,. \tag{3.63}$$

Owing to the fact $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = \gcd(8, 20) = 4$, the theory is also endowed with a $\mathbb{Z}_4^{d\chi}$ chiral symmetry, which is taken to act on χ with a unit charge. It can be checked that this is a genuine symmetry since neither $\mathbb{Z}_4$ nor a subgroup of it can be absorbed in rotations in the centers of $SU(6) \times SU(5)_\chi$. To show that, we try to absorb the elements $e^{i\frac{2\pi\ell}{4}}$, $\ell = 1, 2, 3$, in the centers of $SU(6) \times SU(5)_\chi$:

$$\mathbb{Z}_4: \quad \psi \longrightarrow e^{2\pi i \frac{2m}{6}} \psi = \psi\,, \quad \chi \longrightarrow e^{-2\pi i \frac{2m}{6}} e^{-2\pi i \frac{p'}{5}} \chi = e^{2\pi i \frac{l}{4}} \chi\,. \tag{3.64}$$

No values of m and p' satisfy these equations for $\ell = 1, 2, 3$, and therefore, $\mathbb{Z}_4^{d\chi}$ is a genuine symmetry. An identical procedure is employed in the rest of the theories to ascertain the genuity of discrete chiral symmetries.

To determine the faithful global symmetry, we must find the quotient group by solving the consistency conditions (3.14). This gives $\mathbb{Z}_3 \times \mathbb{Z}_5$ as the group we divide by. Putting everything together and remembering that the theory possesses a $\mathbb{Z}_2^{[1]}$ 1-form center symmetry, we write the faithful global group:

$$G_{\text{global}} = \frac{SU(5)_\chi \times U(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_5} \times \mathbb{Z}_4^{d\chi} \times \mathbb{Z}_2^{(1)} \,. \tag{3.65}$$

The $\mathbb{Z}_2^F$ fermion number symmetry is contained in the generators of the product group $U(1)_A \times \mathbb{Z}_4^{d\chi}$ (notice that the $U(1)_A$ charges of ψ and χ are odd and even, respectively)

$$\begin{aligned} \mathbb{Z}_2^F \subset U(1)_A &: \psi \longrightarrow -\psi\,, \quad \chi \longrightarrow \chi\,, \\ \mathbb{Z}_2^F \subset \mathbb{Z}_4^{d\chi} &: \psi \longrightarrow \psi\,, \quad \chi \longrightarrow -\chi\,. \end{aligned} \tag{3.66}$$

The topological charges of the CFU fluxes are given by:

$$Q_c = \frac{5m^2}{6}\,, m \in \mathbb{Z}_3\,, \quad Q_\chi = \frac{4p'^2}{5}\,, p' \in \mathbb{Z}_5\,, \quad Q_u = s^2\,, s \in \mathbb{Z}_{15}\,, \tag{3.67}$$

and (m, p', s) are chosen to satisfy (3.14). The anomalies of the theory are listed in Table 3.3. It is worth noting that both $\mathbb{Z}_4^{d\chi}$[grav] and $\mathbb{Z}_4^{d\chi}$[CFU] anomalies give at

Anomaly	Equation	Value
$[U(1)_A]^3$	$\kappa_{u^3} = n_\psi q_\psi^3 \dim\psi + n_\chi q_\chi^3 \dim\chi$	-2025
$U(1)_A[SU(5)_\chi]^2$	$q_\chi \dim\chi$	30
$[SU(5)_\chi]^3$	$\dim\chi$	15
$\mathbb{Z}_4^{d\chi}\,[U(1)_A]^2$	$\kappa_{zu^2} = n_\chi q_\chi^2 \dim\chi$	$300 \bmod 4$
$\mathbb{Z}_4^{d\chi}\,[SU(5)_\chi]^2$	$\dim\chi$	$15 \bmod 4$
$U(1)_A$[grav]	$2(n_\psi q_\psi \dim\psi + n_\chi q_\chi \dim\chi)$	90
$\mathbb{Z}_4^{d\chi}$[grav]	$2n_\chi \dim\chi$	$150 \bmod 8$
$[\mathbb{Z}_4^{d\chi}]^3$	(3.29)	$2250 \bmod 24$
$U(1)_A$[CFU]	$q_\chi \dim\chi\, Q_\chi + \kappa_{u^3} Q_u$	$24p'^2 - 2025s^2$
$\mathbb{Z}_4^{d\chi}$[CFU]	$n_\chi T_\chi Q_c + \dim\chi\, Q_\chi + \kappa_{zu^2} Q_{u^2}$	$\frac{50}{3}m^2 + 12p'^2 + 300s^2$

Table 3.3: Anomalies of $SU(6), k=2$.

most a $\mathbb{Z}_2$ phase. Also, this theory cannot be put on $\mathbb{CP}^2$, as there are no solutions to the conditions (3.19).

We first comment on the possibility that the theory flows to a Banks-Zaks fixed point in the IR. The 2-loop beta function of this theory gives $\frac{g_*^2}{4\pi} \approx 8.5 \gg 1$. This value of the coupling constant is too large for perturbation theory to hold. At 3-loops, we obtain $\frac{g_*^2}{4\pi} \approx 0.73$. Also, this coupling-constant value is large, so we cannot conclude that our theory flows to a conformal fixed point in the IR. In the following, we examine the possibilities of fermion composites and SSB.

Matching by composites

Here, we follow the argument in Section 3.3 to show that composite fermions cannot solely match all the UV anomalies. The UV $\mathbb{Z}_4^{d\chi}$[CFU] anomaly of this theory is (unlike the $U(1)_A$[CFU] anomaly, it is important to notice that the color flux contributes to the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly)

$$n_\chi T_\chi Q_c + \dim \chi \, Q_\chi + \kappa_{zu^2} Q_{u^2} = \frac{50}{3} m^2 + 12 p'^2 + 300 s^2 \,. \tag{3.68}$$

In the IR, a set of gauge invariant composite fermions would generate the corresponding $\mathbb{Z}_4^{d\chi}$[CFU] anomaly:

$$D_\chi^{\rm IR} Q_\chi + D_u^{\rm IR} Q_u \tag{3.69}$$

for integers $D_\chi^{\rm IR}$ and $D_u^{\rm IR}$. The integers $D_\chi^{\rm IR}$ and $D_u^{\rm IR}$ are group-theoretical coefficients that are assumed to be found by matching all anomalies of continuous symmetries. In the presence of a CFU background flux, the ratio between the UV and IR partition functions after undergoing a $\mathbb{Z}_4^{d\chi}$ transformation is given by:

$$\frac{\mathcal{Z}^{\rm UV}}{\mathcal{Z}^{\rm IR}} = e^{\frac{i2\pi}{4}\left(\frac{50}{3}m^2 + \left(12 - D_\chi^{\rm IR}\right)p^2 + \left(300 - D_u^{\rm IR}\right)s^2\right)} = e^{\frac{i2\pi}{4}\left(\frac{50}{3}m^2 + d_\chi p'^2 + d_u s^2\right)} \,, \tag{3.70}$$

where $d_\chi = 12 - D_\chi^{\rm IR} \in \mathbb{Z}$ and $d_u = 300 - D_u^{\rm IR} \in \mathbb{Z}$. If there exists a particular solution (m, p', s) of the consistency conditions (3.14) such that no integers d_χ, d_u exists such that

$$\frac{50}{3} m^2 + d_\chi p^2 + d_u s^2 \in 4\mathbb{Z}, \tag{3.71}$$

then we conclude that composite fermions cannot match the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly.

Consider $(m, p, s) = (1, 0, 2/3)$. This is a solution to the consistency conditions and therefore corresponds to a CFU flux. In the presence of this CFU background, the LHS of (3.71) becomes

$$\frac{50}{3} + d_u \frac{4}{9} = \frac{150 + 4 d_u}{9} \,. \tag{3.72}$$

However, $150 + 4d_u \equiv 2 \mod 4$ for any $d_u \in \mathbb{Z}$. Therefore we can conclude that for this theory, composite fermions cannot solely match the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly in the IR.

Matching by a condensate

Without composites, the anomalies are matched by spontaneous symmetry breaking via condensates. First, the 2-fermion condensate cannot match the anomalies, as it breaks $SU(5)_\chi \times U(1)_A$ down to the anomalous subgroup $SU(4) \times U(1)$. Next, consider the operator

$$\mathcal{C}^{(ij)} = \psi^2 \chi^{(i} \chi^{j)} \,, \tag{3.73}$$

where i, j are $SU(5)_\chi$ flavor indices, and in particular, this condensate is in the two-index symmetric irrep of $SU(5)_\chi$. Thus, the condensation of this operator breaks $SU(5)$ to the anomaly-free subgroup $SO(5)$.

Under the action of $U(1)_A$, the condensate transforms as $\mathcal{C}^{ij} = \psi^2\chi^{(i}\chi^{j)} \longrightarrow \mathcal{C}'^{ij} = e^{2\pi(6\beta)}\psi^2\chi^{(i}\chi^{j)}$ where $\beta \in [0,1)$ is the $U(1)_A$ parameter. So it appears that the condensate is invariant under a discrete $\mathbb{Z}_6$ subgroup of $U(1)_A$. But recall that the global symmetry group includes a division by the $\mathbb{Z}_3$ center of the color group. $\mathbb{Z}_3$ is not a subgroup of $SU(5)$, therefore it can only quotient $U(1)_A$, so the parameter β is in fact a $U(1)_A/\mathbb{Z}_3$ parameter and $\beta \in [0, 1/3)$. Therefore the condensate only exhibits an unbroken $\mathbb{Z}_2$ symmetry, which has no global anomaly, and the $U(1)_A$ breaks to a non-anomalous subgroup.

The condensate also breaks $\mathbb{Z}_4^{d\chi}$ down to $\mathbb{Z}_2$, leading to 2 vacua connected via a domain wall. Recalling that the $\mathbb{Z}_4^{d\chi}$[grav] anomaly is only a $\mathbb{Z}_2$ phase, the unbroken subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_4^{d\chi}$ is anomaly free (remember that $\mathbb{Z}_2$ is also free from nonperturbative anomalies). In addition, the $\mathbb{Z}_4^{d\chi}$[CFU] anomaly is valued in $\mathbb{Z}_2$, meaning that the same condensate saturates it. The breaking of $\mathbb{Z}_4^{d\chi}$ down to $\mathbb{Z}_2$ will also automatically match the $[\mathbb{Z}_4^{d\chi}]^3$ anomaly, since $\mathbb{Z}_2$ is anomaly free.

Let us examine the fate of the $U(1)_A$[CFU] anomaly. First, when we turn on the flavor center flux, the breaking of $SU(5)$ into $SO(5)$ matches the anomaly, as the breaking causes the center of $SU(5)$ to break. Next, we solely turn on the color and $U(1)_A$ fluxes. In this case, $s \in \mathbb{Z}_3$, and the breaking of $U(1)_A$ down to $\mathbb{Z}_2$ implies that we are after $\mathbb{Z}_2[\mathbb{Z}_3]^2$ anomaly. The anomaly coefficient can be read from Table 3.3, and according to (3.32), the anomaly is automatically matched since $\gcd(3, 2) = 1$.

We conclude that the global symmetry G^{g} breaks down to $SO(5) \times \frac{(\mathbb{Z}_2 \subset U(1)_A) \times (\mathbb{Z}_2 \subset \mathbb{Z}_4^{d\chi})}{\mathbb{Z}_2}$. The first $\mathbb{Z}_2$ symmetry acts only on ψ, while the second $\mathbb{Z}_2$ acts only on χ. Then, from (3.66), we see that the combination of these symmetries acts like the fermion number, which is left intact in the IR. The extra modding by $\mathbb{Z}_2$ is employed to avoid overcounting.

Since $SO(5)$ is the largest anomaly-free subgroup of $SU(5)$, this breaking pattern minimizes the number of Goldstones and is the most favorable scenario.

3.4.3 $\boxed{SU(6), k=1}$

This theory has 2 flavors of ψ and 10 flavors of χ, and thus, the flavor symmetry is $SU(2)_\psi \times SU(10)_\chi$. The $U(1)_A$ charges are

$$q_\psi = -5\,, \quad q_\chi = 2\,. \tag{3.74}$$

Because $r = \gcd(n_\chi T_\chi, n_\psi T_\psi) = \gcd(40, 16) = 8$, we may be tempted to conclude the theory has a $\mathbb{Z}_8$ chiral symmetry. However, one can show that a $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_{10}$ center of $SU(10)_\chi$ can be used to identify elements of $\mathbb{Z}_8$:

$$\chi \;:\; e^{-2\pi i \frac{p'}{10}} e^{2\pi i \frac{l}{8}} = e^{2\pi i \frac{l'}{8}} \,, \tag{3.75}$$

for $l, l' = 1, 2, ..., 7$. For example, setting $p' = -5$ identifies $\ell = 1$ and $\ell = 5$, etc. In addition, the solutions to the consistency conditions (3.14) yield the division group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Thus, the faithful global symmetry is

$$G^{\rm g} = \frac{SU(2)_\psi \times SU(10)_\chi \times U(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5} \times \mathbb{Z}_4^{d\chi} \times \mathbb{Z}_2^{(1)} \,. \tag{3.76}$$

The β-function indicates that the theory flows to an IR fixed point. At 2 loops, the coupling constant at the fixed point is $\frac{g_*^2}{4\pi} \approx 0.094$. At 3 loops, we obtain $\frac{g_*^2}{4\pi} \approx 0.075$. Both values are much smaller than our threshold value of 0.1, and the 2- and 3-loop analysis is only 10% apart. Also, the 3-loop to the 2-loop ratio in (3.33) is ≈ 0.2. Thus, the fixed point is reliable. As we pointed out above, the lowest-order bosonic operator in this theory, $F_{\mu\nu}\sigma^{\mu\nu}\chi\psi$, necessitates the introduction of a color field to prevent its vanishing due to statistics. This is a dimension-5 operator, and due to the smallness of the coupling constant, we do not expect this operator to condense. Not to mention that this operator by itself is not enough to match the full set of anomalies, and higher-order condensates must also form to match them. We, thus, conclude that the most probable scenario is that the theory flows to a CFT.

3.4.4 $\boxed{SU(10), k=2}$

The theory admits 3 flavors of ψ and 7 flavors of χ. The charges of the fermions under $U(1)_A$ are

$$q_\psi = -14 \,, \quad q_\chi = 9 \,. \tag{3.77}$$

We also have $r = \gcd(N_\psi, N_\chi) = 4$, so that the theory is endowed with a $\mathbb{Z}_4^{d\chi}$ chiral symmetry, which cannot be absorbed in a combination of the centers of the color or flavor groups. After solving the consistency equations, we obtain the faithful global symmetry group

$$G_{\rm global} = \frac{SU(3)_\psi \times SU(7)_\chi \times U(1)_A}{\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_7} \times \mathbb{Z}_4^{d\chi} \times \mathbb{Z}_2^{(1)} \,. \tag{3.78}$$

The theory develops a Bank-Zaks fixed point. The 2 and 3-loop values of the coupling constant at the fixed point are $\frac{g_*^2}{4\pi^2} \approx 0.059$ and $\frac{g_*^2}{4\pi^2} \approx 0.046$, respectively. Also, the 3-loop to the 2-loop ratio in (3.33) is ≈ 0.2. Thus, like $SU(6), k = 1$, this theory is expected to flow to a CFT.

3.5 Bosonic theories

All gauge-invariant operators in this class of theories are bosonic. In the following, we provide a systematic study of this class.

Theory	n_ψ	n_χ	$\mathbb{Z}_p^{d\chi}$	Γ	(q_ψ, q_χ)	2-loop	3-loop	$\frac{\alpha_*\beta_2}{4\pi\beta_1}$
$SU(16), k=4$	3	5	2	$\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_5$	$(-35, 27)$	0.09	0.064	0.62
$SU(20), k=4$	4	6	2	$\mathbb{Z}_{10}\times\mathbb{Z}_4\times\mathbb{Z}_3$	$(-27, 22)$	0.017	0.015	0.11
$SU(28), k=8$	3	4	1	$\mathbb{Z}_7\times\mathbb{Z}_3\times\mathbb{Z}_4$	$(-52, 45)$	0.086	0.051	1.12
$SU(36), k=8$	4	5	1	$\mathbb{Z}_{18}\times\mathbb{Z}_4\times\mathbb{Z}_5$	$(-85, 76)$	0.019	0.016	0.25
$SU(44), k=8$	5	6	1	$\mathbb{Z}_{11}\times\mathbb{Z}_5\times\mathbb{Z}_6$	$(-126, 115)$	0.0002	0.0002	0.003

Table 3.4: A list of conformal bosonic theories.

3.5.1 Conformal theories

We start by listing theories that flow to a conformal fixed point. These theories are displayed in Table 3.4. In each case, the global symmetry is given by

$$G = \frac{SU(n_\psi)\times SU(n_\chi)\times U(1)_A}{\Gamma}\times\mathbb{Z}_p^{d\chi}\times\mathbb{Z}_2^{(1)}\,. \tag{3.79}$$

We also display the coupling constant $\frac{g_*^2}{4\pi^2}$ at the 2- and 3-loop fixed points. The smallness of the coupling constant and its consistency between the 2- and 3-loop calculations is an indicator of the robustness of the fixed point. To quantify this robustness, we may truncate the β-function to the second term in (3.33) and find the fixed point is given by $\alpha_* = -\frac{4\pi\beta_0}{\beta_1}$. The existence of such a fixed point implies that the first and second terms possess comparable magnitudes. Consequently, the ratio between the third and second (or first) term $\frac{\alpha_*\beta_2}{4\pi\beta_1}$ represents the error incurred by neglecting the third term. A low ratio indicates the perturbative nature of the fixed point.

The two theories $(N=20, k=4)$ and $(N=44, k=8)$ have the most reliable fixed points. While the theory $(N=28, k=8)$ has $\frac{\alpha_*\beta_2}{4\pi\beta_1}=1.12$, and its fixed point is under question.

3.5.2 Confining theories

3.5.2.1 $\boxed{SU(8), k=4}$

This theory was studied in [35]. Here, we revisit it in light of the discrete anomalies not discussed in [35]. The theory admits $n_\psi = 1$ and $n_\chi = 3$ flavors of fermions. The fermion charges under $U(1)_A$ are

$$q_\psi = -9\,, \quad q_\chi = 5\,. \tag{3.80}$$

Also, the theory admits a $\mathbb{Z}_2^{d\chi}$ discrete chiral symmetry. Solving the consistency conditions (3.14) yield the faithful global symmetry

$$G^{\rm g} = \frac{SU(3)_\chi\times U(1)_A}{\mathbb{Z}_4\times\mathbb{Z}_3}\times\mathbb{Z}_2^{d\chi}\times\mathbb{Z}_2^{(1)}\,. \tag{3.81}$$

Anomaly	Equation	Value
$[SU(3)_\chi]^3$	$\dim_\chi$	28
$U(1)_A\,[SU(3)_\chi]^2$	$q_\chi \dim_\chi$	140
$\mathbb{Z}_2^{d\chi}\,[SU(3)_\chi]^2$	$\dim_\chi$	28
$\mathbb{Z}_2^{d\chi}\,[U(1)_A]^2$	$\kappa_{zu^2} = q_\chi^2 \dim_\chi n_\chi$	2100
$U(1)_A[\text{grav}]$	$2(q_\psi \dim_\psi + q_\chi \dim_\chi n_\chi)$	192
$[U(1)_A]^3$	$\kappa_{u^3} = q_\chi^3 \dim_\chi n_\chi + q_\psi^3 \dim_\psi$	-15744
$\mathbb{Z}_2^{d\chi}[\text{grav}]$	$2\dim_\chi n_\chi$	168 (trivial)
$U(1)_A[\text{CFU}]$	$q_\chi \dim\chi\, Q_\chi + \kappa_{u^3} Q_u$	$\frac{280p'^2}{3} - 15744s^2$, $p' \in \mathbb{Z}_3$, $s \in \mathbb{Z}_{12}$
$\mathbb{Z}_2^{d\chi}[\text{CFU}]$	$n_\chi T_\chi Q_c + \dim\chi\, Q_\chi + \kappa_{zu^2} Q_u$	$\frac{27m^2}{2} + \frac{56p'^2}{3} + 2100s^2$, $m \in \mathbb{Z}_4$

Table 3.5: Anomalies of $SU(8), k = 4$.

The theory admits many anomalies in Table 3.5. In addition, the theory admits a $\mathbb{Z}_2^{d\chi}[\text{CFU}]$ anomaly, which yields a phase of π upon turning on a flux with, e.g., $(m, p', s) = (1, 0, \frac{1}{4})$, i.e., this is a $\mathbb{Z}_4 \subset U(1)_A$ flux. We also find that there is an anomaly of $\mathbb{Z}_2^{d\chi}$ on a nonspin manifold, as the partition function acquires a phase of π by turning on a pure $\mathbb{Z}_2^{(1)}$ flux on $\mathbb{CP}^2$.

In [35], it was argued that all the anomalies could be matched by condensing two operators:

$$\mathcal{C}_1^i = \psi\chi^i\,, \quad \mathcal{C}_2^{i_4} = \epsilon_{i_1 i_2 i_3}\chi^{i_1}\chi^{i_2}\chi^{i_3}\chi^{i_4}\,. \tag{3.82}$$

Let us review the anomaly matching using these two operators and comment on why they cannot match the discrete anomalies.

Both operators $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ transform in the defining representation of $SU(3)$ and break it down to the anomaly-free $SU(2)$ (it has no Witten anomalies because the dimensions of the representations are even). Yet, the condensation of $\mathcal{C}_1^i$ or $\mathcal{C}_2^i$ leaves behind an unbroken $SU(3)$ generator. We take $\mathcal{C}_1^i \propto \delta_{i,1}$ and $\mathcal{C}_2^i \propto \delta_{i,1}$ and parametrize the $SU(3)$ matrix that corresponds to the unbroken Cartan generator of $SU(3)$ as diag $\left(e^{i4\pi\alpha}, e^{-i2\pi\alpha}, e^{-i2\pi\alpha}\right)$. Then, under $SU(3)_\chi \times U(1)_A \times \mathbb{Z}_2^{d\chi}$, the operators transform as

$$\mathcal{C}_1^i \longrightarrow e^{i4\pi\alpha - i8\pi\beta + in\pi}\mathcal{C}_1^i\,, \quad \mathcal{C}_2^i \longrightarrow e^{i40\pi\beta + i4\pi\alpha}\mathcal{C}_2^i\,, \tag{3.83}$$

where β corresponds to the $U(1)_A$ transformation, whereas $n = 1$ corresponds to the $\mathbb{Z}_2^{d\chi}$ transformation. Taking $\alpha = -\frac{5}{24}$, $\beta = \frac{1}{48}$, and $n = 1$ leaves $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ invariant under the combined transformations of $SU(3)_\chi \times U(1)_A \times \mathbb{Z}_2^{d\chi}$. This superficially hints at an unbroken $\mathbb{Z}_{24}$ symmetry. However, owing to the modding by $\mathbb{Z}_4$ in (3.81), the genuine unbroken subgroup is $\mathbb{Z}_6$. This unbroken symmetry can be written as $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_3$ is a genuine subgroup of $U(1)_A$. This can be seen by setting $n = 0$, then we find that $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ are left invariant by taking $\alpha = -\frac{5}{12}$ and $\beta = \frac{1}{24}$. Remembering the modding by $\mathbb{Z}_4$ in (3.81), we conclude that there is a $\mathbb{Z}_3$ unbroken subgroup of $U(1)_A$.

It is straightforward to calculate the $\mathbb{Z}_3$ anomaly using (3.29) to find that it is non-vanishing, meaning that the condensation of $\mathcal{C}_1^i$ and $\mathcal{C}_2^i$ is not enough to match the complete set of anomalies. The way out is to consider the condensation of the operator

$$\mathcal{C}_3^{(ij)} = \psi^2 \chi^{(i}\chi^{j)}\,, \tag{3.84}$$

which transforms in the 2-index symmetric representation of $SU(3)_\chi$ and breaks it down to $SO(3)$. $U(1)_A$ is broken to $\mathbb{Z}_2$, after taking into account the modding by $\mathbb{Z}_4$ in (3.81). The $\mathbb{Z}_2^{d\chi}$[CFU] anomaly is automatically matched as $U(1)_A$ is broken down to $\mathbb{Z}_2$ (remember, however, that this $\mathbb{Z}_2$ is the fermion number since both fermions carry odd charges under $U(1)_A$, and the fermion number is gauged). Recalling that we had to turn on a $\mathbb{Z}_4 \subset U(1)_A$ flux in the first place to see this anomaly (a π phase), the breaking of $U(1)_A$ to a smaller subgroup than $\mathbb{Z}_4$ (in this case $\mathbb{Z}_2 \subset \mathbb{Z}_4$) trivializes the anomaly. Thus, at this level, one does not need to break $\mathbb{Z}_2^{d\chi}$. This differs from the findings in [35], where it was argued that the CFU anomaly is not trivial. Here, we arrive at a different IR condensate by scrutinizing the discrete subgroups of $U(1)_A$.

What about matching the anomaly of $\mathbb{Z}_2^{d\chi}$ on $\mathbb{CP}^2$? Since this anomaly is valued in $\mathbb{Z}_2$, it can be matched by a TQFT on a nonspin manifold, as was argued in [54]. Yet, another scenario is to form the condensate $\mathcal{C}_1^i$, which breaks $\mathbb{Z}_2^{d\chi}$ to unity (remember that the $\mathbb{Z}_2^{(1)}$ 1-form symmetry is unbroken assuming confinement). Thus, the condensation of both $\mathcal{C}_1^i$ and $\mathcal{C}_3^{(ij)}$ match all anomalies and break the global group down to $SO(3)$, resulting in 2 vacua (because of the breaking of $\mathbb{Z}_2^{d\chi}$) connected via domain walls.

3.5.2.2 $\boxed{SU(8), k=2}$

This case was also considered briefly in [35]. The theory admits 2 flavors of ψ fermions and 6 flavors of χ fermions. The $U(1)_A$ charges of the fermions are

$$q_\psi = -9\,, \quad q_\chi = 5\,. \tag{3.85}$$

Since $\gcd(N_\psi, N_\chi) = 4$, one may naively conclude that the discrete symmetry is $\mathbb{Z}_4$. Yet, two elements of $\mathbb{Z}_4$ are identified with elements in $\mathbb{Z}_2 \subset \mathbb{Z}_6$, where $\mathbb{Z}_6$ is the center of $SU(6)_\chi$. This leaves us with $\mathbb{Z}_2^{d\chi}$ as the genuine discrete group, which we take to act solely on χ. The faithful global symmetry is

$$G^{\rm g} = \frac{SU(2)_\psi \times SU(6)_\chi \times U(1)_A}{\mathbb{Z}_4 \times \mathbb{Z}_6} \times \mathbb{Z}_2^{d\chi} \times \mathbb{Z}_2^{(1)}\,. \tag{3.86}$$

The UV theory has the 't Hooft anomalies in Table 3.6. The $\mathbb{Z}_2^{d\chi}$[CFU] anomaly does not provide new information. However, there is a non-trivial $\mathbb{Z}_2^{d\chi}[\text{CFU}]_{\mathbb{CP}^2}$ anomaly (a π phase) in the background of a CFU configuration with all fluxes turned on, e.g., $(m, p, p', s) = (1, 1, 1, -5/12)$.

Anomaly	Equation	Value
$[SU(6)_\chi]^3$	dim_χ	28
$U(1)_A[\text{grav}]$	$2(q_\psi\text{dim}_\psi + q_\chi\text{dim}_\chi n_\chi)$	384
$\mathbb{Z}_2^{d\chi}[\text{grav}]$	$\text{dim}_\chi n_\chi$	336 (trivial)
$U(1)_A[SU(6)_\chi]^2$	$q_\chi\text{dim}_\chi$	140
$U(1)_A[SU(2)_\psi]^2$	$q_\psi\text{dim}_\psi$	-324
$[U(1)_A]^3$	$q_\psi^3\text{dim}_\psi + q_\chi^3\text{dim}_\chi n_\chi$	-31488
$\mathbb{Z}_2^{d\chi}[SU(6)_\chi]^2$	dim_χ	28 (trivial)
$\mathbb{Z}_2^{d\chi}[U(1)_A]^2$	$q_\psi^2\text{dim}_\psi n_\psi + q_\chi^2\text{dim}_\chi n_\chi$	4200 (trivial)

Table 3.6: Anomalies of $SU(8), k=2$.

The condensation of the operator

$$\mathcal{C}^i_{1\ j} = \psi_j\chi^i \tag{3.87}$$

break $SU(2)_\psi \times SU(6)_\chi$ down to $SU(2)\times SU(4)$. The unbroken $SU(4)$ is anomalous.

Another operator is

$$\mathcal{C}_2^{[i_1 i_2]} = \psi^2\chi^{[i_1}\chi^{i_2]}\,, \tag{3.88}$$

which is neutral under $SU(2)_\psi \times \mathbb{Z}_2^{d\chi}$ but transforms in the 2-index anti-symmetric representation of $SU(6)$ and breaks it down to the anomaly-free $Sp(6)$*. In addition, the condensation of $\mathcal{C}_2^{[i_1 i_2]}$ breaks $U(1)_A$ to the anomaly-free $\mathbb{Z}_2$, after taking into account the modding by $\mathbb{Z}_4$ in (3.86). What about the $\mathbb{Z}_2^{d\chi}[\text{CFU}]_{\mathbb{CP}^2}$ anomaly? Remember that one needs to turn on a configuration with $U(1)_A$ flux in $\mathbb{Z}_{12}$. Since $U(1)_A$ breaks down to $\mathbb{Z}_2$, the anomaly trivilizes. Recall that this $\mathbb{Z}_2$ is the fermion number since both fermions have odd charges under $U(1)_A$, and that the fermion number is gauged. Thus, the condensation of $\mathcal{C}_2^{[i_1 i_2]}$ leaves behind the unbroken $\frac{SU(2)_\psi \times Sp(6)}{\mathbb{Z}_2} \times \mathbb{Z}_2^{d\chi}$ subgroup and matches all anomalies†.

3.5.2.3 $\boxed{SU(12), k=4}$

The number of flavors in this case is $n_\psi = 2$ and $n_\chi = 4$, and the $U(1)_A$ charges are:

$$q_\psi = -10\,, \quad q_\chi = 7\,. \tag{3.89}$$

Since $\gcd(n_\chi T_\chi, n_\psi T_\psi) = \gcd(40, 28) = 4$, one may conclude that the theory is endowed with a $\mathbb{Z}_4$ chiral symmetry that acts on χ. However, this $\mathbb{Z}_4$ is the center

*Alternatively, one could propose the formation of a condensate transforming in the 2-index symmetric representation of $SU(6)$. This condensate, however, would break $SU(6)$ down to $SO(6)$, resulting in a larger number of Goldstones.

†The symplectic group $Sp(2N)$ has a $\mathbb{Z}_2$ center symmetry, see, e.g., [65]. This is why we needed to mod by $\mathbb{Z}_2$ that is common between $Sp(6)$ and $SU(2)_\psi$.

of the $SU(4)_\chi$ flavor symmetry. Therefore, the theory does not possess a discrete chiral symmetry. Solving the consistency conditions (3.14), we find that the faithful global symmetry group is:

$$G_{\rm global} = \frac{SU(2)_\psi \times SU(4)_\chi \times U(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2} \times \mathbb{Z}_2^{(1)} \,. \tag{3.90}$$

This theory is endowed with the anomalies in Table 3.7. The theory does not possess a Witten anomaly of $SU(2)_\psi$ since $\dim_\psi = 66$ is an even number.

Anomaly	Equation	Value
$[U(1)_A]^3$	$q_\chi^3 \dim_\chi n_\chi + q_\psi^3 n_\psi \dim_\psi$	-65448
$U(1)_A[SU(2)_\psi]^2$	$q_\psi \dim_\psi$	-780
$U(1)_A[SU(4)_\chi]^2$	$q_\chi \dim_\chi$	462
$[SU(4)_\chi]^3$	$\dim_\chi$	66
$U(1)_A[\text{grav}]$	$\kappa_{u^3} = q_\chi \dim_\chi n_\chi + q_\psi n_\psi \dim_\psi$	576
$U(1)_A[\text{CFU}]$	$q_\psi \dim_\psi Q_\psi + q_\chi \dim_\chi Q_\chi + \kappa_{u^3} Q_u$	$390p^2 + \frac{693}{2}p'^2 - 65448 s^2,\ p, p' \in \mathbb{Z}_2$

Table 3.7: Anomalies of $SU(12), k = 4$.

The 2-loop and the 3-loop β-functions predict fixed points at $\frac{g_*^2}{4\pi} = 0.514$ and 0.202, respectively. Both values are large for the fixed points to be robust.

In searching for candidates that break the symmetries spontaneously, let us study the bilinear condensate:

$$\mathcal{C}^i_j = \epsilon^{a_1 \ldots a_{12}} \left(f^c_{\mu\nu}\right)^{a_{13}}_{a_2} \sigma^{\mu\nu} \epsilon_{\alpha_1\alpha_2} \psi^{\alpha_1}_{j,\,(a_1 a_{13})} \chi^{\alpha_2,\,i}_{[a_3..a_{12}]}\,, \quad j = 1,2\,, i = 1,2,3,4, \tag{3.91}$$

where, as usual, $a_1, a_2, ..$ are color indices, α_1, α_2 are spinor indices, while j and i are respectively $SU(2)_\psi$ and $SU(4)_\chi$ indices. The transformation of $\mathcal{C}^i_j$ is noteworthy as it occurs in the fundamental representation of $SU(2)\psi$ and the anti-fundamental representation of $SU(4)\chi$. Consequently, upon condensation, it has the potential to break down $SU(2)_\psi \times SU(4)_\chi$ to $SU(2)_V \times SU(2)$. This symmetry-breaking pattern can be explained as follows [57].

To create an invariant potential for the 4×2 matrix $\mathcal{C}^i_j$, we define the 4×4 matrix $\Phi^{i,i'} \equiv \sum_{j=1}^2 \mathcal{C}^i_j \mathcal{C}^{i'}_j$. By considering the effective potential as a trace over quadratic and quartic terms of $\Phi^{i,i'}$, we might initially assume that we can transform to a basis where $\Phi^{i,i'}$ becomes a non-degenerate diagonal matrix. However, this assumption leads to a contradiction because the 4×1 column vectors in $\Phi^{i,i'}$ are dependent due to the construction of $\Phi^{i,i'}$ from a 4×2 matrix. In other words, $\Phi^{i,i'}$ possesses two zero eigenvalues. Hence, we conclude that we can only transform to a basis that diagonalizes $SU(2)\psi \times (SU(2) \subset SU(4)\chi)$. This results in the diagonal (vector-like) matrix $SU(2)_V$, while $SU(4-2) = SU(2) \subset SU(4)\chi$ remains unbroken. Both $SU(2)_V$ and $SU(2) \subset SU(4)\chi$ are subgroups devoid of anomalies. The potential anomaly, namely the Witten anomaly, does not afflict any of these subgroups. The UV particle content ensures that the number of fermions transforming under

$SU(2)\psi$ and $SU(2) \subset SU(4)_\chi$ is dim$\psi = 78$ and dim$\chi = 66$, respectively, both of which are even numbers. Therefore, none of these groups can exhibit Witten anomalies.

Moreover, due to the $\mathbb{Z}_3$ modding in (3.90), the axial symmetry $U(1)_A/\mathbb{Z}_3$ identifies a transformation phase α with $\alpha + \frac{2\pi}{3}$. The charge of the condensate $\mathcal{C}^i_j$ under $U(1)_A$ is -3, leading to the breaking of $U(1)_A$ to unity. Hence, we conclude that the 2-fermion condensate $\mathcal{C}^i_j$ successfully saturates all the anomalies and breaks the global symmetry down to $\frac{SU(2)_V \times (SU(2) \subset SU(4)_\chi)}{\mathbb{Z}_2}$, where we mod by the $\mathbb{Z}_2$ common center of both groups.

3.5.2.4 $\boxed{SU(12), k = 8}$

The number of flavors in this case is $n_\psi = 1$ and $n_\chi = 2$ and the $U(1)_A$ charges are:

$$q_\psi = -10\,, \quad q_\chi = 7\,. \tag{3.92}$$

Given that $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = \gcd(14, 20) = 2$, we may conclude that the theory has a $\mathbb{Z}_2$ discrete chiral symmetry. Yet, one can absorb this $\mathbb{Z}_2$ in the center of $SU(2)_\chi$, leaving behind no genuine discrete symmetry. After solving the consistency conditions, we find that the faithful global symmetry group is:

$$G^{\mathrm{g}} = \frac{SU(2)_\chi \times U(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_2} \times \mathbb{Z}_2^{(1)}\,. \tag{3.93}$$

The theory possesses the anomalies in Table 3.8. The potential Witten anomaly of

Anomaly	Equation	Value
$[U(1)_A]^3$	$\kappa_{u^3} = q_\chi^3 \mathrm{dim}_\chi n_\chi + q_\psi^3 \mathrm{dim}_\psi$	-32724
$U(1)_A[SU(2)_\chi]^2$	$q_\chi \mathrm{dim}_\chi$	462
$U(1)_A[\mathrm{grav}]$	$q_\chi \mathrm{dim}_\chi n_\chi + q_\psi \mathrm{dim}_\psi$	288
$U(1)_A[\mathrm{CFU}]$	$q_\chi \,\mathrm{dim}_\chi\, Q_\chi + \kappa_{u^3} Q_u$	$231p'^2 - 32724s^2$, $p' \in \mathbb{Z}_2$

Table 3.8: Anomalies of $SU(12), k = 8$.

$SU(2)_\chi$ is absent because $\mathrm{dim}_\chi = 66$ is an even number.

The 2-loop and 3-loop β-functions do not predict fixed points, and the theory needs to break its symmetries spontaneously by forming condensates. The operator

$$\mathcal{C}^i_1 = \psi\chi^i\,, \tag{3.94}$$

where the index i is the $SU(2)_\chi$ flavor, breaks the global symmetry down to $U(1)$. To see that, let us fix the vacuum to be $[1 \quad 0]^T$. Then, if a $U(1)$ generator is left unbroken by the vacuum, one should find a nontrivial solution to

$$\exp\left[i2\pi\beta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right] e^{-i6\pi\alpha I_{2\times 2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.95}$$

It is easy to check that the solution $\beta = 3\alpha$ satisfies the above equation, which is the unbroken $U(1)$ direction. The unbroken $U(1)$ symmetry inherits the UV mixed $U(1)_A$[grav] anomaly, and thus, condensing $\mathcal{C}_1^i$ is not enough to match the anomalies.

Another operator that can condense is

$$\mathcal{C}_2 = \epsilon_{ij}\psi\psi\chi^i\chi^j\,, \tag{3.96}$$

with possible insertions of gluon fields. The operator $\mathcal{C}_2$ is singlet under $SU(2)$, but it has a charge -6 under $U(1)_A$. Because of the modding by $\mathbb{Z}_3$ in (3.93), the condensation of $\mathcal{C}_2$ breaks $U(1)_A$ down to $\mathbb{Z}_2$, an anomaly-free subgroup. We conclude that the condensation of $\mathcal{C}_2$ is enough to match the anomalies, a scenario with the minimum number of Goldstones.

3.5.2.5 $\boxed{SU(20), k=8}$

The number of flavors is $n_\psi = 2$ and $n_\chi = 3$, while the $U(1)_A$ charges are:

$$q_\psi = -27\,, \quad q_\chi = 22\,. \tag{3.97}$$

Since $r = \gcd(n_\psi T_\psi, n_\chi T_\chi) = (36, 66) = 2$, we might conclude that the theory possesse a $\mathbb{Z}_2$ chiral symmetry. However, this symmetry can be rotated way in the following way. First, according to our choice, the would-be chiral symmetry acts only on χ. Thus, $(\psi,\chi) \longrightarrow (\psi,-\chi)$ under this $\mathbb{Z}_2$. Next, we apply a transformation by $(-1)^F$, which sends $(\psi,-\chi) \longrightarrow (-\psi,\chi)$. Finally, we apply another transformation by the center of $SU(2)_\psi$, which sends $(-\psi,\chi) \longrightarrow (\psi,\chi)$. This shows that the theory does not possess a discrete chiral symmetry. Finding the solutions to the consistency conditions, the faithful global symmetry group is:

$$G_{\text{global}} = \frac{SU(2)_\psi \times SU(3)_\chi \times U(1)_A}{\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3} \times \mathbb{Z}_2^{(1)}\,. \tag{3.98}$$

The anomalies of this theory are given in Table 3.9.

Anomaly	Equation	Value
$[U(1)_A]^3$	$\kappa_{u^3} = q_\chi^3 \dim_\chi n_\chi + q_\psi^3 \dim_\psi$	-2197500
$U(1)_A\,[SU(2)_\psi]^2$	$q_\psi n_\psi \dim_\psi$	-5670
$U(1)_A\,[SU(3)_\chi]^2$	$q_\chi n_\chi \dim_\chi$	4180
$[SU(3)_\chi]^3$	$\dim_\chi$	190
$U(1)_A\,[\text{grav}]$	$2(q_\chi \dim_\chi n_\chi + q_\psi \dim_\psi)$	2400
$U(1)_A[\text{CFU}]$	$q_\psi \dim_\psi Q_\psi + q_\chi \dim_\chi Q_\chi + \kappa_{u^3} Q_u$	$-2835p^2 + \frac{8360}{3}p'^2 - 2197500s^2$

Table 3.9: Anomalies of $SU(20), k=8$.

Since $SU(2)_\psi$ is an anomaly-free group, it does not need to break. The scenario that gives the lowest number of Goldstones amounts to breaking $SU(3)\chi \times U(1)_A$

to an anomaly-free subgroup. This can be achieved by condensing

$$\mathcal{C}^{(ij)} = \psi^2 \chi^{(i} \chi^{j)}\,, \tag{3.99}$$

which is singlet under $SU(2)_\psi$ and transforms in the 2-index symmetric representation of $SU(3)$ breaking it to $SO(3)$. As before, this condensate also breaks $U(1)_A$ to the anomaly-free subgroup $\mathbb{Z}_2$. Thus, the IR unbroken 0-form symmetry is $\frac{SU(2)\psi\times(\mathbb{Z}_2\subset U(1)_A)}{\mathbb{Z}_2}\times SO(3)$.

3.6 Summary

Theory	Global Symmetries	Condensate(s)	IR Symmetries
$SU(5), k=1$	$\frac{SU(9)_\chi\times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_9}$	$\psi\chi^9\chi^{(i}\chi^{j)}$	$SO(9)\times(\mathbb{Z}_{10}\subset U(1)_A)$
$SU(6), k=1$	$\frac{SU(2)_\psi\times SU(10)_\chi\times U(1)_A}{\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_5}\times\mathbb{Z}_4^{d\chi}$	—	CFT
$SU(6), k=2$	$\frac{SU(5)_\chi\times U(1)_A}{\mathbb{Z}_3\times\mathbb{Z}_5}\times\mathbb{Z}_4^{d\chi}$	$\psi^2\chi^{(i}\chi^{j)}$	$SO(5)\times\frac{(\mathbb{Z}_2\subset U(1)_A)\times(\mathbb{Z}_2\subset\mathbb{Z}_4^{d\chi})}{\mathbb{Z}_2}$
$SU(10), k=2$	$\frac{SU(3)_\psi\times SU(7)_\chi\times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_3\times\mathbb{Z}_7}\times\mathbb{Z}_4^{d\chi}$	—	CFT
$SU(8), k=2$	$\frac{SU(2)_\psi\times SU(6)_\chi\times U(1)_A}{\mathbb{Z}_4\times\mathbb{Z}_6}\times\mathbb{Z}_2^{d\chi}$	$\psi^2\chi^{[i}\chi^{j]}$	$\frac{SU(2)_\psi\times Sp(6)}{\mathbb{Z}_2}\times\mathbb{Z}_2^{d\chi}$
$SU(8), k=4$	$\frac{SU(3)_\chi\times U(1)_A}{\mathbb{Z}_4\times\mathbb{Z}_3}\times\mathbb{Z}_2^{d\chi}$	$\psi\chi^i, \psi^2\chi^{(i}\chi^{j)}$	$SO(3)$
$SU(12), k=4$	$\frac{SU(2)_\psi\times SU(4)_\chi\times U(1)_A}{\mathbb{Z}_6\times\mathbb{Z}_2\times\mathbb{Z}_2}$	$\psi_i\chi^j$	$\frac{SU(2)_V\times(SU(2)\subset SU(4)_\chi)}{\mathbb{Z}_2}$
$SU(12), k=8$	$\frac{SU(2)_\chi\times U(1)_A}{\mathbb{Z}_3\times\mathbb{Z}_2}$	$\epsilon_{ij}\psi^2\chi^i\chi^j$	$\frac{SU(2)_\chi\times(\mathbb{Z}_2\subset U(1)_A)}{\mathbb{Z}_2}$
$SU(16), k=4$	$\frac{SU(3)_\psi\times SU(5)_\chi\times U(1)_A}{\mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_5}\times\mathbb{Z}_2^{d\chi}$	—	CFT
$SU(20), k=8$	$\frac{SU(2)_\psi\times SU(3)_\chi\times U(1)_A}{\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_3}$	$\psi^2\chi^{(i}\chi^{j)}$	$\frac{SU(2)_\psi\times(\mathbb{Z}_2\subset U(1)_A)}{\mathbb{Z}_2}\times SO(3)$
$SU(20), k=4$	$\frac{SU(4)_\psi\times SU(6)_\chi\times U(1)_A}{\mathbb{Z}_{10}\times\mathbb{Z}_4\times\mathbb{Z}_3}\times\mathbb{Z}_2^{d\chi}$	—	CFT
$SU(28), k=8$	$\frac{SU(3)_\psi\times SU(4)_\chi\times U(1)_A}{\mathbb{Z}_7\times\mathbb{Z}_3\times\mathbb{Z}_4}$	—	CFT
$SU(36), k=8$	$\frac{SU(4)_\psi\times SU(5)_\chi\times U(1)_A}{\mathbb{Z}_{18}}$	—	CFT
$SU(44), k=8$	$\frac{SU(5)_\psi\times SU(6)_\chi\times U(1)_A}{\mathbb{Z}_{11}\times\mathbb{Z}_5\times\mathbb{Z}_6}$	—	CFT

Table 3.10: A summary of the 2-index chiral theories, their global symmetries, and their IR realizations. Theories with N even also enjoy a $\mathbb{Z}_2^{(1)}$ 1-form symmetry acting on the Wilson lines. This symmetry is assumed to be unbroken in theories that confine.

In this chapter, we exhaustively scrutinized the 2-index chiral gauge theories. By studying the 2-loop and 3-loop β-functions, we could pinpoint a few theories that may flow to an IR CFT. Theories that do not admit a fixed point break its global symmetries. We considered scenarios that give the minimal number of IR Goldstones, as this lowers the free energy of the theory. We paid particular attention to the anomaly-matching conditions and ensured that the condensates match any discrete subgroup of $U(1)_A$. Our theories, their global symmetries, the proposed IR phase condensates, and the unbroken IR symmetries are shown in Table 3.10. The first 4 theories are fermionic, while the rest are bosonic.

This investigation included a closer examination of the CFU anomalies one of the authors studied in the previous work [35], giving a better interpretation of this class of anomalies in the light of the discrete-anomaly matching conditions. The general finding is that matching the full set of anomalies and, in particular, the anomalies of the discrete subgroups of the axial $U(1)_A$ symmetry necessitates the formation of multiple higher-order condensates. One expects such higher-order condensates to form in strongly-coupled theories. Here, their formation is explained via the constraints of the anomaly-matching conditions. We also employed a systematic approach to search for massless composite fermions that could match the anomalies in the case of fermionic theories. We were not able to find such composites. In one case, we used the CFU anomaly to show that a set of composites cannot solely match this anomaly, hinting at a deeper reason why the composites could not be found.

This chapter provides a systematic approach that can be applied to study other classes of strongly-coupled phenomena, including different chiral gauge theories.

Chapter **4**

Noninvertible symmetries and anomalies from gauging 1-form electric centers

4.1 Introduction

In the previous chapter, we studied anomalies arising from generalized gauge backgrounds and their restrictions on the IR physics. The presence of fractionalized backgrounds, however, has further implications. In particular they imply the presence of non-invertible symmetries. Over the last couple of years, symmetries have expanded their domain to encompass operators that defy the conventional notion of inversion. These are known as noninvertible symmetries. While noninvertible symmetries initially found their roots and applications in the realm of 2-dimensional QFT, see, e.g., [66, 67], their significance in the context of 4-dimensional QFT sparked a deluge of research endeavors in this area (a non-comprehensive list is [68, 69, 70, 71, 72, 73, 74, 22, 23, 75, 76, 77, 78, 79, 24, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95]. Also, see [96, 11] for reviews.)

It is well known that quantum electrodynamics has a classical $U(1)_\chi$ axial symmetry that breaks down because of the Adler–Bell–Jackiw (ABJ) anomaly. However, it was realized in [22, 23] that the axial symmetry does not completely disappear. Instead, it resurfaces as a noninvertible symmetry for each fractional element of the classical $U(1)_\chi$. This profound reinterpretation of symmetries prompted a compelling quest to unearth analogous structures in QFT. [97] established a technique for unveiling noninvertible 0-form symmetries within $SU(N) \times U(1)$ gauge theories in the presence of matter in representation $\mathcal{R}$. This approach employed the Hamiltonian formalism, where the theory was put on a three-dimensional torus $\mathbb{T}^3$, subjecting it to $\mathbb{Z}_N$ magnetic twists along all three spatial directions. Taking the matter to be a single Dirac fermion, this theory is endowed with invertible $\mathbb{Z}^\chi_{2\text{gcd}(T_\mathcal{R},d_\mathcal{R})}$ 0-form chiral symmetry, where $T_\mathcal{R}$ and $d_\mathcal{R}$ are the Dynkin index and dimension of $\mathcal{R}$, respectively. Yet, it was shown that the theory also possesses a noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ 0-form chiral symmetry. Such symmetry acts on the Hilbert space projectively by selecting special sectors characterized by certain magnetic numbers. New noninvertible symmetries were also revealed in [98] in theories with mixed anomalies between $\mathbb{Z}_2^{(1)}$ 1-form and 0-form discrete chiral symmetries.

The topological essence of symmetries, encompassing the noninvertible variants, underscores their sensitivity to the global structure of the gauge group. Consequently, the inquiry arises: how do we identify these noninvertible symmetries within a general gauge group, characterized as either $SU(N)/\mathbb{Z}_p$ or $SU(N) \times U(1)/\mathbb{Z}_p$ where $\mathbb{Z}_p$ is a subgroup of the center symmetry? In this chapter, we answer this question by devising a general method that applies to any theory with a direct multiplication of abelian and semi-simple nonabelian gauge groups quotiented by a discrete center, whether the theory is vector-like or chiral. This is achieved by putting the theory on $\mathbb{T}^3$ and turning on magnetic fluxes in a refined subgroup of $\mathbb{Z}_N$, depending on the matter content as well as the global structure of the gauge group.

In the context of $SU(N)$ gauge theory, the introduction of matter characterized by an N-ality n has the effect of breaking the $\mathbb{Z}_N$ center of the group down to a subgroup $\mathbb{Z}_q$, where q is the greatest common divisor (gcd) of N and n. Our focus is on understanding the noninvertible 0-form symmetries present in the $SU(N)/\mathbb{Z}_p$ gauge theories, where $\mathbb{Z}_p$ is a subgroup of the remaining center $\mathbb{Z}_q$. These theories exhibit both electric $\mathbb{Z}^{(1)}_{q/p}$ and magnetic $\mathbb{Z}^{(1)}_p$ 1-form global symmetries*. To identify the noninvertible symmetries, we initiate the process starting from $SU(N)$ theory endowed with a single Dirac fermion in representation $\mathcal{R}$, which possesses an invertible $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry. We then subject this theory to electric and magnetic twists characterized by elements of $\mathbb{Z}_p$. If the theory exhibits a mixed anomaly between its chiral and electric $\mathbb{Z}^{(1)}_p$ 1-form symmetries, the act of gauging $\mathbb{Z}_p$ effectively reveals the chiral symmetry as noninvertible. The construction of a gauge-invariant operator corresponding to the noninvertible symmetry $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ involves several steps. First, we create a topological operator by integrating the anomalous current conservation law over $\mathbb{T}^3$. The resulting operator is not invariant under $\mathbb{Z}_p$ gauge transformations. Yet, we can restore gauge invariance by summing over all possible $\mathbb{Z}_p$ gauge-transformed operators. This process results in a noninvertible chiral symmetry operator that projects onto specific sectors in the Hilbert space, each characterized by certain 't Hooft lines charged under the magnetic $\mathbb{Z}^{(1)}_p$ 1-form symmetry. $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ can exhibit further anomalies when subjected to twists by the electric $\mathbb{Z}^{(1)}_{q/p}$ 1-form symmetry, implying that states within the Hilbert space of the $SU(N)/\mathbb{Z}_p$ gauge theory will display multiple degeneracies.

We employ a similar approach to identify noninvertible symmetries in $SU(N) \times$

*There are p distinct theories $(SU(N)/\mathbb{Z}_p)_n$, where $n = 0, 1, .., p-1$ are the discrete θ-like parameters [99]. These theories differ by the set of compatible line operators (Wilson, 't Hooft, and dyonic operator). Here, we restrict our analysis to $n = 0$.

$U(1)/\mathbb{Z}_p$ gauge theories, where $\mathbb{Z}_p$ is a subgroup of the electric $\mathbb{Z}_N^{(1)}$ 1-form center symmetry. Unlike in $SU(N)$ theories, the introduction of matter does not reduce the $\mathbb{Z}_N$ center. This is due to the presence of an abelian $U(1)$ sector, which ensures that all matter representations adhere to the cocycle condition. In addition to the 1-form electric center symmetry, this theory is also endowed with a magnetic $U_m^{(1)}(1)$ 1-form symmetry. $SU(N)$ gauge theory with matter exhibits an anomaly between its chiral and $U(1)$ baryon-number symmetries. Gauging the latter transforms the theory into an $SU(N) \times U(1)$ gauge theory and reveals the chiral symmetry $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ as noninvertible. Placing the theory on $\mathbb{T}^3$ enables us to construct the corresponding noninvertible chiral operator by summing over large $U(1)$ gauge transformations with distinct winding numbers. Furthermore, since the theory exhibits a 1-form electric center symmetry, we can decorate the noninvertible operator with $\mathbb{Z}_N$ magnetic twists. If we choose to further gauge a $\mathbb{Z}_p^{(1)}$ subgroup of the electric $\mathbb{Z}_N^{(1)}$ symmetry, thereby resulting in the $SU(N) \times U(1)/\mathbb{Z}_p$ theory, we must ensure that the noninvertible operator remains invariant under $\mathbb{Z}_p$ gauge transformation. This is accomplished by summing over all $\mathbb{Z}_p$ gauge-transformed chiral operators. Once again, we discover that the resultant operator projects onto specific sectors within the Hilbert space, distinguished by the presence of 't Hooft lines charged under $U_m^{(1)}(1)$. The noninvertible symmetry also exhibits a mixed anomaly with the remaining electric $\mathbb{Z}_{N/p}^{(1)}$ global symmetry. The anomaly implies that certain sectors of the theory, designated by certain $\mathbb{Z}_{N/p}^{(1)}$ electric fluxes, exhibit multi-fold degeneracy.

Placing the theory on $\mathbb{T}^3$ offers a distinct advantage: it presents a systematic approach for computing the 't Hooft anomalies inherent to a given theory. Simultaneously, it provides a means to construct the Hilbert space explicitly. In this chapter, we put a significant emphasis on this Hilbert space construction, shedding light on the intricate relationship between Wilson lines, 't Hooft lines, and the noninvertible operator. Specifically, through several illustrative examples, we showcase how the noninvertible chiral operator, within the framework of the Hilbert space and Hamiltonian formalism, acts to annihilate the minimal 't Hooft lines.

We also introduce couplings of gauge theories to axions. The underlying renormalization group invariance of the noninvertible symmetries, along with their associated anomalies, guarantees that the infrared (IR) axion physics faithfully inherits all the characteristics of the theory at the ultraviolet (UV) level. We substantiate this by explicitly constructing noninvertible chiral operators, commencing from the IR anomalous axion current conservation law. In our exploration, we offer concrete illustrations of various UV theories and their corresponding IR axion physics manifestations.

This chapter is organized as follows. In Section 4.2, we provide a concise overview of the essential elements required for the development of noninvertible symmetries. This section encompasses the introduction of our notation, a review of the path-integral formalism on the 4-torus ($\mathbb{T}^4$), 't Hooft twists, and the Hamiltonian formalism on $\mathbb{T}^3$. Moving on to Section 4.3, we proceed to construct noninvertible symmetries within the context of $SU(N)/\mathbb{Z}_p$ theories while also identifying their associated anomalies. This section concludes with the presentation of specific examples of noninvertible symmetries in both vector and chiral gauge theories. In Section 4.4, we replicate the same analysis, this time focusing on $SU(N)\times U(1)/\mathbb{Z}_p$. Two examples are discussed, including the Standard Model (SM), and we demonstrate that the SM lacks noninvertible symmetries within its non-gravitational sector. Finally, this chapter culminates in Section 4.5, where we explore the coupling of gauge theories to axions. We show that noninvertible symmetry operators can also be constructed using the axion anomalous current.

4.2 Preliminaries

In this section, we review the path integral and the Hamiltonian formalisms of gauge theories put on a compact manifold with possible 't Hooft twists, both in space and time directions. Additionally, we examine the global symmetries and anomalies in both formalisms, providing an exploration of these key aspects. We base our formalism and notation on [18, 100, 19, 97], and set the stage for constructing the noninvertible operators we carry out in the subsequent sections. While some results in this section are new, many are a mere review of previous results. Moreover, some details are avoided, referring the reader to the literature for an in-depth discussion. Yet, the information encapsulated here is necessary to make this chapter self-contained.

4.2.1 Twisting in the Path integral

Pure $SU(N)$ theory

We will consider $SU(N)$ pure Yang-Mills (YM) theory on $\mathbb{T}^4$, where $\mathbb{T}^4$ is a 4-torus with periods of length L_μ, $\mu = 1, 2, 3, 4$, with 't Hooft twists labelled by $n_{\mu\nu}$ (see section 2.1.3). It will be useful to break $n_{\mu\nu}$ into spatial (magnetic) m_i and temporal (electric) k_i twists:

$$k_i \equiv n_{i4}\,, \quad n_{ij} \equiv \epsilon_{ijk} m_k\,, \tag{4.1}$$

and $i, j = 1, 2, 3$ or x, y, z. We also use bold-face letters, e.g., $\boldsymbol{k} \equiv (k_1, k_2, k_3)$, to denote 3-dimensional vectors. When applied to the gauge fields on $\mathbb{T}^4$, the twists induce a background with fractional topological charge [18]:

$$Q = \frac{1}{8\pi^2}\int_{\mathbb{T}^4} \mathrm{tr}[f^c \wedge f^c] = -\frac{1}{8N}\epsilon_{\mu\nu\alpha\beta} n_{\mu\nu} n_{\alpha\beta} + \mathbb{Z} = \frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N} + \mathbb{Z}\,, \tag{4.2}$$

where f^c is the field strength of a^c. Notice that the twists $(\boldsymbol{m}, \boldsymbol{k}) \in (\mathbb{Z}\,\mathrm{Mod}\,N)^6$. Adding multiples of N to $\boldsymbol{m}$ or $\boldsymbol{k}$ leaves the cocycle condition intact. However, this has the effect of changing the topological charges by integers. Hence, from here on, we shall take the twists $m_i, k_i \in \mathbb{Z}$, not $\operatorname{Mod} N$. The partition function of the $SU(N)$ gauge theory with given twists $(\boldsymbol{m}, \boldsymbol{k})$ is

$$\mathcal{Z}[\boldsymbol{m}, \boldsymbol{k}]_{SU(N)} = \sum_{\nu\in\mathbb{Z}} \int \left[Da_\mu^c\right]_{(\boldsymbol{m},\boldsymbol{k})} e^{-S_{YM} - i\left(\frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N}+\nu\right)\theta} . \tag{4.3}$$

Here, S_{YM} is the YM action, and the subscript $(\boldsymbol{m}, \boldsymbol{k})$ indicates that the path integral is to be performed with a given set of twisted boundary conditions. Summation over the integer-valued topological sectors, $\nu \in \mathbb{Z}$, is necessary so that the theory satisfies locality (cluster decomposition). *

$SU(N)$ theory with matter

Next, we add matter fields in a representation $\mathcal{R}$ under $SU(N)$. The matter representation has N-ality n. Then, the full $\mathbb{Z}_N$ center breaks down to $\mathbb{Z}_q$, $q = \gcd(N, n)$, i.e., the Wilson lines are charged under $\mathbb{Z}_q^{(1)}$ 1-from center symmetry†. Putting the matter, which, from now on, will be assumed to be fermions, on $\mathbb{T}^4$ modifies the cocycle conditions. Let ψ be a left-handed Weyl fermion transforming under $\mathcal{R}$ of $SU(N)$. Then, the fermion obeys the boundary conditions

$$\psi(x + \hat{e}_\mu L_\mu) = \mathcal{R}(\Omega_\mu(x))\psi(x) . \tag{4.4}$$

The matrix $\mathcal{R}(\Omega_\mu(x))$ is built from Ω_μ, transforming in the defining representation of $SU(N)$, with suitable symmetrization or anti-symmetrization over n indices (the N-ality of the representation) according to the specific representation $\mathcal{R}$. Thus, schematically (ignoring symmetrization over indices)

$$\mathcal{R}(\Omega_\mu) \sim \underbrace{\Omega_\mu ... \Omega_\mu}_{n} . \tag{4.5}$$

$\mathcal{R}(\Omega_\mu)$ must satisfy the cocycle condition

$$\mathcal{R}(\Omega_\mu(x + \hat{e}_\nu L_\nu))\ \mathcal{R}(\Omega_\nu(x)) = \mathcal{R}(\Omega_\nu(x + \hat{e}_\mu L_\mu))\ \mathcal{R}(\Omega_\mu(x)) , \tag{4.6}$$

which, via Eq. (2.36), reveals that the allowed values of the twists $\boldsymbol{m}$ and $\boldsymbol{k}$ are $\frac{N}{q}, \frac{2N}{q}, ...$. Twisting by the center subgroup $\mathbb{Z}_q$ induces a background field with fractional topological charge

$$Q = \frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N} + \mathbb{Z} , \quad \boldsymbol{m}, \boldsymbol{k} \in \frac{N}{q}\mathbb{Z} , \tag{4.7}$$

*We can also preserve locality if we sum over topological sectors that are fixed multiples of some integer k, i.e. $\nu \in k\mathbb{Z}$, see [101, 102] for details. In the rest of this chapter we will take $\nu \in \mathbb{Z}$ for simplicity.

†For example, $SU(2M)$ gauge theory with matter in the 2-index (anti)symmetric representation has a $\mathbb{Z}_2^{(1)}$ center symmetry that acts on Wilson lines.

and the partition function in the presence of matter reads

$$\begin{aligned}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}} &= \sum_{\nu\in\mathbb{Z}}\int\{\left[Da^c_\mu\right][D\,\text{matter}]\}_{(\boldsymbol{m},\boldsymbol{k})}e^{-S_{YM}-S_{\text{matter}}-i\left(\frac{\boldsymbol{k}\cdot\boldsymbol{m}}{N}+\nu\right)\theta}\,,\\ & m_i,k_i\in\frac{N}{q}\mathbb{Z}\,,\quad i=1,2,3\,.\end{aligned} \tag{4.8}$$

In the presence of matter, the theory is endowed with classical nonabelian and abelian flavor symmetries. The $U(1)$ baryon-number symmetry survives the quantum corrections. In contrast, the chiral part of the abelian symmetry, denoted by $U(1)_\chi$, will generally break down to a discrete symmetry because of the Adler–Bell–Jackiw (ABJ) anomaly of $U(1)_\chi$ in the background of color instantons (which have integer topological charges). To fix ideas, we consider a single flavor of a Dirac fermion with classical $U(1)$ baryon number and $U(1)_\chi$ chiral symmetries. We take the $U(1)$ baryon charge of the Dirac fermion to be $+1$. The ABJ anomaly breaks $U(1)_\chi$ down to invertible $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ chiral symmetry, where $T_\mathcal{R}$ is the Dynkin index of the representation. Generalizing the theory to include many flavors is straightforward, and we shall work out examples of this sort later in the chapter. In the presence of the twists $(\boldsymbol{m},\boldsymbol{k})$, there can be an anomaly of $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ in the background of $\mathbb{Z}^{(1)}_q$. The anomaly is a non-trivial phase acquired by $\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}}$ as we apply a transformation by an element of $\mathbb{Z}_{2T_\mathcal{R}}$:

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}}|_{k_i,m_i\in N\mathbb{Z}/q}\longrightarrow e^{i2\pi\ell\frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N}}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}}\,, \tag{4.9}$$

and $\ell=0,1,2,..,T_\mathcal{R}-1$ are the elements of $\mathbb{Z}^\chi_{2T_\mathcal{R}}$. For the smallest twists $m_j=k_j=\frac{N}{q}$ in the j-th direction, we obtain [103]

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}}|_{m_3=k_3=\frac{N}{q}}\longrightarrow e^{i2\pi\ell\frac{N}{q^2}}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}}\,. \tag{4.10}$$

Bearing in mind that $N/q\in\mathbb{Z}$, we can generally absorb the integral part of N/q^2 by adding an integer topological charge, which cannot change the anomaly. Nevertheless, we will retain the phase as indicated in Eq. (4.10). The phase is nontrivial, and hence there is an anomaly, if and only if $\ell\frac{N}{q^2}\notin\mathbb{Z}$. In the next section, we show how to obtain the same anomaly using the Hamiltonian formalism.

We can do more regarding turning on fractional fluxes in $SU(N)$ with matter. Instead of limiting ourselves to $\mathbb{Z}_q$ twists, we can twist with the full $\mathbb{Z}_N$ center symmetry or any subgroup of it provided we also turn on backgrounds of $U(1)$ baryon number symmetry [20, 103]. Let ω_μ denote the $U(1)$ transition functions such that for the $U(1)$ gauge field a_μ, we have $a_\nu(x+\hat{e}_\mu)=\omega_\mu\circ a_\nu(x)\equiv a_\nu(x)-i\omega_\mu^{-1}\partial_\nu\omega_\mu$. Then, Ω_μ and ω_μ obey the cocycle conditions:

$$\begin{aligned}\Omega_\mu(x+\hat{e}_\nu L_\nu)\,\Omega_\nu(x) &= e^{i\frac{2\pi n_{\mu\nu}}{N}}\Omega_\nu(x+\hat{e}_\mu L_\mu)\,\Omega_\mu(x)\,,\\ \omega_\mu(x+\hat{e}_\nu L_\nu)\,\omega_\nu(x) &= e^{-i\frac{2\pi n n_{\mu\nu}}{N}}\omega_\nu(x+\hat{e}_\mu L_\mu)\,\omega_\mu(x)\,,\end{aligned} \tag{4.11}$$

where the N-ality of the matter representation is incorporated in the abelian transition functions. The topological charges of both the nonabelian center and abelian backgrounds read*

$$Q_{SU(N)} = \frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N} + \mathbb{Z}\,, \quad Q_u = \left(\frac{n}{N}\boldsymbol{m} + \boldsymbol{A}\right)\cdot\left(\frac{n}{N}\boldsymbol{k} + \boldsymbol{B}\right)\,, \quad \boldsymbol{m},\boldsymbol{k},\boldsymbol{A},\boldsymbol{B}\in\mathbb{Z}^3\,. \tag{4.12}$$

Here, $\boldsymbol{A}, \boldsymbol{B}$ are arbitrary integral magnetic and electric quantum numbers that we can always turn on since they leave the cocycle condition intact.

$SU(N)\times U(1)$ theory with matter

We may also choose to make $U(1)$ dynamical, which entails summing over small and large gauge transformations of $U(1)$, with the latter implementing integer winding. This results in $SU(N)\times U(1)$ gauge theory with a Dirac fermion in representation $\mathcal{R}$, with N-ality N and baryon-charge $+1$. In this case, the $U(1)$ instantons reduce $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ down to the genuine (invertible) symmetry $\mathbb{Z}^{\chi}_{2\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$, and $d_{\mathcal{R}}$ is the dimension of $\mathcal{R}$. The easiest way to see that is by recalling the partition function under a $U(1)_\chi$ transformation acquires a phase:

$$\exp\left[i2\alpha T_{\mathcal{R}}\int_{\mathbb{T}^4}\frac{\text{tr}\,(f^c\wedge f^c)}{8\pi^2} + i2\alpha d_{\mathcal{R}}\int_{\mathbb{T}^4}\frac{f\wedge f}{8\pi^2}\right]\,, \tag{4.13}$$

where f is the field strength of the $U(1)$ field. Recalling that for the dynamical $SU(N)$ and $U(1)$ fields we have $\int_{\mathbb{T}^4}\frac{\text{tr}(f^c\wedge f^c)}{8\pi^2}\in\mathbb{Z}$, $\int_{\mathbb{T}^4}\frac{f\wedge f}{8\pi^2}\in\mathbb{Z}$, we conclude that only $\mathbb{Z}^{\chi}_{2\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$ survises the chiral transformation. The theory admits Wilson lines:

$$W_{\mu\,,SU(N)} = \text{tr}_{\Box}\left[P e^{i\int_{x_\mu=0}^{x_\mu=L_\mu} a^c_\mu}\Omega_\mu\right]\,, \quad W_{\mu\,,U(1)} = e^{-i\int_{x_\mu=0}^{x_\mu=L_\mu} a_\mu}\omega_\mu\,, \tag{4.14}$$

which are charged under an electric $\mathbb{Z}_N^{(1)}$ 1-form center symmetry. In addition, the theory is endowed with a magnetic $U_m^{(1)}(1)$ 1-form symmetry because of the absence of magnetic monopoles. For the sake of completeness, we also give the partition function of $SU(N)\times U(1)$ theory with matter in the background of given $(\boldsymbol{m},\boldsymbol{k})$ fluxes:

$$\begin{aligned}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)\times U(1)+\text{matter}} \\ = \sum_{\nu,\nu_{U(1)}\in\mathbb{Z}}\int\{\left[Da^c_\mu\right][Da_\mu]\,[D\,\text{matter}]\}_{(\boldsymbol{m},\boldsymbol{k})}e^{-S_{YM}-S_{U(1)}-S_{\text{matter}}} \\ m_i,k_i\in\mathbb{Z}\,,\quad i=1,2,3\end{aligned} \tag{4.15}$$

*Here, we use the construction described in 2.1.3 along with abelian field strengths $f_{12} = \frac{2\pi}{L_1L_2}(\frac{n}{N}m_3 + A_3)$ and $f_{34} = \frac{2\pi}{L_3L_4}(\frac{n}{N}k_3 + B_3)$ in the 1-2 and 3-4 planes, and similar expressions in the rest of the planes. Substituting into $Q_u = \int_{\mathbb{T}^4}\frac{f\wedge f}{8\pi^2}$, we obtain the fractional $U(1)$ topological charge.

and in addition to the $SU(N)$ integer topological charges ν, we included a sum over integer topological charges $\nu_{U(1)}$ of the $U(1)$ sector.

4.2.2 Twisting in the Hamiltonian formalism

Pure $SU(N)$ theory

Let us repeat the above discussion using the Hamiltonian formalism, starting with pure $SU(N)$ YM theory (we use a hat to distinguish an operator in this section.) To this end, we put the gauge theory on a spatial 3-torus $\mathbb{T}^3$ and apply the magnetic $\boldsymbol{m}$ twists along the 3-spatial directions. The transition functions in the defining representation along the spatial directions, denoted by Γ_i, can be chosen to be constant $N \times N$ matrices obeying the cocycle condition

$$\Gamma_i \, \Gamma_j = e^{i\frac{2\pi\epsilon_{ijk}m_k}{N}} \Gamma_j \, \Gamma_i \,. \tag{4.16}$$

Then, one can construct the states of the physical Hilbert space using the temporal gauge condition $a_0^c = 0$. The states can be written using the “position" eigenstates of the gauge fields a_j^c, $j = 1, 2, 3$ (or $i = x, y, z$) as follows:

$$|\psi\rangle_{\boldsymbol{m}} \equiv |a_1^c, a_2^c, a_3^c\rangle_{\boldsymbol{m}} \,, \quad \hat{a}_j^c|a_1^c, a_2^c, a_3^c\rangle_{\boldsymbol{m}} = a_j^c|a_1^c, a_2^c, a_3^c\rangle_{\boldsymbol{m}} \,, \tag{4.17}$$

and the subscript $\boldsymbol{m}$ emphasizes that the Hilbert space is constructed in the background of the magnetic twists. In writing Eq. (4.17), we have put many details under the rug, and the reader is referred to [18, 100, 19] for details. For example, notice that the gauge fields A_i need to respect the twisted boundary conditions (4.16), i.e., they transform according to (2.35) as we traverse any spatial direction on $\mathbb{T}^3$. The theory admits 3 fundamental Wilson lines wrapping the three cycles of $\mathbb{T}^3$; these are given by (2.37) by restricting μ to the spatial directions. The Wilson lines are charged under the $\mathbb{Z}_N^{(1)}$ 1-form symmetry generated by three symmetry generators $\hat{T}_j$, the Gukov-Witten operators, supported on co-dimension 2 surfaces. Thus, we have

$$\hat{T}_j \hat{W}_j = e^{i\frac{2\pi}{N}} \hat{W}_j \hat{T}_j \,, \tag{4.18}$$

and there are $\hat{W}_j^{e_j}$ distinct Wilson lines with N distinct N-alities $e_j = 0, 1, .., N-1$. The center-symmetry generators $\hat{T}_i$ are hard to construct explicitly. However, their explicit form is not important to us. What is important is that they commute with the YM Hamiltonian $\hat{H}$, and thus, $\hat{H}$ and $\hat{T}_i$ can be simultaneously diagonalized. The physical states of the theory $|\psi\rangle_{\text{phy},\boldsymbol{m}}$ are designated by the eigenvalues of $\hat{T}_i$. It can be shown that the action of $\hat{T}_i$ on $|\psi\rangle_{\text{phy},\boldsymbol{m}}$ is given by

$$\hat{T}_j |\psi\rangle_{\text{phy},\boldsymbol{m}} = e^{i\frac{2\pi}{N}e_j - i\theta\frac{m_j}{N}} |\psi\rangle_{\text{phy},\boldsymbol{m}} \,, \tag{4.19}$$

where $e_j, m_j \in \mathbb{Z}_N$ and the θ term ensures that $\hat{T}_i^N |\psi\rangle_{\text{phy},\boldsymbol{m}} = e^{-i\theta m_j} |\psi\rangle_{\text{phy},\boldsymbol{m}}$, and hence, $\hat{T}_i^N$ works as a large gauge transformation. The combination $e_j - \frac{\theta}{2\pi} m_j$ is the $\mathbb{Z}_N$ electric flux in the j-th direction. This is justified as follows. Consider the state $\hat{W}_j|\psi\rangle_{\text{phy},\boldsymbol{m}}$, obtained from $|\psi\rangle_{\text{phy},\boldsymbol{m}}$ by the action of $\hat{W}_j$. Using Eqs. (4.18, 4.19), we find $\hat{T}_j\hat{W}_j|\psi\rangle_{\text{phy},\boldsymbol{m}} = e^{i\frac{2\pi}{N}(e_j+1)-i\theta\frac{m_j}{N}}\hat{W}_j|\psi\rangle_{\text{phy},\boldsymbol{m}}$. Therefore, acting with $\hat{W}_j$ on the state $|\psi\rangle_{\text{phy},\boldsymbol{m}}$ increases e_j by one unit in the j-th direction. Since $\hat{W}_j$ inserts an electric flux tube winding in the j-th direction, the interpretation of e_j as electric flux follows. Notice also that because $\hat{T}_j$ and $\hat{H}$ can be simultaneously diagonalized, we may label the states by the energy and the electric flux:

$$|\psi\rangle_{\text{phy},\boldsymbol{m}} \equiv |E,\boldsymbol{e}\rangle_{\boldsymbol{m}}\,, \quad \boldsymbol{e}\in\mathbb{Z}_N^3\,. \tag{4.20}$$

It is worth spending some time to explain our notation in Eq. (4.20), as we shall use this notation extensively in this chapter. The physical state is labeled by the eigenvalues of a set of commuting operators, here the energy and the electric flux. The $SU(N)$ theory does not admit a 1-form magnetic symmetry, and thus, we cannot label the states by magnetic fluxes. Yet, we can turn on a background magnetic flux $\boldsymbol{m}$, indicated as a subscript; all physical quantities are calculated in this magnetic background. Also, we use the letter $\boldsymbol{m}$ to denote the set of magnetic fluxes we can consistently turn on. Here, we have $\boldsymbol{m}\in\mathbb{Z}^3$.

How can we make sense of the fractional topological charge (4.7) on $\mathbb{T}^3$? We consider the product of $\mathbb{T}^3$ and the time interval $[0,L_4]$ and consider the boundary conditions $\hat{a}_i^c(t=L_4) = C[\boldsymbol{k}]\circ\hat{a}_i^c(t=0)$, where $C[\boldsymbol{k}]$ is an "improper gauge" transformation implementing a twist $\boldsymbol{k}\in\mathbb{Z}^3$ on the gauge fields by an element of the center*. In the presence of the magnetic twists $\boldsymbol{m}$, it can be shown that an application of $C[\boldsymbol{k}]$ results in the topological charge (Pontryagin square) [18, 100, 19]:

$$Q[C[\boldsymbol{k}]] = \int_{\mathbb{T}^3} K(C\circ\hat{a}^c) - K(\hat{a}^c) = \frac{1}{24\pi^2}\int_{\mathbb{T}^3}\text{tr}\left[CdC^{-1}\right]^3 = \frac{\boldsymbol{m}\cdot\boldsymbol{k}}{N} + \mathbb{Z}\,, \tag{4.21}$$

where $K(\hat{a}^c)$ is the topological current density operator $K(\hat{a}^c) = \frac{1}{8\pi^2}\text{tr}\left[\hat{a}^c\wedge\hat{f}^c - \frac{i}{3}\hat{a}^c\wedge\hat{a}^c\wedge\hat{a}^c\right]$, or in terms of the components: $\hat{K}^\mu(a^c) = \frac{1}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}\left(\hat{a}_\nu^{c,m}\partial_\lambda\hat{a}_\sigma^{c,m} - \frac{1}{3}f^{mpq}\hat{a}_\nu^{c,m}\hat{a}_\lambda^{c,p}\hat{a}_\sigma^{c,q}\right)$.

$SU(N)$ theory with matter

Adding fermions of N-ality n changes the center from $\mathbb{Z}_N$ to $\mathbb{Z}_q$, $q=\gcd(N,n)$, and the twists $(\boldsymbol{m},\boldsymbol{k})$ are now in $(N\mathbb{Z}/q)^6$. Otherwise, all the steps used to put the theory on $\mathbb{T}^3$ and construct the Hilbert space carry over. In particular, $\hat{T}_i$ now are the generators of the $\mathbb{Z}_q^{(1)}$ 1-form symmetry, and their action on the physical states in the Hilbert space is given by† (now we turn off the θ angle as we can rotate it

*In fact, C should be designated by both $\boldsymbol{k}$ and the integral instanton number ν; see [18]. However, ν does not play a role in this section.

†It is conceivable to introduce an additional label to signify the distinct symmetries generated by different operators $\hat{T}_j$. For instance, we could designate $\hat{T}_{N,j}$ as the generator of $\mathbb{Z}_N^{(1)}$ and

away via a chiral transformation acting on the fermion)

$$\hat{T}_j|\psi\rangle_{\text{phy},\boldsymbol{m}} = e^{i\frac{2\pi}{q}e_j}|\psi\rangle_{\text{phy},\boldsymbol{m}}\,, \tag{4.22}$$

and the theory has $e_j = 0, 1, 2, .., q-1$ electric flux sectors in each direction $j = 1, 2, 3$. The operators $\hat{T}_j$ act on the spatial Wilson lines in the defining representation of $SU(N)$ as $\hat{T}_j\hat{W}_j = e^{i\frac{2\pi}{q}}\hat{W}_j\hat{T}_j$, and there are q distinct Wilson lines $W_j^{e_j}$. The physical states $|\psi\rangle_{\text{phy},\boldsymbol{m}}$ are simultaneous eigenstates of the Hamiltonian and $\hat{T}_j$ since both operators commute. Thus, we can write the physical states in the magnetic flux background $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$ as

$$|\psi\rangle_{\text{phy},\boldsymbol{m}} = |E, \boldsymbol{e}N/q\rangle_{\boldsymbol{m}}\,, \quad \boldsymbol{e} = (e_1, e_2, e_3) \in \mathbb{Z}_q^3\,, \tag{4.23}$$

and Ne_j/q is the amount of electric flux carried by the state in direction j. We may also say that e_j is the number of electric fluxes in units of N/q. For matter with N-ality $n = 0$, e.g., in the adjoint representation, $q = N$ and we recover what we have said about pure $SU(N)$ gauge theory.

The partition function (4.8) can be written in the Hamiltonian formalism as a trace over states in Hilbert space:

$$\begin{aligned}\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]_{SU(N)+\text{matter}} &= \text{tr}_{\boldsymbol{m}}\left[e^{-L_4\hat{H}}(\hat{T}_x)^{k_x}(\hat{T}_y)^{k_y}(\hat{T}_z)^{k_z}\right]\\ &= \sum_{\boldsymbol{e}\in\{0,1,..,q-1\}^3} e^{i\frac{2\pi \boldsymbol{e}\cdot\boldsymbol{k}}{q}}\,{}_{\boldsymbol{m}}\langle E, \boldsymbol{e}N/q|e^{-L_4\hat{H}}|E, \boldsymbol{e}N/q\rangle_{\boldsymbol{m}}\,,\end{aligned} \tag{4.24}$$

where the subscript $\boldsymbol{m}$ in the trace means that we are considering the states in the background of the magnetic flux $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$. We also used Eqs. (4.22, 4.23), the fact that the states are eigenstates of both the energy and the 1-form center operators.

To detect the anomaly between $\mathbb{Z}^{\chi}_{2T_\mathcal{R}}$ and $\mathbb{Z}_q^{(1)}$ in the Hamiltonian formalism, we first define the operator that implements the discrete chiral symmetry. To this end, we recall that under a chiral $U(1)_A$ rotation, the presence of the ABJ anomaly indicates non-conservation of the corresponding symmetry rotation:

$$\partial_\mu \hat{j}^\mu_A = 2T_\mathcal{R}\partial_\mu\hat{K}^\mu(a^c)\,. \tag{4.25}$$

Yet, we can define a conserved current:

$$\hat{j}^\mu_5 \equiv \hat{j}^\mu_A - 2T_\mathcal{R}\hat{K}^\mu\,, \tag{4.26}$$

and correspondingly a conserved charge:

$$\hat{Q}_5 = \int_{\mathbb{T}^3}\hat{j}^0_5\,. \tag{4.27}$$

$\hat{T}_{q,j}$ as the generator of $\mathbb{Z}_q^{(1)}$. Nonetheless, this approach may lead to increased complexity in our expressions, and we opt not to pursue it. Instead, we will explicitly specify the symmetry in question when discussing these distinct operators.

Therefore, it is natural to define the operator

$$\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \equiv \exp\left[i\frac{2\pi\ell}{2T_{\mathcal{R}}}\hat{Q}_5\right] = \exp\left[i\frac{2\pi\ell}{2T_{\mathcal{R}}}\int_{\mathbb{T}^3}(\hat{j}_A^0 - 2T_{\mathcal{R}}\hat{K}^0(\hat{a}^c))\right], \tag{4.28}$$

for $\ell = 0, 1, .., T_{\mathcal{R}} - 1$, which implements the action of the $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry. $\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}}}$ is invariant under both small and large $SU(N)$ gauge transformations (with integer winding). To find the mixed anomaly between $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ and $\mathbb{Z}^{(1)}_q$, we compute the commutation between $\hat{T}_j$, which implements the action of the electric center symmetry in the j-th direction, and $\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}}}$:

$$\hat{T}_j\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\hat{T}_j^{-1}, \tag{4.29}$$

remembering that the theory is in the background of a magnetic twist $m_j \in \frac{N\mathbb{Z}}{q}$ in the j-th direction*. First, $\hat{T}_j$ commutes with the current j^0_A since the latter is a color singlet operator. However, $\hat{K}^0$ fails to commute with $\hat{T}_j$; the commutation between the two operators is found by recalling that the action of $\hat{T}_j$ is implemented on the gauge fields $\hat{a}^c_j$ as $\hat{a}^c_j = C[k_j] \circ \hat{a}^c_j$. Thus, we find, after making use of (4.21),

$$\begin{aligned} &\hat{T}_j \exp\left[i2\pi\ell\int_{\mathbb{T}^3}\hat{K}^0(\hat{a}^c)\right]\hat{T}_j^{-1} \\ =\;& \exp\left[i2\pi\ell\int_{\mathbb{T}^3}\hat{K}^0\left(C[k_j]\circ\hat{a}^c\right) - \hat{K}^0(\hat{a}^c)\right]\exp\left[i2\pi\ell\int_{\mathbb{T}^3}\hat{K}^0(\hat{a}^c)\right] \\ =\;& \exp\left[i2\pi\ell\frac{m_jk_j}{N}\right]_{m_j,k_j\in\left(\frac{N\mathbb{Z}}{q}\right)^2}\exp\left[i2\pi\ell\int_{\mathbb{T}^3}\hat{K}^0(\hat{a}^c)\right], \end{aligned} \tag{4.30}$$

noting the restriction $m_j, k_j \in \left(\frac{N\mathbb{Z}}{q}\right)^2$ due to the presence of matter; otherwise, we would not satisfy the cocycle condition. Collecting everything and using the minimal twists $m_j = k_j = \frac{N}{q}$ we conclude

$$\hat{T}_j\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\hat{T}_j^{-1} = e^{i2\pi\ell\frac{N}{q^2}}\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}, \tag{4.31}$$

which is exactly the mixed anomaly between the $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral and the $\mathbb{Z}^{(1)}_q$ 1-form center symmetries found in (4.10) from the path integral formalism. The anomaly along with the commutation relations (remember that both $\mathbb{Z}^{(1)}_q$ and $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ are good symmetries of the theory, and hence, the corresponding operators commute with the Hamiltonian)

$$[\hat{H}, \hat{T}_j] = 0, \quad [\hat{H}, \hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}}}] = 0, \tag{4.32}$$

furnishes a finite-dimensional space with a minimum dimension of $q^2/\gcd(q^2, N)$. This means that sectors in Hilbert space exhibit a $q^2/\gcd(q^2, N)$-fold degeneracy.

*Similar to the Footnote †, we could use a label that denotes the specific magnetic flux background when we are dealing with the operator $\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$. This background can be taken in sets such as $\frac{N\mathbb{Z}}{q}$ or $\frac{N\mathbb{Z}}{p}$, among others. However, adopting this approach may introduce unnecessary complexity to our notation. As a result, we have chosen to adopt a more transparent approach: we will explicitly mention the magnetic flux background whenever we discuss this operator.

$SU(N)\times U(1)$ theory with matter

Next, we discuss the Hamiltonian quantization of $SU(N)\times U(1)$ gauge theory with matter fields on $\mathbb{T}^3$ in the background of twists. In this case, we may twist with the full $\mathbb{Z}_N$ center symmetry provided we also turn on a background of $U(1)$. Thus, we replace the cocycle conditions (4.16) with

$$\begin{aligned} \Gamma_i\,\Gamma_j &= e^{i\frac{2\pi\epsilon_{ijk}m_k}{N}}\Gamma_j\,\Gamma_i\,,\\ \omega_i(x+\hat{e}_jL_j)\,\omega_j(x) &= e^{-i\frac{2\pi n\epsilon_{ijk}m_k}{N}}\omega_j(x+\hat{e}_iL_i)\,\omega_i(x)\,, \end{aligned} \tag{4.33}$$

and we included the N-ality of the matter representation n in the cocycle condition of the abelian field. This guarantees that the combined transition functions satisfy the correct cocycle conditions in the presence of matter. Here, we can allow background center fluxes with $(\boldsymbol{m},\boldsymbol{k})\in\mathbb{Z}^6$ for all matter representations, thanks to the $U(1)$ gauge group. We also introduce the operators $\hat{T}_j$ for $SU(N)$ and $\hat{t}_j$ for $U(1)$, $j=1,2,3$. The combinations $\hat{T}_j\hat{t}_j$ are the generators of the electric $\mathbb{Z}_N^{(1)}$ 1-form global symmetry and act on the spatial Wilson lines in (4.14) as: $\hat{T}_jW_{j,SU(N)}=e^{i\frac{2\pi}{N}}W_{j,SU(N)}\hat{T}_j$ and $\hat{t}_jW_{j,U(1)}=e^{-i\frac{2\pi}{N}}W_{j,U(1))}\hat{t}_j$. The action of $\hat{t}_j$ is implemented on the gauge fields, as usual, by improper gauge transformations of $\hat{a}_j$ as $\hat{a}_j=c[k_j]\circ\hat{a}_j$, and amounts to applying nk_j (Mod N) electric twists (notice the appearance of the N-ality). Unlike $\hat{T}_j$, the explicit form of $\hat{t}_j$ is simple:

$$\hat{t}_j\equiv e^{i\lambda_j(x)}\,,\quad \lambda_j(x)=\frac{-2\pi n}{N}\frac{x_jk_j}{L_j}\,. \tag{4.34}$$

Since $\mathbb{Z}_N^{(1)}$ is a good global symmetry, we can choose the states in Hilbert space to be eigenstates of the $\mathbb{Z}_N^{(1)}$ generators $\hat{T}_j\hat{t}_j$:

$$\hat{T}_j\hat{t}_j|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}=e^{i\frac{2\pi e_j}{N}}|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}\,, \tag{4.35}$$

where $e_j=0,1,...,N-1$. Notice that the states are constructed in the “fractional" background magnetic flux $\boldsymbol{m}\in\mathbb{Z}^3$ (remember that in principle $m_i\in\mathbb{Z}$ Mod N, and thus, it implements the fractional magnetic twist. However, we can always add multiples of N to m_i without affecting the cocycle conditions, and hence, we drop the Mod N restriction.) In addition, the theory has a magnetic $U(1)_m^{(1)}$ 1-form global symmetry, which can be used to characterize the physical states by an “integer" value of the magnetic flux. Therefore, a state in the physical Hilbert space can be labeled as

$$|\psi\rangle_{\text{phy}\,,\boldsymbol{m}}=|E,\boldsymbol{e},\boldsymbol{N}\rangle_{\boldsymbol{m}}\,,\quad \boldsymbol{e}\in\mathbb{Z}_N^3\,, \tag{4.36}$$

and $\boldsymbol{N}=(N_x,N_y,N_z)\in\mathbb{Z}^3$ (not Mod N) label the integral magnetic fluxes of the $U(1)$ gauge group. The partition function (4.15) can be written as a trace over states in Hilbert space in $(\boldsymbol{m}, \boldsymbol{k})$ backgrounds as follows:

$$\begin{aligned}\mathcal{Z}[\boldsymbol{m}, \boldsymbol{k}]_{SU(N)\times U(1)+\text{matter}} &= \text{tr}_{\boldsymbol{m}}\left[e^{-L_4\hat{H}}(\hat{T}_x\hat{t}_x)^{k_x}(\hat{T}_y\hat{t}_y)^{k_y}(\hat{T}_z\hat{t}_z)^{k_z}\right] \\ &= \sum_{\boldsymbol{e}\in\{0,1,..,N-1\}^3, \boldsymbol{N}\in\mathbb{Z}^3} e^{i\frac{2\pi \boldsymbol{e}\cdot\boldsymbol{k}}{N}}{}_{\boldsymbol{m}}\langle E, \boldsymbol{e}, \boldsymbol{N}|e^{-L_4\hat{H}}|E, \boldsymbol{e}, \boldsymbol{N}\rangle_{\boldsymbol{m}}\,.\end{aligned} \tag{4.37}$$

We also build the operator that corresponds to the chiral transformation. This construction was detailed in [97], and we do not repeat it here. Instead, we only give a synopsis of the derivation, which is needed in this study. The anomaly equation of the chiral current is

$$\partial_\mu \hat{j}^\mu_A - 2T_{\mathcal{R}}\partial_\mu \hat{K}^\mu(\hat{a}^c) - \frac{2d_{\mathcal{R}}}{8\pi^2}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu\hat{a}^\nu\partial^\lambda\hat{a}^\sigma = 0\,. \tag{4.38}$$

Then, the chiral symmetry operator in the background of the m_j magnetic flux is given by

$$\hat{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} = \exp\left[i\frac{2\pi\ell}{2T_{\mathcal{R}}}\hat{Q}_5\right]\,, \tag{4.39}$$

where the conserved charge $\hat{Q}_5$ is given by

$$\begin{aligned}\hat{Q}_5 &= \int_{\mathbb{T}^3} d^3x\left[\hat{j}^0_\chi - 2T_{\mathcal{R}}K^0(\hat{a}^c) - 2\frac{d_{\mathcal{R}}}{8\pi^2}\epsilon^{ijk}\hat{a}_i\partial_j\hat{a}_k\right] \\ &+ \frac{d_R}{4\pi}(N_z + \frac{n}{N}n_z)\left[\int_0^{L_y}\frac{dy}{L_y}\int_0^{L_z}dz\hat{a}_z(x=0,y,z) + \int_0^{L_x}\frac{dx}{L_x}\int_0^{L_z}dz\hat{a}_z(x,y=0,z)\right] \\ &+ \sum_{\text{cyclic}}(x\to y\to z\to x)\,.\end{aligned} \tag{4.40}$$

The last term comes from carefully treating the boundary term implied from the transition functions $\omega_j(x)$, since, unlike Γ_j, they depend explicitly on x_j, see [97] for details. In addition to the background flux n_j, which introduces the fractional winding number, we also allow integer magnetic winding N_j. Under a transformation with $\hat{t}_j$, the integral of the abelian Chern-Simons term $\hat{K}^0(\hat{a}) = \epsilon^{ijk}\hat{a}_i\partial_j\hat{a}_k$ in the background of the integral M_j and fractional m_j magnetic fluxes transforms as (recall (4.34))

$$\begin{aligned}\hat{t}_j\exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]t_j^{-1} &= \exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(c\circ\hat{a}) - i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]\exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right] \\ &= \left(N_j + \frac{nn_j}{N}\right)\left(\frac{nk_j}{N}\right)\exp\left[i\int_{\mathbb{T}^3}\hat{K}^0(\hat{a})\right]\,.\end{aligned} \tag{4.41}$$

The reader will notice that we switched from the letter $\boldsymbol{m}$, which we use to signify the set of fractional fluxes we can activate, e.g., here we have $\boldsymbol{m}\in\mathbb{Z}^3$, to the letter $\boldsymbol{n}$, which is the actual number of fractional magnetic fluxes we turn on. We shall use the same labeling throughout the chapter.

In the next sections, we use these constructions to argue that $SU(N)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_q$ as well as $SU(N)\times U(1)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_N$ enjoy a noninvertible 0-form chiral symmetry, with a possible mixed anomaly with the 1-form center symmetry.

4.3 $SU(N)/\mathbb{Z}_p$, $\mathbb{Z}_p \subseteq \mathbb{Z}_q$ theories, noninvertible symmetries, and their anomalies

In this section, we direct our attention to YM theories featuring matter fields residing in a particular representation $\mathcal{R}$ and characterized by an N-ality n. Building upon the discussion in the preceding section, it is established that $SU(N)$ gauge theories, when coupled to matter, exhibit an electric $\mathbb{Z}_q^{(1)}$ 1-form center symmetry (recall $q = \gcd(N, n)$). A notable maneuver within this framework involves the gauging of $\mathbb{Z}_q^{(1)}$ or a subgroup of it, leading to $SU(N)/\mathbb{Z}_p$ theory, $\mathbb{Z}_p \subseteq \mathbb{Z}_q$, whose partition function is obtained by summing over integer and fractional topological charge sectors. Thus, gauge transformations with fractional winding numbers are part of the gauge structure, and well-defined operators should be invariant under such gauge transformations. Here, we would like to emphasize that there are p distinct theories: $(SU(N)/\mathbb{Z}_p)_n$, $n = 0, 1, ..., p$, which differ by the admissible genuine (electric, magnetic, or dyonic) line operators. In this chapter, we limit our treatment to $(SU(N)/\mathbb{Z}_p)_{n=0}$, and whenever we mention $SU(N)/\mathbb{Z}_p$, we particularly mean $(SU(N)/\mathbb{Z}_p)_0$. What happens to the invertible $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ discrete chiral symmetry of this theory? As we shall discuss, this symmetry can stay invertible or become noninvertible, depending on whether it exhibits a mixed anomaly with $\mathbb{Z}_p^{(1)}$ symmetry in the original $SU(N)$ theory.

4.3.1 $SU(N)/\mathbb{Z}_q$

We start by discussing noninvertible 0-form chiral symmetries in $SU(N)/\mathbb{Z}_q$ theories, i.e., theories obtained by gauging the full electric $\mathbb{Z}_q^{(1)}$ 1-form center symmetry. Such theories do not possess global electric 1-form symmetry; hence, there are no genuine Wilson lines. This can be understood as follows. We start with pure $SU(N)$ gauge theory, which has an electric $\mathbb{Z}_N^{(1)}$ 1-form symmetry and admits the full spectrum of Wilson lines, i.e., it admits Wilson lines with all N-alities $n = 0, 1, 2, .., N-1$. Gauging a $\mathbb{Z}_q$ subgroup of $\mathbb{Z}_N$, we obtain $SU(N)/\mathbb{Z}_q$ gauge theory. Now, the spectrum of allowed Wilson lines must be invariant under $\mathbb{Z}_q$, forcing us to remove those lines with N-alities that are not multiples of q. The remaining lines in pure $SU(N)/\mathbb{Z}_q$ theory are charged under an electric $\mathbb{Z}_{N/q}^{(1)}$ 1-form symmetry; these are $W_j^{qe_j}$, with $e_j = 0, 1, .., N/q - 1$ and W_j is Wilson line in the defining representation of $SU(N)$. Finally, introducing matter with N-ality q means that those remaining lines can end on the matter and must also be removed from the spectrum. This deprives $SU(N)/\mathbb{Z}_q$ gauge theory with matter from all genuine Wilson lines.

Despite that $SU(N)/\mathbb{Z}_q$ theory with matter does not possess an electric 1-form symmetry, it is endowed with a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form global symmetry. This can be understood, again, starting from the pure $SU(N)/\mathbb{Z}_q$ theory. As we discussed above, the pure theory has an electric $\mathbb{Z}_{N/q}^{(1)}$ 1-form symmetry. The magnetic dual of $SU(N)/\mathbb{Z}_q$ is $SU(N)/\mathbb{Z}_{N/q}$, which admits a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form symmetry. The pure $SU(N)/\mathbb{Z}_q$ theory has q distinct magnetic fluxes ('t Hooft lines) in its spectrum. Let $\mathcal{T}_j$ be the 't Hooft line winding around direction j in the defining representation of $SU(N)$, i.e., it has N-ality 1. Then, the pure $SU(N)/\mathbb{Z}_q$ theory possesses the following set of 't Hooft lines $\mathcal{T}_j^{n_j N/q}$, $n_j = 0, 1, .., q-1$ for $j = 1, 2, 3$, which are mutually local with the set of Wilson lines $W_j^{qe_j}$, $e_j = 0, 1, .., N/q - 1$*. Introducing electric matter removes all Wilson lines (as stated above) but does not alter the magnetic symmetry. Thus, we conclude that $SU(N)/\mathbb{Z}_q$ theory with matter possesses a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form global symmetry acting on a set of 't Hooft lines $\mathcal{T}_j^{n_j N/q}$, $n_j = 0, 1, .., q-1$ for $j = 1, 2, 3$.

We can label the states in the physical Hilbert space of $SU(N)/\mathbb{Z}_q$ theory with matter by both energy and magnetic fluxes since the Hamiltonian commutes with the generators of the magnetic $\mathbb{Z}_q^{m(1)}$ 1-form symmetry†:

$$|\psi\rangle_{\text{phy}} = |E, \boldsymbol{n}N/q\rangle\,, \quad \boldsymbol{n} = (n_x, n_y, n_z) \in (\mathbb{Z}_q)^3\,. \tag{4.42}$$

The partition function of these theories involves summing over sectors with fractional topological charges $N\mathbb{Z}/q^2$ (use Eq.(4.7) and set $k_i = m_i = N/q$), which can be written in the path-integral formalism as (we set the vacuum angle $\theta = 0$) ‡

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_q+\text{matter}} = \sum_{\nu\in\mathbb{Z},(\boldsymbol{m},\boldsymbol{k})\in(N\mathbb{Z}/q)^6} \int \{\left[Da_\mu^c\right] D\left[\text{matter}\right]\}_{(\boldsymbol{m},\boldsymbol{k})} e^{-S_{YM}-S_{\text{matter}}}\,, \tag{4.43}$$

or in the Hamiltonian formalism as

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_q+\text{matter}} = \text{tr}\left[e^{-L_4\hat{H}}\right] = \sum_{\text{physical states}} {}_{\text{phy}}\langle\psi|e^{-L_4\hat{H}}|\psi\rangle_{\text{phy}}\,. \tag{4.44}$$

Our main task is to build a gauge invariant operator that implements the $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ chiral transformation in $SU(N)/\mathbb{Z}_q$ theory with matter. To this end, we use the Hamiltonian formalism of Section 4.2.2, dropping the hats from all operators to reduce clutter. We also use x, y, z to label the three spatial directions. For $\ell \in \mathbb{Z}^\chi_{2T_\mathcal{R}}$, the chiral symmetry operator is given by:

$$U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} = e^{2\pi i \frac{\ell}{2T_\mathcal{R}} \int_{\mathbb{T}^3}\left(j_A^0 - 2T_\mathcal{R} K^0(a^c)\right)}\,. \tag{4.45}$$

*This can be easily seen since $\mathcal{T}_j^{n_j N/q}$ and $W_j^{qe_j}$ satisfy the Dirac quantization condition.

†Recall that the allowed magnetic twists in the $SU(N)$ theory with matter are $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$.

‡We can write this in a Lorentz invariant manner by using $n_{\mu\nu}$ instead of $(\mathbf{m}, \mathbf{k})$, see 4.1.

This operator is invariant under large gauge transformations with integer winding numbers. We will now gauge the $\mathbb{Z}_q^{(1)}$ one-form symmetry. In $SU(N)/\mathbb{Z}_q$ gauge theory with matter, we sum over arbitrary $\mathbb{Z}_q$ twists with fractional topological charges $N\mathbb{Z}/q^2$. We consider the operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ in the presence of magnetic fluxes $\boldsymbol{m} \in (N\mathbb{Z}/q)^3$ (these are the magnetic fluxes that label the physical states in Eq. (4.42).) Let T_x be the generator of an electric $\mathbb{Z}_q$ center twist along the x direction (i.e., a $\mathbb{Z}_q$ gauge transformation), and we take it to have the minimal twist of N/q. It acts on $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ via (recall the discussion around Eq. (4.30))

$$T_x U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} T_x^{-1} = e^{-2\pi i \ell Q} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} = e^{-2\pi i \ell \frac{n_x N}{q^2}} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\,, \quad n_x \in \mathbb{Z}\,. \tag{4.46}$$

n_x counts the magnetic fluxes inserted in the y-z plane in units of N/q. Identical relations to (4.46) hold in the y and z directions. As we saw in the previous section, if $\ell\frac{N}{q^2} \notin \mathbb{Z}$, there is a mixed 't Hooft anomaly between the electric $\mathbb{Z}_q^{(1)}$ 1-form center and the discrete chiral symmetries of $SU(N)$ theory with matter. Eq. (4.46) implies that the operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ is not gauge invariant under a $\mathbb{Z}_q$ gauge transformation as we attempt to gauge $\mathbb{Z}_q^{(1)}$. We can remedy this problem and reconstruct a gauge-invariant operator, denoted by $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}}}$, by summing over all $\mathbb{Z}_q$ gauge transformations generated by T_x, T_y and T_z:

$$\begin{aligned}
\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \equiv & \sum_{p_x,p_y,p_z\in\mathbb{Z}} (T_x)^{p_x}(T_y)^{p_y}(T_z)^{p_z} U_{\mathbb{Z}_{2T_\mathcal{R}}} (T_x)^{-p_x}(T_y)^{-p_y}(T_z)^{-p_z} \\
&= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{p_x,p_y,p_z\in\mathbb{Z}} e^{-2\pi i \frac{\ell N}{q^2}(p_x n_x + p_y n_y + p_z n_z)} \equiv U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{\boldsymbol{p}\in\mathbb{Z}^3} e^{-2\pi i \frac{\ell N}{q^2}\boldsymbol{p}\cdot\boldsymbol{n}} \\
&= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{n_x \ell N}{q^2} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{n_y \ell N}{q^2} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{n_z \ell N}{q^2} - l_z\right).
\end{aligned} \tag{4.47}$$

In the first line, we included a sum over arbitrary powers of T_x, T_y, T_z to enforce the gauge invariance. Then, we used Eq. (4.46) in going from the first to the second line and the Poisson resummation formula in going from the second to the third line. Even though $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ is gauge invariant, it has no inverse; it is, in general, a noninvertible operator that implements the action of $\tilde{\mathbb{Z}}^{\chi}_{2T_\mathcal{R}}$, and we use a tilde to denote the noninvertible nature of symmetries and their operators. The noninvertibility stems from the fact that $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}}}$ works as a projector: the insertion of this operator in the path integral of $SU(N)/\mathbb{Z}_q$ theory with matter projects onto specific topological charge sectors of $SU(N)/\mathbb{Z}_q$, depending on ℓ. This can be seen from the second line in (4.47), which is a sum over Fourier modes that projects in and out sectors, depending on their topological charge, upon acting on them. One can see the projective nature of $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ by inserting it into the partition function (4.44):

$$\langle \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \rangle = \sum_{\text{physical states}} {}_{\text{phy}}\langle \psi | e^{-L_4 \hat{H}} \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} | \psi \rangle_{\text{phy}}\,, \tag{4.48}$$

and then using the physical states defined in Eq. (4.42). We find that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ annihilates sectors with $\frac{n_x \ell N}{q^2} \notin \mathbb{Z}$, etc. We remind that $\frac{n_x N}{q^2}$ is the topological charge (see Eq. (4.7)), which we can write as

$$\frac{n_x N}{q^2} = \underbrace{\frac{n_x N}{q}}_{m_x} \underbrace{\frac{N}{q}}_{k_x} \frac{1}{N}\,, \tag{4.49}$$

and, as we mentioned earlier and emphasize now, n_x is the number of magnetic fluxes in units of N/q. The same applies to the magnetic sectors in the y and z directions. We conclude that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ selects sectors in Hilbert space with certain magnetic fluxes.

We can make the following observations about $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$:

1. If $\ell \in q\mathbb{Z}$, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ is invertible since in this case $\frac{n_{x,y,z}\ell N}{q^2} \in \mathbb{Z}$ for all values of $n_x, n_y, n_z \in \mathbb{Z}$. The invertible subgroup of $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ is $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}/q}$.
2. If $\gcd(\ell N/q, q) = 1$, then we must have $n_x, n_y, n_z \in q\mathbb{Z}$. In other words, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ projects onto untwisted flux sectors. In particular, in the sector given by $n_x, n_y, n_z \in q\mathbb{Z}$, the symmetry operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ act invertibly for all elements of the chiral symmetry $\ell = 1, 2, .., T_{\mathcal{R}}$.
3. If $\gcd(\ell N/q, q) = a \neq 1$ and $\ell < q$, then let $q = aq'$, and we must have $n_{x,y,z} \in q'\mathbb{Z}$. $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ projects onto background fluxes with topological charge $Q \in \mathbb{Z}/q'$, i.e. sectors that have $\mathbb{Z}_{q'}$ twists.
4. The noninvertibility of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ can be seen by multiplying the operator by its "potential inverse" $\overline{\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}}$ to find

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} \times \overline{\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}} \sim \sum_{\boldsymbol{p}\in\mathbb{Z}^3} e^{-2\pi i \frac{\ell N}{q^2}\boldsymbol{p}\cdot\boldsymbol{n}} \equiv \mathcal{C}\,. \tag{4.50}$$

   $\mathcal{C}$ is known as the condensation operator, which can be thought of as a sum over topological surface operators $\exp[-i \oint_{\mathbb{T}^2\subset\mathbb{T}^3} B^{(2)}] = \exp[-i2\pi\mathbb{Z}/q]$ wrapping the three 2-cycles of $\mathbb{T}^3$, and $B^{(2)}$ is the 2-form field of the $\mathbb{Z}_q^{(1)}$ 1-form symmetry.

We use the fact that $SU(N)/\mathbb{Z}_q$ theory possesses a magnetic $\mathbb{Z}_q^{m(1)}$ 1-form global symmetry to make one more observation. Let $\mathcal{T}_j$ be 't Hooft line of N-ality 1 in direction j. Then, the minimal 't Hooft line in $SU(N)/\mathbb{Z}_q$ theory is $\mathcal{T}_j^{N/q}$, i.e., it has N-ality N/q. The minimal line acts on a physical state by increasing its magnetic flux by one in units of N/q*. Now, let us take a theory with $\gcd(N/q, q) = 1$ so

*Similar to the discussion we had after Eq. (4.19), we can also consider the generators of the magnetic 1-form symmetry and argue that $\mathcal{T}_j^{N/q}$ inserts a magnetic flux N/q, as measured by the action of the magnetic 1-form symmetry on the state $\mathcal{T}_j^{N/q}|\psi\rangle_{\text{phy}}$.

that $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ acts projectively on certain states. Then, $|E,(n_x=q,n_y=q,n_z=q)N/q\rangle$ is one of the physical states that survive under the action of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$. We have $\mathcal{T}_x^{N/q}|E,(q,q,q)N/q\rangle = |E,(q+1,q,q)N/q\rangle$. Thus, we immediately see from Eq. (4.47) that

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}\mathcal{T}_x^{N/q}|E,(q,q,q)N/q\rangle = \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}|E,(q+1,q,q)N/q\rangle\rangle = 0\,. \tag{4.51}$$

We write this result as

$$\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}\mathcal{T}_j^{N/q} = 0\,, \quad j=x,y,z\,. \tag{4.52}$$

In other words, the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ annihilates the minimal 't Hooft lines in this theory. It also annihilates all 't Hooft lines $\mathcal{T}_j^{n_j N/q}$, $n_j \neq 0$ Mod q. This is an alternative way to see the projective nature of this operator.

4.3.2 $SU(N)/\mathbb{Z}_p$

Next, we discuss $SU(N)/\mathbb{Z}_p$ theory with matter with N-ality n, and $\mathbb{Z}_p \subseteq \mathbb{Z}_q = \mathbb{Z}_{\gcd(N,n)}$. The partition function of this theory is given by the path integral in Eq. (4.43), now restricting the sum over the electric and magnetic twists $(\boldsymbol{m},\boldsymbol{k}) \in (N\mathbb{Z}/p)^6$. The theory possess an electric $\mathbb{Z}_{q/p}^{(1)}$ 1-form global symmetry. As before, T_x is taken to be the generator of the electric $\mathbb{Z}_q^{(1)}$ symmetry. Then, the electric $\mathbb{Z}_{q/p}^{(1)}$ 1-form global symmetry is generated by T_x^p (as well as T_y^p and T_z^p). The theory has q/p distinct Wilson lines $W_j^{e_j p}$, with $e_j = 0,1,2,..,q/p-1$. These lines are invariant under $\mathbb{Z}_p$, as they should be since $\mathbb{Z}_p$ is gauged. The minimal admissible Wilson line W_j^p carries one electric flux in units of pN/q. In the limiting case $p=q$, the line $W_j^{p=q}$ coincides with the matter content and must be removed from the spectrum of line operators. Therefore, in this case, the theory does not possess a 1-form electric symmetry, as discussed in the previous section.

In addition, the theory has a magnetic $\mathbb{Z}_p^{m(1)}$ 1-form symmetry. If $\mathcal{T}_j$ is the 't Hooft line with N-ality 1, then the minimal admissible 't Hooft line in the theory is $\mathcal{T}_j^{N/p}$, which carries one magnetic flux in units of N/p. There are p distinct 't Hooft lines in the theory $\mathcal{T}_j^{n_j N/p}$, $n_j = 0,1,..,p-1$, which are mutually local with Wilson lines $W_j^{e_j p}$. The Hamiltonian, Wilson lines generators, and the 't Hooft lines generators of this theory can be simultaneously diagonalized. Therefore, the energies and eigenvalues of the set of Wilson and 't Hooft operators can be used to label the physical states of Hilbert space:

$$|\psi\rangle_{\text{phy}} = |E,\boldsymbol{e}pN/q,\boldsymbol{n}N/p\rangle\,, \quad \boldsymbol{e}\in(\mathbb{Z}_{q/p})^3\,,\boldsymbol{n}\in(\mathbb{Z}_p)^3\,. \tag{4.53}$$

Next, we need to build a gauge invariant chiral symmetry operator. Our starting point, as usual, is the operator

$$U_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} = e^{2\pi i\frac{\ell}{2T_{\mathcal{R}}}\int_{\mathbb{T}^3}\left(j_A^0 - 2T_{\mathcal{R}}K^0(a^c)\right)} \tag{4.54}$$

taken in the presence of the fractional magnetic fluxes $\boldsymbol{m} \in (N\mathbb{Z}/p)^3$, which label the Hilbert space in Eq. (4.53). The operator $T_x^{q/p}$ generates the electric $\mathbb{Z}_p^{(1)}$ 1-form symmetry, which is gauged. In other words, $T_x^{q/p}$ implements the twists $\boldsymbol{k} \in (N\mathbb{Z}/p)^3$. In analogy with $SU(N)/\mathbb{Z}_q$ theories, we need to build gauge invariants of the chiral symmetry operator using the building block $T_x^{q/p} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} T_x^{-q/p}$. To compute this block, we use the discussion around Eq. (4.30), taking the minimal twist N/p generated by $T_x^{q/p}$, to obtain

$$T_x^{q/p} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} T_x^{-q/p} = e^{-2\pi i \ell \frac{n_x N}{p^2}} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\,, \quad n_x \in \mathbb{Z}\,, \tag{4.55}$$

and n_x counts the magnetic fluxes in units of N/p. If $\ell \frac{N}{p^2} \notin \mathbb{Z}$, there is a mixed anomaly between $\mathbb{Z}^\chi_{2T_R}$ and the electric $\mathbb{Z}_p^{(1)}$ symmetries in $SU(N)$ theory with matter, and we expect the chiral symmetry becomes noninvertible upon gauging $\mathbb{Z}_p^{(1)}$. The corresponding gauge invariant operator of the $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ symmetry is then given by the summations

$$\begin{aligned}
\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} &= \sum_{p_x,p_y,p_z \in \mathbb{Z}} (T_x)^{qp_x/p} (T_y)^{qp_y/p} (T_z)^{qp_z/p} U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} (T_x)^{-qp_x/p} (T_y)^{-qp_y/p} (T_z)^{-qp_z/p} \\
&= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{p_x,p_y,p_z \in \mathbb{Z}} e^{-2\pi i \frac{\ell N}{p^2}(p_x n_x + p_y n_y + p_z n_z)} \\
&= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell} \sum_{l_x \in \mathbb{Z}} \delta\left(\frac{n_x \ell N}{p^2} - l_x\right) \sum_{l_y \in \mathbb{Z}} \delta\left(\frac{n_y \ell N}{p^2} - l_y\right) \sum_{l_z \in \mathbb{Z}} \delta\left(\frac{n_z \ell N}{p^2} - l_z\right).
\end{aligned} \tag{4.56}$$

This noninvertible operator generalizes (4.47) to any $\mathbb{Z}_p \subseteq \mathbb{Z}_q$, and it projects onto sectors with finer topological charges than the sectors admissible by (4.47). This means there exist sectors where $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ act invertibly for all $\ell = 1, 2, .., T_\mathcal{R}$ if and only if

$$l_x = \frac{n_x N}{p^2} \in \mathbb{Z}\,, \tag{4.57}$$

with similar conditions in the y and z directions. As special cases, we may first set $p = q$ to readily cover (4.47). Also, setting $p = 1$, the operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ becomes invertible, as can be easily seen from the second line in (4.56). Notice that $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ does not act on Wilson lines in this theory, as the noninvertible operator is built from $(T_j)^{qp_j/p}$ and its inverse; thus, one can push a Wilson line through $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ without being affected*. We can write this observation as

$$\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} W_j^{e_j p} = W_j^{e_j p} \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\,, \quad e_j = 0, 1, 2, .., q/p - 1\,, \quad j = x, y, z\,. \tag{4.58}$$

*Although we do not give the explicit form of T_j, it can be thought of as an exponential of an integral of the chromoelectric field over a 2-dimensional submanifold; see [104]. A Wilson line would acquire a phase as we push it past $T_j^{q/p}$ (we use $[a_j^{c,a}(\boldsymbol{x},t), E_k^b(\boldsymbol{y},t)] = i\delta_{jk}\delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ab}$, where a, b are the color indices, along with the Baker-Campbell-Hausdorff formula). It also acquires the negative of the same phase as it is pushed past $T_j^{-q/p}$. Therefore, the phases cancel out, and hence, the result in Eq. (4.58).

This is very different from the action of $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ on 't Hooft lines, as we discussed before.

The procedure employed to construct the noninvertible operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ contains an additional layer of underlying physics. It is essential to keep in mind that this operator is constructed in $SU(N)/\mathbb{Z}_p$ theory, where its creation involved a sum over magnetic $\boldsymbol{m} \in (N\mathbb{Z}/p)^3$ and electric $\boldsymbol{k} \in (N\mathbb{Z}/p)^3$ twists. These twists do not encompass the entire range of permissible twists that can be applied. Recall that the theory encompasses a global $\mathbb{Z}^{(1)}_{q/p}$ symmetry, which affords us the opportunity to introduce the electric twists $\boldsymbol{k} \in (pN\mathbb{Z}/q)^3$. Moreover, we can turn on magnetic twists $\boldsymbol{m} \in (pN\mathbb{Z}/q)^3$, compatible with the cocycle condition*. This broader scope of twists provides a richer set of possibilities within the theory. We recall that T_x^p is the generator of $\mathbb{Z}^{(1)}_{q/p}$ symmetry that implements the twists $k_x \in pN\mathbb{Z}/q$. Then, one can write the partition function of $SU(N)/\mathbb{Z}_p$ theory in these background twists as

$$\begin{aligned}\mathcal{Z}_{SU(N)/\mathbb{Z}_p+\text{matter}}[\boldsymbol{m},\boldsymbol{k}] &= \text{tr}_{\boldsymbol{m}\in(pN\mathbb{Z}/q)^3}\left[e^{-L_4 H}T_x^{k_x p}T_y^{k_y p}T_z^{k_z p}\right] \\ &= \sum_{\boldsymbol{e}\in(\mathbb{Z}_{q/p})^3} e^{-i2\pi\frac{p\boldsymbol{k}\cdot\boldsymbol{e}}{q}}\,{}_{\text{phy}}\langle\psi|e^{-L_4 H}|\psi\rangle_{\text{phy}}|_{\boldsymbol{m}\in(pN\mathbb{Z}/q)^3},\end{aligned} \tag{4.59}$$

and we used Eq. (4.53) along with $T_j^{k_j p}|\psi\rangle_{\text{phy}} = e^{-i2\pi\frac{pk_j e_j}{q}}|\psi\rangle_{\text{phy}}$; see the discussion around Eqs. (4.62, 4.63) below.

Next, consider the commutation relation between T_x^p and $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$, the latter operator is being in the background of the magnetic twist $\boldsymbol{m} \in (pN\mathbb{Z}/q)^3$. Using the discussion and procedure around Eq. (4.30), we obtain

$$T_x^p\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}T_x^{-p} = e^{-2\pi i\ell n_x\frac{p^2N}{q^2}}\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,. \tag{4.60}$$

The failure of the commutation between T_x^p and $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ by the phase $e^{-2\pi i\ell n_x\frac{p^2N}{q^2}}$, assuming $\ell n_x\frac{p^2N}{q^2} \notin \mathbb{Z}$, signals a mixed anomaly between the noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry and the electric $\mathbb{Z}^{(1)}_{q/p}$ 1-form global symmetry. This anomaly means that certain sectors in Hilbert space exhibit degeneracy. Let us analyze this situation more closely. We assume there exists a sector with n_x, n_y, n_z that satisfies Eq. (4.57), and thus, in this sector, the symmetry operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ acts invertibly for all elements $\ell = 1, 2, .., T_{\mathcal{R}}$. Now, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$, being a global symmetry operator, commutes with the Hamiltonian:

$$[\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}, H] = 0\,. \tag{4.61}$$

Likewise, since $\mathbb{Z}^{(1)}_{q/p}$ is a global symmetry, its generators T_j^p commute with the Hamiltonian:

$$[T_j^p, H] = 0\,. \tag{4.62}$$

*Recall that twists in $N\mathbb{Z}/q$ are compatible with the cocycle conditions. Therefore, twists in $pN\mathbb{Z}/q$ are a subset of the allowed twists. Notice that the twists $\boldsymbol{m} \in (pN\mathbb{Z}/q)^3$ provide background magnetic fluxes and do not label the physical states in Hilbert space, Eq. (4.53).

This commutation relation, along with Eq. (4.22), implies that T_j^p acts on physical states in Hilbert space as (the label $\boldsymbol{l} = (l_x, l_y, l_z)$ emphasizes that such states satisfy condition (4.57), such that $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ acts invertibly on such states. Also, we suppressed the detailed dependence on the different quantum numbers to reduce clutter)

$$T_j^p|E, e_j\rangle_{\boldsymbol{l}} = e^{i\frac{2\pi p}{q}e_j}|E, e_j\rangle_{\boldsymbol{l}}\,, \tag{4.63}$$

and that the states are labeled by their energies as well as $e_j = 1, 2, ..., q/p$ distinct labels; these are the eigenvalues (fluxes) of the $\mathbb{Z}^{(1)}_{q/p}$ symmetry operator. The algebra defined by the commutation relations Eqs. (4.61, 4.62), along with the mixed anomaly represented as Eq. (4.60), under the assumption of a nontrivial phase, furnishes a finite-dimensional space with a minimum dimension of $q^2/\gcd(n_x p^2 N, q^2)$ (we take $n_x = n_y = n_z$). The Hilbert space of physical states, which are labeled by q/p distinct fluxes, sit in $q^2/\gcd(n_x p^2 N, q^2)$ orbits, and a rotation by $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell=1}$ links a state with a flux e_j to a state with a flux $e_{j+\gcd(n_j p^2 N, q^2)/(qp)}$ as:

$$\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell=1}|E, e_j\rangle_{\boldsymbol{l}} = |E, e_j + \gcd(n_j p^2 N, q^2)/(qp)\rangle_{\boldsymbol{l}}\,. \tag{4.64}$$

Using the commutation relation (4.61), one immediately sees that the states $|E, e_j\rangle_{\boldsymbol{l}}$ and $|E, e_j + \gcd(n_j p^2 N, q^2)/(qp)\rangle_{\boldsymbol{l}}$ have the same energy*.

In the following subsections, we apply our formalism to examples of theories with fermions in specific representations.

4.3.3 Examples

4.3.3.1 $SU(4n+2)/\mathbb{Z}_2$ and $SU(4n)/\mathbb{Z}_2$ with a Dirac fermion in the 2-index anti-symmetric representation

The $SU(4n+2)/\mathbb{Z}_2$ gauge theory with a 2-index anti-symmetric Dirac fermion (N-ality 2) has a $\mathbb{Z}^\chi_{8n}$ chiral symmetry. The $SU(4n+2)$ theory possesses an electric $\mathbb{Z}^{(1)}_2$ one-form symmetry. In [98], the authors argued that upon gauging $\mathbb{Z}^{(1)}_2$, the odd rotations of $\mathbb{Z}^\chi_{8n}$ become non-invertible. We can show this is the case on $\mathbb{T}^3$ using our construction. Setting $N = 4n + 2$ in (4.47), we obtain

$$\tilde{U}_{\mathbb{Z}_{8n},\ell} = U_{\mathbb{Z}_{8n},\ell}\sum_{l_x\in\mathbb{Z}}\delta\left(\frac{n_x\ell}{2} - l_x\right)\sum_{l_y\in\mathbb{Z}}\delta\left(\frac{n_y\ell}{2} - l_y\right)\sum_{l_z\in\mathbb{Z}}\delta\left(\frac{n_z\ell}{2} - l_z\right)\,. \tag{4.65}$$

For ℓ odd, $\tilde{U}_{\mathbb{Z}_{8n},\ell}$ projects onto untwisted gauge sectors and becomes non-invertible.

*It is helpful to give a numerical example. Take $N = 1000$, $q = 500$, and $p = 20$. Such numbers are contrived and do not necessarily correspond to any realistic theory. Condition (4.57) is satisfied if we take $n_x = 2$. Then, the phase in the anomaly Eq. (4.60) is $e^{-i2\pi/5}$, implying a 5-fold degeneracy. The theory has an electric $\mathbb{Z}^{(1)}_{25}$ 1-form symmetry, and thus, 25 distinct flux states. These states set in 5 different orbits such that the states labeled with e_1, e_6, e_{11}, e_{16}, e_{21} have the same energy, and the states e_2,e_7,...,e_{22}, have the same energy, etc.

The $SU(4n)/\mathbb{Z}_2$ theory with a 2-index anti-symmetric Dirac fermion has a $\mathbb{Z}^{\chi}_{8n-4}$ chiral symmetry. The cocycle conditions, say in the x-direction, must satisfy (see Eq. (4.6))

$$e^{2\pi i \frac{2n_{yz}}{4n}} = 1\,. \tag{4.66}$$

Therefore we must have $n_{yz} \in 2n\mathbb{Z}$. There is no mixed anomaly between $\mathbb{Z}^{\chi}_{8n-4}$ and the electric $\mathbb{Z}^{(1)}_2$ symmetries in the $SU(4n)$ theory since the anomaly phase $\frac{n_{yz}}{2} \in n\mathbb{Z}$ is trivial. Thus, the full chiral symmetry $\mathbb{Z}^{\chi}_{8n-4}$ is invertible. This is also in agreement with [98].

4.3.3.2 $SU(6)/\mathbb{Z}_3$ with a Dirac fermion in the 3-index anti-symmetric representation

This theory has a $\mathbb{Z}^{\chi}_6$ chiral symmetry. What is special about this theory is that its bilinear fermion operator vanishes identically because of Fermi statistics. Moreover, the $SU(6)$ theory exhibits a mixed anomaly between its electric $\mathbb{Z}^{(1)}_3$ 1-form center and chiral symmetries [105, 106]. Assuming confinement, then the chiral symmetry must be broken in the infrared. Yet, this breaking has to be accomplished via higher-order condensate. Using (4.47), we find that the operator corresponding to a chiral transformation in $SU(6)/\mathbb{Z}_3$ theory is

$$\tilde{U}_{\mathbb{Z}_6,\ell} = U_{\mathbb{Z}_6,\ell} \sum_{l_x\in\mathbb{Z}} \delta\left(\frac{2n_x\ell}{3} - l_x\right) \sum_{l_y\in\mathbb{Z}} \delta\left(\frac{2n_y\ell}{3} - l_y\right) \sum_{l_z\in\mathbb{Z}} \delta\left(\frac{2n_z\ell}{3} - l_z\right)\,. \tag{4.67}$$

Hence, for $\ell \in \{1, 2, 4, 5\}$, the operator $\tilde{U}_{\mathbb{Z}_6,\ell}$ projects onto untwisted gauge sectors, and the chiral symmetry operator becomes noninvertible.

4.3.3.3 2-index $SU(6)$ chiral gauge theory

Our next example is a chiral gauge theory. This is $SU(6)$ YM theory with a single left-handed Weyl fermion ψ in the 2-index symmetric representation and 5 flavors of left-handed Weyl fermions χ in the complex conjugate 2-index anti-symmetric representation. The fermion budget ensures the theory is free from gauge anomalies. The theory encompasses continuous global symmetry $SU(5)_\chi \times U(1)_A$, where $SU(5)_\chi$ acts on χ. The charges of ψ and χ under $U(1)_A$ are $q_\psi = -5\,, q_\chi = 2$. The theory is also endowed with a $\mathbb{Z}^{\chi}_4$ chiral symmetry, which is taken to act on χ with a unit charge. It can be checked that this is a genuine symmetry since neither $\mathbb{Z}_4$ nor a subgroup of it can be absorbed in rotations in the centers of $SU(6) \times SU(5)_\chi$. It turns out, see [1] for details (also see [35]), that we must divide the global symmetry by $\mathbb{Z}_3 \times \mathbb{Z}_5$ to remove redundancies. Putting everything together and remembering that the theory possesses an electric $\mathbb{Z}^{(1)}_2$ 1-form center symmetry (since all fermions have N-ality $n = 2$), we write the faithful global group as:

$$G^{\rm g} = \frac{SU(5)_\chi \times U(1)_A}{\mathbb{Z}_3 \times \mathbb{Z}_5} \times \mathbb{Z}^{\chi}_4 \times \mathbb{Z}^{(1)}_2\,. \tag{4.68}$$

This theory has an anomaly between its $\mathbb{Z}_2^{(1)}$ center symmetry and $\mathbb{Z}_4^\chi$ chiral symmetry. To see the anomaly, we recall that we can turn on the magnetic and electric twists $(\boldsymbol{m},\boldsymbol{k})\in(3\mathbb{Z})^6$. This gives the topological charge $Q\in\mathbb{Z}/2$. Thus, under a chiral transformation, the partition function acquires a phase

$$\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]\longrightarrow\exp\left[i\frac{2\pi\ell N_\chi T_\chi Q}{4}\right]\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]=\exp\left[i2\pi\ell/2\right]\mathcal{Z}[\boldsymbol{m},\boldsymbol{k}]\,, \tag{4.69}$$

where $N_\chi=5$ is the number of the χ flavors and $T_\chi=4$ is the Dynkin index of χ. Therefore, we expect that $\mathbb{Z}_4^\chi$ becomes noninvertible in the $SU(6)/\mathbb{Z}_2$ chiral theory. Using (4.47), the noninvertible operator corresponding to a chiral transformation in $SU(6)/\mathbb{Z}_2$ theory is

$$\tilde{U}_{\mathbb{Z}_4,\ell}=U_{\mathbb{Z}_4,\ell}\sum_{l_x\in\mathbb{Z}}\delta\left(\frac{n_x\ell}{2}-l_x\right)\sum_{l_y\in\mathbb{Z}}\delta\left(\frac{n_y\ell}{2}-l_y\right)\sum_{l_z\in\mathbb{Z}}\delta\left(\frac{n_z\ell}{2}-l_z\right)\,. \tag{4.70}$$

Hence, for $\ell\in\{1,3\}$, the operator $\tilde{U}_{\mathbb{Z}_4,\ell}$ projects onto untwisted gauge sectors, and the chiral symmetry operator becomes noninvertible.

4.4 $SU(N)\times U(1)/\mathbb{Z}_p$, $\mathbb{Z}_p\subseteq\mathbb{Z}_N$ theories, noninvertible symmetries, and their anomalies

In this section, we also gauge the $U(1)$ baryon number symmetry. Thus, we are discussing $SU(N)\times U(1)$ gauge theory with a Dirac fermion in a representation $\mathcal{R}$, N-ality n, and $U(1)$ charge $+1$. This theory, as we discussed in Section 4.2, is endowed with an invertible $\mathbb{Z}^\chi_{2\text{gcd}(T_\mathcal{R},d_\mathcal{R})}$ chiral symmetry as well as an electric $\mathbb{Z}_N^{(1)}$ center symmetry acting on its Wilson lines; see Eqs. (4.14). However, in [97], it was shown that $SU(N)\times U(1)$ theories also have noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ chiral symmetry. In the following, we first review the construction of the noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ operator in $SU(N)\times U(1)$ theories, and next, we discuss this operator in $SU(N)\times U(1)/\mathbb{Z}_p$, $\mathbb{Z}_p\subseteq\mathbb{Z}_N$, theories.

4.4.1 $SU(N)\times U(1)$

Our starting point is the $SU(N)\times U(1)$ theory and its $\mathbb{Z}^\chi_{2T_\mathcal{R}}$ operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}=e^{i\frac{2\pi\ell}{2T_\mathcal{R}}Q_5}$, where Q_5 is the conserved chiral charge defined in Eq. (4.40) in the background of the fractional $n_{x,y,z}$ and integer $N_{x,y,z}$ magnetic fluxes in the x,y,z directions. We remind that we can turn on fractional fluxes in $\mathbb{Z}_N$ irrespective of the N-ality of the matter content since we use $U(1)$ transition functions to impose the cocycle condition; see Eq. (4.33). No nontrivial electric twists are applied at this stage, i.e., we take $\boldsymbol{k}\in(N\mathbb{Z})^3$, since our nonabelian gauge group is $SU(N)$ rather than $SU(N)/\mathbb{Z}_p$. The operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ is invariant under $SU(N)$. To see that, we apply a large $SU(N)$ gauge transformation, recalling Eq. (4.21) and setting

$\boldsymbol{k}\in(N\mathbb{Z})^3$, which immediately gives the change in the nonabelian winding number by $Q\in\mathbb{Z}$. In addition, $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ must be invariant under $U(1)$ gauge symmetry. The photon gauge field a_i transforms under $U(1)$ gauge symmetry as $a_j(x+\hat{e}_kL_k)=a_j-\partial_k\xi(x)$, and $\xi(x)$ is a periodic gauge function: $\xi(x+\hat{e}_kL_k)=\xi(x)+2\pi p$, $p\in\mathbb{Z}$. Applying a large $U(1)$ gauge transformation to Q_5, we find (see [97] for the derivation)

$$U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\longrightarrow U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}e^{-2\pi i\ell\left(p_x\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)+p_y\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_y+\frac{nn_y}{N}\right)+p_z\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_z+\frac{nn_z}{N}\right)\right)},\qquad(4.71)$$

where $p_{x,y,z}$ are arbitrary integers corresponding to the $U(1)$ gauge transformation. Eq. (4.71) shows that the operator $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ fails to be gauge invariant under $U(1)$ gauge symmetry. To remedy this problem, we follow the procedure of the previous section and define a new operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ by summing over all gauge-transformations of $U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$:

$$\begin{aligned}\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} &= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\sum_{p_x,p_y,p_z\in\mathbb{Z}} e^{-2\pi i\ell\left(p_x\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)+p_y\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_y+\frac{nn_y}{N}\right)+p_z\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_z+\frac{nn_z}{N}\right)\right)}\\ &= U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\sum_{l_x\in\mathbb{Z}}\delta\left(\ell\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)-l_x\right)\left(\sum_{l_y\in\mathbb{Z}}...\right)\left(\sum_{l_z\in\mathbb{Z}}...\right).\end{aligned}\qquad(4.72)$$

The operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ implements the chiral transformation of the now-noninvertible $\tilde{\mathbb{Z}}^\chi_{2T_\mathcal{R}}$ symmetry, as it acts projectively by selecting certain nonvanishing sectors in Hilbert space labeled by the integers $l_{x,y,z}$, such that for $\ell=1$ we must have

$$l_x=\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)\in\mathbb{Z}\,,\qquad(4.73)$$

with identical expressions for l_y and l_z. Condition (4.73) ensures that all the symmetry elements $\ell=1,2,..,T_\mathcal{R}$ act invertibly on the same admissible sector. To explicitly see the projective nature of $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ on states in Hilbert space, we use the partition function of the $SU(N)\times U(1)$ theory given by Eq. (4.37) (we set the electric flux background $\boldsymbol{k}$=0 and, as usual, we use $\boldsymbol{n}$ to label a specific fractional magnetic flux background: $\boldsymbol{n}=(n_x,n_y,n_z)$) to compute $\langle\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\rangle$:*

$$\langle\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\rangle=\sum_{\boldsymbol{e}\in\mathbb{Z}_N^3,\,\boldsymbol{N}\in\mathbb{Z}^3}{}_{\boldsymbol{n}}\langle E,\boldsymbol{e},\boldsymbol{N}|e^{-L_4\hat{H}}\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}|E,\boldsymbol{e},\boldsymbol{N}\rangle_{\boldsymbol{n}}\,.\qquad(4.74)$$

We immediately see from the Kronecker deltas in Eq. (4.72) that only those sectors with $\boldsymbol{N}$ satisfying Eq. (4.73) are selected.

Turning off the fractional magnetic flux background (i.e., setting $\boldsymbol{n}=0$), the operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ becomes invertible for $\ell\in T_\mathcal{R}\mathbb{Z}/\gcd(T_\mathcal{R},d_\mathcal{R})$. We recognize that

*Recall from our earlier analysis that the theory is endowed with electric $\mathbb{Z}_N^{(1)}$ and magnetic $U_m^{(1)}(1)$ symmetries, and the states of the theory are labeled by the eigenstates of these symmetries, $\boldsymbol{e}$ and $\boldsymbol{N}$, respectively.

we have just recovered the invertible $\mathbb{Z}^{\chi}_{2\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$ subgroup of $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$. Furthermore, setting $\boldsymbol{n}=0$, the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ annihilates all Hilbert space sectors characterized with integral magnetic fluxes $\boldsymbol{N}\notin T_{\mathcal{R}}\mathbb{Z}^3/\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})$. This noninvertible nature of the chiral operator should have been anticipated. When we start with the SU(N) theory with matter, we find an 't Hooft anomaly between its invertible $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ chiral symmetry and $U(1)$ baryon symmetry. This anomaly is valued in $\mathbb{Z}_{T_{\mathcal{R}}/\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})}$. Upon gauging $U(1)$, this anomaly becomes of the ABJ type, and the chiral symmetry becomes noninvertible. Now, If we take the Euclidean version of our theory in the infinite volume limit and apply a $\pi/2$ rotation to $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$, the operator becomes a defect. Alternatively, we may also use the half-gauging procedure to construct this defect, which was done in [97]. Inserting this defect at some position will generally create a domain wall (since it enforces a chiral transformation) dressed with a TQFT that accounts for the noninvertible nature of the defect. It will be interesting to analyze what happens to the domain walls when we turn on an external magnetic field with flux $\boldsymbol{N}\notin T_{\mathcal{R}}\mathbb{Z}^3/\text{gcd}(T_{\mathcal{R}},d_{\mathcal{R}})$.

$SU(N)\times U(1)$ gauge theory has an electric $\mathbb{Z}^{(1)}_N$ 1-form global center symmetry, and the immediate exercise would be checking whether there is a mixed anomaly between the center and the noninvertible chiral symmetries. To this end, we turn on both electric and magnetic twists* $(\boldsymbol{m},\boldsymbol{k})\in\mathbb{Z}^6$, giving rise to nonabelian fractional topological charge $Q_{SU(N)}\in\mathbb{Z}/N$ as well as abelian topological charge $Q_u=\left(\frac{n}{N}\right)^2$; see Eq. (4.12). Using Eqs. (4.21, 4.41), setting $\boldsymbol{k}=(1,0,0)$, we find

$$\begin{aligned} T_xt_x\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}(T_xt_x)^{-1} &= \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}e^{-2\pi i\ell\left(\frac{n_x}{N}-\frac{n}{N}\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x+\frac{nn_x}{N}\right)\right)} \\ &= \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}e^{-i2\pi\ell\left(\frac{n_x-nl_x}{N}\right)}\,, \end{aligned} \tag{4.75}$$

and we used Condition (4.73) to go from the first to the second line. If the phase is nontrivial, then there is a mixed anomaly between the electric $\mathbb{Z}^{(1)}_N$ 1-form center and the 0-form noninvertible $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ symmetries, leading to spectral degeneracy of states (those that already selected by the operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$). The algebra defined by the commutation relations $[H,T_jt_j]=[H,\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}]=0$ along with the mixed anomaly (4.75), under the assumption of a nontrivial phase, furnishes a finite-dimensional space with dimension $N/\text{gcd}(N,n_x-nl_x)$ (we take $n_x=n_y=n_z$). The Hilbert space of physical states, which are labeled by N different electric fluxes $\boldsymbol{e}$, sit in $N/\text{gcd}(N,n_x-nl_x)$ orbits, and a rotation by $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ links a state with a flux e_j to a state with a flux $e_j+\text{gcd}(N,n_j-nl_j)$, i.e., they have the same energy.

4.4.2 $SU(N)\times U(1)/\mathbb{Z}_p$, $\mathbb{Z}_p\subseteq\mathbb{Z}_N$

Next, we study the noninvertible operators in $SU(N)\times U(1)/\mathbb{Z}_p$ gauge theory, where $\mathbb{Z}_p$ is a subgroup of the $\mathbb{Z}_N$ center symmetry. This theory has an electric

*Notice that these electric twists $\boldsymbol{k}\in\mathbb{Z}^3$ are $\boldsymbol{e}$ that label the physical states in Hilbert space: $|E,\boldsymbol{e},\boldsymbol{N}\rangle_{\boldsymbol{n}}$. In principle, k_j should be in $\mathbb{Z}$ Mod N, but, as usual, we drop the modding as this does not affect the cocycle conditions.

$\mathbb{Z}^{(1)}_{N/p}$ 1-form global symmetry acting on the p-th power of the spatial components of the abelian and nonabelian Wilson lines defined in Eq. (4.14):

$$W^{e_j p}_{j,\,SU(N)}\,,W^{e_j p}_{j,\,U(1)}\,,\quad e_j = 1,2,..,N/p,\quad j=1,2,3\,. \tag{4.76}$$

These Wilson lines are invariant under $\mathbb{Z}_p$, as they should be, as this symmetry is gauged. Notice that the allowed abelian probe charges q need to satisfy $q = z_e$, where $z_e = e_j p$ is the N-ality of the nonabelian line. Thus, we can represent the lines in Eq. (4.76) by the pair $(z_e, q = z_e)$. The theory also possesses a magnetic $U^{(1)}_m(1)$ 1-form symmetry acting on 't Hooft lines. Let $z_m = 0,1,..,p-1$, and g be the N-ality of the nonabelian 't Hooft line and the abelian magnetic charge, respectively. Then, the pairs $(z_e, q = z_e)$ and (z_m, g) must satisfy the Dirac quantization condition $e^{i2\pi(-qg+z_e z_m/p)} = 1$ or $z_e z_m - pqg \in p\mathbb{Z}$, which gives a constraint on the magnetic charges: $g = \frac{z_m}{p} + \mathbb{Z}$, i.e., the abelian magnetic charges can be fractional [107]. Another way of putting it is that the presence of the Abelian Wilson lines $W^{e_j p}_{j,U(1)}$ demand that the Abelian 't Hooft lines are $\mathcal{T}^{N_j+n_j/p}_{j,\,U(1)}$, $n_j \in \mathbb{Z}_p$, $N_j \in \mathbb{Z}$, such that the electric and magnetic lines are mutually local. The physical states in Hilbert space are taken to be eigenstates of the commuting set of the Hamiltonian, the generators of electric symmetry, and the generators of magnetic symmetry:

$$\begin{aligned}|\psi\rangle_{\text{phy}\,,\boldsymbol{m}} \;=\; |E, p\boldsymbol{e}, \boldsymbol{n}/p+\boldsymbol{N}\rangle_{\boldsymbol{m}}\,,\quad & e_j = 0,1,..,N/p-1\,,\quad N_j\in\mathbb{Z}\quad n_j = 0,1,..,p-1\,,\\ & j=1,2,3\,,\end{aligned} \tag{4.77}$$

and $\boldsymbol{m}\in\mathbb{Z}^3$ is the fractional magnetic flux background (or background magnetic twist). Remember that, in principle, $\boldsymbol{m}\in(\mathbb{Z}\,\text{Mod}\,N)^3$; however, we drop the modding by N since this cannot affect the cocycle condition. Notice that we can always activate a $\mathbb{Z}_N$ magnetic twist since, as emphasized several times, we use a combination of nonabelian and abelian transition functions. Also, in the special case $p = N$, we should remove the subscript $\boldsymbol{m}$ since, in this case, the Hilbert space is spanned by eigenstates of the full magnetic $\mathbb{Z}_N$ fluxes, i.e., $n_j = 0,1,...,N-1$.

The operator $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ defined in Eq. (4.72) is invariant under both $SU(N)$ and $U(1)$ gauge transformations. However, because we are now gauging $\mathbb{Z}_p$, the operator must also be invariant under $\mathbb{Z}_p$ gauge transformations. Let us recall that $T_j t_j$ is the generator of the electric $\mathbb{Z}^{(1)}_N$ symmetry, and therefore, $(T_j t_j)^{N/p}$ generates the $\mathbb{Z}_p$ symmetry, which must be gauged. The action of $(T_j t_j)^{N/p}$ on $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ can be read from the first line in Eq. (4.75) by applying the operation N/p times:

$$(T_j t_j)^{N/p}\,\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\,(T_j t_j)^{-N/p} = \tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}\, e^{-2\pi i\ell\left(\frac{n_x}{p}-\frac{n}{p}\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x+\frac{n n_x}{N}\right)\right)}\,. \tag{4.78}$$

This relation shows that for a general ℓ, $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$ fails to be gauge invariant under a $\mathbb{Z}_p$ gauge transformation*. Being acquainted with the remedy of this problem, we

*In the special case $p=1$, the phase becomes $e^{2\pi i\ell n\frac{d_{\mathcal{R}}}{T_{\mathcal{R}}}\left(N_x+\frac{n n_x}{N}\right)}$, and using Condition (4.73), the phase trivializes. This shows that this operator is gauge invariant in $SU(N)\times U(1)$ theory, as expected.

use $(T_jt_j)^{N/p}\,\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\,(T_jt_j)^{-N/p}$ as a building block of a gauge invariant operator by summing over gauge transformations of the block. The noninvertible operator is then given by

$$\begin{aligned}
&\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell} &(4.79)\\
&=\sum_{\mathbf{p}\in\mathbb{Z}^3}(T_xt_x)^{\frac{Np_x}{p}}(T_yt_y)^{\frac{Np_y}{p}}(T_zt_z)^{\frac{Np_z}{p}}U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}(T_xt_x)^{-\frac{Np_x}{p}}(T_yt_y)^{-\frac{Np_y}{p}}(T_zt_z)^{-\frac{Np_z}{p}}\\
&=U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\sum_{p_x,p_y,p_z\in\mathbb{Z}}e^{-2\pi i\ell\left(\frac{n_x}{p}-\frac{n}{p}\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)\right)+(x\to y)+(x\to z)}\\
&=U_{\mathbb{Z}_{2T_\mathcal{R}},\ell}\sum_{l_x\in\mathbb{Z}}\delta\left(\frac{\ell n_x}{p}-\frac{\ell n}{p}\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)-l_x\right)\left(\sum_{l_y\in\mathbb{Z}}...\right)\left(\sum_{l_z\in\mathbb{Z}}...\right). &(4.80)
\end{aligned}$$

The operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$ acts invertibly on sectors in Hilbert space that, for $\ell=1$, satisy the condition

$$l_x=\frac{n_x}{p}-\frac{n}{p}\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)\in\mathbb{Z}\,, \tag{4.81}$$

with identical expressions in the y and z directions. The operator $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$, as introduced in Eq. (4.80), within the context of $SU(N)\times U(1)/\mathbb{Z}_p$ gauge theory, is a generalization of the operator defined in Eq. (4.72) for the conventional $SU(N)\times U(1)$ theory. Furthermore, Condition (4.81) represents a broader generalization of Condition (4.73). In the specific scenario where $p=N$ holds, corresponding to the $SU(N)\times U(1)/\mathbb{Z}_N$ theory, Condition (4.81) precisely mirrors the criterion for the absence of a mixed anomaly between the electric $\mathbb{Z}_N^{(1)}$ 1-form global symmetry and the noninvertible chiral symmetry inherent to the $SU(N)\times U(1)$ theory. This correspondence is clear from the first line of Eq. (4.75).

The $SU(N)\times U(1)/\mathbb{Z}_p$ theory exhibits an electric $\mathbb{Z}_{N/p}^{(1)}$ one-form global symmetry, which is generated by the operators $(T_jt_j)^p$. When introducing a background for this symmetry, we uncover a mixed anomaly between the noninvertible chiral symmetry and the $\mathbb{Z}_{N/p}^{(1)}$ symmetry. Sandwiching $\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}$, defined in Eq. (4.80), between $(T_xt_x)^p$ and $(T_xt_x)^{-p}$ and using Eqs. (4.21, 4.41), we find

$$\begin{aligned}
(T_xt_x)^p\tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}(T_xt_x)^{-p} &= \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}e^{-2\pi i\ell\left(\frac{pn_x}{N}-\frac{pn}{N}\frac{d_\mathcal{R}}{T_\mathcal{R}}\left(N_x+\frac{nn_x}{N}\right)\right)}\\
&= \tilde{U}_{\mathbb{Z}_{2T_\mathcal{R}},\ell}e^{-i2\pi l_x\ell\frac{p^2}{N}}\,, &(4.82)
\end{aligned}$$

where we used l_x defined in Eq. (4.81) in going from the first to the second line. When the phase $e^{-i2\pi l_x\ell\frac{p^2}{N}}$ is nontrivial, it signifies the presence of a degeneracy within the spectrum. Notice that the anomaly phase coincides with the phase in Eq. (4.60) if we set $q=N$ in the latter. This should not surprise us since, in this section, we employ the full $\mathbb{Z}_N$ center symmetry, thanks to gauging $U(1)$. The anomaly in (4.82) is valued in $\mathbb{Z}_{N/\gcd(N,p^2l_x)}$ (we take $n_x=n_y=n_z$) indicating a $N/\text{gcd}(N, p^2 l_x)$-fold degeneracy. The Hilbert space of physical states, which are labeled by N/p distinct electric fluxes, sit in $N/\text{gcd}(N, p^2 l_x)$ orbits, and a rotation by $\tilde{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ maps a state with an electric flux pe_j to a state with a flux $p(e_j + \text{gcd}(N, p^2 l_j)/p)$, i.e., they have the same energy.

4.4.3 Examples

4.4.3.1 $SU(4k+2) \times U(1)/\mathbb{Z}_p$ with 2-index antisymmetric fermions

$SU(4k+2) \times U(1)$ theory with a single 2-index anti-symmetric Dirac fermion was considered in [97]. Here, we study this theory when we gauge a $\mathbb{Z}_p \subseteq \mathbb{Z}_N$ subgroup of the center. Numerical scans reveal that condition (4.81) is always satisfied for specific values of n_x and N_x. Also, the anomaly (4.82) is trivial unless both p and l_x are odd; then, the anomaly is valued in $\mathbb{Z}_2$. The Hilbert space is spanned by the physical states

$$
\begin{aligned}
&|\psi\rangle_{\text{phy}\,,\boldsymbol{m}} = |E, p\boldsymbol{e}, \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}} \\
&\qquad e_j = 0, 1, .., (4k+2)/p - 1\,, \quad N_j \in \mathbb{Z}\,, \quad n_j = 0, 1, .., p-1\,, \quad j = 1, 2, 3
\end{aligned}
$$

and the anomaly means that the states live in two orbits such that $|E, p\boldsymbol{e}, \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}$, $|E, p(\boldsymbol{e} + \text{gcd}(N, p^2\boldsymbol{l})/p), \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}$, $|E, p(\boldsymbol{e} + 2\text{gcd}(N, p^2\boldsymbol{l})/p), \boldsymbol{n}/p + \boldsymbol{N}\rangle_{\boldsymbol{m}}$, etc. have the same energy (we take $n_x = n_y = n_z$).

4.4.3.2 The Standard Model

The methods presented in this chapter provide a systematic means to find noninvertible symmetries in any given gauge theory. As an important application, we employ our approach to search for noninvertible symmetries in the nongravitational sector of the Standard Model (SM). SM is based on $su(3) \times su(2) \times u(1)$ Lie algebra. Yet, the faithful gauge group, i.e., the global structure of the group, is to be uncovered. The matter content and charges under the gauge and global symmetries are displayed in Table 4.1, and all fermions are taken to be left-handed Weyls. The anomalies associated with the $U(1)_B$ and $U(1)_L$ symmetries are given by: $U(1)_B \times [SU(2)]^2 = U(1)_L \times [SU(2)]^2 = 1$, $U(1)_B \times [SU(3)]^2 = U(1)_L \times [SU(3)]^2 = 0$, $U(1)_B \times [U(1)]^2 = U(1)_L \times [U(1)]^2 = -18$. Thus, we see that $U(1)_{B-L}$ symmetry is anomaly-free symmetry (we neglect gravity in this context). Under a $U(1)_{B+L}$ rotation, the path integral picks up an ABJ phase

$$
\exp\left(i\alpha \cdot N_f(2c_2(F) - 36c_2(f))\right)\,, \tag{4.83}
$$

where N_f is the number of families, $c_2(F)$ is the second Chern class for $SU(2)$ and $c_2(f)$ is the second Chern class for $U(1)$. The ABJ anomaly breaks the $U(1)_{B+L}$ down to a $\mathbb{Z}^{B+L}_{\text{gcd}(2,36)N_f} = \mathbb{Z}^{B+L}_{2N_f}$ symmetry. Notice that $SU(3)$ does not play a role in the ABJ anomaly.

field	$SU(3)$	$SU(2)$	$U(1)$	$U(1)_B$	$U(1)_L$
q_L	$\square$	$\square$	1	$\frac{1}{3}$	0
l_L	$\mathbf{1}$	$\square$	-3	0	1
$\tilde{e}_R$	$\mathbf{1}$	$\mathbf{1}$	6	0	-1
$\tilde{u}_R$	$\overline{\square}$	$\mathbf{1}$	-4	$-\frac{1}{3}$	0
$\tilde{d}_R$	$\overline{\square}$	$\mathbf{1}$	2	$-\frac{1}{3}$	0
h	$\mathbf{1}$	$\square$	3	0	0

Table 4.1: Matter content and charges of SM: q_L and l_L are the quark and lepton doublets, $\tilde{e}_R, \tilde{u}_R, \tilde{d}_R$ are the electron and up and down quarks singlets, while h is the Higgs doublet. Notice that we take the hyper $U(1)$ charges to be integers, while the matter content has the standard charges under the baryon number $U(1)_B$ and lepton number $U(1)_L$ symmetries.

The matter content is consistent with the existence of an electric $\mathbb{Z}_6^{(1)}$ 1-form global symmetry [107, 108]. The cocycle conditions satisfied by SM on $\mathbb{T}^4$ with a gauged $\mathbb{Z}_6^{(1)}$ are given by [108]:

$$\begin{aligned}
\Omega_{(3)\mu}(x^\nu = L^\nu)\Omega_{(3)\nu}(x^\mu = 0) &= e^{2\pi i \frac{n^{(3)}_{\mu\nu}}{3}} \Omega_{(3)\nu}(x^\mu = L^\mu)\Omega_{(3)\mu}(x^\nu = 0)\,, \\
\Omega_{(2)\mu}(x^\nu = L^\nu)\Omega_{(2)\nu}(x^\mu = 0) &= e^{2\pi i \frac{n^{(2)}_{\mu\nu}}{2}} \Omega_{(2)\nu}(x^\mu = L^\mu)\Omega_{(2)\mu}(x^\nu = 0)\,, \\
\omega_{(1)\mu}(x^\nu = L^\nu)\omega_{(1)\nu}(x^\mu = 0) &= e^{-2\pi i \left(\frac{n^{(3)}_{\mu\nu}}{3} + \frac{n^{(2)}_{\mu\nu}}{2}\right)} \omega_{(1)\nu}(x^\mu = L^\mu)\omega_{(1)\mu}(x^\nu = 0)\,.
\end{aligned} \tag{4.84}$$

$\Omega_{(i)}$, $i = 2, 3$, and $\omega_{(1)}$ are the transition functions of the gauge bundles, $n^{(i)}_{\mu\nu}$ are the 't Hooft twists, and the superscript/subscript $(i) = (3), (2), (1)$ denote the condition for the $SU(3), SU(2), U(1)$ gauge groups respectively. The electric $\mathbb{Z}_6^{(1)}$ symmetry is generated by a combinations of the $SU(3)$ center, $T_j^{(3)}$, the $SU(2)$ center, $T_j^{(2)}$, and the $U(1)$ center t_j, such that the full $\mathbb{Z}_6^{(1)}$ symmetry generator is given by $T_j^{(3)} T_j^{(2)} t_j$, $j = x, y, z$.

The anomalous $U(1)_{B+L}$ current conservation law is given by

$$\partial_\mu j^\mu_{B+L} - 2N_f \partial_\mu K^\mu_{SU(2)}(a^c) + \frac{36 N_f}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma} \partial^\mu a^\nu \partial^\lambda a^\sigma = 0\,, \tag{4.85}$$

where $K^\mu_{SU(2)}$ is the $SU(2)$ topological current. The corresponding unbroken $\mathbb{Z}^{B+L}_{2N_f}$ symmetry operator on $\mathbb{T}^3$ is given by:

$$U_{\mathbb{Z}_{2N_f}, \ell} = \exp\left[i \frac{2\pi \ell}{2N_f} Q_5 \right]\,, \tag{4.86}$$

where the conserved charge Q_5 is given by (here we turn on a $\mathbb{Z}_6$ magnetic twist)

$$\begin{aligned} Q_5 &= \int_{\mathbb{T}^3} d^3x \left[j^0_{B+L} - 2N_f K^0_{SU(2)}(a^c) + \frac{36N_f}{8\pi^2}\epsilon^{ijk} a_i \partial_j a_k \right] \\ & -\frac{18N_f}{4\pi}(N_z + \frac{1}{6}n_z) \left[\int_0^{L_y} \frac{dy}{L_y} \int_0^{L_z} dz a_z(x=0,y,z) + \int_0^{L_x} \frac{dx}{L_x} \int_0^{L_z} dz a_z(x,y=0,z) \right] \\ & + \sum_{\text{cyclic}} (x \to y \to z \to x) \, . \end{aligned} \tag{4.87}$$

Under a $U(1)$ gauge transformation, $U_{\mathbb{Z}_{2N_f},\ell}$ transforms as

$$U_{\mathbb{Z}_{2N_f},\ell} \longrightarrow U_{\mathbb{Z}_{2N_f},\ell} e^{-i2\pi\left(\frac{18\ell N_f}{N_f}\left(N_x + \frac{n_x}{6}\right)\right) + (x\to y) + (x \to z)} = U_{\mathbb{Z}_{2N_f},\ell} \, . \tag{4.88}$$

Therefore, $U_{\mathbb{Z}_{2N_f},\ell}$ is $U(1)$ gauge invariant, as required. Further, we examine $U_{\mathbb{Z}_{2N_f},\ell}$ after gauging the electric $\mathbb{Z}_6^{(1)}$ 1-form center by sandwiching $U_{\mathbb{Z}_{2N_f},\ell}$ between its generators (this is a generalization of Eq. (4.75)):

$$\begin{aligned} T_x^{(3)} T_x^{(2)} t_x U_{\mathbb{Z}_{2N_f},\ell} \left(T_x^{(3)} T_x^{(2)} t_x\right)^{-1} &= \underbrace{e^{-i\frac{2\pi\ell(2N_f)}{2N_f}\frac{n_x^{(2)}}{2}}}_{\text{from } K^0_{SU(2)}(a^c)} \underbrace{e^{i2\pi\ell\frac{36N_f}{2N_f}\left(\frac{1}{6}\right)\left(N_x + \frac{n_x^{(2)}}{2} + \frac{n_x^{(3)}}{3}\right)}}_{\text{from } \epsilon^{ijk}a_i\partial_j a_k} U_{\mathbb{Z}_{2N_f},\ell} \\ &= U_{\mathbb{Z}_{2N_f},\ell} \, . \end{aligned} \tag{4.89}$$

We used Eq. (4.30), setting $k_x = m_x = 1$, to find the first exponent. The second exponent is found by applying Eq. (4.41) and using $n = 1$, $N = 6$. Here, $n_x^{(2)}$, $n_x^{(3)}$, and N_x are the $SU(2)$ and $SU(3)$ fractional twists and $U(1)$ integral magnetic flux, respectively. This analysis shows that SM does not possess noninvertible symmetries in its nongravitational sectors. Our findings are consistent with [93].

4.5 Coupling gauge theories to axions and noninvertible symmetries

In this section, we introduce axions into the game, taking $\mathbb{T}^4$ to be larger than any scale in the theory. To be specific, we take $SU(N)/\mathbb{Z}_p$ or $SU(N) \times U(1)/\mathbb{Z}_p$ gauge theories of the previous sections and follow the setup of [17] by adding a complex scalar Φ that is neutral under the gauge groups but couples to the fermions. Thus, we add the following terms to the Lagrangian:

$$\mathcal{L} \supset |\partial_\mu \Phi|^2 + V(\Phi) + y\Phi\tilde{\psi}\psi + \text{h.c.}\,, \tag{4.90}$$

where $\psi, \tilde{\psi}$ are two left-handed Weyl fermions in representations $\mathcal{R}$ and its complex conjugate $\bar{\mathcal{R}}$, respectively, and y is a Yukawa coupling. The potential of the complex field is $V(\Phi) = \lambda(|\Phi|^2 - v^2)^2$, where λ is $\mathcal{O}(1)$ dimensionless parameter. We take the scalar field v.e.v. $v \gg \Lambda$, where Λ the strong scale of the gauge sector. We shall pretend that we did not know about the noninvertible symmetries or how to construct them, and let us see if we can reproduce them in the IR.

Let us first consider the $SU(N)$ gauge theory before gauging $U(1)$ and the electric $\mathbb{Z}_p^{(1)}$ symmetry. Under $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ and $U(1)$ baryon number, the different fields transform as

$$\begin{aligned} \mathbb{Z}^{\chi}_{2T_{\mathcal{R}}} \quad &: \quad \Phi \longrightarrow e^{i\frac{-2\pi}{T_{\mathcal{R}}}}\Phi\,, \psi \longrightarrow e^{i\frac{2\pi}{2T_{\mathcal{R}}}}\psi\,, \tilde{\psi} \longrightarrow e^{i\frac{2\pi}{2T_{\mathcal{R}}}}\tilde{\psi}\,, \\ U(1) \quad &: \quad \Phi \longrightarrow \Phi\,, \psi \longrightarrow e^{i\alpha}\psi\,, \tilde{\psi} \longrightarrow e^{-i\alpha}\tilde{\psi}\,, \alpha \in [0, 2\pi)\,. \end{aligned} \tag{4.91}$$

If we write Φ as $\Phi = \rho e^{ia}$, where a is the axion *, then a transforms under $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ as

$$a \longrightarrow a - \frac{2\pi}{T_{\mathcal{R}}} \tag{4.92}$$

and notice that the axion is inert under the $\mathbb{Z}_2^F$ fermion number subgroup of $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$.

Next, we consider $SU(N)/\mathbb{Z}_p$ or $SU(N) \times U(1)/\mathbb{Z}_p$ gauge theories with axions. Flowing to an energy scale below v, the radial degree of freedom ρ freezes in, i.e., we set $\rho = v$, and the fermions acquire a mass $\sim yv$ and decouple. What remains is the light degree of freedom, the axion a. However, the axion should reproduce all the UV anomalies. Thus, we can write the following IR effective Lagrangian of a:

$$\mathcal{L}_a = v^2 \left(\partial_\mu a\right)^2 + T_{\mathcal{R}} a \frac{\mathrm{tr}(f^c \wedge f^c)}{8\pi^2} + d_{\mathcal{R}} a \frac{f \wedge f}{8\pi^2}\,. \tag{4.93}$$

Variation of $\mathcal{L}_a$ w.r.t a produces the anomalous current conservation law:

$$\partial_\mu j^\mu_{(a)} - T_{\mathcal{R}} \partial_\mu K^\mu(a^c) - \frac{d_{\mathcal{R}}}{8\pi^2} \epsilon_{\mu\nu\lambda\sigma} \partial^\mu a^\nu \partial^\lambda a^\sigma = 0\,, \tag{4.94}$$

where $j^\mu_{(a)} = v^2 \partial^\mu a$. This is exactly the anomalous current conservation law we had previously, now written down for the axion current. Therefore, everything we said in the previous sections applies here. In particular, we can define an operator of the $\mathbb{Z}^{\chi}_{2T_{\mathcal{R}}}$ symmetry as:

$$\mathcal{U}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell} = \exp\left[i \frac{2\pi\ell}{T_{\mathcal{R}}} \int_{\mathbb{T}^3} (j^0_{(a)} - T_{\mathcal{R}} K^0(a^c)) + ...\right]\,, \tag{4.95}$$

where the dots denote the contribution from the $U(1)$ gauge field (see Eq. (4.40)). We used a calligraphic letter for the operator to emphasize that it is constructed in the IR. Yet, all the anomalies and failure of invariance under gauge symmetries that lead to the noninvertibility of the UV operators apply here as well. Thus, similar to what we did before, we can construct the noninvertible operator $\tilde{\mathcal{U}}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell}$,

*a without any indices represents the axion, while a_μ with an index represents the $U(1)$ gauge field. Apologies for the confusing notation.

which implements the noninvertible symmetry $\tilde{\mathbb{Z}}^{\chi}_{2T_{\mathcal{R}}}$ in the IR. Such operators shall project onto magnetic sectors and also exhibit mixed anomalies with the global 1-form electric center symmetry, exactly as we discussed previously.

It was pointed out in [109] that $SU(N)/\mathbb{Z}_p$ theories with axions have noninvertible symmetries. However, our construction shows that such a conclusion is not general and depends on the UV completion. Consider two theories $SU(4k)/\mathbb{Z}_2$ and $SU(4k+2)/\mathbb{Z}_2$ with a Dirac fermion in the 2-index antisymmetric representation and coupled to a complex scalar field Φ as above. As we flow to the IR, we can construct the operators corresponding to the chiral symmetries. We discussed in Section 4.3.3.1 that $SU(4k)/\mathbb{Z}_2$ theory does not exhibit an anomaly between its chiral symmetry and the 1-form symmetry of the corresponding $SU(4k)$ theory, and hence, the chiral symmetry operator is invertible. Therefore, an axion domain wall (DW), implemented by the action of $\tilde{\mathcal{U}}_{\mathbb{Z}_{8k-4},\ell}$, will not be dressed with TQFT degrees of freedom. On the contrary, $SU(4k+2)/\mathbb{Z}_2$ exhibits an anomaly between its chiral symmetry and the 1-form center of the corresponding $SU(4k+2)$ theory, and thus, the minimal chiral symmetry operator $\tilde{\mathcal{U}}_{\mathbb{Z}_{8k},\ell=1}$ is noninvertible. The axion DW implemented by $\tilde{\mathcal{U}}_{\mathbb{Z}_{8k},\ell=1}$ must be dressed with a fractional quantum Hall TQFT.

We may also consider axions in $SU(N)\times U(1)/\mathbb{Z}_p$ theory of Section 4.4. Everything we said there is transcendent to the IR axion domain walls. In particular, for $p=1$, the operator $\tilde{\mathcal{U}}_{\mathbb{Z}_{2T_{\mathcal{R}}},\ell=1}$ annihilates the Hilbert space sectors characterized by vanishing fractional $\boldsymbol{n}=0$ and integral magnetic fluxes $\boldsymbol{N}\notin T_{\mathcal{R}}\mathbb{Z}^3/\gcd(T_{\mathcal{R}},d_{\mathcal{R}})$. It will be interesting to examine what happens to the axion domain walls of this theory as we place them in such an external magnetic field.

Chapter **5**

Global aspects of 3-form gauge theory: implications for axion-Yang-Mills systems

5.1 Introduction

As we have seen in the last chapter, axion theories come with interesting generalized symmetries. They are an elegant solution to the Strong CP problem and a popular dark matter candidate. It would be desirable to further understand axions and their properties. The natural question to ask is: what can generalized symmetries tell us about axions? In this chapter, we will study the relationship between axions and higher-form gauge theories, in particular a three-form gauge theory.

Three-form gauge theory is a fascinating topic that attracted attention since Lüscher's seminal work [110]. There, it was argued that the non-trivial topology of the 4-D Yang-Mills theory shows up in the infrared as an abelian long-range 3-form gauge field c_3. It does not correspond to a physical massless particle, i.e., a propagating degree of freedom. Nevertheless, it contributes to the theory's vacuum energy, i.e., cosmological constant. In this formulation, the CP-violating θ term can be written as

$$\theta \int_{\mathcal{M}_4} dc_3 \,, \tag{5.1}$$

where c_3 is given in terms of the nonabelian Chern-Simons current density: $c_3 = \mathrm{tr}\left[\frac{1}{3}\left(a_1^{\mathrm{c}}\right)^3 + a_1^{\mathrm{c}} \wedge da_1^{\mathrm{c}}\right]/(8\pi^2)$, a_1^{c} is the color field, and $\mathcal{M}_4$ is the 4-D manifold. The correlator of the derivatives of two Chern-Simons current densities is the topological susceptibility χ of Yang-Mills theory, which develops a pole as the momentum vanishes, the Kogut-Susskind pole [111], corresponding to a pole of the c_3 correlator. This is also known as the Veneziano ghost [112] because the pole appears with the opposite sign compared to those of conventional particles, and it provides an alternative means (to Yang-Mills instantons) to solve the axial $U(1)$ problem. One can write down an effective action that reproduces these findings [113]:

$$S_{\text{IR}} = \frac{1}{2\chi}\int_{\mathcal{M}_4} |dc_3|^2 + \theta\int_{\mathcal{M}_4} dc_3\,, \tag{5.2}$$

where a kinetic-energy term of c_3 is added to give the correct long-range interaction form of the c_3 correlator. The effective action (5.2) yields the vacuum energy [114]:

$$E_0(\theta) = \frac{\chi}{2}\text{min}_k\,(\theta + 2\pi k)^2\;, \quad k \in \mathbb{Z}\,. \tag{5.3}$$

This multi-branch function is periodic in θ and develops cusps at $\theta = \pi, 3\pi, ...$, a result derived by Witten in the large-N limit [115]. While this renowned result lends support to the validity of (5.2), it is important to emphasize that the effective description (5.2) is strictly derived in the large-N limit. See [114] for a review and [116, 117, 118, 119, 120, 121] for further works on 3-form gauge theory, including supersymmetric versions.

When massless quarks are introduced, the θ term can be rotated away, and the theory restores its CP invariance, thus solving the strong CP problem. A similar effect can be achieved by introducing an axion a through the Peccei–Quinn mechanism [122]. This approach employs an anomalous global $U(1)$ symmetry along with a complex scalar field whose phase corresponds to the axion. The complex scalar undergoes spontaneous symmetry breaking at a scale v, which is much higher than the strong-coupling scale Λ. Below Λ, Yang-Mills instantons generate an effective potential for $a + \theta$, which is minimized at a point in the field space restoring the CP symmetry. The same conclusion can be reached using the 3-form gauge theory, as demonstrated by Dvali in [123, 124]. In this framework, c_3 undergoes Higgsing when it absorbs the axion, resulting in c_3 becoming a short-range field. This mechanism can be elegantly seen in the Kalb-Ramond frame, where the axion is dualized to a 2-form gauge theory. This process effectively eliminates the second term in (5.2) and restores the CP invariance.

Since c_3 does not carry a physical degree of freedom, it is reasonable to question whether the 3-form gauge theory is essential for formulating Yang-Mills theory in the deep IR (without or with axions) or if it is merely a redundant description lacking true physical significance. This study offers a new perspective on the validity of the 3-form gauge theory beyond the large-N limit. Under certain conditions, we argue that the 3-form gauge theory is a faithful IR effective description in axion-Yang-Mills systems. This is achieved by examining the global symmetries of such systems as well as certain types of 't Hooft anomaly-matching conditions.

Our understanding of symmetries has undergone a conceptual paradigm shift over the past decade [4]; see [10] for a review and [125, 27, 81, 126, 127] for examples of works that discussed the 3-form symmetries from a modern perspective. In the contemporary paradigm, a p-form global symmetry in 4-D acts on a p-dimensional object and is generated by operators (symmetry defects) living on $(3-p)$-dimensional topological manifolds. Gauging a global symmetry is performed by introducing a background field of the symmetry and including an arbitrary sum over inequivalent classes of this background in the path integral. Moreover, it has been realized recently that the concept of symmetry can be extended to include operations that lack inversions; see, e.g., [128, 23, 22, 83, 76, 97] and the reviews [11, 96]. Our primary goal is to conduct a systematic study of the 3-form gauge theory and apply it to Yang-Mills theory coupled to matter fields and axions, with a focus on its global aspects. By utilizing newly developed mathematical tools, we can thoroughly examine the 3-form effective field theory of axion-Yang-Mills systems. This effective description successfully passes several consistency checks.

This chapter is divided into two main parts. The first part discusses the symmetry aspects of a general low-energy 3-form gauge theory coupled to axion, including the multi-field case. Starting from a low-energy Lagrangian exhibiting a $U(1)^{(2)}$ 2-form along with $U(1)^{(0)}$ 0-form global symmetries, we construct a 3-form gauge theory by gauging the former symmetry. The resulting theory is a topological quantum field theory (TQFT) of the BF type, modified by marginal and irrelevant operators, and describes a set of q domain walls (q is a free parameter) separating q distinct vacua and forming a junction at the locus of an axion string. Generally, the gauge theory exhibits $\mathbb{Z}_q^{(0)}$ 0-form and $\mathbb{Z}_q^{(3)}$ 3-form global symmetries, with a mixed anomaly between them valued in $\mathbb{Z}_q$. This anomaly is matched by breaking the two participating symmetries, leading to q distinct vacua separated by domain walls. Furthermore, we point out that the theory encompasses a gauged $U(1)^{(-1)}$ (-1)-form symmetry, which undergoes spontaneous breaking (Higgsing), signifying that the vacuum energy has no contribution coming from c_3. We also construct the symmetry defects associated with these symmetries in two dual frames: the axion and the Kalb-Ramond frames. We demonstrate the action of such symmetries within an example of a domain-wall system. When the discrete 3-form global symmetry or a subgroup thereof, $\mathbb{Z}_p^{(3)} \subseteq \mathbb{Z}_q^{(3)}$, is gauged, we are left with only q/p distinct vacua.

In the second part of this chapter, we apply this formalism to $SU(N)$ Yang-Mills theory endowed with a single massless Dirac fermion in a representation $\mathcal{R}$ coupled to a neutral complex scalar field. In the UV, this theory exhibits both a $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ 0-form chiral and $\mathbb{Z}_m^{(1)}$ 1-form center symmetries, with a possible mixed anomaly between the two symmetries. Here, $T_\mathcal{R}$ is the Dynkin index of $\mathcal{R}$ and $m = \gcd(N, n)$, where n is the N-ality of $\mathcal{R}$. When matched below the strong-coupling scale, this non-vanishing mixed anomaly necessitates introducing a dynamical c_3. We summarize the idea here, with details provided in the main body of the chapter.

Let v and Λ be the complex scalar vev and the Yang-Mills strong-coupling scales, respectively, and we take $\Lambda \ll v$. To see the $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)} - \mathbb{Z}_m^{(1)}$ mixed anomaly, we turn on a 2-form background field of $\mathbb{Z}_m^{(1)}$. This is achieved, as in [14, 15, 16, 17], by first introducing a pair of 1-form and 2-form $U(1)$ gauge fields (B_1^c, B_2^c) along with the constraint $mB_2^c = dB_1^c$. The field strength of the 1-form field satisfies the quantization condition $\int_{\mathcal{M}_2} dB_1^c \in 2\pi\mathbb{Z}$. Then the constraint implies that B_2^c has a vanishing field strength $dB_2^c = 0$, while its holonomy is fractional $\int_{\mathcal{M}_2} B_2^c \in \frac{2\pi\mathbb{Z}}{m}$. To couple the background field to fermions, in the second step, we enlarge $SU(N)$ to $U(N)$ and embed the $\mathbb{Z}_m^{(1)}$ background field into the $U(1)^{(1)}$ 1-form symmetry of $U(N)$ gauge theory. However, we must ensure that the enlargement from $SU(N)$ to $U(N)$ does not introduce new degrees of freedom, which can be done by postulating that the theory is invariant under an auxiliary 1-form gauge symmetry that acts simultaneously on the $U(N)$ and (B_1^c, B_2^c) fields. As we flow to energy scales $\Lambda \ll E \ll v$, the magnitude of the complex scalar field freezes at v, while the winding of its phase (the axion) leads to the emergence of a $U(1)^{(2)}$ 2-form symmetry that couples to the axion strings. Below Λ, the strong dynamics set in, leading to the confinement of the color field. Here, one is faced with a puzzle: the confinement of the color field means that one should no longer incorporate the nonabelian field in the calculations. If true, the low-energy theory is no longer invariant under the postulated auxiliary 1-form gauge symmetry, indicating that something is missing. The way out is to introduce the dynamical 3-form gauge field c_3 of $U(1)^{(2)}$. The latter transforms non-trivially under the auxiliary gauge symmetry, ensuring the full low-energy effective field theory (in the background of the $\mathbb{Z}_m^{(1)}$ flux) is invariant under this auxiliary symmetry. Moreover, the IR theory reproduces the mixed anomaly, which is an important check on the analysis since anomalies are all-scale phenomena. In this regard, c_3 can be thought of as the long-range tail of the nonabelian dynamics. Even though it does not carry a physical degree of freedom, its presence is essential for the consistency of the theory deep in the IR.

Below the scale, v, the fermions become massive, with a mass of order v, and decouple, leading to the enhancement of $\mathbb{Z}_m^{(1)}$ to $\mathbb{Z}_N^{(1)}$ 1-form symmetry. The groups $U(1)^{(2)}$ and $\mathbb{Z}_N^{(1)}$ constitute a higher-group structure, where the former is the parent and the latter is the daughter symmetries. One may gauge the parent without gauging the daughter, but not conversely. As we flow below Λ, we may freely gauge $U(1)^{(2)}$ and introduce the dynamical c_3 without worrying about $\mathbb{Z}_N^{(1)}$. The latter stays an enhanced symmetry below Λ.

One of our main results is the IR effective field theory at energy scale $E \ll \Lambda$ given by Eq. (5.114), which we display here for convenience:

$$\mathcal{L}_{E\ll\Lambda} = \frac{v^2}{2} da \wedge \star da + \frac{T_{\mathcal{R}} a}{2\pi}\left(dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c\right) + \Lambda^4 \mathcal{K}\left(\frac{dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c}{\Lambda^4}\right), \tag{5.4}$$

where $\mathcal{K}$ is the kinetic energy term of c_3, and the background of $\mathbb{Z}_m^{(1)}$ is activated. This theory exhibits $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)} \times \mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ global symmetries. Dynamically, the IR theory forms axion domain walls, separating $T_{\mathcal{R}}$ distinct minima and breaking $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)}$ and $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ maximally. The enhanced $\mathbb{Z}_N^{(1)}$ symmetry is explicitly broken by higher-order operators down to the genuine $\mathbb{Z}_m^{(1)}$ symmetry, which remains intact.

The field strength of the 3-form gauge field satisfies the quantization condition $\int_{\mathcal{M}_4} dc_3 = 2\pi m$, where m is an integer equivalent to the topological charge of the Yang-Mills instantons. The full partition function of the IR theory (at energy scale $E \ll \Lambda$) includes a sum over all integers m. We may integrate out c_3, and using the Poisson resummation formula, we obtain the Euclidean partition function:

$$Z[a] \sim \sum_{k\in\mathbb{Z}} \exp\left[-i\frac{kN}{4\pi}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c\right] \exp\left[-\int_{\mathcal{M}_4} \frac{v^2}{2} da \wedge \star da + \frac{\Lambda^4}{8\pi^2}\left(T_{\mathcal{R}}a + 2\pi k\right)^2\right] . \tag{5.5}$$

This partition function reproduces the chiral-center anomaly upon shifting $a \to a + \frac{2\pi}{T_{\mathcal{R}}}$. It also displays an infinite number of vacua, with the true vacuum energy given by

$$V(a) \sim \Lambda^4 \text{min}_k \left(T_{\mathcal{R}}a + 2\pi k\right)^2 . \tag{5.6}$$

The potential $V(a)$ has $T_{\mathcal{R}}$ minima at $2\pi\ell/T_{\mathcal{R}}$, $\ell = 0, 1, ..., T_{\mathcal{R}} - 1$, as well as cusps at $a = \pi(2\ell + 1)/T_{\mathcal{R}}$, reflecting two facts. First, the cusps indicate that additional degrees of freedom, not accounted for by $V(a)$, are sandwiched between the true minima of the theory. These are the hadronic walls, which are very thin compared to the thickness of the axion domain walls [129, 130]. Second, a restructuring in the hadronic sector occurs as one goes between one minimum and the other. These results are consistent with the large-N limit (5.3).

There exists a higher group structure between $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ and the enhanced $\mathbb{Z}_N^{(1)}$ symmetries. However, this structure trivialises for the genuine $\mathbb{Z}_m^{(1)} \subset \mathbb{Z}_N^{(1)}$ symmetry. This means we can gauge $\mathbb{Z}_m^{(1)}$ without worrying about $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$. Gauging the former gives $SU(N)/\mathbb{Z}_m$ theory. This theory still exhibits a spontaneously broken IR $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ symmetry due to the formation of domain walls. However, the chiral symmetry $\mathbb{Z}_{T_{\mathcal{R}}}^{(0)}$ becomes noninvertible. This results in dressing the domain walls with an IR TQFT. This intricate structure works as a consistency check on the adequacy of using the 3-form gauge theory to describe the axion-Yang-Mills systems' IR physics.

This chapter is organized as follows. In Section 5.2, we set the stage by considering the field theory of a compact scalar, which possesses two global symmetries: shift and winding symmetries. The theory encounters a mixed 't Hooft anomaly, and thus, the shift symmetry breaks into a discrete group upon gauging the winding symmetry. Next, we couple the gauge field of the winding symmetry, the 3-form gauge field, to the compact scalar and analyze the resulting theory in great detail: we identify the global symmetries, their mixed anomalies, and the noninvertible symmetries within this theory. Section 5.3 is devoted to studying the compact scalar in the dual frame, the Kalb-Ramond gauge theory, while Section 5.4 generalizes these findings to two or more 3-form gauge fields. In Section 5.5, we use the machinery built in the previous sections to examine our proposal that the deep IR regime of the axion-Yang-Mills systems is described by a 3-form gauge theory and apply various checks to this proposal. Finally, we conclude in Section 5.6 with a brief discussion.

5.2 The Axion theory

We consider the 4-D theory of a 2π-periodic scalar field a, the axion, i.e., we identify $a(\mathcal{P}) \equiv a(\mathcal{P}) + 2\pi\mathbb{Z}$ at the spacetime point $\mathcal{P}$. The basic Lagrangian is

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da \,, \tag{5.7}$$

where v is a constant of mass dimension 1. When coupling the axion to a Yang-Mills theory via the Peccei-Quinn mechanism, v is the axion's symmetry-breaking scale. The Lagrangian (5.7) has a global $U(1)^{(0)}$ 0-form shift symmetry acting on the axion as $a \to a+\alpha$, where α is a constant. The corresponding Noether's 1-form current is

$$j_1 = v^2 da \,, \tag{5.8}$$

which is conserved thanks to the equation of motion:

$$d \star j_1^{(0)} = v^2 d \star da = 0 \,. \tag{5.9}$$

The topological symmetry generator (symmetry defect) enacting this transformation is defined on a closed co-dimension-1 manifold $\mathcal{M}_3$:

$$U_\alpha^{(0)}(\mathcal{M}_3) = e^{i\alpha \int_{\mathcal{M}_3} v^2 \star da} \,. \tag{5.10}$$

The superscript emphasizes that the operator implements the action of a 0-form symmetry. We can take $U_\alpha^{(0)}(\mathcal{M}_3)$ to surround the local operator

$$V(\mathcal{M}_0 = \mathcal{P}) = e^{ia(\mathcal{P})} \,, \tag{5.11}$$

and then topologically deforming $U_\alpha^{(0)}(\mathcal{M}_3)$ past V to find

$$U_\alpha^{(0)}(\mathcal{M}_3) V(\mathcal{M}_0) = e^{i\alpha} V(\mathcal{M}_0) \,. \tag{5.12}$$

The axion theory is also endowed with a $U(1)^{(2)}$ 2-form global symmetry with a corresponding 3-form current

$$j_3 = \star da \,, \tag{5.13}$$

which is conserved because of the Bianchi identity:

$$d \star j_3 = d^2 a = 0 \,. \tag{5.14}$$

We can also define the symmetry defect of the $U(1)^{(2)}$ 2-form symmetry by integrating the Hodge-dual of j_3 on a co-dimension-3 manifold $\mathcal{M}_1$ as:

$$U_\beta^{(2)}(\mathcal{M}_1) = e^{i\beta \int_{\mathcal{M}_1} da} \,. \tag{5.15}$$

This symmetry defect acts on the 2-dimensional axion-string worldsheet $\mathcal{M}_2$ [27]. Let $V(\mathcal{M}_2)$ be the axion-string Wilson surface, which has no local description* in terms of the axion field a. Deforming the symmetry defect $U_\beta^{(2)}(\mathcal{M}_1)$ past $V(\mathcal{M}_2)$ transforms the latter by a phase:

$$U_\beta^{(2)}(\mathcal{M}_1)V(\mathcal{M}_2) = e^{i\beta \mathrm{Link}(\mathcal{M}_1,\mathcal{M}_2)}V(\mathcal{M}_2)\,, \tag{5.16}$$

where $\mathrm{Link}(\mathcal{M}_1, \mathcal{M}_2)$ is the linking number between the two manifolds.

There is a mixed 't Hooft anomaly between $U(1)^{(0)}$ and $U(1)^{(2)}$ symmetries. To see it, we examine the commutation relation between the symmetry defects $U_\alpha^{(0)}(\mathcal{M}_3)$ and $U_\beta^{(2)}(\mathcal{M}_1)$. One way to perform the calculations is by foliating $\mathcal{M}_4$ into constant time slices and orienting both $\mathcal{M}_3$ and $\mathcal{M}_1$ to be time-like surfaces:

$$U_\alpha^{(0)}(\mathcal{M}_3(t)) = e^{i\alpha v^2 \int_{\mathcal{M}_3(t)} d^3x \partial_0 a(x,t)}\,, \quad U_\beta^{(2)}(\mathcal{M}_1(t)) = e^{i\beta \int_{\mathcal{M}_1(t)} \partial_i a dx^i}\,, \tag{5.17}$$

where $i \in \{1,2,3\}$. Then, using the equal-time commutation relation $[a(\boldsymbol{x},t), \Pi_a(\boldsymbol{y},t)] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, where $\Pi_a = v^2\partial_0 a$, and employing the Baker-Campbell-Hausdorff formula, we find

$$U_\alpha^{(0)}(\mathcal{M}_3(t))U_\beta^{(2)}(\mathcal{M}_1(t)) = e^{-i\alpha\beta}U_\beta^{(2)}(\mathcal{M}_1(t))U_\alpha^{(0)}(\mathcal{M}_3(t))\,. \tag{5.18}$$

The phase manifests the mixed anomaly: one cannot move the symmetry defects freely without encountering non-trivial phases. The anomaly implies that gauging one symmetry breaks the other into, at most, a discrete subgroup. †

5.2.1 Gauging the $U(1)^{(2)}$ 2-form symmetry: 3-form gauge theory and domain walls

Now, we gauge the $U(1)^{(2)}$ symmetry, meaning that we introduce the 3-form gauge field c_3 of the 2-form symmetry and perform the path integral over c_3. We couple c_3 to its current j_3 by adding the BF term

$$\frac{q}{2\pi} \star j_3 \wedge c_3 = \frac{q}{2\pi} da \wedge c_3 \tag{5.20}$$

to the Lagrangian (5.7). We introduced the positive integer $q \in \mathbb{N}$ as a free parameter of the theory, and its physical significance will be apparent below. The gauge

*It is, however, possible to define an operator $e^{i\int_{\Sigma_3} v^2 \star da}$ on an open surface Σ_3, with the string positioned at $\Sigma_2 = \partial\Sigma_3$, the boundary of Σ_3. This approach mirrors the implicit definition of 't Hooft lines in earlier formulations. A direct definition of the Wilson surface operator as an integral over a closed 2-dimensional surface will be given in Section 5.3 using the dual Kalb-Ramd field.

†The anomaly inflow action is given by

$$S_{\mathrm{inflow}} = \frac{i}{2\pi}\int_X dA_1 \wedge C_3, \tag{5.19}$$

where X is a five-dimensional manifold with the physical spacetime as the boundary, and A_1, C_3 are the background gauge fields for the $U(1)^{(0)}$ and $U(1)^{(2)}$ symmetries respectively [27].

field c_3 transforms as $c_3 \to c_3 + d\lambda_2$ under the $U(1)^{(2)}$ gauge transformation, and via integration by parts, we see that the new term (5.20) is invariant under this transformation. The field strength of c_3 is $f_4 = dc_3$, and it satisfies the quantization condition:

$$\int_{\mathcal{M}_4} f_4 \in 2\pi\mathbb{Z} \tag{5.21}$$

on a closed $\mathcal{M}_4$. The consistency of the theory under $U(1)^{(2)}$ large gauge transformations implies that $d\lambda_2$ satisfies the condition

$$\int_{\mathcal{M}_3} d\lambda_2 \in 2\pi\mathbb{Z}\,. \tag{5.22}$$

In addition, since c_3 is a dynamical field*, we can include a kinetic energy term $\mathcal{K}$ for c_3. The total Lagrangian is†:

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{q}{2\pi} da \wedge c_3 + \Lambda^4 \mathcal{K}\left(\frac{dc_3}{\Lambda^4}\right)\,, \tag{5.23}$$

and we introduced the new scale Λ. More on $\mathcal{K}$ will be discussed momentarily. The presence of the c_3 field reduces the $U(1)^{(0)}$ symmetry to a $\mathbb{Z}_q^{(0)} \subset U(1)^{(0)}$ symmetry; the Lagrangian (5.23) is only invariant under the shift $a \to a + 2\pi/q$. This demonstrates the earlier assertion that the mixed anomaly between the $U(1)^{(2)}$ and $U(1)^{(0)}$ symmetries leads to the latter being broken into a discrete subgroup when the former is gauged‡. The current conservation law (5.9) of the 0-form symmetry is modified to:

$$v^2 d \star da - \frac{q}{2\pi} dc_3 = 0\,. \tag{5.24}$$

The corresponding $\mathbb{Z}_q^{(0)}$ symmetry defect is topological only when we include this combination of fields - (5.10) is modified to

$$U_\alpha^{(0)}(\mathcal{M}_3) = e^{i\alpha \int_{\mathcal{M}_3}\left(v^2 \star da - \frac{q}{2\pi} c_3\right)}\,. \tag{5.25}$$

Using the quantization condition on $d\lambda_2$, i.e., $\int_{\mathcal{M}_3} d\lambda_2 \in 2\pi\mathbb{Z}$, we readily see that $U_\alpha^{(0)}(\mathcal{M}_3)$ is gauge-invariant under a $U(1)^{(2)}$ gauge transformation if and only if $\alpha = 2\pi\ell/q, \ell \in \mathbb{Z}$. This seconds the above assertion that introducing the term (5.20) reduces the $U(1)^{(0)}$ symmetry down to a $\mathbb{Z}_q^{(0)}$ subgroup.

Next, we focus on $\mathcal{K}$, the kinetic energy term of the 3-form gauge field. In the following, it will be helpful to write and analyze the Lagrangian (5.23) in index

*The 3-form gauge field has mass dimension 3.

†In a manifold with boundary, we also need to consider a boundary term so that the variation of the kinetic term at the boundary vanishes; see, e.g., [120] and references therein. We do not run into this subtlety in this analysis.

‡As we shall see, this mixed anomaly is the IR incarnation of the axial-color ABJ anomaly in an axion-Yang-Mills UV complete theory.

notation*:

$$\mathcal{L} = \frac{v^2}{2}\partial_\mu a\,\partial^\mu a - \frac{q}{2\pi}\frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu a)\,c_{\nu\rho\sigma} + \Lambda^4\mathcal{K}\,, \tag{5.26}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and the Greek indices run over $0,1,2,3$. The canonical (quadratic) form of the kinetic energy term is:

$$\mathcal{K}_{\text{can}} = -\frac{1}{2\cdot 4!\Lambda^8}f^{\mu\nu\alpha\beta}f_{\mu\nu\alpha\beta}\,, \tag{5.27}$$

where $f_{\mu\nu\alpha\beta} = \partial_\mu c_{\nu\alpha\beta} - \partial_\nu c_{\mu\alpha\beta} + \partial_\alpha c_{\mu\nu\beta} - \partial_\beta c_{\mu\nu\alpha}$. Since $f_{\mu\nu\alpha\beta}$ is totally anti-symmetric in the 4 indices, we can always write it as

$$f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}f(x)\,, \tag{5.28}$$

for some scalar function $f(x)$. Therfore, $\mathcal{K}_{\text{can}}$ takes the simple form†

$$\mathcal{K}_{\text{can}} = \frac{f^2(x)}{2\Lambda^8}\,. \tag{5.29}$$

The mathematical statement (5.28) is equivalent to saying that the free 3-form field c_3 does not carry propagating degrees of freedom. To see that, use the canonical kinetic term of c_3, ignore the axion in (5.26), and vary the Lagrangian with respect to $c_{\mu\nu\alpha}$ to find

$$\partial_\mu f^{\mu\nu\alpha\beta} = 0\,, \tag{5.30}$$

which admits the general solution $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}f$, where f, in this case, is a constant. The constant field strength of a 3-form gauge field carries no propagating degrees of freedom, much like free electrodynamics in 2-D. In the absence of a, the 4-form field f_4 is a cosmological constant, which is easily seen by substituting $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}f$ into the Lagrangian‡.

From the perspective of effective field theory, the kinetic energy term can take a more generalized form, with $\mathcal{K}$ represented as a polynomial in f:

$$\mathcal{K}\left(\frac{f}{\Lambda^4}\right) = \theta\frac{f}{\Lambda^4} + \frac{f^2}{2\Lambda^8} + c'\frac{f^4}{\Lambda^{16}} + \ldots\,, \tag{5.31}$$

where θ and c' are some real parameters.The first term is topological, and since $\int_{\mathcal{M}^4} f_4 = \int_{\mathcal{M}_4} d^4x f \in 2\pi\mathbb{Z}$, the theory is invariant under the shift $\theta \to \theta + 2\pi$,

*Translating from the d-forms to the index notation, it helps to remember that we are working in Minkowski space, with metric $\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$, such that $\star\star = -1$ for even forms and $\star\star = +1$ for odd forms.

†We used $\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\alpha\beta} = -4!$. We can also express f in terms of $f_{\mu\nu\alpha\beta}$ as $f = \frac{1}{4!}\epsilon^{\mu\nu\alpha\beta}f_{\mu\nu\alpha\beta}$.

‡However, a subtle issue arises because the cosmological constant obtained in this manner does not align with the correct value derived by varying (5.26) with respect to the metric tensor [121]. This discrepancy can be resolved by including boundary terms in (5.26). Nonetheless, to obtain the correct form of the energy-momentum tensor and, consequently, the cosmological constant, we will rely on the variation of the action with respect to the metric tensor, disregarding boundary terms.

and thus, θ is 2π-periodic. This term breaks the CP invariance unless $\theta = \{0, \pi\}$. However, θ can be rotated away by combining $\theta \frac{f}{\Lambda^4}$ with the second term in (5.26), after integrating by parts and shifting $qa \to qa - \theta$. Other possible higher-order kinetic energy terms may involve higher derivatives of f_4. Yet, these terms may be plagued with ghosts [121], and thus, we ignore such terms in our construction.

By rescaling the fields a and f as $a \to av$ and $f \to \Lambda f$, we observe that the Lagrangian (5.26) manifests as a BF theory deformed by marginal terms (the canonical kinetic terms of a and f) and by irrelevant operators (the higher-order terms of f). The full quantum theory guarantees that the system has q degenerate ground states To see that, we may want to find an effective axion potential $V_{\text{eff}}(a)$ by integrating out f_4 and imposing the constraint (5.21). To streamline the analysis, we proceed by disregarding the axion kinetic energy term in (5.23), which is a good approximation assuming $\Lambda \ll v$. Then, the Euclidean partition function reads:

$$Z[a] = \int [Dc_3] \sum_{m \in \mathbb{Z}} \delta\left(2\pi m - \int_{\mathcal{M}_4} f_4\right) e^{-S_E}\,, \tag{5.32}$$

and

$$S_E = -\int_{\mathcal{M}_4} \Lambda^4 \mathcal{K}\left(if_4/\Lambda^4\right) + i\frac{q}{2\pi} a \wedge f_4\,. \tag{5.33}$$

We can further simplify our analysis by recalling that f_4 does not carry propagating degrees of freedom and can be expressed by Eq. (5.28). We take $\mathcal{M}_4$ to be a closed manifold, and therefore, we have $\int_{\mathcal{M}_4} f_4 = \int dV_{\mathcal{M}_4} f(x) = f_0 V_{\mathcal{M}_4}$, where f_0 is the zero mode of $f(x)$ and $V_{\mathcal{M}_4}$ is the 4-volume of $\mathcal{M}_4$, and we assume $\Lambda^4 V_{\mathcal{M}_4} \gg 1$. Focusing only on the zero modes of f and a, we find

$$Z[a] \sim \left[\sum_{m \geq 0} e^{\Lambda^4 V_{\mathcal{M}_4} \mathcal{K}(2\pi i m)} \cos(mqa_0)\right] \times \text{higher modes of}\, a\,, \tag{5.34}$$

where a_0 is the zero mode of a. The higher modes of a are suppressed by inverse powers of $\Lambda^4 V_{\mathcal{M}_4}$ and can be neglected deep in the IR. The effective potential is defined via $V_{\text{eff}}(a) = -V_{\mathcal{M}_4}^{-1} \log Z[a]$, and thus, $V_{\text{eff}}(a_0) \sim \Lambda^4 \mathcal{F}(qa_0)$, where $\mathcal{F}$ is a periodic function with period $2\pi/q$. We conclude that integrating out f_4 yields a periodic potential for the axion that respects the $\mathbb{Z}_q^{(0)}$ shift symmetry, as it should be. Minimizing $V_{\text{eff}}(a_0)$ yields q-degenerate ground states connected via domian walls.

An alternative way to perform the path integral in (5.32) is to use the Poisson resummation formula $\sum_{m \in \mathbb{Z}} \delta\left(2\pi m - \int_{\mathcal{M}_4} f_4\right) = \sum_{k \in \mathbb{Z}} e^{-ik \int_{\mathcal{M}_4} f_4}$. Taking $\mathcal{K}$ in the canonical form $\mathcal{K}_{\text{can}} = f_4^2/(2\Lambda^2)$, focusing on the zero modes, and performing the Gaussian integral, we obtain

$$Z[a] \sim \sum_{k \in \mathbb{Z}} e^{-\frac{\Lambda^4 V_{\mathcal{M}_4}}{8\pi^2}(qa + 2\pi k)^2}\,, \tag{5.35}$$

which, again, is a periodic function that respects the $\mathbb{Z}_q^{(0)}$ shift symmetry. The result (5.35) is remarkable. This partition function displays an infinite number of vacua, most of which are false. The true vacuum energy is given by

$$V(a) \sim \Lambda^4 \text{min}_k \left(qa + 2\pi k\right)^2 . \tag{5.36}$$

This potential has q distinct minima, with cusps at $a = \pi/q$, $3\pi/q$, etc. These cusps appear only upon including the infinite sum over m in (5.34) and performing the Poisson resummation formula. In other words, the cusps are a feature of the full quantum theory. This observation will have far-reaching consequences in axion-Yang-Mills systems.

Let us return to the Lagrangian (5.26) and study its classical aspects. Varying it with respect to $c_{\nu\rho\sigma}$, assuming the general form of $\mathcal{K}$, yields the equation of motion of the 3-form field:

$$\frac{q}{2\pi}\partial_\mu a = -\Lambda^4 \partial_\mu \mathcal{K}'\left(\frac{f}{\Lambda^4}\right) , \tag{5.37}$$

where the prime denotes the derivative with respect to the argument of $\mathcal{K}$. Equation (5.37) is pivotal to our subsequent analysis. Integrating once, we obtain

$$\frac{q}{2\pi}(a - \tilde{a}_0) = -\Lambda^4 \mathcal{K}'\left(\frac{f}{\Lambda^4}\right) , \tag{5.38}$$

for some integration constant $\tilde{a}_0$. Assuming $\mathcal{K}'$ is invertible, we can rearrange the equation of f:

$$f = \Lambda^4 (\mathcal{K}')^{-1}\left(\frac{q}{2\pi\Lambda^4}(\tilde{a}_0 - a)\right) . \tag{5.39}$$

From (5.26), the equation of motion for the axion a is

$$v^2 \partial_\mu \partial^\mu a - \frac{q}{2\pi} f = 0 . \tag{5.40}$$

Here, f acts as the derivative of a classical effective potential for the axion field: in the presence of a classical effective potential for the axion, the equation of motion is $v^2\partial_\mu\partial^\mu a + \frac{\partial V_{\text{cl-eff}}(a)}{\partial a} = 0$. This means we can set $\frac{q}{2\pi} f = -\frac{\partial V_{\text{cl-eff}}(a)}{\partial a}$, and using (5.39), we conclude

$$\frac{\partial V_{\text{cl-eff}}(a)}{\partial a} = -\frac{q}{2\pi} f = -\frac{q}{2\pi}\Lambda^4 (\mathcal{K}')^{-1}\left(\frac{q}{2\pi\Lambda^4}(\tilde{a}_0 - a)\right) . \tag{5.41}$$

Then, one can integrate (5.41) to obtain an expression of $V_{\text{cl-eff}}(a)$. Unlike the effective potential obtained from the full partition function, the classical effective potential does not need to yield q degenerate ground states. The form of the classical effective potential depends on the kinetic energy term for c_3. To elucidate this point, we consider two examples of the resulting $V_{\text{cl-eff}}(a)$: the quadratic and the cosine potentials. The corresponding kinetic energy functions are [123]:

$$V^{\text{quadratic}}_{\text{cl-eff}}(a) \;=\; \frac{\Lambda^4}{2}(a - \tilde{a}_0)^2 \iff \mathcal{K}_{\text{can}}\left(\frac{f}{\Lambda^4}\right) = \frac{q^2 f^2}{2\Lambda^8} , \tag{5.42}$$

and

$$\begin{aligned} V^{\text{cos}}_{\text{cl-eff}}(a) &= \Lambda^4(1-\cos{(n(a-\tilde{a}_0))}) \iff \\ \mathcal{K}_{\text{cos}}\left(\frac{f}{\Lambda^4}\right) &= -1+\sqrt{1-\left(\frac{qf}{2\pi n\Lambda^4}\right)^2}+\frac{qf}{2\pi n\Lambda^4}\arcsin\left(\frac{qf}{2\pi n\Lambda^4}\right) . \end{aligned} \tag{5.43}$$

The integration constant in $\mathcal{K}$ is chosen such that $\mathcal{K}\left(\frac{f}{\Lambda^4}=0\right)$=0.

The kinetic energy term (5.43) is designed to produce $V^{\text{cos}}_{\text{cl-eff}}(a)$, and when $n \in q\mathbb{N}$, it exhibits multiple-of-q minima. These minima are located at values of a satisfying $\frac{\partial V^{\text{cos}}_{\text{cl-eff}}(a)}{\partial a}=0$, and from Eq.(5.41) we see that f vanishes there; the 3-form gauge field c_3 is gapped at these minima. Expanding $\mathcal{K}_{\text{cos}}\left(\frac{f}{\Lambda^4}\right)$ to the leading order in f about one of the minima results, up to a proportionality constant, in the canonical kinetic energy term (5.42). This, however, does not imply that the canonical kinetic energy term fails to produce q-degenerate ground states. As previously discussed, regardless of the form of $\mathcal{K}$, the full partition function of the Lagrangian (5.23), incorporating the quantization condition (5.21), will always lead to q-fold degeneracy in the deep IR regime. Nonetheless, $\mathcal{K}_{\text{cos}}\left(\frac{f}{\Lambda^4}\right)$ proves invaluable as it facilitates connections with textbook examples of axion domain walls. One can think of it as a UV completion of the canonical kinetic energy.

Domain walls. In the following, we proceed to discuss the classical domain wall solutions in the $U(1)^{(2)}$ gauge theory. We shall use the effective cosine potential in (5.43) to carry out our analysis. However, this section's conclusions also hold for arbitrary potential, i.e., for arbitrary forms of $\mathcal{K}$.

The cosine potential yields n vacua $a_\ell=\frac{2\pi\ell}{n}$, where $\ell=0,1,,...,n-1$. There are n domain walls separating the adjacent vacua with a kink-like profile given by (here, we may set $\tilde{a}_0=0$):

$$a(z)=\frac{2\pi\ell}{n}+\frac{4}{n}\text{arctan}\,(e^{nm_a z})\ , \quad -\infty<z<\infty\,, \tag{5.44}$$

and $m_a=\frac{\Lambda^2}{v}$ is related to the axion mass (the actual mass of the axion is obtained after using the canonical kinetic term, which yields the mass $\frac{n\Lambda^2}{v}$). We assumed the walls are space-filling in the x and y directions with a profile along the z-direction, taking $\mathcal{M}_4=\mathbb{R}^4$ for simplicity. We also assumed that the walls are separated by distances much larger than their width $\sim m_a^{-1}$. In the following, the statement $z\to\pm\infty$ means that $|z|\gg m_a^{-1}$ but still away from the adjacent walls. We observe that the reality of the kinetic energy term $\mathcal{K}$, see (5.43), implies the inequality $|f|\leq 2\pi n\Lambda^4/q$. The value of $\mathcal{K}$ attains its minimum value, $\mathcal{K}=1$, at $f=0$, while it is maximized at $|f|=2\pi n\Lambda^4/q$. In addition, the first equality in (5.41) yields:

$$f(z) \;=\; -\frac{2\pi n}{q}\Lambda^4\sin{(na(z))}=-\frac{2\pi n}{q}\Lambda^4\sin{[4\text{arctan}\,(e^{nm_a z})]}\ . \tag{5.45}$$

The theory has a $\mathbb{Z}_q^{(0)}$ 0-form symmetry that acts as $a\to a+\frac{2\pi}{q}$. The invariance of the cosine potential under the 0-form symmetry demands that

$$\frac{n}{q}\in\mathbb{N}\,, \tag{5.46}$$

and thus $n \geq q$. Later, we shall discuss that a stack of n domain walls intersects at the locus of an axion string carrying a charge q under c_3. It is more energetically favorable for the charge-q string to support only $n = q$ domain walls. Nevertheless, we maintain the generality of n and q in our subsequent discussion*.

In the following, the derivatives of $\mathcal{K}$ will prove useful for our study:

$$\mathcal{K}' = \frac{q}{2\pi n\Lambda^4}\text{arcsin}\left(\frac{qf}{2\pi n\Lambda^4}\right), \quad \mathcal{K}'' = \frac{q^2}{(2\pi n\Lambda^4)^2\sqrt{1-\left(\frac{qf}{2\pi n\Lambda^4}\right)^2}}\,. \tag{5.47}$$

Also, two important limiting behaviours of $f(z)$ are worth noticing:

$$\begin{aligned} f(z\to 0) &= \frac{4\pi n^2}{q} z\Lambda^4\,, \\ f(z\to\pm\infty) &= \frac{8\pi n}{q}\Lambda^4 e^{-nm_a|z|}\text{sign}(z)\,. \end{aligned} \tag{5.48}$$

At the wall core, $z = 0$, we find that f attains its mimimum value $f(0) = 0$, where $\mathcal{K}$ is also minimized, while in the vacum $z \to \pm\infty$, we similarly have $f(\pm\infty) = 0$, where again $\mathcal{K}$ is minimized. However, there exists a distance inside the wall, $\pm|z_m|$, at which $|f(z_m)| = \frac{2\pi n}{q}\Lambda^4$, i.e., it is maximized:

$$m_a|z_m| = \frac{1}{n}\log\tan\left(\frac{\pi}{8}\right)\,. \tag{5.49}$$

At $|z_m|$, the kinetic energy $\mathcal{K}$ is also maximized. Thus, $|f(z)|$ monotonically increases from the core of the domain wall until it reaches z_m, after which it starts decreasing exponentially; see Figure 5.1. The exponential decay observed in $f(z)$ is a defining trait of a gapped system. In this scenario, the 4-form field f_4 eats the axion, resulting in its acquisition of mass. This is also evident from

$$\int_{\mathbb{R}^4} f_4 = 0\,, \tag{5.50}$$

a result consistent with the quantization condition (5.21). In contrast, in a gapless system, $|f(z)|$ would remain at the constant value of $\frac{2\pi n}{q}\Lambda^4$ from $\pm z_m$ to infinity. Below, we shall show that $\frac{q}{2\pi}\left(a(\infty) - a(-\infty)\right) = q/n$ is the domain wall charge under a $\mathbb{Z}_q^{(3)}$ 3-form global symmetry. Thus, the stack of n domain walls carries a total charge of q under the 3-form symmetry.

5.2.2 The 3- and (−1)-form symmetries, and their anomalies

In this section, we show that the pure $U(1)^{(2)}$ gauge theory possesses a $U(1)^{(3)}$ 3-form global symmetry, which undergoes a breakdown into a $\mathbb{Z}_q^{(3)}$ symmetry in

*For example, we could have a kinetic energy term $\mathcal{K}$ that corresponds to a more general form of the effective potential $V_{\text{cl-eff}}(a) = \sum_{m\geq 1}\Lambda_m^4\left(1-\cos(mqa)\right)$.

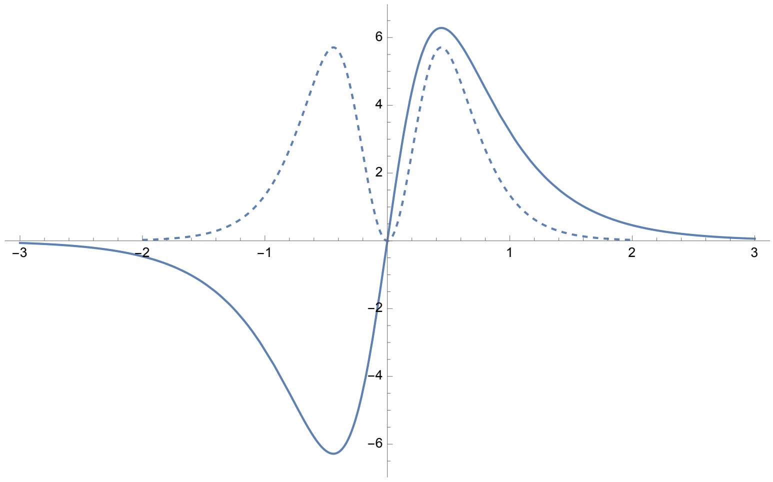

Figure 5.1: The profiles of f (solid line) and $\mathcal{K}_{\cos}$ (dashed line) as functions of the coordinate z (not to scale). We take $q = n = 2$ and set $\Lambda = 1$.

the presence of charge-q matter. In addition, the theory is endowed with a gauged (-1)-form symmetry that can be used to diagnose the existence of a cosmological constant. We further demonstrate that there is a mixed 't Hooft anomaly between $\mathbb{Z}_q^{(3)}$ and $\mathbb{Z}_q^{(0)}$ symmetries. Gauging the former gives rise to a $U(1)^{(2)}/\mathbb{Z}_q$ gauge theory.

3-form global symmetry. To begin, we rewrite Eq. (5.37), the equation of motion of c_3, in vacuum, i.e., setting the left-hand-side to 0, as

$$\mathcal{K}''\left(\frac{f}{\Lambda^2}\right)\partial_\mu f^{\mu\nu\alpha\beta} = 0\,, \tag{5.51}$$

which implies either $\mathcal{K}'' = 0$ or $\partial_\mu f^{\mu\nu\alpha\beta} = 0$. The first equation holds only when $\mathcal{K}$ is extremized, and thus, we concentrate solely on the latter equation that can be rewritten in the d-form language as

$$d \star dc_3 = 0\,. \tag{5.52}$$

This takes the form of the conservation law of the Hodge dual of a 4-form current:

$$d \star j_4 = 0\,, \quad \star j_4 = \frac{\star dc_3}{\Lambda^4} = \frac{\star f_4}{\Lambda^4}\,, \tag{5.53}$$

implying that gauging the $U(1)^{(2)}$ 2-form symmetry in vacuum gives rise to an emergent $U(1)^{(3)}$ 3-form global symmetry. The 3-form symmetry couples to 3-surfaces $\mathcal{M}_3$, such that the Wilson surface operator is given by

$$V(\mathcal{M}_3) = e^{ip\int_{\mathcal{M}_3} c_3}\,, \quad p \in \mathbb{Z}\,. \tag{5.54}$$

The value $p = 1$ gives the fundamental Wilson surface, while values of $p > 1$ are higher representations. The conserved charge of this symmetry $Q^{(3)}$ is given by integrating $\star j_4$ over a 0-dimensional manifold, or in other words, it is the local operator $\star j_4(\mathcal{P}) = \star f_4(\mathcal{P})/\Lambda^4$ at the spacetime point $\mathcal{P}$. Using $\star f_4 = \frac{1}{4!}\epsilon^{\mu\nu\alpha\beta} f_{\mu\nu\alpha\beta}$ along with $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} f$ and $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta} = -4!$, we find

$$Q^{(3)} = \star j_4(\mathcal{P}) = \frac{f(\mathcal{P})}{\Lambda^4}\,, \tag{5.55}$$

and, thus, the generator of the symmetry (symmetry defect) is

$$U^{(3)}_\gamma(\mathcal{M}_0 = \mathcal{P}) = e^{i\gamma\star j_4} = e^{i\gamma f/\Lambda^4}\,, \tag{5.56}$$

where $\gamma \in [0, 2\pi)$. The symmetry defect measures the amount of flux carried by a Wilson surface. Upon pushing $U^{(3)}_\gamma(\mathcal{M}_0)$ past $V(\mathcal{M}_3)$, we obtain the algebra:

$$U^{(3)}_\gamma(\mathcal{M}_0) V(\mathcal{M}_3) = e^{ip\gamma \mathrm{Link}(\mathcal{M}_0,\mathcal{M}_3)} V(\mathcal{M}_3)\,. \tag{5.57}$$

Let us repeat the analysis in the presence of matter. We shall use two approaches. First, we will investigate the 3-form symmetry in the vicinity of the wall but far enough from its core. Next, we shall redo the analysis, this time without making any approximations or assumptions about the nature of the wall. We start with the approximate method, analyzing the situation near the domain walls of the cosine potential we studied above. We assume that we are far from the domain wall core, i.e., we are considering distances $|z| > z_m$, where z_m is given by (5.49), which is the distance at which f and $\mathcal{K}$ are maximized. At z_m, $\mathcal{K}''$ is ill-defined, invalidating our analysis; this is why we need to perform the calculations far from the core. Keeping this constraint in mind, we start by rewriting Eq. (5.37) in the d-form language as:

$$\Lambda^4 \mathcal{K}''(f) d\star dc_3 = -\frac{q}{2\pi} da\,. \tag{5.58}$$

Far from the core, we can safely set $f \cong 0$, as $f(z)$ decays exponentially fast at distances $z > z_m$. We have* $\mathcal{K}''(f \cong 0) = q^2/(4\pi^2 n^2 \Lambda^8)$. Using this information, we can rearrange (5.58) as a conservation law:

$$\begin{aligned} d\star j_4 = 0\,, \quad Q^{(3)} = \star j_4(\mathcal{P}) &= \frac{q^2}{(2\pi n)^2\Lambda^2} \star f_4(\mathcal{P}) + \frac{q}{2\pi} a(\mathcal{P}) \\ &= \frac{q^2}{(2\pi n)^2\Lambda^2} f(\mathcal{P}) + \frac{q}{2\pi} a(\mathcal{P})\,. \end{aligned} \tag{5.59}$$

The symmetry defect is

$$U^{(3)}_\gamma(\mathcal{M}_0 = \mathcal{P}) = e^{i\gamma\star j_4} = e^{i\gamma\left(\frac{q^2 f(\mathcal{P})}{(2\pi n)^2\Lambda^4} + \frac{q a(\mathcal{P})}{2\pi}\right)}\,. \tag{5.60}$$

*Consider if we had employed the quadratic potential, as represented by Eq. (5.42). In such a scenario, we would obtain $\mathcal{K}'' = q^2/\Lambda^8$. The disparity of $(2\pi n)^2$ between these two cases becomes evident when we recognize that the function f reaches its zero precisely at the local minimum of $\mathcal{K}$ in the case of the cosine potential, where we may approximate $\mathcal{K}$ by a quadratic function $\mathcal{K} \cong 1 + \frac{q^2 f^2}{2(2\pi n)^2\Lambda^8}$.

Since $a(\mathcal{P})$ and $a(\mathcal{P})+2\pi$ are identified, $U_\gamma^{(3)}(\mathcal{M}_0=\mathcal{P})$ is meaningful only when $\gamma\in 2\pi\mathbb{Z}/q\mathbb{Z}\equiv 2\pi\mathbb{Z}_q$. The charged object under the global symmetry is still the 3-dimensional Wilson surface $V(\mathcal{M}_3)$ given by (5.54). Now, the algebra of $U_\ell^{(3)}(\mathcal{M}_0)$, $\ell\in\mathbb{Z}_q$, and $V(\mathcal{M}_3)$ is given by:

$$U_\ell^{(3)}(\mathcal{M}_0)V(\mathcal{M}_3)=e^{i\frac{2\pi p\ell}{q}\text{Link}(\mathcal{M}_0,\mathcal{M}_3)}V(\mathcal{M}_3)\,. \tag{5.61}$$

As we emphasized above, expression (5.60) is valid only far from the domain wall core, i.e., (5.60) is consistent with setting $f\cong 0$. However, inspired by the preceding analysis, we can repeat the treatment without making any approximations or assumptions about the nature of the walls. Our central equation, as usual, is (5.37), which we will write as a conservation law*:

$$\partial_\mu Q^{(3)}=0\,,\quad Q^{(3)}(\mathcal{P})=\frac{q}{2\pi}a(\mathcal{P})+\Lambda^4\mathcal{K}'\left(\frac{f(\mathcal{P})}{\Lambda^4}\right)\,, \tag{5.62}$$

with corresponding symmetry defect

$$U_\ell^{(3)}(\mathcal{M}_0)=e^{i\frac{2\pi\ell}{q}\left(\frac{q}{2\pi}a(\mathcal{P})+\Lambda^4\mathcal{K}'\left(\frac{f(\mathcal{P})}{\Lambda^4}\right)\right)}\,,\quad \ell=1,2,..,q\,. \tag{5.63}$$

This is the generator of a $\mathbb{Z}_q^{(3)}$ symmetry for a generic form of the kinetic energy $\mathcal{K}$. It is easy to check that (5.62) reproduces the approximate expression (5.60) of the cosine potential near $f\cong 0$. The charge $Q^{(3)}(\mathcal{P})$ is a constant of motion unless one encounters a domain wall: crossing an elementary wall changes $Q^{(3)}$ by

$$\Delta Q^{(3)}=\frac{q}{2\pi}\left(a(\infty)-a(-\infty)\right)=\frac{q}{n} \tag{5.64}$$

units. In other words, $\Delta Q^{(3)}$ is the domain wall charge under the $\mathbb{Z}_q^{(3)}$ global symmetry. Interestingly, when $n=q$, the most natural scenario, $\Delta Q^{(3)}$ coincides with the concept of topological charge in the theory of solitons. As we shall discuss in Section 5.3, the domain walls intersect in a line; this is the locus of a string. From the flux conservation, this string carries a charge q, evenly distributed among the n intersecting domain walls. We conclude that there are n dynamical walls attached to a string, carrying a total charge $Q^{(3)}=q$ under $\mathbb{Z}_q^{(3)}$.

The 3-form symmetry $\mathbb{Z}_q^{(3)}$ acts on q distinct Wilson surfaces $V(\mathcal{M}_3)=e^{ip\int_{\mathcal{M}_3}c_3}$, $p=1,2,..,q$. When $p\neq 0$ Mod q, the flux carried by these Wilson surfaces cannot be absorbed by the dynamical domain walls since the latter always comes in a stack of a total charge q.

It remains to discuss the fate of the 3-form global symmetry. We start with the $U(1)^{(3)}$ symmetry in the absence of matter, i.e., taking $v\to\infty$. In this case, as we discussed before, the equation of motion of the 3-form gauge field c_3 yields a

*In the cosine potential example, the derivative of $Q^{(3)}$ is ill-defined at the core. Nevertheless, $Q^{(3)}$ is well-defined everywhere.

constant solution: $f_{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta} f$, where f is a constant. Two walls experience a constant force, meaning that the $U(1)^{(3)}$ is unbroken and the Wilson surface $V(\mathcal{M}_3)$ exhibits the "area" law $\langle V(\mathcal{M}_3)\rangle = 0$. Introducing the axion field and the coupling $-\frac{q}{2\pi} da \wedge c_3$ breaks $U(1)^{(3)}$ down to $\mathbb{Z}_q^{(3)}$ and endows the theory with a $\mathbb{Z}_q^{(0)}$ 0-form global symmetry. Now, the 4-form field f_4 is gapped, and the Wilson surfaces exhibit the "perimeter" law $\langle V(\mathcal{M}_3)\rangle \neq 0$, meaning that $\mathbb{Z}_q^{(3)}$ is spontaneously broken. Moreover, if the theory forms domain walls, $\mathbb{Z}_q^{(0)}$ also breaks spontaneously.

The (-1)-form symmetry, its gauging, and the cosmological constant. In addition, the theory possess a $U(1)^{(-1)}$ (-1)-form symmetry. The Bianchi identity $d^2 c_3 = 0$ can be written as the conservation law of the Hodge-dual of a "magnetic" current $\star j_4^m = dc_3 = f_4$. The corresponding symmetry defect is

$$U_\gamma^{(-1)}(\mathcal{M}_4) = e^{i\gamma \int_{\mathcal{M}_4} f_4}\,, \quad \text{noting that} \quad \int_{\mathcal{M}_4} f_4 \in 2\pi\mathbb{Z}\,. \tag{5.65}$$

The operator $U_\gamma^{(-1)}(\mathcal{M}_4)$ does not act on any physical objects directly. However, the two-point correlator $\langle c_3(x) c_3(0)\rangle$ can be used to determine whether the (-1)-form symmetry is preserved or spontaneously broken. A massless pole in the correlator signifies symmetry breaking; if absent, the symmetry remains unbroken [131].

In the absence of axions, the (-1)-form symmetry functions as a global symmetry. The gauge field c_3 is massless, leading to a pole in its two-point correlator, which indicates that the (-1)-form symmetry is spontaneously broken. In this context, c_3 can be viewed as the Nambu-Goldstone field associated with this breaking.

When we couple c_3 to the axion through the term $\frac{q}{2\pi} a \wedge f_4$, the axion can be interpreted as the background gauge field for the (-1)-form symmetry. By introducing a kinetic term for a, we effectively sum over this background gauge field in the path integral, meaning we are gauging the (-1)-form symmetry. As previously discussed, this leads to the axion acquiring mass, which can be understood as a absorbing the would-be Goldstone field c_3 and becoming massive. Consequently, the gauged (-1)-form symmetry is spontaneously broken, and the correlator $\langle c_3(x) c_3(0)\rangle$ no longer exhibits a massless pole.

From our discussion, we find that the (-1)-form symmetry is intricately linked to the presence/absence of a cosmological constant sourced by c_3. The energy-momentum tensor of c_3 can be derived directly from (5.23) by varying with respect to the metric tensor*:

$$T_{\mu\nu} = \eta_{\mu\nu}\Lambda^4 \left[\mathcal{K}\left(\frac{f}{\Lambda^4}\right) - \frac{f}{\Lambda^4}\mathcal{K}'\left(\frac{f}{\Lambda^4}\right)\right]\,, \tag{5.66}$$

which takes the form of a cosmological constant. Without axions, the global (-1)-form symmetry is spontaneously broken and $f =$ constant, meaning that the vacuum energy gets a contribution from the long-range field c_3. Coupling to axions,

*Remember that we use the boundary condition $\mathcal{K}(f=0) = 0$.

the (−1)-form symmetry is gauged and Higgsed. Now, the system is gapped, i.e., $f = 0$ (this is true far from the domain wall core); thus, the vacuum energy does not receive contribution from c_3, and we have $T_{\mu\nu} = 0$.

The mixed 't Hooft anomaly between $\mathbb{Z}_q^{(3)}$ and $\mathbb{Z}_q^{(0)}$ symmetries. An important question is whether there is a mixed anomaly between the two global symmetries $\mathbb{Z}_q^{(3)}$ and $\mathbb{Z}_q^{(0)}$. To answer this question, we examine the commutation relation between $U_\ell^{(3)}(\mathcal{M}_0)$ and $U_{\ell'}^{(0)}(\mathcal{M}_3)$ given by (5.63) and (5.25), respectively. The calculations can be performed, as before, by foliating $\mathcal{M}_4$ into constant time slices and orienting $\mathcal{M}_3$ to be a time-like surface:

$$U_{\ell'}^{(0)}(\mathcal{M}_3(t)) = e^{i\frac{2\pi\ell'}{q}\int_{\mathcal{M}_3(t)}\left(v^2\partial_0 a - \frac{q}{2\pi}\frac{c_{ijk}\epsilon^{ijk}}{3!}\right)}\,. \tag{5.67}$$

Then, using the equal-time commutation relation $[a(\boldsymbol{x},t), \Pi_a(\boldsymbol{y},t)] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$, where $\Pi_a = v^2\partial_0 a - \frac{q}{2\pi}\frac{c_{ijk}\epsilon^{ijk}}{3!}$, we obtain

$$U_{\ell'}^{(0)}(\mathcal{M}_3(t))U_\ell^{(3)}(\mathcal{M}_0(t)) = e^{i\frac{2\pi\ell\ell'}{q}}U_\ell^{(3)}(\mathcal{M}_0(t))U_{\ell'}^{(0)}(\mathcal{M}_3(t))\,. \tag{5.68}$$

Thus, a $\mathbb{Z}_q$-valued mixed anomaly exists between the two symmetries. The anomaly implies one or both symmetries are broken. Our previous discussion reveals that both symmetries are broken in a theory that forms domain walls in the IR.

We further explore the consequences of the mixed anomaly (5.68), now working in a Hamiltonian formalism. Let H be the Hamiltonian of the $U(1)^{(2)}$ gauge theory under study. Since the theory has a $\mathbb{Z}_q^{(3)}$ 3-form global symmetry, its generators commute with the Hamiltonian: $\left[H, U_\ell^{(3)}\right] = 0$, and we can take the physical states in Hilbert space to be simultaneous eigenstates of these two operators. Thus, we have

$$|\psi\rangle_{\text{phy}} = |E(e), e\rangle\,, \quad e = 0, 1, .., q-1\,. \tag{5.69}$$

Here, $E(e)$ labels the energy and e labels the “flux" of the state (under the 3-form symmetry) such that

$$H|E(e), e\rangle = E(e)|E(e), e\rangle\,, \quad U_\ell^{(3)}|E(e), e\rangle = e^{i\frac{2\pi\ell e}{q}}|E(e), e\rangle\,. \tag{5.70}$$

Notice that the state's energy $E(e)$ can also depend on the value of the flux carried by the state.

To show that e labels the flux carried by the state $|E(e), e\rangle$, let us insert a Wilson surface $V(\mathcal{M}_3) = e^{ip\int_{\mathcal{M}_3} c_3}$ in the state $|E(e), e\rangle$ and then measure the new flux; the measurement is performed by acting with $U_{\ell=1}^{(3)}$ on the new state. Since this Wison surface carries a flux p, we expect inserting it increases the state flux by p unit. To confirm this, we perform the operation

$$U_{\ell=1}^{(3)}V(\mathcal{M}_3)|E(e), e\rangle = e^{i\frac{2\pi(e+p)}{q}}V(\mathcal{M}_3)|E(e), e\rangle\,, \tag{5.71}$$

where we used (5.61) and (5.70). The relation (5.71) shows that the state $V(\mathcal{M}_3)|E(e),e\rangle$ carries a flux $e+p$, and thus, indeed, e in (5.69) labels the flux carried by a state as anticipated.

Next, we act with both sides of the anomaly (5.68) on the state $|E(e),e\rangle$ to find that $U^{(0)}_{\ell'=1}|E(e),e\rangle$ is an eigenstate of $U^{(3)}_{\ell=1}$ with eigenvalue $e-1$. Since $\mathbb{Z}_q^{(0)}$ is a symmetry of the theory, we have $[H, U^{(0)}_{\ell'}]=0$, and thus, the state $U^{(0)}_{\ell'=1}|E(e),e\rangle$ has the same energy as $|E(e),e\rangle$. Repeating the statement q times, we conclude that there are q degenerate eigenstates of the same energy, labeled by the q different values of e, as in (5.69). This q-fold degeneracy is a direct consequence of the mixed anomaly (5.68), which is true for both the ground and excited states of the system. In the thermodynamic limit, i.e., as we take the manifold $\mathcal{M}_4$ to be very large, we become interested mainly in the ground states, which stay q-fold degenerate.

Gauging $\mathbb{Z}_q^{(3)}$. Next, we consider the $U(1)^{(2)}/\mathbb{Z}_p$ gauge theory resulting from gauging a subgroup $\mathbb{Z}_p^{(3)} \subseteq \mathbb{Z}_q^{(3)}$. Gauging the discrete symmetry $\mathbb{Z}_p^{(3)}$ means introducing a background gauge field of the symmetry and including a sum over arbitrary insertions of this background in the path integral. We can introduce the $\mathbb{Z}_p^{(3)}$ background F_4 into the path integral by replacing every f_4 in (5.23) by $f_4 \to f_4 + F_4$, where F_4 can be expressed as $pF_4 = dF_3$ and dF_3 satisfies the quantization condition $\int_{\mathcal{M}_4} dF_3 \in 2\pi\mathbb{Z}$. In turn, this implies the quantization of F_4 in units of $1/p$, i.e., $\int_{\mathcal{M}_4} F_4 \in \frac{2\pi}{p}\mathbb{Z}$, such that dF_4=0.

Now consider the second term in (5.23) in the presence of the $\mathbb{Z}_q^{(3)}$ background. Its Euclidean form is $i\frac{q}{2\pi} a \wedge (f_4 + F_4)$. Since, $\int_{\mathcal{M}_4} F_4 \in \frac{2\pi}{p}\mathbb{Z}$, only the shift $a \to a + \frac{2\pi p}{q}$, i.e., $\mathbb{Z}^{(0)}_{q/p}$, survives as a genuine discrete symmetry. Actually, gauging $\mathbb{Z}_p^{(3)}$ renders the symmetry operator of $\mathbb{Z}_q^{(0)}$ a projective operator. To see this, consider the commutation relation between the $\mathbb{Z}_q^{(0)}$ symmetry defect $U_\ell^{(0)}(\mathcal{M}_3)$ and the generator of the $\mathbb{Z}_p^{(3)}$ symmetry $\left[U_1^{(3)}(\mathcal{M}_0)\right]^{q/p}$ in the original $U(1)^{(2)}$ gauge theory. From (5.68) we have

$$\left[U_1^{(3)}(\mathcal{M}_0)\right]^{-q/p} U_\ell^{(0)}(\mathcal{M}_3) \left[U_1^{(3)}(\mathcal{M}_0)\right]^{q/p} = e^{i\frac{2\pi\ell}{p}} U_\ell^{(0)}(\mathcal{M}_3)\,. \tag{5.72}$$

This relation shows that $U_\ell^{(0)}(\mathcal{M}_3)$ fails to be a gauge-invariant operator in the $U(1)^{(2)}/\mathbb{Z}_p$ gauge theory. To remedy the problem, we sum over an arbitrary number of gauge transformations under the operator $\left[U_1^{(3)}(\mathcal{M}_0)\right]^{q/p}$ by defining the new $U(1)^{(2)}/\mathbb{Z}_p$ gauge-invariant symmetry defect:

$$\begin{aligned}\mathcal{U}_\ell^{(0)}(\mathcal{M}_3) &\equiv \sum_{r\in\mathbb{Z}} \left[U_1^{(3)}(\mathcal{M}_0)\right]^{-rq/p} U_\ell^{(0)}(\mathcal{M}_3) \left[U_1^{(3)}(\mathcal{M}_0)\right]^{rq/p} \\ &= U_\ell^{(0)}(\mathcal{M}_3) \sum_{r\in\mathbb{Z}} e^{i\frac{2\pi\ell r}{p}} \\ &= U_\ell^{(0)}(\mathcal{M}_3)\delta_{\ell\in p\mathbb{Z}}\,.\end{aligned} \tag{5.73}$$

Thus, $\mathcal{U}_\ell^{(0)}(\mathcal{M}_3)$ is a projective operator. Recalling $\ell = 0, 1, .., q$, we conclude that the subgroup $\mathbb{Z}_{q/p}^{(0)}$ survives as a genuine symmetry.

The $U(1)^{(2)}/\mathbb{Z}_p$ gauge theory has a remaining $\mathbb{Z}_{q/p}^{(3)}$ global symmetry with symmetry defect given by

$$\left[U_\ell^{(3)}(\mathcal{M}_0)\right]^p = e^{i\frac{2\pi\ell p}{q}\left(\frac{q}{2\pi}a(\mathcal{P})+\Lambda^4\mathcal{K}'\left(\frac{f(\mathcal{P})}{\Lambda^4}\right)\right)}\,, \quad \ell = 1, 2, .., q/p. \tag{5.74}$$

The $\mathbb{Z}_{q/p}^{(3)}$ symmetry acts on the $U(1)^{(2)}/\mathbb{Z}_p$ gauge-invariant Wilson surfaces $V(\mathcal{M}_3) = e^{imp\int_{\mathcal{M}_3} c_3}$, $m \in \mathbb{Z}_{q/p}$. The action of $\mathbb{Z}_{q/p}^{(0)}$ takes us between the q/p vacua of the theory.

Let us discuss the consequences of gauging $\mathbb{Z}_p^{(3)}$ in the Hamiltonian formalism. When $\mathcal{U}_\ell^{(0)}(\mathcal{M}_3)$ acts on the state $|E(e), e\rangle$, with $e = 0, 1, .., q-1$, it annihilates it unless ℓ is a multiple of p:

$$\mathcal{U}_\ell^{(0)}(\mathcal{M}_3)|E(e), e\rangle = \delta_{\ell\in p\mathbb{Z}}U_\ell^{(0)}(\mathcal{M}_3)|E(e), e\rangle\,, \tag{5.75}$$

i.e., only q/p states are not annihilated by $\mathcal{U}_\ell^{(0)}(\mathcal{M}_3)$. When we fully gauge $\mathbb{Z}_q^{(3)}$, i.e., in $U(1)^{(2)}/\mathbb{Z}_q$ gauge theory, we have neither genuine 0-form nor 3-form global symmetries.

What has just happened, especially concerning a generic kinetic energy term $\mathcal{K}$, that leads to the formation of dynamical domain walls? We analyze the situation by considering the cosine potential (5.43), setting $n = q$ for simplicity. We shall also gauge the full $\mathbb{Z}_q^{(3)}$ symmetry. In this case, what we are operationally doing is that we are declaring the equivalence between all the q vacua: $a \equiv a + \frac{2\pi\ell}{q}$, $\ell = 1, 2, .., q$. Therefore, it is more meaningful to define $\varphi \equiv qa$ and replace $V_{\text{eff}}(a) = \Lambda^4(1 - \cos(qa))$ with $V_{\text{eff}}(\varphi) = \Lambda^4(1 - \cos\varphi)$. The latter potential supports a single domain wall interpolating between $\varphi = 0$ and $\varphi = 2\pi$. However, such a wall is unstable quantum mechanically as it decays by instanton effects. We end up with a theory with a unique vacuum, supporting only strings but no domain walls.

5.3 The dual description: the Kalb-Ramond frame

While the axion framework provided valuable insights into the global symmetries within our system, the notion of strings remained implicit. To address this, we transition to the Kalb-Ramond frame [132], where the presence and properties of strings become more evident and accessible.

To dualize the axion Lagrangian (5.7) to a theory of a 2-form Kalb-Ramond field, we add to the Lagrangian (5.7) an extra term [123]:

$$\mathcal{L} = \frac{v^2}{2}da \wedge \star da - \frac{1}{2\pi}b_2 \wedge d^2a\,. \tag{5.76}$$

Here, b_2 is a Lagrange multiplier used to impose the Bianchi identity*- integrating

*The form of the extra term implies that b_2 has mass dimension 2.

out b_2 gives $d^2a = 0$. We can also integrate out a via its equation of motion:

$$2\pi v^2 \star da = db_2 \,. \tag{5.77}$$

Substituting back into (5.76) gives the dual Kalb-Ramond theory:

$$\mathcal{L}_{\text{dual}} = \frac{1}{2(2\pi)^2 v^2} db_2 \wedge \star db_2 \,. \tag{5.78}$$

The Kalb-Ramond field b_2 couples to the 2-dimensional axion-string worldsheet $\mathcal{M}_2$. Thus, the Wilson-like operator of an axion-string is:

$$V(\mathcal{M}_2) = e^{i\int_{\mathcal{M}_2} b_2} \,. \tag{5.79}$$

Unlike $V(\mathcal{M}_2)$, the operator $V(\mathcal{M}_0)$, which in the original theory was given by e^{ia}, has no local description in terms of b_2. Therefore, in the Kalb-Ramond frame, $V(\mathcal{M}_2)$ and $V(\mathcal{M}_0)$ behave respectively like Wilson and 't Hooft operators in electrodynamics. This picture is reversed in the axion frame.

In the dual description, the 0-form and 2-form global symmetry currents are:

$$j_1 = \frac{1}{2\pi} \star db_2, \qquad j_3 = 2\pi v^2 db_2 \,. \tag{5.80}$$

These currents satisfy the conservation laws:

$$d\star j_1 = 0\,, \quad d\star j_3 = 0\,, \tag{5.81}$$

which are the results of the Bianchi identity $d^2 b_2 = 0$ and the equation of motion $d\star db_2 = 0$, respectively. The corresponding $U(1)^{(0)}$ and $U(1)^{(2)}$ symmetry defects are given by

$$U^{(0)}_\alpha(\mathcal{M}_3) = e^{i\alpha\int_{\mathcal{M}_3} \frac{1}{2\pi} db_2}\,, \quad U^{(2)}_\beta(\mathcal{M}_1) = e^{i\beta\int_{\mathcal{M}_1} 2\pi v^2 \star db_2} \,. \tag{5.82}$$

With this dual formulation, we can see the action of $U(1)^{(2)}$ global symmetry- it shifts b_2 by a constant 2-form Λ_2:

$$b_2 \to b_2 + \Lambda_2 \,. \tag{5.83}$$

This transforms the axion string by a $U(1)$ phase:

$$U^{(2)}_\beta(\mathcal{M}_2) V(\mathcal{M}_2) = e^{i\beta \text{Link}(\mathcal{M}_1, \mathcal{M}_2)} V(\mathcal{M}_2) \,. \tag{5.84}$$

5.3.1 Gauging the $U(1)^{(2)}$ 2-form symmetry

Here, we derive the Lagrangian of the dual Kalb-Ramond gauge theory, which results by gauging the $U(1)^{(2)}$ symmetry. Our starting point is the Lagrangian (5.23) after adding the term $-b_2 \wedge d^2a/(2\pi)$ to enforce the Bianchi identity $d^2a = 0$:

$$\mathcal{L} = \frac{v^2}{2} da \wedge \star da - \frac{q}{2\pi} da \wedge c_3 - \frac{1}{2\pi} b_2 \wedge d^2 a + \Lambda^4 \mathcal{K}\left(\frac{dc_3}{\Lambda^4}\right) \,. \tag{5.85}$$

The equation of motion of a is:

$$d \star da - \frac{1}{2\pi} d^2 b_2 - \frac{q}{2\pi} dc_3 = 0\,, \tag{5.86}$$

and integrating once we find

$$\star da = \frac{1}{2\pi v^2}\left(db_2 + qc_3\right)\,. \tag{5.87}$$

Substituting (5.87) into (5.85), we obtain the dual Lagrangian

$$\mathcal{L}_{\text{dual}} = \frac{1}{2(2\pi)^2 v^2}(db_2 + qc_3) \wedge \star(db_2 + qc_3) + \Lambda^4 \mathcal{K}\left(\frac{dc_3}{\Lambda^4}\right)\,. \tag{5.88}$$

In this formulation, the $U(1)^{(2)}$ symmetry is gauged by introducing the 3-form gauge field c_3, which couples minimally to the Kalb-Ramond field b_2. The minimal coupling $db_2 + qc_3$ means that b_2 carries a charge q under c_3. This is also manifest in the fact that the dual Lagrangian is invariant under the $U(1)^{(2)}$ local gauge transformation $c_3 \to c_3 + d\lambda_2$, $b_2 \to b_2 - q\lambda_2$. We may think of b_2 as the Stuckelberg field of c_3; as c_3 eats up the b_2 field, it aquires a mass $\sim \frac{\Lambda^2}{v}$. In the limit $v \to \infty$, the Kalb-Ramond field decouples, leaving us with a pure $U(1)^{(2)}$ gauge theory.

The effect of gauging the $U(1)^{(2)}$ symmetry is that the spectrum of extended operator changes. The Wlison surface operator (5.79) is no longer gauge-invariant under the 2-form gauge symmetry. In the Kalb-Ramond frame, a gauge-invariant operator is

$$e^{i\int_{\mathcal{M}_3} db_2 + qc_3} = e^{i\int_{\mathcal{M}_2 = \partial\mathcal{M}_3} b_2} e^{iq\int_{\mathcal{M}_3} c_3}\,, \tag{5.89}$$

which can be interpreted as a string attached to a stack of domain walls with a cumulative charge of q (the charge under the c_3 field). An elementary axion that winds around this configuration cannot detect a nontrivial phase. This can be envisaged by computing the commutator

$$\left[e^{ia}, e^{i\int_{\mathcal{M}_3} db_2 + qc_3}\right] = 0\,, \tag{5.90}$$

where we used (5.87) along with $[a(\boldsymbol{x},t), \Pi_a(\boldsymbol{y},t)] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$ and $\Pi_a = v^2\partial_0 a - \frac{q}{2\pi}\frac{c_{ijk}\epsilon^{ijk}}{3!}$.

We end this section by discussing the global symmetries in the Kalb-Ramond frame. First, the generator of the $\mathbb{Z}_q^{(0)}$ 0-form symmetry is given by $U_\alpha^{(0)}(\mathcal{M}_3)$ in (5.82). This generator, however, must be invariant under a $U(1)^{(2)}$ gauge transformation $b_2 \to b_2 - q\lambda_2$. Using $\int_{\mathcal{M}_3} d\lambda_2 \in 2\pi\mathbb{Z}$, we find $\alpha = \frac{2\pi\ell}{q}$, $\ell = 1, 2, .., q$, as expected for the $\mathbb{Z}_q^{(0)}$ symmetry. Second, we also have a $\mathbb{Z}_q^{(3)}$ 3-form global symmetry. However, the generator of this symmetry has no local description in the Kalb-Ramond frame. It is crucial to highlight that transitioning between the axion and Kalb-Ramond frames does not eliminate global symmetries. Rather, certain symmetries may not be manifest in a local description.

5.4 Multi 3-form gauge theory

Consider an axion a coupled to 2 distinct 3-form gauge fields c_3 and $\tilde{c}_3$. The Lagrangian reads

$$\mathcal{L} = \frac{v^2}{2}|da|^2 - \frac{q_1}{2\pi} da \wedge c_3 - \frac{q_2}{2\pi} da \wedge \tilde{c}_3 + \Lambda^4 \mathcal{K}\left(\frac{f_4}{\Lambda^4}\right) + \Lambda^4 \tilde{\mathcal{K}}\left(\frac{\tilde{f}_4}{\Lambda^4}\right) , \qquad (5.91)$$

where $f_4 = dc_3$ and $\tilde{f}_4 = d\tilde{c}_3$ are the field strengths of c_3 and $\tilde{c}_3$, respectively, and we assumed that the scale Λ is the same for all the 3-form fields. Notice that we did not include a kinetic-mixing term to avoid complications. The 3-form gauge fields c_3 and $\tilde{c}_3$ are invariant under $U(1)^{(2)} \times U(1)^{(2)}$ gauge transformations $c_3 \to c_3 + d\lambda_2$ and $\tilde{c}_3 \to \tilde{c}_3 + d\tilde{\lambda}_2$, where $\int_{\mathcal{M}_3} d\lambda_2, \int_{\mathcal{M}_3} d\tilde{\lambda}_2 \in 2\pi\mathbb{Z}$, while the field strengths satisfy the quantization conditions

$$\int_{\mathcal{M}_4} f_4 \, , \int_{\mathcal{M}_4} \tilde{f}_4 \in 2\pi\mathbb{Z} \, . \qquad (5.92)$$

The equations of motion of a, c_3, and $\tilde{c}_3$ read

$$v^2 d \star da - \left(\frac{q_1}{2\pi} dc_3 + \frac{q_2}{2\pi} d\tilde{c}_3\right) = 0 \, , \quad \frac{q_1}{2\pi}\partial_\mu a = -\Lambda^4 \partial_\mu \mathcal{K}'\left(\frac{f}{\Lambda^4}\right) \, ,$$
$$\frac{q_2}{2\pi}\partial_\mu a = -\Lambda^4 \partial_\mu \tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda^4}\right) \, . \qquad (5.93)$$

Trading f and $\tilde{f}$ for a classical axion effective potential yields

$$\frac{\partial V_{\text{cl-eff}}(a)}{\partial a} = -\frac{1}{2\pi}\left(q_1 f + q_2 \tilde{f}\right) \, . \qquad (5.94)$$

This relationship asserts that the combination of fields $q_1 f + q_2 \tilde{f}$ vanishes at the extrema of $V_{\text{cl-eff}}(a)$. This implies that this particular combination of the 3-form fields is gapped at the theory's vacua. Conversely, the independent combination $q_2 f - q_1 \tilde{f}$ remains ungapped [123].

The system (5.91) enjoys a multitude of global symmetries. First, the theory is invariant under a $\mathbb{Z}_q^{(0)}$ symmetry that acts on a as $a \to a + \frac{2\pi}{q}$ and $q = \gcd(q_1, q_2)$. The generator of $\mathbb{Z}_q^{(0)}$ is

$$U_\ell^{(0)}(\mathcal{M}_3) = e^{i\frac{2\pi\ell}{q}\int_{\mathcal{M}_3}\left(v^2 \star da - \frac{q_1 c_3 + q_2 \tilde{c}_3}{2\pi}\right)} \, , \quad \ell = 1, 2, ..., q \, , \qquad (5.95)$$

which acts on the local operator $e^{ia(\mathcal{P})}$.

In addition, the system exhibits two independent 3-form global symmetries. To find them, we use two methods. We start with the first method, which was not discussed previously but works as an alternative view on the global 3-form symmetry. In this method, we shift c_3 and $\tilde{c}_3$ by two independent closed but not exact 3-forms Λ_3 and $\tilde{\Lambda}_3$:

$$c_3 \to c_3 + \Lambda_3\,, \quad \tilde{c}_3 \to \tilde{c}_3 + \tilde{\Lambda}_3\,, \tag{5.96}$$

under which the action gets shifted by

$$S \to S + \frac{q_1}{2\pi}\int_{\mathcal{M}_4} da \wedge \Lambda_3 + \frac{q_2}{2\pi}\int_{\mathcal{M}_4} da \wedge \tilde{\Lambda}_3 = S + q_1 k\alpha + q_2 k\beta\,, \tag{5.97}$$

where we defined $\int_{\mathcal{M}_3}\Lambda_3 = \alpha$, $\int_{\mathcal{M}_3}\tilde{\Lambda}_3 = \beta$, and we recalled that $\int_{\mathcal{M}_1} da = 2\pi k, k \in \mathbb{Z}$ since a is a compact scalar. Under the Λ_3 and $\tilde{\Lambda}_3$ shifts, the path integral picks up a phase:

$$Z \to e^{iq_1 k\alpha + iq_2 k\beta} Z\,. \tag{5.98}$$

It is easily seen that there are two combinations of α and β that lead to two independent 3-form global symmetries that leave the action invariant: a $U(1)^{(3)}$ symmetry is obtained by setting $q_1\alpha = -q_2\beta$ and a $\mathbb{Z}_q^{(3)}$ symmetry is obtained upon taking $\alpha, \beta \in \frac{2\pi}{q}\mathbb{Z}$, where $q = \gcd(q_1, q_2)$. These are linearly independent transformations, so there are no redundancies, and the faithful 3-form symmetry group is

$$\mathbb{Z}_q^{(3)} \times U(1)^{(3)}\,. \tag{5.99}$$

Another way to obtain the same result is by combining the equations of motion of c_3 and $\tilde{c}_3$ in the form of two independent conservation laws. Using (5.93) we find

$$\begin{aligned} \partial_\mu\left(\frac{q_1 a + q_2 a}{2\pi} + \Lambda^4\mathcal{K}'\left(\frac{f}{\Lambda}\right) + \Lambda^4\tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right)\right) &= 0\,, \\ \partial_\mu\left(-q_2\Lambda^4\mathcal{K}'\left(\frac{f}{\Lambda}\right) + q_1\Lambda^4\tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right)\right) &= 0\,, \end{aligned} \tag{5.100}$$

from which we define the two symmetry defects:

$$\begin{aligned} U_{\alpha_1}^{(3)}(\mathcal{M}_0) &= e^{i\alpha_1\left(\frac{q_1 a + q_2 a}{2\pi} + \Lambda^4\mathcal{K}'\left(\frac{f}{\Lambda}\right) + \Lambda^4\tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right)\right)} \\ U_{\alpha_2}^{(3)}(\mathcal{M}_0) &= e^{i\alpha_2\left(-q_2\Lambda^4\mathcal{K}'\left(\frac{f}{\Lambda}\right) + q_1\Lambda^4\tilde{\mathcal{K}}'\left(\frac{\tilde{f}}{\Lambda}\right)\right)} \end{aligned} \tag{5.101}$$

While the phase α_2 is an arbitrary $U(1)$ phase, implying that $U_{\alpha_2}^{(3)}$ is the symmetry defec of a $U(1)^{(3)}$ 3-form global symmetry, the single-valuedness of $U_{\alpha_1}^{(3)}$ as $a \sim a + 2\pi$ implies that $\alpha_1 = \frac{2\pi\mathbb{Z}}{q}$, reducing the second symmetry group from $U(1)^{(3)}$ down to $\mathbb{Z}_q^{(3)}$, in accordance with our earlier finding.

The theory also possesses two distinct $U(1)^{(-1)}$ (-1)-form symmetries associated with the Bianchi's identities: $d^2 c_3 = d^2\tilde{c}_3 = 0$. As we mentioned above, only the field combination $q_1 c_3 + q_2\tilde{c}_3$ is gapped while the other combination $-q_2 c_3 + q_1\tilde{c}_3$ remains gappless. This implies that only one of the two (-1)-form symmetries is gauged and spontaneously broken (Higgsed), resulting in the axion acquiring a mass. The other (-1)-form symmetry is a global symmetry, which also exhibits spontaneous breaking, resulting in a massless 3-form gauge field that sources a cosmological constant in the deep IR.

This treatment is easily generalized to any K distinct 3-form fields to find that the full faithful symmetry group is

$$\mathbb{Z}_q^{(0)} \times \mathbb{Z}_q^{(3)} \times \prod_{i=1}^{K-1} U(1)^{(i)(3)} \tag{5.102}$$

where $q = \gcd(q_1, \ldots, q_K)$.

5.5 UV completion: axion-Yang-Mills theory

In this section, we argue that the 3-form gauge theory in either the axion or the Kalb-Ramond frame emerges in the IR from a UV-complete axion-Yang-Mills system*. This conclusion is reached by using effective field theory methods empowered by new 't Hooft anomaly matching conditions. We put the IR effective field theory under scrutiny by testing its adequacy under various checks.

To this end, consider an $SU(N)$ gauge theory endowed with a massless Dirac fermion in a representation $\mathcal{R}$ under $SU(N)$. In addition, consider a complex scalar Φ that is inert under $SU(N)$ but otherwise couples to the Dirac fermion. The Lagrangian of the system reads [17]:

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}\left(f_2^c \wedge \star f_2^c\right) + \bar{\psi}\bar{\sigma}^\mu D_\mu \psi + \bar{\tilde{\psi}}\bar{\sigma}^\mu D_\mu \tilde{\psi} + |d\Phi|^2 - V(\Phi) + y\Phi\tilde{\psi}\psi + \text{h.c.} \tag{5.103}$$

$f_2^c = da_1^c - ia_1^c \wedge a_1^c$ is the field strength of the $SU(N)$ color field a_1^c. ψ and $\tilde{\psi}$ are two left-handed Weyl fermions in representations $\mathcal{R}$ and $\overline{\mathcal{R}}$ under $SU(N)$, respectively, constituting together a single Dirac fermion. The covariant derivative is $D_\mu = \partial_\mu - ia_\mu^c$ and y is the Yukawa coupling. The potential of the complex scalar field is $V(\Phi) = \lambda\left(|\Phi|^2 - v^2/2\right)$, where λ is $\mathcal{O}(1)$ parameter. We take $v \gg \Lambda$, where Λ is the strong scale of the gauge sector. The Lagrangian (5.103) is invariant under two classical 0-form symmetries $U(1)_B^{(0)} \times U(1)_\chi^{(0)}$, the baryon-number and axial symmetries. The ABJ anomaly in the color background breaks $U(1)_\chi^{(0)}$ down to $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$, and we find that the full good global symmetry of the $SU(N)$ axion-YM theory is [20, 41]

$$G^{\text{global}} = \frac{U(1)_B^{(0)} \times \mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}}{\mathbb{Z}_{N/m} \times \mathbb{Z}_2^F} \times \mathbb{Z}_m^{(1)}\,. \tag{5.104}$$

*The reader might object referring to the system we study in this section as UV complete since we use a scalar field that exhibits a Landau pole. Here, by a UV-complete, we just mean a model that couples the axion to Yang-Mills theory and gives the desired symmetries and natural hierarchy of scales.

The 1-form global symmetry $\mathbb{Z}_m^{(1)}$ acts on Wilson's lines of a_1^c, where $m = \gcd(N, n)$ and n is the N-ality of the representation $\mathcal{R}$, i.e., the boxes in the Young tableaux modulo N. The baryon-number $U(1)_B^{(0)}$ and the chiral $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ symmetries act on the local fields as

$$
\begin{aligned}
U(1)_B^{(0)} \quad &: \quad \psi \to e^{i\alpha}\psi\,, \quad \tilde{\psi} \to e^{-i\alpha}\tilde{\psi}\,, \quad \Phi \to \Phi\,,\\
\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)} \quad &: \quad \psi \to e^{i\frac{2\pi\ell}{2T_\mathcal{R}}}\psi\,, \quad \tilde{\psi} \to e^{i\frac{2\pi\ell}{2T_\mathcal{R}}}\tilde{\psi}\,, \quad \Phi \to e^{-i\frac{4\pi\ell}{2T_\mathcal{R}}}\Phi\,,
\end{aligned} \tag{5.105}
$$

and $\ell = 1, 2, .., 2T_\mathcal{R}$ and $T_\mathcal{R}$ is the Dynkin index of $\mathcal{R}$ (in our normalization, $T_\Box = 1$, where $\Box$ is the fundamental representation). The modding by $\mathbb{Z}_{N/m} \times \mathbb{Z}_2^F$ in (5.104) is important to remove redundancies. Here, $\mathbb{Z}_2^F$ is the $(-1)^F$ fermion number subgroup of the Lorentz group*. The complex scalar field can be written as $\Phi = |\Phi|e^{ia}$, where a is the axion. At energy scales $\ll v$, we can set $|\Phi| = v/\sqrt{2}$, and thus, one may only work with the axion as the lightest degree of freedom.

5.5.1 The mixed 't Hooft anomaly and IR Lagrangian

Energy scale $E \gg v$

Among the anomalies of the axion-YM theory, the mixed anomaly between the $\mathbb{Z}_m^{(1)}$ 1-form center and $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ chiral symmetries is essential in connection with the 3-form gauge theory. To see the link, we first review this anomaly from the UV point of view [14, 15, 17, 103]. We shall be general and examine the anomaly between a subgroup of the full center $\mathbb{Z}_m^{(1)}$ and chiral symmetries. We shall also work in the Euclidean space.

To this end, we turn on a background of $\mathbb{Z}_p^{(1)} \subseteq \mathbb{Z}_m^{(1)}$. This can be implemented by introducing the pair of $U(1)$ fields (B_1^c, B_2^c) and the constraint $pB_2^c = dB_1^c$. Demanding the quantization condition $\int_{\mathcal{M}_2} dB_1^c \in 2\pi\mathbb{Z}$ implies the fractional quantization of B_2^c flux: $\int_{\mathcal{M}_2} B_2^c \in \frac{2\pi\mathbb{Z}}{p}$. We couple B_2^c to fermions as follows. First, we enlarge the gauge group from $SU(N)$ to $U(N)$; we introduce the $\hat{a}_1^c$ gauge field of $U(N)$ such that $\hat{a}_1^c \equiv a_1^c + \frac{B_1^c}{p}$ with field strength $\hat{f}_2^c = d\hat{a}_1^c + \hat{a}_1^c \wedge \hat{a}_1^c$. This, in turn, implies the relation $\mathrm{tr}(\hat{f}_2^c) = NB_2^c$. Enlarging the group form $SU(N)$ to $U(N)$ introduces an extra degree of freedom, which can be eliminated by postulating the invariance of the theory under the action of an additional $U(1)^{(1)}$ 1-form gauge symmetry: $\hat{a}_1^c \to \hat{a}_1^c - \lambda_1^c$. This implies that $\hat{f}_2^c$, B_1^c, and B_2^c transform as $\hat{f}_2^c \to \hat{f}_2^c - d\lambda_1^c$, $B_1^c \to B_1^c - p\lambda_1^c$, and $B_2^c \to B_2^c - d\lambda_1^c$, such that the condition $dB_1^c = pB_2^c$ remains invariant.

The mixed anomaly between the $\mathbb{Z}_p^{(1)}$ center and $\mathbb{Z}_{2T_\mathcal{R}}^{\chi(0)}$ chiral symmetries is envisaged by examining the partition function in the background of the $U(N)$ and $\mathbb{Z}_p^{(1)}$

*Notice that the gauge group that faithfully acts on the fermions is $SU(N)/\mathbb{Z}_m$. Thus, the fermions are charged under $\mathbb{Z}_{N/m}$ subgroup of the center of $SU(N)$ gauge group. When N/m is even, the fermion number is a subgroup of $\mathbb{Z}_{N/m}$, i.e., the fermion number is gauged. In this case, all gauge-invariant operators are bosons.

fluxes. In such backgrounds, the topological charge is determined by replacing f_2^c with the combination $\hat{f}_2^c - B_2^c$ in the expression for the topological charge. Importantly, this expression remains invariant under gauge transformations by λ_1^c. Thus, the topological charge is

$$\begin{aligned} Q^c &= \frac{1}{8\pi^2}\int_{\mathcal{M}_4} \mathrm{tr}_\Box \left[\left(\hat{f}_2^c - B_2^c\right)\wedge\left(\hat{f}_2^c - B_2^c\right)\right] \\ &= \frac{1}{8\pi^2}\int_{\mathcal{M}_4} \mathrm{tr}_\Box \left[\hat{f}_2^c \wedge \hat{f}_2^c\right] - \frac{N}{8\pi^2}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c\,, \end{aligned} \tag{5.106}$$

and is fractional. Recalling that $\int_{\mathcal{M}_4} \mathrm{tr}_\Box \left[\hat{f}_2^c \wedge \hat{f}_2^c\right] \in 8\pi^2\mathbb{Z}$, the partition function transforms by the phase $-\frac{N}{8\pi^2}\int_{\mathcal{M}_4} B_2^c \wedge B_2^c = -\frac{N}{8\pi^2 p^2}\int_{\mathcal{M}_4} dB_1^c \wedge dB_1^c \in \frac{N\mathbb{Z}}{p^2}$. This is the mixed anomaly between the $\mathbb{Z}_p^{(1)}$ 1-form center and $\mathbb{Z}_{2T_\mathcal{R}}^{(0)}$ discrete chiral symmetries. The anomaly is nontrivial, provided that p^2 is not a divisor of N. It is important to highlight the group-theoretical result

$$\mathbb{Z}_{m/\gcd(m,m')} \subseteq \mathbb{Z}_{T_\mathcal{R}}\,, \tag{5.107}$$

where we have expressed $N = mm'$. This result can be verified numerically; we shall use it in our analysis below.

Energy scale $\Lambda \ll E \ll v$

At energy scale $\Lambda \ll E \ll v$, the magnitude of Φ freezes and we may set $\Phi \cong \frac{v}{\sqrt{2}}e^{ia}$. Also, the fermions acquire a mass $\sim yv$ and decouple. Then, the effective Lagrangian is:

$$\mathcal{L}_{\Lambda\ll E\ll v} = -\frac{1}{2g^2}\mathrm{tr}\left(f_2^c \wedge \star f_2^c\right) + \frac{v^2}{2}da\wedge\star da + T_\mathcal{R} a q^c\,, \tag{5.108}$$

where q^c the topological charge density: $Q^c = \int_{\mathcal{M}_4} q^c$, and Q^c is given by the expression (5.106). Thus, we have

$$q^c = \frac{1}{8\pi^2}\left[\mathrm{tr}_\Box\left(\hat{f}_2^c\wedge\hat{f}_2^c\right) - NB_2^c\wedge B_2^c\right]\,. \tag{5.109}$$

In particular, one can easily see that the Euclidean version of (5.108) reproduces the anomaly $e^{-i\frac{2\pi N}{p^2}}$ under the transformation $a \to a + \frac{2\pi}{T_\mathcal{R}}$.

In the absence of the center background, the Lagrangian (5.108) is invariant under the global symmetry group*

$$G^{\text{global}} = \mathbb{Z}_{T_\mathcal{R}}^{(0)} \times \left(\mathbb{Z}_N^{(1)}\tilde{\times}U(1)^{(2)}\right)\,. \tag{5.110}$$

The 2-form symmetry $U(1)^{(2)}$ is an emergent winding-number symmetry that acts on axion strings, while $\mathbb{Z}_N^{(1)}$ is an enhanced 1-form symmetry (remember that the UV genuine 1-form symmetry is $\mathbb{Z}_m^{(1)}$) resulting from the decoupling of fermions.

*In fact, $U(1)^{(2)}$ is only approximate global symmetry. See our discussion after Eq. (5.117).

Notice that there can be a higher group structure between $\mathbb{Z}_N^{(1)}$ and $U(1)^{(2)}$ symmetries, and we used the symbol $\tilde{\times}$ to denote this structure. To see it, we activate a background for $\mathbb{Z}_N^{(1)}$ by introducing the pair $(B_1^{(N)}, B_2^{(N)})$ such that $NB_2^{(N)} = dB_1^{(N)}$ and demanding $\int_{M_2} dB_1^{(N)} \in 2\pi\mathbb{Z}$. This, in turn, implies the flux of $B_2^{(N)}$ is fractional: $\int_{M_2} B_2^{(N)} \in \frac{2\pi\mathbb{Z}}{N}$. The pair of fields $(B_1^{(N)}, B_2^{(N)})$ transforms under a $U(1)^{(1)}$ gauge transformation as $B_1^{(N)} \to B_1^{(N)} + N\lambda_1^{(N)}$ and $B_2^{(N)} \to B_2^{(N)} + d\lambda_1^{(N)}$, which leaves the relation $NB_2^{(N)} = dB_1^{(N)}$ invariant. We also introduce C_3, the background gauge field of the global $U(1)^{(2)}$ symmetry.

Inspection of (5.108, 5.109) reveals that the backgrounds of $\mathbb{Z}_N^{(1)}$ and $U(1)^{(2)}$ couple to the axion via the term [95]

$$\mathcal{L} \supset \frac{1}{2\pi} a G_4 \,, \tag{5.111}$$

where G_4 is the field strength of the combined backgrounds. It is given by

$$G_4 = dC_3 - \frac{T_{\mathcal{R}} N}{4\pi} B_2^{(N)} \wedge B_2^{(N)} \,. \tag{5.112}$$

G_4 is invariant under a gauge transformation by $\lambda_1^{(N)}$ provided that C_3 transforms as

$$C_3 \to C_3 + d\lambda_2 + \frac{T_{\mathcal{R}} N}{2\pi} \lambda_1^{(N)} \wedge B_2^{(N)} + \frac{T_{\mathcal{R}} N}{4\pi} \lambda_1^{(N)} \wedge d\lambda_1^{(N)} \,. \tag{5.113}$$

The interplay among C_3, $B_2^{(N)}$, and $\lambda_1^{(N)}$ indicates a higher-group structure, where $\mathbb{Z}_N^{(1)}$ represents the daughter symmetry and $U(1)^{(2)}$ the parent symmetry. Notably, the former cannot exist independently of the latter [26], imposing constraints on the emergent (enhanced) symmetry scales: $E_{\mathbb{Z}_N^{(1)}} \lesssim E_{U(1)^{(2)}}$. This condition aligns well with effective field theory expectations: $E_{\mathbb{Z}_N^{(1)}} \cong \sqrt{\lambda} v$, $E_{\mathbb{Z}_N^{(1)}} \cong yv$, and $\lambda \ll y^2 \ll 1$; see [27] for details.

In a higher-group structure, one cannot gauge the daughter symmetry without gauging the parent. But the reverse is possible. This observation shall play an important role below. Notice that the higher-group symmetry becomes split (trivialized) if one can write G_4 as a total derivative [95]. For example, there is no higher-group structure between the genuine $\mathbb{Z}_m^{(1)}$ symmetry of the UV theory and $U(1)^{(2)}$. To see that, we replace $B_2^{(N)} \wedge B_2^{(N)}$ in Eq. (5.112) by $B_2^c \wedge B_2^c$, where we use the pair (B_1^c, B_2^c) (which satisfies the constraint $mB_2^c = dB_1^c$) to activate the background of $\mathbb{Z}_m^{(1)}$. Thus, $G_4 = dC_3 - \frac{T_{\mathcal{R}} N}{4\pi m^2} dB_1^c \wedge dB_1^c$. Since $\mathbb{Z}_{m/\gcd(m,m')} \subseteq \mathbb{Z}_{T_{\mathcal{R}}}$ (remember that $N = mm'$), we can write $\frac{T_{\mathcal{R}} N}{m^2} = m''$, $m'' \in \mathbb{N}$, and hence, $G_4 = dC_3' \equiv d\left(C_3 - \frac{m''}{4\pi} B_1^c \wedge dB_1^c\right)$, trivializing the higher-group. This observation is important for our subsequent analysis.

Energy scale $E \ll \Lambda$

Next, we flow to the deep IR at energy scale $\ll \Lambda$, where we assume the theory confines, and hence, the color gauge field is gapped. We must write down an effective Lagrangian that captures the UV center-chiral anomaly. This can be achieved by (i) gauging the $U(1)^{(2)}$ symmetry and introducing the dynamical 3-form gauge field c_3 and (ii) replacing $\mathrm{tr}_\square\left(\hat{f}_2^c \wedge \hat{f}_2^c\right)/4\pi$ in Eqs. (5.108, 5.109) by dc_3:

$$\mathcal{L}_{E\ll\Lambda} = \frac{v^2}{2} da \wedge \star da + \frac{T_{\mathcal{R}} a}{2\pi}\left(dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c\right) + \Lambda^4 \mathcal{K}\left(\frac{dc_3 - \frac{N}{4\pi} B_2^c \wedge B_2^c}{\Lambda^4}\right), \tag{5.114}$$

and we added a kinetic energy term for c_3. The field strength of c_3 satisfies the quantization condition $\int_{\mathcal{M}_4} dc_3 \in 2\pi\mathbb{Z}$, which is simply the infrared manifestation of the quantization of topological charges in Yang-Mills theory. The reader will notice that the coupling between a and dc_3 has an extra factor of $T_{\mathcal{R}}$ compared to the coupling in Eq. (5.111). This is because c_3 in Eq. (5.114) is a dynamical rather than a background field, and as the dynamical field absorbs the axion, it should describe the formation of $T_{\mathcal{R}}$ domain walls. As we shall discuss later, at an energy scale below Λ, the theory has enhanced $\mathbb{Z}_N^{(1)}$ 1-form symmetry. As noted above, we are allowed to gauge $U(1)^{(2)}$ without gauging the daughter symmetry $\mathbb{Z}_N^{(1)}$. This is important; otherwise, we would have changed the theory's global structure and run into trouble since $\mathbb{Z}_N^{(1)}$ is not a genuine symmetry of the theory.

The Lagrangian (5.114) must pass several checks. First, it must be invariant under the $\mathbb{Z}_{2T_{\mathcal{R}}}^{\chi(0)}$ chiral symmetry in the absence of the center background, which is evident from the transformation $a \to a + \frac{2\pi}{T_{\mathcal{R}}}$ along with the condition $\int_{\mathcal{M}_4} dc_3 \in 2\pi\mathbb{Z}$. Second, the Lagrangian must be invariant under the same auxiliary $U(1)^{(1)}$ gauge transformation, by λ_1^c, of the UV theory. This is the case provided that c_3 transforms as

$$c_3 \to c_3 + d\lambda_2 + \frac{p'}{2\pi} B_1^c \wedge d\lambda_1^c + \frac{pp'}{4\pi} \lambda_1^c \wedge d\lambda_1^c\,, \tag{5.115}$$

and we wrote $N = pp'$. Also, the Lagrangian (5.114) must reproduce the mixed $\mathbb{Z}_{2T_{\mathcal{R}}}^{\chi(0)}$-$\mathbb{Z}_p^{(1)}$ anomaly of the UV theory. This can be easily verified by observing that the partition function acquires the phase $e^{-i\frac{2\pi N}{p^2}}$ when a is shifted by $a \to a + \frac{2\pi}{T_{\mathcal{R}}}$ in the presence of the center background. In the absence of a center background, the Lagrangian (5.114) exactly matches (5.23) in Section 5.2.1, and everything we said there applies here.

Another check on the validity of (5.114) is to integrate out c_3 along the lines of our discussion that led from Eq. (5.32) to Eq. (5.35). Thus, we sum over arbitrary values of the integers $\int_{\mathcal{M}_4} f_4 \in 2\pi\mathbb{Z}$ and use the Poisson resummation formula $\sum_{m\in\mathbb{Z}} \delta\left(2\pi m - \int_{\mathcal{M}_4} f_4\right) = \sum_{k\in\mathbb{Z}} e^{-ik\int_{\mathcal{M}_4} f_4}$. We also use the change of variables $\hat{f}_4 = f_4 - \frac{N}{4\pi} B_2^c \wedge B_2^c$. Specifying to the canonical kinetic term $\mathcal{K}_{\text{can}}$, performing the Gaussian integral over $\hat{f}_4$, and focusing only on the zero modes, we obtain the

Euclidean partition function

$$Z[a] \sim \sum_{k\in\mathbb{Z}} e^{-i\frac{kN}{4\pi}\int_{\mathcal{M}_4} B_2^c\wedge B_2^c} e^{-\frac{\Lambda^4 V_{\mathcal{M}_4}}{8\pi^2}(T_{\mathcal{R}}a+2\pi k)^2}\,. \tag{5.116}$$

This effective partition function picks up the anomaly $e^{i\frac{N}{4\pi}\int_{\mathcal{M}_4} B_2^c\wedge B_2^c} = e^{-i\frac{2\pi N}{p^2}}$ upon shifting $a \to a + \frac{2\pi}{T_{\mathcal{R}}}$. It also displays the expected structure of the Yang-Mills theory: it has an infinite number of vacua, with the true vacuum energy density given by

$$V(a) = \frac{\Lambda^4}{8\pi^2}\mathrm{min}_k \left(T_{\mathcal{R}}a + 2\pi k\right)^2 \,. \tag{5.117}$$

The potential $V(a)$ has $T_{\mathcal{R}}$ minima with cusps at $a = \pi/T_{\mathcal{R}}, 3\pi/T_{\mathcal{R}}$, etc.. The cusps indicate that the potential $V(a)$ is missing degrees of freedom at these locations. These are the hadronic walls sandwiched between the axion domain walls. An axion wall has width $\sim v/\Lambda^2$, while a hadronic wall is much thinner with width $\sim \Lambda^{-1}$. Including the infinite sum over all the integers $\int_{\mathcal{M}_4} f_4 \in 2\pi m$, $m \in \mathbb{Z}$ was crucial to see these cusps. As emphasized above, the integer m is the IR manifestation of the Yang-Mills instantons' topological charge. Below Λ, the theory is strongly coupled, and the vacuum receives contributions from all the topological charge sectors.

Let us examine the theory's behavior at energy scales $\Lambda \ll E \ll v$, such as at a corresponding temperature. In this case, it suffices to include the contribution from minimal charges $\int_{\mathcal{M}_4} f_4 = m = \pm 1$ in the partition function. Translating this into the language of Yang-Mills instantons, the dilute instanton-gas approximation is reliable at this temperature because it serves as an infrared cut-off on the instanton's scale modulus [133]. Thus, summing over the smallest instantons, which possess topological charges of ± 1, is adequate. Limiting the sum over m to the lowest charge sector means that one can no longer perform the Poisson resummation that leads to (5.116), and thus, one can no longer make sense of (5.117) or the cusps. This is consistent with the expectation that the hadronic walls melt away at a temperature $\sim \Lambda$. Nevertheless, at temperatures in the range $\Lambda \ll T \ll v$, the classical theory (5.114) still possesses $T_{\mathcal{R}}$ vacua with axion domain walls interpolating between them. Notice that in this energy range, c_3 does not strongly fluctuate (since$\int_{\mathcal{M}_4} f_4 = \pm 1$), and we can consider the $U(1)^{(2)}$ 3-form gauge field c_3 as a background rather than a dynamical field. Thus, one may still regard $U(1)^{(2)}$ as an approximate global symmetry. Eventually, the axion domain walls melt at temperature $T \gtrsim v$.

As noted in [110, 114, 123] and discussed in the Introduction, it was recognized that the IR behavior of both pure YM theory in the large-N limit and the axion-YM theory can be effectively described using the 3-form gauge field c_3. Here, incorporating c_3 in our discussion has been essential for aligning with the infrared constraints of the 't Hooft anomaly. Our method provides a systematic approach to argue for a consistent infrared effective field theory of the axion-YM system.

At energy scales below Λ, the theory acquires the global symmetry

$$G^{\text{global}} = \mathbb{Z}^{(0)}_{T_\mathcal{R}} \times \left(\mathbb{Z}^{(1)}_N \tilde{\times} \mathbb{Z}^{(3)}_{T_\mathcal{R}}\right) . \quad (5.118)$$

In general, a higher-group structure may exist between $\mathbb{Z}^{(1)}_N$ and $\mathbb{Z}^{(3)}_{T_\mathcal{R}}$, which becomes apparent when both symmetries' backgrounds are activated. The background of $\mathbb{Z}^{(1)}_N$ was discussed earlier, while that of $\mathbb{Z}^{(3)}_{T_\mathcal{R}}$ can be activated by introducing the pair (F_3, F_4), satisfying the constraint $dF_3 = T_\mathcal{R} F_4$, along with the quantization condition $\int_{\mathcal{M}4} dF_3 \in 2\pi\mathbb{Z}$ [102]. The axion coupled to these backgrounds is represented by

$$\mathcal{L} \supset \frac{a}{2\pi}\left(T_\mathcal{R} dc_3 + dF_3 - \frac{T_\mathcal{R} N}{4\pi} B^{(N)}_2 \wedge B^{(N)}_2\right) . \quad (5.119)$$

Maintaining invariance under a gauge transformation by $\lambda^{(N)}_1$ requires F_3 to transform as

$$F_3 \to F_3 + d\lambda_2 + \frac{T_\mathcal{R} N}{2\pi} d\lambda^{(N)}_1 \wedge B^{(N)}_2 + \frac{T_\mathcal{R} N}{4\pi} d\lambda^{(N)}_1 \wedge d\lambda^{(N)}_1 . \quad (5.120)$$

The interplay among F_3, $B^{(N)}_2$, and $\lambda^{(N)}_1$ indicates a higher-group symmetry $\mathbb{Z}^{(1)}_N \tilde{\times} \mathbb{Z}^{(3)}_{T_\mathcal{R}}$. However, this higher-group structure becomes trivial if $dF_3 - \frac{N T_\mathcal{R}}{4\pi N^2} dB^{(N)}_1 \wedge dB^{(N)}_1$ can be expressed as a total derivative. This holds particularly true for the $\mathbb{Z}^{(1)}_m$ symmetry, as demonstrated above in a similar case involving $U(1)^{(2)}$ and $\mathbb{Z}^{(1)}_m$. Understanding this aspect is pivotal when gauging the genuine center, as this operation should be executed without gauging $\mathbb{Z}^{(3)}_{T_\mathcal{R}}$.

In the IR, the symmetries $\mathbb{Z}^{(0)}_{T_\mathcal{R}}$ and $\mathbb{Z}^{(3)}_{T_\mathcal{R}}$ undergo spontaneous breaking. The enhanced symmetry $\mathbb{Z}^{(1)}_N$ remains unbroken until length scales $\sim yv/\Lambda^2$, at which point it also undergoes explicit breaking due to the heavy fermions that pop up from vacuum as we take the Wilson lines to be larger than $\sim yv/\Lambda^2$. This leaves $\mathbb{Z}^{(1)}_m$ as the sole surviving unbroken symmetry.

5.5.2 $SU(N)/\mathbb{Z}_p$ and noninvertible chiral symmetry

Let us investigate whether our construction yields the desired results when we gauge the genuine center or any of its subgroups, aligning with the well-established findings in the literature [128, 98, 2]. We shall see that the answer is affirmative, lending support to the picture that the deep IR regime of the system is genuinely described by the 3-form gauge theory.

We consider the same axion-YM theory with matter, but now let us gauge a subgroup of the center $\mathbb{Z}^{(1)}_p \subseteq \mathbb{Z}^{(1)}_m$, i.e., we consider $SU(N)/\mathbb{Z}_p$ axion-YM theory with matter*. This theory is constructed by promoting (B^c_1, B^c_2) to dynamical fields

*In principle, there are p distinct theories: $(SU(N)/\mathbb{Z}_p)_k$, $k = 0, 1, .., p-1$ differing by the admissible genuine (electric, magnetic, or dyonic) line operators [99]. In this chapter, we limit our treatment to $(SU(N)/\mathbb{Z}_p)_{k=0}$. The Hilbert space and the noninvertible chiral symmetry in $(SU(N)/\mathbb{Z}_p)_{k=0}$ theory were considered in [2].

(b_1^c, b_2^c) and performing the sum over the fractional instantons in the path integral. Let us define the new 3-form gauge field $\hat{c}_3$:

$$\hat{c}_3 \equiv c_3 - \frac{N}{4\pi p^2} b_1^c \wedge db_1^c \,, \tag{5.121}$$

keeping in mind the quantization condition $\int_{\mathcal{M}_4} dc_3 \in 2\pi\mathbb{Z}$. The Lagrangian of this theory at energy scale $E \ll \Lambda$ reads

$$\mathcal{L}_{E\ll\Lambda}[(b_1^c, b_2^c)] = \frac{v^2}{2} da \wedge \star da + \frac{T_{\mathcal{R}}}{2\pi} a \wedge d\hat{c}_3 + \Lambda^4 \mathcal{K}\left(\frac{d\hat{c}_3}{\Lambda^4}\right) , \tag{5.122}$$

and the partition function is

$$Z = \sum_{(b_1^c, b_2^c)} \int [Dc_3][Da] e^{i\int_{\mathcal{M}_4} \mathcal{L}_{E\ll\Lambda}[(b_1^c, b_2^c)]} \,. \tag{5.123}$$

In the Kalb-Ramond frame, we replace $q \to T_{\mathcal{R}}$ and $c_3 \to \hat{c}_3$ in (5.88). It is important to repeat what we stated above: we can gauge the genuine $\mathbb{Z}_m^{(1)}$ center symmetry or a subgroup thereof without spoiling $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$ since the pair does not constitute a higher-group. On the contrary, gauging the enhanced $\mathbb{Z}_N^{(1)}$ is disastrous: it entails that we also gauge $\mathbb{Z}_{T_{\mathcal{R}}}^{(3)}$, which destroys the domain walls.

The chiral symmetry defect is given by (5.25) after replacing c_3 with $\hat{c}_3$ and summing over (b_1^c, b_2^c):

$$\tilde{U}_\ell^{(0)}(\mathcal{M}_3) \sim \sum_{(b_1^c, b_2^c)} e^{i\frac{2\pi\ell}{T_{\mathcal{R}}}\int_{\mathcal{M}_3} v^2 \star da - i\ell\int_{\mathcal{M}_3}\left(c_3 - \frac{N}{4\pi p^2} b_1^c \wedge db_1^c\right)} \,, \quad \ell = 1, 2, .., T_{\mathcal{R}} \,. \tag{5.124}$$

The new defect $\tilde{U}_\ell^{(0)}(\mathcal{M}_3)$ defines a noninvertible chiral symmetry $\tilde{\mathbb{Z}}_{T_{\mathcal{R}}}^{(0)}$. To simplify the form of $\tilde{U}_\ell^{(0)}(\mathcal{M}_3)$, we write N as $N = pp'$ and assume that $p' = 1$ Mod p. Then, $\frac{N}{4\pi p^2} b_1^c \wedge db_1^c \sim \frac{1}{4\pi p} b_1^c \wedge db_1^c$, i.e., this is an improperly quantized quantum Hall term. We may rewrite it in terms of an auxiliary 1-form gauge field φ_1 that lives on $\mathcal{M}_3$:

$$\tilde{U}_\ell^{(0)}(\mathcal{M}_3) \sim \sum_{(b_1^c, b_2^c)} e^{i\frac{2\pi\ell}{T_{\mathcal{R}}}\int_{\mathcal{M}_3} v^2 \star da - i\ell\int_{\mathcal{M}_3} c_3} \int [D\varphi_1] e^{-i\ell\int_{\mathcal{M}_3}\left(\frac{p}{4\pi}\varphi_1 \wedge d\varphi_1 + \frac{1}{2\pi}\varphi_1 \wedge db_1^c\right)} \tag{5.125}$$

The last term is the minimal abelian TQFT $\mathcal{A}^{p,1}$ discussed in [134]. When ℓ is a multiple of p, the TQFT is trivialized, giving us an invertible symmetry.

5.6 Discussion

In this chapter, we critically evaluated the hypothesis that the 3-form gauge theory offers more than an alternative framework for the deep IR regime of axion-Yang-Mills systems. We commenced by rigorously investigating the 3-form gauge theory coupled to axions, as encapsulated by the Lagrangian (5.23). Our analysis revealed that this theory represents a network of domain walls terminating on an axion string, with particular emphasis placed on its global structure. A dual formulation, the Kalb-Ramond Lagrangian (5.88), was also considered, which describes the same physical phenomenon. Notably, while some symmetry defects are explicit in one formulation, others are evident in the alternative frame. Crucially, in the absence of gravitational effects*, both formulations are equivalent, possessing identical global symmetries.

Subsequently, we examined the $SU(N)$ Yang-Mills theory with a Dirac fermion coupled to an $SU(N)$-neutral complex scalar, highlighting the necessity of an emergent 3-form gauge field for IR matching of the theory's mixed center-chiral anomaly. Consider varying the complex scalar vev such that we go to the limit $v \ll \Lambda$. In this case, the strong dynamics set in well before the axions are amenable to the weak-coupling treatment. In this opposite limit, the theory still forms domain walls leading to $T_{\mathcal{R}}$ distinct vacua, and thus, we do not expect a bulk phase transition to take place as we vary v below or above Λ. We may not rigorously justify the introduction of the 3-form gauge field in this scenario. However, by continuity, we expect that the 3-form gauge theory remains a valid description in the deep IR. This reasoning, combined with the large-N limit analysis discussed in the introduction, supports the notion that the vacuum of Yang-Mills theory should likely be described by a 3-form gauge theory.

Incorporating gravity offers further insights into the significance of the 3-form description, as the cosmological constant in this context can be interpreted as arising from a gauge principle. Theoretically, one can distinguish between a pure cosmological constant and a 3-form gauge theory, as the latter yields a non-vanishing contribution to the trace anomaly proportional to the Gauss-Bonnet invariant [117, 135]. Irrespective of this subtle effect, Brown and Teitelboim realized that the action (5.2) taken as a starting point with no reference to its UV completion leads to the quantum creation of closed membranes localized on the boundary of $\mathcal{M}_4$ [136, 137]. As the membranes are produced, the vacuum energy density associated with c_3 decreases, reducing the effective value of the cosmological constant. This idea, when refined, may lead to a solution to the cosmological constant problem [138, 139]. Also, connections between the QCD vacuum and the cosmological constant problem were discussed in [140, 141].

As discussed in this chapter, introducing the 3-form gauge field c_3 was necessary to match the chiral-center anomaly in an axion-Yang-Mills system. Eventually, c_3 eats the axion, becoming a short-range field, and the cosmological constant vanishes. Yet, one can think of an alternative scenario with an axion, two distinct Yang-Mills fields, and a chiral-center anomaly. In this case, two 3-form gauge fields are anticipated. One combination of these fields eats the axion, while an orthogonal combination remains gapless. The latter can source a cosmological constant. Intriguingly, in this scenario, the infrared cosmological constant can be

*Inclusion of gravity may differentiate between the axion and Kalb-Ramond frames, see [117].

considered a by-product of the 't Hooft anomaly-matching condition. However, in such a scenario, global symmetries should only be considered approximate since exact global symmetries are forbidden in quantum gravity, see, e.g., [142, 143, 144].

The method presented in this chapter can be extended in various ways. One immediate application is to address the problem of multi-flavor quarks by incorporating the 3-form gauge theory description in the chiral Lagrangian. Another venue is applying our formalism to the Standard Model (SM) and its variants, potentially through coupling with axions. It is well-established that the SM exhibits a $\mathbb{Z}_6^{(1)}$ 1-form symmetry, and the true gauge group might be modded by $\mathbb{Z}_6$ or a subgroup thereof (see, e.g., [107, 108, 145, 146, 147, 148, 149]). Exploring whether and how a 3-form gauge theory may emerge deep in the IR of the SM and its extensions and whether the modded discrete group could play a significant role in this formalism will be an exciting avenue of research. Additionally, linking the emergent 3-form to the observed cosmological constant presents another intriguing possibility. These investigations are worthy of future exploration.

Appendix A

’t Hooft Fluxes for $SU(2)$ and $SU(3)$

For the gauge group $SU(N)$, the weights of the defining representation $\nu_{c,A}$ (labelled by the superscript $A = 1, \ldots, N$ are normalized such that

$$\nu_{c,A} \cdot \nu_{c,B} = \delta_{AB} - \frac{1}{N} \tag{A.1}$$

We can express the generators of the Cartan subalgebra in terms of the weights of the defining representation:

$$H^a_{c,AB} = \delta_{AB}(\nu_{c,A})^a \tag{A.2}$$

where $a = 1, \ldots, N-1$ indicates the a-th component of $(\nu_{c,A})^a$. We can write $\mathbf{H_c}$ as a vector of matrices. Then a given element of the $\mathbb{Z}_N$ center of $SU(N)$ is generated by

$$e^{2\pi i m_1 \mathbf{H_c}\cdot \nu_\mathbf{c}} = e^{-2\pi i m_1 \frac{1}{N}} 1_N \tag{A.3}$$

For $SU(2)$, the normalized Cartan generator is given by

$$H_c = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{A.4}$$

with weights $\nu_{c,1} = \frac{1}{\sqrt{2}}, \nu_{c,2} = \frac{1}{\sqrt{2}}$. A.3 is indeed satisfied by either choice of weight. For $SU(3)$, the normalized Cartan generators are given by

$$H_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \tag{A.5}$$

The weights of the defining representation are:

$$\nu_{c,1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \qquad \nu_{c,2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \qquad \nu_{c,3} = \begin{pmatrix} 0 \\ -\frac{2}{\sqrt{6}} \end{pmatrix} \tag{A.6}$$

A quick calculation shows that all three weights satisfy A.3.

Appendix **B**

Obtaining the discrete chiral symmetry

In this appendix, we show that there is a discrete symmetry $\mathbb{Z}_r$, where $r = \gcd(N_\psi, N_\chi)$, that acts on χ. To this end, we conisder the groups $\mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$ and $U(1)_A$ we discussed in the text. Under $U(1)_A \times \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$, ψ transforms as

$$\psi \longrightarrow e^{2\pi i a q_\psi \alpha} e^{2\pi i p_\psi \frac{l}{N_\psi p_\psi + N_\chi p_\chi}} \psi \,, \tag{B.1}$$

where $l \in \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$, a is a charge factor, and $\alpha \in [0,1)$. This transformation leaves ψ invariant if

$$a q_\psi \alpha + p_\psi \frac{l}{N_\psi p_\psi + N_\chi p_\chi} = k_1 \in \mathbb{Z} \implies \alpha = \frac{k_1}{a q_\psi} - \frac{p_\psi l}{a q_\psi (N_\psi p_\psi + N_\chi p_\chi)} \,. \tag{B.2}$$

Note that k_1 can be freely chosen. Then, χ transforms under $U(1)_A \times \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$ as:

$$\begin{aligned}
\chi \quad &\longrightarrow e^{2\pi i a q_\chi \alpha} e^{2\pi i p_\chi \frac{l}{N_\psi p_\psi + N_\chi p_\chi}} \chi = e^{2\pi i \left(a q_\chi \left(\frac{k_1}{a q_\psi} - \frac{p_\psi l}{a q_\psi (N_\psi p_\psi + N_\chi p_\chi)} \right) + p_\chi \frac{l}{N_\psi p_\psi + N_\chi p_\chi} \right)} \chi \\
&= e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 + \frac{l}{N_\psi p_\psi + N_\chi p_\chi} \left(p_\chi - p_\psi \frac{q_\chi}{q_\psi} \right) \right)} \chi = e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 + \frac{l}{(N_\psi p_\psi + N_\chi p_\chi) q_\psi} \left(-p_\chi \frac{N_\chi}{r} - p_\psi \frac{N_\psi}{r} \right) \right)} \chi \,, \\
&= e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 - \frac{l}{r q_\psi} \right)} \chi \,,
\end{aligned} \tag{B.3}$$

where $r = \gcd(N_\psi, N_\chi)$ and we used $q_\psi = -\frac{N_\chi}{r}$ and $q_\chi = \frac{N_\psi}{r}$. We can rewrite $l = m_1 + m_2 r$, where $m_1 = 0, 1, \ldots, r-1$ and $m_2 \in \mathbb{Z}$. Bezout's theorem also tells us that since $r = \gcd(N_\psi, N_\chi)$, there are integers k_1, k_2 such that $m_2 r = k_1 N_\psi + k_2 N_\chi$. Applying this to the transformation of χ gives us:

$$\begin{aligned}
\chi \quad &\longrightarrow e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 - \frac{l}{r q_\psi} \right)} \chi = e^{2\pi i \left(\frac{q_\chi}{q_\psi} k_1 - \frac{m_1 + m_2 r}{r q_\psi} \right)} \chi = e^{2\pi i \left(\frac{m_1 + m_2 r}{N_\chi} - \frac{N_\psi}{N_\chi} k_1 \right)} \chi \\
&= e^{2\pi i \frac{m_1}{N_\chi}} e^{2\pi i \frac{m_2 r - N_\psi k_1}{N_\chi}} \chi = e^{2\pi i \frac{m_1}{N_\chi}} e^{2\pi i k_2} \chi = e^{2\pi i \frac{m_1}{N_\chi}} \chi \,.
\end{aligned} \tag{B.4}$$

Since $m_1 = 0, 1, \ldots, r-1$, there are only r distinct transformations generated by $U(1)_A \times \mathbb{Z}_{N_\psi p_\psi + N_\chi p_\chi}$, and the symmetry group that acts on χ is $\mathbb{Z}_r$. For our purposes, we will assume that under $\mathbb{Z}_r$, χ transforms with charge 1 (in principle, we could fix any charge). Finally, one needs to check whether this $\mathbb{Z}_r$ is a genuine symmetry in the sense that it cannot be absorbed in the center of color or flavor groups. This will be done on a case-by-case basis.

Appendix **C**

Anomaly Descent

It turns out that all the information from anomalies is captured by a five dimensional inflow action. To see this, first note that we can interpret the anomaly as a gauge variation in the effective action:

$$\delta_\epsilon \widetilde{W}[A] = \int d^4x\, \epsilon^\alpha(x)\mathcal{A}_\alpha(x) \tag{C.1}$$

We can also write this as

$$\mathcal{A}_\alpha(x) = -\left(D_\mu \frac{\delta}{\delta A_\mu(x)}\right)_\alpha \widetilde{W}[A] \tag{C.2}$$

Since the anomaly can be expressed as a derivative, its form is constrained (similar to how $\nabla \cdot \nabla f = 0$ for a scalar function f). This constraint is known as the *Wess-Zumino consistency condition.*

One way to derive the Wess-Zumino condition is by considering the BRST transformations on the effective action. Recall that the BRST operator, s, anti-commutes with gauge fields and acts on the gauge field A_μ via

$$sA_\mu = D_\mu w \tag{C.3}$$

where w^α is a ghost field, i.e. a gauge transformation with the ghost field as a parameter. It follows that the field strength transforms as

$$sF = [F, w] \tag{C.4}$$

s acts on the ghost field via

$$sw^\alpha = -\frac{1}{2}C_{\alpha\beta\gamma}w^\beta w^\gamma \tag{C.5}$$

Then when acting on the effective action,

$$\begin{aligned} s\widetilde{W}[A] &= \int d^4x\, (D_\mu w(x))^\alpha \frac{\delta}{\delta A^\alpha_\mu(x)} \widetilde{W}[A] \\ &= \int d^4x\, w^\alpha(x) \left(-D_\mu \frac{\delta}{\delta A_\mu(x)} \right)_\alpha \widetilde{W}[A] \\ &= \int d^4x\, w^\alpha(x) \mathcal{G}_\alpha \widetilde{W}[A] \\ &= \int d^4x\, w^\alpha(x) \mathcal{A}_\alpha(x) \\ &\equiv \mathcal{A}[w, A] \end{aligned}$$

where we introduced a dependence on A to remind ourselves that the anomaly depends on the gauge field. In general the above derivation holds for any functional F of the gauge fields:

$$sF[A] = \int d^4x\, w^\alpha(x) \mathcal{G}_\alpha F[A] \tag{C.6}$$

Since $s^2 = 0$, we have

$$\begin{aligned} 0 = s\mathcal{A}[w, A] &= \int d^4x\, s(w^\alpha(x)\mathcal{A}_\alpha(x) \\ &= \int d^4x \Big((sw^\alpha(x))\mathcal{A}_\alpha(x))\mathcal{A}_\alpha(x) - w^\alpha(x)(s\mathcal{A}_\alpha(x)) \Big) \\ &= \int d^4x \Big(-\frac{1}{2} C_{\alpha\beta\gamma} w^\beta(x) w^\gamma(x) \mathcal{A}_\alpha(x) \\ &\qquad\qquad - w^\alpha(x) \int d^4y\, w^\beta(y) \mathcal{G}_\beta(y) \mathcal{A}_\alpha(x) \Big) \\ &= \int d^4x\, d^4y \left(-\frac{1}{2} w^\alpha(x) w^\beta(y) \right) \Big(C_{\alpha\beta\gamma} \delta^{(4)}(x-y) \mathcal{A}_\gamma(x) \\ &\qquad\qquad + \mathcal{G}_\beta(y)\mathcal{A}(x) - \mathcal{G}_\alpha(x)\mathcal{A}_\beta(y) \Big) \end{aligned}$$

where we used the last line via anti-commutation of the ghost fields. This is the Wess-Zumino consistency condition:

$$s\mathcal{A}[w, A] = 0 \tag{C.7}$$

Therefore the anomaly $\mathcal{A}[w, A]$ is a BRST-closed functional of ghost number one. If the anomaly is BRST-closed but not exact, we say the anomaly is *relevant* - it cannot be cancelled out by a local counterterm. If the anomaly is BRST-exact, then it is *irrelevant* [21].

It turns out that the Wess-Zumino condition is sufficient in determining the full form of the anomaly up to an overall constant. We include a calculation in section C.0.1. In other words, solutions to the Wess-Zumino condition give us the anomaly.

There is a way to generate solutions to the Wess-Zumino condition via a process known as anomaly descent. The 6-dimensional characteristic polynomial is given by

$$P_3 = \mathrm{tr}\, F^3 = d\omega_5 \tag{C.8}$$

It can be expressed as the derivative of the five-dimensional Chern-Simons action:

$$\omega_5(A) = \mathrm{tr}\left(iA \wedge dA \wedge dA + \frac{3}{2}A^3 \wedge dA - \frac{3i}{5}A^5\right) \tag{C.9}$$

The characteristic polynomial is gauge invariant, so the gauge variation of the Chern-Simons form satisfies:

$$0 = \delta P_3 = \delta(d\omega_5) = d(\delta\omega_5) \tag{C.10}$$

We see that $\delta\omega_5$ is locally exact - we can express it as the derivative of a 4-form on local coordinate patches:

$$\delta\omega_5 = dQ_4 \tag{C.11}$$

where $Q_4 = Q_4(v, A)$ is linear in the gauge parameter. Let us assume that the manifold is compact, so Q_4 is defined globally. We can also choose the topology such that every closed form is exact [21].

Recall that the BRST transformation is a gauge transformation with the ghost field w as a parameter - therefore we have

$$s\omega_5 = dQ_4(w, A) \tag{C.12}$$

Using the property $s^2 = 0$ and $ds = -sd$

$$0 = s(s\omega_5) = s(dQ_4) = -d(sQ_4) \tag{C.13}$$

Therefore $sQ_4 = dq_3$ is exact. As the manifold is compact, we have

$$s\int_{\mathcal{M}} Q_4 = \int_{\mathcal{M}} dq_3 = 0 \tag{C.14}$$

So we see that $\int_{\mathcal{M}} Q_4$ satisfies the Wess-Zumino condition. Note that the anomaly is defined up to BRST-exact terms - this corresponds to the fact that the anomaly is defined up to adding local counterterms to the action [21].

C.0.1 Computing the anomaly from the Wess-Zumino condition

We can compute the full expression of the anomaly using the Wess-Zumino consistency condition. For ease of calculation, let us redefine the fields in our calculation to absorb the factors of i:

$$\mathbf{A} = -iA, \qquad \mathbf{F} = -iF = d\mathbf{A} + \mathbf{A}^2, \qquad D = d + \mathbf{A} \qquad v = -i\omega \tag{C.15}$$

Here we used the shorthand $\mathbf{A}^2 = \mathbf{A} \wedge \mathbf{A}$. Using $\mathcal{A}[\omega, A] = \frac{-1}{24\pi^2} \int \mathrm{tr}_{\mathcal{R}} \omega dAdA + O(A^3)$, the anomaly can be written as:

$$\mathcal{A}[\omega, \mathbf{A}] = \frac{i}{24\pi^2} \int \mathrm{tr}_{\mathcal{R}} v d\mathbf{A} d\mathbf{A} + O(\mathbf{A}^3) \tag{C.16}$$

Since the ghost fields ω^α anticommute amongst themselves, and so do the dx^μ, we will make the choice that dx^μ anticommute with the ghost fields as well:

$$v dx^\mu = -dx^\mu v \tag{C.17}$$

Then the BRST transformation of the one-form $\mathbf{A}$ is given by:

$$s\mathbf{A} = s\mathbf{A}_\mu dx^\mu = (\partial_\mu v + [\mathbf{A}_\mu, v])dx^\mu = -dv - \mathbf{A}v - v\mathbf{A} = -dv - \{\mathbf{A}, v\} \tag{C.18}$$

The rescaled field strength and ghost field transform as:

$$s\mathbf{F} = [\mathbf{F}, v] \qquad\qquad sv = -vv \tag{C.19}$$

where we used the shorthand $vv = [v, v]$ for two ghost fields.

As the anomaly comes from a variation of the effective action, it must be of the form $\int \mathrm{tr}\, v(\ldots)$. The ghost field v and gauge field $\mathbf{A}$ both have scaling dimension 1, so the term $\mathrm{tr}\, v d\mathbf{A} d\mathbf{A}$ has scaling dimension 5. The BRST operator increases the scaling dimension by 1, so $s\mathrm{tr}\, v d\mathbf{A} d\mathbf{A}$ has scaling dimension 6. In order to satisfy the consistency condition $s\mathcal{A}[v, \mathbf{A}] = 0$, $\mathrm{tr}\, v d\mathbf{A} d\mathbf{A}$ has to be cancelled by terms higher order in the gauge field with scaling dimension 5. We also know that the anomaly is proportional to the anti-symmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, so we can express these terms as 4-forms. Therefore up to adding a BRST exact form, the most general form of the anomaly is:

$$\mathcal{A}[v, \mathbf{A}] = \frac{i}{24\pi^2} \int \mathrm{tr}\, v(d\mathbf{A}d\mathbf{A} + b_1 \mathbf{A}^2 d\mathbf{A} + b_2 \mathbf{A} d\mathbf{A}\mathbf{A} + b_3 d\mathbf{A}\mathbf{A}^2 + c\mathbf{A}^4) \tag{C.20}$$

Requiring the anomaly to be BRST closed, $s\mathcal{A}[v, \mathbf{A}] = 0$, will fix the coefficients in the anomaly.

Before computing the $s\mathcal{A}[v, \mathbf{A}]$, it will be helpful to first compute:

$$s\mathbf{A} = -dv - \{\mathbf{A}, v\} \tag{C.21}$$

$$\begin{aligned} s\mathbf{A}^2 &= (s\mathbf{A})\mathbf{A} - \mathbf{A}(s\mathbf{A}) \\ &= (-dv - \{\mathbf{A}, v\})\mathbf{A} - \mathbf{A}(-dv - \{\mathbf{A}, v\}) \\ &= -\mathbf{A}v\mathbf{A} - v\mathbf{A}^2 + \mathbf{A}^2 v + \mathbf{A}v\mathbf{A} - dv\mathbf{A} + \mathbf{A}dv \\ &= [\mathbf{A}, dv] + [\mathbf{A}^2, v] \end{aligned}$$

$$
\begin{aligned}
s\mathbf{A}^3 &= (s\mathbf{A}^2)\mathbf{A} + \mathbf{A}^2(s\mathbf{A}) \\
&= ([\mathbf{A}, dv] + [\mathbf{A}^2, v])\mathbf{A} + \mathbf{A}^2(-dv - \{\mathbf{A}, v\}) \\
&= \mathbf{A}dv\mathbf{A} - dv\mathbf{A}^2 - \mathbf{A}^2v\mathbf{A} - v\mathbf{A}^3 - \mathbf{A}^2dv - \mathbf{A}^3v - \mathbf{A}^2v\mathbf{A} \\
&= -dv\mathbf{A}^2 + \mathbf{A}dv\mathbf{A} - \mathbf{A}^2dv - \left\{\mathbf{A}^2, v\right\}
\end{aligned}
$$

$$
\begin{aligned}
s\mathbf{A}^4 &= (s\mathbf{A}^3)\mathbf{A} - \mathbf{A}^3(s\mathbf{A}) \\
&= (-dv\mathbf{A}^2 + \mathbf{A}dv\mathbf{A} - \mathbf{A}^2dv - \left\{\mathbf{A}^2, v\right\})\mathbf{A} - \mathbf{A}^3(-dv - \{\mathbf{A}, v\}) \\
&= -dv\mathbf{A}^3 + \mathbf{A}dv\mathbf{A}^2 - \mathbf{A}^2dv\mathbf{A} - \mathbf{A}^3v\mathbf{A} - v\mathbf{A}^4 + \mathbf{A}^3dv + \mathbf{A}^4v + \mathbf{A}^3v\mathbf{A} \\
&= -dv\mathbf{A}^3 + \mathbf{A}dv\mathbf{A}^2 - \mathbf{A}^2dv\mathbf{A} + \mathbf{A}^3dv + [\mathbf{A}^4, v]
\end{aligned}
$$

$$
\begin{aligned}
s(d\mathbf{A}) = s(\mathbf{F} - \mathbf{A}^2) &= s\mathbf{F} - s\mathbf{A}^2 \\
&= [\mathbf{F}, v] - ([\mathbf{A}, dv] + [\mathbf{A}^2, v]) \\
&= [d\mathbf{A}, v] - [\mathbf{A}, dv]
\end{aligned}
$$

Now let us compute the consistency condition $s\mathcal{A}[v, \mathbf{A}] = 0$. First let us consider:

$$
\begin{aligned}
s\mathrm{tr}\, v\mathbf{A}^4 &= \mathrm{tr}\,(sv)\mathbf{A}^4 - \mathrm{tr}\, v(s\mathbf{A}^4) \\
&= -\mathrm{tr}\, v^2\mathbf{A}^4 - \mathrm{tr}\, v[\mathbf{A}^4, v] + \dots \\
&= -\mathrm{tr}\, v^2\mathbf{A}^4 - \mathrm{tr}\, v\mathbf{A}^4v + \mathrm{tr}\, v^2\mathbf{A}^4 + \dots \\
&= \mathrm{tr}\, v^2\mathbf{A}^4 + \dots
\end{aligned}
$$

where the $\dots$ are the terms containing both v and dv, and we the last term has a positive sign because

$$
\begin{aligned}
\mathrm{tr}\, v\mathbf{A}^4v &= \mathrm{tr}\, v_{\mu_1}(\mathbf{A}^4)_{\mu_2\mu_3\mu_4\mu_5}w_{\mu_6}dx^{\mu_1}\dots dx^{\mu_6} \\
&= \mathrm{tr}\, v_{\mu_6}v_{\mu_1}(\mathbf{A}^4)_{\mu_2\mu_3\mu_4\mu_5}dx^{\mu_1}\dots dx^{\mu_6} && \text{by cyclicity of trace} \\
&= (-1)^5\mathrm{tr}\, v_{\mu_6}v_{\mu_1}(\mathbf{A}^4)_{\mu_2\mu_3\mu_4\mu_5}dx^{\mu_6}dx^{\mu_1}\dots dx^{\mu_5} \\
&= -\mathrm{tr}\, v^2\mathbf{A}^4
\end{aligned}
$$

There can be no other term $\propto \mathrm{tr}\, v^2\mathbf{A}^4$ in $s\mathcal{A}[v, \mathbf{A}]$ since every other term involves at least one derivative. Therefore we have $c = 0$.

Computing the other terms in $s\mathcal{A}[v, \mathbf{A}]$ yields:

$$
\begin{aligned}
s\mathrm{tr}\, v\mathbf{A}^2d\mathbf{A} &= \mathrm{tr}\,\Big((sv)\mathbf{A}^2d\mathbf{A} - v(s\mathbf{A}^2)d\mathbf{A} - v\mathbf{A}^2s(d\mathbf{A})\Big) \\
&= \mathrm{tr}\,\Big(-v^2\mathbf{A}^2d\mathbf{A} - v([\mathbf{A}, dv] + [\mathbf{A}^2, v])d\mathbf{A} - v\mathbf{A}^2([d\mathbf{A}, v] - [\mathbf{A}, dv])\Big) \\
&= \mathrm{tr}\,(-v^2\mathbf{A}^2d\mathbf{A} - v\mathbf{A}dvd\mathbf{A} + vdv\mathbf{A}d\mathbf{A} - v\mathbf{A}^2vd\mathbf{A} + v^2\mathbf{A}2d\mathbf{A} \\
&\qquad - v\mathbf{A}^2d\mathbf{A}v + v\mathbf{A}^2vd\mathbf{A} + v\mathbf{A}^3dv - v\mathbf{A}^2dv\mathbf{A}) \\
&= \mathrm{tr}\,(v^2\mathbf{A}^2d\mathbf{A}v - v\mathbf{A}^2dv\mathbf{A} + v\mathbf{A}^3dv) + \mathrm{tr}\,(vdv\mathbf{A}d\mathbf{A} - v\mathbf{A}dvd\mathbf{A})
\end{aligned}
$$

$$\begin{aligned}
s\mathrm{tr}\, v\mathbf{A}d\mathbf{A}\mathbf{A} &= \mathrm{tr}\,\Big(-v^2\mathbf{A}d\mathbf{A}\mathbf{A} - v(s\mathbf{A})d\mathbf{A}\mathbf{A} + v\mathbf{A}(sd\mathbf{A})\mathbf{A} + v\mathbf{A}d\mathbf{A}(s\mathbf{A})\Big) \\
&= -v^2\mathbf{A}d\mathbf{A}\mathbf{A} - v(-dv - \{\mathbf{A}, v\})d\mathbf{A}\mathbf{A} \\
&\qquad + v\mathbf{A}([d\mathbf{A}, v] - [\mathbf{A}, dv])\mathbf{A} + v\mathbf{A}d\mathbf{A}(-dv - \{\mathbf{A}, v\}) \\
&= -v^2\mathbf{A}d\mathbf{A}\mathbf{A} + vdvd\mathbf{A}\mathbf{A} + v\mathbf{A}vd\mathbf{A}\mathbf{A} + v^2\mathbf{A}d\mathbf{A}\mathbf{A} \\
&\quad + v\mathbf{A}d\mathbf{A}v\mathbf{A} - v\mathbf{A}vd\mathbf{A}\mathbf{A} - v\mathbf{A}^2dv\mathbf{A} + v\mathbf{A}dv\mathbf{A}^2 \\
&\quad - v\mathbf{A}d\mathbf{A}dv - v\mathbf{A}d\mathbf{A}\mathbf{A}v - v\mathbf{A}d\mathbf{A}v\mathbf{A} \\
&= \mathrm{tr}\,(v^2\mathbf{A}d\mathbf{A}\mathbf{A} - v\mathbf{A}^2dv\mathbf{A} + v\mathbf{A}dv\mathbf{A}^2) + \mathrm{tr}\,(vdv\mathbf{A}\mathbf{A} + v\mathbf{A}d\mathbf{A}dv)
\end{aligned}$$

$$\begin{aligned}
s\mathrm{tr}\, vd\mathbf{A}\mathbf{A}^2 &= \mathrm{tr}\,\Big(-v^2\mathbf{A}d\mathbf{A}\mathbf{A}^2 - v(sd\mathbf{A})\mathbf{A}^2 - vd\mathbf{A}(s\mathbf{A}^2)\Big) \\
&= \mathrm{tr}\,\Big(-v^2d\mathbf{A}\mathbf{A}^2 - v([\mathbf{A}, dv] + [\mathbf{A}^2, v])\mathbf{A}^2 - vd\mathbf{A}([\mathbf{A}, dv] + [\mathbf{A}^2, v])\Big) \\
&= \mathrm{tr}\,(-v^2d\mathbf{A}\mathbf{A}^2 - vd\mathbf{A}v\mathbf{A}^2 + v^2d\mathbf{A}\mathbf{A}^2 + v\mathbf{A}dv\mathbf{A}^2 - vdv\mathbf{A}^3 \\
&\quad - vd\mathbf{A}\mathbf{A}dv + vd\mathbf{A}dv\mathbf{A} - vd\mathbf{A}\mathbf{A}^2v + vd\mathbf{A}v\mathbf{A}^2) \\
&= \mathrm{tr}\,(v^2d\mathbf{A}\mathbf{A}^2 + v\mathbf{A}dv\mathbf{A}^2 - vdv\mathbf{A}^3) + \mathrm{tr}\,(vd\mathbf{A}dv\mathbf{A} - vd\mathbf{A}\mathbf{A}dv)
\end{aligned}$$

Here we divided the terms into those containing only one derivative and those containing two derivatives. $s\mathrm{tr}\, vd\mathbf{A}d\mathbf{A}$ can only give terms involving two derivatives, the terms with one derivative must cancel each other or add up to an exact form. If we set

$$b \equiv b_1 = -b_2 = b_3 \tag{C.22}$$

then combining the three terms in the anomaly gives:

$$\begin{aligned}
s\mathrm{tr}\,(vd\mathbf{A}^3) &= s\mathrm{tr}\,(v\mathbf{A}^2d\mathbf{A} - v\mathbf{A}d\mathbf{A}\mathbf{A} + vd\mathbf{A}\mathbf{A}^2) \\
&= \mathrm{tr}\,(v^2\mathbf{A}^2d\mathbf{A}v - v\mathbf{A}^2dv\mathbf{A} + v\mathbf{A}^3dv) + \mathrm{tr}\,(vdv\mathbf{A}d\mathbf{A} - v\mathbf{A}dvd\mathbf{A}) \\
&\quad - \mathrm{tr}\,(v^2\mathbf{A}d\mathbf{A}\mathbf{A} - v\mathbf{A}^2dv\mathbf{A} + v\mathbf{A}dv\mathbf{A}^2) + \mathrm{tr}\,(vdvd\mathbf{A}\mathbf{A} + v\mathbf{A}d\mathbf{A}dv) \\
&\quad + \mathrm{tr}\,(v^2d\mathbf{A}\mathbf{A}^2 + v\mathbf{A}dv\mathbf{A}^2 - vdv\mathbf{A}^3) + \mathrm{tr}\,(vd\mathbf{A}dv\mathbf{A} - vd\mathbf{A}\mathbf{A}dv) \\
&= d\mathrm{tr}\, v^2\mathbf{A}^3 + \mathrm{tr}\,(-vdvd\mathbf{A}^2 - dvvd\mathbf{A}^2 - v\mathbf{A}dvd\mathbf{A}) \\
&= d\mathrm{tr}\, v^2\mathbf{A}^3 + \mathrm{tr}\,\Big(-2dvdv\mathbf{A}^2 + dv\mathbf{A}dv\mathbf{A} + 2d(vdv\mathbf{A}^2) - d(v\mathbf{A}dv\mathbf{A})\Big)
\end{aligned}$$

The exact forms integrate to zero, and we also have

$$\mathrm{tr}\, dv\mathbf{A}dv\mathbf{A} = -\mathrm{tr}\, dv\mathbf{A}dv\mathbf{A} = 0 \tag{C.23}$$

since dv$\mathbf{A}$is an anti-commuting 2-form. So the consistency condition becomes:

$$s\mathcal{A}[v, \mathbf{A}] = \frac{i}{24\pi^2}\int \Big(s\mathrm{tr}\, vd\mathbf{A}d\mathbf{A} - 2b\mathrm{tr}\, dvdv\mathbf{A}^2\Big) \tag{C.24}$$

Finally we can compute $s\mathrm{tr}\, vd\mathbf{A}d\mathbf{A}$ to determine b:

$$\begin{aligned}
s\mathrm{tr}\, vd\mathbf{A}d\mathbf{A} &= \mathrm{tr}\,\Big(-v^2 d\mathbf{A}d\mathbf{A} - v(sd\mathbf{A})d\mathbf{A} - vd\mathbf{A}(sd\mathbf{A})\Big)\\
&= \mathrm{tr}\,\Big(-v^2 d\mathbf{A}d\mathbf{A} - v([d\mathbf{A}, v] - [\mathbf{A}, dv])d\mathbf{A} - vd\mathbf{A}([d\mathbf{A}, v] - [\mathbf{A}, dv])\Big)\\
&= \mathrm{tr}\,(-v^2 d\mathbf{A}d\mathbf{A} - vd\mathbf{A}vd\mathbf{A} + v^2 d\mathbf{A}d\mathbf{A} + v\mathbf{A}dvd\mathbf{A} - vdv\mathbf{A}d\mathbf{A}\\
&\qquad\qquad - vd\mathbf{A}d\mathbf{A}v + vd\mathbf{A}vd\mathbf{A} + vd\mathbf{A}\mathbf{A}dv - vd\mathbf{A}dv\mathbf{A})\\
&= \mathrm{tr}\,(v^2 d\mathbf{A}d\mathbf{A} + dvvd\mathbf{A}\mathbf{A} - vdv\mathbf{A}d\mathbf{A} + v\mathbf{A}dvd\mathbf{A} - vd\mathbf{A}dv\mathbf{A})\\
&= \mathrm{tr}\,\Big(d(v^2 d\mathbf{A}\mathbf{A}) + vdvd\mathbf{A}\mathbf{A} - vdv\mathbf{A}d\mathbf{A} + d(v\mathbf{A}dv\mathbf{A}) - dv\mathbf{A}dv\mathbf{A}\Big)\\
&= \mathrm{tr}\,\Big(vdvd\mathbf{A}^2 + d(\ldots)\Big)\\
&= \mathrm{tr}\,\Big(dvdv\mathbf{A}^2 - d(vdv\mathbf{A}^2) + d(\ldots)\Big)
\end{aligned}$$

where to go from the fifth line to the sixth line we used the fact that $\mathrm{tr}\, dv\mathbf{A}dv\mathbf{A} = 0$. Inserting this expression back into the consistency condition, we have:

$$0 = s\mathcal{A}[v, \mathbf{A}] = \frac{i}{24\pi^2}\int \mathrm{tr}\,\Big(dvdv\mathbf{A}^2 - 2bdvdv\mathbf{A}^2\Big) \tag{C.25}$$

$$\implies b = \frac{1}{2} \tag{C.26}$$

Therefore the full form of the anomaly is

$$\mathcal{A}[v, \mathbf{A}] = \frac{i}{24\pi^2}\int \mathrm{tr}\, v\left(d\mathbf{A}d\mathbf{A} + \frac{1}{2}dA^3\right) = \frac{i}{24\pi^2}\int \mathrm{tr}\, v\, d\left(\mathbf{A}d\mathbf{A} + \frac{1}{2}A^3\right) \tag{C.27}$$

Replacing the ghost field ω with the transformation parameter ϵ and in terms of the original gauge field A, the anomaly is given by:

$$\mathcal{A}[\epsilon, A] = -\frac{1}{24\pi^2}\int \mathrm{tr}\, \epsilon d\left(AdA - \frac{i}{2}A^3\right) \tag{C.28}$$

$$\mathcal{A}_\alpha(x) = -\frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^\beta_\nu \partial_\rho A^\gamma_\sigma - \frac{i}{4}A^\beta_\nu [A_\rho, A_\sigma]^\gamma\right) D^{\mathcal{R}}_{\alpha\beta\gamma} \tag{C.29}$$

Appendix D

The 3-loop β-function and the IR fixed points

The 3-loop β function is given by (see [150, 151, 152])

$$
\begin{aligned}
\beta(g) =& -\beta_0 \frac{g^3}{(4\pi)^2} - \beta_1 \frac{g^5}{(4\pi)^4} - \beta_2 \frac{g^7}{(4\pi)^6} \ , \\
\beta_0 =& \frac{11}{6} C_2(G) - \sum_{\mathcal{R}} \frac{1}{3} T_{\mathcal{R}} n_{\mathcal{R}} \ , \\
\beta_1 =& \frac{34}{12} C_2^2(G) - \sum_{\mathcal{R}} \left\{ \frac{5}{6} n_{\mathcal{R}} C_2(G) T_{\mathcal{R}} + \frac{n_{\mathcal{R}}}{2} C_2(\mathcal{R}) T_{\mathcal{R}} \right\} \ , \\
\beta_2 =& \frac{2857}{432} C_2^3(G) - \sum_{\mathcal{R}} \frac{n_{\mathcal{R}} T_{\mathcal{R}}}{4} \left[-\frac{C_2^2(\mathcal{R})}{2} + \frac{205 C_2(G) C_2(\mathcal{R})}{36} + \frac{1415 C_2^2(G)}{108} \right] \\
&+ \sum_{\mathcal{R},\mathcal{R}'} \frac{n_{\mathcal{R}} n'_{\mathcal{R}} T_{\mathcal{R}} T_{\mathcal{R}'}}{16} \left[\frac{44 C_2(\mathcal{R})}{18} + \frac{158 C_2(G)}{54} \right] \ .
\end{aligned}
\tag{D.1}
$$

Here, G denotes the adjoint representation, and $n_{\mathcal{R}}$ is the number of the Weyl flavors in representation $\mathcal{R}$. Also, $C_2(\mathcal{R})$ is the quadratic Casimir operator of representation $\mathcal{R}$, defined as

$$
t^a_{\mathcal{R}} t^a_{\mathcal{R}} = C_2(\mathcal{R}) \mathbf{1}_{\mathcal{R}} \ . \tag{D.2}
$$

We reserve $C_2(G)$ for the quadratic Casimir of the adjoint representation. $T_{\mathcal{R}}$ is the Dynkin index of $\mathcal{R}$, which is defined by

$$
\mathrm{tr}\left[t^a_{\mathcal{R}} t^b_{\mathcal{R}} \right] = T_{\mathcal{R}} \delta^{ab} \ . \tag{D.3}
$$

From Eqs. (D.2) and (D.3), we easily obtain the useful relation

$$
T_{\mathcal{R}} \mathrm{dim}_G = C_2(\mathcal{R}) \mathrm{dim}_{\mathcal{R}} \ , \tag{D.4}
$$

where $\mathrm{dim}_{\mathcal{R}}$ is the dimension of $\mathcal{R}$.

In particular, we have $C_2(G) = 2N$, $\dim_G = N^2 - 1$, $T_\psi = N + 2$, $\dim_\psi = \frac{N(N+1)}{2}$, $C_2(\psi) = \frac{2(N+2)(N-1)}{N}$, $T_\chi = N - 2$, $\dim_\chi = \frac{N(N-1)}{2}$, $C_2(\chi) = \frac{2(N-2)(N+1)}{N}$. Then, the values of β_0 to β_2 are

$$
\begin{aligned}
\beta_0 &= \frac{1}{3}\left[11N - \frac{2}{k}(N^2 - 8)\right] \ , \\
\beta_1 &= \frac{2\left(-48 + 76N^2 + 17kN^3 - 8N^4\right)}{3kN} \ , \\
\beta_2 &= \frac{1}{54k^2N^2}\Big[2857k^2N^5 + N(-8448 + 12448N^2 - 2584N^4 + 145N^6) \\
&\quad - 2k(864 + 3948N^2 - 8945N^4 + 988N^6)\Big] \ .
\end{aligned}
\tag{D.5}
$$

Assuming that $\beta_0 > 0$ and $\beta_1 < 0$, the theory develops an IR fixed point to 2-loops. The value of the coupling constant at the fixed point is

$$
\alpha_* \equiv \frac{g_*^2}{4\pi} = -\frac{4\pi\beta_0}{\beta_1} = \frac{2\pi N\left(16 + 11kN - 2N^2\right)}{48 - 76N^2 - 17kN^3 + 8N^4} \ . \tag{D.6}
$$

To assess the stability of this fixed point, we can examine the roots of the β-function when the 3-loop term is taken into account.

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Colophon

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