Asymptotic results concerning heat content and spectra of the Laplacian

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Abstract: We investigate the relationship between analytical quantities associated with the Laplacian on a domain $\Omega \subset \mathbb{R}^d$ and the geometry of Ω. In particular, we prove new results concerning small-time asymptotics for the heat content of polygons contained inside larger polygons with Neumann boundary conditions imposed. We also prove some new results concerning the asymptotic behaviour of minimisers to spectral shape optimisation problems for Neumann, and consequently Robin, eigenvalues of the Laplacian under perimeter and diameter constraint. Moreover, we consider some related spectral shape optimisation problems for mixed Dirichlet-Neumann, so-called Zaremba, eigenvalues of the Laplacian.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Some of the work presented in this thesis has been published, accepted in journals, or is available as preprints in public repositories – the relevant publications are listed below.

- **Sam Farrington**, Katie Gittins. *Heat flow in polygons with reflecting edges*. Integral Equations and Operator Theory 95.4 (2023).

  [Both authors contributed equally to this work]

- **Sam Farrington**. *On the Isoperimetric and Isodiametric Inequalities and the Minimisation of Eigenvalues of the Laplacian*. Journal of Geometric Analysis 35.2 (2025).

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Firstly, to my old friends Connah, Joe, Lewis and Mark for bringing a healthy dosage of joy and reality away from my studies. I have no doubt that we will have many more great times in the future on various, and arguably slightly bizarre, travels around the UK and abroad.

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Thirdly, to the many great members of St Mary's Ultimate Frisbee (SMUF). Time may have come for me to hang up my boots, but it has been an honour to play for such a wonderful club over the past eight years. May SMUF continue to reign supreme on the college frisbee scene for many more years to come.

Fourthly, to my loved ones. To my parents, Ben and Jan, who have always been there through thick and thin, and have been a steadfast rock of support throughout my life. You both have been, and continue to be, such marvellous parents and no amount of words will ever suffice to represent just how much you have done for me. To my sister Emma for her brutal honesty and support. I hope this thesis does not at all disprove that I am, and always will be, your silly little brother. And, to my wonderfully talented girlfriend Emma[1], who has been at my side from day one of my PhD. You have been patient beyond belief through my many mad ramblings and various panics! I truly love you Emma and I am so excited for our future together.

To end, I would like to pay tribute to those who I have lost whilst undertaking my PhD studies.

Firstly, to my late grandfather Edward Herbert Farrington, from whom I hoped to have gained some conjuring powers and sleight of hand to put to use in mathematics. May you rest in eternal peace.

Secondly, to my late grandmother Mary Valerie Farrington (née Poole), who no doubt would've shown boundless enthusiasm towards the contents of this thesis. Not because she was a mathematician or a scientist, but because that was just who she was – a truly loving person fascinated by the interests of others. She embodied the best of humanity and I hope that I too can have the same zeal to make the world that little bit better. Thank you for all the wonderful memories Mary, may you rest in eternal peace.

[1]These two Emmas are indeed two very different people!

You're entirely bonkers. But I'll tell you a secret: all the best people are.

— from *Alice's Adventures in Wonderland* by Lewis Carroll

Dedicated to

My grandfather Gump, more formally Professor David Williams F.R.S., who I hope is proud of my 'silly sums'

Contents

List of Figures

List of Tables

Chapter 1

Introduction

This thesis focuses on problems at the interface of analysis and geometry, namely: spectral shape optimisation and small-time heat flow asymptotics. Problems at this interface have long been studied in the mathematical literature and can be traced back to mythological beginnings. In Virgil's Aeneid [Vir09, p. 23] one may read the passage:

> *"At last they landed, where from far your eyes*
> *May view the turrets of new Carthage rise;*
> *There bought a space of ground, which (Byrsa call'd,*
> *From the bull's hide) they first inclos'd, and wall'd."*
>
> – The Aeneid, Virgil

This passage refers to Queen Dido fleeing to near Carthage, nowadays a suburb of Tunis in Tunisia, after her husband was killed by her brother. Upon arrival, after some bargaining, she is able to obtain as much land as she may enclose within a bull's hide. As the tale goes, Queen Dido and her party made the hide into thin strips to maximise the amount of land they could enclose. Mathematically, we may reformulate this into the question: among all domains of a given perimeter, which one has the largest area? The well-known answer is of course the ball. This is the famous isoperimetric problem that has a long and illustrious history, which is far too lengthy to tell here. We refer the reader to the wonderful articles of Bandle [Ban17] and Blåsjö [Blå05] for more information on this.

A particularly interesting proof of the isoperimetric inequality in any dimension for measurable sets of a given perimeter is by Preunkert [Pre04], who generalised the earlier work of Ledoux [Led94]. The proof uses the heat semigroup $(T_t)_{t\geq 0}$ on $L^2(\mathbb{R}^d)$, for which $T_t f$, $f \in L^2(\mathbb{R}^d)$, is the unique solution to

$$\begin{cases} \dfrac{\partial}{\partial t} u(t;x) = \Delta u(t;x), & t > 0,\ x \in \mathbb{R}^d, \\ \lim_{t\downarrow 0} \|u(t;\cdot) - f\|_{L^2(\mathbb{R}^d)} = 0. \end{cases} \tag{1.0.1}$$

We use the analyst's convention for the Laplacian in this thesis, that is

$$\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \tag{1.0.2}$$

rather than the geometer's Laplacian which has a minus sign in the definition. In [MirPPP07] it is shown that for any Borel set $D \subset \mathbb{R}^d$ of finite perimeter,

$$H_D(t) := \int_D T_t \mathbb{1}_D = V(D) - P(D)\frac{t^{1/2}}{\pi^{1/2}} + o(t^{1/2}) \tag{1.0.3}$$

as $t \downarrow 0$, where $V(D)$ denotes the volume of D and $P(D)$ denotes the perimeter of D. Here, we call $H_D(t)$ the open heat content of D. Physically, $H_D(t)$ measures the amount of heat in D at time $t \geq 0$ given that at time $t = 0$ the temperature is one in D and zero outside D.

In two dimensions, if D is a polygon, van den Berg and Gittins [BerGit16] showed that one can refine the small-time asymptotic expansion in (1.0.3) to

$$H_D(t) = V(D) - P(D)\frac{t^{1/2}}{\pi^{1/2}} + \left(\sum_{\alpha\in\mathcal{A}} a(\alpha)\right) t + o(t^\infty) \tag{1.0.4}$$

as $t \downarrow 0$, where $\mathcal{A}$ is the collection of interior angles of D and $a : (0, 2\pi) \to \mathbb{R}$ is an explicitly known function. Other similar small-time asymptotic expansions for the heat content of polygons for the heat equation with Dirichlet boundary conditions have been computed in [BerSri90; BerGG20]. In this thesis, we also compute small-time asymptotic expansions for the heat content of polygons, but instead with respect to the Neumann heat semigroup $(S_t)_{t\geq 0}$ on $L^2(\Omega)$ for a larger polygon $\Omega \supset D$. For $f \in L^2(\Omega)$, $S_t f$ is the unique solution to

$$\begin{cases} \dfrac{\partial}{\partial t} u(t;x) = \Delta u(t;x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial}{\partial n} u(t;x) = 0, & t > 0,\ x \in \partial\Omega, \\ \lim_{t\downarrow 0} \|u(t;\cdot) - f\|_{L^2(\Omega)} = 0, \end{cases} \tag{1.0.5}$$

where $\frac{\partial}{\partial n}$ denotes the inwards pointing normal derivative on the boundary. We the define the heat content of D with respect to Ω as

$$H_{D\subset\Omega}(t) := \int_D S_t \mathbb{1}_D. \tag{1.0.6}$$

The primary result in this thesis for the heat content of polygons is Theorem 4.2.1, which roughly reads

$$H_{D\subset\Omega}(t) = V(D) - P(D,\Omega)\frac{t^{1/2}}{\pi^{1/2}} + (\text{angular contributions})t + o(t^\infty) \tag{1.0.7}$$

as $t \downarrow 0$, where $P(D,\Omega)$ is the perimeter of D inside Ω. Our methods work even when Ω is unbounded, e.g. an infinite sector. We also discuss some generalisations for when Ω has interior angles of angle 2π and when D becomes more complicated, i.e. when vertices have a large number of incident edges. These results come from the paper [FarGit23], which was a joint work between the author and his PhD supervisor.

Another famous isoperimetric problem was put forward by Lord Rayleigh on page 284 of his book ‘A Theory of Sound’ [Str77]. In it he conjectured the following:

> *“If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.”*
>
> – John William Strutt, 3rd Baron Rayleigh

In modern mathematical speak, Lord Rayleigh’s conjecture may be expressed as: the ball minimises the first Dirichlet eigenvalue over all bounded domains in $\mathbb{R}^2$ of a given area. Here, for a given bounded domain $\Omega \subset \mathbb{R}^2$, its first Dirichlet eigenvalue $\lambda_1(\Omega)$ is given by the smallest value of $\lambda > 0$ such that

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.0.8}$$

admits a non-trivial (weak) solution. Lord Rayleigh's conjecture was independently proven by Faber [Fab23] and Krahn [Kra25] in the 1920's. In fact, they showed that the result is true in any dimension and the result is now colloquially known as the Faber-Krahn inequality. The Faber-Krahn inequality is an example of a spectral shape optimisation problem.

In this thesis, we consider spectral shape optimisation problems for Neumann eigenvalues of bounded convex domains $\Omega \subset \mathbb{R}^d$ under perimeter and diameter constraint. These eigenvalues arise as the values $\mu \geq 0$ for which the partial differential equation

$$\begin{cases} -\Delta u(x) = \mu u(x), & x \in \Omega, \\ \frac{\partial}{\partial n} u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.0.9}$$

has a non-trivial (weak) solution. The Neumann eigenvalues form a countable collection that only accumulates at $+\infty$ and we denote them (counting multiplicities) by

$$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots . \tag{1.0.10}$$

Letting $D(\Omega)$ denote the diameter of Ω, we prove that the minimisation problems

$$\min\left\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex, } P(\Omega) = 1\right\} \tag{1.0.11}$$

and

$$\min\left\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex, } D(\Omega) = 1\right\}, \tag{1.0.12}$$

for any $d \geq 2$, have minimisers for $k \in \mathbb{N}$ sufficiently large. Moreover, we show that any sequence of minimisers converges to the ball as $k \to +\infty$ with respect to the Hausdorff distance and give an asymptotic rate for this convergence, see Theorems

3.1.1 and 3.1.3. Such results relate to the isoperimetric and isodiametric inequalities via the well-known Weyl asymptotic law

$$\mu_k(\Omega) \sim 4\pi^2 \left(\frac{k}{\omega_d V(\Omega)}\right)^{2/d} \tag{1.0.13}$$

as $k \to +\infty$, where ω_d is the volume of the d-dimensional unit ball, in the sense that one would want to maximise volume in order to minimise Neumann eigenvalues under perimeter or diameter constraint. This is of course a naïve assertion that is not wholly true, as we shall discuss later on in Theorems 3.1.2 and 3.1.6. We also show how analogous results to those in the Neumann case carry over to spectral minimisation problems for Robin and mixed Dirichlet-Neumann, so-called Zaremba, eigenvalues. The results on asymptotic spectral shape optimisation given in this thesis is a mildly expanded version of the paper [Far25] written by the author of this thesis.

Our work on the spectral shape optimisation of Neumann eigenvalues here is complementary to a recent paper by Bogosel, Henrot and Michetti [BogHM24], in which the authors give a detailed analysis of minimisation problems (1.0.11) and (1.0.12) in dimension two. We provide a brief exposition of this paper in Section 3.2.3 due to its relevance to our results.

1.1 Structure of this thesis

We now give an overview of the outline of the contents of this thesis. Due to the differing literature for small-time heat content asymptotics and asymptotic spectral shape optimisation, we defer referring to the relevant literature fully until the appropriate chapters of the thesis.

In Chapter 2, we summarise the key components from convex geometry, partial differential equations, probability theory and spectral theory required to understand the proofs of the results in this thesis. Most of the material in this chapter may be found in a standard graduate textbook on the relevant areas, and where material is non-standard we either provide an explicit reference or do the proof ourselves. This chapter also contains most of the notation that is used throughout the thesis.

In Chapter 3, we study the results of [Far25] which concern the asymptotic behaviour of minimisers of eigenvalues of the Laplacian under diameter and perimeter constraint. This work closely relates to the papers [BucFre13; Ber15], which study such problems in the case of Dirichlet boundary conditions and were the initial inspiration for the ideas in this chapter. Our primary contribution is extending some of the results in [Ber15] to the case of Neumann boundary conditions.

In Chapter 4, we study the results of [FarGit23] concerning the small-time asymptotics for the heat content of bounded polygonal subdomains of larger polygonal domains under Neumann boundary conditions. This work extends that of the papers [BerSri90; BerGit16; BerGG20] from Dirichlet and open boundary conditions to the case of Neumann boundary conditions.

In Chapter 5, we conclude with some further possible avenues for research related to the problems considered in this thesis as well as some conjectural remarks on some of the possible answers. Of particular interest in this conclusion is unifying the ideas of shape optimisation and heat content to look into shape optimisation for heat content.

Chapter 2

Preliminary material

We aim to keep the presentation of results self-contained as far as is reasonably possible throughout this thesis. In this spirit, we now provide some standard reference material that will be used throughout the following chapters. Most of the material presented can be found in graduate textbooks on convex geometry, partial differential equations, probability theory and spectral theory. The main basis for our exposition here are the books [Eva10; Gru07; LevPM23; RogWil00] and the paper [GalMcK72]. Our exposition is intentionally kept light for conciseness.

2.1 Some quick spectral theory

Here we provide a quick overview of necessary pre-requisite spectral theory needed in this thesis. For clarity to the reader, given a bounded open subset Ω of $\mathbb{R}^d$:

- $L^2(\Omega)$ is the usual Hilbert space of real-valued square integrable functions equipped with the inner product

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, \mathrm{d}x; \tag{2.1.1}$$

- $H^1(\Omega)$ is the usual Hilbert space of real-valued square integrable functions with square integrable first order weak partial derivatives equipped with the inner product

$$\langle f, g\rangle_{H^1(\Omega)} = \int_\Omega \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x + \langle f, g\rangle_{L^2(\Omega)}; \tag{2.1.2}$$

- and, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ given by the closure of the space

$$C_0^\infty(\Omega) := \{\phi \in C^\infty(\Omega) : d(\mathrm{supp}(\phi), \partial\Omega) > 0\} \tag{2.1.3}$$

with respect to the norm on $H^1(\Omega)$. Here $\mathrm{supp}(\phi)$ is the support of the function ϕ defined as

$$\mathrm{supp}(\phi) := \overline{\{x \in \Omega : \phi(x) \neq 0\}}. \tag{2.1.4}$$

One may also equivalentally define $H_0^1(\Omega)$, when Ω is Lipschitz, as functions in $H^1(\Omega)$ who trace is zero on the boundary $\partial\Omega$ via the trace theorem for Sobolev spaces, see for example [Eva10, §5.5].

2.1.1 Dirichlet eigenvalues

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and consider the Helmholtz equation with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1.5}$$

It is well-known in this case that the associated Dirichlet Laplacian $-\Delta_\Omega^D$ acting on $L^2(\Omega)$ has discrete spectrum consisting of a countable collection of Dirichlet eigenvalues

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \cdots, \tag{2.1.6}$$

that accumulates only at $+\infty$. These eigenvalues have associated eigenfunctions $u_1, u_2, u_3, \ldots \in H_0^1(\Omega)$, which can be chosen so they form an orthonormal basis of $L^2(\Omega)$, and any eigenpair $(\lambda_k(\Omega), u_k)$ weakly satisfies the partial differential equation in (2.1.5). Dirichlet eigenvalues also admit the variational characterisation

$$\lambda_k(\Omega) = \min_{\substack{S \subset H_0^1(\Omega) \\ \dim(S)=k}} \max_{\substack{u \in S \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega |u|^2} \tag{2.1.7}$$

and satisfy the Weyl law

$$\lambda_k(\Omega) \sim 4\pi^2 \left(\frac{k}{\omega_d V(\Omega)}\right)^{2/d} \tag{2.1.8}$$

as $k \to +\infty$, where $V(\Omega)$ denotes the d-dimensional volume of Ω and

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})} \tag{2.1.9}$$

denotes the volume of the d-dimensional unit ball.

We also define the Dirichlet eigenvalue counting function by

$$\mathcal{N}_\Omega^D(\alpha) := \#\{k \in \mathbb{N} : \lambda_k(\Omega) < \alpha\}. \tag{2.1.10}$$

The Dirichlet eigenvalue counting function satisfies the Weyl law

$$\mathcal{N}_\Omega^D(\alpha) \sim \frac{\omega_d V(\Omega)}{(2\pi)^d} \alpha^{d/2} \tag{2.1.11}$$

as $\alpha \to +\infty$.

There are lots of results linking the geometry of Ω to the Dirichlet eigenvalues, see [Hen06] for a good overview, which are far too numerous to list in this thesis. The most fundamental result to our discussion is the Faber-Krahn inequality, as mentioned in the introduction, which asserts that the ball minimises the first Dirichlet eigenvalue over all bounded domains of a given volume.

Theorem 2.1.1 (Faber-Krahn, see [Fab23; Kra25]). *For any bounded open set $\Omega \subset \mathbb{R}^d$, we have that*

$$\lambda_1(\Omega) \cdot V(\Omega)^{2/d} \geq \lambda_1(\mathbb{B}^d) \cdot (\omega_d)^{2/d}, \tag{2.1.12}$$

where $\mathbb{B}^d$ is the d-dimensional unit ball.

Dirichlet eigenvalues also have three key properties that are of importance to us here, namely:

- Reverse monotonicity under inclusion of domains, i.e. $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$ whenever $\Omega_1 \subset \Omega_2$;
- (-2)-scaling under homothety, i.e, $\lambda_k(\alpha\Omega) = \alpha^{-2}\lambda_k(\Omega)$ for all $\alpha \in (0, +\infty)$ and $k \in \mathbb{N}$;
- and, invariance under isometries of $\mathbb{R}^d$, i.e. $\lambda_k(\Omega) = \lambda_k(f(\Omega))$ for any isometry $f : \mathbb{R}^d \to \mathbb{R}^d$ and $k \in \mathbb{N}$.

All of the above properties may be obtained directly from the variational characterisation of Dirichlet eigenvalues, see equation (2.1.7).

2.1.2 Neumann eigenvalues

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and consider the Helmholtz equation with Neumann boundary conditions:

$$\begin{cases} -\Delta u(x) = \mu u(x), & x \in \Omega, \\ \dfrac{\partial}{\partial n} u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1.13}$$

It is well-known in this case that the associated Neumann Laplacian $-\Delta_\Omega^N$ acting on $L^2(\Omega)$ has discrete spectrum consisting of a countable collection of Neumann eigenvalues

$$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots \tag{2.1.14}$$

that accumulates only at $+\infty$. These eigenvalues have associated eigenfunctions $u_1, u_2, u_3, \ldots \in H^1(\Omega)$, which can be chosen so they form an orthonormal basis of $L^2(\Omega)$, and any eigenpair $(\mu_k(\Omega), u_k)$ weakly satisfies the partial differential equation in (2.1.13). Neumann eigenvalues also admit the variational characterisation

$$\mu_k(\Omega) = \min_{\substack{S \subset H^1(\Omega) \\ \dim(S) = k}} \max_{\substack{u \in S \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega |u|^2} \tag{2.1.15}$$

and satisfy the Weyl law

$$\mu_k(\Omega) \sim 4\pi^2 \left(\frac{k}{\omega_d V(\Omega)} \right)^{2/d} \tag{2.1.16}$$

as $k \to +\infty$.

We also define the Neumann eigenvalue counting function by

$$\mathcal{N}_\Omega^N(\alpha) := \#\{k \in \mathbb{N} : \mu_k(\Omega) < \alpha\}. \tag{2.1.17}$$

The Neumann eigenvalue counting function satisfies the Weyl law

$$\mathcal{N}_\Omega^N(\alpha) \sim \frac{\omega_d V(\Omega)}{(2\pi)^d} \alpha^{d/2} \tag{2.1.18}$$

as $\alpha \to +\infty$, which is the same as for the Dirichlet eigenvalue counting function in (2.1.11).

As in the case of Dirichlet eigenvalues, there is a plethora of literature on the interplay between the geometry of Ω and its Neumann eigenvalues. Again, see [Hen06] for a good overview of this. The result in this direction that is most pertinent to us is the Payne-Weinberger inequality, which provides a lower bound on the first non-trivial Neumann eigenvalue of a bounded convex domain in terms of its diameter.

Theorem 2.1.2 (Payne-Weinberger inequality, see [PayWei60] and [Beb03])**.** *For a given bounded convex domain $\Omega \subset \mathbb{R}^d$, one has that*

$$\mu_2(\Omega) D(\Omega)^2 > \pi^2, \tag{2.1.19}$$

where $D(\Omega)$ is the diameter of Ω.

Three key things for us to note about the properties of Neumann eigenvalues in this thesis are the following:

- Reverse monotonicity under inclusion of domains is not necessarily true, i.e. we may have $\mu_k(\Omega_1) < \mu_k(\Omega_2)$ when $\Omega_1 \subset \Omega_2$ – see Figure 2.1 for an example of this;

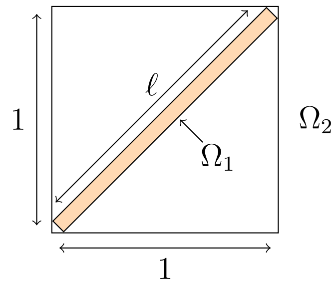

Figure 2.1: Illustration of a case of lack of reverse domain monotonicity for Neumann eigenvalues in dimension two. Here Ω_2 is the unit square with $\mu_2(\Omega_2) = \pi^2$ and Ω_1 is a suitably thin rectangle of length $\ell \approx \sqrt{2}$ contained in Ω_2 for which $\mu_2(\Omega_2) = \pi^2\ell^{-2} \approx \pi^2/2$.

- but, they do have the same (-2)-scaling under homothety as Dirichlet eigenvalues, i.e, $\mu_k(\alpha\Omega) = \alpha^{-2}\mu_k(\Omega)$ for all $\alpha \in (0, +\infty)$ and $k \in \mathbb{N}$;
- and, they are also invariant under isometries of $\mathbb{R}^d$, i.e. $\mu_k(\Omega) = \mu_k(f(\Omega))$ for any isometry $f : \mathbb{R}^d \to \mathbb{R}^d$ and $k \in \mathbb{N}$.

The lack of reverse domain monotonicity for Neumann eigenvalues makes them more challenging to deal with in regards to spectral shape optimisation problems. We remark that some recent results have been proven with respect to reverse domain monotonicity (up to a constant) of Neumann eigenvalues, see [Fun23].

2.1.3 A few words on Pólya's conjecture

In 1954, Pólya conjectured [Pól54] that the asymptotic term in Weyl's law is in fact an upper bound for the Dirichlet eigenvalue counting function and a lower bound for the Neumann eigenvalue counting function, that is

$$\mathcal{N}_\Omega^D(\alpha) \leq \frac{\omega_d V(\Omega)}{(2\pi)^d}\alpha^{d/2} \leq \mathcal{N}_\Omega^N(\alpha) \tag{2.1.20}$$

for all $\alpha > 0$. In terms of the eigenvalues, this may be equivalently reformulated as

$$\mu_{k+1}(\Omega) \leq 4\pi^2 \left(\frac{k}{\omega_d V(\Omega)}\right)^{2/d} \leq \lambda_k(\Omega). \tag{2.1.21}$$

Pólya's conjecture is known to hold for tiling domains, see [Pól61; Kel66]. Of pertinence to us in this thesis is that the conjecture holds for cuboids in any dimension as cuboids are tiling domains. More recently, in [FilLPS23], Filonov, Levitin, Polterovich and Sher proved that Pólya's conjecture holds true for the Dirichlet eigenvalue counting function of balls in any dimension and the Neumann eigenvalue counting function of balls in two dimensions. We use the latter in the proof of Theorem 3.1.1 and estimating the value N_2 in Table 3.1.

In 1912, Weyl himself conjectured [Wey12] the following refinement to Weyl's law:

$$\mathcal{N}_\Omega^D(\alpha) = \frac{V(\Omega)}{(2\pi)^d}\omega_d\alpha^{d/2} - \frac{P(\Omega)}{4\cdot(2\pi)^{d-1}}\omega_{d-1}\alpha^{(d-1)/2} + o(\alpha^{(d-1)/2}), \tag{2.1.22}$$

$$\mathcal{N}_\Omega^N(\alpha) = \frac{V(\Omega)}{(2\pi)^d}\omega_d\alpha^{d/2} + \frac{P(\Omega)}{4\cdot(2\pi)^{d-1}}\omega_{d-1}\alpha^{(d-1)/2} + o(\alpha^{(d-1)/2}) \tag{2.1.23}$$

as $\alpha \to +\infty$. In the literature, this is called the two-term Weyl asymptotic. The conjecture is known to hold when Ω satisfies a certain dynamical condition, see [Ivr80]. In particular, it is known that convex analytic domains and polygons satisfy this dynamical condition and hence their respective Dirichlet and Neumann eigenvalue counting functions satisfy the two-term Weyl asymptotic, see [SafVas97] and the references therein. For such domains, we then know that their Dirichlet and Neumann eigenvalue counting functions satisfy Pólya's conjecture for $\alpha > 0$ sufficiently large by the two-term Weyl asymptotics.

We refer the reader to [FilLPS23] for more information on Pólya's conjecture.

2.1.4 Other boundary conditions

In this thesis we also consider eigenvalues of the Laplacian with Robin and mixed Dirichlet-Neumann, so-called Zaremba, boundary conditions. As in the previous subsection, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.

Firstly, consider the Helmholtz equation on Ω with Robin boundary conditions of constant parameter $\beta \in (0, +\infty)$:

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ \dfrac{\partial}{\partial n} u(x) = \beta u(x), & x \in \partial\Omega. \end{cases} \tag{2.1.24}$$

The associated Robin Laplacian $-\Delta_\Omega^\beta$ acting on $L^2(\Omega)$ has discrete spectrum consisting of a countable collection of Robin eigenvalues

$$0 < \lambda_1^\beta(\Omega) \le \lambda_2^\beta(\Omega) \le \lambda_3^\beta(\Omega) \le \cdots \tag{2.1.25}$$

that accumulates only at $+\infty$. These eigenvalues have associated eigenfunctions $u_1, u_2, u_3, \ldots \in H^1(\Omega)$, which can be chosen so they form an orthonormal basis of $L^2(\Omega)$, and any eigenpair $(\lambda_k^\beta(\Omega), u_k)$ weakly satisfies the partial differential equation in (2.1.24). Robin eigenvalues admit the variational characterisation

$$\lambda_k^\beta(\Omega) = \min_{\substack{S \subset H^1(\Omega) \\ \dim(S)=k}} \max_{\substack{u \in S \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 + \beta \int_{\partial\Omega} |u|^2}{\displaystyle\int_\Omega |u|^2}. \tag{2.1.26}$$

Now, consider the Helmholtz equation on Ω with Dirichlet boundary conditions on a relatively open subset of the boundary $\Gamma \subset \partial\Omega$ and Neumann boundary conditions on $\partial\Omega \backslash \Gamma$:

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \Gamma, \\ \dfrac{\partial}{\partial n} u(x) = 0, & x \in \partial\Omega \backslash \Gamma. \end{cases} \tag{2.1.27}$$

The associated Zaremba Laplacian $-\Delta_\Omega^\Gamma$, with the aforementioned mixed Dirichlet-Neumann boundary conditions, acting on $L^2(\Omega)$ has discrete spectrum consisting of a countable collection of Zaremba eigenvalues

$$0 < \lambda_1^\Gamma(\Omega) \le \lambda_2^\Gamma(\Omega) \le \lambda_3^\Gamma(\Omega) \le \cdots \tag{2.1.28}$$

that accumulates only at $+\infty$. These eigenvalues have associated eigenfunctions $u_1, u_2, u_3, \ldots \in H^1_{0,\Gamma}(\Omega)$, which can be chosen so they form an orthonormal basis of $L^2(\Omega)$, and any eigenpair $(\lambda_k^\Gamma(\Omega), u_k)$ weakly satisfies the partial differential equation in (2.1.27). Here $H^1_{0,\Gamma}(\Omega)$ is the Sobolev space given by the closure of the space

$$C^\infty_{0,\Gamma}(\Omega) := \{\phi \in C^\infty(\Omega) : d(\operatorname{supp}(\phi), \Gamma) > 0\} \tag{2.1.29}$$

with respect to the norm on $H^1(\Omega)$. Equivalently, $H^1_{0,\Gamma}(\Omega)$ is the space of functions in $H^1(\Omega)$ whose trace is zero on Γ. These eigenvalues admit the variational characterisation

$$\lambda^\Gamma_k(\Omega) = \min_{\substack{S \subset H^1_{0,\Gamma}(\Omega) \\ \dim(S)=k}} \max_{\substack{u \in S \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega |u|^2}. \tag{2.1.30}$$

Both Robin and Zaremba eigenvalues satisfy the Dirichlet-Neumann bracketing relation

$$\mu_k(\Omega) \leq \lambda^\beta_k(\Omega), \lambda^\Gamma_k(\Omega) \leq \lambda_k(\Omega), \tag{2.1.31}$$

and hence they also satisfy Weyl's law. This bracketing relation can be proven directly from the variational characterisations of these eigenvalues.

One may define the respective Robin and Zaremba eigenvalue counting functions totally analogously to those of the Dirichlet and Neumann eigenvalue counting functions, but we do not need them in this thesis.

2.2 Convex geometry and the Hausdorff distance

We now present some results from convex geometry and Hausdorff convergence of bounded convex domains. Our exposition is somewhat non-canonical as the results are usually stated for compact convex domains and we introduce a slightly modified notion of Hausdorff convergence, which we call $\sim$-Hausdorff convergence. These choices are stylistic and make the statement of results in Chapter 3 more concise. Most of the material presented here may be found in the book by Gruber [Gru07].

Throughout this section and in Chapter 3, $\mathcal{A}^d$ will denote the collection of bounded convex subdomains of $\mathbb{R}^d$.

2.2.1 Geometric definitions and inequalities

The three principle geometric quantities used in this thesis are volume, perimeter and diameter, for which we now give a formal definition. We also define the inradius and circumradius of a convex domain here. Let $\Omega \in \mathcal{A}^d$, then:

- The volume of Ω, denoted by $V(\Omega)$, is the d-dimensional Lebesgue measure of Ω.

- The perimeter of Ω, denoted by $P(\Omega)$, is the $(d-1)$-dimensional Hausdorff measure of the topological boundary $\partial\Omega$ of Ω, i.e. $P(\Omega) := \mathcal{H}^{d-1}(\partial\Omega)$.

- The diameter of Ω, denoted by $D(\Omega)$, is defined as

$$D(\Omega) := \sup_{x,y\in\Omega} \|x-y\|_2, \tag{2.2.1}$$

  where $\|\cdot\|_2$ is the usual Euclidean norm on $\mathbb{R}^d$.

- The inradius of Ω, denoted by $\rho(\Omega)$, is defined as

$$\rho(\Omega) := \sup\{r > 0 : \exists x \in \Omega \text{ with } B_r(x) \subset \Omega\}, \tag{2.2.2}$$

  where $B_r(x)$ is the d-dimensional open ball of radius $r > 0$ centred at the point $x \in \mathbb{R}^d$.

- The circumradius of Ω, denoted by $R(\Omega)$, is defined as

$$R(\Omega) := \inf\{r > 0 : \exists x \in \mathbb{R}^d \text{ with } B_r(x) \supset \Omega\}. \tag{2.2.3}$$

We note that volume, perimeter, diameter, inradius and circumradius are all monotone with respect to the inclusion of domains in $\mathcal{A}^d$. The key property to highlight here is the monotonicity of perimeter which is not true for inclusion of Lipschitz domains generally – consider for example a dented ball. Moreover, these quantities satisfy the following scaling relations under homothety,

$$V(\alpha\Omega) = \alpha^d V(\Omega),\ P(\alpha\Omega) = \alpha^{d-1} P(\Omega),\ D(\alpha\Omega) = \alpha D(\Omega),$$
$$\rho(\alpha\Omega) = \alpha\rho(\Omega),\ R(\alpha\Omega) = \alpha R(\Omega) \tag{2.2.4}$$

for $\alpha \in (0, +\infty)$ and they are all invariant under isometries of $\mathbb{R}^d$.

We now state two famous geometric inequalities that are the protagonists of Chapter 3 of this thesis. We will later use quantitative versions of these inequalities in the proofs of the main results in Chapter 3 but we omit their statement for now.

Theorem 2.2.1 (Isoperimetric inequality)**.** *For any* $\Omega \in \mathcal{A}^d$

$$P(\Omega) \geq d(\omega_d)^{1/d} V(\Omega)^{(d-1)/d}, \tag{2.2.5}$$

with equality if and only if Ω *is a ball.*

Theorem 2.2.2 (Isodiametric inequality)**.** *For any* $\Omega \in \mathcal{A}^d$

$$V(\Omega) \leq 2^{-d}\omega_d D(\Omega)^d, \tag{2.2.6}$$

with equality if and only if Ω *is a ball.*

We now state two inequalities that are of great use to us in this thesis to give lower bounds on the inradius of a bounded convex domain. Firstly, an inequality due to Osserman.

Proposition 2.2.3 ([Oss79, §C, Thm. 12, Cor. 2])**.** *For any* $\Omega \in \mathcal{A}^d$*,*

$$\rho(\Omega) \geq P(\Omega)^{-1} V(\Omega). \tag{2.2.7}$$

And, secondly an inequality derived from Osserman's inequality by van den Berg in [Ber15].

Proposition 2.2.4 ([Ber15, Lemma 3])**.** *For any* $\Omega \in \mathcal{A}^d$*,*

$$\rho(\Omega) \geq 2^{d-1}(d\omega_d)^{-1} D(\Omega)^{1-d} V(\Omega). \tag{2.2.8}$$

To end this subsection, we define the fattening and erosion of a convex domain in $\mathcal{A}^d$. These will come of use in obtaining upper bounds for the Neumann eigenvalue counting function of a domain $\Omega \in \mathcal{A}^d$ in Chapter 3.

For $\Omega \in \mathcal{A}^d$ we define its fattening[1] by $\delta > 0$ as

$$\Omega \oplus \delta\mathbb{B}^d := \{x + y : x \in \Omega,\ y \in \delta\mathbb{B}^d\} \equiv \{x \in \mathbb{R}^d : d(x, \Omega) < \delta\} \tag{2.2.9}$$

and its erosion by $\delta > 0$ as

$$\Omega \ominus \delta\mathbb{B}^d := \{x : B_\delta(x) \subset \Omega\} \equiv \{x \in \Omega : d(x, \partial\Omega) \geq \delta\}. \tag{2.2.10}$$

We remark that both the fattening and erosion of a convex domain are themselves convex, which can be readily seen from the definition. Of importance to us later on is that we have a suitably good geometric control on the volume of fattenings and erosions of bounded convex domains. We now give two results regarding this.

Theorem 2.2.5 (Minkowski-Steiner, see [Gru07, §6])**.** *There exist functions $W_2, W_3, \ldots, W_{d-1} : \mathcal{A}^d \to [0, +\infty)$, called the quermassintegrals of Ω, such that*

$$V(\Omega \oplus \delta\mathbb{B}^d) = V(\Omega) + P(\Omega)\delta + \sum_{j=1}^{d-1} \binom{d}{j} W_j(\Omega)\delta^j + \omega_d\delta^d \tag{2.2.11}$$

for any $\delta > 0$ and any $\Omega \in \mathcal{A}^d$.

The quermassintegrals are monotone with respect to inclusion of domains in $\mathcal{A}^d$ and invariant under isometries of $\mathbb{R}^d$. Moreover, under homothety, $W_j(\alpha\Omega) = \alpha^{d-j}W_j(\Omega)$. This will be of great importance in the proofs in Section 3.2.4.

Theorem 2.2.6 (Erosion formula)**.** *For any $\Omega \in \mathcal{A}^d$ and $\delta > 0$, one has that*

$$V(\Omega \ominus \delta\mathbb{B}^d) \geq V(\Omega) - P(\Omega)\delta. \tag{2.2.12}$$

Proof. This result is seemingly folkloric in the literature. The only full reference for this result the author is aware of is Remark 5.7 in [GitLén20], which we now outline.

[1] For those familiar with convex geometry, one may notice that our definition of the fattening of a convex domain is a special case of the Minkowski sum of convex domains.

We note in [GitLén20] the authors consider bounded C^2 convex domains but, as they point out, it holds more generally for bounded convex domains.

We have that

$$\begin{cases} V(\Omega \ominus \delta\mathbb{B}^d) > 0, & 0 < \delta < \rho(\Omega), \\ V(\Omega \ominus \delta\mathbb{B}^d) = 0, & \delta \geq \rho(\Omega), \end{cases} \tag{2.2.13}$$

and so it suffices to consider the case $0 < \delta < \rho(\Omega)$. From the work of Matheron in [Mat78, §2], we know that

$$\frac{\mathrm{d}}{\mathrm{d}\delta} V(\Omega \ominus \delta\mathbb{B}^d) = -P(\Omega \ominus \delta\mathbb{B}^d) \tag{2.2.14}$$

for $0 < \delta < \rho(\Omega)$ and so

$$\begin{aligned} V(\Omega \ominus \delta\mathbb{B}^d) &= V(\Omega) - \int_0^\delta \mathrm{d}r\, P(\Omega \ominus r\mathbb{B}^d) \\ &\geq V(\Omega) - \int_0^\delta \mathrm{d}r\, P(\Omega) \\ &= V(\Omega) - P(\Omega)\delta. \end{aligned} \tag{2.2.15}$$

Here we have used that $\Omega \ominus \delta\mathbb{B}^d$ is itself convex and a subset of Ω, and so $P(\Omega \ominus \delta\mathbb{B}^d) \leq P(\Omega)$ by the monotonicity of perimeter under inclusion of bounded convex domains. □

Remark 2.2.7. One may obtain Osserman's inequality, Proposition 2.2.3, as a corollary of Theorem 2.2.6 by taking $\delta = \rho(\Omega)$ and noting that $V(\Omega \ominus \rho(\Omega)\mathbb{B}^d) = 0$. However, Osserman obtained their inequality by other means as a corollary of a different inequality in [Oss79].

2.2.2 Hausdorff convergence

We now move on to defining the key notions of Hausdorff convergence and $\sim$-Hausdorff convergence.

Defining the equivalence relation $\sim$ on $\mathcal{A}^d$ by $\Omega_1 \sim \Omega_2$ if there exists an isometry $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Omega_1 = f(\Omega_2)$, we denote the associated quotient space of equivalence classes $[\Omega]$ by $\mathcal{A}^d/_\sim$. Any function $F : \mathcal{A}^d \to \mathbb{R}$ that is invariant under any isometry of $\mathbb{R}^d$ also naturally induces a well-defined function $\widetilde{F} : \mathcal{A}^d/_\sim \to \mathbb{R}$ in the obvious way, i.e. $\widetilde{F}([\Omega]) = F(\Omega')$ for any $\Omega' \in [\Omega]$. Notationally, we will often omit the tilde and simply write F to also mean $\widetilde{F}$.

On $\mathcal{A}^d/_\sim$ we can define the $\sim$-Hausdorff distance $d_H : \mathcal{A}^d/_\sim \times \mathcal{A}^d/_\sim \to [0, +\infty)$ by

$$d_H([\Omega_1], [\Omega_2]) := \inf_{\substack{\Omega_1' \in [\Omega_1] \\ \Omega_2' \in [\Omega_2]}} d^H(\Omega_1', \Omega_2'), \tag{2.2.16}$$

where $d^H : \mathcal{A}^d \times \mathcal{A}^d \to [0, +\infty)$ is the usual Hausdorff distance given by

$$d^H(\Omega_1', \Omega_2') := \min\left\{\sup_{x \in \Omega_1'} d(x, \Omega_2'), \sup_{y \in \Omega_2'} d(y, \Omega_1')\right\}. \tag{2.2.17}$$

The moral of the definition of the $\sim$-Hausdorff distance is that it solely detects convergence in geometry rather than the usual Hausdorff distance which also detects convergence in the physical distance between domains.

Definition 2.2.8. We say that a sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{A}^d$:

- *$\sim$-Hausdorff converges* to $\Omega \in \mathcal{A}^d$ as $n \to +\infty$ if $d_H([\Omega_n], [\Omega]) \to 0$ as $n \to +\infty$.
- *Hausdorff converges* to $\Omega \in \mathcal{A}^d$ as $n \to +\infty$ if $d^H(\Omega_n, \Omega) \to 0$ as $n \to +\infty$.

We then say a function $F : \mathcal{A}^d \to \mathbb{R}$ is continuous if for any sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{A}^d$ that Hausdorff converges to some $\Omega \in \mathcal{A}^d$ as $n \to +\infty$, we have $F(\Omega_n) \to F(\Omega)$ as $n \to +\infty$. Any such continuous function which is also invariant under isometries satisfies the following result.

Proposition 2.2.9. *Any continuous function $F : \mathcal{A}^d \to \mathbb{R}$ which is invariant under isometries of $\mathbb{R}^d$ has that*

$$F(\Omega_n) \to F(\Omega) \tag{2.2.18}$$

as $n \to +\infty$ if $(\Omega)_{n\geq 1}$ $\sim$-Hausdorff converges to Ω as $n \to +\infty$.

It is well-known that volume, perimeter, diameter, inradius, circumradius and quermassintegrals are all continuous as functions $\mathcal{A}^d \to (0, +\infty)$. Moreover, it is well-known that Dirichlet and Neumann eigenvalues are continuous as functions $\mathcal{A}^d \to (0, +\infty)$, see for example [Ber15] and [Ros04]. As all the aforementioned quantities are also invariant under isometries of $\mathbb{R}^d$, Proposition 2.2.9 holds for these quantities.

Alongside continuity, it is useful to have compactness when dealing with shape optimisation problems. Effectively, in the best case scenario, the simplest way to determine existence of optimisers is through (a version of) the extreme value theorem. We now provide a mildly adapted statement of Blaschke's selection theorem from the conventional one, see [Gru07, Thm. 6.3], that is convenient for our own purposes and give its proof.

Theorem 2.2.10 (Blaschke's selection theorem)**.** *Any sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{A}^d$ with $V(\Omega_n) \geq c$ and $D(\Omega_n) \leq C$ for all $n \in \mathbb{N}$ for some constants $c, C > 0$ has a $\sim$-Hausdorff convergent subsequence. If in addition, $\Omega_n \subset B$ for all $n \in \mathbb{N}$ for some suitably large ball B then $(\Omega_n)_{n\geq 1}$ has a Hausdorff convergent subsequence.*

Proof. From Proposition 2.2.4, the isodiametric inequality and the geometric constraints on the Ω_n in the statement of the proposition, we have that

$$\rho(\Omega_n) \geq 2^{-d}(d\omega_d)^{-1} D(\Omega_n)^{1-d} V(\Omega_n) \geq \rho^* \tag{2.2.19}$$

for some $\rho^* > 0$. Hence, as the diameters of the Ω_n are uniformly bounded, we can find a suitably large compact convex set $K' \subset \mathbb{R}^d$ such that we can arrange the Ω_n so that

$$B(0; \rho^*) \subset \Omega_n \subset K' \tag{2.2.20}$$

for each $n \in \mathbb{N}$. By the classical form of Blaschke's selection theorem, see [Gru07, Thm. 6.3], the sequence $\left(\overline{\Omega_n}\right)_{n\geq 1}$ has a Hausdorff convergent subsequence $\left(\overline{\Omega_{n_k}}\right)_{k\geq 1}$ converging to some compact convex set K as $k \to +\infty$. By the properties of Hausdorff convergence of compact convex sets, we necessarily have that $B(0; \rho^*) \subset K$ and so K has non-empty interior. Denoting the interior of K by Ω, our rearranged version of the sequence $(\Omega_n)_{n\geq 1}$ has a Hausdorff convergent subsequence $(\Omega_{n_k})_{k\geq 1}$

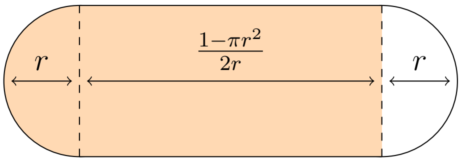

Figure 2.2: Two dimensional convex domain of unit volume with a subdomain highlighted in orange for which the inequality in (2.2.21) is sharp. The analagous picture in higher dimensions also gives sharpness.

converging to Ω as $k \to +\infty$. It is then immediate that the unmodified sequence $(\Omega_n)_{n\geq 1}$ has a $\sim$-Hausdorff convergent subsequence. The last part of the theorem follows precisely in the same way but for which we do not have to rearrange the sequence of domains. $\square$

We now provide a quantitative control on the Hausdorff distance between a convex domain $\Omega_1 \in \mathcal{A}^d$ and a convex subdomain $\Omega_2 \subset \Omega_1$ in terms of their difference in volume.

Lemma 2.2.11. *For any $\Omega_1, \Omega_2 \in \mathcal{A}^d$ with $\Omega_2 \subset \Omega_1$,*

$$d^H(\Omega_1, \Omega_2) \leq \left(\frac{2}{\omega_d}\right)^{1/d} (V(\Omega_1) - V(\Omega_2))^{1/d}. \tag{2.2.21}$$

Moreover, this inequality is sharp in the sense that for any $\epsilon > 0$, one may find a convex domain $\Omega_1 \in \mathcal{A}^d$ and a convex subdomain $\Omega_2 \subset \Omega_1$ with $V(\Omega_1) - V(\Omega_2) = \epsilon$ such that equality holds in (2.2.21)*.*

Proof. Let $x \in \Omega_1$ and let $\mathrm{HP}(x)$ be the set of all $(d-1)$-dimensional hyperplanes Π which pass through x. Each such hyperplane Π defines two disjoint open half-spaces Π^+ and Π^-. We do not worry about how we order these two half-spaces. We define the function $g : \Omega_1 \to (0, +\infty)$ by

$$g(x) := \inf_{\Pi \in \mathrm{HP}(x)} \min\left\{V(\Omega_1 \cap \Pi^+), V(\Omega_1 \cap \Pi^-)\right\}. \tag{2.2.22}$$

By definition it is clear

$$g(x) \geq \frac{1}{2}\omega_d d(x, \partial\Omega)^d \tag{2.2.23}$$

and moreover that

$$\Omega_1^\epsilon := \{x \in \Omega_1 : g(x) > \epsilon\} \subset \Omega_2 \tag{2.2.24}$$

for all $\Omega_2 \subset \Omega_1$ with $V(\Omega_1) - V(\Omega_2) \leq \epsilon$. Then we see that

$$\left\{x \in \Omega_1 : d(x, \partial\Omega_1) > \left(\frac{2\epsilon}{\omega_d}\right)^{1/d}\right\} \subset \Omega_1^\epsilon. \tag{2.2.25}$$

Thus, for any such aforementioned Ω_2 one must have

$$d^H(\Omega_1, \Omega_2) \leq d^H(\Omega_1, \Omega_1^\epsilon) \leq \left(\frac{2\epsilon}{\omega_d}\right)^{1/d}. \tag{2.2.26}$$

Taking $\epsilon = V(\Omega_1) - V(\Omega_2)$ gives the result. The fact that the inequality is sharp in dimension two and higher can be seen by conjoining two half-balls with a suitably long cylinder, see Figure 2.2. In dimension one, it is obviously sharp as any bounded convex set is simply an interval. □

To end the subsection, we state one further good property of Hausdorff convergence of convex sets with regards to monotonicity. This property is stated in [Ber15] but we give a proof of it here for completeness. We first require the following proposition.

Proposition 2.2.12. *Let $\Omega \in \mathcal{A}^d$ and suppose that $0 \in \Omega$ and that $B_{\rho(\Omega)}(0) \subset \Omega$, then*

$$\Omega \oplus \delta\mathbb{B}^d \subset \left(1 + \frac{\delta}{\rho(\Omega)}\right)\Omega \tag{2.2.27}$$

for any $\delta > 0$.

Proof. Let $x \in (\Omega \oplus \delta\mathbb{B}^d)\backslash\overline{\Omega}$, and let $y \in \partial\Omega$ be the unique point such that

$$\{y\} = \partial\Omega \cap \{\lambda x : \lambda \in [0,1]\}. \tag{2.2.28}$$

By construction

$$\frac{\|x\|_2}{\|y\|_2} = 1 + \frac{\|x-y\|_2}{\|y\|_2}. \tag{2.2.29}$$

Note that we do not necessarily have that $\|x-y\|_2 \leq \delta$, so the proof is not immediate. Let E be the convex set

$$E := \{y + \lambda(z-y) : \lambda \in (0, +\infty),\ z \in B_{\rho(\Omega)}(0)\}. \tag{2.2.30}$$

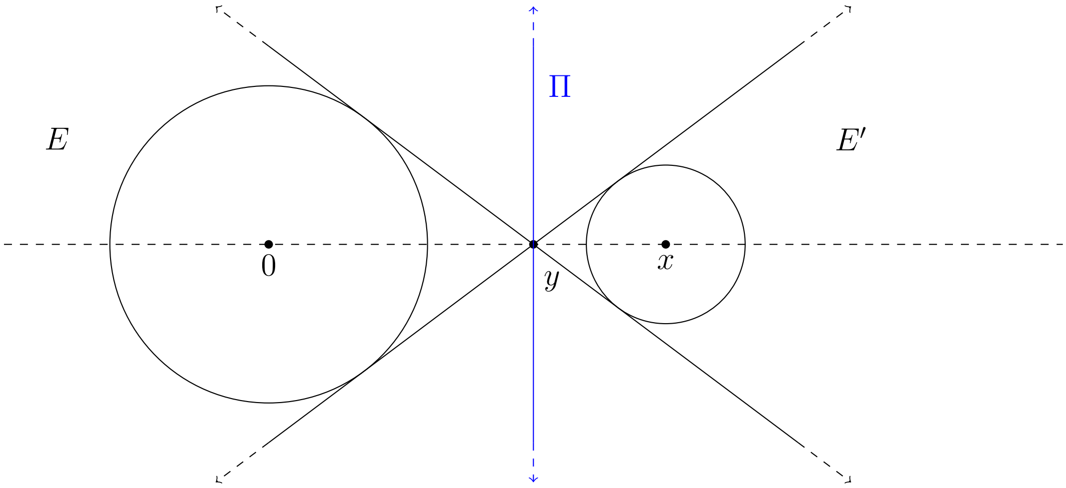

Figure 2.3: Visualisation of the sets constructed in the proof of Proposition 2.2.12.

That is, E is the smallest infinite open cone with vertex y that contains $B_{\rho(\Omega)}(0)$. Further, let E' be the reflection of E across the hyperplane Π that is orthogonal to $\{\lambda x : \lambda \in [0,1]\}$ and contains y, i.e.

$$E' = \{y - \lambda(z-y) : \lambda \in (0,+\infty),\, z \in B_{\rho(\Omega)}(0)\}. \tag{2.2.31}$$

See Figure 2.3 for a visualisation of these sets. By construction we know that $x \in E'$ and that $E' \cap \overline{\Omega} = \emptyset$, as otherwise this would imply that y lies in Ω and not on $\partial\Omega$.

Let $\epsilon = d(x, \partial E')$, then we know that $\epsilon \le d(x,\partial\Omega) < \delta$. By simple geometric considerations we see that

$$\frac{\|x-y\|_2}{\|y\|_2} = \frac{d(x,\partial E')}{\rho(\Omega)} < \frac{\delta}{\rho(\Omega)}. \tag{2.2.32}$$

Hence, combining (2.2.29) and (2.2.32) one obtains the result. □

We have the following consequence of this result.

Corollary 2.2.13. *Suppose that $(\Omega_n)_{n\ge1}$ is a sequence in $\mathcal{A}^d$ Hausdorff converging to $\Omega \in \mathcal{A}^d$, then the Ω_n and Ω may be arranged so that*

$$\left(1 + \frac{d^H(\Omega_n,\Omega)}{2\rho(\Omega)}\right)^{-1} \Omega \subset \Omega_n \subset \left(1 + \frac{d^H(\Omega_n,\Omega)}{2\rho(\Omega)}\right) \Omega \tag{2.2.33}$$

for n sufficiently large.

Proof. The proof is immediate from the equivalent definition of the Hausdorff distance

$$d^H(\Omega_n, \Omega) = \max\left\{\sup\{\delta_1 > 0 : \Omega_n \subset \Omega \oplus \delta_1 \mathbb{B}^d\}, \sup\{\delta_2 > 0 : \Omega \subset \Omega_n \oplus \delta_2 \mathbb{B}^d\}\right\} \tag{2.2.34}$$

and that $\rho(\Omega_n) \geq \rho(\Omega)/2$ for n sufficiently large. □

Remark 2.2.14. By the scaling properties of eigenvalues under homothety it is often to convenient to work under the assumption that if a sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{A}^d$ Hausdorff converges to $\Omega \in \mathcal{A}^d$ as $n \to +\infty$, then one has that $\Omega_n \subset \Omega$ for all n sufficiently large. The above corollary indeed allows us to do this, up to homothety, and we use this in Section 3.3.2.

Other equivalent definitions of Hausdorff convergence may be found in the book of Gruber [Gru07]. One of use to us is that

$$d^H(\Omega_1, \Omega_2) = d^H(\partial\Omega_1, \partial\Omega_2) \tag{2.2.35}$$

for any $\Omega_1, \Omega_2 \in \mathcal{A}^d$.

2.3 The heat equation

In this section, we provide the necessary pre-requisite material for the proofs in Chapter 4 of this thesis. The most important section to make the reader aware of is Section 2.3.2 in which we outline a geometric construction for the Neumann heat kernel on polygonal domains given in [GalMcK72]. We then go on to give the probabilistic interpretation of heat kernels with the necessary material needed to understand the proofs in this thesis.

As a quick digression before going forward, a given fundamental solution to the heat equation on an arbitrary manifold is not a priori unique. However, there is a unique minimal fundamental solution which we refer to as the heat kernel of the manifold, unless we are specifically referring to a fundamental solution with prescribed boundary conditions, e.g. Neumann boundary conditions.

Such minimal fundamental solutions are always constructable by taking a nested sequence of relatively compact open sets which exhaust the manifold and considering the pointwise limit of their Dirichlet heat kernels, see [Dod83]. Moreover, and most crucially, these minimal fundamental solutions have the all important relevant probabilistic interpretation in terms of Brownian motion, see [Hsu02, §4]. For us in this thesis, the (Neumann) heat kernels we work with are all unique and the subtleties of non-uniqueness shall not bother us in our work here, but is good to be aware of this and our nomenclature.

2.3.1 Spectral theory and the heat kernel

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. The unique smooth heat kernel for the heat equation in Ω with Neumann boundary conditions is given by

$$\eta_\Omega(t;x,y) = \sum_{j=1}^{\infty} e^{-\mu_j(\Omega)t} u_j(x) u_j(y) \tag{2.3.1}$$

where $\mu_1(\Omega), \mu_2(\Omega), \ldots$ are the Neumann eigenvalues of Ω and $u_1, u_2, \ldots : \Omega \to \mathbb{R}$ are an associated $L^2(\Omega)$-orthonormalised basis of eigenfunctions. For any $f \in L^2(\Omega)$, one may obtain the unique solution $u_f(t;x)$ to the heat equation in Ω with Neumann boundary conditions on $\partial\Omega$ with initial datum f, i.e.

$$\begin{cases} \dfrac{\partial}{\partial t} u_f(t;x) = \Delta u_f(t;x), & t \in (0,+\infty),\ x \in \Omega, \\ \dfrac{\partial}{\partial n} u_f(t;x) = 0, & t \in (0,+\infty),\ x \in \partial\Omega, \\ \lim_{t\downarrow 0} \|u_f(t;\cdot) - f(\cdot)\|_{L^2(\Omega)} = 0, \end{cases} \tag{2.3.2}$$

by

$$u_f(t;x) = \int_\Omega \eta_\Omega(t;x,y) f(y)\,\mathrm{d}y \equiv \sum_{k=1}^{\infty} e^{-\mu_k(\Omega)t} u_k(x) \langle f, u_k \rangle_{L^2(\Omega)}. \tag{2.3.3}$$

One gets the entirely analogous representation for the heat kernel in the case of Dirichlet, Robin or Zaremba boundary conditions in terms of the respective eigenvalues and eigenfunctions.

2.3.2 A geometric construction of the Neumann heat kernel on polygonal domains

The spectral resolution of the Neumann heat kernel in equation (2.3.1) is not, a priori, easy to work with as we generally do not explicitly know either the eigenvalues or eigenfunctions of a given bounded domain. In this subsection, we outline a construction of the Neumann heat kernel on polygonal domains due to Gallavotti and McKean in [GalMcK72]. We favour this construction to others as it is bespoke for polygonal domains and it is very geometric in flavour, which makes it more intuitive to work with. Using this construction, we are able prove some locality principles for solutions to the Neumann heat equation in polygonal domains in Section 4.4.

Half-plane

The simplest two-dimensional space on which one may construct the Neumann heat kernel is the upper half-space $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$[2]. One may reflect $\overline{\mathbb{H}}$ to obtain $\mathbb{R}^2$ in the sense that we have a projection $\Psi : \mathbb{R}^2 \to \overline{\mathbb{H}}$ given by

$$\Psi(x, y) = (x, |y|). \tag{2.3.4}$$

Neumann boundary conditions correspond to a fluxless boundary condition and so we should lose no heat through the boundary. In this vein, any heat that would bleed into $\mathbb{R}^2$ if there was no boundary condition imposed should remain in $\mathbb{H}$. Using this logic, we see that the Neumann heat kernel for $\mathbb{H}$ should look like

$$\eta_{\mathbb{H}}(t; x, y) = \sum_{z \in \Psi^{-1}(\{y\})} p_{\mathbb{R}^2}(t; x, z), \tag{2.3.5}$$

where

$$p_{\mathbb{R}^2}(t; x, y) = \frac{1}{4\pi t} e^{-\frac{\|x-y\|_2^2}{4t}} \tag{2.3.6}$$

[2] $\mathbb{H}$ is not the hyperbolic space here, just the honest upper half-space.

is the heat kernel on $\mathbb{R}^2$. It is easy to verify that this is indeed the Neumann heat kernel for $\mathbb{H}$.

Infinite wedge

One may apply the logic of reflection, also known as the method of images, from the case of the half-space to the slightly more complicated case of infinite wedges. Let W_θ be the infinite wedge of angle $\theta \in (0, 2\pi)$, which is given in polar coordinates by

$$W_\theta = \{(r, \phi) : r > 0,\ 0 < \phi < \theta\}. \tag{2.3.7}$$

One may view W_θ as a submanifold of the manifold

$$W_\infty := \{(r, \phi) : r > 0,\ -\infty < \phi < +\infty\} \tag{2.3.8}$$

endowed with the usual polar coordinate metric. Using this intepretation we can see that W_∞ may be obtained by reflected copies of W_θ in the following sense: $W_\infty = \bigcup_{k\in\mathbb{Z}} W_\theta^{(k)}$, where

$$W_\theta^{(k)} := \{(r, \phi) : r > 0,\ k\theta \le \phi \le (k+1)\theta\}, \tag{2.3.9}$$

and there is an associated family of projections $\Psi_k : W_\theta^{(k)} \to W_\theta^{(0)}$ given by

$$\Psi_k(r, \phi) = \begin{cases} (r, \phi - k\theta), & k \in 2\mathbb{Z}, \\ (r, (k+1)\theta - \phi), & k \in \mathbb{Z}\backslash(2\mathbb{Z}). \end{cases} \tag{2.3.10}$$

Combining these projections one may construct a well-defined projection $\Psi : W_\infty \to W_\theta^{(0)}$ given by $\Psi(r, \phi) = \Psi_k(r, \phi)$ if $(r, \phi) \in W_\theta^{(k)}$. The Neumann heat kernel on W_θ can then be obtained from the heat kernel p_{W_∞} on W_∞ by

$$\eta_{W_\theta}(t; x, y) = \sum_{z\in\Psi^{-1}(\{y\})} p_{W_\infty}(t; x, z) \tag{2.3.11}$$

for $t > 0$ and $x, y \in W_\theta$.

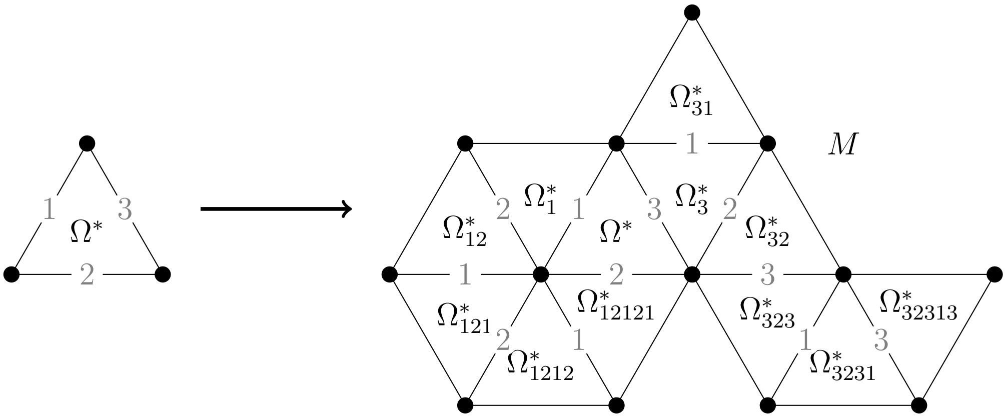

Figure 2.4: Visualisation of part of the manifold generated by an equilateral triangle Ω^*. Note that conventionally we would identify Ω^*_{12121} with Ω^*_2 but we treat them as distinct here. (This figure is an adaptation of Figures 2 and 3 in [GalMcK72].)

Polygonal domains

Again, one may go further and use the preceding ideas to construct the Neumann heat kernel on polygonal domains.

Let Ω be a, possibly unbounded, polygonal domain whose edges are labelled 1 through n. Then for each string $a_1 a_2 \cdots a_n$ with $a_i \in \{1, 2, \ldots, n\}$ and $a_{i+1} \neq a_i$ we can obtain a copy $\overline{\Omega}_{a_1 a_2 \cdots a_n}$ of $\overline{\Omega}$ by reflecting $\overline{\Omega}$ across the sides $a_1, a_2, \ldots, a_n$ successively. We also have the copy of $\overline{\Omega}$ obtained by carrying out no reflection and simply denote it by $\overline{\Omega}$. We declare each of these copies of $\overline{\Omega}$ to be different and the collection K of all these copies is a covering of $\overline{\Omega}$. From K, we can obtain the open manifold $M := K \backslash \{\text{images of vertices of } \Omega\}$, which we view as a flat Riemannian manifold and call it the manifold generated by Ω. For M, there is the obvious projection $\Psi_M : M \to \Omega^* := (\Omega \cup \partial\Omega) \backslash \mathcal{V}$, where $\mathcal{V}$ denotes the vertices of Ω, given in the same way as in the previous two cases via the natural projections $\Omega_{a_1 a_2 \cdots a_n} \to \Omega$. Figure 2.4 shows a visualisation of the manifold generated by an equilateral triangle.

In the same way as before, one then obtains that the Neumann heat kernel on Ω is given by

$$\eta_\Omega(t; x, y) = \sum_{z \in \Psi_M^{-1}(\{y\})} p_M(t; x, z) \tag{2.3.12}$$

for $t > 0$, $x, y \in \Omega$, where p_M is the heat kernel on the manifold M.

2.3.3 A probabilistic interpretation

We briefly give the probabilistic interpretation of Neumann heat kernels on polygonal domains and heat kernels on open manifolds. For our purposes, we do not need a fully detailed understanding of the theory of the associated stochastic processes so we only cover the parts relevant to us. Some particularly relevant references to which the reader can refer here are [McK69; GalMcK72; Hsu84; Bel75; Hsu02; RogWil00]. Perhaps our greatest omission here is any theory on filtrations, that is nested families $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras such that $\mathcal{F}_t$ tracks all the information known about a given stochastic process up to a time $t \geq 0$. Though filtrations are interesting in their own right, they do not yield anything of use to our analysis here.

Also, as a small but important remark, we consider stochastic processes associated with the parabolic operator $-\frac{\partial}{\partial t} + \Delta$ rather than the standard probabilistic parabolic operator $-\frac{\partial}{\partial t} + \frac{1}{2}\Delta$. This only means that our processes run with twice the clock as those in the usual probability literature.

Neumann heat kernels

Firstly, we consider stochastic processes associated with the Neumann heat kernel on a polygonal domain Ω.

Fix $x \in \overline{\Omega}$ and define the probability space $(C_x([0, +\infty), \overline{\Omega}), \mathcal{F}, \mathbb{P}^{\Omega}_x)$, where:

- $C_x([0, +\infty), \overline{\Omega})$ is the space of continuous paths $w : [0, +\infty) \to \overline{\Omega}$ with $w(0) = x$;

- $\mathcal{F}$ is the σ-algebra generated by all sets of the form

$$\{w \in C_x([0, +\infty), \overline{\Omega}) : w(t_1) \in A_1, w(t_2) \in A_2, \ldots, w(t_n) \in A_n\}, \qquad (2.3.13)$$

  where $0 \leq t_1 < t_2 < \cdots < t_n$ and $A_1, A_2, \ldots, A_n \subset \overline{\Omega}$ are Borel sets;

- and, $\mathbb{P}_x^\Omega$ is a probability measure on $\mathcal{F}$ uniquely determined by

$$\begin{aligned}&\mathbb{P}_x^\Omega(\{w \in C_x([0,+\infty),\overline{\Omega}) : w(t_1) \in A_1, w(t_2) \in A_2, \dots, w(t_n) \in A_n\}) \\ &\quad = \int_{A_1} \mathrm{d}x_1 \cdots \int_{A_n} \mathrm{d}x_n \, \eta_\Omega(t_1; x, x_1)\eta_\Omega(t_2 - t_1; x_1, x_2) \cdots \eta_\Omega(t_n - t_{n-1}; x_{n-1}, x_n).\end{aligned} \tag{2.3.14}$$

We call a stochastic process $(X_t)_{t\geq 0}$ on $(C_x([0,+\infty),\overline{\Omega}), \mathcal{F}, \mathbb{P}_x^\Omega)$ a reflecting Brownian motion on Ω starting at $x \in \overline{\Omega}$ if

$$\mathbb{P}_x^\Omega(X_t \in A) = \int_A \mathrm{d}y \, \eta_\Omega(t; x, y) \tag{2.3.15}$$

for any Borel set $A \subset \Omega$ and any $t > 0$. In this case, we call (2.3.15) the law of $(X_t)_{t\geq 0}$. The existence of such a process may be ascertained by simply taking the canonical process $X_t(w) = w(t)$.

For an open subset $E \subset \overline{\Omega}$ in the subspace topology, we define the exit time of $(X_t)_{t\geq 0}$ from E as

$$\tau_E := \inf\{t \geq 0 : X_t \notin E\}. \tag{2.3.16}$$

It is well-known that, see for example Theorem 76.1 in [RogWil00],

$$\{\tau_E \leq t\}, \{\tau_E > t\} \in \mathcal{F} \tag{2.3.17}$$

and so we have well-defined probabilities for such exit times.

Heat kernels on manifolds

Now we consider stochastic processes associated with heat kernels on an open manifold M.

Fix $x \in M$, and define the probability space $(C_x([0,+\infty), M), \mathcal{G}, \mathbb{P}_x^M)$, where:

- $C_x([0,+\infty), M)$ is the space of continuous paths $w : [0,+\infty) \to M$ with $w(0) = x$;

- $\mathcal{G}$ is the σ-algebra generated by sets of the form

$$\{w \in C_x([0,+\infty), M) : w(t_1) \in A_1, w(t_2) \in A_2, \ldots, w(t_n) \in A_n\} \tag{2.3.18}$$

  where $0 \leq t_1 < t_2 < \cdots < t_n$ and $A_1, A_2, \ldots, A_n \subset M$ are Borel sets;

- and, $\mathbb{P}_x^M$ is a probability measure on $\mathcal{G}$ uniquely determined by

$$\begin{aligned}\mathbb{P}_x^M&(\{w \in C_x([0,+\infty), M) : w(t_1) \in A_1, w(t_2) \in A_2, \ldots, w(t_n) \in A_n\}) \\ &= \int_{A_1} \mathrm{dvol}_M(x_1) \cdots \int_{A_n} \mathrm{dvol}_M(x_n) \Big[p_M(t_1; x, x_1) p_M(t_2 - t_1; x_1, x_2) \cdots \\ &\qquad\qquad \cdots p_M(t_n - t_{n-1}; x_{n-1}, x_n) \Big].\end{aligned} \tag{2.3.19}$$

We call a stochastic process $(B_t)_{t\geq 0}$ on $(C_x([0,+\infty), \overline{\Omega}), \mathcal{F}, \mathbb{P}_x^\Omega)$ a Brownian motion on M starting at $x \in M$ if

$$\mathbb{P}_x^M(B_t \in A) = \int_A \mathrm{dvol}_M(y)\, p_M(t; x, y) \tag{2.3.20}$$

for any Borel set $A \subset M$ and any $t > 0$. We call (2.3.20) the law of $(B_t)_{t\geq 0}$. The existence of such a process again may be ascertained by simply taking the canonical process $B_t(w) = w(t)$.

For an open submanifold $N \subset M$ we define the exit time of Brownian motion $(B_t)_{t\geq 0}$ starting at $x \in M$ from N as

$$\tau_N := \inf\{t \geq 0 : B_t \notin N\}. \tag{2.3.21}$$

As in the case of reflecting Brownian motions,

$$\{\tau_N \leq t\}, \{\tau_N > t\} \in \mathcal{G} \tag{2.3.22}$$

and so we have well-defined probabilities for such exit times. Moreover, we have that

$$\mathbb{P}_x^M(B_t \in A; \tau_N > t) = \int_A \mathrm{dvol}_M(y)\, q_N(t; x, y) \tag{2.3.23}$$

for any Borel set $A \subset M$ and $t > 0$, where $q_N(t; x, y)$ is the Dirichlet heat kernel (minimal fundamental solution to the heat equation) on N, see [Hsu02, §4].

The construction of Gallavotti and McKean given in the previous subsection can be applied to the processes themselves rather than just the kernels. In fact, this is how they prove the whole construction does indeed work. Let M be the manifold generated by a polygonal domain Ω and $\Psi_M : M \to \overline{\Omega}$ be the associated projection. They prove that for an M-valued Brownian motion $(B_t)_{t\geq 0}$ its image under this projection $(\Psi_M(B_t))_{t\geq 0}$ is identical in law to a reflecting Brownian motion on Ω. That is to say

$$\mathbb{P}_x^{\Omega}(X_t \in A) = \mathbb{P}_x^M(B_t \in \Psi_M^{-1}(A)) \tag{2.3.24}$$

for any Borel set $A \subset \overline{\Omega}$, $x \in \Omega$ and $t > 0$. We observe that this the complete analogue of (2.3.12).

We will not use any further probability theory until Section 4.4, where the aforementioned ideas will come into play.

2.4 Remarks on asymptotic notation

In Chapter 3 of this thesis we use the following notation for functions $f, g : \mathbb{N} \to \mathbb{R}_{>0}$ or $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ by

- $f(k) \sim g(k)$ if

$$\lim_{k\to+\infty} \frac{f(k)}{g(k)} = 1. \tag{2.4.1}$$

- $f(k) \ll g(k)$ or $f(k) = o(g(k))$ if

$$\lim_{k\to+\infty} \frac{f(k)}{g(k)} = 0. \tag{2.4.2}$$

- $f(k) \lesssim g(k)$ or $f(k) = O(g(k))$ if

$$\limsup_{k\to+\infty} \frac{f(k)}{g(k)} < +\infty. \tag{2.4.3}$$

In Chapter 4 of this thesis, we use the following asymptotic notation for functions $f, g : [0, +\infty) \to \mathbb{R}$:

- $f(t) = O(g(t))$ if
$$\limsup_{t \downarrow 0} \frac{|f(t)|}{|g(t)|} < +\infty. \tag{2.4.4}$$
- $f(t) = o(g(t))$ if
$$\limsup_{t \downarrow 0} \frac{|f(t)|}{|g(t)|} = 0. \tag{2.4.5}$$
- $f(t) = o(t^\infty)$ if $f(t) = o(t^n)$ for all $n \in \mathbb{N}$.

In Chapter 5 of this thesis, we will use the additional following asymptotic notation for functions $f, g : [0, +\infty) \to \mathbb{R}$:

- $f(t) \asymp g(t)$ if $f(t) = O(g(t))$ and $g(t) = O(f(t))$.

Chapter 3

Spectral minimisation problems

In this chapter, we present the results of the paper of [Far25] which is a published work by the author of this thesis. The results concern the minimisation of eigenvalues of the Neumann, Robin and Zaremba Laplacians under perimeter and diameter constraint. We also briefly present complementary material from a recent paper of Bogosel, Henrot and Michetti [BogHM24], where the same, and similar, problems were studied. The author became aware of this paper whilst at a conference in late 2023, where he and Professor Antoine Henrot both gave presentations on some of the material presented in this chapter and subsequently compared methods.

The prerequisite material for this chapter is covered in Sections 2.1 and 2.2, and the notation is taken from these sections. As in these sections, we assume that Ω is an open, bounded subset of $\mathbb{R}^d$ and any other assumptions on Ω will be made clear where necessary.

3.1 Motivations and related literature

3.1.1 Origins and naïve assertions

The origin of the paper [Far25] comes from the author's time as a Master's student, in which he studied the paper [BucFre13]. In this paper, the asymptotic behaviour of minimisers to

$$\inf\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^2,\ P(\Omega) = 1\right\} \tag{3.1.1}$$

as $k \to +\infty$ was considered and it was shown that any sequence of minimisers $\sim$-Hausdorff converges the ball of unit perimeter in this limit. The elegance of the proof of this result in part comes from a result in [BerIve13], whereby it was shown that any minimiser of (3.1.1) can be taken to be convex and so (3.1.1) can be reduced to the minimisation problem

$$\inf\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex},\ P(\Omega) = 1\right\}. \tag{3.1.2}$$

The reason that one may do this reduction is that for an arbitrary bounded open set $\Omega \subset \mathbb{R}^2$ with $P(\Omega) = 1$, one may rearrange the connected components of Ω such that the open convex envelope $\widetilde{\Omega}$ of Ω has a lower perimeter than Ω, see [BerIve13]. Moreover, $\Omega \subset \widetilde{\Omega}$ and so $\lambda_k(\widetilde{\Omega}) \leq \lambda_k(\Omega)$.

Once you are in the realm of convex domains you have very good properties to work with in terms of spectral shape optimisation problems. The proof of the result follows from being able to exploit the reverse domain monotonicity of Dirichlet eigenvalues, Weyl's law and a lower bound on Dirichlet eigenvalues in terms of their volume due to Li and Yau [LiYau83].

Taking a naïve standpoint, Weyl's law and the isoperimetric inequality suggest that minimisers to (3.1.1) should be close to the ball for k large. This, at least (incredulously!) heuristically, explains the asymptotic behaviour of the minimisers from the result of Bucur and Freitas.

A higher dimensional analogue of the result of Bucur and Freitas was proven by van den Berg in [Ber15]. van den Berg proved that for any $d \geq 3$ fixed, there exists a minimiser to

$$\inf\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex},\ P(\Omega) = 1\right\}, \tag{3.1.3}$$

for all $k \geq 1$. Moreover, any such sequence of minimisers $\sim$-Hausdorff converges to the ball of unit perimeter as $k \to +\infty$. If we remove the convexity condition, we cannot use the idea that convexification reduces perimeter as in dimension three and higher this is not necessarily true. As far as the author is aware, the asymptotic behaviour of minimisers to the problem in the general case is unknown, see for example [BucFre13, Remark 1.3].

In [Ber15], van den Berg also considered the minimisation problem

$$\inf\left\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \ D(\Omega) = 1\right\}. \tag{3.1.4}$$

It was shown that this problem admits a minimiser for all $k \geq 1$ and any sequence of minimisers $\sim$-Hausdorff converges to the ball of unit diameter as $k \to +\infty$. A very detailed analysis for this problem in dimension two was done by Bogosel, Henrot and Lucardesi in [BogHL18].

Recall our aforementioned naïve standpoint:

> *the isoperimetric/isodiametric inequality and Weyl's law suggest that the ball asymptotically minimises Dirichlet eigenvalues of bounded domains under perimeter/diameter constraint.*

Our primary goal in this chapter is to take this naïve standpoint and determine if one can prove analogous results to those proven in [BucFre13] and [Ber15] in the case of Neumann eigenvalues.

3.1.2 Main results

If we consider the analogous minimisation problem to (3.1.1) for Neumann eigenvalues, i.e.

$$\inf\left\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ Lipschitz}, \ P(\Omega) = 1\right\}, \tag{3.1.5}$$

one immediately encounters a problem. Without any other constraints one immediately obtains that this infimum is zero by the considering the disjoint union of k balls of perimeter $1/k$. This infimum is attained and there are infinitely many minimisers. If one instead considers this minimisation over the space of simply connected domains, the infimum is also zero but it is not attained for $k \geq 2$ and so the minimisation problem has no minimisers. The fact that the infimum is zero in this case can be shown by considering k disjoint balls of suitably small perimeter connected by suitably thin tubes. This by virtue naturally leads us to consider the minimisation problem

$$\inf \left\{ \mu_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex}, \; P(\Omega) = 1 \right\} \tag{3.1.6}$$

in order to hope to have a well-posed minimisation problem that is non-trivial. We note that this infimum is non-zero for $k \geq 2$ by the Payne-Weinberger inequality since a perimeter constraint implies a diameter constraint for bounded convex domains in two dimensions. However, the Payne-Weinberger inequality also tells us no minimiser exists for $k = 2$ so the existence of a minimiser is non-trivial! In [BerBG16], the authors prove that for all $k \geq 3$ there exists a minimiser to

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ rectangle}, \; P(\Omega) = 1\}, \tag{3.1.7}$$

and any sequence of minimisers $\sim$-Hausdorff converges to the square as $k \to +\infty$. Our first result is that this holds more generally in the setting of bounded convex domains.

Theorem 3.1.1. *For all $k \geq 3$, there exists a minimiser to*

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex}, \; P(\Omega) = 1\}. \tag{3.1.8}$$

Moreover, any sequence $(\Omega_k^*)_{k\geq 3}$ of such minimisers $\sim$-Hausdorff converges to the ball B of unit perimeter as $k \to +\infty$. The convergence satisfies the asymptotic estimate

$$d_H([\Omega_k^*], [B]) = O(k^{-1/8}). \tag{3.1.9}$$

In dimension three and higher, unlike in the case of dimension two, we do not have the existence of minimisers under perimeter constraint. More formally:

Theorem 3.1.2. *Fix $d \geq 3$. For all $k \geq 1$*

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \ P(\Omega) = 1\} = 0, \tag{3.1.10}$$

Hence, no minimisers exist for any $k \geq 2$.

Proof. Consider the sequence of cuboids

$$\Omega_\epsilon = (0, \epsilon) \times \cdots \times (0, \epsilon) \times \Big(0, \tfrac{1}{2(d-1)}(\epsilon^{2-d} - 2\epsilon)\Big) \tag{3.1.11}$$

for $0 < \epsilon < 2^{-1/(d-1)}$, which satisfy $P(\Omega_\epsilon) = 1$. It is easy to determine that

$$\mu_k(\Omega_\epsilon) \leq 4(d-1)^2 \frac{\pi^2 k^2}{(\epsilon^{2-d} - 2\epsilon)^2} \to 0 \tag{3.1.12}$$

as $\epsilon \downarrow 0$ for $k \in \mathbb{N}$ fixed. The Payne-Weinberger inequality asserts that $\mu_k(\Omega) > 0$ for all $k \geq 2$ for a given bounded convex domain $\Omega \subset \mathbb{R}^d$. And, hence no minimisers can exist for any $k \geq 2$. □

In contrast to the case of perimeter constraint, under diameter constraint one eventually has existence of minimisers in any dimension over the collection of bounded convex domains. As in the perimeter case, we require the convexity condition to make the minimisation problem well-posed.

Theorem 3.1.3. *Let $d \geq 2$ be given. Then there exists a constant $N_d \in \mathbb{N}$ such that for all $k \geq N_d$, there exists a minimiser to*

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \ D(\Omega) = 1\}. \tag{3.1.13}$$

Moreover, any such sequence $(\Omega_k^*)_{k \geq N_d}$ of minimisers $\sim$-Hausdorff converges to the ball B of unit diameter as $k \to +\infty$. The convergence satisfies the asymptotic estimate

$$d_H([\Omega_k^*], [B]) = O(k^{-1/(2d^2)}). \tag{3.1.14}$$

Remark 3.1.4. As in the perimeter case, in dimension $d = 2$ we have existence for all $k \geq N_2 = 3$ by Theorem 2.4 in [BogHM24]. One may obtain indicative upper

d	2	3	4	5	6
N_d upper bound	2.51×10^4	3.01×10^9	2.78×10^{16}	2.71×10^{29}	1.78×10^{42}
Optimal N_d	3	Unknown	Unknown	Unknown	Unknown

Table 3.1: Numerical approximate upper bounds on N_d.

bounds for N_d, see Table 3.1, using Proposition 3.2.3 and comparing to Neumann eigenvalues of the ball of unit diameter. See the Appendix A for more information on this.

The result of Theorem 3.1.2 that

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \ P(\Omega) = 1\} = 0 \tag{3.1.15}$$

for all $k \in \mathbb{N}$ for any $d \geq 3$ is somewhat dissatisfying. One always can get arbitrarily close to the infimum by taking a d-dimensional cuboid that is suitably long in one direction. Instead, what if we don't allow ourselves to do this but do allow an arbitrary amount of stretching eventually, i.e. for k sufficiently large. In an effort to formalise this idea, we state the following question. We first recall that $\mathcal{A}^d$ is the collection of all bounded convex subdomains of $\mathbb{R}^d$.

Question 3.1.5. *Does there exist a nested family $\mathcal{A}^d_1 \subset \mathcal{A}^d_2 \subset \mathcal{A}^d_3 \subset \cdots$ of subsets of $\mathcal{A}^d$ with $\bigcup_{k\geq 1} \mathcal{A}^d_k = \mathcal{A}^d$ such that there exists a minimiser to*

$$\inf\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d_k, \ P(\Omega) = 1\}. \tag{3.1.16}$$

for k sufficiently large? Can we give any sufficient conditions on the sets $\mathcal{A}^d_k$? And, do these minimisers $\sim$-Hausdorff converge to the ball of unit perimeter as $k \to +\infty$?

Such a question is non-canonical in the literature as we allow the collection of shapes we are optimising over to evolve with k. But, the premise of it enables us to ask that: if we do not permit domains that are 'too degenerate', then can we have minimisers? Our methods in this thesis allow us to give positive answers to all aspects of this question. We now state our main result concerning these questions.

Theorem 3.1.6. *Let $d \geq 3$ and $f : \mathbb{N} \to (0, +\infty)$ with $1 \ll f(k) \ll k^{1/d(d-1)}$ be given. Then there exists a constant $N_{d,f} \in \mathbb{N}$ such that for all $k \geq N_{d,f}$, there exists a minimiser to*

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \ P(\Omega) = 1, \ D(\Omega) \leq f(k)\}. \tag{3.1.17}$$

Moreover, any such sequence $(\Omega_k^*)_{k \geq N_{d,f}}$ of minimisers $\sim$-Hausdorff converges to the ball B of unit perimeter as $k \to +\infty$. The convergence satisfies the asymptotic estimate

$$d_H([\Omega_k^*], [B]) = O(k^{-1/(2d^2)}), \tag{3.1.18}$$

which is independent of f in terms of rate of convergence.

Another natural minimisation problem arises when one considers optimal subdomains of a given convex domain. This is the so-called 'interior problem', see [CavFHLLS23]. Using the methods presented in this chapter one may prove the following result.

Theorem 3.1.7. *Fix $\Omega' \in \mathcal{A}^d$. Then there exists a constant $N \in \mathbb{N}$, depending on Ω', such that for all $k \geq N$, there exists a minimiser to*

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \ \Omega \subset \Omega'\}. \tag{3.1.19}$$

Moreover, any such sequence $(\Omega_k^*)_{k \geq N}$ of minimisers Hausdorff converges to Ω' as $k \to +\infty$. The convergence satisfies the asymptotic estimate

$$d^H(\Omega_k^*, \Omega') = O(k^{-1/(2d^2)}). \tag{3.1.20}$$

Remark 3.1.8. In dimension two, there are estimates on N given in Theorem 3.1.7 in terms of the geometry of Ω', see [CavFHLLS23] and Section 3.2.3.

3.1.3 Related literature

The asymptotic behaviour of optimisers to spectral shape optimisation problems has also been studied in other contexts for differing geometric constraints and spectral functionals. Here we give a brief overview of some related results and remarks on their differences to our own here.

Let $\mathcal{R}^d$ denote the space of d-dimensional cuboids, that is the space of all sets of the form $(0, a_1) \times \cdots \times (0, a_d)$, $a_1, \ldots, a_d \in (0, +\infty)$, up to a rigid transformation. For such domains one can gain a very strong control on the Dirichlet and Neumann counting functions owing to the fact that they may be written as lattice point counting problems, see [Mar20] for a good overview. This strong control has been utilised to prove results in asymptotic spectral shape optimisation. The following was proven by Gittins and Larson in [GitLar17].

Theorem 3.1.9 ([GitLar17, Adapted from Thms 1.1 & 1.2])**.** *Let $d \geq 2$ be given. Then:*

- *For any $k \geq 1$ there exists a minimiser R_k^* to*

$$\inf\{\lambda_k(R) : R \in \mathcal{R}^d,\, V(R) = 1\}. \tag{3.1.21}$$

  Moreover, any sequence $(R_k^*)_{k\geq 1}$ of minimisers $\sim$-Hausdorff converges to the d-dimensional unit cube as $k \to +\infty$.

- *For any $k \geq 1$ there exists a maximiser S_k^* to*

$$\sup\{\mu_k(S) : S \in \mathcal{R}^d,\, V(S) = 1\}. \tag{3.1.22}$$

  Moreover, any sequence $(S_k^*)_{k\geq 1}$ of maximisers $\sim$-Hausdorff converges to the d-dimensional unit cube as $k \to +\infty$.

One should remark that prior to the work of Gittins and Larson, the above result was known in the Dirichlet case in dimension two [AntFre13] and in dimension three [BerGit17] and in the Neumann case in dimension two [BerBG16]. It is also worth noting the above theorem can be proven using the more general results of Marshall in [Mar20].

In Theorem 3.1.9, under volume constraint one maximises Neumann eigenvalues rather than minimises them. This in contrast to Theorems 3.1.1, 3.1.3 and 3.1.6

where we minimise Neumann eigenvalues under perimeter and diameter constraint. Under volume constraint, we have that

$$\inf\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ bounded convex, } V(\Omega) = 1\} = 0 \tag{3.1.23}$$

for any $k \geq 1$ and any $d \geq 2$ and so the minimisation problem is ill-posed. The fact that the infimum in (3.1.23) is zero can be seen by considering the sequence of cuboids

$$(0, \epsilon^{1-d}) \times (0, \epsilon) \times \cdots \times (0, \epsilon) \subset \mathbb{R}^d. \tag{3.1.24}$$

This makes the diameter and perimeter constraints more interesting as a bounded convex domain having small volume does not necessarily imply it has large Neumann eigenvalues, unlike in the case of Dirichlet eigenvalues.

In this thesis, our control on the Neumann and Dirichlet counting functions is not good enough to obtain asymptotic results concerning the problems

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ bounded convex, } V(\Omega) = 1\}, \tag{3.1.25}$$

$$\sup\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ bounded convex, } V(\Omega) = 1\}. \tag{3.1.26}$$

Moreover, as far as the author is aware, the asymptotic behaviour of optimisers to these problems is unknown. However, extremal problems under volume constraint for averages of eigenvalues have been considered in the literature.

In [Fre17], Freitas considers extremal problems for the average of the first k Dirichlet eigenvalues under volume constraint, and also perimeter constraint. Due to the relevance to the results of this paper, we also note that in the perimeter case Freitas proves that any sequence of the associated minimisers $\sim$-Hausdorff converges to the ball as $k \to +\infty$. Freitas also discusses the analogues of these problems for the average of the first k Neumann eigenvalues in Section 5 of [Fre17].

Riesz means of eigenvalues have also been studied and results concerning the asymptotic behaviour of optimisers to a problem similar to (3.1.25) have been obtained.

For $\gamma \geq 0$ we define the Dirichlet Riesz mean by

$$\mathcal{R}_{\Omega}^{D,\gamma}(\Lambda) := \sum_{\{k:\lambda_k(\Omega)<\Lambda\}} (\Lambda - \lambda_k(\Omega))^{\gamma} \tag{3.1.27}$$

The Riesz mean $\mathcal{R}_{\Omega}^{D,\gamma}(\Lambda)$ can be viewed as an average of the Dirichlet eigenvalue counting function $\mathcal{N}_{\Omega}^{D}$. Note that for $\gamma = 0$, $\mathcal{R}_{\Omega}^{D,\gamma} = \mathcal{N}_{\Omega}^{D}$. Moreover, due to this, minimising Dirichlet eigenvalues is morally the same idea as maximising the Riesz mean.

It was shown in [FraLar20, Cor. 1.3] that for any $\gamma \geq 1$ fixed, there exists a maximiser $\Omega^*_{\gamma,\Lambda}$ to

$$\sup\{\mathcal{R}_{\Omega}^{D,\gamma}(\Lambda) : \Omega \subset \mathbb{R}^d \text{ bounded convex}, V(\Omega) = 1\} \tag{3.1.28}$$

for all $\Lambda > 0$. Moreover, letting $\Omega^*_{\gamma,\Lambda}$ denote any choice of such maximiser, one has that $\Omega^*_{\gamma,\Lambda}$ $\sim$-Hausdorff converges to the ball of unit volume as $\Lambda \to +\infty$.

This fits with the idea that in the regime of volume constraint, one wants to minimise perimeter to minimise large Dirichlet eigenvalues, see the two-term Weyl asymptotic (2.1.22). For further recent results on Riesz means and their associated asymptotic spectral shape optimisation, we refer the reader to [Lar19] and [FraLar24].

3.2 Proof of main results and other applications

In this section we prove the results stated in Section 3.1.2 and consider an application of our methods presented to the geometric stability of Weyl's law, see Theorem 3.2.13.

3.2.1 A warm-up with Dirichlet

We will start by considering a proof of a result concerning Dirichlet eigenvalues. The purpose of this is to illustrate how the properties of Dirichlet eigenvalues are used in such a proof and see where problems may arise when we wish to work with Neumann eigenvalues. The following theorem is a slight modification of a result due to van den Berg in [Ber15] and effectively follows the same proof.

Theorem 3.2.1. *Let $d \geq 2$ be fixed. Then for all $k \geq 1$ there exists a minimiser Ω_k^* to*

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ open}, \, D(\Omega) = 1\}, \tag{3.2.1}$$

which can be taken to be convex. Moreover, any sequence $(\Omega_k^*)_{k\geq 1}$ of such minimisers $\sim$-Hausdorff converges to the ball of unit diameter as $k \to +\infty$.

Proof. Firstly, we argue that we can reduce this to the minimisation problem

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \, D(\Omega) = 1\}. \tag{3.2.2}$$

Let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set with $D(\Omega) = 1$. The open convex envelope $\widetilde{\Omega}$ of Ω, that is the smallest open convex set containing Ω, also has $D(\widetilde{\Omega}) = 1$ and that $\Omega \subset \widetilde{\Omega}$. Thus, by the reverse monotonicity of Dirichlet eigenvalues $\lambda_k(\widetilde{\Omega}) \leq \lambda_k(\Omega)$. Moreover, if $\widetilde{\Omega}\backslash\Omega$ contains an open set then $\lambda_k(\widetilde{\Omega}) < \lambda_k(\Omega)$, see Proposition 3.2.2 in [LevPM23], and so the improvement is strict. If $\widetilde{\Omega}\backslash\Omega$ does not contain an open set then we do not necessarily have a strict inequality, see Remark 3.2.2. But, we do have that for all $x \in \widetilde{\Omega}\backslash\Omega$ and any $r > 0$, $B_r(x) \cap \Omega \neq \emptyset$. This gives that $d^H(\widetilde{\Omega}, \Omega) = 0$. Thus, we just simply identify the two if this is the case and any asymptotic convergence behaviour remains the same. Hence, we may reduce our minimisation problem to the case

$$\inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex}, \, D(\Omega) = 1\}. \tag{3.2.3}$$

Existence is immediate from the continuity of Dirichlet eigenvalues under Hausdorff convergence of convex domains, the Faber-Krahn inequality and Blaschke's selection theorem.

The bound of Li and Yau in [LiYau83, Cor. 1] reads as follows:

$$\lambda_k(\Omega) \geq \frac{d}{d+2} \cdot 4\pi^2 \left(\frac{k}{\omega_d V(\Omega)}\right)^{2/d}. \tag{3.2.4}$$

In particular if we fix B to the be the ball of unit diameter then we know that

$$\frac{d}{d+2}\cdot 4\pi^2\left(\frac{k}{\omega_d V(\Omega_k^*)}\right)^{2/d} \leq \lambda_k(\Omega_k^*) \leq \lambda_k(B) = 4\pi^2\left(\frac{k}{\omega_d V(B)}\right)^{2/d} + o(k^{2/d}). \quad (3.2.5)$$

Thus, dividing through by $k^{2/d}$ and taking the limsup as $k \to +\infty$ we see that

$$\liminf_{k\to+\infty} V(\Omega_k^*) \geq \left(\frac{d}{d+2}\right)^{d/2} V(B) > 0. \quad (3.2.6)$$

In particular, the sequence $(\Omega_k^*)_{k\geq1}$ is non-degenerate. By Proposition 2.2.4, this implies that we have a lower bound on the inradii of the Ω_k^* for k sufficiently large and so invoking Blaschke's selection theorem, we have a $\sim$-Hausdorff convergent subsequence, which we also denote by $(\Omega_k^*)_{k\geq1}$, converging to some $\Omega_\infty \in \mathcal{A}^d$ as $k \to +\infty$. By the definition of $\sim$-Hausdorff convergence, for any $\delta > 0$ we may assume that

$$\Omega_k^* \subset \Omega_\infty \oplus \delta\mathbb{B}^d \quad (3.2.7)$$

for k sufficiently large, translating the Ω_k^* if necessary. Letting B be the ball of diameter one, using the reverse monotonicity of Dirichlet eigenvalues under inclusion of domains

$$\lambda_k(\Omega_\infty \oplus \delta\mathbb{B}^d) \leq \lambda_k(\Omega_k^*) \leq \lambda_k(B) \quad (3.2.8)$$

for k sufficiently large. Dividing through by $k^{2/d}$ and using Weyl's law one obtains

$$V(\Omega_\infty \oplus \delta\mathbb{B}^d) \geq V(B). \quad (3.2.9)$$

Since $\delta > 0$ was arbitrary, we see that $V(\Omega_\infty) = V(B)$ and so Ω_∞ must be a ball of diameter one by the isodiametric inequality. Since the subsequence was arbitrary, we are done. □

Remark 3.2.2. If $\Omega \neq \widetilde{\Omega}$ but $\widetilde{\Omega}\backslash\Omega$ does not contain an open set, then one does not necessarily have that $\lambda_k(\widetilde{\Omega}) < \lambda_k(\Omega)$. This can be seen by setting Ω to be the union of the nodal domains of an eigenfunction $u : \widetilde{\Omega} \to \mathbb{R}$ corresponding to a Courant-sharp eigenvalue $\lambda_k(\widetilde{\Omega})$. In this case, one has that $\lambda_k(\Omega) = \lambda_k(\widetilde{\Omega})$.

By a Courant-sharp eigenvalue $\lambda_k(\widetilde{\Omega})$ here, we mean that there exists an associated eigenfunction $u_k : \widetilde{\Omega} \to \mathbb{R}$ such that the set

$$\widetilde{\Omega} \setminus \{x \in \Omega : u_k(x) = 0\} \tag{3.2.10}$$

has precisely k connected components. It cannot have strictly more than k connected components by Courant's nodal domain theorem, see [LevPM23, §4], and hence the term Courant-sharp.

3.2.2 A bound on the Neumann counting function

We now present a family of upper bounds on the Neumann eigenvalue counting function for bounded convex domains. The proof is originally inspired by the proof of Proposition A.1 in [GitLén20], where the authors give an upper bound on the Neumann eigenvalue counting function for bounded C^2 convex domains. Moreover, one should note that the idea of the proof is very classical and can be attributed back to the proof of Weyl's law in book of Courant and Hilbert [CouHil53].

Although our bound is more general than the bound in [GitLén20], our bound is weaker and less general than others in the literature, see Remark 3.2.6.

Before stating our family of bounds, we make a notational remark that $\lceil x \rceil$ means the smallest integer greater than or equal to $x \in \mathbb{R}$.

Proposition 3.2.3. *For any $n \in \mathbb{N}$, $\Omega \in \mathcal{A}^d$ and $\alpha > 0$,*

$$\mathcal{N}_\Omega^N(\alpha) \leq \frac{nV(\Omega)}{(\mu_{n+1}^*)^{d/2}} \alpha^{d/2} + r_n(\Omega; \alpha), \tag{3.2.11}$$

where:

$$\begin{aligned} r_n(\Omega;\alpha) = &\left(\frac{\kappa_n}{(\mu_{n+1}^*)^{1/2}}\right)^{d-1} (2\kappa_n + 3) d^{1/2} P(\Omega) \alpha^{(d-1)/2} \\ &+ \sum_{j=2}^{d-1} \binom{d}{j} (4d)^{j/2} \left(\frac{\kappa_n}{(\mu_{n+1}^*)^{1/2}}\right)^{d-j} W_j(\Omega) \alpha^{(d-j)/2} + (4d)^{d/2} \omega_d; \end{aligned} \tag{3.2.12}$$

μ_{n+1}^* denotes the $(n+1)$-th Neumann eigenvalue of the d-dimensional unit cube; $\kappa_n := \left\lceil \pi^{-1} (d\mu_{n+1}^*)^{1/2} \right\rceil$; and, the $W_j(\Omega)$ are the quermassintegrals as given in Theorem 2.2.5.

Moreover, the remainder $r_n(\Omega;\alpha)$ *is monotone with respect to inclusion of convex domains, i.e.* $r_n(\Omega_1;\alpha) \leq r_n(\Omega_2;\alpha)$ *for all* $\alpha > 0$ *whenever* $\Omega_1 \subset \Omega_2$.

Proof. Fix $\delta > 0$ and $n \in \mathbb{N}$. For $m \in \mathbb{Z}^d$, let $Q_{m,\delta} := \delta\left(m + (0,1)^d\right)$. We first note that

$$\mathcal{N}_\Omega^N(\delta^{-2}\mu_{n+1}^*) \leq n \tag{3.2.13}$$

by the definition of the Neumann eigenvalue counting function and the scaling property of Neumann eigenvalues under homothety.

Setting

$$\mathcal{I}_\delta := \{m \in \mathbb{Z}^d : Q_{m,\delta} \cap \Omega = Q_{m,\delta}\}, \tag{3.2.14}$$

we immediately see that

$$\#\mathcal{I}_\delta \leq \delta^{-d} V(\Omega) \tag{3.2.15}$$

as for any $m \in \mathcal{I}_\delta$ we must have $Q_{m,\delta} \subset \Omega$. We then define $\Omega_\delta^i := \bigcup_{m\in\mathcal{I}_\delta} Q_{m,\delta}$.

Taking $\kappa_n \in \mathbb{N}$ as in the statement of the proposition, let

$$\mathcal{J}_\delta := \{m \in \mathbb{Z}^d : Q_{m,\delta/\kappa_n} \cap \Omega \neq \varnothing, Q_{m,\delta/\kappa_n} \cap \Omega_\delta^i = \varnothing\} \tag{3.2.16}$$

and set

$$\Omega_{\delta/\kappa_n}^o := \Omega \cap \left(\bigcup_{m\in\mathcal{J}_\delta} Q_{m,\delta/\kappa_n}\right). \tag{3.2.17}$$

As $\kappa_n \in \mathbb{N}$, we have $(\delta/\kappa_n)\mathbb{Z} \supset \delta\mathbb{Z}$ and so by construction $\Omega_\delta^i \cap \Omega_{\delta/\kappa_n}^o = \varnothing$ and $\Omega_\delta^i \cup \Omega_{\delta/\kappa_n}^o = \Omega$ up to a set of measure zero.

We now argue that for any $m \in \mathcal{J}_\delta$, one necessarily has that $Q_{m,\delta/\kappa_n}$ is a subset of the region

$$\partial\Omega_{\delta,\kappa_n} := \{x \in \Omega : d(x,\partial\Omega) \leq (2+1/\kappa_n)\delta d^{1/2}\} \cup \{x \in \mathbb{R}^d \backslash \Omega : d(x,\partial\Omega) \leq 2\delta d^{1/2}/\kappa_n\}, \tag{3.2.18}$$

this will allow us to bound the cardinality of $\mathcal{J}_\delta$. We argue this by contradiction.

Firstly, suppose that

$$Q_{m,\delta/\kappa_n} \cap \{x \in \Omega : d(x,\partial\Omega) > (2+1/\kappa_n)\delta d^{1/2}\} \neq \varnothing. \tag{3.2.19}$$

Then we must have that $d(m\delta/\kappa_n, \partial\Omega) > 2\delta d^{1/2}$. Letting $m^* \in \mathbb{Z}^d$ be the unique integer lattice point so that $Q_{m^*,\delta} \supset Q_{m,\delta/\kappa_n}$. One then can observe that $\|m\delta/\kappa_n - m^*\delta\|_2 \leq \delta d^{1/2}$ and so necessarily $d(m^*\delta, \partial\Omega) > d^{1/2}\delta$. However, this implies that $m^* \in \mathcal{I}_\delta$ which in turn implies that $m \notin \mathcal{J}_\delta$ and we arrive at a contradiction.

Secondly, suppose that

$$Q_{m,\delta/\kappa_n} \cap \{x \in \mathbb{R}^d \backslash \Omega : d(x, \partial\Omega) > 2\delta d^{1/2}/\kappa_n\} \neq \varnothing. \tag{3.2.20}$$

Then one has that $d(m\delta/\kappa_n, \partial\Omega) > \delta d^{1/2}/\kappa_n$, but again this contradicts $m \in \mathcal{J}_\delta$ as otherwise one would have $Q_{m,\delta/\kappa_n} \cap \Omega = \varnothing$.

Thus, we indeed must have $Q_{m,\delta/\kappa_n} \subset \partial\Omega_{\delta,\kappa_n}$ whenever $m \in \mathcal{J}_\delta$.

Using the Minkowski-Steiner formula, see Theorem 2.2.5, to estimate the volume of

$$\{x \in \mathbb{R}^d \backslash \Omega : d(x, \partial\Omega) \leq 2\delta d^{1/2}/\kappa_n\} \tag{3.2.21}$$

and the erosion formula, see Theorem 2.2.6, to estimate the volume of

$$\{x \in \Omega : d(x, \partial\Omega) \leq (2 + 1/\kappa_n)\delta d^{1/2}\}, \tag{3.2.22}$$

we obtain that

$$\begin{aligned} V(\partial\Omega_{\delta,\kappa_n}) \leq (2 + 3/\kappa_n) d^{1/2} P(\Omega)\delta + \sum_{j=2}^{d-1} \binom{d}{j} (4d)^{j/2} (\kappa_n)^{-j} W_j(\Omega)\delta^j \\ + (4d)^{d/2}\omega_d(\kappa_n)^{-d}\delta^d. \end{aligned} \tag{3.2.23}$$

We can then immediately obtain a bound on the cardinality of $\mathcal{J}_\delta$:

$$\begin{aligned} \#\mathcal{J}_\delta &\leq (\kappa_n)^d \delta^{-d} V(\partial\Omega_{\delta,\kappa_n}) \\ &\leq (\kappa_n)^{d-1}(2\kappa_n + 3) d^{1/2} P(\Omega)\delta^{-d+1} \\ &\quad + \sum_{j=2}^{d-1} \binom{d}{j} (4d)^{j/2} (\kappa_n)^{d-j} W_j(\Omega)\delta^{-d+j} + (4d)^{d/2}\omega_d. \end{aligned} \tag{3.2.24}$$

Observing that $\Omega \cap Q_{m,\delta/\kappa_n}$ is convex, by our choice of κ_n and the Payne-Weinberger inequality, see Theorem 2.1.2,

$$\mu_2(\Omega \cap Q_{m,\delta/\kappa_n}) > \frac{\pi^2(\kappa_n)^2}{d\delta^2} \geq \delta^{-2}\mu^*_{n+1} \tag{3.2.25}$$

for any $m \in \mathcal{J}_\delta$. And so $\mathcal{N}^N_{\Omega\cap Q_{m,\delta/\kappa_n}}(\delta^{-2}\mu^*_{n+1}) = 1$ for any $m \in \mathcal{J}_\delta$. By the variational characterisation of Neumann eigenvalues, see (2.1.15), it is straightforward to verify that

$$\mu_k(\Omega^i_\delta \cup \Omega^o_{\delta/\kappa_n}) \le \mu_k(\Omega) \tag{3.2.26}$$

for all $k \in \mathbb{N}$, which implies

$$\mathcal{N}^N_{\Omega^i_\delta\cup\Omega^o_{\delta/\kappa_n}}(\alpha) \ge \mathcal{N}^N_\Omega(\alpha) \tag{3.2.27}$$

for all $\alpha > 0$. So, it suffice to bound the Neumann eigenvalue counting function of $\Omega^i_\delta \cup \Omega^o_{\delta/\kappa_n}$. Taking $\delta = \alpha^{-1/2}(\mu^*_{n+1})^{1/2}$ and using the bounds on $\#\mathcal{I}_\delta$ and $\#\mathcal{J}_\delta$, we have

$$\begin{aligned}\mathcal{N}^N_{\Omega^i_\delta\cup\Omega^o_{\delta/\kappa_n}}(\alpha) &\le \sum_{m\in\mathcal{I}_\delta}\mathcal{N}^N_{Q_{m,\delta}}(\alpha) + \sum_{n\in\mathcal{J}_\delta}\mathcal{N}^N_{Q_{n,\delta/\kappa_n}}(\alpha)\\ &\le \frac{nV(\Omega)}{(\mu^*_{n+1})^{d/2}}\alpha^{d/2} + \left(\frac{\kappa_n}{(\mu^*_{n+1})^{1/2}}\right)^{d-1}(2\kappa_n+3)d^{1/2}P(\Omega)\alpha^{(d-1)/2}\\ &\quad + \sum_{j=2}^{d-1}\binom{d}{j}(4d)^{j/2}\left(\frac{\kappa_n}{(\mu^*_{n+1})^{1/2}}\right)^{d-j}W_j(\Omega)\alpha^{(d-j)/2} + (4d)^{d/2}\omega_d\\ &= \frac{nV(\Omega)}{(\mu^*_{n+1})^{d/2}}\alpha^{d/2} + r_n(\Omega;\alpha).\end{aligned} \tag{3.2.28}$$

The monotonicity of perimeter and the quermassintegrals $W_j(\Omega)$ with respect to inclusion of bounded convex domains gives the monotonicity of the remainder $r_n(\Omega;\alpha)$ with respect to inclusion of bounded convex domains, which completes the proof. □

Remark 3.2.4 (A remark à la Pólya)**.** As an observation, from the bound in Proposition 3.2.3 we see that

$$\frac{n}{(\mu^*_{n+1})^{d/2}} \ge \frac{\omega_d}{(2\pi)^d} \tag{3.2.29}$$

by Weyl's law. Rearranging this gives

$$\mu^*_{n+1} \le 4\pi^2\left(\frac{n}{\omega_d}\right)^{2/d}. \tag{3.2.30}$$

This is precisely Pólya's conjectured bound for the unit cube, see Section 2.1.3. One can of course modify the proof of Proposition 3.2.3 to prove Pólya's bound for any convex tiling domain of unit volume, which is already known in the more general case of tiling domains from [Kel66].

On may remove the dependence on the parameter $n \in \mathbb{N}$ in Proposition 3.2.3 as follows:

Corollary 3.2.5. *For any convex domain* $\Omega \in \mathcal{A}^d$

$$\mathcal{N}_\Omega^N(\alpha) \leq \frac{\omega_d V(\Omega)}{(2\pi)^d}\alpha^{d/2} + r(\Omega;\alpha), \tag{3.2.31}$$

where $r(\Omega;\alpha)$ *is monotone with respect to inclusion of bounded convex domains and* $r(\Omega;\alpha) = O(\alpha^{(2d-1)/4})$.

Proof. To obtain this bound we let the free parameter $n \in \mathbb{N}$ in the statement of Proposition 3.2.3 vary with α. By considering upper bounds on the Neumann eigenvalue counting function of the d-dimensional unit cube, one can observe that

$$\frac{n}{(\mu_{n+1}^*)^{d/2}} \leq (2\pi)^{-d}\omega_d + O(n^{-1/d}). \tag{3.2.32}$$

Moreover, from the definition of κ_n in the statement of Proposition 3.2.3 we have that

$$\left(\frac{\kappa_n}{(\mu_{n+1}^*)^{1/2}}\right)^{d-1}(2\kappa_n+3) = O(n^{1/d}). \tag{3.2.33}$$

So, taking

$$n(\alpha) \sim V(\Omega)^{d/2}P(\Omega)^{-d/2}\alpha^{d/4} \tag{3.2.34}$$

one gets the desired result as one can impart uniform control on κ_n in terms of the geometry of Ω via

$$\kappa_{n(\alpha)} \sim \frac{2d^{1/2}V(\Omega)^{1/2}}{(\omega_d)^{1/d}P(\Omega)^{1/2}}\alpha^{1/4} \leq C_d V(\Omega)^{1/(2d)}\alpha^{1/4} \tag{3.2.35}$$

as $\alpha \to +\infty$, where $C_d > 0$ is a constant depending only on only on the dimension d. Here we have used the isoperimetric inequality to get the bound on the right-hand side of (3.2.35). This choice of asymptotic behaviour for $n(\alpha)$ is natural as $n(\alpha)$ is dimensionless and $\alpha^{-1/2}$ scales like a length. □

Remark 3.2.6. Our bound in Corollary 3.2.5 is certainly not state of the art in comparison to others in the literature. For example, in [NetSaf05, Thm. 1.3] the authors obtain the bound

$$\mathcal{N}_\Omega^N(\alpha) \leq \frac{\omega_d V(\Omega)}{(2\pi)^d}\alpha^{d/2} + O(\alpha^{(d-1)/2}\log\alpha) \tag{3.2.36}$$

and for much more general domains than just convex ones. However, our bound is easier to work with and the monotonicity of the remainder is very advantageous for the proofs in Section 3.2.4. This is why we prefer to work with our own bound rather than others in the literature.

3.2.3 Existence in dimension two

We now briefly present the work in [BogHM24] regarding the existence of minimisers in dimension two in Theorems 3.1.1 and 3.1.3.

Understanding the existence of such minimisers relies on understanding the behaviour of Neumann eigenvalues of bounded two-dimensional convex domains as they collapse to a line segment. For Dirichlet eigenvalues this is essentially trivial as one can bound the first Dirichlet eigenvalue from below by $4\pi\rho(\Omega)^{-2}$, see [Her60]. In the case of Neumann eigenvalues, it is much more subtle. For a given $\Omega \in \mathcal{A}^2$, let I_Ω denote its projection onto $\mathbb{R}$ in the first coordinate, i.e. $I_\Omega = \Psi(\Omega)$ where $\Psi(x,y) = x$. As Ω is convex, $I_\Omega \subset \mathbb{R}$ is simply a bounded open interval.

We define $h_\Omega^+, h_\Omega^- : I_\Omega \to \mathbb{R}$ so that

$$\Omega \equiv \{(x,y) : x \in I_\Omega,\ h^-(x) < y < h^+(x)\}, \tag{3.2.37}$$

and set

$$h_\Omega(x) = h_\Omega^+(x) - h_\Omega^-(x). \tag{3.2.38}$$

We call h_Ω the height profile of Ω and we can make the quick observation that h is non-negative, continuous, concave and bounded. From here onwards, we will fix $I_\Omega = (0,1)$ for any Ω.

Bogosel, Henrot and Michetti in [BogHM24] introduce the function space

$$\mathcal{U} = \{h \in L^\infty(I_\Omega) : 0 \le h \le 1,\ h \text{ concave and continuous}\}. \tag{3.2.39}$$

For any $h \in \mathcal{U}$, they define the Sturm-Liouville eigenvalues $0 \le \mu_1(h) \le \mu_2(h) \le \mu_3(h) \le \cdots$ of the system

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}(x)\right) = h(x)\mu(h)u(x), \quad x \in I_\Omega, \\ h(0)\dfrac{\mathrm{d}u}{\mathrm{d}x}(0) = h(1)\dfrac{\mathrm{d}u}{\mathrm{d}x}(1) = 0. \end{cases} \tag{3.2.40}$$

These eigenvalues admit the variational characterisation

$$\mu_k(h) = \inf_{\substack{V \subset H^1(I_\Omega) \\ \dim(V)=k \\ V \perp_{L^2} h}} \sup_{\substack{u \in V \\ u \neq 0}} \frac{\displaystyle\int_{I_\Omega} \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 h}{\displaystyle\int_{I_\Omega} |u|^2 h}, \tag{3.2.41}$$

where the infimum is taken over all k-dimensional subspaces of $H^1(I_\Omega)$ that are L^2 orthogonal to h.

In Lemma 2.3 of [BogHM24], they show that

$$\inf\{\mu_k(h) : h \in \mathcal{U}\} = \pi^2 k^2 \tag{3.2.42}$$

for all $k \ge 1$ and this infimum is attained by $h \equiv 1$ in each case. Moreover, they prove the following results.

Proposition 3.2.7 ([BogHM24, Lemma 2.2])**.** *Let $(h_\epsilon)_{\epsilon>0}$ be a sequence in U converging in $L^2((0,1))$ to some $h \in \mathcal{U}$ as $\epsilon \downarrow 0$. For any decomposition $h_\epsilon = h_\epsilon^+ - h_\epsilon^-$ into a sum of two nonnegative bounded concave functions we set*

$$\Omega_{\epsilon h_\epsilon} := \{(x,y) : 0 < x < 1,\ -\epsilon h_\epsilon^-(x) < y < \epsilon h_\epsilon^+(x)\}. \tag{3.2.43}$$

Irrespective of this choice of decomposition we have

$$\liminf_{\epsilon \downarrow 0} \mu_{k+1}(\Omega_{\epsilon h_\epsilon}) \ge \mu_k(h). \tag{3.2.44}$$

This proposition shows that the $\mu_k(h)$ asymptotically bounds $\mu_{k+1}(\Omega_n)$ from below for any sequence of two dimensional convex domains $(\Omega_n)_{n\geq 1}$ with $I_{\Omega_n} = (0,1)$ in $\mathcal{A}^2$ collapsing to I_{Ω_n} as $n \to +\infty$.

Combining the above proposition with (3.2.42), one has the existence of minimisers to

$$\inf\{\mu_k(\Omega) : \Omega \in \mathcal{A}^2,\, D(\Omega) = 1\} \tag{3.2.45}$$

if there exists a domain $\Omega' \in \mathcal{A}^2$ with $D(\Omega') = 1$ such that $\mu_k(\Omega) \leq \pi^2(k-1)^2$. By explicitly choosing a domain, one may show that this is true for all $k \geq 3$, see Theorem 2.4 in [BogHM24]. This is indeed the best one can do as the Payne-Weinberger shows a minimiser cannot exist for μ_2 in this case.

Similarly to the diameter case, one may show that

$$\inf\{\mu_k(\Omega) : \Omega \in \mathcal{A}^2,\, P(\Omega) = 1\} \tag{3.2.46}$$

admits a minimiser for all $k \geq 3$, see Theorem 2.5 in [BogHM24].

In two dimensions, in the case of the interior problem, see Theorem 3.1.7, using different ideas one can show that minimisers exist if and and only if there exists a subdomain $\Omega \subset \Omega'$, such that

$$\mu_k(\Omega) \leq \frac{\pi^2(k-1)^2}{D(\Omega')^2}, \tag{3.2.47}$$

see Theorem 2.4 in [CavFHLLS23].

3.2.4 Proof of main results

The proofs of Theorems 3.1.1, 3.1.3, 3.1.6 and 3.1.7 mainly follow from the following proposition, regarding asymptotic lower bounds for Neumann eigenvalues, in combination with Corollary 3.2.5. The proposition itself can be proven using Proposition 3.2.3 or Corollary 3.2.5. We choose to use Proposition 3.2.3 as a stylistic choice.

Proposition 3.2.8. *For any $A > 0$ and any $f : \mathbb{N} \to (0, +\infty)$ such that $c \leq f(k) \ll$*

$k^{1/(d(d-1))}$ *for some* $c > 0$,

$$\liminf_{k\to+\infty} k^{-2/d}\left[\inf\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, V(\Omega) \le A,\, D(\Omega) \le f(k)\}\right] \ge \frac{4\pi^2}{(\omega_d A)^{2/d}}. \quad (3.2.48)$$

Proof. Let $k, n \in \mathbb{N}$ and $\epsilon > 0$ be fixed, and let $\Omega \in \mathcal{A}^d$ with $V(\Omega) \le A$ and $D(\Omega) \le f(k)$. It is straightforward to see that any such Ω can necessarily be contained in a ball of diameter $2f(k)$. Thus, from the bound in Proposition 3.2.3, using the monotonicity of the remainder we see that

$$k \le \mathcal{N}^N_\Omega(\mu_k(\Omega) + \epsilon) \le \frac{nV(\Omega)}{(\mu^*_{n+1})^{d/2}}(\mu_k(\Omega) + \epsilon)^{d/2} + r_n(B_k; \mu_k(\Omega) + \epsilon), \qquad (3.2.49)$$

where B_k is the ball of diameter $2f(k)$. Since our choice of Ω was arbitrary and $V(\Omega) \le A$, we have

$$1 \le \frac{nA}{k(\mu^*_{n+1})^{d/2}}(m_k + \epsilon)^{d/2} + k^{-1}r_n(B_k; m_k + \epsilon), \qquad (3.2.50)$$

where

$$m_k = \inf\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, V(\Omega) \le A,\, D(\Omega) \le f(k)\}. \qquad (3.2.51)$$

Setting $\overline{m}_k = k^{-2/d}(m_k + \epsilon)$, writing out the right-hand side of (3.2.50),

$$\begin{aligned} 1 &\le \frac{nA}{(\mu^*_{n+1})^{d/2}}(\overline{m}_k)^{d/2} + C_{d,n}k^{-1/d}P(B_k)(\overline{m}_k)^{(d-1)/2} \\ &\quad + \sum_{j=2}^{d-1} C_{d,n,j}W_j(B_k)k^{-j/d}(\overline{m}_k)^{(d-j)/2} + C_d k^{-1} \\ &=: p_{n,k}(\overline{m}_k), \end{aligned} \qquad (3.2.52)$$

for some constants $C_d, C_{d,n}, C_{d,n,j} > 0$ whose dependence is denoted in the subscript.

By the scaling properties of perimeter and quermassintegrals,

$$k^{-1/d}P(B_k),\, k^{-j/d}W_j(B_k) \to 0 \qquad (3.2.53)$$

as $k \to +\infty$. Hence, for any $0 < \delta < 1$ there exists $k_\delta \in \mathbb{N}$ such that for all $k \ge k_\delta$

$$1 \le p_{n,k}(\overline{m}_k) \le \frac{nA}{(\mu^*_{n+1})^{d/2}}(\overline{m}_k)^{d/2} + \delta\sum_{j=1}^{d}(\overline{m}_k)^{(d-j)/2} =: q_{n,\delta}(\overline{m}_k). \qquad (3.2.54)$$

Letting $\gamma_{n,\delta}$ be the unique positive solution to $q_{n,\delta}(x) = 1$, one can immediately see that $\overline{m}_k \geq \gamma_{n,\delta}$ for any $k \geq k_\delta$ as $q_{n,\delta} : (0, +\infty) \to \mathbb{R}$ is strictly monotone increasing for each $n \in \mathbb{N}$ and $\delta > 0$. Since $\delta > 0$ was arbitrary

$$\liminf_{k \to +\infty} \overline{m}_k \geq \lim_{\delta \downarrow 0} \gamma_{n,\delta} = \left(\frac{(\mu^*_{n+1})^{d/2}}{nA} \right)^{2/d} = \frac{\mu^*_{n+1}}{n^{2/d} A^{2/d}}, \tag{3.2.55}$$

and so

$$\liminf_{k \to +\infty} k^{-2/d} m_k \geq \frac{\mu^*_{n+1}}{n^{2/d} A^{2/d}}. \tag{3.2.56}$$

Taking the limit as $n \to +\infty$ gives the result by Weyl's law. □

Remark 3.2.9. The condition $f(k) = o(k^{1/d(d-1)})$ is sharp in the statement of the proposition. For a proof of this, see Proposition 3.2.14.

One can also prove the following result completely analogously to that of Proposition 3.2.8.

Proposition 3.2.10. *For any $A > 0$ and any $f : \mathbb{N} \to (0, +\infty)$ such that $c \leq f(k) \ll k^{1/2}$ for some $c > 0$,*

$$\liminf_{k \to +\infty} k^{-1} \left[\inf \left\{ \mu_k(\Omega) : \Omega \in \mathcal{A}^2,\ V(\Omega) \leq A,\ P(\Omega) \leq f(k) \right\} \right] \geq 4\pi A^{-1}. \tag{3.2.57}$$

We are now ready to give proofs of the results in Section 3.1.2.

Proof of Theorem 3.1.1. Existence of minimisers for all $k \geq 3$ comes directly from Theorem 2.5 in [BogHM24], as discussed Section 3.2.3. So, it suffices to prove the asymptotic behaviour of any sequence $(\Omega^*_k)_{k \geq 3}$ of minimisers as $k \to +\infty$.

Let B be the two-dimensional ball of unit perimeter. From Weyl's law we know that

$$\mu_k(B) \sim \frac{4\pi k}{V(B)} \tag{3.2.58}$$

and, from Proposition 3.2.10, for any $0 < \epsilon < V(B)$ and $0 < \delta < 1$

$$\inf \left\{ \mu_k(\Omega) : \Omega \in \mathcal{A}^2,\ V(\Omega) \leq V(B) - \epsilon,\ P(\Omega) = 1 \right\} \geq (1 - \delta) \frac{4\pi k}{(V(B) - \epsilon)} \tag{3.2.59}$$

for k sufficiently large. Combining (3.2.58) and (3.2.59), we see that for any $0 < \epsilon <$

$V(B)$, for k sufficiently large

$$\mu_k(B) < \inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^2,\, V(\Omega) \le V(B) - \epsilon,\, P(\Omega) = 1\right\}. \tag{3.2.60}$$

Hence, one must have that $V(\Omega_k^*) > V(B) - \epsilon$ for k sufficiently large. Since $0 < \epsilon < V(B)$ was arbitrary, we see that $V(\Omega_k^*) \to V(B)$ as $k \to +\infty$. Bonnesen's quantitative isoperimetric inequality, see [Bon24; Kri27], asserts that

$$\pi^2(R(\Omega) - \rho(\Omega))^2 \le P(\Omega)^2 - 4\pi V(\Omega) = 4\pi(V(B) - V(\Omega)). \tag{3.2.61}$$

for any $\Omega \in \mathcal{A}^2$ with $P(\Omega) = 1$. This quantitative isoperimetric inequality allows one to deduce a bound on the $\sim$-Hausdorff distance between Ω_k^* and B by

$$d_H([\Omega_k^*], [B]) \le R(\Omega_k^*) - \rho(\Omega_k^*) \tag{3.2.62}$$

as we necessarily have

$$B_{\rho(\Omega_k^*)}(0) \subset B \subset B_{R(\Omega_k^*)}(0) \tag{3.2.63}$$

by the monotonicity of perimeter with respect to bounded convex domains. From, (3.2.61) and (3.2.62), and knowing that $V(\Omega_k^*) \to V(B)$ as $k \to +\infty$, we immediately deduce that $(\Omega_k^*)_{k\ge1}$ must $\sim$-Hausdorff converge towards the ball of unit perimeter as $k \to +\infty$.

We now move on to estimate the rate of this $\sim$-Hausdorff convergence. From the bound in Corollary 3.2.5 and validity of Pólya's conjecture for Neumann eigenvalues of two-dimensional balls, see Section 2.1.3,

$$\mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \mathcal{N}_{\Omega_k^*}^N(\mu_k(\Omega_k^*)) \ge \frac{V(B)}{4\pi}\mu_k(\Omega_k^*) - \frac{V(\Omega_k^*)}{4\pi}\mu_k(\Omega_k^*) - r(\Omega_k^*; \mu_k(\Omega_k^*)). \tag{3.2.64}$$

As this remainder is monotone with respect to inclusion of bounded convex domains and Ω_k^* $\sim$-Hausdorff converges to B as $k \to +\infty$, we may pick a function $s : [0, +\infty) \to [0, +\infty)$ with $r(\Omega_k^*; \alpha) \le s(\alpha)$ for all $k \in \mathbb{N}$ and $\alpha > 0$ that satisfies the asymptotic estimate $s(\alpha) = O(\alpha^{3/4})$. Hence,

$$\mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \mathcal{N}_{\Omega_k^*}^N(\mu_k(\Omega_k^*)) \ge \frac{[V(B) - V(\Omega_k^*)]}{4\pi}\mu_k(\Omega_k^*) - s(\mu_k(\Omega_k^*)) \tag{3.2.65}$$

By the optimality of the Ω_k^*, we must not have that $\mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \mathcal{N}_{\Omega_k^*}^N(\mu_k(\Omega_k^*)) > 0$ for any $k \in \mathbb{N}$. Thus, we may obtain

$$V(B) \geq V(\Omega_k^*) \geq V(B) - \frac{4\pi}{\mu_k(\Omega_k^*)} \cdot s(\mu_k(\Omega_k^*)) = V(B) - O(k^{-1/4}) \tag{3.2.66}$$

as we know

$$\mu_k(\Omega_k^*) \sim \frac{4\pi k}{V(B)}. \tag{3.2.67}$$

Hence, combining equations (3.2.61), (3.2.62) and (3.2.66), $d_H([\Omega_k^*], [B]) = O(k^{-1/8})$ which completes the proof. □

Proof of Theorem 3.1.3. In dimension two, the proof of existence of minimisers for all $k \geq 3$ and not for $k = 2$ follows from Theorem 2.4 in [BogHM24], as discussed in the Section 3.2.3. In higher dimensions, we cannot use the same trick as in two-dimensions as collapsing sequences of convex domains of unit diameter do not necessarily collapse to a line segment. Instead we show that minimisers must eventually exist from the asymptotic result in Proposition 3.2.8.

Let B be the d-dimensional ball of unit diameter. Weyl's law tells us that

$$\mu_k(B) \sim \frac{4\pi^2}{(\omega_d V(B))^{2/d}} k^{2/d} \tag{3.2.68}$$

and from Proposition 3.2.8 we see that for any $0 < \epsilon < V$ and $0 < \delta < 1$, for k sufficiently large

$$\inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, V(\Omega) \leq V(B) - \epsilon,\, D(\Omega) \leq 1\right\} \geq (1-\delta)\frac{4\pi^2}{(\omega_d(V(B)-\epsilon))^{2/d}} k^{2/d} \tag{3.2.69}$$

Thus, for any $0 < \epsilon < V$, for k sufficiently large

$$\begin{aligned} \mu_k(B) &< \inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, V(\Omega) \leq V(B) - \epsilon,\, D(\Omega) \leq 1\right\} \\ &\leq \inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, V(\Omega) \leq V(B) - \epsilon,\, D(\Omega) = 1\right\}. \end{aligned} \tag{3.2.70}$$

Hence,

$$\begin{aligned} &\inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, D(\Omega) = 1\right\} \\ &\qquad = \inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d,\, V(\Omega) \geq V(B) - \epsilon,\, D(\Omega) = 1\right\}. \end{aligned} \tag{3.2.71}$$

By our version of Blaschke's selection theorem and the continuity of Neumann eigenvalues under Hausdorff convergence of domains, a minimiser must necessarily exist for k sufficiently large.

As $0 < \epsilon < V(B)$ was arbitrary in (3.2.70), it is clear that one necessarily has for any sequence $(\Omega_k^*)_{k\geq 1}$ of minimisers, $V(\Omega_k^*) \to V(B)$ as $k \to +\infty$. Using the quantitative isodiametric inequality [MagPP14, Thm. 2],

$$d_H([\Omega_k^*], [B]) = O\left((V(B) - V(\Omega_k^*))^{1/d}\right). \tag{3.2.72}$$

Thus, we immediately have that $(\Omega_k^*)_{k\geq 1}$ must $\sim$-Hausdorff converge to B as $k \to +\infty$.

As before, we may estimate the rate of this $\sim$-Hausdorff convergence. From the bound in Corollary 3.2.5, we see that

$$\mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \mathcal{N}_{\Omega_k^*}^N(\mu_k(\Omega_k^*)) \geq \mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \frac{\omega_d V(\Omega_k^*)}{(2\pi)^d}(\mu_k(\Omega_k^*))^{d/2} - r(\Omega_k^*; \mu_k(\Omega_k^*)) \tag{3.2.73}$$

As the remainder r is monotone with respect to inclusion of bounded convex domains and the Neumann eigenvalue counting function of the ball satisfies the two-term Weyl asymptotic formula, we may obtain

$$\mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \mathcal{N}_{\Omega_k^*}^N(\mu_k(\Omega_k^*)) \geq \frac{\omega_d(V(B) - V(\Omega_k^*))}{(2\pi)^d}(\mu_k(\Omega_k^*))^{d/2} + s\left(\mu_k(\Omega_k^*)\right), \tag{3.2.74}$$

for some function $s : [0, +\infty) \to [0, +\infty)$ satisfying the asymptotic estimate $s(\alpha) = O\left(\alpha^{(2d-1)/4}\right)$.

We must not have that $\mathcal{N}_B^N(\mu_k(\Omega_k^*)) - \mathcal{N}_{\Omega_k^*}^N(\mu_k(\Omega_k^*)) > 0$ by the optimality of the Ω_k^*. Thus,

$$V(B) - V(\Omega_k^*) = O\left(\mu_k(\Omega_k^*)^{-1/4}\right) = O\left(k^{-1/(2d)}\right), \tag{3.2.75}$$

as we know that

$$\mu_k(\Omega_k^*) \sim 4\pi^2 \left(\frac{k}{\omega_d V(B)}\right)^{2/d}. \tag{3.2.76}$$

Combining equations (3.2.72) and (3.2.75), we obtain

$$d_H([\Omega_k^*],[B]) = O(k^{-1/(2d^2)}) \tag{3.2.77}$$

as desired. □

Proof of Theorem 3.1.6. As the assumptions on the function $f : \mathbb{N} \to (0,+\infty)$ in Theorem 3.1.6 are the same as those in Proposition 3.2.8, following the same lines of argument of the proof of Theorem 3.1.3, one can prove the existence part of Theorem 3.1.6 analogously. Moreover, one may show that any sequence of minimisers $(\Omega_k^*)_{k\geq 1}$ must necessarily have $V(\Omega_k^*) \to V(B)$, where B is the ball of unit perimeter, as $k \to +\infty$ in the same way as in the proof of Theorem 3.1.3. Hence, $(\Omega_k^*)_{k\geq 1}$ must $\sim$-Hausdorff converge to B as $k \to +\infty$ by the isoperimetric inequality.

Our estimate on the asymptotic convergence rate clearly depends on d and our choice of f if we try to use the bounds from Corollary 3.2.5. However, we know $\limsup_{k\to+\infty} D(\Omega_k^*) < +\infty$ from Proposition 2.2.3. So then we can just resort to the same methods as in the diameter case to determine the asymptotic rate alongside using the quantitative isoperimetric inequality result due to Fuglede in [Fug89]. □

Proof of Theorem 3.1.7. A simple application of Proposition 3.2.8 shows that for any $\epsilon > 0$

$$\liminf_{k\to+\infty} k^{-2/d}\Big[\inf\Big\{\mu_k(\Omega) : \Omega \subset \Omega',\, \Omega \in \mathcal{A}^d,\, V(\Omega) \leq V(\Omega') - \epsilon\Big\}\Big] \geq \frac{4\pi^2}{(\omega_d(V(\Omega')-\epsilon))^{2/d}}. \tag{3.2.78}$$

In the same way as in the proof of Theorem 3.1.3, one can show by comparing (3.2.78) with Weyl's law for $\mu_k(\Omega')$, that minimisers must exist for k sufficiently large. Moreover, Weyl's law for $\mu_k(\Omega')$ and (3.2.78) also imply that for any sequence of minimisers $(\Omega_k^*)_{k\geq 1}$ we have that $V(\Omega_k^*) \to V(\Omega')$ as $k \to +\infty$, as $\epsilon > 0$ was arbitrary. Hence, using Lemma 2.2.11, one immediately deduces that $(\Omega_k^*)_{k\geq 1}$ must Hausdorff converge to Ω' as $k \to +\infty$.

Completely analogously to the proof of Theorem 3.1.3, one may deduce that

$$V(\Omega') - V(\Omega) = O(k^{-1/(2d)}) \tag{3.2.79}$$

and from Lemma 2.2.11 one may deduce that

$$d^H(\Omega_k^*, \Omega') = O(k^{-1/(2d^2)}) \tag{3.2.80}$$

as desired. □

3.2.5 Geometric stability of Weyl's law

In the spirit of Theorem 3.1.6, one sees that if one has a sequence of domains $(\Omega_k)_{k\geq 1}$ in $\mathcal{A}^d$ with $V(\Omega_k) = A > 0$ and $D(\Omega_k) \ll k^{1/d(d-1)}$, then

$$\mu_k(\Omega_k) \geq 4\pi^2 \left(\frac{k}{\omega_d A}\right)^{2/d} + o(k^{2/d}). \tag{3.2.81}$$

The question then becomes can we replace the inequality with an equality in this case? This is in fact true for Dirichlet and Neumann eigenvalues as we now show. First we prove an elementary lower bound on the Dirichlet eigenvalue counting function of a bounded convex domain.

Lemma 3.2.11. *For any $n \in \mathbb{N}, \Omega \in \mathcal{A}^d$ and $\alpha > 0$,*

$$\mathcal{N}_\Omega^D(\alpha) \geq \frac{nV(\Omega)}{(\lambda_n^*)^{d/2}}\alpha^{d/2} - \frac{2nd^{1/2}P(\Omega)}{(\lambda_n^*)^{(d-1)/2}}\alpha^{(d-1)/2}, \tag{3.2.82}$$

where λ_n^* is the n-th Dirichlet eigenvalue d-dimensional of the unit cube.

Proof. Let $\epsilon > 0$. For $m \in \mathbb{Z}^d$ and $\delta > 0$, define $Q_{m,\delta}$ and $\mathcal{I}_\delta$ as in the proof of Proposition 3.2.3. It is clear that for a given $m \in \mathbb{Z}^d$ if $Q_{m,\delta} \cap \{x \in \Omega : d(x, \partial\Omega) \geq 2\delta d^{1/2}\} \neq \emptyset$ then $m \in \mathcal{I}_\delta$. Hence, we obtain that

$$\#\mathcal{I}_\delta \geq \delta^{-d}|\{x \in \Omega : d(x, \partial\Omega) \geq 2\delta d^{1/2}\}| \geq \delta^{-d}V(\Omega) - 2d^{1/2}\delta^{-d+1}P(\Omega). \tag{3.2.83}$$

By the variational characterisation of Dirichlet eigenvalues, it suffices to bound the Dirichlet eigenvalue counting function of $\bigcup_{m\in\mathcal{I}_\delta} Q_{m,\delta}$ from below. Hence, taking

$\delta = \alpha^{-1/2}(\lambda_n^* + \epsilon)^{1/2}$, noting that $\mathcal{N}^D_{Q_{m,\delta}}(\delta^{-2}(\lambda_n^* + \epsilon)) \geq n$, and using the estimate on $\#\mathcal{I}_\delta$ from (3.2.83)

$$\mathcal{N}^D_\Omega(\alpha) \geq \sum_{m \in \mathcal{I}_\delta} \mathcal{N}^D_{Q_{m,\delta}}(\alpha) \geq \frac{nV(\Omega)}{(\lambda_n^* + \epsilon)^{d/2}}\alpha^{d/2} - \frac{2nd^{1/2}P(\Omega)}{(\lambda_n^* + \epsilon)^{(d-1)/2}}\alpha^{(d-1)/2}. \tag{3.2.84}$$

Taking $\epsilon \downarrow 0$ completes the proof. □

Proposition 3.2.12. *For any $A > 0$ and $f : \mathbb{N} \to (0, +\infty)$ with $c \leq f(k) \ll k^{1/d}$, we have that*

$$\limsup_{k \to +\infty} k^{-2/d} \sup\left\{\lambda_k(\Omega) : \Omega \in \mathcal{A}^d,\ V(\Omega) \geq A,\ P(\Omega) \leq f(k)\right\} \leq \frac{4\pi^2}{(\omega_d A)^{2/d}}, \tag{3.2.85}$$

provided that the set is non-empty for k sufficiently large.

Proof. For an arbitrary $\Omega \in \mathcal{A}^d$ with $V(\Omega) \geq A$ and $P(\Omega) \leq f(k)$, observe that

$$\begin{aligned} k \geq \mathcal{N}^D_\Omega(\lambda_k(\Omega)) &\geq \frac{nV(\Omega)}{(\lambda_n^*)^{d/2}}\lambda_k(\Omega)^{d/2} - \frac{2nd^{1/2}P(\Omega)}{(\lambda_n^*)^{(d-1)/2}}\lambda_k(\Omega)^{(d-1)/2} \\ &\geq \frac{nA}{(\lambda_n^*)^{d/2}}\lambda_k(\Omega)^{d/2} - \frac{2nd^{1/2}f(k)}{(\lambda_n^*)^{(d-1)/2}}\lambda_k(\Omega)^{(d-1)/2}, \end{aligned} \tag{3.2.86}$$

using the bound from Lemma 3.2.11. Setting

$$M_k := \sup\left\{\lambda_k(\Omega) : \Omega \in \mathcal{A}^d,\ V(\Omega) \geq A,\ P(\Omega) \leq f(k)\right\}, \tag{3.2.87}$$

we have that

$$k \geq \frac{nA}{(\lambda_n^*)^{d/2}}\lambda_k(\Omega)^{d/2} - \frac{2nd^{1/2}f(k)}{(\lambda_n^*)^{(d-1)/2}}(M_k)^{(d-1)/2} \tag{3.2.88}$$

by the definition of M_k. Now, by taking the supremum over the right-hand side of (3.2.88) and dividing through by k,

$$1 \geq \frac{nA}{(\lambda_n^*)^{d/2}}\left(\frac{M_k}{k^{2/d}}\right)^{d/2} - \frac{2nd^{1/2}f(k)}{k^{1/d}(\lambda_n^*)^{(d-1)/2}}\left(\frac{M_k}{k^{2/d}}\right)^{(d-1)/2}. \tag{3.2.89}$$

Since $f(k) \ll k^{1/d}$, we see that $k^{-1/d}f(k)$ is a bounded sequence and so taking $n = 1$, we have that there exist constants $C_1, C_2 > 0$ such that

$$C_1\left(\frac{M_k}{k^{2/d}}\right)^{d/2} - C_2\left(\frac{M_k}{k^{2/d}}\right)^{(d-1)/2} - 1 \leq 0. \tag{3.2.90}$$

From which it immediately follows that there exists a constant $C > 0$ such that

$$M_k \leq C k^{2/d}. \tag{3.2.91}$$

Now in view of equation (3.2.89), we have

$$1 \geq \frac{nA}{(\lambda_n^*)^{d/2}} \left(\frac{M_k}{k^{2/d}}\right)^{d/2} - \frac{2nd^{1/2} f(k)}{k^{1/d}(\lambda_n^*)^{(d-1)/2}} C^{(d-1)/2}, \tag{3.2.92}$$

Taking the limsup as $k \to +\infty$ and rearranging yields that

$$\limsup_{k\to+\infty} k^{-2/d} M_k \leq \frac{\lambda_n^*}{n^{2/d} A^{2/d}}. \tag{3.2.93}$$

Since $n \in \mathbb{N}$ was arbitrary,

$$\limsup_{k\to+\infty} k^{-2/d} M_k \leq 4\pi^2 (\omega_d A)^{-2/d} \tag{3.2.94}$$

using Weyl's law, which completes the proof. □

We now give a formal statement of our intuition earlier.

Theorem 3.2.13. *Let $(\Omega_k)_{k\geq1} \in \mathcal{A}^d$ be a sequence of bounded domains of volume $V(\Omega_k) = A > 0$ satisfying $D(\Omega_k) \ll (k^{1/d(d-1)})$, then*

$$\lambda_k(\Omega_k) \sim \mu_k(\Omega_k) \sim 4\pi^2 \left(\frac{k}{\omega_d A}\right)^{2/d}. \tag{3.2.95}$$

Proof. Noting that the condition $D(\Omega_k) \ll k^{1/(d(d-1))}$ as $k \to +\infty$ implies that $P(\Omega_k) \ll k^{1/d}$ as $k \to +\infty$ and that by classical variational arguments $\mu_k(\Omega_k) \leq \lambda_k(\Omega_k)$, combining a simple application of Propositions 3.2.8 and 3.2.12 gives the result. □

The condition on the growth of the diameter of the sequence of domains in Theorem 3.2.13 is in fact sharp as we now show.

Proposition 3.2.14. *For all $\epsilon > 0$ there exists a sequence $(\Omega_k)_{k\geq1}$ in $\mathcal{A}^d$ with $V(\Omega_k) = 1$ and $D(\Omega_k) = \epsilon k^{1/d(d-1)}$ such that* (3.2.95) *fails to hold.*

Proof. It suffices to construct a sequence of domains $(\Omega_\alpha)_{\alpha>0}$ with $V(\Omega_\alpha) = 1$ and $D(\Omega_\alpha) = \epsilon\alpha^{1/2(d-1)}$ such that

$$\limsup_{\alpha\to+\infty} \frac{\mathcal{N}^D_{\Omega_\alpha}(\alpha)}{\alpha^{d/2}} < \frac{\omega_d}{(2\pi)^d} < \liminf_{\alpha\to+\infty} \frac{\mathcal{N}^N_{\Omega_\alpha}(\alpha)}{\alpha^{d/2}} \tag{3.2.96}$$

for any given $\epsilon > 0$. Indeed, if we let $\alpha_k = 4\pi^2(k/\omega_d)^{2/d}$ we obtain that

$$\limsup_{k\to+\infty} \frac{\mathcal{N}^D_{\Omega_{\alpha_k}}(4\pi^2(k/\omega_d)^{2/d})}{k} < 1 < \liminf_{k\to+\infty} \frac{\mathcal{N}^N_{\Omega_{\alpha_k}}(4\pi^2(k/\omega_d)^{2/d})}{k} \tag{3.2.97}$$

which contradicts (3.2.95).

Our candidate for such a sequence is the sequence of cuboids given by

$$\Omega_\alpha := (0, \epsilon\alpha^{1/2(d-1)}) \times \cdots \times (0, \epsilon\alpha^{1/2(d-1)}) \times (0, \epsilon^{1-d}\alpha^{-1/2}), \tag{3.2.98}$$

for which $D(\Omega_\alpha) = d^{1/2}\epsilon\alpha^{2/(d-1)}$.

The Dirichlet eigenvalue counting function of Ω_α evaluated at α is explicitly given by

$$\mathcal{N}^D_{\Omega_\alpha}(\alpha) = \#\left\{i_1,\dots,i_{d-1}, j \in \mathbb{N} : \sum_{k=1}^{d-1} \frac{\pi^2 i_k^2}{\epsilon^2\alpha^{1/(d-1)}} + \pi^2 j^2 \epsilon^{2(d-1)}\alpha < \alpha\right\}. \tag{3.2.99}$$

We may rewrite this as

$$\mathcal{N}^D_{\Omega_\alpha}(\alpha) = \sum_{j=1}^{\lceil(\pi\epsilon^{d-1})^{-1}\rceil-1} \#\left\{i_1,\dots,i_{d-1} \in \mathbb{N} : \pi^2\sum_{k=1}^{d-1} i_k^2 < \alpha^{d/(d-1)}\epsilon^2(1 - \pi^2 j^2\epsilon^{2(d-1)})\right\}. \tag{3.2.100}$$

Taking $\epsilon^{d-1} = (\pi M)^{-1}$ for some $M \in \mathbb{N}$, this rearranges to

$$\mathcal{N}^D_{\Omega_\alpha}(\alpha) = \sum_{j=1}^{M-1} \#\left\{i_1,\dots,i_{d-1} \in \mathbb{N} : \pi^2\sum_{k=1}^{d-1} i_k^2 < \frac{\alpha^{d/(d-1)}}{(\pi M)^{2/(d-1)}}\left(1 - \frac{j^2}{M^2}\right)\right\}. \tag{3.2.101}$$

The summands are simply the Dirichlet eigenvalue counting function of the $(d-1)$-dimensional unit cube evaluated at $\frac{\alpha^{d/(d-1)}}{(\pi M)^{2/(d-1)}}\left(1 - \frac{j^2}{M^2}\right)$ and so they satisfy the Pólya bound, see Section 2.1.3,

$$\begin{aligned}
\#\left\{i_1,\ldots,i_{d-1}\in\mathbb{N}:\pi^2\sum_{k=1}^{d-1}i_k^2<\frac{\alpha^{d/(d-1)}}{(\pi M)^{2/(d-1)}}\left(1-\frac{j^2}{M^2}\right)\right\}\\
\leq \frac{\omega_{d-1}}{(2\pi)^{d-1}}\left(\frac{\alpha^{d/(d-1)}}{(\pi M)^{2/(d-1)}}\left(1-\frac{j^2}{M^2}\right)\right)^{(d-1)/2}\\
=\frac{\omega_{d-1}}{(2\pi)^{d-1}}\cdot\frac{\alpha^{d/2}}{\pi M}\left(1-\frac{j^2}{M^2}\right)^{(d-1)/2}.
\end{aligned} \tag{3.2.102}$$

Hence, we know that

$$\mathcal{N}^D_{\Omega_\alpha}(\alpha)\leq\frac{\omega_{d-1}}{(2\pi)^{d-1}}\cdot\frac{\alpha^{d/2}}{\pi M}\sum_{j=1}^{M-1}\left(1-\frac{j^2}{M^2}\right)^{(d-1)/2}. \tag{3.2.103}$$

By a quick inspection,

$$S_M:=\frac{1}{M}\sum_{j=1}^{M-1}\left(1-\frac{j^2}{M^2}\right)^{(d-1)/2}<\int_0^1 \mathrm{d}x\,(1-x^2)^{(d-1)/2} \tag{3.2.104}$$

as the sum strictly under approximates the integral. From equation 3.251.1 in [GraRyz07], we have that

$$\int_0^1 \mathrm{d}x\,(1-x^2)^{(d-1)/2}=\frac{1}{2}\mathcal{B}\left(\frac{1}{2},\frac{d+1}{2}\right), \tag{3.2.105}$$

where $\mathcal{B}(x,y)$ is the beta function given by

$$\mathcal{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.2.106}$$

Using that

$$\frac{\omega_d}{\omega_{d-1}}=\frac{\Gamma\left(\frac{d+1}{2}\right)\pi^{1/2}}{\Gamma\left(\frac{d}{2}+1\right)} \tag{3.2.107}$$

and that $\Gamma\left(\frac{1}{2}\right)=\pi^{1/2}$, we deduce

$$\int_0^1 \mathrm{d}x\,(1-x^2)^{(d-1)/2}=\frac{\omega_d}{2\omega_{d-1}}. \tag{3.2.108}$$

Thus, we know that

$$\frac{\mathcal{N}^D_{\Omega_\alpha}(\alpha)}{\alpha^{d/2}}\leq\frac{S_M\omega_{d-1}}{\pi(2\pi)^{d-1}}<\frac{\omega_d}{(2\pi)^d} \tag{3.2.109}$$

and so

$$\limsup_{\alpha\to+\infty}\frac{\mathcal{N}^D_{\Omega_\alpha}(\alpha)}{\alpha^{d/2}}<\frac{\omega_d}{(2\pi)^d} \tag{3.2.110}$$

for each $M \in \mathbb{N}$. Since M was arbitrary we have the result in the Dirichlet case.

The Neumann case can be done in the same way except we strictly over-estimate the integral in (3.2.104) in this case. Indeed, using the same substitution as before, one may obtain that

$$\begin{aligned} \mathcal{N}^N_{\Omega_\alpha}(\alpha) &= \sum_{j=0}^{M-1} \# \left\{ i_1, \ldots, i_{d-1} \in \mathbb{N} \cup \{0\} : \pi^2 \sum_{k=1}^{d-1} i_k^2 < \frac{\alpha^{d/(d-1)}}{(\pi M)^{2/(d-1)}} \left(1 - \frac{j^2}{M^2}\right) \right\} \\ &\geq \frac{\omega_{d-1}}{(2\pi)^{d-1}} \cdot \frac{\alpha^{d/2}}{\pi M} \sum_{j=0}^{M-1} \left(1 - \frac{j^2}{M^2}\right)^{(d-1)/2} \end{aligned} \tag{3.2.111}$$

using the Pólya bound on the Neumann counting function of the $(d-1)$-dimensional unit cube and the desired conclusion follows analogously. □

3.3 Some results for mixed boundary conditions

Dirichlet and Neumann eigenvalues are lower and upper bounds respectively for Robin eigenvalues with constant positive parameter and mixed Dirichlet-Neumann, so-called Zaremba, eigenvalues. Due to this, we explore to what extent the results in Section 3.1.2 carry over to these eigenvalues. This section is by no means a complete analysis in this direction but rather some partial answers and indicators to some potentially interesting future directions.

3.3.1 Robin eigenvalues

The analogous results to Theorems 3.1.1, 3.1.3, 3.1.6 and 3.1.7 hold for Robin eigenvalues with parameter $\beta \in (0, +\infty)$. The only difference is that we have the existence of minimisers for all $k \geq 1$. For the sake of exposition and brevity we omit restatement of the results for Robin eigenvalues with positive parameter. However, we briefly justify why we have existence of minimisers for all $k \geq 1$.

As a remark, we restrict ourselves to the case $\beta > 0$ due to a limitation in our methods, namely we simply use the bracketing relation $\lambda_k^\beta(\Omega) \geq \mu_k(\Omega)$ and the results in the Neumann case to get the desired asymptotic result. In the negative parameter case $\beta \in (-\infty, 0)$, we do not have the aforementioned bracketing relation and we are unable to say anything for this case here.

Proposition 3.3.1. *Let $\beta \in (0, +\infty)$ be fixed. Then:*

- *For all $k \geq 1$, there exists a minimiser to*

$$\inf\left\{\lambda_1^\beta(\Omega) : \Omega \in \mathcal{A}^d,\ P(\Omega) = 1\right\}. \tag{3.3.1}$$

- *For all $k \geq 1$, there exists a minimiser to*

$$\inf\left\{\lambda_1^\beta(\Omega) : \Omega \in \mathcal{A}^d,\ D(\Omega) = 1\right\}. \tag{3.3.2}$$

Proof. We only prove the case of perimeter as the case of diameter can be done totally analogously.

Fix $k \geq 1$, using Proposition 2.3 in [AntFK13], we see that

$$\inf\left\{\lambda_k^\beta(\Omega) : \Omega \in \mathcal{A}^d,\ V(\Omega) \leq \epsilon\right\} \uparrow +\infty \tag{3.3.3}$$

as $\epsilon \downarrow 0$. Hence, there exists $\epsilon_0 > 0$ such that

$$\inf\left\{\lambda_k^\beta(\Omega) : \Omega \in \mathcal{A}^d,\ P(\Omega) = 1\right\} = \inf\left\{\lambda_k^\beta(\Omega) : \Omega \in \mathcal{A}^d,\ P(\Omega) = 1,\ V(\Omega) \geq \epsilon_0\right\}. \tag{3.3.4}$$

By Proposition 2.2.3 and our version of Blaschke's selection theorem any sequence attaining the infimum on the right-hand side of (3.3.4) has a $\sim$-Hausdorff convergent subsequence. Hence, using lower semi-continuity of Robin eigenvalues under $\sim$-Hausdorff convergence of bounded convex domains, see for example [Cit19, Prop. 3.1.], one obtains that a minimiser exists for all $k \geq 1$. □

Remark 3.3.2. One may notice that the existence is better in this case than the Neumann case in terms of that we have minimisers under perimeter constraint in dimension three and higher. However, using the methods presented in this thesis we are unable to determine whether such minimisers should converge to the ball as $k \to +\infty$. We defer any further discussion until Chapter 5.

3.3.2 Zaremba eigenvalues

A significant challenge presented by shape optimisation problems for Zaremba eigenvalues is having a canonical way of prescribing where to place the Dirichlet and Neumann boundary conditions. To remedy this we consider a subset $\mathcal{A}^d(M)$ of $\mathcal{A}^d$ with some given parameter $M > 0$.

Let $\wp$ be the canonical projection $\mathbb{R}^d \to \mathbb{R}^{d-1}$ which omits the final coordinate. Throughout the rest of this chapter, $\wp$ will denote this projection. Given $\Omega \in \mathcal{A}^d$, its image under $\wp$, denoted $\wp(\Omega)$, is a convex domain in $\mathbb{R}^{d-1}$. For each $x' \in \wp(\Omega)$ we can define two functions $h^+, h^- : \wp(\Omega) \to \mathbb{R}$ by

$$h^+(x') = \sup\{y \in \mathbb{R} : (x', y) \in \Omega\}, \quad h^-(x') = \inf\{y \in \mathbb{R} : (x', y) \in \Omega\}. \tag{3.3.5}$$

We call h^+ and h^- the upper and lower profiles of Ω and as functions they are concave and convex respectively. These functions are well-defined as any line passing through a bounded convex domain intersects the boundary precisely twice. Given $M > 0$, we say that $\Omega \in \mathcal{A}^d$ is a convex M-Lip domain if h^+ and h^- are both M-Lipschitz and agree on the boundary of $\wp(\Omega)$, denoted $\partial\wp(\Omega)$. We denote the collection of all convex M-Lip domains in $\mathbb{R}^d$ by $\mathcal{A}^d(M)$. We define the upper boundary of Ω by $\Gamma^+ := \Gamma^+(\Omega) := \{(x', h^+(x')) : x' \in \wp(\Omega)\} \subset \partial\Omega$ and define the lower boundary $\Gamma^- := \Gamma^-(\Omega)$ analogously.

We then define the Zaremba eigenvalues for $\Omega \in \mathcal{A}^d(M)$ by the eigenvalues associated with the Laplacian acting on $L^2(\Omega)$ with Dirichlet boundary conditions on Γ^- and Neumann boundary conditions on Γ^+, see Section 2.1. From here forward, we denote these eigenvalues by

$$0 < \zeta_1(\Omega) \leq \zeta_2(\Omega) \leq \zeta_3(\Omega) \leq \cdots, \tag{3.3.6}$$

and they have the variational characterisation

$$\zeta_k(\Omega) = \inf_{\substack{S \subset H^1_{0,\Gamma^-}(\Omega) \\ \dim(S) = k}} \sup_{\substack{u \in S \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_\Omega |u|^2}. \tag{3.3.7}$$

Now we have defined $\mathcal{A}^d(M)$ and the associated Zaremba eigenvalues for domains in $\mathcal{A}^d(M)$, we are ready to state our main results in this extended example.

Theorem 3.3.3. *Fix $d \geq 2$ and $M > 0$. For all $k \geq 1$ there exists a minimiser Ω_k^* to the problem*

$$\inf\{\zeta_k(\Omega) : \Omega \in \mathcal{A}^d(M),\ P(\Omega) = 1\}. \tag{3.3.8}$$

Moreover, any sequence $(\Omega_k^*)_{k\geq 1}$ of minimisers satisfies

$$V(\Omega_k^*) \to \sup\{V(\Omega) : \Omega \in \mathcal{A}^d(M),\ P(\Omega) = 1\} \tag{3.3.9}$$

as $k \to +\infty$ and has a subsequence $(\Omega_{k_j}^*)_{j\geq 1}$ which $\sim$-Hausdorff converges to a domain attaining the supremum in (3.3.9) as $j \to +\infty$, i.e. it $\sim$-Hausdorff converges to a solution of the isoperimetric problem over $\mathcal{A}^d(M)$ as $j \to +\infty$.

Remark 3.3.4. Our main problem is knowing that the solution to the isoperimetric problem over $\mathcal{A}^d(M)$ is unique. We give a discussion about what we know about solutions to the isoperimetric problem over $\mathcal{A}^d(M)$ later in this section.

In the same way as one proves Theorem 3.3.3, one can also deduce the analogous result in the case of diameter constraint.

Theorem 3.3.5. *Fix $d \geq 2$ and $M > 0$. For all $k \geq 1$ there exists a minimiser Ω_k^* to the problem*

$$\inf\{\zeta_k(\Omega) : \Omega \in \mathcal{A}^d(M),\ D(\Omega) = 1\}. \tag{3.3.10}$$

Moreover, any sequence $(\Omega_k^*)_{k\geq 1}$ of minimisers satisfies

$$V(\Omega_k^*) \to \sup\{V(\Omega) : \Omega \in \mathcal{A}^d(M),\ D(\Omega) = 1\} \tag{3.3.11}$$

as $k \to +\infty$ and has a subsequence $(\Omega_{k_j}^*)_{j\geq 1}$ which $\sim$-Hausdorff converges to a domain attaining the supremum in (3.3.11) as $j \to +\infty$, i.e. it $\sim$-Hausdorff converges to a solution of the isodiametric problem over $\mathcal{A}^d(M)$ as $j \to +\infty$.

Remark 3.3.6. Again, we do not know if the solution to the isodiametric problem over $\mathcal{A}^d(M)$ is unique.

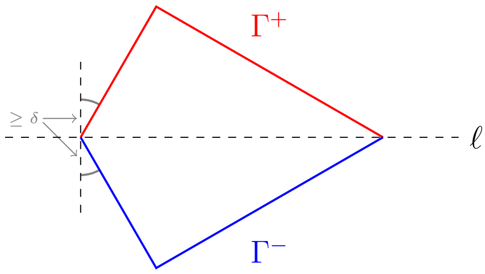

Figure 3.1: An example of symmetric Zaremba boundary conditions on a kite about its axis of symmetry, with Dirchlet boundary conditions denoted in blue and Neumann boundary conditions denoted in red.

To make our results clearer, let us illuminate Theorem 3.3.3 through an example in two dimensions.

Example 3.3.7. Fix $0 < \delta \leq \frac{\pi}{4}$. Let $\Omega \subset \mathbb{R}^2$ be a kite of unit perimeter, let ℓ be the line of symmetry of Ω and assume that the angles that ℓ passes through are less than or equal to $\pi - 2\delta$, see Figure 3.1 for an example of this. The collection of such kites is closed in the Hausdorff metric. Partition the boundary of the kite into two disjoint relatively open components Γ^+ and Γ^- which lie on either side of ℓ and, up to a set of measure zero, cover $\partial\Omega$. Then one can define the Zaremba Laplacian for kites in the way described earlier in Section 2.1. Then, arguing as in the proof of Theorem 3.3.3, this gives that for $k \geq 1$ there exists a minimiser Ω_k^* of the k-th Zaremba eigenvalue among such kites with unit perimeter, and the isoperimetric problem for kites implies that any sequence of such optimisers must $\sim$-Hausdorff converge to the square of unit perimeter as $k \to +\infty$.

As a corollary, one can carry out the same for rhombii where ℓ is the line of symmetry passing through the smallest opposite pair of interior angles. Then under perimeter constraint, again one has existence of optimisers for $k \geq 1$ and that the optimisers necessarily $\sim$-Hausdorff converge to the square of unit perimeter as $k \to +\infty$.

As we shall soon argue, for any $k \geq 1$ and $d \geq 3$,

$$\inf\left\{\zeta_k(\Omega) : \Omega \in \bigcup_{M>0} \mathcal{A}^d(M),\ P(\Omega) = 1\right\} = 0 \tag{3.3.12}$$

and so without a uniform M-Lipschitz constraint the conclusion of Theorem 3.3.3 fails to hold. Moreover, for any $k \geq 1$ and $d \geq 3$,

$$\inf\left\{\mu_k(\Omega) : \Omega \in \mathcal{A}^d(M),\ P(\Omega) = 1\right\} = 0 \tag{3.3.13}$$

for all $k \geq 1$ and so the Zaremba eigenvalues behave fundamentally differently to Neumann eigenvalues over the collection $\mathcal{A}^d(M)$.

We now briefly illustrate (3.3.12) and (3.3.13) as an example when $d = 3$, the higher dimensional cases can be done similarly.

Example 3.3.8. Let $0 < \epsilon < 1$ and set $R_\epsilon = (0, \epsilon) \times (0, \epsilon)$. Define $f_{M,\epsilon} : R_\epsilon \to \mathbb{R}$ by $f_{M,\epsilon}(x, y) = \min\{Md((x, y), \partial R_\epsilon), \epsilon^{-1}\}$ and let

$$\Omega_{M,\epsilon} := \{(x, y, z) : (x, y) \in R_\epsilon,\ 0 < z < f_{M,\epsilon}(x, y)\}, \tag{3.3.14}$$

which lies in $\mathcal{A}^3(M)$. Let $u_j(x, y, z) = \sin(\pi(j + 1/2)\epsilon z)$ for $1 \leq j \leq k$ and let $S_k = \operatorname{span}\{u_1, \ldots, u_k\}$. Note that the collection $\{u_1, \ldots, u_k\}$ is a linearly independent set and so S_k can be used as a test space in the variational characterisation of the k-th Zaremba eigenvalue for $\Omega_{M,\epsilon}$. Then we see that

$$\zeta_k(\Omega_{M,\epsilon}) \leq \sup_{0\neq u\in S_k} \frac{\displaystyle\int_{\Omega_{M,\epsilon}} |\nabla u|^2}{\displaystyle\int_{\Omega_{M,\epsilon}} |u|^2} = \sup_{0\neq u\in S_k} \frac{\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 \mathbb{1}_{\Omega_{M,\epsilon}}}{\displaystyle\int_{\mathbb{R}^3} |u|^2 \mathbb{1}_{\Omega_{M,\epsilon}}}. \tag{3.3.15}$$

By the dominated convergence theorem

$$\sup_{0\neq u\in S_k} \frac{\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 \mathbb{1}_{\Omega_{M,\epsilon}}}{\displaystyle\int_{\mathbb{R}^3} |u|^2 \mathbb{1}_{\Omega_{M,\epsilon}}} \to \sup_{0\neq u\in S_k} \frac{\displaystyle\int_{R_\epsilon\times(0,\epsilon^{-1})} |\nabla u|^2}{\displaystyle\int_{R_\epsilon\times(0,\epsilon^{-1})} |u|^2} = \pi^2(k + 1/2)^2\epsilon^2 \tag{3.3.16}$$

as $M \to +\infty$. Moreover, $P(\Omega_{M,\epsilon}) \to 4 + 2\epsilon^2$ as $M \to +\infty$. Since $0 < \epsilon < 1$ was arbitrary, by the properties of Zaremba eigenvalues under homothety, i.e. $\zeta_k(s\Omega) = s^{-2}\zeta_k(\Omega)$ for any $s > 0$, we see that (3.3.12) indeed holds for any $k \geq 1$ when $d = 3$. Note that $\mathcal{A}^d(M)$ is closed under homothety here.

Now set $S_\epsilon := (0, \epsilon^{-1}) \times (0, \epsilon)$ and set $g_{M,\epsilon} : S_\epsilon \to (0, +\infty)$ by $g_{M,\epsilon}(x, y) :=$

$Md((x,y),\partial S_\epsilon)$ and define the domain Ω_ϵ by

$$\Omega_\epsilon := \{(x,y,z) : (x,y) \in S_\epsilon,\ -g_{M,\epsilon}(x,y) < z < g_{M,\epsilon}(x,y)\}. \tag{3.3.17}$$

We have that $\Omega_\epsilon \in \mathcal{A}^3(M)$ and that $P(\Omega_\epsilon) = 2\sqrt{1+M^2}$ for all $0 < \epsilon < 1$. Letting $u_j(x,y,z) = \cos(\pi j \epsilon x)$ for $1 \le j \le k$, one sees that the collection $\{u_1, \dots, u_k\}$ is a linearly independent set. Denoting $S_k = \operatorname{span}\{u_1, \dots, u_k\}$, we can use S_k as a test space in the variational characterisation of the k-th Neumann eigenvalue for Ω_ϵ.

Doing this, where we L^2-normalise functions in S_k by assumption to remove the denominator, we obtain

$$\begin{aligned}
\mu_k(\Omega_\epsilon) &\le \sup_{\substack{u=\alpha_1 u_1+\cdots+\alpha_k u_k \in S_k \\ \|u\|_{L^2(\Omega_\epsilon)}=1}} \int_{\Omega_\epsilon} |\nabla u|^2 \\
&= \sup_{\substack{u=\alpha_1 u_1+\cdots+\alpha_k u_k \in S_k \\ \|u\|_{L^2(\Omega_\epsilon)}=1}} \int_0^{\epsilon^{-1}} dx \int_0^{\epsilon} dy \int_{-g_{M,\epsilon}(x,y)}^{g_{M,\epsilon}(x,y)} dz \left| \nabla \sum_j \alpha_j u_j(x,y,z) \right|^2 \\
&\le M\epsilon^2 \sup_{\substack{u=\alpha_1 u_1+\cdots+\alpha_k u_k \in S_k \\ \|u\|_{L^2(\Omega_\epsilon)}=1}} \int_0^{\epsilon^{-1}} dx \left| \nabla \sum_j \alpha_j \cos(\pi j \epsilon x) \right|^2 \\
&= M\pi^2 k^2 \epsilon^4.
\end{aligned} \tag{3.3.18}$$

Since $0 < \epsilon < 1$ was arbitrary, by the scaling properties of Neumann eigenvalues under homothety, i.e. $\mu_k(s\Omega) = s^{-2}\mu_k(\Omega)$ for any $s > 0$, we see that (3.3.13) indeed holds for any $k \ge 1$ when $d = 3$. Again, note that $\mathcal{A}^d(M)$ is closed under homothety here.

We now turn our attention to proving Theorems 3.3.3 and 3.3.5.

We begin the section by showing $\mathcal{A}^d(M)$ is closed in the Hausdorff topology provided that one does not have degeneracy of the volume in the limit. Then we use the definition of $\mathcal{A}^d(M)$ to prove the continuity of these Zaremba eigenvalues in the Hausdorff topology and then prove a Li-Yau type lower bound for these eigenvalues.

Both the proofs of the continuity and the lower bound require the use of Sobolev extension operators and the choice of definition of $\mathcal{A}^d(M)$ will become more apparent throughout this section.

Properties of $\mathcal{A}^d(M)$

Lemma 3.3.9. *If $(\Omega_n)_{\geq 1}$ is a sequence in $\mathcal{A}^d(M)$ that Hausdorff converges to $\Omega \in \mathcal{A}^d$ as $n \to +\infty$, then $\Omega \in \mathcal{A}^d(M)$.*

Proof. It suffices to prove the result in the case of sequences $(\Omega_n)_{n\geq 1}$ for which $\Omega_n \subset \Omega$ for each $n \in \mathbb{N}$, see Remark 2.2.14.

Let $h_n^+ : \wp(\Omega_n) \to \mathbb{R}$ be the upper height function of Ω_n and h^+ the upper height function of Ω. Now fix $x', y' \in \wp(\Omega)$ and let $\epsilon = \frac{1}{2}\min\{d(x', \partial\wp(\Omega)), d(y', \partial\wp(\Omega))\}$. Then $\wp(\Omega_n)$ Hausdorff converges to $\wp(\Omega)$ as $n \to +\infty$ so we have that $B(x', \epsilon), B(y', \epsilon) \subset \wp(\Omega_n)$ for n sufficiently large. Now also for n sufficiently large, $d(\partial\Omega, \partial\Omega_n) < \epsilon$ by (2.2.35). As $\Omega_n \subset \Omega$, there exist sequences $(x'_n, h_n^+(x'_n)), (y'_n, h_n^+(y'_n)) \in \partial\Omega_n$ converging to $(x', h^+(x'))(y', h^+(y')) \in \partial\Omega$ as $n \to +\infty$. In particular, these sequences can be chosen so that

$$\|(x'_n, h_n^+(x'_n)) - (x', h^+(x'))\|_2, \|(y'_n, h_n^+(y'_n)) - (y', h^+(y'))\|_2 \leq d^H(\partial\Omega_n, \partial\Omega). \quad (3.3.19)$$

Then

$$\begin{aligned} |h^+(x') - h^+(y')| &\leq |h^+(x') - h_n^+(x'_n)| + |h_n^+(x'_n) - h_n^+(y'_n)| + |h_n^+(y'_n) - h^+(y')| \\ &\leq 2d^H(\partial\Omega_n, \partial\Omega) + M\|(x'_n, 0) - (y'_n, 0)\|_2. \end{aligned} \quad (3.3.20)$$

Taking the limit as $n \to +\infty$ we see that h^+ is M-Lipschitz. Similarly one can show that h^-, the lower height function of Ω, is M-Lipschitz. The fact that h^+ and h^- agree on the boundary $\partial\wp(\Omega)$ is easy to argue by contradiction. □

Lemma 3.3.10. *If $(\Omega_n)_{n\geq 1}$ is a sequence in $\mathcal{A}^d(M)$ Hausdorff converging to a domain $\Omega \in \mathcal{A}^d(M)$ as $n \to +\infty$, then $\Gamma_n^- := \Gamma^-(\Omega_n)$ Hausdorff converges to $\Gamma^- := \Gamma^-(\Omega)$ as $n \to +\infty$.*

Proof. As in the proof of Lemma 3.3.9, we may assume that $\Omega_n \subset \Omega$ for each $n \geq 1$. For $\delta > 0$ define the compact subset

$$K_\delta := \{(x', y) \in \wp(\Omega) \times \mathbb{R} : h^-(x') + \delta \leq y \leq h^+(x') - \delta\} \subset \Omega. \tag{3.3.21}$$

Then for n sufficiently large, $K_\delta \subset \Omega_n$. Fix $(x', h^-(x')) \in \Gamma^-(\Omega)$, then let x'_δ be the closest point in $\wp(K_\delta)$ to x'. Then clearly $\|x' - x'_\delta\|_2 \leq d^H(K_\delta, \Omega)$ and so

$$\begin{aligned} |h^-(x') - h_n^-(x'_\delta)| &\leq |h^-(x') - h^-(x'_\delta)| + |h^-(x'_\delta) - h_n^-(x'_\delta)| \\ &\leq M d^H(K_\delta, \Omega) + \delta. \end{aligned} \tag{3.3.22}$$

Since $\delta > 0$ was arbitrary we see that $\sup_{x \in \Gamma^-(\Omega)} \inf_{y \in \Gamma^-(\Omega_n)} \|x - y\|_2 \to 0$ as $n \to +\infty$. One can then deduce that $\sup_{x \in \Gamma^-(\Omega_n)} \inf_{y \in \Gamma^-(\Omega)} \|x - y\|_2 \to 0$ as $n \to +\infty$ similarly. □

Continuity of the ζ_k

We now move on to prove the continuity of these Zaremba eigenvalues in the Hausdorff topology.

In [Chs75], Chenais proved the continuity of solutions to the Neumann problem for domains satisfying a uniform cone condition with respect to the Hausdorff metric. A crucial part of Chenais' proof is to show that over such a collection of domains there exists a uniform constant such that there exists a Sobolev extension operator $H^1(\Omega) \to H^1(\mathbb{R}^d)$ whose norm is at most this constant. Then from the continuity of the solutions to the Neumann problem, one can prove the continuity of Neumann eigenvalues with respect to the Hausdorff metric, see [Hen06, §3].

The issue that arises in the Zaremba problem is that one wants to extend by zero on the Dirichlet parts of the boundary and extend non-trivially along the Neumann parts of the boundary. This is an inherently tricky situation as these are not naturally compatible demands. Our definition of $\mathcal{A}^d(M)$ allows us to define an extension operator which for any $\Omega \in \mathcal{A}^d(M)$ extends any $u \in H^1_{0,\Gamma^-}(\Omega)$ by zero below Γ^- and 'into H^1 above Γ^+'. Moreover, we can uniformly bound such operators over $\mathcal{A}^d(M)$.

For a precise formulation of this see Corollary 3.3.12. Then using these extension operators and similar arguments to Chenais, we prove the continuity of Zaremba eigenvalues over the collection $\mathcal{A}^d(M)$ with respect to the Hausdorff metric.

Lemma 3.3.11 ([FraLW23, Lemma 2.91]). *There exists a constant $C_M > 0$ depending only on $M > 0$ such that for any M-Lipschitz function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, there exists a Sobolev extension operator $\mathcal{E} : H^1(\Omega_f) \to H^1(\mathbb{R}^d)$, where*

$$\Omega_f := \{(x', y) \in \mathbb{R}^{d-1} \times \mathbb{R} : y < f(x')\}, \tag{3.3.23}$$

with $\|\mathcal{E}[u]\|_{L^2(\mathbb{R}^d \setminus \Omega_f)} \leq \sqrt{2}\|u\|_{L^2(\Omega_f)}$ and $\|\nabla \mathcal{E}[u]\|_{L^2(\mathbb{R}^d \setminus \Omega_f)} \leq C_M \|\nabla u\|_{L^2(\Omega_f)}$ for any $u \in H^1(\Omega_f)$. Explicitly, we have that

$$\mathcal{E}[u](x', y) = \begin{cases} u(x', y), & y < f(x'), \\ u(x', -y + 2f(x')), & y > f(x'). \end{cases} \tag{3.3.24}$$

A detailed analysis of this Sobolev extension operator is not necessary for our means, the only important point for us here is the following immediate corollary.

Corollary 3.3.12. *There exists a constant $C_M > 0$ depending only on $M > 0$ such that for any $\Omega \in \mathcal{A}^d(M)$ there exists an extension operator $\mathcal{E}_\Omega : H^1_{0,\Gamma^-}(\Omega) \to H^1_0(\Omega_\infty)$, where*

$$\Omega_\infty = \{(x', y) \in \wp(\Omega) \times \mathbb{R} : y > h_-(x')\}, \tag{3.3.25}$$

with $\|\mathcal{E}_\Omega[u]\|_{L^2(\mathbb{R}^d \setminus \Omega)} \leq \sqrt{2}\|u\|_{L^2(\Omega)}$ and $\|\nabla \mathcal{E}_\Omega[u]\|_{L^2(\mathbb{R}^d \setminus \Omega)} \leq C_M \|u\|_{L^2(\Omega)}$ for any $u \in H^1_{0,\Gamma^-}(\Omega)$.

Proof. Take any $\phi \in C^\infty_{0,\Gamma^-}(\Omega) \cap C^\infty(\overline{\Omega})$. By a theorem of McShane in [McS34], we can extend $h^+ : \wp(\Omega) \to \mathbb{R}$ to an M-Lipschitz function $\widetilde{h}^+ : \mathbb{R}^{d-1} \to \mathbb{R}$. Defining $\Omega_{\widetilde{h}^+}$ as in (3.3.23), by extending by zero $\phi \in H^1(\Omega_{\widetilde{h}_+})$, and by the definition of $\mathcal{E}$ in (3.3.24) it is clear that one must have $\mathcal{E}[\phi] \in H^1_0(\Omega_\infty)$. Define $\mathcal{E}_\Omega[\phi]$ in this way. Then by the density of $C^\infty_{0,\Gamma^-}(\Omega) \cap C^\infty(\overline{\Omega})$ in $H^1_{0,\Gamma^-}(\Omega)$, the result immediately follows. $\square$

Lemma 3.3.13. *Fix $k \geq 1$. If $(\Omega_n)_{n\geq 1}$ is a sequence in $\mathcal{A}^d(M)$ Hausdorff converging to $\Omega \in \mathcal{A}^d(M)$ as $n \to +\infty$, then $\zeta_k(\Omega_n) \to \zeta_k(\Omega)$ as $n \to +\infty$.*

Proof. Since we know that Ω_n Hausdorff converges to Ω as $n \to +\infty$, there exists $\beta_n \to 1$ such that $\beta_n\Omega_n \subseteq \Omega$, up to a possible translation, for n sufficiently large.

From here onwards, we follow the ideas of the proof of Proposition IV.1 in [Chs75]. Fix $f \in L^2(\Omega)$. By the Riesz-Fréchet representation theorem there exists a unique $u_n \in H^1_{0,\Gamma^-}(\beta_n\Omega_n)$ such that

$$\int_\Omega \mathbb{1}_{\beta_n\Omega_n}\nabla u_n \cdot \nabla\phi + \int_\Omega \mathbb{1}_{\beta_n\Omega_n} u_n\phi = \int_\Omega \mathbb{1}_{\beta_n\Omega_n} f\phi, \quad \text{for all } \phi \in C^\infty_{0,\Gamma^-}(\beta_n\Omega_n) \tag{3.3.26}$$

with $\|u_n\|_{H^1(\beta_n\Omega_n)} = \|f\|_{L^2(\beta_n\Omega_n)} \leq \|f\|_{L^2(\Omega)}$.

Then we see that we can extend each $u_n \in H^1_{0,\Gamma^-}(\beta_n\Omega_n)$ via $\mathcal{E}_{\Omega_n}$, as defined in Corollary 3.3.12, to a function $\bar{u}_n \in H^1_{0,\Gamma^-}(\Omega)$ with $\|\bar{u}_n\|_{H^1(\Omega)} \leq C_M\|f\|_{L^2(\Omega)}$.

By the Banach-Alaoglu theorem, up to a subsequence, $\bar{u}_n \rightharpoonup u$ in $H^1_{0,\Gamma}(\Omega)$ as $n \to +\infty$. We now show that u must be the unique solution to

$$\int_\Omega \nabla u \cdot \nabla\phi + \int_\Omega u\phi = \int_\Omega f\phi, \quad \text{for all } \phi \in C^\infty_{0,\Gamma^-}(\Omega). \tag{3.3.27}$$

Fix $\phi \in C^\infty_{0,\Gamma^-}(\Omega)$. Then by Lemma 3.3.10, we see that the support of ϕ is at a positive distance from $\Gamma^-(\Omega_n)$ for n sufficiently large, and so $\phi|_{\beta_n\Omega_n} \in H^1_{0,\Gamma^-}(\beta_n\Omega_n)$ for n sufficiently large. Thus, for n sufficiently large

$$\int_\Omega \mathbb{1}_{\beta_n\Omega_n}\nabla\bar{u}_n \cdot \nabla\phi + \int_\Omega \mathbb{1}_{\beta_n\Omega_n}\bar{u}_n\phi = \int_\Omega \mathbb{1}_{\beta_n\Omega_n} f\phi. \tag{3.3.28}$$

Following the arguments in [Chs75, Prop IV.1], if we take the limit $n \to +\infty$,

$$\int_\Omega \nabla u \cdot \nabla\phi + \int_\Omega u\phi = \int_\Omega f\phi. \tag{3.3.29}$$

Since $\phi \in C^\infty_{0,\Gamma^-}(\Omega)$ was arbitrary, u is indeed the solution to (3.3.27) as desired.

Moreover, $\bar{u}_n \to u$ in $L^2(\Omega)$ by the Rellich–Kondrachov compactness theorem since $\bar{u}_n \rightharpoonup u$ in $H^1_{0,\Gamma^-}(\Omega)$. Now following the proof of Theorem 2.3.2. in [Hen06], we see that $\zeta_k(\beta_n\Omega_n) \to \zeta_k(\Omega)$. Then noting that $\zeta_k(\beta_n\Omega_n) = (\beta_n)^{-2}\zeta_k(\Omega_n)$ we obtain the result. □

Proof of Theorems 3.3.3 and 3.3.5

With the continuity of Zaremba eigenvalues over $\mathcal{A}^d(M)$ in hand, we now prove the existence of minimisers using the extension operator from Corollary 3.3.12.

Lemma 3.3.14. *For each $k \geq 1$ there exists a minimiser Ω_k^* to* (3.3.8).

Proof. Let $\Omega \in \mathcal{A}^d(M)$ and suppose $\delta = \|h^+ - h^-\|_{L^\infty(\wp(\Omega))}$. Then one sees that, up to a possible translation, $\Omega \subset \wp(\Omega) \times (0,\delta)$. We can extend the first Zaremba eigenfunction of Ω to the Sobolev space $H^1_{0,\wp(\Omega)\times\{0\}}(\wp(\Omega)\times(0,\delta))$. Hence, from the variational characterisation of the first Zaremba eigenvalue

$$\widetilde{\zeta}_1(\wp(\Omega) \times (0,\delta)) \leq C_M\zeta_1(\Omega), \tag{3.3.30}$$

where $\widetilde{\zeta}_1(\wp(\Omega)\times(0,\delta))$ is the first eigenvalue of the Zaremba Laplacian on $\wp(\Omega)\times(0,\delta)$ with Dirichlet boundary conditions on $\wp(\Omega) \times \{0\}$.

By separation of variables one can deduce that

$$\widetilde{\zeta}_1(\wp(\Omega) \times (0,\delta)) = \mu_1(\wp(\Omega)) + \frac{\pi^2}{4\delta^2} = \frac{\pi^2}{4\delta^2}. \tag{3.3.31}$$

And so we see that

$$\zeta_k(\Omega) \geq \zeta_1(\Omega) \geq \frac{\pi^2}{4C_L\delta^2} \uparrow +\infty \tag{3.3.32}$$

as $\delta \downarrow 0$. Hence, we must have that δ is uniformly bounded from below and so the inradii of the sets must be uniformly bounded from below.

Let $(\Omega_n)_{n\geq1}$ be a minimising sequence for the infimum in (3.3.8). As the inradii are uniformly bounded from below and the volume is bounded from above by the isoperimetric inequality, Lemma 3.3.9 and Blaschke's selection theorem assert that $(\Omega_n)_{n\geq1}$ has a $\sim$-Hausdorff convergent subsequence which we also denote by $(\Omega_n)_{n\geq1}$ $\sim$-Hausdorff converging to some $\Omega \in \mathcal{A}^d(M)$. Since the Zaremba eigenvalues are continuous with respect to the Hausdorff distance, see Lemma 3.3.13, and thus the $\sim$-Hausdorff distance, $\zeta_k(\Omega_n) \to \zeta_k(\Omega)$ as $n \to +\infty$ and we are done. □

We now give a lower bound for Zaremba eigenvalues in the spirit of the classical Li-Yau bound, see [LiYau83, Cor. 1], for Dirichlet eigenvalues. We note that for $\Omega \in \mathcal{A}^d(M)$, $V(\wp(\Omega))$ will denote the $(d-1)$-dimensional volume of $\wp(\Omega)$ for the rest of this section.

Lemma 3.3.15. *There exists a constant $C_{d,M} > 0$, depending only on $d \geq 2$ and $M > 0$, such that for any $\epsilon > 0$*

$$\zeta_k(\Omega) \geq \frac{C_{d,M} k^{2/d}}{(V(\Omega) + \epsilon V(\wp(\Omega))^{d/(d-1)})^{2/d}} - ((d-1)M^2 + 1)\frac{1}{\epsilon^2 V(\wp(\Omega))^{2/(d-1)}} \quad (3.3.33)$$

for all $\Omega \in \mathcal{A}^d(M)$.

Proof. Let $\epsilon > 0$. Fix $\Omega \in \mathcal{A}^d(M)$ and define the set

$$\Omega^\epsilon = \{(x', y) \in \wp(\Omega) \times \mathbb{R} : h^-(x') < y < h^+(x') + \epsilon\}, \quad (3.3.34)$$

and further define the function $\chi_\epsilon : \wp(\Omega) \times \mathbb{R} \to [0, 1]$ by

$$\chi_\epsilon(x', y) := \begin{cases} 1, & y \leq h^+(x'), \\ 1 - \frac{(y - h^+(x'))}{\epsilon}, & h^+(x') < y < h^+(x') + \epsilon, \\ 0, & y \geq h^+(x') + \epsilon. \end{cases} \quad (3.3.35)$$

Let $\mathcal{E} := \mathcal{E}_\Omega$ be the Sobolev extension operator given in Corollary 3.3.12. For any $u \in H^1_{0,\Gamma^-}(\Omega)$, we have that $\chi_\epsilon \mathcal{E}[u] \in H^1_0(\Omega^\epsilon)$. Moreover, let $S_k = \{u_1, \ldots, u_k\}$ denote the span of the first k $L^2(\Omega)$-orthonormalised Zaremba eigenfunctions of $-\Delta^Z_\Omega$. Then the collection $\{\chi_\epsilon \mathcal{E}[u_1], \ldots, \chi_\epsilon \mathcal{E}[u_k]\} \subset H^1_0(\Omega^\epsilon)$ is linearly independent and so we consider the span of these functions as a trial space into the variational formulation for the k-th Dirichlet eigenvalue of Ω^ϵ.

Before proceeding let us make some relevant observations. Namely that, for any $u \in H^1_{0,\Gamma^-}(\Omega)$: $\|\mathcal{E}[u]\|_{L^2(\Omega^\epsilon)} \geq \|u\|_{L^2(\Omega)}$ since $\mathcal{E}[u] \equiv u$ in Ω; $\|\mathcal{E}[u]\|_{L^2(\mathbb{R}^d \setminus \Omega)} \leq \sqrt{2}\|u\|_{L^2(\Omega)}$; and, $\|\nabla \mathcal{E}[u]\|_{L^2(\mathbb{R}^d \setminus \Omega)} \leq C_M \|\nabla u\|_{L^2(\Omega)}$ as stated in Corollary 3.3.12.

By repeated use of the uniform bounds given in Corollary 3.3.12 and removing the denominator from the variational characterisation of the k-th Dirichlet eigenvalue of Ω^ϵ by L^2-normalising functions in S_k in the definition of the maximum, we have that

$$\begin{aligned}
\lambda_k(\Omega^\epsilon) &\leq \sup_{\substack{u\in S_k\\ \|u\|_{L^2(\Omega)}=1}} \int_{\Omega^\epsilon} |\nabla(\chi_\epsilon \mathcal{E}[u])|^2 \\
&\leq \sup_{\substack{u\in S_k\\ \|u\|_{L^2(\Omega)}=1}} \left\{\int_\Omega |\nabla u|^2 + \int_{\Omega^\epsilon\setminus\Omega} |\chi_\epsilon \nabla\mathcal{E}[u] + \mathcal{E}[u]\nabla\chi_\epsilon|^2\right\} \\
&\leq \sup_{\substack{u\in S_k\\ \|u\|_{L^2(\Omega)}=1}} \left\{\int_\Omega |\nabla u|^2 + 2\int_{\Omega^\epsilon\setminus\Omega} |\chi_\epsilon \nabla\mathcal{E}[u]|^2 + 2\int_{\Omega^\epsilon\setminus\Omega} |\mathcal{E}[u]\nabla\chi_\epsilon|^2\right\} \\
&\leq \sup_{\substack{u\in S_k\\ \|u\|_{L^2(\Omega)}=1}} \left\{\int_\Omega |\nabla u|^2 + 2\int_{\mathbb{R}^d\setminus\Omega} |\nabla\mathcal{E}[u]|^2 + 2((d-1)M^2+1)\epsilon^{-2}\int_{\mathbb{R}^d\setminus\Omega} |\mathcal{E}[u]|^2\right\} \\
&\leq \sup_{\substack{u\in S_k\\ \|u\|_{L^2(\Omega)}=1}} \left\{(1+2C_M)\int_\Omega |\nabla u|^2 + 2\sqrt{2}((d-1)M^2+1)\epsilon^{-2}\right\} \\
&\leq C'_M\left(\sup_{\substack{u\in S_k\\ \|u\|_{L^2(\Omega)}=1}} \left\{\int_\Omega |\nabla u|^2\right\} + ((d-1)M^2+1)\epsilon^{-2}\right) \\
&= C'_M(\zeta_k(\Omega) + ((d-1)M^2+1)\epsilon^{-2}).
\end{aligned} \tag{3.3.36}$$

By the classical Dirichlet eigenvalue lower bound of Li and Yau [LiYau83, Cor. 1]

$$\lambda_k(\Omega^\epsilon) \geq \frac{d}{d+2}\cdot\frac{4\pi^2 k^{2/d}}{(\omega_d V(\Omega^\epsilon))^{2/d}}. \tag{3.3.37}$$

Now, observing that $V(\Omega^\epsilon) = V(\Omega) + \epsilon V(\wp(\Omega))$, we obtain

$$\zeta_k(\Omega) \geq \frac{C_{d,L}k^{2/d}}{(V(\Omega)+\epsilon V(\wp(\Omega)))^{2/d}} - ((d-1)M^2+1)\epsilon^{-2}. \tag{3.3.38}$$

Taking $\epsilon = V(\wp(\Omega))^{1/(d-1)}\epsilon'$ for some $\epsilon' > 0$, the result immediately follows. □

Before proving Theorem 3.3.3, we now briefly look at the isoperimetric problem

$$\sup\left\{V(\Omega) : \Omega\in\mathcal{A}^d(M),\, P(\Omega)=1\right\} \tag{3.3.39}$$

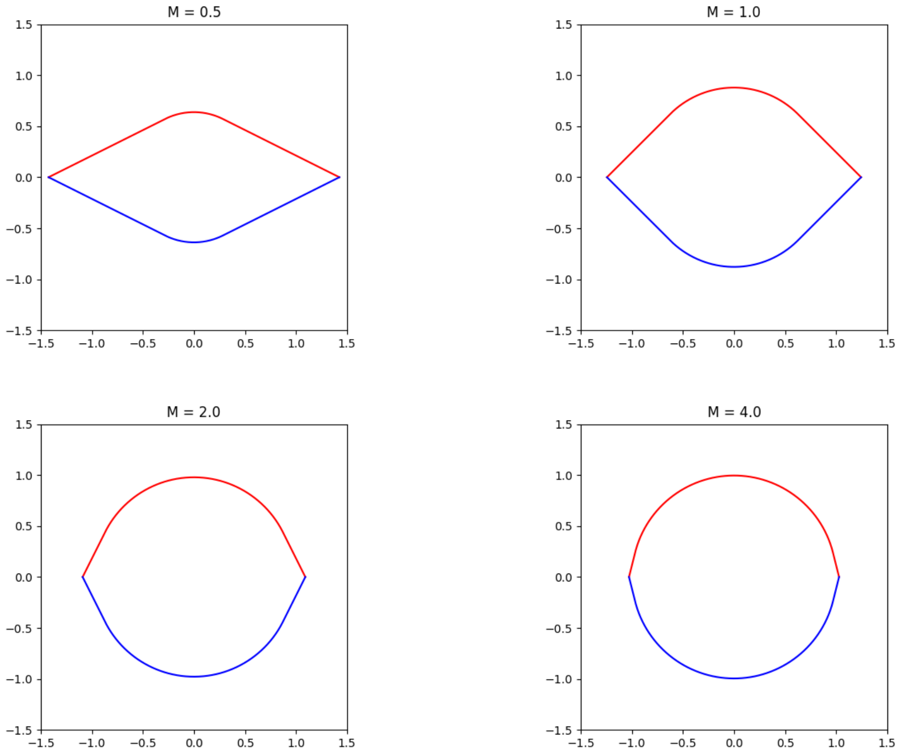

Figure 3.2: Numerically computed optimal solutions to the isoperimetric problem with perimeter 2π over $\mathcal{A}^2(M)$ with $M = 0.5$ (top left), $M = 1$ (top right), $M = 2$ (bottom left) and $M = 4$ (bottom right).

for domains in $\mathcal{A}^d(M)$. It is clear that there exists a solution to the isoperimetric problem over $\mathcal{A}^d(M)$, however we cannot say too much immediately about such solutions as balls do not lie in $\mathcal{A}^d(M)$. We now give some remarks on properties of solutions to (3.3.39).

By the results of Fuglede in [Fug89], for $M > 0$ large one can note that any solution to the isoperimetric problem over $\mathcal{A}^d(M)$ must be (quantifiably) close to the ball of the same perimeter. Moreover, for any $\Omega \in \mathcal{A}^d(M)$, its Steiner symmetrisation $\Omega^\#$ about the hyperplane $\{x_d = 0\}$ defined by

$$\Omega^\# := \{(x', y) \in \wp(\Omega) \times \mathbb{R} : -h(x') < y < h(x')\}, \tag{3.3.40}$$

where $h(x') := (h^+(x') - h^-(x'))/2$, also lies in $\mathcal{A}^d(M)$. Thus by standard properties of Steiner symmetrisation, see [Hen06, §2.2], $V(\Omega^\#) = V(\Omega)$ and $P(\Omega^\#) \leq P(\Omega)$, with equality if and only if $\Omega^\#$ and Ω are isometric. Hence, any solution to the isoperimetric problem over $\mathcal{A}^d(M)$ is necessarily symmetric about the hyperplane $\{x_d = 0\}$.

As far as the author is aware, it is not known whether the solution to the isoperimetric problem over $\mathcal{A}^d(M)$ is unique. In dimension two, it appears to be unique and the author has numerically computed solutions to the isoperimetric problem for $\mathcal{A}^2(M)$, see Figure 3.2.

We now show for any $\Omega \in \mathcal{A}^d(M)$ that the condition $P(\Omega) = 1$ imposes constraints on $V(\wp(\Omega))$, which is the final ingredient needed to prove Theorem 3.3.3.

Lemma 3.3.16. *Fix $\Omega \in \mathcal{A}^d(M)$ and suppose that $P(\Omega) = 1$, then*

$$\frac{1}{2\sqrt{M^2+1}} \leq V(\wp(\Omega)) \leq \frac{1}{2}. \tag{3.3.41}$$

Proof. For $\Omega \in \mathcal{A}^d(M)$ observe that

$$2V(\wp(\Omega))\sqrt{1+M^2} \geq P(\Omega) = \int_{\wp(\Omega)} \sqrt{1+|\nabla h^+|^2} + \sqrt{1+|\nabla h^-|^2} \geq 2V(\wp(\Omega)) \tag{3.3.42}$$

and the result immediately follows. □

Proof of Theorem 3.3.3. With our previous results in hand, we now follow the outline of the proof of Theorem 1.1 in [BucFre13] to prove Theorem 3.3.3.

We already know the existence of minimisers to (3.3.8) from Lemma 3.3.14 and so it suffices to prove just the asymptotic behaviour of the minimisers as $k \to +\infty$.

Let $(\Omega_k^*)_{k\geq 1}$ be any sequence of minimisers to (3.3.8) and let $\Omega' \in \mathcal{A}^d(M)$ with $P(\Omega') = 1$ be fixed. Using Lemma 3.3.15 and Lemma 3.3.16, taking $\epsilon > 0$,

$$\begin{aligned} \zeta_k(\Omega) &\geq \frac{C_{d,M}k^{2/d}}{(V(\Omega)+\epsilon V(\wp(\Omega))^{d/(d-1)})^{2/d}} - ((d-1)M^2+1)\frac{1}{\epsilon^2 V(\wp(\Omega))^{2/(d-1)}} \\ &\geq \frac{C_{d,M}k^{2/d}}{(V(\Omega)+\epsilon V(\wp(\Omega))^{d/(d-1)})^{2/d}} - ((d-1)M^2+1)\frac{(4+4M^2)^{1/(d-1)}}{\epsilon^2} \end{aligned} \tag{3.3.43}$$

for any $\Omega \in \mathcal{A}^d(M)$ with $P(\Omega) = 1$. Then observe that

$$\begin{aligned}\frac{C_{d,M}k^{2/d}}{(V(\Omega_k^*) + \epsilon V(\wp(\Omega_k^*))^{d/(d-1)})^{2/d}} \\ -((d-1)M^2+1)\frac{(4+4M^2)^{1/(d-1)}}{\epsilon^2} &\le \zeta_k(\Omega_k^*) \\ &\le \zeta_k(\Omega') \\ &= \frac{4\pi^2 k^{2/d}}{(\omega_d V(\Omega'))^{2/d}} + o(k^{2/d})\end{aligned} \tag{3.3.44}$$

Dividing through by $k^{2/d}$ and taking the limsup, we have, using Lemma 3.3.16 again,

$$\begin{aligned}\liminf_{k\to+\infty}\left[V(\Omega_k^*) + \frac{\epsilon}{2^{d/(d-1)}}\right]^{2/d} &\ge \liminf_{k\to+\infty}\left[V(\Omega_k^*)^{2/d} + \epsilon V(\wp(\Omega_k^*))^{d/(d-1)}\right]^{2/d} \\ &\ge \frac{C_{d,M}}{4\pi^2}(\omega_d V(\Omega'))^{2/d}. \qquad (3.3.45) \\ &> 0.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we see that the sequence of minimisers is non-degenerate, i.e. $\liminf_{k\to+\infty} V(\Omega_k^*) > 0$. Knowing the non-degeneracy, by Proposition 2.2.3, we see that

$$\liminf_{k\to+\infty}\rho(\Omega_k^*) \ge \liminf_{k\to+\infty} P(\Omega_k^*)^{-1}V(\Omega_k^*) \ge \liminf_{k\to+\infty} V(\Omega_k^*) > 0. \tag{3.3.46}$$

Proposition 2.2.4 tells us that

$$D(\Omega) \le 2d(\omega_{d-1})^{-1}\rho(\Omega)^{1-d}V(\Omega), \tag{3.3.47}$$

and so the diameters of the Ω_k^* are uniformly bounded from above. Thus, by Dirichlet-Neumann bracketing, i.e. $\mu_k(\Omega) \le \zeta_k(\Omega)$, and Proposition 3.2.8, we can deduce in the same way as in the proof of Theorem 3.1.3, that

$$V(\Omega_k^*) \to \sup\{V(\Omega) : \Omega \in \mathcal{A}^d(M),\ P(\Omega) = 1\} \tag{3.3.48}$$

as $k \to +\infty$.

Our version of Blaschke's selection theorem then tells us that the sequence $(\Omega_k^*)_{k\ge1}$ has a $\sim$-Hausdorff convergent subsequence whose limit point is necessarily a solution to the isoperimetric problem over $\mathcal{A}^d(M)$ by the continuity of volume under Hausdorff convergence of bounded convex domains. □

The proof of Theorem 3.3.5 follows entirely analogously to the proof of Theorem 3.3.3 by noting that the condition $D(\Omega) = 1$ implies that $V(\wp(\Omega)) \leq 2^{-(d-1)}\omega_{d-1}$ via the $(d-1)$-dimensional isodiametric inequality.

Chapter 4

Heat content for polygons with reflection

In this chapter, we state and prove the results of the paper [FarGit23], which was a joint work between the author and his supervisor. The work builds on those of van den Berg, Gilkey, Gittins and Srisatkunarajah, see [BerSri90; BerGit16; BerGG20], in which precise small-time asymptotics for the heat content of polygons were given for Dirichlet and open boundary conditions. Our work here is the first case of small-time asymptotics for the heat content of polygons with Neumann boundary conditions imposed.

We have one piece of additional notation for this chapter that was not previously defined. For two Lipschitz domains $D, \Omega \subset \mathbb{R}^2$, we define the perimeter of D inside Ω by

$$P(D, \Omega) = \mathcal{H}^1((\partial D) \cap \Omega), \tag{4.0.1}$$

where we recall that $\mathcal{H}^1$ is the one-dimensional Hausdorff measure here.

Unlike in the previous chapter, D will no longer to refer to the diameter of a domain here but rather a bounded polygonal domain in $\mathbb{R}^2$. The diameter of domains will not be needed in this chapter so this should not cause any confusion.

4.1 Introduction and related literature

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain such that the interior angles θ at vertices of $\partial\Omega$ satisfy $0 < \theta < 2\pi$. Let $D \subset \Omega$ be a polygonal subdomain. We consider the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \tag{4.1.1}$$

on Ω with Neumann boundary condition imposed on $\partial\Omega$, that is

$$\frac{\partial}{\partial n} u(t; x) = 0, \;\; x \in \partial\Omega, \, t > 0, \tag{4.1.2}$$

where n is the inward-pointing unit normal to $\partial\Omega$ (defined up to a set of measure zero), and initial datum

$$\lim_{t \downarrow 0} u(t; x) = \mathbb{1}_D(x), \;\; x \in \Omega \setminus \partial D. \tag{4.1.3}$$

Here $\mathbb{1}_D$ is the indicator function of D. We denote the unique, smooth solution to this problem by $u_{D\subset\Omega}$. Physically, $u_{D\subset\Omega}(t; x)$ represents the temperature at $x \in \Omega$ at time $t \geq 0$ when the initial temperature is 1 in D and 0 in $\Omega \setminus D$ and the total heat contained in Ω is conserved due to the Neumann (fluxless or insulated) boundary condition. The solution $u_{D\subset\Omega}$ can be obtained from the unique Neumann heat kernel $\eta_\Omega(t; x, y)$ for Ω by

$$u_{D\subset\Omega}(t; x) = \int_\Omega dy \, \eta_\Omega(t; x, y) \mathbb{1}_D(y) = \int_D dy \, \eta_\Omega(t; x, y). \tag{4.1.4}$$

We are interested in the heat content of D, that is the amount of heat inside D at time $t \geq 0$, which is given by

$$H_{D\subset\Omega}(t) := \int_D dx \, u_{D\subset\Omega}(t; x). \tag{4.1.5}$$

The function $H_{D\subset\Omega}(t)$ is smooth from the smoothness of $u_{D\subset\Omega}$ and other properties can be deduced from the $L^2(\Omega)$-eigenfunction expansion of $\eta_\Omega(t; x, y)$. Let $\mu_j(\Omega)$, $j = 1, 2, \ldots,$ denote the Neumann eigenvalues of Ω with corresponding eigenfunctions u_j that are orthonormalised in $L^2(\Omega)$. Recall that the Neumann heat kernel on Ω

has the following expansion:

$$\eta_\Omega(t;x,y) = \sum_{j=1}^{\infty} e^{-\mu_j(\Omega)t} u_j(x)u_j(y). \tag{4.1.6}$$

The heat content of D inside of Ω can then be written as

$$H_{D\subset\Omega}(t) = \sum_{j=1}^{\infty} e^{-\mu_j(\Omega)t} \left(\int_D dx\, u_j(x)\right)^2, \tag{4.1.7}$$

from which we observe that $t \mapsto H_{D\subset\Omega}(t)$ is strictly decreasing on $(0,+\infty)$,

$$\frac{\partial}{\partial t} H_{D\subset\Omega}(t) = \sum_{j=1}^{\infty} -\mu_j(\Omega) e^{-\mu_j(\Omega)t} \left(\int_D dx\, u_j(x)\right)^2 < 0, \tag{4.1.8}$$

and strictly convex on $(0,+\infty)$,

$$\frac{\partial^2}{\partial t^2} H_{D\subset\Omega}(t) = \sum_{j=1}^{\infty} (\mu_j(\Omega))^2 e^{-\mu_j(\Omega)t} \left(\int_D dx\, u_j(x)\right)^2 > 0. \tag{4.1.9}$$

Since $\mu_1(\Omega) = 0$ and $u_1(x) = V(\Omega)^{-1/2}$, we see that

$$\lim_{t\to+\infty} H_{D\subset\Omega}(t) = \lim_{t\to+\infty} \left(\left(\int_D dx\, u_1(x)\right)^2 + \sum_{j=2}^{\infty} e^{-\mu_j(\Omega)t} \left(\int_D dx\, u_j(x)\right)^2 \right) = \frac{V(D)^2}{V(\Omega)}. \tag{4.1.10}$$

In the rest of this chapter, we consider the small-time asymptotic behaviour of $H_{D\subset\Omega}(t)$.

The heat content of polygonal subdomains in, possibly unbounded, domains $\Omega \subset \mathbb{R}^2$ can be defined analogously to (4.1.5) by considering the heat equation on Ω with some boundary condition imposed on $\partial\Omega$ (when the latter is non-empty). The small-time asymptotics for such cases have been obtained in [BerSri90], [BerGit16], and [BerGG20] and we summarise these below.

In [BerSri90], the authors consider the Dirichlet case, that is $\Omega = D$, D has initial temperature 1 and Dirichlet boundary condition imposed on ∂D for all $t > 0$. The Dirichlet heat content of D is defined as

$$Q_D(t) := \int_D dx \int_D dy\, q_D(t;x,y), \tag{4.1.11}$$

where $q_D(t;x,y)$ is the Dirichlet heat kernel of D, and the authors obtain that

$$Q_D(t) = V(D) - \frac{2P(D)}{\pi^{1/2}} t^{1/2} + \left(\sum_{\gamma \in \mathcal{A}} f(\gamma) \right) t + O\left(e^{-C_1/t}\right), \tag{4.1.12}$$

where: $\mathcal{A}$ is the collection of interior angles at the vertices of D; $C_1 > 0$ is a constant depending only on D; and, $f : (0, 2\pi) \to \mathbb{R}$ is given by

$$f(\gamma) := \int_0^\infty d\theta \, \frac{4 \sinh\left((\pi - \gamma)\theta\right)}{\sinh(\pi\theta)\cosh(\gamma\theta)}. \tag{4.1.13}$$

We note that in [BerSri90], the authors include the case where interior angles can be equal to 2π. One can also introduce this for Neumann boundary conditions under a suitable generalisation of the boundary condition, see Section 4.6.2.

In [BerGit16], the authors consider the open case, that is $\Omega = \mathbb{R}^2$, D has initial temperature 1 and $\mathbb{R}^2 \setminus \overline{D}$ has initial temperature 0. The open heat content of D is defined as

$$H_D(t) := \int_D dx \int_D dy \, p_{\mathbb{R}^2}(t;x,y), \tag{4.1.14}$$

where $p_{\mathbb{R}^2}(t;x,y) = (4\pi t)^{-1} e^{-\|x-y\|_2^2/(4t)}$ is the heat kernel for $\mathbb{R}^2$. The authors obtain that

$$H_D(t) = V(D) - \frac{P(D)}{\pi^{1/2}} t^{1/2} + \left(\sum_{\gamma \in \mathcal{A}} a(\gamma) \right) t + O\left(e^{-C_2/t}\right), \tag{4.1.15}$$

where: $\mathcal{A}$ is as above; $C_2 > 0$ is a constant depending only on D; and, $a : (0, 2\pi) \to \mathbb{R}$ is given by

$$a(\gamma) := \begin{cases} \frac{1}{\pi} + \left(1 - \frac{\gamma}{\pi}\right) \cot\gamma, & \gamma \in (0,\pi) \cup (\pi, 2\pi), \\ 0, & \gamma = \pi. \end{cases} \tag{4.1.16}$$

In [BerGG20], the authors consider the Dirichlet-open case, that is $\Omega = \mathbb{R}^2 \setminus \partial D_-$, where $\partial D_- \subset \partial D$ is some collection of edges, D has initial temperature 1, $\mathbb{R}^2 \setminus \overline{D}$ has initial temperature 0 and Dirichlet boundary condition imposed on ∂D_- for all $t > 0$. The corresponding heat content of D is defined as

$$G_{D,\partial D_-}(t) := \int_D dx \int_D dy \, q_{\mathbb{R}^2 \setminus \partial D_-}(t;x,y), \tag{4.1.17}$$

where $q_{\mathbb{R}^2 \setminus \partial D_-}(t;x,y)$ is the respective Dirichlet heat kernel. The authors obtain that

$$G_{D,\partial D_-}(t) = V(D) - \frac{[P(D) + 2\mathcal{H}^1(\partial D_-)]}{\pi^{1/2}} t^{1/2} + \left(\sum_{\gamma \in \mathcal{A}_1} a(\gamma) + \sum_{\gamma \in \mathcal{A}_2} f(\gamma) + \sum_{\gamma \in \mathcal{A}_3} g(\gamma) \right) t + O\left(e^{-C_3/t}\right), \tag{4.1.18}$$

where: $\mathcal{A}_1$ is the collection of interior angles at vertices where two open edges intersect; $\mathcal{A}_2$ is the collection of interior angles at vertices where two Dirichlet edges intersect; $\mathcal{A}_3$ is the collection of interior angles at vertices where a Dirichlet edge and an open edge intersect; $C_3 > 0$ is a constant only depending on D; a and f are as in (4.1.13) and (4.1.16) respectively; and $g : (0, 2\pi) \to \mathbb{R}$ is given by

$$g(\gamma) := -\frac{3}{4} + \int_0^\infty d\theta \, \frac{4\sinh^2\left(\left(\pi - \frac{\gamma}{2}\right)\theta\right) - \sinh^2\left((\pi - \gamma)\theta\right)}{\sinh^2\left(\frac{\pi}{2}\theta\right)\cosh(\pi\theta)}. \tag{4.1.19}$$

4.2 Results for Neumann boundary conditions

From here onwards Ω will be a bounded polygonal domain in $\mathbb{R}^2$ unless otherwise stated.

In the Dirichlet, open, and Dirichlet-open cases any boundary conditions are only imposed on a subset of ∂D and so the results only ever depend on the geometry of D. For the problem we consider here, this is not the case and we also have the relative geometry of D with respect to Ω to consider (see Figure 4.1 for an example setup of the problem we are considering). Thus, in order to state our main result, we require some additional terminology and notation. Let $\mathcal{V}$ denote the union of the vertices of D and the vertices of Ω lying on ∂D. Moreover, let $\mathcal{E}$ denote the collection of edges of parts of the boundary of ∂D between vertices in $\mathcal{V}$. We call edges in $\mathcal{E}$ that lie in Ω (except, perhaps, the endpoints of the edges) open edges and edges in $\mathcal{E}$ that lie on $\partial\Omega$ Neumann edges. Throughout this work we assume each vertex in $\mathcal{V}$ is of one of the following types.

(i) We say a vertex in $\mathcal{V}$ is an open vertex if it lies in Ω and it has two incident

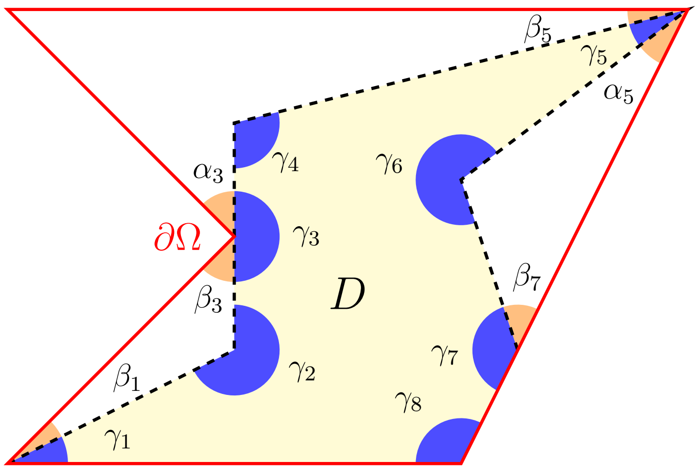

Figure 4.1: An example setup for our problem with D highlighted in yellow and the angles formed by D and Ω labelled at the vertices.

edges in $\mathcal{E}$ which are open. Let $\mathcal{A}$ denote the collection of interior angles at all such open vertices.

(ii) We say a vertex in $\mathcal{V}$ is a Neumann–Open–Neumann (NON) vertex if it lies on $\partial\Omega$ and has two incident edges in $\mathcal{E}$, one of which is open and the other is Neumann. Let $\mathcal{B}$ denote the collection of pairs (γ, β) at all such NON vertices where γ denotes the interior angle of D at the vertex and β denotes the exterior angle relative to Ω at the vertex (see Figure 4.2).

(iii) We say a vertex in $\mathcal{V}$ is a Neumann–Open–Open–Neumann (NOON) vertex if it lies on $\partial\Omega$ and it has two or four incident edges in $\mathcal{E}$, for which two are open and the rest are Neumann. Let $\mathcal{C}$ denote the collection of triples (γ, β, α) at all such NOON vertices where γ denotes the angle between the open edges, which we call the middle angle at the vertex, and β and α denote the two other angles between open and Neumann edges at the vertex (see Figure 4.2). The ordering of α and β does not matter but we only have one triple for each NOON vertex. Note that NOON vertices either have that γ is an interior angle of D, or β and α are both interior angles of D.

(iv) We say a vertex in $\mathcal{V}$ is a Neumann vertex if it lies on $\partial\Omega$ and has two incident edges in $\mathcal{E}$ which are both Neumann.

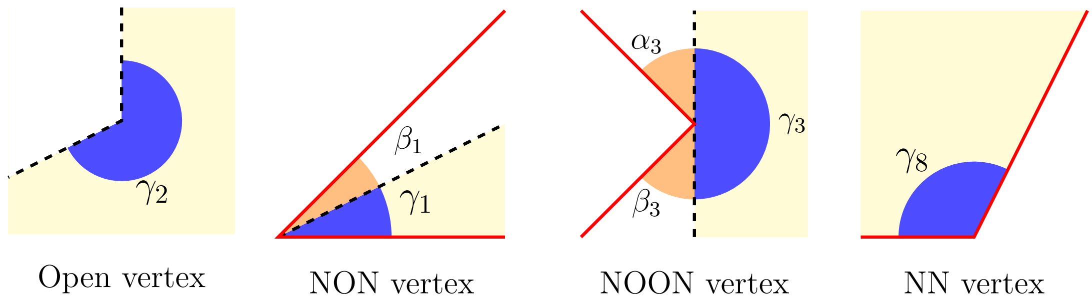

Figure 4.2: An example of each of the four possible types of vertices that can arise in the problem. They are magnifications of the vertices in Figure 4.1 (see angle subscripts). See Figure 4.3 for an example of the other type of NOON vertex.

Theorem 4.2.1. *There exists a constant $C_{D\subset\Omega} > 0$, depending only on D and Ω, such that*

$$H_{D\subset\Omega}(t) = V(D) - P(D,\Omega)\frac{t^{1/2}}{\pi^{1/2}} + \left(\sum_{\gamma\in\mathcal{A}} a(\gamma) + \sum_{(\gamma,\beta)\in\mathcal{B}} b(\gamma,\beta) + \sum_{(\gamma,\beta,\alpha)\in\mathcal{C}} c(\gamma,\beta,\alpha)\right) t + O\left(e^{-C_{D\subset\Omega}/t}\right), \tag{4.2.1}$$

where: $a : (0, 2\pi) \to \mathbb{R}$ is defined as above in (4.1.16)*; $b(\gamma,\beta)$ is given by*

$$b(\gamma,\beta) := \int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh\left((\gamma-\beta)\,\theta\right) - \cosh\left(\left(\frac{\pi}{2}-\gamma-\beta\right)\theta\right)}{2\sinh\left((\gamma+\beta)\,\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)}; \tag{4.2.2}$$

and $c(\gamma,\beta,\alpha)$ is given by

$$c(\gamma,\beta,\alpha) := b(\gamma+\beta,\alpha) + b(\gamma+\alpha,\beta) + \int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\left[\cosh\left((\beta+\alpha)\,\theta\right) - \cosh\left((\beta-\alpha)\,\theta\right)\right]}{\sinh\left((\gamma+\beta+\alpha)\,\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)}. \tag{4.2.3}$$

It was observed in [BerGit16] that the function $a(\gamma)$, which corresponds to the contribution from the angles at an open vertex, is symmetric with respect to π. We observe that the function $b(\gamma,\beta)$, which corresponds to the contribution from the angles at a NON vertex, is symmetric in β and γ, and that the function $c(\gamma,\beta,\alpha)$, which corresponds to the contribution from the angles at a NOON vertex, is symmetric in α and β as mentioned before in (iii). By these symmetry properties of $a(\gamma)$

and $b(\gamma,\beta)$ we have the following consequence of Theorem 4.2.1. If D_1, D_2 are two polygonal subdomains of a bounded polygonal domain Ω such that $D_2 = \Omega \setminus \overline{D_1}$, then

$$\left|[H_{\Omega,D_1}(t) - V(D_1)] - [H_{\Omega,D_2}(t) - V(D_2)]\right| = O(e^{-C/t}) \tag{4.2.4}$$

for some constant $C > 0$. Moreover, if $V(D_1) = V(D_2)$, then the heat contents of D_1, D_2 have the same long-time asymptotic behaviour and the same small-time asymptotic behaviour up to an exponentially small remainder. Analogous statements also hold when D_1 has NOON vertices due to properties of $c(\gamma,\beta,\alpha)$ (see Theorem 4.5.9).

Considering reflections of NON and NOON wedges with respect to a Neumann edge motivates the following relations between $a(\gamma)$, $b(\gamma,\beta)$, and $c(\gamma,\beta,\alpha)$.

Proposition 4.2.2. *For $\gamma,\beta,\alpha \in (0,\pi)$, we have the following.*

(i) If $\gamma+\beta<\pi$, then $c(2\gamma,\beta,\beta) = 2b(\gamma,\beta)$;

(ii) $b(\gamma,\pi-\gamma) = \frac{1}{2}a(2\gamma)$;

(iii) If $\gamma+\beta+\alpha = \pi$ and $\alpha \le \beta$, then $2c(\gamma,\beta,\alpha) = 2a(\gamma) + 2k(2\alpha,\gamma,\gamma)$ and $2c(\gamma,\beta,\alpha) = a(2\alpha) + a(2\beta) + 2k(\gamma,2\alpha,2\beta)$, where the function $k(\alpha,\theta,\sigma)$ is given in [BerGit16] as part of the open case when more than two open edges meet at a vertex, and is defined as

$$\begin{aligned} k(\alpha,\theta,\sigma) := \frac{1}{2\pi}\Big[&-(\sigma+\theta+\alpha-\pi)\cot(\sigma+\theta+\alpha) - (\alpha-\pi)\cot(\alpha) \\ &+ (\sigma+\alpha-\pi)\cot(\sigma+\alpha) + (\theta+\alpha-\pi)\cot(\theta+\alpha)\Big] \end{aligned} \tag{4.2.5}$$

for $\sigma+\theta+\alpha \neq \pi$, $\alpha \neq \pi$, $\sigma+\alpha \neq \pi$, $\theta+\alpha\neq\pi$. In any of the remaining cases, such as $\alpha=\pi$, $k(\alpha,\theta,\sigma)$ is defined by taking appropriate limits via l'Hôpital's rule.

Remark 4.2.3. The identities in the previous proposition ultimately arise from how one can obtain Neumann heat kernels from the method of images. In particular, (ii) and (iii) arise from the Neumann heat kernel for the half-plane and so we have the relation with the angular contributions in the open case.

Proof. We prove (i) by direct calculation, namely

$$\begin{aligned}
b(\gamma,\beta) &= \int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh\left(\left(\gamma-\beta\right)\theta\right)-\cosh\left(\left(\frac{\pi}{2}-\gamma-\beta\right)\theta\right)}{2\sinh\left(\left(\gamma+\beta\right)\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)} \\
&= \int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh\left(\left(\gamma-\beta\right)\theta\right)-\cosh\left(\left(\frac{\pi}{2}-\gamma-\beta\right)\theta\right)}{2\sinh\left(\left(\gamma+\beta\right)\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)}\cdot\frac{2\cosh((\gamma+\beta)\theta)}{2\cosh((\gamma+\beta)\theta)} \\
&= \int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\left(\cosh(2\gamma\theta)+\cosh(2\beta\theta)-1\right)-\cosh\left(\left(\frac{\pi}{2}-2\gamma-2\beta\right)\theta\right)}{2\sinh\left(2\left(\gamma+\beta\right)\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)} \\
&= b(2\gamma+\beta,\beta)+\int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\left[\cosh\left(2\beta\theta\right)-1\right]}{2\sinh\left(2\left(\gamma+\beta\right)\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)} \\
&= \frac{1}{2}c(2\gamma,\beta,\beta).
\end{aligned} \tag{4.2.6}$$

For (ii) and (iii) we require an additional identity. For $|z|<|\pi|$, we have that by formula 3.511.9 in [GraRyz07]

$$\int_0^\infty d\theta\, \frac{\cosh(\frac{\pi}{2}\theta)\left(\cosh(z\theta)-1\right)}{\sinh(\pi\theta)\sinh(\frac{\pi}{2}\theta)} = \frac{2}{\pi}\int_0^\infty dx\, \frac{\sinh^2(\frac{z}{\pi}x)}{\sinh^2(x)} = \frac{1}{\pi}-\frac{z}{\pi}\cot(z). \tag{4.2.7}$$

We observe that (ii) clearly holds for $\gamma=\frac{\pi}{2}$. The case $\gamma\neq\frac{\pi}{2}$ is immediate from (4.2.7) by

$$\begin{aligned}
b(\gamma,\pi-\gamma) &= \int_0^\infty d\theta\, \frac{\cosh(\frac{\pi}{2}\theta)\left(\cosh((\pi-2\gamma)\theta)-1\right)}{2\sinh(\pi\theta)\sinh(\frac{\pi}{2}\theta)} \\
&= \frac{1}{2\pi}+\frac{1}{2}(1-\tfrac{2\gamma}{\pi})\cot(2\gamma) \\
&= \frac{1}{2}a(2\gamma).
\end{aligned} \tag{4.2.8}$$

For (iii) it is sufficient to show that this identity holds when $2\gamma+2\alpha\neq\pi$, $2\alpha\neq\pi$, $\gamma+2\alpha\neq\pi$. By four uses of (4.2.7), we see that

$$
\begin{aligned}
a(\gamma) + k(2\alpha, \gamma, \gamma) = a(\gamma) + \frac{1}{2\pi}\Big[&- (\gamma + \alpha - \beta)\cot(\gamma + \alpha - \beta) \\
&- (\gamma + \beta - \alpha)\cot(\gamma + \beta - \alpha) \\
&+ 2(\beta - \alpha)\cot(\beta - \alpha)\Big] \\
= &\int_0^\infty d\theta\, \frac{\cosh(\frac{\pi}{2}\theta)\,(\cosh((\gamma - \beta + \alpha)\theta) - 1)}{2\sinh(\pi\theta)\sinh(\frac{\pi}{2}\theta)} \\
&+ \int_0^\infty d\theta\, \frac{\cosh(\frac{\pi}{2}\theta)\,(\cosh((\gamma + \beta - \alpha)\theta) - 1)}{2\sinh(\pi\theta)\sinh(\frac{\pi}{2}\theta)} \\
&+ \int_0^\infty d\theta\, \frac{\cosh(\frac{\pi}{2}\theta)\,(\cosh(\pi - \gamma)\theta) - \cosh(\beta - \alpha)\theta))}{\sinh(\pi\theta)\sinh(\frac{\pi}{2}\theta)} \\
= &\, b(\gamma + \beta, \alpha) + b(\gamma + \alpha, \beta) \\
&+ \int_0^\infty d\theta\, \frac{\cosh(\frac{\pi}{2}\theta)\,(\cosh(\beta + \alpha)\theta) - \cosh(\beta - \alpha)\theta))}{\sinh(\pi\theta)\sinh(\frac{\pi}{2}\theta)} \\
= &\, c(\gamma, \beta, \alpha),
\end{aligned} \tag{4.2.9}
$$

as desired. The other identity for $c(\gamma, \beta, \alpha)$ follows similarly and agian makes use of (4.2.7). □

Inverse problems for the small-time asymptotic expansion of the Dirichlet heat content of polygons have been investigated in [BerDK14; MeyMcD17; Bro18]. For the setting under consideration here, we can obtain polygonal subdomains of the unit square which have the same small-time heat content expansion up to an exponentially small remainder, see Figure 4.3.

Given a rectangle R with polygonal subdomain D, the idea is to reflect both of R and D with respect to one of the edges of R. We then take the union of R with its image under this reflection, call this R_1, and the union of D with its image under this reflection and call it D_1. Then we observe that D_1 is a polygonal subdomain of R_1 and

$$
|H_{R_1,D_1}(t) - 2H_{R,D}(t)| = O(e^{-C/t}). \tag{4.2.10}
$$

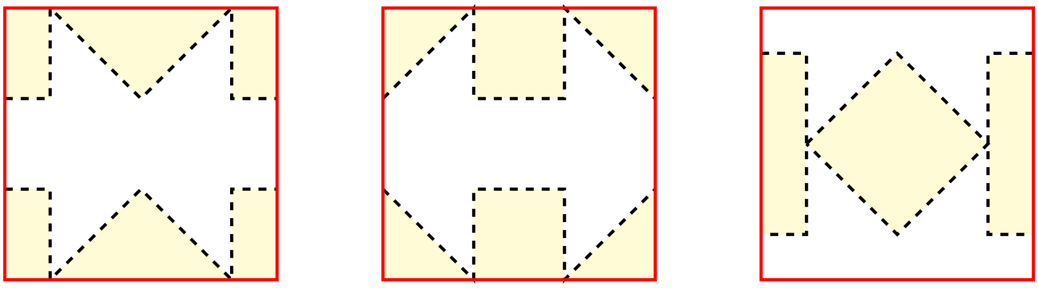

Figure 4.3: Three polygonal subdomains of the square with the same area and same small-time heat content expansion, up to an exponentially small remainder.

4.3 Outline of proof of main theorem

The proof of Theorem 4.2.1 follows the strategy employed in the papers [BerSri90], [BerGit16], and [BerGG20]. The main idea is to partition D in such a way that we can model the heat content contribution of each part of the partition by the heat content of the same part in a different ambient space to Ω for which a suitable, locally comparable heat kernel is known explicitly.

4.3.1 Construction of the partition

Let $\mathcal{V}$ be as defined in the last section and $\mathcal{V}'$ denote the vertices of Ω. Our first goal is to create an open sector, or the union of two open sectors at some NOON vertices, based at each vertex in $v \in \mathcal{V}$ given by $S_R(v) := B_R(v) \cap D$ for some $R > 0$, where $B_R(v)$ is the open ball in $\mathbb{R}^2$ centred at v of radius R. We define the quantities

$$R_1 := \frac{1}{2} \inf_{u \in \mathcal{V} \cup \mathcal{V}'} \inf_{\substack{v \in \mathcal{V} \cup \mathcal{V}' \\ v \neq u}} \|u - v\|_2, \qquad R_2 := \frac{1}{2} \inf_{u \in \mathcal{V} \setminus \mathcal{V}'} d(u, \partial\Omega), \tag{4.3.1}$$

and set $R := \min\{R_1, R_2\}$. The definition of R_1 ensures two things: $S_R(u) \cap S_R(v) = \emptyset$ for $u, v \in \mathcal{V}$ with $u \neq v$ and that $S'_R(u) \cap S'_v(R) = \emptyset$ also in this case, where $S'_R(v) := B_R(v) \cap \Omega$. The second point here is crucial for our comparisons. The definition of R_2 also allows us to do the required comparisons and calculations specifically for open vertices.

The next part of the partition we want to define is that of rectangles lying inside D

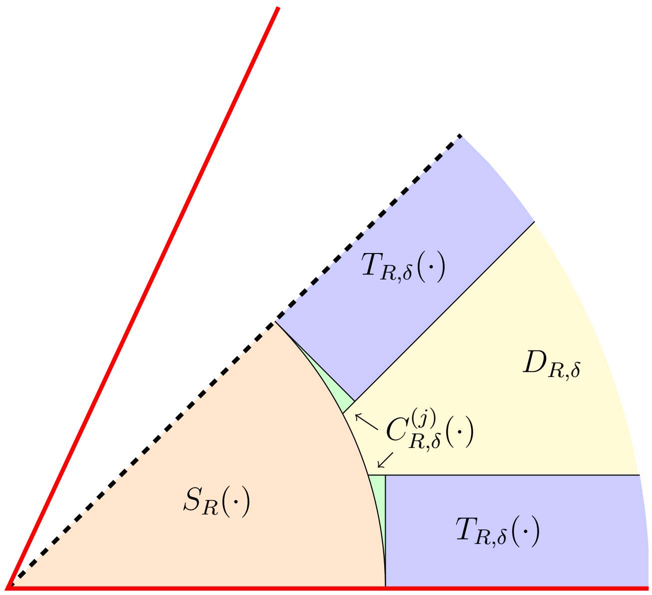

Figure 4.4: Visualisation of the partition constructed for D near a NON vertex. The partition looks similar for NN and NOON vertices.

for which one of its sides lies on ∂D. Let $\mathcal{E}$ be as defined in the last section. For $\bar{e} \in \mathcal{E}$ with length $\ell(\bar{e})$, we want to have a rectangle of width $\ell(\bar{e}) - 2R$ and height $\delta > 0$ with a side of length $\ell(\bar{e}) - 2R$ lying on $\bar{e}$ and the rectangle lying inside D. We denote this rectangle by $T_{R,\delta}(\bar{e})$.

In order to apply our comparisons in the proceeding sections, we require that no two of these rectangles intersect so we must define a suitable choice of $\delta > 0$. Denote the collection of interior angles of D by $\mathcal{A}_1$ and the collection of exterior angles of D relative to Ω at NON and NOON vertices by $\mathcal{A}_2$. Then define the quantities

$$\gamma_1 := \frac{1}{2}\underset{\kappa \in \mathcal{A}_1}{\operatorname{argmin}} \left(\sin \tfrac{\kappa}{2}\right), \tag{4.3.2}$$

and

$$\gamma_2 := \frac{1}{2}\underset{\kappa \in \mathcal{A}_2}{\operatorname{argmin}} \left(\sin \tfrac{\kappa}{2}\right). \tag{4.3.3}$$

We set $\delta_1 := R\sin(\gamma_1)$ and $\delta_2 := R\sin(\gamma_2)$. Then we determine δ by setting $\delta := \min\{\delta_1, \delta_2\}$. The definition of δ_1 ensures that these rectangles do not overlap and the definition of δ_2 allows us to make our comparisons.

Now let $D_{R,\delta} := \{x \in D : d(x, \partial D) > \delta \text{ and } d(x, \mathcal{V}) > R\}$. We observe that

$$D \neq \left(\bigcup_{v \in \mathcal{V}} S_R(v) \right) \cup \left(\bigcup_{\bar{e} \in \mathcal{E}} T_{R,\delta}(\bar{e}) \right) \cup D_{R,\delta} \tag{4.3.4}$$

even up to a set of measure zero so we have one final model space to consider. The remainder is the union of disjoint cusps which up to rigid planar motions are expressible as $\{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < R, |x| > R, 0 < x_2 < \delta\}$. The definition of δ ensures no two of these cusps overlap. Each sector $S_R(v)$ has two such cusps associated with it which we denote by $C^{(1)}_{R,\delta}(v)$ and $C^{(2)}_{R,\delta}(v)$. Then, up to a set of measure zero, we have that

$$D = \left(\bigcup_{v \in \mathcal{V}} S_R(v) \cup C^{(1)}_{R,\delta}(v) \cup C^{(2)}_{R,\delta}(v) \right) \cup \left(\bigcup_{\bar{e} \in \mathcal{E}} T_{R,\delta}(\bar{e}) \right) \cup D_{R,\delta}. \tag{4.3.5}$$

A visualisation of the partition we have constructed is given in Figure 4.4.

4.3.2 Results from model spaces

Now we have the partition defined in the last subsection, Theorem 4.2.1 follows immediately from the following theorem, which is subsequently proved in detail in Sections 4.4 and 4.5.

Theorem 4.3.1. *Using the notation in Sections 4.2 and 4.3.1, we have the following results:*

(i)

$$\int_{D_{R,\delta}} dx \int_D dy\, \eta_\Omega(t; x, y) = V(D_{R,\delta}) + O\left(e^{-\delta^2/8t}\right). \tag{4.3.6}$$

(ii) If $\bar{e} \in \mathcal{E}$ lies on $\partial\Omega$, i.e. is a Neumann edge, then

$$\int_{T_{R,\delta}(\bar{e})} dx \int_D dy\, \eta_\Omega(t; x, y) = V(T_{R,\delta}(\bar{e})) + O\left(e^{-\delta^2/8t}\right). \tag{4.3.7}$$

(iii) If $\bar{e} \in \mathcal{E}$ lies in Ω, i.e. is an open edge, then

$$\int_{T_{R,\delta}(\bar{e})} dx \int_D dy\, \eta_\Omega(t; x, y) = V(T_{R,\delta}(\bar{e})) - \frac{(\ell(\bar{e}) - 2R)}{\pi^{1/2}} t^{1/2} + O\left(e^{-\delta^2/8t}\right). \tag{4.3.8}$$

(iv) If $C_{R,\delta}$ is a cusp lying adjacent to an edge $\bar{e} \in \mathcal{E}$ lying on $\partial\Omega$, i.e. is a Neumann cusp, then we have that

$$\int_{C_{R,\delta}} dx \int_D dy\, \eta_\Omega(t;x,y) = V(C_{R,\delta}) + O\left(e^{-\delta^2/8t}\right). \tag{4.3.9}$$

(v) If $C_{R,\delta}$ is a cusp lying adjacent to an edge $\bar{e} \in \mathcal{E}$ lying in Ω, i.e. is an open cusp, then we have that

$$\int_{C_{R,\delta}} dx \int_D dy\, \eta_\Omega(t;x,y) = V(C_{R,\delta}) - \frac{R}{\pi^{1/2}} t^{1/2} \int_1^\infty \frac{dw}{w^2} \int_0^1 dy\, \frac{y}{(1-y^2)^{\frac{1}{2}}} e^{-\frac{R^2y^2w^2}{4t}} + O\left(e^{-\delta^2/8t}\right). \tag{4.3.10}$$

(vi) For $v \in \mathcal{V}$ an NN vertex

$$\int_{S_R(v)} dx \int_D dy\, \eta_\Omega(t;x,y) = V(S_R(v)) + O\left(e^{-\delta^2/8t}\right). \tag{4.3.11}$$

(vii) For $v \in \mathcal{V}$ a NON vertex with interior angle γ and exterior angle β,

$$\begin{aligned}\int_{S_R(v)} dx \int_D dy\, \eta_\Omega(t;x,y) = {} & V(S_R(v)) - \frac{R}{\pi^{1/2}} t^{1/2} + b(\gamma,\beta)t \\ & + \frac{R}{\pi^{1/2}} t^{1/2} \int_1^\infty \frac{dw}{w^2} \int_0^1 dy\, \frac{y}{(1-y^2)^{\frac{1}{2}}} e^{-R^2y^2w^2/4t} \\ & + O\left(e^{-C_2/t}\right),\end{aligned} \tag{4.3.12}$$

where $C_2 > 0$ is a constant depending on R, γ and β.

(viii) For $v \in \mathcal{V}$ a NOON vertex with middle angle γ and exterior angles β, α,

$$\begin{aligned}\int_{S_R(v)} dx \int_D dy\, \eta_\Omega(t;x,y) = {} & V(S_R(v)) - \frac{2R}{\pi^{1/2}} t^{1/2} + c(\gamma,\beta,\alpha)t \\ & + \frac{2R}{\pi^{1/2}} t^{1/2} \int_1^\infty \frac{dw}{w^2} \int_0^1 dy\, \frac{y}{(1-y^2)^{\frac{1}{2}}} e^{-R^2y^2w^2/4t} \\ & + O\left(e^{-C_3/t}\right),\end{aligned} \tag{4.3.13}$$

where $C_3 > 0$ is a constant depending on R, γ, β and α.

The key point to note is that: sectors $S_R(v)$ at NN vertices have two associated Neumann cusps which have trivial contribution; sectors $S_R(v)$ at NON vertices have one associated Neumann cusp and one associated open cusp which cancels out the term

$$\frac{R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{\frac{1}{2}}}e^{-R^2y^2w^2/4t}; \tag{4.3.14}$$

and, sectors $S_R(v)$ at NOON vertices have two associated open cusps which cancel out the term

$$\frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{\frac{1}{2}}}e^{-R^2y^2w^2/4t}. \tag{4.3.15}$$

Analogous results also hold for the case of an open vertex with two neighbouring cusps. Indeed, we recall the following results from [BerGit16].

Lemma 4.3.2 (Open vertex with two open cusps [BerGit16, Lem. 9 & §4.2])**.** *We have that*

$$\begin{aligned}\int_0^R dr\,r\int_0^\gamma d\phi\int_0^\infty dr_0\,r_0\int_0^\gamma d\phi_0\,p_{\mathbb{R}^2}(t;r,\phi,r_0,\phi_0) + 2\int_{C_{R,\delta}} dx\int_D dy\,p_{\mathbb{R}^2}(t;x,y)\\ = V(S_R(v)) + 2V(C_{R,\delta}) - \frac{2R}{\pi^{1/2}}t^{1/2} + a(\gamma)t + O\left(te^{-R^2C_{\gamma,\delta}/t}\right),\end{aligned} \tag{4.3.16}$$

where $C_{\gamma,\delta}>0$ is a constant depending only on γ,δ.

When one sums up all the heat content contributions from each part of the partition we get the desired form of Theorem 4.2.1.

4.4 Locality principles

The computations in Theorem 4.3.1 are done explicitly where we replace Ω and η_Ω with some other ambient space Ω' and its (Neumann) heat kernel $\eta_{\Omega'}$. In order to justify this switch rigorously, we need to prove that we can indeed do this. This is the idea of locality principles.

Locality principles for heat kernels are well-known. In fact, for Neumann heat kernels on polygonal domains they are already known, see for example [NurRS19]. One may use their methods to obtain remainders with behaviour $O(t^{-1/2}e^{-C/t})$, which are of course superpolynomial, i.e. $o(t^{\infty})$. Their proofs are very analytic in flavour and much more general than we require. Below, we provide the mild improvement to the remainder, that of $O(e^{-C/t})$, in our setting. Our proof is based on probabilistic methods rather than any technical heat kernel estimates.

Lemma 4.4.1. *Let Ω be a polygonal domain and $\delta > 0$ fixed.*

(i) Let $x \in \Omega$ with $d(x, \partial\Omega) \geq \delta$, then we have that for any Borel sets $A_1 \subset \Omega$ and $A_2 \subset \mathbb{R}^2$ with $B_\delta(x) \cap A_1 = B_\delta(x) \cap A_2$,

$$\left| \int_{A_1} dy\, \eta_\Omega(t; x, y) - \int_{A_2} dy\, p_{\mathbb{R}^2}(t; x, y) \right| \leq 4e^{-\delta^2/8t}. \tag{4.4.1}$$

(ii) Let $\bar{e}$ be an edge of $\partial\Omega$ and $x \in \Omega$ with $d(x, \bar{e}) \leq \delta$ and $d(x, \partial D \backslash \bar{e}) \geq \delta$. Let $\mathbb{H}_{\bar{e}}$ denote the half-space with $\bar{e} \subset \partial\mathbb{H}_{\bar{e}}$ and $x \in \mathbb{H}_{\bar{e}}$. Then we have that for any Borel sets $A_3 \subset \Omega$ and $A_4 \subset \mathbb{H}_{\bar{e}}$ with $A_3 \cap \Omega \cap B_\delta(x) = A_4 \cap \mathbb{H}_{\bar{e}} \cap B_\delta(x)$,

$$\left| \int_{A_3} dy\, \eta_\Omega(t; x, y) - \int_{A_4} dy\, \eta_{\mathbb{H}_{\bar{e}}}(t; x, y) \right| \leq 4e^{-\delta^2/8t}. \tag{4.4.2}$$

Here $\eta_{\mathbb{H}_{\bar{e}}}(t; x, y)$ is the Neumann heat kernel of the half-space $\mathbb{H}_{\bar{e}}$.

(iii) Let v be a vertex of $\partial\Omega$ with interior angle γ. Let W_γ be the infinite wedge of angle γ with vertex at v and suppose that $B_{2\delta}(v) \cap \Omega = B_{2\delta}(v) \cap W_\gamma$. Then for any Borel sets $A_5 \subset \Omega$ and $A_6 \subset W_\gamma$ with $A_5 \cap \Omega \cap B_{2\delta}(x) = A_6 \cap W_\gamma \cap B_{2\delta}(x)$,

$$\left| \int_{A_5} dy\, \eta_\Omega(t; x, y) - \int_{A_6} dy\, \eta_{W_\gamma}(t; x, y) \right| \leq 4e^{-\delta^2/8t}. \tag{4.4.3}$$

Here $\eta_{W_\gamma}(t; x, y)$ is the Neumann heat kernel of the infinite sector W_γ

Proof. Recall from Section 2.3.2 that one may construct the Neumann heat kernel on a polygonal domain Ω in terms of a projection of the heat kernel of a Riemannian manifold M generated by Ω. Moreover, the projection $(\Psi_M(B_t))_{t\geq 0}$ of a Brownian motion $(B_t)_{t\geq 0}$ on M onto $\overline{\Omega}$ is identical in law to a reflecting Brownian motion on Ω. Through this construction, we do the necessary comparisons at the manifold level.

Let $N_1 \subset M_1$ and $N_2 \subset M_2$ be two subdomains of some Riemannian manifolds M_1 and M_2. Suppose there is an isometry $h : N_1 \to N_2$, then we have that

$$\mathbb{P}_x^{M_1}(B_t^{(1)} \in A, \tau_{N_1} > t) = \mathbb{P}_{h(x)}^{M_2}(B_t^{(2)} \in h(A), \tau_{N_2} > t), \tag{4.4.4}$$

where: $A \subset M_1$ is a Borel set, $(B_t^{(1)})_{t\geq 0}$ is a Brownian motion on M_1 started at $x \in M_1$; $(B_t^{(2)})_{t\geq 0}$ is a Brownian motion on M_2 started at $h(x) \in M_2$; and, τ_{N_1} and τ_{N_2} are the first exit times of $(B_t^{(1)})_{t\geq 0}$ and $(B_t^{(2)})_{t\geq 0}$ from N_1 and N_2 respectively.

For the analyst, the reason why (4.4.4) holds is that

$$\mathbb{P}_x^{M_1}(B_t^{(1)} \in A, \tau_{N_1} > t) = \int_A q_{N_1}(t; x, y) \,\mathrm{dvol}_{M_1}(x) \tag{4.4.5}$$

where q_{N_1} is the Dirichlet heat kernel on N_1. Then we know that the Laplacian commutes with isometries and (4.4.4) immediately follows.

For us, the key point is that the manifold M generated by a polygonal domain Ω is everywhere locally isometric to $\mathbb{R}^2$. This means we have a direct comparison to $\mathbb{R}^2$ or the half-space when we are away from vertices of Ω.

Throughout the rest of this proof, M_1 will always be the manifold generated by Ω and M_2 will either be $\mathbb{R}^2$ or another manifold generated by a polygonal domain.

For notational purposes, for a polygonal domain Ω we define the set

$$F(y, \delta, \Omega) := \{x \in \overline{\Omega} : |x - y| < \delta\} \tag{4.4.6}$$

and for a reflecting Brownian motion $(X_t)_{t\geq 0}$ on Ω starting at $x \in \Omega$ we define $\tau_X(y, \delta)$ as the first exit time of $(X_t)_{t\geq 0}$ from $F(y, \delta, \Omega)$, that is

$$\tau_X(y, \delta) := \inf\{t \geq 0 : |X_t - y| \geq \delta\}. \tag{4.4.7}$$

If $y = x$, then we simply denote this quantity by $\tau_X(\delta)$. We define the analogous quantities for Brownian motions on $\mathbb{R}^2$.

For (i), we have $M_2 = \mathbb{R}^2$ and let $(B_t^{(2)})_{t\geq 0}$ be a Brownian motion on M_2.

Let $(X_t)_{t\geq 0} := (\Psi_{M_1} \circ B_t^{(1)})_{t\geq 0}$ be a reflecting Brownian motion on Ω given as the

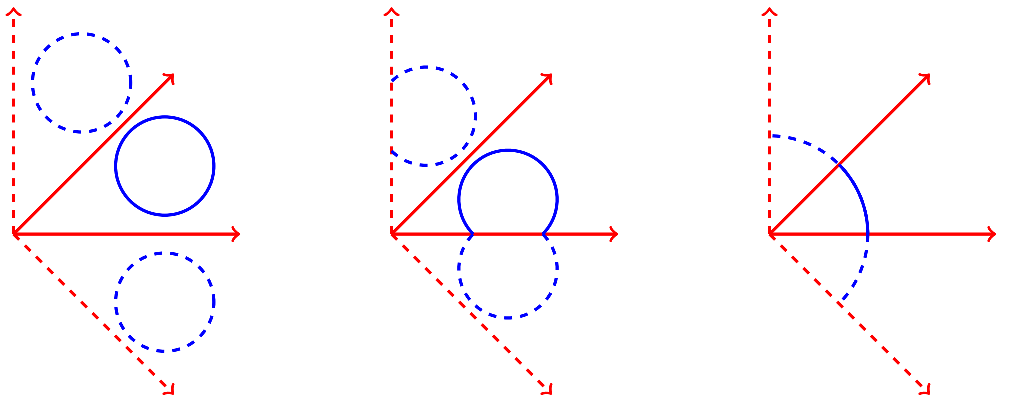

Figure 4.5: Local illustrations of the preimages of subsets of a polygonal domain Ω, whose exit times we are interested in, on the manifold generated by Ω.

projection of a Brownian motion $(B_t^{(1)})_{t\geq 0}$ on M_1. Via the construction of $(X_t)_{t\geq 0}$ we immediately have that

$$\mathbb{P}_x^{\Omega}(X_t \in A_1, \tau_X(\delta) > t) = \mathbb{P}_x^{M_1}(B_t^{(1)} \in \Psi_{M_1}^{-1}(A_1), \tau_{N_1} > t) \tag{4.4.8}$$

where N_1 is the connected component of $\Psi_{M_1}^{-1}(F(x, \delta, \Omega))$ containing x.

Now N_1 is isometric to the ball $N_2 = B_\delta(x) \subset \mathbb{R}^2$, see Figure 4.5(i), and so we deduce that

$$\begin{aligned}\mathbb{P}_x^{M_1}(B_t^{(1)} \in \Psi_{M_1}^{-1}(A_1), \tau_{N_1} > t) &= \mathbb{P}_x^{\mathbb{R}^2}(B_t^{(2)} \in A_2, \tau_{N_2} > t) \\ &= \mathbb{P}_x^{\mathbb{R}^2}(B_t^{(2)} \in A_2, \tau_{B^{(2)}}(\delta) > t),\end{aligned} \tag{4.4.9}$$

by the restriction of the exit time and that $A_1 \cap B_\delta(x) = A_2 \cap B_\delta(x)$. Hence,

$$\mathbb{P}_x^{\Omega}(X_t \in A_1, \tau_X(\delta) > t) = \mathbb{P}_x^{\mathbb{R}^2}(B_t^{(2)} \in A_2, \tau_{B^{(2)}}(\delta) > t). \tag{4.4.10}$$

We can further deduce that

$$\mathbb{P}_x^{\Omega}(\tau_X(\delta) \leq t) = \mathbb{P}_x^{\mathbb{R}^2}(\tau_{B^{(2)}}(\delta) \leq t) \tag{4.4.11}$$

by taking $A_1 = \Omega$ and $A_2 = \mathbb{R}^2$ in (4.4.10).

From [BerSri90, §3], we know the bound $\mathbb{P}_x^{\mathbb{R}^2}(\tau_{B^{(2)}}(\delta) \leq t) \leq 4e^{-\delta^2/8t}$ and so

$$\begin{aligned}\int_{A_1} dy\, \eta_\Omega(t;x,y) &= \mathbb{P}_x^\Omega(X_t \in A_1) \\ &= \mathbb{P}_x^\Omega(X_t \in A_1, \tau_X(\delta) > t) + \mathbb{P}_x^\Omega(X_t \in A_1, \tau_X(\delta) \leq t) \\ &\leq \mathbb{P}_x^{\mathbb{R}^2}(B_t^{(2)} \in A_2, \tau_{B^{(2)}}(\delta) > t) + \mathbb{P}_x^\Omega(\tau_X(\delta) \leq t) \\ &\leq \mathbb{P}_x^{\mathbb{R}^2}(B_t^{(2)} \in A_2) + 4e^{-\delta^2/8t} \\ &= \left(\int_{A_2} dy\, p_{\mathbb{R}^2}(t;x,y)\right) + 4e^{-\delta^2/8t}.\end{aligned} \tag{4.4.12}$$

Reversing the roles of $(X_t)_{t\geq 0}$ and $(B_t^{(2)})_{t\geq 0}$, we obtain the result.

For (ii) the strategy is essentially the same as for (i) except we are comparing $(X_t)_{t\geq 0}$ with a reflecting Brownian motion $(Y_t)_{t\geq 0}$ on the half plane $\mathbb{H}_{\bar{e}}$. The key subtlety is that the manifold generated by $\mathbb{H}_{\bar{e}}$ is simply $M_2 = \mathbb{R}^2$ again and $(B_t^{(2)})_{t\geq 0}$ will again denote a Brownian motion in the plane. Except we see that $\Psi_{M_1}^{-1}(F(x,\delta,\Omega))$ is now comprised of the disjoint union of sets which are each the union of two overlapping balls of radius δ, see Figure 4.5(ii), and we take N_1 to be the connected component of $\Psi_{M_1}^{-1}(F(\delta,x,\Omega))$ containing x.

Arguing in the same way as in (i),

$$\mathbb{P}_x^\Omega(X_t \in A_3, \tau_X(\delta) > t) = \mathbb{P}_x^{\mathbb{H}_{\bar{e}}}(Y_t \in A_4, \tau_Y(\delta) > t) \tag{4.4.13}$$

and hence we see that $\mathbb{P}_x^\Omega(\tau_X(\delta) \leq t) = \mathbb{P}_x^{\mathbb{H}_{\bar{e}}}(\tau_Y(\delta) \leq t)$. It is easy to see that

$$\mathbb{P}_x^{\mathbb{H}_{\bar{e}}}(\tau_Y(\delta) \leq t) \leq \mathbb{P}_x^{\mathbb{R}^2}(\tau_{B^{(2)}}(\delta) \leq t) \leq 4e^{-\delta^2/8t}. \tag{4.4.14}$$

Hence following the argument as in (4.4.12) replacing $(B_t^{(2)})_{t\geq 0}$ with $(Y_t)_{t\geq 0}$, we obtain the result.

For (iii) again the strategy is essentially the same as for (i) but now M_2 is the manifold generated by the wedge W_γ, N_1 and N_2 are the connected components of $\Psi_{M_1}^{-1}(F(v,2\delta,\Omega))$ and $\Psi_{M_2}^{-1}(F(v,2\delta,W_\gamma))$ containing x respectively, see Figure 4.5(iii), and we are comparing $(X_t)_{t\geq 0}$ with a reflecting Brownian motion $(Y_t)_{t\geq 0}$ on the infinite wedge W_γ. But N_1 and N_2 are clearly isometric and thus one determines immediately as above that

$$\mathbb{P}_x^{\Omega}(X_t \in A_5, \tau_X(v, 2\delta) > t) = \mathbb{P}_x^{W_\gamma}(Y_t \in A_6, \tau_Y(v, 2\delta) > t) \tag{4.4.15}$$

and hence $\mathbb{P}_x^{\Omega}(\tau_X(v, 2\delta) \leq t) = \mathbb{P}_x^{W_\gamma}(\tau_Y(v, 2\delta) \leq t)$.

The radial component of a reflecting Brownian motion in the Neumann wedge W_γ is the same as that of a Brownian motion on $\mathbb{R}^2$. So, since $x \in F(v, \delta, W_\gamma)$, we see that $\mathbb{P}_x^{W_\gamma}(\tau_Y(v, 2\delta) \leq t) \leq 4e^{-\delta^2/8t}$. Hence following the argument as in (4.4.12) replacing $(B_t^{(2)})_{t\geq 0}$ with $(Y_t)_{t\geq 0}$, one obtains the result. □

4.5 Model computations

In this section, we prove some explicit heat content calculations that will act as our model heat content contributions. This will complete the proof of Theorem 4.3.1 and give the explicit coefficients with respect to the geometry that we are interested in.

Full justification of why these approximations are valid was given in terms of the locality principles in the last section.

4.5.1 Interior approximations

From the construction of the partition in Section 4.3.1, and using the same notation, we see that $D_{R,\delta}$ is compactly contained in D and thus in Ω. From Lemma 4.4.1, we know we can replace the Neumann heat kernel η_D by the heat kernel $p_{\mathbb{R}^2}$ on $\mathbb{R}^2$ in our model computation.

Thus, the model content contribution for $D_{R,\delta}$ is given by

$$\int_{D_{R,\delta}} dx \int_D dy \, p_{\mathbb{R}^2}(t; x, y) \tag{4.5.1}$$

where $p_{\mathbb{R}^2}(t; x, y)$ is the heat kernel for $\mathbb{R}^2$.

For any $x \in D_{R,\delta}$, we have $d(x, \partial D) \geq \delta$ and so

$$\begin{aligned} 1 \geq \int_D dy\, p_{\mathbb{R}^2}(t;x,y) &= \int_{\mathbb{R}^2} dy\, p_{\mathbb{R}^2}(t;x,y) - \int_{\mathbb{R}^2\setminus D} dy\, p_{\mathbb{R}^2}(t;x,y) \\ &= 1 - (4\pi t)^{-1} \int_{\mathbb{R}^2\setminus D} dy\, e^{-\|x-y\|_2^2/(4t)} \\ &\geq 1 - (4\pi t)^{-1} e^{-\frac{\delta^2}{8t}} \int_{\mathbb{R}^2} dy\, e^{-\|x-y\|_2^2/(8t)} \\ &= 1 - 2e^{-\delta^2/8t}. \end{aligned} \tag{4.5.2}$$

This is the analogue of the 'principle of not feeling the boundary' that was introduced in [Kac66], see also [Ber13, Prop. 9(i)]. The idea is that in small-time on the interior we should not detect any heat loss. Hence, as in [BerGit16, Lem. 4], it follows that

$$\left| \int_{D_{R,\delta}} dx \int_D dy\, p_{\mathbb{R}^2}(t;x,y) - V(D_{R,\delta}) \right| \leq 2e^{-\delta^2/8t}, \tag{4.5.3}$$

which, in combination with Lemma 4.4.1, gives the proof of Theorem 4.3.1(i).

4.5.2 Boundary approximations

For an edge $\bar{e} \in \mathcal{E}$ lying in Ω, in small-time the rectangle $T_{R,\delta}(\bar{e})$ should almost look like it is living in $\mathbb{R}^2$ but instead in the case that the initial datum is the indicator function of the half-plane containing $T_{R,\delta}(\bar{e})$ whose boundary contains $\bar{e}$. Indeed, Lemma 4.4.1 justifies this. Up to a rigid planar motion, this reads:

Lemma 4.5.1 ([BerGit16, §4.1])**.** *Let $T = (0, \ell(\bar{e}) - 2R) \times (0, \delta)$. Then we have that*

$$\int_T dx \int_{\mathbb{H}} dy\, p_{\mathbb{R}^2}(t;x,y) = V(T) - \frac{(\ell(\bar{e}) - 2R)}{\pi^{1/2}} t^{1/2} + O\left(t^{1/2} e^{-\delta^2/8t}\right), \tag{4.5.4}$$

where $\mathbb{H}$ denotes the upper half-space $\mathbb{R} \times \mathbb{R}_{>0}$.

The above gives the validity of Theorem 4.3.1(iii). The same can be done for a cusp $C_{R,\delta}$ lying adjacent to an open edge, which gives the result of Theorem 4.3.1(v).

Lemma 4.5.2 ([BerGG20, Lem. 2.3]).

$$\int_{C_{R,\delta}} dx \int_{\mathbb{H}} dy\, p_{\mathbb{R}^2}(t;x,y) = V(C_{R,\delta}) - \frac{R}{\pi^{1/2}} t^{1/2} \int_1^\infty \frac{dw}{w^2} \int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}} e^{-R^2y^2w^2/4t} + O\left(t^{1/2} e^{-\delta^2/4t}\right). \tag{4.5.5}$$

The formulae (ii) and (iv) in Theorem 4.3.1 follow directly from Lemma 4.4.1 and using the same arguments above but with the Neumann heat kernel in the half-space, which of course yields trivial heat content contributions up to an exponentially small remainder.

4.5.3 Model computations near vertices

We now move on to compute the model computations for vertices of D that lie on $\partial\Omega$. For $0 < \gamma < 2\pi$, we define the infinite wedge of angle γ to be $W_\gamma := \{(r,\phi) : r > 0, 0 < \phi < \gamma\}$ in polar coordinates. The Green's function for the heat equation with Neumann boundary conditions on W_γ is the solution to

$$\begin{cases} sG_{W_\gamma} - \dfrac{\partial^2}{\partial r^2} G_{W_\gamma} - \dfrac{1}{r}\dfrac{\partial}{\partial r} G_{W_\gamma} - \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \phi^2} G_{W_\gamma} = \dfrac{1}{r}\delta(r-r_0)\delta(\phi-\phi_0), \\ \dfrac{\partial}{\partial \phi} G_{W_\gamma} = 0, \ \phi = 0, \gamma, \end{cases} \tag{4.5.6}$$

where δ is the Dirac delta distribution. This Green's function can be computed explicitly to be

$$G_{W_\gamma}(s; r, \phi, r_0, \phi_0) = \frac{1}{\pi^2} \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s}) K_{i\theta}(r_0\sqrt{s}) \Phi_\gamma(\theta, \phi, \phi_0), \tag{4.5.7}$$

where

$$\Phi_\gamma(\theta,\phi,\phi_0) = \cosh\left((\pi - |\phi_0 - \phi|)\,\theta\right) + \frac{\sinh(\pi\theta)}{\sinh(\gamma\theta)} \cosh\left((\phi + \phi_0 - \gamma)\,\theta\right) + \frac{\sinh\left((\pi-\gamma)\,\theta\right)}{\sinh(\gamma\theta)} \cosh\left((\phi - \phi_0)\,\theta\right), \tag{4.5.8}$$

see for example [NurRS24, Appendix A] and references therein. Here the K_ν are modified Bessel functions of the second kind, that is the unique solution to the equation

$$z^2 K_\nu''(z) + z K_\nu'(z) - (z^2 + \nu^2) K_\nu(z) = 0. \tag{4.5.9}$$

We note that (4.5.7) expresses the Green's function of an infinite wedge as a Kontorovich-Lebedev transform. This approach was used by D. B. Ray to compute the angular contribution to the small-time asymptotic expansion of the Dirichlet heat trace for a polygon (see the footnote on page 44 of [McKSin67]) as well as in [BerGG20; BerSri88; BerSri90].

The unique Neumann heat kernel η_{W_γ} on W_γ is given by the inverse Laplace transform of G_{W_γ} i.e.

$$\eta_{W_\gamma}(t; r, \phi, r_0, \phi_0) = \mathcal{L}^{-1}\left\{G_{W_\gamma}(s; r, \phi, r_0, \phi_0)\right\}(t), \tag{4.5.10}$$

and will use this form of the Neumann heat kernel going forward to compute the heat content contributions in small-time.

The Neumann heat kernel on W_γ may be expressed explicitly in terms of Bessel functions and Neumann eigenfunctions of the interval $(0, \gamma)$, see for example [Che83, p. 592], but we are unable to obtain the necessary small-time computations using this representation of the Neumann heat kernel on W_γ.

Neumann-Neumann (NN) vertices

By noting that the only solution to the heat equation on W_γ with Neumann boundary condition and initial datum $\mathbb{1}_{W_\gamma}$ is $u = \mathbb{1}_{W_\gamma}$, it is immediate that the heat content of $S_R(v)$ when v is an NN vertex is $V(S_R(v))$.

However, this route is not as illuminating as to our method when the computations become non-trivial. So we recall the relevant computations below to warm-up for the more challenging computations ahead. The following lemma in combination with Lemma 4.4.1 gives a proof of Theorem 4.3.1(vi).

Lemma 4.5.3. *Let v be a NN vertex of angle $\gamma \in (0, 2\pi)$. Then*

$$\begin{aligned}
&\int_{S_R(v)} \mathrm{d}x \int_{W_\gamma} \mathrm{d}y \, \eta_{W_\gamma}(t; x, y) \\
&\quad = \int_0^R r\, dr \int_0^\gamma d\phi \int_0^\infty r_0 \, dr_0 \int_0^\gamma d\phi_0 \, \eta_{W_\gamma}(t, r, \phi, r_0, \phi_0) \\
&\quad = \frac{1}{2}\gamma R^2 \\
&\quad = V(S_R(v)).
\end{aligned} \tag{4.5.11}$$

for all $t, R > 0$.

Proof. Observe that we are computing the quantity

$$\begin{aligned}
&\int_0^R r\, dr \int_0^\gamma d\phi \int_0^\infty r_0 \, dr_0 \int_0^\gamma d\phi_0 \, \mathcal{L}^{-1}\left\{ G_{W_\gamma}(s, r, \phi, r_0, \phi_0) \right\}(t) \\
&= \int_0^R r\, dr \int_0^\gamma d\phi \int_0^\infty r_0 \, dr_0 \int_0^\gamma d\phi_0 \, \mathcal{L}^{-1}\left\{ \frac{1}{\pi^2} \int_0^\infty d\theta \, K_{i\theta}(r\sqrt{s}) K_{i\theta}(r_0\sqrt{s}) \Phi_\gamma(\theta, \phi, \phi_0) \right\}(t)
\end{aligned} \tag{4.5.12}$$

and that by Fubini's theorem we can rearrange the integrals.

One can easily compute that

$$\int_0^\gamma d\phi \int_0^\gamma d\phi_0 \, \Phi_\gamma(\theta, \phi, \phi_0) = \frac{2\gamma}{\theta} \sinh(\pi\theta) \tag{4.5.13}$$

and hence we now want to compute the quantity

$$\frac{2\gamma}{\pi^2} \mathcal{L}^{-1}\left\{ \int_0^R r\, dr \int_0^\infty r_0 \, dr_0 \int_0^\infty \frac{d\theta}{\theta} K_{i\theta}(r\sqrt{s}) K_{i\theta}(r_0\sqrt{s}) \sinh(\pi\theta) \right\}(t). \tag{4.5.14}$$

This quantity has been computed before in [BerSri90, §2] and is $\frac{1}{2}\gamma R^2$ as desired. For the sake of completeness we give the calculation here.

Combining formulae 6.561.16 and 8.332.3 in [GraRyz07], we see that

$$\int_0^\infty dr \, r K_{i\theta}(r\sqrt{s}) = \frac{\pi\theta}{2s \sinh\left(\frac{\pi}{2}\theta\right)}. \tag{4.5.15}$$

Formula 6.794.2 in [GraRyz07] reads

$$\int_0^\infty d\theta \, K_{i\theta}(r\sqrt{s}) \cosh\left(\tfrac{\pi}{2}\theta\right) = \frac{\pi}{2}. \tag{4.5.16}$$

Hence, applying (4.5.15) and (4.5.16) successively we see that

$$\begin{aligned}
&\frac{2\gamma}{\pi^2}\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_0^\infty r_0\,dr_0\int_0^\infty \frac{d\theta}{\theta}\,K_{i\theta}(r\sqrt{s})K_{i\theta}(r_0\sqrt{s})\sinh(\pi\theta)\right\}(t)\\
&\qquad=\frac{2\gamma}{\pi}\mathcal{L}^{-1}\left\{\frac{1}{s}\int_0^R r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\cosh(\tfrac{\pi}{2}\theta)\right\}(t) \qquad (4.5.17)\\
&\qquad=\gamma\mathcal{L}^{-1}\left\{\frac{1}{s}\int_0^R r\,dr\right\}(t)=\frac{1}{2}\gamma R^2,
\end{aligned}$$

as desired. □

Neumann-Open-Neumann (NON) and Neumann-Open-Open-Neumann (NOON) vertices

The case of Neumann-Open-Neumann (NON) and Neumann-Open-Open-Neumann (NOON) vertices is much more involved and will require the following technical tools that we shall now prove.

Lemma 4.5.4. *Let $\mathcal{T} := \{(\sigma,\rho,\lambda) : 0<\lambda<\rho<\sigma<2\pi\}$. Let $\xi:\mathcal{T}\to\mathbb{R}$ be a function in only ρ and λ, denoted $\xi(\rho,\lambda)$, such that $0\le\xi(\rho,\lambda)<2\sigma$.*

(i) We have that the collection $\{A_N,B_N\}_{N\in\mathbb{Z}_{\ge0}}$, where

$$A_N := \{(\sigma,\rho,\lambda)\in\mathcal{T} : |\tfrac{\pi}{2}-(2N+1)\sigma+\xi(\rho,\lambda)|<\sigma\} \qquad (4.5.18)$$

and

$$B_N := \{(\sigma,\rho,\lambda)\in\mathcal{T} : \tfrac{\pi}{2}+\xi(\rho,\lambda)=2(N+1)\sigma\}, \qquad (4.5.19)$$

forms a covering of $\mathcal{T}$. Moreover, for $N\ge1$ fixed, any $(\sigma,\rho,\lambda)\in A_N$ or $(\sigma,\rho,\lambda)\in B_N$, and any $1\le n\le N$, we have that $0<\frac{\pi}{2}-2n\sigma+\xi(\rho,\lambda)<\frac{\pi}{2}$.

(ii) If $\xi(\rho,\lambda)>0$ then we have that the collection $\{C_N,D_N\}_{N\in\mathbb{Z}_{\ge0}}$, where

$$C_N := \{(\sigma,\rho,\lambda)\in\mathcal{T} : |\tfrac{\pi}{2}-(2N-1)\sigma-\xi(\rho,\lambda)|<\sigma\} \qquad (4.5.20)$$

and

$$D_N := \{(\sigma,\rho,\lambda)\in\mathcal{T} : \tfrac{\pi}{2}-\xi(\rho,\lambda)=2N\sigma\}, \qquad (4.5.21)$$

forms a covering of $\mathcal{T}$*. Moreover, for* $N \geq 1$ *fixed, any* $(\sigma, \rho, \lambda) \in C_N$ *or* $(\sigma, \rho, \lambda) \in D_N$*, and any* $1 \leq n \leq N$*, we have that* $0 < \frac{\pi}{2} - 2(n-1)\sigma - \xi(\rho, \lambda) < \frac{\pi}{2}$.

Proof. For (i) we have that $(\sigma, \rho, \lambda) \in A_N$ if and only if $-\sigma < \frac{\pi}{2} - (2N+1)\sigma + \xi(\lambda, \sigma) < \sigma$ which occurs if and only if $2N\sigma < \frac{\pi}{2} + \xi(\rho, \lambda) < 2(N+1)\sigma$. Now pick an arbitrary point $(\sigma, \rho, \lambda) \in \mathcal{T}$. Suppose that $\frac{\pi}{2} + \xi(\rho, \lambda) \neq 2(M+1)\sigma$ for any $M \in \mathbb{Z}_{\geq 0}$. Then clearly there exists a $K \in \mathbb{Z}_{\geq 0}$ such that $2K\sigma < \frac{\pi}{2} + \xi(\rho, \lambda) < 2(K+1)\sigma$ and hence $(\sigma, \rho, \lambda) \in A_K$. If we have that $\frac{\pi}{2} + \xi(\rho, \lambda) = 2(K'+1)\sigma$ for some $K' \in \mathbb{Z}_{\geq 0}$, then $(\sigma, \rho, \lambda) \in B_{K'}$. Since our choice of point (σ, ρ, λ) was arbitrary, we indeed have that $\{A_N, B_N\}_{N \in \mathbb{Z}_{\geq 0}}$ is a covering of $\mathcal{T}$.

Now fix $N \geq 1$. Let us choose an arbitrary $(\sigma, \rho, \lambda) \in A_N$. For $1 \leq n \leq N$, we have that

$$\frac{\pi}{2} - 2n\sigma + \xi(\rho, \lambda) \leq \frac{\pi}{2} - 2\sigma + \xi(\rho, \lambda) < \frac{\pi}{2} \tag{4.5.22}$$

and that

$$\frac{\pi}{2} - 2n\sigma + \xi(\rho, \lambda) \geq \frac{\pi}{2} - 2N\sigma + \xi(\rho, \lambda) > 2N\sigma - 2N\sigma = 0. \tag{4.5.23}$$

Now let us choose $(\sigma, \rho, \lambda) \in B_N$. For $1 \leq n \leq N$, we have that

$$\frac{\pi}{2} - 2n\sigma + \xi(\rho, \lambda) \leq \frac{\pi}{2} - 2\sigma + \xi(\rho, \lambda) < \frac{\pi}{2} \tag{4.5.24}$$

and that

$$\frac{\pi}{2} - 2n\sigma + \xi(\rho, \lambda) \geq \frac{\pi}{2} - 2N\sigma + \xi(\rho, \lambda) = 2(N+1)\sigma - 2N\sigma = 2\sigma > 0. \tag{4.5.25}$$

So we have proven (i).

We prove (ii) in a similar fashion. We have that $(\sigma, \rho, \lambda) \in C_N$ if and only if $2(N-1)\sigma < \frac{\pi}{2} - \xi(\rho, \lambda) < 2N\sigma$. Now pick an arbitrary point $(\sigma, \rho, \lambda) \in \mathcal{T}$. Suppose that $\frac{\pi}{2} - \xi(\rho, \lambda) \neq 2M\sigma$ for any $M \in \mathbb{Z}_{\geq 0}$. Then clearly there exists a $K \in \mathbb{Z}_{\geq 0}$ such that $2(K-1)\sigma < \frac{\pi}{2} - \xi(\rho, \lambda) < 2K\sigma$ and hence $(\sigma, \rho, \lambda) \in C_K$. If we have that $\frac{\pi}{2} - \xi(\rho, \lambda) = 2K'\sigma$ for some $K' \in \mathbb{Z}_{\geq 0}$, then $(\sigma, \rho, \lambda) \in D_{K'}$. Since our choice of point (σ,ρ,λ) was arbitrary, we indeed have that $\{C_N, D_N\}_{N\in\mathbb{Z}_{\geq 0}}$ is a covering of $\mathcal{T}$.

Now fix $N \geq 1$. Let us choose an arbitrary $(\sigma,\rho,\lambda) \in C_N$. For $1 \leq n \leq N$, we have that

$$\tfrac{\pi}{2} - 2(n-1)\sigma - \xi(\rho,\lambda) \leq \tfrac{\pi}{2} - \xi(\rho,\lambda) < \tfrac{\pi}{2} \tag{4.5.26}$$

and that

$$\tfrac{\pi}{2} - 2(n-1)\sigma - \xi(\rho,\lambda) \geq \tfrac{\pi}{2} - 2(N-1)\sigma - \xi(\rho,\lambda) > 2(N-1)\sigma - 2(N-1)\sigma = 0. \tag{4.5.27}$$

Now let us choose an $(\sigma,\rho,\lambda) \in D_N$. For $1 \leq n \leq N$, we have that

$$\tfrac{\pi}{2} - 2(n-1)\sigma - \xi(\rho,\lambda) \leq \tfrac{\pi}{2} - \xi(\rho,\lambda) < \tfrac{\pi}{2} \tag{4.5.28}$$

and that

$$\tfrac{\pi}{2} - 2(n-1)\sigma - \xi(\rho,\lambda) \geq \tfrac{\pi}{2} - 2(N-1)\sigma - \xi(\rho,\lambda) = 2N\sigma - 2(N-1)\sigma = 2\sigma > 0. \tag{4.5.29}$$

So we have proven (ii). □

Before moving forward, let us give some known results for small-time asymptotics for inverse Laplace transforms.

Lemma 4.5.5. *(i) For $a, b \in \mathbb{R}$ such that $|a| < |b|$, we have that*

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s})\frac{\cosh(a\theta)-1}{\theta\sinh(b\theta)}\right\}(t) = O\left(te^{-R^2/(4t)}\right). \tag{4.5.30}$$

(ii) For $-\frac{\pi}{2} < a < \frac{\pi}{2}$, we have that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s})\frac{\sinh(a\theta)}{\theta}\right\}(t) = O\left(te^{-R^2\cos^2(a)/(4t)}\right). \tag{4.5.31}$$

(iii) For $a > 0$, we have that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s})\frac{\tanh(a\theta)}{\theta}\right\}(t) = O\left(te^{-R^2/(4t)}\right). \tag{4.5.32}$$

Proof. Proofs of (ii) and (iii) can be found in [BerSri90, §2] on pages 47 and 48. For (i), note that if $|a| < |b|$ then

$$\int_0^\infty d\theta \left| \frac{\cosh(a\theta) - 1}{\theta \sinh(b\theta)} \right| < +\infty, \tag{4.5.33}$$

and we may obtain the result following similar methods to those in [BerGG20, §4], as we now show.

From equation 3.547.4 in [GraRyz07], one has that

$$K_{i\theta}(r\sqrt{s}) = \int_0^\infty \mathrm{d}w\, \cos(w\theta) e^{-r\sqrt{s}\cosh w}. \tag{4.5.34}$$

Moreover, from formula 5.6.3 in [ErdMOT54]

$$\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-(as)^{1/2}}\right\}(t) = \frac{2}{\pi^{1/2}}\int_{(a/(4t))^{1/2}}^\infty \mathrm{d}z\, e^{-z^2}, \tag{4.5.35}$$

which yields that

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-r\sqrt{s}\cosh w}\right\}(t) &= \frac{2}{\pi^{1/2}}\int_{(r\cosh w)/(4t)^{1/2}}^\infty \mathrm{d}z\, e^{-z^2} \\ &= \operatorname{erfc}\left((r\cosh w)/(4t)^{1/2}\right), \end{aligned} \tag{4.5.36}$$

where erfc is the complementary error function given by

$$\operatorname{erfc}(x) = \frac{2}{\pi^{1/2}}\int_x^\infty \mathrm{d}z\, e^{-z^2}. \tag{4.5.37}$$

Thus, we have that

$$\begin{aligned} &\left|\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s})\frac{\cosh(a\theta)-1}{\theta\sinh(b\theta)}\right\}(t)\right| \\ &\quad = \left|\int_R^\infty r\,dr\int_0^\infty \mathrm{d}w\, \operatorname{erfc}\left((r\cosh w)/(4t)^{1/2}\right)\int_0^\infty d\theta\, \cos(w\theta)\frac{\cosh(a\theta)-1}{\theta\sinh(b\theta)}\right| \\ &\quad \le \int_R^\infty r\,dr\int_0^\infty \mathrm{d}w\, \operatorname{erfc}\left((r\cosh w)/(4t)^{1/2}\right)\int_0^\infty d\theta\, \left|\frac{\cosh(a\theta)-1}{\theta\sinh(b\theta)}\right| \\ &\quad = C_{a,b}\int_R^\infty r\,dr\int_0^\infty \mathrm{d}w\, \operatorname{erfc}\left((r\cosh w)/(4t)^{1/2}\right), \end{aligned} \tag{4.5.38}$$

where

$$C_{a,b} := \int_0^\infty d\theta \left| \frac{\cosh(a\theta) - 1}{\theta \sinh(b\theta)} \right| < +\infty, \tag{4.5.39}$$

by assumption. By the observation that $\operatorname{erfc}(z) \leq e^{-z^2}$ for any $z \geq 0$, we get

$$\begin{aligned}\int_R^\infty r\,dr \int_0^\infty \mathrm{d}w \operatorname{erfc}\left((r\cosh w)/(4t)^{1/2}\right) &\leq \int_R^\infty r\,dr \int_0^\infty \mathrm{d}w\, e^{-(r\cosh w)^2/(4t)} \\ &= 2t \int_0^\infty \frac{\mathrm{d}w}{(\cosh w)^2} e^{-(R\cosh w)^2/(4t)} \\ &\leq 2te^{-R^2/(4t)} \int_0^\infty \frac{\mathrm{d}w}{(\cosh w)^2} \\ &= O\left(te^{-R^2/(4t)}\right).\end{aligned} \tag{4.5.40}$$

Combining (4.5.38) and (4.5.40) gives (i) as desired. □

Using the previous two lemmata we can now prove the following crucial lemma, which is done in the spirit of the proof that the remainder is exponentially small in [BerSri90].

Lemma 4.5.6. *Let $(\sigma, \rho, \lambda) \in \mathcal{T}$ and ξ be as in Lemma 4.5.4.*

(i) Then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s}) \frac{\cosh((\frac{\pi}{2} - \sigma + \xi(\rho,\lambda))\theta) - 1}{\theta\sinh(\sigma\theta)}\right\}(t) \\ = O\left(te^{-R^2 C_1^{\sigma,\rho,\lambda}/(4t)}\right),\end{aligned} \tag{4.5.41}$$

where $C_1^{\sigma,\rho,\lambda} > 0$ is a constant depending only on σ, ρ and λ.

(ii) If $\xi(\rho, \lambda) > 0$, then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr \int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s}) \frac{\cosh((\frac{\pi}{2} + \sigma - \xi(\rho,\lambda))\theta) - 1}{\theta\sinh(\sigma\theta)}\right\}(t) \\ = O\left(te^{-R^2 C_2^{\sigma,\rho,\lambda}/(4t)}\right),\end{aligned} \tag{4.5.42}$$

where $C_2^{\sigma,\rho,\lambda} > 0$ is a constant depending only on σ, ρ and λ.

Proof. Let us prove (i). Let $\{A_N, B_N\}_{N\in\mathbb{Z}_{\geq 0}}$ be the covering of $\mathcal{T}$ as given in Lemma 4.5.4(i). For $(\sigma, \rho, \lambda) \in A_0$, the result is immediate by Lemma 4.5.5(i) and for $(\sigma, \rho, \lambda) \in B_0$ we see that

$$\frac{\cosh((\frac{\pi}{2} - \sigma + \xi(\rho,\lambda))\theta) - 1}{\theta\sinh(\sigma\theta)} = \frac{\tanh(\frac{\sigma}{2}\theta)}{\theta} \tag{4.5.43}$$

and the result is immediate from Lemma 4.5.5(iii). Let $N \geq 1$. Suppose $(\sigma, \rho, \lambda) \in A_N$ and observe that

$$\frac{\cosh((\frac{\pi}{2} - \sigma + \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)} = 2\sum_{n=1}^{N} \frac{\sinh((\frac{\pi}{2} - 2n\sigma + \xi(\rho,\lambda))\theta)}{\theta} + \frac{\cosh((\frac{\pi}{2} - (2N+1)\sigma + \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)}, \tag{4.5.44}$$

then the result comes from applying Lemmata 4.5.5(i) and 4.5.5(ii). Suppose $(\sigma, \rho, \lambda) \in B_N$, then we can observe that

$$\frac{\cosh((\frac{\pi}{2} - \sigma + \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)} = 2\sum_{n=1}^{N} \frac{\sinh((\frac{\pi}{2} - 2n\sigma + \xi(\rho,\lambda))\theta)}{\theta} + \frac{\tanh(\frac{\sigma}{2}\theta)}{\theta}, \tag{4.5.45}$$

and the result comes from applying Lemmata 4.5.5(ii) and 4.5.5(iii). Since $\{A_N, B_N\}_{N\in\mathbb{Z}_{\geq 0}}$ is a covering of $\mathcal{T}$ we are done.

Now let us prove (ii). Let $\{C_N, D_N\}_{N\in\mathbb{Z}_{\geq 0}}$ be the covering of $\mathcal{T}$ as given in Lemma 4.5.4(ii). For $(\sigma, \rho, \lambda) \in C_0$, the result is immediate by Lemma 4.5.5(i) and for $(\sigma, \rho, \lambda) \in D_0$ we see that

$$\frac{\cosh((\frac{\pi}{2} + \sigma - \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)} = \frac{\tanh(\frac{\sigma}{2}\theta)}{\theta} \tag{4.5.46}$$

and the result is immediate from Lemma 4.5.5(iii). Let $N \geq 1$. Suppose $(\sigma, \rho, \lambda) \in C_N$ and observe that

$$\frac{\cosh((\frac{\pi}{2} + \sigma - \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)} = 2\sum_{n=1}^{N} \frac{\sinh((\frac{\pi}{2} - 2(n-1)\sigma - \xi(\rho,\lambda))\theta)}{\theta} + \frac{\cosh((\frac{\pi}{2} - (2N-1)\sigma - \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)}, \tag{4.5.47}$$

then the result comes from applying Lemmata 4.5.5(i) and 4.5.5(ii). Suppose $(\sigma, \rho, \lambda) \in D_N$, then we can observe that

$$\frac{\cosh((\frac{\pi}{2} + \sigma - \xi(\rho,\lambda))\theta) - 1}{\theta \sinh(\sigma\theta)} = 2\sum_{n=1}^{N} \frac{\sinh((\frac{\pi}{2} - 2(n-1)\sigma - \xi(\rho,\lambda))\theta)}{\theta} + \frac{\tanh(\frac{\sigma}{2}\theta)}{\theta}, \tag{4.5.48}$$

and the result comes from applying Lemmata 4.5.5(ii) and 4.5.5(iii). Since $\{C_N, D_N\}_{N\in\mathbb{Z}_{\geq 0}}$ is a covering of $\mathcal{T}$ we are done. □

Remark 4.5.7. From the proof it is clear that for (i) and (ii), if $\xi(\rho,\lambda)$ depends only on ρ, then the constants $C_1^{\sigma,\rho,\lambda} > 0$ and $C_2^{\sigma,\rho,\lambda} > 0$ depend only on σ and ρ. Moreover, in case (i) if $\xi(\rho,\lambda) = 0$ then $C_1^{\sigma,\rho,\lambda} > 0$ depends only on σ.

With this toolbox now in hand we are ready to compute the model computations for the NON and NOON vertex cases. First, let us state a result that was computed in [BerSri90, §2] (see equations (2.10) and (2.14) there) and will be used in what follows:

$$\begin{aligned}\mathcal{L}^{-1}&\left\{\int_0^R r\,dr\int_0^\infty r_0\,dr_0\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})K_{i\theta}(r_0\sqrt{s})\frac{2\sinh^2(\frac{\pi}{2}\theta)}{\pi^2\theta^2}\right\}(t)\\&=\mathcal{L}^{-1}\left\{\frac{1}{s}\int_0^R r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\frac{\sinh(\frac{\pi}{2}\theta)}{\pi\theta}\right\}(t)\\&=\frac{R}{\pi^{1/2}}t^{1/2}-\frac{R}{\pi^{1/2}}t^{1/2}\int_1^\infty\frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t}.\end{aligned}\tag{4.5.49}$$

The following Theorem in combination with Lemma 4.4.1 gives a proof of Theorem 4.3.1(vii).

Theorem 4.5.8. *Let v be a NON vertex with interior angle γ and exterior angle β. Under the variable change $\gamma=\rho$, $\beta=\sigma-\rho$, we have that $0<\rho<\sigma<2\pi$ and the model heat content contribution from $S_R(v)$ is*

$$\begin{aligned}\int_0^R r\,dr&\int_0^\rho d\phi\int_0^\infty r_0\,dr_0\int_0^\rho d\phi_0\,\eta_{W_\sigma}(t;r,\phi,r_0,\phi_0)\\&=\frac{1}{2}\rho R^2-\frac{R}{\pi^{1/2}}t^{1/2}+\widehat{b}(\sigma,\rho)t\\&\quad+\frac{R}{\pi^{1/2}}t^{1/2}\int_1^\infty\frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t}+O\left(te^{-R^2C_{\sigma,\rho}/4t}\right)\\&=V(S_R(v))-\frac{R}{\pi^{1/2}}t^{1/2}+b(\gamma,\beta)t\\&\quad+\frac{R}{\pi^{1/2}}t^{1/2}\int_1^\infty\frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t}+O\left(te^{-R^2C_{\gamma,\beta}/4t}\right),\end{aligned}\tag{4.5.50}$$

where

$$\widehat{b}(\sigma,\rho)=\int_0^\infty d\theta\,\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta)-\cosh\left(\left(\frac{\pi}{2}-\sigma\right)\theta\right)}{2\sinh(\sigma\theta)\sinh(\frac{\pi}{2}\theta)}\tag{4.5.51}$$

and $C_{\sigma,\rho}, C_{\gamma,\beta}>0$ are constants depending only on $\sigma,\rho>0$ and $\gamma,\beta>0$ respectively.

Proof. As in the case of the NN wedge, we compute the angular terms first. Again we can do this by Fubini's theorem. With standard hyperbolic trigonometric identities, one can show that

$$\begin{aligned}
&\int_0^\rho d\phi \int_0^\rho d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0)\\
&\quad = \frac{2\rho}{\theta}\sinh(\pi\theta) + \frac{2}{\theta^2}(\cosh((\pi-\rho)\theta) - \cosh(\pi\theta)) + \frac{2\sinh((\pi-\sigma)\theta)}{\theta^2\sinh(\sigma\theta)}(\cosh(\rho\theta)-1)\\
&\qquad + \frac{\sinh(\pi\theta)}{\theta^2\sinh(\sigma\theta)}(\cosh((\sigma-2\rho)\theta) + \cosh(\sigma\theta) - 2\cosh((\sigma-\rho)\theta))\\
&\quad = \frac{2\rho}{\theta}\sinh(\pi\theta) - \frac{2}{\theta^2}\sinh^2(\tfrac{\pi}{2}\theta) + \frac{2\sinh(\frac{\pi}{2}\theta)}{\theta^2\sinh(\sigma\theta)}\Big\{\cosh\left(\tfrac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta)\\
&\qquad - \cosh\left(\left(\tfrac{\pi}{2}-\sigma\right)\theta\right)\Big\}.
\end{aligned} \tag{4.5.52}$$

We know how to treat the first two terms in the last line of the equation above from equations (4.5.17) and (4.5.49), so we only need to treat the third term. Applying the identity in equation (4.5.15) twice, we see that

$$\begin{aligned}
&\mathcal{L}^{-1}\Big\{2\int_0^\infty r\,dr\int_0^\infty r_0\,dr_0\int_0^\infty d\theta\, K_{i\theta}(r\sqrt{s})K_{i\theta}(r_0\sqrt{s})\sinh(\tfrac{\pi}{2}\theta)\\
&\quad\times\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta) - \cosh\left(\left(\frac{\pi}{2}-\sigma\right)\theta\right)}{\pi^2\theta^2\sinh(\sigma\theta)}\Big\}(t)\\
&\qquad = \mathcal{L}^{-1}\left\{\frac{1}{2s^2}\int_0^\infty d\theta\,\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta) - \cosh\left(\left(\frac{\pi}{2}-\sigma\right)\theta\right)}{\sinh(\sigma\theta)\sinh(\frac{\pi}{2}\theta)}\right\}\\
&\qquad = \frac{t}{2}\int_0^\infty d\theta\,\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta) - \cosh\left(\left(\frac{\pi}{2}-\sigma\right)\theta\right)}{\sinh(\sigma\theta)\sinh(\frac{\pi}{2}\theta)}.
\end{aligned} \tag{4.5.53}$$

Thus, by the linearity of the inverse Laplace transform, we have that

$$\begin{aligned}
&\int_0^R r\,dr\int_0^\rho d\phi\int_0^\infty r_0\,dr_0\int_0^\rho d\phi_0\,\eta_{W_\sigma}(t,r,\phi,r_0,\phi_0)\\
&\quad = \frac{1}{2}\rho R^2 - \frac{R}{\pi^{1/2}}t^{1/2} + \widehat{b}(\sigma,\rho)t - S_1(t) + \frac{R}{\pi^{1/2}}t^{1/2}\int_1^\infty\frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t},
\end{aligned} \tag{4.5.54}$$

where

$$S_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta)-\cosh\left(\left(\frac{\pi}{2}-\sigma\right)\theta\right)}{\pi\theta\sinh(\sigma\theta)}\right\}(t). \tag{4.5.55}$$

Thus, it suffices to show that $S_1(t)$ is exponentially small as $t \downarrow 0$.

Observe that

$$\begin{aligned}
&\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-2\rho)\theta)-\cosh\left(\left(\frac{\pi}{2}-\sigma\right)\theta\right)}{\pi\theta\sinh(\sigma\theta)}\\
&\quad=\frac{\cosh((\frac{\pi}{2}-\sigma+2\rho)\theta)+\cosh((\frac{\pi}{2}+\sigma-2\rho)\theta)-2\cosh((\frac{\pi}{2}-\sigma)\theta)}{2\pi\theta\sinh(\sigma\theta)}\\
&\quad=\frac{\cosh((\frac{\pi}{2}-\sigma+2\rho)\theta)-1}{2\pi\theta\sinh(\sigma\theta)}+\frac{\cosh((\frac{\pi}{2}+\sigma-2\rho)\theta)-1}{2\pi\theta\sinh(\sigma\theta)}\\
&\qquad-\frac{\cosh((\frac{\pi}{2}-\sigma)\theta)-1}{\pi\theta\sinh(\sigma\theta)}.
\end{aligned} \tag{4.5.56}$$

Hence, we have that

$$\begin{aligned}
S_1(t) = {} & \mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\frac{\cosh((\frac{\pi}{2}-\sigma+2\rho)\theta)-1}{2\pi\theta\sinh(\sigma\theta)}\right\}(t)\\
&+\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\frac{\cosh((\frac{\pi}{2}+\sigma-2\rho)\theta)-1}{2\pi\theta\sinh(\sigma\theta)}\right\}(t)\\
&-\mathcal{L}^{-1}\left\{\frac{1}{s}\int_R^\infty r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\frac{\cosh((\frac{\pi}{2}-\sigma)\theta)-1}{\pi\theta\sinh(\sigma\theta)}\right\}(t).
\end{aligned} \tag{4.5.57}$$

Using Lemmata 4.5.6(i) and 4.5.6(ii), we see immediately that $S_1(t) = O\left(te^{-R^2C_{\sigma,\rho}/t}\right)$ where $C_{\sigma,\rho} > 0$ is a constant depending only on σ and ρ.

Undoing the variable substitution one sees that, $\widehat{b}(\sigma,\rho) = b(\gamma,\beta)$ with b as defined in Theorem 4.2.1 and $\frac{1}{2}\rho R^2 = \frac{1}{2}\gamma R^2 = V(S_R(v))$, which concludes the proof. □

The proof of the NOON case becomes even more complicated but the idea is precisely the same. The main difference is we get a cross-term we need to deal with but we can still handle it using Lemma 4.5.6.

The following Theorem in combination with Lemma 4.4.1 gives a proof of Theorem 4.3.1(viii).

Theorem 4.5.9. *Let v be a NOON vertex with middle angle γ and exterior angles β and α. Under the variable change $\gamma = \rho - \lambda$, $\beta = \sigma - \rho$ and $\alpha = \lambda$, we have that $0 < \lambda < \rho < \sigma < 2\pi$.*

The model heat content contribution from $S_R(v)$ when γ is an interior angle of D is

$$\begin{aligned}
&\int_0^R r\,dr \int_\lambda^\rho d\phi \int_0^\infty r_0\,dr_0 \int_\lambda^\rho d\phi_0\, \eta_{W_\sigma}(t;r,\phi,r_0,\phi_0) \\
&\quad = \frac{1}{2}(\rho-\lambda)R^2 - \frac{2R}{\pi^{1/2}}t^{1/2} + \widehat{c}(\sigma,\rho,\lambda)t \\
&\qquad + \frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t} + O\left(te^{-R^2C_{\sigma,\rho,\lambda}/4t}\right) \\
&\quad = V(S_R(v)) - \frac{2R}{\pi^{1/2}}t^{1/2} + c(\gamma,\beta,\alpha)t \\
&\qquad + \frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t} + O\left(te^{-R^2C_{\gamma,\beta,\alpha}/4t}\right),
\end{aligned} \tag{4.5.58}$$

and when β and α are interior angles of D is

$$\begin{aligned}
&\int_0^R r\,dr \int_{(0,\lambda)\cup(\rho,\sigma)} d\phi \int_0^\infty r_0\,dr_0 \int_{(0,\lambda)\cup(\rho,\sigma)} d\phi_0\, \eta_{W_\sigma}(t;r,\phi,r_0,\phi_0) \\
&\quad = \frac{1}{2}(\lambda+\sigma-\rho)R^2 - \frac{2R}{\pi^{1/2}}t^{1/2} + \widehat{c}(\sigma,\rho,\lambda)t \\
&\qquad + \frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t} + O\left(te^{-R^2C_{\sigma,\rho,\lambda}/4t}\right) \\
&\quad = V(S_R(v)) - \frac{2R}{\pi^{1/2}}t^{1/2} + c(\gamma,\beta,\alpha)t \\
&\qquad + \frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t} + O\left(te^{-R^2C_{\gamma,\beta,\alpha}/4t}\right).
\end{aligned} \tag{4.5.59}$$

Here,

$$\begin{aligned}
\widehat{c}(\sigma,\rho,\lambda) &= \widehat{b}(\sigma,\rho) + \widehat{b}(\sigma,\lambda) \\
&\quad - \int_0^\infty d\theta\, \frac{\cosh\left(\frac{\pi}{2}\theta\right)\left(\cosh((\sigma-\rho-\lambda)\theta) - \cosh((\sigma-\rho+\lambda)\theta)\right)}{\sinh(\sigma\theta)\sinh(\frac{\pi}{2}\theta)}
\end{aligned} \tag{4.5.60}$$

and $C_{\sigma,\rho,\lambda}, C_{\gamma,\beta,\alpha} > 0$ are constants depending only on $\sigma,\rho,\lambda > 0$ and $\gamma,\beta,\alpha > 0$ respectively.

Proof. First let us consider the case where γ is an interior angle of D.

Again, as in the case of the NN and NON wedges, we begin with the integrals over the angles and save the radial terms for later in order get the integrals into a form where we can use Lemma 4.5.6.

We first observe that by Fubini's theorem

$$\int_\lambda^\rho d\phi \int_\lambda^\rho d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) = \int_0^\rho d\phi \int_0^\rho d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) + \int_0^\lambda d\phi \int_0^\lambda d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) - 2\int_0^\lambda d\phi \int_0^\rho d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) \tag{4.5.61}$$

and we have treated the first two terms on the right-hand side in the NON wedge case. So only the third term needs to be treated. We have that

$$\begin{aligned}
&\int_0^\lambda d\phi \int_0^\rho d\phi_0\, \cosh((\pi - |\phi-\phi_0|)\theta) \\
&= \frac{2\lambda}{\theta}\sinh(\pi\theta) + \frac{1}{\theta^2}\Big[\cosh((\pi-\lambda)\theta) + \cosh((\pi-\rho)\theta) - \cosh(\pi\theta) \\
&\quad - \cosh((\pi+\lambda-\rho)\theta)\Big] \\
&= \frac{2\lambda}{\theta}\sinh(\pi\theta) + \frac{1}{2\theta^2\sinh(\sigma\theta)}\Big[\sinh((\pi+\sigma-\lambda)\theta) - \sinh((\pi-\sigma-\lambda)\theta) \\
&\quad + \sinh((\pi+\sigma-\rho)\theta) - \sinh((\pi-\sigma-\rho)\theta) - \sinh((\pi+\sigma)\theta) \\
&\quad + \sinh((\pi-\sigma)\theta) - \sinh((\pi+\sigma-\rho+\lambda)\theta) + \sinh((\pi-\sigma-\rho+\lambda)\theta)\Big],
\end{aligned} \tag{4.5.62}$$

as well as

$$\begin{aligned}
&\frac{\sinh(\pi\theta)}{\sinh(\sigma\theta)}\int_0^\lambda d\phi \int_0^\rho d\phi_0\, \cosh((\phi+\phi_0-\sigma)\theta) \\
&= \frac{\sinh(\pi\theta)}{\theta^2\sinh(\sigma\theta)}\Big[\cosh((\sigma-\rho-\lambda)\theta) + \cosh(\sigma\theta) - \cosh((\sigma-\lambda)\theta) \\
&\quad - \cosh((\sigma-\rho)\theta)\Big] \\
&= \frac{1}{2\theta^2\sinh(\sigma\theta)}\Big[2\sinh(\pi\theta)\cosh((\sigma-\rho-\lambda)\theta) + \sinh((\pi+\sigma)\theta) \\
&\quad + \sinh((\pi-\sigma)\theta) - \sinh((\pi+\sigma-\lambda)\theta) - \sinh((\pi-\sigma+\lambda)\theta) \\
&\quad - \sinh((\pi+\sigma-\rho)\theta) - \sinh((\pi-\sigma+\rho)\theta)\Big],
\end{aligned} \tag{4.5.63}$$

and that

$$\begin{aligned}
&\frac{\sinh((\pi-\sigma)\theta)}{\sinh(\sigma\theta)}\int_0^\lambda d\phi\int_0^\rho d\phi_0\,\cosh((\phi-\phi_0)\theta)\\
&=\frac{\sinh((\pi-\sigma)\theta)}{\theta^2\sinh(\sigma\theta)}\Big[\cosh(\lambda\theta)+\cosh(\rho\theta)-\cosh((\rho-\lambda)\theta)-1\Big]\\
&=\frac{1}{2\theta^2\sinh(\sigma\theta)}\Big[\sinh((\pi-\sigma+\lambda)\theta)+\sinh((\pi-\sigma-\lambda)\theta)\\
&\quad+\sinh((\pi-\sigma+\rho)\theta)+\sinh((\pi-\sigma-\rho)\theta)-\sinh((\pi-\sigma+\rho-\lambda)\theta)\\
&\quad-\sinh((\pi-\sigma-\rho+\lambda)\theta)-2\sinh((\pi-\sigma)\theta)\Big].
\end{aligned}\tag{4.5.64}$$

Summing (4.5.62), (4.5.63), and (4.5.64) we obtain that

$$\begin{aligned}
\int_0^\lambda d\phi\int_0^\rho d\phi_0\,\Phi_\sigma(\theta,\phi,\phi_0)&=\frac{2\lambda}{\theta}\sinh(\pi\theta)+\frac{1}{\theta^2\sinh(\sigma\theta)}\bigg(\sinh(\pi\theta)\cosh((\sigma-\rho-\lambda)\theta)\\
&\quad-\frac{1}{2}\sinh((\pi+\sigma-\rho+\lambda)\theta)-\frac{1}{2}\sinh((\pi-\sigma+\rho-\lambda)\theta)\bigg)\\
&=\frac{2\lambda}{\theta}\sinh(\pi\theta)+\frac{\sinh(\pi\theta)}{\theta^2\sinh(\sigma\theta)}\bigg(\cosh((\sigma-\rho-\lambda)\theta)\\
&\quad-\cosh((\sigma-\rho+\lambda)\theta)\bigg).
\end{aligned}\tag{4.5.65}$$

Hence, using the identity (4.5.17) and the identity (4.5.15) twice, we have that

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_0^\infty r_0\,dr_0\int_0^\lambda d\phi\int_0^\rho d\phi_0\,G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t)\\
&=\frac{1}{2}\lambda R^2-S_2(t)+t\int_0^\infty d\theta\,\frac{\cosh(\frac{\pi}{2}\theta)\,(\cosh((\sigma-\rho-\lambda)\theta)-\cosh((\sigma-\rho+\lambda)\theta))}{2\sinh(\sigma\theta)\sinh(\frac{\pi}{2}\theta)}
\end{aligned}\tag{4.5.66}$$

where

$$\begin{aligned}
S_2(t)=\mathcal{L}^{-1}\bigg\{&\frac{1}{s}\int_R^\infty r\,dr\int_0^\infty d\theta\,K_{i\theta}(r\sqrt{s})\\
&\times\frac{\cosh\left(\frac{\pi}{2}\theta\right)(\cosh((\sigma-\rho-\lambda)\theta)-\cosh((\sigma-\rho+\lambda)\theta))}{\pi\theta\sinh(\sigma\theta)}\bigg\}(t).
\end{aligned}\tag{4.5.67}$$

Now observing that

$$\begin{aligned}
&\frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-\rho-\lambda)\theta)}{\pi\theta\sinh(\sigma\theta)} - \frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh((\sigma-\rho+\lambda)\theta)}{\pi\theta\sinh(\sigma\theta)} \\
&\qquad = \frac{\cosh((\frac{\pi}{2}+\sigma-\rho-\lambda)\theta)-1}{2\pi\theta\sinh(\sigma\theta)} + \frac{\cosh((\frac{\pi}{2}-\sigma+\rho+\lambda)\theta)-1}{2\pi\theta\sinh(\sigma\theta)} \\
&\qquad\quad - \frac{\cosh((\frac{\pi}{2}+\sigma-\rho+\lambda)\theta)-1}{2\pi\theta\sinh(\sigma\theta)} - \frac{\cosh((\frac{\pi}{2}-\sigma+\rho-\lambda)\theta)-1}{2\pi\theta\sinh(\sigma\theta)}
\end{aligned} \tag{4.5.68}$$

by using Lemmata 4.5.6(i) and 4.5.6(ii) we have $S_2(t) = O\left(te^{-R^2C_{\sigma,\rho,\lambda}/t}\right)$ where $C_{\sigma,\rho,\lambda} > 0$ is a constant depending only on σ, ρ and λ.

Now by the linearity of the inverse Laplace transform we have that

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_\lambda^\rho d\phi\int_0^\infty r_0\,dr_0\int_\lambda^\rho d\phi_0\, G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t) \\
&\quad = \mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_0^\rho d\phi\int_0^\infty r_0\,dr_0\int_0^\rho d\phi_0\, G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t) \\
&\quad + \mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_0^\lambda d\phi\int_0^\infty r_0\,dr_0\int_0^\lambda d\phi_0\, G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t) \\
&\quad - 2\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_0^\lambda d\phi\int_0^\infty r_0\,dr_0\int_0^\rho d\phi_0\, G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t) \\
&\quad = \frac{1}{2}(\rho-\lambda)R^2 - \frac{2R}{\pi^{1/2}}t^{1/2} + \widehat{c}(\sigma,\rho,\lambda)t \\
&\quad + \frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t} + O\left(te^{-R^2C_{\sigma,\rho,\lambda}/4t}\right).
\end{aligned} \tag{4.5.69}$$

Undoing the variable substitution one sees that, $\widehat{c}(\sigma,\rho,\lambda) = c(\gamma,\beta,\alpha)$ with c as defined in Theorem 4.2.1 and $\frac{1}{2}(\rho-\lambda)R^2 = \frac{1}{2}\gamma R^2 = V(S_R(v))$.

The case where β and α are interior angles of D follows on from this. Note that by Fubini's theorem we have that

$$\begin{aligned}
&\int_{(0,\lambda)\cup(\rho,\sigma)} d\phi\int_{(0,\lambda)\cup(\rho,\sigma)} d\phi_0\,\Phi_\sigma(\theta,\phi,\phi_0) \\
&\qquad = \int_0^\sigma d\phi\int_0^\sigma d\phi_0\,\Phi_\sigma(\theta,\phi,\phi_0) + \int_\lambda^\rho d\phi\int_\lambda^\rho d\phi_0\,\Phi_\sigma(\theta,\phi,\phi_0) \\
&\qquad - 2\int_\lambda^\rho d\phi\int_0^\sigma d\phi_0\,\Phi_\sigma(\theta,\phi,\phi_0).
\end{aligned} \tag{4.5.70}$$

We see from (4.5.65) that

$$\int_\lambda^\rho d\phi\int_0^\sigma d\phi_0\,\Phi_\sigma(\theta,\phi,\phi_0) = \frac{2(\rho-\lambda)}{\theta}\sinh(\pi\theta). \tag{4.5.71}$$

Hence by (4.5.17) we have

$$\mathcal{L}^{-1}\left\{2\int_0^R r\,dr\int_\lambda^\rho d\phi\int_0^\infty r_0\,dr_0\int_0^\sigma d\phi_0\,G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t)=(\rho-\lambda)R^2, \quad (4.5.72)$$

and by Lemma 4.5.3 we have

$$\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_0^\sigma d\phi\int_0^\infty r_0\,dr_0\int_0^\sigma d\phi_0\,G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t)=\frac{\sigma}{2}R^2. \quad (4.5.73)$$

Thus, we have

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_{(0,\lambda)\cup(\rho,\sigma)} d\phi\int_0^\infty r_0\,dr_0\int_{(0,\lambda)\cup(\rho,\sigma)} d\phi_0\,G_{W_\sigma}(s,r,\phi,r_0,\phi_0)\right\}(t)\\
&\quad=\frac{1}{2}(\sigma+\rho-\lambda-2(\rho-\lambda))R^2-\frac{2R}{\pi^{1/2}}t^{1/2}+\widehat{c}(\sigma,\rho,\lambda)t\\
&\quad\quad+\frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty\frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t}+O\left(te^{-R^2C_{\sigma,\rho,\lambda}/4t}\right)\\
&\quad=\frac{1}{2}(\beta+\alpha)R^2-\frac{2R}{\pi^{1/2}}t^{1/2}+c(\gamma,\beta,\alpha)t\\
&\quad\quad+\frac{2R}{\pi^{1/2}}t^{1/2}\int_1^\infty\frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t}+O\left(te^{-R^2C_{\gamma,\beta,\alpha}/4t}\right),
\end{aligned} \quad (4.5.74)$$

which completes the proof. □

4.6 Extensions of Theorem 4.2.1

In this section, we discuss some further extensions of Theorem 4.2.1. We did not include them in the original statement of the theorem to keep the exposition of the proof as simple as possible.

4.6.1 Vertices with an arbitrary number of incident edges

The computations in the previous section can be extended to vertices with an arbitrary number of incident edges in $\mathcal{E}$. The case of open vertices lying in Ω with an arbitrary number of incident edges which are all open is dealt with in [BerGit16] and the case for NOON vertices with four incident edges is answered in Theorem

4.5.9. Here we outline the case when the vertex lies on the Neumann boundary $\partial\Omega$. The notation used below is the same as that used in the proofs of Theorems 4.5.8 and 4.5.9.

We choose such a vertex $v \in \mathcal{V}$ with interior angle with respect to Ω denoted $\sigma \in (0, 2\pi)$. Translating v to the origin and rotating as necessary, choose $R > 0$ sufficiently small so that

$$B_R(0) \cap D = B_R(0) \cap \left(\bigcup_{i=1}^{k} W_{\lambda_i}^{\rho_i} \right) \tag{4.6.1}$$

where

$$W_{\lambda_i}^{\rho_i} := \{(r, \phi) : r > 0, \lambda_i < \phi < \rho_i\} \tag{4.6.2}$$

and $0 \leq \lambda_1 < \rho_1 < \lambda_2 < \rho_2 < \cdots < \lambda_k < \rho_k \leq \sigma$.

Then, as in Section 4.3.1, we set $S_R(v) = B_R(0) \cap D$ and we model the heat content contribution from $S_R(v)$ by

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \mathcal{L}^{-1} \left\{ \int_0^R r\, dr \int_{\lambda_i}^{\rho_i} d\phi \int_0^\infty r_0\, dr_0 \int_{\lambda_j}^{\rho_j} d\phi_0\, G_{W_\gamma}(s; r, \phi, r_0, \phi_0) \right\} (t), \tag{4.6.3}$$

which, splitting the second sum, is equal to

$$\begin{aligned} &= \sum_{i=1}^{k} \mathcal{L}^{-1} \left\{ \int_0^R r\, dr \int_{\lambda_i}^{\rho_i} d\phi \int_0^\infty r_0\, dr_0 \int_{\lambda_i}^{\rho_i} d\phi_0\, G_{W_\gamma}(s; r, \phi, r_0, \phi_0) \right\} (t) \\ &\quad + \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\ j \neq i}} \mathcal{L}^{-1} \left\{ \int_0^R r\, dr \int_{\lambda_i}^{\rho_i} d\phi \int_0^\infty r_0\, dr_0 \int_{\lambda_j}^{\rho_j} d\phi_0\, G_{W_\gamma}(s; r, \phi, r_0, \phi_0) \right\} (t). \end{aligned} \tag{4.6.4}$$

For $0 \leq \lambda < \rho \leq \sigma$, we define

$$\widehat{f}(\sigma, \rho, \lambda) := \begin{cases} 0, & \lambda = 0,\ \rho = \sigma, \\ \widehat{b}(\sigma, \rho), & \lambda = 0,\ \rho < \sigma, \\ \widehat{b}(\sigma, \sigma - \lambda), & \lambda > 0,\ \rho = \sigma, \\ \widehat{c}(\sigma, \rho, \lambda), & \lambda > 0,\ \rho < \sigma, \end{cases} \tag{4.6.5}$$

where $\widehat{b}$ and $\widehat{c}$ are defined in the statements of Theorems 4.5.8 and 4.5.9 respectively.

Then, from Lemma 4.5.3 and Theorems 4.5.8 and 4.5.9, we know that

$$\begin{aligned}
&\int_0^R r\,dr \int_\lambda^\rho d\phi \int_0^\infty r_0\,dr_0 \int_\lambda^\rho d\phi_0\, \eta_{W_\sigma}(t;r,\phi,r_0,\phi_0) \\
&\quad = \frac{1}{2}(\rho-\lambda)R^2 - \frac{(2-\mathbb{1}_{\{0\}}(\lambda)-\mathbb{1}_{\{\sigma\}}(\rho))R}{\pi^{1/2}}t^{1/2} + \widehat{f}(\sigma,\rho,\lambda)t \\
&\qquad + (2-\mathbb{1}_{\{0\}}(\lambda)-\mathbb{1}_{\{\sigma\}}(\rho))\frac{R}{\pi^{1/2}}t^{1/2}\int_1^\infty \frac{dw}{w^2}\int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}}e^{-R^2y^2w^2/4t} \\
&\qquad + O(e^{-C/t}),
\end{aligned} \tag{4.6.6}$$

for some constant $C > 0$ depending on R, σ, ρ, and λ (note that when $\lambda = 0$ and $\rho = \sigma$ we consider a Neumann vertex so there is no remainder as in Lemma 4.5.3).

We now come to deal with the cross terms which conveniently are not too difficult to deal with from our previous work.

Suppose, without loss of generality, that $0 \leq \lambda < \rho < \lambda' < \rho' \leq \sigma$. By Fubini's theorem we observe that

$$\begin{aligned}
\int_\lambda^\rho d\phi \int_{\lambda'}^{\rho'} d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) &= \int_0^\rho d\phi \int_0^{\rho'} d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) + \int_0^\lambda d\phi \int_0^{\lambda'} d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) \\
&\quad - \int_0^\lambda d\phi \int_0^{\rho'} d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) - \int_0^\rho d\phi \int_0^{\lambda'} d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0).
\end{aligned} \tag{4.6.7}$$

Using (4.5.65) on each of these four double integrals, we see that

$$\begin{aligned}
\int_\lambda^\rho d\phi \int_{\lambda'}^{\rho'} d\phi_0\, \Phi_\sigma(\theta,\phi,\phi_0) &= \frac{\sinh(\pi\theta)}{\theta^2\sinh(\sigma\theta)}\Big(\cosh((\sigma-\rho'-\rho)\theta) - \cosh((\sigma-\rho'+\rho)\theta) \\
&\qquad + \cosh((\sigma-\lambda'-\lambda)\theta) - \cosh((\sigma-\lambda'+\lambda)\theta) \\
&\qquad - \cosh((\sigma-\rho'-\lambda)\theta) + \cosh((\sigma-\rho'+\lambda)\theta) \\
&\qquad - \cosh((\sigma-\lambda'-\rho)\theta) + \cosh((\sigma-\lambda'+\rho)\theta)\Big) \\
&=: \frac{\sinh(\pi\theta)}{\theta^2\sinh(\sigma\theta)}\widehat{g}(\theta;\sigma,\rho,\lambda,\rho',\lambda').
\end{aligned} \tag{4.6.8}$$

Using similar arguments as in the proof of Theorem 4.5.9 and repeated use of Lemma

4.5.6, one can deduce that

$$\mathcal{L}^{-1}\left\{\int_0^R r\,dr\int_\lambda^\rho d\phi\int_0^\infty r_0\,dr_0\int_{\lambda'}^{\rho'} d\phi_0\, G_{W_\gamma}(s;r,\phi,r_0,\phi_0)\right\}(t)$$
$$= \int_0^\infty d\theta\,\frac{\cosh(\frac{\pi}{2}\theta)\widehat{g}(\theta;\sigma,\rho,\lambda,\rho',\lambda')}{2\sinh(\sigma\theta)\sinh(\frac{\pi}{2}\theta)} + O(e^{-C'/t}) \tag{4.6.9}$$
$$=: \widehat{h}(\sigma,\rho,\lambda,\rho',\lambda') + O(e^{-C'/t}),$$

where $C' > 0$ is a constant depending on R and the angles $\sigma, \lambda, \rho, \lambda', \rho'$.

Using (4.6.6) and (4.6.9), we see that the model heat content contribution of $S_R(v)$ is

$$\begin{aligned} V(S_R(v)) &- \frac{(2k - \mathbb{1}_{\{0\}}(\lambda_1) - \mathbb{1}_{\{\sigma\}}(\rho_k))R}{\pi^{1/2}} t^{1/2} \\ &+ \left(\sum_{i=1}^k \widehat{f}(\sigma,\rho_i,\lambda_i) + \sum_{i=1}^k \sum_{\substack{1\le j\le k\\ j\ne i}} \widehat{h}(\sigma,\rho_i,\lambda_i,\rho_j,\lambda_j)\right) t \\ &+ (2k - \mathbb{1}_{\{0\}}(\lambda_1) - \mathbb{1}_{\{\sigma\}}(\rho_k))\frac{R}{\pi^{1/2}} t^{1/2} \int_1^\infty \frac{dw}{w^2}\int_0^1 dy\,\frac{y}{(1-y^2)^{1/2}} e^{-R^2y^2w^2/4t} \\ &+ O(e^{-C''/t}) \end{aligned} \tag{4.6.10}$$

for some constant $C'' > 0$ depending on R and the angles $\sigma, \lambda_1, \rho_1, \ldots, \lambda_k, \rho_k$.

To extend Theorem 4.2.1 to include vertices with an arbitrary number of incident edges in $\mathcal{E}$, one would need to construct a partition analogously to one required to prove Theorem 4.2.1.

We remark that we believe these ideas can be used for such wedges where instead of Neumann boundary conditions we have Dirichlet boundary conditions or mixed Dirichlet-Neumann boundary conditions. This would give rise to analogues of Theorem 4.2.1 in the Dirichlet-Open-Open-Dirichlet (DOOD) case, the Dirichlet-Open-Open-Neumann (DOON) case, and so on. It would suffice to do it only in the Dirichlet case as the mixed Dirichlet-Neumann case may be obtained by the method of images.

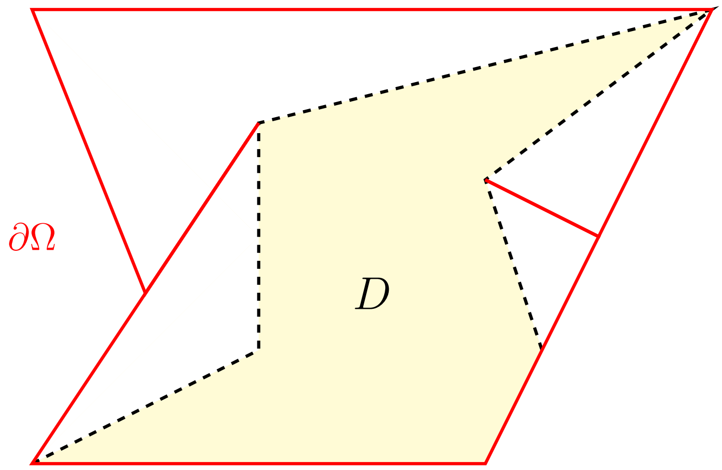

Figure 4.6: An example setup for our problem when one has interior angles of angle 2π as a modification of Figure 4.1.

4.6.2 Interior angles of angle 2π

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain as before but now suppose we have an interior angle with angle 2π, see for example Figure 4.6. All the model computations for a vertex of D lying at a vertex of Ω with interior angle 2π carry over immediately without any need for further calculations. The only moot point is whether the locality principles still hold.

One may construct the Neumann heat kernel on such polygonal domains via the same construction as given in Section 2.3.2 and from this one can determine the locality principles in the same way as before. The construction relies on viewing Ω in terms of the manifold metric

$$\widehat{d}(x,y) := \inf\{\ell(\gamma) : \gamma \text{ is piecewise } C^1 \text{ from } x \text{ to } y\} \tag{4.6.11}$$

on Ω where $\ell(\gamma)$ denotes the length of γ. The topology induced on Ω by this metric is precisely the Euclidean topology however the completion $\widehat{\Omega}$ of Ω in this metric is not a subset of $\mathbb{R}^2$, rather an abstract space, or more conveniently a manifold. We have a new boundary $\partial\widehat{\Omega} := \widehat{\Omega} \setminus \Omega$ for which the normal derivative $\widehat{n}$ on the boundary can be defined almost everywhere in the obvious way. The heat equation with Neumann boundary conditions and initial datum $f \in L^2(\Omega)$ then becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}(t;x) = \Delta u(t;x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \widehat{n}}(t;x) = 0, & t > 0,\ x \in \partial\widehat{\Omega} \text{ a.e.}, \\ \lim_{t\downarrow 0} \|u(t;\cdot) - f(x)\|_{L^2(\Omega)} = 0. \end{cases} \tag{4.6.12}$$

One may use the classical energy method to determine the uniqueness of solutions to the Neumann heat equation via a generalised Gauss-Green formula due to Chen, see [Chn93, Thm 4.5], in this case.

One may also wish Ω to be unbounded, and possibly also have interior angles of 2π. The same manifold construction of the Neumann heat kernel works in this case and one has all the same model computations. For more information on Neumann heat kernels and reflecting Brownian motion in such cases we refer the reader to [Chn93] as one requires much more probabilistic machinery than we mention in this thesis. A remark in this direction is also given in Appendix B of [FarGit23].

4.6.3 Zaremba polygons

The methods presented in this thesis can also be used to compute small-time heat content asymptotics for polygons with Zaremba boundary conditions. Let D be a bounded polygonal subdomain and $\Gamma \subset \partial D$ be the union of some of the edges of ∂D. Let $u_D^\Gamma(t;x)$ be the solution of

$$\begin{cases} \dfrac{\partial}{\partial t} u(t;x) = \Delta u(t;x), & t \in (0,+\infty),\, x \in D, \\ u(t;x) = 0, & t \in (0,+\infty),\, x \in \Gamma, \\ \dfrac{\partial}{\partial n} u(t;x) = 0, & t \in (0,+\infty),\, x \in \partial D \backslash \Gamma \\ \lim_{t\downarrow 0} \|u(t;\cdot) - \mathbb{1}_D\|_{L^2(D)} = 0, \end{cases} \tag{4.6.13}$$

and define the Zaremba heat content as

$$Q_D^\Gamma(t) := \int_D \mathrm{d}x\, u_D^\Gamma(t;x). \tag{4.6.14}$$

Recall the function

$$f(\alpha) := \int_0^\infty d\theta \, \frac{4\sinh\left((\pi - \alpha)\,\theta\right)}{\sinh\left(\pi\theta\right)\cosh\left(\alpha\theta\right)}. \tag{4.6.15}$$

Theorem 4.6.1. *We have the following small-time asymptotic expansion for the Zaremba heat content of a polygon:*

$$\begin{aligned} Q_D^\Gamma(t) = V(D) - 2\mathcal{H}^1(\Gamma)\frac{t^{1/2}}{\pi^{1/2}} + \left(\sum_{\alpha\in\mathcal{A}_{DD}} f(\alpha) + \frac{1}{2}\sum_{\alpha\in\mathcal{A}_{DN}} f(2\alpha) \right) t \\ + O(e^{-C_D/t}), \end{aligned} \tag{4.6.16}$$

where: $\mathcal{A}_{DD}$ is the collection of interior angles of D where two Dirichlet edges meet; $\mathcal{A}_{DN}$ is the collection of interior angles of D where a Neumann and a Dirichlet edge meet; and, $C_D > 0$ is a constant depending on D.

Proof. The proof is a hybridisation of that of van den Berg and Srisatkunarajah in [BerSri90] and the methods presented in this thesis. We outline the proof of the result here but omit most of the details to avoid any repetition of previous ideas.

Firstly, we may construct a partition of D as described in [BerSri90] that allows us to divide D into regions where we can make justifiable model computations.

Secondly, the model computations are the obvious ones, e.g. an infinite wedge with Dirichlet boundary conditions for vertices where two Dirichlet edges meet. The only one that is different to any presented either here or in [BerSri90] is the Dirichlet-Neumann wedge calculation. However, by the method of images the heat content of a Dirichlet-Neumann wedge of angle $0 < \alpha < 2\pi$ is the precisely half that of the Dirichlet-Dirichlet wedges with angle 2α. The only moot point is that, of course, $0 < 2\alpha < 4\pi$ and so the Dirichlet-Dirichlet wedge under consideration may not be a subdomain of $\mathbb{R}^2$. But this is not a problem at all! And, one may obtain the calculation for such a wedge in exactly the same way as presented in [BerSri90].

Finally, the justification of the model computations comes from an analogous proof the locality principles in Section 4.4 in this case. Namely, one may consider the Dirichlet heat kernel $q_N(t; x, y)$ on a submanifold $N \subset M$, where M is the manifold generated by D, where N is given by copies of D obtained by reflecting only over sides not in Γ. Then, similarly to the Neumann case, the Zaremba heat kernel $q_D^\Gamma(t;x,y)$, with the appropriate mixed boundary conditions, for D may be obtained as

$$q_D^\Gamma(t;x,y) = \sum_{z\in\Psi_M^{-1}(\{y\})} q_N(t;x,z) \tag{4.6.17}$$

in the same way as in the paper [GalMcK72]. Arguing as in the proof of Lemma 4.4.1 one obtains the necessary locality principles.

Important to note is that the stochastic process $(X_t)_{t\geq 0}$ associated with $q_D^\Gamma(t;x,y)$ is a reflecting Brownian motion on D that is killed on Γ, i.e. the governing law is

$$\mathbb{P}_x^D(X_t \in A, \tau_\Gamma > t) = \int_D \mathrm{d}y\, q_D^\Gamma(t;x,y), \tag{4.6.18}$$

where

$$\tau_\Gamma := \inf\{t \geq 0 : X_t \in \Gamma\}. \tag{4.6.19}$$

This is the subtle difference for proving the desired locality principles in this case but the proof is effectively entirely the same. □

Chapter 5

Conclusions and further directions

In this thesis, we have considered the asymptotic behaviour of minimisers to spectral shape optimisation problems and small-time heat-flow asymptotics for the Neumann Laplacian. In particular, we built upon previously known results for the Dirichlet Laplacian and considered problems relating to Zaremba Laplacians in both cases.

We now give an extended discussion of possible future directions for research arising from the topics discussed in this thesis, and give some examples for a few of these directions.

5.1 Spectral shape optimisation

We begin by stating some spectral shape optimisation problems that are natural extensions to the ones we considered in Chapter 3. First we recall Remark 3.3.2, which concerns the asymptotic behaviour of minimisers to the problem

$$\inf\left\{\lambda_k^\beta(\Omega) : \Omega \subset \mathbb{R}^d \text{ convex, } P(\Omega) = 1\right\} \tag{5.1.1}$$

as $k \to +\infty$ when $d \geq 3$. We proved that minimisers do indeed exist for all $k \geq 1$. In the $\beta = 0$ case, i.e. the Neumann case, one does not have existence of minimisers for any $k \geq 2$ and in the $\beta = +\infty$ case, i.e. the Dirichlet case, we have existence for minimisers for all $k \geq 1$ and the minimisers $\sim$-Hausdorff converge to the ball of unit perimeter as $k \to +\infty$.

In the case of rectangles and volume constraint, Freitas and Kennedy showed [FreKen21] that despite the existence of minimisers for all $k \geq 1$ to the problem

$$\inf \left\{ \lambda_k^\beta(\Omega) : \Omega \subset \mathbb{R}^2 \text{ rectangle}, \ V(\Omega) = 1 \right\}, \tag{5.1.2}$$

the diameter of the minimisers diverges as $k \to +\infty$. So although our existence is the same as in the Dirichlet case, we have the degeneracy of the Neumann case appearing as $k \to +\infty$.

In dimension three and higher, for bounded convex domains, a bound on perimeter does not imply a bound on diameter. Hence, it would be interesting to know if the minimisers for Robin eigenvalues in the case of perimeter constraint do indeed diverge in diameter or not. A study in the case of cuboids in dimension three following the ideas in [FreKen21] would be a good start in this direction.

Also, as alluded to earlier in the thesis, the minimisation problem

$$\inf \left\{ \lambda_k(\Omega) : \Omega \subset \mathbb{R}^d, \ P(\Omega) = 1 \right\}, \tag{5.1.3}$$

where we drop the convexity condition, has a minimiser but the asymptotic behaviour of these minimisers is unknown, see Remark 1.3 in [BucFre13]. The natural conjecture to make is that the minimisers should $\sim$-Hausdorff converge to the ball of unit perimeter as $k \to +\infty$. Indeed, this result is known to hold for the average of the first k Dirichlet eigenvalues, see [Fre17].

We note that if one replaces the perimeter constraint with a diameter constraint, we settled this problem with a mild extension of the work of van den Berg in [Ber15], see Theorem 3.2.1.

5.2 Shape optimisation of heat content

In an attempt to unify the work on the topics of heat content and shape optimisation, which were fairly distinct in this thesis, we now discuss shape optimisation for heat content.

There is over 50 years of mathematical literature on the topic, but it is somewhat sparsely spread out and usually comes as a consequence of much more general inequalities. We discuss how these problems are potentially of interest in their own right.

5.2.1 Luttinger's inequality

For a measurable set $E \subset \mathbb{R}^d$, we define its Schwarz rearrangement as the ball E^* with the same volume as E centred at the origin. For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define it Schwarz rearrangement (also known as the symmetric decreasing rearrangement), by

$$f^*(x) := \int_0^{+\infty} \mathrm{d}t\, \mathbb{1}_{\{y\in\mathbb{R}^d : f(y)>t\}^*}(x). \tag{5.2.1}$$

The well-known Riesz rearrangement inequality states that

$$\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, f(y)k(x-y)g(y) \leq \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, f^*(x)k(x-y)g^*(y) \tag{5.2.2}$$

for any $f, g : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ measurable and $k : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ a radially symmetric decreasing function. For the open heat content, as observed in [Pre04], the Riesz inequality immediately implies that

$$\begin{aligned} H_E(t) &= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, \mathbb{1}_E(x) p_{\mathbb{R}^2}(t;x,y) \mathbb{1}_E(y) \\ &\leq \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, \mathbb{1}_{E^*}(x) p_{\mathbb{R}^2}(t;x,y) \mathbb{1}_{E^*}(y) \\ &= H_{E^*}(t) \end{aligned} \tag{5.2.3}$$

for all $t \geq 0$. That is to say, the ball maximises the open heat content over all measurable sets of a given volume for all $t \geq 0$. As alluded to in the introduction of this thesis, if you combine this result with the work in [MirPPP07], one obtains a proof of the isoperimetric inequality for measurable sets.

For a bounded open set $\Omega \subset \mathbb{R}^d$, we define its Dirichlet heat content by

$$Q_\Omega(t) := \int_\Omega \mathrm{d}x \int_\Omega \mathrm{d}y \, q_\Omega(t; x, y), \tag{5.2.4}$$

where $q_\Omega(t; x, y)$ is the Dirichlet heat kernel on Ω. Using a generalised version of the Riesz inequality, an inequality due to Friedberg and Luttinger proven in [FriLut76], and representing the Dirichlet heat semigroup via a Trotter product formula, Luttinger in [Lut73] gives a proof that

$$Q_\Omega(t) \leq Q_{\Omega^*}(t) \tag{5.2.5}$$

for all $t \geq 0$. That is to say, the ball maximises the Dirichlet heat content over bounded open sets of a given volume for all $t \geq 0$. Noting that

$$\lambda_1(\Omega) = \lim_{t \to +\infty} -t^{-1} \log Q_\Omega(t) \tag{5.2.6}$$

one immediately obtains the Faber-Krahn inequality. Moreover, if one takes the first moment of the Dirichlet heat content you get the torsional rigidity $T(\Omega)$ of Ω, i.e.

$$T(\Omega) = \int_0^\infty Q_\Omega(t). \tag{5.2.7}$$

Thus, Luttinger's inequality also implies the Saint-Venant inequality

$$T(\Omega) \leq T(\Omega^*), \tag{5.2.8}$$

which was known earlier to be true by the work of Pólya in [Pól48]. We should digress that conventionally the torsional rigidity is defined as the L^1 norm over Ω of the weak solution to

$$\begin{cases} -\Delta u(x) = 1, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{5.2.9}$$

Given that u may obtained via the resolvent formula

$$u(x) = \int_\Omega \mathrm{d}x \int_0^\infty \mathrm{d}t \, q_\Omega(t; x, y), \tag{5.2.10}$$

one can immediately see these two definitions of torsional rigidity are equivalent by Tonelli's theorem.

Although the author is not aware of any explicit reference in the literature, as an immediate corollary of Luttinger's methods one may prove an analogue of (5.2.5) for triangles and quadrilaterals via Steiner symmetrisation. That is, the equilateral triangle maximises the Dirichlet heat content amongst all triangles of the same area for all $t \geq 0$ and the square maximises the Dirichlet heat content among all quadrilaterals of the same area for all $t \geq 0$. Hence, one may obtain a proof of the analogues of the Faber-Krahn and Saint-Venant inequalities for triangles and quadrilaterals.

Some recent attempts in the direction of proving the analogous inequality to (5.2.5) for the class of N-gons using these heat flow ideas was done by Bogosel, Bucur and Fragalà in [BogBF24]. This would resolve the Pólya-Szegő conjecture for the first Dirichlet eigenvalue, see [PólSze51, p. 159], which states that the regular N-gon minimises the first Dirichlet eigenvalue among all N-gons of a given area.

One can go further and obtain more inequalities regarding the exit times of Brownian motion from a domain using the ideas of Luttinger, see for example [McD13, §2], and prove analogous results to those of (5.2.5) on spheres and hyperbolic space, see [BurSch01]. We also remark that shape optimisation for the Dirichlet heat content with obstacles was recently done by Li in [Li22] and other similar problems for the open heat content were studied in the PhD thesis of Preunkert [Pre04].

We now turn our attention to the Robin heat content with positive parameter $\beta \in (0, +\infty)$ and give a conjecture in this direction. In this case, the heat content of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ is given by

$$Q_\Omega^\beta(t) := \int_\Omega \mathrm{d}x \int_\Omega \mathrm{d}y \, q_\Omega^\beta(t; x, y), \tag{5.2.11}$$

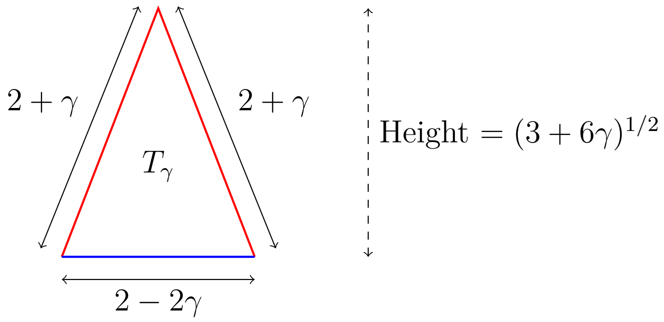

Figure 5.1: Visualisation of the triangle T_γ with the Dirichlet boundary conditions in blue and the Neumann boundary conditions in red.

where $q_\Omega^\beta(t;x,y)$ is the associated Robin heat kernel on Ω. Given that the Faber-Krahn and Saint-Venant inequalities hold for the Robin Laplacian with positive parameter, see [Bos86; Dan06; BucGia15], it is natural to make the following conjecture for the Robin heat content.

Conjecture 5.2.1. *Fix $\beta \in (0,+\infty)$. For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, for all $t \geq 0$*

$$Q_\Omega^\beta(t) \leq Q_{\Omega^*}^\beta(t). \tag{5.2.12}$$

5.2.2 Zaremba heat content

All the aforementioned heat content optimisation problems have static solutions in the sense that one has the same maximiser for the heat content for all time $t \geq 0$. Under mixed Dirichlet-Neumann boundary conditions this not necessarily the case and equally one may not have existence at all.

For $-1/2 < \gamma < 1$, let T_γ the isosceles triangle with base length $\ell_\gamma = 2 - 2\gamma$ and other sides of length $2 + \gamma$. By construction we see that $P(T_\gamma) = 6$. We consider the heat content of T_γ, denoted by $Q_\gamma(t)$, with Dirichlet boundary conditions on the base and Neumann boundary conditions on the other two sides, see Section 4.6.3 for more information on the definition of this heat content and see Figure 5.1 for a visualisation of T_γ and the Zaremba boundary conditions imposed. For a fixed $t \geq 0$, we consider the maximisation problem

$$\sup\{Q_\gamma(t) : -1/2 < \gamma < 1\}. \tag{5.2.13}$$

We recall that for γ fixed, one may show that $Q_\gamma(t)$ is strictly convex and strictly decreasing in the same manner to that of the heat content problems described in Chapter 4.

Proposition 5.2.2. *For all $t \geq 0$, the maximisation problem* (5.2.13) *has a maximiser $T_{\gamma_t^*}$. Moreover, we may choose the maximisers such that $(\gamma_t^*)_{t\geq 0}$ is continuous as a function $[0,+\infty) \to (-1/2,1)$ and letting $f(t) = Q_{T_{\gamma_t^*}}(t)$ we have that $f : [0,+\infty) \to (0,+\infty)$ is strictly decreasing and convex, and* $\lim_{t\to+\infty} f(t) = 0$.

Proof. As in the Neumann case in Chapter 4, we may express $Q_\gamma(t)$ as

$$Q_\gamma(t) = \sum_{j=1}^{\infty} e^{-\zeta_j(T_\gamma)t} \left| \left\langle \mathbb{1}_{T_\gamma}, v_j^{(\gamma)} \right\rangle_{L^2(T_\gamma)} \right|^2, \tag{5.2.14}$$

where the $\zeta_j(T_\gamma)$ are the Zaremba eigenvalues of T_γ and the $v_j^{(\gamma)}$ are the associated $L^2(\Omega)$-orthonormalised basis of eigenfunctions. From Bessel's inequality we may obtain

$$\begin{aligned} Q_{T_\gamma}(t) &= \sum_{j=1}^{\infty} e^{-\zeta_j(T_\gamma)t} \left| \left\langle \mathbb{1}_{T_\gamma}, v_j^{(\gamma)} \right\rangle_{L^2(T_\gamma)} \right|^2 \\ &\leq e^{-\zeta_1(T_\gamma)t} \sum_{j=1}^{\infty} \left| \left\langle \mathbb{1}_{T_\gamma}, v_j^{(\gamma)} \right\rangle_{L^2(T_\gamma)} \right|^2 \\ &\leq V(T_\gamma) e^{-\zeta_1(T_\gamma)t}. \end{aligned} \tag{5.2.15}$$

Let $Q^* := Q_{T_0}(t)$, and $a_t < 0 < b_t$ such that $V(T_{a_t}) = V(T_{b_t}) = Q^*$, then we must have any maximiser necessarily lies in $[a_t, b_t] \subset (-1/2, 1)$.

For each $t \geq 0$, one may find $M_t > 0$ such that $T_\gamma \in \mathcal{A}^2(M_t)$ for all $\gamma \in [a_t, b_t]$, where $\mathcal{A}^2(\cdot)$ is the space defined at the start of Section 3.3.2. In fact such an $M > 0$ exists for any compact subset of $(-1/2, 1)$. Thus, combining the proof of Lemma 3.3.13, in which we prove the continuity of the associated resolvents, and the First Trotter-Kato Approximation Theorem, see [EngNag00, Thm 4.8], one gets the continuity of the associated heat semigroups and thus of the heat contents with respect to γ for $t \geq 0$ fixed.

Since $[a_t, b_t]$ is compact and $Q_{T_\gamma}(t)$ is continuous with respect to γ for $t \geq 0$ fixed, a maximiser exists for all $t \geq 0$ by the extreme value theorem.

The continuity of the path $(\gamma_t^*)_{t\geq 0}$ comes immediately from the continuity of the heat contents $Q_\gamma(t)$ with respect to γ. One argue this by contradiction.

$f(t)$ is convex as it is the supremum of convex functions and the fact that it is strictly decreasing condition comes directly from

$$f(t_2) = Q_{\gamma_{t_2}^*}(t_2) < Q_{\gamma_{t_2}^*}(t_1) \leq Q_{\gamma_{t_1}^*}(t_1) = f(t_1) \tag{5.2.16}$$

for any $0 \leq t_1 < t_2$. The limit $f(t) \to 0$ as $t \to +\infty$ is self-evident. □

For $t = 0$, the maximiser is simply T_0, i.e. $\gamma_t^* = 0$, by the isoperimetric inequality for triangles. However, we may now use our small-time heat content asymptotics for Zaremba polygons in Section 4.6.3 to deduce that for $t > 0$ small, the equilateral triangle is not a minimiser.

Proposition 5.2.3. *We have that*

$$\gamma_t^* = \frac{8t^{1/2}}{9\pi^{1/2}} + o(t^{1/2}) \tag{5.2.17}$$

as $t \downarrow 0$.

Proof. From Theorem 4.6.1, we may deduce that

$$Q_{T_\gamma}(t) = (1-\gamma)(3+6\gamma)^{1/2} - (4-4\gamma)\frac{t^{1/2}}{\pi^{1/2}} + O_\gamma(t), \tag{5.2.18}$$

as $t \downarrow 0$, where the remainder can be taken to be uniform for $|\gamma|$ small. Taking the derivative of this small-time expansion with respect to γ yields

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} Q_{T_\gamma}(t) = -\frac{9\gamma}{2(3+6\gamma)^{1/2}} + \frac{4t^{1/2}}{\pi^{1/2}} + O_\gamma(t). \tag{5.2.19}$$

Noting that

$$\frac{9\gamma}{2(3+6\gamma)^{1/2}} = \frac{9}{2}\gamma + O(\gamma^2) \tag{5.2.20}$$

and any maximum of the heat content has $\frac{\mathrm{d}}{\mathrm{d}\gamma} Q_{T_\gamma}(t) = 0$, we may deduce

$$\gamma_t^* = \frac{8t^{1/2}}{9\pi^{1/2}} + o(t^{1/2}) \tag{5.2.21}$$

as desired. □

The question now arises as to what happens to γ_t^* for $t \geq 0$ large? We can answer this as a consequence of the following lemma.

Lemma 5.2.4. *Denoting the first Zaremba eigenvalue of T_γ under the aforementioned mixed boundary conditions by $\zeta_1(T_\gamma)$, we have*

$$\frac{\lambda_1(\mathbb{B}^2)}{(2+\gamma)^2} \leq \zeta_1(T_\gamma) \leq \frac{\lambda_1(\mathbb{B}^2)}{3+6\gamma}. \tag{5.2.22}$$

In particular, $\zeta_1(T_\gamma) \downarrow \frac{1}{9}\lambda_1(\mathbb{B}^2)$ as $\gamma \uparrow 1$ and the minimisation problem

$$\inf\{\zeta_1(T_\gamma) : -1/2 < \gamma < 1\} \tag{5.2.23}$$

does not have a minimiser.

Proof. The proof is straightforward. By monotonicity considerations, $\zeta_1(T_\gamma)$ is greater than the first Zaremba eigenvalue $\widehat{\zeta}_1(S_\gamma)$ of the sector

$$S_\gamma := \left\{(r,\phi) : 0 < r < 2+\gamma,\ 0 < \phi < 2\arcsin\left(\frac{1-\gamma}{2+\gamma}\right)\right\} \tag{5.2.24}$$

with Dirichlet boundary conditions on the arc and Neumann boundary conditions on the straight edges. By a separation of variables argument, one may deduce that

$$\widehat{\zeta}_1(S_\gamma) = (2+\gamma)^{-2}\lambda_1(\mathbb{B}^2). \tag{5.2.25}$$

For the other inequality, we define the sector

$$S'_\gamma := \left\{(r,\phi) : 0 < r < (3+6\gamma)^{1/2},\ 0 < \phi < 2\arcsin\left(\frac{1-\gamma}{2+\gamma}\right)\right\}, \tag{5.2.26}$$

for which we can deduce, again by separation of variables and monotonicity, that

$$\zeta_1(T_\gamma) \leq \widehat{\zeta}_1(S'_\gamma) = \frac{\lambda_1(\mathbb{B}^2)}{3+6\gamma}, \tag{5.2.27}$$

which completes the proof. □

Corollary 5.2.5. *We have that $\gamma_t^* \to 1$ as $t \to +\infty$.*

Proof. Suppose $\gamma_t^* \in (-1/2, 1-\delta)$ for all $t \geq 0$ for some $0 < \delta < 3/2$. Using (5.2.15) and our assertion on the γ_t^*,

$$Q_{\gamma_t^*}(t) \leq V(T_\gamma)e^{-(\lambda_1(\mathbb{B}^2)t)/(3-\delta)^2} \leq \sqrt{3}e^{-(\lambda_1(\mathbb{B}^2)t)/(3-\delta)^2} \tag{5.2.28}$$

Moreover, from letting $\gamma' = 1-\epsilon$ we see

$$Q_{\gamma'}(t) \asymp e^{-\zeta_1(T_{\gamma'})t} \leq e^{-(\lambda_1(\mathbb{B}^2)t)/(9-\epsilon)}. \tag{5.2.29}$$

So taking $0 < \epsilon < 3/2$ sufficiently small, to be precise $\epsilon < 6\delta - \delta^2$,

$$Q_{\gamma_t^*}(t) < Q_{\gamma'}(t) \tag{5.2.30}$$

for t sufficiently large. Thus, we must have $\gamma_t^* \in (1-\delta, 1)$ for t sufficiently large, which completes the proof as $0 < \delta < 3/2$ was arbitrary. □

It is reasonable to conceive that the sequence $(\gamma_t^*)_{t\geq 0}$ should in fact be monotone, i.e, $0 \leq \gamma_{t_1} \leq \gamma_{t_2} \leq 1$ for all $0 \leq t_1 \leq t_2$. And indeed, we know this to be true in small time.

An interesting question is: which γ maximises the torsional rigidity

$$\int_0^\infty Q_\gamma(t)\,\mathrm{d}t \tag{5.2.31}$$

in this case? Also, we could ask: do different γ maximise other moments of the heat content, i.e. what about maximising

$$\int_0^\infty t^{s-1}Q_\gamma(t)\,\mathrm{d}t \tag{5.2.32}$$

for $0 < s < +\infty$? And what happens to such maximisers as $s \to +\infty$? Such moments of the heat content have meaningful probabilistic interpretations in terms of expectation of powers of exit times, see [McD13, §2] for more information on this. We leave these as open problems here.

The moral of the story is the optimal solutions $T_{\gamma_t^*}$ (may chosen so that they) continuously flow from the solution to the isoperimetric problem to the degenerate limit attaining the infimum to the minimisation problem of the first Zaremba eigenvalue, possibly flowing via maximisers of (5.2.32). And therefore shape optimisation problems for heat content marries up various well-known shape optimisation problems.

We briefly remark that if we instead consider the case where we have the same arrangement of Zaremba boundary conditions but insist that $V(T_\gamma) = 1$ rather than $P(T_\gamma) = 6$, the maximisation problem is ill-posed. This can be seen by taking the Neumann sides to be arbitrarily long and carrying out some similar computations to the ones in Lemma 5.2.4.

A boundary obstacle problem

To end this thesis, we give a suggestion of a further shape optimisation problem concerning Zaremba heat contents. This problem arose on a walk with Professor Michiel van den Berg through Québec city whilst at a conference in 2023.

Given a fixed smooth connected domain $\Omega \subset \mathbb{R}^2$, a fixed $0 < \ell < P(\Omega)$ and $t \geq 0$ fixed, which connected relatively open set $\Gamma_t^* \subset \partial\Omega$ with $\mathcal{H}^1(\Gamma_t^*) = \ell$ maximises the Zaremba heat content of $Q_\Omega^{\Gamma_t}(t)$, defined analogously to (4.6.14), among all such Γ_t?

Of course if Ω is a ball, then this problem is trivial but the ball is not the interesting case due to its incredibly high symmetry. Instead, if Ω is an ellipse, then we should get an interplay between where Γ_t^* is located and the curvature of $\partial\Omega$. Heuristically, the placement of Γ_t^* tells us which connected part of $\partial\Omega$ of a given length ℓ kills (on average) a reflecting Brownian motion on Ω the least over the time interval $[0, t]$.

Appendix A

Python code for computations

In this appendix we provide the Python code used to create Table 3.1 and Figure 3.2 and give a mathematical outline for how the code works in each case.

A.1 Estimating upper bounds for N_d

We now give the Python code and outline of how the estimates of N_d in Table 3.1 were obtained.

If we take $n = 1$ in the Neumann eigenvalue counting function bound from Proposition 3.2.3, then we have $\mu_2^* = \pi^2$ and $\kappa_1 = \lceil d^{1/2} \rceil$ and the bound reads

$$\begin{aligned} \mathcal{N}_\Omega^N(\alpha) \leq \frac{V(\Omega)}{\pi^d}\alpha^{d/2} + \left(\frac{\lceil d^{1/2} \rceil}{\pi}\right)^{d-1} \left(2\lceil d^{1/2} \rceil + 3\right) d^{1/2} P(\Omega) \\ + \sum_{j=1}^{d-1} \binom{d}{j} (4d)^{j/2} \left(\frac{\lceil d^{1/2} \rceil}{\pi}\right)^{d-j} W_j(\Omega)\alpha^{(d-j)/2} + (4d)^{d/2}\omega_d \end{aligned} \tag{A.1.1}$$

for any bounded convex domain $\Omega \subset \mathbb{R}^d$. Under the additional imposition that $D(\Omega) = 1$, we see that Ω, up to a suitable translation, may be contained in the unit ball $\mathbb{B}^d$ of diameter two. By the monotonicity of quermassintegrals and that

$P(\mathbb{B}^d) = d\omega_d$ and $W_j(\mathbb{B}^d) = \omega_d$ for all j, the bound becomes

$$\begin{aligned} \mathcal{N}_\Omega^N(\alpha) \leq \frac{V(\Omega)}{\pi^d}\alpha^{d/2} + \omega_d \Bigg[& \underbrace{\left(\frac{\lceil d^{1/2} \rceil}{\pi}\right)^{d-1} \left(2\lceil d^{1/2}\rceil + 3\right) d^{3/2}}_{=:r_1(\alpha)} \\ & + \underbrace{\sum_{j=2}^{d} \binom{d}{j} (4d)^{j/2} \left(\frac{\lceil d^{1/2} \rceil}{\pi}\right)^{d-j} \alpha^{(d-j)/2}}_{=:r_2(\alpha)} \Bigg]. \end{aligned} \tag{A.1.2}$$

The Neumann eigenvalue counting function of the d-dimensional ball of unit diameter $B = \frac{1}{2}\mathbb{B}^d$ satisfies the two term-Weyl asymptotic and so for α sufficiently large we know that

$$\mathcal{N}_B^N(\alpha) \geq \frac{\omega_d V(B)}{(2\pi)^d}\alpha^{d/2} = \frac{(\omega_d)^2}{(4\pi)^d}\alpha^{d/2}, \tag{A.1.3}$$

which is the conjectured Pólya inequality and is known to be true for two-dimensional balls. By virtue of the two-term Weyl asymptotic, we assume, for simplicity and to reduce the computational expense of the calculation, that Pólya's inequality is indeed true for d-dimensional balls for the Neumann eigenvalue counting function.

If we know that

$$\begin{aligned} \mathcal{N}_B^N(\alpha) > & \inf_{\substack{\Omega \in \mathcal{A}^d \\ D(\Omega)=1}} \left[\frac{V(\Omega)}{\pi^d}\alpha^{d/2} + \omega_d[r_1(\alpha) + r_2(\alpha)]\right] \\ & = \omega_d[r_1(\alpha) + r_2(\alpha)] \end{aligned} \tag{A.1.4}$$

for all $\alpha > \alpha^*$, then a minimiser necessarily exists for

$$k > \mathcal{N}_B^N(\alpha^*) \geq \frac{(\omega_d)^2}{(4\pi)^d}(\alpha^*)^{d/2} =: k^* \tag{A.1.5}$$

as the Neumann eigenvalues of the ball are smaller than any collapsing sequence of bounded domains of diameter one. So our code consists of estimating this α^* to then obtain an estimate for k^* which our estimated bound on N_d.

```python
#Import the relevant variables and functions from the 'math' module
from math import ceil,sqrt,pi,gamma,log10

```

```python
#Import the binomial coefficent nCk as bc(n,k)
from math import comb as bc

#Function to return the kappa_1 variable
def kappa(d):
    return ceil(sqrt(d))

#Function to determine the volume of the d-dimensional unit ball
↪ omega_d
def omega(d):
    return (pi**(d/2)) / (gamma(1+(d/2)))

#Function to compute the remainder r_1(alpha)
def r1(a,d):
    return ((kappa(d)/pi)**(d-1)) * (2*kappa(d)+3) * (d**(3/2)) *
    ↪ (a**((d-1)/2))

#Function to compute the remainder r_2(alpha)
def r2(a,d):
    return sum(bc(d,j) * ((4*d)**(j/2)) * ((kappa(d)/pi)**(d-j)) *
    ↪ (a**((d-j)/2)) for j in range(2,d+1))

#Function for the conjectured Pólya lower bound for the d-dimensional
↪ ball
def polya(a,d):
    return ((omega(d)**2) * (a**(d/2))) / (((4*pi)**d))

#Function for finding the difference epsilon(alpha)
def epsilon(a,d):
    return polya(a,d) - omega(d) * (r1(a,d) + r2(a,d))
```

```python

#Function for returning an estimate alpha* with jumps of size N starting at a0
def est_alpha(d,N,a0):
    a = a0
    while epsilon(a,d) < 0:
        a += N
    return a

#Function for estimating Nd by refining the search each time and then uses the Pólya
#Estimate on the final step
def Nd_estimate(d):
    a0 = 0
    for j in range(2*d):
        N = 10**(2 * d - j)
        a0 = est_alpha(d,N,a0) - N
    return int(polya(est_alpha(d,1,a0),d)) + 1

#Function convert a number n to 3 significant figures in standard form
def standard_form(n):
    r = int(log10(n))
    m = n/(10**r)
    return str(round(m,2)) + " * 10^" + str(r)

#Determine and print the estimates for Nd in dimensions 2 to 6.
for d in range(2,7):
    print("d = " + str(d) + ",", "Nd = " + standard_form(Nd_estimate(d)))
```

The output of the above code is as follows, which is written in Table 3.1.

```
d = 2, Nd = 2.51 * 10^4
d = 3, Nd = 3.01 * 10^9
d = 4, Nd = 2.78 * 10^16
d = 5, Nd = 2.71 * 10^29
d = 6, Nd = 1.78 * 10^42
```

A.2 Numerically estimating isoperimetric solutions over $\mathcal{A}^2(M)$

We now give the Python code for the numerical solutions to the isoperimetric problem over $\mathcal{A}^2(M)$ in Figure 3.2. We use the module `cvxpy` which is well placed to solve such optimisation problems and (very) roughly outline how these problems are solved numerically.

Let $0 < a < \pi$ and $M > 0$ be given. We consider the problem of finding a an M-Lipschitz function $f : [0, a] \to \mathbb{R}$ with $f(0) = f(a) = 0$ and

$$\int_0^a \sqrt{1 + (f'(x))^2}\, \mathrm{d}x \leq \pi, \tag{A.2.1}$$

i.e. the length of the curve $(x, f(x))$ is at most π, that maximises

$$\int_0^a f(x)\, \mathrm{d}x \tag{A.2.2}$$

among all such functions. Numerically we can do this by discretising the interval $[0, 1]$ into n points $\{x_j\}_{0 \leq j \leq n}$ given by $x_j = j/n$. Our free variables are $\{y_j\}_{0 \leq j \leq n}$ and our choices of the y_j define a function $f : [0, a] \to \mathbb{R}$ via linear interpolation. Note that we must have $y_0 = y_n = 0$ to satisfy the endpoint conditions so there are $n - 2$ free variables here.

The M-Lipschitz condition then becomes

$$\max_{0\le j\le n-1}\left|\frac{y_{j+1}-y_j}{n}\right| \le M, \tag{A.2.3}$$

the length condition (A.2.1) becomes

$$\sum_{j=0}^{n-1}\left[\frac{1}{n^2}+(y_{j+1}-y_j)^2\right]^{1/2} \le \pi \tag{A.2.4}$$

and the function we want to maximise (A.2.2) becomes

$$\sum_{j=0}^{n-1}\frac{y_j+y_{j+1}}{n}. \tag{A.2.5}$$

Once the problem is converted into this numerical setup, `cvxpy` has a method of finding the numerically optimal such $\{y_j\}_{0\le j\le n}$ for this problem.

The aforementioned method only works for $0<a<\pi$ fixed so we additionally discretise in terms of the parameter a, i.e. consider the points $\{a_j\}_{1\le j\le n}$, and numerically compute the solution above for each a_j. Then we maximise over all the a_j to obtain our numerically computed optimal set in $\mathcal{A}^2(M)$. In code, this works as follows:

```python
import numpy as np
import cvxpy as cp

# Problem Data
m = 400 #number of based lengths to sample
n = 400 # points in interval [0,1] to discretise y at
L = np.pi #Half the length of the boundary of the set
M = 1.0 #M-Lipschitz condition to impose

X = np.linspace(0,L,num=m,endpoint=False) #Sample lengths of base to
↪   try
Y = [0.0]
opt = []
```

```

for i in range(1,len(X)):

    #Define variables for optimisation
    y = cp.Variable(n+1) # y[i] = y(i/n) for i=0,1,...,n
    x = X[i]*(np.arange(n+1) / n)
    dy = y[1:] - y[:-1]
    dx = x[1:] - x[:-1]

    # Define objective function for optimisation problem
    objective = X[i]*(((cp.sum(y[:-1]) + cp.sum(y[1:])) / 2) / n)

    # Define constraints for optimisation
    constraints = [

        #Fix end points
        y[0] == 0, y[-1] == 0,

        #Ensure solution has boundary at most length L
        cp.sum(cp.norm2(cp.vstack([dy, dx]), axis=0)) <= L,

        #Ensure solution is M-Lipschitz
        cp.max(cp.abs(cp.vstack([dy/dx]))) <= M,
    ]

    # Create an instance of the problem in cvxpy
    prob = cp.Problem(cp.Maximize(objective), constraints)

    # Solve Problem
    prob.solve(verbose=False)
```

```

    #Determine if solution with base length X[i] is best so far
    if prob.value > np.max(Y):

        #Store (x,y) coordinates, base length and domain area
        #For optimal problem so far
        opt = [x,y.value,X[i],prob.value]

    #Add in base length to list for later comparison
    Y.append(X[i]*(((np.sum(y.value[:-1]) + np.sum(y.value[1:])) / 2)
    ↪ / n))
```

One may then plot optimal solutions using the `matplotlib` module as follows:

```
import matplotlib.pyplot as plt

s = opt[2]/2 #How much to shift solution to centre domain at (0,0)
plt.plot(opt[0]-s,opt[1],'r-') #Plot upper boundary in red
plt.plot(opt[0]-s,-opt[1],'b-') #Plot lower boundary in blue

# Fix axes
plt.axis(xmin=-1.5,xmax=1.5,ymin=-1.5,ymax=1.5)
ax = plt.gca()
ax.set_aspect('equal', adjustable='box')

#Title plot
plt.title("M = "+str(M))

#Show plot
plt.show()
```

Bibliography

[AntFK13] P. R. S. Antunes, P. Freitas and J. B. Kennedy. 'Asymptotic behaviour and numerical approximation of optimal eigenvalues of the Robin Laplacian'. In: *ESAIM Control Optim. Calc. Var.* 19.2 (2013), pp. 438–459.
`https://doi.org/10.1051/cocv/2012016`
`doi:10.1051/cocv/2012016`.

[AntFre13] P. R. S. Antunes and P. Freitas. 'Optimal spectral rectangles and lattice ellipses'. In: *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 469.2150 (2013), pp. 20120492, 15.
`https://doi.org/10.1098/rspa.2012.0492`
`doi:10.1098/rspa.2012.0492`.

[Ban17] C. Bandle. 'Dido's problem and its impact on modern mathematics'. In: *Notices Amer. Math. Soc.* 64.9 (2017), pp. 980–984.
`https://doi.org/10.1090/noti1576`
`doi:10.1090/noti1576`.

[Beb03] M. Bebendorf. 'A note on the Poincaré inequality for convex domains'. In: *Z. Anal. Anwendungen* 22.4 (2003), pp. 751–756.
`https://doi.org/10.4171/ZAA/1170`
`doi:10.4171/ZAA/1170`.

[Bel75] B. A. Bellot. *On reflected Brownian motion in two dimensions.* Thesis (Ph.D.)–University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 1975, p. 69. http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:7615109.

[Ber13] M. van den Berg. 'Heat flow and perimeter in $\mathbb{R}^m$'. In: *Potential Anal.* 39.4 (2013), pp. 369–387. https://doi.org/10.1007/s11118-013-9335-z doi:10.1007/s11118-013-9335-z.

[Ber15] M. van den Berg. 'On the minimization of Dirichlet eigenvalues'. In: *Bull. Lond. Math. Soc.* 47.1 (2015), pp. 143–155. https://doi.org/10.1112/blms/bdu106 doi:10.1112/blms/bdu106.

[BerBG16] M. van den Berg, D. Bucur and K. Gittins. 'Maximising Neumann eigenvalues on rectangles'. In: *Bull. Lond. Math. Soc.* 48.5 (2016), pp. 877–894. https://doi.org/10.1112/blms/bdw049 doi:10.1112/blms/bdw049.

[BerDK14] M. van den Berg, E. B. Dryden and T. Kappeler. 'Isospectrality and heat content'. In: *Bull. Lond. Math. Soc.* 46.4 (2014), pp. 793–808. https://doi.org/10.1112/blms/bdu035 doi:10.1112/blms/bdu035.

[BerGG20] M. van den Berg, P. B. Gilkey and K. Gittins. 'Heat flow from polygons'. In: *Potential Anal.* 53.3 (2020), pp. 1043–1062. https://doi.org/10.1007/s11118-019-09797-5 doi:10.1007/s11118-019-09797-5.

[BerGit16] M. van den Berg and K. Gittins. 'On the heat content of a polygon'. In: *J. Geom. Anal.* 26.3 (2016), pp. 2231–2264. https://doi.org/10.1007/s12220-015-9626-2 doi:10.1007/s12220-015-9626-2.

[BerGit17] M. van den Berg and K. Gittins. 'Minimizing Dirichlet eigenvalues on cuboids of unit measure'. In: *Mathematika* 63.2 (2017), pp. 469–482. https://doi.org/10.1112/S0025579316000413 doi:10.1112/S0025579316000413.

[BerIve13] M. van den Berg and M. Iversen. 'On the minimization of Dirichlet eigenvalues of the Laplace operator'. In: *J. Geom. Anal.* 23.2 (2013), pp. 660–676. https://doi.org/10.1007/s12220-011-9258-0 doi:10.1007/s12220-011-9258-0.

[BerSri88] M. van den Berg and S. Srisatkunarajah. 'Heat equation for a region in $\mathbf{R}^2$ with a polygonal boundary'. In: *J. London Math. Soc. (2)* 37.1 (1988), pp. 119–127. https://doi.org/10.1112/jlms/s2-37.121.119 doi:10.1112/jlms/s2-37.121.119.

[BerSri90] M. van den Berg and S. Srisatkunarajah. 'Heat flow and Brownian motion for a region in $\mathbf{R}^2$ with a polygonal boundary'. In: *Probab. Theory Related Fields* 86.1 (1990), pp. 41–52. https://doi.org/10.1007/BF01207512 doi:10.1007/BF01207512.

[Blå05] V. Blåsjö. 'The isoperimetric problem'. In: *Amer. Math. Monthly* 112.6 (2005), pp. 526–566. https://doi.org/10.2307/30037526 doi:10.2307/30037526.

[BogBF24] B. Bogosel, D. Bucur and I. Fragalà. 'The nonlocal isoperimetric problem for polygons: Hardy-Littlewood and Riesz inequalities'. In: *Math. Ann.* 389.2 (2024), pp. 1835–1882. https://doi.org/10.1007/s00208-023-02683-x doi:10.1007/s00208-023-02683-x.

[BogHL18] B. Bogosel, A. Henrot and I. Lucardesi. 'Minimization of the eigenvalues of the Dirichlet-Laplacian with a diameter constraint'. In: *SIAM J. Math. Anal.* 50.5 (2018), pp. 5337–5361. https://doi.org/10.1137/17M1162147 doi:10.1137/17M1162147.

[BogHM24] B. Bogosel, A. Henrot and M. Michetti. 'Optimization of Neumann eigenvalues under convexity and geometric constraints'. In: *SIAM J. Math. Anal.* 56.6 (2024), pp. 7327–7349. https://doi.org/10.1137/24M1641099 doi:10.1137/24M1641099.

[Bon24] T. Bonnesen. 'Über das isoperimetrische Defizit ebener Figuren'. In: *Math. Ann.* 91.3-4 (1924), pp. 252–268. https://doi.org/10.1007/BF01556082 doi:10.1007/BF01556082.

[Bos86] M-H. Bossel. 'Membranes élastiquement liées: extension du théorème de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger'. In: *C. R. Acad. Sci. Paris Sér. I Math.* 302.1 (1986), pp. 47–50.

[Bro18] M. Brown. *Heat Content In Polygons.* Honors Theses. 444. Advisors: E. B. Dryden, J. J. Langford. https://digitalcommons.bucknell.edu/honors_theses/444. 2018.

[BucFre13] D. Bucur and P. Freitas. 'Asymptotic behaviour of optimal spectral planar domains with fixed perimeter'. In: *J. Math. Phys.*

54.5 (2013), pp. 053504, 6.
https://doi.org/10.1063/1.4803140
doi:10.1063/1.4803140.

[BucGia15] D. Bucur and A. Giacomini. 'The Saint-Venant inequality for the Laplace operator with Robin boundary conditions'. In: *Milan J. Math.* 83.2 (2015), pp. 327–343.
https://doi.org/10.1007/s00032-015-0243-0
doi:10.1007/s00032-015-0243-0.

[BurSch01] A. Burchard and M. Schmuckenschläger. 'Comparison theorems for exit times'. In: *Geom. Funct. Anal.* 11.4 (2001), pp. 651–692.
https://doi.org/10.1007/PL00001681
doi:10.1007/PL00001681.

[CavFHLLS23] L. Cavallina, K. Funano, A. Henrot, A. Lemenant, I. Lucardesi and S. Sakaguchi. *Two extremum problems for Neumann eigenvalues.* 2023.
https://arxiv.org/abs/2312.13747.
arXiv:2312.13747 [math.SP].

[Che83] J. Cheeger. 'Spectral geometry of singular Riemannian spaces'. In: *J. Differential Geom.* 18.4 (1983), pp. 575–657.
http://projecteuclid.org/euclid.jdg/1214438175.

[Chn93] Z-Q Chen. 'On reflecting diffusion processes and Skorokhod decompositions'. In: *Probab. Theory Related Fields* 94.3 (1993), pp. 281–315.
https://doi.org/10.1007/BF01199246
doi:10.1007/BF01199246.

[Chs75] D. Chenais. 'On the existence of a solution in a domain identification problem'. In: *J. Math. Anal. Appl.* 52.2 (1975), pp. 189–219.

https://doi.org/10.1016/0022-247X(75)90091-8
doi:10.1016/0022-247X(75)90091-8.

[Cit19] S. Cito. 'Existence and regularity of optimal convex shapes for functionals involving the Robin eigenvalues'. In: *J. Convex Anal.* 26.3 (2019), pp. 925–943.

[CouHil53] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I.* Interscience Publishers, Inc., New York, 1953, pp. xv+561.

[Dan06] D. Daners. 'A Faber-Krahn inequality for Robin problems in any space dimension'. In: *Math. Ann.* 335.4 (2006), pp. 767–785.
https://doi.org/10.1007/s00208-006-0753-8
doi:10.1007/s00208-006-0753-8.

[Dod83] J. Dodziuk. 'Maximum principle for parabolic inequalities and the heat flow on open manifolds'. In: *Indiana Univ. Math. J.* 32.5 (1983), pp. 703–716.
https://doi.org/10.1512/iumj.1983.32.32046
doi:10.1512/iumj.1983.32.32046.

[EngNag00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations.* Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Springer-Verlag, New York, 2000, pp. xxii+586.

[ErdMOT54] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi. *Tables of integral transforms. Vol. I.* Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1954, pp. xx+391.

[Eva10] L. C. Evans. *Partial differential equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. `https://doi.org/10.1090/gsm/019` `doi:10.1090/gsm/019`.

[Fab23] G. Faber. 'Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt'. In: *Sitz. Ber. Bayer. Akad. Wiss.* (1923), pp. 169–172.

[Far25] S. Farrington. 'On the Isoperimetric and Isodiametric Inequalities and the Minimisation of Eigenvalues of the Laplacian'. In: *J. Geom. Anal.* 35.2 (2025), Paper No. 62. `https://doi.org/10.1007/s12220-024-01887-0` `doi:10.1007/s12220-024-01887-0`.

[FarGit23] S. Farrington and K. Gittins. 'Heat flow in polygons with reflecting edges'. In: *Integral Equations Operator Theory* 95.4 (2023), Paper No. 27, 37. `https://doi.org/10.1007/s00020-023-02749-0` `doi:10.1007/s00020-023-02749-0`.

[FilLPS23] N. Filonov, M. Levitin, I. Polterovich and D. A. Sher. 'Pólya's conjecture for Euclidean balls'. In: *Invent. Math.* 234.1 (2023), pp. 129–169. `https://doi.org/10.1007/s00222-023-01198-1` `doi:10.1007/s00222-023-01198-1`.

[FraLar20] R. L. Frank and S. Larson. 'Two-term spectral asymptotics for the Dirichlet Laplacian in a Lipschitz domain'. In: *J. Reine Angew. Math.* 766 (2020), pp. 195–228.

https://doi.org/10.1515/crelle-2019-0019
doi:10.1515/crelle-2019-0019.

[FraLar24] R. L. Frank and S. Larson. *Riesz means asymptotics for Dirichlet and Neumann Laplacians on Lipschitz domains*. 2024. https://arxiv.org/abs/2407.11808. arXiv:2407.11808 [math.SP].

[FraLW23] R. L. Frank, A. Laptev and T. Weidl. *Schrödinger operators: eigenvalues and Lieb-Thirring inequalities*. Vol. 200. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2023, pp. xiii+507.
https://doi.org/10.1017/9781009218436
doi:10.1017/9781009218436.

[Fre17] P. Freitas. 'Asymptotic behaviour of extremal averages of Laplacian eigenvalues'. In: *J. Stat. Phys.* 167.6 (2017), pp. 1511–1518.
https://doi.org/10.1007/s10955-017-1789-8
doi:10.1007/s10955-017-1789-8.

[FreKen21] P. Freitas and J. B. Kennedy. 'Extremal domains and Pólya-type inequalities for the Robin Laplacian on rectangles and unions of rectangles'. In: *Int. Math. Res. Not. IMRN* 18 (2021), pp. 13730–13782.
https://doi.org/10.1093/imrn/rnz204
doi:10.1093/imrn/rnz204.

[FriLut76] R. Friedberg and J. M. Luttinger. 'A new rearrangement inequality for multiple integrals'. In: *Arch. Rational Mech. Anal.* 61.1 (1976), pp. 45–64.
https://doi.org/10.1007/BF00251862
doi:10.1007/BF00251862.

[Fug89] B. Fuglede. ‘Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbf{R}^n$’. In: *Trans. Amer. Math. Soc.* 314.2 (1989), pp. 619–638.
https://doi.org/10.2307/2001401
doi:10.2307/2001401.

[Fun23] K. Funano. ‘A note on domain monotonicity for the Neumann eigenvalues of the Laplacian’. In: *Illinois J. Math.* 67.4 (2023), pp. 677–686.
https://doi.org/10.1215/00192082-10972651
doi:10.1215/00192082-10972651.

[GalMcK72] G. Gallavotti and H. P. McKean Jr. ‘Boundary conditions for the heat equation in a several-dimensional region’. In: *Nagoya Math. J.* 47 (1972), pp. 1–14.
http://projecteuclid.org/euclid.nmj/1118798680.

[GitLar17] K. Gittins and S. Larson. ‘Asymptotic behaviour of cuboids optimising Laplacian eigenvalues’. In: *Integral Equations Operator Theory* 89.4 (2017), pp. 607–629.
https://doi.org/10.1007/s00020-017-2407-5
doi:10.1007/s00020-017-2407-5.

[GitLén20] K. Gittins and C. Léna. ‘Upper bounds for Courant-sharp Neumann and Robin eigenvalues’. In: *Bull. Soc. Math. France* 148.1 (2020), pp. 99–132.
https://doi.org/10.24033/bsmf.2800
doi:10.24033/bsmf.2800.

[GraRyz07] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Seventh. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX). Elsevier/Academic Press, Amsterdam, 2007, pp. xlviii+1171.

[Gru07] P. M. Gruber. *Convex and discrete geometry*. Vol. 336. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin, 2007, pp. xiv+578.

[Hen06] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006, pp. x+202.

[Her60] J. Hersch. 'Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum'. In: *Z. Angew. Math. Phys.* 11 (1960), pp. 387–413. https://doi.org/10.1007/BF01604498 doi:10.1007/BF01604498.

[Hsu02] E. P. Hsu. *Stochastic analysis on manifolds*. Vol. 38. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, pp. xiv+281. https://doi.org/10.1090/gsm/038 doi:10.1090/gsm/038.

[Hsu84] P. Hsu. *Reflecting Brownian Motion, Boundary Local Time and the Neumann Problem*. Thesis (Ph.D.)–Stanford University. ProQuest LLC, Ann Arbor, MI, 1984, p. 108. http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:8420554.

[Ivr80] V. Ja. Ivriĭ. 'The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary'. In: *Funktsional. Anal. i Prilozhen.* 14.2 (1980), pp. 25–34.

[Kac66] M. Kac. ‘Can one hear the shape of a drum?’ In: *Amer. Math. Monthly* 73.4 (1966), pp. 1–23.
`https://doi.org/10.2307/2313748`
`doi:10.2307/2313748`.

[Kel66] R. Kellner. ‘On a theorem of Pólya’. In: *Amer. Math. Monthly* 73 (1966), pp. 856–858.
`https://doi.org/10.2307/2314181`
`doi:10.2307/2314181`.

[Kra25] E. Krahn. ‘Über eine von Rayleigh formulierte Minimaleigenshaft des Kreises’. In: *Math. Ann.* 94 (1925), pp. 97–100.

[Kri27] N. Kritikos. ‘Über konvexe Flächen und einschließende Kugeln’. In: *Math. Ann.* 96.1 (1927), pp. 583–586.
`https://doi.org/10.1007/BF01209189`
`doi:10.1007/BF01209189`.

[Lar19] S. Larson. ‘Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains’. In: *J. Spectr. Theory* 9.3 (2019), pp. 857–895.
`https://doi.org/10.4171/JST/265`
`doi:10.4171/JST/265`.

[Led94] M. Ledoux. ‘Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space’. In: *Bull. Sci. Math.* 118.6 (1994), pp. 485–510.

[LevPM23] M. Levitin, D. Mangoubi and I. Polterovich. *Topics in Spectral Geometry*. preliminary version dated May 29, 2023.
`https://michaellevitin.net/Book/`.

[Li22] L. Li. ‘On the placement of an obstacle so as to optimize the Dirichlet heat content’. In: *SIAM J. Math. Anal.* 54.3 (2022), pp. 3275–3291.

https://doi.org/10.1137/21M1433411
doi:10.1137/21M1433411.

[LiYau83] P. Li and S. T. Yau. 'On the Schrödinger equation and the eigenvalue problem'. In: *Comm. Math. Phys.* 88.3 (1983), pp. 309–318.
http://projecteuclid.org/euclid.cmp/1103922378.

[Lut73] J. M. Luttinger. 'Generalized isoperimetric inequalities'. In: *J. Mathematical Phys.* 14 (1973), pp. 586–593.
https://doi.org/10.1063/1.1666363
doi:10.1063/1.1666363.

[MagPP14] F. Maggi, M. Ponsiglione and A. Pratelli. 'Quantitative stability in the isodiametric inequality via the isoperimetric inequality'. In: *Trans. Amer. Math. Soc.* 366.3 (2014), pp. 1141–1160.
https://doi.org/10.1090/S0002-9947-2013-06126-0
doi:10.1090/S0002-9947-2013-06126-0.

[Mar20] N. F. Marshall. 'Stretching convex domains to capture many lattice points'. In: *Int. Math. Res. Not. IMRN* 10 (2020), pp. 2918–2951.
https://doi.org/10.1093/imrn/rny102
doi:10.1093/imrn/rny102.

[Mat78] G. Matheron. 'La formule de Steiner pour les érosions'. In: *J. Appl. Probability* 15.1 (1978), pp. 126–135.
https://doi.org/10.2307/3213242
doi:10.2307/3213242.

[McD13] P. McDonald. 'Exit times, moment problems and comparison theorems'. In: *Potential Anal.* 38.4 (2013), pp. 1365–1372.
https://doi.org/10.1007/s11118-012-9318-5
doi:10.1007/s11118-012-9318-5.

[McK69] H. P. McKean Jr. *Stochastic integrals*. Vol. No. 5. Probability and Mathematical Statistics. Academic Press, New York-London, 1969, pp. xiii+140.

[McKSin67] H. P. McKean Jr. and I. M. Singer. 'Curvature and the eigenvalues of the Laplacian'. In: *J. Differential Geometry* 1.1 (1967), pp. 43–69. http://projecteuclid.org/euclid.jdg/1214427880.

[McS34] E. J. McShane. 'Extension of range of functions'. In: *Bull. Amer. Math. Soc.* 40.12 (1934), pp. 837–842. https://doi.org/10.1090/S0002-9904-1934-05978-0 doi:10.1090/S0002-9904-1934-05978-0.

[MeyMcD17] R. Meyerson and P. McDonald. 'Heat content determines planar triangles'. In: *Proc. Amer. Math. Soc.* 145.6 (2017), pp. 2739–2748. https://doi.org/10.1090/proc/13397 doi:10.1090/proc/13397.

[MirPPP07] M. Miranda Jr., D. Pallara, F. Paronetto and M. Preunkert. 'Short-time heat flow and functions of bounded variation in $\mathbf{R}^N$'. In: *Ann. Fac. Sci. Toulouse Math. (6)* 16.1 (2007), pp. 125–145. http://afst.cedram.org/item?id=AFST_2007_6_16_1_125_0.

[NetSaf05] Yu. Netrusov and Yu. Safarov. 'Weyl asymptotic formula for the Laplacian on domains with rough boundaries'. In: *Comm. Math. Phys.* 253.2 (2005), pp. 481–509. https://doi.org/10.1007/s00220-004-1158-8 doi:10.1007/s00220-004-1158-8.

[NurRS19] M. Nursultanov, J. Rowlett and D. A. Sher. 'How to Hear the Corners of a Drum'. In: *2017 MATRIX Annals*. Springer International Publishing, 2019, pp. 243–278.

https://doi.org/10.10072F978-3-030-04161-8_18
doi:10.1007/978-3-030-04161-8_18.

[NurRS24] M. Nursultanov, J. Rowlett and D. A. Sher. ‘The heat kernel on curvilinear polygonal domains in surfaces’. In: *Ann. Math. Québec* (2024).
https://doi.org/10.1007/s40316-024-00237-4
doi:10.1007/s40316-024-00237-4.

[Oss79] R. Osserman. ‘Bonnesen-style isoperimetric inequalities’. In: *Amer. Math. Monthly* 86.1 (1979), pp. 1–29.
https://doi.org/10.2307/2320297
doi:10.2307/2320297.

[PayWei60] L. E. Payne and H. F. Weinberger. ‘An optimal Poincaré inequality for convex domains’. In: *Arch. Rational Mech. Anal.* 5 (1960), pp. 286–292.
https://doi.org/10.1007/BF00252910
doi:10.1007/BF00252910.

[Pól48] G. Pólya. ‘Torsional rigidity, principal frequency, electrostatic capacity and symmetrization’. In: *Quart. Appl. Math.* 6 (1948), pp. 267–277.
https://doi.org/10.1090/qam/26817
doi:10.1090/qam/26817.

[Pól54] G. Pólya. *Mathematics and Plausible Reasoning, Volumes 1 & 2.* Princeton: Princeton University Press, 1954.
https://doi.org/10.1515/9780691218304
doi:doi:10.1515/9780691218304.

[Pól61] G. Pólya. ‘On the eigenvalues of vibrating membranes’. In: *Proc. London Math. Soc. (3)* 11 (1961), pp. 419–433.

https://doi.org/10.1112/plms/s3-11.1.419
doi:10.1112/plms/s3-11.1.419.

[PólSze51] G. Pólya and G. Szegő. *Isoperimetric Inequalities in Mathematical Physics*. Vol. No. 27. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1951, pp. xvi+279.

[Pre04] M. Preunkert. 'A semigroup version of the isoperimetric inequality'. In: *Semigroup Forum* 68.2 (2004), pp. 233–245.
https://doi.org/10.1007/s00233-003-0004-1
doi:10.1007/s00233-003-0004-1.

[RogWil00] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 1*. Cambridge Mathematical Library. Foundations, Reprint of the second (1994) edition. Cambridge University Press, Cambridge, 2000, pp. xx+386.
https://doi.org/10.1017/CBO9781107590120
doi:10.1017/CBO9781107590120.

[Ros04] M. Ross. 'The Lipschitz continuity of Neumann eigenvalues on convex domains'. In: *Hokkaido Math. J.* 33.2 (2004), pp. 369–381.
https://doi.org/10.14492/hokmj/1285766171
doi:10.14492/hokmj/1285766171.

[SafVas97] Yu. Safarov and D. Vassiliev. *The asymptotic distribution of eigenvalues of partial differential operators*. Vol. 155. Translations of Mathematical Monographs. Translated from the Russian manuscript by the authors. American Mathematical Society, Providence, RI, 1997, pp. xiv+354.
https://doi.org/10.1090/mmono/155
doi:10.1090/mmono/155.

[Str77] J. W. Strutt. *The Theory of Sound*. Macmillan and Co, London, 1877, Vol. I, xlii+480.

https://doi.org/10.1017/CBO9781139058087
doi:10.1017/CBO9781139058087.

[Vir09] Virgil. *The Aeneid*. Translated by John Dryden. The Floating Press, Auckland, 2009.

[Wey12] H. Weyl. 'Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)'. In: *Math. Ann.* 71.4 (1912), pp. 441–479.
https://doi.org/10.1007/BF01456804
doi:10.1007/BF01456804.