From Amplitude Bootstrap to Cosmological Correlators

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Abstract

In this thesis, we explore how to adapt the amplitude bootstrap techniques from Minkowski space to (Anti) de-Sitter space. We begin by reviewing the use of physical principles to bootstrap amplitudes in flat spacetime. Building on this foundation, the first part of the thesis examines the relationship between enhanced soft limits and effective field theories in de-Sitter space. Specifically, we analyze the soft limits of theories with Lagrangians that exhibit hidden shift symmetries, demonstrating that these theories indeed possess enhanced soft limits up to six points. In the second part, we focus on spinning particles. Starting with the four-point gluon wavefunction coefficient, we use the double copy idea by squaring the gluon result. Then, by combining this with the bootstrap techniques, we compute the four-graviton wavefunction coefficient in de-Sitter space. In the final part of the thesis, we investigate the Mellin-Momentum representation of AdS amplitudes. This representation, which resembles the analytic structure of the S-matrix, enables us to introduce a novel and efficient algorithm for bootstrapping n-point amplitudes, incorporating the modern on-shell amplitude approach. We then compute gluon and gravity amplitudes up to five points. Chapter 3 of this thesis is the reproduction of the work presented in [1], Chapter 4 is the reproduction of [2], Chapter 5 is the reproduction of [3, 4].

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification.

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CHAPTER 1

Introduction

Quantum field theory, as the mathematical framework for understanding fundamental physics, has achieved tremendous success over the last century. Scattering amplitudes are a cornerstone of Quantum Field Theory (QFT), possessing both theoretical and experimental significance in predicting collider results. In recent decades, there has been mounting evidence to suggest that fields in QFT are auxiliary objects that sometimes obscure the underlying simplicity of nature. Instead, one should focus on the observables themselves and employ physical principles to bootstrap them directly. Over the past decades, there has been significant progress in bootstrapping the S-matrix using fundamental physical principles such as Lorentz invariance, locality, and unitarity [5–7].

Gravity stands out as one of the most compelling examples in this regard. Einstein Gravity is notoriously challenging to compute, even at the perturbation level. Conversely, the modern scattering amplitudes approach in flat space has achieved tremendous success by employing the on-shell approach to compute Gravity amplitudes. Moreover, the BCFW recursion [8] significantly enhances accessibility to higher-point tree-level Gravity amplitudes, requiring only three-point amplitude as input. This eliminates the need to know the infinite expansion of the Einstein-

Hilbert action.

However, our universe is not flat, such comprehension in curved space is still in its early stages, yet undeniably crucial for understanding our Universe—especially the uniqueness of Einstein Gravity in curved space. As a small step towards unravelling Gravity in curved space, we explore this bootstrap approach in the following maximally symmetric spacetime, (Anti)de-Sitter space. However, defining the S-matrix of QFT in curved spacetime is a challenge. In Anti de Sitter (AdS) space, the gauge-gravity duality allows us to obtain the correlation function of a Conformal Field Theory (CFT) on the boundary [9]. In addition, QFT in de Sitter (dS) space offers powerful tools for computing cosmological observables, an active area of research reviewed in [10]. To be more precise, Generalizing this from flat space to curved space, by replacing Lorentz invariance with conformal invariance, presents considerable challenges. Recent progress has been made in momentum space and cosmology to tackle this problem [11–17], which is known as the Cosmological Bootstrap. However, cosmological correlators are not invariant under field redefinition/gauge transformation [18]. While factorization is manifest in momentum space, the pole structure is significantly more intricate, and the special conformal generator in momentum space is a second-order differential operator, making it challenging to implement conformal symmetry. While significant progress has been made in the cosmological bootstrap program [10–15, 19–21], exploration of spinning particles and amplitudes beyond four-point remains very limited [22–31]. Building on insights from flat space, it becomes evident that grasping the structure of amplitudes in curved space requires a deep understanding of higher-point amplitudes.

To begin our exploration of cosmological correlators using the scattering amplitude approach, it's essential to first examine the fundamental differences between these observables. In collider physics, we have the ability to control the initial states and measure the out states, thus exerting control over the entire scattering process and then measuring the probability of the scattering process. However, in cosmology, we lack observables that living at the beginning of time. Instead, we can only access the spatial boundary correlators that lives in the future boundary of dS, which represents a crucial distinction. Unlike in Minkowski space, where we have bulk observables, in cosmology, our observables are confined to the boundary.

The study of perturbative gravitational observables analogous to scattering amplitudes is far less understood in curved backgrounds. Of particular interest are boundary correlators of gravitons in Anti-de Sitter space (AdS) and de Sitter space (dS), which play a prominent role in the AdS/CFT correspondence [9] and cosmology [32–38], respectively. In the context of cosmology, these quantities are known as wavefunction coefficients [39] and cosmological correlators (or in-in correlators) can be obtained by squaring wavefunctions and computing expectation values [34, 40]. While there has been impressive progress in computing supergravity correlators in AdS using conformal bootstrap techniques [41,42], it is not straightforward to adapt these methods to more realistic models in four dimensional de Sitter space (dS_4). But the wavefunction coefficients can be computed from Wick-rotated EAdS Witten diagrams in momentum space [35]. Moreover, perturbative calculations in (A)dS encounter similar difficulties to those in flat space but are even more challenging due to the intrinsic complexity of working in curved backgrounds. Indeed, the tree-level wavefunction of four gravitons in dS_4 was only determined in full generality recently [43] (see for [44] for earlier partial results). Despite the fact that Witten diagrams give hundreds of thousands of terms, the final result was only about a page in length. This simplification was achieved by using a powerful set of constraints including the flat space limit [32, 45], Cosmological Optical Theorem (COT) [15, 19] and Manifestly Local Test (MLT) [14], which are part of a broader arsenal of techniques collectively known as the cosmological boostrap [10].

Now, we start with some important lessons learned from the study of scattering amplitudes in Minkowski space, and in this thesis, we will seek to generalize these concepts to curved backgrounds.

There is a deep relation between soft limits of scattering amplitudes and hidden symmetries. For example, the soft theorems of graviton amplitudes [46–48] encode extended BMS symmetry [49, 50], while soft limits of pion amplitudes encode spontaneously broken chiral symmetry of QCD [51]. Pions are the Goldstone bosons

associated with spontaneous symmetry breaking and are described by a low-energy effective action known as the non-linear sigma model (NLSM) [52–54]. Of particular interest for this paper is a property of NLSM amplitudes known as the Adler zero [55], which is an example of an enhanced soft limit. A scattering amplitude is said to exhibit an enhanced soft limit when it scales like $\mathcal{O}(p^\sigma)$, where p is the soft momentum and σ is an integer greater than the expectation based on counting the number of derivatives per field in the Lagrangian. For the NLSM, $\sigma = 1$. More generally, σ can be no higher than three and the cases $\sigma = 2, 3$ correspond to the Dirac-Born-Infeld (DBI) and special Galileon theories, respectively [56, 57]. Enhanced soft limits arise from cancellations among Feynman diagrams of different topology and are a consequence of symmetries [57, 58]. In the NLSM, this is just an ordinary shift symmetry but in the other two cases the symmetries are higher shift symmetries which are nontrivially realised from the point of view of the Lagrangian and are often referred to as hidden symmetries.
Soft limits also play an important role in cosmology. For example in the context of inflation, where the early universe is approximately described by de Sitter space (dS), they provide constraints relating higher-point correlators to conformal transformations of lower-point correlators [34, 59, 60], and certain inflationary 3-point functions can be deduced from soft limits of 4-point de Sitter correlators [61–65]. Lagrangians for DBI and sGal theories were also recently deduced from higher shift symmetries in dS [66]. These Lagrangians have nontrivial masses and curvature corrections away from the flat space limit. As we will see in this paper, the NLSM can be trivially uplifted to dS space since curvature corrections would break the shift symmetry. It is therefore natural to ask if the wavefunction coefficients of these theories (which can be computed from Witten diagrams ending on the future boundary of dS [32, 34–36, 67]) exhibit enhanced soft limits analogous to their scattering amplitudes in the flat space limit.

Moving on to spinning particles, computing gravitational scattering amplitudes using standard Feynman diagram techniques is a formidable task due to the enormous number of terms that arise. On the other hand, modern approaches make use of a remarkable relation known as the double copy, which allows one to reduce gravitational calculations to much simpler calculations in gauge theory [68–77]. Roughly speaking, it relates gravitational amplitudes to the square of gauge theory amplitudes. The double copy was first discovered in string theory, but applies to general theories of gravity coupled to matter, providing deep theoretical insights into the mathematical structure of gauge theory and gravity as well as powerful new computational tools which have important applications to the study of gravitational waves. For a review of recent developments, see [78, 79].

After exploring various attempts to compute wavefunction coefficients and ultimately understand cosmological correlators, we have encountered several drawbacks that complicate our efforts to comprehend physics in curved space in perturbation calculation, unlike the simple analytic structure of perturbative S-matrix in flat space. To overcome these challenges, we propose studying a new representation for the AdS amplitude: Mellin-Momentum amplitude. With such a new representation, we will be able to introduce a novel and efficient algorithm for bootstrapping n-point amplitudes, incorporating the modern on-shell amplitude approach.

In Chapter 2 of this thesis, we will begin by reviewing the amplitude bootstrap in Minkowski space, followed by the connection of soft theorems and effective field theories. We will then explore topics such as color/kinematic duality and the double copy. Finally, we will delve into the basic properties of computing cosmological correlators.

In Chapter 3, we will review the method of expressing Witten diagrams in terms of boundary conformal generators. This formalism enables us to analyze the soft limit in general conformal dimensions and spacetime dimensions. This is demonstrated for NLSM, DBI and sGal at four points and NLSM and DBI for six points.

In Chapter 4, we will explore the combination of the double copy technique with the bootstrap approach to compute gravity amplitudes in (A)dS. We will begin by reviewing the basic concepts of the bootstrap approach for wavefunction coefficients. Then, we will use the double copy method to construct our ansatz, followed by employing the bootstrap approach to determine the remaining structure and ultimately obtain the four-graviton wavefunction coefficient.

In Chapter 5, we introduce a Mellin-Momentum representation for studying cosmological correlators. This representation enables us to closely mimic the amplitude bootstrap procedures in Minkowski space. We demonstrate how to recursively build gluon and graviton amplitudes up to 5 points using this formalism. Finally, we explore the application of the double copy technique within this framework and explain how our results can be easily mapped back to momentum space.

We conclude this thesis in Chapter 6.

CHAPTER 2

Preliminaries

In this chapter, we review some basic properties of scattering amplitudes and cosmological correlators. Specifically, we begin with an exploration of amplitude bootstrap in Minkowski space, followed by a discussion on soft theorems for Effective Field Theories (EFTs) and the double copy method for gravity. Finally, in the last section, we review the basics of computing cosmological correlators.

2.1 Amplitude bootstrap

There have been numerous efforts to understand and bootstrap the S-matrix from basic physical principles [5,7][1]. In this section, we will review some examples of bootstrapping Yang-Mills (YM) and General Relativity (GR) in general dimensions. The amplitudes will be a function of n momenta p_i^μ and obey the following momentum conservation due to translation invariance,

$$\sum_{i=1}^{n} p_i^\mu = 0, \tag{2.1}$$

[1]See also [80]for any mass and any spin, and [81] which discusses the existence of spin $\frac{3}{2}$ particles requiring supersymmetry.

and the inner dot product is contracting with $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ and the external particles obey the on-shell condition $p_i^2 = \eta_{\mu\nu} p_i^\mu p_i^\nu = 0$ for massless particles. For spinning particles, the amplitude will also be dressed up with polarization data, massless spin-1: ε_a^μ, massless spin-2: $\varepsilon_a^{\mu\nu} = \varepsilon_a^\mu \varepsilon_a^\nu$. Firstly, the principles for bootstrap include: the amplitudes should be Lorentz invariant, and the pole structure of the amplitude should be consistent with factorization and unitarity. To make the Lorentz symmetry manifest, the variables can only be inner dot product of momentum and for all the spinning amplitude we impose the following gauge condition as $\varepsilon_i \cdot \varepsilon_i = \varepsilon_i \cdot p_i = 0$ but leaving one degree of freedom left to be fixed by gauge Ward identity:

$$A_n|_{\varepsilon_i \to p_i} = 0. \tag{2.2}$$

Let us start with three-point amplitude. First, we consider the one-derivative massless spin-1 theory. We can readily enumerate all the possible terms fixed by Lorentz invariance and gauge condition with unfixed coefficients,

$$A_3 = c_1 \varepsilon_1 \cdot \varepsilon_3 p_3 \cdot \varepsilon_2 + c_2 \varepsilon_1 \cdot \varepsilon_2 p_1 \cdot \varepsilon_3 + c_3 \varepsilon_2 \cdot \varepsilon_3 p_2 \cdot \varepsilon_1. \tag{2.3}$$

Next, we can impose the gauge Ward identity above and solving the constraints, we can easily fix the coefficient and obtain:

$$A_3 = \varepsilon_1 \cdot \varepsilon_3 p_3 \cdot \varepsilon_2 + \varepsilon_1 \cdot \varepsilon_2 p_1 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 p_2 \cdot \varepsilon_1. \tag{2.4}$$

However, this amplitude is actually not allowed. If we consider the exchange of particles $1 \leftrightarrow 2$, we observe that the amplitude transforms as $A_3 \to -A_3$, violating Bose symmetry. This aligns with the fact that the photon does not have self-interaction. To obtain a non-vanishing amplitude, we require more than just the kinematic data from above. By assigning each particle with a color structure, we ensure that it now obeys Bose symmetry,

$$A_3 = f_{abc}(\varepsilon_1 \cdot \varepsilon_3 p_3 \cdot \varepsilon_2 + \varepsilon_1 \cdot \varepsilon_2 p_1 \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 p_2 \cdot \varepsilon_1), \tag{2.5}$$

where f_{abc} is fully antisymmetric, so the amplitude is now even under the exchange of two bosons. This is of course the same amplitude as using the usual Yang-Mills Feynman rules.

We can now conduct a similar exercise for the two-derivative massless spin-2 theory.

$$\begin{aligned} M_3 = & c_1(\varepsilon_1 \cdot \varepsilon_3)^2(p_1 \cdot \varepsilon_2)^2 + c_2(\varepsilon_1 \cdot \varepsilon_2)^2(p_1 \cdot \varepsilon_3)^2 + c_3(\varepsilon_2 \cdot \varepsilon_3)^2(p_2 \cdot \varepsilon_1)^2 \\ & + c_4(\varepsilon_{12,13} p_1 \cdot \varepsilon_3 p_1 \cdot \varepsilon_2) + c_5(\varepsilon_{13,23} p_2 \cdot \varepsilon_1 p_1 \cdot \varepsilon_2) + c_6(\varepsilon_{12,23} p_1 \cdot \varepsilon_3 p_2 \cdot \varepsilon_1), \end{aligned} \tag{2.6}$$

where we used the shorthand notation $\varepsilon_{ij,kl} = \varepsilon_i \cdot \varepsilon_j \varepsilon_k \cdot \varepsilon_l$. Similarly, applying diffeomorphism invariant/gauge Ward identity we can fix all the coefficients and the result can be nicely written as

$$M_3 = A_3^2. \tag{2.7}$$

Now we can move on to the four-point amplitude, locality implies that the tree-level amplitude has simple poles corresponding to propagators going on-shell while unitarity implies that the amplitude factorizes into lower points on-shell amplitude when the exchanged particles are on-shell:

$$A_4 \xrightarrow{P^2 \to 0} \sum_h A_3^{-h} \frac{1}{P^2} A_3^h \tag{2.8}$$

where $P^2 = (p_1^\mu + p_2^\mu)^2$ is the exchanged momentum. It's worth noting that we are sending $(p_1 + p_2)^2 \to 0$, but not demanding that $p_1^\mu \to -p_2^\mu$, otherwise this result will become just a special kinematic configuration. Now we can write down the ansatz for 4-point Yang-Mills amplitude, incorporating both the pole structure and terms without a pole, simply by dimensional counting we see that for contact terms (with no pole structure) the only Lorentz invariant quantity we can write down is the last term, and we simply enumerate all the possible terms and sum them up with unfixed coefficient as before,

$$A_4 = c_s \sum_h A_3^{-h} \frac{1}{s} A_3^h + c_s \sum_{i,j,k,l=1}^{4} C_{ij,kl} \varepsilon_{ij,kl} + \mathcal{P}(2,3,4) \tag{2.9}$$

with $\mathcal{P}(2,3,4)$ denotes permutation to obtain other channels and $\varepsilon_{ij,kl} := \varepsilon_i \cdot \varepsilon_j \varepsilon_k \cdot \varepsilon_l$, $c_s = f_{a_1 a_2 b} f_{b a_3 a_4}$. By dimensional analysis the contact terms can not depend on any momentum and s, t, u are the usual Mandelstam variables:

$$\begin{aligned} s &= (p_1 + p_2)^\mu (p_1 + p_2)_\mu = 2p_1^\mu p_{2\mu}, \\ t &= (p_1 + p_4)^\mu (p_1 + p_4)_\mu = 2p_1^\mu p_{4\mu}, \\ u &= (p_1 + p_3)^\mu (p_1 + p_3)_\mu = 2p_1^\mu p_{3\mu}, \end{aligned} \tag{2.10}$$

and the polarization sum is given by[2]:

$$\sum_{h=\pm} \varepsilon_\mu(p,h) \varepsilon_\nu(p,h)^* = \eta_{\mu\nu}. \tag{2.11}$$

Finally, just like the 3-point amplitude studied before, we can demand gauge Ward identity to the four-point result to determine all the unfixed coefficients, this gives the same amplitude as from standard Feynman rules calculation,

$$\begin{aligned} A_4 =& \frac{c_s}{s} [\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4 (t-u) + \varepsilon_1 \cdot \varepsilon_2 (p_1 \cdot \varepsilon_3 p_2 \cdot \varepsilon_4 - p_2 \cdot \varepsilon_3 p_1 \cdot \varepsilon_4) \\ &+ \varepsilon_3 \cdot \varepsilon_4 (p_3 \cdot \varepsilon_1 p_4 \cdot \varepsilon_2 - p_4 \cdot \varepsilon_1 p_3 \cdot \varepsilon_2) + (p_2 \cdot \varepsilon_1 \varepsilon_2 - p_1 \cdot \varepsilon_2 \varepsilon_1) \cdot (p_4 \cdot \varepsilon_3 \varepsilon_4 - p_3 \cdot \varepsilon_4 \varepsilon_3)] \\ &+ \varepsilon_1 \cdot \varepsilon_3 \varepsilon_2 \cdot \varepsilon_4 - \varepsilon_1 \cdot \varepsilon_4 \varepsilon_2 \cdot \varepsilon_3 + \mathcal{P}(2,3,4) \end{aligned} \tag{2.12}$$

Now we can repeat the same exercise for gravity, for two-derivative theory, we first down the structure of the amplitude determined by unitarity and again enumerate all the possible contact terms by dimensional analysis and Lorentz invariance,

$$M_4 = \sum_h M_3^{-h} \frac{1}{s} M_3^h + \sum \varepsilon_{ab,cd,ef} (C_1 \varepsilon_m \cdot p_i \varepsilon_n \cdot p_j + C_2 \varepsilon_m \cdot \varepsilon_n p_i \cdot p_j) + \mathcal{P}(2,3,4), \tag{2.13}$$

with $\mathcal{P}(2,3,4)$ denotes permutation to obtain other channels and the second sum should run over all the possible terms. Now the polarization sum for spin-2 in $d+1$

[2] Note that we did not keep the term with reference momentum as when contracting with conserved current, they completely drop out.

dimension is given by:

$$\sum_{h=\pm} \varepsilon_{\mu\nu}(k,h)\varepsilon_{\rho\sigma}(k,h)^* = \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma} + \frac{1}{2}\eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{d-1}\eta_{\mu\nu}\eta_{\rho\sigma}. \tag{2.14}$$

Finally, we should use diffeomorphism to fix all the unknown coefficients above, the full expression for 4-graviton amplitude is rather lengthy, so we will not keep it here. Instead, we will use double copy to express the four-graviton amplitude in terms of gluon amplitude in a very compact form.

2.2 Soft theorems and EFTs

In this section, our focus shifts to a set of exceptional scalar effective field theories characterized by Lorentz-invariant S-matrices. These theories can be classified based on their soft properties, which are intricately connected to symmetry in the conventional field theory framework. We begin by revisiting these properties and closely follow the treatment in [1, 57].

Considering scalar field theory with the following global shift symmetry:

$$\phi(x) \to \phi(x) + a \tag{2.15}$$

The field ϕ is a Goldstone boson and we can insert the Noether current associated with the shift symmetry into the vacuum state and the one particle state of the Goldstone boson $\langle\phi(p)|$:

$$\langle\phi(p)| J^\mu(x) |0\rangle = ip^\mu F e^{ip\cdot x}, \tag{2.16}$$

where the right-hand-side is fixed by Lorentz invariance and current conservation (since $p_\mu p^\mu = 0$) up to a dimensionless overall constant F. Inserting the current between incoming and outgoing states then gives

$$\langle out| J^\mu(0) |in\rangle = -\frac{p^\mu}{p^2} F \langle out + \phi(p)|\, in\rangle + R^\mu(p), \tag{2.17}$$

where p^μ is the difference between the momenta of the in and out states. Similar to the off-shell amplitude, the first term on the right hand side contains a pole from a Goldstone boson and there should be a second term that is with no pole structure $R^\mu(p)$. Multiplying by p^μ and the Left-hand side due to the current is conserved so is zero,

$$\langle out + \phi(p)|\, in\rangle = \frac{1}{F} p \cdot R. \tag{2.18}$$

From this, we immediately see that the amplitude for ϕ production vanishes in the soft limit, if the regular term $R^\mu(p)$ has no singular structure:

$$\lim_{p\to 0} \langle out + \phi(p)|\, in\rangle = \mathcal{O}(p). \tag{2.19}$$

This is the famous Adler zero [55]. In fact, the $R^\mu(p)$ has no pole structure in p require the absence of the cubic interaction. Consider the Goldstone has ϕ^3 interaction, then the propagator will develop the following pole $\frac{1}{(p_1+p)^2} \to \frac{1}{2p_1\cdot p}$, so for fixed direction of p_μ the $p \cdot R$ could cancel the pole and becomes finite term. From the amplitude point of view, this is also saying that the cubic interaction can not have vanishing soft limit.

We can also consider a scalar theory with a higher shift symmetry similarly:

$$\delta\phi = \theta_{\mu_1 \ldots \mu_k} x^{\mu_1} \ldots x^{\mu_k} + \ldots, \tag{2.20}$$

where θ is a constant and the ellipsis denote field-dependent terms that we will not need to consider. Following a similar argument above, and Fourier transforming $x^{\mu_1} \ldots x^{\mu_k}$ to momentum space, implying a higher-order Adler zero (The construction of pole structure terms and no pole terms are the same, expect now that we have more index from the current so one need build the numerator with more p_μ also obey the condition for higher form current):

$$\lim_{p\to 0} \langle out + \phi(p)|\, in\rangle = \mathcal{O}\left(p^{k+1}\right). \tag{2.21}$$

The non- linear sigma model (NLSM), the Dirac-Born-Infeld (DBI) theory, and special Galileon (sGal) theories correspond to $k = 0, 1, 2$, respectively. Translating this into amplitude gives,

$$\lim_{p\to 0} A(p) = \mathcal{O}\left(p^{k+1}\right) \tag{2.22}$$

From a scattering amplitude point of view, this behavior arises from nontrivial cancellations among Feynman diagrams and is therefore referred to as an enhanced soft limit.

2.2.1 Soft Bootstrap

In the following, we will utilize the soft limit as input to directly bootstrap the amplitude. We will observe that these results align with those obtained from Lagrangians exhibiting shift symmetry.

Firstly, when considering only scalars, our variables are limited to momentum. Due to Lorentz symmetry, at the level of three-point, the scalar amplitude can only be a constant thanks to momentum conservation and massless condition $p_i^2 = p_i \cdot p_j = 0$. Consequently, there are no vanishing soft limits. Proceeding to the four-point scenario, once again, the constant amplitude would seem like the most straightforward choice. However, as there are no vanishing soft limits, let's begin with the case of two derivatives. For identical scalars, we easily observe:

$$A_4 = s + t + u = 0. \tag{2.23}$$

So such amplitude vanishes for two derivative scalar. However, if the scalars have a color structure then

$$A_4^{NLSM} = c_s s + c_t t + c_u u, \tag{2.24}$$

where $c_s = f_{a_1 a_2 b} f_{b a_3 a_4}$ with f_{abc} being the $SU(N)$ flavour group structure constant. Now, this amplitude is non-trivial and has the desired vanishing soft limits. Similarly, if we consider the four-derivatives and six-derivatives, it's very straightforward to write down that

$$\begin{aligned} A_4^{DBI} &= s^2 + t^2 + u^2, \\ A_4^{sGal} &= s^3 + t^3 + u^3. \end{aligned} \tag{2.25}$$

At the level of four-point, their soft limit is trivial to see, so we move on to the six-point, our strategy would be similar to our bootstrap approach before, firstly write down all the terms that satisfied the factorization,

$$A_6 \xrightarrow{P^2 \to 0} A_4 \frac{1}{P^2} A_4, \tag{2.26}$$

with P being the exchanged momentum. Hence we consider the color-ordered amplitude for 6-point NLSM and all the terms that needed to have the right factorization limits,

$$A_6^{NLSM} = 4\frac{s_{13}s_{46}}{s_{123}} + cyc(a \to a+2) + \dots \tag{2.27}$$

where the factorization pole $s_{ijk\dots} = (p_i + p_j + p_k + \dots)^2$ and $\dots$ are the left unknown contact terms. By dimensional analysis, the $\dots$ can only be represented as $\sum c_{ij} s_{ij}$, which is the sum over all possible two-derivative terms. Finally, the most non-trivial step in our soft bootstrap approach involves demanding the amplitude to vanish in the soft limit:

$$\lim_{p \to 0} A_6^{NLSM} = 0. \tag{2.28}$$

For example, taking $p_1 \to 0$:

$$\lim_{p_1 \to 0} A_6^{NLSM} = 4\frac{s_{35}s_{62}}{s_{612}} - c_{35}s_{35}, \tag{2.29}$$

where we have implicitly used momentum conservation to set a minimal set of variables, and this fixes $c_{35} = 4$. Repeating the same exercise for other legs or simply by permutation, we completely determine the 6-point amplitude. From a Feynman diagrams point of view, this implies that the six-point exchange diagrams have a non-trivial relation with the contact diagrams. Hence, from a Lagrangian point of view, this implies a connection between quartic terms and sixth-order terms at the perturbation expansion. Indeed, one can keep repeating the same exercise to higher points and this should match with amplitude from the NLSM Lagrangian,

$$\begin{aligned}\mathcal{L}_{\text{NLSM}} =&\text{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right),\\ =&-\text{Tr}\left[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \Phi^2\partial_\mu\Phi\partial^\mu\Phi + \left(\Phi^4\partial_\mu\Phi\partial^\mu\Phi + \tfrac{1}{2}\Phi^2\partial_\mu\Phi\Phi^2\partial^\mu\Phi\right) + \mathcal{O}(\Phi^6)\right],\end{aligned} \tag{2.30}$$

with $U = \exp(i\phi)$. Since Lagrangian is not unique but up to free equation of motion and field redefinition. So a simpler way to do a comparison would be to compute the amplitude from the Lagrangian.

Moving on to the DBI theory, we can repeat the same process and set the six-point amplitude to vanish at order $\mathcal{O}(p^2)$, and this gives,

$$\begin{aligned}A_6^{DBI} =&2\frac{(s_{12}s_{23}+s_{23}s_{31}+s_{31}s_{12})(s_{45}s_{56}+s_{56}s_{64}+s_{64}s_{45})}{s_{123}} + Perms\\ &+3s_{12}s_{34}s_{56} + Perms.\end{aligned} \tag{2.31}$$

Such amplitude is described by the DBI Lagrangian,

$$\begin{aligned}\mathcal{L}_{DBI} =&\frac{1}{\lambda}(\sqrt{1-\lambda(\partial\phi)^2}-1),\\ =&-\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{8}(\partial\phi)^4 - \frac{\lambda^2}{16}(\partial\phi)^6 + \dots\end{aligned} \tag{2.32}$$

We could extend our analysis to higher derivative theories for sGal. However, the expression for the six-point amplitude and Lagrangian become considerably lengthy, so we will not record it here. Nonetheless, it's worth noting that the bootstrap procedure remains unchanged and applicable in these cases as well.

We reiterate that the information regarding the shift symmetry is encoded within the S-matrix. Specifically, at the 6-point level, the exchange diagrams and contact diagrams in the soft limit must cancel each other out, ensuring that the soft limit evaluates to zero.

2.3 Color/Kinematic duality and double copy

In this section, we will review a remarkable property of the scattering amplitudes known as color-kinematics (CK) duality. This states that gauge theory amplitudes can be written in such a way that kinematic numerators obey relations analogous to Jacobi relations for their color factors [82]. Using this decomposition, it is then possible to obtain gravitational amplitudes by replacing color factors with another set of kinematic numerators, implying a general relation between gauge and gravitational scattering amplitudes known as the double copy, which was first seen in the context string amplitudes in the form of the KLT relations [69].

We will start with double copy of states, starting with the Yang-Mills ε_μ and consider the tensor product:

$$\varepsilon^\mu \tilde{\varepsilon}^\nu = \underbrace{\frac{1}{2}\left(\varepsilon^\mu \tilde{\varepsilon}^\nu + \varepsilon^\nu \tilde{\varepsilon}^\mu - \frac{2}{d-2}\eta^{\mu\nu}\right)}_{\text{graviton}} + \underbrace{\frac{1}{2}\left(\varepsilon^\mu \tilde{\varepsilon}^\nu - \varepsilon^\nu \tilde{\varepsilon}^\mu\right)}_{\text{B-field}} + \underbrace{\left(\frac{1}{d-2}\eta^{\mu\nu}\right)}_{\text{dilaton}}, \tag{2.33}$$

where the first one is the graviton and is symmetric and traceless, and the second one is the antisymmetric B-field which will only be non-zero if we consider two different polarizations. The last one is the trace term which is referred to as dilaton.

So with such mapping in mind, the double copy of pure Yang-Mills theory in d-dimension will usually give the so-called $\mathcal{N}=0$ supergravity, whose action:

$$S = \int d^{d+1}x\sqrt{-g}\left[-\frac{1}{2}R + \frac{1}{2(d-1)}\partial^\mu\phi\partial_\mu\phi + \frac{1}{6}e^{-4\phi/(d-1)}H^{\lambda\mu\nu}H_{\lambda\mu\nu}\right]. \tag{2.34}$$

where ϕ is the dilaton and $H_{\mu\nu}$ is the field strength of the two-index anti symmetric tensor $B_{\mu\nu}$. A few comments are in order. In the explicit examples we consider below, we focus solely on the tree-level gravity amplitude. Due to dilaton conservation, all dilaton contributions completely decouple from the graviton S-matrix. However, when extending to loop-level, the dilaton state will mix with the gravity amplitude [83], requiring techniques like generalized unitarity cuts to extract pure Einstein Gravity [84].

Now we can start with explicit examples, as we have shown before the three-point graviton amplitude is simply the square of gluon amplitude,

$$M_3 = A_3^2, \tag{2.35}$$

where we have set the coupling constant to be 1. Moving on to 4-point, the color-dressed gluon amplitude can be written as,

$$A_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}. \tag{2.36}$$

where n_i are the kinematic numerators and defined by (2.12), and c_i are color factors obeying the Jacobi relation:

$$c_s + c_t + c_u = 0. \tag{2.37}$$

If we express c_t in terms of c_s and c_u using (2.37), then (2.36) can be written as

$$A_4 = c_s A_{1234} - c_u A_{1342}, \tag{2.38}$$

where the color-ordered amplitudes are given by

$$\begin{aligned} A_{1234} &= \frac{n_s}{s} - \frac{n_t}{t}, \\ A_{1324} &= \frac{n_t}{t} - \frac{n_u}{u}. \end{aligned} \tag{2.39}$$

The numerators are related by exchanges:

$$n_t = -n_s\big|_{2\leftrightarrow 4}, \qquad n_u = -n_s\big|_{2\leftrightarrow 3}, \tag{2.40}$$

and obey an analog of the Jacobi relation in (2.37):

$$n_s + n_t + n_u = 0, \tag{2.41}$$

which is known as the kinematic Jacobi relation and encodes color/kinematics duality [70]. The double copy states that gravitational amplitudes can be obtained from color-dressed gluon amplitudes by replacing the color factors with kinematic numerators:

$$M_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}, \tag{2.42}$$

where we have set the gravitational coupling to 1.

Generalized dimensional reduction [85] of the above gluon and graviton amplitudes implies a double copy for scalars exchanging gluons and gravitons, respectively. The basic idea is that d-dimensional scalars arise from $(d+1)$-dimensional polarisation vectors which point along the internal direction and are therefore orthogonal to d-dimensional momenta. In particular, writing the gravity polarisations in terms of polarisation vectors and taking the polarisation vectors to satisfy $\epsilon_a^\mu \epsilon_{b,\mu} = 1$ and $k_a^\mu \epsilon_{b,\mu} = 0$ (where $a \neq b$ are particle labels), the first line of (2.39) reduces to

$$A_\phi^{1234} = \frac{t-u}{s} - \frac{u-s}{t}, \tag{2.43}$$

which describes massless adjoint scalars exchanging a gluon. From this expression and (2.40) we can then read off that $n_s = t-u$, $n_t = u-s$, and $n_u = s-t$. Squaring the numerators according to (2.42) and noting that $s+t+u=0$ then gives

$$M_4^\phi = -4\left(\frac{tu}{s} + \frac{us}{t} + \frac{st}{u}\right), \tag{2.44}$$

which describes massless scalars exchanging a graviton and agrees with the generalized dimensional reduction of (2.42). Note that the scalar amplitudes live in the same spacetime dimension as the gluon and graviton amplitudes, which is why we refer to this as generalized dimensional reduction.

The double copy has been shown to hold for any multiplicity at tree-level [73, 74] and to a very high order at loop level [75–77]. We review the n-point statement here, for the n-point YM amplitude

$$A_n = \sum_i \frac{c_i n_i}{D_i} \tag{2.45}$$

where D_i is the propagator, if the kinematic numerators obey the same Jacobi identity as the color, then we replace the color factor with the kinematic numerator and obtain:

$$M_n = \sum_i \frac{n_i^2}{D_i} \tag{2.46}$$

Remarkably, this yields the n-point gravity amplitude for Einstein gravity! It's intriguing to understand why such a construction is correct from the bootstrap procedure. Firstly, the 'squaring' expression clearly preserves Lorentz invariance and has the correct pole structure. As a two-derivative theory, the only thing left is diffeomorphism symmetry, which, as we will see now, is simply a consequence of color/kinematic duality.

We know that the gauge symmetry from Yang-Mills, $\varepsilon_\mu \to \varepsilon_\mu + p_\mu$, is invariant. This implies

$$n_i \to n_i + \delta_i, \qquad \delta_i = n_i|_{\varepsilon_\mu \to p_\mu}. \tag{2.47}$$

Then the invariant of the amplitude implies that,

$$\sum_i \frac{c_i \delta_i}{D_i} = 0. \tag{2.48}$$

We do not need to know the explicit expression for δ_i, but the only possible identity needed is the Jacobi identity of the color structure. So, if another function also satisfies the same Jacobi identity, in other words, if we have color/kinematic duality, this implies that,

$$\sum_i \frac{n_i \delta_i}{D_i} = 0. \tag{2.49}$$

Now we are ready to perform the same analysis to the double-copy expression, the gravity amplitude should obey linearized diffeomorphisms

$$\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + p_{(\mu} q_{\nu)} \tag{2.50}$$

with q_μ the reference momentum obeying $p_\mu q^\mu = 0$, and the parenthesis denote symmetrization of spacetime indices. Finally, for the double copy expression eq((2.46)) under the linearized diffeomorphism gives,

$$\begin{aligned} \mathcal{M}_n =& \sum_i \frac{n_i^2}{D_i}, \\ \mathcal{M}_n \to& \mathcal{M}_n - i\left\{\sum_i \frac{\delta_i \tilde{n}_i|_{\tilde{\varepsilon}\to q}}{D_i} + \sum_i \frac{n_i|_{\varepsilon\to q}\,\tilde{\delta}_i}{D_i}\right\}, \end{aligned} \tag{2.51}$$

with the last two terms vanishing due to equation (2.49) as we discussed above, the double copy expression indeed exhibits diffeomorphism symmetry. From a bootstrap perspective, we clearly see why the double copy gives the correct Gravity amplitude! Generally speaking, gauge symmetry + color/kinematic duality ⇒ diffeomorphism symmetry.

2.4 Review on Cosmological correlators

Considering particles as the irreducible representation of the Poincare group, the Minkowski space story above is pretty beautiful and well-understood by now. From the effective field theory perspective, any new physics beyond this framework will require a new energy scale. For example, to have a UV complete tree-level four-graviton scattering, one can show that with the new energy scale and hence higher dimensional operators, the natural candidate for such amplitude would be the Virasoro-Shapiro Amplitude [86] with the corresponding energy scale being the string scale. However, accessing such high energy scales may not be feasible in the near future. On the other hand, there is a natural energy scale—the Hubble scale during inflation—where even the theoretical understanding of the observables is still premature, and experimentally it has the potential to be measured in the future. In this thesis, we will focus on de-sitter correlators while the inflationary correlators can be obtained by understanding the perturbed dS correlators.

2.4.1 dS correlators

We will now switch our attention to cosmological correlators. We will work in the Poincaré patch of dS_4 with unit radius:

$$\mathrm{d}s^2 = (1/\eta)^2(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2), \tag{2.52}$$

where $-\infty < \eta < 0$ is the conformal time and $\vec{x}$ denotes the Euclidean boundary directions, with individual components x^i, $i = 1, 2, 3$ and we set hubble constant H to be 1. Cosmological correlators (or in-in correlators) can be computed as follows:

$$\left\langle \phi(\vec{k}_1)...\phi(\vec{k}_n) \right\rangle = \frac{\int \mathcal{D}\phi\, \phi(\vec{k}_1)...\phi(\vec{k}_n)\, |\Psi[\phi]|^2}{\int \mathcal{D}\phi\, |\Psi[\phi]|^2}, \tag{2.53}$$

where ϕ represents the value of a generic bulk field in the future boundary Fourier transformed to momentum space, $\vec{k}_a$ are boundary momenta, and $\Psi[\phi]$ is the cosmological wavefuntion, which is a functional of ϕ. For simplicity, we are considering a scalar field but in general, we should integrate over the boundary values of all the bulk fields, including the metric.

The wavefunction can be perturbatively expanded as follows:

$$\ln \Psi[\phi] = -\sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \psi_n \left(\vec{k}_1, ... \vec{k}_n \right) \phi(\vec{k}_1)...\phi(\vec{k}_n), \tag{2.54}$$

where the wavefunction coefficients ψ_n can be expressed as

$$\psi_n = \delta^d(\vec{k}_T) \left\langle\left\langle \mathcal{O}(\vec{p}_1) ... \mathcal{O}(\vec{p}_n) \right\rangle\right\rangle, \tag{2.55}$$

where $\vec{k}_T = \vec{k}_1 + ... + \vec{k}_n$ and the object in double brackets can be treated as a CFT correlator in the future boundary [11, 13, 16, 32, 87–90]. Note that momentum is conserved along the boundary but the total energy defined as

$$E = \sum_{a=1}^{n} k_a, \tag{2.56}$$

where $k_a = |\vec{k}_a|$, is not conserved. The wavefunction coefficients in (2.55) can be computed by analytically continuing AdS Witten diagrams [18, 35] and will be our main focus in the thesis. To be more precise, one should wick rotate $\eta \to iz$ and $R_{dS} \to -iR_{AdS}$ [91]. In practice, we will drop the momentum conserving delta function when referring to the wavefunction coefficients. We will also analytically continue to Euclidean AdS when performing conformal time integrals.

For spinning fields we define the wavefunction coefficients in the helicity basis,

$$\ln \Psi[\gamma] = -\sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \psi_n^{h_1 \ldots h_n} \left(\vec{k}_1, \ldots \vec{k}_n\right) \gamma^{h_1}(\vec{k}_1) \ldots \gamma^{h_n}(\vec{k}_n), \tag{2.57}$$

where h_a are helicities and are summed over. In order to apply the bootstrap methods outlined later in this section it is necessary to additionally define the so called "trimmed" wavefunction coefficients [92],

$$\psi_n^{h_1 \ldots h_n}(\vec{k}_1 \ldots \vec{k}_n) = \sum_{\text{contractions}} \left[\epsilon_1^{h_1} \ldots \epsilon_n^{h_n} \left(\vec{k}_1\right)^{\alpha_1} \ldots \left(\vec{k}_n\right)^{\alpha_n}\right] \tilde{\psi}_n(\vec{k}_1, \ldots, \vec{k}_n), \tag{2.58}$$

where $\left(\vec{k}_a\right)^{\alpha_a}$ denotes the tensor product of α_a copies of $\vec{k}_a$, whose indices contract with those of the polarisation tensors on the left. The sum tells us that generically each wavefunction coefficient will contain several such trimmed terms and each one of these must be determined individually in the bootstrap approach. In the next subsections, we will describe how to compute wavefunction coefficients using Witten diagrams.

Witten Diagrams

Next, our goal is to explain how to compute wavefunction coefficients. For gluon we will use Feynman rules in axial gauge in AdS momentum space first derived in [37, 93] with

$$\mathrm{d}s^2 = (1/z)^2(\mathrm{d}z^2 + \mathrm{d}\vec{x}^2), \tag{2.59}$$

where we have set the AdS radius R to be 1. For notational simplicity, we will adopt conventions where factors of i will not appear in the Feynman rules. For gluons in axial gauge, it has the following bulk-to-bulk propagators in momentum space:

$$G^A_{ij}(z,z',\vec{k}) = -\int_0^\infty \omega d\omega \frac{z^{\frac{1}{2}} J_{\frac{1}{2}}(\omega z) J_{\frac{1}{2}}(\omega z')(z')^{\frac{1}{2}}}{k^2+\omega^2} H_{ij}, \tag{2.60}$$

where $\vec{k}$ is the momentum flowing through the propagator along the boundary directions, $k = |\vec{k}|$, J_ν is a Bessel function of the first kind, and

$$H_{ij} = \eta_{ij} + \frac{k_i k_j}{\omega^2}, \tag{2.61}$$

where η_{ij} is the Euclidean boundary metric. Note that we have Wick rotated $\eta \to iz$, where $0 < z < \infty$, in order to make conformal time integrals manifestly convergent. The bulk-to-boundary propagator is given by

$$G^A_i(z,\vec{k}) = \epsilon_i \sqrt{\frac{2k}{\pi}} z^{\frac{1}{2}} K_{\frac{1}{2}}(kz), \tag{2.62}$$

where $\vec{k}$ and $\vec{\epsilon}$ are the boundary momentum and polarisation vector, respectively, which satisfy $\epsilon \cdot \epsilon = \epsilon \cdot k = 0$ (where the dot denotes an inner product of 3-vectors), and K_ν is a modified Bessel function of the second kind.

The color-ordered Feynman vertices for gluons have the same structure as in flat space but the indices only run over the boundary directions in axial gauge. In more detail, the three and four-point vertices are

$$\begin{aligned} V_{jkl}(\vec{k}_1,\vec{k}_2,\vec{k}_3) &= \left(\eta_{jk}(\vec{k}_1-\vec{k}_2)_l + \eta_{kl}(\vec{k}_2-\vec{k}_3)_j + \eta_{lj}(\vec{k}_3-\vec{k}_1)_k\right), \\ V_{jklm} &= 2\eta_{jl}\eta_{km} - \left(\eta_{jk}\eta_{lm} + \eta_{jm}\eta_{kl}\right), \end{aligned} \tag{2.63}$$

where we have set the gluon coupling $g = \sqrt{2}$ for convenience. When computing color-ordered 4-point wavefunctions, it will be convenient to split the 4-point contact diagram into an s and t-channel contribution. After dressing the second line of (2.63) with polarisations we then get the following quantities:

$$\begin{aligned} V^s_c &= \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3, \\ V^t_c &= \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4. \end{aligned} \tag{2.64}$$

Finally, we note that for each interaction vertex, we must perform an integral over the AdS radius along with the measure $\sqrt{\det g} = z^{-4}$. In practice, there will be additional factors of z coming from the inverse metrics used to contract indices.

The bulk-to-boundary and bulk-to-bulk propagators for gravitons in axial gauge are given by

$$G^{\gamma}_{ij}(z,\vec{k}) = \epsilon_{ij}\sqrt{\frac{2}{\pi}}z^{-2}(kz)^{\frac{3}{2}}K_{\frac{3}{2}}(kz), \tag{2.65}$$

$$G^{\gamma}_{ij,kl}\left(z,z',\vec{k}\right) = \frac{-\left(zz'\right)^{-\frac{1}{2}}}{2}\int_0^{\infty} d\omega J_{\frac{3}{2}}(\omega z)J_{\frac{3}{2}}\left(\omega z'\right)\frac{\omega\left(H_{ik}H_{jl}+H_{il}H_{jk}-H_{ij}H_{kl}\right)}{k^2+\omega^2}, \tag{2.66}$$

where $\epsilon_{ij} = \epsilon_i\epsilon_j$ is a graviton polarisation. The Feynman rules for scalars coupled to gluons and gravitons can then be deduced by setting $\epsilon_a \cdot \epsilon_b = 1$ and $\epsilon_a \cdot k_b = 0$, where $a \neq b$ and the polarisations correspond to external scalars. For example, the scalar bulk-to-boundary propagator is

$$G^{\phi}(z,\vec{k}) = \sqrt{\frac{2}{\pi}}z^{3/2}k^{\nu}K_{\nu}(kz), \tag{2.67}$$

where $\nu = 1/2$ for conformally coupled scalars (which descend from gluons) and $\nu = 3/2$ for massless scalars (which descend from gravitons). In general, the $\nu = \Delta - d/2$ is related to the conformal mass of the scalar:

$$m^2 = \Delta(\Delta - d). \tag{2.68}$$

Clearly, for massless scalar the conformal mass is simply zero, while for conformally coupled scalar $\nu = 1/2$ means $m^2 = -2$, and it implies the stress tensor of the theory is traceless and hence enjoys conformal symmetry.

Moreover, the three-point scalar-scalar-gluon vertex can be deduced from the first line of (2.63) by dressing two of the legs with polarisations and performing the generalized dimensional reduction procedure described above:

$$v_i(\vec{k}_1,\vec{k}_2,\vec{k}_3) = (k_1 - k_2)_i, \tag{2.69}$$

where leg three is a gluon with index i.

CHAPTER 3

EFTs and Soft theorems

In this chapter, we explore the relationship of soft limits and hidden symmetries in de-Sitter space. To analyze soft limits in general spacetime and conformal dimensions, we first reformulate the Witten diagram calculation in terms of conformal generators in future boundary acting on contact diagrams. We begin by reviewing the exceptional scalar theory in dS, characterized by Lagrangians with shift symmetries. Then we use enhanced soft limits to fix the masses and 4-point couplings of the NLSM, DBI, and sGal theories in dS, and comment on the the double copy of 4-point wavefunction coefficients. Finally, we illustrate how our method extends to higher points, providing explicit examples with the 6-point coupling of the NLSM and DBI theory.

3.1 Review

In this section, we will review the Lagrangians for the NLSM, DBI, and sGal theories in dS which is a generalization of the flat space story reviewed in section 2.2 and explain how to compute cosmological wavefunction coefficients in terms of conformal generators in future boundary acting on contact diagrams. This is referred as the differential representation of the Witten diagram [94–98].

3.1.1 de Sitter Lagrangians

It is easy to write down the Lagrangian for the NLSM in dS_{d+1}:

$$\frac{\mathcal{L}_{\mathrm{NLSM}}}{\sqrt{-g}} = \mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right), \quad U = \exp\left(i\phi\right), \tag{3.1}$$

where ϕ is in the adjoint of an $SU(N)$ flavour symmetry. No masses or curvature corrections are allowed because they would spoil the shift symmetry in (2.15). Later on we will deduce this fact from enhanced soft limits of the wavefunction coefficients. While the previous parametrization make the shift symmetry manifest, in practice, it is also convenient to use the parametrization $U = (\mathbb{I}+\Phi)(\mathbb{I}-\Phi)^{-1}$ when we do perturbation expansion. Expanding the Lagrangian in Φ then gives,

$$\frac{\mathcal{L}_{\mathrm{NLSM}}}{\sqrt{-g}} = -\mathrm{Tr}\left[\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \Phi^2\partial_\mu\Phi\partial^\mu\Phi + \left(\Phi^4\partial_\mu\Phi\partial^\mu\Phi + \tfrac{1}{2}\Phi^2\partial_\mu\Phi\Phi^2\partial^\mu\Phi\right) + \mathcal{O}(\Phi^6)\right]. \tag{3.2}$$

The Lagrangians for the DBI and sGal theory does not trivially lift to dS and was recently derived from the following shift symmetry [66]:

$$\delta\phi = \theta_{A_1\ldots A_k} X^{A_1}\ldots X^{A_k} + \ldots, \tag{3.3}$$

where X^A are embedding coordinates satisfying $-\left(X^1\right)^2 + \sum_{A=2}^{d+2}\left(X^A\right)^2 = 1$ and the ellipsis denote field-dependent terms. This symmetry fixes the mass to be $m^2 = -\kappa(\kappa+d)$. In the DBI case ($\kappa = 1$), the resulting action is quite simple and given by

$$\frac{\mathcal{L}_{\mathrm{DBI}}}{\sqrt{-g}} = \frac{1}{(1-\phi^2)^{\frac{d+1}{2}}}\sqrt{1 - \frac{\nabla\phi\cdot\nabla\phi}{1-\phi^2}}, \tag{3.4}$$

where $\nabla\phi\cdot\nabla\phi = \partial_\mu\phi\partial^\mu\phi$. In the sGal case ($\kappa = 2$) the Lagrangian is very nontrivial:

$$\frac{\mathcal{L}_{\text{sGal}}}{\sqrt{-g}} = \Bigg[\sum_{j=0}^{d} \frac{(1+\phi)^{d+1-j} + (-1)^j(1-\phi)^{d+1-j}}{2^{j+1}(1-\phi^2)^{\frac{d+4}{2}}\Gamma(j+3)} \left((j+1)f_{j+1}(\phi) - (j+2)f_j(\phi)\right)\partial^\mu\phi\partial^\nu\phi X^{(j)}_{\mu\nu}(\phi) \\ - \frac{2}{d+2}\left(1 - \frac{(1+\phi)^{d+2} + (1-\phi)^{d+2}}{2(1-\phi^2)^{\frac{d+2}{2}}}\right)\Bigg], \tag{3.5}$$

where $X^{(j)}_{\mu\nu}$ is defined recursively as $X^{(n)}_{\mu\nu} = -n(\nabla_\mu\nabla^\alpha\phi)X^{(n-1)}_{\alpha\nu} + g_{\mu\nu}(\nabla^\alpha\nabla^\beta\phi)X^{(n-1)}_{\alpha\beta}$ with $X^0_{\mu\nu} = g_{\mu\nu}$, and

$$f_j(\phi) = {}_2F_1\left(\frac{d+4}{2}, \frac{j+1}{2}; \frac{j+3}{2}; \frac{\nabla\phi\cdot\nabla\phi}{4(1-\phi^2)}\right). \tag{3.6}$$

In the remainder of this paper, we will demonstrate that the masses and couplings of these theories can be fixed by demanding that the wavefunction coefficients have vanishing soft limits analogous to (2.21).

3.1.2 Boundary Conformal generators

To study soft limits of wavefunction coefficients, it is natural to work in dS momentum space [16, 37, 45], which is also the standard language used for cosmology (see [11–13, 15, 32, 38, 88, 99–105] for some recent developments). Another technique we will employ is to express the wavefunction coefficients in terms of boundary conformal generators acting on contact diagrams [94, 95, 97, 106–110]. Soft limits can then computed by Taylor expanding bulk-to-boundary propagators in the contact diagram, acting on them with boundary conformal generators, and using the equations of motion to remove terms which are not linearly independent. Starting with a general effective action with unfixed masses and couplings (including curvature corrections), we then find that imposing enhanced soft limits of the tree-level 4-point wavefunction coefficients fixes all the masses and 4-point couplings for the DBI and sGal theories in agreement with the Lagrangians constructed in [66]. For the NLSM, we find that enhanced soft limits forbid mass terms or curvature corrections, so the Lagrangian can be trivially lifted from flat space. These results, in turn, allow us to fix all the parameters of the generalized double copy prescription proposed in [110], which relates the 4-point tree-level wavefunction coefficient of the NLSM model to those of the DBI and sGal theories [1]. Above four points, there must be non-trivial cancellations between contact and exchange Witten diagrams in order to have enhanced soft limits. Since lower-point couplings feed into the exchange diagrams, in principle this allows us to fix all higher-point couplings using a bootstrap procedure, which we demonstrate for the NLSM and DBI theory at six points. The method can also be applied to the sGal theory above four points, but the Witten diagrams become very numerous so we save that for future work. In this section, we will review how to represent the wavefunction coefficient in terms of boundary conformal generators. To start with, The scalar operators $\mathcal{O}$ in dS have scaling dimension Δ, and are dual to scalar fields ϕ in the bulk with mass,

$$m^2 = \Delta(d - \Delta). \tag{3.7}$$

In the previous subsection, we claimed that shift symmetries fix $m^2 = -\kappa(\kappa + d)$ where $\kappa = 0, 1, 2$ for the NLSM, DBI, and sGal theories, respectively. The corresponding scaling dimensions are therefore $\Delta = d + \kappa$. We will show that these values are required by enhanced soft limits of the wavefunction coefficients.

The bulk-to-boundary propagators in this background satisfy the free equations of motion $(\mathcal{D}_k^2 + m^2)\phi^\nu = 0$, where

$$\mathcal{D}_k^2 = \eta^2\partial_\eta^2 + (1-d)\eta\partial_\eta + \eta^2 k^2, \tag{3.8}$$

with $k = |\vec{k}|$. The solutions are given by

$$\phi^\nu(k,\eta) = (-1)^{\nu-\frac{1}{2}}\sqrt{\frac{\pi}{2}}k^\nu \eta^{d/2} H_\nu(-k\eta), \tag{3.9}$$

where $\nu = \Delta - d/2$, H_ν is a Hankel function of the second kind, and the normalization

[1]The double copy was first proposed in the context of scattering amplitudes, relating graviton amplitudes to the square of gluon amplitudes [70, 75]. For recent work on the double copy for (A)dS correlators see for example [21, 25, 28, 97, 98, 111–118].

is chosen for convenience. We then define an n-point contact diagram as follows:

$$\mathcal{C}_n^{\Delta} = \int \frac{d\eta}{\eta^{d+1}} \prod_{a=1}^{n} \phi_a, := \int \frac{d\eta}{\eta^{d+1}} U_{1,n}(\eta) \tag{3.10}$$

where a labels an external leg, k_a is the magnitude of the boundary momentum of that leg, and $\phi_a = \phi^{\nu}(k_a, \eta)$. From now on we use the short-handed notation for the product of the bulk-to-boundary propagator $U_{1,n}(\eta)$ from leg 1 to leg n.

As shown in [94], soft limits of wavefunction coefficients take a particularly simple form when Witten diagrams are expressed in terms of certain differential operators constructed from boundary conformal generators acting on contact diagrams. The boundary conformal generators are given by

$$\begin{aligned} P^i &= k^i, \\ D &= k^i \partial_i + (d - \Delta), \\ K_i &= k_i \partial^j \partial_j - 2k^j \partial_j \partial_i - 2(d - \Delta)\partial_i, \\ M_{ij} &= k_i \partial_j - k_j \partial_i, \end{aligned} \tag{3.11}$$

where $\partial_i = \frac{\partial}{\partial k^i}$. We will collectively denote the generators by $\mathcal{D}^A \in \{P^i, M_{ij}, D, K_i\}$, where A is an adjoint index. Note that wavefunction coefficients satisfy the following conformal Ward identities:

$$\sum_{a=1}^{n} \mathcal{D}_a^A \Psi_n = 0. \tag{3.12}$$

Using boundary conformal generators we can define the following differential operators which will play an important role throughout the paper (This is the same operator as the Casimir operator up to mass term):

$$\mathcal{D}_a \cdot \mathcal{D}_b = \frac{1}{2} \left(P_a^i K_{bi} + K_{ai} P_b^i - M_{a,ij} M_b^{ij} \right) + D_a D_b, \tag{3.13}$$

where $\mathcal{D}_a$ is a boundary conformal generator defined in terms of the boundary momentum associated with leg a. Acting on a pair of bulk-to-boundary propagators (3.9) associated with legs a and b, the operator in (3.13) satisfies the following useful identity:

$$(\mathcal{D}_a \cdot \mathcal{D}_b)\,(\phi_a\phi_b) = \eta^2[\partial_\eta\phi_a\partial_\eta\phi_b + (\vec{k}_a \cdot \vec{k}_b)\phi_a\phi_b]. \tag{3.14}$$

Hence acting with $\mathcal{D}_a \cdot \mathcal{D}_b$ on a pair of bulk-to-boundary propagators is equivalent (up to a sign) to acting with a single $\nabla_a \cdot \nabla_b$, where ∇_a is a bulk covariant derivative acting on leg a. To simplify notation we will define $\hat{s}_{ab} = \mathcal{D}_a \cdot \mathcal{D}_b$.

In section 3.3 we will also consider exchange diagrams so we need to define bulk-to-bulk propagators, $G_\nu(k, \eta, \tilde{\eta})$. For our purposes, we will only need to use the following property:

$$[(\mathcal{D}_1 + \ldots + \mathcal{D}_p)^2 + m^2]^{-1}\mathcal{C}_n^{\Delta} = \int \frac{d\eta}{\eta^{d+1}}\frac{d\tilde{\eta}}{\tilde{\eta}^{d+1}} U_{p+1,n}(\eta) G_\nu(k_{1\ldots p}, \eta, \tilde{\eta}) U_{1,p}(\tilde{\eta}). \tag{3.15}$$

This follows from the equation of motion

$$(\mathcal{D}_k^2 + m^2)G_\nu = \eta^{d+1}\delta(\eta - \tilde{\eta}), \tag{3.16}$$

and the following identity:

$$(\mathcal{D}_{1\ldots p}^2 U_{1,p})U_{p+1,n} = (\mathcal{D}_1 + \ldots + \mathcal{D}_p)^2 U_{1,n}, \tag{3.17}$$

where in the left-hand side $\mathcal{D}_{1\ldots p}^2$ is defined in (3.8) with $k = |\vec{k}_1 + \ldots + \vec{k}_p|$ and $p < n$. For more details, see for example section 2.2 of [94].

3.2 Four-point soft limits

In this section, we will fix the masses and 4-point couplings of the NLSM, DBI, and sGal theories in de Sitter space from enhanced soft limits of their wavefunction coefficients. Our strategy will be to express the Witten diagrams in terms of differential operators acting on a contact diagram and then take the soft limit of a bulk-to-boundary propagator in the contact diagram. The soft limit of bulk-to-boundary propagators can be read off from the series expansion of (3.9) which is schematically given by

$$\phi^{\nu}(k,\eta) \sim \sum_{n=0}^{\infty} \left(a_{2n} + b_{2n} k^{2\Delta-d}\right) k^{2n}. \tag{3.18}$$

We can see that the second series has $k^{2(\Delta-d/2+n)}$ terms which are subleading for positive $\Delta \geq d/2$, which is the main interest in this thesis. In each case of interest, the enhanced soft limits will fix $\Delta = d+\kappa$ where κ is the order of the shift symmetry in the Lagrangian. This sets $\nu = d/2+\kappa$ and ensures that the second series does not contribute to the soft limit. We therefore take the soft limit of the wavefunction to be

$$\begin{aligned} \phi^{\nu}(k,\eta) &= \frac{\mathcal{N}}{\eta^{\Delta-d}} \left(1 + \frac{\eta^2 k^2}{2(2\Delta-d-2)}\right) + \mathcal{O}(k^4), \\ \text{where } \mathcal{N} &= \frac{\Gamma\left(\Delta - d/2\right) 2^{\Delta-d/2-1/2}}{\sqrt{\pi}}. \end{aligned} \tag{3.19}$$

This formula can then be used to study the soft limit of wavefunction coefficients.

3.2.1 NLSM

The effective Lagrangian for the NLSM takes the following form at 4-points:

$$\frac{\mathcal{L}_4^{NLSM}}{\sqrt{-g}} = -\mathrm{Tr}\{\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + \tfrac{1}{2}m^2\Phi^2 + \Phi^2\nabla\Phi\cdot\nabla\Phi + \tfrac{1}{4}C\Phi^4\}, \tag{3.20}$$

where we leave the mass and curvature correction C unfixed. Note that the 2-derivative interaction comes from the naive uplift from flat space and we normalize the coupling to one. The corresponding tree-level flavour-ordered 4-point wavefunction coefficient can be obtained from two Witten diagrams and is given by [110]

$$\begin{aligned} \Psi_4^{NLSM} &= -\delta^3(\vec{k}_T)\left(2\hat{s}_{13} + C - m^2\right)\mathcal{C}_4^{\Delta}, \\ &= -\delta^3(\vec{k}_T)\int \frac{d\eta}{\eta^{d+1}}\left[2\eta^2\left(\vec{k}_1\cdot\vec{k}_3\phi_1\phi_3 + \dot{\phi}_1\dot{\phi}_3\right)\phi_2\phi_4 + (C+\Delta(\Delta-d))\phi_1\phi_2\phi_3\phi_4\right]. \end{aligned} \tag{3.21}$$

If we take a soft limit of $\vec{k}_1$, we find that

$$\begin{aligned}\lim_{\vec{k}_1\to 0}\Psi_4^{NLSM} &= \mathcal{N}\delta^3(\vec{k}_T)\int\frac{d\eta}{\eta^{d+1}}\left[\eta^2\frac{2(\Delta-d)}{\eta^{\Delta-d+1}}\phi_2\dot{\phi}_3\phi_4+\frac{C+\Delta(\Delta-d)}{\eta^{\Delta-d}}\phi_2\phi_3\phi_4\right]+\mathcal{O}(k_1),\\ &= \mathcal{N}\delta^3(\vec{k}_T)\int\frac{d\eta}{\eta^{\Delta+1}}\left[2(\Delta-d)\eta\phi_2\dot{\phi}_3\phi_4+\left(C+\Delta(\Delta-d)\right)\phi_2\phi_3\phi_4\right]+\mathcal{O}(k_1),\\ &= \mathcal{N}\delta^3(\vec{k}_T)\left[2(\Delta-d)D_3+C-\Delta(\Delta-d)\right]\int\frac{d\eta}{\eta^{\Delta+1}}\phi_2\phi_3\phi_4+\mathcal{O}(k_1),\end{aligned} \tag{3.22}$$

where in the final line we have used the definition of the dilatation operator acting on the bulk-to-boundary propagator. We see from (3.22) that the soft limit will vanish to $\mathcal{O}(k_1)$ if $\Delta = d$ and $C = 0$, i.e. if we have a massless scalar and no curvature corrections in agreement with (3.2). We can also see from (3.21) that it is not possible for the soft limit to vanish at higher order since there is no way to cancel the $\vec{k}_1\cdot\vec{k}_3$ term given that the bulk-to-boundary propagators only depend on magnitudes of momenta. Hence, the wavefunction coefficient is simply

$$\Psi_4^{NLSM} = -2\delta^3(\vec{k}_T)\hat{s}_{13}\mathcal{C}_4^{\Delta=d}. \tag{3.23}$$

3.2.2 DBI

At 4-points, the DBI theory can be described by the following general effective Lagrangian (modulo integration by parts and free equations of motion):

$$\frac{\mathcal{L}_4^{DBI}}{\sqrt{-g}} = -\{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{8}(\nabla\phi\cdot\nabla\phi)^2+\tfrac{1}{4!}C\phi^4\}, \tag{3.24}$$

where the 4-derivative interaction (whose coupling constant are set to one) arises from the naive uplift from flat space and we leave the mass and curvature correction C unfixed. The tree-level 4-point wavefunction coefficient can be computed from Witten diagrams and is given by [110]

$$\Psi_4^{DBI} = -\delta^3\left(\vec{k}_T\right)\left(\hat{s}_{12}^2+\hat{s}_{13}^2+\hat{s}_{14}^2+C\right)\mathcal{C}_4^{\Delta}. \tag{3.25}$$

More explicitly, the action of $\hat{s}_{12}^2$ on bulk-to-boundary propagators is given by

$$\hat{s}_{12}^2\phi_1\phi_2 = \eta^4\Big[(\vec{k}_1\cdot\vec{k}_2)^2\phi_1\phi_2 + 2\vec{k}_1\cdot\vec{k}_2\dot{\phi}_1\dot{\phi}_2 + \ddot{\phi}_1\ddot{\phi}_2, \\ + \frac{1}{\eta}\left(2\vec{k}_1\cdot\vec{k}_2\left(\phi_1\dot{\phi}_2 + \dot{\phi}_1\phi_2\right) - k_1^2\phi_1\dot{\phi}_2 - k_2^2\dot{\phi}_1\phi_2 + \dot{\phi}_1\ddot{\phi}_2 + \ddot{\phi}_1\dot{\phi}_2\right) \\ + \frac{1}{\eta^2}\left((2-d)\vec{k}_1\cdot\vec{k}_2\phi_1\phi_2 + \dot{\phi}_1\dot{\phi}_2\right)\Big]. \tag{3.26}$$

We then insert the soft limit for ϕ_1 from equation (3.19).

To fix Δ and C we need to expand the integrand to $\mathcal{O}\left(k_1^2\right)$ and use the equations of motion and integration by parts to eliminate terms which are not independent. One option is to use the equations of motion of the bulk-to-boundary propagators to remove any explicit dependence on k_2^2 in (3.26) (k_2 will still enter in the arguments of ϕ_2). Alternatively, we can apply the equations of motion to leave only terms containing ϕ_2 and $\dot{\phi}_2$ along with factors of k_2^2. This second approach is equivalent to using the identity $H_{\nu-1}(x) = -H_{\nu+1}(x) + \frac{2\nu}{x}H_\nu(x)$ on the Hankel functions which appear in the derivatives of propagators to leave only two independent functions. Removing the explicit dependence on k_2^2 in the first term of (3.25) and summing over cyclic permutations then gives

$$\lim_{\vec{k}_1\to 0}\Psi_4^{DBI} = \mathcal{N}\delta^3\left(\vec{k}_T\right)\int\frac{d\eta}{\eta^{\Delta+1}}\Big[(\Delta-d-1)\left((\Delta-d)\eta^2\ddot{\phi}_2 - 2\eta^3\vec{k}_1\cdot\vec{k}_2\dot{\phi}_2\right)\phi_3\phi_4 \\ + \text{Cyc.}[234] + \left(\Delta(\Delta-d)(4\Delta-3d-1)+C\right)\phi_2\phi_3\phi_4 + \mathcal{O}(k_1^2)\Big], \tag{3.27}$$

where we used the following identity to remove the $\dot{\phi}_a$ terms ($a\in\{2,3,4\}$) at $\mathcal{O}(k_1^0)$:

$$\int\frac{d\eta}{\eta^{\Delta+1}}\eta\partial_\eta\left(\prod_{i=2}^{n}\phi_i\right) \sim \Delta\int\frac{d\eta}{\eta^{\Delta+1}}\left(\prod_{i=2}^{n}\phi_i\right). \tag{3.28}$$

In deriving the above formula, we discarded a total derivative term. This term actually gives divergent contributions at $\eta = 0$ and therefore needs to be regulated, however, these contributions are analytic in at least two momenta and therefore correspond to contact terms which have delta function support when Fourier transformed to position space [32].

From (3.27), we see that the soft limit vanishes to $\mathcal{O}(k_1^2)$ if $\Delta = d+1$ and $C = -(d+1)(d+3)$. Plugging these values into (3.25) gives

$$\Psi_4^{DBI} = -\delta^3\left(\vec{k}_T\right)\left(\hat{s}_{12}^2 + \hat{s}_{13}^2 + \hat{s}_{14}^2 - (d+1)(d+3)\right)\mathcal{C}_4^{\Delta=d+1}. \tag{3.29}$$

Moreover, (3.24) becomes

$$\frac{\mathcal{L}_4^{DBI}}{\sqrt{-g}} = -\{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi - \tfrac{d+1}{2}\phi^2 + \tfrac{1}{8}(\nabla\phi\cdot\nabla\phi)^2 - \tfrac{(d+1)(d+3)}{4!}\phi^4\}. \tag{3.30}$$

From (3.26) we can see that it is not possible for the soft limit to vanish beyond $\mathcal{O}(k_1^2)$ since this term contains a piece proportional to $(\vec{k}_1\cdot\vec{k}_2)^2$ but the soft limit of Witten diagrams coming from the ϕ^4 interaction will only depend on the magnitude k_1. We also note that while the $\mathcal{O}(k_1)$ contribution to the wavefunction coefficient is needed to fix Δ, once this is fixed only the leading soft limit is needed to fix C. This appears to be a general feature in de Sitter space, in contrast to flat space where all the subleading data is needed to fix coefficients.

Let us now compare to the Lagrangian in (3.4) which was derived from shift symmetries. Expanding it to a quartic order gives

$$\begin{aligned}\frac{\mathcal{L}_{\text{DBI}}}{\sqrt{-g}} &= \frac{1}{(1-\phi^2)^{(d+1)/2}}\sqrt{1-\frac{\nabla\phi\cdot\nabla\phi}{1-\phi^2}},\\ &= -\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi - \frac{d+1}{2}\phi^2 + \frac{1}{8}(\nabla\phi\cdot\nabla\phi)^2 - \frac{(d+1)(d+3)}{4!}\phi^4 + \mathcal{O}(\phi^6)\right),\end{aligned} \tag{3.31}$$

where we have used integration by parts and the free equation of motion $\nabla^2\phi = m^2\phi$ to remove a $(\nabla\phi\cdot\nabla\phi)\phi^2$ term. This precisely matches (3.30), which was derived from enhanced soft limits.

3.2.3 sGal

At 4-points, the sGal theory can be described by the following effective action modulo integration by parts and free equations of motion:

$$\frac{\mathcal{L}_4^{sGal}}{\sqrt{-g}} = -\{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{8}(\nabla_\mu\nabla_\nu\phi)^2\nabla\phi\cdot\nabla\phi+\tfrac{1}{8}B(\nabla\phi\cdot\nabla\phi)^2+\tfrac{1}{4!}C\phi^4\}, \quad (3.32)$$

where the 6-derivative term uplifts from flat space and we have normalized its coupling to one while the remaining interaction terms are curvature corrections with unfixed coefficients. The 4-point wavefunction coefficient can be computed from Witten diagrams and is given by [110]

$$\Psi_4^{sGal} = \delta^3(\vec{k}_T)[(\hat{s}_{12}^3+\hat{s}_{13}^3+\hat{s}_{14}^3)+(d-B)(\hat{s}_{12}^2+\hat{s}_{13}^2+\hat{s}_{14}^2)-C]\mathcal{C}_4^{\Delta}. \quad (3.33)$$

The $\hat{s}_{ab}^3$ terms are quite lengthy and can be found in Appendix A.1. The $\hat{s}_{ab}^2$ terms were already considered in the previous subsection.

We will now expand the integrand up to $\mathcal{O}(k_1^2)$ and present the soft limit in parts. After substituting (3.19) we apply equations of motion to eliminate any explicit dependence on k_2^2 in the $\hat{s}_{12}^3$ term and sum over permutations to obtain

$$\begin{aligned}\lim_{\vec{k}_1\to 0}\Psi_4^{sGal} = -\mathcal{N}(\Delta-d-2)\delta^3\left(\vec{k}_T\right)\int\frac{d\eta}{\eta^{\Delta+1}}\Bigg[\eta\Bigg(&(\Delta-d-1)(\Delta-d)\eta^2\\ &+\frac{k_1^2}{2\Delta-d-2}(\Delta-d-3)(\Delta-d-4)\Bigg)\dddot{\phi}_2\\ -3\vec{k}_1\cdot\vec{k}_2\eta^4\ddot{\phi}_2+3(\vec{k}_1\cdot\vec{k}_2)^2\eta^5\dot{\phi}_2\Bigg]&\phi_3\phi_4+\text{Cyc.}[234]+\mathcal{O}(k_1^3)+\ldots,\end{aligned} \quad (3.34)$$

where the ellipsis represents terms that can also arise from 4-derivative and ϕ^4 interactions. We must then set $\Delta = d+2$ in order for the terms displayed above to vanish. When this is substituted into the remaining terms they simplify significantly and we obtain

$$\lim_{\vec{k}_1\to 0}\Psi_4^{sGal} = -\mathcal{N}(B+2d+2)\delta^3\left(\vec{k}_T\right)\int\frac{d\eta}{\eta^{\Delta+1}}\eta^2\left(2\ddot{\phi}_2-2\eta(\vec{k}_1\cdot\vec{k}_2)\dot{\phi}_2+\eta^2(\vec{k}_1\cdot\vec{k}_2)^2\phi_2\right)\phi_3\phi_4 + \text{Cyc.}[234]+\mathcal{O}(k_1^3)+\ldots, \tag{3.35}$$

where the ellipsis denotes terms that can also arise from ϕ^4 interactions. After setting $B=-2(d+1)$ the above terms vanish and the soft limit of the wavefunction coefficient reduces to

$$\lim_{\vec{k}_1\to 0}\Psi_4^{sGal} = \mathcal{N}(4(d+2)^2-C)\delta^3\left(\vec{k}_T\right)\int\frac{d\eta}{\eta^{\Delta+1}}\frac{4+2d+\eta^2k_1^2}{2(d+2)}\phi_2\phi_3\phi_4+\mathcal{O}(k_1^3), \tag{3.36}$$

which fixes $C=4(d+2)^3$. The wavefunction coefficient with $\mathcal{O}(k_1^3)$ soft behavior is therefore

$$\Psi_4^{sGal} = \delta^3\left(\vec{k}_T\right)\left(\hat{s}_{12}^3+\hat{s}_{13}^3+\hat{s}_{14}^3+(3d+2)\left(\hat{s}_{12}^2+\hat{s}_{13}^2+\hat{s}_{14}^2\right)-4(d+2)^3\right)\mathcal{C}_4^{\Delta=d+2}. \tag{3.37}$$

We can see from equations (3.35) and (3.36) that once Δ is fixed, we can fix B and C using only the leading order soft limit.

Moreover, we find that the Lagrangian in (3.32) is given by

$$\frac{\mathcal{L}_4^{sGal}}{\sqrt{-g}} = -\{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi-(d+2)\phi^2+\tfrac{1}{8}(\nabla_\mu\nabla_\nu\phi)^2\nabla\phi\cdot\nabla\phi-\tfrac{d+1}{4}(\nabla\phi\cdot\nabla\phi)^2+\tfrac{(d+2)^3}{6}\phi^4\}. \tag{3.38}$$

Let us compare the above Lagrangian to the one derived from hidden symmetry. Expanding (3.5) to quartic order gives

$$\begin{aligned}\frac{\mathcal{L}^{\text{sGal}}}{\sqrt{-g}} = -\Bigg(&\frac{1}{2}\nabla\phi\cdot\nabla\phi-(d+2)\phi^2-\frac{1}{4!}2(d+2)(d(d+4)+12)\phi^4\\ &+\frac{1}{4!}(d(3d+8)+28)\phi^2\nabla\phi\cdot\nabla\phi+\frac{d+4}{96}(\nabla\phi\cdot\nabla\phi)^2+\frac{2-d}{24}\phi\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi\\ &-\frac{1}{96}\nabla\phi\cdot\nabla\phi\,(\nabla_\mu\nabla_\nu\phi)^2+\frac{1}{48}\nabla^\mu\phi\nabla^\nu\phi\nabla_\sigma\nabla_\mu\phi\nabla^\sigma\nabla_\nu\phi\Bigg)+\mathcal{O}(\phi^6),\end{aligned} \tag{3.39}$$

where we have used the free equation of motion $\nabla^2\phi = m^2\phi = -2(d+2)\phi$. We can then use integration by parts and free equations of motion to bring this to the form in (3.32). In more detail, the final term in (3.39) can be written as

$$\partial^\mu\phi\partial^\nu\phi\nabla^\sigma\nabla_\mu\phi\nabla_\sigma\nabla_\nu\phi \sim -\frac{1}{2}\left((\nabla\phi\cdot\nabla\phi)\nabla^\sigma\nabla^\mu\phi\nabla_\sigma\nabla_\mu\phi + (\nabla\phi\cdot\nabla\phi)\partial^\nu\phi\nabla^2\nabla_\nu\phi\right), \tag{3.40}$$

where we applied integration by parts on ∇^σ. The second term on the right-hand side can then be reduced to lower-derivative terms by noting that

$$\begin{aligned}\nabla_\sigma\nabla^\sigma\partial_\nu\phi &= \nabla_\sigma\nabla_\nu\partial^\sigma\phi,\\ &= \nabla_\nu\nabla^2\phi + [\nabla_\nu\nabla_\sigma]\partial^\sigma\phi,\\ &= m^2\partial_\nu\phi + R_{\mu\nu}\partial^\mu\phi,\\ &= -(d+4)\partial_\nu\phi.\end{aligned} \tag{3.41}$$

Similarly, using integration by parts and free equations of motion, the two-derivative term in the first line of (3.39) can be reduced to a ϕ^4 term, and the second four-derivative term in the second line of (3.39) can be written in the form $(\nabla\phi\cdot\nabla\phi)^2$ plus a ϕ^4 term. In the end, we are left with three interaction terms:

$$\frac{\mathcal{L}_{\rm int}^{\rm sGal}}{\sqrt{-g}} = -\frac{1}{48}(\nabla\phi\cdot\nabla\phi)\nabla^\alpha\nabla^\beta\phi\nabla_\alpha\nabla_\beta\phi + \frac{d+1}{24}(\nabla\phi\cdot\nabla\phi)^2 - \frac{1}{36}(d+2)^3\phi^4 + \mathcal{O}(\phi^6). \tag{3.42}$$

After multiplying by 6 (equivalent to rescaling the six-derivative coupling) this indeed matches the interaction terms in (3.38), which were deduced from enhanced soft limits.

In conclusion, we matched the two Lagrangian modulo to integration by part and equation of motion, which means the Lagrangian is not unique but expected when we are working on off-shell object. This also motivates us to work on the on-shell object in the later chapter.

3.2.4 Double Copy

In flat space, the scattering amplitudes of the NLSM, DBI, and sGal theories enjoy double copy relations [85], which are made manifest using a formulation based on scattering equations [119, 120]. Scattering equations in (A)dS were later formulated in [94, 95, 106, 107] and used to explore the double copy for effective scalar theories in [110] (the double copy for effective scalar theories in AdS was also explored from various other points of view in [97,98,116]). In more detail, a generalised double copy for 4-point wavefunction coefficients was proposed in terms of unfixed parameters encoding masses and curvature corrections. In this subsection, we will explain how to fix these parameters using our results on enhanced soft limits.

Let us briefly review the representation of tree-level wavefunction coefficients in terms of scattering equations and the generalized double copy at 4 points. We will focus on effective scalar theories with mass $m^2 = \Delta(d-\Delta)$. The discussion will be very schematic but the interested reader can find more details in [110]. A tree-level n-point wavefunction coefficient can be written as an integral over n-punctures on the sphere:

$$\Psi_n = \delta^d(\vec{k}_T) \int_\gamma \prod_{a \neq e,f,g}^{n} \mathrm{d}\sigma_a \, S_a^{-1} (\sigma_{ef}\sigma_{fg}\sigma_{ge})^2 \, \mathcal{I}_n \mathcal{C}_n^{\Delta}, \tag{3.43}$$

where $\sigma_{ab} = \sigma_a - \sigma_b$. The three punctures denoted e, f, g are fixed and $\mathcal{I}_n$ is a theory-dependent integrand, which in general is a differential operator acting on an n-point contact diagram $\mathcal{C}_n^{\Delta}$. Since the integrand is constructed from $\hat{s}_{ab}$ operators it can in principle have ordering ambiguities, although they do not arise for scalar theories with polynomial interactions [95]. The contour γ encircles the poles where differential operators S_a vanish when acting on everything to the right. The operators are defined as

$$S_a = \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{\alpha_{ab}}{\sigma_{ab}} \tag{3.44}$$

where $\alpha_{ab} = 2\hat{s}_{ab} + \mu_{ab}$ with $\mu_{aa\pm1} = -m^2$ and zero otherwise. In practice, it is not known how to explicitly solve the equations that determine these poles, dubbed the cosmological scattering equations, but the integral can be mapped to a sum of Witten diagrams using the global residue theorem.

For the NLSM at 4-points, the following integrand was proposed in [110]:

$$\mathcal{I}_4^{NLSM} = \lambda^2 \text{PT} \left(\text{Pf}'A\right)^2 + c\text{PT}\ \text{Pf}X|_{\text{conn}}\, \text{Pf}'A, \tag{3.45}$$

where $\text{PT} = (\sigma_{12}...\sigma_{n1})^{-1}$, $\text{Pf}'A$ is related to the Pfaffian of an operator-valued matrix whose off-diagonal elements are $A_{rs} = \alpha_{rs}/\sigma_{rs}$, $\text{Pf}X$ is the Pfaffian of a matrix whose off-diagonal elements are $X_{rs} = 1/\sigma_{rs}$, and $\text{Pf}\,X|_{\text{conn}}$ refers to the sum over connected perfect matchings which arise in $\text{Pf}X$. The first term on the right-hand side of (3.45) represents the naive uplift from flat space while the second term encodes a potential curvature correction. Evaluating the contour integral in (3.43) then gives

$$\Psi_4^{NLSM} = -\delta^3(\vec{k}_T)\left(2\lambda^2 \hat{s}_{13} - c - m^2\right)\mathcal{C}_4^{\Delta}. \tag{3.46}$$

Comparing this to the wavefunction coefficient with enhanced soft limits in (3.23) then fixes the mass and coefficients as follows:

$$\lambda = 1, \ \ c = m = 0. \tag{3.47}$$

For the DBI and sGal theories at four-points the following integrand was proposed in [110]:

$$\mathcal{I}_4^{(6)} = a(\text{Pf}'A)^3(\text{Pf}'A + m^2\ \text{Pf}X|_{\text{conn}}) + b(\text{Pf}'A)^2(\text{Pf}'A\text{Pf}X + m^2\text{PT}) + c\text{PT}\ \text{Pf}X|_{\text{conn}}\,\text{Pf}'A, \tag{3.48}$$

where $a = 0$ for the DBI theory (note that in the above equation a, b, c are understood to be coefficients rather than labels of external legs). For both theories, c is a curvature correction while b is also a curvature correction in the sGal theory. Note that (3.48) can be obtained from (3.45) via the following replacement:

$$\lambda^2\text{PT} \to a\text{Pf}'A\left(\text{Pf}'A + m^2\ \text{Pf}X|_{\text{conn}}\right) + b\left(\text{Pf}'A\text{Pf}X + m^2\text{PT}\right). \tag{3.49}$$

In addition to performing this replacement, we are also free to change the value of the mass and coefficient c in (3.45) so that they do not necessarily have the same value as the NLSM. In the flat space limit (where curvature corrections and masses are set to zero), this replacement encodes the double copy of NLSM amplitudes to DBI and sGal amplitudes. In curved background, we, therefore, refer to it as a generalized double copy.

After specifying a simple prescription to avoid potential ordering ambiguities of the integrand in (3.48), the contour integral in (3.43) gives

$$\Psi_4^{(6)} = \delta^3(\vec{k}_T)[\frac{8a}{3}(\hat{s}_{12}^3 + \hat{s}_{13}^3 + \hat{s}_{14}^3) + 2(b - am^2)(\hat{s}_{12}^2 + \hat{s}_{13}^2 + \hat{s}_{14}^2) + \frac{1}{3}am^6 - bm^4 + c]\mathcal{C}_4^{\Delta}. \tag{3.50}$$

Comparing this to the wavefunction coefficient for the DBI theory derived from enhanced soft limits in (3.29) then fixes the parameters as follows:

$$a = 0, \;\; b = -\frac{1}{2}, \;\; c = \frac{1}{2}\left(d^2 + 6d + 5\right), \;\; m^2 = -(d+1). \tag{3.51}$$

Moreover, comparing (3.50) to the wavefunction coefficient for the sGal theory in (3.37) implies that

$$a = \frac{3}{8}, \;\; b = \frac{1}{4}\left(3d - 2\right), \;\; c = -8(d+2)^2, \;\; m^2 = -2(d+2). \tag{3.52}$$

In summary, the parameters of the generalized double copy for four-point wavefunction coefficients can be fully fixed by enhanced soft limits. In the next section we will show that enhanced soft limits also fix higher-point wavefunction coefficients, so it would be interesting to see if the double copy prescription can be extended to higher points as well.

3.3 Higher Points

In this section, we will show that all 6-point couplings of the NLSM and DBI theory in dS can also be fixed from enhanced soft limits of wavefunction coefficients. The method we develop can also be applied to the sGal theory, but at six points its Lagrangian has 13 interaction vertices going up to ten derivatives so Witten diagram calculations become very tedious. We will therefore leave that case for future work.

3.3.1 NLSM

We start with the NLSM, which is very simple but nicely illustrates the procedure for fixing higher-point couplings. At six points, the most general Lagrangian is given by

$$\begin{aligned}\frac{\mathcal{L}_6^{NLSM}}{\sqrt{-g}} = \mathrm{Tr}\Bigg[&-\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 - \Phi^2\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4}C\Phi^4 \\ &-A\left(\Phi^4\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}\Phi^2\partial_\mu\Phi\Phi^2\partial^\mu\Phi\right) - \frac{1}{6}F\Phi^6\Bigg],\end{aligned} \tag{3.53}$$

where the Φ^4 and Φ^6 terms are curvature corrections. We have already fixed $m = 0$ and $C = 0$ from the enhanced soft limit at four points. The coefficient A can be fixed by the flat space limit but we will deduce it along with F from enhanced soft limits at six points. The six-point wavefunction coefficient was already computed from Witten diagrams in [110] and takes the form

$$\Psi_6^{NLSM} = \delta^3(\vec{k}_T)\left[\left(\frac{\hat{s}_{13}\hat{s}_{46}}{\hat{s}_{123}} + A\,\hat{s}_{13} + \mathrm{Cyc.}[i \to i{+}2]\right) + F\right]\mathcal{C}_6^{\Delta=d}, \tag{3.54}$$

where we've used the shorthand and $\hat{s}_{abc} = \mathcal{D}_a\cdot\mathcal{D}_b + \mathcal{D}_b\cdot\mathcal{D}_c + \mathcal{D}_c\cdot\mathcal{D}_a$. The first term in parenthesis comes from an exchange diagram with two 4-point vertices. It was obtained using integration by parts to move all derivatives with respect to conformal time onto the external propagators. In this form, the expression is free or ordering ambiguities since $[\hat{s}_{abc}, \hat{s}_{ab}]\mathcal{C}^\Delta = 0$.

If we take $\vec{k}_1$ soft, all operators of the form $\mathcal{D}_1\cdot\mathcal{D}_a$ will vanish up to $\mathcal{O}(k_1)$ when acting on the contact diagram $\mathcal{C}^\Delta$ as in (3.22) since $\Delta = d$. Hence two of the channels in (3.54) drop out immediately and it reduces to

$$\lim_{\vec{k}_1\to 0}\Psi_6^{NLSM} = \delta^3(\vec{k}_T)\left[\left(\frac{\hat{s}_{35}\hat{s}_{62}}{\hat{s}_{612}} + A\,\hat{s}_{35}\right) + F\right]\mathcal{C}_6^{\Delta=d}. \tag{3.55}$$

Noting that $\lim_{\vec{k}_1\to 0}\hat{s}_{612} = \hat{s}_{62}$ when $\Delta = d$, we then can see the soft limit vanishes if $A = -1$ and $F = 0$, in agreement with (3.2).

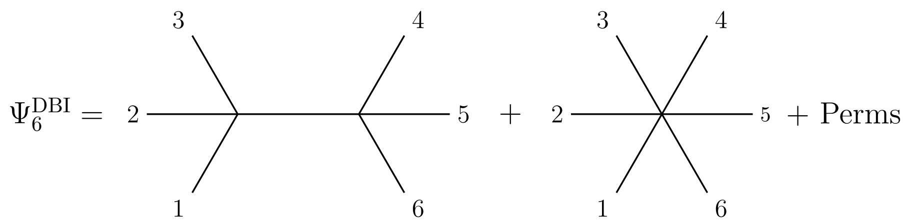


Figure 3.1: Witten diagrams contributing the 6-point sGal wavefunction coefficient.

In summary, we see that the enhanced soft limit arises via cancellations between exchange and contact diagrams, fixing higher-point couplings in terms of lower-point couplings. In this way, we can in principle bootstrap all tree-level wavefunction coefficients and reconstruct the Lagrangian.

3.3.2 DBI

We now consider the following 6-point effective Lagrangian:

$$\frac{\mathcal{L}_6^{DBI}}{\sqrt{-g}} = \frac{\mathcal{L}_4^{DBI}}{\sqrt{-g}} + \frac{A}{48}(\nabla\phi\cdot\nabla\phi)^3 + \frac{B}{16}(\nabla\phi\cdot\nabla\phi)^2\phi^2 + \frac{C}{6!}\phi^6, \tag{3.56}$$

where the 4-point Lagrangian was fixed by enhanced soft limits in (3.30). The coefficient A can be determined by the flat space limit but we will fix it along with the other coefficients from enhanced soft limits. First, we compute the 6-point wavefunction coefficient from Witten diagrams, which are depicted in Figure 3.1.

To compute the exchange diagrams, first consider the 4-point vertex on the left of the exchange diagram in Figure 3.1 which is illustrated in Figure 3.2:

$$\Psi_L = \left(\hat{s}_{12}\hat{s}_{3L} + \hat{s}_{23}\hat{s}_{1L} + \hat{s}_{31}\hat{s}_{2L} - (d+1)(d+3)\right), \tag{3.57}$$

which is understood to act on a 6-point contact diagram in combination with a bulk-to-bulk propagator and another 4-point vertex. We can then use the conformal Ward identities at the vertex $\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 = -\mathcal{D}_L$ to get

$$\Psi_L = \big(-2\left(\hat{s}_{12}\hat{s}_{23} + \hat{s}_{23}\hat{s}_{31} + \hat{s}_{31}\hat{s}_{12}\right) + m^2(\hat{s}_{12} + \hat{s}_{23} + \hat{s}_{31}) - (d+1)(d+3)\big), \tag{3.58}$$

where $-m^2 = \Delta(\Delta - d) = d+1$. Combining this with the rest of the Witten diagram

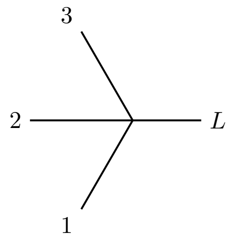
$\Psi_L =$ 

Figure 3.2: Four-point vertex contributing to 6-point exchange diagram.

and summing over permutations then gives

$$\Psi^{DBI}_{6,\,\mathrm{exch}} = \delta^3(\vec{k}_T)\frac{1}{(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2}\Big(2\left(\hat{s}_{12}\hat{s}_{23}+\hat{s}_{23}\hat{s}_{31}+\hat{s}_{31}\hat{s}_{12}\right)$$
$$+\,(d+1)(\hat{s}_{12}+\hat{s}_{23}+\hat{s}_{31}+(d+3))\Big)\times(123)\to(456))\mathcal{C}_6^{\Delta=d+1}+\text{perms}, \tag{3.59}$$

where the permutation sum is over 10 inequivalent factorization channels. Note that this expression is free of ordering ambiguities. Moreover, it is straightforward to read off the contact Witten diagrams from (3.56):

$$\Psi^{DBI}_{6,\,\mathrm{cont}} = \delta^3(\vec{k}_T)\left[A\left(\hat{s}_{12}\hat{s}_{34}\hat{s}_{56}+\text{perms}\right)+B\left(\hat{s}_{12}\hat{s}_{34}+\text{perms}\right)+C\right]\mathcal{C}_6^{\Delta=d+1}, \tag{3.60}$$

where we sum over inequivalent permutations giving 61 terms.

Let us now expand the wavefunction coefficient to $\mathcal{O}(k_1^2)$. To this order, the 4-point vertex in (3.58) is given by

$$\Psi_L = -((\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2)\left(\hat{s}_{12}+\hat{s}_{31}+\frac{1}{2}(d-1)\right)+\mathcal{O}(k_1^2), \tag{3.61}$$

As a result, the left vertex of the numerator of the exchange diagram in Figure 3.1 can be written as

$$\begin{aligned}&\Big(2\left(\hat{s}_{12}\hat{s}_{23}+\hat{s}_{23}\hat{s}_{31}+\hat{s}_{31}\hat{s}_{12}\right)+(d+1)(\hat{s}_{12}+\hat{s}_{23}+\hat{s}_{31}+(d+3))\Big)\\&\quad=((\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2)\left(\hat{s}_{12}+\hat{s}_{31}+\frac{1}{2}(d-1)\right)+\mathcal{O}(k_1^2).\end{aligned} \tag{3.62}$$

Hence, in the soft limit, we can use the left part of the numerator to cancel all the propagators in the denominator and are left with a cubic polynomial in $\hat{s}_{ij}$. We then apply conformal Ward identities to cancel exchange and contact contributions, mimicking the analogous cancellation of terms that arise in the flat space limit using momentum conservation.

We then use integration by parts and equations of motion to write the conformal time integrand in terms of linearly independent terms, as before. In the present case, the procedure is somewhat complicated so we provide more details in Appendix A.2. In the end, we find that the soft limit of the 6-point wavefunction coefficient vanishes to $\mathcal{O}(k_1^2)$ if and only if $A = 3, B = d+1, C = 2(d+1)(9-d^2)$. Since Δ was already fixed from the 4-point soft limit, these values can be deduced by considering only the leading order soft limit at six points. We therefore find that the 6-point effective Lagrangian can be written as

$$\frac{\mathcal{L}_6^{DBI}}{\sqrt{-g}} = -\frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{d+1}{2}\phi^2 - \frac{1}{8}(\nabla\phi\cdot\nabla\phi)^2 + \frac{(d+1)(d+3)}{4!}\phi^4 - \frac{1}{16}(\nabla\phi\cdot\nabla\phi)^3 + \frac{d+1}{16}(\nabla\phi\cdot\nabla\phi)^2\phi^2 + \frac{2(d+1)(9-d^2)}{6!}\phi^6. \tag{3.63}$$

On the other hand, expanding the Lagrangian in (3.5) to the sixth order (without applying equations of motion) gives

$$\begin{aligned}\frac{\mathcal{L}_6^{DBI}}{\sqrt{-g}} = &-\frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{d+1}{2}\phi^2 - \frac{1}{8}(\nabla\phi\cdot\nabla\phi)^2 - \frac{1}{4}(d+3)(\nabla\phi\cdot\nabla\phi)\phi^2 \\ &+ \frac{3(d+1)(d+3)}{4!}\phi^4 - \frac{1}{16}(\nabla\phi\cdot\nabla\phi)^3 - \frac{3(d+5)}{16}(\nabla\phi\cdot\nabla\phi)^2\phi^2 \\ &- \frac{3(d+3)(d+5)}{48}(\nabla\phi\cdot\nabla\phi)\phi^4 + \frac{15(d+1)(d+3)(d+5)}{6!}\phi^6.\end{aligned} \tag{3.64}$$

Matching the two Lagrangians using integration by parts and equations of motion is very tedious, so we instead verify that they give the same 6-point wavefunction coefficient in Appendix A.3.

3.4 Conclusions

In this Chapter, we have found evidence that the link between hidden symmetries and enhanced soft limits for scattering amplitudes in flat space extends to wavefunction coefficients in de Sitter space. In more detail, we have shown that enhanced soft limits fix the masses and couplings (including curvature corrections) of scalar effective field theories in agreement with the Lagrangians recently derived for the DBI and sGal theories from hidden symmetries in [66]. We have carried out these calculations up to six points in the NLSM and DBI theory and four points in the sGal theory. At six points, the enhanced soft limits arise from cancellations between exchange and contact Witten diagrams, allowing us to fix all 6-point couplings in terms of 4-point couplings. In principle, this procedure can be extended to any number of points allowing us to reconstruct the entire tree-level wavefunction coefficient, or equivalently the entire Lagrangian.

There are a number of future directions. First of all, it would be interesting to extend our calculations to any number of points. This would involve writing down the most general effective action that reduces to the known one in the flat space limit, computing the tree-level wavefunction coefficients up to a given number of points using Witten diagrams, fixing the couplings from enhanced soft limits, and showing that the result agrees with the Lagrangians recently derived from hidden shift symmetries. If this were possible, it would be very significant because it would allow us to prove the relation between enhanced soft limits and hidden symmetries in dS. The difficulty with this approach is that the number of Witten diagrams quickly becomes very large at higher points. A more efficient method for fixing higher-point couplings from enhanced soft limits would therefore be very welcome.

CHAPTER 4

Gravity amplitudes in (A)dS from Double Copy

In this chapter, we will combine the bootstrap techniques used in [43] with the double copy, leading to a further reduction of the 4-graviton wavefunction down to only a few lines. Starting with the tree-level wavefunction for four gluons, we will apply a squaring procedure inspired by the double copy for flat space amplitudes. The resulting formula for the s-channel contribution to the wavefunction in (4.22) can be written in two lines and satisfies the flat space limit [32, 45], Cosmological Optical Theorem (COT) [15,19] and Manifestly Local Test (MLT) [14], we will make use of the following cosmological bootstrap techniques which will be described in greater detail in section 4.1. Moreover, it captures the vast majority of the hundreds of thousands of terms that arise from Witten diagrams. The full result for the s-channel contribution to the graviton wavefunction in (4.23) can then be obtained by noting that the double copy ansatz contains spurious poles which can be cancelled by adding a simple two-line correction whose structure is fixed by the MLT. Moreover, this correction can be deduced by looking at scalars exchanging a graviton. Using the double copy as a starting point, no new corrections arise after generalizing this example to the gravitational case. Hence, while we do not yet have a systematic understanding of the double copy in (A)dS, it appears to be a very useful tool in the study of gravitational correlators.

4.1 Bootless Bootstrap review

In this section, we will review some facts of bootstrap approach to study cosmological correlators which will be useful later on.

While it is relatively straightforward to compute 4-point gluon wavefunctions using Witten diagrams, doing so for gravitons is very challenging due to the large number of terms that arise. It was shown in [43] that the graviton trispectrum (four-point graviton correlators) is completely fixed up to arbitrary (non-local) field redefinitions by the combination of the flat space limit [32,45], the Cosmological Optical Theorem (COT) [15, 19] and the Manifestly Local Test (MLT) [14]. These tools have been established over the last few years as key ingredients in the Cosmological Bootstrap [10]. A consequence of this is that any expression that satisfies the COT and has the correct flat space limit can be combined with the MLT to give the graviton trispectrum. To aid the reader we will briefly review these three tests. In particular, these tests will be refered as the bootless bootstrap as it does not rely on the special conformal ward identities (boots symmetry).

Fields in the Bunch-Davies vacuum in the infinite past of de Sitter behave just like flat space fields. As a result, wavefunction coefficients contain the flat space amplitude within them as the residue of the total energy pole. For Einstein gravity (being a two derivative theory) this means that

$$\lim_{E\to 0} \psi_4^{\gamma} \propto \frac{k_1 k_2 k_3 k_4}{E^3}\mathcal{M}_4, \tag{4.1}$$

where $E = k_1 + k_2 + k_3 + k_4$ and $\mathcal{M}_4$ is the 4-graviton amplitude. While a naive squaring of the tree-level 4-point gluonic wavefunction satisfies the correct flat space limit [28], it does not satisfy the COT, which we describe in the next paragraph. To remedy this, we will instead consider squaring the numerators in the conformal time integrand.

As a consequence of unitary time evolution in the bulk de Sitter space time, all wavefunction coefficients satisfy the so-called COT. This relationship relates exchange diagrams to simpler diagrams involving one fewer exchanged particle. In the case of gravity, this relationship can be expressed as

$$\psi_4^{h_1h_2h_3h_4}(k_1,k_2,k_3,k_4,k_s,k_t)+\psi_4^{h_1h_2h_3h_4}(-k_1,-k_2,-k_3,-k_4,k_s,-k_t)^*=$$
$$\sum_h P^h(k_s)\left[\psi_3^{h_1h_2h}(k_1,k_2,k_s)-\psi_3^{h_1h_2h}(k_1,k_2,-k_s)\right]\left[\psi_3^{h_3h_4h}(k_3,k_4,k_s)-\psi_3^{h_3h_4h}(k_3,k_4,-k_s)\right], \tag{4.2}$$

where $k_s = |\vec{k}_1+\vec{k}_2|$ and $k_t = |\vec{k}_1+\vec{k}_4|$ and h being the helicity. Note that the k_t dependence on the left-hand side is encoded by the polarization sum on the right-hand side. We also note that there will be some dependence on the directions of the momenta through the polarisation tensors but this has been left implicit due to the convention that they are unchanged when we adjust the energies [102]. As was noticed in [14] this is sufficient to fix all of the partial energy poles and so any result satisfying both this and the flat space limit will be equal to the full answer up to sub-leading total energy poles.

Finally, any four-point[1] interaction arising from a Lagrangian with no inverse Laplacian acting on single fields (such as that arising from Einstein gravity) must generate a wavefunction coefficient that satisfies the so-called MLT:

$$\lim_{k_1\to 0}\partial_{k_1}\tilde{\psi}_4(k_1,k_2,k_3,k_4,k_s,k_t)=0, \tag{4.3}$$

which is true even away from physical momentum configurations (unlike, for example, the soft theorems). The tilde indicates that this applies to the trimmed wavefunction coefficients (defined in (2.58)) as the kinematics of the polarisation tensors can introduce poles in the wavefunction that violate the assumptions that go into the MLT. As we will see later, our double copy prescription will satisfy the flat space limit, COT, and MLT, but will contain spurious poles requiring us to add a simple correction whose structure will be fixed by the MLT.

Recent developments in related direction include geometric approaches [38, 99],

[1] Equivalent results exist for more general interactions but the expression given here is explicitly for a 4-point wavefunction coefficient.

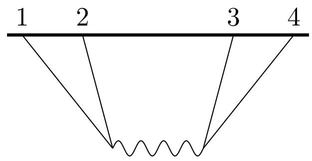


Figure 4.1: Witten diagrams for conformally coupled scalars exchanging a gluon.

and methods based on factorisation [11, 13, 88, 121], unitarity [14, 15, 19, 101–103], Mellin-Barnes representations [100,104], recursion relations [122–124], color/kinematics duality [21, 28, 97, 98, 108, 113, 116, 117], scattering equations [94, 95, 106, 107], and the double copy [25, 110–112, 114, 115, 125, 126].

4.2 Scalar Wavefunctions

In this section, we will derive a compact new formula for the 4-point wavefunction of minimally coupled scalars exchanging a graviton starting from the wavefunction for conformally coupled scalars exchanging a gluon. This will be a warm-up for obtaining the 4-point graviton wavefunction from the gluonic one in the next section. Indeed, the scalar wavefunctions we derive in this section can be obtained via generalised dimensional reduction of the spinning ones.

4.2.1 Ansatz

Let us begin with conformally coupled scalars exchanging a gluon. We will consider the color-ordered wavefunction analogous to the first line of (2.39) in flat space and take the scalars to be in the adjoint representation of the gauge group. Using the Feynman rules in 2.4.1, the s-channel Witten diagram depicted in Figure 4.1 is given by

$$\psi_{\phi,A}^{(s)} = \int \frac{d\omega\, \omega}{k_s^2 + \omega^2} dz\, dz'\, (KKJ)_{12}^{1/2}(z)(KKJ)_{34}^{1/2}(z')N_s^{\phi}, \tag{4.4}$$

where the integrals over ω, z, and z' are from zero to infinity, $k_s = |\vec{k}_1 + \vec{k}_2|$,

$$N_s^{\phi} = v_{12}^i v_{34}^j H_{ij}, \tag{4.5}$$

$v_{12}^i = (\vec{k}_1 - \vec{k}_2)^i$, $H_{ij} = \eta_{ij} + \frac{\vec{k}_{12}^i \vec{k}_{12}^j}{\omega^2}$, and $\vec{k}_{ab} = \vec{k}_a + \vec{k}_b$. The KKJ integrals are given by

$$(KKJ)_{ab}^{\nu} = \frac{2}{\pi}(k_a k_b z)^{\nu} z K_{\nu}(k_a z) K_{\nu}(k_b z) J_{\nu}(\omega z). \tag{4.6}$$

We have also dropped the overall factor of i.

The numerator N_s^{ϕ} can be thought of as the analog of the kinematic numerator n_s in (2.39). By analogy to (2.42) a natural guess for the double copy is

$$\psi_{\phi,\mathrm{DC}}^{(s)} \stackrel{?}{=} \int \frac{d\omega\, \omega}{k_s^2 + \omega^2} dz\, dz'\, (KKJ)_{12}^{3/2}(z)(KKJ)_{34}^{3/2}(z') \left(N_s^{\phi}\right)^2, \tag{4.7}$$

where we have replaced $\nu = \frac{1}{2}$ Bessel functions with $\nu = \frac{3}{2}$ Bessel functions, as expected for minimally coupled scalars and gravitons, and squared the numerator. While this guess has the correct flat space limit, it does not satisfy the COT in (4.2). Looking at the graviton bulk-to-bulk propagator in (2.66) then motivates the following ansatz:

$$\psi_{\phi,\mathrm{DC}}^{(s)} = \int \frac{d\omega\, \omega}{k_s^2 + \omega^2} dz\, dz'\, (KKJ)_{12}^{3/2}(z)(KKJ)_{34}^{3/2}(z') \left(\left(N_s^{\phi}\right)^2 - \frac{1}{2}\tilde{v}_{12}^{ij} H_{ij} \tilde{v}_{34}^{kl} H_{kl} \right), \tag{4.8}$$

where $\tilde{v}_{12}^{ij} = v_{12}^i v_{12}^j$. While the second term in parenthesis is similar in structure to the third term in (2.66), it is constructed from scalar-scalar-gluon vertices. We will say more about the double copy origin of this term in section 4.3.2. In Appendix B.2, we evaluate the integrals in (4.8) and obtain a more explicit formula:

$$\begin{aligned}\psi_{\phi,\mathrm{DC}}^{(s)} = {}& \frac{1}{3}k_s^4 f_{2,2}\Pi_{2,2} - \frac{1}{3}k_s^2 k_{12}k_{34} f_{2,1}\Pi_{2,1} + \frac{1}{2}f_{2,0}\frac{k_{12}^2\alpha^2 k_{34}^2\beta^2}{k_s^4} \\ & - \frac{1}{2}f_{2,1}\left(\left(k_{12}^2 + \alpha^2 - k_s^2 - \frac{k_{12}^2\alpha^2}{k_s^2}\right)\frac{k_{34}^2\beta^2}{k_s^2} + \frac{k_{12}^2\alpha^2}{k_s^2}\left(k_{34}^2 + \beta^2 - k_s^2 - \frac{k_{34}^2\beta^2}{k_s^2}\right)\right),\end{aligned} \tag{4.9}$$

where $k_{ij} = k_i + k_j$, $\alpha = k_1 - k_2$, $\beta = k_3 - k_4$, $\Pi_{2,2}$ and $\Pi_{2,1}$ are polarisation sums given in (B.1.7), and $f_{2,2}$, $f_{2,1}$, and $f_{2,0}$ are conformal time integrals given in (B.2.11).

4.2.2 Corrections

While the ansatz in (4.8) has the correct flat space limit and satisfies the COT, after integration we find that it contains spurious poles in k_{12} and k_{34} (The only physical pole one should have is $\frac{1}{E_t}$ and $\frac{1}{E_L E_R}$). These can be canceled by adding the following simple correction:

$$\psi^{(s)}_{\rm sp} = -\frac{1}{2}\left(\frac{2k_1k_2k_3k_4}{(k_{12}+k_{34})^2}\left(\frac{\alpha^2}{k_{34}}+\frac{\beta^2}{k_{12}}\right)+\frac{\alpha^2 k_3k_4}{k_{34}}+\frac{\beta^2 k_1k_2}{k_{12}}\right). \tag{4.10}$$

In fact, this is the unique correction that cancels the spurious poles without affecting the flat space limit or COT, modulo adding terms which do not contain spurious poles. This ambiguity can be fixed by the MLT, which is satisfied by (4.8) but not (4.10).

Following the procedure in [43], we will construct an ansatz for the missing terms and fix it by enforcing the MLT. As was shown in [127, 128], the most general tree-level wavefunction coefficient for interactions involving Einstein gravity is a rational function of the energies. Moreover, the correction terms can have at worst E^{-2} poles and no other singularities so as not to affect the flat space limit or COT [2]. Scale invariance also forces any correction term to scale like momentum cubed so the most general correction must have the form

$$\psi^{(s)}_{\rm MLT} = \frac{{\rm Poly}^{(5)}(k_1,k_2,k_3,k_4,k_s,k_t,k_u)}{E^2}, \tag{4.11}$$

where $\text{Poly}^{(5)}$ is a general polynomial with homogeneity degree 5 under rescaling momenta. We can simplify this by noting that we are only adjusting the s-channel and so anything that we add must respect the s-channel symmetries. To encode the $k_1 \leftrightarrow k_2$ and $k_3 \leftrightarrow k_4$ exchange symmetry we express this polynomial as a function of the combinations k_{12}, k_1k_2, k_{34}, k_3k_4 and k_s^2. The remaining dependence on k_t and k_u can only be through the combination $k_t^2 - k_u^2$, which picks up a minus sign under $k_1 \leftrightarrow k_2$ and so must be multiplied by something else that also behaves in

[2] We are free to add in some non-local field redefinitions with k_s poles, as was shown in [43], but these are always present and so will be ignored in our ansatz.

this way. Therefore,

$$\psi^{(s)}_{\text{MLT}} = \frac{\text{Poly}^{(5)}(k_{12}, k_1k_2, k_{34}, k_3k_4, k_s^2) + A_1\alpha\beta(k_t^2 - k_u^2)E + A_2(k_t^2 - k_u^2)^2 E}{E^2} + \left(\vec{k}_1, \vec{k}_2\right) \leftrightarrow \left(\vec{k}_3, \vec{k}_4\right),$$

where the contribution at the end is required to recover the s-channel symmetry that is not explicit in the construction of this polynomial. This ansatz has a total of 17 free coefficients.

On fixing the free coefficients such that (4.12) combines with (4.10) to satisfy the MLT we find

$$\begin{aligned}\psi^{(s)}_{\text{MLT}} = A(k_1^3 + k_2^3 + k_3^3 + k_4^3) + \frac{1}{2E}\Big((k_1k_3 + k_2k_4)(k_1k_4 + k_2k_3) - 2(\alpha^2 k_1k_2 + \beta^2 k_3k_4) \\ + (\alpha^2 k_3k_4 + \beta^2 k_1k_2) - 3(k_{34}^2 k_1k_2 + k_{12}^2 k_3k_4) + 2(k_{12}^2 k_1k_2 + k_{34}^2 k_3k_4) + 6k_1k_2k_3k_4 \\ + k_{12}k_{34}(E^2 - 2(k_1k_2 + k_3k_4))\Big),\end{aligned} \tag{4.12}$$

where A is a free coefficient that corresponds to the field redefinition $\phi \to \phi + A\phi^3$, where ϕ is the external scalar field. Choosing $A = -7/2$ then gives the compact form

$$\begin{aligned}\psi^{(s)}_{\text{MLT}} = \frac{5k_1k_2k_3k_4}{E} + \frac{E}{2}(k_{12}k_{34} - 4k_1k_2 - 4k_3k_4) - \frac{1}{E}(k_1k_2 - k_3k_4)(\alpha^2 - \beta^2) \\ - 3(\alpha^2 k_{12} + \beta^2 k_{34}).\end{aligned} \tag{4.13}$$

In summary, we find that the s-channel contribution to the wavefunction for minimally coupled scalars exchanging a graviton can be written as

$$\psi^{(s)}_{\phi,\gamma} = \psi^{(s)}_{\phi,\text{DC}} + \psi^{(s)}_{\text{sp}} + \psi^{(s)}_{\text{MLT}}, \tag{4.14}$$

where the three terms on the right-hand-side are given by (4.8), (4.10), and (4.13). It would be interesting that the three terms are coming from a more general double copy procedure, but the structure of terms does not suggest such form. The full

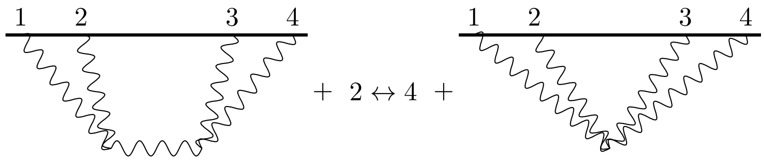
$$\psi_A = $$ 

Figure 4.2: Witten diagrams for four-point gluon wavefunction coefficient.

wavefunction can then be obtained by summing over all three channels, where the contributions from the t and u channels can be obtained from (4.14) by exchanging $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$. More explicitly, plugging in (4.9) we obtain

$$\begin{aligned}\psi_{\phi,\gamma}^{(s)} = &\frac{1}{3}k_s^4 f_{2,2}\Pi_{2,2} - \frac{1}{3}k_s^2 k_{12}k_{34}f_{2,1}\Pi_{2,1} + \frac{1}{2}k_{12}k_{34}(k_{12}k_{34}+k_s^2)f_{2,1}\left(-\frac{\alpha^2+\beta^2}{k_s^2} + 3\frac{\alpha^2}{k_s^2}\frac{\beta^2}{k_s^2}\right) \\ &- \frac{1}{2}\frac{k_s^2}{E}(k_1k_2+k_3k_4+E^2)\frac{\alpha^2}{k_s^2}\frac{\beta^2}{k_s^2} - \frac{1}{2E}(k_1k_2-k_3k_4)(\alpha^2-\beta^2) - \frac{5}{2}(k_{12}\alpha^2+k_{34}\beta^2) \\ &+ \frac{5k_1k_2k_3k_4}{E} + \frac{E}{2}\left(k_{12}k_{34} - 4k_1k_2 - 4k_3k_4\right).\end{aligned} \tag{4.15}$$

This agrees up to field redefinition with the result previously obtained in [43] using bootstrap methods.

4.3 Spinning Wavefunctions

In this section, we will generalize the procedure in the previous section to deduce the tree-level 4-point graviton wavefunction from gluons, arriving at a compact new formula.

4.3.1 Ansatz and Corrections

Let us start with the s-channel contribution to the 4-point color-ordered gluon wavefunction, depicted in Figure 4.2. Using the Feynman rules in section 2.4.1 we obtain

$$\begin{aligned}\psi_A^{(s)} = &\int \frac{dzdz'd\omega\omega}{(k_s^2+\omega^2)}\left(KKJ\right)_{12}^{1/2}(z)\left(KKJ\right)_{34}^{1/2}(z')V_{12}^i H_{ij} V_{34}^j \\ &+ \int dzdz'\delta(z-z')\left(KK\right)_{12}^{1/2}(z)V_c^s\left(KK\right)_{34}^{1/2}(z')\end{aligned} \tag{4.16}$$

where $(KK)^{1/2}_{ab}(z) = \sqrt{k_a k_b} z K_{1/2}(k_a z) K_{1/2}(k_b z)$,

$$\begin{aligned} V^i_{ab} &= \epsilon_a \cdot \epsilon_b (\vec{k}_a - \vec{k}_b)^i + 2\epsilon_a \cdot \vec{k}_b \epsilon^i_b - 2\epsilon_b \cdot \vec{k}_a \epsilon^i_a, \\ V^s_c &= \epsilon_1 \cdot \epsilon_3 \, \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \, \epsilon_2 \cdot \epsilon_3. \end{aligned} \tag{4.17}$$

The second term in (4.16) arises from a bulk contact interaction so we have written it as integral over two bulk points with a delta function. To combine it with the first term, use the orthogonality of Bessel functions

$$\delta(z - z') = \int d\omega \, \omega \, (zz')^{1/2} J_{1/2}(\omega z) J_{1/2}(\omega z'). \tag{4.18}$$

We then obtain

$$\psi^{(s)}_A = \int \frac{d\omega \, \omega}{k_s^2 + \omega^2} dz \, dz' \, (KKJ)^{1/2}_{12}(z)(KKJ)^{1/2}_{34}(z') N_s, \tag{4.19}$$

where the numerator N_s is

$$N_s = V^i_{12} H_{ij} V^j_{34} + V^s_c (\omega^2 + k_s^2). \tag{4.20}$$

By analogy with the scalar double copy ansatz in (4.8), a natural guess for gravitons is

$$\psi^{(s)}_{\gamma,\mathrm{DC}} \overset{?}{=} \int \frac{d\omega \, \omega}{k_s^2 + \omega^2} dz \, dz' \, (KKJ)^{3/2}_{12}(z)(KKJ)^{3/2}_{34}(z') \left(N_s^2 - \frac{1}{2} \tilde{V}^{ij}_{12} H_{ij} \tilde{V}^{kl}_{34} H_{kl} \right), \tag{4.21}$$

where $\tilde{V}^{ij}_{ab} = V^i_{ab} V^j_{ab}$. While this ansatz satisfies the COT, the second term spoils the flat space limit. This can be remedied by adding one more term to the integrand:

$$\begin{aligned} \psi^{(s)}_{\gamma,\mathrm{DC}} = \int & \frac{d\omega \, \omega}{k_s^2 + \omega^2} dz \, dz' \, (KKJ)^{3/2}_{12}(z)(KKJ)^{3/2}_{34}(z') \\ & \times \left(N_s^2 - \frac{1}{2} \tilde{V}^{ij}_{12} H_{ij} \tilde{V}^{kl}_{34} H_{kl} + \frac{1}{2} (\epsilon_1 \cdot \epsilon_2)^2 (\epsilon_3 \cdot \epsilon_4)^2 (\omega^2 + k_s^2)^2 \right). \end{aligned} \tag{4.22}$$

In the next subsection, we will explain how the second and third terms arise from the double copy.

As before, we must add terms to cancel spurious poles and satisfy the MLT. Remarkably, these turn out to be identical to the scalar case after dressing with polarisations. In the end, we find that the s-channel contribution to the 4-point graviton wavefunction is given by

$$\psi_{\gamma}^{(s)} = \psi_{\gamma,\mathrm{DC}}^{(s)} + (\epsilon_1 \cdot \epsilon_2)^2 (\epsilon_3 \cdot \epsilon_4)^2 \left(\psi_{\mathrm{sp}}^{(s)} + \psi_{\mathrm{MLT}}^{(s)}\right), \tag{4.23}$$

where the terms on the right-hand-side are given in (4.22), (4.10), and (4.13). The full wavefunction can then be obtained by summing over all three channels, where the contributions from the t and u channels can be obtained from (4.23) by exchanging $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$. This non-trivially agrees with the result previously obtained in [43] using bootstrap methods, but now provides a more compact expression which exposes the underlying double copy structure.

In Appendix B.2 we evaluate the integrals in (4.22) to obtain the following more explicit formula:

$$\begin{aligned}\psi_{\gamma}^{(s)} = (\epsilon_1 \cdot \epsilon_2)^2(\epsilon_3 \cdot \epsilon_4)^2\psi_{\mathrm{DC}}^{(s)} &+ \left(8(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)W_s k_s^2 \Pi_{1,1} + 16 W_s^2\right) f_{2,2} \\ &- (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)k_{12}k_{34}\left(8W_s\Pi_{1,0} + \alpha\beta V_c^s\right) f_{2,1} + \left((V_c^s)^2 + \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)^2(\epsilon_3 \cdot \epsilon_4)^2\right) f_a \\ &+ \left((\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)(\vec{k}_1 - \vec{k}_2) \cdot (\vec{k}_3 - \vec{k}_4) + 4W_s\right) V_c^s f_b,\end{aligned} \tag{4.24}$$

where f_a and f_b are given in (B.2.13), V_c^s is given in (4.17), $\psi_{DC}^{(s)}$ is defined in (4.22), and the other tensor structure is given by

$$\begin{aligned}W_s = (\epsilon_1 \cdot \epsilon_2)\left[(k_1 \cdot \epsilon_3)(k_2 \cdot \epsilon_4) - (k_2 \cdot \epsilon_3)(k_1 \cdot \epsilon_4)\right] + (\epsilon_3 \cdot \epsilon_4)\left[(k_3 \cdot \epsilon_1)(k_4 \cdot \epsilon_2) - (k_4 \cdot \epsilon_1)(k_3 \cdot \epsilon_2)\right] \\ + \left[(k_2 \cdot \epsilon_1)\epsilon_1 - (k_1 \cdot \epsilon_2)\epsilon_2\right] \cdot \left[(k_4 \cdot \epsilon_3)\epsilon_4 - (k_3 \cdot \epsilon_4)\epsilon_3\right].\end{aligned} \tag{4.25}$$

Notice that the tensor structure $(\epsilon_1 \cdot \epsilon_3)(\epsilon_3 \cdot \epsilon_2)(\epsilon_2 \cdot \epsilon_4)(\epsilon_4 \cdot \epsilon_1)$, which appeared in the result presented in [43], is absent in (4.24) but this merely reflects a different choice of tensors to represent the answer. This contribution is instead captured by the $(V_c^s)^2$ and $V_c^s(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)$ terms as well as a modification to the $(\epsilon_1 \cdot \epsilon_2)^2(\epsilon_3 \cdot \epsilon_4)^2$

term.

4.3.2 Double Copy Structure

The simple ansatz in (4.22) captures most of the terms in the 4-point graviton wavefunction. Comparing this to (2.42), we see that the analog of the graviton numerator in the s-channel is

$$N_s^{\gamma} = N_s^2 - \frac{1}{2}\tilde{V}_{12}^{ij} H_{ij} \tilde{V}_{34}^{kl} H_{kl} + \frac{1}{2}\left(\epsilon_1 \cdot \epsilon_2\right)^2 \left(\epsilon_3 \cdot \epsilon_4\right)^2 \left(k_s^2 + \omega^2\right)^2, \tag{4.26}$$

where N_s is the s-channel gluon numerator:

$$N_s = V_{12}^{\underline{i}} V_{34}^{j} H_{\underline{i}j} + \left(k_s^2 + \omega^2\right) \epsilon_{12}^{\underline{ij}} \epsilon_{34}^{kl} \eta_{\underline{i}[k} \eta_{l]\underline{j}}, \tag{4.27}$$

where $\epsilon_{12}^{\underline{ij}} = \epsilon_1^{\underline{i}} \epsilon_2^{\underline{j}}$ and we have added an underscore to indices associated with the left side of the s-channel Witten diagrams, i.e. legs 1 and 2.

Naively squaring the gluon numerator gives tensors which contract indices on the left with indices on the right:

$$N_s^2 = \tilde{V}_{12}^{\underline{ij}} \tilde{V}_{34}^{kl} T_{\underline{ij}kl} + 2\left(k_s^2 + \omega^2\right) \epsilon_{12}^{\underline{ij}} V_{12}^{\underline{k}} \epsilon_{34}^{lm} V_{34}^{n} T_{\underline{ijk}lmn} + \left(k_s^2 + \omega^2\right)^2 \epsilon_{12}^{\underline{ij}} \epsilon_{12}^{\underline{kl}} \epsilon_{34}^{mn} \epsilon_{34}^{pq} T_{\underline{ijkl}mnpq}, \tag{4.28}$$

where

$$\begin{aligned} T_{\underline{ij}kl} &= H_{\underline{i}k} H_{\underline{j}l}, \\ T_{\underline{ijk}lmn} &= \eta_{\underline{i}[l} \eta_{m]\underline{j}} H_{\underline{k}n}, \\ T_{\underline{ijkl}mnpq} &= \eta_{\underline{i}[m} \eta_{n]\underline{j}} \eta_{\underline{k}[p} \eta_{q]\underline{l}}. \end{aligned} \tag{4.29}$$

On the other hand, we can also consider an alternative prescription for squaring the tensor structures where indices on the left are never contracted with indices on the right:

$$\tilde{N}_s^2 \equiv \tilde{V}_{12}^{\underline{ij}} \tilde{V}_{34}^{kl} \tilde{T}_{\underline{ij}kl} + 2\left(k_s^2 + \omega^2\right) \epsilon_{12}^{\underline{ij}} V_{12}^{\underline{k}} \epsilon_{34}^{lm} V_{34}^{n} \tilde{T}_{\underline{ijk}lmn} + \left(k_s^2 + \omega^2\right)^2 \epsilon_{12}^{\underline{ij}} \epsilon_{12}^{\underline{kl}} \epsilon_{34}^{mn} \epsilon_{34}^{pq} \tilde{T}_{\underline{ijkl}mnpq}, \tag{4.30}$$

where

$$\begin{aligned} \tilde{T}_{\underline{ij}kl} &= H_{\underline{ij}}H_{kl}, \\ \tilde{T}_{\underline{ijk}lmn} &= 0, \\ \tilde{T}_{\underline{ijkl}mnpq} &= \lambda\eta_{\underline{ij}}\eta_{\underline{kl}}\eta_{mn}\eta_{pq}. \end{aligned} \tag{4.31}$$

Note that λ in the third line is an unfixed relative coefficient and the second line vanishes because there are an odd number of indices with or without underscores so there is no way to contract all of them.

Hence we find that there are two ways to define the double copy of the gluon numerator. Moreover, consistency with the flat space limit and the COT implies that both are required. Indeed, (4.26) can be written as

$$N_s^{\gamma} = N_s^2 - \frac{1}{2}\tilde{N}_s^2, \tag{4.32}$$

where we set $\lambda = -1$ in (4.31). This can be also written in terms of asymmetric products of deformed numerators:

$$N_s^{\gamma} = \frac{1}{2}\left(N_{12}^{-}N_{34}^{+} + N_{12}^{+}N_{34}^{-}\right), \tag{4.33}$$

where

$$\begin{aligned} N_{12}^{\pm} &= N_s + \frac{i}{\sqrt{2}}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\left(\omega^2 + k_s^2\right) \pm \frac{1}{\sqrt{2}}\tilde{V}_{12}^{ij}H_{ij}, \\ N_{34}^{\pm} &= N_s - \frac{i}{\sqrt{2}}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\left(\omega^2 + k_s^2\right) \pm \frac{1}{\sqrt{2}}\tilde{V}_{34}^{ij}H_{ij}. \end{aligned} \tag{4.34}$$

It would be interesting to explore if these numerators encode some analog of color/kinematics duality.

This story can easily be adapted to the case of external scalars using generalized dimensional reduction, i.e. setting $\epsilon_a\cdot k_b = 0$ and $\epsilon_a\cdot\epsilon_b = 1$ for $a \neq b$. Indeed, dimensional reduction of the gluon numerator in (4.27) gives

$$N_s^{\phi} = v_{12}^{i}v_{34}^{j}H_{\underline{ij}}, \tag{4.35}$$

which agrees with (4.5). Applying the two double copy prescriptions described above then gives

$$\left(N_s^{\phi}\right)^2 = \tilde{v}_{12}^{ij}\tilde{v}_{34}^{kl}T_{\underline{ij}kl}, \quad \left(\tilde{N}_s^{\phi}\right)^2 = \tilde{v}_{12}^{ij}\tilde{v}_{34}^{kl}\tilde{T}_{\underline{ij}kl}. \tag{4.36}$$

Note that generalized dimensional reduction of the third term in (4.30) gives $\left(k_s^2+\omega^2\right)^2$, but this doesn't affect the COT or flat space limit after summing over all three channels, so this can be discarded. After doing so, we obtain the second term in (4.36). The scalar-graviton numerator in (4.8) can then be written as

$$N_s^{\phi,\gamma} = \left(N_s^{\phi}\right)^2 - \frac{1}{2}\left(\tilde{N}_s^{\phi}\right)^2, \tag{4.37}$$

which can in turn be expressed in terms of deformed numerators as follows:

$$N_s^{\phi,\gamma} = \frac{1}{2}\left(N_{12}^{\phi-}N_{34}^{\phi+} + N_{12}^{\phi+}N_{34}^{\phi-}\right), \tag{4.38}$$

where

$$N_{12}^{\phi\pm} = N_s^{\phi} \pm \frac{1}{\sqrt{2}}\tilde{v}_{12}^{ij}H_{ij}, \quad N_{34}^{\phi\pm} = N_s^{\phi} \pm \frac{1}{\sqrt{2}}\tilde{v}_{34}^{ij}H_{ij}. \tag{4.39}$$

4.4 Conclusions

In this chapter, we derive a compact new expression for the tree-level wavefunction of four gravitons in dS$_4$. The starting point is to write the s-channel contribution to the 4-point wavefunction for gluons as a conformal time integral, square the numerator while replacing $\nu = 1/2$ Bessel functions with $\nu = 3/2$ Bessel functions, and sum over all three channels. After doing so, we obtain a four-line formula for the s-channel contribution to the graviton wavefunction in (4.23) which agrees with the much lengthier result previously obtained in [43], up to field redefinitions.

In summary, the double copy leads to significant simplifications of the 4-point graviton wavefunction in dS$_4$, although we do not yet have a systematic understanding of how it works. In particular, it would be interesting to see if there is some modification of our double copy prescription which doesn't give rise to spurious poles.

Another interesting future direction would be to extend our calculations to higher points and loop-level, where expect color/kinematics duality to play an essential role. Indeed, for flat space gluon amplitudes with more than four external legs, one must perform generalized gauge transformations in order to obtain numerators that obey kinematic Jacobi relations before squaring them to obtain graviton amplitudes.

CHAPTER 5

Mellin-Momentum space

In this chapter, we turn our focus to Mellin-Momentum space to compute wavefunction coefficients and ultimately understand cosmological correlators. We begin by outlining the motivation behind introducing this formalism and proceed to define the Mellin-Momentum amplitude in terms of wavefunction coefficients. Additionally, we review a novel set of Feynman rules for gluon scattering before delving into an exploration of the analytic structure of the amplitude. Later on, we introduce a novel and efficient algorithm for bootstrapping n-point amplitudes, incorporating the modern on-shell amplitude approach. The key concept involves recycling lower-point on-shell amplitudes to recursively construct higher-point amplitudes. Then, by taking the residue of OPE poles, we fix the amplitude up to contact terms. Finally, by comparing with the soft and flat space limits, we determine all the contact terms. It is crucial to emphasize that our algorithm is entirely automated and requires no guesswork, enabling possible exploration of unknown higher-point functions. Our method proves valuable not only for comprehending the structure of higher-point Mellin-Momentum amplitudes but also for easily mapping the result from amplitude to Cosmological correlator.

5.1 Mellin-Momentum space

5.1.1 Motivation for on-shell amplitude

Beginning with the Schwinger-Dyson equations in quantum field theory, we then consider theories with a continuous symmetry and the associated Noether current. This leads to the well-known Ward-Takahashi identity:

$$\partial^{\mu}\left\langle J_{\mu}\left(\vec{x}_1\right) O\left(\vec{x}_2\right) \cdots O\left(\vec{x}_n\right)\right\rangle=-\sum_{a=2}^{n} \delta\left(\vec{x}_1-\vec{x}_a\right)\left\langle O\left(\vec{x}_2\right) \cdots O\left(\vec{x}_a\right) \cdots O\left(\vec{x}_n\right)\right\rangle . \tag{5.1}$$

The right-hand side, characterized by a delta function support, will be referred to as local terms, as it represents a local effect that is non-zero only when two operators are close to each other.[1]

Unlike the scattering amplitude or S-matrix in flat space, the cosmological correlator or wavefunction coefficient is not a unique object. Under a field redefinition, the correlator remains invariant only up to local terms (such as those with delta function support). This is expected, as for S-matrix the right-hand side of equation 5.1 is zero due to LSZ reduction formula, ensuring its invariance and uniqueness. To be more precise, the LSZ formulae take the correlators to amplitudes by Fourier transforming to momentum space and amputating external legs, in the process all contact terms are dropped since they don't have the right pole structure. For example, consider the wavefunction coefficient or boundary correlator in momentum space, which is not invariant under field redefinition [12, 32]. Specifically, let's examine a massless free scalar theory in $d=3$. Under a field redefinition:

$$\phi \rightarrow \phi+\alpha \phi^{3}, \tag{5.2}$$

the four-point correlator changes to

$$\langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle \rightarrow \langle\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle-\frac{1}{3}\alpha\sum_{i=1}^{4}(k_i^3). \tag{5.3}$$

[1]These are also known as contact terms, although they are distinct from the bulk contact interactions in correlators.

These are the local terms or boundary contact terms in momentum space since the term is analytic in two of its momentum [18]. More generally, computing the same correlator using different coordinates, gauges, or employing free equations of motion during the derivation, can yield different results, up to local terms. In simpler, lower-point examples, identifying such local terms is feasible. However, in higher-point or more complex scenarios, this process becomes considerably more challenging. In this chapter, we aim to construct higher-point correlators recursively using lower-point data. This approach raises the question of which local terms should be retained and which should be disregarded when constructing the higher-point correlators.

Another crucial observation is that in a theory with shift symmetry, the correlator should exhibit enhanced soft limits when taking one of the external leg to be soft [1]. However, in cases such as the simplest Nonlinear Sigma Model (NLSM), there exist local terms that do not vanish in the soft limit. It's important to note that the presence of non-zero terms in the soft limit does not imply a lack of protection by shift symmetry. Instead, this discrepancy arises because the Noether current associated with the shift symmetry conserves only up to local terms for correlators, precisely due to the Ward identity mentioned above Eq(5.1).

None of these problems arise in flat space when working with the S-matrix, thanks to the LSZ reduction formula. In a similar spirit, to overcome all the challenges mentioned above, we introduce the on-shell amplitude Mellin-Momentum amplitude below.

Significant progress has been made in this topic on defining such invariant observables. See defining the S-matrix in AdS boundary [129], On-shell correlators in maximally symmetric space [98], and refining the S-matrix in de-Sitter space to enhance crossing and positivity [130, 131]. While much of these research focuses on scalar theory, here in this chapter we focus on spinning particles, and how to compute spinning correlators recursively and efficiently. Our approach shares the same spirit with these prior works, so it would be very interesting to combine these works to better improve other properties of the observables.

5.1.2 Definition of Mellin-Momentum amplitude

In this chapter we will use a slightly different notation closer to scattering amplitude as follows, this helps to distinguish the Mellin-Momentum amplitude in this chapter from the wavefunction coefficients/correlators computed in the previous chapters. To set up our notation, we will be working on the Poincare patch of AdS_{d+1} space with metric:

$$\tilde{g}_{mn}dx^m dx^n = \tfrac{\mathcal{R}^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu), \tag{5.4}$$

with $0 < z < \infty$ and $\mathcal{R} = 1$. The equation of motion (EoM) operator in momentum space is defined as:

$$\begin{aligned}
&\mathcal{D}_k^\Delta \phi_\Delta(k, z) = 0, \\
&\mathcal{D}_k^\Delta \equiv z^2 k^2 - z^2 \partial_z^2 - (1-d) z \partial_z + \Delta(\Delta - d),
\end{aligned} \tag{5.5}$$

with Δ the scaling dimension and $k = |\vec{k}|$ the norm of the boundary momentum. The solution for bulk-to-boundary propagator is given by

$$\phi_\Delta(k, z) = \sqrt{\frac{2}{\pi}} z^{d/2} k^{\Delta - \frac{d}{2}} \mathcal{K}_{\Delta - \frac{d}{2}}(zk) \tag{5.6}$$

with $\mathcal{K}_{\Delta-\frac{d}{2}}(zk)$ being the Bessel K function of the second kind. In the next step, we will consider the Mellin-Fourier transform $\phi(x, z) \sim e^{ik\cdot x} z^{-2s+d/2} \phi(s, k)$:

$$\phi_\Delta(k, z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} z^{-2s+d/2} \phi_\Delta(s, k), \tag{5.7}$$

$$\phi_\Delta(s, k) = \frac{\Gamma\left(s + \frac{1}{2}\left(\frac{d}{2} - \Delta\right)\right) \Gamma\left(s - \frac{1}{2}\left(\frac{d}{2} - \Delta\right)\right)}{2\Gamma\left(\Delta - \frac{d}{2} + 1\right)} \left(\frac{k}{2}\right)^{-2s + \Delta - \frac{d}{2}} \tag{5.8}$$

We can now start to define our observables. The observables in AdS - boundary correlators or wavefunction coefficient in dS (after Wick rotation) can be treated as the boundary correlators of CFT in momentum space,

$$\Psi_n = \delta^d(\vec{k}_T) \langle \mathcal{O}(\vec{k}_1) \ldots \mathcal{O}(\vec{k}_n) \rangle, \tag{5.9}$$

where $\vec{k}_T = \vec{k}_1 + \ldots \vec{k}_n$. The definition of Mellin-Momentum amplitude $\mathcal{A}(zk, s)$ is given as

$$\Psi_n = \int [ds_i] \int \frac{dz}{z^{d+1}} \mathcal{A}_n(zk, s) \prod_{i=1}^{n} \phi_\Delta(s_i, k_i) z^{-2s_i + d/2}, \tag{5.10}$$

with $\int [ds_i] = \prod_{i=1}^{n} \int_{-i\infty}^{+i\infty} \frac{ds_i}{(2\pi i)}$. The amplitude obeys the following on-shell condition (The Mellin transform of the EoM Eq(5.5) above):

$$(z^2 k^2 + (d/2 - \Delta)^2 - 4s^2)\phi_\Delta(s, k) = 0, \tag{5.11}$$

It's worth noting that momentum k will always be associated with a factor z to capture scale invariance, and the amplitude will also depend on the differential operator of z as we will see later.
We can also integrate out the z variable at every vertex [104]:

$$\int \frac{dz}{z^{d+1+b}} z^{\sum_{i=1}^{n}(-2s_i + d/2)} = \delta(d + b + \sum_{i=1}^{n}(2s_i - d/2)), \tag{5.12}$$

where b is counting the extra factor of z due to scale invariance, this is referred to Mellin delta function, similar to the momentum conservation that captures translation invariant.

LSZ reduction formula and local terms: The boundary correlator in momentum space is not invariant under field redefinition [12, 32]. For instance, consider a massless free scalar theory in $d = 3$, where under the field redefinition $\phi \to \phi + \alpha\phi^3$, the relevant action changes to $\alpha\phi^2 \partial_m \phi \partial^m \phi$ and hence the four-point correlator changes to $\alpha \sum_{i=1}^{4}(k_i^3)$. These are boundary contact terms in momentum space. For Mellin-Momentum amplitude, under such field redefinition, by simply replacing the relevant action with Mellin-Fourier transform we can easily see that it changes to $\alpha \sum_{i=2}^{4}(z^2 k_1 \cdot k_i + (d/2 - 2s_1)(d/2 - 2s_i))$. Importantly, by using boundary momentum conservation, Mellin delta function and on-shell condition Eq(5.11), it vanishes and hence the Mellin-Momentum amplitude is invariant under field redefinition.

These observations can be explained by LSZ reduction formula in AdS with similar argument as QFT in Minkowski space: *contact term is not singular on the on-shell poles.* Contact term of boundary correlator in position space take the following general form:

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle \delta^d(x_i - x_j), \tag{5.13}$$

which vanishes unless the two operators collide. We now consider the Mellin-Fourier transform $\phi(x,z) \sim e^{ik\cdot x} z^{-2s+d/2} \phi(s,k)$. Due to the delta function, it becomes a $n-1$ point correlator. Hence, the contact term can not be written as Eq(5.10) where the definition needs $\prod_{i=1}^{n} \phi(s_i, k_i)$.

> Therefore, *contact terms in a boundary correlator do not contribute to the Mellin-Momentum amplitude $\mathcal{A}_n$.*

This is the main reason why we claim that Mellin-Momentum amplitude should be treated as amputated amplitude in AdS.

5.1.3 Spinning particles

Next, for the spinning particle, we rescale the field accordingly, such that gluon behaves like a $\Delta = d-1$ scalar, while the graviton behaves like a massless scalar. To be more specific, for Yang-Mills, we have $\mathbf{A}_m = (\mathcal{R}/z)A_m$, where $\mathbf{F}_{mn} = \partial_m \mathbf{A}_n - \partial_n \mathbf{A}_m - i[\mathbf{A}_m, \mathbf{A}_n]$ is the usual field strength. The graviton will be parametrized as $g_{mn} = \tilde{g}_{mn} + \frac{\mathcal{R}^2}{z^2} h_{mn}$. Note here after the rescaling A_m and h_{mn} are dimensionless fields. We can then expand this in Einstein field equation and obtain:

$$\mathcal{D}_k^{d-1} A_\mu(k,z) = 0, \tag{5.14}$$

$$\mathcal{D}_k^{d} h_{\mu\nu}(k,z) = 0. \tag{5.15}$$

Clearly, the solutions are just scalar propagators dressed up with boundary polarization.

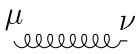
$$A_\mu(k,z) = \varepsilon_\mu \phi_{d-1}(k,z), \tag{5.16}$$

$$h_{\mu\nu}(k,z) = \varepsilon_{\mu\nu} \phi_d(k,z). \tag{5.17}$$

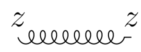
With spinning particles being simply scalar dressed up with polarization, we can now easily use the same definition for scalar of Mellin-Momentum amplitude. Note that our definition here for the bulk-to-boundary propagator is slightly different from (2.62) the wavefunction coefficient calculation before. This difference arises due to the rescaling $\mathbf{A}_m = (\mathcal{R}/z)A_m$, which is crucial for defining the spinning Mellin-Momentum amplitude.

5.1.4 Feynman rules for Mellin-Momentum amplitude

In this section, we will derive a new set of Feynman rules for gluons that make the flat space structure manifest. We will employ the boundary transverse gauge, $k_\mu \cdot A^\mu = 0$, which allows us to have only scalar-like propagators [123]. By directly solving the equation of motion, we found the Feynman rules are identical to flat space in Coulomb gauge with the simple replacement of the EoM operator $\frac{1}{s}$ with $\frac{1}{\mathcal{D}_k^{d-1}}$:

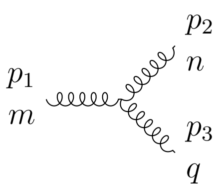


$$: G_{\mu\nu} = \frac{\Pi_{\mu\nu}}{\mathcal{D}_k^{d-1}},$$

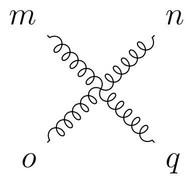


$$: G_{zz} = \frac{1}{z^2 k^2}, \tag{5.18}$$

where $\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is the spin-1 projection tensor. Similarly, after employing the Mellin-Fourier transform $\phi(x,z) \sim e^{ik\cdot x} z^{-2s+d/2}\phi(k,s)$, it's easy to derive the color ordered vertex and see that they are the same as flat space with:



$$: V_{mnq}(p_1,p_2,p_3) = \frac{i}{2}\left(\eta_{mn}(p_1-p_2)_q + \eta_{nq}(p_2-p_3)_m + \eta_{qm}(p_3-p_1)_n\right),$$



$$: V_{mnqo} = i\left(\frac{1}{2}\eta_{mq}\eta_{no} - \frac{1}{4}(\eta_{mn}\eta_{qo} + \eta_{mo}\eta_{nq}),\right) \tag{5.19}$$

where $p^m = (i(-2s+d/2), zk^\mu)$. So we can easily uplift any Feynman diagrams from flat space to AdS amplitude. The crucial distinction between flat space and AdS is the k_I pole, which is non-physical in flat space. However, this pole in AdS is precisely the signal of a CFT, which is necessary for the CFT to have an infinite expansion (corresponding to infinite number of descendent operators) in the OPE limit [88]. We have also used these Feynman rules to compare with the bootstrap result above, finding perfect agreement.

5.2 Factorization limit and OPE limit for scalar four-point amplitude

In this section, we will adopt the amplitude bootstrap approach to derive the 4-point scalar exchanging spinning particle in AdS. We will demonstrate that due to the simple analytic structure of Mellin-Momentum amplitude, we can replicate the success in the flat space amplitude analysis. We will focus on the spin-1 and spin-2 cases, since they encode the most non-trivial information in YM and gravity amplitude.

At the level of 3-point, by imposing the gauge condition $\varepsilon_i \cdot k_i = 0$, dimensional analysis and boundary Lorentz invariance, we can write down the on-shell 3-point amplitude for two scalars and one spin-ℓ particle in general dimension:

$$\mathcal{A}_3 = z^\ell (k_1 \cdot \varepsilon_3)^\ell. \tag{5.20}$$

In general, tree-level Mellin-Momentum amplitudes have two types of poles: factorization poles $\mathcal{D}^{\Delta}_{k_s}$ and OPE poles $\vec{k}_s$, where $\vec{k}_s = \vec{k}_1 + \vec{k}_2$. We will begin with spin-1 exchange as an example to demonstrate how studying the pole structure can determine the 4-point amplitude. In the factorization limit:

$$\begin{aligned}\mathcal{A}_4^J \xrightarrow{\mathcal{D}^{d-1}_{k_s}\to 0} \frac{\sum_h \mathcal{A}_3^J \mathcal{A}_3^J}{\mathcal{D}^{d-1}_{k_s}} &= \sum_h \mathcal{A}_3^J (\mathcal{D}^{d-1}_{k_s})^{-1} \mathcal{A}_3^J \\ &= \frac{z^2 \Pi_{1,1}}{\mathcal{D}^{d-1}_{k_s}},\end{aligned} \tag{5.21}$$

where the sum runs over the possible helicities (guarantee by unitarity) and J denotes conserved current being exchanged. The polarization sums are detailed in Appendix[C.1]. We explicitly write out the middle step to stress that the inverse operator should act on the three-point amplitude and the inverse is explicitly defined in Appendx[C.2].

We now turn to the OPE limit. In position space, we take position 1 close to position 2 and position 3 close position 4, which corresponding to taking $\vec{k}_s \to 0$ in momentum space, and the singularities arising from this limit will be referred as OPE poles. This limit is determined by the Conformal Partial Wave (CPW) [100][2]

$$\begin{aligned}&\lim_{\vec{k}_s \to 0} \langle \mathcal{O}(\vec{k}_1)\mathcal{O}(\vec{k}_2)\mathcal{O}(\vec{k}_3)\mathcal{O}(\vec{k}_4)\rangle \\ \sim&\langle \mathcal{O}(\vec{k}_1)\mathcal{O}(\vec{k}_2)\mathcal{O}(\vec{k}_s;\hat{\partial}_\varepsilon)\rangle\langle \tilde{\mathcal{O}}(-\vec{k}_s;\varepsilon)\mathcal{O}(\vec{k}_3)\mathcal{O}(\vec{k}_4)\rangle.\end{aligned} \tag{5.22}$$

Putting the 3-point data Eq(5.20) into the equation above implies that,

$$\mathcal{A}_4^J \xrightarrow{\vec{k}_s \to 0} \mathcal{O}(k_s^0). \tag{5.23}$$

This is all we need from CPW. The leading term in OPE limit from Eq(5.21) is controlled by,

$$\frac{z^2(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{16k_s^2(s_{12} - d/2)(s_{12} - 1)}, \tag{5.24}$$

where $s_{12} = s_1 + s_2$. In the denominator, we replaced the z derivative with Mellin variables because in the OPE limit it behaves like contact interaction. Then by using EoM Eq(5.11) and Mellin delta function, it becomes:

$$-\frac{(s_1 - s_2)(s_3 - s_4)}{z^2 k_s^2}. \tag{5.25}$$

Amazingly, the Mellin variables in the denominator cancel exactly. On the other hand, Eq(5.23) tells us that pole $(k_s)^{-2}$ must cancel exactly. Hence, this term should

[2] where $\hat{\partial}_\varepsilon$ is the Todorov operator on the boundary, and $\tilde{\mathcal{O}}$ is the shadow operator.

then be subtracted (to have the correct OPE limit), therefore completely fixed the 4-point amplitude:

$$\mathcal{A}_4^J = \frac{z^2\Pi_{1,1}}{\mathcal{D}_{k_s}^{d-1}} + \Pi_{1,0}. \tag{5.26}$$

Next, we will turn to minimally coupled scalars exchanging graviton. The factorization limit demands that

$$\mathcal{A}_4^T \xrightarrow{\mathcal{D}_{k_s}^d \to 0} \frac{\sum_h \mathcal{A}_3^T \mathcal{A}_3^T}{\mathcal{D}_{k_s}^d} = \frac{z^4\Pi_{2,2}}{\mathcal{D}_{k_s}^d}, \tag{5.27}$$

where T denotes the stress tensor being exchanged. In the OPE limit, the situation becomes slightly more complicated because of the additional terms. Details can be found in Appendix[C.1] along with the definition of $\Pi_{2,i}$.

In the end, we fixed the four-point scalar with graviton exchange as follows:

$$\mathcal{A}_4^T = \frac{z^4\Pi_{2,2}}{\mathcal{D}_{k_s}^d} + z^2\Pi_{2,1} + \Pi_{2,0}. \tag{5.28}$$

We have also verified this formula with literature, details can be found in Appendix[C.2].

5.3 Amplitude Bootstrap

We follow the same logic of amplitude bootstrap as in Minkowski space. Thus, our only input for the AdS amplitudes is the following 3-pt on-shell Yang-Mills amplitude:

$$\mathcal{A}_3 = z(\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot k_1 + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot k_2 + \varepsilon_3 \cdot \varepsilon_1 \varepsilon_2 \cdot k_3). \tag{5.29}$$

Subsequently, the n-point amplitudes are determined by factorization, soft limits (OPE limits), and flat space limits, based on their pole structures.

Mellin-Momentum amplitudes only exhibit two types of poles. By examining the pole structure of the propagator we found that the general structure of n-point amplitude is given as follows:

$$\mathcal{A}_n = \sum_{\text{Channels}} \frac{a_1(12\dots n)}{\mathcal{D}^{\Delta}_{k_I}\mathcal{D}^{\Delta}_{k_J}\dots\mathcal{D}^{\Delta}_{k_M}} + \frac{a_2(12\dots n)}{\mathcal{D}^{\Delta}_{k_J}\dots\mathcal{D}^{\Delta}_{k_M}} + \dots + \frac{b_{\ell_I,\dots,\ell_M}(12\dots n)}{k_I^{2\ell_I}k_J^{2\ell_J}\dots k_M^{2\ell_M}} + \dots + c(12\dots n), \tag{5.30}$$

Here the I refers to a general subset of external momenta, $\mathcal{D}^{\Delta}_{k_I}$ is a bulk-to-bulk propagator with the sum of momenta in the set I flowing through it, and $1/k_I^2 \equiv 1/|\sum_{x\in I}\vec{k}_x|^2$. As we will see later, poles in k_I are required by the OPE. The order of these poles can go up to $\ell_{\max} = 1$ for gluons and $\ell_{\max} = 2$ for gravitons. The coefficients appearing in the numerators of this ansatz are then fixed by imposing various constraints coming from factorization, the OPE, and generalized dimensional reduction. We briefly describe each of them below.

Factorization: Unitarity implies that the amplitude will factorize into lower point on-shell amplitudes on the factorization pole $\frac{1}{\mathcal{D}^{\Delta}_{k_I}}$ [15, 101, 102]

$$\mathcal{A}_n \to \sum_h \mathcal{A}_a^h \frac{1}{\mathcal{D}^{\Delta}_{k_I}} \mathcal{A}_{n-a+2}^{-h}. \tag{5.31}$$

This step will fix all the a_i terms in our ansatz.

Internal Soft limit (OPE limit): When two operators get close to each other, it implies that the internal soft momentum $k_I^2 \to 0$, whose behaviour is then controlled by lower-point amplitudes from the usual OPE analysis [62, 88, 100, 105], and it implies the residue of the OPE poles must be zero:

$$\operatorname*{Res}_{k_I^2\to 0} \mathcal{A}_n = 0. \tag{5.32}$$

Remarkably, with this condition, all the b_i terms in our ansatz are simply determined by taking the residue of a_i:

$$b(12\dots n) = -\operatorname*{Res}_{k_I^2\to 0}\left(\frac{a_1(12\dots n)}{\mathcal{D}^{\Delta}_{k_I}\mathcal{D}^{\Delta}_{k_J}\dots\mathcal{D}^{\Delta}_{k_M}} + \dots\right). \tag{5.33}$$

Flat space limit: In the flat space limit, the particles are ignoring the curvature correction and the Lorentz symmetry will emerge and give us the S-matrix in flat space [45, 132–134]:

$$\lim_{z\to\infty} \mathcal{A}_n \to A_n. \tag{5.34}$$

This step fixes all the contact terms c_i in our ansatz. For $n \geq 5$ gluon amplitudes, the n-point functions are fully determined by the $(n-1)$ point amplitudes through factorization and the residue of the OPE poles. There are no contact terms beyond 4-point, as this would result in an incorrect flat space limit.

Gravity is slightly more subtle, but the same procedure applies, leaving us with only the unfixed contact interaction. Subsequently, these contact terms are fully determined by the flat space limit and external soft limit. To elaborate, we perform dimensional reduction on n-point graviton amplitudes, setting: $\varepsilon_i \cdot \varepsilon_j = 1, \varepsilon_i \cdot k = 0, \varepsilon_j \cdot k = 0$ resulting in two scalar and $n-2$ graviton amplitudes. Then by taking the momentum of the scalar soft $k_i \to 0$, this amplitude will vanish due to shift symmetry[3], which is the Alder zero in curved space [1, 110]. Our procedure can be fully automated to n-point as we will now demonstrate.

5.4 Gluon amplitudes

Following our amplitude bootstrap procedure, we now start with 3-point Yang-Mills amplitude in Eq (5.29) to bootstrap the color ordered four-point amplitude:

$$\mathcal{A}_4 = \frac{a(1,2,3,4)}{\mathcal{D}_{k_s}^{d-1}} + \frac{b(1,2,3,4)}{k_s^2} + c(1,2,3,4) + [t], \tag{5.35}$$

where $[t]$ denotes the t-channel contribution and $k_s^2 = k_{12}^2 = (\vec{k}_1 + \vec{k}_2)^2$. Then by factorization:

$$a(1,2,3,4) = \sum_{h=\pm} \mathcal{A}_3(1,2,-k_s^h)\mathcal{A}_3(k_s^{-h},3,4), \tag{5.36}$$

[3] To be more precise, after dimensional reduction we obtain $\langle \phi\phi hhh\ldots\rangle$ where the scalar will couple with graviton as $\nabla^m\phi\nabla^n\phi h_{mn}$, which enjoys a protected shift symmetry $\phi \to \phi + C$

where the sum over of helicity is given by:

$$\sum_{h=\pm} \varepsilon_\mu(k,h)\varepsilon_\nu(k,h)^* = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \equiv \Pi_{\mu\nu}. \tag{5.37}$$

Simply taking the residue of the OPE pole, we obtain:

$$b(1,2,3,4) = -\operatorname*{Res}_{k_s^2\to 0} \frac{a(1,2,3,4)}{\mathcal{D}^{d-1}_{k_s}}. \tag{5.38}$$

where $\lim_{k_s^2\to 0} \frac{1}{\mathcal{D}^{\Delta}_{k_s}} \to \frac{1}{2(d-2s_1-2s_2)(s_1+s_2-1)} + O(k_s^2)$. Finally, $c(1,2,3,4)$ is fixed by the flat space limit, which is simply the four-point contact terms. This result matches with our new Feynman rules result in 5.1.4 and literature [3, 135].

We now turn to the five-point ansatz:

$$\mathcal{A}_5 = \frac{a_1(1,2,3,4,5)}{\mathcal{D}^{d-1}_{k_{12}}\mathcal{D}^{d-1}_{k_{45}}} + \frac{a_2(1,2,3,4,5)}{\mathcal{D}^{d-1}_{k_{12}}} + \frac{b(1,2,3,4,5)}{k_{12}^2 k_{45}^2} + Cyc, \tag{5.39}$$

where Cyc means sum over cyclic permutation, and factorization fixes all the a terms by recycling the four-point Eq (5.35):

$$\begin{aligned} a_1(1,2,3,4,5) &= \sum_h a(1,2,3,-k_I)\cdot \mathcal{A}_3(k_I,4,5), \\ a_2(1,2,3,4,5) & \\ &= \sum_h \mathcal{A}_3(1,2,-k_I)\cdot \left(\frac{b(k_I,3,4,5)}{k_{45}^2} + c(k_I,3,4,5) + [t]\right). \end{aligned} \tag{5.40}$$

Everything is fixed by the four-point on-shell amplitude so far. The final step is again the OPE poles, and we simply need to consider the residue on double OPE limits, which takes a very simple form:

$$\begin{aligned} &- b(1,2,3,4,5) \\ &= \operatorname*{Res}_{\substack{k_{12}^2\to 0 \\ k_{45}^2\to 0}} \left(\frac{a_1(1,2,3,4,5)}{\mathcal{D}^{d-1}_{k_{12}}\mathcal{D}^{d-1}_{k_{45}}} + \frac{a_2(1,2,3,4,5)}{\mathcal{D}^{d-1}_{k_{12}}} + \frac{a_2(4,5,1,2,3)}{\mathcal{D}^{d-1}_{k_{45}}}\right) \\ &= \frac{(s_1-s_2)(s_4-s_5)}{z^3}\varepsilon_1\cdot\varepsilon_2\varepsilon_4\cdot\varepsilon_5\varepsilon_3\cdot(k_4+k_5). \end{aligned} \tag{5.41}$$

We have obtained the full color-ordered five-point amplitude. It is easy to verify that the flat space limit is correct. Additionally, we have confirmed that it exactly matches the results from the new Feynman rules calculation and the literature [135, 136] after applying the mapping in Section 5.9.

5.5 Four Graviton amplitude

The four-graviton correlator in dS_4 was first computed in [43]. In this section, we will demonstrate how our bootstrap method enables efficient computation of this amplitude in AdS_{d+1}, beginning with the three-point amplitude:

$$\mathcal{M}_3 = \mathcal{A}_3^2. \tag{5.42}$$

Our four-point ansatz is given as:

$$\mathcal{M}_4 = \frac{a(1,2,3,4)}{\mathcal{D}^d_{k_s}} + \frac{b(1,2,3,4)}{k_s^{2m}} + c(1,2,3,4) + \mathcal{P}(2,3,4), \tag{5.43}$$

where remember that m can be 1 or 2 and $\mathcal{P}(2,3,4)$ denotes permutation to obtain other channels. Similar to the spin-1 case, factorization and OPE limit gives:

$$\begin{aligned} a(1,2,3,4) &= \sum_{h=\pm} \mathcal{M}_3(1,2,-k_s)\cdot\mathcal{M}_3(k_s,3,4), \\ b(1,2,3,4) &= -\operatorname*{Res}_{k_s^2\to 0} \frac{a(1,2,3,4)}{\mathcal{D}^d_{k_s}}, \end{aligned} \tag{5.44}$$

where the spin-2 polarization sum is given by,

$$\begin{aligned} &\sum_{h=\pm} \varepsilon_{\mu\nu}(k,h)\varepsilon_{\rho\sigma}(k,h)^* \\ =&\frac{1}{2}\Pi_{\mu\rho}\Pi_{\nu\sigma} + \frac{1}{2}\Pi_{\mu\sigma}\Pi_{\rho\nu} - \frac{1}{d-1}\Pi_{\mu\nu}\Pi_{\rho\sigma}. \end{aligned} \tag{5.45}$$

Now we are only left with contact terms in our bootstrap ansatz. Since Einstein Gravity is a two derivatives theory, the structure of contact terms is quite simple and can only have up to two derivatives[4]. We can classify them into two types:

$$c(1,2,3,4) = c_0(1,2,3,4) + c_1(1,2,3,4). \tag{5.46}$$

For c_0, it originates from a part of the graviton propagator, which is necessary for the amplitude to have Lorentz invariance in flat space limit and the curvature correction can easily be determined by external soft limit,

$$\begin{aligned} &c_0(1,2,3,4) \\ =&\varepsilon_{12,34}^2 \frac{8d(s_1-s_2)(s_3-s_4) - 4(s_1+s_3-s_2-s_4)^2 + d^2}{16(d-1)}, \end{aligned} \tag{5.47}$$

where we used notation $\varepsilon_{12,34} \equiv \varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4$. Finally, c_1 is the four-point contact interaction, whose general form will be:

$$c_1(1,2,3,4) = \varepsilon_{ab,cd,ef}(C_1 z^2 \varepsilon_m \cdot k_i \varepsilon_n \cdot k_j + C_2 \varepsilon_m \cdot \varepsilon_n \mathcal{D}^d_{ks}), \tag{5.48}$$

which is simply two/zero derivatives contact interaction. We can readily determine its coefficient using the flat space limit [5]. This completely determines the four-graviton amplitude and matched with [3], and [2, 43] after using the mapping in section 5.9.

[4] The most streamlined approach to deriving them, without any guesswork, would be to subtract their flat space limit and add the remaining terms into the ansatz. However, these terms will include unfixed sub-leading curvature corrections. Subsequently, the external soft limit will resolve all curvature corrections, similar to enhanced soft limits in EFT [1, 110]

[5] One might be concerned that the two-derivative contact interaction could have $1/R^2$ correction, which vanishes in the flat space limit. However, this term would not exhibit the correct behavior under external soft limits.

5.6 Five Graviton amplitude

To demonstrate the power of our algorithm, we will use this method to bootstrap the first five-graviton amplitude in AdS_{d+1}, the five-point ansatz is given by:

$$\mathcal{M}_5 = \frac{a_1(1,2;3;4,5)}{\mathcal{D}^d_{k_{12}}\mathcal{D}^d_{k_{45}}} + \frac{a_2(1,2;3,4,5)}{\mathcal{D}^d_{k_{12}}} \\ + \frac{b_1(1,2;3;4,5)}{k_{12}^{2m_1}k_{45}^{2m_2}} + \frac{b_2(1,2;3,4,5)}{k_{12}^{2m}} + c(1,2,3,4,5) + Perm. \tag{5.49}$$

We can recycle our four-point result to obtain all the terms with factorization poles:

$$a_1(1,2;3;4,5) = \sum_h a(1,2,3,-k_I)\cdot\mathcal{M}_3(k_I,4,5), \\ a_2(1,2;3,4,5) \\ = \sum_h \mathcal{M}_3(1,2,-k_I)\cdot\left(\frac{b(k_I,3,4,5)}{k_{45}^{2m}} + c(k_I,3,4,5) + \mathcal{P}(3,4,5)\right). \tag{5.50}$$

To simplify the calculation we first fix the terms with double OPE poles, which is the same procedure as Yang-Mills[6]:

$$-b_1(1,2;3;4,5) \\ = \operatorname*{Res}_{\substack{k_{12}^2\to 0\\ k_{45}^2\to 0}} \left(\frac{a_1(1,2;3;4,5)}{\mathcal{D}^d_{k_{12}}\mathcal{D}^d_{k_{45}}} + \frac{a_2(1,2;3,4,5)}{\mathcal{D}^d_{k_{12}}} + \frac{a_2(4,5;1,2,3)}{\mathcal{D}^d_{k_{45}}}\right). \tag{5.51}$$

However, unlike Yang-Mills, gravity includes terms with single OPE poles. Furthermore, as multiple channels contribute to the same single OPE poles, it becomes imperative to combine different channels. Strikingly, we can still resolve this issue

[6]Technically, since m could be 1 or 2, there are four terms with double poles for b_1, but since they can all be obtained by taking residue on both poles, we keep this notation for convenience.

simply by taking residues:

$$
\begin{aligned}
&- b_2(1,2;3,4,5) \\
=&\operatorname*{Res}_{k_{12}^2\to 0}\left\{\frac{a_1(1,2;3;4,5)}{\mathcal{D}^d_{k_{12}}\mathcal{D}^d_{k_{45}}}+\frac{b_1(1,2;3;4,5)}{k_{12}^{2m_1}k_{45}^{2m_2}}+Cyc(3,4,5)\right. \\
&\left.+\frac{a_2(1,2;3,4,5)}{\mathcal{D}^d_{k_{12}}}+\left(\frac{a_2(3,4;5,1,2)}{\mathcal{D}^d_{k_{34}}}+Cyc(3,4,5)\right)\right\}.
\end{aligned} \tag{5.52}
$$

Finally, we are left with only contact terms, which can be resolved by following the same procedure as for the four-point case,

$$
\begin{aligned}
c_0(1,2;3,4,5) =&\frac{d^2-4(s_1-s_2)^2-2d(s_1+s_2)}{16(d-1)} \\
&\times \varepsilon_{12,12,34,35,45}.
\end{aligned} \tag{5.53}
$$

Similar to four-point, the five-point contact interaction takes the following form, and we are left only with coefficients that can be readily determined by comparison with the flat space amplitude,

$$
c_1 =\varepsilon_{ab,cd,ef,rs}(C_1 z^2\varepsilon_m\cdot k_i\varepsilon_n\cdot k_j+C_2\varepsilon_m\cdot\varepsilon_n\mathcal{D}^d_{k_{ij}}). \tag{5.54}
$$

Summing over permutation for all the terms above, we obtained the first five-graviton amplitude in AdS_{d+1} and it shares the similar analytic structure of S-matrix in flat space.

5.7 Yang-Mills Amplitude and Color/Kinematics Duality

In the following sections, we will tackle the most interesting cases of AdS amplitudes: Yang-Mills and Gravity. While spinor-helicity techniques have shown to be very efficient in the flat space amplitude bootstrap, it is still unclear how to achieve similar simplicity in the AdS context. So in the rest of this chapter, our strategy will be writing the known expression for Yang-Mills from [123] into Mellin-Momentum space, and exploiting double copy to compute the gravity amplitude.

The 4-point Yang-Mills amplitude we can explicitly write down,

$$\begin{aligned}\mathcal{A}_4 =& \frac{z^2\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Pi_{1,1}+z^2W_s}{\mathcal{D}_{k_s}^{d-1}}+\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Pi_{1,0}+V_c^s\\ &-[(12)\to(23)],\end{aligned} \tag{5.55}$$

where

$$\begin{aligned}4W_s =& \varepsilon_1\cdot\varepsilon_2(k_1\cdot\varepsilon_3k_2\cdot\varepsilon_4-k_2\cdot\varepsilon_3k_1\cdot\varepsilon_4)\\ &+\varepsilon_3\cdot\varepsilon_4(k_3\cdot\varepsilon_1k_4\cdot\varepsilon_2-k_4\cdot\varepsilon_1k_3\cdot\varepsilon_2)\\ &+(k_2\cdot\varepsilon_1\varepsilon_2-k_1\cdot\varepsilon_2\varepsilon_1)\cdot(k_4\cdot\varepsilon_3\varepsilon_4-k_3\cdot\varepsilon_4\varepsilon_3),\end{aligned} \tag{5.56}$$

and the s-channel contact diagram is

$$4V_c^s =\varepsilon_1\cdot\varepsilon_3\varepsilon_2\cdot\varepsilon_4-\varepsilon_1\cdot\varepsilon_4\varepsilon_2\cdot\varepsilon_3. \tag{5.57}$$

We can now extract the kinematic numerator from the expression above:

$$n_s =\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4(z^2\Pi_{1,1}+\Pi_{1,0}\mathcal{D}_{k_s}^d)+z^2W_s+V_c^s\mathcal{D}_{k_s}^d, \tag{5.58}$$

where we have replaced the EoM operator Eq(5.5) with the different conformal dimension for Gravity. This seems to be an unavoidable procedure in curved space and we will discuss more about it in section5.10. The reason we keep the kinematic numerator as an operator form is so that when we perform the double copy, we can simply cancel it with the propagator in the denominator. However, the EoM operator itself by definition Eq(5.10) is equivalent to $\mathcal{D}_{k_s}^d = z^2k_s^2+4s_{12}s_{34}$. It's noteworthy that after this replacement the kinematic numerator is free of the OPE pole k_s now. Other channels can be obtained by permutation:

$$n_t = n_s\big|_{(234)\to(423)}, \quad n_u = n_s\big|_{(234)\to(342)}. \tag{5.59}$$

It's easy to verify that the kinematic Jacobi identity is satisfied, similar to the case of flat space amplitude [21, 27, 28, 108, 113, 117]:

$$n_s + n_t + n_u = 0. \tag{5.60}$$

5.8 Graviton Amplitude And Double copy

Firstly, we revisit the 3-point gravity amplitude:

$$\mathcal{M}_3 = z^2(\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot k_1 + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot k_2 + \varepsilon_3 \cdot \varepsilon_1 \varepsilon_2 \cdot k_3)^2. \tag{5.61}$$

This has a manifestly double copy structure [27, 111, 112, 125, 126, 137, 138] with the appropriate normalization:

$$\mathcal{M}_3 = (\mathcal{A}_3)^2. \tag{5.62}$$

This relation has no ordering ambiguity and is valid in general dimensions. However, this is not the full story. As double copy of pure Yang-Mills should give graviton coupled to dilaton and antisymmetric tensor. Considering the tensor product of the polarization, we decompose it into a transverse and traceless tensor (graviton) and a trace (dilaton),

$$\varepsilon^\mu \varepsilon^\nu = \frac{1}{2}(\varepsilon^\mu \varepsilon^\nu + \varepsilon^\nu \varepsilon^\mu - \frac{2}{d-1}\Pi^{\mu\nu}) + \frac{1}{d-1}\Pi^{\mu\nu}, \tag{5.63}$$

where $\Pi_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is the projection tensor. This predicts a new interaction between two graviton $h_{\mu\nu}$ and one dilaton ϕ (where dilaton is identified as $\varepsilon^+\varepsilon^-$):

$$\begin{aligned} \mathcal{M}_3(1_h, 2_h, 3_\phi) &= (\mathcal{A}_3)^2|_{\varepsilon_3^\mu \varepsilon_3^\nu \to \Pi^{\mu\nu}}, \\ &= z^2(\varepsilon_1 \cdot \varepsilon_2)^2 k_1^\mu k_1^\nu \Pi_{\mu\nu}. \end{aligned} \tag{5.64}$$

This amplitude has a vanishing flat space limit as expected. Moving to four-point, with the color/kinematic duality satisfying in Eq(5.60), we can replace the color factor with the kinematic numerator [2, 25, 27, 115, 139],

$$\mathcal{M}_4 = \frac{n_s^2}{\mathcal{D}_{k_s}^d} + \frac{n_t^2}{\mathcal{D}_{k_t}^d} + \frac{n_u^2}{\mathcal{D}_{k_u}^d}. \tag{5.65}$$

However, with the new 3-point interaction found above, the double copy result will include four external graviton exchanging dilaton. It would be very interesting to check whether this corresponds to the four-graviton amplitude in a dilaton-graviton theory. We leave this exploration to the future. Instead, here we will extract Einstein gravity from double copy result above by using a similar unitarity method in flat space. A similar situation happened for pure gravity at loop level and massive scalar [83, 84, 140–143]. To project out the dilaton scalar degree of freedom, we can demand the factorization only has graviton propagation:

$$\mathcal{M}_4^{EG} = \frac{n_s^2}{\mathcal{D}_{k_s}^d} + \frac{n_t^2}{\mathcal{D}_{k_t}^d} + \frac{n_u^2}{\mathcal{D}_{k_u}^d} - \tilde{\mathcal{M}}_{\text{AdS}}. \tag{5.66}$$

So we subtracted out the dilaton state,

$$\begin{aligned}\tilde{\mathcal{M}}_{\text{AdS}} =& (\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot \varepsilon_4)^2 \Big(\frac{z^4 \Pi_{2,2}^{\text{Tr}}}{\mathcal{D}_{k_s}^d} - \Pi_{2,0} + \Pi_{1,0}^2 \mathcal{D}_{k_s}^d\Big) \\ &+ \mathcal{P}(2,3,4),\end{aligned} \tag{5.67}$$

where $\mathcal{P}(2,3,4)$ denotes sum over permutation to obtain the t-channel and u-channel. The first correction term is the dilaton exchange diagram and the last two can be understood as the conformal structure of the graviton propagator. This formula is the first 4-point gravity amplitude in AdS_{d+1} and takes on a remarkably simple form. In particular, this formula exhibits flat space structure and explains the origin of the complex contact interaction terms as simply zero/two derivative scalar contact interaction like flat space amplitude. As a result, one can simply replace the flat space amplitude by Eq(5.73) to obtain AdS amplitude. We have explicitly verified that it matches with [43] in $d = 3$ by reverting back to momentum space.

5.9 Mapping to Momentum space

Ultimately, our interest still lies in correlators composed of pure kinematic momentum. Mellin-Momentum amplitude not only serves as a convenient framework for understanding amplitude structure but also serves as a useful computational tool for cosmological correlators. In this section, we will verify all expressions in the chapter by mapping them back to momentum space. We will provide a straightforward algebraic algorithm to demonstrate that this transition is transparent and simple for n-point without doing any integrals. For Yang-Mills in $d = 3$, Mellin variable s_i by definition gives $s_i \to \frac{zk_i}{2} + \frac{1}{4}$, then the n-point amplitude can be mapped as follows:

$$\begin{aligned} \text{Amplitudes} &\to \text{Correlators} \\ z^{4-n}A(k,\varepsilon) &\to \frac{A(k,\varepsilon)}{E_t} \\ \frac{z^{6-n}A(k,\varepsilon)}{\mathcal{D}_{k_I}^{d-1}} &\to \frac{A(k,\varepsilon)}{E_t E_I E_{t-I}} \\ \frac{z^{2n_{\mathcal{D}}+4-n}A(k,\varepsilon)}{\mathcal{D}_{k_I}^{d-1}\mathcal{D}_{k_J}^{d-1}\dots\mathcal{D}_{k_M}^{d-1}} &\to \sum_{\sigma} \frac{1}{E_t E_I}\frac{A(k,\varepsilon)}{\mathcal{D}_{k_J}^{d-1}\dots\mathcal{D}_{k_M}^{d-1}}, \end{aligned} \tag{5.68}$$

where the sum over σ represents summing over all the possible permutation on $I, J, \dots, M$ and $n_{\mathcal{D}}$ is the number of $1/\mathcal{D}$. Our notation are total energy pole $E_t = \sum_{i=1}^{n} k_i$, and sub-total energy pole E_I. Such recursive integral is not too surprising for Yang-Mills, given their Weyl invariance in $3+1$ dimension [99, 121], so we simply obtained this mapping from the same observation as flat space wavefunction recursion. However, for Gravity in $d = 3$, we found a similar recursion for n-point scalar integral as well! This mapping is obtained based on the representation of the bulk-to-bulk propagator in (C.3.17) which allows us to write all the exchange diagram in terms of contact diagrams,

$$\begin{aligned}
\text{Amplitudes} &\rightarrow \text{Correlator} \\
z^2 M(k,\varepsilon) &\rightarrow M(k,\varepsilon)\mathcal{C}_1 \\
\prod_{m=1}^{l}(-2s_m + d/2)\, M(k,\varepsilon) &\rightarrow M(k,\varepsilon)\mathcal{C}_2^l \\
\frac{M(k,\varepsilon)}{\mathcal{D}^d_{k_I}} &\rightarrow M(k,\varepsilon)\mathcal{I}_I \\
\frac{M(k,\varepsilon)}{\mathcal{D}^d_{k_J}\dots\mathcal{D}^d_{k_M}} &\rightarrow M(k,\varepsilon)\mathcal{I}_{J\dots M}.
\end{aligned} \tag{5.69}$$

The first contact integral with no derivatives is:

$$\mathcal{C}_1^{(n)} = \left(\sum_{m=0}^{n-2} m! \sum_{1\le i_1<\dots<i_{m+2}}^{n} \frac{k_{i_1}\dots k_{i_{m+2}}}{E_t^{m+1}}\right) - E_t. \tag{5.70}$$

For the number of derivatives greater than two ($l \geq 2$),

$$\mathcal{C}_2^{l;(n)} = \left(\sum_{m=0}^{n-l}(2l-4+m)! \sum_{\substack{l+1\le i_{l+1}< \\ \dots<i_{m+l}}}^{n} \frac{k_{i_{l+1}}\dots k_{i_{m+l}}}{E_t^{(2l-4)+m+1}}\right)(-1)^l k_1^2 \dots k_l^2, \tag{5.71}$$

where when $m = 0$, the the numerator above E_t is just 1. The n-point exchange integral can be recursively obtained by simply taking the residue of the above contact integral:

$$\begin{aligned}
\mathcal{I}_I &\equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi i}\frac{p^{-2}}{k_I^2+p^2}\bar{\mathcal{C}}(k_1,\dots,ip)\bar{\mathcal{C}}(ip,\dots,k_n), \\
\mathcal{I}_{IJ\dots M} &\equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi i}\frac{p^{-2}}{k_I^2+p^2}\bar{\mathcal{C}}(k_1,\dots,ip)\bar{\mathcal{I}}_{J\dots M}(ip,\dots,k_n),
\end{aligned} \tag{5.72}$$

where we define a shifted function: $\bar{\mathcal{C}} = \mathcal{C}(k_1,\dots,ip) - \mathcal{C}(k_1,\dots,-ip)$ and $\bar{\mathcal{I}} = \mathcal{I}(ip\dots,k_n) - \mathcal{I}(-ip\dots,k_n)$. We believe these cover all n-point scalar integrals for Gravity in $d = 3$. We will provide more details and examples at five-point in Appendix C.3. Therefore, if one is provided with an n-point Mellin-Momentum amplitude, one can simply follow the map to obtain the wavefunction coefficients, requiring only the computation of a finite number of residues without the need for any time integrals. Then one can use the recent transitioning from AdS to dS [104, 144, 145], to obtain cosmological correlators.

5.10 Remarks

Remark on flat space structure: Given the simplicity of the Mellin-Momentum amplitude and its resemblance to its flat space counterpart, we can define an uplifting operation as follows:

$$\begin{aligned}\mathcal{U}:\{&\epsilon_i\cdot\epsilon_j\to\varepsilon_i\cdot\varepsilon_j,\epsilon_i\cdot k_j\to z\varepsilon_i\cdot k_j,\\ &\frac{(T-U)^2}{S}\to\frac{z^4\Pi_{2,2}}{\mathcal{D}^{\Delta}_{k_s}}+\Pi_{2,1}+\Pi_{2,0},\\ &\frac{T-U}{S}\to\frac{z^2\Pi_{1,1}}{\mathcal{D}^{\Delta}_{k_s}}+\Pi_{1,0},S\to z^2k_s^2+4s_{12}s_{34}\},\end{aligned}\tag{5.73}$$

which uplifts all the scattering variables in $(d+1)$ Minkowski space to the AdS ones. In hindsight, the last three steps are essentially stating that we should replace Lorentz-invariant quantities with conformally-invariant ones. It is noteworthy that all the examples considered in this chapter adhere to this form,

$$\mathcal{A}^{\text{AdS}}=\mathcal{U}(A^{\text{Mink}}).\tag{5.74}$$

It would be interesting to compare this operation with weight-shifting operators approach: [27, 121, 126, 146, 147].

Remark on flat space limit: Based on the previous work [45, 133], it is easy to guess that the flat space limit of the Mellin-Momentum amplitude can be obtained by taking the scaling limit of the Mellin variables $s_i\to\infty$, and then replacing them with the corresponding norm of momentum:

$$\lim_{\mathcal{R}_{\text{AdS}}\to\infty,s_i\to\infty}\mathcal{A}^{\text{AdS}}\xrightarrow{s_i\to\frac{zk_i}{2}}A^{\text{Mink}}\delta^{d+1}(\sum_i^n\vec{k}_i).\tag{5.75}$$

Under the scaling limit, the Mellin mode in z direction behaves like a Fourier mode and hence the delta function in flat space naturally arises from combining the boundary momentum conservation with the Mellin delta function. One might try to prove this formula following the discussion in [134].

Remark on Double Copy in curved space: In our proposal, the squaring process naturally mimics the Double Copy structure in the S-matrix, while the rest of the bootstrap procedure aims to probe the extra structure in curved space. The additional constraint we need to construct the Double Copy to Gravity might be a generic feature in curved spacetime. The color/kinematic duality and double copy for Non-Linear Sigma Models(NLSM) was studied in [1, 98, 108, 110, 116]. Moreover, in [98] the authors showed that the duality holds off-shell at symmetric spacetime manifold. However, the process of replacing color with kinematic is blind to the extra quantum number of conformal dimension, which has different values in AdS for NLSM, and sG [1, 148].

5.11 Conclusions

In this chapter, we show that the analytic structure of n-point Mellin-Momentum amplitudes is remarkably simple, and can be computed recursively like flat space amplitudes, which confirms the proposal that Mellin-Momentum amplitudes in (A)dS should play a similar role as S-matrix in flat space. Pragmatically, it is easy to check that our new result for five-graviton amplitude by bootstrap method is correct by construction, as any mistake would result in non-physical poles. This will give us the Gravity Quadrispectrum for cosmological correlators.

It would also be interesting to extend the Gravity calculation to loop-level. Once employing our bootstrap approach to determine the Mellin-Momentum amplitude, we are then left with scalar loop integrals. Particularly, it was shown in [149] that the scalar loop integral for the in-in correlator is closer to the S-matrix in flat space.

CHAPTER 6

Conclusions

In this thesis, we investigate various aspects of incorporating scattering amplitude techniques to study wavefunction coefficients and ultimately understand cosmological correlators. Our exploration starts with the exceptional scalar theory, for which we discovered a nontrivial connection between soft limit and shift symmetry in de-Sitter space. Moving on to the spinning particles, we explore whether the complicated four-point graviton correlators can be expressed as the square of much simpler gluon correlators, which is referred to as the double copy structure for S-matrix in flat space. It's still not clear whether we can achieve a similar statement as flat space, but instead, we combine the double copy result with bootstrap method to reproduce the four-graviton correlators. Despite the successful attempts in our previous approach, we also realize that this is still not ideal since even all the simple examples at four-points require much more work and calculation compared to flat space. We also constantly find that there are boundary contact terms that are not zero but need to be identified. All the subtleties will clearly be much more significant when we go to higher points. In order to avoid all these problems, in the last chapter we introduce the on-shell amplitude in Mellin-momentum space which circumvents the problems.

In Chapter 3, we show that the enhanced soft limit can fix the couplings of EFTs up to six-points, and we expect this will go on to higher point. However, this is still not a proof to all couplings. Another approach for fixing all couplings of the DBI theory from enhanced soft limits is suggested by the following observation. The DBI Lagrangian in dS can be written in the form:

$$\frac{\mathcal{L}_{DBI}}{\sqrt{-g}} = \frac{\sqrt{1-X-Y}}{(1-Y)^{d/2+1}} = L(X,Y), \tag{6.1}$$

where $X = \nabla\phi \cdot \nabla\phi$ and $Y = \phi^2$, which is a solution to the following simple differential equation:

$$(1-X-Y)\frac{\partial L}{\partial X} + \frac{L}{2} = 0. \tag{6.2}$$

In [56], the flat space analogue of this differential equation (which corresponds to setting $Y = 0$) was deduced from general arguments about enhanced soft limits of the S-matrix. Given the simplicity of the DBI Lagrangian in dS, it seems plausible that these arguments can be generalised to dS.

This leads us to the next question: how do we prove that higher shift symmetries in dS imply enhanced soft limits of the wavefunction coefficients without using Lagrangians? The analogous proof in flat space, which was sketched in section 2.2, relied heavily on the definition of the S-matrix, and does not immediately lift to wavefunction coefficients or CFT correlators. But with the new representation we discussed in Chapter 5, it seems plausible to repeat the proof to de-Sitter space, we hope to gain a deeper understanding of this issue in the future.

Moving on to the double copy structure of gravity amplitudes, we still don't fully understand the story in (A)dS space. However, with the new dilaton interaction discovered in Section 5.8, it is crucial to understand the role of the dilaton in these amplitudes. In particular, when the dilaton state appears in the four-graviton amplitude, the cutting rules become simpler and can be seen as simply the square of the Yang-Mills result. This suggests that incorporating the dilaton degree of freedom might result in a simpler gravity amplitude in de-Sitter space. If this statement turns out to be true, it would be very surprising that adding a new degree of freedom actually results in a simpler gravity amplitude. Such a statement is completely obscure from the Lagrangian point of view, like in $\mathcal{N} = 4$ Super Yang-Mills, where the superamplitude in flat space is simpler than the Yang-Mills amplitude itself [51]. This could perhaps provide us with a different perspective on understanding gravity theory in a curved background.

Given the bootstrap method we have presented which makes the flat space structure of the AdS amplitude manifest in Mellin-Momentum space, our dream for the future is to discover an n-point formula akin to the Park-Taylor formula in the S-matrix [150]. The Parke-Taylor formula transformed the thousands of Feynman diagrams for n-gluon scattering in a specific helicity configuration into a simple, compact formula, which revolutionized the field of scattering amplitudes. To achieve this, we first need to understand better the spinor-helicity representation [32] in Mellin-Momentum space and make the on-shell degrees of freedom manifest. We hope to report progress on this in the future.
More generally, Our bootstrap approach does not rely on the spacetime symmetry [12] but simply by demanding the correlators in differential representation have the correct pole structure and have the correct limits. So the AdS study here is just the simplest example of curved space. Therefore, there is potential for its implementation in more general curved backgrounds, such as FLRW spacetime or even black hole backgrounds. In practice, the simpler next setup would be to understand or bootstrap the inflationary correlators using our method.

APPENDIX A

Appendix for Chapter 3

A.1 4-point sGal Soft Limit

This appendix includes some extra details of the calculations in section 3.2.3. In particular, we will explain how to evaluate the $\hat{s}^3_{ab}$ terms in (3.33). This is done using the definitions in (3.11) along with their known action on bulk-to-boundary propagators [110]:

$$\begin{aligned} D\mathcal{K}_\nu &= \eta\tfrac{\partial}{\partial\eta}\mathcal{K}_\nu, \qquad & P^i\mathcal{K}_\nu &= k^i\mathcal{K}_\nu, \\ K_i\mathcal{K}_\nu &= \eta^2 k_i\mathcal{K}_\nu, \qquad & M_{ij}\mathcal{K}_\nu &= 0. \end{aligned} \tag{A.1.1}$$

To evaluate the action of $\hat{s}^3_{ab}$ we also need

$$\begin{aligned}
K_\alpha(k_i\phi) &= \eta^2 k_\alpha k_i \phi - 2\eta\delta_{i\alpha}\dot{\phi}, \\
K_\alpha\dot{\phi} &= k_\alpha(\eta^2\dot{\phi} + 2\eta\phi), \\
K_\alpha(k_i k_j\phi) &= \eta^2 k_\alpha k_i k_j\phi - 2(\delta_{\alpha i}k_j + \delta_{\alpha j}k_i)(\phi + \eta\dot{\phi}) + 2k_\alpha\delta_{ij}\phi, \\
K_\alpha\ddot{\phi} &= k_\alpha(\eta^2\ddot{\phi} + 4\eta\dot{\phi} + 2\phi), \\
D\dot{\phi} &= \eta\ddot{\phi} + \dot{\phi}, \\
M_{12}\left(\vec{k}_1\cdot\vec{k}_2 f(k_1,k_2)\right) &= 2(d-1)\vec{k}_1\cdot\vec{k}_2 f(k_1,k_2), \\
M_{12}\left((\vec{k}_1\cdot\vec{k}_2)^2 f(k_1,k_2)\right) &= 4\left(d(\vec{k}_1\cdot\vec{k}_2)^2 - k_1^2k_2^2\right) f(k_1,k_2),
\end{aligned} \tag{A.1.2}$$

where $f(k_1, k_2)$ is some function depending only on the magnitudes of the momenta.

The action of the cubic operator is then given by

$$\begin{aligned}
\hat{s}^3_{12}\phi_1\phi_2 = \eta^6\Bigg[&(\vec{k}_1\cdot\vec{k}_2)^3\phi_1\phi_2 + 3(\vec{k}_1\cdot\vec{k}_2)^2\dot{\phi}_1\dot{\phi}_2 + 3(\vec{k}_1\cdot\vec{k}_2)\ddot{\phi}_1\ddot{\phi}_2 + \dddot{\phi}_1\dddot{\phi}_2 \\
&+ \frac{3}{\eta}\Big(2(\vec{k}_1\cdot\vec{k}_2)^2(\dot{\phi}_1\phi_2 + \phi_1\dot{\phi}_2) + (\vec{k}_1\cdot\vec{k}_2)\left(-k_1^2\phi_1\dot{\phi}_2 - k_2^2\dot{\phi}_1\phi_2 + 3(\ddot{\phi}_1\dot{\phi}_2 + \dot{\phi}_1\ddot{\phi}_2)\right) \\
&\quad - k_1^2\dot{\phi}_1\ddot{\phi}_2 - k_2^2\ddot{\phi}_1\dot{\phi}_2 + \dddot{\phi}_1\ddot{\phi}_2 + \ddot{\phi}_1\dddot{\phi}_2\Big) \\
&+ \frac{1}{\eta^2}\Bigg((10-3d)(\vec{k}_1\cdot\vec{k}_2)^2\phi_1\phi_2 + 2(\vec{k}_1\cdot\vec{k}_2)\Big(2(\ddot{\phi}_1\phi_2 + \phi_1\ddot{\phi}_2) + (29-3d)\dot{\phi}_1\dot{\phi}_2 \\
&\quad - (k_1^2+k_2^2)\phi_1\phi_2\Big) + 2k_1^2k_2^2\phi_1\phi_2 \\
&\quad - k_1^2(5\dot{\phi}_1\dot{\phi}_2 + 4\phi_1\ddot{\phi}_2) - k_2^2(5\dot{\phi}_1\dot{\phi}_2 + 4\ddot{\phi}_1\phi_2) + \dddot{\phi}_1\dot{\phi}_2 + \dot{\phi}_1\dddot{\phi}_2\Bigg) \\
&+ \frac{1}{\eta^3}\Big(4(3-d)\vec{k}_1\cdot\vec{k}_2(\dot{\phi}_1\phi_2 + \phi_1\dot{\phi}_2) + (d-6)k_1^2\phi_1\dot{\phi}_2 + (d-6)k_2^2\dot{\phi}_1\phi_2 \\
&+ 3(\ddot{\phi}_1\dot{\phi}_2 + \dot{\phi}_1\ddot{\phi}_2) \\
&+ \frac{1}{\eta^4}\left((d-2)^2\vec{k}_1\cdot\vec{k}_2\phi_1\phi_2 + \dot{\phi}_1\dot{\phi}_2\right)\Bigg].
\end{aligned} \tag{A.1.3}$$

We can then compute the soft limit:

$$\lim_{\vec{k}_1\to 0} \hat{s}_{12}^3\phi_1\phi_2 = \eta^6\Bigg(\dddot{\phi}_1 \dddot{\phi}_2 + \frac{3}{\eta}\left(-k_2^2\ddot{\phi}_1\dot{\phi}_2 + \dddot{\phi}_1\ddot{\phi}_2 + \ddot{\phi}_1\dddot{\phi}_2\right),$$
$$+ \frac{1}{\eta^2}\left(-k_2^2(5\dot{\phi}_1\dot{\phi}_2 + 4\ddot{\phi}_1\phi_1) + \dddot{\phi}_1\dot{\phi}_2 + 9\ddot{\phi}_1\ddot{\phi}_2 + \dot{\phi}_1\dddot{\phi}_2)\right),$$
$$+ \frac{1}{\eta^3}\left((d-6)k_2^2\dot{\phi}_1\dot{\phi}_2 + 3(\ddot{\phi}_1\dot{\phi}_2 + \dot{\phi}_1\ddot{\phi}_2)\right) + \frac{1}{\eta^4}\dot{\phi}_1\dot{\phi}_2\Bigg) + \mathcal{O}(k_1). \tag{A.1.4}$$

This can also be expressed in terms of boundary generators as was done in previous work [110]. For example, the leading soft limit of (3.33) is given by

$$\lim_{\vec{k}_1\to 0} \Psi_4^{sGal} = -\mathcal{N}\Big((\Delta-d)(\Delta-d-1)(\Delta-d-2)(D_2^3+D_3^3+D_4^3)$$
$$-(\Delta-d)(\Delta-d-1)(B+2+2d)(D_2^2+D_3^2+D_4^2)$$
$$+\Delta(\Delta-d)(d(\Delta^2+\Delta-4) - B(2+\Delta) - \Delta^3 + 4\Delta - 4) - C\Big)$$
$$\int \frac{d\eta}{\eta^{\Delta+1}}\phi_2\phi_3\phi_4 + \mathcal{O}(k_1). \tag{A.1.5}$$

A.2 6-point DBI Soft Limit

In this Appendix, we will provide more details about the calculation in section 3.3.2. In particular, we will present an algorithm for systematically applying equivalence relations to express the 6-point tree-level wavefunction coefficient in terms of linearly independent terms. This allows us to fix all the couplings from enhanced soft limits. The equivalence relations are

- conformal Ward identities in terms of the $\hat{s}_{ab}$ operators,
- boundary momentum conservation,
- equations of motion for the bulk-to-boundary wavefunctions,
- integration by parts identities/ addition of a total derivative to the integrand.

Note that we neglect any boundary contributions that may come from integration by parts since they have delta function support when Fourier transformed to position space. Although the relations implied by conformal Ward identities can also be obtained from a combination of the other three equivalence relations, in practice we use all four in such a way as to remove the need for guesswork. In particular, we apply momentum conservation, equations of motion, and integration by parts relations in a particular order such that the latter can be constructed systematically.

After fixing Δ from the enhanced soft limit at four points, it is sufficient to work to leading order in the soft momentum in order to fix the 6-point couplings. The procedure for fixing these couplings is then given below:

1. Write the soft limit of an exchange diagram as a contact diagram by cancelling numerator and denominator in this limit (see (3.61)).

2. Sum all diagrams over permutations to obtain the wavefunction coefficient. The wavefunction coefficient is now of the form $f(\hat{s}_{ab})\mathcal{C}_6$, where f is a polynomial up to cubic order in the $\hat{s}_{ab}$.

3. Apply the conformal Ward identities to eliminate one leg and one $\hat{s}_{ab}$, mimicking the use of momentum conservation needed to demonstrate enhanced limits of amplitudes in flat space. We choose to eliminate leg n and $\hat{s}_{n-2\,n-1}$ using $\hat{s}_{an} = -\sum_{b=1}^{n-1} s_{ab}$ and $\left(\sum_{a=1}^{n-1} \mathcal{D}_a\right)^2 = \hat{s}_{nn}$. At each stage we can also apply $\hat{s}_{aa} \sim -m^2$. Note that this will remove any derivatives acting on the field ϕ_n. It will not however remove all occurrences of $\vec{k}_{n-2} \cdot \vec{k}_{n-1}$ in the integrand since they can also appear from the successive action of $\hat{s}_{a\,n-2}\hat{s}_{a\,n-1}$, for example. This means that we can still apply boundary momentum conservation to eliminate quantities that are not independent.

4. Use (3.19) to finish taking the soft limit and use the propagator equation of motion to remove factors of k_a^2.

5. Use boundary momentum conservation to remove $\vec{k}_{n-2} \cdot \vec{k}_{n-1}$. This will reintroduce the magnitudes k_a^2 (including k_n) so we again apply equations of motion such that the integrand contains only functions not linked by equations of motion.

6. The equations of motion will introduce derivatives of ϕ_n so use integration by parts to remove $\ddot{\phi}_n$ and then $\dot{\phi}_n$. This step can be done systematically by identifying terms of the form $\int d\eta\, g(\eta, \vec{k}_a, \partial_\eta^l \phi_{b\neq n})\partial_\eta^m \phi_n$ for some function g and deriving the appropriate total derivative which contains it.

7. The wavefunction coefficient can now be seen to vanish for specific choices of the coefficients A, B, C in (3.56).

Finally, we note that operators that are quadratic or cubic in leg 1 can be written as combinations of operators that are at most linear in leg 1, up to $\mathcal{O}(k_1)$. It is this property for example that allowed us to obtain equation (3.61). We also observe that

$$\begin{aligned} \hat{s}_{12}^3 \mathcal{C}_6^{\Delta=d+1} &= \left((d^2+d+1)\hat{s}_{12} + d(d+1)\right)\mathcal{C}_6^{\Delta=d+1} + \mathcal{O}(k_1^2), \\ \hat{s}_{12}\hat{s}_{13}\hat{s}_{23}\mathcal{C}_6^{\Delta=d+1} &= \left(\hat{s}_{23}^2 + \hat{s}_{13}\hat{s}_{23} - (d+1)\hat{s}_{12} + d\hat{s}_{23}\right)\mathcal{C}_6^{\Delta=d+1} + \mathcal{O}(k_1^2). \end{aligned} \tag{A.2.6}$$

In principal, we could also use these properties to solve for the unknown coefficients without needing to consider the full integrand.

A.3 Matching 6-point Wavefunctions

We will now show that the wavefunction coefficient obtained from the Lagrangian in (3.64) gives the same wavefunction coefficient as the one obtained from enhanced soft limits. Applying the free equation of motion to rewrite the $(\nabla\phi\cdot\nabla\phi)\phi^4$ as a ϕ^6 interaction gives

$$\begin{aligned} \frac{\mathcal{L}_6^{DBI}}{\sqrt{-g}} = &-\frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{d+1}{2}\phi^2 - \frac{1}{8}(\nabla\phi\cdot\nabla\phi)^2 \\ &- \frac{1}{4}(d+3)(\nabla\phi\cdot\nabla\phi)\phi^2 + \frac{3(d+1)(d+3)}{4!}\phi^4 - \frac{3}{48}(\nabla\phi\cdot\nabla\phi)^3 \\ &- \frac{3(d+5)}{16}(\nabla\phi\cdot\nabla\phi)^2\phi^2 + \frac{6(d+1)(d+3)(d+5)}{6!}\phi^6. \end{aligned} \tag{A.3.7}$$

We then obtain the following contribution from 6-point contact Witten diagrams:

$$\begin{aligned}&\Psi^{DBI}_{6,\,\mathrm{cont}}\\ =&\delta^3\left(\vec{k}_T\right)\left[3\left(\hat{s}_{12}\hat{s}_{34}\hat{s}_{56}+\text{perms}\right)-(5+d)\left(\hat{s}_{12}\hat{s}_{34}+\text{perms}\right)+6(1+d)(3+d)(5+d)\right]\mathcal{C}_6^{\Delta=d+1},\end{aligned} \tag{A.3.8}$$

where the terms are summed over all inequivalent permutations. Moreover, we find the following contribution from exchange diagrams:

$$\begin{aligned}\Psi^{DBI}_{6,\,\mathrm{exch}}=&\frac{\delta^3\left(\vec{k}_T\right)}{(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2}\Big[\hat{s}_{12}\hat{s}_{3L}+\text{Cyc.}[123]-3(1+d)(3+d)\\ &-(d+3)\left(\hat{s}_{12}+\hat{s}_{23}+\hat{s}_{31}+\mathcal{D}_L\cdot(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)\right)\Big]\times(123)\leftrightarrow(456)\mathcal{C}_6^{\Delta=d+1}+\text{perms}.\end{aligned} \tag{A.3.9}$$

Next we use the conformal Ward identity at the vertex $-\mathcal{D}_L=\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3$ to express the terms quadratic in boundary conformal generators terms as an inverse propagator plus a constant:

$$\begin{aligned}\Psi^{DBI}_{6,\,\mathrm{exch}}=&\frac{\delta^3\left(\vec{k}_T\right)}{(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2}\Big[\hat{s}_{12}\hat{s}_{3L}+\text{Cyc.}[123]-3(1+d)(3+d)\\ &-(d+3)\left(\frac{1}{2}[(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2]+2(d+1)\right)\Big]\times(123)\leftrightarrow(456)\mathcal{C}_6^{\Delta=d+1}+\text{perms},\end{aligned} \tag{A.3.10}$$

where we have used $\mathcal{D}_a^2\sim-m^2$ to simplify the constant. This can be identified as the exchange diagram from (3.59) plus a new contact contribution:

$$\Psi^{DBI}_{6,\,\mathrm{exch}}=\frac{\delta^3\left(\vec{k}_T\right)\Psi_L\Psi_R}{(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)+m^2}\mathcal{C}_6^{\Delta=d+1}+\tilde{\Psi}^{DBI}_{6,\,\mathrm{cont}}, \tag{A.3.11}$$

where

$$\tilde{\Psi}^{DBI}_{6,\,\mathrm{cont}}=\delta^3\left(\vec{k}_T\right)\left\{\frac{1}{2}(d+3)(\Psi_L+\Psi_R)+\frac{1}{4}(d+3)^2\left[(\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3)^2+m^2\right]\right\}\mathcal{C}_6^{\Delta=d+1}+\text{perms}. \tag{A.3.12}$$

We now work with the new contact contribution, summing over the 10 factorisation channels and comparing to the form in (A.3.8). To do this, we want the quadratic term to be expressed as a sum of terms each with 4 distinct labels. We therefore use the conformal Ward identities to write $\mathcal{D}_L = \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6$ to get

$$\Psi_L = \hat{s}_{12}(\hat{s}_{34} + \hat{s}_{35} + \hat{s}_{36}) + \text{Cyc.}[123] - (1+d)(3+d), \tag{A.3.13}$$

and analogously for Ψ_R. We can see that the quadratic term from $\Psi_L + \Psi_R$ will contain 18 terms so the sum over 10 channels will give a permutation-invariant sum of 180 terms. Since there are 45 unique $\hat{s}_{ab}\hat{s}_{cd}$, this gives us a symmetry factor of 4. A similar analysis of the linear terms from $(\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3)^2$ gives a symmetry factor of 4 as well. We can therefore express the new contact contribution as

$$\begin{aligned}&\tilde{\Psi}^{DBI}_{6,\,\text{cont}}\\ =&\delta^3\left(\vec{k}_T\right)\left[2(d+3)(\hat{s}_{12}\hat{s}_{34} + \text{Perms}) + (d+3)^2(\hat{s}_{12} + \text{Perms}) - 5(d+1)(d+3)^2\right]\mathcal{C}_6^{\Delta=d+1}.\end{aligned} \tag{A.3.14}$$

Noting that $(\hat{s}_{12} + \text{perms}) = 3m^2 = -3(d+1)$, this becomes

$$\tilde{\Psi}^{DBI}_{6,\,\text{cont}} = \delta^3\left(\vec{k}_T\right)\left[2(d+3)(\hat{s}_{12}\hat{s}_{34} + \text{perms}) - 8(d+1)(d+3)^2\right]\mathcal{C}_6^{\Delta=d+1}. \tag{A.3.15}$$

We can then combine this with equation (A.3.8) to give

$$\Psi^{DBI}_{6,\,\text{cont}} = \delta^3\left(\vec{k}_T\right)\left[(d+1)(\hat{s}_{12}\hat{s}_{34} + \text{Perms}) + 2(d+1)(9-d^2)\right]\mathcal{C}_6^{\Delta=d+1}, \tag{A.3.16}$$

matching the result obtained from the enhanced soft limit. This wavefunction coefficient therefore also corresponds to the one obtained from (3.63).

APPENDIX B

Appendix for Chapter 4

B.1 Notation and Conventions

In this Appendix we will summarise our conventions and collect various useful definitions that are used throughout the paper. When performing conformal time integrals, we Wick-rotate to Euclidean AdS_4 with unit radius, whose metric is given by

$$\mathrm{d}s^2 = (1/z)^2(\mathrm{d}z^2 + \mathrm{d}\vec{x}^2), \tag{B.1.1}$$

where $0 < z < \infty$ is the radial coordinate and x^i with $i \in \{1, 2, 3\}$ are the boundary coordinates. This is obtained from (2.59) by taking $\eta \to iz$ and dropping an overall minus sign. Moreover, we Fourier transform wavefunction coefficients to momentum space along the boundary directions and our Fourier convention is

$$f(\vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \equiv \int_{\vec{k}} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}}. \tag{B.1.2}$$

We use Greek indices, $\mu, \nu \ldots$ to label the components of 4-vectors and Latin indices from the middle of the alphabet, $i,\ j \ldots$ to label the components of 3-vectors. Latin letters from the start of the alphabet, $a,\ b \ldots$ are reserved for labeling particles.

The three-momenta $\vec{k}_a$ have components k_a^i and norms $k_a = |\vec{k}_a|$. The corresponding massless four-momenta have components $k_a^\mu = (k_a, k_a^i)$. We define $k_{ab} = k_a + k_b$ as well as

$$k_s = |\vec{k}_1 + \vec{k}_2|, \qquad k_t = |\vec{k}_1 + \vec{k}_4|, \qquad k_u = |\vec{k}_1 + \vec{k}_3|. \tag{B.1.3}$$

Using three-momentum conservation $\sum_{a=1}^4 \vec{k}_a = 0$, these satisfy

$$k_s^2 + k_t^2 + k_u^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2. \tag{B.1.4}$$

We also define several combinations of these energies:

$$E = k_{12}+k_{34}, \quad E_L = k_s+k_{12}, \quad E_R = k_s+k_{34}, \quad \alpha = k_1-k_2, \quad \beta = k_3-k_4. \tag{B.1.5}$$

We work in axial gauge where polarisation tensors only have components along boundary directions. The polarisation vectors for gluons are denoted as ϵ_i and satisfy $\epsilon_a \cdot \epsilon_a = \epsilon_a \cdot k_a = 0$, where the dot denotes the product of three-vectors using the Euclidean boundary metric η_{ij}. Graviton polarisations can then be written in terms of polarisation vectors as $\epsilon_{ij} = \epsilon_i \epsilon_j$, which automatically encodes the transverse and traceless conditions. Waveunctions with external scalars can be obtained from spinning wavefunctions by the taking polarisations to satisfy $\epsilon_a \cdot \epsilon_b = 1$ and $\epsilon_a \cdot k_b = 0$ with $a \neq b$. The resulting scalar wavefunctions still live in the boundary of dS_4, so we refer to this procedure as generalised dimensional reduction.

We use the following formulae for gluon polarisation sums, which were first defined in [13]:

$$\begin{aligned} \Pi_{1,1} &= \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2) + k_s^2(k_u^2 - k_t^2)}{k_s^4}, \\ \Pi_{1,0} &= \frac{(k_1 - k_2)(k_3 - k_4)}{k_s^2}. \end{aligned} \tag{B.1.6}$$

The analogous formulae for gravitons are

$$\Pi_{2,2} = \frac{3}{2k_s^4}(\vec{k}_1 - \vec{k}_2)^i(\vec{k}_1 - \vec{k}_2)^j(\pi_{il}\pi_{jm} + \pi_{im}\pi_{jl} - \pi_{ij}\pi_{lm})(\vec{k}_3 - \vec{k}_4)^l(\vec{k}_3 - \vec{k}_4)^m,$$

$$\begin{aligned}&\Pi_{2,1}\\ &= \frac{3}{2k_s^2 k_{12}k_{34}}(\vec{k}_1 - \vec{k}_2)^i(\vec{k}_1 - \vec{k}_2)^j(\pi_{il}\hat{k}_j\hat{k}_m + \pi_{jm}\hat{k}_i\hat{k}_l + \pi_{im}\hat{k}_j\hat{k}_l + \pi_{jl}\hat{k}_i\hat{k}_m)(\vec{k}_3 - \vec{k}_4)^l(\vec{k}_3 - \vec{k}_4)^m,\end{aligned} \tag{B.1.7}$$

where $\pi_{ij} = \eta_{ij} - \hat{k}_i\hat{k}_j$ and $\hat{k}_i = \frac{(\vec{k}_1+\vec{k}_2)_i}{k_s}$. Note that (B.1.6) and (B.1.7) are defined in the s-channel. The equivalent expressions in the t- and u-channels can be obtained with the substitutions $2 \leftrightarrow 4$ and $2 \leftrightarrow 3$, respectively.

B.2 Integrals

In this Appendix, we will explain how to evaluate the integrals in (4.8) and (4.22). First note that both of these formulae contain the following tensor structure:

$$\begin{aligned} 2H_{il}H_{jm} - H_{ij}H_{lm} &= \pi_{il}\pi_{jm} + \pi_{im}\pi_{jl} - \pi_{ij}\pi_{lm} \\ &\quad + \frac{k_s^2+\omega^2}{\omega^2}\left(\pi_{il}\hat{k}_j\hat{k}_m + \pi_{im}\hat{k}_j\hat{k}_l + \pi_{jm}\hat{k}_i\hat{k}_l + \pi_{jl}\hat{k}_i\hat{k}_m\right) \\ &\quad - \frac{k_s^2+\omega^2}{\omega^2}\left(\pi_{ij}\hat{k}_l\hat{k}_m + \pi_{lm}\hat{k}_i\hat{k}_j\right) + \left(\frac{k_s^2+\omega^2}{\omega^2}\right)^2 \hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m. \end{aligned} \tag{B.2.8}$$

We have performed this decomposition in such a way that the first two lines encode the polarisation sums $\Pi_{2,2}$ and $\Pi_{2,1}$ in (B.1.7). The final line is written in such a way as to get a convenient set of integrals.

After performing the decomposition in (B.2.8), we obtain integrals of the following general form:

$$\begin{aligned} f_A &= \int_0^\infty \frac{d\omega\,\omega}{\omega^2+k_s^2}\int dz\,dz'(KKJ)_{12}^{3/2}(z)(KKJ)_{34}^{3/2}(z')\mathcal{I}_A \\ &= \frac{2}{\pi}\int_0^\infty \frac{d\omega\,\omega^4}{\omega^2+k_s^2}\frac{(k_{12}^2+\omega^2+2k_1k_2)}{(k_{12}^2+\omega^2)^2}\frac{(k_{34}^2+\omega^2+2k_3k_4)}{(k_{34}^2+\omega^2)^2}\mathcal{I}_A, \end{aligned} \tag{B.2.9}$$

with the following set of integrands:

$$\mathcal{I}_{2,2} = 1, \quad \mathcal{I}_{2,1} = \frac{\omega^2 + k_s^2}{\omega^2}, \quad \mathcal{I}_{2,0} = \left(\frac{\omega^2 + k_s^2}{\omega^2}\right)^2, \\ \mathcal{I}_a = (\omega^2 + k_s^2)^2, \quad \mathcal{I}_b = (\omega^2 + k_s^2). \tag{B.2.10}$$

The first three evaluate to

$$\begin{aligned} f_{2,2} &= \frac{2k_1k_2k_3k_4\,(E_LE_R + Ek_s)}{E_L^2E^3E_R^2} + \frac{k_1k_2\,(E_Lk_{34} + Ek_s)}{E_L^2E^2E_R} + \frac{k_3k_4\,(Ek_s + E_Rk_{12})}{E_LE^2E_R^2} + \frac{E_LE_R - k_s^2}{E_LEE_R}, \\ f_{2,1} &= \frac{2k_1k_3k_4k_2}{E^3k_{12}k_{34}} + \frac{k_1k_2}{E^2k_{12}} + \frac{k_3k_4}{E^2k_{34}} + \frac{1}{E}, \\ f_{2,0} &= \frac{k_{12}k_{34} + k_s^2}{k_{12}k_{34}} f_{2,1} - \frac{k_s^2}{E}\left(\frac{k_1k_2}{k_{12}^3k_{34}} + \frac{2k_1k_3k_4k_2}{k_{12}^3k_{34}^3} + \frac{k_3k_4}{k_{12}k_{34}^3}\right), \end{aligned} \tag{B.2.11}$$

where E, E_L, and E_R are defined in (B.1.5). The last two integrals are divergent:

$$\begin{aligned} f_a &= \frac{2}{\pi}\left(\frac{\Lambda^3}{3} - \Lambda(k_{12}^2 + k_{34}^2 - k_s^2 + 2(k_1k_2 + k_3k_4))\right) + \text{finite}, \\ f_b &= \frac{2}{\pi}\Lambda + \text{finite}, \end{aligned} \tag{B.2.12}$$

where Λ is a cut-off on the ω integral. On the the other hand, the divergent pieces are analytic in at least two of the momenta and therefore correspond to boundary contact terms. Moreover they become imaginary after analytically continuing back to de Sitter so won't contribute to the in-in correlator. Dropping these divergences then gives

$$\begin{aligned} f_a &= \left(k_{12}k_{34} + k_s^2\right) f_b + \frac{1}{E}\left(2k_1k_2k_3k_4 - k_1k_2(2E^2 + k_{12}^2) - k_3k_4(2E^2 + k_{34}^2) - 2k_{12}k_{34}E^2 + E^4\right), \\ f_b &= \left(\frac{2k_1k_2k_3k_4}{E^3} + k_1k_2\frac{k_{34} + E}{E^2} + k_3k_4\frac{k_{12} + E}{E^2} + \frac{k_{12}k_{34} - E^2}{E}\right). \end{aligned} \tag{B.2.13}$$

APPENDIX C

Appendix for Chapter 5

C.1 Polarization sums

In this appendix, we provide the details of the polarization sums employed in this letter. Following the boundary transverse gauge [123]: (This is the same as in QFT textbook [151] with Coulomb gauge.)

$$\sum_{h=\pm} \varepsilon_\mu(k,h)\varepsilon_\nu(k,h)^* = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \equiv \Pi_{\mu\nu}, \tag{C.1.1}$$

$$\sum_{h=\pm} \varepsilon_{\mu\nu}(k,h)\varepsilon_{\rho\sigma}(k,h)^* = \frac{1}{2}\Pi_{\mu\rho}\Pi_{\nu\sigma} + \frac{1}{2}\Pi_{\mu\sigma}\Pi_{\rho\nu} - \frac{1}{d-1}\Pi_{\mu\nu}\Pi_{\rho\sigma}, \tag{C.1.2}$$

which are transverse and traceless projection tensor. Let's return to QED in Coulomb gauge for a moment. The polarization tensor above which appear in the photon propagator is not Lorentz invariant on its own, but we can restore Lorentz invariance to obtain the covariant photon propagator. This is the same logic that we use to derive all of the polarization sums below by demanding conformal invariance.

Finally, let us explicitly write out the polarization sums at 4-point, see also [11, 13, 121] for the case of conformally coupled scalar.

$$\Pi_{1,1} \equiv \frac{1}{4}(k_1^\mu - k_2^\mu)\Pi_{\mu\nu}(k_3^\nu - k_4^\nu) = \frac{1}{4}(k_1 - k_2)\cdot(k_3 - k_4) + \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{4k_s^2}, \tag{C.1.3}$$

$$\Pi_{1,0} \equiv -\frac{(s_1 - s_2)(s_3 - s_4)}{z^2 k_s^2}. \tag{C.1.4}$$

Next, we write the spin-2 polarization sums in a way that makes its double copy structure clear.

$$\Pi_{2,2} \equiv \frac{1}{16}(k_1^\mu - k_2^\mu)(k_1^\nu - k_2^\nu)(\frac{1}{2}\Pi_{\mu\rho}\Pi_{\nu\sigma} + \frac{1}{2}\Pi_{\mu\sigma}\Pi_{\rho\nu} - \frac{1}{d-1}\Pi_{\mu\nu}\Pi_{\rho\sigma})(k_3^\rho - k_4^\rho)(k_3^\sigma - k_4^\sigma) \tag{C.1.5}$$

$$= \Pi_{1,1}^2 - \Pi_{2,2}^{\mathrm{Tr}}, \tag{C.1.6}$$

$$\Pi_{2,1} \equiv 2\Pi_{1,1}\Pi_{1,0}, \tag{C.1.7}$$

$$\Pi_{2,2}^{\mathrm{Tr}} \equiv \frac{(k_1^\mu - k_2^\mu)\Pi_{\mu\nu}(k_1^\nu - k_2^\nu)(k_3^\rho - k_4^\rho)\Pi_{\rho\sigma}(k_3^\sigma - k_4^\sigma)}{16(d-1)}, \tag{C.1.8}$$

$$\begin{aligned}\Pi_{2,0} \equiv &-\frac{d(k_1^2 - k_2^2)(k_3^2 - k_4^2)(s_1 - s_2)(s_3 - s_4)}{4(d-1)k_s^4} + \frac{(d-2)z^2(k_1^2 - k_2^2)(k_3^2 - k_4^2)(s_1 - s_2)(s_3 - s_4)}{4(d-1)k_s^2(d - 2s_{12})(d - 2s_{34})} \\ &+ \frac{(k_1^2 - k_2^2)(s_1 - s_2)(d^2 - 8(s_3^2 + s_4^2))}{8(d-1)k_s^2(d - 2s_{12})} + \frac{(k_3^2 - k_4^2)(s_3 - s_4)(d^2 - 8(s_1^2 + s_2^2))}{8(d-1)k_s^2(d - 2s_{34})} \\ &+ \frac{(z^2 k_s^2 + 4s_{12}s_{34})}{(d-1)} + \frac{4(s_1 - s_2)^2 + 4(s_3 - s_4)^2 - d^2}{16(d-1)}.\end{aligned} \tag{C.1.9}$$

Both $\Pi_{2,1}$ and $\Pi_{2,0}$ can be determined in the same way as we obtained $\Pi_{1,0}$ in the main text. Note that there are still terms with Mellin variables in the denominator which naively violate locality. However, they will all cancel after using the Mellin delta function.

C.2 Back to Momentum space and Cosmological Correlators

In this letter, we focused on the analytic structure of Mellin-Momentum amplitude. However, it is also important to stress that we can easily obtain the actual observables: Cosmological correlators. As a non-trivial example, we will give a detailed translation from the Four-point gravity amplitude to the gravity Trispectrum [43]. Expanding out the full expression from Eq(5.66):

$$\begin{aligned}\mathcal{M}_4 = &\frac{(\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 z^2\Pi_{1,1}+z^2W_s)^2-(\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 z^2)^2\Pi^{\text{Tr}}_{2,2}}{\mathcal{D}^d_{k_s}}+(\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4)^2\Pi_{2,0}\\ &+2(\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 z^2\Pi_{1,1}+z^2W_s)(\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Pi_{1,0}+V^s_c)\\ &+((V^s_c)^2+2\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4 V^s_c\Pi_{1,0})(z^2k_s^2+4s_{12}s_{34})+\mathcal{P}(2,3,4).\end{aligned} \tag{C.2.10}$$

First of all, we want to emphasize that unlike the usual bulk calculation on spinning particles in AdS which involves complicated bulk integral in axial gauge [20, 135], all of our calculations are just scalar integrals, which can be easily automated by Mathematica. Now by inverse Mellin transform:

$$\mathcal{I}_{2,2}=\frac{z^4\Pi_{2,2}}{\mathcal{D}^d_{k_s}}\to\Pi_{2,2}\int\frac{dz}{z^{d+1}}(z^2\phi_1\phi_2)(\mathcal{D}^d_{k_s})^{-1}(z^2\phi_3\phi_4), \tag{C.2.11}$$

$$\mathcal{I}_{2,1}=\Pi_{2,1}\to\frac{\Pi_{1,1}}{k_s^2}\int\frac{dz}{z^{d+1}}z^2(\partial_{z1}-\partial_{z2})(\partial_{z3}-\partial_{z4})\phi_1\phi_2\phi_3\phi_4. \tag{C.2.12}$$

where ∂_{zi} means the ∂_z acting on the corresponding leg only. The inversion is defined via the standard Green function:

$$(\mathcal{D}(z))^{-1}\mathcal{O}(z)=\int\frac{dy}{y^{d+1}}G(z,y)\mathcal{O}(y), \tag{C.2.13}$$

$$\mathcal{D}(z)G(z,y)=z^{d+1}\delta(z-y). \tag{C.2.14}$$

We will evaluate the integral in $d=3$,

$$
\begin{aligned}
&\mathcal{I}_{2,2}\\
=&\Pi_{2,2}\int\frac{dz}{z^{d+1}}(z^2\phi_1\phi_2)G(k_s,z,z')(z'^2\phi_3\phi_4)\\
=&\Pi_{2,2}\left(\frac{2k_1k_2k_3k_4\left(E_LE_R+Ek_s\right)}{E_L^2E^3E_R^2}+\frac{k_1k_2\left(E_Lk_{34}+Ek_s\right)}{E_L^2E^2E_R}+\frac{k_3k_4\left(Ek_s+E_Rk_{12}\right)}{E_LE^2E_R^2}+\frac{E_LE_R-k_s^2}{E_LEE_R}\right),
\end{aligned}
\tag{C.2.15}
$$

$$
\mathcal{I}_{2,1}=\frac{\Pi_{1,1}(k_1-k_2)(k_3-k_4)}{k_s^2}\left(\frac{2k_1k_3k_4k_2}{E^3}+\frac{k_1k_2k_{34}}{E^2}+\frac{k_3k_4k_{12}}{E^2}+\frac{k_{12}k_{34}}{E}\right).
\tag{C.2.16}
$$

The integral for $\Pi_{2,0}$ clearly involve more z derivatives, but it is essentially just contact diagram, we will not present the integrated expression here, but we have explicitly verified that agree with [43]. In particular, we matched our $\Pi_{2,0}$ with $f^{(s)}_{(2,0)}(E_LE_R-sk_T)\Pi^{(s)}_{2,0}+f_c$ in Eq(2.39) [43].

Moving forward, we can utilize the formula in [13, 43], which establishes a connection between the wavefunction coefficient and In-In correlator, this will give us the graviton trispecturm.

C.3 Scalar Integrals for Gravity

The scalar integrals for Yang-Mills in $d=3$ are simply plane waves, so we focus on Gravity here. We will be using the following representation of Bulk-to-Bulk propagator [93]:

$$
G(k,z_1,z_2)=\int_0^\infty\frac{dp}{2\pi i}\frac{-p^{d+1-2\Delta}}{k^2+p^2}\left(\phi_\Delta(z_1,ip)-\phi_\Delta(z_1,-ip)\right)\left(\phi_\Delta(z_2,ip)-\phi_\Delta(z_2,-ip)\right),
\tag{C.3.17}
$$

where $\phi_\Delta(z,k)=z^{d/2}k^{\Delta-d/2}K_{\Delta-d/2}(kz)$ is the usual Bulk to Boundary propagator. This makes the recursive relation (5.72) manifest. For example, the scalar integral with two propagators for 5-point graviton is:

$$\frac{1}{\mathcal{D}^d_{k_{12}}\mathcal{D}^d_{k_{45}}} \to \int_{-\infty}^{\infty} \frac{dp_1}{2\pi i} \frac{p_1^{-2}}{k_{12}^2+p_1^2} \bar{\mathcal{C}}_1(k_1,k_2,ip_1)\bar{\mathcal{I}}^{(4)}_{45}(ip_1,k_3,k_4,k_5)$$

$$= \underset{p_1,p_2}{\text{Res}} \frac{64k_3^3p_1^4p_2^4\left(k_1^2+4k_2k_1+k_2^2+p_1^2\right)\left(k_4^2+4k_5k_4+k_5^2+p_2^2\right)}{\pi^2\left((k_1+k_2)^2+p_1^2\right)^2\left(k_{12}^2+p_1^2\right)\left(k_3^2+(p_{12}^m)^2\right)^2\left((k_4+k_5)^2+p_2^2\right)^2\left(k_{45}^2+p_2^2\right)\left(k_3^2+(p_{12})^2\right)^2} \tag{C.3.18}$$

with $p_{12}^m = p_1 - p_2$. In the second step we can simply recycle the three-point contact and the four-point exchange results, and we are left with taking a few simple residues of p_1, p_2. This completes the mapping to momentum space without doing any integrals.

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