Spectral theory of random cusped hyperbolic surfaces

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Abstract

The aim of this thesis is to study the spectral theory of random finite-area non-compact hyperbolic surfaces, focusing on the spectral gap. We study the size of the spectral gap for two different models of random surfaces: random covers and the Weil-Petersson model.

First we show that for any non-compact finite-area hyperbolic surface X, there is a constant $C > 0$ such that a uniformly random degree-n cover X_n has no eigenvalues below $\frac{1}{4} - C\frac{(\log\log\log n)^2}{\log\log n}$, other than those of X, with probability tending to 1 as $n \to \infty$.

Secondly, we show that for any $\varepsilon > 0$, $\alpha \in [0, \frac{1}{2})$, as $g \to \infty$ a generic finite-area genus g hyperbolic surface with $n = O(g^{\alpha})$ cusps, sampled with probability arising from the Weil-Petersson metric on moduli space, has no non-zero eigenvalue of the Laplacian below $\frac{1}{4} - \left(\frac{2\alpha+1}{4}\right)^2 - \varepsilon$. For $\alpha = 0$ this gives a spectral gap of size $\frac{3}{16} - \varepsilon$ and for any $\alpha < \frac{1}{2}$ gives a uniform spectral gap of explicit size.

Spectral theory of random cusped hyperbolic surfaces

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Contents

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Section 3 is based on joint work with Michael Magee [HM23].

Statement of copyright

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Dedication

This thesis is dedicated to my beloved Grandmother Elizabeth Harrison.

1 Introduction

A hyperbolic surface is a smooth, connected, orientable Riemannian surface with constant Gaussian curvature -1. Let X be a finite-area non-compact hyperbolic surface. The L^2-spectrum of the Laplacian Δ_X, denoted $\mathrm{spec}(\Delta_X)$, consists of:

- A simple eigenvalue at 0 and possibly finitely many eigenvalues in $\left(0, \frac{1}{4}\right)$. Such eigenvalues are often called *small* or *exceptional eigenvalues.*
- Absolutely continuous spectrum $\left[\frac{1}{4}, \infty\right)$ with multiplicity equal to the number of cusps of X.
- Possibly infinitely many discrete eigenvalues in $\left[\frac{1}{4}, \infty\right)$, embedded in the absolutely continuous spectrum. Such eigenvalues are called *embedded eigenvalues.*

The spectral gap of Δ_X, occasionally denoted here by $SG(X)$, refers to the gap between the simple zero eigenvalue and the remaining spectrum. As such, $\frac{1}{4}$ is the optimal spectral gap for a finite-area non-compact surface. $\frac{1}{4}$ is also the bottom of the spectrum of the Laplacian on the universal cover $\mathbb{H}$.

The spectral gap is an important quantity, governing the exponential rate of mixing of the geodesic flow on T^1X [Ra87] and providing error terms in prime geodesic counting [Hu59].

The main aim of this thesis is to study the following question.

Question 1.1. *Does a random finite-area non-compact hyperbolic surface have a large spectral gap?*

We study Question 1.1 for two different models of random surfaces, random covers §1.2 and Weil-Petersson random surfaces §1.3.

1.1 Motivation

We outline some motivation for studying Question 1.1.

Geometry

Let X be a finite-area hyperbolic surface which may be compact or non-compact. For any subset $A \subset X$ we define $h^*(A) \stackrel{\text{def}}{=} \frac{\text{length}(\partial A)}{\text{Vol}(A)}$ where ∂A is the boundary of A. If either quantity is undefined for some A then $h^*(A) \stackrel{\text{def}}{=} \infty$. The Cheeger constant of X is then

$$h(X) \stackrel{\text{def}}{=} \inf_{\substack{S \subset X \\ \text{Vol}(S) \leqslant \frac{1}{2}\text{Vol}(X)}} h^*(S).$$

The Cheeger constant roughly quantifies how difficult it is to separate the surface into reasonably sized pieces with a short curve. The Cheeger-Buser inequalities say that the spectral gap is comparable to the Cheeger constant. In particular, the Cheeger inequality [Ch70] says that $\text{SG}(X) \geqslant \frac{h(X)^2}{4}$, whilst the Buser inequality [Bu82] says that there exists $C > 0$ such that $\text{SG}(X) \leqslant C \cdot h(X)$.

Another natural measure of connectivity is the diameter of a surface. This is well defined for compact surfaces and for non-compact surfaces one can consider the diameter of the ε-thick part $X_{\geqslant\varepsilon}$. A lower bound for spectral gap of the surface provides an upper bound for the diameter [Ma20]. In particular, if $SG(X) \geqslant \frac{1-\delta^2}{4}$ then

$$\text{diam}(X_{\geqslant\varepsilon}) \leqslant \frac{2}{1-\delta}\log(\text{vol}(X)) + \frac{4}{1-\delta}\log\log(\text{vol}(X)) + O_{\varepsilon,\delta}(1).$$

So one can view the spectral gap as a measure of connectivity; a surface with a large spectral gap is highly connected.

Random regular graphs

There is a close analogue between large-genus random hyperbolic surfaces and random regular graphs. The spectrum of a graph $\mathcal{G}$ with n vertices is the set of eigenvalues of its adjacency matrix $A_{\mathcal{G}}$. When $\mathcal{G}$ is d-regular, the eigenvalues are given by

$$\lambda_0 = d \geqslant \lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_{n-1}.$$

$\lambda_0 \neq \lambda_1$ if and only if $\mathcal{G}$ is connected, in which case there is a spectral gap $\lambda_0 - \lambda_1$. The spectral gap of $\mathcal{G}$ governs the rate at which a random walk on $\mathcal{G}$ converges to the uniform measure, graphs with a large spectral gap are highly desirable for real-world applications.

However, there is a bound on what can be achieved; Alon and Boppana [Ni91] proved that for any sequence $\mathcal{G}_n$ of d-regular graphs on n vertices has $\lambda_1 \geqslant 2\sqrt{d-1} - o(1)$ as $n \to \infty$. A connected d-regular graph with all non-trivial eigenvalues in $\left[-2\sqrt{d-1}, 2\sqrt{d-1}\right]$ is called a *Ramanujan graph* after Lubotzky, Phillips and Sarnak [LPS88]. In [Al86], Alon conjectured that for any $\epsilon > 0$, a random d-regular graph on n vertices has no non-trivial eigenvalues with absolute value above $2\sqrt{d-1} + \varepsilon$ as $n \to \infty$. In other words, almost all d-regular graphs have almost optimal spectral gap. Alon's conjecture was proved by Friedman [Fr08].

It was conjectured [MN08, Sa04] that the distribution of the second largest eigenvalue of a random d-regular graph, after re-scaling by $n^{\frac{2}{3}}$, is the same as the distribution of the largest eigenvalue of the Gaussian Orthogonal Ensemble. This would mean that there is a constant $C_{n,d}$ such that $n^{\frac{2}{3}}\left(\lambda_1 - 2\sqrt{d-1}\right) - C_{n,d}$ has Tracy-Widom distribution. If $C_{n,d}$ is of order 1, this would imply that λ_1 fluctuates at scales $O\left(n^{-\frac{2}{3}}\right)$. If $C_{n,d} = 0$ then this would imply that slightly more than half of all d-regular graphs are Ramanujan. An important first step towards this conjecture is determining the optimal error bound $\varepsilon = \varepsilon(n)$ in Alon's conjecture. It was shown by Bordenave [Bo20], that one can take $\varepsilon = \text{const} \cdot \left(\frac{\log \log n}{\log n}\right)^2$. Subsequently, it was shown by Huang and Yau [HY21] that one can take $\varepsilon = O\left(n^{-c}\right)$ for some $c > 0$.

Friedman conjectured that an extension of Alon's conjecture holds for random covers of finite graphs [Fr03]. Given any finite graph $\mathcal{G}$, one can define a notion of a degree-n cover G_n of $\mathcal{G}$. Eigenvalues of $\mathcal{G}_n$ which do not appear as eigenvalues of $\mathcal{G}$ are called new-eigenvalues. It was conjectured by Friedman that for any fixed finite graph $\mathcal{G}$ and for any $\varepsilon > 0$, a uniformly random degree-n cover $\mathcal{G}_n$ has no new-eigenvalues with absolute value above $\rho\left(\tilde{G}\right) + \varepsilon$ with probability tending to 1 as $n \to \infty$. Here $\rho\left(\tilde{\mathcal{G}}\right)$ is the spectral radius of the adjacency operator on $l^2\left(\tilde{\mathcal{G}}\right)$, where $\tilde{\mathcal{G}}$ is the universal cover of G. Friedman's conjecture was proved by Bordenave and Collins [BC19].

Ramanujan graphs of fixed degree with number of vertices tending to infinity were constructed by Lubotzky, Phillips and Sarnak [LPS88] and independently by Margulis [Ma88]. Marcus, Spielman and Srivastava [MSS15] proved the existence of bipartite Ramanujan graphs of all degrees $d \geqslant 3$ by proving a variant of a conjecture of Bilu and Linial [BL06]. In particular, they prove that every finite graph G has a degree-2 cover which has no new-eigenvalues above $\rho\left(\tilde{G}\right)$.

Selberg's eigenvalue conjecture

Spectral theory of the Laplacian on certain arithmetic hyperbolic surfaces has important consequences in Number Theory, see e.g. [Sa03]. Let $N \geqslant 1$, the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N is

$$\Gamma(N) = \{T \in \mathrm{SL}_2(\mathbb{Z}) \mid T \equiv I \mod N\}.$$

Consider the quotient $X(N) \stackrel{\text{def}}{=} \Gamma(N)\backslash\mathbb{H}$. Letting $\lambda_1(X(N))$ denote the first non-zero eigenvalue of the Laplacian on $X(N)$, in [Se65] Selberg made the following conjecture.

Conjecture 1.2. *For every $N \geqslant 1$,*

$$\lambda_1(X(N)) \geqslant \frac{1}{4}.$$

Selberg's Conjecture would imply the existence of surfaces with *optimal* spectral gap in unbounded volume. Conjecture 1.2 remains open however there have been a number of results in this direction. Selberg proved in [Se65] that Conjecture 1.2 holds with the bound $\frac{3}{16}$. After many intermediate results [GJ78, Iw89, LRS95, Sa95, Iw96, KS02], the best known result is the following due to Kim and Sarnak [Ki03].

Theorem 1.3 ([Ki03]). *For every $N \geqslant 1$,*

$$\lambda_1(X(N)) \geqslant \frac{975}{4096}.$$

Buser's Conjecture

On a compact hyperbolic surface, the spectrum of the Laplacian consists of eigenvalues

$$0 = \lambda_0(X) < \lambda_1(X) \leq \cdots \leq \lambda_i(X) \leq \cdots$$

with $\lambda_i(X) \to \infty$ as $i \to \infty$ and the spectral gap is $\lambda_1(X)$. In this case, there is an asymptotic upper bound on λ_1 in large genus.

Theorem 1.4 ([Hu74]). *Let Y_i be a sequence of compact hyperbolic surfaces with genera $g(i) \to \infty$ as $i \to \infty$. Then*

$$\limsup_{i\to\infty} \lambda_1(Y_i) \leqslant \frac{1}{4}.$$

This is analogous to the Alon-Boppana bound for a d-regular graphs. It was conjectured by Buser in [Bu84] that $\frac{1}{4}$ can in fact be attained asymptotically.

Conjecture 1.5 ([Bu84]). *There exists a sequence of compact hyperbolic surfaces $\{Y_i\}_{i\in\mathbb{N}}$ with genera $g(i) \to \infty$ as $i \to \infty$ and*

$$\lim_{i\to\infty} \lambda_1(Y_i) = \frac{1}{4}.$$

Buser showed in [Bu84], using Selberg's $\frac{3}{16}$ Theorem and work of Jacquet-Langlands [JL70][1] that there exists compact hyperbolic Y_i surfaces with genera $g(i) \to \infty$ as $i \to \infty$ and $\lambda_1(Y_i) \geqslant \frac{3}{16}$. In fact, Conjecture 1.2 implies Conjecture 1.5. Buser, Burger and Dodzuik [BBD87] proved the slightly weaker $\lambda_1(Y_i) \geqslant c$ where c can be arbitrarily close to $\frac{3}{16}$, by using a more geometric approach (rather than Jacquet-Langlands). In particular, they proved the following.

Lemma 1.6 ([BBD87]). *Let X be a finite area hyperbolic surface with an even number of cusps $\{C_i\}$. It is possible to deform the surface X in a certain way to a finite area hyperbolic surface with boundary, where each cusp becomes a bounding geodesic of length t, and then*

[1] By the work of Jacquet-Langlands [JL70], to each principal congruence subgroup $\Gamma(N)$ one can associate a cocompact $\tilde{\Gamma}(N) \subset \mathrm{SL}_2(\mathbb{R})$ with the property that $\lambda_1\left(\mathbb{H}/\tilde{\Gamma}_N\right) \geqslant \lambda_1(X(N))$. For large enough N, $\mathbb{H}/\tilde{\Gamma}_N$ is a compact hyperbolic surface whose genus is an increasing function of N.

glue the geodesic corresponding to C_{2i-1} to the one corresponding to C_{2i} to form a family of closed hyperbolic surfaces $X^{(t)}$ such that

$$\limsup_{t\to 0} \lambda_1(X^{(t)}) \geq \inf\left(\mathrm{spec}(\Delta_X) \cap (0,\infty)\right).$$

Therefore Conjecture 1.5 would follow from the existence of a sequence of finite-area non-compact hyperbolic surfaces X_i with an even number of cusps and $\mathrm{Vol}\,(X_i) \to \infty$ as $i \to \infty$ and $\inf\left(\mathrm{spec}(\Delta_{X_i}) \cap (0,\infty)\right) \to \frac{1}{4}$. Further progress towards Conjecture 1.5 ran parallel with progress towards progress towards Conjecture 1.2, outlined in the previous section, by either Jacquet-Langlands or Lemma 1.6.

1.2 Random covers

In this subsection, we outline our results for the random covering model which is obtained as follows. Take a fixed hyperbolic surface X and for $n \in \mathbb{N}$, consider all degree-n Riemannian covers X_n sampled uniformly at random. Since $\mathrm{spec}(\Delta_X) \subset \mathrm{spec}\,(\Delta_{X_n})$ as multi-sets for any degree-n cover X_n, it is natural to restrict attention to *new eigenvalues*, i.e. eigenvalues in $\mathrm{spec}\,(\Delta_{X_n})$ which do not arise by a lifting from X.

Spectral gaps for random covers were first studied in [MN20] for Schottky surfaces and [MNP20] for compact surfaces. The first result we highlight is the following.

Theorem 1.7 ([MNP20, Theorem 1.5])**.** *Let Y be a compact hyperbolic surface. For any $\varepsilon > 0$ a uniformly random degree-n cover Y_n has no new eigenvalues below $\frac{3}{16} - \varepsilon$ with probability tending to 1 as $n \to \infty$.*

It is conjectured in [MNP20] that the same result holds with $\frac{3}{16}$ replaced with $\frac{1}{4}$.

Following an intermediate result [MN20], Magee and Naud prove in [MN21] that for X conformally compact, a uniformly random cover X_n has no new resonances in any compact set $\mathcal{K} \subset \{s \mid \mathrm{Re}(s) > \frac{\delta}{2}\}$ with probability tending to 1 as $n \to \infty$, where δ is the Hausdorff dimension of the limit set of Γ_X. For L^2-spectral gaps this result is optimal.

We want to study the case where the base surface X is finite-area and non-compact. The first result of the thesis is the following, joint with Michael Magee.

Theorem 1.8 ([HM23, Theorem 1.1]). *Let X be a finite-area non-compact hyperbolic surface. For any $\varepsilon > 0$, a uniformly random degree-n cover has no new eigenvalues below $\frac{1}{4} - \varepsilon$ with probability tending to 1 as $n \to \infty$.*

Theorem 1.8 is an analogue of Alon and Friedman's conjectures for finite-area non-compact hyperbolic surfaces. As a corollary, taking X to be the thrice punctured sphere which has $\lambda_1 > \frac{1}{4}$ and using Lemma 1.6 we obtain a proof of Buser's Conjecture.

Corollary 1.9. *Conjecture 1.5 is true.*

An alternative proof of Conjecture 1.5 is given in [LM22]. In fact, it is shown in [LM22] that every compact hyperbolic surface has a sequence of degree-n covers with no new eigenvalues below $\frac{1}{4} - o_{n\to\infty}(1)$. In particular, the surfaces in Corollary 1.9 can be taken to be arithmetic.

An outstanding open problem is whether there exists a sequence of hyperbolic surfaces $\{X_i\}_{i\in\mathbb{N}}$ with $\mathrm{Vol}(X_i) \to \infty$ with $\lambda_1(X_i) \geqslant \frac{1}{4}$. This would, for example, follow from Selberg's Conjecture 1.2. Such surfaces would be analogous to Ramanujan graphs. It is interesting to ask whether the analogue of the Bilu-Linial conjecture (§1.1) holds in this setting, that is, does every finite area hyperbolic surface have a degree-2 cover with no new-eigenvalues below $\frac{1}{4}$?

In analogy with random regular graphs §1.1, it is natural to conjecture that after some suitable re-scaling, the bottom of the new-spectrum of a random cover of a finite-area non-compact hyperbolic surface has a limiting distribution[2]. As such, it is desirable to determine the optimal rate at which one can allow $\varepsilon = \varepsilon(n)$ to tend to 0 as $n \to \infty$ in Theorem 1.8. We prove a result in this direction.

Theorem 1.10. *Let X be a finite-area non-compact hyperbolic surface. There is a constant $C > 0$ such that a uniformly random degree-n cover has no new eigenvalues below*

$$\frac{1}{4} - C\frac{(\log\log\log n)^2}{\log\log n},$$

[2] For random covers of compact surfaces (or even random unitary bundles over non-compact finite-area surfaces) it is perhaps natural to also conjecture that, after suitable re-scaling, λ_1 has Tracy-Widom distribution. The finite-area non-compact case is more subtle since it could well be the case that there are no non-zero eigenvalues with positive probability.

with probability tending to 1 *as* $n \to \infty$*.*

Theorem 1.10 follows from effectivising the arguments of [HM23] and applying subsequent powerful results of Bordenave and Collins [BC23]. In this thesis we will prove Theorem 1.10 but stress that the method follows [HM23].

Remark 1.11. Let $X = \Gamma\backslash\mathbb{H}$ be a finite-area non-compact hyperbolic surface. Then Γ is a finitely generated free group with generators $\gamma_1, \ldots, \gamma_r$ for some r. We can equip $\operatorname{Hom}(\Gamma, \mathrm{U}(n))$ with a probability measure by choosing the image of the generators $\varphi(\gamma_1), \ldots, \varphi(\gamma_r)$ in $\mathrm{U}(n)$ independently with Haar probability. Given $\varphi \in \operatorname{Hom}(\Gamma, \mathrm{U}(n))$, Let $\rho_\varphi : \Gamma \to \mathrm{U}(n)$ be the random $\mathbb{C}^n$ representation obtained via $\mathrm{std}_n \circ \varphi$ where std_n is the standard representation. We consider the associated (random) unitary bundle E_φ and the Laplacian Δ_φ on sections of E_φ. Then $\operatorname{spec}(\Delta_\phi) \cap [0, \frac{1}{4})$ consists of finitely many eigenvalues with finite multiplicity. This setting was studied by Zargar [Za22] who proved the analogue of Theorem 1.8 in the unitary case. For random unitary bundles, via the same methods of Theorem 1.10 we can obtain a better rate.

Theorem 1.12 ([Hi23])*. For any finite-area non-compact hyperbolic surface X, there exists a constant $c > 0$ such that a random unitary bundle E_ϕ over X of rank n has*

$$\inf \operatorname{spec}\Delta_\phi \geqslant \frac{1}{4} - c\frac{(\log\log n)^2}{\log n},$$

with probability tending to 1 *as* $n \to \infty$

1.3 Weil-Petersson random surfaces

Another model of random surfaces we are interested in is the Weil-Petersson model [GPY11, Mi13]. Consider the moduli space $\mathcal{M}_{g,n}$ with probability measure arising from the Weil-Petersson metric §4.2. One can think of $\mathcal{M}_{g,n}$ as the space of all hyperbolic metrics that a surface with genus g and n cusps can wear, up to isometry. Mirzakhani was the first to prove a spectral gap for Weil-Petersson random surfaces.

Theorem 1.13 ([Mi13, Theorem 4.8])**.** *The Weil-Petersson probability that a genus g compact hyperbolic surface has a non-zero Laplacian eigenvalue below* $\frac{1}{4}\left(\frac{\log(2)}{2\pi+\log(2)}\right)^2 \approx$

0.0024 *tends to zero as* $g \to \infty$.

Subsequently, Mirzakhani's result was improved to $\frac{3}{16} - \varepsilon$ independently by Wu-Xue [WX21] and Lipnowski-Wright [LW21] and recently to $\frac{2}{9} - \varepsilon$ by Anantharaman and Monk [AM23].

We study random non-compact surfaces in the Weil-Petersson model. Here, there is flexibility for the rate at which g and n grow and it is interesting to ask what effect this has on the spectrum of a Weil-Petersson random hyperbolic surface. Actually, if n is allowed to grow faster than g, the spectral gap necessarily shrinks to 0.

Theorem 1.14 ([Zo87, Theorem 2])**.** *There is a constant $C > 0$ such that for any $X \in \mathcal{M}_{g,n}$,*

$$\mathrm{SG}\,(X) \leqslant C\frac{g+1}{n}.$$

In this thesis, we study the case where the number of cusps grows slowly with the genus. We prove the following.

Theorem 1.15. *For any $0 \leqslant \alpha < \frac{1}{2}$, if $n = O\left(g^{\alpha}\right)$ then for any $\varepsilon > 0$ the Weil-Petersson probability that a genus g non-compact finite-area surface with n cusps has a non-zero Laplacian eigenvalue below $\frac{1}{4} - \left(\frac{2\alpha+1}{4}\right)^2 - \varepsilon$ tends to zero as $g \to \infty$.*

If $\alpha = 0$, i.e. the number of cusps is bounded, then Theorem 1.15 returns a spectral gap of size $\frac{3}{16} - \varepsilon$ as obtained in the works [LW21, WX21]. An undesirable feature of Theorem 1.15 is that the spectral gap goes to 0 as $\alpha \to \frac{1}{2}$. However, one can extend Mirzakhani's methods, Theorem 1.13, to the case where $n = o\left(\sqrt{g}\right)$ to obtain the bound $\mathrm{SG}\,(X) \geqslant \frac{1}{4}\left(\frac{\log(2)}{2\pi+\log(2)}\right)^2$ for $X \in \mathcal{M}_{g,n}$ with high probability, c.f. [SW22].

Subsequently, it was shown by Shen and Wu that if n grows faster than $\sqrt{g}$, a Weil-Petersson random surface has an arbitrarily small spectral gap.

Theorem 1.16 ([SW22])**.** *Let $n : \mathbb{N} \to \mathbb{N}$ be any function with $\frac{n(g)}{\sqrt{g}} \to \infty$ and $\frac{n(g)}{g} \to 0$ as $g \to \infty$. Then for any $\varepsilon > 0$, the Weil-Petersson probability that $X \in \mathcal{M}_{g,n}$ satisfies*

$$SG\,(X) \leqslant \varepsilon$$

tends to 1 *as* $g \to \infty$.

It is natural to ask how many small eigenvalues does a random surface with many cusps have? In a recent joint work with Joe Thomas, we study this problem and show the following.

Theorem 1.17 ([HT23]). *Let $g \geqslant 0$ be fixed. For any $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that a Weil-Petersson random surface $X \in \mathcal{M}_{g,n}$ has at least cn Laplacian eigenvalues below ε with probability tending to 1 as $n \to \infty$.*

By a result of Ballmann, Mathiesen and Mondal [BMM17] it is known that every surface in $\mathcal{M}_{g,n}$ has at most $2g+n-2$ exceptional eigenvalues and therefore Corollary 1.17 is optimal up to a multiplicative factor. We do not prove Theorem 1.17 in this thesis.

1.4 Other related works

Brooks and Makover in [BM04] were the first to study spectral gaps of random surfaces. They considered a combinatorial model of random surfaces, showing the existence of a non-explicit uniform spectral gap with high probability. They considered a random closed surface formed by gluing $2n$ copies of an ideal hyperbolic triangle with gluing determined by a random trivalent ribbon graph and then applying a compactification procedure. They proved the existence of a non-explicit constant $C > 0$ such that the first non-zero eigenvalue is greater than C with probability tending to 1 as $n \to \infty$. Other works on the Brooks-Makover model include [Ga06, BCP21, SW22A].

As described in §1.1, one can view spectral gap as a measure of connectivity. Other notions of connectivity have been studied by probabilistic methods with great success. For the diameter, by comparing with balls in the hyperbolic plane, one can show that $\log g$ is the asymptotically minimal diameter of a genus g compact hyperbolic surface. It was shown by Budzinski, Curien and Petri [BCP21a] that

$$\lim_{g\to\infty} \min_{X\in\mathcal{M}_g} \frac{\operatorname{diam}(X)}{\log g} = 1.$$

For the Cheeger constant, it was shown by Budzinski, Curien and Petri [BCP22] that

$$\limsup_{g\to\infty} \sup_{X\in\mathcal{M}_g} h(X) \leqslant \frac{2}{\pi}.$$

The Cheeger constant of $\mathbb{H}$ is equal to 1, asymptotically attained by large discs so there there is a gap between the maximal Cheeger constant of a large volume compact surface and its universal cover.

We briefly mention that there has been great recent progress on deterministic upper bounds for λ_1 on compact hyperbolic surfaces with some very different perspectives, namely conformal bootstrap [Bo22, KMP23] and linear programming [FBP23]. We refer the reader to the cited articles for an account of the literature here.

Other related work on spectral theory on random hyperbolic surfaces includes the study of Laplacian eigenfunctions [GMST21, Th22], quantum ergodicity [LS20], local Weyl law [Mo21] and Gaussian Orthogonal Ensemble energy statistics [Ru22, Na22].

Plan of the thesis

First, the necessary background is introduced in Section 2. In Section 3 we prove Theorem 1.10. Next, In Section 4.2 we prove Theorem 1.15. Finally in Section 5 we discuss some problems on embedded eigenvalues. The results of Section 4.2 rely on some estimates for Weil-Petersson volumes which are proven in the Appendix A.

2 Background

In this section we introduce some background on the geometry and spectral theory of non-compact hyperbolic surfaces.

2.1 Hyperbolic surfaces

Consider the upper half plane

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\},$$

with metric given by

$$\frac{dx^2 + dy^2}{y^2}.$$

The orientation preserving isometry group of $\mathbb{H}$ is $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \pm I$, which acts by Möbius transformations. The elements of $\mathrm{PSL}_2(\mathbb{R})$ can be classified as follows.

Definition 2.1. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ with $\gamma \neq I$.

1. γ is parabolic if and only if $|a+b| = 2$,
2. γ is hyperbolic if and only if $|a+b| > 2$
3. γ is elliptic if and only if $|a+b| < 2$.

A hyperbolic surface is smooth, connected, orientable Riemannian surface with constant curvature -1. Any hyperbolic surface can be realized as a quotient $\Gamma\backslash\mathbb{H}$ where Γ is a discrete, torsion free subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

A parabolic cylinder is the quotient of $\mathbb{H}$ by a parabolic cyclic group. We define a cusp to be the small end of a parabolic cylinder, with boundary the unique closed horocycle of length 1. We can identify any cusp $\mathcal{C}$ with

$$\mathcal{C} \overset{\text{def}}{=} (1, \infty) \times S^1,$$

with the metric

$$\frac{dr^2 + dx^2}{r^2},$$

where $(r, x) \in (1, \infty) \times S^1$. By [Bu92, Lemma 4.4.6], in any finite-area hyperbolic surface, cusps must be pairwise disjoint.

We shall closely follow [Iw02, Section 2.2]. Let $X = \Gamma_X\backslash\mathbb{H}$ be a finite-area non-compact hyperbolic surface so that Γ_X is a finitely generated free group. For $w \in \mathbb{H}$, the Dirichlet domain centered at w is defined by

$$\mathcal{F}_w \overset{\text{def}}{=} \{z \in \mathbb{H} \mid d(z, w) \leqslant d(z, \gamma w) \text{ for all } \gamma \in \Gamma_X\}. \tag{2.1}$$

We write $\mathcal{F}$ to denote some Dirichlet fundamental domain for Γ_X. Since $\mathcal{F}$ is a non-compact polygon, it has some of its vertices on $\mathbb{R} \cup \infty$ in $\mathbb{H} \cup \partial\mathbb{H}$. We call such a vertex a cuspidal vertex. By e.g. [Iw02, Proposition 2.4], we can ensure that the cuspidal vertices are distinct modulo Γ_X. The sides of $\mathcal{F}$ can be arranged in pairs so that the side pairing motions generate Γ_X. The two sides of $\mathcal{F}$ meeting at a cuspidal vertex have to be pairs since the cuspidal vertices are distinct modulo Γ_X. The side-pairing motion has to fix the vertex and is therefore a parabolic element of Γ_X. This gives rise to a cusp in the quotient $\Gamma_X \backslash \mathbb{H}$ and each cuspidal vertex corresponds to a unique cusp in this way. We label the cuspidal vertices by $\mathfrak{a}_1, ..., \mathfrak{a}_n \in \operatorname{cusp}(X)$. We denote the stabilizer subgroup of the vertex $\mathfrak{a}_i$ by

$$\Gamma_{\mathfrak{a}_i} \stackrel{\text{def}}{=} \{\gamma \in \Gamma_X \mid \gamma\mathfrak{a}_i = \mathfrak{a}_i\}.$$

Each $\Gamma_{\mathfrak{a}_i}$ is an infinite cyclic group generated by the parabolic element $\gamma_{\mathfrak{a}_i}$, which is the side-pairing motion at the vertex $\mathfrak{a}_i$. There exists $\sigma_{\mathfrak{a}_i} \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\sigma_{\mathfrak{a}_i}^{-1}\gamma_{\mathfrak{a}_i}\sigma_{\mathfrak{a}_i} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

$\sigma_{\mathfrak{a}_i}$ is determined up to right multiplication by a translation. $\sigma_{\mathfrak{a}_i}$ is determined up to right multiplication by a translation. We choose $\sigma_{\mathfrak{a}_i}$ so that for each $l \geqslant 1$, the semi-strip

$$P(l) \stackrel{\text{def}}{=} \{z \in \mathbb{H} \mid 0 < x < 1, y \geqslant l\},$$

is mapped into $\mathcal{F}$ by $\sigma_{\mathfrak{a}_i}$.

Definition 2.2. For $i = 1, \ldots, n$ and $l \geqslant 1$, we define

$$D_{\mathfrak{a}_i}(l) \stackrel{\text{def}}{=} \sigma_{\mathfrak{a}_i} P(l),$$

and

$$D(l) \stackrel{\text{def}}{=} \mathcal{F} \backslash \bigsqcup_{i=1}^{n} D_{\mathfrak{a}_i}(l).$$

$D_{\mathfrak{a}_i}(l)$ is the part of the fundamental domain in the ith cusp bounded below by the length $\frac{1}{l}$ horocycle and $D(l)$ is a pre-compact region of $\mathcal{F}$. By e.g. [Bu92, Lemma 4.4.6], the cusps $D_{\mathfrak{a}_i}(1)$ are pairwise disjoint and since $l \geqslant 1$, $D_{\mathfrak{a}_i}(l) \cap D_{\mathfrak{a}_j}(l) = \emptyset$ for $i \neq j$ and we can partition the fundamental domain as

$$\mathcal{F} = D(l) \sqcup \bigsqcup_{i=1}^{n} D_{\mathfrak{a}_i}(l).$$

2.2 Spectral theory

For an unbounded linear operator A on an Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(A) \subset \mathcal{H}$, the *spectrum* $\mathrm{spec}(A)$ of A is the set of $\lambda \in \mathbb{C}$ for which $A - \lambda \mathrm{Id} : \mathcal{D}(A) \to \mathcal{H}$ fails to have a bounded inverse. In this section we define the Laplacian Δ on hyperbolic surfaces and describe the spectrum $\mathrm{spec}(\Delta)$.

2.2.1 Laplacian on non-compact finite-area surfaces

The Laplacian on $C^{\infty}(\mathbb{H})$, denoted $\Delta_{\mathbb{H}}$, is given by

$$\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Since any surface can be realized as $X = \Gamma \backslash \mathbb{H}$, we can use the Laplacian on $\mathbb{H}$ to define our Laplacian on X. Since $\Delta_{\mathbb{H}}$ is invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$, it descends to an operator on $C_c^{\infty}(X)$. It extends uniquely to a non-negative unbounded self-adjoint operator on $L^2(X)$. We let Δ_X denote the Laplacian on X and write $\mathrm{spec}(\Delta_X)$ for the spectrum of Δ_X. We write $\lambda_j(X)$ to denote the jth smallest non-zero eigenvalue of Δ_X *if it exists*.

We briefly describe the spectral decomposition of $L^2(X)$. Our reference here is [Iw02]. Let $\mathcal{B}(X)$ denote the space of smooth, bounded functions on X. Let $\psi \in C_0^{\infty}(\mathbb{R}_{\geqslant 0})$ and $\mathfrak{a} \in \mathrm{Cusp}(X)$. We define the incomplete Eisenstein series

$$E_{\mathfrak{a}}(z \mid \psi) \stackrel{\text{def}}{=} \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \psi\left(\mathrm{Im} \sigma_{\mathfrak{a}}^{-1} z\right), \tag{2.3}$$

which converges absolutely and gives a function in $\mathcal{B}(X)$[3]. Let $\mathcal{E}(X) \subset \mathcal{B}(X)$ denote the space of incomplete Eisenstein series.

[3] Setting $\psi = y^s$ for $s > 1$, in 2.3 we obtain the Eisenstein series $E_{\mathfrak{a}}(z, s)$ which is Laplacian eigenfunction with eigenvalue $s(1-s)$ but is not square-integrable.

Any function $f \in \mathcal{B}(X)$ has a Fourier expansion

$$f\left(\sigma_{\mathfrak{a}}(x+iy)\right) = \sum_{m \in \mathbb{Z}} f_{\mathfrak{a},m}(y)\, e^{2\pi m i x},$$

where

$$f_{\mathfrak{a},m}(y) = \int_0^1 f\left(\sigma_{\mathfrak{a}}(x+iy)\right) e^{2\pi m i x} dx.$$

One can check that for any $\mathfrak{a} \in \mathrm{Cusp}(X)$ and $\psi \in C_0^{\infty}(\mathbb{R}_{\geqslant 0})$,

$$\langle f, E_{\mathfrak{a}}(* \mid \psi)\rangle = \int_0^{\infty} f_{\mathfrak{a},m}(y)\, \bar{\psi}(y)\, \frac{dy}{y^2}.$$

In particular, f is orthogonal to $\mathcal{E}(X)$ if and only if for each $\mathfrak{a}$, $f_{\mathfrak{a},m}(y) = 0$ for y a.e. We write $\mathcal{C}(X)$ to denote the smooth, bounded functions of X such that $f_{\mathfrak{a},m}(y) = 0$ a.e. for every $\mathfrak{a} \in \mathrm{Cusp}(X)$. Then

$$L^2(X) = \overline{\mathcal{C}(X)} \oplus \overline{\mathcal{E}(X)},$$

where the bar means completion with respect to the L^2-norm. The spectral decomposition of Δ in $\overline{\mathcal{C}(X)}$ consists entirely of eigenvalues, known as *cusp forms.* There are examples of surfaces with infinitely many cusp forms e.g. principal congruence covers, although there is a priori no reason that $\overline{\mathcal{C}(X)}$ must even be non-empty. This problem is discussed in more detail in Section 5. The spectral decomposition of Δ in $\overline{\mathcal{E}(X)}$ consists of absolutely continuous spectrum in $\left[\frac{1}{4}, \infty\right)$ with multiplicity equal to the number of cusps and finitely many eigenvalues below $\frac{1}{4}$, always including the trivial eigenvalue $\lambda_0 = 0$. Any eigenvalue of Δ in $\overline{\mathcal{E}(X)}$ is called a *residual eigenvalue.* Here, residual refers to the fact that the discrete part of $\overline{\mathcal{E}(X)}$ is spanned by residues of Eisenstein series. Any eigenvalue above $\frac{1}{4}$ is called an *embedded eigenvalue*, since it is embedded in the continuous spectrum. An embedded eigenvalue is necessarily a cusp form but there can also be cusp forms below $\frac{1}{4}$.

3 Spectral gaps for random covers

The material of this chapter is based on [HM23] joint with Michael Magee and [Hi23]. The aim of this chapter is to prove the following, c.f. Theorem 1.10.

Theorem 3.1. *Let X be a finite-area non-compact hyperbolic surface. There is a constant $C>0$ such that a uniformly random degree-n cover has no new eigenvalues below*

$$\frac{1}{4}-C\frac{(\log\log\log n)^{2}}{\log\log n},$$

with probability tending to 1 *as* $n\to\infty$.

3.1 Outline of the proof

We say an event A depending on a parameter n happens *asymptotically almost surely* and write a.a.s. if the probability A holds tends to 1 as $n\to\infty$.

We first outline the proof for ε fixed, which is [HM23, Theorem 1], and then explain the extra steps to obtain the rates $\varepsilon=\varepsilon(n)$ in Theorem 3.1.

Set up and approach

Let X be a fixed finite-area non-compact surface. Let $\phi\in\mathrm{Hom}(\Gamma,S_n)$ be a uniformly random permutation and X_ϕ the corresponding cover. Our aim is to demonstrate that for any $\varepsilon>0$, a.a.s. for every $s\in\left[\frac{1}{2}+\sqrt{\varepsilon},1\right]$ there exists a bounded operator

$$R_{X_\phi}(s):L^2_{\text{new}}(X_\phi)\to H^2_{\text{new}}(X_\phi),$$

with

$$\left(\Delta_{X_\phi}-s(1-s)\right)R_{X_\phi}(\lambda)=\mathrm{Id}_{L^2_{\text{new}}(X_\phi)}. \tag{3.1}$$

Since (3.1) implies that $\left(\Delta_{X_\phi}-s(1-s)\right):H^2_{\text{new}}(X_\phi)\to L^2_{\text{new}}(X_\phi)$ is onto and therefore, by self-adjointness, has no non-trivial kernel in $H^2_{\text{new}}(X_\phi)$, it would follow that a.a.s. Δ_{X_ϕ} has no new eigenvalues $s(1-s)\leqslant\frac{1}{4}-\varepsilon$.

Our approach is to construct an approximate inverse $\mathbb{M}_\phi(s):L^2_{\text{new}}(X_\phi)\to H^2_{\text{new}}(X_\phi)$, i.e. a bounded operator with the property that

$$\left(\Delta_{X_\phi}-s(1-s)\right)\mathbb{M}_\phi(\lambda)=\mathrm{Id}_{L^2_{\text{new}}(X_\phi)}+\mathbb{L}_\phi(s),$$

where $\mathbb{L}_{\phi}(s) : L^2_{\text{new}}(X_{\phi}) \to L^2_{\text{new}}(X_{\phi})$. If one can show that

$$\|\mathbb{L}_{\phi}(\lambda)\|_{L^2_{\text{new}}(X_{\phi})} < 1, \tag{3.2}$$

then 1 is not in the spectrum of $\mathbb{L}_{\phi}(\lambda)$, in particular

$$\left(\text{Id}_{L^2_{\text{new}}(X_{\phi})} + \mathbb{L}_{\phi}(\lambda)\right)^{-1} : L^2_{\text{new}}(X_{\phi}) \to L^2_{\text{new}}(X_{\phi}),$$

exists as a bounded operator. One can then take

$$R_{X_{\phi}}(\lambda) \stackrel{\text{def}}{=} \mathbb{M}_{\phi}(\lambda)\left(\text{Id}_{L^2_{\text{new}}(X_{\phi})} + \mathbb{L}_{\phi}(\lambda)\right)^{-1}.$$

in (3.1). The problem is then reduced to showing that (3.2) holds a.a.s (for an appropriate choice of $\mathbb{M}_{\phi}(\lambda)$).

Building the approximate inverse

We build $\mathbb{M}_{\phi}(s)$ is by patching together a 'cuspidal parametrix' $\mathbb{M}^{\text{cusp}}_{\phi}(s)$ based on a model resolvent in the cusps and an an interior parametrix $\mathbb{M}^{\text{int}}_{\phi}(s)$ that localizes to a compact part of X_{ϕ}. We then let

$$\mathbb{M}_{\phi}(s) = \mathbb{M}^{\text{int}}_{\phi}(s) + \mathbb{M}^{\text{cusp}}_{\phi}(s)$$

and we get a resulting splitting

$$\mathbb{L}_{\phi}(s) = \mathbb{L}^{\text{int}}_{\phi}(s) + \mathbb{L}^{\text{cusp}}_{\phi}(s).$$

In Section 3.4 we show that the term $\mathbb{M}^{\text{cusp}}_{\phi}(s)$ can be designed so that $\|\mathbb{L}^{\text{cusp}}_{\phi}(s)\| \leq \frac{1}{8}$ (or any small number) for any ϕ, i.e. $\|\mathbb{L}^{\text{cusp}}_{\phi}(s)\|$ *can be bounded deterministically*, and will not cause issues in obtaining $\|\mathbb{L}_{\phi}(s)\| < 1$. This is achieved by ensuring $\mathbb{M}^{\text{cusp}}_{\phi}(s)$ localises sufficiently high up into the cusp, essentially above height $\frac{1}{\sqrt{\varepsilon}}$.

The term $\mathbb{M}^{\text{int}}_{\phi}(s)$ is based on averaging the resolvent kernel of the hyperbolic plane over the fundamental group of Γ (suitably twisting by ϕ) to obtain an integral operator on $L^2_{\text{new}}(X_{\phi})$. The problem with this is that the averaging will not obviously converge, so we have to multiply the hyperbolic resolvent kernel by a radial cutoff that localizes to radii $\leq T+1$ to get a priori convergence for all $s \in \left(\frac{1}{2}, 1\right]$. This gives us that $\mathbb{M}_{\phi}^{\mathrm{int}}(s)$ is bounded (Lemma 3.13).

The effect of this cutoff is that the error term $\mathbb{L}_{\phi}^{\mathrm{int}}(s)$ is an integral operator with smooth kernel.

Bounding $\mathbb{L}_{\phi}^{\mathrm{int}}(s)$

We prove that we can unitarily conjugate $\mathbb{L}_{\phi}^{\mathrm{int}}(s)$ to

$$\sum_{\gamma \in S} a_{\gamma}(s) \otimes \rho_{\phi}(\gamma)$$

acting on $L^2(\mathcal{F}) \otimes V_n^0$, where $\mathcal{F}$ is a Dirichlet fundamental domain for Γ and $\left(\rho_{\phi}, V_n^0\right)$ is the standard $n-1$ dimensional irreducible representation of S_n. The $a_{\gamma}(s)'s$ are compact operators on $L^2(\mathcal{F})$ and S is a finite set, which is fixed depending on the cut off T. In particular there are only finitely many $\gamma \in \Gamma$ for which $a_{\gamma}(s)$ is non-zero.

Because Γ is a free group of rank d, picking $\phi \in \mathrm{Hom}(\Gamma, S_n)$ is the same as picking d permutations indepentantly and uniformly at random. The breakthrough results of Bordenave and Collins from [BC19] allow us to control the norm of

$$\|\mathbb{L}_{\phi}^{\mathrm{int}}(s)\|_{L^2_{\mathrm{new}}(X_{\phi})} = \left\| \sum_{\gamma \in S} a_{\gamma}(s) \otimes \rho_{\phi}(\gamma) \right\|_{L^2(\mathcal{F}) \otimes V_n^0},$$

a.a.s. In particular, if instead, the $a_{\gamma}(s)$ were matrices in $M_r(\mathbb{C})$ for some fixed finite r, the work of Bordenave and Collins from [BC19] would tell us that for any $\kappa > 0$, a.a.s.

$$\left\| \sum_{\gamma \in S} a_{\gamma}(s) \otimes \rho_{\phi}(\gamma) \right\|_{\mathbb{C}^r \otimes V_n^0} \leq \left\| \sum_{\gamma \in \Gamma} a_{\gamma}(s) \otimes \rho_{\infty}(\gamma) \right\|_{\mathbb{C}^r \otimes l^2(\Gamma)} + \kappa \tag{3.3}$$

where $\rho_{\infty} : \Gamma \to \mathrm{End}\left(\ell^2(\Gamma)\right)$ is the right regular representation. This is where the fact that Γ is free is exploited.

Because the $a_{\gamma}(s)$ are in reality compact operators on Hilbert spaces we can approximate by finite rank operators to the same effect. The rank of the matrices we need to take to achieve (3.3) depends on the cutoff T through the size of the set S.

Crucially, we understand the operator in the right hand side of (3.3) well: it can be unitarily conjugated to an operator on $L^2(\mathbb{H})$ that is the composition of multiplication with a cutoff (with norm ≤ 1) and an integral operator with real-valued radial kernel. This latter operator is self-adjoint we can use the theory of the Selberg transform to estimate its norm in Lemma 3.11. By choosing T sufficiently large in the beginning, we can force the norm in the right hand side of (3.3) to be as small as we like, for $s > \frac{1}{2} + \varepsilon$. Consequently, we can control $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ for *any fixed* $s > \frac{1}{2} + \varepsilon$ a.a.s.

We now want to be able to control $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ for *all* $s > \frac{1}{2} + \varepsilon$ a.a.s. To do this, we show that $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ does not fluctuate much on small intervals (Lemma 3.16). We can then split up the interval $\left(\frac{1}{2} + \varepsilon, 1\right)$ into a fine enough grid so that controlling $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ at each point s reduces to controlling $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ at a finite number of points in the grid. We can then apply an intersection bound to control $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ at every point in the (finite) grid a.a.s, thus bounding $\|\mathbb{L}_\phi^{\text{int}}(s)\|$ on $(\frac{1}{2} + \varepsilon, 1]$ a.a.s.

In total, we have shown $\|\mathbb{L}_\phi(s)\| < 1$ for every $s > \frac{1}{2} + \varepsilon$ a.a.s. giving the desired result.

Effetiving previous arguments

We now describe how this argument is made effective, i.e. how one can allow ε to depend on n. We want to repeat the argument to control $\|\mathbb{L}_\phi(s)\|$ a.a.s. when $s \in \left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$ where $\kappa(n) \to 0$ as $n \to \infty$. To achieve this we need to take the cut-off $T = T(n)$ to grow depending on $\kappa(n)$ and we need to cut off the cusps at a height proportional to $\kappa(n)$. Now our operator $\mathbb{L}_\phi^{\text{int}}(s)$ is conjugated to

$$\sum_{\gamma \in S(n)} a_\gamma \otimes \rho_\phi(\gamma) \tag{3.4}$$

where $S(n)$ now grows as $n \to \infty$. We again want to approximate each a_γ by a finite rank operator and compare (3.4) to

$$\sum_{\gamma \in S(T)} b_\gamma \otimes \rho_\phi(\gamma),$$

where $b_\gamma \in M_m(\mathbb{C})$. However since $S = S(n)$ is now growing as $n \to \infty$, we need to take larger and larger (depending on n) finite rank approximations of a_γ. In total, we need to control the norm of a matrix polynomial with coefficients in $M_m(\mathbb{C})$ where $m = m(n)$, where the number of terms of the polynomial also depends on n. To do this we apply recent work of Bordenave and Collins [BC23] which, together with an effective linearisation procedure (c.f. § 3.3), provides an effective version of (3.3). In order to access their results, we also need to control the size of $S = S(n)$ and the largest wordlength of any $\gamma \in S$. This is achieved in Section 3.7.

3.2 Set up

Throughout the rest of this section, $X = \Gamma \backslash \mathbb{H}$ will be a fixed non-compact finite-area surface. We consider all degree-n Riemannian covers of X, sampled uniformly at random. We stress that the covers we consider need not be connected, however will be connected with high probability, which we will observe shortly.

There is a 1-to-1 correspondence between degree-n covers X_ϕ of $X = \Gamma\backslash\mathbb{H}$ with labelled fiber $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$ and $\phi \in \mathrm{Hom}(\Gamma, S_n)$. Given a degree-$n$ cover X_n, fix a point $x_0 \in X$ and assume the the fiber above x_0 is labelled by $[n]$. The monodromy map

$$\phi : \pi_1(X, x_0) \to S_n,$$

which describes how the fiber of x_0 is permuted when following lifts of closed curves from X to X_n, uniquely determines X_n. Given $\phi \in \mathrm{Hom}(\Gamma, S_n)$ we can build a cover with monodromy ϕ by

$$X_\phi \stackrel{\text{def}}{=} \mathbb{H} \times [n] / \sim$$

where $(z, i) \sim (\gamma z, \phi(\gamma) i)$ for $\gamma \in \Gamma$.

Since X is non-compact, Γ is necessarily a free group, freely generated by some

$$\gamma_1, \ldots, \gamma_d \in \Gamma,$$

and choosing $\phi \in \mathrm{Hom}(\Gamma, S_n)$ is the same as choosing

$$\sigma_i \stackrel{\mathrm{def}}{=} \phi(\gamma_i), \quad i = 1, \ldots, d$$

independently and uniformly at random in S_n. The surface X_ϕ is connected if and only if Γ acts transitively on $[n]$ via ϕ. By a theorem of Dixon [Di69], two independent and uniformly random permutations in S_n generate S_n or A_n a.a.s and it follows that a uniformly random cover X_ϕ is connected a.a.s.

Let $V_n \stackrel{\mathrm{def}}{=} \ell^2([n])$ and $V_n^0 \subset V_n$ the subspace of functions on $[n]$ with zero mean. The representation of S_n on $\ell^2([n])$ is its standard representation by 0-1 matrices and the subspace V_n^0 is an irreducible subspace of dimension $(n-1)$: we write

$$\rho_\phi : \Gamma \to \mathrm{End}(V_n^0)$$

for the random representation of Γ induced by the random ϕ.

3.2.1 Function spaces

We define $L^2_{\mathrm{new}}(X_\phi)$ to be the space of L^2 functions on X_ϕ orthogonal to all lifts of L^2 functions from X. Then

$$L^2(X_\phi) \cong L^2_{\mathrm{new}}(X) \oplus L^2(X).$$

Fix $\mathcal{F}$ to be a Dirichlet fundamental domain for X (2.1). Let $C^\infty\left(\mathbb{H}; V_n^0\right)$ denote the smooth V_n^0-valued functions on $\mathbb{H}$. There is an isometric linear isomorphism between

$$C^\infty(X_\phi) \cap L^2_{\mathrm{new}}(X_\phi),$$

and the space of smooth V_n^0-valued functions on $\mathbb{H}$ satisfying

$$f(\gamma z) = \rho_\phi(\gamma) f(z), \tag{3.5}$$

for all $\gamma \in \Gamma$, with finite norm

$$\|f\|_{L^2(\mathcal{F})}^2 \stackrel{\text{def}}{=} \int_F \|f(z)\|_{V_n^0}^2 d\mu_{\mathbb{H}}(z) < \infty.$$

We denote the space of such functions by $C_\phi^\infty\left(\mathbb{H}; V_n^0\right)$. The completion of $C_\phi^\infty\left(\mathbb{H}; V_n^0\right)$ with respect to $\|\bullet\|_{L^2(\mathcal{F})}$ is denoted by $L_\phi^2\left(\mathbb{H}; V_n^0\right)$; the isomorphism above extends to one between $L_{\text{new}}^2(X_\phi)$ and $L_\phi^2\left(\mathbb{H}; V_n^0\right)$.

Let $C_{c,\phi}^\infty\left(\mathbb{H}; V_n^0\right)$ denote the subset of $C_\phi^\infty\left(\mathbb{H}; V_n^0\right)$ consisting of functions which are compactly supported modulo Γ. We let $H_\phi^2\left(\mathbb{H}; V_n^0\right)$ denote the completion of $C_{c,\phi}^\infty\left(\mathbb{H}; V_n^0\right)$ with respect to the norm

$$\|f\|_{H_\phi^2(\mathbb{H}; V_n^0)}^2 \stackrel{\text{def}}{=} \|f\|_{L^2(\mathcal{F})}^2 + \|\Delta f\|_{L^2(\mathcal{F})}^2.$$

We let $H^2(X_\phi)$ denote the completion of $C_c^\infty(X_\phi)$ with respect to the norm

$$\|f\|_{H^2(X_\phi)}^2 \stackrel{\text{def}}{=} \|f\|_{L^2(X_\phi)}^2 + \|\Delta f\|_{L^2(X_\phi)}^2.$$

Viewing $H^2(X_\phi)$ as a subspace of $L^2(X_\phi)$, we let

$$H_{\text{new}}^2(X_\phi) \stackrel{\text{def}}{=} H^2(X_\phi) \cap L_{\text{new}}^2(X_\phi).$$

There is an isometric isomorphism between $H_{\text{new}}^2(X_\phi)$ and $H_\phi^2\left(\mathbb{H}; V_n^0\right)$ that intertwines the two relevant Laplacian operators.

3.3 Random matrix theory

In this section we introduce the necessary random matrix theory results. Recall Γ is a free group on d generators $\gamma_1, \ldots, \gamma_d$. The wordlength $\text{wl}(\gamma)$ is the length of γ as a reduced word in $\gamma_1, \ldots, \gamma_d, \gamma_1^{-1}, \ldots, \gamma_d^{-1}$. Let $\rho_\infty : \Gamma \to \text{End}\left(l^2(\Gamma)\right)$ denote the right regular representation of Γ.

As described in §3.1, we make essential use of the fact, due to Bordenave and Collins [BC19], that for any $m \in \mathbb{N}$ and any finitely suppported map $\gamma \mapsto a_\gamma \in M_m(\mathbb{C})$, a uniformly random $\phi_n \in \mathrm{Hom}(\Gamma, S_n)$ satisfies

$$\|\sum_{\gamma\in\Gamma} a_\gamma \otimes \rho_\phi(\gamma)\|_{\mathbb{C}^m\otimes V_n^0} \leqslant (1+o_{n\to\infty}(1))\,\|\sum_{\gamma\in\Gamma} a_\gamma\otimes\rho_\infty(\gamma)\|_{\mathbb{C}^m\otimes l^2(\Gamma)}, \tag{3.6}$$

a.a.s. To obtain precise rates as in Theorem 1.10, it is necessary to have an effective version of (3.6).

By the linearisation trick [Pi96, HT05], proving 3.6 is equivalent to proving 3.6 for every linear polynomial. More precisely for any $m \in \mathbb{N}$ and $a_0, a_1, \ldots a_d \in M_m(\mathbb{C})$ with $a_0 = a_0^*$, a uniformly random $\phi_n \in \mathrm{Hom}(\Gamma, S_n)$ satisfies

$$\begin{aligned} &\left\| a_0 \otimes \mathrm{Id}_{V_n^0} + \sum_{i=1}^d \left(a_i \otimes \rho_\phi(\gamma_i) + a_i^* \otimes \rho_\phi\left(\gamma_i^{-1}\right)\right)\right\|_{\mathbb{C}^m\otimes V_n^0} \\ &\leqslant (1+o_{n\to\infty}(1)) \left\| a_0\otimes \mathrm{Id}_{\ell^2(\Gamma)} + \sum_{i=1}^d \left(a_i\otimes\rho_\infty(\gamma_i) + a_i^*\otimes\rho_\infty\left(\gamma_i^{-1}\right)\right)\right\|_{\mathbb{C}^m\otimes\ell^2(\Gamma)}, \end{aligned} \tag{3.7}$$

a.a.s. The idea is that one can replace a polynomial of large degree (i.e. a_γ supported on long words in Γ) with a polynomial of smaller degree (to eventually a_γ only supported on generators and their inverses, i.e. a linear polynomial) at the cost of replacing $M_m(\mathbb{C})$ by $M_m(\mathbb{C}) \otimes M_k(\mathbb{C})$ for some k. Since (3.7) holds for matrices of every size, the statements are equivalent. The benefit is that a statement like (3.7) is often easier to prove.

Again, for our purposes, we need a quantitative version of (3.6). We rely heavily on a quantitative version (3.7) due to Bordenave and Collins [BC23].

Theorem 3.2 ([BC23, Corollary 1.4])**.** *Let $m \leqslant n^{\sqrt{\log n}}$ and $a_0, a_1, \ldots a_d \in M_m(\mathbb{C})$ with $a_0 = a_0^*$. Then there exists a constant $c_1 > 0$ such that for a uniformly random $\phi \in \mathrm{Hom}(\Gamma, S_n)$, with probability at least $1 - \frac{c_1}{\sqrt{n}}$,*

$$\begin{aligned} &\left\| a_0 \otimes \mathrm{Id}_{V_n^0} + \sum_{i=1}^d \left(a_i \otimes \rho_\phi(\gamma_i) + a_i^* \otimes \rho_\phi\left(\gamma_i^{-1}\right)\right)\right\|_{\mathbb{C}^m\otimes V_n^0} \\ &\leqslant \left\| a_0\otimes \mathrm{Id}_{\ell^2(\Gamma)} + \sum_{i=1}^d \left(a_i\otimes\rho_\infty(\gamma_i) + a_i^*\otimes\rho_\infty\left(\gamma_i^{-1}\right)\right)\right\|_{\mathbb{C}^m\otimes\ell^2(\Gamma)} \left(1 + \frac{c_1}{(\log n)^{\frac{1}{4}}}\right). \end{aligned}$$

To pass from (3.2) to a quantitative version of (3.6), we use an effective linearization proved in [BC23, Section 8].

In [BC23, Section 8], the authors considered operators of the form $\sum_{\gamma\in B_l} a_\gamma \otimes \rho_\phi(\gamma)$ where B_l is the ball of size l in the word metric of Γ with our fixed choice of generators. In our case, the operators we want to consider will be of the form $\sum_{\gamma\in S} a_\gamma \otimes \rho_\phi(\gamma)$ where $S \subset B_l$ where $|S|$ is roughly of size l which shall give us a quantitative saving. This is only a minor adaptation to the arguments in [BC23, Section 8], however since this is a key point for our method, we include the details.

We say that a subset $S \subset \Gamma$ is symmetric if $g \in S$ implies $g^{-1} \in S$.

Lemma 3.3. *Let $l \geqslant 2$ be an even integer and let $S \subset B_l$. Consider $(a_g)_{g\in S}$ with $a_g \in M_m(\mathbb{C})$. Then there exists a symmetric set $S_1 \subset B_{\frac{l}{2}}$ with $|S_1| \leqslant 4|S|$, $(b_g)_{g\in S_1}$ with $b_g \in M_m(\mathbb{C}) \otimes M_{2|S_1|}(\mathbb{C})$ and $\theta \geqslant 0$ such that for any unitary representation (ρ, V) of Γ,*

$$\| \sum_{\gamma\in S} a_\gamma \otimes \rho(\gamma) \|_{\mathbb{C}^m\otimes V} = \| \sum_{\gamma\in S_1} b_\gamma \otimes \rho(\gamma) \|^2_{\mathbb{C}^m\otimes \mathbb{C}^{2|S_1|}\otimes V} - \theta,$$

where

$$\theta \leqslant 4|S| \, \| \sum_{\gamma\in S} a_\gamma \otimes \rho_\infty(\gamma) \|_{\mathbb{C}^m \otimes l^2(\Gamma)}.$$

Proof. We consider a set $S_1 \subset B_{\frac{l}{2}}$ such that

$$S \subset \left\{ g^{-1}h \mid g, h \in S_1 \right\}.$$

We claim we can choose S_1 so that

$$|S_1| \leqslant 4|S|.$$

Indeed if $w \in S \cap B_{\frac{l}{2}}$, we can just take w and the identity to be in S_1. If $w \in S$ has wordlength $> \frac{l}{2}$, then it can be written as $g^{-1}h$ for two words $g, h \in B_{\frac{l}{2}}$ and we add both words to S_1. We make S_1 symmetric by including the inverses of any word already added, at worst doubling the size of S_1.

Note that we can enlarge S to a symmetric set without changing the size of S_1, since S_1 is symmetric. After possibly replacing $M_m(\mathbb{C})$ with $M_m(\mathbb{C}) \otimes M_2(\mathbb{C})$ and enlarging S to a symmetric set, we can assume that the symmetry condition $a_\gamma = a^*_{\gamma^{-1}}$ holds, in particular $P \overset{\text{def}}{=} \sum_{\gamma\in S} a_\gamma \otimes \rho(\gamma)$ is self-adjoint e.g. [BC23, Proof of Theorem 1.1]. Explicitly, by considering the operator $\hat{P} = \sum_{\gamma\in S} \hat{a}_\gamma \otimes \rho(\gamma) \in M_m(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \mathrm{End}(V)$ with

$$\hat{a}_\gamma = \begin{pmatrix} 0 & a_\gamma \\ a^*_{\gamma^{-1}} & 0 \end{pmatrix},$$

$\hat{P}$ satisfies the symmetry condition and $\|\hat{P}\|_{\mathbb{C}^m\otimes\mathbb{C}^2\otimes V} = \|P\|_{\mathbb{C}^m\otimes V}$.

We now follow [BC23, Proof of Lemma 8.1]. Consider the element $\tilde{a} \in M_m(\mathbb{C}) \otimes M_{|S_1|}(\mathbb{C})$ defined by $(\tilde{a}_{g,h})_{g,h\in S_1}$,

$$\tilde{a}_{g,h} = \frac{1}{\#\left\{(g',h') \in S_1 \times S_1 \mid (g')^{-1} h' = g^{-1}h\right\}} a_{g^{-1}h},$$

when $g^{-1}h \in S$ and $\tilde{a}_{g,h} = 0$ otherwise. Then

$$\sum_{\substack{g,h\in S_1 \\ g^{-1}h = w\in S}} \tilde{a}_{g,h} = a_w.$$

We have

$$\|\tilde{a}\|^2 \leqslant \|\sum_{g,h\in S_1} \tilde{a}_{g,h}\tilde{a}^*_{g,h}\| \leqslant \|\sum_{w\in S} a_w a_w^*\| \leqslant \|\sum_{w\in S} a_w \otimes \rho_\infty(w)\|^2.$$

The operator $\tilde{a} + \|\tilde{a}\|\mathrm{Id}_{m|S_1|}$ is positive semi-definite and we let $\tilde{b} \in M_m(\mathbb{C}) \otimes M_{|S_1|}(\mathbb{C})$ be its self-adjoint square root. For $g \in S_1$ we define

$$b_g \overset{\text{def}}{=} \tilde{b}\left(\mathrm{Id}_m \otimes e_{g,\emptyset}\right) \in M_m(\mathbb{C}) \otimes M_{|S_1|}(\mathbb{C}),$$

where $e_{g,h} \overset{\text{def}}{=} \delta_g \otimes \delta_h \in M_{B_{\frac{L}{2}}}(\mathbb{C})$ and $\emptyset$ is the unit in Γ. Then defining

$$Q \overset{\text{def}}{=} \sum_{g\in S_1} b_g \otimes \rho(g),$$

we have

$$Q^*Q = \sum_{g,h\in S_1} \left(\mathrm{Id}_m \otimes e_{\emptyset,g}\right) \tilde{b}^2 \left(\mathrm{Id}_m \otimes e_{h,\emptyset}\right) \otimes \rho(g^{-1}h)$$
$$= \sum_{g,h\in S_1} e_{\emptyset,\emptyset} \otimes \left(\tilde{a}_{g,h} + \|\tilde{a}\| \mathbf{1}_{g=h}\mathrm{Id}_m\right) \otimes \rho(g^{-1}h)$$
$$= e_{\emptyset,\emptyset} \otimes \left(\sum_{\gamma\in S} a_\gamma \otimes \rho\left(\gamma\right) + \theta \mathrm{Id}_{\mathbb{C}^m\otimes V} \right),$$

where

$$\theta \leqslant |S_1| \, \| \sum_{\gamma\in S} a_\gamma \otimes \rho_\infty\left(\gamma\right) \| \leqslant 4\,|S|\, \| \sum_{\gamma\in S} a_\gamma \otimes \rho_\infty\left(\gamma\right) \|.$$

It follows that

$$\|Q\|^2 = \| \sum_{\gamma\in S} a_\gamma \otimes \rho\left(\gamma\right) + \theta \mathrm{Id}_{\mathbb{C}^m\otimes V} \|_{\mathbb{C}^m\otimes V} = \| \sum_{\gamma\in S} a_\gamma \otimes \rho\left(\gamma\right) \|_{\mathbb{C}^m\otimes V} + \theta.$$

□

We can iterate this process to obtain the following, c.f. [BC23, Lemma 8.2].

Lemma 3.4. *Let $l \geqslant 2$ be an integer, $S \subset B_l$ and let $v = \lceil \log_2 l \rceil$. Then for each $k \in \{0,\ldots,v\}$ there is:*

- *An integer $n_k \geqslant 1$ with $n_v \leqslant 2l\,|S|^{\lceil \log_2 l\rceil}\, l^{(\lceil \log_2 l\rceil - 1)}$.*
- *A symmetric set $S_k \subset B_{2^{v-k}}$ with $S_0 = S$, $|S_k| \leqslant \min\left\{4^k\,|S|\,, |B_{2^{v-k}}|\right\}$.*
- *A set $\left(a_g^k\right)_{g\in S_k}$ with $a_g^k \in M_m\left(\mathbb{C}\right) \otimes M_{n_k}\left(\mathbb{C}\right)$.*
- *A constant $\theta_k \geqslant 0$ such that for $k \geqslant 1$,*

$$\theta_k \leqslant \Big\| \sum_{\gamma\in S_{k-1}} a_\gamma^{k-1} \otimes \rho_\infty\left(\gamma\right) \Big\|_{\mathbb{C}^m\otimes\mathbb{C}^{n_{k-1}}\otimes l^2(\Gamma)} |S_k|\,,$$

such that for any unitary representation (ρ, V) of Γ,

$$\| \sum_{\gamma\in S_{k-1}} a_g^{k-1} \otimes \rho(\gamma) \|_{\mathbb{C}^m\otimes\mathbb{C}^{n_{k-1}}\otimes V} = \| \sum_{\gamma\in S_k} a_g^k \otimes \rho(\gamma) \|^2_{\mathbb{C}^m\otimes\mathbb{C}^{n_k}\otimes V} - \theta_k.$$

Proof. This is a straightforward consequence of iterating the procedure of Lemma 3.3. We have

$$n_v \leqslant \prod_{i=1}^{v} 2\left|S_i\right| \leqslant \prod_{i=1}^{v} 2 \cdot 4^i \cdot |S| = 2^v 4^{\frac{v(v-1)}{2}} |S|^v ,$$

where $v = \lceil \log_2 l \rceil$ which gives

$$n_v \leqslant 2l \left|S\right|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)},$$

as claimed. ☐

As a consequence we obtain the following, c.f. [BC23, Lemma 8.3]

Lemma 3.5. *Let $l \geqslant 2$ be an integer, $S \subset B_l$ and set $v = \lceil \log_2 l \rceil$. Consider $\left(a_g^v\right)_{g \in S_v}$ as in Lemma 3.4 and denote $a_0 = a_\emptyset^v$, $a_i = a_{\gamma_i}^v$ for $1 \leqslant i \leqslant 2d$. Let (ρ, V) be any unitary representation of Γ. We have that for $0 < \epsilon < 1$, if*

$$2\epsilon l^2 \left|S\right|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)} < 1$$

and

$$\|a_0 \otimes \mathrm{Id}_V + \sum_{i=1}^{2d} a_i \otimes \rho(\gamma_i)\|_{\mathbb{C}^m \otimes \mathbb{C}^{n_v} \otimes V} \leqslant \|a_0 \otimes \mathrm{Id}_{l^2(\Gamma)} + \sum_{i=1}^{2d} a_i \otimes \rho_\infty(\gamma_i)\|_{\mathbb{C}^m \otimes \mathbb{C}^{n_v} \otimes l^2(\Gamma)} (1+\epsilon),$$

then

$$\| \sum_{\gamma \in S} a_\gamma \otimes \rho\left(\gamma\right) \|_{\mathbb{C}^m \otimes V} \leqslant \| \sum_{\gamma \in S} a_\gamma \otimes \rho_\infty\left(\gamma\right) \|_{\mathbb{C}^m \otimes l^2(\Gamma)} \left(1 + 2\epsilon l^2 \left|S\right|^{\lceil \log_2 l \rceil} l^{(\lceil \log_2 l \rceil - 1)}\right).$$

Proof. For $k \in \{1, \ldots, v\}$, let $a_g^k \in M_m\left(\mathbb{C}\right) \otimes M_{n_k}\left(\mathbb{C}\right)$ for $g \in S_k$ be as given by Lemma 3.4. For some $k \in \{1, \ldots, v\}$, assume that for some $0 < \epsilon_k < 1$,

$$\| \sum_{\gamma \in S_k} a_\gamma^k \otimes \rho\left(\gamma\right) \| \leqslant \| \sum_{\gamma \in S_k} a_\gamma^k \otimes \rho_\infty\left(\gamma\right) \| \left(1 + \epsilon_k\right).$$

Then by Lemma 3.4 applied twice,

$$\begin{aligned}
\| \sum_{\gamma\in S_{k-1}} a_\gamma^{k-1}\otimes\rho(\gamma)\| - \| \sum_{\gamma\in S_{k-1}} a_\gamma^{k-1}\otimes\rho_\infty(\gamma)\| &= \| \sum_{\gamma\in S_k} a_\gamma^{k}\otimes\rho(\gamma)\|^2 - \| \sum_{\gamma\in S_k} a_\gamma^{k}\otimes\rho_\infty(\gamma)\|^2 \\
&\leqslant \epsilon_k(1+2\epsilon_k)\left(\| \sum_{\gamma\in S_{k-1}} a_\gamma^{k-1}\otimes\rho_\infty(\gamma)\| + \theta_k\right) \\
&\leqslant 4\cdot 4^k|S|\,\epsilon_k \| \sum_{\gamma\in S_{k-1}} a_\gamma^{k-1}\otimes\rho_\infty(\gamma)\|.
\end{aligned} \tag{3.8}$$

By assumption, $\epsilon_v = \epsilon < 1$ and then by setting $\epsilon_{k-1} \overset{\text{def}}{=} 4\cdot 4^k|S|\,\epsilon_k$ (recalling $\theta_k \leqslant 4^k|S|\,\|\sum_{\gamma\in S_{k-1}} a_\gamma^{k-1}\otimes\rho_\infty(\gamma)\|$ from Lemma 3.4), By the definition of ϵ_{k-j} we see

$$\epsilon_0 = \epsilon\prod_{i=1}^{v} 4\cdot 4^i|S| \leqslant 2\epsilon l^2|S|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil -1)},$$

If one picks $2\epsilon l^2|S|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil -1)} < 1$ then this ensures that $\epsilon_{k-j} < 1$ for $j = 1,\dots,k$ and we can apply the inequality (3.8) inductively starting from $k = v$ to $k = 1$ provided that each subsequent $\epsilon_{k-j} < 1$. □

By applying Lemma 3.5 and Theorem 3.2 have the following corollary.

Corollary 3.6. *Let m and l satisfy*

$$2ml\,|S|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil -1)} \leqslant n^{\sqrt{\log n}}.$$

Let $S\subset B_l$ be a finite set whose size satisfies

$$2c_1 l^2|S|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil -1)} \leqslant (\log(n))^{\frac14},$$

where c_1 is the constant in Theorem 3.6. Let $\gamma\mapsto a_\gamma\in M_m(\mathbb{C})$ be any map supported in S. For a uniformly random $\varphi\in \mathrm{Hom}(\Gamma,S_n)$, with probability at least $1-\frac{c_1}{\sqrt{n}}$ one has

$$\| \sum_{\gamma\in S} a_\gamma\otimes\rho_\varphi(\gamma)\|_{\mathbb{C}^m\otimes V_n^0} \leqslant \| \sum_{\gamma\in S} a_\gamma\otimes\rho_\infty(\gamma)\|_{\mathbb{C}^m\otimes l^2(\Gamma)}\left(1+c_1\frac{2l^2|S|^{\lceil \log_2 l\rceil} l^{(\lceil \log_2 l\rceil -1)}}{(\log(n))^{\frac14}}\right).$$

3.4 Cusp parametrix

In this subsection we introduce the cuspidal part of the parametrix. We make the assumption that X has only one cusp to simplify notation. We identify the cusp $\mathcal{C}$ with

$$\mathcal{C} \stackrel{\text{def}}{=} (1, \infty) \times S^1,$$

with the metric

$$\frac{dr^2 + dx^2}{r^2}, \tag{3.9}$$

where $(r, x) \in (1, \infty) \times S^1$. For each $n \in \mathbb{N}$ we will define the cutoff functions $\chi^+_{\mathcal{C},n}, \chi^-_{\mathcal{C},n} : \mathcal{C} \to [0, 1]$ to be functions that are identically zero in a neighborhood of $\{1\} \times S^1$, identically equal to 1 in a neighborhood of $\{\infty\} \times S^1$, such that

$$\chi^+_{\mathcal{C},n}\chi^-_{\mathcal{C},n} = \chi^-_{\mathcal{C},n}. \tag{3.10}$$

We extend $\chi^{\pm}_{\mathcal{C},n}$ by 0 to functions on X. Let $\kappa : \mathbb{N} \to (0, \infty)$ be some given function. Later on (Lemma 3.17) we shall pick a specific function $\kappa(n)$, which will essentially be the rate we can take $\inf \operatorname{spec}_{\text{new}}(\Delta_\phi) \to \frac{1}{4}$ a.a.s.. As indicated by the subscript, the functions $\chi^+_{\mathcal{C},n}, \chi^-_{\mathcal{C},n}$ will depend on n through the function $\kappa(n)$. We lift $\chi^{\pm}_{\mathcal{C},n}$ through the covering map to obtain functions $\chi^{\pm}_{\mathcal{C},n,\phi}$ on X_ϕ. Indeed, the cusp of X splits in X_ϕ into several regions of the form

$$(1, \infty) \times \mathbb{R}/m\mathbb{Z}, \tag{3.11}$$

with $m \in \mathbf{N}$, and with the same metric (3.9). In these coordinates the covering map sends

$$\pi_\phi : (r, x + m\mathbb{Z}) \mapsto (r, x + \mathbb{Z}).$$

In particular, it preserves the r coordinate. We then define.

$$\chi^{\pm}_{\mathcal{C},n,\phi} \stackrel{\text{def}}{=} \chi^{\pm}_{\mathcal{C}} \circ \pi_\phi : X_\phi \to [0, 1],$$

where $\pi_\phi : X_\phi \to X$ is the covering map.

Lemma 3.7. *Given $\kappa : \mathbb{N} \to (0, \infty)$, for each $n \in \mathbb{N}$ we can choose $\chi^{\pm}_{\mathcal{C},n}$ as above so that*

$$\|\nabla \chi^{+}_{\mathcal{C},n,\phi}\|_{\infty}, \|\Delta \chi^{+}_{\mathcal{C},n,\phi}\|_{\infty} \leq \frac{\kappa(n)}{30}.$$

Proof. One can find a $\tau_0 > 1$ and a smooth function $\chi^{+}_{\mathcal{C},0} : [0, \infty) \to [0, 1]$ with $\chi^{+}_{\mathcal{C},0} \equiv 0$ for τ in $[0, 1]$, $\chi^{+}_{\mathcal{C},0} \equiv 1$ for $\tau \geq \tau_0$ such that

$$\sup_{[0,\infty)} |(\chi^{+}_{\mathcal{C},0})'|, \sup_{[0,\infty)} |(\chi^{+}_{\mathcal{C},0})''| \leq 1.$$

Then defining

$$\chi^{+}_{\mathcal{C},n}(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } t \in [0, 1] \\ \chi^{+}_{\mathcal{C},0}\left(\frac{\kappa(n)}{60}(t-1)+1\right) & \text{for } t \in (1, \infty) \end{cases},$$

we have

$$\sup_{[0,\infty)} |(\chi^{+}_{\mathcal{C},n})'|, \sup_{[0,\infty)} |(\chi^{+}_{\mathcal{C},n})''| \leq \frac{\kappa(n)}{60}. \tag{3.12}$$

Note that $\chi^{+}_{\mathcal{C},n}(\tau) \equiv 1$ for $\tau \geqslant \tau_n \stackrel{\text{def}}{=} \frac{60}{\kappa(n)}(\tau_0 - 1) + 1$. Let $\mathcal{C}'$ be any cusp region of X_ϕ as in (3.11). Using the change of coordinates $r = e^\tau$ we view $\mathcal{C}'$ as

$$(0, \infty)_\tau \times \mathbb{R}/m\mathbb{Z},$$

with the metric $(d\tau)^2 + e^{-2\tau}(dx)^2$ where x is the coordinate in $\mathbb{R}/m\mathbb{Z}$. In these coordinates, one can calculate directly from the formula for the metric that

$$\|\nabla \chi^{+}_{\mathcal{C},\phi}\|(\tau, x) = |[\chi^{+}_{\mathcal{C}}]'(\tau)|,$$

and

$$|\Delta \chi^{+}_{\mathcal{C},\phi}|(\tau, x) = |[\chi^{+}_{\mathcal{C}}]''(\tau) - [\chi^{+}_{\mathcal{C}}]'(\tau)|.$$

It follows from (3.12) that

$$\|\nabla \chi^{+}_{\mathcal{C},n}\|_{\infty} = \sup_{[0,\infty)} |(\chi^{+}_{\mathcal{C},n})'| \leqslant \frac{\kappa(n)}{30},$$

and

$$\|\Delta\chi^+_{\mathcal{C},n}\|_\infty = \sup_{[0,\infty)} |(\chi^+_{\mathcal{C},n})'' - (\chi^+_{\mathcal{C},n})'| \leqslant \frac{\kappa(n)}{30}.$$

If one chooses $\chi^-_{\mathcal{C},n}$ to be a function with $\chi^-_{\mathcal{C}}(\tau) \equiv 0$ for $\tau \leq \tau_n$ and $\chi^-_{\mathcal{C}}(\tau) \equiv 1$ for $\tau \geq 2\tau_n$, (3.10) is satisfied and the lemma is proved. □

Let $\mathcal{C}_\phi$ denote the subset of X_ϕ that covers $\mathcal{C}$. We extend $\mathcal{C}$ to the parabolic cylinder

$$\tilde{\mathcal{C}} \overset{\text{def}}{=} (0,\infty) \times S^1,$$

with the same metric (3.9), and let $\tilde{\mathcal{C}}_\phi$ be the corresponding extension of $\mathcal{C}_\phi$. Let $H^2\left(\tilde{\mathcal{C}}_\phi\right)$ denote the completion of $C_c^\infty\left(\tilde{\mathcal{C}}_\phi\right)$ with respect to the given norm

$$\|f\|^2_{H^2} \overset{\text{def}}{=} \|f\|^2_{L^2} + \|\Delta f\|^2_{L^2}.$$

The Laplacian $\Delta = \Delta_{\tilde{\mathcal{C}}_\phi}$ extends uniquely from $C_c^\infty\left(\tilde{\mathcal{C}}_\phi\right)$ to a self-adjoint unbounded operator on $L^2\left(\tilde{\mathcal{C}}_\phi\right)$ with domain $H^2\left(\tilde{\mathcal{C}}_\phi\right)$.

Lemma 3.8. *For any $f \in H^2(\tilde{\mathcal{C}}_\phi)$, we have $\langle \Delta f, f\rangle \geq \frac{1}{4}\|f\|^2$.*

Proof. It suffices to prove this for $\tilde{\mathcal{C}}_\phi$ replaced by $(0,\infty)\times\mathbb{R}/m\mathbb{Z}$ with the metric (3.9) i.e. with only one connected component. Then changing coordinates to τ we are working in the region $(-\infty,\infty)\times\mathbb{R}/m\mathbb{Z}$ with the metric $(d\tau)^2 + e^{-2\tau}(dx)^2$. The corresponding volume form is $e^{-\tau}d\tau\wedge dx$ and the Laplacian is given by $\Delta = -e^\tau\frac{\partial}{\partial\tau}e^{-\tau}\frac{\partial}{\partial\tau} - e^{2\tau}\frac{\partial^2}{\partial\theta^2}$. Now suppose $f \in C_c^\infty\left((-\infty,\infty)\times\mathbb{R}/m\mathbb{Z}\right)$. We calculate

$$e^{-\tau/2}\Delta e^{\tau/2} = -\frac{\partial^2}{\partial\tau^2} + \frac{1}{4} - e^{2\tau}\frac{\partial^2}{\partial\theta^2},$$

so if $g = e^{-\tau/2}f \in C_c^\infty\left((-\infty,\infty)\times\mathbb{R}/m\mathbb{Z}\right)$ we have

$$\begin{aligned}\int \Delta[f]\bar{f}e^{-\tau}d\tau\wedge dx &= \int_{-\infty}^{\infty}\int_0^n [e^{-\tau/2}\Delta e^{\tau/2}]\,(g)\,\bar{g}d\tau\wedge dx\\ &\geq \frac{1}{4}\int_{-\infty}^{\infty}\int_0^n g\bar{g}d\tau\wedge dx = \frac{1}{4}\int_{-\infty}^{\infty}\int_0^n f\bar{f}e^{-\tau}d\tau\wedge dx.\end{aligned}$$

The inequality here used integrating by parts. The inequality obtained now extends to $H^2\left(\tilde{\mathcal{C}}_\phi\right)$ by density of $C_c^\infty(\tilde{\mathcal{C}}_\phi)$ and continuity of $\langle \Delta f, f\rangle$. $\square$

Lemma 3.8 implies that the resolvent operator

$$R_{\tilde{\mathcal{C}}_\phi}(s) \stackrel{\text{def}}{=} (\Delta - s(1-s))^{-1} : L^2\left(\tilde{\mathcal{C}}_\phi\right) \to H^2\left(\tilde{\mathcal{C}}_\phi\right),$$

is a holomorphic family of bounded operators in $\mathrm{Re}(s) > \frac{1}{2}$, each a bijection to their image. This gives an a priori bound for the resolvent: using

$$(\Delta - s(1-s))\, R_{\tilde{\mathcal{C}}_\phi}(s) f = f,$$

and Lemma 3.8 we obtain that for $f \in L^2\left(\tilde{\mathcal{C}}_\phi\right)$ and $s \in (\frac{1}{2}, \infty)$

$$\|R_{\tilde{\mathcal{C}}_\phi}(s) f\|_{L^2} \le \left(\frac{1}{4} - s(1-s)\right)^{-1} \|f\|_{L^2}. \tag{3.13}$$

Since

$$\Delta R_{\tilde{\mathcal{C}}_\phi}(s) f = f + s(1-s) R_{\tilde{\mathcal{C}}_\phi}(s) f,$$

we obtain for $s \in (\frac{1}{2}, \infty)$

$$\begin{aligned}
\|\Delta R_{\tilde{\mathcal{C}}_\phi}(s) f\|_{L^2} &\le \|f\|_{L^2} + s(1-s)\|R_{\tilde{\mathcal{C}}_\phi}(s) f\|_{L^2} \\
&\le \left(1 + \frac{s(1-s)}{\frac{1}{4} - s(1-s)}\right) \|f\|_{L^2} \\
&= \frac{1}{1 - 4s(1-s)} \|f\|_{L^2}.
\end{aligned} \tag{3.14}$$

We now define the cusp parametrix as

$$\mathbb{M}_\phi^{\text{cusp}}(s) \stackrel{\text{def}}{=} \chi^+_{\mathcal{C},n,\phi} R_{\tilde{\mathcal{C}}_\phi}(s) \chi^-_{\mathcal{C},n,\phi}. \tag{3.15}$$

Here,

- (multiplication by) $\chi^-_{\mathcal{C},n,\phi}$ is viewed as an operator from $L^2(X_\phi)$ to $L^2(\tilde{\mathcal{C}}_\phi)$ by mapping first to $L^2(\mathcal{C}_\phi)$ and then extending by zero. This is a bounded linear operator.

- $R_{\tilde{\mathcal{C}}_\phi}(s)$ is a bounded operator from $L^2(\tilde{\mathcal{C}}_\phi)$ to $H^2(\tilde{\mathcal{C}}_\phi)$.
- (multiplication by) $\chi^+_{\mathcal{C},n,\phi}$ is viewed as a operator from $H^2(\tilde{\mathcal{C}}_\phi)$ to $H^2(X_\phi)$, using that $\chi^+_{\mathcal{C},n,\phi}$ localizes to $\mathcal{C}_\phi$ and then extending by zero. This operator is bounded because derivatives of $\chi^+_{\mathcal{C},n,\phi}$ are bounded and compactly supported.

Hence

$$\mathbb{M}^{\text{cusp}}_\phi(s) : L^2(X_\phi) \to H^2(X_\phi)$$

is a bounded operator.

The covering map $\mathcal{C}_\phi \to \mathcal{C}$ extends in an obvious way to a covering map $\tilde{\mathcal{C}}_\phi \to \tilde{\mathcal{C}}$ that intertwines the two Laplacian operators. This, together with the fact that multiplication by $\chi^-_{\mathcal{C},n,\phi}$ and $\chi^+_{\mathcal{C},n,\phi}$ leave invariant the subspaces of functions lifted through the covering map, one sees that

$$\mathbb{M}^{\text{cusp}}_\phi(s)\left(L^2_{\text{new}}\left(X_\phi\right)\right) \subset H^2_{\text{new}}\left(X_\phi\right).$$

Because $\chi^+_{\mathcal{C},n,\phi}\chi^-_{\mathcal{C},n,\phi} = \chi^-_{\mathcal{C},n,\phi}$,

$$\left(\Delta - s(1-s)\right)\mathbb{M}^{\text{cusp}}_\phi(s) = \chi^-_{\mathcal{C},n,\phi} + \left[\Delta, \chi^+_{\mathcal{C},n,\phi}\right] R_{\tilde{\mathcal{C}}_\phi}(s)\chi^-_{\mathcal{C},n,\phi} = \chi^-_{\mathcal{C},n,\phi} + \mathbb{L}^{\text{cusp}}_\phi(s), \quad (3.16)$$

where

$$\mathbb{L}^{\text{cusp}}_\phi(s) \stackrel{\text{def}}{=} \left[\Delta, \chi^+_{\mathcal{C},n,\phi}\right] R_{\tilde{\mathcal{C}}_\phi}(s)\chi^-_{\mathcal{C},n,\phi},$$

and $[A,B] \stackrel{\text{def}}{=} AB - BA$ denotes the commutator of linear maps. Here again we view $\chi^-_{\mathcal{C},n,\phi}$ and $R_{\tilde{\mathcal{C}}_\phi}(s)$ as above, and $\left[\Delta, \chi^+_{\mathcal{C},n,\phi}\right] : H^2(\tilde{\mathcal{C}}_\phi) \to L^2(X_\phi)$. This means that $\mathbb{L}^{\text{cusp}}_\phi(s)$ is an operator on $L^2(X_\phi)$. By similar arguments to before, using that $[\Delta, \chi^+_{\mathcal{C},n,\phi}]$ only involves radial derivatives (since $\chi^+_{\mathcal{C},n,\phi}$ is radial), we obtain

$$\mathbb{L}^{\text{cusp}}_\phi(s)\left(L^2_{\text{new}}(X_\phi)\right) \subset L^2_{\text{new}}(X_\phi).$$

Lemma 3.9. *For $s \in \left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$, the operator $\mathbb{L}^{\text{cusp}}_\phi(s)$ is a self-adjoint, bounded operator on $L^2\left(X_\phi\right)$ with operator norm*

$$\|\mathbb{L}^{\text{cusp}}_\phi(s)\|_{L^2} \leq \frac{1}{8}.$$

Proof. As an operator on $H^2(\tilde{\mathcal{C}}_\phi)$

$$\left[\Delta, \chi^+_{\mathcal{C},n,\phi}\right] = \left(\Delta\chi^+_{\mathcal{C},n,\phi}\right) - 2\left(\nabla\chi^+_{\mathcal{C},n,\phi}\right) \cdot \nabla.$$

The first summand is a multiplication operator; for $f \in H^2\left(\tilde{\mathcal{C}}_\phi\right)$ we have

$$\|(\Delta\chi^+_{\mathcal{C},n,\phi})f\|_{L^2} \leq \|(\Delta\chi^+_{\mathcal{C},n,\phi})\|_\infty \|f\|_{L^2} \tag{3.17}$$

and by Schwarz inequality if $\|f\|_{H^2} \leq 1$ then

$$\begin{aligned}
\|(\nabla\chi^+_{\mathcal{C},n,\phi}) \cdot \nabla f\|_{L^2} &\leq \|\nabla\chi^+_{\mathcal{C},n,\phi}\|_\infty \|\nabla f\|_{L^2} \\
&= \|\nabla\chi^+_{\mathcal{C},n,\phi}\|_\infty \langle \Delta f, f\rangle^{\frac{1}{2}} \leq \|\nabla\chi^+_{\mathcal{C},n,\phi}\|_\infty \|\Delta f\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}} \\
&\leq \|\nabla\chi^+_{\mathcal{C},n,\phi}\|_\infty.
\end{aligned} \tag{3.18}$$

The two estimates (3.17), (3.18) hence show that $[\Delta, \chi^+_{\mathcal{C},n,\phi}]$ has norm bounded by $\|(\Delta\chi^+_{\mathcal{C},n,\phi})\|_\infty + 2\|\nabla\chi^+_{\mathcal{C},n,\phi}\|_\infty$ as a map $H^2\left(\tilde{\mathcal{C}}_\phi\right) \to L^2\left(X_\phi\right)$. Since multiplication by $\chi^-_{\mathcal{C},n,\phi}$ has norm ≤ 1 from L^2 to L^2, and $R_{\tilde{\mathcal{C}}_\phi}(s)$ has norm from $L^2(\tilde{\mathcal{C}}_\phi)$ to $H^2(\tilde{\mathcal{C}}_\phi)$ bounded by $\frac{5}{4\kappa(n)}$ for $s \in \left[\frac{1}{2} + \sqrt{\kappa\left(n\right)}, 1\right]$ by (3.13) and (3.14). In particular, for $s \in \left[\frac{1}{2} + \sqrt{\kappa\left(n\right)}, 1\right]$,

$$\|\mathbb{L}^{\text{cusp}}_\phi(s)\|_{L^2_{\text{new}}(X_\phi)} \leq \left(\|(\Delta\chi^+_{\mathcal{C},n,\phi})\|_\infty + 2\|\nabla\chi^+_{\mathcal{C},n,\phi}\|_\infty\right) \cdot \frac{5}{4\kappa\left(n\right)} \leqslant \frac{1}{8}, \tag{3.19}$$

by applying Lemma 3.7. $\square$

3.5 Operators on $\mathbb{H}$

3.5.1 Resolvent on $\mathbb{H}$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}$, let

$$R_{\mathbb{H}}(s) : L^2\left(\mathbb{H}\right) \to L^2(\mathbb{H}), R_{\mathbb{H}}(s) \overset{\text{def}}{=} \left(\Delta_{\mathbb{H}} - s(1-s)\right)^{-1},$$

be the resolvent on the upper half plane. Letting $r(x,y) \overset{\text{def}}{=} d_{\mathbb{H}}(x,y)$, $R_{\mathbb{H}}(s)$ is an integral operator with radial kernel $R_{\mathbb{H}}(s;r)$ given by

$$R_{\mathbb{H}}(s;r) \overset{\text{def}}{=} \frac{1}{4\pi}\int_0^1 \frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^s}dt. \tag{3.20}$$

For $t \in (0,1)$, we have

$$\begin{aligned}
\frac{\partial}{\partial r}\frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^s} &= -s\sinh\left(\frac{r}{2}\right)\cosh\left(\frac{r}{2}\right)\frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^{s+1}},\\
\frac{\partial}{\partial s}\frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^s} &= \log\left(\frac{t(1-t)}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)}\right)\frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^s},\\
\frac{\partial^2}{\partial s\partial r}\frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^s} &= -\sinh\left(\frac{r}{2}\right)\cosh\left(\frac{r}{2}\right)\frac{t^{s-1}(1-t)^{s-1}}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)^{s+1}}\cdot\left[1+s\log\left(\frac{t(1-t)}{\left(\cosh^2\left(\frac{r}{2}\right)-t\right)}\right)\right].
\end{aligned}$$

Each of these are smooth in $(s,r,t) \in [\frac{1}{2},1]\times[1,\infty)\times(0,1)$. Because for s,r in a fixed compact set of $\left[\frac{1}{2},1\right]\times[1,\infty)$, these all have absolute values bounded above by integrable functions of $t \in (0,1)$, we can interchange derivatives and integrals to bound $R_{\mathbb{H}}$. Firstly, there is a constant $C>0$ such that for $r_0 \geq 1$ and $s \in [\frac{1}{2},1]$ we have

$$\left|R_{\mathbb{H}}(s;r)\right|, \left|\frac{\partial R_{\mathbb{H}}}{\partial r}(s;r)\right| \leq Ce^{-sr_0}. \tag{3.21}$$

Secondly, there is a constant $C'>0$ such that for any $T>1$ and $r \in [1,T+1]$ and all $s \in [\frac{1}{2},1]$,

$$\left|\frac{\partial R_{\mathbb{H}}}{\partial s}(s;r)\right|, \left|\frac{\partial^2 R_{\mathbb{H}}}{\partial s\partial r}(s;r_0)\right| \leq C'. \tag{3.22}$$

3.5.2 Integral operators

If $k_0 : [0,\infty) \to \mathbb{R}$ is smooth and compactly supported, which will suffice here, then one can construct a kernel

$$k(x,y) \overset{\text{def}}{=} k_0(d_{\mathbb{H}}(x,y))$$

with corresponding integral operator $C^\infty(\mathbb{H}) \to C^\infty(\mathbb{H})$

$$K[f](x) \stackrel{\text{def}}{=} \int_{y\in\mathbb{H}} k(x,y)f(y)d\mathbb{H}(y)$$

where $d\mathbb{H}$ is the hyperbolic area form on $\mathbb{H}$. Such an operator commutes with the Laplacian on $\mathbb{H}$ and hence preserves its generalized eigenspaces. If $f \in C^\infty(\mathbb{H})$ is a generalized eigenfunction of Δ with eigenvalue $\frac{1}{4}+\xi^2$, $\xi \geq 0$, then by [Be16, Thm. 3.7, Lemma 3.9] (cf. also Selberg's original article [Se56])

$$K[f] = h(\xi)f$$

where

$$h(\xi) = \sqrt{2}\int_{-\infty}^{\infty} e^{i\xi u}\int_{|u|}^{\infty} \frac{k_0(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho)-\cosh(u)}}d\rho du.$$

By our assumptions on k_0 the integral above is convergent. Since $L^2(\mathbb{H})$ has a generalized eigenbasis of C^∞ eigenfunctions of the Laplacian, by Borel functional calculus K extends from e.g. $C_c^\infty(\mathbb{H})$ to a self-adjoint operator on $L^2(\mathbb{H})$ with operator norm

$$\|K\|_{L^2(\mathbb{H})} = \sup_{\xi\geq 0}|h(\xi)|. \tag{3.23}$$

3.5.3 Interior parametrix on $\mathbb{H}$

Let $\chi_0 : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\chi_0(t) = \begin{cases} 1 & \text{if } t \leqslant 0, \\ 0 & \text{if } t \geqslant 1. \end{cases}.$$

For $T > 0$, we define a smooth cutoff function χ_T by

$$\chi_T(t) \stackrel{\text{def}}{=} \chi_0(t-T).$$

We then define the operator $R_{\mathbb{H}}^{(T)}(s) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ to be the integral operator with radial kernel

$$R_{\mathbb{H}}^{(T)}(s;r) \stackrel{\text{def}}{=} \chi_T(r) R_{\mathbb{H}}(s;r).$$

In radial coordinates the Laplacian on $\mathbb{H}$ is given by [Bo16, pg. 50]

$$-\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r}\frac{\partial}{\partial r} - \frac{1}{\sinh^2 r}\frac{\partial^2}{\partial \theta^2}.$$

We now perform the following calculation, writing Δ_x for the Laplacian acting on coordinate x:

$$\begin{aligned} [\Delta_x - s(1-s)]\, R_{\mathbb{H}}^{(T)}(s;r) &= [\Delta_x - s(1-s)]\,(\chi_T(r) R_{\mathbb{H}}(s;r)) \\ &= \left[-\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r}\frac{\partial}{\partial r}, \chi_T\right] R_{\mathbb{H}}(s;r) + \delta_{r=0} \end{aligned} \tag{3.24}$$

which is understood in a distributional sense. We further calculate

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{\tanh r}\frac{\partial}{\partial r}, \chi_T\right] = -\frac{\partial^2}{\partial r^2}[\chi_T] - 2\frac{\partial}{\partial r}[\chi_T]\frac{\partial}{\partial r} - \frac{1}{\tanh r}\frac{\partial}{\partial r}[\chi_T]. \tag{3.25}$$

Combining (3.24) and (3.25) we expect an identity of operators

$$[\Delta - s(1-s)]\, R_{\mathbb{H}}^{(T)}(s) = 1 + \mathbb{L}_{\mathbb{H}}^{(T)}(s) \tag{3.26}$$

where we define $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$ to be the integral operator with radial kernel

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s;r) \stackrel{\text{def}}{=} \left(-\frac{\partial^2}{\partial r^2}[\chi_T] - \frac{1}{\tanh r}\frac{\partial}{\partial r}[\chi_T]\right) R_{\mathbb{H}}(s;r) - 2\frac{\partial}{\partial r}[\chi_T]\frac{\partial R_{\mathbb{H}}}{\partial r}(s;r). \tag{3.27}$$

The identity (3.26) will be established in Lemma 3.12 below. The following estimates can be easily obtained from (3.21) and (3.22).

Lemma 3.10. *We have*

1. *For $T > 0$ and $s \in [\frac{1}{2}, 1]$, $\mathbb{L}^{(T)}(s; \bullet)$ is smooth and supported in $[T, T+1]$.*

2. *There is a constant $C > 0$ such that for any $T > 0$ and $s \in [\frac{1}{2}, 1]$ we have*

$$|\mathbb{L}_{\mathbb{H}}^{(T)}(s; r_0)| \leq Ce^{-sr_0}$$

3. *There is a constant $C > 0$ such that for any $T > 0$, $s \in \left[\frac{1}{2}, 1\right]$ and $r_0 \in [T, T+1]$*

$$\left| \frac{\partial \mathbb{L}_{\mathbb{H}}^{(T)}}{\partial s}(s_0; r_0) \right| \leq C.$$

We can now bound bound the operator norm of $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$.

Lemma 3.11. *There is a constant $C > 0$ such that for any $T > 0$ and $s \in [\frac{1}{2}, 1]$ the operator $\mathbb{L}_{\mathbb{H}}^{(T)}(s)$ extends to a bounded operator on $L^2(\mathbb{H})$ with operator norm*

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})} \leqslant CTe^{\left(\frac{1}{2}-s\right)T}.$$

Proof. We apply (3.23) which tells us

$$\begin{aligned}
\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2} &= \sup_{\xi \geq 0} \left| \sqrt{2} \int_{-\infty}^{\infty} e^{i\xi u} \int_{|u|}^{\infty} \frac{\mathbb{L}_{\mathbb{H}}^{(T)}(s; \rho) \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \right| \\
&\leqslant \sqrt{2} \int_{-\infty}^{\infty} \int_{|u|}^{\infty} \frac{|\mathbb{L}_{\mathbb{H}}^{(T)}(s; \rho)| \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \\
&\leqslant 2\sqrt{2}C \int_{0}^{T+1} \int_{\max(|u|,T)}^{T+1} \frac{e^{-s\rho} \sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \\
&\leqslant C' e^{-sT} \int_{0}^{T+1} \int_{\max(|u|,T)}^{T+1} \frac{\sinh(\rho)}{\sqrt{\cosh(\rho) - \cosh(u)}} d\rho du \\
&= C' e^{-sT} \int_{0}^{T+1} \int_{\cosh \max(|u|,T)}^{\cosh(T+1)} \frac{dy}{\sqrt{y - \cosh(u)}} du \\
&= C'' e^{-sT} \int_{0}^{T+1} \Big[\sqrt{\cosh(T+1) - \cosh u} \\
&\qquad - \sqrt{\cosh \max(|u|, T) - \cosh |u|} \Big] du \\
&\leqslant C''' T e^{\left(\frac{1}{2}-s\right)T}
\end{aligned}$$

where the third inequality used Lemma 3.10. ∎

We will need to ensure that, for example,

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})} < \frac{1}{5},$$

for $s \in \left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$. This means we have to take $T = T(n)$ such that,

$$Te^{-T\sqrt{\kappa(n)}} < \frac{1}{5} \tag{3.28}$$

for all sufficiently large n. We will eventually take $\kappa(n) = \frac{4(\log T)^2}{T^2}$ which ensures (3.28).

The following lemma shows that smoothly cutting off $R_{\mathbb{H}}(s)$ at radius T does not significantly affect its mapping properties.

Lemma 3.12. *For any $T > 0$ and $s \in [\frac{1}{2}, 1]$, for any compact $K \subset \mathbb{H}$, there is $C = C(s, K, T) > 0$ such that:*

1. *For any $f \in C_c^\infty(\mathbb{H})$ with $\mathrm{supp}(f) \subset K$ we have $R_{\mathbb{H}}^{(T)}(s)f \in H^2(\mathbb{H})$ and*

$$\|R_{\mathbb{H}}^{(T)}(s)f\|_{H^2} \leq C(s, K, T)\|f\|_{L^2}.$$

2. *Furthermore, with f as above*

$$(\Delta - s(1-s))R_{\mathbb{H}}^{(T)}(s)[f] = f + \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f]$$

in the sense of equivalence of L^2 functions.

Proof. Suppose that compact K is given and $f \in C_c^\infty(\mathbb{H})$ with $\mathrm{supp}(f) \subset K$. For $y \in K$ we have $R_{\mathbb{H}}^{(T)}(s; x, y) = 0$ unless

$$x \in K'(T, K) \stackrel{\text{def}}{=} \{\, x \,:\, d(x, K) \leq T + 1 \,\}$$

with K' compact. Therefore using the usual Hilbert-Schmidt inequality we obtain

$$\begin{aligned}
&\|R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})}^2 \\
&= \int_{x\in K'} \left| \int_{y\in K} R_{\mathbb{H}}^{(T)}(s;x,y) f(y) d\mathbb{H}(y) \right|^2 d\mathbb{H}(x) \\
&\leq \int_{x\in K'} \left(\int_{y\in K} R_{\mathbb{H}}^{(T)}(s;x,y)^2 d\mathbb{H}(y) \right) \left(\int_{y\in K} |f(y)|^2 d\mathbb{H}(y) \right) d\mathbb{H}(x). \qquad (3.29)
\end{aligned}$$

Recall that we write $r = d_{\mathbb{H}}(x,y)$, hence the inner integral can be written in polar coordinates as

$$\begin{aligned}
\int_{y\in K} R_{\mathbb{H}}^{(T)}(s;x,y)^2 d\mathbb{H}(y) &= \int_0^{2\pi} \int_0^{\infty} R_{\mathbb{H}}^{(T)}(s;r)^2 \sinh r \, dr \, d\theta \\
&\leq 2\pi \int_0^{M} R_{\mathbb{H}}^{(T)}(s;r)^2 \sinh r \, dr \qquad (3.30)
\end{aligned}$$

for $M = M(K,T)$. Because $\chi_T \equiv 1$ near 0, the type of singularity that $R_{\mathbb{H},n}^{(T)}(s;r)$ has at $r=0$ is exactly the same as the type of singularity of $R_{\mathbb{H}}(s;r)$ near $r=0$; namely by [Bo16, (4.2)]

$$R_{\mathbb{H}}^{(T)}(s;r) = -\frac{1}{4\pi} \log\left(\frac{r}{2}\right) + O(1) \qquad (3.31)$$

as $r \to 0$. The function $R_{\mathbb{H}}^{(T)}(s;r)$ is smooth away from 0. Hence, since $\left(\log\left(\frac{r}{2}\right)\right)^2 \sinh r \to 0$ as $r \to 0$, $R_{\mathbb{H}}^{(T)}(s;r)$ is in particular square integrable on $[0,M]$. This gives from (3.29)

$$\|R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})}^2 \leq \int_{x\in K'} C(s,K,T) \|f\|_{L^2(\mathbb{H})}^2 d\mathbb{H}(x) \leq C'(s,K,T) \|f\|_{L^2(\mathbb{H})}^2. \qquad (3.32)$$

We now aim for a bound on $\|\Delta R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})}^2$ so as to prove $R_{\mathbb{H}}^{(T)}(s)[f] \in H^2(\mathbb{H})$ and bound its H^2-norm.

Let $g \in C_c^{\infty}(\mathbb{H})$ be a test function and $f \in C_c^{\infty}(\mathbb{H})$ with support as above. Consider

$$\langle R_{\mathbb{H}}^{(T)}(s)[f], \Delta g \rangle = \int_{x\in\mathbb{H}} \left(\int_{y\in\mathbb{H}} R_{\mathbb{H}}^{(T)}(s;x,y) f(y) d\mathbb{H}(y) \right) \Delta \overline{g}(x) d\mathbb{H}(x).$$

Because f and g are compactly supported and the singularity of $R_{\mathbb{H}}^{(T)}(x,y)$ is locally L^1

using (3.31), we can use Fubini to get

$$\langle R_{\mathbb{H}}^{(T)}(s)[f], \Delta g\rangle = \int_{y\in\mathbb{H}} \left(\int_{x\in\mathbb{H}} R_{\mathbb{H}}^{(T)}(s;x,y)\Delta\bar{g}(x)d\mathbb{H}(x) \right) f(y)d\mathbb{H}(y). \tag{3.33}$$

We use hyperbolic polar coordinates for the inner integral, writing $r = d(x,y)$ and θ for polar angle, $G_y(r,\theta) \stackrel{\text{def}}{=} \bar{g}(x)$, and the inner integral is understood as an improper integral as follows:

$$\begin{aligned}
&\int_{x\in\mathbb{H}} R_{\mathbb{H}}^{(T)}(s;x,y)\Delta\bar{g}(x)d\mathbb{H}(x) \\
=& -\lim_{\epsilon\to 0}\int_{\epsilon}^{\infty}\int_0^{2\pi} R_{\mathbb{H}}^{(T)}(s;\tilde{r}) \left(\frac{\partial}{\partial r}\left(\sinh r \frac{\partial G_y}{\partial r}\right)(\tilde{r},\tilde{\theta}) + \frac{1}{(\sinh r)^2}\frac{\partial^2 G_y}{\partial\theta^2}(\tilde{r},\tilde{\theta}) \right) d\tilde{\theta}d\tilde{r} \\
=& -\lim_{\epsilon\to 0}\int_{\epsilon}^{\infty}\int_0^{2\pi} R_{\mathbb{H}}^{(T)}(s;\tilde{r}) \left(\frac{\partial}{\partial r}\left(\sinh r \frac{\partial G_y}{\partial r}\right)(\tilde{r},\tilde{\theta}) \right) d\tilde{\theta}d\tilde{r} \\
=& -\lim_{\epsilon\to 0}\int_0^{2\pi}\int_{\epsilon}^{\infty} R_{\mathbb{H}}^{(T)}(s;\tilde{r}) \left(\frac{\partial}{\partial r}\left(\sinh r \frac{\partial G_y}{\partial r}\right)(\tilde{r},\tilde{\theta}) \right) d\tilde{r}d\tilde{\theta} \\
=& \lim_{\epsilon\to 0} R_{\mathbb{H}}^{(T)}(s;\epsilon)\sinh\epsilon \int_0^{2\pi} \frac{\partial G_y}{\partial r}(\epsilon,\tilde{\theta})d\tilde{\theta} \\
&\quad + \lim_{\epsilon\to 0}\int_0^{2\pi}\int_{\epsilon}^{\infty} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s;\tilde{r})\sinh\tilde{r}\frac{\partial G_y}{\partial r}(\tilde{r},\tilde{\theta})d\tilde{r}d\tilde{\theta} \\
=& \lim_{\epsilon\to 0}\int_0^{2\pi}\int_{\epsilon}^{\infty} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s;\tilde{r})\sinh\tilde{r}\frac{\partial G_y}{\partial r}(\tilde{r},\tilde{\theta})d\tilde{r}d\tilde{\theta}
\end{aligned} \tag{3.34}$$

where the last equality used (3.31) with smoothness of G_y. Now a similar calculation gives

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^{2\pi}\int_{\epsilon}^{\infty} \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s;\tilde{r})\sinh r\frac{\partial G_y}{\partial r}(\tilde{r},\tilde{\theta})d\tilde{r}d\tilde{\theta} \\
=& -\lim_{\epsilon\to 0}\frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s;\epsilon)\sinh\epsilon\int_0^{2\pi} G_y(\epsilon,\tilde{\theta})d\tilde{\theta} \\
&- \lim_{\epsilon\to 0}\int_0^{2\pi}\int_{\epsilon}^{\infty}\frac{\partial}{\partial r}\left(\sinh r\frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}\right)(s;\tilde{r})G_y(\tilde{r},\tilde{\theta})d\tilde{r}d\tilde{\theta} \\
=& \bar{g}(y) - \lim_{\epsilon\to 0}\int_0^{2\pi}\int_{\epsilon}^{\infty}\frac{\partial}{\partial r}\left(\sinh r\frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}\right)(s;\tilde{r})G_y(\tilde{r},\tilde{\theta})d\tilde{r}d\tilde{\theta}.
\end{aligned} \tag{3.35}$$

The second equality used [Bo16, pg. 66]

$$\frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}(s;\epsilon) = -\frac{1}{2\pi\epsilon} + O(1)$$

as $\epsilon \to 0$ with $\lim_{\epsilon\to 0} G_y(\epsilon, \tilde{\theta}) = \bar{g}(y)$. Now we note

$$-\frac{1}{\sinh r}\frac{\partial}{\partial r}\left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}\right)(s;\tilde{r}) = \Delta R_{\mathbb{H}}^{(T)}(s;\tilde{r})$$

and so using (3.24), (3.25) and (3.27) we get, for $\tilde{r} > 0$,

$$-\frac{1}{\sinh r}\frac{\partial}{\partial r}\left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}\right)(s;\tilde{r}) = s(1-s)R_{\mathbb{H}}^{(T)}(s;\tilde{r}) + \mathbb{L}_{\mathbb{H}}^{(T)}(s;\tilde{r}).$$

Therefore

$$\begin{aligned}
&- \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^\infty \frac{\partial}{\partial r}\left(\sinh r \frac{\partial R_{\mathbb{H}}^{(T)}}{\partial r}\right)(s;\tilde{r})G_y(\tilde{r},\tilde{\theta})d\tilde{r}d\tilde{\theta} \\
&= \lim_{\epsilon\to 0}\int_{d(x,y)>\epsilon}\left(s(1-s)R_{\mathbb{H}}^{(T)}(s;\tilde{r}) + \mathbb{L}_{\mathbb{H}}^{(T)}(s;d(x,y))\right)\bar{g}(x)d\mathbb{H}(x) \\
&= \int_{x\in\mathbb{H}}\left(s(1-s)R_{\mathbb{H}}^{(T)}(s;d(x,y)) + \mathbb{L}_{\mathbb{H}}^{(T)}(s;d(x,y))\right)\bar{g}(x)d\mathbb{H}(x) \qquad (3.36)
\end{aligned}$$

and this last integral is easily seen to converge by working in polar coordinates centered at y and using $g \in C_c^\infty(\mathbb{H})$ and (3.31).

Now combining (3.34), (3.35), and (3.36) gives, for (3.33),

$$\begin{aligned}
&\langle R_{\mathbb{H}}^{(T)}(s)[f], \Delta g\rangle \\
&= \int_{y\in\mathbb{H}} f(y)\bar{g}(y)d\mathbb{H}(y) \\
&\quad + \int_{y\in\mathbb{H}}\left(\int_{x\in\mathbb{H}}\left(s(1-s)R_{\mathbb{H}}^{(T)}(s;d(x,y)) + \mathbb{L}_{\mathbb{H}}^{(T)}(s;d(x,y))\right)\bar{g}(x)d\mathbb{H}(x)\right)f(y)d\mathbb{H}(y) \\
&= \langle f, g\rangle + \langle s(1-s)R_{\mathbb{H}}^{(T)}(s)[f], g\rangle + \langle \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f], g\rangle.
\end{aligned}$$

Note that by (3.32) and Lemma 3.11 all functions above are in $L^2(\mathbb{H})$. This identity now clearly extends to any $g \in H^2(\mathbb{H})$ and now self-adjointness of $\Delta_{\mathbb{H}}$ on $H^2(\mathbb{H})$ gives that $R_{\mathbb{H}}^{(T)}(s)[f] \in H^2(\mathbb{H})$ and moreover

$$(\Delta - s(1-s))R_{\mathbb{H}}^{(T)}(s)[f] = f + \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f]$$

in the sense of elements of $L^2(\mathbb{H})$. This proves the second part of the lemma.

We now rewrite this identity as

$$\Delta R_{\mathbb{H}}^{(T)}(s)[f] = f + s(1-s)R_{\mathbb{H}}^{(T)}(s)[f] + \mathbb{L}_{\mathbb{H}}^{(T)}(s)[f]$$

and using Lemma 3.11 and (3.32) now gives

$$\|\Delta R_{\mathbb{H}}^{(T)}(s)[f]\|_{L^2(\mathbb{H})} \leq c(s,K,T)\|f\|_{L^2(\mathbb{H})}.$$

Combining this with (3.32), this proves the first part of the lemma. □

3.6 Interior parametrix

To build our interior parametrix, we define,

$$R_{\mathbb{H},n}^{(T)}(s;x,y) \stackrel{\text{def}}{=} R_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_n^0},$$
$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s;x,y) \stackrel{\text{def}}{=} \mathbb{L}_{\mathbb{H}}^{(T)}(s;x,y)\mathrm{Id}_{V_n^0},$$

and $R_{\mathbb{H},n}^{(T)}(s), \mathbb{L}_{\mathbb{H},n}^{(T)}(s)$ as the corresponding integral operators. The relevant properties are summarized in the following Lemma.

Lemma 3.13. *For all $s \in [\frac{1}{2}, 1]$,*

1. *The integral operator $R_{\mathbb{H},n}^{(T)}(s)(1-\chi_{\mathcal{C},n}^{-})$ is well-defined on $C_{c,\phi}^{\infty}(\mathbb{H}; V_n^0)$ and extends to a bounded operator*

$$R_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right) : L_{\phi}^{2}\left(\mathbb{H}; V_n^0\right) \to H_{\phi}^{2}\left(\mathbb{H}; V_n^0\right).$$

2. *The integral operator $\mathbb{L}_{\mathbb{H},P,n}^{(T)}(s)(1-\chi_{\mathcal{C},n}^{-})$ is well-defined on $C_{c,\phi}^{\infty}\left(\mathbb{H}; V_n^0\right)$ and and extends to a bounded operator on $L_{\phi}^{2}(\mathbb{H}; V_n^0)$.*

3. *We have*

$$\left[\Delta - s(1-s)\right] R_{\mathbb{H},P,\phi}^{(T)}(s)(1-\chi_{\mathcal{C},n}^{-}) = (1-\chi_{\mathcal{C},n}^{-}) + \mathbb{L}_{\mathbb{H},P,n}^{(T)}(s)(1-\chi_{\mathcal{C},n}^{-}) \qquad (3.37)$$

as an identity of operators on $L_{\phi}^{2}(\mathbb{H}; V_n^0)$.

Proof. Suppose first that $f \in C^{\infty}_{c,\phi}(\mathbb{H}; V_n^0)$ (i.e. automorphic, smooth, and compactly supported modulo Γ). We have for $x \in F$

$$R^{(T)}_{\mathbb{H},n}(s)(1-\chi^{-}_{\mathcal{C},n})[f](x) \stackrel{\text{def}}{=} \int_{y\in\mathbb{H}} R^{(T)}_{\mathbb{H}}(s;x,y)(1-\chi^{-}_{\mathcal{C},n}(y))f(y)d\mathbb{H}(y). \tag{3.38}$$

The integrand here is non-zero unless $d(x,y) \le T+1$ and y is in the the support of $1-\chi^{-}_{\mathcal{C}}$, which is a union of the Γ-translates of a compact set K of of $\bar{F}$. There is compact $K_1 = K_1(T) \subset \bar{F}$ and finite set $S = S(T)$ such that the integrand in (3.38) is supported on the compact set $K_2 \stackrel{\text{def}}{=} \cup_{\gamma\in S}\gamma^{-1}K$ and the whole integral is zero unless $x \in K_1$ (given $x \in \bar{F}$ to begin with). A proof of this fact is given in Lemma 3.14.

Let ψ be a smooth function that is $\equiv 1$ in $K_2 \cup K$, valued in $[0,1]$ and compactly supported. Let $\{e_i : i \in [n-1]\}$ denote an orthonormal basis for V_n^0 and let

$$f_i \stackrel{\text{def}}{=} \langle f, e_i\rangle \in C^{\infty}(\mathbb{H}).$$

The above shows that for $x \in F$ we have

$$\begin{aligned} R^{(T)}_{\mathbb{H},n}(s)(1-\chi^{-}_{\mathcal{C},n})[f](x) &= R^{(T)}_{\mathbb{H},n}(s)(1-\chi^{-}_{\mathcal{C},n})[\psi f](x) \\ &= \sum_{i=1}^{n-1} R^{(T)}_{\mathbb{H}}(s)\left[\left(1-\chi^{-}_{\mathcal{C},n}\right)\psi f_i\right](x)e_i \end{aligned} \tag{3.39}$$

hence

$$\|R^{(T)}_{\mathbb{H},n}(s)(1-\chi^{-}_{\mathcal{C},n})[f](x)\|^2_{V_n^0} = \sum_{i=1}^{n-1}\left|R^{(T)}_{\mathbb{H}}(s)\left[\left(1-\chi^{-}_{\mathcal{C},n}\right)\psi f_i\right](x)\right|^2,$$
$$\|\Delta R^{(T)}_{\mathbb{H},n}(s)(1-\chi^{-}_{\mathcal{C},n})[f](x)\|^2_{V_n^0} = \sum_{i=1}^{n-1}\left|\Delta R^{(T)}_{\mathbb{H}}(s)\left[\left(1-\chi^{-}_{\mathcal{C},n}\right)\psi f_i\right](x)\right|^2.$$

Each function $\left(1-\chi^{-}_{\mathcal{C},n}\right)\psi f_i$ is smooth here and has has compact support depending only on T and $\chi^{-}_{\mathcal{C},n}$.

Now using Lemma 3.12 Part 1, the fact that $\left(1-\chi^{-}_{\mathcal{C},n}\right)$ is valued in $[0,1]$, using ψ is supported only on finitely many Γ-translates of F, together with automorphy of f, we get by integrating over F

$$\|R_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f]\|_{L^2(\mathcal{F})}^2 \leq C\sum_i \|\psi f_i\|_{L^2(\mathbb{H})}^2 \leq C'\|f\|_{L^2(\mathcal{F})}^2,$$
$$\|\Delta R_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f]\|_{L^2(\mathcal{F})}^2 \leq C\sum_i \|\psi f_i\|_{L^2(\mathbb{H})}^2 \leq C'\|f\|_{L^2(\mathcal{F})}^2,$$

where C, C' depend on s, T. Now this bound clearly extends to $f \in L_\phi^2(\mathbb{H}; V_n^0)$. This proves the first statement of the lemma.

The statement that $\mathbb{L}_{\mathbb{H},n}^{(T)}(s)$ is well-defined and bounded on $L_\phi^2(\mathbb{H}; V_n^0)$ is just an easier version of the previous proof using Lemma 3.11 instead of Lemma 3.12. This gives the second part of the lemma. We note that we also obtain

$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f] = \sum_{i=1}^{n-1} \mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left[\left(1-\chi_{\mathcal{C},n}^{-}\right)\psi f_i\right] e_i \tag{3.40}$$

analogously to (3.39).

Now going back to (3.39) and using Lemma 3.12 Part 2 give, considering

$$\begin{aligned}
&(\Delta - s(1-s))R_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f] \\
=&\sum_{i=1}^{n-1}(\Delta - s(1-s))R_{\mathbb{H}}^{(T)}(s)\left[\left(1-\chi_{\mathcal{C},n}^{-}\right)\psi f_i\right] e_i \\
=&\sum_{i=1}^{n-1}\left(\left(1-\chi_{\mathcal{C},n}^{-}\right)\psi f_i + \mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left[\left(1-\chi_{\mathcal{C},n}^{-}\right)\psi f_i\right]\right) e_i \\
=&\left(1-\chi_{\mathcal{C},n}^{-}\right) f + \mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f]\,.
\end{aligned}$$

On the other hand, the fact that all functions at the two ends of the string of equalities above satisfy the automorphy equation (3.5) almost everywhere implies that indeed

$$(\Delta - s(1-s))R_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f] = \left(1-\chi_{\mathcal{C},n}^{-}\right) f + \mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f]$$

as equivalence classes of measurable functions on $\mathbb{H}$. This proves the final part of the lemma. $\square$

We define our interior parametrix

$$\mathbb{M}_{\phi}^{\text{int}}(s) : L^2_{\text{new}}\left(X_\phi\right) \to H^2_{\text{new}}\left(X_\phi\right),$$

to be the operator corresponding under $L^2_{\text{new}}\left(X_\phi\right) \cong L^2_\phi\left(\mathbb{H}; V_n^0\right)$ and $H^2_{\text{new}}\left(X_\phi\right) \cong H^2_\phi\left(\mathbb{H}; V_n^0\right)$ to the integral operator $R^{(T)}_{\mathbb{H},n}(s)\left(1 - \chi^-_{\mathcal{C},n}\right)$. Then by defining

$$\mathbb{M}_\phi\left(s\right) = \mathbb{M}_{\phi}^{\text{int}}(s) + \mathbb{M}_{\phi}^{\text{cusp}}(s),$$

we obtain, using (3.16),

$$\begin{aligned}\left(\Delta_{X_\phi} - s(1-s)\right)\mathbb{M}_\phi\left(s\right) &= \left(1 - \chi^-_{\mathcal{C},n,\phi}\right) + \mathbb{L}_{\phi}^{\text{int}}(s) + \chi^-_{\mathcal{C},n} + \chi^+_{\mathcal{C},n,\phi} R_{\tilde{C},\phi}\left(s\right)\chi^-_{\mathcal{C},n,\phi} \\ &= 1 + \mathbb{M}_{\phi}^{\text{int}}(s) + \mathbb{M}_{\phi}^{\text{cusp}}(s). \qquad (3.41)\end{aligned}$$

3.7 Probabilistic bounds on operator norms

In this section we prove the probabilistic estimates needed for the proofs of Theorem 3.1. Some constants in this section will be important and others will not. We often write C to denote some positive constant (which may only depend on possibly the choice of base surface X) whose value we do not need to track and warn the reader that the precise value of C may change from line to line. Important constants will be indicated by a subscript or given a numerical value.

3.7.1 Preliminaries

Throughout this subsection, let $\kappa : \mathbb{N} \to (0, \infty)$ be given and let $\chi^{\pm}_{\mathcal{C},n}$ be chosen as to satisfy the conclusion of Lemma 3.7. The purpose of this subsection is to ensure that our random operators $\mathbb{M}^{\text{int}}(s)$ are of the correct form as to apply Corollary 3.6.

Let $f \in C_{\phi}^{\infty}\left(\mathbb{H}; V_n^0\right)$ with $\|f\|_{L^2(\mathcal{F})}^2 < \infty$. We have

$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)[f](x) = \int_{y\in\mathbb{H}} \mathbb{L}_{\mathbb{H},n}^{(T)}(s)\,(s;x,y)\left(1-\chi_{\mathcal{C},n}^{-}(y)\right) f(y)$$
$$= \sum_{\gamma\in\Gamma}\int_{y\in F} \mathbb{L}_{\mathbb{H},n}^{(T)}(s)\,(s;\gamma x,y)\,\rho_{\phi}\left(\gamma^{-1}\right)\left(1-\chi_{\mathcal{C},n}^{-}(y)\right) f(y). \quad (3.42)$$

We have an isomorphism of Hilbert spaces

$$L_{\phi}^{2}\left(\mathbb{H};\mathbb{C}^{n}\right) \cong L^{2}\left(\mathcal{F}\right)\otimes V_n^0;$$
$$f \mapsto \sum_{e_i}\langle f|_F, e_i\rangle_{V_n^0}\otimes e_i.$$

Conjugating by this isomorphism,

$$\mathbb{L}_{\mathbb{H},n}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right) \cong \mathcal{L}_{n,\phi}(s) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma} a_{\gamma,n}^{(T)}(s)\otimes\rho_{\phi}\left(\gamma^{-1}\right),$$

where

$$a_{\gamma,n}^{(T)}(s) : L^2\left(\mathcal{F}\right)\to L^2\left(\mathcal{F}\right)$$
$$a_{\gamma,n}^{(T)}(s)[f](x) \stackrel{\text{def}}{=} \int_{y\in F}\mathbb{L}_{\mathbb{H}}^{(T)}\,(s;\gamma x,y)\left(1-\chi_{\mathcal{C},n}^{-}(y)\right) d\mathbb{H}(y).$$

Note that for any $n\in\mathbb{N}$,$T>1$, $s\in\left[\frac{1}{2},1\right]$ and $\gamma\in\Gamma$, the operator $a_{\gamma,n}^{(T)}(s)$ is an Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded only depending on X. Indeed by Lemma 3.10, we have

$$\int_{x,y\in F}\left|\mathbb{L}_{\mathbb{H}}^{(T)}\,(s;\gamma x,y)\left(1-\chi_{\mathcal{C},n}^{-}(y)\right)\right|^2 d\mathbb{H}(x)d\mathbb{H}(y)\leqslant C\mathrm{Vol}\left(X\right)^2.$$

It is crucial that the map $\gamma\mapsto a_{\gamma,n}^{(T)}(s)$ has finite support S whose size we can control. We also need to bound the wordlength of any $\gamma\in S$ to apply Theorem 3.6. This is achieved in the following lemma.

Lemma 3.14. *Given n and $T>0$, there is a finite set $S(T)\subset\Gamma$ which contains the* *support of the map $\gamma \mapsto a^{(T)}_{\gamma,n}(s)$ for any any $s > \frac{1}{2}$. There is a constant $C > 0$ such that*

$$|S(T)| \leqslant C\kappa(n)^2 e^{2T}, \tag{3.43}$$

and if $\gamma \in S(T)$ then its word-length $\mathrm{wl}(\gamma)$ satisfies

$$\mathrm{wl}(\gamma) \leqslant C\kappa(n)^2 e^{2T}. \tag{3.44}$$

Proof. We define

$$K_n \stackrel{\text{def}}{=} \mathrm{Supp}\left(1 - \chi^-_{\mathcal{C},n}\right) \subset F.$$

Recall from that $D_{\mathfrak{a}}(L)$ is the region of the fundamental domain $\mathcal{F}$ with $y \geqslant L$. By the definition of $\chi^-_{\mathcal{C},n}$ (Section 3.4), we have

$$K_n \subset F \backslash D_{\mathfrak{a}}\left(\frac{C}{\kappa(n)}\right),$$

for some constant. We have that

$$\mathcal{F}\backslash D_{\mathfrak{a}}\left(\frac{C}{\kappa(n)}\right) = (\mathcal{F}\backslash D_{\mathfrak{a}}(1)) \bigsqcup \left(D(1) \backslash D_{\mathfrak{a}}\left(\frac{C}{\kappa(n)}\right)\right).$$

Recall that $D(1) \backslash D_{\mathfrak{a}}\left(\frac{C}{\kappa(n)}\right)$ is the region of the cusp $\mathfrak{a}$ bounded by the length 1 and the length $\frac{C}{\kappa(n)}$ horocycle. The diameter of $(\mathcal{F}\backslash D_{\mathfrak{a}}(1))$ is bounded by a constant depending only on X. The diameter of $D(1) \backslash D_{\mathfrak{a}}\left(\frac{C}{\kappa(n)}\right)$ is bounded above by $\log\left(\frac{C}{\kappa(n)}\right) + 2$. It follows that

$$\mathrm{diam}(K_n) \leqslant C + \log\left(\frac{1}{\kappa(n)}\right).$$

Then for $x \in F$, by Lemma 3.10, the expression

$$\mathbb{L}^{(T)}_{\mathbb{H}}(s; \gamma x, y)\left(1 - \chi^-_{\mathcal{C},n}(y)\right)$$

is non-zero only when $y \in K_n$ and $d(\gamma x, y) \leqslant T + 1$. Recall that $\mathcal{F}$ is a Dirichlet domain about some point w, we can assume $w \in K_n$. Then

$$\begin{aligned} d(\gamma x, w) &\leqslant d(\gamma x, y) + d(w, y) \\ &\leqslant T + 1 + \operatorname{diam}(K_n). \end{aligned}$$

Then since $\mathcal{F}$ is a Dirichlet domain about w,

$$\begin{aligned} d(\gamma w, w) &\leqslant d(\gamma w, \gamma x) + d(\gamma x, w) = d(w, x) + d(\gamma x, w) \leqslant 2d(\gamma x, w) \\ &\leqslant 2\left(C + \log\left(\frac{1}{\kappa(n)}\right) + T\right). \end{aligned}$$

Then we can employ a lattice point count to deduce that

$$\begin{aligned} |S(T)| &\leqslant \#\{\gamma \in \Gamma \mid d(\gamma w, w) \leqslant C + 2\log\kappa(n) + 2T\} \\ &\leqslant C\exp\left(2\left(C + \log\left(\frac{1}{\kappa(n)}\right) + T\right)\right) \leqslant C\frac{e^{2T}}{\kappa(n)^2}, \end{aligned}$$

proving (3.43).

We now show property (3.44) holds. We assumed that $\mathcal{F}$ is a Dirichlet domain for Γ, we can also assume that $\mathcal{F}$ is such that the set of side pairings $\{h_1, \ldots, h_k, h_1^{-1}, \ldots, h_k^{-1}\}$ for $\mathcal{F}$ contain our choice of generators $\gamma_1, \ldots, \gamma_d$ and their inverses. We let $\overline{\mathrm{wl}}(\gamma)$ denote the minimal length of γ as a word in $\{h_1, \ldots, h_k, h_1^{-1}, \ldots, h_k^{-1}\}$. Since any h_i or its inverse h_i^{-1} is a finite word in $\gamma_1, \ldots, \gamma_d, \gamma_1^{-1}, \ldots, \gamma_d^{-1}$ it follows that there is a constant $C > 0$ with

$$\mathrm{wl}(\gamma) \leqslant C\overline{\mathrm{wl}}(\gamma).$$

We now set about bounding

$$\sup_{\gamma \in S(T)} \overline{\mathrm{wl}}(\gamma).$$

By the previous argument, if $\gamma \in S(T)$ then

$$\gamma\mathcal{F} \cap B(w, \operatorname{diam}(K_n) + T + 1) \neq \emptyset. \tag{3.45}$$

We claim that if γ satisfies (3.45) and $\overline{\mathrm{wl}}(\gamma) \geqslant 1$, then there is a γ' with $\overline{\mathrm{wl}}(\gamma) = \overline{\mathrm{wl}}(\gamma') - 1$

which satisfies (3.45). The case $\overline{\mathrm{wl}}(\gamma) = 1$ is clear since $w \in \mathcal{F}$. For $l > 1$ let Γ_l denote the elements of Γ with $\overline{\mathrm{wl}}(\gamma) = l$. Since $\left\{h_1, \ldots, h_k, h_1^{-1}, \ldots, h_k^{-1}\right\}$ are side pairings for the Dirichlet domain F, we see that see that

$$\bigcup_{\gamma \in \Gamma} \gamma\mathcal{F} \backslash \left(\bigcup_{\gamma \in \Gamma_l} \gamma\mathcal{F} \right) = \left(\bigcup_{i<l} \bigcup_{\gamma \in \Gamma_i} \gamma\mathcal{F} \right)^{\circ} \sqcup \left(\bigcup_{i>l} \bigcup_{\gamma \in \Gamma_i} \gamma\mathcal{F} \right)^{\circ}.$$

is disconnected. Here U°denotes the interior of U. Therefore if there claim were not true, then one could find an $l > 1$ with

$$\bigcup_{\gamma \in \Gamma_l} \gamma\mathcal{F} \cap B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \neq \emptyset, \tag{3.46}$$

such that

$$\bigcup_{\gamma \in \Gamma_{l-1}} \gamma\mathcal{F} \cap B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) = \emptyset,$$

in particular,

$$B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \subset \bigcup_{\gamma \in \Gamma} \gamma\mathcal{F} \backslash \left(\bigcup_{\gamma \in \Gamma_{l-1}} \gamma\mathcal{F} \right).$$

Then since the ball of radius r in the hyperbolic plane is connected and the identity in Γ satisfies (3.45),

$$B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \subset \left(\bigcup_{i<l-1} \bigcup_{\gamma \in \Gamma_i} \gamma\mathcal{F} \right)^{\circ}.$$

This gives a contradiction to (3.46) and the claim follows. It follows that if γ satisfies (3.45) then $\overline{\mathrm{wl}}(\gamma)$ is bounded above by the number of $\gamma \in \Gamma$ which satisfy (3.45). Then by the argument that led to (3.14),

$$\sup_{\gamma \in S(T)} \mathrm{wl}(\gamma) \leqslant C \# \left\{ \gamma \in \Gamma \mid \gamma\mathcal{F} \cap B\left(w, \operatorname{diam}\left(K_n\right) + T + 1\right) \neq \emptyset \right\} \leqslant C \frac{e^{2T(n)}}{\kappa(n)^2},$$

and the claim is proved. $\square$

Currently, our operators $\sum_{\gamma \in S} a_{\gamma,n}^{(T)}(s) \otimes \rho_{\phi}\left(\gamma^{-1}\right)$ whose norm we wish to bound are almost of the form of Corollary 3.6 except $a_{\gamma,n}^{(T)}(s) : L^2(\mathcal{F}) \to L^2(\mathcal{F})$ are not matrices. However each $a_{\gamma,n}^{(T)}(s)$ is compact so can be approximated by finite rank operators. We need an effective version of this whilst having control over the rank in terms of the error.

Lemma 3.15. *Let $s \in \left[\frac{1}{2}, 1\right]$ be given. For every $n \in \mathbb{N}$ and $T > 1$, there exists a finite dimensional subspace $W \subset L^2(X)$ with $|W| \leqslant C\left(S(T)\right)^3$ for some constant C and finite rank operators $b_{\gamma,n}^{(T)} : W \to W$ for each $\gamma \in S(T)$ such that*

$$\|b_{\gamma,n}^{(T)}(s) - a_{\gamma,n}^{(T)}(s)\|_{L^2(\mathcal{F})} \leqslant \frac{1}{20|S(T)|},$$

Proof. Let $\gamma \in S(T)$, then since $a_{\gamma}^{(T)}(s)$ is compact, it has a singular value decomposition

$$a_{\gamma,n}^{(T)}(s) = \sum_{i \in \mathbb{N}} s_n\left(a_{\gamma,n}^{(T)}(s)\right)\langle \cdot, e_i\rangle f_i,$$

where $\{e_i\}_{i\in\mathbb{N}}$ and $\{f_i\}_{i\in\mathbb{N}}$ are orthonormal systems in $L^2(\mathcal{F})$ and $\{s_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative real numbers. Then by defining

$$b_{\gamma,n}^{(T)}(s) \stackrel{\text{def}}{=} \sum_{i=1}^{r} s_i\left(a_{\gamma}^{(T)}(s)\right)\langle \cdot, e_i\rangle f_i,$$

we see that $b_{\gamma}^{T}(s) : W_\gamma \to W_\gamma$ where $|W_\gamma| \leqslant 2r$ and

$$\|b_{\gamma,n}^{(T)}(s) - a_{\gamma,n}^{(T)}(s)\| \leqslant s_{r+1}(A).$$

We want r to be such that

$$s_{r+1}\left(a_{\gamma,n}^{(T)}(s)\right) \leqslant \frac{1}{20|S(T)|}. \tag{3.47}$$

We have

$$\sum_{i=1}^{\infty} s_i\left(a_{\gamma,n}^{(T)}(s)\right)^2 = \|a_{\gamma,n}^{(T)}(s)\|_{\text{H.S}}^2 \leqslant C,$$

Then

$$\begin{aligned} 0 \leqslant \sum_{i=r+1}^{\infty} s_i\left(a_{\gamma,n}^{(T)}(s)\right)^2 &= \|a_{\gamma,n}^{(T)}(s)\|_{\text{H.S}}^2 - \sum_{i=1}^{r} s_i\left(a_{\gamma,n}^{(T)}(s)\right)^2 \\ &\leqslant C - r s_r\left(a_{\gamma,n}^{(T)}(s)\right)^2. \end{aligned}$$

In particular,

$$s_r\left(a_{\gamma,n}^{(T)}(s)\right)\leqslant\sqrt{\frac{C}{r}}.$$

Taking $r\geqslant 400\cdot C\cdot S(T)^2$ guarantees that (3.47) is satisfied. Then $|W_\gamma|\leqslant CS(T)^2$ for each $\gamma\in S(T)$ and taking

$$W=\bigcup_{\gamma\in S(T)}W_\gamma,$$

gives the conclusion. □

Finally we prove a simple deviations bound.

Lemma 3.16. *There exists a constant $c_2>0$ depending only on X such that for any $T>1$, any $\gamma\in S(T)$ and $s_1,s_2\in\left[\frac{1}{2},1\right]$,*

$$\|a_{\gamma,n}^{(T)}(s_1)-a_{\gamma,n}^{(T)}(s_2)\|_{L^2(\mathcal{F})}\leq c_2|s_1-s_2|.$$

Proof. The operator

$$a_{\gamma,n}^{(T)}(s_1)-a_{\gamma,n}^{(T)}(s_2)$$

is an integral operator with kernel

$$\left(\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)-\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)\right)\left(1-\chi_{\mathcal{C},n}^{-}(y)\right).$$

We have for any $T>1$, $\gamma\in S(T)$, by Lemma 3.13,

$$\begin{aligned}\left|\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)-\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)\right|&\leqslant\sup_{s\in\left[\frac{1}{2},1\right]}\left|\frac{\partial}{\partial s}\mathbb{L}_{\mathbb{H}}^{(T)}(s;\gamma x,y)\right||s_1-s_2|\\&\leqslant C\left|s_1-s_2\right|.\end{aligned}$$

Then we see

$$\|a_\gamma^{(T)}(s_1)-a_\gamma^{(T)}(s_2)\|_{L^2(\mathcal{F})}\leqslant\|a_\gamma^{(T)}(s_1)-a_\gamma^{(T)}(s_2)\|_{\mathrm{H.S.}}\leqslant c_2\left|s_1-s_2\right|,$$

for some constant $c_2>0$. □

3.7.2 Random operator bounds

We are now in a position to apply the results of Section 3.3 to our random operators $\mathcal{L}_{n,\phi}(s)$.

Lemma 3.17. *With notations as above, Taking $T=\frac{\sqrt{\log\log n}}{24}$ and $\kappa(n)=\frac{4\cdot 24^2(\log\log\log n)^2}{\log\log n}$, we have that with probability tending to 1 as $n\to\infty$*

$$\sup_{s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]}\|\mathcal{L}_{n,\phi}(s)\|_{L^2(\mathcal{F})\otimes V_n^0}<\frac{3}{5}.$$

Proof. Let $T=\frac{\sqrt{\log\log n}}{24}$, $\kappa(n)=\frac{4\cdot 24^2(\log\log\log n)^2}{\log\log n}$ and let $s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]$ be fixed. Then by Lemma 3.15, there exists a finite dimensional subspace $W\subset L^2(X)$ with $m=|W|\leqslant C\frac{e^{3T}}{\kappa(n)^3}$ and operators $b_\gamma^{(T)}:W\to W$ for each $\gamma\in S(T)$ such that

$$\|b_{\gamma,n}^{(T)}(s)-a_{\gamma,n}^{(T)}(s)\|_{L^2(\mathcal{F})}\leqslant\frac{1}{20\,|S\,(T)|}.$$

It follows that

$$\|\mathcal{L}_{n,\phi}(s)-\sum_{\gamma\in S(T)}b_{\gamma,n}^{(T)}(s)\otimes\rho_\phi(\gamma)\,\|_{L^2(\mathcal{F})\otimes V_n^0}\leqslant\frac{1}{20}. \tag{3.48}$$

We want to apply Corollary 3.6, to bound

$$\|\sum_{\gamma\in S(T)}b_{\gamma,n}^{(T)}(s)\otimes\rho_\phi(\gamma)\,\|_{\mathbb{C}^m\otimes V_n^0},$$

leading us to require that

$$2ml\,|S|^{\lceil\log_2 l\rceil}\,l^{(\lceil\log_2 l\rceil-1)}\leqslant n^{\sqrt{\log n}},$$

and

$$l^2\,|S|^{\lceil\log_2 l\rceil}\,l^{(\lceil\log_2 l\rceil-1)}\leqslant(\log(n))^{\frac{1}{4}}.$$

Since $m\leqslant C\frac{e^{3T}}{\kappa(n)^3}$ and $l,|S|\leqslant C\frac{e^{2T}}{\kappa(n)^2}$, c.f. Lemma 3.14 and Lemma 3.15, it is a simple calculation to check both inequalities are satisfied if one takes $T=\frac{\sqrt{\log\log n}}{24}$ and $\kappa(n)=$

$\frac{4\cdot 24^2(\log\log\log n)^2}{\log\log n}$. We learn that with probability at least $1-\frac{c_1}{\sqrt{n}}$, we have

$$\begin{aligned}\|\sum_{\gamma\in S} b_{\gamma,n}^{(T)}(s)\otimes\rho_\phi(\gamma)\|_{\mathbb{C}^m\otimes V_n^0} &\leqslant \|\sum_{\gamma\in S} b_{\gamma,n}^{(T)}(s)\otimes\rho_\infty(\gamma)\|_{\mathbb{C}^m\otimes l^2(\Gamma)}\left(1+\frac{l^2\,|S|^{\lceil\log_2 l\rceil}\,l^{\frac{3}{2}(\lceil\log_2 l\rceil-1)}}{n^{\frac{1}{30d+100}}}\right)\\ &= \|\sum_{\gamma\in S} b_{\gamma,n}^{(T)}(s)\otimes\rho_\infty(\gamma)\|_{\mathbb{C}^m\otimes l^2(\Gamma)}\left(1+o(1)\right).\end{aligned}$$

We have an isometric linear isomorphism

$$\begin{aligned}L^2(\mathcal{F})\otimes\ell^2(\Gamma)&\cong L^2(\mathbb{H}),\\ f\otimes\delta_\gamma&\mapsto f\circ\gamma^{-1},\end{aligned}$$

(with $f\circ\gamma^{-1}$ extended by zero from a function on $\gamma\mathcal{F}$). Under this isomorphism, the operator $\sum_{\gamma\in S} a_{\gamma,n}^{(T)}(s)\otimes\rho_\infty\left(\gamma^{-1}\right)$ is conjugated to

$$\mathbb{L}_{\mathbb{H}}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right): L^2(\mathbb{H})\to L^2(\mathbb{H})$$

from Section 3.5. Since $\left(1-\chi_{\mathcal{C},n}^{-}\right)$ is valued in $[0,1]$, multiplication by it has operator norm ≤ 1 on $L^2(\mathbb{H})$, we see that

$$\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\left(1-\chi_{\mathcal{C},n}^{-}\right)\|_{L^2(\mathbb{H})}\leqslant\|\mathbb{L}_{\mathbb{H}}^{(T)}(s)\|_{L^2(\mathbb{H})}<CT(n)\,e^{-T(n)\left(\frac{1}{2}-s\right)}.$$

Since $s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]$ and $\kappa(n)=\frac{4(\log T(n))^2}{T(n)^2}$, we have

$$T(n)\,e^{-T(n)\left(\frac{1}{2}-s\right)}\leqslant T(n)\,e^{-2\log(T(n))}=o(1).$$

Then by Lemma 3.11 we have

$$\|\sum_{\gamma\in S} a_{\gamma,n}^{(T)}(s)\otimes\rho_\infty(\gamma^{-1})\|_{L^2(\mathcal{F})\otimes l^2(\Gamma)}<\frac{1}{10}, \tag{3.49}$$

for sufficiently large n. By the argument that led to (3.48), we see

$$\| \sum_{\gamma \in S} b_{\gamma,n}^{(T)} \otimes \rho_\infty \left(\gamma^{-1}\right) - \sum_{\gamma \in S} a_{\gamma,n}^{(T)}(s) \otimes \rho_\infty(\gamma^{-1}) \|_{L^2(\mathcal{F}) \otimes l^2(\Gamma)} < \frac{1}{20}. \tag{3.50}$$

Then by (3.48), (3.49) and (3.50), for our fixed choice of s,

$$\|\mathcal{L}_{n,\phi}(s)\|_{L^2(\mathcal{F}) \otimes V_n^0} < \frac{2}{5},$$

with probability at least $1 - \frac{c_1}{\sqrt{n}}$.

We now use a finite net argument to control all $s \in \left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$ uniformly. Let $\mathcal{Y}$ be a finite set of points in $\left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$ so that each point of $\left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$ is within

$$\frac{1}{5|S(T)|c_2},$$

of some element of $\mathcal{Y}$, where c_2 is the constant in Lemma 3.16. We can pick $\mathcal{Y}$ so that $|\mathcal{Y}| \leqslant 5c_2 |S(T)| \leqslant C\frac{e^{2T}}{\kappa(n)^2}$. Then by applying an intersection bound, the probability that

$$\|\mathcal{L}_{n,\phi}(s)\|_{L^2(\mathcal{F}) \otimes V_n^0} < \frac{2}{5}$$

for *every point* $s \in \mathcal{Y}$ is bounded below by

$$1 - C\frac{e^{2T}}{\sqrt{n}\kappa(n)^2} \geqslant 1 - C'\frac{e^{\sqrt{\log\log n}}}{\sqrt{n}}, \tag{3.51}$$

which tends to 1 as $n \to \infty$ and

$$\sup_{s \in \mathcal{Y}} \|\mathcal{L}_{n,\phi}(s)\|_{L^2(\mathcal{F}) \otimes \mathbb{C}^n} \leqslant \frac{2}{5},$$

a.a.s. Finally, for $s_1, s_2 \in \left[\frac{1}{2} + \sqrt{\kappa(n)}, 1\right]$,

$$\mathcal{L}_{n,\phi}(s_1) - \mathcal{L}_{n,\phi}(s_2) = \sum_{\gamma \in S(T)} \left[a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\right] \otimes \rho_\phi\left(\gamma^{-1}\right). \tag{3.52}$$

Then by Lemma 3.16, for some constant $c_2 > 0$ we have

$$\|a_\gamma^{(T)}(s_1) - a_\gamma^{(T)}(s_2)\|_{L^2(\mathcal{F})} \leq c_2|s_1 - s_2|,$$

for all $\gamma \in S(T)$ and $s_1, s_2 \in [s_0, 1]$. We see that,

$$\|\mathcal{L}_{n,\phi}(s_1) - \mathcal{L}_{n,\phi}(s_2)\|_{L^2(\mathcal{F})\otimes\mathbb{C}^n} \leq |S(T)|\, c_2|s_1 - s_2|.$$

Then by the choice of $\mathcal{Y}$, it follows that

$$\sup_{s\in\mathcal{Y}} \|\mathcal{L}_{n,\phi}(s)\| \leqslant \frac{2}{5} \implies \sup_{s\in\left[\frac{1}{2}+\sqrt{\kappa(n)},1\right]} \|\mathcal{L}_{n,\phi}(s)\| \leqslant \frac{3}{5}.$$

Since the prior happens with probability tending to 1 as $n \to \infty$, the first claim is proved.

□

3.8 Proofs of Theorem 3.1

It is now straightforward to conclude Theorem 3.1. Recall that our parametrix is defined by

$$\mathbb{M}_\phi(s) \stackrel{\text{def}}{=} \mathbb{M}_\phi^{\text{int}}(s) + \mathbb{M}_\phi^{\text{cusp}}(s),$$

then $\mathbb{M}_\phi(s) : L^2_{\text{new}}(X_\phi) \to H^2_{\text{new}}(X_\phi)$ is a bounded operator and

$$\left(\Delta_{X_\phi} - s(1-s)\right)\mathbb{M}_\phi(s) = 1 + \mathbb{L}_\phi^{\text{int}}(s) + \mathbb{L}_\phi^{\text{cusp}}(s),$$

by Section 3.6. We proved in Lemma 3.17 that there is a constant c_3 (whose precise value can be read off in Lemma 3.17) such that a.a.s.

$$\|\mathbb{L}_\phi^{\text{int}}(s)\| \leqslant \frac{3}{5},$$

for all $s \in \left[\frac{1}{2} + \sqrt{c_3 \frac{\log\log\log n}{\sqrt{\log\log n}}}, 1\right]$. Then since by (3.19)

$$\|\mathbb{L}_\phi^{\text{cusp}}(s)\| \leqslant \frac{1}{8},$$

we have a.a.s.

$$\sup_{s\in\left[\frac{1}{2}+\sqrt{c_3\frac{\log\log\log n}{\sqrt{\log\log n}}},1\right].}\|\mathbb{L}_\phi^{\text{int}}(s)+\mathbb{L}_\phi^{\text{cusp}}(s)\|\leqslant\frac{4}{5}.$$

This implies that a.a.s.

$$\mathbb{M}_\phi(s)\left(1+\mathbb{L}_\phi^{\text{int}}(s)+\mathbb{L}_\phi^{\text{cusp}}(s)\right)^{-1}$$

exists as a bounded operator $L^2_{\text{new}}(X_\phi)\to H^2_{\text{new}}(X_\phi)$ for every $s\in\left[\frac{1}{2}+\sqrt{c_3\frac{\log\log\log n}{\sqrt{\log\log n}}},1\right]$, giving a bounded right inverse for $\left(\Delta_{X_\phi}-s(1-s)\right)$. It follows that a.a.s. $\left(\Delta_{X_\phi}-s(1-s)\right)$ maps $H^2_{\text{new}}(X_\phi)$ onto $L^2_{\text{new}}(X_\phi)$ for every $s\in\left[\frac{1}{2}+\sqrt{c_3\frac{\log\log\log n}{\sqrt{\log\log n}}},1\right]$ and since it is self-adjoint for $s\in[\frac{1}{2},1]$, it cannot have any kernel in $H^2_{\text{new}}(X_\phi)$. Therefore a.a.s. Δ_{X_ϕ} cannot have any new eigenvalues below

$$\frac{1}{4}-c_3\frac{(\log\log\log n)^2}{\log\log n}.$$

4 Spectral gaps for Weil-Petersson random surfaces

The material in this chapter is based on [Hi22]. The main Theorem of this section is the following, c.f. Theorem 1.15

Theorem 4.1. *For any $0\leqslant\alpha<\frac{1}{2}$, if $n=O(g^\alpha)$ then for any $\varepsilon>0$ the Weil-Petersson probability that a genus g non-compact finite-area surface with n cusps has a non-zero Laplacian eigenvalue below $\frac{1}{4}-\left(\frac{2\alpha+1}{4}\right)^2-\varepsilon$ tends to zero as $g\to\infty$.*

Overview of proof

Our method is based on the approach of [WX21, LW21], for compact surfaces. Both [WX21] and [LW21], rely on Selberg's trace formula, e.g. [Bu92, 9.5.3] to relate the spectrum of the of a surface to its length spectrum. In the non-compact finite-area setting, there is a version of Selberg's trace formula, e.g. [Iw02, Theorem 10.2], but it is more complicated with additional terms related to the absolutely continuous spectrum. Instead of dealing with these additional terms directly, we prove a trace inequality which allows us to discard them. In Section 4.1 we prove that if a surface $X\in\mathcal{M}_{g,n}$ has $\lambda_1(X)\leqslant\frac{3}{16}$ then $\lambda_1(X)$

satisfies an inequality (Theorem 4.2) involving the set of oriented primitive closed geodesics $\mathcal{P}(X)$, which closely resembles the form of Selberg's trace formula for compact surfaces, up to well behaved error terms depending only the topology of the surface. Roughly we prove that there are strictly positive functions R and f such that

$$R\left(\lambda_1\left(X\right), g, n\right) \leqslant \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{l_\gamma\left(X\right)}{2 \sinh\left(\frac{k l_\gamma(x)}{2}\right)} f\left(k l_\gamma\left(X\right)\right), \tag{4.1}$$

where $l_\gamma\left(X\right)$ is the length of the geodesic $\gamma \in \mathcal{P}\left(X\right)$. The proof of Theorem 4.2 relies on results from [Ga02]. The function R is large for small $\lambda_1\left(X\right)$ and bounding the Weil-Petersson expectation of the right hand side of (4.1) will allow us to conclude Theorem 1.15 through Markov's inequality.

In Section 4.3 we set about bounding the Weil-Petersson expectation of (4.1). To do this we consider separately the contribution of simple and non-simple geodesics $\gamma \in \mathcal{P}\left(X\right)$ and extend an argument of Wu-Xue [WX21] to deal with non-compact surfaces. We explain the methods of Section 4.3 in more detail in Section 4.3.1.

4.1 Analytic preparations

In this section we prove a version of Selberg's trace formula, using a pre-trace inequality in place of the usual pre-trace formula.

In Section 4.1.1 we exhibit a family of test functions f_T where $T = 4\log g$, and f_T is a non-negative, even, smooth function with support exactly $(-T, T)$ whose Fourier transform $\hat{f}_T$ is non-negative on $\mathbb{R} \cup i\mathbb{R}$ with $\hat{f}_T\left(\frac{i}{2}\right) = O\left(g^2\right)$. The family of test functions f_T is defined by (4.3) with $T = 4\log g$.

The goal of this section is to prove the following.

Theorem 4.2. *For $g \geqslant 2$, let f_T be the test function defined by (4.3) with $T = 4\log g$. For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that for any non-compact finite-area* *surface X with genus g, $n = o\left(g^{\frac{1}{2}}\right)$ cusps and $\lambda_1(X) \leqslant \frac{3}{16}$,*

$$C(\varepsilon)\log(g)\, g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1(X)}} \leqslant \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)} f_T\left(kl_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + O(ng). \tag{4.2}$$

Remark 4.3. Given $\kappa > 0$, we could have stated Theorem 4.2 with the hypothesis $\lambda_1(X) \leqslant \frac{1}{4} - \kappa$, (the statement is almost the same except the constant $C(\varepsilon)$ will also depend on κ) however our geometric estimates (Section 4.3) are not strong enough to prove a spectral gap larger than $\frac{3}{16}$. We therefore state Theorem 4.2 with the hypothesis $\lambda_1(X) \leqslant \frac{3}{16}$ to simplify notation.

Throughout Section 4.1 we let $X = \Gamma_X \backslash \mathbb{H}$ be a fixed non-compact finite-area hyperbolic surface with genus g and $n = o\left(g^{\frac{1}{2}}\right)$ cusps and, for the sake of argument, $\lambda_1(X) \leqslant \frac{3}{16}$.

4.1.1 Test functions

In this subsection we introduce the family of test functions used in Theorem 4.2.

Proposition 4.4. *There exists an $f_1 \in C_c^\infty(\mathbb{R})$ with*

1. $\mathrm{Supp}(f_1) = (-1, 1)$.
2. *f_1 is non-negative and even.*
3. *The Fourier transform $\hat{f}_1$ satisfies $\hat{f}_1(\xi) \geqslant 0$ for $\xi \in \mathbb{R} \cup i\mathbb{R}$.*
4. *f_1 is non-increasing in $[0, 1)$.*

Proposition 4.4 is based on [MNP20, Section 2.2], with the extra condition (4) for convenience later on.

Proof of Proposition 4.4. Let ψ_0 be an even, C^∞, real valued non-negative function whose support is exactly $(-\frac{1}{2}, \frac{1}{2})$ which is non-increasing in $[0, \frac{1}{2})$. Let

$$f_1(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \psi_0(x-t)\psi_0(t)dt.$$

It is proved in [MNP20, Section 2.2] that f_1 satisfies $(1)-(3)$. It remains to check (4). Since f_1 is even we have $f_1'(0) = 0$. If $0 < x < \frac{1}{2}$, one can calculate that

$$f_1'(x) = \int_0^{\frac{1}{2}-x} (\psi_0(x-z) - \psi_0(x+z))\, \psi_0'(z)dz + \int_{\frac{1}{2}-x}^{\frac{1}{2}} \psi_0(x-z)\psi_0'(z)dz.$$

Since ψ_0 is positive, even and non-increasing in $[0, \frac{1}{2})$, we have $\psi_0'(z) \leqslant 0$ and $\psi_0(x-z) - \psi_0(x+z) \geqslant 0$ for all $0 \leqslant z \leqslant \frac{1}{2} - x$, so the first integrand is non-positive. The second integrand is also non-positive since ψ_0 is non-negative. Therefore $f_1'(x) \leqslant 0$ in $[0, \frac{1}{2})$. If $\frac{1}{2} \leqslant x < 1$, then

$$f_1'(x) = \int_{x-\frac{1}{2}}^{\frac{1}{2}} \psi_0'(t)\psi_0(x-t)dt \leqslant 0,$$

and f_1 is non-increasing in $[0, 1)$. □

From here on in, we fix such a function f_1. For $T > 1$ we define

$$f_T(x) \stackrel{\text{def}}{=} f_1\left(\frac{x}{T}\right). \tag{4.3}$$

Then by Proposition 4.4, for each $T > 1$, f_T is a non-negative, even, smooth function with support exactly $(-T, T)$ whose Fourier transform $\hat{f}_T$ is non-negative on $\mathbb{R} \cup i\mathbb{R}$. We also have that f_T is non-increasing in $[0, T)$.

Let k_T denote the inverse Abel transform of f_T, i.e.

$$k_T(\rho) \stackrel{\text{def}}{=} \frac{-1}{\sqrt{2}\pi} \int_\rho^\infty \frac{f_T'(u)}{\sqrt{\cosh u - \cosh \rho}} du. \tag{4.4}$$

We see that k_T is smooth, $\operatorname{Supp}(k_T) \subseteq [0, T)$ and since f_T is non-increasing in $[0, T)$, k_T is non-negative.

We now have a fixed family of test functions f_T for $T > 1$. We conclude this subsection by stating a lower bound on $\hat{f}_T$ in $i\mathbb{R}$ from [MNP20].

Lemma 4.5 ([MNP20, Lemma 2.4])**.** *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for all $t \in \mathbb{R}_{\geqslant 0}$ and for all $T > 1$ the Fourier transform $\hat{f}_T$ satisfies*

$$\hat{f}_T(it) \geqslant T C_\varepsilon e^{T(1-\varepsilon)t}. \tag{4.5}$$

[MNP20, Lemma 2.4] applies for any function satisfying properties $(1)-(3)$ from Proposition 4.4 so it also applies here. Lemma 4.5 tells us that small values of λ_1 imply large values of $\hat{f}_T\left(i\sqrt{\frac{1}{4}-\lambda_1}\right)$.

4.1.2 Eigenfunction estimates

Now we have a family of test functions, we proceed with the proof of Theorem 4.2. For $z, w \in \mathbb{H}$, $T > 1$ we define

$$k_T(z,w) \stackrel{\text{def}}{=} k_T\left(d(z,w)\right).$$

Let $r : [0,\infty) \to \mathbb{C}$ be the function given by

$$r(x) = \begin{cases} i\sqrt{\frac{1}{4}-x} & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ \sqrt{x-\frac{1}{4}} & \text{if } x > \frac{1}{4}. \end{cases}$$

Let $u_j \in L^2(X)$ denote the normalized eigenfunction of the Laplacian on X corresponding to the eigenvalue λ_j. Our starting point is the following.

Lemma 4.6 (Pre-trace inequality [Ga02, Proposition 5.2]). *For all $T > 1$ and $z \in \mathbb{H}$ we have that*

$$\sum_{j:\lambda_j<\frac{1}{4}} \hat{f}_T\left(r\left(\lambda_j\right)\right)\left|u_j(z)\right|^2 \leqslant \sum_{\gamma\in\Gamma_X} k_T\left(z,\gamma z\right). \tag{4.6}$$

Lemma 4.6 is immediately deduced from [Ga02, Proposition 5.2], using the fact that $\hat{f}_T$ is non-negative on $\mathbb{R}\cup i[0,\frac{1}{2}]$ (the image of $[0,\infty)$ under r). We refer to the left hand side of (4.6) as the spectral side and the right hand side as the geometric side. We prove Theorem 4.2 by integrating (4.6). We cannot integrate (4.6) over the full fundamental domain as the contribution of the parabolic elements

$$\sum_{\{\gamma\in\Gamma_X\setminus\{\mathrm{Id}\}\,|\,|\mathrm{tr}(\gamma)|=2\}} k_T\left(z,\gamma z\right),$$

is not absolutely integrable over the fundamental domain $\mathcal{F}$. We get around this by integrating over the region $D(l)$, as defined in Definition 2.2, with $l = 2$ (the choice $l = 2$ could be replaced by any fixed $l > 1$). This leads to another issue: we could potentially lose information on the spectral side after integrating. This could happen if an eigenfunction concentrated outside $D(2)$. The following lemma resolves this issue. From now on we write $D = D(2)$.

Lemma 4.7 ([Ga02, Lemma 4.1])**.** *For any $\kappa > 0$, there is a constant $c(\kappa) > 0$ such that for any u_j with $\lambda_j \leqslant \frac{1}{4} - \kappa$, we have*

$$\int_D |u_j(z)|^2 d\mu(z) \geqslant c(\kappa).$$

The constant c does not depend on the surface X.

The upshot is that when we integrate (4.6) over D, we obtain something bounded on the geometric side and we get a definite contribution from each eigenvalue on the spectral side.

Remark 4.8. [Ga02, Lemma 4.1] is stated for quotients of $\mathbb{H}$ by geometrically finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$. The proof extends trivially to all finite-area non-compact surfaces, as noted in [Ga02, Footnote 10].

4.1.3 Proof of Theorem 4.2

We conclude this section by proving Theorem 4.2.

Proof of Theorem 4.2. Recall that X is a finite-area non-compact hyperbolic surface with genus g, $n = o\left(g^{\frac{1}{2}}\right)$ cusps. We write $\lambda_j = \lambda_j(X)$ and recall that X has first non-zero Laplacian eigenvalue $\lambda_1 \leqslant \frac{3}{16}$. Let $T = 4\log g$. By Lemma 4.6,

$$\sum_{j:\lambda_j<\frac{1}{4}} \hat{f}_T(r(\lambda_j))\,|u_j(z)|^2 \leqslant \sum_{\gamma\in\Gamma_X} k_T(z,\gamma z). \tag{4.7}$$

Since $\hat{f}_T$ is non-negative on $i\mathbb{R}$, $\hat{f}_T \circ r$ is non-negative on $[0,\frac{1}{4}]$ and (4.7) still holds if we reduce the sum to just λ_0 and λ_1. Integrating (4.7) over D, we get

$$\hat{f}_T(r(\lambda_0))\int_D |u_0(z)|^2 d\mu(z) + \hat{f}_T(r(\lambda_1))\int_D |u_1(z)|^2 d\mu(z) \leqslant \int_D \sum_{\gamma\in\Gamma_X} k_T(z,\gamma z)\,d\mu(z). \tag{4.8}$$

First we look at the spectral side. The eigenvalue $\lambda_0 = 0$ corresponds to the constant eigenfunction

$$u_0(z) = \frac{1}{\sqrt{\mathrm{Vol}(X)}}.$$

We have

$$\hat{f}_T(r(\lambda_0)) \int_D |u_0(z)|^2 d\mu(z) = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)} \hat{f}_T\left(\frac{i}{2}\right).$$

Recall that

$$D = \mathcal{F} \backslash \bigsqcup_{i=1}^{n} D_{\mathfrak{a}_i}(2).$$

Since $D_{\mathfrak{a}_i}(2)$ is isometric to $\{z \in \mathbb{H} \mid 0 < x < 1, y \geqslant 2\}$, $\mathrm{Vol}(D_{\mathfrak{a}_i}(2)) = \frac{1}{2}$ for each i. By Gauss-Bonnet, $\mathrm{Vol}(X) = 2\pi(2g - 2 + n)$ and we see that

$$\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(X)} = \frac{2\pi(2g-2+n) - \frac{n}{2}}{2\pi(2g-2+n)} = 1 + O\left(\frac{n}{g}\right).$$

For the contribution of λ_1, by Lemma 4.7 with $\kappa = \frac{1}{16}$, there is a constant $c > 0$ with

$$\hat{f}_T(r(\lambda_1)) \int_D |u_1(z)|^2 d\mu(z) \geqslant c \hat{f}_T(r(\lambda_1)). \tag{4.9}$$

Let $\varepsilon > 0$ be given, then since $\lambda_1 \leqslant \frac{3}{16}$, $r(\lambda_1) = i\sqrt{\frac{1}{4} - \lambda_1}$, then by Lemma 4.5, there is a constant $C_\varepsilon > 0$ with

$$\hat{f}_T(r(\lambda_1)) \geqslant T C_\varepsilon e^{T(1-\varepsilon)\sqrt{\frac{1}{4} - \lambda_1}}. \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we see there exists a constant $C(\varepsilon) > 0$ with

$$TC(\varepsilon) e^{T(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}} + \left(1 + O\left(\frac{n}{g}\right)\right) \hat{f}_T\left(\frac{i}{2}\right) \leqslant \int_D \sum_{\gamma \in \Gamma_X} k_T(z, \gamma z)\, d\mu(z). \tag{4.11}$$

We now look at the geometric side. We arrange the sum in the geometric side into the contribution from the identity, parabolic and hyperbolic elements to obtain

$$\begin{aligned}\int_D \sum_{\gamma\in\Gamma_X} k_T\left(z,\gamma z\right) d\mu(z) &= \sum_{\gamma\in\Gamma_X}\int_D k_T\left(z,\gamma z\right) d\mu(z)\\ &= \int_D k_T\left(z,z\right) d\mu(z) + \sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}}\int_D k_T\left(z,\gamma z\right) d\mu(z)\\ &\quad + \sum_{\{\gamma\in\Gamma_X\setminus\{\mathrm{Id}\} \mid |\mathrm{tr}(\gamma)|=2\}}\int_D k_T\left(z,\gamma z\right) d\mu(z).\end{aligned}$$

Interchanging summation and integration is justified since D is a compact region and k_T is supported in $[0,T)$, then for each $z\in D$, $\#\{\gamma\in\Gamma_X \mid d(z,\gamma z)<T\}$ is finite and the summation is over finitely many terms.

First we treat the contribution of the identity. Since $k_T(z,w)=k_T\left(d(z,w)\right)$,

$$\int_D k_T\left(z,z\right) d\mu(z) = \mathrm{Vol}(D)k_T(0).$$

A calculation involving the Abel Transform, see for example the proof of [Bu92, Theorem 9.5.3], gives that

$$k_T(0)=\frac{1}{4\pi}\int_{-\infty}^{\infty} r\hat{f}_T(r)\tanh(\pi r)dr.$$

We calculate

$$\begin{aligned}\int_{-\infty}^{\infty} r\hat{f}_T(r)\tanh(\pi r)dr &= T\int_{-\infty}^{\infty} r\hat{f}_1\left(Tr\right)\tanh(\pi r)dr\\ &= \frac{1}{T}\int_{-\infty}^{\infty} r'\hat{f}_1\left(r'\right)\tanh\left(\frac{\pi r'}{T}\right)dr'\\ &\leqslant \frac{2}{T}\int_0^{\infty} r'\hat{f}_1\left(r'\right)dr' \ll \frac{1}{T},\end{aligned}$$

where the last line follows from the fact that f_1 is compactly supported, thus $\hat{f}_1$ is a Schwartz function and decays faster that the inverse of any polynomial. Since $\mathrm{Vol}(D)=2\pi\left(2g-2+n\right)-\frac{n}{2}$, and X has $o\left(g^{\frac{1}{2}}\right)$ cusps, this tells us that

$$\int_D k_T\left(z,z\right) d\mu(z) = O\left(g\right). \tag{4.12}$$

Now we look at the hyperbolic terms. By the non-negativity of k_T,

$$\sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}} \int_D k_T\left(z,\gamma z\right) d\mu(z) \leqslant \sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}} \int_{\mathcal{F}} k_T\left(z,\gamma z\right) d\mu(z).$$

By arranging the sum into conjugacy classes and unfolding the integral, one can compute that

$$\sum_{\{\gamma\in\Gamma_X \mid |\mathrm{tr}(\gamma)|>2\}} \int_{\mathcal{F}} k_T\left(z,\gamma z\right) d\mu(z) = \sum_{\gamma\in\mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{l_\gamma\left(X\right)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)} f\left(kl_\gamma\left(X\right)\right). \tag{4.13}$$

This computation is carried out in detail in [Iw02, Section 10.2].

It remains to bound the contribution of the parabolic elements. Any $\gamma\in\Gamma_X\backslash\{\mathrm{Id}\}$ with $|\mathrm{tr}(\gamma)|=2$ is conjugate to $\gamma_{\mathfrak{a}_i}^l$ for some unique pair $i\in\{1,\ldots,n\}$ and $l\in\mathbb{Z}\backslash\{0\}$. Since the centralizer of $\gamma_{\mathfrak{a}_i}^l$ in Γ_X is $\Gamma_{\mathfrak{a}_i}$, we see

$$\sum_{\{\gamma\in\Gamma_X\backslash\{\mathrm{Id}\} \mid |\mathrm{tr}(\gamma)|=2\}} \int_D k_T\left(z,\gamma z\right) d\mu(z) = \sum_{i=1}^{n} \sum_{l\in\mathbb{Z}^*} \sum_{\tau\in\Gamma_{\mathfrak{a}_i}\backslash\Gamma} \int_D k_T\left(z,\tau^{-1}\gamma_{\mathfrak{a}_i}^l\tau z\right) d\mu(z).$$

Since k_T and $d\mu$ are invariant under isometries, by unfolding the integral, denoting $\Gamma\cdot D \stackrel{\mathrm{def}}{=} \cup_{\gamma\in\Gamma}\gamma D$, we calculate

$$\sum_{\tau\in\Gamma_{\mathfrak{a}_i}\backslash\Gamma} \int_D k_T\left(z,\tau^{-1}\gamma_{\mathfrak{a}_i}^l\tau z\right) d\mu(z) = \int_{\Gamma_{\mathfrak{a}_i}\backslash\Gamma\cdot D} k_T\left(z,\gamma_{\mathfrak{a}_i}^l z\right) d\mu(z).$$

We can choose a fundamental domain $\tilde{\mathcal{F}}_i$ for the action of $\Gamma_{\mathfrak{a}_i}$ on $\Gamma\cdot D$ so that

$$\tilde{\mathcal{F}}_i \subseteq \sigma_{\mathfrak{a}_i}\{z\in\mathbb{H} \mid 0<x\leqslant 1, 0<y\leqslant 2\},$$

and we see, recalling that $\sigma_{\mathfrak{a}_i}^{-1}\gamma_{\mathfrak{a}_i}\sigma_{\mathfrak{a}_i}\left(z\right)=z+1$,

$$\sum_{\tau\in\Gamma_{\mathfrak{a}_i}\backslash\Gamma}\int_D k_T(z,\tau^{-1}\gamma_{\mathfrak{a}_i}^l\tau z)d\mu(z) = \int_{\tilde{\mathcal{F}}_i} k_T(z,\gamma_{\mathfrak{a}_i}^l z)d\mu(z)$$
$$= \int_{\sigma_{\mathfrak{a}_i}^{-1}(\tilde{\mathcal{F}}_i)} k_T(z,z+l)d\mu(z)$$
$$\leqslant \int_{x=0}^{x=1}\int_{y=0}^{y=2} k_T(z,z+l)d\mu(z).$$

We sum over the parabolic conjugacy classes to calculate,

$$\begin{aligned}\sum_{\{\gamma\in\Gamma_X\backslash\{\mathrm{Id}\}||\mathrm{tr}(\gamma)|=2\}}\int_D k_T(z,\gamma z)d\mu(z) &\leqslant n\sum_{l\in\mathbb{Z}^*}\int_0^1\int_0^2 k_T(z,z+l)d\mu(z)\\ &= n\sum_{l\in\mathbb{Z}^*}\int_0^2 k_T\left(\operatorname{arcosh}\left(1+\frac{l^2}{2y^2}\right)\right)y^{-2}dy\\ &= n\sum_{l\in\mathbb{N}}\frac{\sqrt{2}}{l}\int_{\min\left\{\operatorname{arcosh}\left(1+\frac{l^2}{8}\right),T\right\}}^T\frac{k_T(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho)-1}}d\rho.\end{aligned} \tag{4.14}$$

On the second line we used that $\cosh d(z,z+l) = 1+\frac{l^2}{2y^2}$ and on the third line we used the change of variables $\rho = \operatorname{arcosh}\left(1+\frac{l^2}{2y^2}\right)$ and that $\operatorname{Supp}(k_T)\subseteq[0,T)$. When $\operatorname{arcosh}\left(1+\frac{l^2}{8}\right)\leqslant T$, we use that f_T is the Abel transform of k_T to see that

$$\int_{\min\left\{\operatorname{arcosh}\left(1+\frac{l^2}{8}\right),T\right\}}^T\frac{k_T(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho)-1}}d\rho \leqslant \int_0^T\frac{k_T(\rho)\sinh(\rho)}{\sqrt{\cosh(\rho)-1}}d\rho = f_T(0) = f_1(0).$$

If $\operatorname{arcosh}\left(1+\frac{l^2}{8}\right)\leqslant T$ then the contribution to the sum (4.14) is 0 and we conclude that

$$\sum_{\{\gamma\in\Gamma_X\backslash\{\mathrm{Id}\}||\mathrm{tr}(\gamma)|=2\}}\int_D k_T(z,\gamma z)d\mu(z)\leqslant 2nf_1(0)\sum_{l=1}^{\lfloor\sqrt{8\cosh T}\rfloor}\frac{1}{l}\leqslant 2nf_1(0)\log\left(2\sqrt{2}e^{\frac{T}{2}}\right).$$

Thus combining (4.11), (4.12), (4.13) and (4.1.3), we conclude that

$$
\begin{aligned}
&TC(\varepsilon)e^{T(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}} + \left(1+O\left(\frac{n}{g}\right)\right)\hat{f}_T\left(\frac{i}{2}\right) \\
&\leqslant \sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}\frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(X)}{2}\right)}f_T\left(kl_\gamma(X)\right) + 2nf_1(0)\log\left(2\sqrt{2}e^{\frac{T}{2}}\right) + O\left(g\right).
\end{aligned}
$$

Recalling that $T = 4\log g$, since f_T is even,

$$
\hat{f}_T\left(\frac{i}{2}\right) = \int_0^\infty 2\cosh\left(\frac{x}{2}\right)f_T(x)dx = O\left(g^2\right),
$$

and we deduce that

$$
C(\varepsilon)\log(g)\,g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1}} \leqslant \sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}\frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)}f_T\left(kl_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + O\left(ng\right),
$$

as claimed. $\square$

Remark 4.9. By considering only the zero eigenvalue, the proof of Theorem 4.2 gives that there exists a constant $\nu \geqslant 0$ such that for sufficiently large g and for any $X \in \mathcal{M}_{g,n}$,

$$
\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}\frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)}f_T\left(kl_\gamma(X)\right) - \hat{f}_T\left(\frac{i}{2}\right) + \nu ng \geqslant 0.
$$

This fact will be important in Section 4.4 when we want to apply Markov's inequality to the above quantity, viewed as a random variable on $\mathcal{M}_{g,n}$.

4.2 Weil-Petersson model

In this subsection, we introduce the necessary background on moduli space and the Weil-Petersson metric needed for the remainder of the chapter. We refer to the survey of Wright [Wr20] for a more detailed exposition.

4.2.1 Moduli space

Let $\Sigma_{g,c,d}$ denote a topological surface with genus g, c labeled punctures and d labeled boundary components where $2g+n+d \geqslant 3$. A marked surface of signature (g,c,d) is a pair (X, φ) where X is a hyperbolic surface and $\varphi : \Sigma_{g,c,d} \to X$ is a homeomorphism. Given $(l_1, ..., l_d) \in \mathbb{R}^d_{>0}$, we define the Teichmüller space $\mathcal{T}_{g,c+d}(l_1, \ldots, l_d)$ by

$$\mathcal{T}_{g,c,d}(l_1, ..., l_d) \stackrel{\text{def}}{=} \left\{ \begin{array}{c} \text{Marked surfaces } (X,\varphi) \text{ of signature } (g,c,d) \\ \text{with labelled totally geodesic boundary components} \\ (\beta_1,...,\beta_d) \text{ with lengths } (l_1,...,l_d) \end{array} \right\} / \sim,$$

where $(X_1, \varphi_1) \sim (X_2, \varphi_2)$ if and only if there exists an isometry $m : X_1 \to X_2$ such that φ_2 and $m \circ \varphi_1$ are isotopic. Let $\text{Homeo}^+(\Sigma_{g,c,d})$ denote the group of orientation preserving homeomorphisms of $\Sigma_{g,c,d}$ which leave every boundary component setwise fixed and do not permute the punctures. Let $\text{Homeo}^+_0(\Sigma_{g,c,d})$ denote the subgroup of homeomorphisms isotopic to the identity. The mapping class group is defined as

$$\text{MCG}_{g,c,d} \stackrel{\text{def}}{=} \text{Homeo}^+(\Sigma_{g,c,d}) / \text{Homeo}^+_0(\Sigma_{g,c,d}).$$

$\text{Homeo}^+(\Sigma_{g,c,d})$ acts on $\mathcal{T}_{g,c,d}(l_1, ..., l_d)$ by pre-composition of the marking, and $\text{Homeo}^+_0(\Sigma_{g,c,d})$ acts trivially, hence $\text{MCG}_{g,c,d}$ acts on $\mathcal{T}_{g,c,d}(l_1, ..., l_d)$ and we define the moduli space $\mathcal{M}_{g,c,d}(l_1, ..., l_d)$ by

$$\mathcal{M}_{g,c,d}(l_1, ..., l_d) \stackrel{\text{def}}{=} \mathcal{T}_{g,c,d}(l_1, ..., l_d) / \text{MCG}_{g,c,d}.$$

By convention, a geodesic of length 0 is a cusp and we suppress the distinction between punctures and boundary components in our notation by allowing $l_i \geqslant 0$. In particular,

$$\mathcal{M}_{g,c+d} = \mathcal{M}_{g,c,d}(0, \ldots, 0).$$

4.2.2 Weil-Petersson metric

By the work of Goldman [Go84], the space $\mathcal{T}_{g,n}(\underline{l})$ carries a natural symplectic structure known as the Weil-Petersson symplectic form and is denoted by ω_{WP}. In the case where $\underline{l} = \underline{0}$, this agrees with the form arising from the Weil-Petersson Kähler metric on $\mathcal{T}_{g,n}$[4]. It

[4] The cotangent space $T^*_{(X,\varphi)}\mathcal{T}_{g,n}$ at $(X, \varphi) \in \mathcal{T}_{g,n}$ can be identified with the space of quadratic differentials $\mathcal{Q}(X)$. The space $\mathcal{Q}(X)$ has an inner product $\langle , \rangle_{\text{WP}}$, the Weil-Petersson inner product, inducing a Riemannian metric on $\mathcal{T}_{g,n}$; the Weil-Petersson metric. The Weil-Petersson sympletic form ω_{WP} is the form dual to $\text{Im}\langle , \rangle_{\text{WP}}$.

is invariant under the action of the mapping class group and descends to a symplectic form on the quotient $\mathcal{M}_{g,n}(\underline{l})$. The form ω_{WP} induces the volume form

$$\mathrm{dVol}_{WP} \stackrel{\text{def}}{=} \frac{1}{(3g-3+n)!} \bigwedge_{i=1}^{3g-3+n} \omega_{WP},$$

which is also invariant under the action of the mapping class group and descends to a volume form on $\mathcal{M}_{g,n}(\underline{l})$. We write dX as shorthand for dVol_{WP}. We let $V_{g,n}(\underline{l})$ denote $\mathrm{Vol}_{WP}(\mathcal{M}_{g,n}(\underline{l}))$, the total volume of $\mathcal{M}_{g,n}(\underline{l})$, which is finite. We write $V_{g,n}$ to denote $V_{g,n}(\underline{0})$.

By work of Wolpert [Wo85], the Weil-Petersson symplectic form has a simple form in Fenchel-Nielsen coordinates. Let $\Sigma_{g,c,d}$ be as before, a topological surface with with genus g, c labeled punctures and d labeled boundary components where $2g+n+d \geqslant 3$. A pants decomposition of $\Sigma_{g,c,d}$ is a collection of disjoint simple closed curves $\{\alpha_i\}_{i=1}^{3g-3+c+d}$ such that cutting the surface along all curves gives a disjoint collection of topological pants. For a marked surface (X,φ), let $l_{\alpha_i}(X)$ to be the length of the unique geodesic in the free homotopy class of $\varphi(\alpha_i)$ and $\tau_{\alpha_i}(X)$ be the corresponding twist parameter. Then if $c+d=n$,

$$\begin{aligned}
&\mathcal{T}_{g,n}(\underline{l}) \cong \mathbb{R}_{+}^{3g-3+n} \times \mathbb{R}^{3g-3+n} \\
&(X,\varphi) \mapsto \left(l_{\alpha_1}(X), \ldots, l_{\alpha_{3g-3+n}}(X), \tau_{\alpha_1}(X), \ldots, \tau_{\alpha_{3g-3+n}}(X)\right).
\end{aligned}$$

Then by a Theorem of Wolpert [Wo85, Theorem 1.3],

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3+n} dl_i \wedge d\tau_i,$$

i.e. the Weil-Petersson symplectic form is the standard sympletic form in Fenchel-Neilsen coordinates.

As in [Mi13, WX21, LW21], we define a probability measure on $\mathcal{M}_{g,n}$ by normalizing

dVol_{WP}. Indeed, for any Borel subset $\mathcal{B} \subseteq \mathcal{M}_{g,n}$,

$$\mathbb{P}_{WP}^{g,n}[\mathcal{B}] \stackrel{\text{def}}{=} \frac{1}{V_{g,n}} \int_{\mathcal{M}_{g,n}} \mathbf{1}_{\mathcal{B}} dX,$$

where

$$\mathbf{1}_{\mathcal{B}}(X) = \begin{cases} 0 & \text{if } x \notin \mathcal{B}, \\ 1 & \text{if } x \in \mathcal{B}. \end{cases}$$

is the indicator function on $\mathcal{B}$. We write $\mathbb{E}_{WP}^{g,n}$ to denote expectation with respect to $\mathbb{P}_{WP}^{g,n}$.

4.2.3 Mirzakhani's integration formula

We recall Mirzakhani's integration formula from [Mi07]. We define a multi-curve to be an ordered k-tuple $(\gamma_1, ..., \gamma_k)$ of disjoint, simple, non-peripheral closed curves. Let $\Gamma = [\gamma_1, ..., \gamma_k]$ denote the homotopy class of a multi-curve. The mapping class group $\mathrm{MCG}_{g,n}$ acts naturally on homotopy classes of multi-curves and we denote the orbit containing Γ by

$$\mathcal{O}_\Gamma = \{(g \cdot \gamma_1, ..., g \cdot \gamma_k) \mid g \in \mathrm{MCG}_{g,n}\}.$$

Given a function $F : \mathbb{R}_{\geqslant 0}^k \to \mathbb{R}_{\geqslant 0}$, define $F^\Gamma : \mathcal{M}_{g,n} \to \mathbb{R}$ by

$$F^\Gamma(X) = \sum_{(\alpha_1,...,\alpha_k) \in \mathcal{O}_\Gamma} F(l_{\alpha_1}(X), ..., l_{\alpha_k}(X)),$$

where $l_{\alpha_i}(X)$ is defined for $(X, \varphi) \in \mathcal{T}_{g,n}$ as the length of the geodesic in the homotopy class of $\varphi(\alpha_i)$. Note that the function F^Γ is well defined on $\mathcal{M}_{g,n}$ since we are summing over the orbit $\mathcal{O}_\Gamma$. Let $S_{g,n}(\Gamma)$ denote the result of cutting the surface $S_{g,n}$ along $(\gamma_1, ..., \gamma_k)$, then $S_{g,n}(\Gamma) = \sqcup_{i=1}^s S_{g_i,n_i}$ for some $\{(g_i, n_i)\}_{i=1}^s$. Each γ_i gives rise to two boundary components γ_i^1 and γ_i^2 of $S_{g,n}(\Gamma)$. Given $\underline{x} = (x_1, ..., x_k) \in \mathbb{R}_{\geqslant 0}^k$, let $\mathcal{M}(S_{g,n}(\Gamma); l_\Gamma = \underline{x})$ be the moduli space of hyperbolic surfaces homeomorphic to $S_{g,n}(\Gamma)$ such that for $1 \leqslant i \leqslant k$, $l_{\gamma_i^1} = l_{\gamma_i^2} = x_i$. Let $\underline{x}^{(i)}$ denote the tuple of coordinates x_j of $\underline{x}$ such that γ_j is a boundary component of S_{g_i,n_i}. We have that

$$\mathcal{M}\left(S_{g,n}\left(\Gamma\right);l_{\Gamma}=\underline{x}\right)=\prod_{i=1}^{s}\mathcal{M}_{g_i,n_i}\left(\underline{x}^{(i)}\right),$$

and we define

$$V_{g,n}\left(\Gamma,\underline{x}\right)\stackrel{\text{def}}{=}\mathrm{Vol}_{WP}\left(\mathcal{M}\left(S_{g,n}\left(\Gamma\right);l_{\Gamma}=\underline{x}\right)\right)=\prod_{i=1}^{s}V_{g_i,n_i}\left(\underline{x}^{(i)}\right).$$

In terms of the above notation we have the following.

Theorem 4.10 (Mirzakhani's Integration Formula [Mi07, Theorem 7.1])**.** *Given* $\Gamma=[\gamma_1,...,\gamma_k]$*,*

$$\int_{\mathcal{M}_{g,n}}F^{\Gamma}\left(X\right)dX=C_{\Gamma}\int_{\mathbb{R}^k_{\geqslant 0}}F\left(x_1,...,x_k\right)V_{g,n}\left(\Gamma,\underline{x}\right)x_1\cdots x_k dx_1\cdots dx_k,$$

where the constant $C_{\Gamma}\in(0,1]$ *only depends on* Γ*. Moreover, if* $g>2$ *and* $\Gamma=[\gamma]$ *where* γ *is a simple, non-separating closed curve, then* $C_{\Gamma}=\frac{1}{2}$*.*

4.3 Geometric estimates

Recall that the family of test functions f_T in Theorem 4.2 is defined in (4.3) with $T=4\log g$. For $X\in\mathcal{M}_{g,n}$, $\gamma\in\mathcal{P}(X)$, $k\in\mathbb{N}$, we shall denote

$$H_{X,k}(\gamma)\stackrel{\text{def}}{=}\frac{l_{\gamma}\left(X\right)}{2\sinh\left(\frac{kl_{\gamma}(x)}{2}\right)}f_T\left(kl_{\gamma}\left(X\right)\right).$$

The goal of this section is to prove the following.

Theorem 4.11. *For* $0\leqslant\alpha<\frac{1}{2}$*, let* $n=O\left(g^{\alpha}\right)$*. For any* $\varepsilon_1>0$ *there exists a constant* $c_1\left(\varepsilon_1\right)>0$*, independent of* α*, with*

$$\mathbb{E}^{g,n}_{WP}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)\right]\ll n^2g+\log\left(g\right)^5\cdot g+c_1\left(\varepsilon_1\right)\left(\log g\right)^{\beta+1}\cdot n^2\cdot g^{1+4\epsilon_1},$$

where $\beta>0$ *is a universal constant.*

Throughout Section 4.3 we shall always have $n = O(g^\alpha)$ for fixed $0 \leqslant \alpha < \frac{1}{2}$.

Remark 4.12. The proof of Theorem 4.11 closely follows [WX21, Chapters 6 & 7], making the necessary adaptations to the case of surfaces with cusps. We therefore omit some arguments that are identical in the compact and non-compact case and instead refer the reader to the relevant place.

4.3.1 Method

We prove Theorem 4.11 by considering separately the contribution of different types of geodesics. As in [WX21], we introduce the following notation.

Definition 4.13. For $X \in \mathcal{M}_{g,n}$ we define

1. $\mathcal{P}^s_{sep}(X) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{P}(X) \mid \gamma \text{ is simple and separating}\}$.
2. $\mathcal{P}^s_{nsep}(X) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{P}(X) \mid \gamma \text{ is simple and non-separating}\}$.
3. $\mathcal{P}^{ns}(X) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{P}(X) \mid \gamma \text{ is non-simple}\}$.

Notice that $\mathcal{P}(X) = \mathcal{P}^s_{sep}(X) \sqcup \mathcal{P}^s_{nsep}(X) \sqcup \mathcal{P}^{ns}(X)$. We partition the sum $\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma)$ as

$$\begin{aligned}\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} H_{X,k}(\gamma) &= \sum_{\gamma \in \mathcal{P}(X)} H_{X,1}(\gamma) + \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=2}^{\infty} H_{X,k}(\gamma) \\ &= \sum_{\gamma \in \mathcal{P}^s_{sep}(X)} H_{X,1}(\gamma) + \sum_{\gamma \in \mathcal{P}^s_{nsep}(X)} H_{X,1}(\gamma) + \sum_{\gamma \in \mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \\ &\quad + \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=2}^{\infty} H_{X,k}(\gamma).\end{aligned}$$

Subtracting $\hat{f}\left(\frac{i}{2}\right)$ and taking Weil-Petersson expectations, we see

$$\begin{aligned}
&\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)\right] \\
&\leqslant \underbrace{\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)\right]}_{(a)}+\underbrace{\left|\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)}H_{X,1}(\gamma)\right]-\hat{f}\left(\frac{i}{2}\right)\right|}_{(b)} \\
&\quad+\underbrace{\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\right]}_{(c)}+\underbrace{\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^{ns}(X)}H_{X,1}(\gamma)\right]}_{(d)}. \qquad (4.15)
\end{aligned}$$

The remainder of this section is dedicated to bounding terms $(a)-(d)$, from which Theorem 4.11 will follow.

- Since terms (a) and (b) depend on simple geodesics, we can bound them by applying Mirzakhani's integration formula directly.

- To bound (c) we consider geodesics with length <1 and length $\geqslant 1$ separately. The contribution of geodesics with length $\geqslant 1$ can be bounded deterministically. Any geodesic with length <1 must be simple, by e.g. [Bu92, Theorem 4.2.4], so we can apply Mirzakhani's integration formula directly to bound their contribution.

- To bound (d), we cannot apply Mirzakhani's integration formula directly since the geodesics are not simple. Instead, we pass from non-simple geodesics to subsurfaces with simple geodesic boundary and apply Mirzakhani's integration formula to the simple boundary geodesics.

4.3.2 Contribution of simple separating geodesics

In this subsection we bound term (a) in (4.15), the contribution of simple separating geodesics. In particular, we prove the following.

Lemma 4.14.

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)\right]\ll n^2 g.$$

Proof. We have

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma)\right] = \frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma) dX. \tag{4.16}$$

We shall apply Mirzakhani's integration formula, Theorem 4.10, to bound the integral in (4.16). Recall that $S_{g,n}$ is a topological surface with genus g and n labeled punctures. For $0 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor$, $0 \leqslant j \leqslant n$, let $\alpha_{i,j}$ be a simple closed curve in $S_{g,n}$ which separates $S_{g,n}$ into subsurfaces $S_{i,j+1}$ and $S_{g-i,n-j+1}$, each with one boundary component and j and $n-j$ punctures respectively. Then $\alpha_{i,j}$ partitions the punctures into two disjoint subsets I and J of size j and $n-j$ respectively. Let $[\alpha_{i,j}]$ denote the homotopy class of $\alpha_{i,j}$. The orbit $\mathrm{MCG}_{g,n} \cdot [\alpha_{i,j}]$ is determined by the set $\{(i, j+1, I), (g-i, n-j+1, J)\}$, since the mapping class group does not permute the punctures. Therefore given i and j, there are $\binom{n}{j}$ $\mathrm{MCG}_{g,n}$-orbits of simple separating closed curves on $S_{g,n}$ which separate off a subsurface with genus i and with j punctures. Recalling that

$$H_{X,1}(\gamma) = \frac{l_\gamma(X)}{2\sinh\left(\frac{l_\gamma(X)}{2}\right)} f_T\left(l_\gamma(X)\right),$$

we now apply Mirzakhani's integration formula, Theorem 4.10, to see

$$\begin{aligned}
&\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{\gamma\in\mathcal{P}_{sep}^{s}(X)} H_{X,1}(\gamma) dX \\
\leqslant &\sum_{\substack{0\leqslant i\leqslant g, 0\leqslant j\leqslant n \\ 2\leqslant 2i+j\leqslant 2g+n-2}} \int_0^\infty \binom{n}{j} \frac{x^2}{\sinh\left(\frac{x}{2}\right)} f_T(x) \frac{V_{i,j+1}\left(\underline{0}_j, x\right) V_{g-i,n-j+1}\left(\underline{0}_{n-j}, x\right)}{V_{g,n}} dx.
\end{aligned}$$

By Lemma A.1,

$$V_{a,b}(\underline{0}_{b-1}, x) \leqslant \frac{2\sinh\left(\frac{x}{2}\right)}{x} V_{a,b},$$

giving

$$\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}}\sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)dX$$
$$\leqslant\frac{4}{V_{g,n}}\left(\sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}}\binom{n}{j}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}}\right)\int_0^\infty\sinh\left(\frac{x}{2}\right)f_T(x)dx.$$

Since f_T is bounded independently of T and supported in $[0,T)$, we see

$$\int_0^\infty\sinh\left(\frac{x}{2}\right)f_T(x)dx\ll e^{\frac{T}{2}}.$$

By Lemma A.4,

$$\sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}}\frac{n!}{j!\,(n-j)!}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}}\ll\frac{n^2}{g},$$

giving

$$\mathbb{E}^{g,n}_{WP}\left[\sum_{\gamma\in\mathcal{P}^s_{sep}(X)}H_{X,1}(\gamma)\right]\ll\frac{n^2}{g}\cdot e^{\frac{T}{2}}\ll n^2g,$$

as claimed. □

4.3.3 Contribution of simple non-separating geodesics

In this subsection we deal with the contribution of simple non-separating geodesics (term (b) in (4.15)). We prove the following.

Lemma 4.15.

$$\left|\mathbb{E}^{g,n}_{WP}\left[\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)}H_{X,1}(\gamma)\right]-\hat{f}\left(\frac{i}{2}\right)\right|\ll n^2g+n\cdot\log(g)^2\cdot g.$$

Proof. Let α_0 be an unoriented simple non-separating closed curve in $S_{g,n}$. There is just one $\mathrm{MCG}_{g,n}$-orbit of simple non-separating closed curves on $S_{g,n}$ and we have

$$\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)} H_{X,1}(\gamma)dX = 2\sum_{\gamma\in\mathrm{MCG}_{g,n}\cdot\alpha_0} H_{X,1}(\gamma),$$

where the factor of 2 occurs since geodesics in $\mathcal{P}(X)$ are oriented. Applying Mirzakhani's integration formula, we get

$$\int_{\mathcal{M}_{g,n}}\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)} H_{X,1}(\gamma)dX = \frac{1}{2}\int_0^\infty \frac{x^2}{\sinh(\frac{x}{2})} f_T(x)V_{g-1,n+2}\left(\underline{0}_n, x, x\right)dx,$$

where the factor $\frac{1}{2}$ occurs since α_0 is simple and non-separating, c.f. Theorem 4.10. By Theorem A.3,

$$V_{g-1,n+2} = V_{g,n}\cdot\left(1+O\left(\frac{n^2}{g}\right)\right).$$

Then we have, by applying Lemma A.1,

$$\frac{V_{g-1,n+2}\left(\underline{0}_n,x,x\right)}{V_{g,n}} = \left(\frac{2\sinh\frac{x}{2}}{x}\right)^2\left(1+O\left(\frac{n^2+nx^2}{g}\right)\right).$$

This gives

$$\begin{aligned}&\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}}\sum_{\gamma\in\mathcal{P}^s_{nsep}(X)}\frac{l_\gamma(X)}{\sinh\left(\frac{l_\gamma(X)}{2}\right)} f_T\left(l_\gamma\left(X\right)\right)dX\\ &= \int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x)\left(1+O\left(\frac{n^2+nx^2}{g}\right)\right)dx.\end{aligned}$$

Since $\hat{f_T}\left(\frac{i}{2}\right)$ is even,

$$\hat{f_T}\left(\frac{i}{2}\right) = \int_0^T 2\cosh\left(\frac{x}{2}\right) f_T(x)dx,$$

and we have

$$\begin{aligned}
&\left|\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}_{nsep}^{s}(X)} H_{X,1}(\gamma)\right] - \hat{f}_T\left(\frac{i}{2}\right)\right| \\
=&\left|\int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x)\left(1+O\left(\frac{1+n^2+nx^2}{g}\right)\right)dx - \int_0^T 2\cosh\left(\frac{x}{2}\right) f_T(x)dx\right| \\
\ll&\left|\int_0^T 2\left(\sinh\left(\frac{x}{2}\right)-\cosh\left(\frac{x}{2}\right)\right)\cdot f_T(x)dx\right| + \left|\int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x)\left(\frac{n^2+nx^2}{g}\right)dx\right|.
\end{aligned}$$

Using that $2\left(\cosh\left(\frac{x}{2}\right)-\sinh\left(\frac{x}{2}\right)\right)=e^{-x}$,

$$\left|\int_0^T 2\left(\sinh\left(\frac{x}{2}\right)-\cosh\left(\frac{x}{2}\right)\right)\cdot f_T(x)dx\right| \ll 1.$$

Recalling $T=4\log g$, we calculate

$$\left|\int_0^T 2\sinh\left(\frac{x}{2}\right) f_T(x)\left(\frac{1+n^2+n^2x}{g}\right)dx\right| \ll \frac{e^{\frac{T}{2}}\left(n^2+nT^2\right)}{g} \ll n^2 g + n\cdot\log(g)^2\cdot g,$$

and

$$\left|\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}_{nsep}^{s}(X)} H_{X,1}(\gamma)\right] - \hat{f}_T\left(\frac{i}{2}\right)\right| \ll n^2 g + n\cdot\log(g)^2\cdot g,$$

as claimed. □

4.3.4 Iterates of primitive geodesics

We now look at the contribution of iterates of primitive geodesics (term (c) in (4.15)). The aim of this subsection is to prove the following.

Lemma 4.16.

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty} H_{X,k}(\gamma)\right] \ll \log(g)^2\cdot g.$$

In order to prove Lemma 4.16, we need the following soft geodesic counting bound.

Lemma 4.17. *For any $X\in\mathcal{M}_{g,n}$ and any $L>0$ we have*

$$\#\{\gamma\in\mathcal{P}(X) \mid 1\leqslant l_\gamma(X)\leqslant L\} \ll g e^L.$$

Proof. Let $\#_0(X, L)$ denote the number of closed geodesics on X with length $\leqslant L$ which are not iterates of closed geodesics of length $\leqslant 2\text{arcsinh}(1)$. An immediate adaptation of the proof of [Bu92, Lemma 6.6.4] using the non-compact version of the collar lemma ([Bu92, Lemma 4.4.6]) tells us that

$$\#_0(X, L) \leqslant \left(g - 1 + \frac{n}{2}\right) e^{L+6}.$$

[Bu92, Lemma 4.4.6] also tells us that the number of primitive geodesics on X with length $\leqslant 4\text{arcsinh}(1)$ is bounded above by $3g - 3 + n$. Using that $n = o\left(\sqrt{g}\right)$, we conclude that

$$\#\{\gamma \in \mathcal{P}(X) \mid 1 \leqslant l_\gamma(X) \leqslant L\} \leqslant \left(g - 1 + \frac{n}{2}\right) e^{L+6} + 3g - 3 + n \ll g e^L,$$

as claimed. $\square$

We now proceed with the proof of Lemma 4.16.

Proof of Lemma 4.16. Let $X \in \mathcal{M}_{g,n}$. We write

$$\sum_{\gamma \in \mathcal{P}(X)} \sum_{k=2}^{\infty} H_{X,k}(\gamma) = \sum_{\{\gamma \in \mathcal{P}(X) | l_\gamma(X) < 1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma) + \sum_{\{\gamma \in \mathcal{P}(X) | l_\gamma(X) \geqslant 1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma).$$

By Lemma 4.16,

$$\#\{\gamma \in \mathcal{P}(X) \mid 1 \leqslant l_\gamma(X) \leqslant L\} \ll g e^L.$$

We then have

$$\begin{aligned}
\sum_{\{\gamma \in \mathcal{P}(X) | l_\gamma(X) \geqslant 1\}} \sum_{k=2}^{\infty} H_{X,k}(\gamma) &\ll \sum_{\{\gamma \in \mathcal{P}(X) | 1 \leqslant l_\gamma(X) \leqslant \frac{T}{2}\}} l_\gamma(X) e^{-l_\gamma(X)} \\
&\leqslant \sum_{m=1}^{\lfloor \frac{T}{2} \rfloor} m e^{-m} \cdot \#\{\gamma \in \mathcal{P}(X) \mid m \leqslant l_\gamma(X) \leqslant m+1\} \\
&\ll g \sum_{m=1}^{\lfloor \frac{T}{2} \rfloor} m \ll (\log g)^2 \cdot g.
\end{aligned}$$

Taking Weil-Petersson expectations, we see

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\right]=\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\right]+O\left((\log g)^2 g\right). \tag{4.17}$$

For each $\gamma\in\mathcal{P}(X)$,

$$H_{X,k}(\gamma)=\frac{l_\gamma(X)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)}f_T\left(kl_\gamma(X)\right)\leqslant f(0),$$

and if $k\geqslant\frac{T}{l_\gamma(X)}$ then $f_T\left(kl_\gamma(X)\right)=0$. This tells us that

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\right]\leqslant f(0)\cdot T\cdot\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}}\frac{1}{l_\gamma(X)}\right]. \tag{4.18}$$

It remains to bound

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}}\frac{1}{l_\gamma(X)}\right].$$

Any geodesic $\gamma\in\mathcal{P}(X)$ with length $l_\gamma(X)\leqslant 1<4\text{arcsinh}1$ must be simple by e.g. [Bu92, Theorem 4.2.4]. Therefore we can apply Mirzakhani's integration formula to get

$$\begin{aligned}\mathbb{E}_{WP}^{g,n}\left[\sum_{\{\gamma\in\mathcal{P}(X)|l_\gamma(X)<1\}}\frac{1}{l_\gamma(X)}\right]&\leqslant\frac{1}{V_{g,n}}\int_0^1 V_{g-1,n+2}(\underline{0}_n,t,t)dt\\&\quad+\sum_{\substack{0\leqslant i\leqslant g,0\leqslant j\leqslant n\\2\leqslant 2i+j\leqslant 2g+n-2}}\frac{n!}{j!\,(n-j)!}\cdot\frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}}\\&\ll\frac{V_{g-1,n+2}}{V_{g,n}}+\frac{n^2}{g}\ll 1,\end{aligned} \tag{4.19}$$

where on the last line we applied Lemma A.4 and Theorem A.3. Thus combining (4.17), (4.18) and (4.19) we see

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=2}^{\infty}H_{X,k}(\gamma)\right]\ll(\log g)^2\cdot g,$$

as required. $\square$

4.3.5 Non-simple geodesics

We now need to deal with the contribution of the non-simple primitive geodesics, (term (d) in (4.15)). In this subsection we shall prove the following

Lemma 4.18. *There is a constant $\beta_1 > 0$ such that for any $\varepsilon_1 > 0$ there is a constant $c_1(\varepsilon_1) > 0$ such that*

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma)\right] \ll (\log g)^6 \cdot g + c_1(\varepsilon_1)(\log g)^{\beta_1} \cdot n^2 \cdot g^{1+4\varepsilon_1}.$$

We prove Lemma 4.18 through a sequence of lemmas. Before we give a brief outline of the method, we need the concept of a filling closed curve.

Definition 4.19. Let X be a finite-area hyperbolic surface with possible boundary. A closed curve $\eta \subset Y$ is filling if the complement $Y\backslash\eta$ is a disjoint union of disks and cylinders such that every cylinder either deformation retracts to a boundary component of Y or is a neighbourhood of a cusp. We let $\#_{\text{fill}}(X, L)$ denote the number of oriented filling geodesics on X with lengths $\leqslant L$.

Idea of the proof of Lemma 4.18

We shall extend the method of [WX21, Section 7] to non-compact surfaces. The basic idea is as follows.

- Given a surface $X \in \mathcal{M}_{g,n}$ and a geodesic $\gamma \in \mathcal{P}^{ns}(X)$, we construct a subsurface $X(\gamma)$ of X with geodesic boundary (of controlled length) which is filled by γ. The multiplicity of the map $\gamma \mapsto X(\gamma)$ is bounded by the number of filling geodesics of $X(\gamma)$. This allows us to write

$$\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \leqslant \sum_{\substack{Y \text{ subsurface of } X \\ Y \text{ has geodesic boundary}}} \; \sum_{\text{filling geodesics } \gamma \text{ on } Y} H_{X,1}(l_X(\gamma)).$$

- We control the length of a filling geodesic in terms of $l_X(\partial Y)$ in Lemma 4.21 and apply [WX21, Theorem 4] to bound the number of filling geodesics on a subsurface and show that there is an explicit function A, supported in $[0,2T)$, with

$$\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \leqslant \sum_{\substack{Y \text{ subsurface of } X\\ Y \text{ has geodesic boundary}}} A\left(l_X\left(\partial Y\right)\right).$$

- Since the boundary of each subsurface Y consists of simple closed geodesics, we can apply Mirzakhani's integration formula to bound the Weil-Petersson expectation of

$$\sum_{\{Y\text{subsurface of } X \text{ with geodesic boundary}\}} A\left(l_X\left(\partial Y\right)\right).$$

Definition 4.20. Let $X \in \mathcal{M}_{g,n}$ be a hyperbolic surface and let $\gamma \subset X$ be a non-simple closed geodesic. Let $N_\delta(\gamma)$ denote the δ-neighborhood of γ where δ is sufficiently small to ensure that $N_\delta(\gamma)$ deformation retracts to γ and that the boundary $\partial N_\delta(\gamma)$ is a disjoint union of simple closed curves. We define $X(\gamma)$ to be the connected subsurface obtained from $N_\delta(\gamma)$ as follows: for each boundary component $\xi \in N_\delta(\gamma)$,

- If ξ bounds a disc we fill the disc into $N_\delta(\gamma)$.
- If ξ is homotopically non-trivial we shrink it to the unique simple closed geodesics in its free homotopy class and deform $N_\delta(\gamma)$ accordingly.
- If two different components ξ, ξ' deform to the same geodesic then we do not glue them together, we view $X(\gamma)$ as an open subsurface of X.
- If ξ is freely homotopic to a closed horocycle bounding a cusp C_i we fill the cusp into $N_\delta(\gamma)$.

After deforming $N_\delta(\gamma)$ in this way we obtain the surface $X(\gamma)$.

The construction of $X(\gamma)$ allows us to control $\mathrm{Vol}\left(X(\gamma)\right)$ and the length of $\partial X(\gamma)$ in terms of $l_\gamma(X)$, as summarized by the following lemma. Bounding $\mathrm{Vol}\left(X(\gamma)\right)$ corresponds to bounding the Euler characteristic of $X(\gamma)$ by Gauss-Bonnet.

Lemma 4.21. *Let $X \in \mathcal{M}_{g,n}$ and γ be a non-simple closed geodesic on X. The subsurface $X(\gamma)$ of X satisfies*

1. γ is a filling geodesic of $X(\gamma)$.

2. The length of the boundary satisfies

$$l\left(\partial X(\gamma)\right) \leqslant 2l_{\gamma}(X).$$

3. The volume satisfies

$$\mathrm{Vol}\left(X(\gamma)\right) \leqslant 4l_{\gamma}(X).$$

Lemma 4.21 is proved in [NWX20, Proposition 47] for compact surfaces. The proof in our case is identical. This leads us to make the following definition.

Definition 4.22. With $T = 4\log g$, $X \in \mathcal{M}_{g,n}$, we define

$$\mathrm{Sub}(X) \stackrel{\text{def}}{=} \{Y \subset X \mid Y \text{ is a connected subsurface of } X \text{ with geodesic boundary}\},$$

and

$$\mathrm{Sub}_T(X) \stackrel{\text{def}}{=} \{Y \in \mathrm{Sub}(X) \mid l(\partial Y) \leqslant 2T, \mathrm{Vol}(Y) \leqslant 4T\},$$

where we allow two distinct simple closed geodesics on the boundary of Y to be a single simple closed geodesic in X.

Lemma 4.21 tells us that for any $X \in \mathcal{M}_{g,n}$, any non-simple geodesic γ with length $\leqslant T$ fills a subsurface $X(\gamma) \in \mathrm{Sub}_T(X)$. If any other $\gamma' \in \mathcal{P}(X)$ satisfies $X(\gamma') = X(\gamma)$ then γ' is also a filling geodesic of $X(\gamma)$ with length $\leqslant T$. We have

$$\{\gamma' \in \mathcal{P}^{ns}(X) \mid X(\gamma') = X(\gamma)\} \subseteq \{\text{oriented filling geodesics of } X(\gamma) \text{ with length } \leqslant T\}. \tag{4.20}$$

Therefore we will need to control the number of non-simple geodesics which fill a given subsurface. This is achieved by the following theorem.

Theorem 4.23 ([WX21, Theorem 4])**.** *Let $m = 2g'-2+n' \geqslant 1$. For any $\varepsilon_1 > 0$ there exists a constant $c(\varepsilon_1, m)$ only depending on ε_1 and m such that for any $X \in \mathcal{M}_{g',n'}(x_1, ..., x_{n'})$ where $x_i \geqslant 0$, we have*

$$\#_{\text{fill}}(X, L) \leqslant c(\epsilon_1, m) \cdot e^{L - \frac{1-\varepsilon_1}{2}\sum_{i=1}^{n} x_i}.$$

Remark 4.24. [WX21, Theorem 4] is stated in for surfaces without cusps, i.e. $x_i > 0$, however the extension to $x_i \geqslant 0$ is immediate. Indeed, [WX21, Theorem 4] follows from [WX21, Theorem 38] and [WX21, Lemma 10]. [WX21, Theorem 38] already holds for non-compact surfaces and it is straightforward to check that the basic counting result [WX21, Lemma 10] generalizes to non-compact surfaces.

We can now pass from non-simple geodesics to subsurfaces with geodesic boundary. This is done in the following lemma, proved in [WX21, Proposition 30] for $X \in \mathcal{M}_g$. The proof is identical in our case.

Lemma 4.25. *For any $\varepsilon_1 > 0$, $X \in \mathcal{M}_{g,n}$, there exists a constant $c_1(\varepsilon_1)$ only depending on ε_1 such that*

$$\sum_{\gamma \in \mathcal{P}^{ns}(X)} H_{X,1}(\gamma) \ll Te^{T} \sum_{\substack{Y \in \mathrm{Sub}_T(X) \\ |\chi(Y)| \geqslant 34}} e^{-\frac{l(\partial Y)}{4}} + c_1(\varepsilon_1)\, T \sum_{\substack{Y \in \mathrm{Sub}_T(X) \\ 1 \leqslant |\chi(Y)| \leqslant 33}} e^{\frac{T}{2} - \frac{1-\epsilon_1}{2} l(\partial Y)}. \tag{4.21}$$

Remark 4.26. The difference between the first and second term arises because we apply Theorem 4.23 to subsurfaces with $1 \leqslant |\chi(Y)| \leqslant 34$ whereas we only apply a soft geodesic counting result, $\#_{\text{fill}}(X, L) \leqslant \text{Area}(X) \cdot e^{L+6}$, to subsurfaces with $|\chi(Y)| \geqslant 34$. The reason for this is that it is not clear how badly the constant $c(\epsilon_1, m)$ from Theorem 4.23 depends on the Euler characteristic m so we can only apply Theorem 4.23 to subsurfaces with uniformly bounded Euler characteristic. As a consequence of forthcoming calculations, the Weil-Petersson expectation of the number of subsurfaces $Y \in \mathrm{Sub}_T(X)$ with $|\chi(Y)| \geqslant k$ is sufficiently small for any $k \geqslant 34$ so that we can accept the loss from the soft geodesic counting.

For the remainder of the section, we assume that g is sufficiently large so that for $Y \in \mathrm{Sub}_T(X)$, the map $Y \mapsto \partial Y$ is injective. This is justified since any two distinct subsurfaces in $Y_1, Y_2 \in \mathrm{Sub}_T(X)$ with $\partial Y_1 = \partial Y_2$ must satisfy $Y_1 \cup Y_2 = X$, giving

$$\mathrm{Vol}\,(X) = 2\pi\,(2g-2+n) \leqslant \mathrm{Vol}\,(Y_1) + \mathrm{Vol}\,(Y_2) \leqslant 8T = 32\log g,$$

which is not possible for sufficiently large g.

We now want to apply Mirzakhani's integration formula to bound the Weil-Petersson expectation of the right hand side of (4.21). We introduce the following notation.

Notation 4.27. Let $X \in \mathcal{M}_{g,n}$. For a subsurface $Y_0 \in \mathrm{Sub}_T(X)$, we write

$$Y_0 = Y_0\,(q, (g_0, a_0, n_0)\,, \{(g_1, a_1, n_1)\,, \ldots, (g_q, a_q, n_q)\}) = Y_0\left(q, \underline{g}, \underline{a}, \underline{n}\right),$$

to indicate that Y_0 has the following properties.

- Y_0 is homeomorphic to $S_{g_0,k+a_0}$where $k > 0$.
  - Y_0 has a_0 cusps and k simple geodesic boundary components. There are $n_0 \geqslant 0$ pairs of simple geodesics in Y_0 which correspond to a single simple closed geodesic in X.
  - The interior of its complement $X \backslash Y_0$ consists of $q \geqslant 1$ components $Y_1, ..., Y_q$ where Y_i is homeomorphic to S_{g_i,n_i+a_i}. We observe that $n_i \geqslant 1$ and
    i) $\sum_{i=1}^{q} 2g_i - 2 + n_i + a_i = 2g - 2 + n - |\chi\,(Y_0)|\,.$
    ii) $\sum_{i=1}^{q} n_i = k - 2n_0.$
    iii) $\sum_{j=1}^{q} a_j = n - a_0.$

Given $X \in \mathcal{M}_{g,n}$ and a choice of marking, any $Y_0(q, \underline{a}, \underline{n}, \underline{g}) \in \mathrm{Sub}_T(X)$ is freely homotopic to the image under the marking of a subsurface $Y \subset S_{g,n}$ where Y is in the $\mathrm{MCG}_{g,n}$-orbit of a subsurface $\tilde{Y}_0 = \tilde{Y}_0(q, \underline{a}, \underline{n}, \underline{g}) \subset S_{g,n}$ (with $\tilde{Y}_0$ homeomorphic to $S_{g_0,k+a_0}$, where $S_{g,n} \backslash \tilde{Y}_0$ has q components $\tilde{Y}_1, ..., \tilde{Y}_q$ with $\tilde{Y}_i$ homeomorphic to S_{g_i,n_i+a_i} with n_i boundary components and a_i punctures). We write $\left[\tilde{Y}_0\right]$ to denote the homotopy class of $\tilde{Y}_0$. Since the mapping class group does not permute the punctures of $S_{g,n}$, the number of distinct $\mathrm{MCG}_{g,n}$-orbits of subsurfaces corresponding to a given choice of $q, (g_0, a_0, n_0)\,, \{(g_1, a_1, n_1)\,, \ldots, (g_q, a_q, n_q)\}$

is bounded above by

$$\frac{n!}{a_0! \cdot \dots \cdot a_q!}.$$

Lemma 4.28.

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in Sub_T(X)\\ |\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right] \ll \frac{(\log g)^5}{g^3}. \tag{4.22}$$

Proof. We start by bounding the contribution of a given $\mathrm{MCG}_{g,n}$-orbit to (4.22). Let g_0, a_0, k be fixed with $m = 2g_0 - 2 + k + a_0 \geqslant 34$. By Gauss-Bonnet, we have that $m \leqslant \frac{4T}{2\pi} \leqslant \frac{5}{2}\log g$. For $n_0, n_1, \dots, n_q, a_1, \dots, a_q, g_1, \dots, g_q \geqslant 0$ with $\sum_{i=1}^q n_i = k - 2n_0$ and $\sum_{j=1}^q a_j = n - a_0$, we have

$$\begin{aligned}
&\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{[Y]\in \mathrm{MCG}_{g,n}\cdot[\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]}\left(l_X\left(\partial Y\right)\right) dX\\
=&\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{[\partial Y]\in \mathrm{MCG}_{g,n}\cdot[\partial\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]}\left(l_X\left(\partial Y\right)\right) dX,
\end{aligned}$$

since the map $Y \mapsto \partial Y$ is injective. By applying Mirzakhani's integration formula, one can compute that

$$\begin{aligned}
&\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}} \sum_{[Y]\in \mathrm{MCG}_{g,n}\cdot[\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})]} e^{-\frac{l(\partial Y)}{4}} \mathbf{1}_{[0,2T]}\left(l_X\left(\partial Y\right)\right) dX\\
\ll& e^{\frac{7}{2}T}\frac{V_{g_0,k+a_0}V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{V_{g,n}\cdot n_0!n_1!\cdots n_q!}.
\end{aligned}$$

A near identical computation is carried out in detail in [WX21, Proposition 31] so we omit it here. We now sum over the $\mathrm{MCG}_{g,n}$-orbits to bound the contribution of subsurfaces in

$\mathrm{Sub}_T(X)$ with a given Euler characteristic. We calculate

$$
\begin{aligned}
&\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in \mathrm{Sub}_T(X)\\ Y\cong S_{g_0,k+a_0}}} e^{-\frac{l(\partial Y)}{4}}\right]\\
\leqslant &\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2a_0}\sum_{\mathcal{A}}\frac{1}{V_{g,n}}\cdot\binom{n}{a_0,\dots,a_q}\cdot\int_{\mathcal{M}_{g,n}}\sum_{[Y]\in \mathrm{MCG}_{g,n}\cdot[\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})]} e^{-\frac{l(\partial Y)}{4}}\mathbf{1}_{[0,2T]}\left(l_X\left(\partial Y\right)\right)dX\\
\ll &e^{\frac{7}{2}T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2a_0}\sum_{\{(g_j,n_j,q_j)\}_{j=1}^{q}\in\mathcal{A}}\binom{n}{a_0,\dots,a_q}\cdot\frac{V_{g_0,k+a_0}V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{V_{g,n}\cdot n_0!n_1!\cdots n_q!},
\end{aligned}
$$

where for a given n_0 and q, the summation is over the set of "admissible triples" $\mathcal{A}$, whose elements we denote by $\{(g_j,n_j,q_j)\}_{j=1}^{q}$, which we define to be the set of $\{(g_1,a_1,n_1),\dots,(g_q,a_q,n_q)\}$ where $g_j,a_j\geqslant 0$, $n_j\geqslant 1$ and $2g_j+a_j+n_j\geqslant 3$ such that

i) $\sum_{i=1}^{q}(2g_i-2+n_i+a_i)=2g-2+n-m.$

ii) $\sum_{i=1}^{q}n_i=k-2n_0.$

iii) $\sum_{j=1}^{q}a_j=n-a_0.$

Recalling that $34\leqslant m=2g_0-2+k+a_0\leqslant\frac{5}{2}\log g$ is fixed, we apply lemma A.5 to see

$$
\begin{aligned}
&\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2n_0}\sum_{\{(g_j,n_j,a_j)\}_{j=1}^{q}\in\mathcal{A}}\frac{n!}{a_0!\cdot\dots\cdot a_q!}\cdot\frac{V_{g_0,k+a_0}V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{V_{g,n}\cdot n_0!n_1!\cdots n_q!}\\
\ll &\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2n_0}(2g_0+k+a_0-3)!\cdot\frac{n^{a_0}}{g^m}\ll\frac{k^2\left(2g_0+k+a_0-3\right)!}{g^m}.
\end{aligned}
$$

Summing over the possible values of g_0, a_0 and k, we calculate

$$
\begin{aligned}
&\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\|\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right]\\
\ll& e^{\frac{7}{2}T}\sum_{0\leqslant a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\ \sum_{1\leqslant k\leqslant\lceil\frac{4T}{2\pi}\rceil+2-a_0}\ \sum_{34\leqslant 2g_0-2+k+a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\Y\cong S_{g_0,k+a_0}}} e^{-\frac{l(\partial Y)}{4}}\right]\\
\ll& e^{\frac{7}{2}T}\sum_{0\leqslant a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\ \sum_{1\leqslant k\leqslant\lceil\frac{4T}{2\pi}\rceil+2-a_0}\ \sum_{34\leqslant 2g_0-2+k+a_0\leqslant\lceil\frac{4T}{2\pi}\rceil}\frac{k^2\,(2g_0+a_0+k-3)!n^{a_0}}{g^{2g_0+a_0+k-2}}\\
\ll& T^5e^{\frac{7T}{2}}\frac{1}{g^{2g_0+\frac{a_0}{2}+k-2}}\ll\frac{T^5e^{\frac{7T}{2}}}{g^{18}},
\end{aligned}
$$

since $2g_0+a_0+k\geqslant 36$ guarantees that $2g_0+\frac{a_0}{2}+k\geqslant 18$. Recalling that $T=4\log g$, we conclude that

$$
\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\|\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right]\ll\frac{(\log g)^5}{g^3},
$$

as required. $\square$

Lemma 4.29. *There is a constant $\beta>0$ such that for any $\varepsilon_1>0$,*

$$
\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in Sub_T(X)\\1\leqslant|\chi(Y)|\leqslant 33}} e^{\frac{T}{2}-\frac{1-\epsilon_1}{2}l(\partial Y)}\right]\ll(\log g)^{\beta}\cdot n^2\cdot g^{1+4\varepsilon_1}.
$$

Proof. Let $\varepsilon_1>0$, $g_0\geqslant 0, a_0\geqslant 0$ and $k\geqslant 1$ be fixed with $1\leqslant m=2g_0-2+k+a_0\leqslant 33$. The computation in [WX21, Proposition 34] gives that there exists a fixed $\beta>0$ [5] with

$$
\begin{aligned}
&\frac{1}{V_{g,n}}\int_{\mathcal{M}_{g,n}}\sum_{\tilde{Y}\in\mathrm{MCG}_{g,n}\cdot\tilde{Y}_0(q,\underline{a},\underline{n},\underline{g})} e^{\frac{T}{2}-\frac{1-\varepsilon_1}{2}l_X(\partial\tilde{Y})}\mathbf{1}_{[0,2T]}\left(l_X\left(\partial\tilde{Y}\right)\right)dX\\
\ll&\frac{T^{\beta}e^{\frac{T}{2}+\varepsilon_1T}}{V_{g,n}n_0!\cdots n_q!}V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}.
\end{aligned}
$$

[5] Note the value of β in [WX21, Proposition 34] is 66 and corresponds to the choice to consider $|\chi(Y)|\leqslant 16$ as opposed to our choice of 33. Here we could for example take $\beta<135$. Fixed powers of $\log g$ will be negligible in the final calculations.

Then we see that

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ Y\cong S_{g_0,k+a_0}}} e^{\frac{T}{2}-\frac{1-\varepsilon_1}{2}l(\partial Y)}\right] \ll T^\beta e^{\frac{T}{2}+\varepsilon_1 T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2a_0}\sum_{\mathcal{A}}\frac{n!}{a_0!\cdots a_q!}\frac{V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{n_0!\cdots n_q! V_{g,n}},$$

where, as before, for given n_0 and q the summation is over the set $\mathcal{A}$ of "admissible triples" $\{(g_j,n_j,q_j)\}_{j=1}^q$ where $g_j,a_j\geqslant 0$, $n_j\geqslant 1$ and $2g_j+a_j+n_j\geqslant 3$ such that $\sum_{i=1}^q 2g_i-2+n_i+a_i=2g-2+n-m$, $\sum_{i=1}^q n_i=k-2n_0$ and $\sum_{j=1}^q a_j=n-a_0$. We apply Lemma A.5 to calculate that

$$\begin{aligned}&T^\beta e^{\frac{T}{2}+\varepsilon_1 T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2n_0}\sum_{\mathcal{A}}\frac{n!}{a_0!\cdots a_q!}\frac{V_{g_1,n_1+a_1}\cdots V_{g_q,n_q+a_q}}{n_0!\cdots n_q! V_{g,n}}\\ \ll& T^\beta e^{\frac{T}{2}+\varepsilon_1 T}\sum_{n_0=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{q=1}^{k-2n_0}\frac{n^{a_0}}{g^{2g_0+a_0+k-2}}\ll T^\beta e^{\frac{T}{2}+\epsilon_1 T}\frac{n^{a_0}}{g^{2g_0+a_0+k-2}}.\end{aligned}$$

We sum over possible values of g_0,a_0 and k to see that

$$\begin{aligned}\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\ 1\leqslant|\chi(Y)|\leqslant 33}} e^{\frac{T}{2}-\frac{1-\epsilon_1}{2}l(\partial Y)}\right]&\ll\sum_{\substack{(g_0,a_0,k)\\ 3\leqslant 2g_0+a_0+k\leqslant 35}} T^\beta e^{\frac{T}{2}+\varepsilon_1 T}\frac{n^{a_0}}{g^{2g_0+a_0+k-2}}\\ &\ll T^\beta e^{\frac{T}{2}+\varepsilon_1 T}\cdot\frac{n^2}{g}\ll(\log g)^\beta n^2 g^{1+4\varepsilon_1},\end{aligned}$$

as claimed. □

We can now prove Lemma 4.18.

Proof of Lemma 4.18. Combining Lemma 4.25, Lemma 4.28 and Lemma 4.29 we deduce that for any $\varepsilon_1 > 0$ there exists a constant $c_1(\varepsilon_1)$ such that

$$\begin{aligned}
&\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}^{ns}(X)} H_{X,1}(\gamma)\right]\\
&\ll e^T T\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\|\chi(Y)|\geqslant 34}} e^{-\frac{l(\partial Y)}{4}}\right] + c_1(\varepsilon_1)\, T\mathbb{E}_{WP}^{g,n}\left[\sum_{\substack{Y\in\mathrm{Sub}_T(X)\\1\leqslant|\chi(Y)|\leqslant 33}} e^{\frac{T}{2}-\frac{1-\varepsilon_1}{2}l(\partial Y)}\right]\\
&\ll (\log g)^6 g + c_1(\varepsilon_1)(\log g)^{\beta+1} n^2 g^{1+4\varepsilon_1},
\end{aligned}$$

concluding the proof. □

4.3.6 Proof of Theorem 4.11

Finally we conclude the section with the proof of Theorem 4.11.

Proof of Theorem 4.11. By Lemma 4.14, Lemma 4.15, Lemma 4.16 and Lemma 4.18 together with (4.15) we see that there is a constant β such that for any $\varepsilon_1 > 0$ there exists a constant $c_1(\varepsilon_1)$ with

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty} H_{X,k}(\gamma) - \hat{f}_T\left(\frac{i}{2}\right)\right] \ll n^2 g + \log(g)^6 g + c_1(\varepsilon_1)(\log g)^{\beta+1} n^2 g^{1+4\epsilon_1}.$$

□

4.4 Proof of Theorem 1.15

We now conclude with the proof of Theorem 1.15.

Proof of Theorem 1.15. Let $n = O(g^{\alpha})$ for some $0 \leqslant \alpha < \frac{1}{2}$ and let $0 < \varepsilon < \min\left\{\frac{1}{4}, \frac{1}{2} - \alpha\right\}$ be given. For $X \in \mathcal{M}_{g,n}$, we define

$$\tilde{\lambda}_1(X) \stackrel{\text{def}}{=} \begin{cases} \lambda_1(X) & \text{if it exists,}\\ \frac{1}{4} & \text{otherwise.}\end{cases}$$

Our aim is to prove that

$$\mathbb{P}_{WP}^{g,n}\left[\tilde{\lambda}_1\left(X\right)\leqslant\frac{1}{4}-\frac{\left(2\alpha+1\right)^2}{16}-\varepsilon\right]\to 0,$$

as $g\to\infty$. By Remark 4.9, there exists a constant $\nu\geqslant 0$ such that for g sufficiently large,

$$\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}\frac{l_\gamma\left(X\right)}{2\sinh\left(\frac{kl_\gamma(x)}{2}\right)}f_T\left(kl_\gamma\left(X\right)\right)-\hat{f}_T\left(\frac{i}{2}\right)+\nu ng\geqslant 0,$$

for any $X\in\mathcal{M}_{g,n}$. By Theorem 4.11, for any $\varepsilon_1>0$ there is constant $c_1\left(\varepsilon_1\right)>0$ with

$$\mathbb{E}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)+\nu ng\right]\ll n^2g+\log\left(g\right)^6g+c_1\left(\varepsilon_1\right)\left(\log g\right)^{\beta+1}n^2g^{1+4\varepsilon_1},$$

where $\beta>0$ is a universal constant. Taking $\varepsilon_1<\frac{\varepsilon}{8}$ and applying Markov's inequality,

$$\mathbb{P}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)+\nu ng>n^2g^{1+\varepsilon}\right]\ll_\epsilon\left(1+\frac{\log\left(g\right)^6}{n^2}+\left(\log g\right)^{\beta+1}\right)g^{-\frac{\varepsilon}{2}}.$$

However, if $X\in\mathcal{M}_{g,n}$ has $\lambda_1(X)\leqslant\frac{1}{4}-\frac{(2\alpha+1)^2}{16}-\varepsilon$, then since $\alpha\in[0,\frac{1}{2})$ this guarantees that $\lambda_1(X)\leqslant\frac{3}{16}$ and we can apply Theorem 4.2 to see

$$C(\varepsilon)\log\left(g\right)g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1(X)}}\leqslant\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)+O\left(ng\right).$$

But since $\varepsilon<\frac{1}{2}-\alpha$,

$$\sqrt{\frac{1}{4}-\lambda_1(X)}\geqslant\frac{2\alpha+1}{4}+\varepsilon,$$

and we deduce that

$$C(\varepsilon)\log\left(g\right)g^{4(1-\varepsilon)\sqrt{\frac{1}{4}-\lambda_1(X)}}\geqslant C(\varepsilon)\log\left(g\right)g^{(1-\varepsilon)((2\alpha+1)+4\varepsilon)}\gg_\epsilon g^{2\alpha+1+2\varepsilon-4\varepsilon^2}>n^2g^{1+\varepsilon},$$

for sufficiently large g. On the last line we used that $\varepsilon<\frac{1}{4}$ and that $n=O\left(g^\alpha\right)$. We deduce that

$$\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)>n^2g^{1+\varepsilon},$$

for sufficiently large g. This tells us that for g sufficiently large,

$$\begin{aligned}\mathbb{P}_{WP}^{g,n}\left[\tilde{\lambda}_1(X)\leqslant\frac{1}{4}-\frac{(2\alpha+1)^2}{16}-\varepsilon\right]&\leqslant\mathbb{P}_{WP}^{g,n}\left[\sum_{\gamma\in\mathcal{P}(X)}\sum_{k=1}^{\infty}H_{X,k}(\gamma)-\hat{f}_T\left(\frac{i}{2}\right)+\nu ng>n^2g^{1+\varepsilon}\right]\\&\ll_\varepsilon\left(1+\frac{\log(g)^6}{n}+(\log g)^{\beta+1}\right)g^{-\frac{\varepsilon}{2}}\to 0,\end{aligned}$$

as $g\to\infty$. □

5 Further problems

In this section, we highlight some further interesting problems on the spectral theory of random cusped hyperbolic surfaces.

5.0.1 Embedded eigenvalues

A fascinating, fundamental open problem is whether a finite-area non-compact hyperbolic surface X has to have infinitely many cusp forms [PS85]. Since any eigenvalue above $\frac{1}{4}$ is necessarily a cusp form, this is equivalent to whether $\operatorname{spec}(\Delta_X)$ contains infinitely many L^2-eigenvalues. In fact, weaker forms of this question are still open, which we now explain.

Let $\lambda_j=\frac{1}{4}+t_j^2\in\operatorname{spec}(\Delta_X)$ be the L^2 eigenvalues of Δ_X and define the counting function

$$N_X(T)\stackrel{\text{def}}{=}\#\{j\mid r_j\leqslant T\}.$$

We also define

$$M_X(T)\stackrel{\text{def}}{=}\frac{1}{4\pi}\int_{-T}^{T}-\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it\right)dt.$$

Roughly, $M_X(T)$ counts the number of poles of the scattering determinant φ in $\operatorname{Re}(s)<\frac{1}{2}$ up to height at most T and up to error $O(T)$ [Iw02, Section 11.1]. The Weyl law for a finite area hyperbolic surface X reads

$$N_X(T) + M_X(T) \sim \frac{\mathrm{Vol}(X)}{4\pi} T^2. \tag{5.1}$$

A surface X is said to be *essentially cuspidal* if the contribution to the Weyl law (5.1) is dominated by eigenvalues, i.e.

$$N_X(T) \sim \frac{\mathrm{Vol}(X)}{4\pi} T^2.$$

Selberg proved for principal congruence covers $X(N)$ that

$$M_{X(N)}(T) \ll T \log T,$$

from which it follows that

$$N_{X(N)}(T) = \frac{\mathrm{Vol}(X(N))}{4\pi} T^2 + O(T \log T).$$

Selberg conjectured that every surface is essentially cuspidal. Despite being widely believed to be false, Selberg's conjecture is still open. Phillips and Sarnak make a very different conjecture [Sa03, Conjecture 1].

Conjecture 5.1. *Let $2g + n - 3 > 0$ and $n > 0$.*

1. *The generic X in any $\mathcal{T}_{g,n}$ is not essentially cuspidal.*

2. *Except in the case where $g = n = 1$, the generic $X \in \mathcal{T}_{g,n}$ has only finitely many eigenvalues.*

Here generic is meant in the topological sense. The case of the once punctured torus is omitted since every $X \in \mathcal{T}_{1,1}$ has a symmetry of order 2 and the functions which are odd with respect to this symmetry are all cuspidal.

Great progress on this problem was made in a series of works of Phillips and Sarnak [PS85, PS85b, PS92] and Wolpert [Wo92, Wo94]. These works show that the first part of Conjecture 5.1 is true under assumptions on the multiplicities of eigenvalues on $X(N)$.

By the work of Phillips and Sarnak, together with work of Luo [Lu01], if the multiplicities $m(\lambda)$ of eigenvalues λ of $X(N)$ are uniformly bounded (for each fixed N) then part (1) of Conjecture 5.1 holds. By the work of Wolpert, if every eigenvalue on the thrice punctured sphere $X(2)$ is simple (i.e. has multiplicity one) then part (1) of Conjecture 5.1 holds.

It is interesting to ask whether probabilistic methods could be used to approach this problem. We make the following conjecture.

Conjecture 5.2. *A random degree-n cover of a non-compact finite-area hyperbolic surface has only finitely many new embedded eigenvalues with probability tending to 1 as $n \to \infty$.*

In the next section we briefly highlight some (non)-examples around Conjecture 5.2.

5.0.2 Examples

Flat bundles

We recall the construction described in Remark 1.11. Let $X = \Gamma\backslash\mathbb{H}$ be a non-compact finite-area hyperbolic surface. Given $\varphi \in \mathrm{Hom}(\Gamma, \mathrm{U}(n))$, one can consider the associated $\mathbb{C}^n$ bundle X_φ and its Laplacian Δ_φ on L^2-sections. Since Γ is a free group, any $\varphi \in \mathrm{Hom}(\Gamma, \mathrm{U}(n))$ is determined by the images of the generators $\gamma_1, \ldots, \gamma_d$ of Γ. We obtain a probability measure on $\mathrm{Hom}(\Gamma, \mathrm{U}(n))$ by picking the images of $\gamma_1, \ldots, \gamma_d$ independently with Haar probability.

One might try to adapt the approach of [HM23] to say something about Conjecture 5.2. Letting $\gamma_{\mathfrak{a}_1}, \ldots, \gamma_{\mathfrak{a}_k}$ denote the generators of the stability groups of the cusps of X, if $\varphi(\gamma_{\mathfrak{a}_1}), \ldots, \varphi(\gamma_{\mathfrak{a}_k})$ do not have 1 as an eigenvalue, then the twisted Laplacian Δ_φ has purely discrete spectrum [Se89], in particular it must have infinitely many eigenvalues (although they are not embedded). In fact, this happens almost surely for the random ρ_φ. It is known by work of Collins and Male [CM14] that $(\rho_\varphi, \mathbb{C}^n)$ strongly converge to $\left(\lambda, l^2(\Gamma)\right)$ with probability tending to 1 as $n \to \infty$. Any approach to Conjecture 5.2 using strong convergence must be wary of this fact.

Covers

The fact that if a unitary representation $\rho : \Gamma \to \mathrm{End}(V)$ has the property that $\rho(\gamma_{\mathfrak{a}_1}), \ldots, \rho(\gamma_{\mathfrak{a}_k})$ have no fixed vectors then the twisted Laplacian Δ_ρ has purely discrete spectrum yields other examples of surfaces with infinitely many cusp forms.

Example 5.3 ([Ve90])**.** Let X be a non-compact finite-area hyperbolic surface and let X_ϕ be a cover of X such that X_ϕ and X have the same number of cusp, then X_ϕ has infinitely many cusp forms.

This follows since the Eisenstein series in the cover are all lifts, in the sense that

$$E_{\mathfrak{a}_i,\phi}(s) = \eta(s) E_{\mathfrak{a}_i}(s), \tag{5.2}$$

for some holomorphic function η (which can be explicitly given). One can verify (5.2) by hand from the definition of the Eisenstein series. The conclusion then follows from the Weyl law (5.1) since (5.2) shows that X_ϕ has no new residual eigenvalues.

A Volume estimates

The purpose of this appendix is to prove the necessary Weil-Petersson volume estimates used in the proof of Theorem 4.11. Similar estimates can be found in e.g. [Mi13, MP19, NWX20, GMST21, LW21].

We need the following lemma in the proof of Lemma 4.14 and Lemma 4.15.

Lemma A.1. *Let $x_1, \ldots, x_n \geqslant 0$. For $g, n \geqslant 0$, $2g-2+n > 0$ we have*

$$\frac{V_{g,n}(x_1, ..., x_n)}{V_{g,n}} \leqslant \prod_{i=1}^{n} \frac{\sinh\left(\frac{x_i}{2}\right)}{\left(\frac{x_i}{2}\right)},$$

and

$$\frac{V_{g,n}\left(\underline{0}_{n-2}, x_1, x_2\right)}{V_{g,n}} = \frac{4\sinh\left(\frac{x_1}{2}\right) \cdot \sinh\left(\frac{x_2}{2}\right)}{x_1 \cdot x_2}\left(1 + O\left(\frac{n\left(x_1^2 + x_2^2\right)}{g}\right)\right),$$

as $g \to \infty$, where the implied constant is independent of n.

Remark A.2. Lemma A.1 is due to Mirzakhani and Petri [MP19, Proposition 3.1]. The proof of the second statement is identical to the proof of [NWX20, Lemma 20], if one uses [LW21, Theorem A.1] in place of [Mi13, Page 286].

We require estimates for $V_{g,n}$ where the number of cusps n is allowed to grow with the genus g. The starting point is the following theorem of Mirzakhani and Zograf.

Theorem A.3 ([MZ15, Theorem 1.8])**.** *There exists a constant $B > 0$ such that if $n = o\left(g^{\frac{1}{2}}\right)$, we have*

$$V_{g,n(g)} = \frac{B}{\sqrt{g}} \left(2g-3+n\left(g\right)\right)! \left(4\pi^2\right)^{2g-3+n(g)} \left(1 + O\left(\frac{1+n\left(g\right)^2}{g}\right)\right),$$

as $g \to \infty$.

In order to control the contribution of simple separating geodesics, in Lemma 4.14 we need the following lemma.

Lemma A.4. *If $n = o\left(g^{\frac{1}{2}}\right)$, then*

$$\sum_{\substack{0 \leqslant i \leqslant g, 0 \leqslant j \leqslant n \\ 2 \leqslant 2i+j \leqslant 2g+n-2}} \binom{n}{j} \cdot \frac{V_{i,j+1} V_{g-i,n-j+1}}{V_{g,n}} \ll \frac{1+n^2}{g}.$$

The case that n is fixed is treated in [Mi13, Lemma 3.3]. The fact that the number of cusps is growing with genus and the presence of the multiplicity $\binom{n}{j}$ presents the new difficulty here.

In the following, we shall frequently apply Stirling's approximation which tells us that there exist constants $1 < c_1 < c_2 < 2$ with

$$c_1 \cdot \sqrt{2\pi w} \left(\frac{w}{e}\right)^w < w! < c_2 \cdot \sqrt{2\pi w} \left(\frac{w}{e}\right)^w, \tag{A.1}$$

for all $w \geqslant 1$.

Proof of Lemma A.4. By Theorem A.3, since $n = o\left(\sqrt{g}\right)$, we have

$$V_{g,n(g)} = \frac{B}{\sqrt{g}} \left(2g-3+n\right)! \left(4\pi^2\right)^{2g-3+n} \left(1 + O\left(\frac{1+n^2}{g}\right)\right). \tag{A.2}$$

By [Mi13, Lemma 3.2, part 3] we have that for $a, b \geqslant 0$, $2a+b \geqslant 1$,

$$V_{a,b+4} \leqslant V_{a+1,b+2}. \tag{A.3}$$

Applying (A.3) iteratively, for $j \geqslant 1$,

$$V_{i,j+1} \leqslant V_{i+\lfloor\frac{j-1}{2}\rfloor, j+1-2\lfloor\frac{j-1}{2}\rfloor}.$$

We can then apply Theorem A.3 to see that

$$V_{i,j+1}V_{g-i,n-j+1} \ll \left(4\pi^2\right)^{2g+n} \frac{(2i+j-2)!}{\sqrt{i+\max\left\{\lfloor\frac{j-1}{2}\rfloor, 0\right\}}} \cdot \frac{(2g-2i+n-j-2)!}{\sqrt{g-i+\max\left\{\lfloor\frac{n-j-1}{2}\rfloor, 0\right\}}}. \tag{A.4}$$

We also observe that

$$\frac{\sqrt{g}}{\sqrt{g-i+\max\left\{\lfloor\frac{n-j-1}{2}\rfloor, 0\right\}} \cdot \sqrt{i+\max\left\{\lfloor\frac{j-1}{2}\rfloor, 0\right\}}} \ll 1. \tag{A.5}$$

Then applying (A.2), (A.4) and (A.5),

$$\begin{aligned}
&\sum_{\substack{0\leqslant i\leqslant g, 0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \frac{n!}{j!\,(n-j)!} \cdot \frac{V_{i,j+1}V_{g-i,n-j+1}}{V_{g,n}}\\
\ll &\sum_{\substack{0\leqslant i\leqslant g, 0\leqslant j\leqslant n\\ 2\leqslant 2i+j\leqslant 2g+n-2}} \frac{n!}{j!\,(n-j)!} \frac{(2i+j-2)!\,(2g-2i+n-j-2)!}{(2g+n-3)!}.
\end{aligned}$$

If $i=0$ then $j \geqslant 2$ and we have

$$\begin{aligned}
\sum_{j=2}^{n} \frac{n!}{j!\,(n-j)!} \cdot \frac{(j-2)!\,(2g+n-j-2)!}{(2g+n-3)!} &= \sum_{j=2}^{n-4} \frac{n!}{j\,(j-1)\,(n-j)!} \cdot \frac{(2g+n-j-2)!}{(2g+n-3)!}\\
&\ll \frac{n^2}{g} + \sum_{j=3}^{n-4} \frac{n^j}{g^{j-1}} \ll \frac{n^2}{g},
\end{aligned}$$

since $n = o\left(\sqrt{g}\right)$. By symmetry, the same calculation holds for the case that $i = g$.

Similarly, if $i = 1$ then $j \geqslant 0$ and we calculate

$$\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} \cdot \frac{j!\,(2g+n-j-4)!}{(2g+n-3)!} \ll \sum_{j=0}^{n} \frac{n^j}{g^{j+1}} \ll \frac{1}{g}.$$

The same calculation holds in the case that $i = g-1$ by symmetry. If $2 \leqslant i \leqslant g-2$ then we claim that

$$\frac{n!}{j!\,(n-j)!} \frac{(2i+j-2)!\,(2g-2i+n-j-2)!}{(2g+n-3)!} \ll g^{-3}. \tag{A.6}$$

It is a straightforward calculation to check that (A.6) holds in the case that $i = 2$, $j = 0$ and $i = 2, j = 1$. Now let $L = 2i + j$. Then if $6 \leqslant L \leq n$,

$$\frac{n!}{j!\,(n-j)!} \frac{(2i+j-2)!\,(2g-2i+n-j-2)!}{(2g+n-3)!} \ll \frac{L! \cdot n^L}{g^L} \ll \sqrt{L} \left(\frac{Ln}{ge}\right)^L,$$

by Stirling's approximation. If $L = 6$ then

$$\sqrt{L} \left(\frac{Ln}{ge}\right)^L \ll \left(\frac{n}{g}\right)^6 \ll g^{-3}.$$

If $6 < L \leqslant n-1$ then

$$\begin{aligned}\sqrt{L} \left(\frac{Ln}{ge}\right)^L = \sqrt{L} \left(\frac{Ln}{ge}\right)^L \left(\frac{6n}{ge}\right)^6 \cdot \left(\frac{eg}{6n}\right)^6 &\ll \sqrt{L} g^{-3} \left(\frac{L}{6}\right)^6 \left(\frac{Ln}{ge}\right)^{L-6} \\ &\leqslant L^{\frac{13}{2}} e^{6-L} g^{-3} \cdot \frac{n^2}{g} \ll g^{-3}.\end{aligned}$$

If $n \leqslant L \leqslant \frac{1}{2}(2g+n-2)$, then since

$$\binom{n}{i} \leqslant 2^n,$$

we have

$$\begin{aligned}\frac{n!}{j!\,(n-j)!} \frac{(2i+j-2)!\,(2g-2i+n-j-2)!}{(2g+n-3)!} &\ll 2^n \frac{L!\,(2g+n-2-L)!}{(2g+n-3)!} \\ &\leqslant \frac{2^n n!\,(2g-2-n)!}{(2g+n-3)!} \ll \left(\frac{2n}{g}\right)^n \ll g^{-3}.\end{aligned}$$

By symmetry, the case that $2i+j \geqslant \frac{1}{2}(2g+n-2)$ is treated analogously. This establishes the claim (A.6). We can now use the rough bound

$$\#\{(i,j) \in \mathbb{Z}_{\geqslant 0} \mid 2 \leqslant i \leqslant g-2, 0 \leqslant j \leqslant n, 2 \leqslant 2i+j \leqslant 2g+n-2\} \ll ng,$$

to deduce that

$$\sum_{\substack{2 \leqslant i \leqslant g-2, 0 \leqslant j \leqslant n \\ 2 \leqslant 2i+j \leqslant 2g+n-2}} \frac{n!}{j!\,(n-j)!} \frac{(2i+j-2)!\,(2g-2i+n-j-2)!}{(2g+n-3)!} \ll \frac{n}{g^2},$$

and the result follows. □

In order to deal with the contribution of non-simple geodesics, we needed the following Lemma.

Lemma A.5. *Let $n = o\left(\sqrt{g}\right)$ and let g_0, a_0, n_0 and k be given with $m = 2g_0 + a_0 + k - 2 \leqslant 3\log g - 2$. For $1 \leqslant q \leqslant k - 2n_0$,*

$$\sum_{\{(g_j,a_j,n_j)\}_{i=1}^{q} \in \mathcal{A}} \frac{n!}{a_0! \cdots a_q!} \cdot \frac{V_{g_0,n_0+a_0} \cdots V_{g_q,n_q+a_q}}{V_{g,n}} \ll (2g_0+k+a_0-3)! \frac{n^{a_0}}{g^m},$$

where the summation is taken over the set $\mathcal{A}$ of all "admissible triples" $\{(g_1,a_1,n_1), \ldots, (g_q,a_q,n_q)\}$ where $g_j, a_j \geqslant 0$, $n_j \geqslant 1$ and $2g_j + a_j + n_j \geqslant 3$ such that

i) $\sum_{i=1}^{q} (2g_i - 2 + n_i + a_i) = 2g - 2 + n - m$,

ii) $\sum_{i=1}^{q} n_i = k - 2n_0$,

iii) $\sum_{j=1}^{q} a_j = n - a_0$.

This is similar to estimates proved in [WX21] but here we need the number of cusps to grow with genus and we have the extra multiplicity

$$\frac{n!}{a_0! \cdot \cdots \cdot a_q!}.$$

We take a similar approach as in the proof of Lemma A.4. Lemma A.5 relies on a lot of computations which, for the sake of readability, are done separately in Lemma A.6.

Proof of Lemma A.5 given Lemma A.6. By [Mi13, Lemma 3.2, part 3] we see that for each $a_i + n_i \geqslant 2$, we have

$$V_{g_i,a_i+n_i} \leqslant V_{g_i+\lfloor\frac{a_i+n_i-2}{2}\rfloor, a_i+n_i-2\lfloor\frac{a_i+n_i-2}{2}\rfloor}.$$

This allows us to apply Theorem A.3 which tells us that there exists $C_1 > 0$ with

$$V_{g_1,n_1+a_1} \cdots V_{g_q,n_q+a_q} \leqslant C_1^q \prod_{j=1}^{q} \frac{\left(4\pi^2\right)^{2g_j+a_j+n_j-3} (2g_j + a_j + n_j - 3)!}{\sqrt{g_j + \max\left\{\lfloor\frac{a_j+n_j-2}{2}\rfloor, 0\right\}}}, \tag{A.7}$$

where since $V_{0,3} = 1$ we interpret the product in (A.7) as only over triples with $g_j + \max\left\{\lfloor\frac{a_j+n_j-2}{2}\rfloor, 0\right\} > 0$. We also see by Theorem A.3 that

$$V_{g_0,a_0+k} \leqslant C_1 \left(4\pi^2\right)^{2g_0+a_0+k-3} (2g_0 + a_0 + k - 3)!, \tag{A.8}$$

and

$$V_{g,n} = \frac{B}{\sqrt{g}} (2g - 3 + n(g))! \left(4\pi^2\right)^{2g-3+n(g)} \left(1 + O\left(\frac{1 + n(g)^2}{g}\right)\right). \tag{A.9}$$

We introduce the notation $\overline{a_j + n_j} \overset{\text{def}}{=} \max\left\{\lfloor\frac{a_j+n_j-2}{2}\rfloor, 0\right\}$. By applying (A.7), (A.8) and (A.9) and noting that $n_i! \geqslant 1$ for each i, we calculate that

$$\begin{aligned}
&\sum_{\mathcal{A}} \frac{n!}{a_0! \cdot \cdots \cdot a_q!} \cdot \frac{V_{g_0,n_0+k} \cdot V_{g_1,n_1+a_1} \cdot \cdots \cdot V_{g_q,n_q+a_q}}{V_{g,n} \cdot n_0! n_1! \cdots n_q!} \\
&\ll (2g_0 + k + a_0 - 3)! \sum_{\mathcal{A}} \frac{C_1^q \sqrt{g}}{\prod_{j=1}^{q} \sqrt{g_j + \overline{a_j + n_j}}} \frac{n!}{\prod_{j=0}^{q} a_j!} \frac{\prod_{j=1}^{q} (2g_j + a_j + n_j - 3)!}{(2g + n - 3)!}.
\end{aligned}$$

The result then follows from the fact that

$$\sum_{\mathcal{A}} \frac{C_1^q \sqrt{g}}{\prod_{j=1}^{q} \sqrt{g_j + \overline{a_j + n_j}}} \frac{n!}{\prod_{j=0}^{q} a_j!} \frac{\prod_{j=1}^{q} (2g_j + a_j + n_j - 3)!}{(2g + n - 3)!} \ll \frac{n^{a_0}}{g^m},$$

which is proved in Lemma A.6. ☐

We now need to prove Lemma A.6, which is purely computational.

Lemma A.6. *Let $n = o\left(\sqrt{g}\right)$ and let g_0, a_0, n_0 and k be given with $m = 2g_0 + a_0 + k - 2 \leqslant$*

$3 \log g - 2$ *and* $1 \leqslant q \leqslant k - 2n_0$. *With* $\mathcal{A}$ *as in Lemma A.5, we have*

$$\sum_{\mathcal{A}} \frac{C_1^q \sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + \overline{\overline{a_j + n_j}}}} \frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q (2g_j + a_j + n_j - 3)!}{(2g + n - 3)!} \ll \frac{n^{a_0}}{g^m}. \tag{A.10}$$

In the proof of Lemma A.6, we will frequently apply the following observation: if $x_i \geqslant 0$ with $\sum_{i=1}^s x_i = A$, then

$$\prod_{i=1}^s x_i! \leqslant A!, \tag{A.11}$$

which can be seen by the fact that the multinomial coefficient $\binom{A}{x_1,\ldots,x_s}$ is bounded below by 1.

Proof. We first note that $q \leqslant 3 \log g$. For $\{(g_1, a_1, n_1), \ldots, (g_q, a_q, n_q)\} \in \mathcal{A}$, we claim that if $\max_{1 \leqslant i \leqslant q} (2g_i + a_i + n_i - 3) \leqslant 2g + n - 3 - m - 8q$ then

$$\frac{C_1^q \sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + \overline{\overline{a_j + n_j}}}} \frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q (2g_j + a_j + n_j - 3)!}{(2g + n - 3)!} \ll g^{-\frac{7}{2}q}. \tag{A.12}$$

This estimate is analogous to (A.6). Once we have established (A.12) we shall apply a rough counting argument to bound the contribution of such terms to the sum (A.10).

Let $\max_{1 \leqslant i \leqslant q} (2g_i + a_i + n_i - 3) = 2g + n - 3 - m - L$. First we treat the case that $L \geqslant \frac{1}{2} (2g + n - m - 3)$. We apply Stirling's approximation (A.1) to see that

$$\begin{aligned}
\frac{(2g_i + n_i + a_i - 3)!}{\sqrt{g_j + \overline{\overline{a_j + n_j}}}} &< c_2 \frac{\sqrt{2\pi (2g_i + n_i + a_i - 3)}}{\sqrt{g_j + \overline{\overline{a_j + n_j}}}} \cdot \left(\frac{2g_i + a_i + n_i - 3}{e} \right)^{2g_i + a_i + n_i - 3} \\
&< 4\sqrt{\pi} \cdot \left(\frac{2g_i + a_i + n_i - 3}{e} \right)^{2g_i + a_i + n_i - 3}.
\end{aligned} \tag{A.13}$$

Applying Stirling's approximation again, we see that

$$\begin{aligned}
\frac{\sqrt{g}}{(2g + n - 3)!} &> \frac{1}{c_2} \frac{\sqrt{g}}{\sqrt{2\pi (2g + n - 3)}} \cdot \left(\frac{e}{2g + n - 3} \right)^{2g + n - 3} \\
&\gg \left(\frac{e}{2g + n - 3} \right)^{2g + n - 3}.
\end{aligned} \tag{A.14}$$

We also note that

$$\frac{n!}{a_0! \cdots a_q!} \leqslant q^n, \tag{A.15}$$

by the multinomial theorem. By (A.13), (A.14) and (A.15),

$$\begin{aligned} &\frac{C_1^q \sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + a_j + n_j}} \frac{n!}{\prod_{j=0}^q a_j!} \frac{\prod_{j=1}^q (2g_j + a_j + n_j - 3)!}{(2g + n - 3)!} \\ \ll &q^n \cdot \left(4C_1\sqrt{\pi}\right)^q \frac{\prod_{j=1}^q (2g_j + a_j + n_j - 3)^{(2g_j + a_j + n_j - 3)}}{(2g + n - 3)^{(2g+n-3)}}. \end{aligned} \tag{A.16}$$

We now bound the expression in (A.16). Given s integers $x_i > 0$, Jensen's inequality for concave functions applied to the function $\log x$ tells us that

$$\frac{\sum_{i=1}^s x_i \log x_i}{\sum_{i=1}^s x_i} \leqslant \log\left(\frac{\sum_{i=1}^s x_i^2}{\sum_{i=1}^s x_i}\right).$$

If $\sum_{i=1}^s x_i = A$ and $\max_{1 \leqslant i \leqslant s} x_i = B$, then

$$\begin{aligned} \sum_{i=1}^s x_i \log x_i &\leqslant A \log\left(\frac{\sum_{i=1}^s x_i^2}{\sum_{i=1}^s x_i}\right) \\ &\leqslant A \log B, \end{aligned}$$

and by exponentiating, we conclude that

$$\prod_{i=1}^s x_i^{x_i} \leqslant B^A. \tag{A.17}$$

Note that (A.17) also holds if instead we just require $x_i \geqslant 0$ since we can apply Jensen's inequality with only the non-zero terms. Recall that $\max_{1 \leqslant i \leqslant q}(2g_i + a_i + n_i - 3) = 2g + n - 3 - m - L$ for $L \geqslant \frac{1}{2}(2g + n - m - 3)$. Since $\sum_{i=1}^q (2g_i - 2 + n_i + a_i) = 2g - 2 + n - m$, then in particular, $\sum_{i=1}^q (2g_i - 3 + n_i + a_i) \leqslant 2g + n - m - 3$ and we can apply (A.17) to

(A.16) to calculate that

$$
\begin{aligned}
& q^n \cdot \left(4C_1\sqrt{\pi}\right)^q \frac{\prod_{j=1}^q (2g_j+a_j+n_j-3)^{(2g_j+a_j+n_j-3)}}{(2g+n-3)^{(2g+n-3)}} \\
\ll & q^n \cdot \left(4C_1\sqrt{\pi}\right)^q \cdot \frac{(2g+n-3-m-L)^{(2g+n-3)}}{(2g+n-3)^{(2g+n-3)}} \\
\leqslant & q^n \cdot 2^{3q} C_1^q \left(\frac{1}{2}\right)^{2g+n-3} \leqslant q^n \left(\frac{1}{2}\right)^{2g+n-3-3q-q\log_2 C_1}.
\end{aligned}
$$

Since $q \leqslant 3\log g$ and $n = o\left(\sqrt{g}\right)$,

$$
q^n \left(\frac{1}{2}\right)^{2g+n-3-3q-q\log_2 C_1} \ll \left(\frac{1}{2}\right)^g = g^{-\frac{g}{\log_2 g}} \ll g^{-\frac{7}{2}q}.
$$

This justifies the claim in the case that $L \geqslant \frac{1}{2}(2g+n-m-3)$.

In order to treat the remaining cases, we first make the following observation. Recalling that $\sum_{j=1}^q (2g_j+a_j+n_j-3) = 2g+n-m-3-(q-1)$ and that $\overline{a_j+n_j} \overset{\text{def}}{=} \max\left\{\lfloor \frac{a_j+n_j-2}{2} \rfloor, 0\right\}$, we see that

$$
\sum_{j=1}^q \left(g_j + \overline{a_j+n_j}\right) \geqslant \frac{1}{2}\sum_{j=1}^q (2g_j+a_j+n_j-3) \geqslant \frac{2g+n-m-3-(q-1)}{2}.
$$

For any q positive integers x_i, we have

$$
\prod_{i=1}^q x_i \geqslant \sum_{i=1}^q x_i - (q-1).
$$

Then

$$
\prod_{j=1}^q \left(g_j + \overline{a_j+n_j}\right) \geqslant \frac{2g+n-m-3-(q-1)}{2} - q - 1 \gg g,
$$

since $n = o\left(\sqrt{g}\right)$ and $q, m = O(\log g)$. We see that

$$
\frac{\sqrt{g}}{\prod_{j=1}^q \sqrt{g_j + \overline{a_j+n_j}}} \ll 1,
$$

and therefore

$$\begin{aligned}&\frac{C_1^q\sqrt{g}}{\prod_{j=1}^q\sqrt{g_j+a_j+n_j}}\frac{n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{(2g+n-3)!}\\ &\ll\frac{C_1^q n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{(2g+n-3)!}.\end{aligned} \tag{A.18}$$

The expression in (A.18) will be easier to work with for the remaining cases. Recalling that $\max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)=2g+n-3-m-L$, we now treat the case that $8q\leqslant L\leqslant n-a_0$. Since $\max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)=2g+n-3-m-L$, this forces $\max_{1\leqslant i\leqslant q}a_i\geqslant n-a_0-L$. Indeed if $\max_{1\leqslant i\leqslant q}a_i<n-a_0-L$ we would have that

$$\max_{1\leqslant i\leqslant q}(2g_i+n_i)>2g-2g_0-n_0,$$

which is not possible. Since there is an $1\leqslant i\leqslant q$ such that $2g_i+a_i+n_i-3=2g+n-3-m-L$ and we have $\sum_{j=1,j\neq q}^q(2g_i+a_i+n_i-3)=L-(q-1)\leqslant L$, we apply (A.11) to see that

$$\begin{aligned}\prod_{j=1}^q(2g_j+a_j+n_j-3)!&=(2g+n-3-m-L)!\prod_{j=1,j\neq i}^q(2g_j+a_j+n_j-3)!\\ &\leqslant L!\,(2g+n-3-m-L)!.\end{aligned} \tag{A.19}$$

We then use the rough bound

$$\frac{n!}{\prod_{j=0}^q a_j!}\leqslant\frac{n!}{(\max_{1\leqslant i\leqslant q}a_i)!}\leqslant\frac{n!}{(n-a_0-L)!}\ll n^{a_0+L}, \tag{A.20}$$

together with (A.19), to see that

$$\frac{n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{(2g+n-3)!}\ll\frac{n^{a_0+L}L!\,(2g+n-3-m-L)!}{(2g+n-3)!}\ll\frac{n^{a_0+L}}{g^{m+L}}L!. \tag{A.21}$$

By applying Stirling's approximation (A.1),

$$\frac{n^{a_0+L}}{g^{m+L}}L!\ll\sqrt{L}\left(\frac{n\cdot L}{e\cdot g}\right)^L\cdot\frac{n^{a_0}}{g^m}.$$

If $L = 8q$ then since $n = o\left(\sqrt{g}\right)$ and $q \leqslant 3\log g$,

$$C_1^q \sqrt{L}\left(\frac{n\cdot L}{e\cdot g}\right)^L \ll C_1^q g^{-4q}\left(8q\right)^{8q+\frac{1}{2}} \ll g^{\frac{-7q}{2}}. \tag{A.22}$$

Now if $8q < L \leqslant n - a_0$,

$$\begin{aligned}
C_1^q \frac{n^{a_0}}{g^m}.\sqrt{L}\left(\frac{n\cdot L}{e\cdot g}\right)^L &\ll C_1^q\frac{\sqrt{L}}{\sqrt{g}}\left(\frac{n\cdot L}{e\cdot g}\right)^L\cdot\left(\frac{n\cdot 8q}{e\cdot g}\right)^{8q}\cdot\left(\frac{e\cdot g}{n\cdot 8q}\right)^{8q}\\
&\ll g^{\frac{-7q}{2}}\cdot\left(\frac{L}{8q}\right)^{8q}\cdot\left(\frac{n\cdot L}{e\cdot g}\right)^{L-8q}\\
&\leqslant g^{\frac{-7q}{2}}\cdot e^{L-8q}\cdot\left(\frac{n\cdot L}{e\cdot g}\right)^{L-8q} \ll g^{\frac{-7q}{2}},
\end{aligned}$$

which justifies the claim (A.12) in the case that $8q \leqslant L \leqslant n - a_0$. Finally we treat the case that $8q < n - a_0 < L \leqslant \frac{2g+n-3-m}{2}$. We calculate, with (A.19) and (A.15), that

$$\begin{aligned}
\frac{C_1^q n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q\left(2g_j+a_j+n_j-3\right)!}{\left(2g+n-3\right)!} &\ll \frac{C_1^q\cdot q^n\cdot L!\left(2g+n-m-3-L\right)!}{\left(2g+n-3\right)!}\\
&\ll \frac{C_1^q\cdot q^n\left(n-a_0\right)!\left(2g+a_0-m-3\right)!}{\left(2g+n-m\right)!}\\
&\ll \frac{g^{3\log C_1}\left(3\log g\right)^{n+1}n^n}{\left(2g\right)^n} \ll g^{-\frac{7}{2}q},
\end{aligned} \tag{A.23}$$

which justifies the claim (A.12) for $8q < n - a_0 < L \leqslant \frac{2g+n-3-m}{2}$. Note that in the case that $n \leqslant 8q - n_0$ we can simply apply the argument in (A.23) with $L \geqslant 8q$. The claim (A.12) is now proved.

Now we have established (A.12), we apply the very rough bound for the size of the set $\mathcal{A}$,

$$|\mathcal{A}| \ll g^{3q},$$

together with (A.12) to calculate

$$\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q\in\mathcal{A}\\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)\leqslant 2g+n-2-m-8q}} \frac{C_1^q\sqrt{g}}{\prod_{j=1}^q\sqrt{g_j+a_j+n_j}}\frac{n!\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{\prod_{j=0}^q a_j!\,(2g+n-3)!}$$
$$\ll\frac{n^{a_0}}{g^m}\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q\in\mathcal{A}\\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)\leqslant 2g+n-2-m-8q}} g^{-\frac{7}{2}q}\ll|\mathcal{A}|\cdot\frac{n^{a_0}}{g^m}\cdot g^{-\frac{7}{2}q}\ll\frac{n^{a_0}}{g^m}\cdot g^{-\frac{q}{2}}. \tag{A.24}$$

We now consider the sum

$$\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q\in\mathcal{A}\\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)>2g+n-3-m-8q}} \frac{C_1^q\sqrt{g}}{\prod_{j=1}^q\sqrt{g_j+a_j+n_j}}\frac{n!\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{\prod_{j=0}^q a_j!\,(2g+n-3)!}. \tag{A.25}$$

Let $\max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)=2g+n-3-m-L$. Since $2g_j+a_j+n_j-3\geqslant 0$ and $\sum_{j=1}^q 2g_j+a_j+n_j-3=2g+n-m-3-(q-1)$, we see that $L\geqslant q-1$. By the same arguments as in (A.21) and (A.22), if $q-1\leqslant L\leqslant 8q\leqslant 24\log g$ then

$$\frac{n!}{\prod_{j=0}^q a_j!}\frac{\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{(2g+n-3)!}\ll\frac{n^{a_0+L}}{g^{m+L}}L!\ll\frac{n^{a_0}}{g^m}\cdot g^{-\frac{L}{4}}. \tag{A.26}$$

We now bound the number of $\{(g_1,a_1,n_1),\dots,(g_q,a_q,n_q)\}\in\mathcal{A}$ with $\max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)$ $=2g+n-3-m-L$. Assume we have that $2g_1+a_1+n_1-3=2g+n-3-m-L$. The remaining $q-1$ triples satisfy

$$\sum_{2\leqslant i\leqslant q}(2g_i+a_i+n_i)=L+3(q-1)\,.$$

Since $\sum_{i=1}^q n_i=k-2n_0$ and $\sum_{j=1}^q a_j=n-a_0$, the triple (g_1,a_1,n_1) is determined by the choice of $\{(g_2,a_2,n_2),\dots,(g_q,a_q,n_q)\}$. Then the number of $\{(g_i,a_i,n_i)\}_{i=1}^q\in\mathcal{A}$ with $\max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)=2g+n-3-m-L$ is therefore bounded above by

$$\binom{L+6(q-1)}{3(q-1)}. \tag{A.27}$$

Therefore combining (A.18), (A.26) and (A.27) we see that the sum (A.25) satisfies

$$\sum_{\substack{\{(g_i,a_i,n_i)\}_{i=1}^q\in\mathcal{A}\\ \max_{1\leqslant i\leqslant q}(2g_i+a_i+n_i-3)>2g+n-3-m-8q}} \frac{C_1^q\sqrt{g}}{\prod_{j=1}^q\sqrt{g_j+a_j+n_j}}\frac{n!\prod_{j=1}^q(2g_j+a_j+n_j-3)!}{\prod_{j=0}^q a_j!\,(2g+n-3)!}$$
$$\ll\frac{n^{a_0}}{g^m}\sum_{L=q-1}^{8q}\binom{L+6(q-1)}{3(q-1)}\frac{C_1^q}{g^{\frac{L}{4}}}\ll\frac{n^{a_0}}{g^m}. \tag{A.28}$$

Combining (A.24) and (A.28), the result follows. □

References

[Al86] N. Alon, Eigenvalues and expanders, Combinatorica 6 (1986), no. 2, 83–96, Theory of computing (Singer Island, Fla., 1984). 8

[AM23] N. Anantharaman and L. Monk, Friedman-Ramanujan functions in random hyperbolic geometry and application to spectral gaps. Preprint, arXiv:2304.02678. (2023) 14

[BBD87] P. Buser, M. Burger, and J. Dodziuk, Riemann surfaces of large genus and large λ1, Geometry and analysis on manifolds (Katata/Kyoto, 1987), Lecture Notes in Math., vol. 1339, Springer, Berlin, 1988, pp. 54–63. 10

[Bo20] C. Bordenave. A new proof of Friedman's second eigenvalue theorem and its extension to random lifts. Ann. Sci. Ec. Norm. Supér. (4) , 53(6):1393–1439, (2020). 8

[Bo16] D. Borthwick, Spectral theory of infinite-area hyperbolic surfaces, second ed., Progress in Mathematics, vol. 318, Birkhäuser/Springer, [Cham], (2016). 42, 45, 46

[Bo22] J. Bonifacio. Bootstrapping closed hyperbolic surfaces. J. High Energy Phys., (3):Paper No. 093, 18, 2022 16

[BC19] C. Bordenave and B. Collins. Eigenvalues of random lifts and polynomials of random permutation matrices. Annals of Mathematics, 190(3):811–875, (2019). 8, 23, 27

[BC23] C. Bordenave and B. Collins, Norm of matrix-valued polynomials in random unitaries and permutations, Preprint, arXiv:2304.05714 (2023). 13, 25, 28, 29, 30, 31, 32

[BCP21] T. Budzinski, N. Curien, and B. Petri. The diameter of random Belyi surfaces. Algebr. Geom. Topol., 21(6):2929–2957, (2021). 15

[BCP21a] T. Budzinski, N. Curien, and B. Petri. On the minimal diameter of closed hyperbolic surfaces. Duke Math. J. 170 (2) 365 - 377 (2021). 15

[BCP22] T. Budzinski, N. Curien, and B. Petri. On Cheeger constants of hyperbolic surfaces. arXiv:2207.00469 (2022). 15

[Be16] N. Bergeron, The spectrum of hyperbolic surfaces, Universitext, Springer, Cham; EDP Sciences, Les Ulis, (2016), Appendix C by Valentin Blomer and Farrell Brumley, Translated from the 2011 French original by Brumley. 41

[BL06] Y. Bilu and N. Linial. Lifts, discrepancy and nearly optimal spectral gap. Combinatorica, 26(5):495–519, (2006). 9

[BM04] R. Brooks and E. Makover. Random construction of Riemann surfaces. J. Differential Geom. 68(1):121–157, (2004). 15

[BMM17] W. Ballmann, H.Matthiesen, and S. Mondal, Small eigenvalues of surfaces of finite type, Compositio Mathematica 153 (2017), no. 8, 1747–1768. 15

[Bu82] P. Buser. A note on the isoperimetric constant. Annales scientifiques de l'École Normale Supérieure, Series 4, Volume 15 (1982) no. 2, pp. 213-230. 7

[Bu84] P. Buser. On the bipartition of graphs, Discrete Appl. Math. 9 (1984), no. 1, 105–109. 10

[Bu92] P. Buser. Geometry and spectra of compact Riemann surfaces. Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, (2010). Reprint of the 1992 edition. 17, 18, 62, 69, 78, 83, 84

[Ch70] J. Cheeger, A Lower Bound for the Smallest Eigenvalue of the Laplacian, in Problems in Analysis, A Symposium in Honor of S. Bochner, Princeton University Press, 1970, pp. 195-199) 7

[CM14] B. Collins and C. Male. The strong asymptotic freeness of Haar and deterministic matrices. Ann. Sci. Ec. Norm. Supér. (4) , 47(1):147–163, 2014. 98

[Di69] J. D. Dixon, The probability of generating the symmetric group, Math. Z. 110 (1969), 199–205. 26

[FBP23] M. Fortier-Bourque, B. Petri. Linear programming bounds for hyperbolic surfaces. arXiv:2302.02540 (2023). 16

[Fr03] J. Friedman, Relative expanders or weakly relatively Ramanujan graphs, Duke Math. J. 118 (2003), no. 1, 19–35. 8

[Fr08] J. Friedman. A proof of Alon's second eigenvalue conjecture and related problems. Mem. Amer. Math. Soc., 195(910):viii+100, (2008). 8

[Ga02] A. Gamburd. On the spectral gap for infinite index "congruence" subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Israel J. Math 127 (2002): 157-200. 63, 66, 67

[Ga06] A. Gamburd. Poisson-Dirichlet distribution for random Belyi surfaces. Ann. Probab., 34(5):1827–1848, (2006). 15

[GMST21] C. Gilmore, E. Le Mason, T. Sahlsten and J. Thomas. Short geodesic loops and L^p norms of eigenfunctions on large genus random surfaces. Geom. Funct. Anal. 31, 62–110 (2021). 16, 99

[GPY11] L. Guth, H. Parlier, and R. Young, Pants decompositions of random surfaces, Geometric and Functional Analysis 21 (2011), 1069–1090. 13

[GJ78] S. Gelbart and H. Jacquet. A relation between automorphic representations of GL_2 and GL_3, Ann. Sci. Ecole Norm. Sup. (4) 11 (1978), no. 4, 471–542. 9

[Go84] W. Goldman. The symplectic nature of fundamental groups of surfaces. Advances in Mathematics 54 (1984) 200 – 225. 73

[HM23] W. Hide and M. Magee, Near optimal spectral gaps for hyperbolic surfaces, Ann. of Math., (2) 198(2): 791-824 (2023). 3, 12, 13, 20, 21, 98

[Hi22] W. Hide, Spectral Gap for Weil–Petersson Random Surfaces with Cusps, International Mathematics Research Notices, (2022). 62

[Hi23] W. Hide. Effective lower bounds for spectra of random covers and random unitary bundles. arXiv:2305.04584 (2023). 13, 20

[HT05] U. Haagerup and S. Thorbjørnsen, A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group, Ann. of Math. (2) 162, 2 (2005), pp. 711--775 28

[HT23] W. Hide and J. Thomas. Large-n asymptotics for Weil-Petersson volumes of moduli spaces of bordered hyperbolic surfaces. Preprint, arXiv:2312.11412. (2023). 15

[Hu59] H. Huber, Zur analytischen Theorie hyperbolischer Raumformen und Bewegungsgruppen. Math. Ann. 138, 1–26 (1959). 6

[Hu74] H. Huber, Über den ersten Eigenwert des Laplace-Operators auf kompakten Riemannschen Flächen, Comment. Math. Helv. 49 (1974), 251–259. 10

[HY21] J. Huang and H.-T. Yau. Spectrum of random d-regular graphs up to the edge. arXiv:2102.00963, (2021). 8

[Iw89] H. Iwaniec, Selberg's lower bound of the first eigenvalue for congruence groups, Number theory, trace formulas and discrete groups (Oslo, 1987), Academic Press, Boston, MA, (1989), pp. 371–375. 9

[Iw96] H. Iwaniec, The lowest eigenvalue for congruence groups, Topics in geometry, Progr. Nonlinear Differential Equations Appl., vol. 20, Birkhäuser Boston, Boston, MA, (1996), pp. 203–212. 9

[Iw02] H. Iwaniec. Spectral methods of automorphic forms, volume 53 of Graduate Studies in Mathematics. American Mathematical Society (2002). 17, 19, 62, 70, 96

[JL70] H. Jacquet and R. P. Langlands, Automorphic forms on GL(2), Lecture Notes in Mathematics, Vol. 114, Springer-Verlag, BerlinNew York, 1970. 10

[Ki03] H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, J. Amer. Math. Soc. 16 (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. 9

[KMP23] P. Kravchuk, D. Mazac, and S. Pal. Automorphic spectra and the conformal bootstrap. Comm. Amer. Math. Soc. 4 (2024), 1-63 16

[KS02] H. Kim and F. Shahidi. Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL2, Ann. of Math. (2) 155 (2002), no. 3, 837–893, With an appendix by Colin J. Bushnell and Guy Henniart. 9

[LM22] L. Louder, M. Magee with Appendix by W. Hide, and M. Magee. Strongly convergent unitary representations of limit groups, Preprint, arXiv:2210.08953, (2022). 12

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak, Ramanujan graphs, Combinatorica 8 (1988), no. 3, 261–277 8, 9

[LRS95] W. Luo, Z. Rudnick and P. Sarnak. On Selberg's eigenvalue conjecture, Geom. Funct. Anal. 5 (1995), no. 2, 387–401. 9

[LS20] E. Le Masson and T. Sahlsten. Quantum ergodicity for Eisenstein series on hyperbolic surfaces of large genus, Math. Ann. (2023). 16

[Lu01] W. Luo, Nonvanishing of L-Values and the Weyl Law. Ann. of Math., (2) 154, No. 2 (2001). 98

[LW21] M. Lipnowski and A. Wright, Towards optimal spectral gap in large genus, Ann. Probab. 52(2): 545-575 (2024). 14, 62, 74, 99, 100

[Ma20] M. Magee, Letter to Bram Petri, https://www.mmagee.net/diameter.pdf, 2020. 7

[MN20] M. Magee and F. Naud. Explicit spectral gaps for random covers of Riemann surfaces. Publ. Math. Inst. Hautes Etudes Sci (2020). 11

[MN21] M. Magee and F. Naud. Extension of Alon's and Friedman's conjectures to Schottky surfaces, (2021). Preprint, arXiv:2106.02555. 11

[MNP20] M. Magee, F. Naud, and D. Puder. A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \varepsilon$, (2020). Preprint, arXiv:2003.10911. 11, 64, 65, 66

[MSS15] A. Marcus, D. Spielman and N. Srivastava, Interlacing families I: Bipartite Ramanujan graphs of all degrees, Ann. of Math., 182 (2015) 307–325. 9

[Ma88] G. Margulis, Explicit group theoretic constructions of combinatorial schemes and their applications to the design of expanders and concentrators, J. Prob. of Info. Trans., (1988) 39–46. 9

[MN08] S. Miller and T. Novikoff. The distribution of the largest nontrivial eigenvalues in families of random regular graphs. Experiment. Math., 17(2):231–244, 2008 8

[Mi07] M. Mirzakhani, Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces, Invent. Math. 167 (2007), no. 1, 179–222. 75, 76

[MP19] M. Mirzakhani and B. Petri, Lengths of closed geodesics on random surfaces of large genus, Comment. Math. Helv. 94 (2019), no. 4, 869–889. 99

[Mi13] M. Mirzakhani. Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus, J. Differential. Geom. 94 (2013), no. 2, 267–300. 13, 74, 99, 100, 101, 104

[MZ15] M. Mirzakhani and P. Zograf, Towards large genus asymptotics of intersection numbers on moduli spaces of curves, Geom. Funct. Anal. 25 (2015), no. 4, 1258–1289. 100

[Mo21] L. Monk, Benjamini-Schramm convergence and spectrum of random hyperbolic surfaces of high genus, Anal. PDE (2021). 16

[Na22] F. Naud, Random covers of compact surfaces and smooth linear spectral statistics, arXiv:2209.07941 (2022) 16

[Ni91] A. Nilli, On the second eigenvalue of a graph, Discrete Math. 91 (1991), no. 2, 207–210. 8

[NWX20] X. Nie, Y. Wu and Y. Xue. Large genus asymptotics for lengths of separating closed geodesics on random surfaces, (2020), Preprint, arXiv:2009.07538. 87, 99, 100

[PS85] R.S Phillips, P. Sarnak. On cusp forms for cofinite subgroups of $\mathrm{PSL}(2, \mathbb{R})$. Invent. Math. 80, 339–364 (1985). 96, 97

[PS85b] R.S Phillips, P. Sarnak. The weyl theorem and the deformation of discrete groups. Comm. Pure and Appl. Math, 38 (1985), 853-866. 97

[PS92] R.S Phillips, P. Sarnak. Perturbation theory for the Laplacian on automorphic functions. J. Amer. Math. Soc. 5 (1992), 1-32. 97

[Pi96] G. Pisier, A simple proof of a theorem of Kirchberg and related results on C^*-norms, J. Operator Theory 35, 2 (1996), pp. 317--335. 28

[Ra87] M. Ratner, The rate of mixing for geodesic and horocycle flows. Ergodic Theory and Dynamical Systems. (1987) ;7(2):267-288. 6

[Ru22] Z. Rudnick, GOE statistics on the moduli space of surfaces of large genus, Geom. Funct. Anal. 33 (2023), no. 6, 1581–1607. 16

[Sa95] P. Sarnak. Selberg's eigenvalue conjecture, Notices Amer. Math. Soc. 42 (1995), no. 11, 1272–1277. 9

[Sa03] P. Sarnak. Spectra of Hyperbolic Surfaces, Bull. Amer. Math. Soc. Vol. 40, No. 4, (2003), 441-478. 9, 97

[Sa04] P. Sarnak, What is . . . an expander? Notices Amer. Math. Soc. 51 (2004), no. 7, 762–763. 8

[Se56] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. (N.S.) 20 (1956), 47–87. 41

[Se65] A. Selberg, On the estimation of Fourier coefficients of modular forms, Proc. Sympos. Pure Math., Vol. VIII, Amer. Math. Soc., Providence, R.I., (1965) pp. 1–15. 9

[Se89] A. Selberg, Göttingen lectures, Collected Works, Vol. I, Springer-Verlag, Berlin, 1989, pp. 626–674 98

[SW22] Y. Shen, Y. Wu. Arbitrarily small spectral gaps for random hyperbolic surfaces with many cusps, (2022). Preprint, arXiv:2203.15681. 14

[SW22A] Y. Shen, Y. Wu, The Cheeger Constants of Random Belyi Surfaces, International Mathematics Research Notices, (2022). 15

[Th22] J. Thomas, Delocalisation of eigenfunctions on large genus random surfaces, Israel Journal of Mathematics 250 (2022), 53–83. 16

[Wr20] A. Wright, A tour through Mirzakhani's work on moduli spaces of Riemann surfaces, Bull. Amer. Math. Soc. (N.S.) 57 (2020), no. 3, 359– 408. 72

[Wo85] S. Wolpert. On the Weil-Petersson geometry of the moduli space of curves. American Journal of Mathematics , Aug., 1985, Vol. 107, No. 4 (1985). 74

[Wo92] S. Wolpert. Spectral limits for hyperbolic surfaces, II. Invent. Math., Vol. 108, (1992), 67-89, 91-129. 97

[Wo94] S. Wolpert. Disappearance of Cusp Forms in Special Families. Annals of Mathematics, Second Series, Vol. 139, No. 2 (1994). 97

[WX21] Y. Wu, Y. Xue, Random hyperbolic surfaces of large genus have first eigenvalues greater than $\frac{3}{16}-\epsilon$. Geom. Funct. Anal. 32, 340–410 (2022). 14, 62, 63, 74, 77, 85, 86, 88, 90, 92, 103

[Ve90] Venkov, A.B. A remark on the discrete spectrum of the automorphic laplacian for a generalized cycloidal subgroup of a general fuchsian group. J Math Sci 52, 3016–3021 (1990). 99

[Za22] M. Zargar, Random flat bundles and equidistribution, Preprint, arXiv:2210.09547 (2022). 13

[Zo87] P. Zograf, Small eigenvalues of automorphic Laplacians in spaces of parabolic forms. J Math Sci 36, 106–114 (1987).

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