Statistical Inference for a Virtual Age Reliability Model

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A Thesis presented for the degree of
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Dedicated to

My mother and father

for their prayers, love and support

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Abstract

During the lifetime of a system, repairs may be performed when the system fails. It is most common to assume either perfect repair or minimal repair. However, a repair actually will sometimes be between minimal repair and perfect repair, which is called imperfect repair. The Kijima type I virtual age model can be used to model these types of repairable systems. This model contains a parameter which reflects the restoration level after each repair.

This thesis considers statistical inference for the Kijima type I model, which deals with repairable systems that can be restored to the operating state through system replacement or repair after the system fails. We present Bayesian analysis for the Kijima type I virtual age model, including consideration of the system's overall time to failure if a given number of repairs is possible. We use both Bayesian analysis, which specifies a single prior distribution, and a robust Bayesian analysis approach. A set of prior distributions is used in robust Bayesian analysis in order to deal with uncertainty regarding prior knowledge of the Kijima type I model parameters in a flexible way and to enhance the objectivity of the analysis in an imprecise Bayesian framework by computing predictive posterior distribution bounds for the reliability function of the system.

Finally, we discuss the use of the developed methods to decide about optimal replacement. Optimal replacement is the methodology of replacing a system component at the most advantageous or efficient moment to increase its performance and minimize overall expected costs. Two policies are introduced with cost functions

based on time and number of failures to make a decision on optimal replacement time or optimal number of failures of the system under the Kijima type I model using the Weibull distribution. These policies illustrate how the Bayesian and robust Bayesian analysis can be used for inferences about the optimal replacement and the expected total cost.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification, and it is all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

1.1 Overview

In reality, many engineered objects (EOs) [5], such as production systems, cars, and aircraft, are repairable. A repairable system is one that can be returned to a satisfactory state using any method other than replacing the entire system after it fails to function [2, 16]. As the costs related to the usage of some engineered objects have become higher, it is important to ensure the continued functioning of these systems in terms of cost and reliability. Failure of some parts in systems, such as aircraft and patient monitoring systems, can have dangerous or even deadly consequences. Therefore, maintaining their functionality through high-quality repairs is important to prevent such severe outcomes.

Modelling work supposes that the reliability of a repairable system is based on both the ageing and the effectiveness of the repair. Most studies assumed that repair effectiveness is presented as perfect repair, called "as good as new" (AGAN), or minimal repair, called "as bad as old" (ABAO), where the corresponding probabilistic models for successive failures are the Renewal Processes (RP) and the Non-Homogeneous Poisson Processes (NHPP), respectively [21, 76]. Repair efficiency typically lies somewhere between these two extreme scenarios, as it is often better than minimal repair but is not necessarily perfect, commonly known as imperfect repair [15, 21, 76].

Modelling the effectiveness of imperfect repair is important in various scenarios, particularly when planning maintenance strategies or estimating system lifetimes. Examples of such systems include components in infrastructure such as nuclear power plants or transportation systems such as aeroplanes and trains [21]. Also, railway track repair, another example, can benefit from an imperfect repair model due to the variety of failure modes and repairable nature of track components [52, 64]. These systems may still appear to be in normal working condition even at the end of their planned life. Some reliability requirements are important to justify the extension of their functioning life. One approach is to account for the effectiveness of repair actions. These repairs, carried out after failures, aim to restore the system to a functional state. Modelling the effects of these repair actions is of great practical interest and is an important first step in assessing corrective maintenance efficiency [21].

Several models to describe imperfect repair have been introduced. The most commonly used models are two virtual age models, which Kijima introduced [39], namely the Kijima type I and Kijima type II models. Doyen and Gaudoin [21] have introduced two other virtual age models, namely the Arithmetic Reduction of Age model (ARA) and the Arithmetic Reduction of Intensity model (ARI). Virtual age models describe system performance after the repair action process to make an inference about a system's status after maintenance [25].

This thesis focuses on the Kijima type I model combined with the Weibull distribution, which deals with repairable systems that can be restored to the operating state after repair when a system fails. We will use Bayesian inference for this model because the engineers' prior knowledge of the failure process can significantly improve the model's inferences by identifying prior assumptions or beliefs about the model's parameters and updating it with observed failure data in order to obtain the posterior distribution and make inferences. Modelling and inference for the Kijima type I model are typically complex and challenging. Most literature uses numerical methods to deal with the complex nature of virtual age models to make inferences. We develop an analytical method and also employ a Markov Chain Monte Carlo (MCMC) approach, using Bayesian methods, for estimating Kijima type I model parameters. The results are then compared with those obtained from Maximum Likelihood Estimation (MLE). To predict the system reliability with the uncertainty until the time to replace it when the number of failures is known, we develop a novel Bayesian method for the Kijima type I model using the posterior predictive empirical reliability function.

In addition to the Bayesian analysis, which specifies a single prior distribution, we also present a robust Bayesian analysis approach, which suggests using a set of prior distributions. Consequently, we need to define a set of prior distributions, which lead to a set of posterior inferences. In this context, this approach is attractive because it allows for modelling partial or incomplete prior knowledge of failure distribution parameters in a flexible way using bounds to enhance the objectivity of the analysis in an imprecise Bayesian framework. Therefore, we propose a novel approach to the robust Bayesian analysis of the Kijima type I model based on a set of priors. After identifying a set of prior distributions, we will compute posterior bounds to achieve robust inferences.

In order to decide whether to replace or repair a system, it is necessary to take both corrective and preventive replacement costs into account. These costs play a crucial role in evaluating the consequences of system cost and reliability. In this study, we analyse two commonly used maintenance or replacement strategies to determine the optimal time and the number of failures for the replacement of the system. We will use the Kijima type I model to make a decision and analyze optimal replacement policies. In this thesis, we will employ our developed inference methods to examine both situations by incorporating Bayesian and robust Bayesian inferences and making decisions. The following are some of the accomplishments of this thesis:

1. Developing a novel method using Bayesian analysis to predict the total predictive system reliability for a given number of repairs.

2. Developing a novel approach using robust Bayesian analysis. This approach uses a set of prior distributions to flexibly model partial or incomplete prior knowledge of the Kijima type I model parameters. By defining a set of prior distributions for the effective repair parameter and the scale parameter with

the effective age parameter, we compute the posterior predictive empirical reliability bounds to achieve robust inferences.

3. Developing and predicting the optimal replacement time and the number of repairs needed based on Bayesian methods and robust Bayesian analysis. This analysis includes both corrective and preventive replacement costs, which are important for evaluating system cost and reliability.

1.2 Outline of this thesis

The organization of this thesis is as follows: In Chapter 2, preliminary concepts relevant to the topics studied in this study are introduced and summarized. The chapter offers an overview of various lifetime distributions that are applicable in the context of reliability applications and some general Bayesian concepts. Also, we provide an overview and discussion of virtual age models, including those for perfect, minimal, and imperfect maintenance, which can be described by virtual age models.

In Chapter 3, we consider the Kijima type I model and a Weibull lifetime distribution for the virtual age model inferences. We present a Bayesian method for this model to make inferences as an alternative method to frequentist techniques. We develop an analytical method using the Bayesian method for the Kijima type I model parameters, and another method uses MCMC. Also, we develop a novel Bayesian method for the Kijima type I model to predict the system's reliability using posterior predictive reliability after each repair and the posterior predictive empirical reliability function to predict the total system reliability with the uncertainty until the time needed to replace it when the number of failures is known. A simulation analysis is then conducted to investigate the performance of the presented methods. This chapter was presented at the 15th International Conference of the ERCIM WG on Computational and Methodological Statistics (CMStatistics, 17 - 19 December 2022).

Chapter 4 presents robust Bayesian statistical inference by introducing a set of prior distributions for some parameters of the Kijima type I model using the Weibull distribution. The robustness was evaluated using the upper and lower bounds of the empirical posterior predictive reliability function based on the introduced set of priors using the MCMC method. Then, to study the effect of the set of prior distributions, a simulation investigation is conducted.

In Chapter 5, we illustrate the use of the developed methods for optimal replacement policies using the Kijima I model. We illustrate how Bayesian and robust Bayesian methods can be applied to infer the optimal replacement time and the expected total cost.

Part of the results from Chapter 4 and 5 were presented at the 12th IMA International Conference on Modelling in Industrial Maintenance and Reliability (MIMAR, 4 - 6 July 2023).

Finally, Chapter 6 summarizes the conclusions of this thesis. Also, we point to some future work to extend what was presented in this thesis.

Chapter 2

Background

This chapter introduces the relevant concepts underlying the research in this thesis. First, we present a general outline of system reliability and discuss some of the common lifetime distributions that are usually used. Then, we provide an overview of virtual age models and define the Kijima type I model combined with the Weibull distribution. Statistical inference for this model is the main topic of this thesis. Finally, we introduce an overview of Bayesian inference.

2.1 Introduction

A system's reliability is generally defined as its ability to successfully perform its intended function during a period of time [49]. In reality, the longer a system can perform its intended function, the more reliable it is. Reliability deals with design and analysis activities that extend a component's life by controlling or eliminating potential failure conditions, such as reducing harmful environmental impacts, reducing loads and pressures applied to a system during its use, and providing preventive maintenance programs to reduce the number of failures [55].

In reality, if a system is repairable, it can be restored to operating condition after failure by some method other than replacing the entire system. Examples of repairable systems include mechanical systems, software systems, medical technology, and manufacturing processes [45]. To extend their functioning life, corrective maintenance must be carried out to restore the system's functioning whenever it fails [21]. Repair, or corrective maintenance (CM), restores the functional state of a system so that the system can return to normal functioning. Therefore, corrective maintenance plays a significant role in increasing the life span of a repairable system, which has a considerable impact on a system's overall reliability [38].

One approach to achieving system reliability is by considering the impact of repair actions or corrective maintenance. Repair activities are undertaken following a failure with the objective of restoring the system to a state where it can resume its intended function. The efficiency of maintenance is subject to several models: (i) "as bad as old" (ABAO), also known as minimal repair, where the corresponding model is a Non-Homogeneous Poisson Process (NHPP); and (ii) "as good as new" (AGAN), also known as perfect repair where the Renewal Processes (RP) is the corresponding model. The ABAO approach assumes that the system will return to the same state as before the failure. The AGAN assumption is that the process of repair results in a state that is equivalent to a new system. Maintenance models that address imperfect repair indicate that the system state after repair is between AGAN and ABAO. Virtual-age models such as the Kijima type I model are examples of such models [17, 21].

2.2 Weibull distribution

The Weibull distribution [57] is commonly used for modelling reliability and for modelling time to failure. It has been extensively employed for modelling data in engineering, reliability, and biological studies [57]. Suppose that T, representing time to failure, has the Weibull distribution [49], then, T has the probability density function

$$f(t) = \lambda\beta t^{\beta-1}\exp\left(-\lambda t^{\beta}\right) , \tag{2.2.1}$$

for $t > 0$, with shape parameter $\beta > 0$ and scale parameter $\lambda > 0$. The reliability function of the Weibull distribution is

$$R(t) = P(T > t) = \exp(-\lambda t^{\beta}) , \tag{2.2.2}$$

and the hazard rate is

$$h(t) = \frac{f(t)}{R(t)} = \lambda\beta t^{\beta-1} \ . \tag{2.2.3}$$

The hazard rate $h(t)$ determines whether a system's failure rate decreases or increases with age [61]. The hazard rate is a strictly increasing function if $\beta > 1$, and it is strictly decreasing if $\beta < 1$. When $\beta = 1$, the Weibull distribution has a constant hazard rate and is equivalent to the Exponential distribution [49, 51]. The Lognormal and Gamma distributions are among other distributions used for failure data analysis, but we do not consider them in this thesis.

2.3 Virtual age models

Assume that a repairable system begins operating at time $t = 0$. The time t will be called the *calendar age* of the repairable system since it began operation. The *virtual age* of a system is the age that describes the state of a system after a repair action [25]. Modelling work supposes that the *virtual age* involves quantifying the effect of repair on the lifetime of the system, which assesses the system's condition relative to its operational and repair history, while the *calendar age* refers to the time that has passed since the system started functioning [23]. It is clear that the *virtual age* depends on how well previous repairs were done. There are some types of repair quality that can be taken into account in the *virtual age*. In case of minimal repair, the *virtual age* is equivalent to the *calendar age*. In the case of perfect repair, the *virtual age* of a system resets to zero when a repair takes place, and the state of the system is as if it were new. The imperfect maintenance scenario determines that after a repair, the *virtual age* returns to a time between zero and the *calendar age*, and the state of the system improves to some intermediate level between as good as new (AGAN) and as bad as old (ABAO) [20, 25].

2.3.1 Notation

The failure events that occur randomly during time can be described by a point process N_t, $t \geq 0$, where N_t is the number of failures observed during a period of time $[0, t]$ [18, 23]. In this thesis, it is assumed that repairs are done after every failure and that corrective maintenance times are negligible. The failure intensity, which completely determines the distribution of the random processes that define the failure times of a repairable system, is represented as follows:

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_{t^-} = 1 | H_{t^-}), \quad t \geq 0 \ , \tag{2.3.4}$$

where H_{t^-} represents the past of the failure process up to time t, and N_{t^-} represents the number of events before time t.

Prior to the occurrence of the first failure, the intensity of failure is supposed to be a continuous function of time, $\lambda(t)$, referred to as the initial intensity. Typically, it is assumed that the system under consideration will experience continuous deterioration before failure, so the initial intensity will increase with time [17].

2.3.2 Basic models of repairable systems

2.3.2.1 As good as new

The Renewal Process (RP) can be used to model a repairable system where the times between failures are independent random variables with identical distributions. The times can represent a failure of any repairable system, for example, a production system. When the system fails, it is immediately replaced with an identical system or repaired to its original state. This means that the process considers the system to be restored and returns to a state identical to the new system (as good as new) after repair [23, 32]. The failure intensity for a renewal process is

$$\lambda_t = \lambda(t - T_{N_{t^-}}) \ ,$$

where N_{t^-} represents the number of failures during time interval $[0, t]$ and $T_{N_{t^-}}$ indicates the last renewal time, so $t - T_{N_{t^-}}$ is the duration that has passed since the previous renewal.

2.3.2.2 As bad as old

The as bad as old repair usually assumes that the repair process does not change the state of the system before its failure and is usually known as minimal repair. This means that a repair does not affect the subsequent random behaviour of the system as if no failure had taken place [23, 78]. In this case, the random process is assumed to be a Non-Homogeneous Poisson Process (NHPP), with failure intensity being a continuous time function represented as

$$\lambda_t = \lambda(t) \ .$$

Based on this equation, the failure intensity after the repair remains the same as before the repair. A minimal repair does not affect the system's future behaviour as if no failure takes place [21].

2.3.3 Kijima models

Kijima and Sumita [41] introduced the concept of the virtual age, and Kijima [39] was the first to introduce virtual age models. Kijima virtual age models are based on the idea that repair actions reduce the age of a system [25]. Modelling work supposes that at any given failure time t, a corresponding virtual age of the system is linked with it, which is a function of the number of repairs and the failure time. Kijima's models have become popular for reliability applications because they allow the effect of repairs to be modelled from an as-good-as-new Renewal Process (RP) to an as-bad-as-old Non-Homogeneous Poisson Process [25, 73].

Modelling work supposes a repairable system observed from the starting of the operation $T_0 = 0$, where T_i denotes the time of the i^{th} failure or repair and the times between failures are given by $x_1,\ x_2, \ldots,\ x_n$ [53, 46]. Let V_n represent the virtual age of the system immediately after the $(n)^{th}$ repair [39]. If $V_{n-1} = y$, then the n^{th} failure time has cumulative distribution function

$$F_{T_n}(x|y) = \frac{F(x+y) - F(y)}{1 - F(y)} \tag{2.3.5}$$

where $F_{T_n}(x|y)$ represents the conditional CDF after each repair. At the n^{th} failure, the real age for any system is denoted by S_n, which is given by

$$S_n = \sum_{i=1}^{n} x_i \ , \tag{2.3.6}$$

where $S_0 = 0$.

Kijima introduced two models of the virtual age [39]. The first one, called the Kijima type I model, assumes that the n^{th} repair has the ability to remove the damages that occurred between the $(n-1)^{th}$ and the n^{th} repair. The virtual age after the n^{th} repair is

$$V_n = V_{n-1} + qx_n \ , \tag{2.3.7}$$

where when the system is new, $V_0 = 0$. The virtual age after the first repair is $V_1 = qx_1$. The virtual age after the second repair will be $V_2 = q(x_1 + x_2)$. V_n represents the virtual age that is instantaneously subsequent to the n^{th} repair, and $q \in [0, 1]$ is a parameter that represents the effect of a repair. Modelling work supposes that when $q = 0$, this model indicates perfect maintenance (AGAN), and when $q = 1$, it represents minimal maintenance (ABAO). Furthermore, in the case of $0 < q < 1$, the model corresponds to imperfect repairs [37].

The Kijima type II virtual age model assumes that the n^{th} corrective maintenance has the ability to remove the cumulative damage for both current and previous failures. The virtual age is

$$V_n = q(V_{n-1} + x_n) \ , \tag{2.3.8}$$

where q is the parameter that describe the efficiency of the n^{th} repair, $0 \leq q \leq 1$ and when the system is new, $V_0 = 0$. The virtual age after the first repair is $V_1 = qx_1$, and after the second repair, it is $V_2 = q(qx_1 + x_2)$. Therefore, V_n can be expressed as

$$V_n = q(q^{n-1}x_1 + q^{n-2}x_2 + ... + x_n) = \sum_{i=1}^{n} q^{n+1-i} x_i \ . \tag{2.3.9}$$

To understand the concepts related to the virtual age of the Kijima type I model, it is critical to analyze the relationship between real age and virtual age while varying

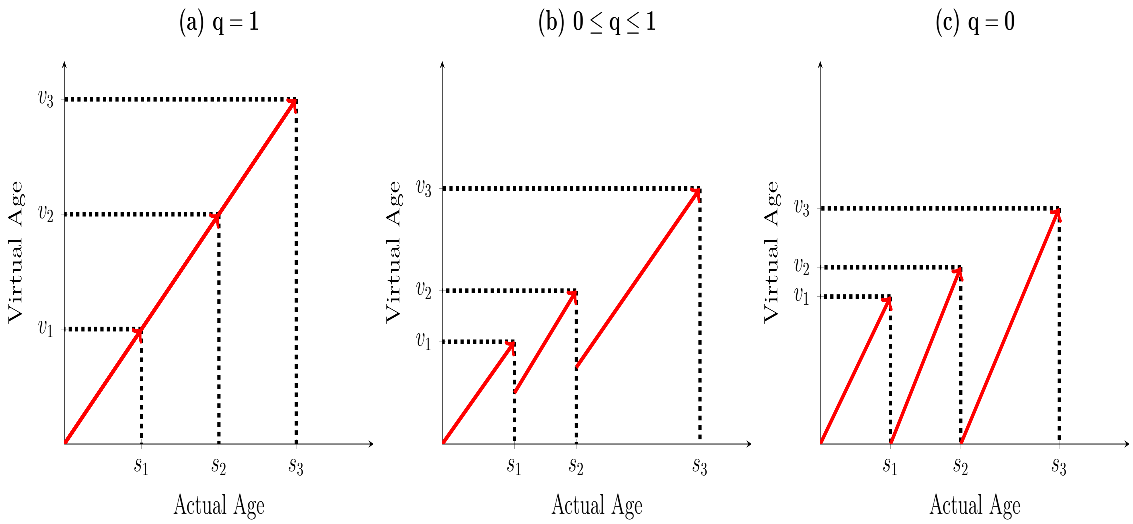


Figure 2.1: Virtual age against real age for varying q values [74]

the value of q. Figure 2.1 shows the relationship between real age and virtual age according to q based on the Kijima type I model, where S_n represents the time of the n^{th} failure (real age), and V_n represents the virtual age directly before n^{th} failure. Figure 2.1 (a) corresponds to minimal repair, where the virtual age continues to increase as the actual age increases, which means that nothing changes after each repair. Therefore, the virtual age is equal to the actual age. Figure 2.1 (b) represents imperfect repair where the virtual age has been reduced between minimal and perfect repair. Figure 2.1 (c) represents perfect repair as $q = 0$, where the virtual age is reduced to zero after each repair [74].

The Kijima type I model assumes that corrective maintenance will only deal with the damages and the wearout created during the last operation period. On the other hand, the Kijima type II model assumes that corrective maintenance could address all the wear out as well as damages that are accumulated till the current failure time [53]. The Kijima type I and type II models can be applied to a single repairable system to analyze the effects of repairs over time. Additionally, the Kijima models can be extended to analyze multiple identical systems [53]. This thesis focuses on scenarios where multiple samples are collected from identical repairable systems, which will be discussed in Chapter 3.

2.3.3.1 Kijima type I model with Weibull distribution

For $i = 1, 2, \ldots, n$, consider the virtual age V_i, and consider the observed time between failures $x_1, x_2, \ldots, x_n$. The time T_i is distributed according to the following conditional cumulative distribution function (CDF)

$$F_{T_i}(x|v_{i-1}) = \frac{F(x+v_{i-1}) - F(v_{i-1})}{1 - F(v_{i-1})} \ , \tag{2.3.10}$$

where $F(t)$ is the CDF of the time to failure of a new system [39]. By assuming a Weibull distribution for T_i with scale parameter λ and and shape parameter β, as specified in Section 2.2, and using Equation (2.3.10), the i^{th} failure time T_i has the following CDF

$$\begin{aligned} F_{T_i}(x|v_{i-1}) &= \frac{\exp\left(-\lambda(v_{i-1})^\beta\right) - \exp\left(-\lambda(x+v_{i-1})^\beta\right)}{\exp\left(-\lambda(v_{i-1})^\beta\right)} \\ &= 1 - \exp\left(-\lambda(x+v_{i-1})^\beta\right)\exp(\lambda(v_{i-1})^\beta) \\ &= 1 - \exp\left[-\lambda\left((x+v_{i-1})^\beta - (v_{i-1})^\beta\right)\right] \ . \end{aligned} \tag{2.3.11}$$

Therefore, T_i has conditional probability density function

$$f_{T_i}(x|v_{i-1}) = \beta\lambda(x+v_{i-1})^{\beta-1}\exp\left[-\lambda\left((x+v_{i-1})^\beta - (v_{i-1})^\beta\right)\right] \quad , t \geq 0 \ . \tag{2.3.12}$$

By introducing the Kijima type I, where $v_{i-1} = q\sum_{j=0}^{i-1} x_j$, into Equation (2.3.11), the CDF of T_i is

$$F_{T_i}(x|v_{i-1}) = 1 - \exp\left[-\lambda\left((x + q\sum_{j=0}^{i-1} x_j)^\beta - (q\sum_{j=0}^{i-1} x_j)^\beta\right)\right] \ , \tag{2.3.13}$$

the reliability function of T_i is

$$R_{T_i}(x|v_{i-1}) = \exp\left[-\lambda\left((x + q\sum_{j=0}^{i-1} x_j)^\beta - (q\sum_{j=0}^{i-1} x_j)^\beta\right)\right] \ , \tag{2.3.14}$$

and the probability density function (PDF) of T_i is

$$f_{T_i}(x|v_{i-1}) = \beta\lambda(x + q\sum_{j=0}^{i-1} x_j)^{\beta-1} \exp\left[-\lambda\left((x + q\sum_{j=0}^{i-1} x_j)^{\beta} - (q\sum_{j=0}^{i-1} x_j)^{\beta}\right)\right] , \tag{2.3.15}$$

with the Weibull scale parameter $\lambda > 0$, the Weibull shape parameter $\beta > 0$, and repair efficiency parameter $0 \leq q \leq 1$.

Generally, these functions are used to study lifetime involving the repairable system based on Kijima type I models. Therefore, they can be used to construct theoretical results, such as inferences for repairable systems based on Bayesian methods. In this thesis, the Kijima type I model functions presented in this section will be used for Bayesian inferences, which will be further discussed in Chapter 3.

2.4 Bayesian inference

2.4.1 Introduction

The term "Bayesian" is derived from Reverend Thomas Bayes, who proposed in 1763 what is presently referred to as Bayes' theorem [49]. Since that time, scientists have used Bayesian approaches for statistical inference. Mostly, subtle and unexpected difficulties with other statistical inference methods have caused the continued renewal and development of the Bayesian procedure of reasoning [49]. The Bayesian method is one of the two main methodologies for statistical inference and has practical applications in statistical fields and related areas. In the Bayesian approach, the model parameters are considered random variables, and it is necessary to define a joint prior distribution that characterizes the understanding of the model's unknown parameters in addition to the data and the model. By employing Bayes' theorem, the prior information is combined with the likelihood function to generate a posterior distribution [43, 49].

2.4.2 Bayesian statistics

In Bayesian statistics, modelling work supposes that parameters are treated as unknown random quantities whose prior distribution demonstrates the current knowledge about the parameter. Assume $t_1, t_2, \ldots, t_n$ be the sample data D and $\theta = (\theta_1, \theta_2, ..., \theta_p)$ denote the parameters of the model [49]. Bayes' theorem states that

$$g(\theta|D) = \frac{\left[\prod_{i=1}^{n} f(t_i|\theta)\right] g(\theta)}{f(D)} = \frac{f(D|\theta)g(\theta)}{f(D)} \ , \tag{2.4.16}$$

where $f(D)$ is the marginal distribution of the sample such that

$$f(D) = \begin{cases} \displaystyle\int_{\theta} f(D|\theta)g(\theta)d\theta, & \text{if } \theta \text{ is continuous} \\ \displaystyle\sum_{\theta} f(D|\theta)g(\theta), & \text{if } \theta \text{ is discrete} \end{cases} \tag{2.4.17}$$

where $g(\theta|D)$ is the posterior probability density of the parameter θ given a data set D, and $g(\theta)$ is the prior probability density of the parameter θ, which represents information about the parameter before the data is observed. $f(D)$ is the marginal probability density of the data, which is represented as a normalizing constant [49], and $f(D|\theta)$ is the likelihood function. The likelihood is the function through which the sample data D changes the prior beliefs about θ, and it may be considered as the function that represents the information about θ which is in the data [49].

Since the factor $f(D)$ does not depend on θ, and D is fixed, Bayesian updating can be expressed according to

$$g(\theta|D) \propto g(\theta)\ell(\theta|D) \ . \tag{2.4.18}$$

Equation (2.4.18) presents that the posterior distribution is proportional to the product of the likelihood function and the prior distribution. The constant of proportionality, which is the marginal distribution of the data, is important to be sure that the posterior distribution integrates or sums to one [29, 49].

In Bayesian decision theory, various loss functions are employed for Bayesian decisions, including the squared error loss function and the absolute error loss function.

These functions are used to make optimal decisions by minimizing the expected loss. The loss function, often represented as $L(\theta, \hat{\theta})$, measures the loss when a decision or action is made using the estimated parameter value $\hat{\theta}$, while the actual parameter value is θ [49]. The squared error is the most common loss function [49], which is

$$L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2 \ .$$

The Bayesian estimator refers to an estimator that aims to minimize the risk or loss associated with the posterior distribution. For a squared-error loss function, the value $\hat{\theta}$, which minimizes the expected loss, is the posterior mean in the Bayes estimator for θ [49].

Predictive distributions quantify uncertainty about a future observation and play an important role in statistics and related topics such as machine learning. In Bayesian statistics, the posterior predictive distribution is the conditional distribution of a future observation given data observations [26].

The posterior predictive distribution for T is conditional on the observed data D, and can be obtained as [16, 29]

$$f(t|D) = \begin{cases} \displaystyle\int_{\theta} f(t|\theta) g(\theta|D) d\theta, & \text{if } \theta \text{ is continuous} \\ \displaystyle\sum_{\theta} f(t|\theta) g(\theta|D), & \text{if } \theta \text{ is discrete} \end{cases} \tag{2.4.19}$$

2.4.3 Markov Chain Monte Carlo (MCMC)

The Markov Chain Monte Carlo (MCMC) method is useful for approximating complex posterior distributions [1]. It is based on the idea that a Markov chain converges to a stationary distribution. The MCMC method is used to simulate a Markov chain that converges to a target distribution, which is the desired posterior distribution of the parameter of interest. The Gibbs Sampling and Metropolis-Hastings methods are two of the most commonly used MCMC methods. The Gibbs Sampling method involves sampling from the conditional distributions of each parameter of interest. The Metropolis-Hastings method, which was presented in 1970 by Hastings [35], is a random walk algorithm that uses an acceptance-rejection step to sample from the posterior distribution. It proposes a new state of the Markov Chain and then decides whether or not to accept it based on the acceptance probability. Both methods are useful for obtaining a sample from the posterior distribution of the parameters. However, if there is no closed form or known conditional distribution for a parameter, Gibbs sampling is not applicable, and Metropolis-Hastings (MH) might be used [28].

The random walk chain of the Metropolis-Hastings algorithms is suitable for a diverse range of Bayesian inference problems. With the Metropolis-Hastings algorithm, samples from the conditional posterior distribution are sampled by proposing new values from a proposal distribution and these values are accepted according to an acceptance proportion. Assume we wish to generate a random value for θ from a posterior distribution $g(\theta|D)$ [1]. We can use random walk Metropolis-Hastings to generate a sample $(\theta_1,\ \theta_2, ...)$, where the algorithm is as follows [1, 69]

1. Start with initial value θ_0.
2. Generate value θ^* from a candidate proposal distribution $g(\theta|\theta_i)$
3. Calculate
$$R = \frac{g(\theta^*|D)}{g(\theta_i|D)}$$
4. Generate u from $U(0, 1)$
5. If $u < R$ then set $\theta_{i+1} = \theta^*$; otherwise set $\theta_{i+1} = \theta_i$
6. Repeat Steps 2-5 for a chosen number of iterations.

Using this algorithm, samples are obtained from the posterior distribution, and the quantities of interest are obtained by taking their average.

In this thesis, we are going to use Bayesian techniques, as described in this section, to transform the prior information about the parameters in the Kijima type I model with Weibull distribution to a posterior distribution given data. We are going to use these principles directly for the Kijima type I model in Chapter 3.

Chapter 3

Bayesian inference for Kijima type I model

For the statistical inference using repair models, NHPP and RP models have been widely presented, but fewer papers address statistical inference for virtual age models [15]. The maximum likelihood estimator (MLE) is the most studied for the virtual age models. [78] conducted a study that presented a method for obtaining MLE by solving the derivatives of the log-likelihood function for the Kijima type I model. Because there are no closed-form solutions for some of these equations, the authors proposed an algorithm based on Monte Carlo simulation. Wang and Yang [77] introduced a method of nonlinear constrained programming to get the maximum likelihood estimates of the Kijima type I and II models, where negative log-likelihood is the objective function, and model parameters are the constrained variables. A solution for Kijima models using MLEs for multiple and single repairable systems was proposed by Mettas and Zhao [53]. Tanwar and Bolia [56] presented the MLE for parameters of the Kijima type I and II models in the cases of corrective and preventive maintenance. The estimated parameters are used to approximate the expected number of failures for the suggested methodologies through Monte Carlo simulation.

Some papers have recently been presented on Bayesian analysis using the virtual age models of repairable systems. Chukova et al. [13] compared the Markov Chain Monte Carlo (MCMC) estimation method to the MLE estimation after modelling imperfect repair under the Weibull, Gamma, and Exponential lifetime distributions. In this study, it was determined that the MLE and Bayesian approaches yield similar outcomes when applied to large sample sizes. Also, a Bayesian study was performed for the arithmetic reduction of age (ARA) model proposed by Doyen and Gaudoin [21] by Corset et al. [15] using a variety of potential priors. The MCMC method is used to obtain Bayesian estimates, and the outcomes are contrasted with MLE. Egemen [22] suggested a unifying virtual age model that combines perfect, imperfect and minimal repairs and introduces generalizations of existing repair models. Egemen also considered a Bayesian framework for statistical inference for this model, and the posterior and posterior predictive analyses are conducted using MCMC. He dealt with the effective age as an unobservable parameter and used the power law intensity function.

While these studies have been made by using both MLE and Bayesian inference to virtual age models, such as the Kijima type I model, these studies have not addressed the challenge of estimating total predictive system reliability with a given number of repairs affecting the ageing of the system as far as we know. This is important because the number of repairs influences system ageing and, consequently, its reliability, which is an important consideration for effective maintenance planning.

To examine a system's reliability, the Weibull distribution is often appropriate [57] since it can capture a wide range of failure patterns. In this chapter, we focus on the Kijima type I model and a Weibull lifetime distribution for the virtual age model. Section 2.3 briefly introduced the Kijima type I model. This is one of the important methods for analyzing the performance of repairable systems in engineering asset management after repair. In reality, the quality and efficiency of repairs impact the reliability and functional continuity of systems. Poor repair efficiency can result in higher costs and reduced system reliability, while effective repairs can minimize costs and enhance reliability. In industries, reducing costs while improving repair effectiveness is important. Therefore, actions can be taken to manage the cost and effectiveness of repairs, especially in industries dealing with critical repairable systems. The Bayesian method provides a flexible framework for modelling and incorporating the performance of repairable systems into analyses based on the Kijima type I model, which contributes to making decisions to improve system life and reduce costs.

In this chapter, we present a Bayesian method that enables the use of prior information and the data on failure processes using the Kijima type I model. Section 3.1 presents the likelihood function and the posterior distribution for the model employed in this thesis, specifically the Kijima type I model. Most literature uses numerical methods because of the Kijima type I model's mathematical complexity. We present an analytic method using the Bayesian method for the Kijima type I model's parameters, and the results are compared with the MLE in Section 3.2. Also, a Markov Chain Monte Carlo (MCMC) approach is used, using continuous priors for Bayesian analysis in Section 3.3, where the results will be illustrated with a suitable and poor choice of priors. In Section 3.4, we present the reliability of a system and analyse predictive reliability changes after each repair. In Section 3.5, we develop a novel method using the posterior predictive empirical reliability function to predict the total system reliability with the uncertainty until the time needed to replace it when the number of failures is known.

3.1 Likelihood and posterior function for the Kijima type I model

According to Yañez et al. [78], and Mettas and Zhao [53], two different expressions of the likelihood function can be obtained for the Kijima type I model based on the experiment of data collection. The first expression is the likelihood function based on the failure-terminated data, which we will use in this thesis. This method is suitable when failure data are observed up to the time of the n^{th} failure occurrence, where the number of failures n is determined beforehand. The likelihood function is given by

$$\ell = \prod_{i=1}^{n} f_{T_i}(x_i|v_{i-1}) \ , \tag{3.1.1}$$

The Kijima type I model with Weibull lifetime distribution, which is given in

Equation (2.3.15), has three parameters: the Weibull scale parameter λ, the Weibull shape parameter β, and the repair efficiency parameter q. Based on the failure-terminated data, the likelihood function is

$$\begin{aligned}\ell(\lambda,\ \beta,\ q|D) &= \prod_{i=1}^{n} \beta\lambda(x_i + q\sum_{j=0}^{i-1} x_j)^{\beta-1} \exp\left[-\lambda\left((x_i + q\sum_{j=0}^{i-1} x_j)^{\beta} - (q\sum_{j=0}^{i-1} x_j)^{\beta}\right)\right] \\ &= \lambda^n\beta^n \prod_{i=1}^{n}(x_i + q\sum_{j=0}^{i-1} x_j)^{\beta-1} \exp\left[-\lambda\sum_{i=1}^{n}\left((x_i + q\sum_{j=0}^{i-1} x_j)^{\beta} - (q\sum_{j=0}^{i-1} x_j)^{\beta}\right)\right] .\end{aligned} \tag{3.1.2}$$

Assume the scenario where M samples are collected from an identical system, and each sample experiences a different number of failures due to different operational conditions and usage patterns. In order to represent the observations, a list of vectors D_M is used, where each vector represents the sequential failure times of length n_m for each sample. The likelihood function is

$$\begin{aligned}\ell(\lambda,\ \beta,\ q|D_M) &= \prod_{m=1}^{M}\left(\lambda^{n_m}\beta^{n_m}\prod_{i=1}^{n_m}(x_{mi} + q\sum_{j=0}^{i-1} x_{mj})^{\beta-1}\times\right. \\ &\qquad \left.\exp\left[-\lambda\sum_{i=1}^{n_m}\left((x_{mi} + q\sum_{j=0}^{i-1} x_{mj})^{\beta} - (q\sum_{j=0}^{i-1} x_{mj})^{\beta}\right)\right]\right) \\ &= \lambda^{\sum_{m=1}^{M} n_m}\beta^{\sum_{m=1}^{M} n_m}\prod_{m=1}^{M}\left(\prod_{i=1}^{n_m}(x_{mi} + q\sum_{j=0}^{i-1} x_{mj})^{\beta-1}\right)\times \\ &\qquad \exp\left[-\lambda\sum_{m=1}^{M}\sum_{i=1}^{n_m}\left((x_{mi} + q\sum_{j=0}^{i-1} x_{mj})^{\beta} - (q\sum_{j=0}^{i-1} x_{mj})^{\beta}\right)\right]\end{aligned} \tag{3.1.3}$$

where $m = 1,\ ......,\ M$ and x_{mi} is the i^{th} failure time of the m^{th} sample from identical system.

For a given continuous prior joint density $g(\lambda, \beta, q)$, the joint posterior distribution of the three parameters $(\lambda,\ \beta,\ q)$ is

$$g(\lambda, \beta, q|D_m) = \frac{g(\lambda, \beta, q)\ell(\lambda, \beta, q|D_m)}{\int_0^1\int_0^{\infty}\int_0^{\infty} g(\lambda, \beta, q)\ell(\lambda, \beta, q|D_m)d\lambda d\beta dq}\ , \tag{3.1.4}$$

where $\ell(\lambda, \beta, q|D_m)$ is the likelihood function of the Kijima type I model, the normalization factor is $\left(\int_0^1 \int_0^\infty \int_0^\infty g(\lambda, \beta, q)\ell(\lambda, \beta, q|D_m)\, d\lambda\, d\beta\, dq\right)^{-1}$. The posterior distribution can be expressed as

$$g(\lambda, \beta, q|D_m) \propto g(\lambda, \beta, q)\ell(\lambda, \beta, q|D_m) \ . \tag{3.1.5}$$

From the joint posterior distribution, the marginal posterior distribution of each parameter can be obtained by integrating or summing with respect to the other two parameters and is shown in the following three Equations [62]:

$$g(\lambda|\beta, q, D_m) \propto g(\lambda)\ell(\lambda, \beta, q|D_m) \ , \tag{3.1.6}$$

$$g(\beta|\lambda, q, D_m) \propto g(\beta)\ell(\lambda, \beta, q|D_m) \ , \tag{3.1.7}$$

$$g(q|\beta, \lambda, D_m) \propto g(q)\ell(\lambda, \beta, q|D_m) \ . \tag{3.1.8}$$

Bayesian estimation of model parameters relies on calculating the expected value of the posterior distribution using squared error loss where, in some situations, closed-form expressions are unavailable. In such cases, numerical estimations can be done using different sampling techniques. We will use the Markov Chain Monte Carlo (MCMC), implemented by the Metropolis-Hastings method as described in Subsection 2.4.3 where $\theta = (\lambda,\ \beta,\ q)$.

Two approaches are employed to analyze the Kijima type I model using Bayesian methodology. The first analysis involves utilizing analytical solutions, where closed-form expressions for estimations are attainable within a Bayesian framework with discrete priors in Section 3.2. In the second approach, a Markov Chain Monte Carlo (MCMC) method is used, using continuous priors for Bayesian analysis in Section 3.3. To elucidate the performance of the first approach, we compare the results obtained from the analytical method with discrete priors to the results derived from the MLE method, which is the most studied estimation method in the literature. These approaches will be illustrated with a suitable and poor choice of priors.

3.2 Mixed priors for the parameters: continuous and discrete

When the scale parameter λ, shape parameter β, and effective age parameter q are unknown, determining a general joint prior for λ, β, q may cause complexities in Bayesian inference. To deal with this issue, we employ a similar approach to Soland's method [72] and extend it for the Kijima type I model. Soland introduced a family of joint prior distributions, where Soland chose a continuous conditional prior for the scale parameter λ and a discrete prior for the shape parameter β for the Weibull distribution. For the Kijima type I model, we take this concept further by introducing a discrete prior for the q parameter as an extension of the Soland method.

A shape parameter $\beta < 1$ indicates a decreasing failure rate, whereas a shape parameter $\beta > 1$ indicates an increasing failure rate. Assume that experts can give reasonable values for β with the probability for each value based on prior knowledge of the underlying failure of the system. Thus, assume that $\beta \in \{\beta_1, \beta_2, ..., \beta_S\}$ with prior probability $p'_1, p'_2, ..., p'_S$, respectively, where $0 \leq p'_s \leq 1$ and $\sum_{s=1}^{S} p'_s = 1$, that is

$$P(\beta = \beta_s) = p'_s, \quad s = 1, 2, ..., S \; . \tag{3.2.9}$$

When there is no information about which value of β is more likely to represent prior knowledge, a discrete uniform distribution can be used.

The q represents the repair efficiency where its value is between 0 and 1. When q has a discrete prior, the experts can choose the values and the probability of each value based on the knowledge of the efficiency of the repair activities. Assume that $q \in \{q_1, q_2, ..., q_K\}$ with prior probability $q'_1, q'_2, ..., q'_K$, respectively, where $0 \leq q'_k \leq 1$ and $\sum_{k=1}^{K} q'_k = 1$, that is

$$P(q = q_k) = q'_k, \quad k = 1, 2, ..., K \; . \tag{3.2.10}$$

Again, if the experts are unsure which value is more likely for the effective repair parameter, the same probability values will be used for each value (a discrete uniform distribution).

Finally, assume that the conditional prior of λ given $\beta = \beta_s$ and $q = q_k$ be Gamma(a, b) with pdf

$$g(\lambda|\beta = \beta_s, q = q_k) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}\exp(-\lambda b) \tag{3.2.11}$$

The Gamma prior distribution for λ is chosen to attain a conjugate prior. In Equation 3.2.11, the parameters a and b of the Gamma distribution are constants, not varying with β and q. Therefore, the prior distribution of λ is independent of β and q. This independence is chosen to simplify the Bayesian inference.

3.2.1 Posterior distribution and Bayesian estimators

The likelihood function given in Equation (3.1.3) can be rewritten as follows. For convenience, denote

$$N = \sum_{m=1}^{M} n_m \quad , \tag{3.2.12}$$

$$u_{sk} = \prod_{m=1}^{M}\left(\prod_{i=1}^{n_m}(x_{mi} + q_k\sum_{j=0}^{i-1} x_{mj})^{\beta_s - 1}\right), \tag{3.2.13}$$

$$r_{sk} = \sum_{m=1}^{M}\sum_{i=1}^{n_m}\left((x_{mi} + q_k\sum_{j=0}^{i-1} x_{mj})^{\beta_s} - (q_k\sum_{j=0}^{i-1} x_{mj})^{\beta_s}\right) \tag{3.2.14}$$

Then, the likelihood function is

$$\ell(\lambda, \beta_s, q_k|D_m) = \lambda^N \beta_s^N u_{sk}\exp(-\lambda r_{sk}) \ . \tag{3.2.15}$$

Assume that λ, β and q are a priori independent. Multiplying the likelihood by the prior of λ, β and q, we obtain the joint posterior distribution, which is

$$\begin{aligned} g(\lambda, \beta_s, q_k|D_m) &\propto \ell(\lambda, \beta_s, q_k|D_m) g(\lambda|\beta = \beta_s, q = q_k) P(\beta) P(q) \\ &= \lambda^N \beta_s^N u_{sk} \exp(-\lambda r_{sk}) \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-\lambda b) p'_s q'_k \\ &= \beta_s^N u_{sk} \frac{b^a}{\Gamma(a)} \lambda^{N+a-1} \exp\left[-\lambda(b + r_{sk})\right] p'_s q'_k \ . \end{aligned} \tag{3.2.16}$$

Multiplying the likelihood by the conditional prior given in Equation (3.2.11), we obtain the conditional posterior of λ given $\beta = \beta_s,\ q = q_k$ which is

$$g(\lambda|\beta = \beta_s, q = q_k, D_m) = \beta_s^N u_{sk} \frac{b^a}{\Gamma(a)} \lambda^{N+a-1} \exp\left[-\lambda(b + r_{sk})\right] \tag{3.2.17}$$

From Equation (3.2.17) the conditional posterior distribution of λ given $\beta = \beta_s,\ q = q_k$ is Gamma$(N + a, r_{sk} + b)$. Let A be the normalizing constant given in Equation (3.2.19) and integrate Equation (3.2.16) with respect to λ, the marginal joint posterior distribution of (β, q) is

$$\begin{aligned} p_{\beta,q}(\beta_s, q_k|D_m) &= A \beta_s^N u_{sk} \frac{b^a}{\Gamma(a)} p'_s q'_k \int_0^\infty \lambda^{N+a-1} \exp\left[-\lambda(b + r_{sk})\right] d\lambda \\ &= A \frac{\beta_s^N u_{sk} b^a p'_s q'_k \Gamma(N + a)}{\Gamma(a)(r_{sk} + b)^{N+a}} \ , \end{aligned} \tag{3.2.18}$$

where A is the normalising constant, which is given by

$$A = \left(\sum_{s=1}^{S} \sum_{k=1}^{K} \frac{\beta_s^N u_{sk} b^a p'_s q'_k \Gamma(N + a)}{\Gamma(a)(r_{sk} + b)^{N+a}} \right)^{-1} \ . \tag{3.2.19}$$

From Equations (3.2.17) and (3.2.18), the marginal posterior distribution of λ is

$$g(\lambda|D_m) = \sum_{s=1}^{S} \sum_{k=1}^{K} g(\lambda|\beta = \beta_s, q = q_k, D_m) p_{\beta,q}(\beta_s, q_k|D_m) \tag{3.2.20}$$

The parameter value that minimizes the expected squared error loss of the posterior is the posterior mean, as briefly explained in the Subsection 2.4.2. Therefore, the Bayesian estimators for λ, β and q are

$$\begin{aligned}\hat{\lambda} = E[\lambda|D_m] &= \int_0^{\infty} \lambda g(\lambda|D_m)d\lambda \\ &= \sum_{s=1}^{S}\sum_{k=1}^{K} p_{\beta,q}(\beta_s, q_k|D_m) \int_0^{\infty} \lambda g(\lambda|\beta=\beta_s, q=q_k, D_m)d\lambda \\ &= \sum_{s=1}^{S}\sum_{k=1}^{K} p_{\beta,q}(\beta_s, q_k|D_m)\frac{N+a}{r_{sk}+b}\end{aligned} \tag{3.2.21}$$

$$\hat{\beta} = E[\beta|D_m] = \sum_{s=1}^{S}\sum_{k=1}^{K} \beta_s p_{\beta,q}(\beta_s, q_k|D_m) \tag{3.2.22}$$

$$\hat{q} = E[q|D_m] = \sum_{s=1}^{S}\sum_{k=1}^{K} q_k p_{\beta,q}(\beta_s, q_k|D_m) \tag{3.2.23}$$

From Equations (2.3.12) and (3.2.16) the posterior predictive distribution for T_i is specified by pdf

$$\begin{aligned}f_{T_i}(x|x_{i-1}, D_m) &= \sum_{s=1}^{S}\sum_{k=1}^{K} \int_0^{\infty} g(\lambda, \beta_s, q_k|D_m) f_{T_i}(x|v_{i-1})d\lambda \\ &= A\sum_{s=1}^{S}\sum_{k=1}^{K} \beta_s^N u_{sk}\frac{b^a}{\Gamma(a)}p'_s q'_k \beta_s(x + q_k\sum_{j=0}^{i-1}x_j)^{\beta_s-1}\int_0^{\infty}\left[\lambda^{N+a-1}\exp\left(-\lambda\left[b+r_{sk}\right]\right)\times\right. \\ &\left.\exp\left(-\lambda\left[(x+q_k\sum_{j=0}^{i-1}x_j)^{\beta_s} - (q_k\sum_{j=0}^{i-1}x_j)^{\beta_s}\right]\right)\right]d\lambda \\ &= A\sum_{s=1}^{S}\sum_{k=1}^{K} \frac{\beta_s^N u_{sk} b^a p'_s q'_k \beta_s (x + q_k\sum_{j=0}^{i-1}x_j)^{\beta_s-1}\Gamma(N+a)}{\Gamma(a)\left[(x+q_k\sum_{j=0}^{i-1}x_j)^{\beta_s} - (q_k\sum_{j=0}^{i-1}x_j)^{\beta_s} + b + r_{sk}\right]^{N+a+1}}\end{aligned} \tag{3.2.24}$$

3.2.2 Simulation study

This subsection presents the results of a simulation study to illustrate the methodology presented in Section 3.2, where we used continuous and discrete priors to estimate the parameters. Also, the results of Bayesian estimators will be compared with the MLE. We generate different samples of the Kijima type I model of sizes $n_m \times M$, combinations of $n_m \in [25,\ 75,\ 100,\ 200]$ and $M \in [20,\ 50,\ 100,\ 250,\ 400]$

where n_m is the number of failure times for each sample M for two scenarios: high and low-quality repair efficiency. For high-quality repair efficiency, we assume that $q = 0.25$, and we assume that $q = 0.85$ for low-quality repair efficiency. We use $\lambda = 3$, and $\beta = 2$ for both scenarios. For both repair efficiency scenarios, the estimations are carried out based on two cases of the selection of the hyperparameters a and b and the values of β_s and q_k. In Case 1, a good prior is selected, where the hyperparameters and the values of β_s and q_k are selected so that the mean of each prior is near the values of actual parameters. In Case 2, a poor prior is selected, with hyperparameters and values of β_s and q_k selected such that the mean value of each prior is far from the values of actual parameters.

In Case 1, we assume that the values of the Gamma hyper-parameters are $a = 12$ and $b = 4$, where the mean of the Gamma distribution is $\frac{a}{b} = \frac{12}{4} = 3$ which is the actual value of λ. Also, assume that the prior probabilities for $\beta_s = (1.90,\ 1.92,\ 1.94,\ 1.96,\ 1.98,\ 2.00,\ 2.02,\ 2.04,\ 2.06,\ 2.08)$ is a discrete Uniform denoted as $p'_s = \frac{1}{S}$, where S represents the total number of β values and the prior probabilities for $q_k = (0.21,\ 0.22,\ 0.23,\ 0.24,\ 0.25,\ 0.26,\ 0.27,\ 0.28,\ 0.29.\ 0.30)$ when $q = 0.25$ and $q_k = (0.80,\ 0.81,\ 0.82,\ 0.83,\ 0.84,\ 0.85,\ 0.86,\ 0.87,\ 0.88,\ 0.89)$ when $q = 0.85$ is a discrete Uniform denoted as $p'_s = \frac{1}{K}$, where K represents the total number of q values. However, in Case 2, assume that the values of the Gamma hyper-parameters are $a = 2$ and $b = 5$, where the mean of Gamma distribution is 0.4 which is far from the actual value of λ. Also, assume that the prior probabilities for $\beta_s = (1.25,\ 1.35,\ 1.45,\ 1.55,\ 1.65,\ 1.75,\ 1.85,\ 1.95,\ 2.05,\ 2.15)$ is a discrete Uniform denoted as $p'_s = \frac{1}{S}$ and the prior probabilities for $q_k = (0.230,\ 0.244,\ 0.258,\ 0.272,\ 0.286,\ 0.300,\ 0.314,\ 0.328,\ 0.342.\ 0.356)$ when $q = 0.25$ and $q_k = (0.84,\ 0.85,\ 0.86,\ 0.87,\ 0.88,\ 0.89,\ 0.90,\ 0.91,\ 0.92,\ 0.93)$ when $q = 0.85$ is a discrete Uniform denoted as $p'_s = \frac{1}{K}$ values.

The Bayesian estimators developed in Section 3.2 for λ, β and q are used for this simulation and compared with the MLE for each case. The results are given in Tables 3.1 for high-quality repair efficiency scenario and 3.2 for low-quality repair efficiency scenario, where "BE Case 1" represents the Bayesian estimation for Case 1, and "BE Case 2" represents the Bayesian estimation for Case 2. Also, Figure

3.1 and Figure 3.2 show the estimated results for each scenario when $q = 0.25$ and $q = 0.85$, where the red horizontal line represents the true value and the lines of different colours with points describe different numbers of sequential failure times for each sample process n_m with increasing M. The results presented in the table and shown in the figures show that as we increase the number of sequential failure times for each process n_m, the results change hardly. However, by increasing the number of sequential processes M, the estimated values for λ and q change and get closer to the true value, whereas $\hat{\beta}$ shows less change.

For parameter estimates, in terms of the closest value to the true values, Case 1 gives the best estimator for the parameters, followed by the MLE and Case 2. For the estimation of λ and q, the prior has impacted the results for both repair efficiency scenarios. We can see in Figure 3.1 for high-quality repair efficiency and in Figure 3.2 for low-quality repair efficiency that the estimation of Bayesian estimators with good prior selection in Case 1 is closer to the true values than the estimators with a poor prior selection in Case 2, especially for smaller sample sizes. As a result, the resulting estimations may not be satisfying if the informative prior choice is not appropriate. However, for the estimation of β, the prior has less impact, especially when $q = 0.25$.

True values: $\lambda = 3$, $\beta = 2$, and $q = 0.25$										
n_m	M	**MLE**			**BE Case** 1			**BE Case** 2		
		$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$
25	20	2.4884	1.9773	0.3195	2.8069	1.9966	0.2716	2.3429	2.0240	0.3270
	50	2.7772	2.0031	0.2715	2.8386	1.9990	0.2665	2.5430	1.9956	0.3056
	100	2.8717	1.9915	0.2569	2.8727	1.9893	0.2586	2.7119	1.9800	0.2797
	250	2.8988	2.0059	0.2582	2.8942	2.0040	0.2599	2.8336	2.0047	0.2668
	400	2.9905	2.0008	0.2503	2.9862	1.9997	0.2514	2.9448	1.9942	0.2569
75	20	2.5137	1.9753	0.3228	2.8416	2.0038	0.2740	2.3916	2.0075	0.3279
	50	2.7927	1.9860	0.2759	2.8619	1.9926	0.2683	2.5664	1.9691	0.3120
	100	2.8802	2.0245	0.2555	2.8758	2.0218	0.2580	2.7534	2.0311	0.2683
	250	2.9225	2.0034	0.2577	2.9184	2.0026	0.2591	2.8592	1.9997	0.2675
	400	3.0357	2.0069	0.2457	3.0313	2.0062	0.2466	3.0390	2.0361	0.2394
100	20	2.5652	1.9842	0.3115	2.8497	2.0065	0.2720	2.4049	2.0046	0.3261
	50	2.8014	1.9735	0.2773	2.8751	1.9807	0.2688	2.5664	1.9559	0.3151
	100	2.8639	2.0061	0.2610	2.8672	2.0058	0.2622	2.7081	1.9931	0.2854
	250	2.9159	1.9981	0.2596	2.9115	1.9974	0.2609	2.7995	1.9660	0.2845
	400	3.0222	2.0099	0.2467	3.0180	2.0093	0.2476	3.0566	2.0485	0.2340
200	20	2.6217	1.9899	0.3013	2.8661	2.0059	0.2698	2.4480	1.9931	0.3247
	50	2.8419	2.0030	0.2645	2.8721	2.0046	0.2628	2.5861	1.9776	0.3087
	100	2.8589	2.0055	0.2617	2.8611	2.0054	0.2627	2.7169	1.9999	0.2836
	250	2.9280	1.9978	0.2582	2.9234	1.9974	0.2593	2.7406	1.9545	0.2684
	400	3.0222	1.9971	0.2494	3.0207	1.9974	0.2497	2.8406	1.9504	0.2503

Table 3.1: Comparing MLE and Bayesian estimation of λ, β, and q for both Cases when $q = 0.25$

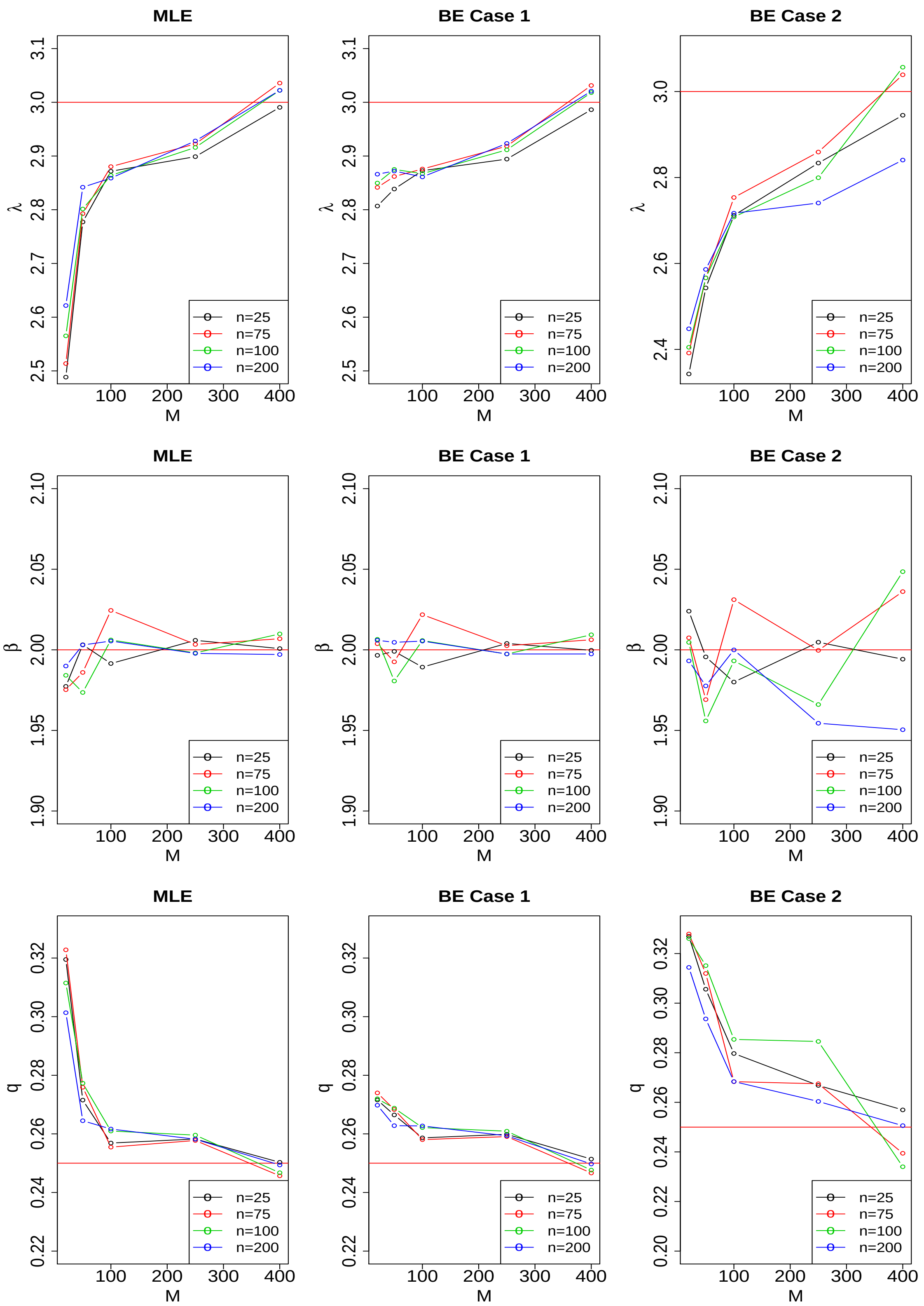


Figure 3.1: The impact of different sample sizes using MLE and Bayesian for both cases when $q = 0.25$.

True values: $\lambda = 3$, $\beta = 2$, and $q = 0.85$										
n_m	M	**MLE**			**BE Case 1**			**BE Case 2**		
		$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$
25	20	2.5446	2.0249	0.9778	2.9303	2.0156	0.8486	2.6638	2.1136	0.8271
	50	2.7636	2.0176	0.9001	2.9181	2.0190	0.8481	2.7883	2.0812	0.8258
	100	2.8119	1.9881	0.8996	2.9187	2.0039	0.8485	2.8687	2.0398	0.8269
	250	2.8365	1.9991	0.8991	2.9286	2.0146	0.8535	2.8991	2.0486	0.8298
	400	2.9632	1.9950	0.8649	2.9960	2.0005	0.8499	2.9759	2.0430	0.8171
75	20	2.5703	2.0140	1.0000	2.9634	2.0265	0.8491	2.7259	2.0949	0.8275
	50	2.7343	1.9952	0.9467	2.9557	2.0132	0.8493	2.8603	2.0500	0.8265
	100	2.8446	2.0255	0.8636	2.8911	2.0288	0.8464	2.8970	2.0260	0.8489
	250	2.8428	2.0007	0.9021	2.9499	2.0100	0.8560	2.8904	2.0500	0.8211
	400	3.0002	2.0003	0.8518	3.0078	2.0016	0.8486	2.9760	2.0497	0.7966
100	20	2.5750	2.0116	1.0000	2.9573	2.0264	0.8487	2.7769	2.0768	0.8280
	50	2.7225	1.9811	0.9694	2.9912	2.0001	0.8503	2.8561	2.0430	0.8269
	100	2.8146	2.0087	0.8936	2.9244	2.0164	0.8492	2.8445	2.0499	0.8244
	250	2.8349	1.9972	0.9096	2.9582	2.0058	0.8580	2.8858	2.0500	0.8168
	400	2.9994	2.0061	0.8462	2.9987	2.0063	0.8470	2.9723	2.0500	0.7946
200	20	2.5958	2.0058	1.0000	2.9709	2.0206	0.8486	2.8555	2.0501	0.8291
	50	2.7743	2.0064	0.9116	2.9420	2.0133	0.8484	2.8092	2.0498	0.8246
	100	2.8115	2.0083	0.8951	2.9326	2.0133	0.8501	2.8222	2.0500	0.8206
	250	2.8416	1.9972	0.9066	2.9670	2.0020	0.8597	2.8826	2.0500	0.8042
	400	2.9834	1.9944	0.8660	3.0127	1.9981	0.8512	3.0191	2.0315	0.7964

Table 3.2: Comparing MLE and Bayesian estimation of λ, β, and q for both Cases when $q = 0.85$

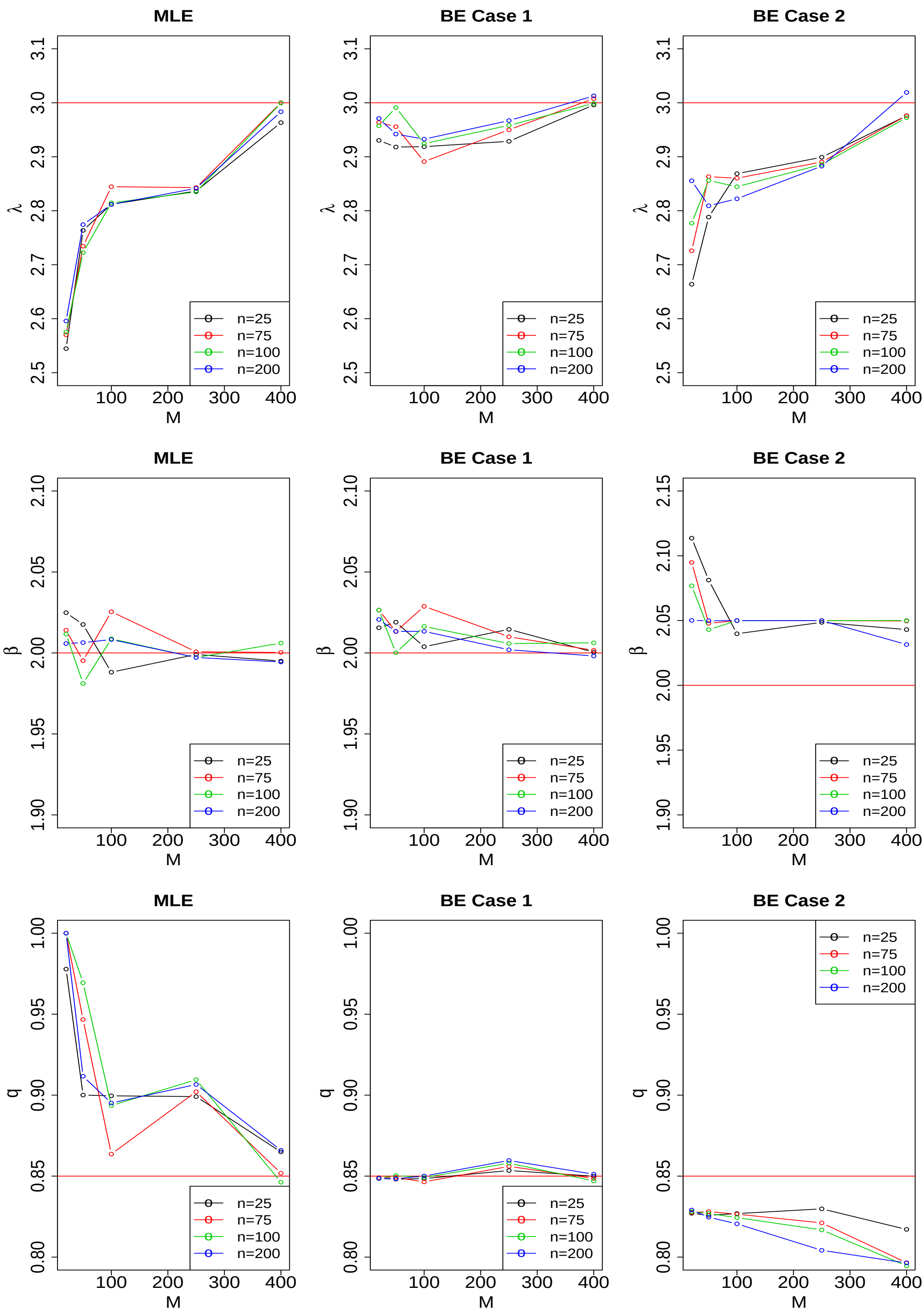


Figure 3.2: The impact of different sample sizes using MLE and Bayesian for both cases when $q = 0.85$.

3.3 Continuous priors for the parameters

In this section, the MCMC method, which is explained in Section 2.4.3, will be used to estimate the scale, shape and repair efficiency parameters of the Kijima type I model, along with the system reliability. Specifically, we will focus on continuous probability distributions, called continuous priors, to represent expert beliefs regarding the parameters.

The Gamma distribution, defined on the interval $[0, \infty]$, is a conjugate prior for the scale parameter λ when other parameters are fixed [49]. Thus, a Gamma distribution is chosen as the prior distribution for the scale parameter $\lambda > 0$, with hyperparameters $a > 0$ and $b > 0$. The probability density function of the Gamma(a, b) distribution is

$$\pi_\lambda(\lambda) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}\exp(-\lambda b) \ , \tag{3.3.25}$$

with Gamma function $\Gamma(a) = \int_0^\infty t^{a-1}\exp(-t)dt$.

A shape parameter $\beta < 1$ indicates a decreasing failure rate, whereas a shape parameter $\beta > 1$ indicates an increasing failure rate. Experts can give an expected interval for β based on prior knowledge of the underlying failure of the system. Therefore, we consider it reasonable to express prior knowledge by choosing a Uniform distribution as the prior distribution for the shape parameter β with the interval $[\beta_1, \beta_2]$. The pdf of the Uniform(β_1, β_2) distribution is

$$\pi_\beta(\beta) = \frac{1}{\beta_2 - \beta_1} \qquad \beta \in [\beta_1,\ \beta_2] \ . \tag{3.3.26}$$

The parameter q represents the repair efficiency where its value is between 0 and 1. The Beta distribution is defined on the interval $[0, 1]$, so we select a Beta distribution as the prior distribution for the effective age parameter q, with hyperparameter $c,\ d > 0$. Assume that the experts can choose the values of hyperparameters c and d based on the knowledge of the efficiency of the repair activities. The pdf of Beta(c, d) distribution is

$$\pi_q(q) = \frac{1}{B(c,d)} q^{c-1}(1-q)^{d-1} \ , \tag{3.3.27}$$

with Beta function $B(c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}$, where $\Gamma(.)$ denots the Gamma function.

Assume that $\lambda,\ \beta$ and q are a priori independent, the prior joint distribution is

$$\begin{aligned}
\pi(\lambda,\ \beta,\ q) &= \pi_\lambda(\lambda)\pi_\beta(\beta)\pi_q(q) \\
&= \frac{b^a}{\Gamma(a)}\lambda^{a-1}\exp(-\lambda b)\frac{1}{\beta_2-\beta_1}\frac{1}{B(c,d)}q^{c-1}(1-q)^{d-1} \\
&\propto \lambda^{a-1}\exp(-\lambda b)\frac{1}{\beta_2-\beta_1}q^{c-1}(1-q)^{d-1} \ .
\end{aligned} \tag{3.3.28}$$

3.3.1 Posterior distribution

Considering Equation (3.3.28) and the likelihood function in Equation (3.1.3), the joint posterior distribution for $\lambda,\ \beta$ and q, given data is specified by the pdf

$$\begin{aligned}
g(\lambda,\beta,q|D_m) &\propto \ell(\lambda,\beta,q|D_m)\pi(\lambda,\ \beta,\ q) \\
&= \lambda^N\beta^N u\exp(-\lambda r)\lambda^{a-1}\exp(-\lambda b)\frac{1}{\beta_2-\beta_1}q^{c-1}(1-q)^{d-1}
\end{aligned} \tag{3.3.29}$$

where $u = \prod_{m=1}^{M}\left(\prod_{i=1}^{n_m}(x_{mi} + q\sum_{j=0}^{i-1}x_{mj})^{\beta-1}\right)$, $N = \sum_{m=1}^{M} n_m$ and

$$r = \sum_{m=1}^{M}\sum_{i=1}^{n_m}\left((x_{mi} + q\sum_{j=0}^{i-1}x_{mj})^\beta - (q\sum_{j=0}^{i-1}x_{mj})^\beta\right) \tag{3.3.30}$$

The Bayesian estimator is the expected value of the posterior distribution by using the mean squared error. Analytical derivation of the estimators is not possible based on this posterior distribution when we use continuous priors, so a numerical method is needed for estimation. There are many different sampling methods for numerical estimates. We will use the Markov Chain Monte Carlo (MCMC) method, implemented by the Metropolis-Hastings Algorithm, to derive a numerical estimation of the three parameters $\lambda,\ \beta$ and q. Also, the reliability function and posterior predictive distribution can not be solved theoretically because of the difficulty of the integration approaches, as shown in Appendix B. Therefore, a numerical method is needed to solve this problem.

3.3.2 Experimental analysis

This subsection presents the results of an experimental analysis to illustrate the impact of the methodology presented in Section 3.3, where we used continuous priors for each parameter. We generate different samples using high and low-quality repair efficiency from the Kijima type I model of sizes $n_m \times M$, where $n_m \in [10,\ 20]$ and $M \in [\,10,\ 20]$. We use the same λ, β and q values, which are presented in Subsection 3.2.2 for both repair efficiency scenarios. For $q = 0.25$ and $q = 0.85$, the estimations are carried out based on two cases of the selection of the hyperparameters of each prior distribution. Assume that a good prior is selected in Case 1. For each prior distribution, hyperparameters are selected so that the means of the priors are close to the actual values of the parameters. Conversely, assume that a poor prior is selected in Case 2. For each prior distribution, hyperparameters are selected so that the mean of the priors is far from the actual parameters' values. Assume that the values of the hyperparameters are selected and given in Table 3.3 for both Cases, where "Beta scenario 1" in the fourth column represents the chosen hyperparameter values of the Beta distribution for a high-quality repair efficiency scenario, and "Beta scenario 2" in the fifth column represents the chosen hyperparameter values of the Beta distribution for a low-quality repair efficiency scenario.

Distributions	Gamma		Uniform		Beta scenario 1		Beta scenario 2	
Hypereparameters	a	b	β_1	β_2	c	d	c	d
Case 1	12	4	1	3	1	3	6	1
Case 2	20	5	1	5	1	4	5	2

Table 3.3: Hyperparameters of priors.

Since the posterior distributions are difficult or impossible to analyze with standard analytical techniques, the Metropolis-Hastings (MCMC) method is used to derive the sample from the posterior distribution and estimate the parameters. 30000 observations are generated from the posterior distribution for λ, β, and q. We consider the first 5000 observations to be burn-in and discard them, while every other observation is used to make an inference. Figure 3.3 shows the histogram and trace plot for Case 1, and Figure 3.4 shows the histogram and trace plot for Case 2 when $n_m = 10$ and $M = 10$, where the red line represents the true value and blue line represent estimated value. The histograms show that in both cases, the posterior distributions of λ, β, and q show a noticeable peak around the estimated value presented in Table 3.4. In Case 1, the central point is closer than Case 2 to the true values of each parameter. The diagnostic trace plot for each parameter indicates that the chains are generally stationary and mixing well, and there are no signs or evidence that chains are not convergent. The other histograms and trace plots for each data sample size and both Cases are shown in Appendix A, which show similar results, where the histogram central point for each parameter becomes closer to the true value when we have larger sample sizes.

The estimated parameters results are given in Table 3.4 and 3.2 based on Case 1 and Case 2 for both high and low-quality repair efficiency. The results presented in the tables show that as we increase the number of sequential failure times for each process n_m, the results hardly change. However, by increasing the number of sequential processes M, the estimated values for each parameter change and get closer to the true value. Also, for both repair efficiency scenarios, the estimated parameters with good prior selection in Case 1 are closer to the true values than the estimators with poor prior selection in Case 2, especially for smaller sample sizes. Therefore, for parameter estimators, in terms of the closest value to the true values based on the two Cases, Case 1 gives a better estimator for the parameters than Case 2. In addition, when data sample sizes increase, Case 1 and Case 2 give more similar results, especially when we have high repair efficiency.

True values: $\lambda = 3$, $\beta = 2$, and $q = 0.25$							
n_m	M	**Case** 1			**Case** 2		
		$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$
10	10	4.0409	2.4163	0.2079	5.0872	2.4957	0.1657
	20	3.1759	1.9459	0.2246	3.6283	1.9510	0.1835
20	10	4.0619	2.1818	0.1779	4.9013	2.2188	0.1438
	20	3.0449	2.0080	0.2553	3.4265	2.0026	0.2223

Table 3.4: Estimated values of λ, β and q using MCMC when $q = 0.25$.

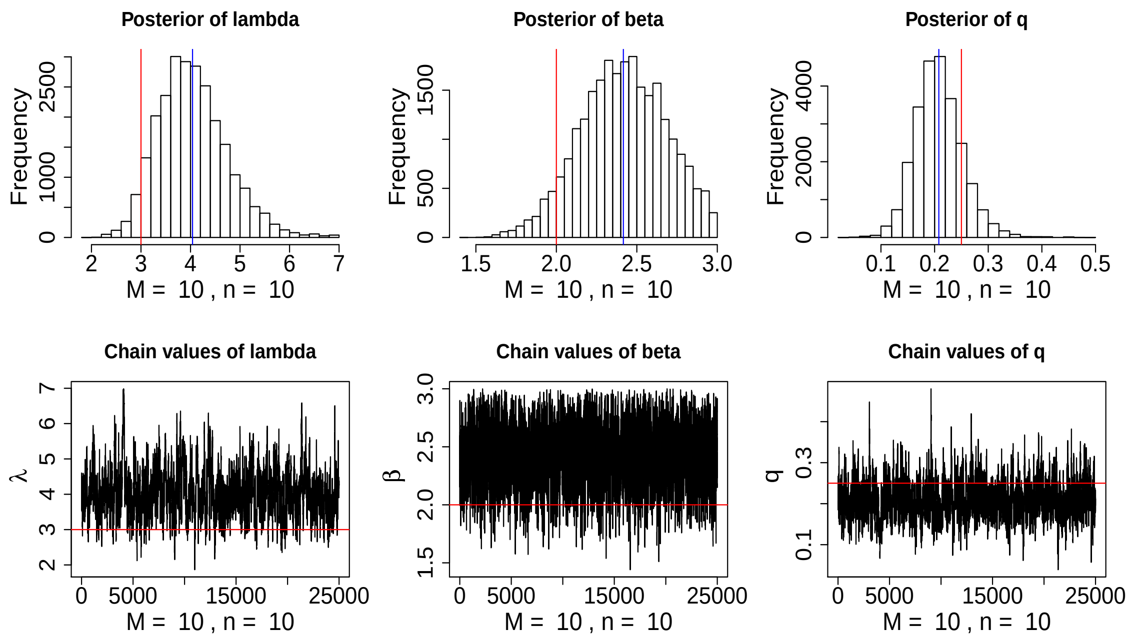


Figure 3.3: Histogram and trace plot of generated draws of λ, β and q for case 1 when $q = 0.25$. The red line represents the true value, and the blue line represents the estimated value for each parameter.

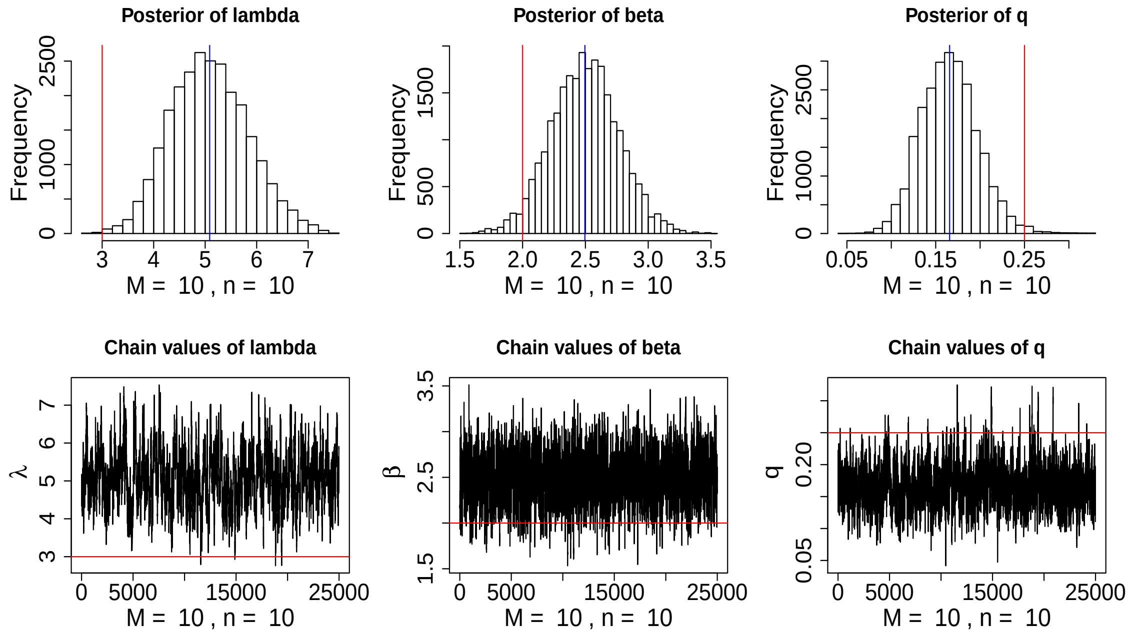


Figure 3.4: Histogram and trace plot of generated draws of λ, β and q for Case 2 when $q = 0.25$.The red line represents the true value, and the blue line represents the estimated value for each parameter.

True values: $\lambda = 3$, $\beta = 2$, and $q = 0.85$							
n_m	M	**Case** 1			**Case** 2		
		$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{q}$
10	10	4.0758	2.4130	0.6394	5.4209	2.7169	0.4736
	20	3.2719	1.9486	0.7935	3.6063	2.0075	0.6761
20	10	3.9569	2.1020	0.6620	4.6766	2.2227	0.5237
	20	3.1937	2.0336	0.8005	3.5068	2.0798	0.6956

Table 3.5: Estimated values of λ, β and q using MCMC when $q = 0.85$

3.4 The system reliability function after each repair

Repair activities play a crucial role in keeping and returning system functionality. Modelling work supposes that modelling repair effects provide an understanding of how these repairs impact the system's reliability. In the reliability of a system, we will analyse how the predictive reliability changes after each repair in this section. This analysis will present helpful insights into system performance after each repair and draw inferences based on the posterior predictive reliability function derived from each repair event.

Under the Kijima type I model, repairable systems are restored to a level between the new and old states. Modelling work of the Kijima type I model supposes that each repair could improve the system's performance. Therefore, the system's reliability after each repair will increase and will be returned to the virtual age $v_{i-1} = qt_{i-1}$ based on the repair efficiency. The posterior predictive reliability function will be used to predict the system's reliability after each repair based on the posterior distribution presented in Section 3.2 and Section 3.3. Therefore, from Equation (2.3.14) and (3.2.16) in Section 3.2 the predictive reliability function for T_i is

$$\begin{aligned}
R_{T_i}(x|x_{i-1}, D_m) &= \sum_{s=1}^{S}\sum_{k=1}^{K}\int_0^\infty g(\lambda, \beta_s, q_k|D_m) R_{T_i}(x|v_{i-1}) d\lambda \\
&= A\sum_{s=1}^{S}\sum_{k=1}^{K} \beta_s^N u_{sk} \frac{b^a}{\Gamma(a)} p'_s q'_k \int_0^\infty \left[\lambda^{N+a-1} \exp\left(-\lambda\left[b + r_{sk}\right]\right) \times\right. \\
&\quad \left. \exp\left(-\lambda\left[(x + q_k\sum_{j=0}^{i-1} x_j)^{\beta_s} - (q_k\sum_{j=0}^{i-1} x_j)^{\beta_s}\right]\right)\right] d\lambda \\
&= A\sum_{s=1}^{S}\sum_{k=1}^{K} \beta_s^N u_{sk} \frac{b^a}{\Gamma(a)} p'_s q'_k \times \int_0^\infty \left[\lambda^{N+a-1} . \times\right. \\
&\quad \left. \exp\left(-\lambda\left[\left((x + q_k\sum_{j=0}^{i-1} x_j)^{\beta_s} - (q_k\sum_{j=0}^{i-1} x_j)^{\beta_s}\right) + b + r_{sk}\right]\right)\right] d\lambda
\end{aligned} \tag{3.4.31}$$

Multiplying by

$$\frac{\Gamma(N+a)\left[\left((x + q_k\sum_{j=0}^{i-1} x_j)^{\beta_s} - (q_k\sum_{j=0}^{i-1} x_j)^{\beta_s}\right) + r_{sk} + b\right]^{N+a}}{\Gamma(N+a)\left[\left((x + q_k\sum_{j=0}^{i-1} x_j)^{\beta_s} - (q_k\sum_{j=0}^{i-1} x_j)^{\beta_s}\right) + r_{sk} + b\right]^{N+a}} \tag{3.4.32}$$

to simplify and solve the integral, so the predictive reliability function is

$$R_{T_i}(x|x_{i-1}, D_m) = A\sum_{s=1}^{S}\sum_{k=1}^{K}\frac{\beta_s^N u_{sk} b^a p'_s q'_k \Gamma(N+a)}{\Gamma(a)\left[\left((x+q_k\sum_{j=0}^{i-1}x_j)^{\beta_s}-(q_k\sum_{j=0}^{i-1}x_j)^{\beta_s}\right)+r_{sk}+b\right]^{N+a}} \tag{3.4.33}$$

where A is the normalising constant, which is given in Equation (3.2.19). Also, from Equation (2.3.14) and (3.3.29) in Section 3.3 the predictive reliability function for T_i is

$$\begin{aligned}
R_{T_i}(x_i|x_{i-1}, D_m) &= \int_0^1\int_{\beta_1}^{\beta_2}\int_0^\infty R(x_i|v_{i-1})g(\lambda,\beta,q|D)\,d\lambda d\beta dq \\
&= A\int_0^1\int_{\beta_1}^{\beta_2}\int_0^\infty \exp\left[-\lambda\left((x_i+q\sum_{j=0}^{i-1}x_j)^\beta-(q\sum_{j=0}^{i-1}x_j)^\beta\right)\right]\lambda^N\times \\
&\quad \beta^N u\exp(-\lambda r)\lambda^{a-1}\exp(-\lambda b)q^{c-1}(1-q)^{d-1}d\lambda d\beta dq \\
&= A\int_0^1\int_{\beta_1}^{\beta_2}\beta^N u q^{c-1}(1-q)^{d-1}\int_0^\infty \lambda^{N+a-1}\times \\
&\quad \exp\left[-\lambda\left(r+b+\left((x_i+q\sum_{j=0}^{i-1}x_j)^\beta-(q\sum_{j=0}^{i-1}x_j)^\beta\right)\right)\right]d\lambda d\beta dq \\
&= A\int_0^1\int_{\beta_1}^{\beta_2}\beta^N u q^{c-1}(1-q)^{d-1}\times \\
&\quad \frac{\Gamma(N+a)}{\left[r+b+\left((x_i+q\sum_{j=0}^{i-1}x_j)^\beta-(q\sum_{j=0}^{i-1}x_j)^\beta\right)\right]^{N+a}}d\beta dq
\end{aligned} \tag{3.4.34}$$

where A is the normalising constant that is given in Equation (3.4.35).

$$\begin{aligned}
A^{-1} &= \int_0^1\int_{\beta_1}^{\beta_2}\int_0^\infty g(\lambda,\beta,q|D)\,d\lambda d\beta dq \\
&= \int_0^1\int_{\beta_1}^{\beta_2}\beta^N u q^{c-1}(1-q)^{d-1}\int_0^\infty \lambda^{N+a-1} \\
&\quad \exp\left[-\lambda\left(r+b\right)\right]d\lambda d\beta dq \\
&= \int_0^1\int_{\beta_1}^{\beta_2}\beta^N u q^{c-1}(1-q)^{d-1}\frac{\Gamma(N+a)}{[r+b]^{N+a}}d\beta dq
\end{aligned} \tag{3.4.35}$$

The reliability function can not be solved analytically because of the difficulty of the integration approaches when continuous priors are used. Therefore, a numerical method is needed to solve this problem.

This section uses the posterior predictive distribution to generate samples, and the results are used to plot the posterior predictive reliability function after each repair. As a result of this, it will be possible to see an overview of the system reliability behaviour after each repair process to make inferences. It assists organizations in evaluating the repair methods using reliability data. The preferred repair method may be adopted if the system's probability of functioning effectively is enhanced through repair to a level comparable to, or nearly equivalent to, that of a new system. In contrast, low reliability after repair can indicate a need for better training or more advanced repair methods. The posterior predictive reliability proposed in this section is illustrated by examples. We illustrate this by using the method presented in Section 3.2 in Example 3.4.1 and the methods presented in Section 3.3 in Example 3.4.2.

Example 3.4.1

This example comprises two cases based on repair efficiency q. In Case 1, we simulate data assuming higher quality repair efficiency. In Case 2, we modify the data to assume lower quality repair efficiency and investigate its effect on a system's predictive reliability functions.

This example illustrates predictive reliability functions after each repair based on two cases, employing the posterior predictive distribution for the method presented in Section 3.2. We generate data using the Kijima type I model where, in Case 1, we assume that $\lambda = 3$, $\beta = 2$ and $q = 0.25$ and we assume that $q = 0.85$ in Case 2 and keeping the same values for λ and β as in Case 1. Assume that we generate data where $n_m = 10$ and $M = 10$ where data for Case 1 and Case 2 are presented in Table 3.6 and Table 3.7, respectively.

Assume prior λ has Gamma$(12, 4)$ distribution and assume that the prior distribution for $\beta_s = (1.90,\ 1.92,\ 1.94,\ 1.96,\ 1.98,\ 2.00,\ 2.02,\ 2.04,\ 2.06,\ 2.08)$ is a discrete Uniform denoted as $p'_s = \frac{1}{S}$, where S represents the total number of β values in Case 1 and Case 2. Also, assume that the prior distribution for $q_k = (0.21,\ 0.22,\ 0.23,\ 0.24,\ 0.25,\ 0.26,\ 0.27,\ 0.28,\ 0.29.\ 0.30)$ in Case 1 and

m	Failure Time									
	1	2	3	4	5	6	7	8	9	10
1	0.665	1.096	1.333	1.378	1.841	1.878	1.898	2.026	2.161	2.725
2	0.750	0.954	1.207	1.734	1.755	1.777	2.270	2.322	2.509	2.652
3	0.771	0.908	1.289	1.656	1.826	1.983	2.456	2.728	2.851	2.952
4	0.422	1.575	1.932	2.262	2.320	2.626	2.703	2.727	2.739	3.212
5	0.732	0.949	1.003	1.447	2.025	2.131	2.303	2.361	2.374	2.823
6	0.408	0.485	1.041	1.405	1.495	1.505	1.523	1.628	1.845	2.445
7	0.061	0.600	1.311	1.981	2.332	2.395	2.644	2.651	3.007	3.163
8	0.504	1.113	1.224	1.404	1.761	1.872	2.198	2.219	2.293	2.409
9	0.709	1.659	2.079	2.443	2.635	3.027	3.211	3.397	3.472	3.474
10	0.476	0.995	1.334	1.484	2.092	2.447	2.732	2.998	3.099	3.263

Table 3.6: Failure times data where $q = 0.25$ (Case 1)

m	Failure Time									
	1	2	3	4	5	6	7	8	9	10
1	0.665	0.905	1.018	1.036	1.300	1.316	1.324	1.384	1.448	1.780
2	0.750	0.837	0.957	1.264	1.273	1.282	1.560	1.582	1.673	1.742
3	0.771	0.823	1.022	1.215	1.294	1.368	1.635	1.775	1.834	1.883
4	0.422	1.368	1.532	1.687	1.710	1.857	1.891	1.901	1.906	2.156
5	0.732	0.826	0.847	1.095	1.436	1.484	1.566	1.592	1.598	1.845
6	0.408	0.438	0.829	1.029	1.069	1.073	1.081	1.129	1.239	1.608
7	0.061	0.566	1.060	1.462	1.639	1.667	1.788	1.792	1.977	2.052
8	0.504	0.917	0.963	1.046	1.238	1.289	1.463	1.472	1.507	1.563
9	0.709	1.372	1.579	1.758	1.846	2.046	2.134	2.223	2.259	2.259
10	0.476	0.818	0.998	1.068	1.443	1.631	1.778	1.915	1.964	2.047

Table 3.7: Failure times data where $q = 0.85$ (Case 2)

$q_k = (0.80, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89)$ in Case 2 is a discrete Uniform denoted as $p'_s = \frac{1}{K}$, where K represents the total number of q values.

Assume that five failure times are simulated from the posterior predictive distribution to plot the system reliability function, where we assume that the system kept operating after the fifth failure until the system ended. Figure 3.5 and Figure 3.6 show the predictive reliabilities of a system $R_{T_i}(x|x_{i-1}, D_m)$ under repair efficiency for Case 1 and Case 2 respectively. The blue line is the system reliability function after each repair action, where we have five failures. Also, the endpoint of each curve represents the failure time after each repair.

In Case 1, where high-quality repair is assumed, the system reliability increases after each repair action, as shown in Figure 3.5, indicating that the repair actions have restored the system's functionality to a level comparable to or nearly equivalent to a new one. On the other hand, In Case 2, where low-quality repair is assumed, the system reliability hardly increases after each repair action, as shown in Figure 3.6, indicating that the repair actions have restored the system's functionality to a level that is better slightly than before failure. Also, it's noted that the system reliability decreases as the number of repairs increases after each repair because of the cumulative effect of performance degradation over the lifetime, suggesting that the system becomes less capable of reaching optimal functionality with each repair.

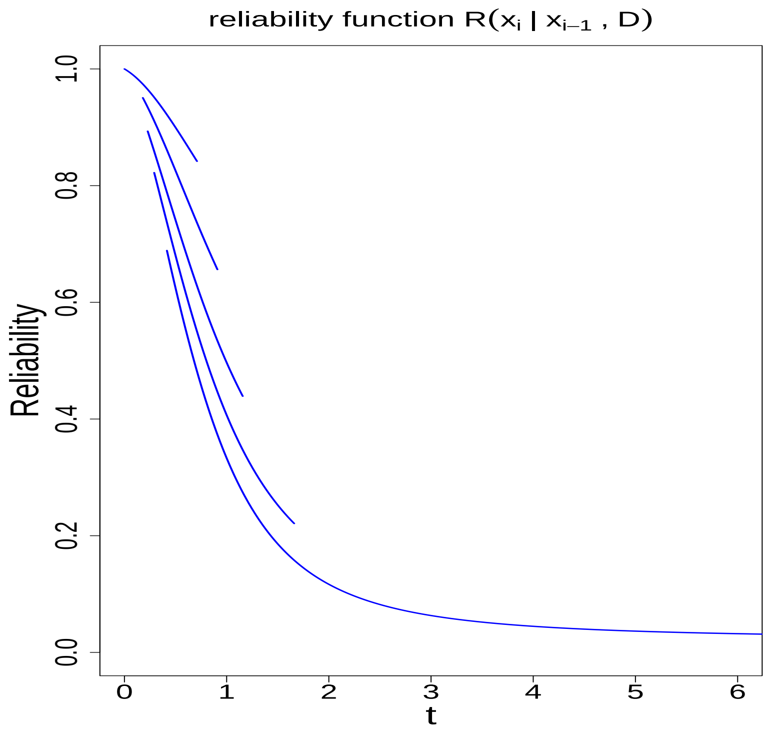


Figure 3.5: The system reliability function after each failure using Case 1

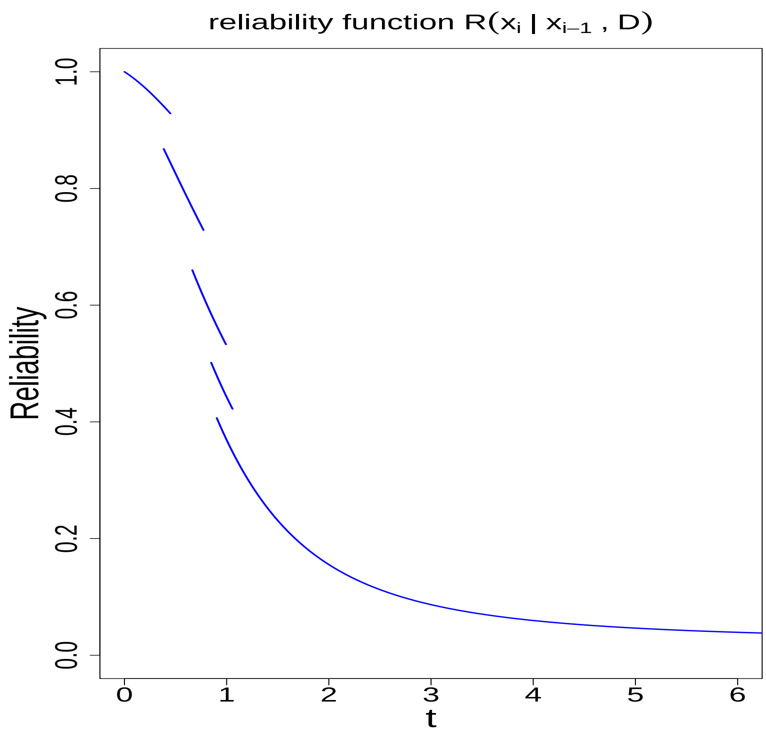


Figure 3.6: The system reliability function after each failure using Case 2

Example 3.4.2

In this example, we generate data based on high-quality and low-quality repair efficiency, As illustrated in the previous Example 3.4.1.

This example illustrates predictive reliability functions after each repair based on two cases, employing the posterior predictive distribution for the method presented in Section 3.3. Also, we generate data using the Kijima type I model where, in Case 1, we assume that $\lambda = 3$, $\beta = 2$ and $q = 0.25$ and we assume that $q = 0.85$ in Case 2 and keeping the same values for λ and β as in Case 1. Assume that we generate data where $n_m = 10$ and $M = 10$ where data for Case 1 and Case 2 are presented in Table 3.6 and Table 3.7, respectively.

Assume that Gamma$(12, 4)$, $U[1, 4]$ and Beta$(1, 3)$ are chosen as prior distributions for λ, β and q respectively in Case 1. In Case 2, we only modify the hyperparameters of Beta distribution and assume Beta$(6, 1)$.

We simulate five failure times from the posterior predictive distribution using MCMC. These simulated failure times are then used to plot the reliability function using numerical integration in R, where we assume that the system kept operating after the fifth failure until the system ended. Figure 3.7 and Figure 3.8 show the predictive reliabilities of a system $R_{T_i}(x|x_{i-1},\ D_m)$ under repair efficiency for Case 1 and Case 2 respectively. The blue line is the system reliability function after each repair action, where we have five failures. Also, the endpoint of each curve represents the failure time after each repair. In Case 1, where high-quality repair is assumed, the system reliability increases after each repair action, as shown in Figure 3.7, indicating that the repair actions have restored the system's functionality to a level comparable to or nearly equivalent to a new one. On the other hand, In Case 2, where low-quality repair is assumed, the system reliability hardly increases after each repair action, as shown in Figure 3.8, indicating that the repair actions have restored the system's functionality to a level that is larger slightly than before failure. Also, it's noted that the system reliability decreases as the number of repairs increases after each repair because of the cumulative effect of performance degradation over the lifetime, suggesting that the system becomes less capable of reaching optimal functionality with each repair.

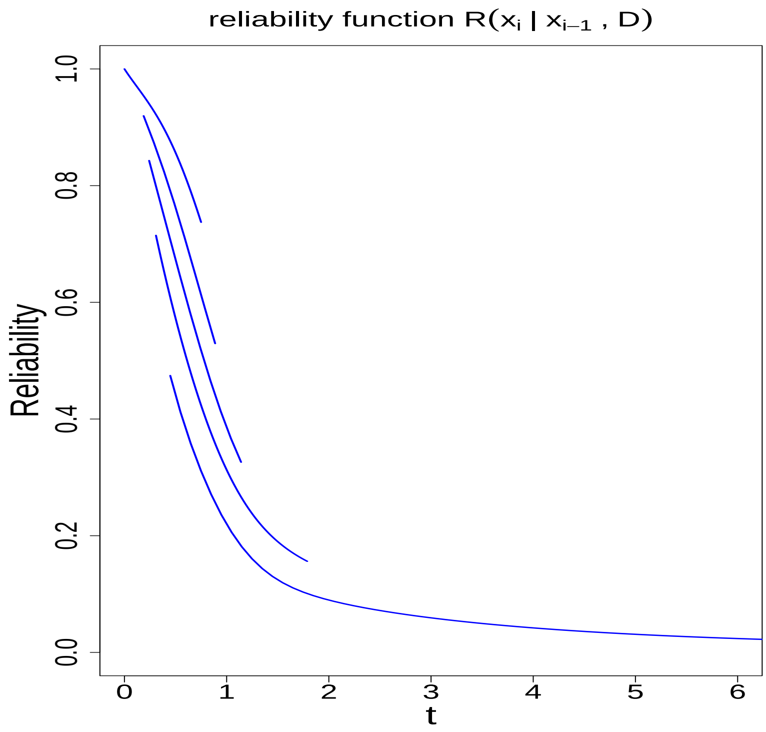


Figure 3.7: The system reliability function after each failure using Case 1

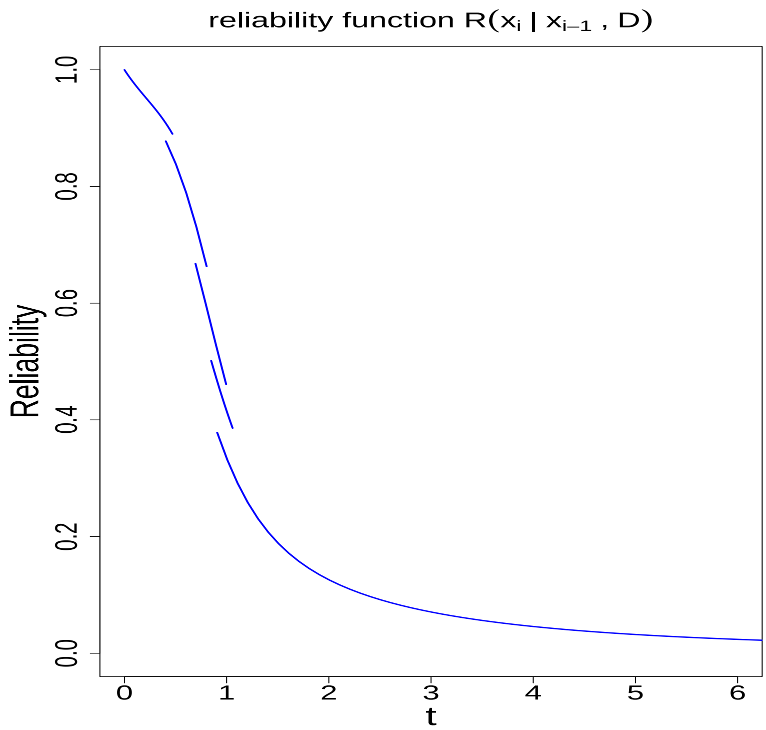


Figure 3.8: The system reliability function after each failure using Case 2

3.5 A system reliability function for a given number of repairs

In the previous Section 3.4, we analysed the predictive reliability function of the system separately after each failure time, considering the effective age parameter q. This analysis provided helpful insights into system performance after each repair and allowed for inferences based on the posterior predictive reliability function derived from each repair event. In this section, the total system reliability when a total number of repairs is given will be explored. We will use the posterior predictive empirical reliability function to predict the total system reliability until the time needed to replace it if the number of failures is known. Modelling work supposes that the number of repairs is considered in this section to model the impact of maintenance activities on the total reliability of the system. As far as we know, no studies on virtual age modelling analysis using the total system reliability for a given number of repairs have been reported in the literature so far.

The empirical reliability function estimator [16] has been widely used for inferences on the reliability function. Let n denote the number of repairs, and $t_1,\ t_2,\ \ldots, t_n$ be the times of repair. The empirical reliability function of T is

$$P(T > t|t_{i-1}, D_m) = \frac{1}{n}\sum_{i=1}^{n} I(T_i > t)\ . \tag{3.5.36}$$

This function estimates the probability that the time T_i exceeds a specific value t. The sum in this equation counts the number of observations greater than t. Then, we divide the number counted by the total number of repairs to obtain the predictive probability of observations greater than t.

This section uses the posterior predictive distribution to generate samples, and the results are used to plot the empirical posterior predictive reliability function by taking the mean or median of different empirical posterior predictive reliability functions. The interval for the posterior predictive empirical reliability function is also derived by taking the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of different empirical posterior predictive reliability functions. As a result of this, it will be possible to see an overview of the overall system reliability behaviour of the number of repair processes and the level of uncertainty measured as a result of the confidence bounds to make inferences and decisions.

The empirical posterior predictive reliability helps predict the probability of a system performing within time, considering a specific number of repairs. It assists organizations in evaluating the overall risk associated with this particular repair count, contributing to the development of safer and more efficient systems. Additionally, it helps decision-makers to make more informed decisions, including choices related to system replacement after a certain number of repairs. For example, this may involve using optimal replacement policies, a topic that will be explored in Chapter 5. The empirical posterior predictive reliability proposed in this section is illustrated by examples using the presented methods in Section 3.2 and Section 3.3.

Example 3.5.1

Based on repair efficiency q, this example comprises two cases. We simulate data in Case 1, assuming high repair efficiency, and in Case 2, assuming low repair efficiency, to investigate its effect on a system's predictive reliability functions.

The example illustrates empirical predictive reliability functions for different numbers of repair actions based on the two cases, employing the posterior predictive distribution for the method presented in Section 3.2. Also, we generate data using the Kijima type I model where, in Case 1, we assume that $\lambda = 3$, $\beta = 2$ and $q = 0.25$ and we assume that $q = 0.85$ in Case 2 and keeping the same values for λ and β as in Case 1. Assume that we generate data where $n_m = 10$ and $M = 10$ where data for Case 1 and Case 2 are presented in Table 3.8 and Table 3.9, respectively.

In this example, we will use the same prior information illustrated earlier in Example 3.4.1 for λ, β and q. We generated 100 different samples from the posterior predictive distribution with different numbers of repair actions and derived the posterior predictive empirical reliability function of each sample. Thus, we will obtain 100 posterior predictive empirical reliability functions at each failure time t. Then, we take the average of these estimates at each time t, which is presented by the blue step line in Figure 3.9 to Figure 3.12 for Case 1 and Case 2 respectively, to show

m	Failure Time									
	1	2	3	4	5	6	7	8	9	10
1	0.665	1.096	1.333	1.378	1.841	1.878	1.898	2.026	2.161	2.725
2	0.750	0.954	1.207	1.734	1.755	1.777	2.270	2.322	2.509	2.652
3	0.771	0.908	1.289	1.656	1.826	1.983	2.456	2.728	2.851	2.952
4	0.422	1.575	1.932	2.262	2.320	2.626	2.703	2.727	2.739	3.212
5	0.732	0.949	1.003	1.447	2.025	2.131	2.303	2.361	2.374	2.823
6	0.408	0.485	1.041	1.405	1.495	1.505	1.523	1.628	1.845	2.445
7	0.061	0.600	1.311	1.981	2.332	2.395	2.644	2.651	3.007	3.163
8	0.504	1.113	1.224	1.404	1.761	1.872	2.198	2.219	2.293	2.409
9	0.709	1.659	2.079	2.443	2.635	3.027	3.211	3.397	3.472	3.474
10	0.476	0.995	1.334	1.484	2.092	2.447	2.732	2.998	3.099	3.263

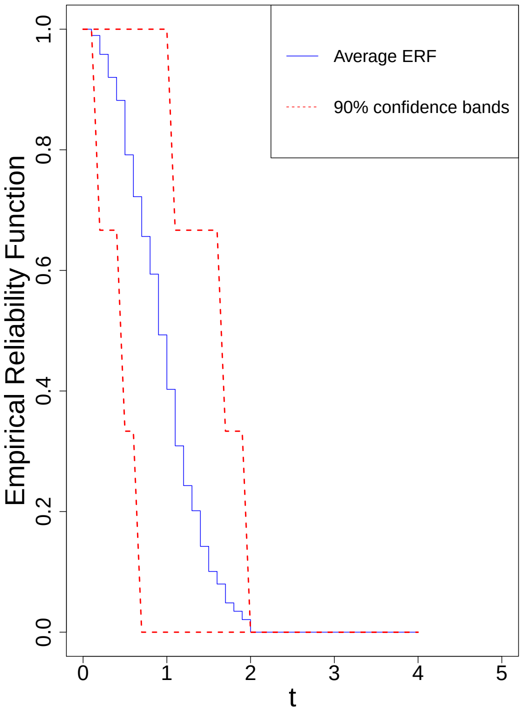
Table 3.8: Failure times data where $q = 0.25$ (Case 1)

m	Failure Time									
	1	2	3	4	5	6	7	8	9	10
1	0.665	0.905	1.018	1.036	1.300	1.316	1.324	1.384	1.448	1.780
2	0.750	0.837	0.957	1.264	1.273	1.282	1.560	1.582	1.673	1.742
3	0.771	0.823	1.022	1.215	1.294	1.368	1.635	1.775	1.834	1.883
4	0.422	1.368	1.532	1.687	1.710	1.857	1.891	1.901	1.906	2.156
5	0.732	0.826	0.847	1.095	1.436	1.484	1.566	1.592	1.598	1.845
6	0.408	0.438	0.829	1.029	1.069	1.073	1.081	1.129	1.239	1.608
7	0.061	0.566	1.060	1.462	1.639	1.667	1.788	1.792	1.977	2.052
8	0.504	0.917	0.963	1.046	1.238	1.289	1.463	1.472	1.507	1.563
9	0.709	1.372	1.579	1.758	1.846	2.046	2.134	2.223	2.259	2.259
10	0.476	0.818	0.998	1.068	1.443	1.631	1.778	1.915	1.964	2.047

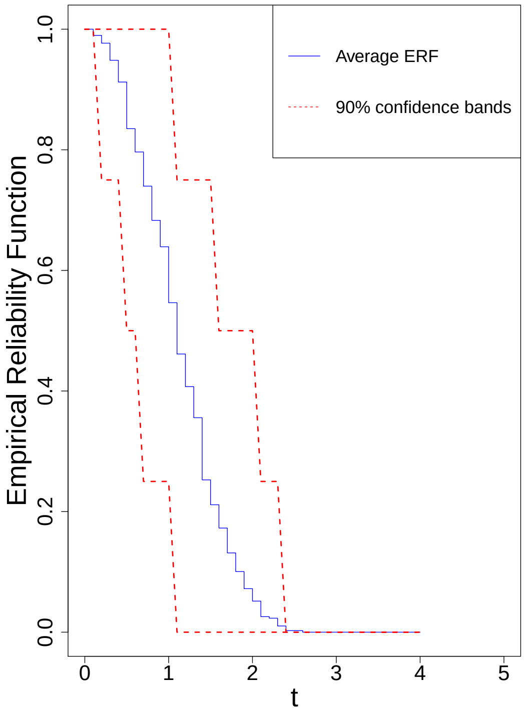
Table 3.9: Failure times data where $q = 0.85$ (Case 2)

the estimate of the empirical posterior predictive reliability function. The two red dashed lines present the 90% confidence bands for the posterior predictive empirical reliability function for each time t. These are derived by excluding the highest 5% and the lowest 5% of values of 100 empirical posterior predictive reliability functions. The blue line represents the mean of several different empirical predictive reliability functions. The resulting function has many steps because of the averaging process across all these functions at each time point t. On the other hand, the boundaries of the confidence intervals for the empirical posterior predictive reliability function, which are represented by red dashed lines, have steps equal to the number of repairs, n, because at any given time, t, the function can only assume one of the possible values, each with a probability of $\frac{1}{n}$ for each observation.

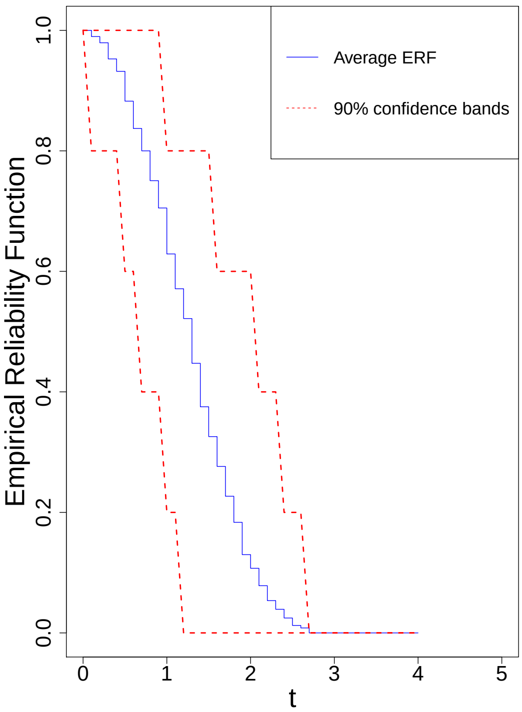
In Case 1, because of high-quality repair actions, Figure 3.9 shows the posterior predictive empirical reliability function drops to zero faster when we have a small number of repairs, which means the system generally maintains an increase in the probability of reliability over time when the number of repairs is increased. In case 2, due to ineffective or low-quality repair actions, Figure 3.10 shows the posterior predictive empirical reliability function drops to zero with a slight change when the number of repairs is increased. That means that the system fails fast, and the system functionality generally stays similar over time. Therefore, Figures 3.9 and 3.10 show that the posterior predictive empirical reliability function drops to zero faster in Case 2 than in Case 1 for each number of repair actions. The upper and lower bands of the posterior predictive empirical reliability function represent the uncertainty for each Case. Wider bands indicate greater uncertainty about reliability estimates at each time point t, while narrower bands suggest greater confidence in reliability estimates. The posterior predictive empirical reliability function bounds drop to zero faster in Case 2 than in Case 1 for each number of repair actions, which is shown in Figures 3.9 and 3.10.



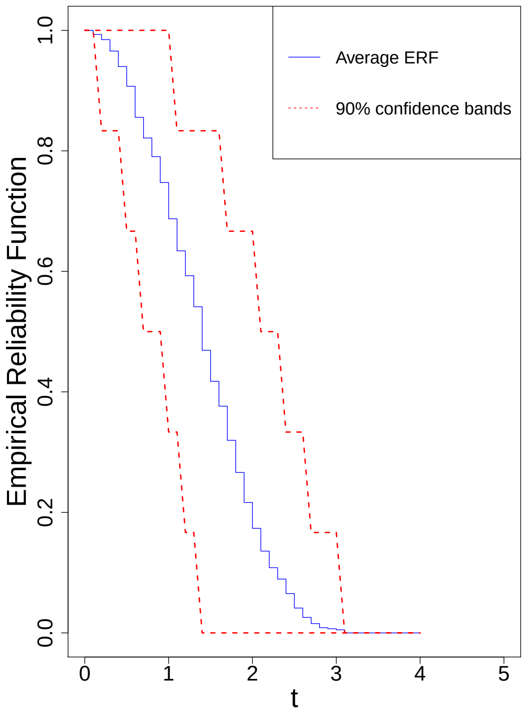
(a) Three repair actions



(b) Four repair actions

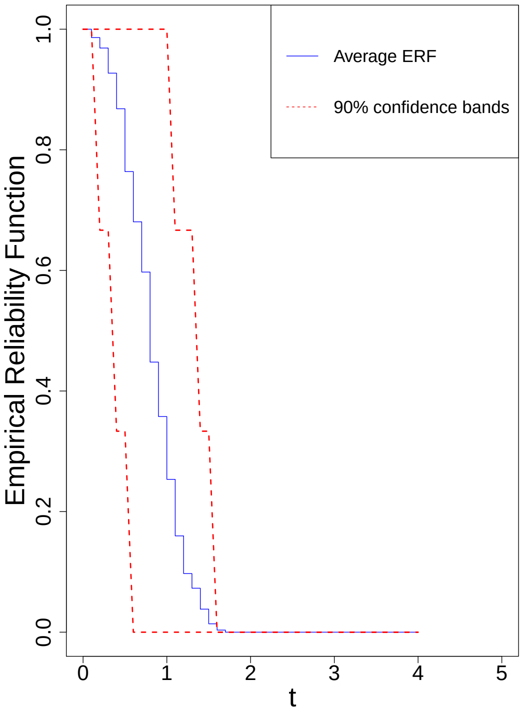


(c) Five repair actions

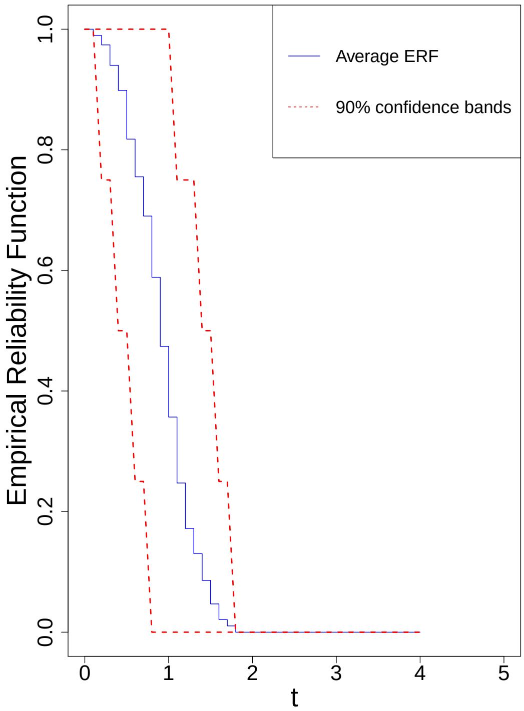


(d) Six repair actions

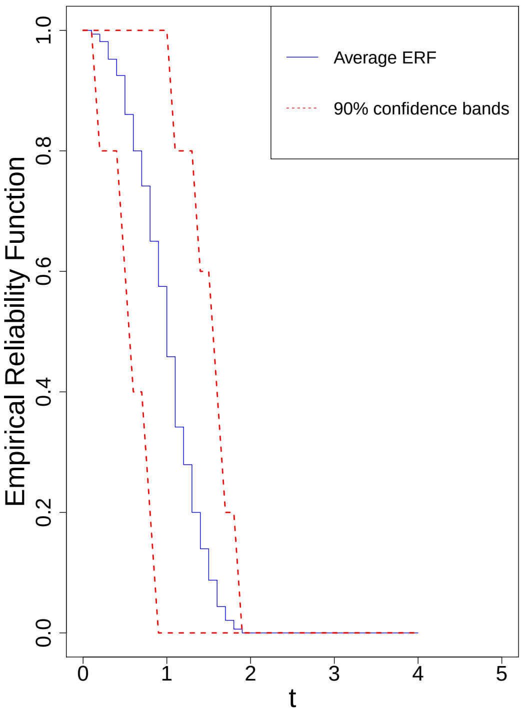
Figure 3.9: Different posterior predictive empirical reliability functions based on the numbers of repair actions with the confidence band using conditional and discrete priors Case 1.



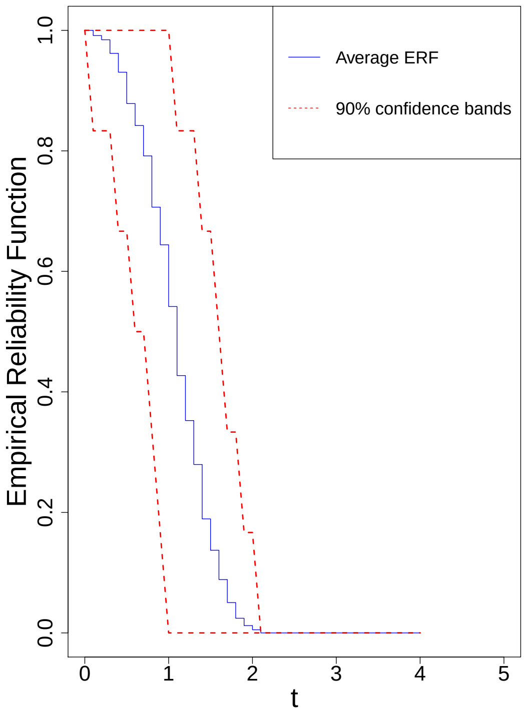
(a) Three repair actions



(b) Four repair actions



(c) Five repair actions



(d) Six repair actions

Figure 3.10: Different posterior predictive empirical reliability functions based on the numbers of repair actions with the confidence band using conditional and discrete priors Case 2.

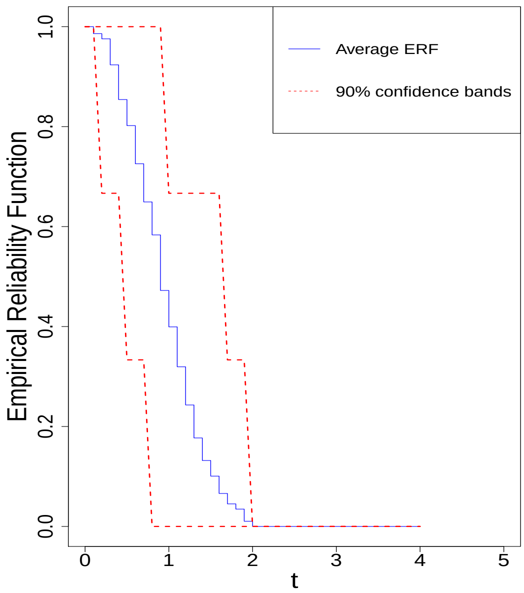
Example 3.5.2

This example comprises two cases based on repair quality. In Case 1, we simulate data assuming higher quality repair efficiency. In Case 2, we modify the data to assume lower quality repair efficiency and investigate its effect on a system's predictive empirical reliability functions with lower and upper bands for different numbers of repair actions.

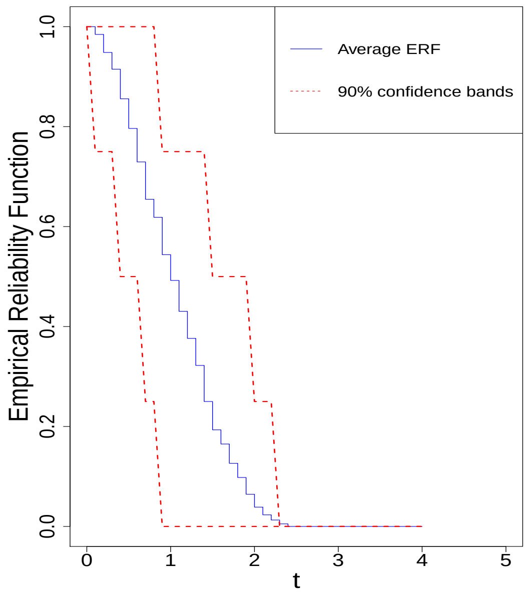
The example illustrates empirical predictive reliability functions for different numbers of repair actions based on the two cases, employing the posterior predictive distribution for the method presented in Section 3.3. Also, we generate data using the Kijima type I model where, in Case 1, we assume that $\lambda = 3$, $\beta = 2$ and $q = 0.25$ and we assume that $q = 0.85$ in Case 2 and keeping the same values for λ and β as in Case 1. Assume that we generate data where $n_m = 10$ and $M = 10$ where data for Case 1 and Case 2 are presented in Table 3.8 and Table 3.9, respectively.

In this example, the same priors information illustrated earlier in Example 3.4.2 are used for λ, β and q. We generated 100 different samples from the posterior predictive distribution with different numbers of repair actions and derived the posterior predictive empirical reliability function of each sample. Thus, we will obtain 100 posterior predictive empirical reliability functions at each failure time t. Then, we take the average of these estimates at each time t, which is presented by the blue step line in Figure 3.11 and Figure 3.12 for Case 1 and Case 2 respectively, to show the estimated posterior predictive empirical reliability function. The two red dashed lines present the 90% confidence bands for the posterior predictive empirical reliability function for each time t. These are derived by excluding the highest 5% and the lowest 5% of values of 100 empirical reliability functions.

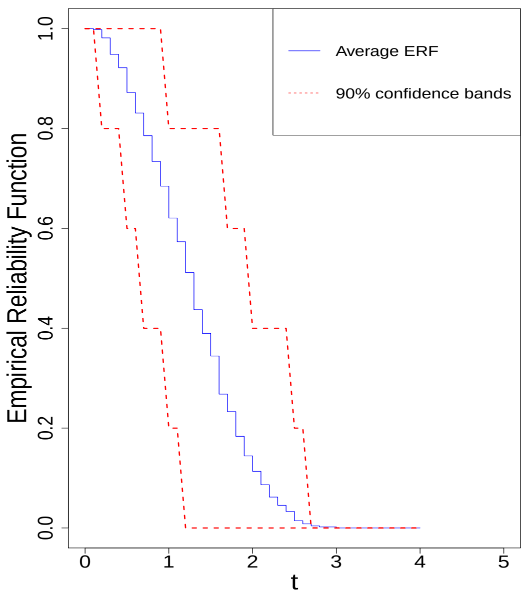
In Case 1, because of high-quality repair actions, Figure 3.11 shows the posterior predictive empirical reliability function drops to zero faster when we have a small number of repairs, which means the system generally maintains an increase in the probability of reliability over time when the number of repairs is increased. In Case 2, due to ineffective or low-quality repair actions, Figure 3.12 shows the posterior predictive empirical reliability function drops to zero with a slight change when the number of repairs is increased. That means that the system fails fast, and the system functionality generally stays similar over time. Therefore, Figures 3.11 and 3.12 show that the posterior predictive empirical reliability function with the 90% confidence bands drops to zero faster in Case 2 than in Case 1 for each number of repair actions.



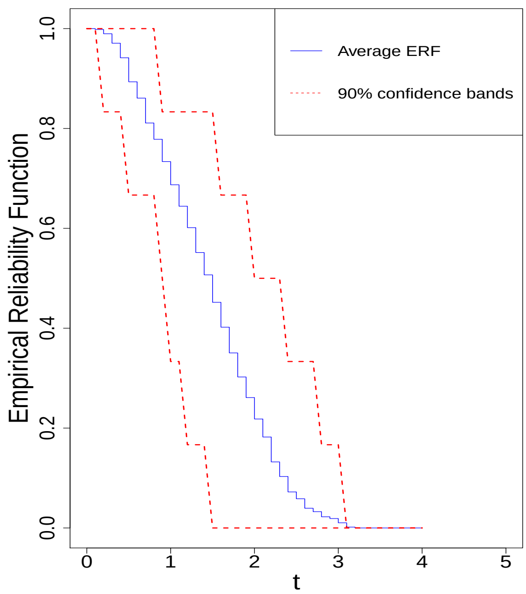
(a) Three repair actions



(b) Four repair actions

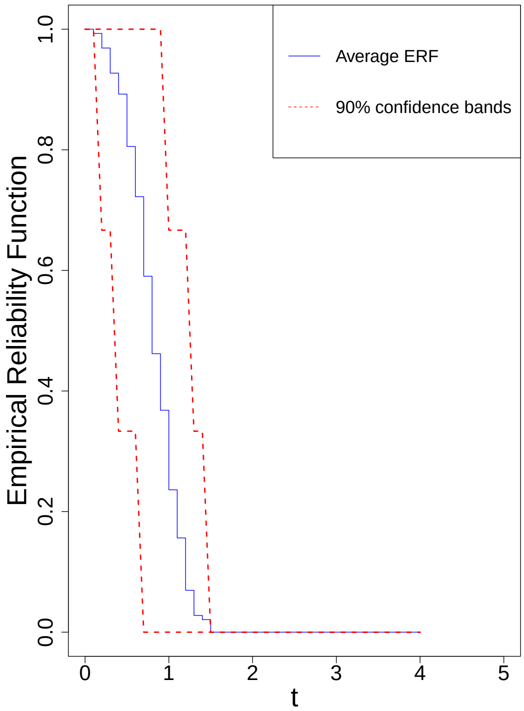


(c) Five repair actions

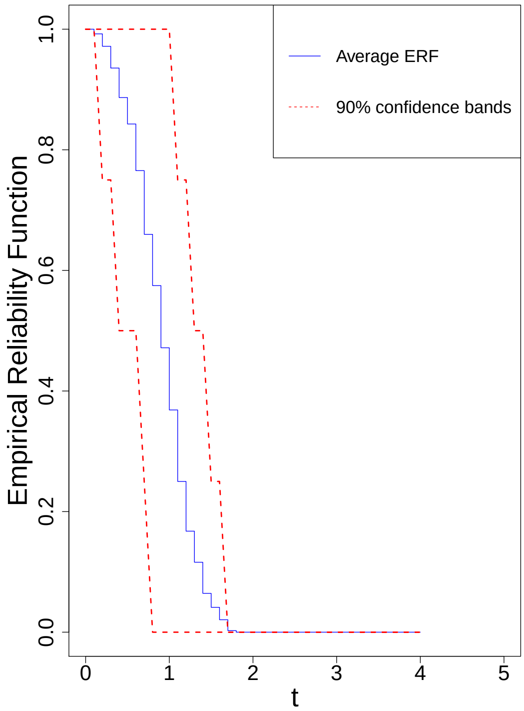


(d) Six repair actions

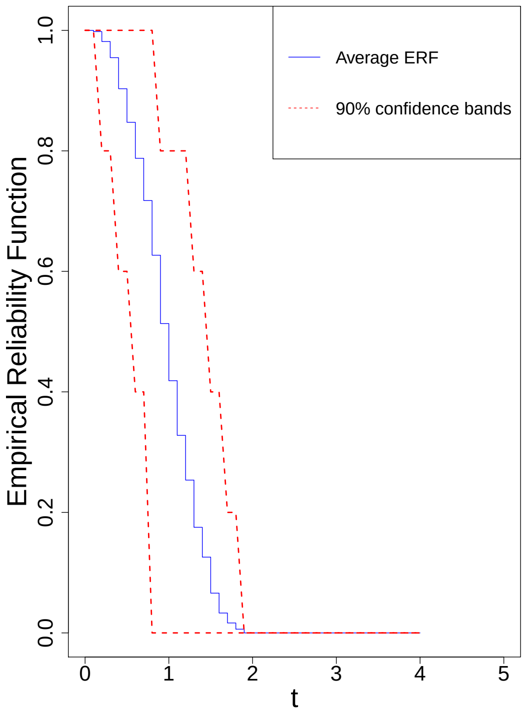
Figure 3.11: Different posterior predictive empirical reliability functions based on the numbers of repair actions with the confidence band using continuous priors Case 1.



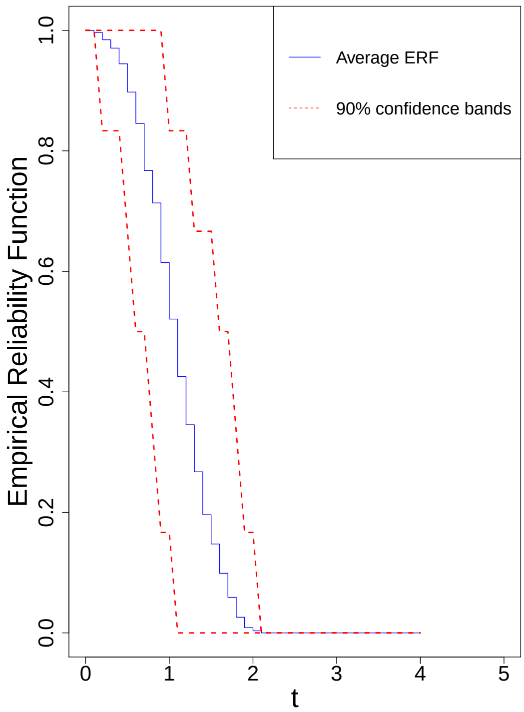
(a) Three repair actions



(b) Four repair actions



(c) Five repair actions



(d) Six repair actions

Figure 3.12: Different posterior predictive empirical reliability functions based on the numbers of repair actions with the confidence band using continuous priors Case 2.

3.6 Concluding remarks

A Bayesian method using the Kijima type I model with Weibull distribution has been presented in this chapter. The Kijima type I model is used to analyse the performance of repairable systems in engineering asset management after repair. The parameters are estimated utilizing analytical solutions when closed-form expressions for estimations are attainable within a Bayesian framework with discrete priors. Alternatively, a Markov Chain Monte Carlo (MCMC) method can be used to estimate the parameters using continuous priors for Bayesian analysis. The estimations are explored based on two cases of the selection of the hyperparameters of each prior distribution. In case 1, a good prior is selected. Hyperparameters of each prior distribution are selected so that the mean of each prior is near the values of actual parameters. In case 2, a poor prior is selected. To evaluate the performance of the analytical solutions, we compare the results obtained from the analytical method with discrete priors to the results derived from the MLE method for high and low-quality repair efficiency using the parameter q. For parameter estimates, in terms of the closest value to the true values, Case 1 gives the best estimator for the parameters, followed by the MLE and then Case 2. The selection of prior has impacted the estimated results, especially for the scale parameter λ and repair efficiency parameter q. A bad prior choice negatively impacts the results more than MLE because of its biased influence, while a good prior improves outcomes by effectively incorporating relevant prior knowledge with the data, leading to more accurate results compared to those obtained using MLE.

The reliability of a system has been derived, and the changes to the predictive reliability have been analysed after each repair based on Bayesian methodology and simulated data using the Kijima type I model. The posterior predictive reliability of a system has been estimated based on the MCMC method using continuous priors for the parameters and analytical solutions with discrete priors. Based on repair efficiency q, high-quality and lower-quality repair data are used. In high-quality repair efficiency data, the system's predictive reliability increases after each repair action, indicating that the repair actions have restored the system's functionality to a level comparable to or nearly equivalent to a new one. On the other hand, In lower-quality data, where low-quality repair is assumed, the system reliability hardly increases after each repair action, indicating that the repair actions have restored the system's functionality to a slightly better level than before failure.

We develop and illustrate a novel method using the posterior predictive empirical reliability function to predict the total system reliability until the time needed to replace it when the number of failures is known and illustrate it. When a total number of repairs is given, the total system reliability with the interval has been illustrated for different numbers of repair actions using analytical solutions with discrete priors and the MCMC method using continuous priors for the parameters. High-quality and lower-quality repair data are used based on repair efficiency q to investigate its effect on a system's predictive empirical reliability functions with lower and upper bands for different numbers of repair actions. When the repair efficiency has high quality, the posterior predictive empirical reliability function drops to zero faster when we have a small number of repairs, which means the system generally maintains an increase in the probability of reliability over time when the number of repairs is increased. When the quality of repair efficiency is low, the posterior predictive empirical reliability function drops to zero with a slight change when the number of repairs is increased. That means that the system fails fast, and the system functionality generally stays similar over time.

Chapter 4

Robust Bayesian inference for Kijima type I model

4.1 Introduction

Bayesian analysis for the Kijima type I model parameters with the Weibull distribution allows experts to include opinions. In reality, the expert, such as a repair technician, formulates their opinions in the form of a prior distribution for the parameters. However, without training, expecting an expert to formulate their beliefs in the form of a specific prior distribution may be unreasonable. This training is part of the elicitation process [14, 27], which aims to include an expert's previous beliefs about the parameters into a mathematical distribution. One must obtain information from an expert after training and then fit distributions to achieve a distribution that suitably represents their prior beliefs [27].

In reality, there are many repair technician experts in any area, so we could ask many repair experts to introduce a specific prior distribution. Many experts often disagree, making arriving at a precise prior difficult [70]. To have a separate model for each repair expert could be one solution that can be used to collect different prior distributions, but this would quickly become unmanageable. However, it is generally impossible to identify a unique prior that fully aligns with different experts' opinions. The significant consequence of this uncertainty is the impact of inaccurately specified priors on quantities of interest, such as posterior set probabilities and means [70].

Clearly, there is uncertainty regarding the specification of this prior knowledge. To deal with this uncertainty, the concept of Bayesian robustness has been introduced [8]. The concept of robust Bayes analysis, also known as Bayesian sensitivity analysis, explores the robustness of outcomes obtained through Bayesian analysis in the face of uncertain aspects of the analysis [6, 8, 36]. Robust Bayesian methods expand the Bayesian approach by acknowledging that the Bayesian approach can take different forms of uncertainty inference depending on subjective inputs, such as prior information [65, 68]. The concept of robust Bayes relies on the experts' uncertainty by defining a set of prior distributions that reflect their opinion [70]. Therefore, we get a set of posterior distributions when each prior is combined with the likelihood function. This set of posteriors can then be used for inferences, resulting in bounds on inferences such as probabilities and other quantities of interest [8].

The idea of bounding probability dates back to the 19th century and was first proposed by Boole [10]. Since then, many papers have been presented on the bounding of probability methods for many areas of statistics. However, the use of a set of priors was suggested first by Good [30, 31]. After Good's papers, Berger [6, 7] developed robust Bayesian analysis and discussed the use of a set of priors. The underlying idea behind using a set of priors is that no single distribution can adequately represent uncertainty, but the set of distributions may form a reasonable model for expressing uncertainty.

In Chapter 3 of this thesis, we used MCMC to get the uncertainty for the empirical posterior predictive reliability function using a single prior. However, there is still a possibility for sensitivity to a single prior distribution for the Kijima type I model parameters. In order to investigate the influence of the prior uncertainty, we can employ a set of prior distributions, which leads to a robust Bayesian analysis. Therefore, this chapter will present robust Bayesian statistical inferences using a set of the parameters' prior distributions. The set of priors will reflect imprecision in posterior predictive system reliability bounds. This allows for modelling partial or imperfect prior knowledge on the Kijima type I model in a flexible way using robust Bayesian analysis.

This chapter is organised as follows: Section 4.2 presents an overview of the selection of the set of priors. A set of prior distributions for the repair effectiveness parameter q is introduced in Section 4.3, leaving the other parameters fixed. Different sets of prior distributions for the parameter q are studied with two cases of repair quality. Also, we will introduce a set of prior distributions on the repair effectiveness parameter q and the scale parameter of Weibull distribution λ in Section 4.4. We study different sets of prior distributions for the parameters λ and q with two cases of repair quality.

4.2 Selecting the set of prior distributions

When we select a set of priors, the problem is which prior distributions are reasonable to include in the set. In some sense, a reasonable prior distribution can be thought of as one that accurately represents the experts' prior beliefs. Berger [8] indicates that the size of the set of prior distributions should represent the experts' prior knowledge when performing a robust Bayesian analysis. If the experts have considerable prior knowledge, choosing a concise set of priors to reflect uncertainty is recommended. On the other hand, if the experts have minimal or no prior knowledge about the parameters, a set of priors should include all, or approximately all, priors, as suggested by Berger. This means there is no longer a precise prior distribution but rather a class of priors reflecting the uncertainties of all experts.

Also, when choosing a class of priors, Berger suggested that one should select the class that can be easily worked with and which is easy to elicit [8]. Furthermore, the class of prior distributions should include as many reasonable priors as possible in order to guarantee robustness. [8].

Modelling work supposes that after selecting a set of prior distributions denoted by $\mathcal{M}^{(0)}$, the Bayesian robustness is examined in terms of its influence on the posterior and the resulting posterior inferences (for example, probabilities, means, etc.), where the set of posterior inferences, denoted by $\mathcal{M}^{(n)}$, is investigated as the prior distribution changes in class $\mathcal{M}^{(0)}$. According to Berger [8], any prior in $\mathcal{M}^{(0)}$ can be chosen when the range of results is small (according to statisticians and expert judgements) since all of them lead to similar results. This means that the result is robust. When the range of results is large, more information is required to get a smaller class $\mathcal{M}^{(n)}$, such as adding and collecting more data. As a result, the range of the inference will reflect the robustness and uncertainty of the inference when selecting a specific prior distribution from the set of priors $\mathcal{M}^{(0)}$ [70].

In the following sections, a set of prior distributions for the repair effectiveness parameter q is introduced in Section 4.3, and we will introduce a set of prior distributions for the repair effectiveness parameter q and λ in Section 4.4.

4.3 Set of prior distributions for repair effectiveness parameter

Using the posterior predictive empirical reliability function enables the system reliability function to be calculated based on the fixed prior hyperparameters $(c,\ d)$ and data where the Weibull distribution scale parameter λ and the Weibull distribution shape parameter β are fixed. $c,\ d > 0$ are the hyperparameters of the Beta distribution, where this distribution is the prior distribution of the repair efficiency parameter q. In this section, we discuss the use of sets of priors $\mathcal{M}^{(0)}$ on the repair effectiveness parameter q, which allows for uncertain and incomplete prior knowledge. Modelling work supposes that the set of priors $\mathcal{M}^{(0)}$ is defined by modifying the Beta$(c,\ d)$ distribution within the set of prior parameters. The goal is to study the robustness of uncertainty results using the upper and lower posterior predictive reliability functions depending on a set of priors and data.

In the uncertainty of prior information, we will take a set that contains all possible prior distributions over all or approximately all expected q values, but by guessing the lower c_1 and upper c_2 values of hyperparameters c with w a fixed constant number to control the distribution's shape and the strength of the prior beliefs it represents. This effectively creates a constraint that shapes the parameter space of the Beta distribution from optimistic to pessimistic in terms of repair efficiency, which helps quantify the uncertainty in prior beliefs. The set of prior distributions on q is given by

$$\mathcal{M}^{(0)} = \{Beta(c,d): \ c_1 \leq c \leq c_2, \ d = w - c, \ \text{for given } c + d = w\} \ . \qquad (4.3.1)$$

When the interval of c is $c \in [c_1, \ c_2]$ then by substituting c into the expression $c+d = w$, the interval of d is $d \in [w-c_2, \ w-c_1]$. The set of the posterior probability distributions, $\mathcal{M}^{(n)}$, is calculated by multiplying the likelihood function and each prior distribution within $\mathcal{M}^{(0)}$ for q.

Due to the complexity of the posterior predictive empirical reliability function of the Kijima type I model, which is presented in Equation (B.2.3), It is inherently difficult to establish the proof of upper and lower posterior predictive empirical reliability functions based on prior parameters. Therefore, we present a possible argument for lower and upper posterior predictive empirical reliability function.

In the Kijima type I model, modelling work supposes that $q = 1$ corresponds to minimal repair, and $q = 0$ corresponds to perfect repair. It suggests that system reliability is bounded between these extreme cases, with the upper bound corresponding to situations where the system is repaired perfectly after failures and the lower bound corresponding to situations where the system is repaired minimally after failures. Therefore, the upper bound of the system reliability is obtained when the repair is perfect $q = 0$. On the other hand, the lower bound of system reliability is obtained when the repair is minimal $q = 1$.

We have a set $\mathcal{M}^{(0)}$ of priors for effective repair parameters in this section, focusing on a set of Beta distributions. Based on the Beta prior distribution of q, the expected value is $\frac{c}{w}$ where $w = c + d$. This framework allows for representing beliefs about repair efficiency, ranging from pessimistic to optimistic, based on the expected values of the Beta distributions. When we have an upper expected value belief of Beta, we represent a pessimistic case of repair efficiency, while when we have a low expected value belief of Beta, we represent an optimistic case of repair efficiency. Therefore, by assuming that c is an interval where $c \in [c_1, c_2]$ and $c + d = w$, the prior parameter pair that maximizes $E[q]$ is $\frac{c_2}{w}$, and the prior parameter pair that minimizes $E[q]$ is $\frac{c_1}{w}$. Thus, a prior parameter in $\mathcal{M}^{(0)}$ that minimizes the system reliability is (c_2, d) where $d = w - c_2$ and a prior parameter in $\mathcal{M}^{(0)}$ that maximizes the system reliability is (c_1, d) where $d = w - c_1$. The lower and upper posterior predictive empirical reliability functions of T are

$$\underline{P}(T > t|t_{i-1}, D_m) = \inf_{P \in P^*} (P(T > t|t_{i-1}, D_m)) = P(T > t|t_{i-1}, c_2, d, D_m) \ , \quad (4.3.2)$$

$$\overline{P}(T > t|t_{i-1}, D_m) = \sup_{P \in P^*} (P(T > t|t_{i-1}, D_m)) = P(T > t|t_{i-1}, c_1, d, D_m) \ , \quad (4.3.3)$$

where P^* is a set of posterior predictive empirical reliability function solutions.

There are no conjugate prior distributions or closed-form solutions for the parameter q and system reliability. We need a numerical method to get the upper and lower estimates of system reliability based on the posterior set. Therefore, we estimate the parameters and posterior predictive empirical reliability functions using MCMC.

4.3.1 Simulation study of prior assumptions regarding repair effectiveness parameter

The aim of this simulation is to illustrate the upper and lower empirical predictive reliability functions based on the set of prior distributions method presented in Section 4.3 using the Kijima type I model based on two cases of repair efficiency q. We simulate data by assuming higher quality repair efficiency in Case 1 and lower quality repair efficiency in Case 2 with a different set of priors and investigate its effect on a system's lower and upper predictive empirical reliability functions.

We generate data using the Kijima type I model using Monte Carlo simulation where, in Case 1, we assume $\lambda = 3,\ \beta = 2$ and $q = 0.25$ and we assume $q = 0.85$ in Case 2 and keeping the same values for λ and β as in Case 1. Assume we generate data where $n_m = 10$ and $M \in [5,\ 10]$ where data for Case 1 and Case 2 are presented in Table 3.8 and Table 3.9, respectively, where we take the first five rows when $M = 5$ and the all the rows when $M = 10$.

Assume that λ and β are fixed, and a set of Beta distributions is chosen as prior distributions for q in Case 1 and 2. Assume that $w = 5.5$ and we take three sets of prior distributions with different hyperparameter intervals, given by

$$\mathcal{M}_1^{(0)} = \{Beta(c,d): \ 0.5 \le c \le 5, \ d = 5.5 - c,\} \ ,$$
$$\mathcal{M}_2^{(0)} = \{Beta(c,d): \ 1.5 \le c \le 4, \ d = 5.5 - c,\} \ ,$$
$$\mathcal{M}_3^{(0)} = \{Beta(c,d): \ 2 \le c \le 3.5, \ d = 5.5 - c,\} \ .$$

Based on the mean $\frac{c}{w}$ of the Beta distribution, the first set is the widest, representing the lowest certainty about q, while the second set is narrower, suggesting a medium level of certainty. The third set is the narrowest, indicating the lowest uncertainty about q.

$n_m = 10$	M	$\mathcal{M}_1^{(0)}$		$\mathcal{M}_2^{(0)}$		$\mathcal{M}_3^{(0)}$	
		$B(0.5,5)$	$B(5,0.5)$	$B(1.5,4)$	$B(4,1.5)$	$B(2,3.5)$	$B(3.5,2)$
Case 1							
$\hat{q}$	5	0.2932	0.3734	0.3101	0.3542	0.3190	0.3452
	10	0.2733	0.3114	0.2815	0.3028	0.2860	0.2984
Case 2							
$\hat{q}$	5	0.7032	0.9357	0.7436	0.8696	0.7649	0.8410
	10	0.7459	0.9298	0.7745	0.8695	0.7902	0.8471

Table 4.1: Estimated values of q using MCMC with different sets of $Beta$ priors for q

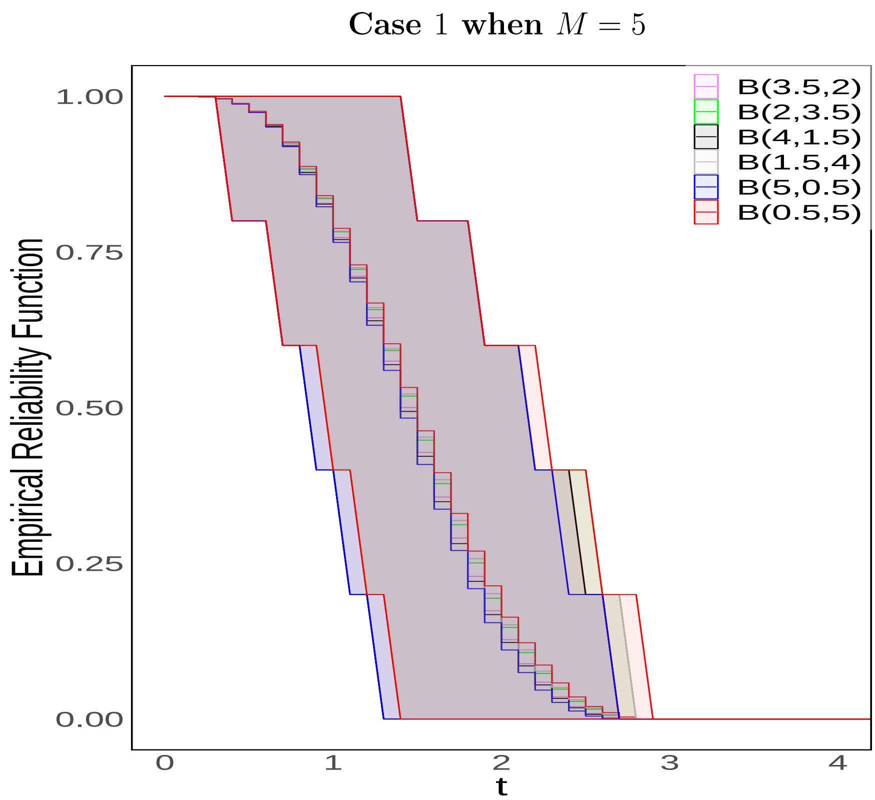


Figure 4.1: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and five repair actions

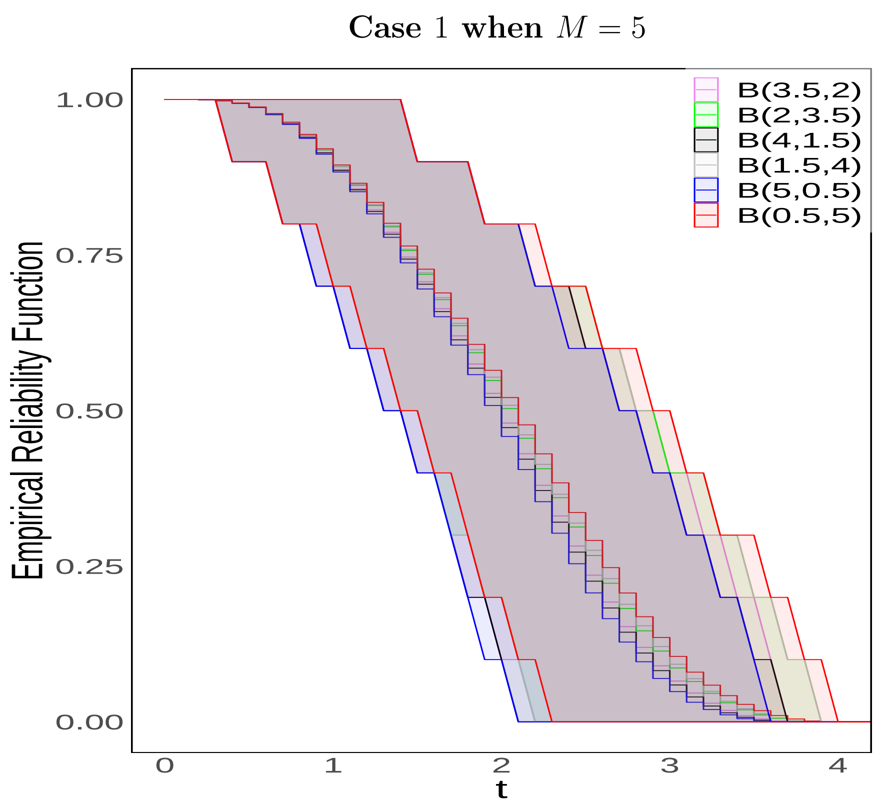


Figure 4.2: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and ten repair actions

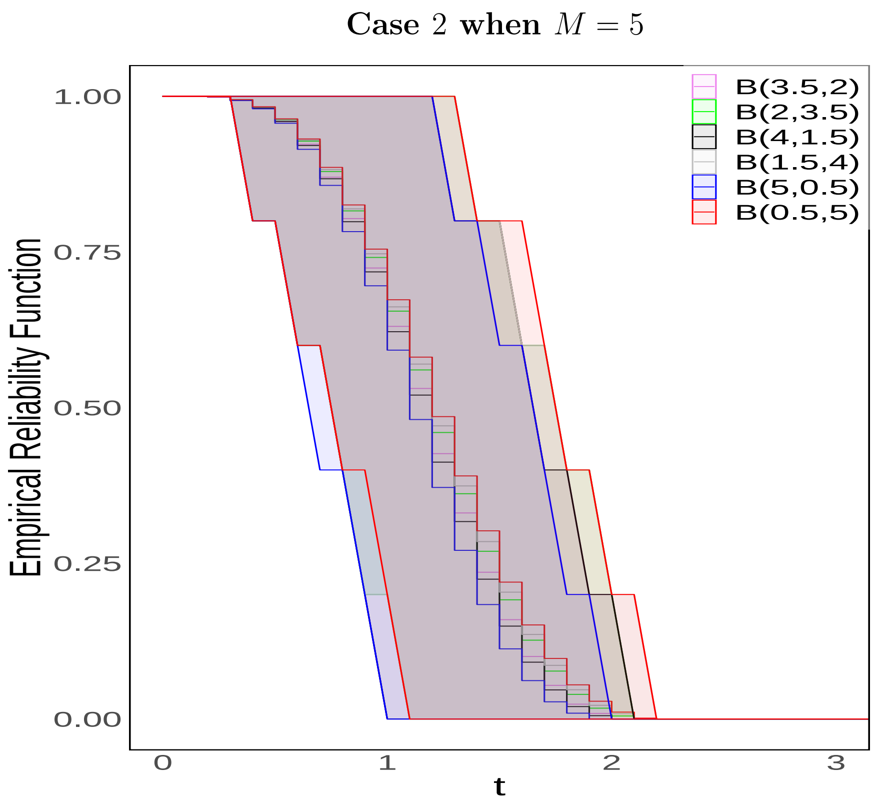


Figure 4.3: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and five repair actions

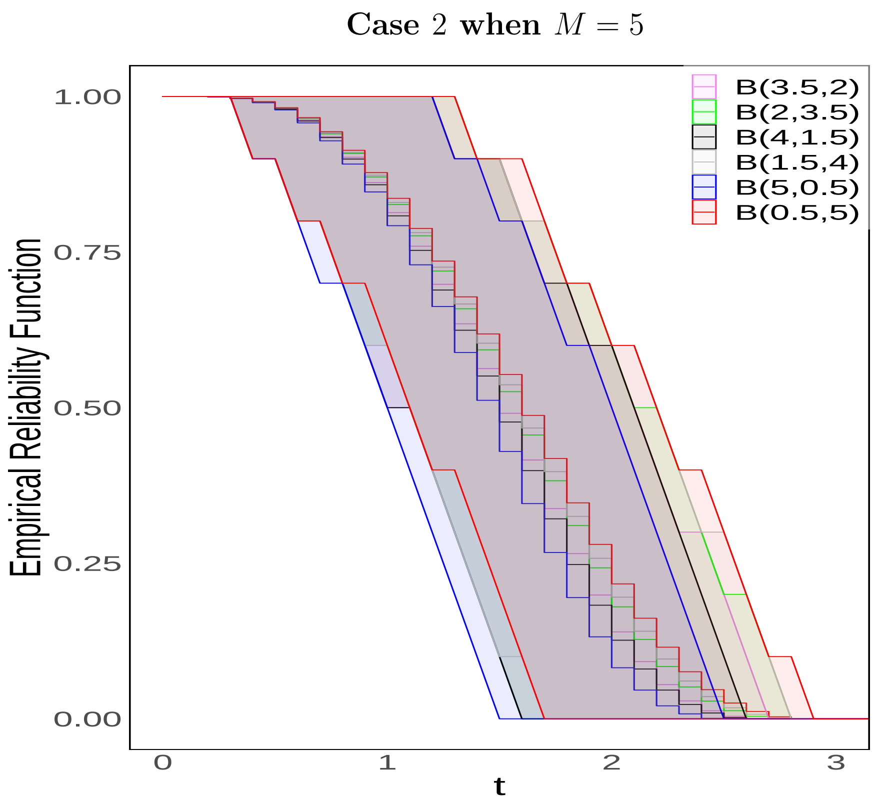


Figure 4.4: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and ten repair actions

Case 1 when $M = 10$

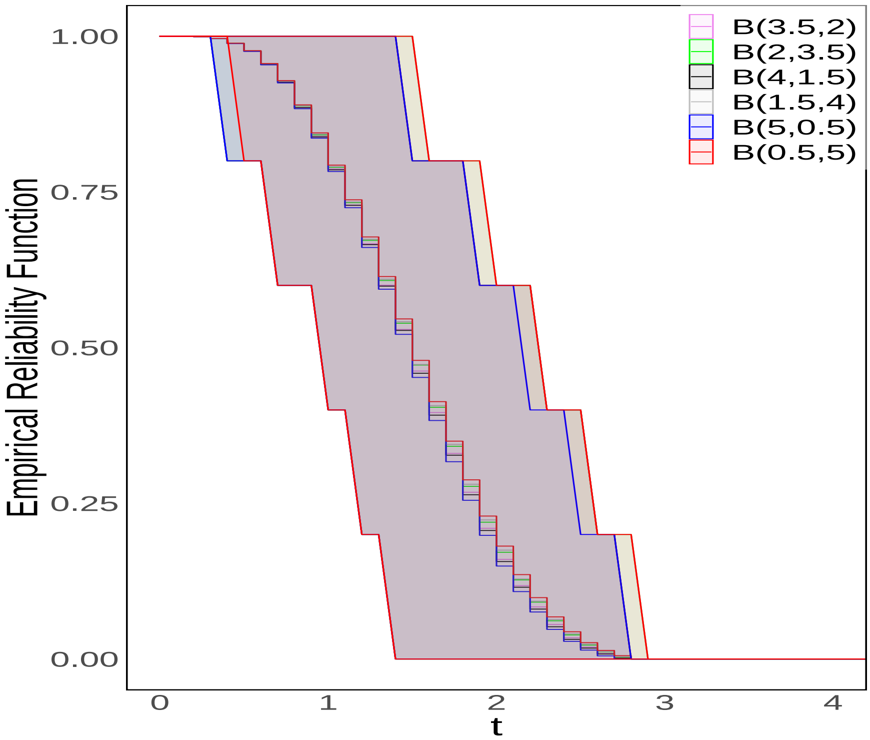


Figure 4.5: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and five repair actions

Case 1 when $M = 10$

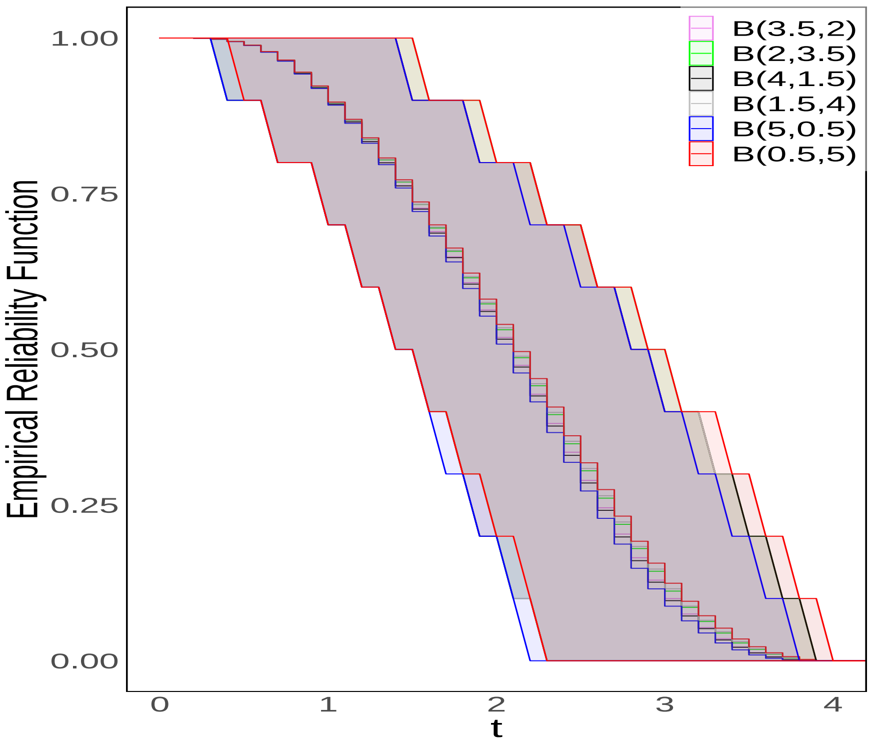


Figure 4.6: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and ten repair actions

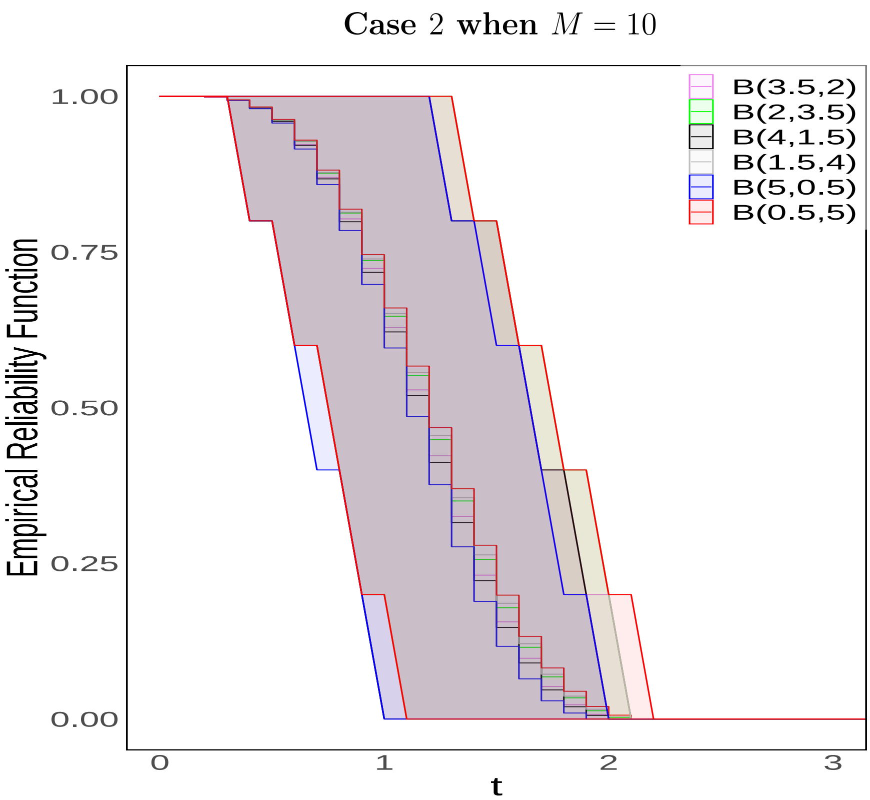


Figure 4.7: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and five repair actions

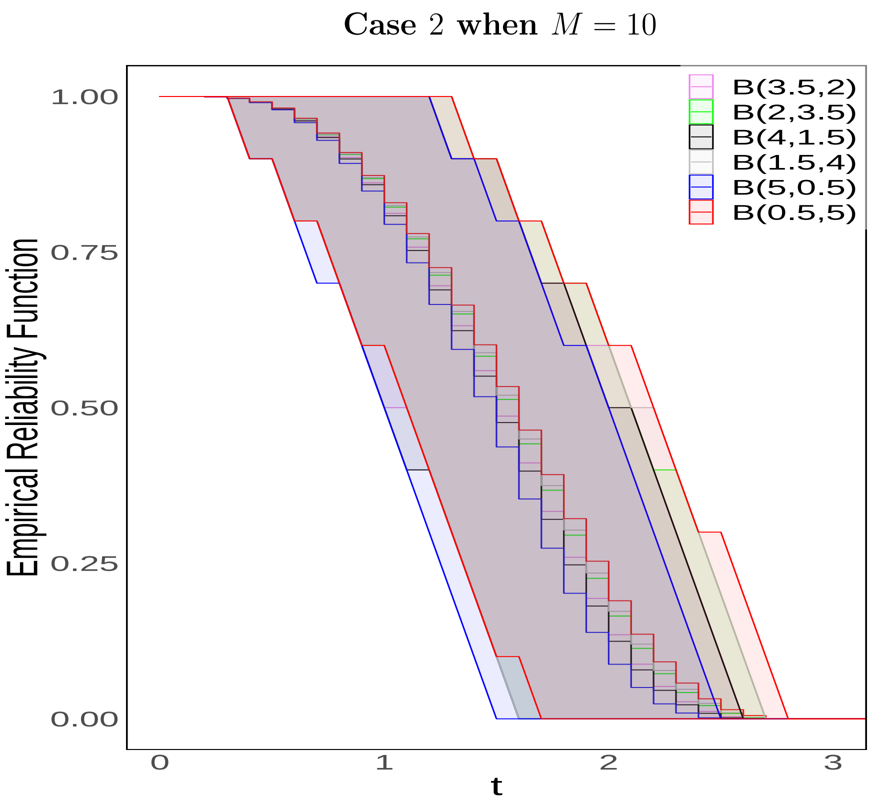


Figure 4.8: Upper and lower posterior predictive empirical reliability function with different sets of beta priors for q and ten repair actions

In Table 4.1, The Metropolis-Hastings (MCMC) is used to estimate the parameter q using squared error loss based on each upper and lower value of c hyperparameter within each specified Beta distribution priors set. Three different sets of $Beta(c, d)$ priors are assumed for q where $c + d = 5.5$ to quantify the uncertainty in prior beliefs with different hyperparameter intervals of c for each set. We get the upper estimated value of q when the prior is $Beta(c = 5, d = 0.5)$ and the lower estimated value of q when the prior is $Beta(c = 0.5, d = 5)$ for the first set and for both cases based on repair efficiency quality data. Also, in the second and third sets of Beta priors, the upper q estimated value is obtained when we have the upper values of c and the lower q estimated value is obtained when we have the lower values of c. The difference between the estimated value of upper and lower q increases when the set increases and decreases when the data increases.

We generated 100 different samples from the posterior predictive distribution based on each set of priors for each case. Then, we took the average of these estimates at each time t. Thus, we obtain 100 posterior predictive empirical reliability functions at each failure time t for each upper and lower. Then, we take the average of these estimates at each time t. The upper and lower posterior predictive empirical reliability functions are obtained using $Beta(0.5, 5)$ and $Beta(5, 0.5)$, respectively, for the first set. In Figures 4.1 to 4.8, the red line represents the upper estimated predictive empirical reliability function using $Beta(0.5, 5)$, and the blue line represents the lower predictive empirical reliability function using $Beta(5, 0.5)$. Also, the grey and black lines show the upper and lower estimated posterior predictive empirical reliability functions, respectively, for the second set, and the green and violet lines show the upper and lower posterior predictive empirical reliability functions, respectively, for the third set. These lines reflect the range of the estimated empirical predictive reliability function based on the posterior estimate.

Based on the first set, which represented the lowest certainty, Figures 4.1 to 4.8 show results with the widest range between upper and lower estimated predictive empirical reliability functions, which indicates the lowest level of robustness. For the second set, which has a range of prior assumptions suggesting stronger certainty than the first set, the figures show a smaller difference between the upper and lower estimated predictive empirical reliability functions, implying an enhancement in reliability as uncertainty decreases. The third set, representing the strongest level of certainty among all three sets, resulted in the lowest difference between upper and lower estimated predictive empirical reliability functions. Thus, the impact of prior certainty levels on the robustness of estimated upper and lower predictive empirical reliability functions indicates more robust results when dealing with a narrower set of priors.

To provide a more comprehensive understanding of the variability and uncertainty in the posterior predictive empirical reliability functions, we have plotted ribbons representing a 90% interval. To plot the interval, we calculated the 5^{th} and 95^{th} percentiles of the reliability estimates at each failure time t across generated samples. The 90% interval offers a measure of the spread of predictions, ensuring that the majority of the posterior predictive distribution is covered. By showing these intervals, we enhance the interpretability of the results by highlighting the uncertainty and variability in predictions of the upper and lower posterior predictive empirical reliability functions based on each set of prior distributions. The ribbons in Figures 4.1 to 4.8 clearly show the confidence we can have in the reliability estimates at different time points, with the shaded regions indicating the interval within which 90% of the predictions of posterior predictive empirical reliability functions lie for each set of the prior distributions. Wider ribbons indicate greater uncertainty about reliability estimates at each time point t, while narrower bands suggest greater confidence in reliability estimates.

From Table 4.1, where we have different $M = 5,\ M = 10$, the difference between the upper and lower values for parameter q become smaller when we increase M for both cases. Also, each set's upper and lower lines become closer when we increase the amount of data as shown in Figures 4.1 to 4.4 when $M = 5$ and in Figures 4.5 to 4.8 when $M = 10$ for Case 1 and 2. That means that the imprecision in Kijima type I's lower and upper predictive empirical reliability function decreases when we have more data. Consequently, the robustness of inferences improves. This robustness suggests that any prior from $\mathcal{M}^{(0)}$ yields similar results across the evolved set $\mathcal{M}^{(n)}$, thus enhancing the confidence in the reliability of these inferences.

4.4 Sets of priors for scale and repair effectiveness parameters

Using the posterior predictive empirical reliability function enables the system reliability function to be calculated based on the fixed prior parameters $(a,\ b,\ c,\ d)$ and data when the shape parameter β is fixed, $a,\ b > 0$ are the parameters of Gamma prior distribution for λ and $c,\ d > 0$ are the parameters of Beta prior distribution for q. In this section, we discuss the use of sets of priors $\mathcal{M}^{(0)}$ on the repair effectiveness parameter q and scale parameter λ, which allows for uncertain and incomplete prior knowledge. Modelling work supposes that the set of priors $\mathcal{M}^{(0)}$ is defined by modifying $Beta(c,\ d)$ and $Gamma(a,\ b)$ within the set of prior parameters. The goal is to study the robustness of uncertainty results using the upper and lower posterior predictive reliability functions depending on the prior parameter combinations and data.

In this section, not only does the hyperparameter c take an interval but also a and b, which helps to identify the uncertainty in prior beliefs based on the prior distributions of λ and q. Let the hyperparameters be $a \in [a_1,\ a_2]$, $b \in [b_1,\ b_2]$, $c \in [c_1,\ c_2]$ and $c + d = w$, so we have a set of priors on λ and q given by

$$\mathcal{M}^{(0)} = \{\pi(\lambda|a, b)\pi(q|c) :\ a_1 \leq a \leq a_2,\ b_1 \leq b \leq b_2,\ \ c_1 \leq c \leq c_2,\ c + d = w\}\ , \tag{4.4.4}$$

when the interval of c is $c \in [c_1,\ c_2]$ then by substituting c into the expression $c + d = w$, the interval of d is $d \in [w - c_2,\ w - c_1]$. The set of the posterior probability distributions, $\mathcal{M}^{(n)}$, is calculated by multiplying the likelihood function and each prior distribution within $\mathcal{M}^{(0)}$ for λ and q.

Based on prior parameters, proving upper and lower posterior predictive empirical reliability functions is inherently difficult due to the complexity of the posterior predictive empirical reliability function of the Kijima type I model, which is presented in Equation (B.3.5). Therefore, we present an argument supporting lower and upper posterior predictive empirical reliability functions. Based on the Weibull distribution [49], the expected lifetime of a system is

$$E[T] = \frac{1}{\lambda^{\frac{1}{\beta}}}\Gamma(1 + \frac{1}{\beta}) \tag{4.4.5}$$

From equation (4.4.5), the scale parameter is in the denominator, which means that when β is fixed, the expected lifetime for a system $E[T]$ is minimized when λ is maximized, while the expected lifetimes for a system $E[T]$ is maximized when λ is minimized. Also, in general, $q = 1$ corresponds to a minimal repair, whereas $q = 0$ corresponds to a perfect repair, which suggests that system reliability is bounded between these extreme cases, as described in Section 4.3. Therefore, the upper bound of the system reliability is obtained when λ is minimized, and the repair is perfect $q = 0$. On the other hand, the lower bound of system reliability is obtained when λ is maximized, and the repair is minimal $q = 1$.

We have a set $\mathcal{M}^{(0)}$ of priors for the scale and effective repair parameters in this section. We have a set of priors for λ based on Gamma distribution and a set of priors for q based on Beta distribution. Based on the Gamma prior distribution of λ, the expected value is $\frac{a}{b}$, which means that the upper prior belief about λ corresponds to the prior upper expected value of the Gamma distribution, while the lower prior belief about λ corresponds to the prior lower expected value. Therefore, by assuming that a and b are values within the intervals $[a_1, a_2]$ and $[b_1, b_2]$ respectively, the prior parameter pair that maximizes $E[\lambda]$ is $\frac{a_2}{b_1}$, and the prior parameter pair that minimizes $E[\lambda]$ is $\frac{a_1}{b_2}$. Also, the expected value is $\frac{c}{w}$ based on the Beta prior distribution of q where $w = c + d$ which means that an upper expected value represents prior information of a pessimistic case of repair efficiency, whereas a lower expected value represents prior information of an optimistic case of repair efficiency. Therefore, by assuming that c is an interval where $c \in [c_1, c_2]$ and $c + d = w$, the prior parameter pair that minimizes $E[q]$ is $\frac{c_1}{w}$, while the prior parameter pair that maximizes $E[q]$ is $\frac{c_2}{w}$. Thus, assuming prior independence of the parameters, a prior parameter combination in $\mathcal{M}^{(0)}$ that minimizes the system reliability is (a_2, b_1, c_2, d) where $d = w - c_2$ and a prior parameter combination in $\mathcal{M}^{(0)}$ that maximizes the system reliability is (a_1, b_2, c_1, d) where $d = w - c_1$. The lower and upper posterior predictive empirical reliability functions of T are

$$\underline{P}(T > t|t_{i-1}, D_m) = \inf_{P \in P^*} (P(T > t|t_{i-1}, D_m)) = P(T > t|t_{i-1}, a_2, b_1, c_2, d, D_m) \ , \tag{4.4.6}$$

$$\overline{P}(T > t|t_{i-1}, D_m) = \sup_{P \in P^*} (P(T > t|t_{i-1}, D_m)) = P(T > t|t_{i-1}, a_1, b_2, c_1, d, D_m) \ , \tag{4.4.7}$$

where P^* is a set of posterior predictive empirical reliability function solutions.

Again, there are no conjugate prior distributions or closed-form solutions for the parameters. We need a numerical method to get the upper and lower estimates of system reliability based on the posterior set. Therefore, we estimate the parameters and posterior predictive empirical reliability functions using MCMC.

4.4.1 Simulation study of prior assumptions regarding scale and repair effectiveness parameters

The aim of this simulation is to investigate the set of prior distributions method introduced in this Section 4.4 using the Kijima type I model with two cases based on the repair efficiency parameter with different sets of priors and investigate its effect on a system's lower and upper predictive empirical reliability functions.

We generate data using the Kijima type I model using Monte Carlo simulation where, in Case 1, we assume higher quality for repair efficiency where $\lambda = 3,\ \beta = 2$ and $q = 0.25$ and we assume lower quality for repair efficiency where $q = 0.85$ in Case 2 and keeping the same values for λ and β as in Case 1. Assume we generate data where $n_m = 10$ and $M \in [5,\ 10]$ where data for Case 1 and Case 2 are presented in Table 3.8 and Table 3.9, respectively, where we take the first five rows when $M = 5$ and all the rows when $M = 10$.

Assume that β is fixed, a set of Gamma distributions is chosen as prior distributions for λ and a set of Beta distributions is chosen as prior distributions for q in Case 1 and 2. Assume we select $w = 5.5$ and the interval for the first set $a \in [5,\ 20]$, $b \in [1,\ 10]$ and $c \in [0.5,\ 5]$ and interval for the second set $a \in [3,\ 12]$, $b \in [5,\ 15]$

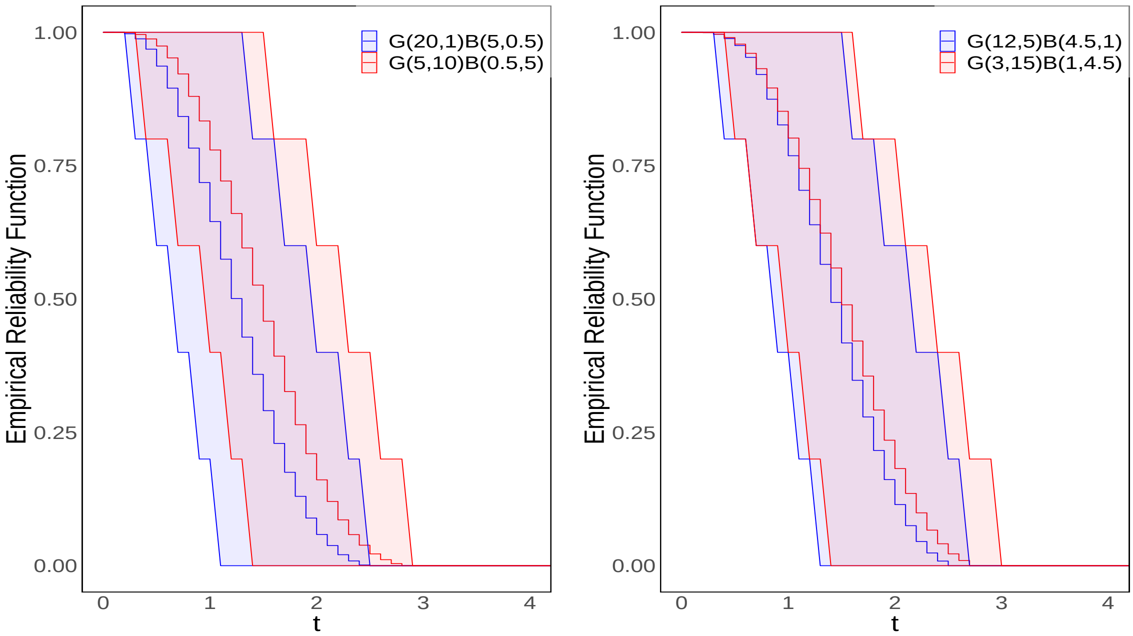
and $c \in [1,\ 4.5]$. The sets of priors for λ and q given by

$$\mathcal{M}_1^{(0)} = \{\pi(\lambda|a,b)\pi(q|c) :\ 5 \leq a \leq 20,\ 1 \leq b \leq 10,\ \ 0.5 \leq c \leq 5,\ d = 5.5 - c\}\ ,$$
$$\mathcal{M}_2^{(0)} = \{\pi(\lambda|a,b)\pi(q|c) :\ 3 \leq a \leq 12,\ 5 \leq b \leq 15,\ \ 1.5 \leq c \leq 4.5,\ d = 5.5 - c\}\ .$$

$n_m = 10$	M	**First set**		**Second set**	
		$G(5,10)B(0.5,5)$	$G(20,1)B(5,0.5)$	$G(3,15)B(1,4.5)$	$G(12,5)B(4.5,1)$
Case 1					
$\hat{\lambda}$	5	3.3728	4.0592	2.9539	3.3926
	10	3.2960	3.6914	3.0198	3.2240
$\hat{q}$	5	0.2693	0.2720	0.3269	0.3402
	10	0.2486	0.2538	0.2897	0.2958
Case 2					
$\hat{\lambda}$	5	3.7136	4.05901	3.6279	3.8503
	10	3.5756	3.6801	3.4955	3.5225
$\hat{q}$	5	0.7524	0.8636	0.7604	0.8458
	10	0.7497	0.8357	0.7586	0.8308

Table 4.2: Estimated values using MCMC with different sets of priors for λ and q

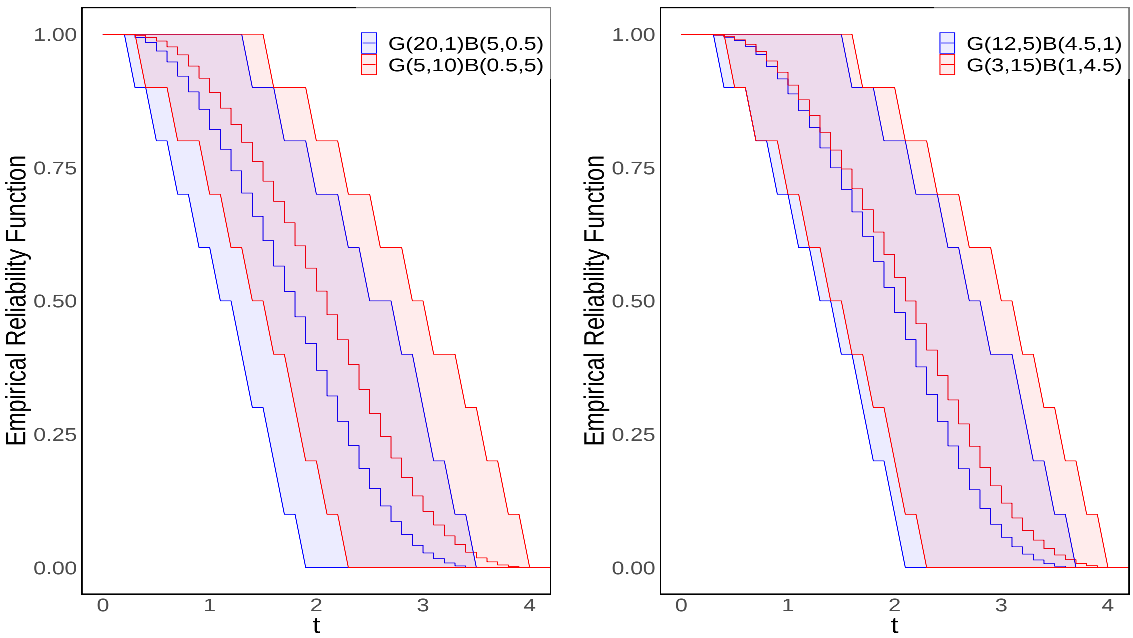
Case 1 when $M = 5$



(a) First set (b) Second set

Figure 4.9: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and five repair actions.

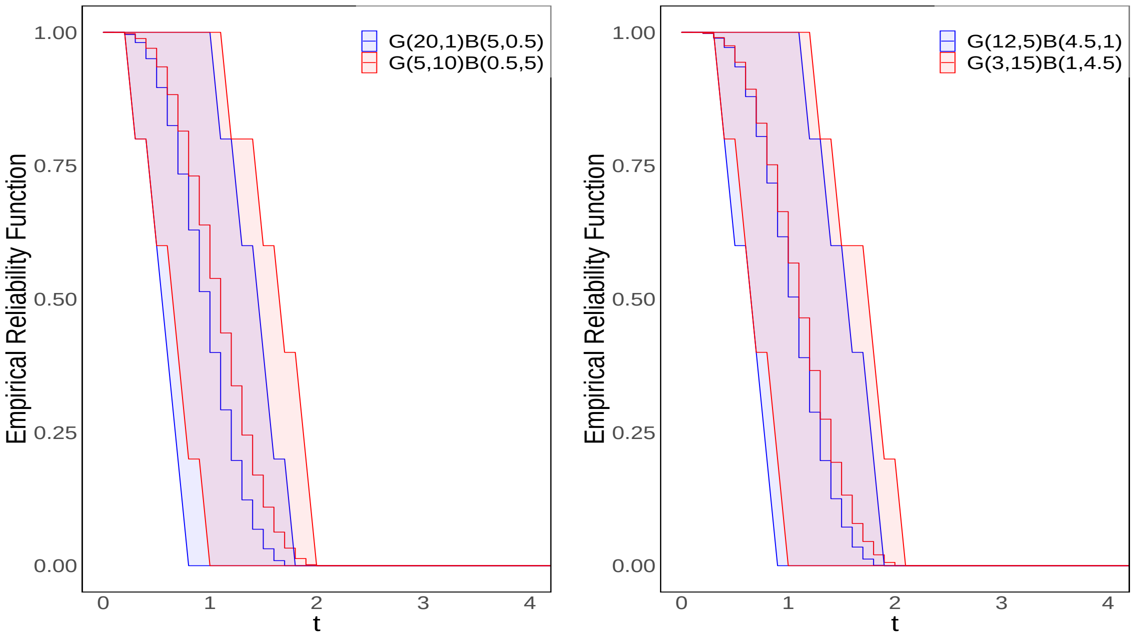
Case 1 when $M = 5$



(a) First set (b) Second set

Figure 4.10: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and ten repair actions.

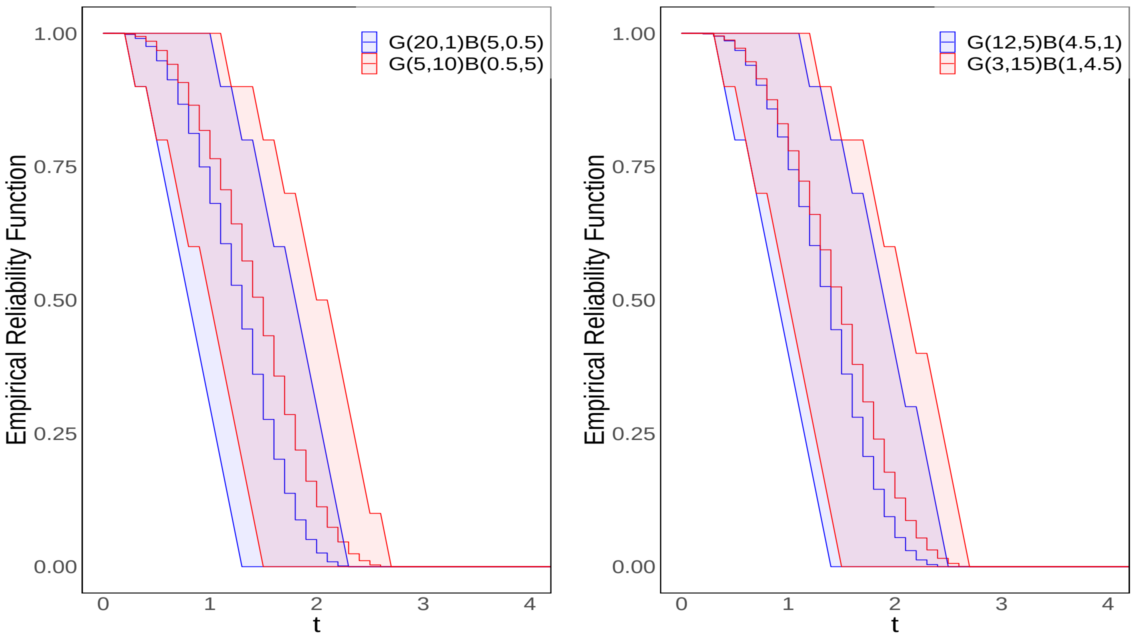
Case 2 **when** $M = 5$



(a) First set (b) Second set

Figure 4.11: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and five repair actions.

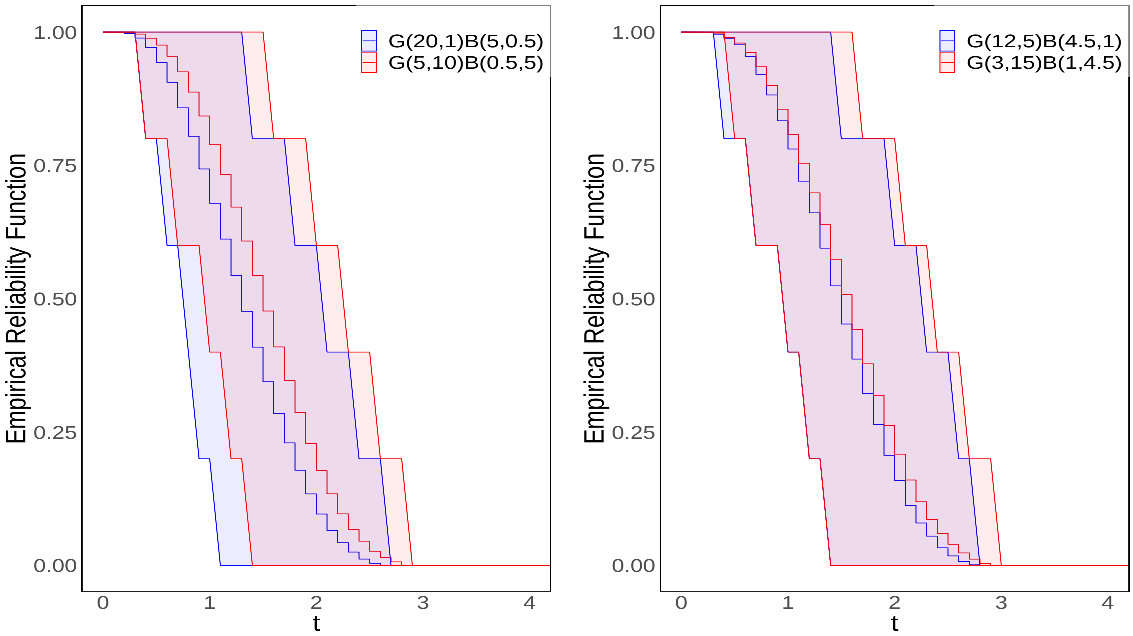
Case 2 **when** $M = 5$



(a) First set (b) Second set

Figure 4.12: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and ten repair actions.

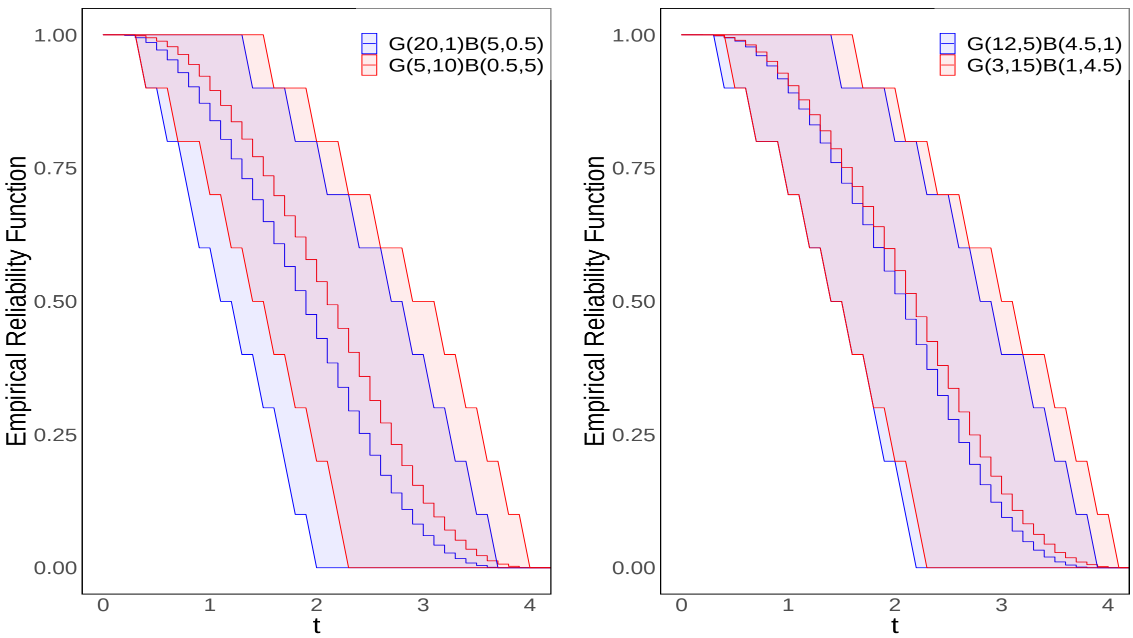
Case 1 when $M = 10$



(a) First set (b) Second set

Figure 4.13: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and five repair actions.

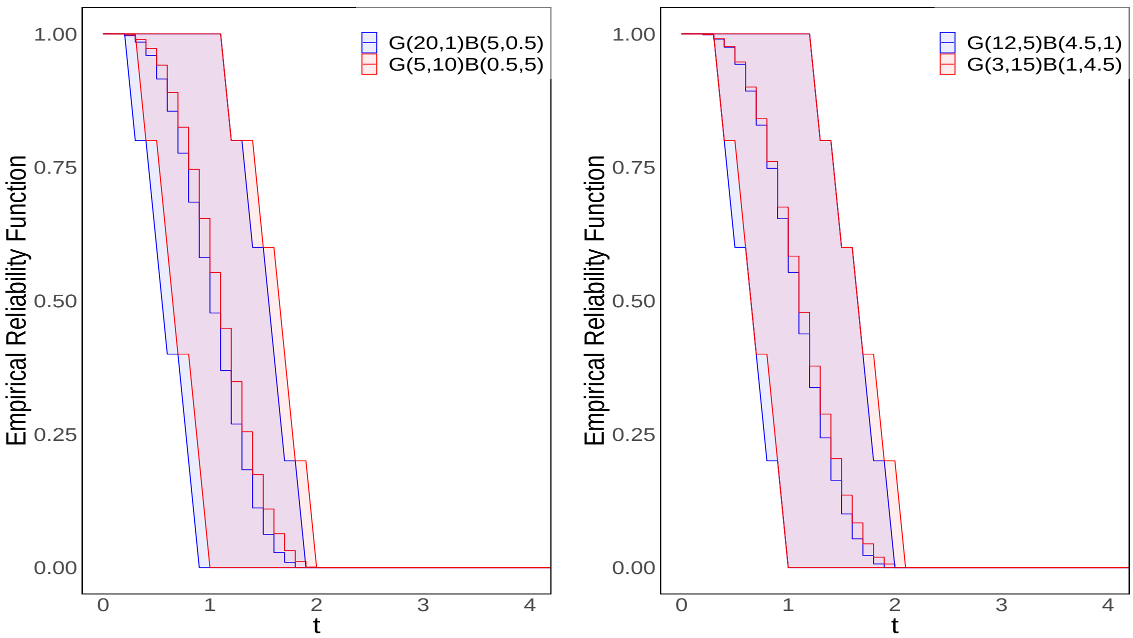
Case 1 when $M = 10$



(a) First set (b) Second set

Figure 4.14: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and ten repair actions.

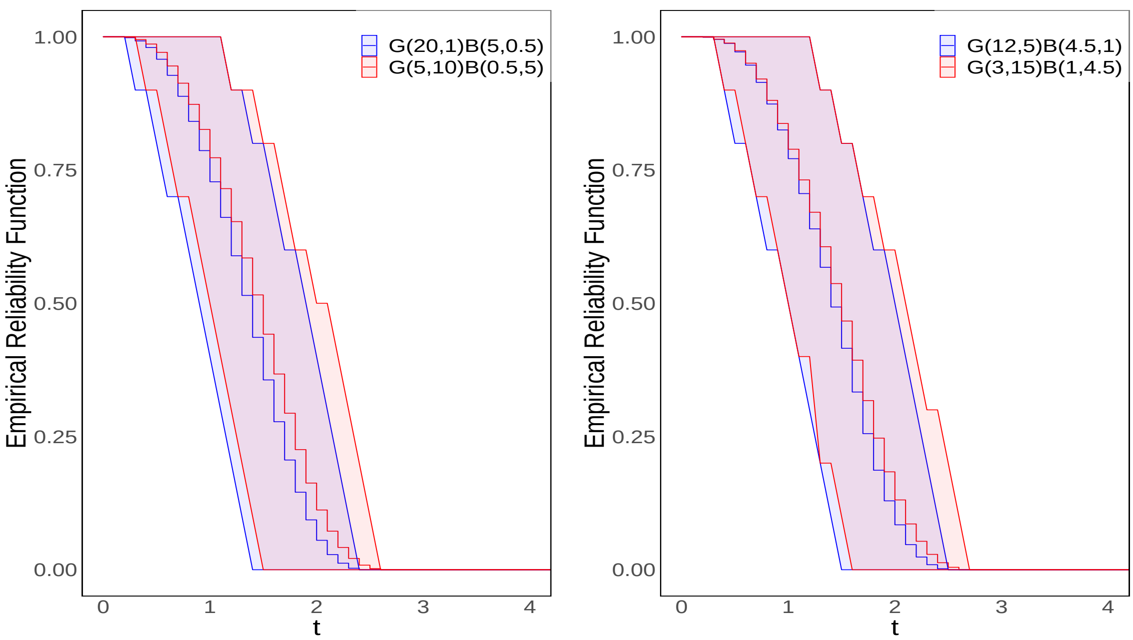
Case 2 when $M = 10$



(a) First set (b) Second set

Figure 4.15: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and five repair actions.

Case 2 when $M = 10$



(a) First set (b) Second set

Figure 4.16: Upper and lower posterior predictive empirical reliability function with different sets of priors for λ and q and ten repair actions.

In Table 4.2, for the estimation of the parameters, λ and q, the Metropolis-Hastings (MCMC) method is used with squared error loss based on the upper and lower values of hyperparameters in the specified priors. Two sets of priors are selected. we get the upper values for q and λ when the priors are $Gamma(a = 20, b = 1)$ for λ and $Beta(c = 5, d = 0.5)$ for q while the lower values for q and λ obtained withe priors $Gamma(a = 5, b = 10)$ for λ and $Beta(c = 0.5, d = 5)$ for q using the first set and for Case 1 and 2. Also, in the second set, the upper λ and q values are obtained when we have the upper values of c and a the lower value of b while the lower λ and q values are obtained when we have the lower values of c and a the upper value of b. The difference between the estimated value of upper and lower λ and q decreases when the data increases.

The upper and lower posterior predictive empirical reliability functions are obtained using $Gamma(5, 10)$ and $Beta(0.5, 5)$ and $Gamma(20, 1)$ $Beta(5, 0.5)$, respectively, for the first set and $Gamma(3, 15)$ $Beta(1, 4.5)$ and $Gamma(12, 5)$ $Beta(4.5, 1)$, respectively, for the second set. We generated 100 different samples from the posterior predictive distribution based on each set of priors for each case. Then, we took the average of these estimates at each time t. Thus, we will obtain 100 posterior predictive empirical reliability functions at each failure time t for each upper and lower. Then, we take the average of these estimates at each time t. In Figures 4.9 to 4.16, the red line represents the upper predictive empirical reliability function and the blue line represents the lower predictive empirical reliability function using each set. These lines reflect the range of the estimated empirical predictive reliability function based on the posterior estimate.

To comprehensively understand the variability and uncertainty in the posterior predictive empirical reliability functions, we have plotted ribbons representing a 90% interval by calculating the 5^{th} and 95^{th} percentiles of generated samples. The ribbons in Figures 4.9 to 4.16 clearly illustrate the confidence in the reliability estimates at different time points, with the shaded regions indicating the interval within which 90% of the predictions of posterior predictive empirical reliability functions lie for each set of prior distributions. Wider ribbons indicate lower certainty about reliability estimates at each time point t, while narrower ribbons suggest a strong level of certainty in the estimates.

From Table 4.2, where we have different $M = 5,\ M = 10$, the difference between the upper and lower values for parameter q become closer when we increase M for both cases. Also, each set's upper and lower lines become closer when we increase the data, which are shown in Figures 4.9 to 4.12 when $M = 5$ and in Figures 4.13 to 4.16 when $M = 10$ for Case 1 and 2. That means that the robustness based on the lower and upper predictive empirical reliability function of Kijima type I increase when we have more data. This robustness suggests that any prior from $\mathcal{M}^{(0)}$ yields similar results within the set $\mathcal{M}^{(n)}$, thus improving confidence in the accuracy of these inferences.

4.5 Concluding remarks

A robust Bayesian method that enables the use of sets of prior information to integrate prior uncertainties using the Kijima type I model has been presented in this chapter. A set of prior distributions for the repair effectiveness parameter q is introduced in Section 4.3, and we introduced a set of prior distributions for the repair effectiveness parameters q and λ in Section 4.4. The imprecision was evaluated using the upper and lower empirical posterior predictive reliability functions. Then, to study the effect of the uncertainty of prior distributions, a simulation investigation is conducted with different sets of prior distributions for the parameters, which are studied in two cases and different sample sizes. The upper and lower empirical posterior predictive reliability functions are estimated using MCMC with respect to the different sets of priors. The results have shown that the imprecision in Kijima type I's lower and upper predictive empirical system reliability decreases while the robustness increases when we have more data. Also, these results helped us to confidently infer that any prior in $\mathcal{M}^{(0)}$ can be chosen when we have more data since the range becomes smaller, which leads to similar results.

As with any statistical method developed for real-world applications, the real value of the method presented in this chapter is important to be shown by real-world applications. With the same model assumptions as for classical inference methods, no additional modelling effort is required; instead, the question is how the uncertainty about the prior distributions can be used to support real-world decisions. Optimal replacement policies will be presented in the following chapter, which explores this important aspect.

Chapter 5

Optimal replacement decisions

5.1 Introduction

In this chapter, we illustrate how predictive inference can be used to infer the optimal replacement policy using the Kijima type I model based on Bayesian approaches described in Chapter 3 and Chapter 4. In particular, we focus on determining the optimal replacement of the item depending on time and the number of repairs. This includes the cost of the replacement item and repair cost.

The failure to perform maintenance adequately to maintain technological systems can lead to significant risks. In reality, for instance, according to Kutor et al. [42], research conducted by the World Health Organization reveals that preventable factors account for approximately 80 per cent of all medical item failures, with inadequate maintenance alone contributing to around 60 per cent of performance-related issues. Furthermore, between 1994 and 2004, maintenance issues were implicated in up to 42 per cent of fatal airline accidents in the United States alone [33].

The systems used to produce goods and provide services constitute the vast majority of the cost of most industries. These systems are susceptible to degradation and accidents resulting from operational and environmental conditions [76]. Maintenance actions are implemented to ensure system availability and efficiency. In reality, maintenance might be expensive in terms of both resources and materials [66]. Maintenance costs constitute a significant portion of the overall operating costs for manufacturers, accounting for approximately 15 to 60 per cent of total production costs [54]. It is important to establish an effective maintenance strategy that takes into account the required resources and production plans. Such a strategy directly results in the reduced occurrence of system failures and decreases the costs of systems maintenance [76].

When a piece of machinery becomes older, it enters a phase where its failure rate increases, known as the wear-out phase. During this period, the item experiences age-related failures, resulting in a decrease in its resistance to failure and an increase in failure rate over time. These failures are caused by processes such as crack growth, corrosion, wear, and creep, which are all time-dependent forms of physical degradation [67]. In the area of maintenance, a preventive replacement strategy requires making decisions for the optimal time for item replacement. The replacement of an item is an important aspect of maintenance decision-making within a time-based preventive maintenance program. This task aims to ensure that the system remains in a condition that satisfies various predetermined performance levels. Most literature discusses two primary strategies for preventive replacement: age-based and constant time interval or periodic replacements [34]. In the age-based strategy, replacement occurs after a time of continuous operation without failure. If the item fails before time has elapsed, replacement takes place at the time of failure. On the other hand, the periodic replacement policy involves performing preventive maintenance on an item at regular time intervals. The timing of maintenance is not dependent on the unit's failure history but instead follows a fixed schedule. Additionally, the item is repaired when failures occur between the scheduled maintenance intervals [59]. This approach removes the need to keep records of item usage. However, the system requires ongoing inspections to detect any failures that may arise between constant time replacements. Furthermore, this strategy has the disadvantage of replacing relatively good items more frequently than necessary [67].

In order to determine the optimal time or the number of repairs for replacing a specific system, it is necessary to specify a cost function that considers both corrective and preventive replacement costs. Corrective replacement refers to instances where the system unexpectedly stops functioning, requiring the customer to seek assistance from maintenance services to resolve the issue. This type of maintenance is unplanned and usually more expensive than planned maintenance. On the other hand, preventive maintenance is performed on a regular basis according to predetermined replacement schedules. The advantages of preventive maintenance include enhanced system reliability, reduced replacement costs, decreased system downtime, and improved spare inventory management [50].

The study of repairable systems is an important field within reliability engineering, as most items used in manufacturing applications can be repaired when they fail [11]. Optimal replacement of systems has long been a subject of interest in academic research. Many replacement policies have been proposed in the literature for models involving repairable systems. For instance, Barlow and Hunter [3] examined a periodic replacement policy, while Park [63] suggested a policy that replaces the system at its n^{th} failure and minimally repaired until the n^{th} failure. Muth [58] presented a replacement policy where the system is replaced at the first failure after a specified time point and minimally repaired in case of failure before that time point. Nakagawa and Kowada [60] analyzed a replacement policy based on both the number of minimal repairs performed and the system's age. Boland and Proschan [9] analyzed a periodic replacement policy for a model with minimal repair and increasing repair cost dependent on the number of corrective maintenance activities in each cycle.

Recently, there has been an increased focus on imperfect maintenance [66]. Imperfect maintenance involves a wide range of models where the repair lies between minimal repair and perfect repair. One particular subset of imperfect maintenance models is virtual age models [11]. In a study conducted by Cassady et al. [11], simulation modelling and analysis were employed, employing the concept of Kijima type I and Kijima type II models. They explored the impact of imperfect repair on item availability and proposed a generic availability function that can be used to determine optimum item replacement intervals based on expected cost per unit of time. Makis and Jardine [48] offered new insights into the optimal replacement strategy for a system. They made the assumption that the replacement cost remained constant while the repair cost was set by the age of the system and the number of failures. By considering certain conditions related to costs and the failure rate, Makis and Jardine [48] showed that the most efficient policy for minimizing the average cost per unit of time is a control-limit kind. This means the system is replaced at the n^{th} failure only if its age exceeds a critical value that depends on n. Love et al. [47] presented a policy iteration algorithm that generates a sequence of enhancing control-limit policies using the Kijima type I model, and the algorithm was further investigated and improved by Dimitrakos and Kyriakidis [19].

Finkelstein [24] examines a system that gets repaired only when it fails. The author constructed a model to determine the cost of repair based on the repair level and explored the optimization of the repair level for the system. Lim et al. [44] investigate maintenance based on the age of the system, with replacement occurring at a specific maintenance age. They supposed that either a minimal or perfect repair is conducted upon failure. Sheu et al. [71] differentiate between small and large failures. After a small failure, the system is repaired, and it is replaced after a certain number of small failures, or at the occurrence of a large failure, or when a specific time is reached, whichever comes first where failures can only be detected through maintenance, and the length of the maintenance interval depends on the number of minor failures. Chang [12] also considers the occurrence of both minor failures that need minimal repairs and extreme breakdowns that require corrective replacement where preventive replacement is performed either at a specific age or upon completion of a designated working time.

Kijima et al. [40] examined the general repair model, which involves bringing the system to a better state through repair. This model contains two cases, namely minimal repair and perfect repair. Kijima et al. [40] made the assumption that the costs of repair and replacement remain constant. They focused on a periodic replacement problem, where the system is replaced at scheduled times, and whenever it fails, the repair takes place. They derived the average cost per unit of time and proposed an approximation method to determine the optimal replacement period.

Yevkin and Krivtsov [79] conducted a study on optimal maintenance policies within the context of the Kijima type I model, considering underlying Weibull distributions. They investigated different replacement policies using two approaches. The first approach involved an approximate formula for the expected number of failures, while the second used the Monte Carlo method. The study compared these policies across various model parameter values and demonstrated the effectiveness of the proposed methods.

A replacement strategy requires making a decision on the optimal time for system replacement. The objective is to minimize the cost to determine the optimal system replacement policies based on the virtual age model using the Bayesian approach. A key challenge in this thesis involves decision-making aimed at minimizing the cost function, which includes both corrective and preventive replacement costs, using the Kijima type I virtual age model under Weibull distribution.

In this chapter, we consider two different replacement policies based on the Kijima type I model and the statistical models we developed in Chapters 3 and 4. The statistical methods we have developed in this thesis contain a Bayesian and robust Bayesian approach, which enhances the robustness based on the prior assumptions. In this chapter, we minimize the expected total cost to get the optimal time or number of repairs for system replacement using the Bayesian method, and we establish boundaries for the expected costs of the optimal replacement time and number of failures based on the Kijima type I model data, using the robust Bayesian lower and upper methods.

This chapter is arranged as follows. In Section 5.2, we consider a replacement policy based on the time in which the new approaches can be applied to support replacement decisions, followed by examples illustrating the proposed method. Section 5.3 considers a replacement policy based on the number of repairs in which our new approaches can be applied to support replacement decisions, followed by examples illustrating the proposed method. Some concluding remarks are presented In Section 5.4.

5.2 Policy A: Replacement decision depending on time

Using the Kijima type I model, modelling work supposes that the system can be replaced by a new one with a cost C_0 or repaired with a cost $C_1 < C_0$. The system can be replaced at an optimal time or corrected at a failure time. As described by [2], we can minimize the average total cost C_τ and decide on the optimal replacement time by selecting replacement policies. This section uses the policy which states that we replace at age τ and perform repairs up to age τ [3]. Using the renewal argument [4], the average total cost per unit of time is

$$C_\tau = \frac{C_0 + C_1 W(\tau)}{\tau} \tag{5.2.1}$$

where $W(\tau)$ is expected number of repairs during $[0, \tau]$ and the preventive replacement time τ is constant. Based on this policy, replacements of a system are implemented periodically, while when a system fails, the repair is performed with the repair time being negligible in the time interval. The goal is to minimise the expected cost function in Equation (5.2.1) in order to get the optimal replacement time τ^*.

It can be seen from Equation (5.2.1) that $W(\tau)$ is required to compute C_τ. We use the posterior predictive distribution to simulate different samples and then apply the Monte Carlo method to calculate the average expected number of repairs $W(\tau)$ for each time period τ because there is no analytical solution for $W(\tau)$ due to the complexity of Kijima type I. Thus, the optimal replacement time τ^* that minimizing C_τ is

$$\tau^* = \arg\min_{\tau}(C_\tau) \ . \tag{5.2.2}$$

We also present a 90% interval for the cost function by calculating the 5^{th} and 95^{th} percentiles based on generated samples to understand the variability and uncertainty of cost functions.

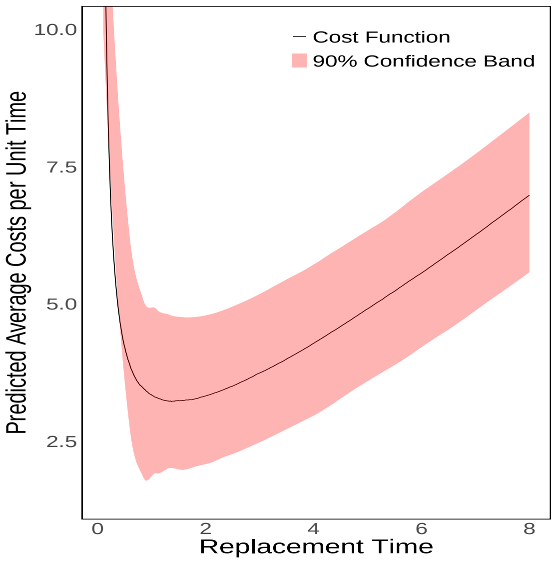
Example 5.2.1

In this example, we illustrate the optimal replacement time corresponding to the minimal expected cost and related expected number of repairs, and we investigate the influence of variation in the replacement costs C_0. By sampling from the posterior predictive distribution, we use the MC method to calculate $W(\tau)$ and predict the expected cost C_τ per unit of time with data from Table 3.8, where $n_m = 10$

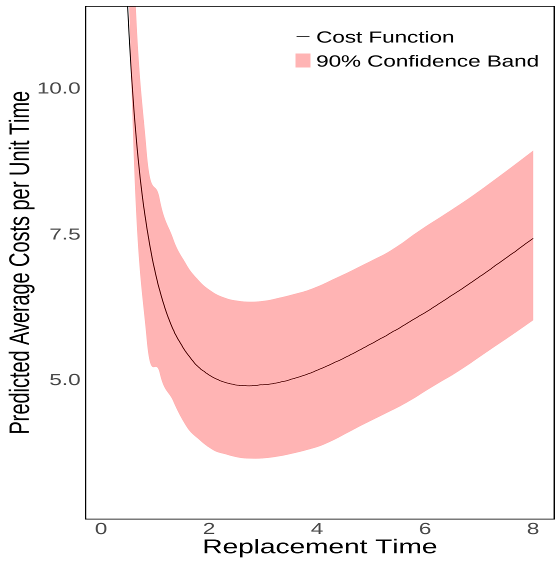
and $M = 10$ and $\lambda = 3,\ \beta = 2$ and $q = 0.25$. We also present a 90% interval by calculating the 5^{th} and 95^{th} percentiles of generated samples from the posterior predictive distribution to understand the variability and uncertainty of cost functions comprehensively. In Table 5.1, by assuming $C_1 = 1$ with different replacement costs C_0, we get the minimal predicted expected cost per unit of time. When $C_0 = 1.5$, $C_{\tau^*} = 3.2221$ and the expected number of repairs is 2.8982 in the first column from Table 5.1. The optimal expected cost C_{τ^*} and the optimal expected number of repairs is reached at $\tau^* = 1.365$. Thus, it is recommended that the system be replaced at time $\tau^* = 1.365$. Figure 5.1 represents the curves with 90% interval of the cost function of policy A with different values of the replacement cost C_0. When the replacement cost increases, the predicted expected cost and the optimal replacement time will increase, as shown in Table 5.1 and Figure 5.1 because as the replacement cost increases, it is recommended to do more repairs to avoid the high expense of the replacement cost, which leads to an increase in the expected cost and replacement time

true values: $\lambda = 3,\ \beta = 2$ and $q = 0.25$				
$C_0 =$	1.5	5	10	20
τ^*	1.365	2.660	3.885	5.280
$W(\tau^*)$	2.8982	8.4442	14.8552	25.3542
C_{τ^*}	3.2221	5.0554	6.3977	8.5898

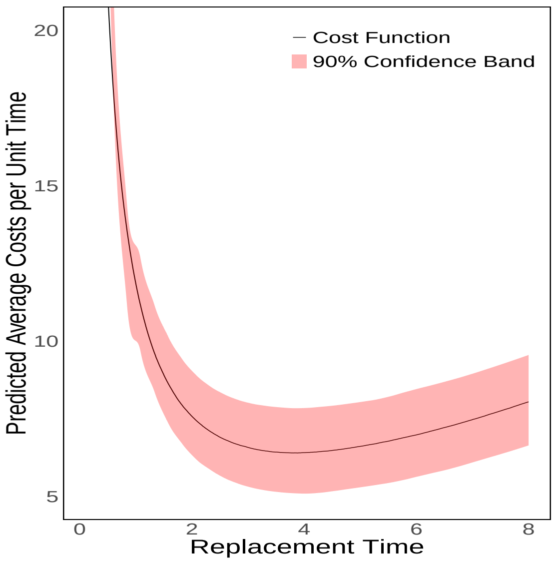
Table 5.1: Predictive total cost with different replacement cost



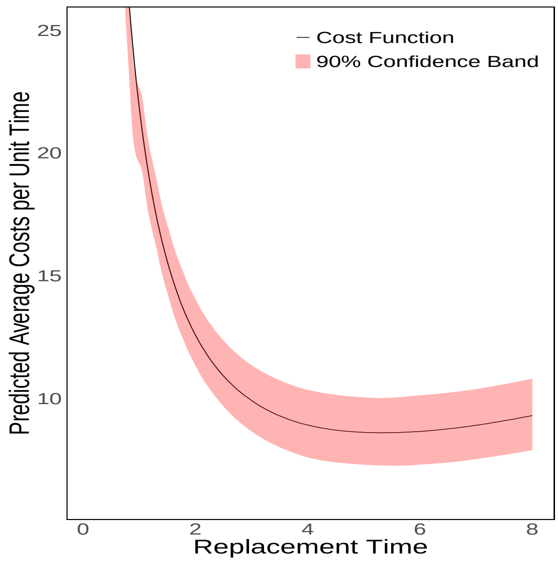
(a) $C_0 = 1.5$



(b) $C_0 = 5$



(c) $C_0 = 10$



(d) $C_0 = 20$

Figure 5.1: Predictive total cost and replacement time

Example 5.2.2

This example illustrates the change of the optimal replacement time corresponding to the minimal expected cost and related expected number of repairs by variation of the parameter β and q values and various values of the replacement costs. The same method in Example 5.2.1 is used to find the minimal predictive cost with different replacement costs. When we increase both β and q parameters, the predictive expected cost increases while the expected number of repairs and the replacement time decrease, as shown in Tables 5.1 - 5.3. For example, when we have $q = 0.25$ and $C_0 = 1.5$, we get the minimal predicted expected cost per unit of time. $C_{\tau^*} = 3.2221$ when the expected number of repairs is 2.8982 that is shown in Table 5.1. The optimal expected cost C_{τ^*} and the optimal expected number of repairs is reached at time $\tau^* = 1.365$. Thus, it is recommended that the system be replaced at $\tau^* = 1.365$, where the expected cost will be 3.2221 and prepare for approximately 3 failures. This information helps to plan repairs and costs and determine the optimal time for replacement rather than continuing the repair. However, when we have $q = 0.55$ and $C_0 = 1.5$, we get the minimal predicted expected cost that is $C_{\tau^*} = 3.6667$ when the expected number of repairs is 1.9284 which is presented in Table 5.2. The optimal expected cost C_{τ^*} and the optimal expected number of repairs is reached at time $\tau^* = 0.935$. This means that the expected cost will increase, and the expected number of repairs and replacement time will decrease when the quality of repair efficiency decreases, and the parameter value is close to 1.

		$q = 0.55$		
$C_0 =$	1.5	5	10	20
τ^*	0.935	1.635	2.310	3.200
$W(\tau^*)$	1.9284	5.4034	10.4530	19.8450
C_{τ^*}	3.6667	6.3629	8.8541	12.4515
		$q = 0.85$		
τ^*	0.725	1.315	1.865	2.570
$W(\tau^*)$	1.3944	4.6754	9.5650	18.505
C_{τ^*}	3.9922	7.3577	10.4906	14.9824

Table 5.2: Predictive total cost with different replacement cost and q values

$\beta = 1.1$				
$C_0 =$	1.5	5	10	20
τ^*	3.000	8.060	13.280	21.955
$W(\tau^*)$	9.3312	29.6428	53.4540	96.9842
C_{τ^*}	3.6104	4.2981	4.7781	5.3283
$\beta = 1.5$				
$C_0 =$	1.5	5	10	20
τ^*	1.460	3.505	4.915	7.600
$W(\tau^*)$	3.9590	12.2236	20.0036	38.2402
C_{τ^*}	3.7390	4.9140	6.1044	7.6631

Table 5.3: Predictive total cost with different replacement cost and β values

5.2.1 Robust Bayesian replacement decision

Using the replacement policy presented in this section, we will use the robust Bayesian methods with the repair costs, replacement costs and a scheduled removal that gives the optimal replacement time. However, robust Bayesian analysis deals with a set of cost functions. Therefore, we have a set of optimal replacement times D instead of only one optimal replacement time. Modelling work supposes one method to deal with the uncertainty issue is using the Γ-minimax criterion, which was investigated by Vidakovic [75]. It corresponds to making the most pessimistic decision. Using the Γ-minimax criterion, we will make a robust decision by selecting the optimal replacement time that minimizes the upper bound of the cost function using a set of prior distributions.

In this subsection, we will use the lower and upper of the robust Bayesian set introduced in Sections 4.3 and Section 4.4 for the age replacement policy. The upper and lower of each hyperparameter of the prior set in the robust Bayesian analysis will lead to the upper and lower bounds for the expected total cost for age replacement of the system. Also, the robust Bayesian will lead to the upper and lower bounds for the optimal replacement time of the item based on the lower and upper optimal expected total cost. The optimal decision of replacement time that minimizes the upper bound of the cost functional is given by

$$\tau^* = \arg\min_{\tau \in D}(\overline{C}_\tau) \tag{5.2.3}$$

where $\overline{C}_\tau$ is the upper cost function.

Example 5.2.1.1

This example clarifies the optimal replacement time related to the upper and lower average total costs with varying replacement cost values, using the method developed in Section 4.3 and the expected cost function in this section. Assume that $n_m = 10$ and $M = 10$, $\lambda = 3$, $\beta = 2$ and $q = 0.25$ where the data is given in Table 3.8 and the set of priors is $\mathcal{M}^{(0)} = \{Beta(c,d) : \ 0.5 \leq c \leq 5, \ d = 5.5 - c, \}$. By sampling from the posterior predictive distribution using each extreme prior, we use the MC method to calculate $W(\tau)$ as explained in Section 5.2 and predict the expected cost C_τ with a 90% interval of the expected cost function. In Table 5.4, when we use the upper value of the hyperparameter c, the minimal upper cost $\overline{C}_{\tau^*}$ is shown with the replacement time $\underline{\tau^*}$, while the minimal lower cost $\underline{C}_{\tau^*}$ with replacement time $\overline{\tau^*}$ is shown when we use the lower value of the hyperparameter c. When we have the replacement cost $C_0 = 1.5$, the lower optimal minimal average predictive cost is 3.2533, which is reached at 1.365 units of time, while the upper optimal minimal average predictive cost is 3.3541, which is reached at 1.230 units of time as presented in Table 5.4 and shown in Figure 5.2 with 90% intervals of the lower and upper expected cost functions which present the variability and uncertainty for cost functions. Therefore, the results of this study indicate that the optimal replacement interval is between 1.230 and 1.365 units of time when $C_0 = 1.5$. This recommended replacement time boundary allows management to choose the replacement time at the most convenient and effective cost. By using robust Bayesian decisions, the management can use the minimum of the maximum expected predicted total cost, representing the worst-case scenario, as the optimal replacement time. That means the management will choose a decision that reduces sensitivity to unpredictable situations. Management can generally determine the optimal replacement time, which is 1.230 units of time, based on the minimum of the upper-cost function when $C_0 = 1.5$. Furthermore, when we have the replacement cost $C_0 = 10$, the lower optimal minimal average predictive cost is 6.6580, which is reached at 3.550 units of time, while the upper optimal minimal average predictive cost is 6.9889, which is reached at 3.360 units of time as presented in Table 5.4 and shown in Figure 5.3 with 90% ribbons of the lower and upper expected cost functions. Therefore, the results indicate that the optimal replacement interval is between 3.360 and 3.550 units of time when $C_0 = 10$. The management can generally determine the optimal replacement time, which is 3.360 units of time as the optimal time for replacement, based on the minimum of the upper-cost function when $C_0 = 10$. It is clear that when the replacement cost increases, the optimal replacement time increases to avoid the high cost of the replacement.

true values: $\lambda = 3$, $\beta = 2$ and $q = 0.25$				
C_0	$\overline{C}_{\tau^*}$	$\underline{C}_{\tau^*}$	$\underline{\tau}^*$	$\overline{\tau}^*$
1.5	3.3541	3.2533	1.230	1.365
5	5.2652	5.0373	2.375	2.545
10	6.9889	6.6580	3.360	3.550
20	9.4605	8.9775	4.800	5.095

Table 5.4: Upper and lower predictive expected total cost with replacement time

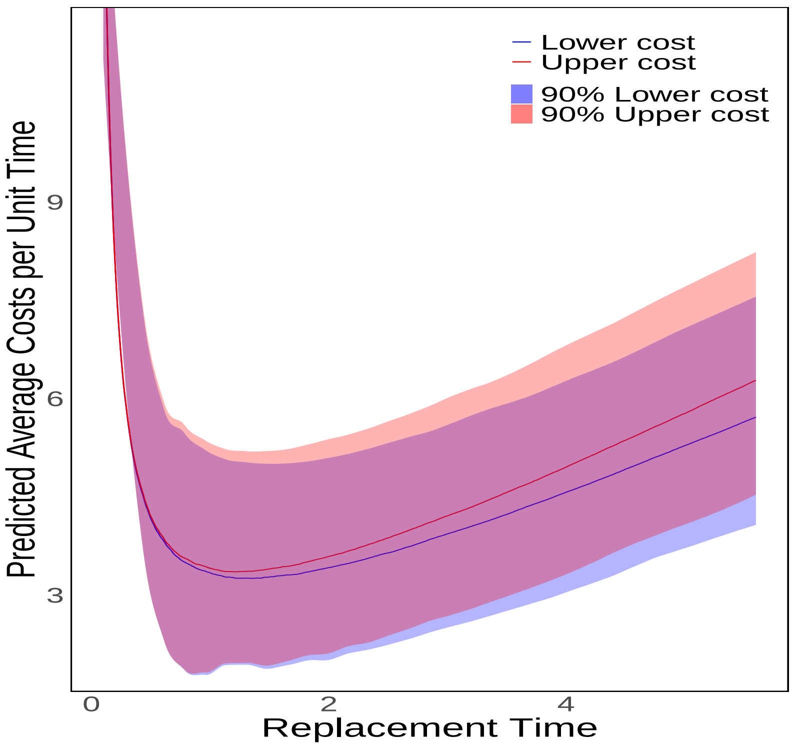


Figure 5.2: upper and lower predictive total cost and replacement time when $C_0 = 1.5$

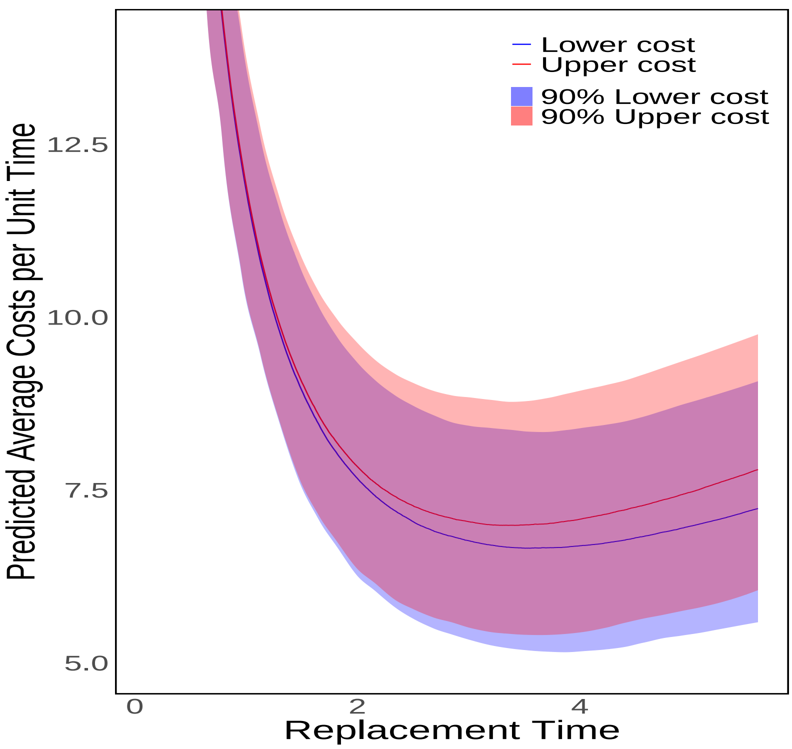


Figure 5.3: upper and lower predictive total cost and replacement time when $C_0 = 10$

Example 5.2.1.2

In this example, we will illustrate the optimal replacement time corresponding to the upper and lower average costs with varying values of the replacement cost, using the developed method in Section 4.4 and the expected cost function in this section. Assume that $n_m = 10$ and $M = 10$, $\lambda = 3$, $\beta = 2$ and $q = 0.25$ where the data is given in Table 3.8 and the set of priors is $\mathcal{M}^{(0)} = \{\pi(\lambda)\pi(q) : \; 5 \leq a \leq 20, \; 1 \leq b \leq 10, \; 0.5 \leq c \leq 5, \; d = 5.5 - c\}$. By sampling from the posterior predictive distribution using the extreme values of the priors, we use the MC method to calculate $W(\tau)$ and predict the expected cost C_τ with a 90% interval of the expected cost function. In Table 5.5, when we use the hyperparameters $a = 20$, $b = 1$ and $c = 5$, the minimal upper cost $\overline{C}_{\tau^*}$ is shown with the replacement time $\underline{\tau^*}$, while the minimal lower cost $\underline{C}_{\tau^*}$ with replacement time $\overline{\tau^*}$ is shown when we use the hyperparameters $a = 5$, $b = 10$ and $c = 0.5$. When we have the replacement cost $C_0 = 1.5$, the lower optimal minimal average predictive cost is 3.3783, which is reached at 1.360 units of time, while the upper optimal minimal average predictive cost is 3.5582, which is reached at 1.265 units of time as presented in Table 5.4 and shown in Figure 5.4 with 90% confidence bands which present the variability and uncertainty for cost functions. Therefore, the results of this study indicate that the optimal replacement interval is between 1.265 and 1.360 units of time when $C_0 = 1.5$. Based on the minimum of the upper-cost function when $C_0 = 1.5$, the management can generally determine the optimal replacement time, which is 1.265 units of time as the optimal time for replacement. Also, when we have the replacement cost $C_0 = 10$, the lower optimal minimal average predictive cost is 6.7155, which is reached at 3.495 units of time, while the upper optimal minimal average predictive cost is 7.0492, which is reached at 3.790 units of time as shown in Figure 5.5 with 90% confidence bands and presented in Table 5.5. Therefore, the results indicate that the optimal replacement interval is between 3.495 and 3.790 units of time when $C_0 = 10$. The management can generally determine the optimal replacement time, which is 3.495 units of time as the optimal time for replacement, based on the minimum of the upper-cost function when $C_0 = 10$.

true values: $\lambda = 3,\ \beta = 2$ and $q = 0.25$				
C_0	$\overline{C}_{\tau^*}$	$\underline{C}_{\tau^*}$	$\underline{\tau}^*$	$\overline{\tau}^*$
1.5	3.5582	3.3783	1.265	1.360
5	5.3912	5.1326	2.525	2.595
10	7.0492	6.7155	3.495	3.790
20	9.4240	8.9763	4.995	5.205

Table 5.5: Upper and lower predictive expected total cost with replacement time

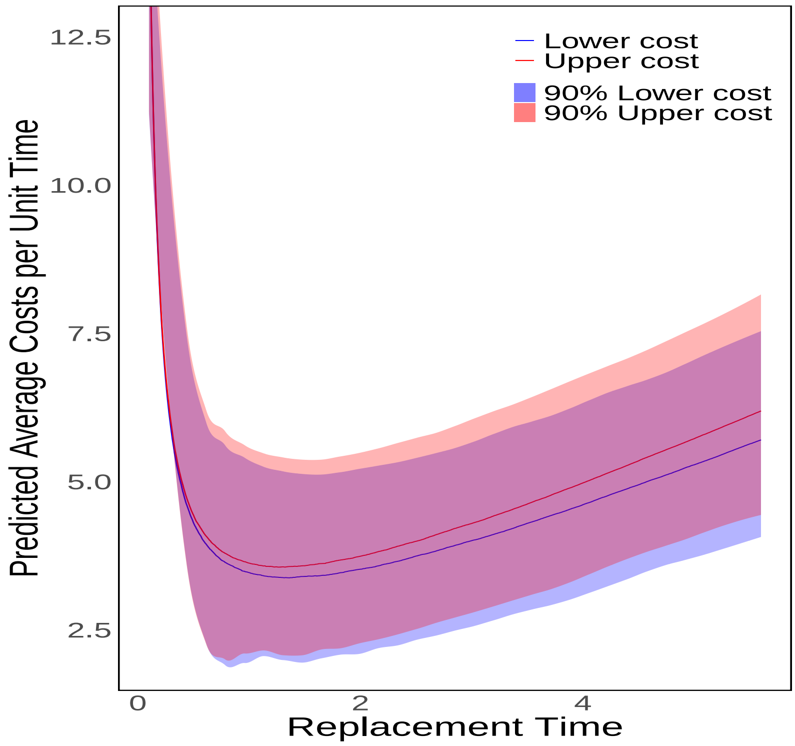


Figure 5.4: upper and lower predictive total cost and replacement time when $C_0 = 1.5$

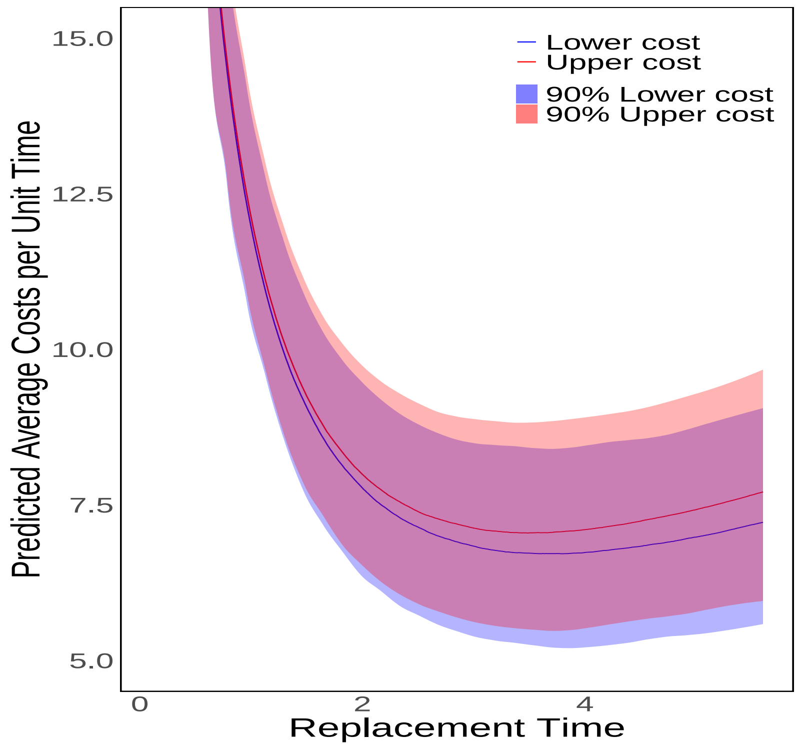


Figure 5.5: upper and lower predictive total cost and replacement time when $C_0 = 10$

5.3 Policy B: Replacement decision depending on number of repairs

Using the Kijima type I model, modelling work supposes the system can be replaced by a new one with a cost C_0 or repaired with a cost $C_1 < C_0$. The system can be replaced at an optimal number of failures or repaired at each failure before replacement. This section uses the policy which states that we replace at the n^{th} failure and perform repairs for the first $n-1$ [63]. The average total cost per unit of time is given by

$$C_\tau = \frac{C_0 + C_1(n-1)}{\tau_n} \tag{5.3.4}$$

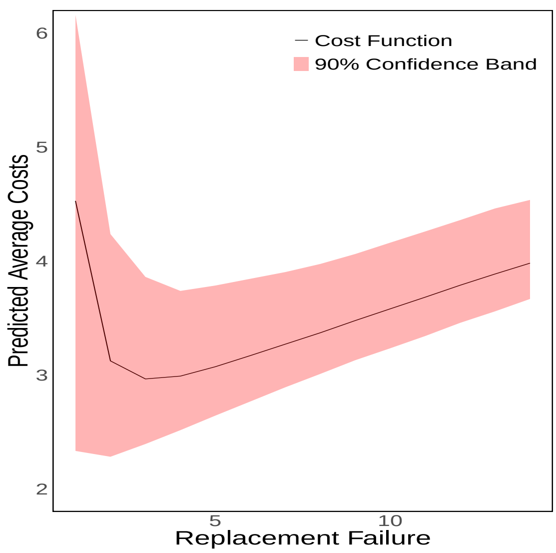
where τ_n is the expected length of the cycle. In Equation (5.3.4), the total cost is minimized with respect to n [79]. The goal is to minimise the expected cost function 5.3.4 in order to get the optimal replacement failure n^*. We use the posterior predictive distribution to simulate different samples and then apply the Monte Carlo method to calculate the average predictive expected cost C_τ based on the τ_n of each sample for each number of repairs n. Then, we show a 90% interval by calculating the 5^{th} and 95^{th} percentiles based on generated samples to understand the variability and uncertainty of cost functions.

Example 5.3.1

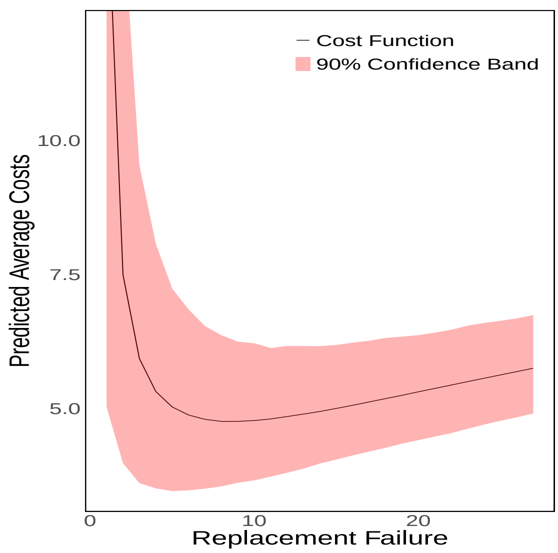
In this example, by using policy B, we illustrate the optimal replacement failure corresponding to the minimal expected cost and related expected length of the cycle, and we investigate the influence of variation in the replacement costs C_0. By sampling from the posterior predictive distribution, the predictive average expected cost is calculated using the MC method for different numbers of failure samples n for each replacement cost. We also present a 90% interval by calculating the 5^{th} and 5^{th} percentiles of samples generated from the posterior predictive distribution. This interval provides a comprehensive understanding of the variability and uncertainty in the cost functions. The average length of cycle τ_n^* of the repair is calculated at the optimal failure n^* corresponding to the predictive minimal expected cost when a system is replaced by a new one. Assume $\lambda = 3$, $\beta = 2$ and $q = 0.25$ where the data is given in Table 3.8 when $n_m = 10$ and $M = 10$. Also, let $C_1 = 1$ with different values of C_0. When we have $C_0 = 1.5$, we get the minimal expected cost $C_{\tau^*} = 2.9308$ at the $n^* = 3$ given in Table 5.6 first column. Therefore, the system is recommended to be replaced at $n^* = 3$ where the cycle length is $\tau_n^* = 1.3049$. This means that the company or management is recommended to repair the first two failures and perform replacement at failure number three. Figure 5.6 represents the curves with 90% interval of the cost function of policy B with different values of C_0. When the replacement cost increases, the predicted expected cost and optimal replacement failure will increase, as presented in Table 5.6 and shown in Figure 5.6, which is as expected when the replacement cost increases, it is recommended to do more repairs to avoid the high expense of the replacement cost, which leads to an increase in the expected cost and replacement time.

true values: $\lambda = 3$, $\beta = 2$ and $q = 0.25$				
$C_0 =$	1.5	5	10	20
n^*	3	8	14	26
C_{τ^*}	2.9308	4.7178	6.2775	8.5013
τ_n^*	1.3049	2.6240	3.7292	5.3436

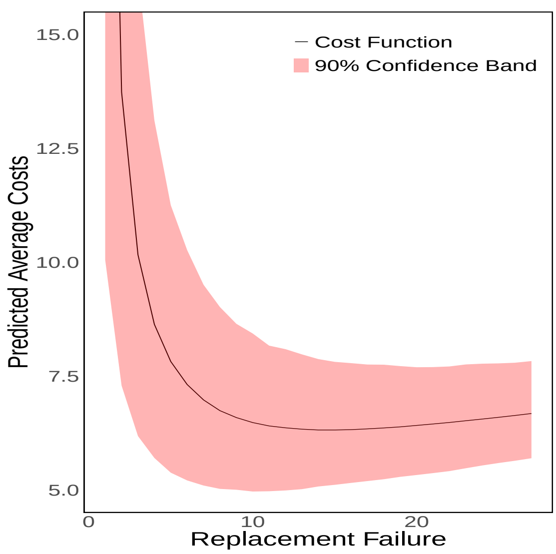
Table 5.6: Predictive total cost with different replacement cost



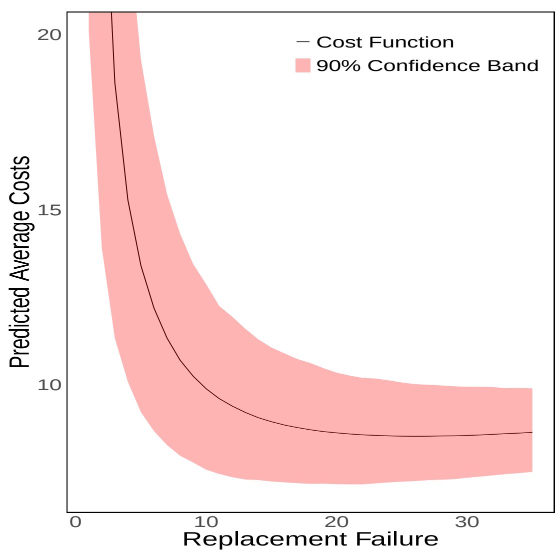
(a) $C_0 = 1.5$



(b) $C_0 = 5$



(c) $C_0 = 10$



(d) $C_0 = 20$

Figure 5.6: Predictive total cost and number of failures

Example 5.3.2

This example analyses the change of the optimal replacement failure corresponding to the minimal expected cost and related expected length of the cycle by variation of the parameter β and q values and various values of the replacement costs. The same method in Example 5.3.1 is used to find the optimal failure corresponding to predictive minimal expected cost with different replacement costs. When we increase q parameters, the predictive expected cost increases while the expected length of the cycle and the replacement failure decrease, as shown in Tables 5.6 to 5.8. For example, when we have $q = 0.25$ and $C_0 = 1.5$, we get the minimal predicted expected cost that is $C_{\tau^*} = 2.9308$ when the length of the cycle is 1.3049 that is shown in Table 5.6. The optimal expected cost C_{τ^*} and the expected length of the cycle are reached at $n^* = 3$. Also, when we have $C_0 = 1.5$ and $q = 0.55$, we get the minimal predicted expected cost that is $C_{\tau^*} = 3.3966$ when the length of the cycle is 0.8574 which is presented in Table 5.2. The optimal expected cost C_{τ^*} and the expected length of the cycle are reached at $n^* = 2$. Therefore, when the repair efficiency parameter is close to 1, the expected cost will increase, and the expected replacement failure and the cycle length decrease. Also, when the β value increases, the expected cost will increase except when $c_0 = 1.5$, and the optimal replacement n^* and the expected cycle length decrease.

$q = 0.55$				
$C_0 =$	1.5	5	10	20
n^*	2	6	11	21
C_{τ^*}	3.3966	6.2539	8.8024	12.4830
τ_n^*	0.8574	1.6651	2.3230	3.2414
$q = 0.85$				
$C_0 =$	1.5	5	10	20
n^*	2	5	10	19
C_{τ^*}	3.6036	7.1086	10.2736	14.5982
τ_n^*	0.8073	1.3291	1.8946	2.5982

Table 5.7: Predictive total cost with different replacement cost and q values

5.3.1 Robust Bayesian replacement decision

Using the replacement policy presented in this section, we will use robust Bayesian methods with the corrective maintenance costs, replacement costs and a scheduled removal that gives the optimal replacement failure. However, robust Bayesian analysis deals with a set of cost functions. Therefore, we have a set of optimal replacement failures N instead of only one optimal failure. Modelling work supposes that one method to deal with the uncertainty issue is using the Γ-minimax criterion, as discussed in Subsection 5.2.1. Thus, the robust decision will be made by selecting the replacement failure that minimizes the upper bound of the cost function using a set of prior distributions.

In this subsection, we will use the lower and upper of the robust Bayesian set presented in Sections 4.3 and 4.4 for the failure replacement policy. The upper and lower of each hyperparameter of the prior set in robust Bayesian will lead to the upper and lower bounds for the expected total cost for failure replacement of the system. Also, the robust Bayesian will lead to the upper and lower bounds for the optimal replacement failure of the system based on the lower and upper optimal expected total cost. The optimal decision of replacement failure that minimizes the

$\beta = 1.1$				
$C_0 =$	1.5	5	10	20
n^*	10	29	51	113
C_{τ^*}	3.5594	4.2793	4.7720	5.3174
τ_n^*	3.1686	7.9082	12.7631	25.0037
$\beta = 1.5$				
$C_0 =$	1.5	5	10	20
n^*	4	11	22	40
C_{τ^*}	3.3236	4.6382	6.0575	7.6340
τ_n^*	1.4982	3.2167	5.2118	7.8088

Table 5.8: Predictive total cost with different replacement cost and β values

upper bound of the cost functional is given by

$$n^* = \arg\min_{n \in N}(\overline{C}_\tau) \tag{5.3.5}$$

where $\overline{C}_\tau$ is the upper cost function.

Example 5.3.1.1

This example clarifies the optimal replacement time related to the upper and lower average total costs with varying replacement cost values, using the developed method in Section 4.3 and the expected cost function in this section. Assume that $n_m = 10$ and $M = 10$, $\lambda = 3$, $\beta = 2$ and $q = 0.25$ where the data is given in Table 3.8 and the set of priors is $\mathcal{M}^{(0)} = \{Beta(c, d) : \ 0.5 \leq c \leq 5, \ d = 5.5 - c, \}$. By sampling from the posterior predictive distribution using each extreme prior, we use the MC method to predict the expected cost C_τ with a 90% interval of the expected cost function. In Table 5.9, when we use the upper value of the hyperparameter c, the minimal upper expected cost $\overline{C}_{\tau^*}$ is shown with the replacement failure $\underline{n^*}$, while the minimal lower expected cost $\underline{C}_{\tau^*}$ with replacement failure $\overline{n^*}$ is shown when we use the lower value of the hyperparameter c. When we have the replacement cost $C_0 =$

1.5, the lower optimal minimal average predictive cost is 2.9936, which is reached at failure 3, while the upper optimal minimal average predictive cost is 3.0822, which is reached at failure 3 as presented in Table 5.9 and shown in Figure 5.7 with 90% confidence bands which present the variability and uncertainty for cost functions. Since the results of this study indicate that the optimal replacement failure occurs at 3 when $C_0 = 1.5$ for both upper and lower costs, the company or management can generally determine 3 failures as the optimal point for replacement when $C_0 = 1.5$ with expected cost 3.0822, which represents the worst-case scenario. This helps the management to make cost plans more robust and flexible. Furthermore, when we have the replacement cost $C_0 = 10$, the lower optimal minimal average predictive cost is 6.5631, which is reached at failure 14, while the upper optimal minimal average predictive cost is 6.8877, which is reached at failure 14 as shown in Figure 5.8 with 90% confidence bands and presented in Table 5.9. Since the results of this study indicate that the optimal replacement failure occurs at 14 when $C_0 = 10$ for both upper and lower costs, the company or management can generally determine 14 failures as the optimal point for replacement when $C_0 = 10$ with expected cost 5.1147. This means that when the replacement cost increases, both upper and lower expected costs increase, as does the optimal number of repairs. Thus, in order to avoid the high replacement cost, more repairs are recommended, which leads to an increase in expected costs, as presented in Table 5.9.

true values: $\lambda = 3$, $\beta = 2$ and $q = 0.25$				
C_0	$\overline{C}_{\tau^*}$	$\underline{C}_{\tau^*}$	$\underline{n^*}$	$\overline{n^*}$
1.5	3.0822	2.9936	3	3
5	5.1147	4.9013	8	8
10	6.8877	6.5631	14	14
20	9.3796	8.8965	26	26

Table 5.9: Upper and lower predictive expected total cost with replacement failure

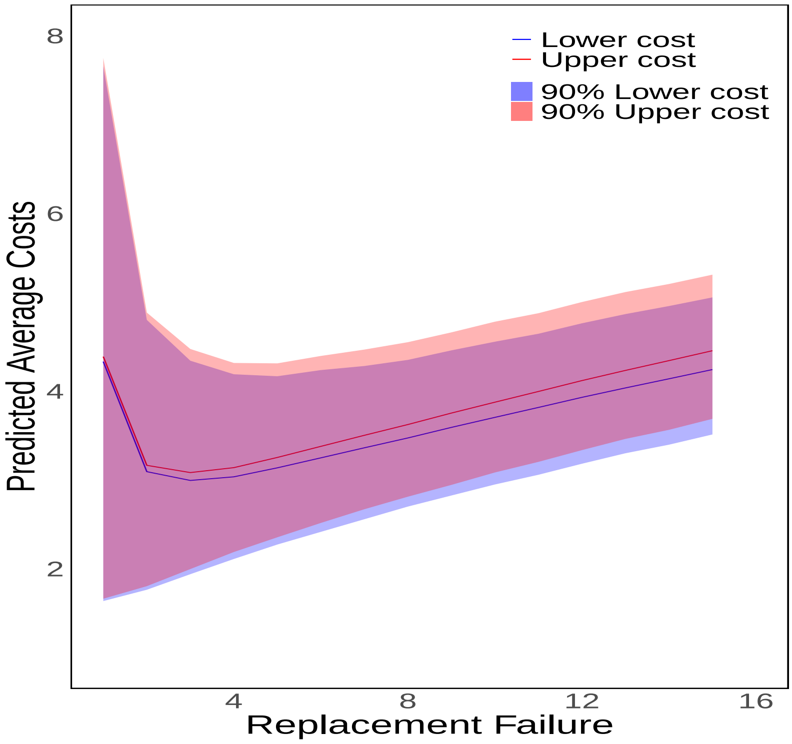


Figure 5.7: upper and lower predictive total cost and replacement failure when $C_0 = 1.5$

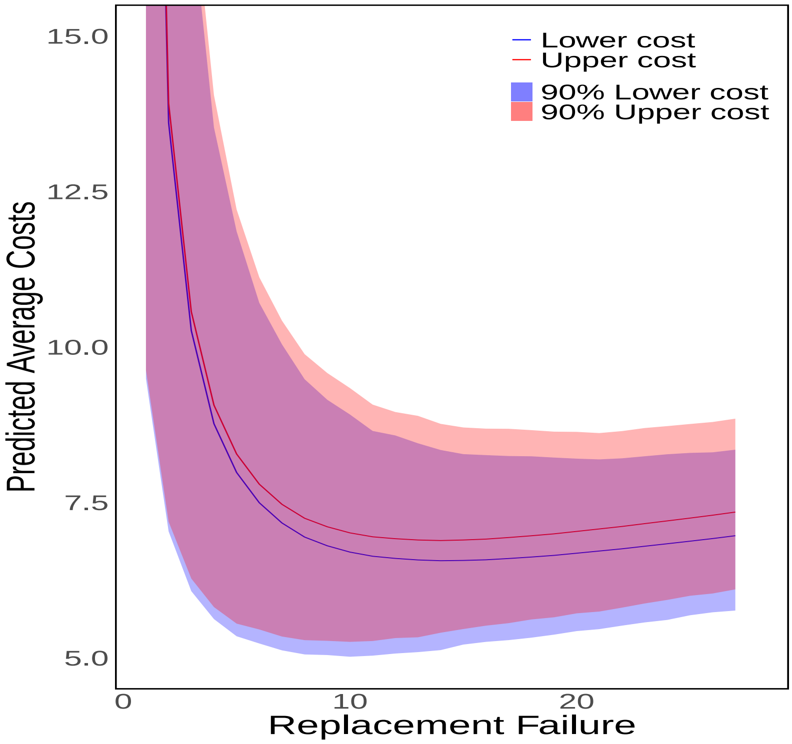


Figure 5.8: upper and lower predictive total cost and replacement failure when $C_0 = 10$

Example 5.3.1.2

In this example, we will illustrate the optimal replacement time corresponding to the upper and lower average costs with varying values of the replacement cost, using the developed method in Section 4.4 and the expected cost function in this section. Assume that $n_m = 10$ and $M = 10$, $\lambda = 3$, $\beta = 2$ and $q = 0.25$ where the data is given in Table 3.8 and the set of priors is $\mathcal{M}^{(0)} = \{\pi(\lambda)\pi(q) : \ 5 \leq a \leq 20, \ 1 \leq b \leq 10, \ 0.5 \leq c \leq 5, \ d = 5.5 - c\}$. By sampling from the posterior predictive distribution using each extreme prior, we use the MC method to predict the expected cost C_τ with a 90% interval of the expected cost function. In Table 5.10, when we use the hyperparameters $a = 20$, $b = 1$ and $c = 5$, the minimal upper expected cost $\overline{C}_{\tau^*}$ is shown with the replacement failure $\underline{n^*}$, while the minimal lower expected cost $\underline{C}_{\tau^*}$ with replacement failure $\overline{n^*}$ is shown when we use the hyperparameters $a = 5$, $b = 10$ and $c = 0.5$. When we have the replacement cost $C_0 = 1.5$, the lower optimal minimal average predictive cost is 3.1248, which is reached at failure 3, while the upper optimal minimal average predictive cost is 3.2924, which is reached at failure 3 as presented in Table 5.10 and shown in Figure 5.9 with 90% confidence bands which present the variability and uncertainty for cost functions. Since the results of this study indicate that the optimal replacement failure occurs at 3 when $C_0 = 1.5$ for both upper and lower costs, the company or management can generally determine 3 failures as the optimal point for replacement when $C_0 = 1.5$. Moreover, when we have the replacement cost $C_0 = 10$, the lower optimal minimal average predictive cost is 6.6213, which is reached at failure 15, while the upper optimal minimal average predictive cost is 6.9539, which is reached at failure 15 as shown in Figure 5.10 with 90% confidence bands and presented in Table 5.10. Since the results of this study indicate that the optimal replacement failure occurs at 15 when $C_0 = 10$ for both upper and lower costs, the company or management can generally determine 15 failures as the optimal point for replacement when $C_0 = 10$.

Furthermore, when we have the replacement cost $C_0 = 5$, the lower optimal minimal average predictive cost is 4.9986, which is reached at failure 8, while the upper optimal minimal average predictive cost is 5.2538, which is reached at failure 9 as presented in Table 5.10. Therefore, the results of this study indicate that the optimal replacement interval is between 8 and 9 failures when $C_0 = 5$. This recommended replacement failure boundary allows management to choose the replacement failure at the most convenient and effective cost. By using robust Bayesian decisions, the management can use the minimum of the maximum expected predicted cost, representing the worst-case scenario, as the optimal replacement failure. That means the management or company will choose a decision that reduces sensitivity to unpredictable situations. The management can generally determine the optimal replacement failure, which is the 9 failure as the optimal failure for replacement, based on the minimum of the upper-cost function when $C_0 = 5$. Therefore, based on the increase in replacement costs, the number of repairs increases with the expected costs using the worst-case scenario. These results indicate performing more repairs to avoid the high replacement cost ultimately leads to an increase in the expected costs, as presented in Table 5.10.

true values: $\lambda = 3$, $\beta = 2$ and $q = 0.25$				
C_0	$\overline{C}_{\tau^*}$	$\underline{C}_{\tau^*}$	$\underline{n^*}$	$\overline{n^*}$
1.5	3.2924	3.1248	3	3
5	5.2538	4.9986	8	9
10	6.9539	6.6213	15	15
20	9.3442	8.9036	27	27

Table 5.10: Upper and lower predictive expected total cost with replacement failure

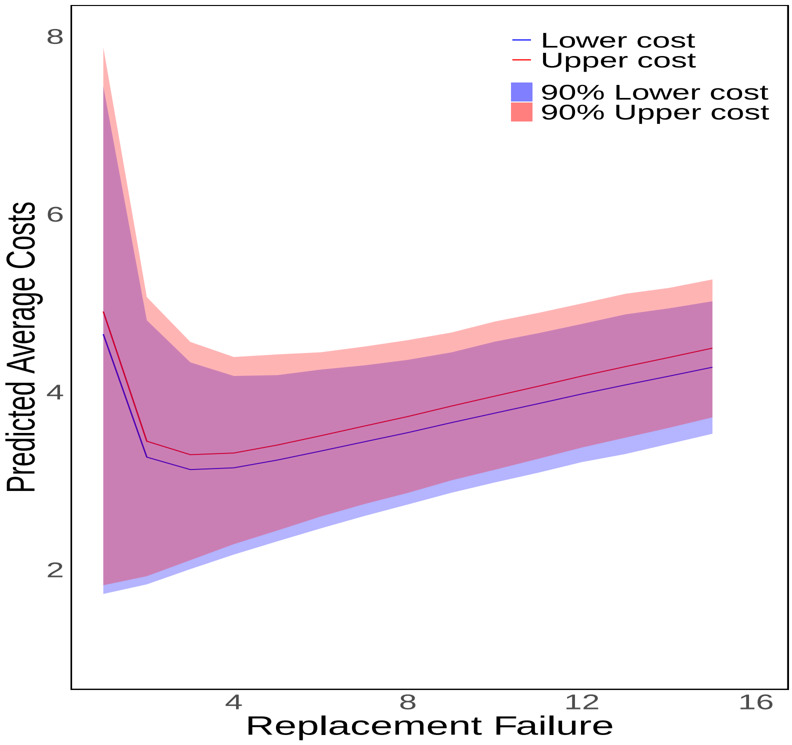


Figure 5.9: upper and lower predictive total cost and replacement failure when $C_0 = 1.5$

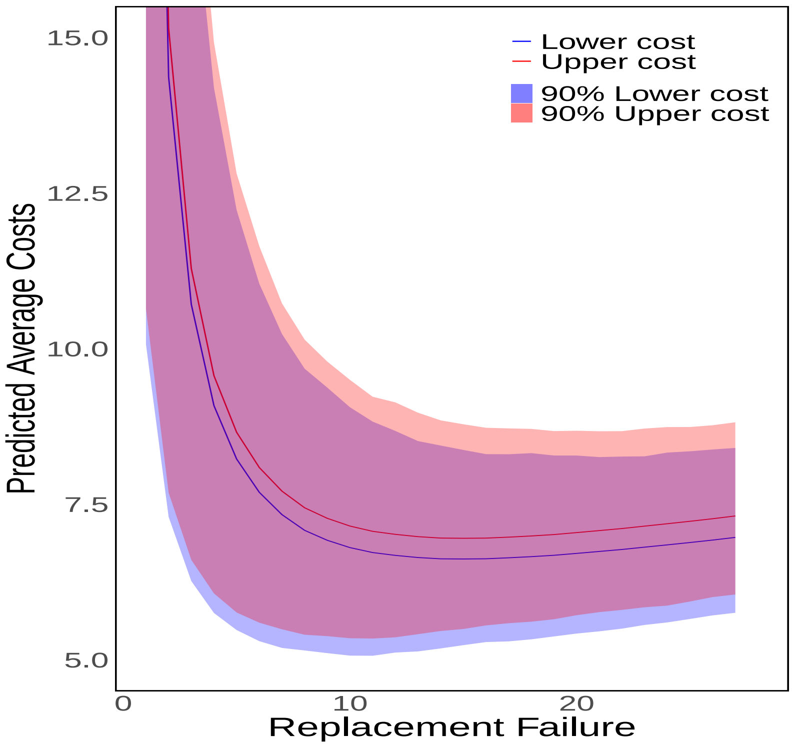


Figure 5.10: upper and lower predictive total cost and replacement failure when $C_0 = 10$

5.4 Concluding remarks

In this chapter, we have presented an application of the Kijima type I mode in order to help decision-makers using Bayesian and robust Bayesian inference with decision theory. To the best of our knowledge, this is a novel application of the Kijima type one virtual age model. To conduct this analysis, we assigned a cost function for decision-makers concerning the replacement policy.

We introduced two replacement policies depending on the time and number of repairs. Based on different replacement costs, the results indicate that when the replacement cost increases, the expected cost and the optimal replacement time will increase. Additionally, with varying repair efficiency values, the outcomes show that the expected cost increases, while the expected number of repairs and replacement time decreases when the repair efficiency parameter is close to 1 or minimal repair.

The robust Bayesian analysis leads to the upper and lower bounds for the optimal replacement time and optimal replacement failure of the system based on the lower and upper optimal expected total cost. This cost boundary allows management to choose the optimal replacement time at the most convenient and effective cost. By using robust Bayesian decisions, the management can use the minimum of the maximum expected predicted cost, representing the worst-case scenario, as the optimal replacement failure. That means the management or company will choose a decision that reduces sensitivity to unpredictable situations. Therefore, to make a robust decision, we use the Γ-minimax criterion, which corresponds to making the most pessimistic decision.

Chapter 6

Conclusions

This chapter summarizes the main results of this thesis and provides some suggestions for related future research. We presented a new method using the posterior predictive empirical reliability function to predict the total system reliability until the time needed to replace it when the number of failures is known. We also presented a robust Bayesian statistical inference development to reflect the uncertainty of prior distribution on posterior predictive system reliability. We further presented an important study on these methods to support management decisions on system replacement.

In Chapter 3, we used Bayesian analyses of the Kijima type I model with Weibull distribution to make inferences as an alternative method to frequentist techniques. We developed an analytical method using the Bayesian method for the Kijima type I model parameters, and another method used MCMC. Also, we develop a Bayesian method for the Kijima type I using the posterior predictive reliability to predict system reliability after each repair and the posterior predictive empirical reliability function with uncertainty to predict the total system reliability until the time needed to replace it when the number of failures is given. We then conducted a simulation to investigate the performance of the presented methods. The analysis of simulations was conducted on different sample sizes, considering both high and low-quality repair efficiency as well as prior selection. The results obtained from the analytical method with different prior choices are compared with MLE. The results from the simulation study showed that when the prior choice is accurate for the parameters, the estimated results increased the accuracy better than MLE in terms of the closest value to the true values. Moreover, we analysed the predictive system reliability function after each failure and the total time until failure given the number of repairs based on high-quality repair efficiency and lower-quality repair efficiency data. We found in high-quality repair, the system's predictive reliability increases after each repair action, indicating that the repair actions have restored the system's functionality to a level comparable to or nearly equivalent to a new one. However, in low-quality repair, the system's predictive reliability hardly increases after each repair action, which means that the repair actions have restored the system's functionality to a slightly better level than before failure. Further, when the repair efficiency has high quality, the predictive system reliability function of total time until failure given the number of repairs drops to zero faster when we have a small number of repairs, which means the system generally maintains an increase in the probability of reliability over time when the number of repairs is increased. However, when the quality of repair efficiency is low, the predictive system reliability function of total time until failure, given the number of repairs, drops to zero with a slight change when the number of repairs is increased. That means that the system fails fast, and the system functionality generally stays similar over time.

In Chapter 4, we presented a robust Bayesian statistical inference method to integrate prior uncertainties for the Kijima type I model using the Weibull distribution. The method includes robustness based on the imprecision, which was evaluated using the upper and lower empirical posterior predictive reliability functions, which provide robustness against the model's prior assumptions by introducing a set of prior. We introduced a set of prior distributions for the repair efficiency parameter q, leaving the other parameters fixed, and we also introduced a set of prior distributions for the repair efficiency parameter q and the scale parameter λ. Simulations were conducted on different sample sizes, considering both high and low-quality repair efficiency. We evaluated the robustness against the uncertainty of prior by employing the upper and lower empirical posterior predictive reliability functions using the MCMC method in the context of different prior sets. The results have shown that the imprecision in Kijima type I's lower and upper predictive empirical reliability function decreases while the robustness increases when we have more data. Also, these results increase confidence to infer that any prior in $\mathcal{M}^{(0)}$ can be chosen since any prior from $\mathcal{M}^{(0)}$ yields similar results within the set $\mathcal{M}^{(n)}$.

In Chapter 5, we illustrated the use of the developed methods. We focused on making a decision of optimal replacement under Kijima's type I model based on two policies with cost functions. We illustrated how Bayesian and robust Bayesian analysis can be applied to infer the optimal replacement and the expected total cost. We calculated the expected total cost using the Bayesian method. The results show that the expected cost increases while the expected number of repairs and replacement time decreases when the repair efficiency decreases. Thus, high-quality repair efficiency that restores systems to approximately new states is important for minimizing expected costs and extending the lifetime of the system. Also, the results indicate that as replacement costs increase, the expected cost and optimal replacement time will also increase because as the replacement cost increases, it is usually recommended to do more repairs to avoid the high cost of the replacement, which leads to an increase in the expected cost and replacement time. In robust Bayesian analysis, we use the Γ-minimax criterion to make a robust decision, which corresponds to making the most pessimistic decision since we have a set of expected total costs, which helps the management to reduce sensitivity to unpredictable situations.

There are many interesting and challenging topics based on the work presented in this thesis. In this thesis, we employed the Kijima type I model with the Weibull distribution. Future studies may consider alternative virtual age models, such as Kijima's type II model [39] and the Arithmetic Reduction of Age model [21]. Furthermore, using these methods to evaluate the system performance using both corrective and preventive maintenance data is a more challenging topic. Because of the complexity of these data, it will be more challenging to model the reliability of the system.

In robust Bayesian inferences, we introduced a set of prior distributions for the repair effectiveness parameter q, as well as for both the repair effectiveness parameter q and the scale parameter λ. Also, we introduced the concept of robustness inferences by defining the range between upper and lower empirical reliability functions. Another interesting challenge for future research involves introducing a set of priors for the repair effectiveness parameter q, the scale parameter λ, and the shape parameter β together. Also, robustness inferences might be studied more in the future by defining another robustness criterion, which allows for a better analysis of robustness against the uncertainty of priors.

Appendix A

Trace and histogram plots of MCMC simulation study

This Appendix presents the diagnostics for each parameter λ, β and q. The histogram plots and trace plots for each parameter for the experimental analysis in Section 3.3.2 are given.

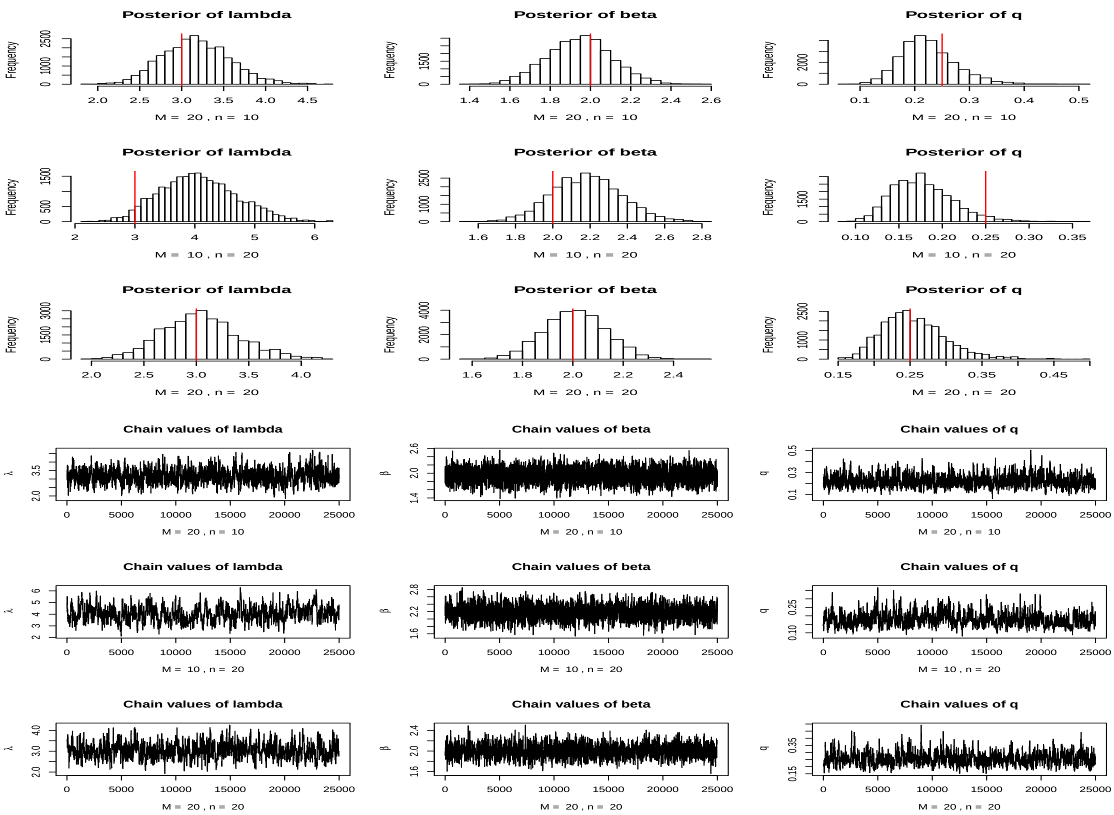


Figure A.1: Histogram and trace plot of generated draws of λ, β and q for Case 1 when $q = 0.25$.

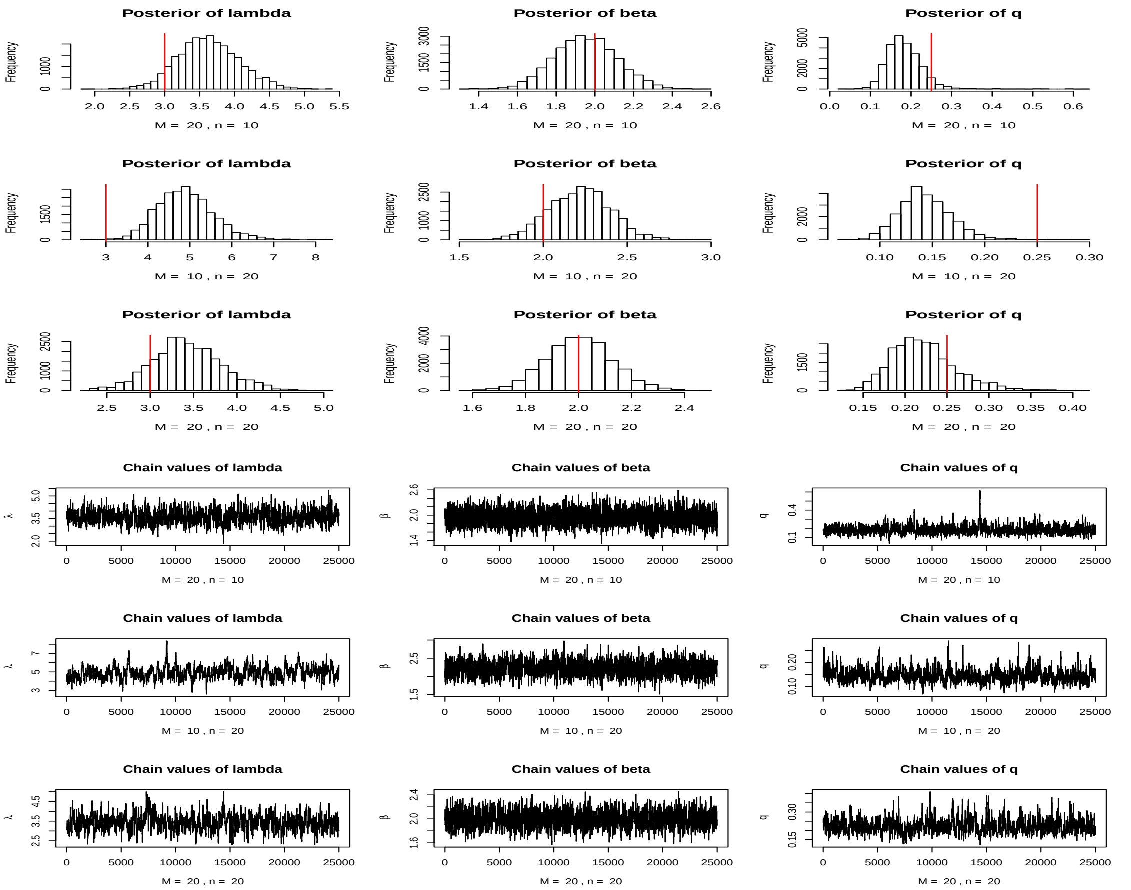
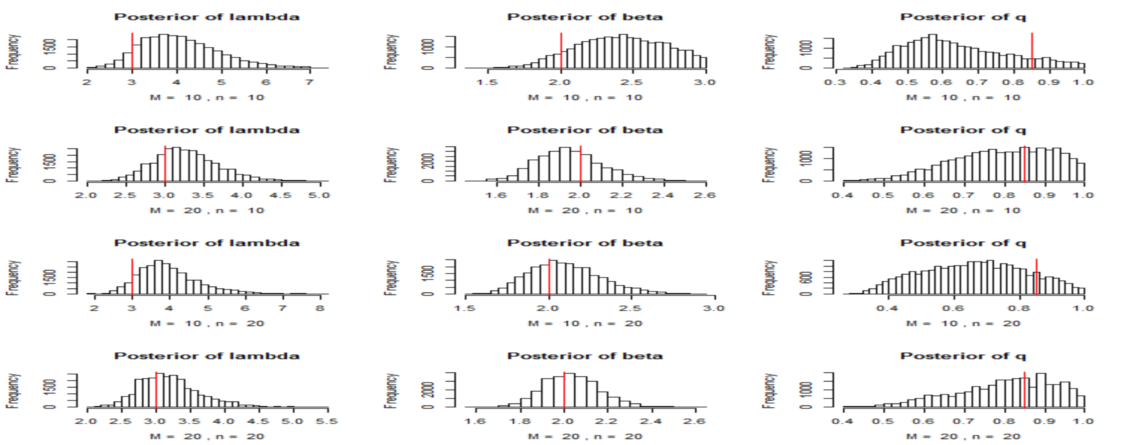


Figure A.2: Histogram and trace plot of generated draws of λ, β and q for Case 2 when $q = 0.25$..



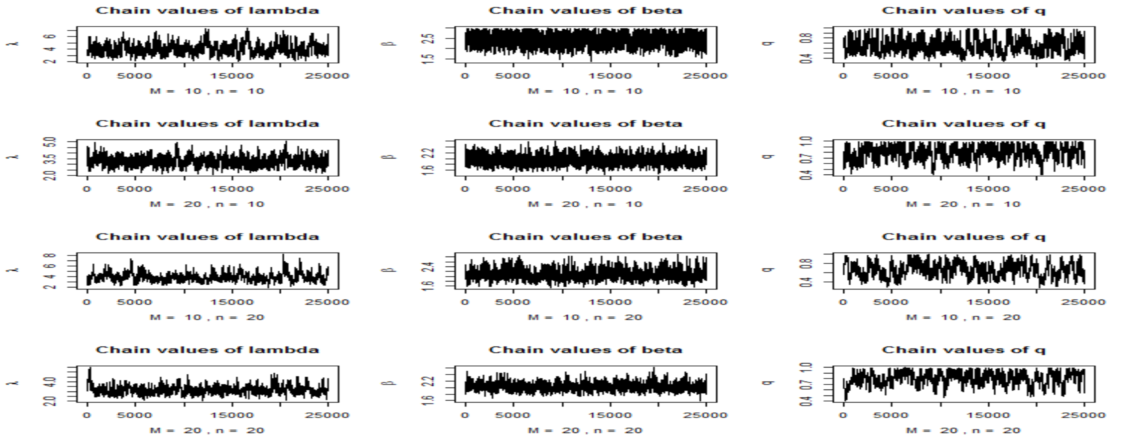


Figure A.3: Histogram and trace plot of generated draws of λ, β and q for Case 1 when $q = 0.85$..

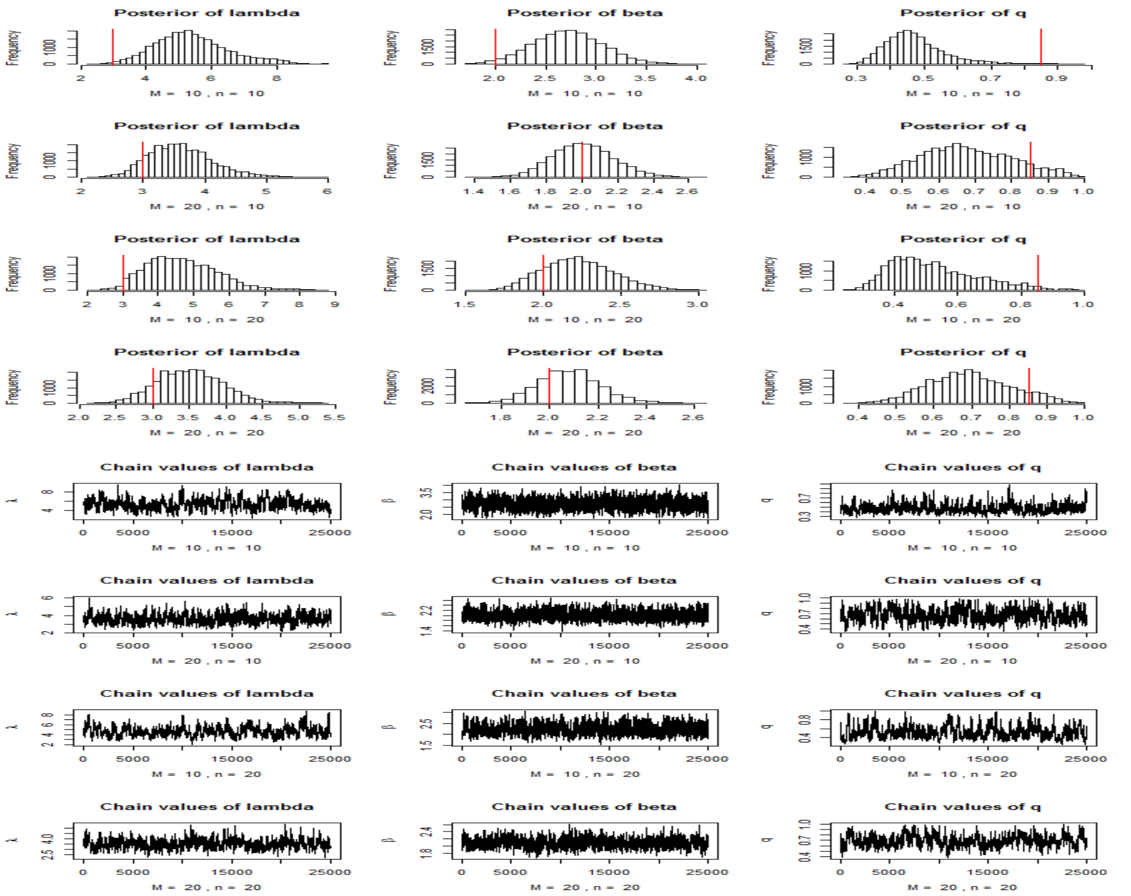


Figure A.4: Histogram and trace plot of generated draws of λ, β and q for Case 2 when $q = 0.85$..

Appendix B

Posterior predictive distribution and reliability function corresponding to continuous priors

B.1 Posterior predictive distribution

From Equation (2.3.12) and the posterior distribution in Equation (3.3.29), the posterior predictive distribution is

$$\begin{aligned} f(x_i|x_{i-1}, D_m) &= \int_0^1 \int_{\beta_1}^{\beta_2} \int_0^\infty f(x_i|v_{i-1}) g(\lambda, \beta, q|D)\, d\lambda d\beta dq \\ &= A \int_0^1 \int_{\beta_1}^{\beta_2} \int_0^\infty \lambda\beta(x_i + q\sum_{j=0}^{i-1} x_j)^{\beta-1} \exp\left(-\lambda\left[(x_i + q\sum_{j=0}^{i-1} x_j)^\beta - (q\sum_{j=0}^{i-1} x_j)^\beta\right]\right) \\ &\quad \lambda^N \beta^N u \exp(-\lambda r) \lambda^{a-1} \exp(-\lambda b) q^{c-1} (1-q)^{d-1} d\lambda d\beta dq \end{aligned} \tag{B.1.1}$$

where A is the normalising constant that is given in Equation (B.1.2).

$$
\begin{aligned}
A^{-1} &= \int_0^1 \int_{\beta_1}^{\beta_2} \int_0^{\infty} g(\lambda, \beta, q|D)\, d\lambda d\beta dq \\
&= \int_0^1 \int_{\beta_1}^{\beta_2} \beta^N u q^{c-1}(1-q)^{d-1} \int_0^{\infty} \lambda^{N+a-1} \exp\left[-\lambda\left(r+b\right)\right] d\lambda d\beta dq \\
&= \int_0^1 \int_{\beta_1}^{\beta_2} \beta^N u q^{c-1}(1-q)^{d-1} \frac{\Gamma(N+a)}{\left[r+b\right]^{N+a}} d\beta dq
\end{aligned}
\tag{B.1.2}
$$

The posterior predictive distribution cannot be solved analytically, so a numerical method is needed to solve this problem.

B.2 Reliability function with fixed λ and β

This Appendix presents the complex posterior predictive empirical reliability function, which is discussed in Section 4.3.

$$
\begin{aligned}
R_{T_i}(x_i|x_{i-1}, D_m) &= \int_0^1 R(x_i|v_{i-1}) g(\lambda, \beta, q|D)\, dq \\
&= A \int_0^1 \exp\left[-\lambda\left((x_i + q\sum_{j=0}^{i-1} x_j)^{\beta} - (q\sum_{j=0}^{i-1} x_j)^{\beta}\right)\right] \lambda^N \times \\
&\qquad \beta^N u \exp(-\lambda r) q^{c-1}(1-q)^{d-1} dq
\end{aligned}
\tag{B.2.3}
$$

where r and u are given in Equation (3.3.30) and A is the normalising constant that is

$$
\begin{aligned}
A^{-1} &= \int_0^1 g(\lambda, \beta, q|D)\, dq \\
&= \int_0^1 \lambda^N \beta^N u \exp(-\lambda r) q^{c-1}(1-q)^{d-1}
\end{aligned}
\tag{B.2.4}
$$

The predictive posterior reliability cannot be solved analytically, so a numerical method is needed to solve this problem.

B.3 Reliability function with fixed β

This Appendix presents the complex posterior predictive empirical reliability function, which is discussed in Section 4.4.

$$
\begin{aligned}
R_{T_i}(x_i|x_{i-1}, D_m) &= \int_0^1 \int_0^\infty R(x_i|v_{i-1}) g(\lambda, \beta, q|D)\, d\lambda dq \\
&= A \int_0^1 \int_0^\infty \exp\left[-\lambda \left((x_i + q\sum_{j=0}^{i-1} x_j)^\beta - (q\sum_{j=0}^{i-1} x_j)^\beta \right)\right] \lambda^N \times \\
&\quad \beta^N u \exp(-\lambda r) \lambda^{a-1} \exp(-\lambda b) q^{c-1} (1-q)^{d-1} d\lambda dq \\
&= A \int_0^1 \beta^N u q^{c-1} (1-q)^{d-1} \int_0^\infty \lambda^{N+a-1} \times \\
&\quad \exp\left[-\lambda \left(r + b + \left((x_i + q\sum_{j=0}^{i-1} x_j)^\beta - (q\sum_{j=0}^{i-1} x_j)^\beta \right)\right)\right] d\lambda dq \\
&= A \int_0^1 \beta^N u q^{c-1} (1-q)^{d-1} \times \\
&\quad \frac{\Gamma(N+a)}{\left[r + b + \left((x_i + q\sum_{j=0}^{i-1} x_j)^\beta - (q\sum_{j=0}^{i-1} x_j)^\beta \right)\right]^{N+a}} dq
\end{aligned}
\tag{B.3.5}
$$

where r and u are given in Equation (3.3.30) and A is the normalising constant that is

$$
\begin{aligned}
A^{-1} &= \int_0^1 \int_0^\infty g(\lambda, \beta, q|D)\, d\lambda dq \\
&= \int_0^1 \beta^N u q^{c-1} (1-q)^{d-1} \int_0^\infty \lambda^{N+a-1} \exp\left[-\lambda\left(r+b\right)\right] d\lambda dq \\
&= \int_0^1 \beta^N u q^{c-1} (1-q)^{d-1} \frac{\Gamma(N+a)}{[r+b]^{N+a}} dq
\end{aligned}
\tag{B.3.6}
$$

The predictive posterior reliability cannot be solved analytically, so a numerical method is needed to solve this problem.

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