Resurgence and modularity in string theory

TREILIS, RUDOLFS

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Rudolfs Treilis

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Abstract

This thesis provides an exploration of the interplay between resurgence analysis and modular invariance in the context of string theory. We focus on two particular applications. Firstly, we analyse a class of modular invariant functions called generalised Eisenstein series that play an important rôle in string perturbation theory at genus-one, as well as in the low energy effective action for Type IIB supergravity. By extending this space of functions to a broader family, we show how a subtle asymptotic analysis via Cheshire-cat resurgence allows us to recover from perturbative data interesting non-perturbative corrections, which can be interpreted as D-$\bar{D}$-brane instantons. These results are based on papers [1,2]. Secondly, we consider a related problem in the study of $\mathcal{N} = 4$ maximally supersymmetric $SU(N)$ Yang-Mills theory. By studying certain integrated four-point correlation functions, we show how the large-N expansion at fixed gauge coupling, τ, of such physical quantities yields modular invariant transseries, and we demonstrate the necessity of including non-perturbative, exponentially suppressed terms at large-N, which holographically originate from (p, q)-string world-sheet instantons. These results are based on paper [3]. The thesis furthermore includes a short overview of resurgence analysis, as well as some relevant aspects of the theory of modular functions such as their representation in terms of Poincaré series and $\mathrm{SL}(2, \mathbb{Z})$ spectral theory.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

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Dedication

Šī disertācija ir veltīta maniem vecākiem Jolantai un Modrim.
Es esmu jums mūžīgi pateicīgs par atbalstu un mīlestību, ko esat man devuši.

CHAPTER 1

Introduction

There is a long history for the presence of the infinite in physical theories. The Greek philosopher Zeno was fascinated by the question of how motion is possible at all - if an arrow is to travel between a marksman and a target, at some point it must reach the middle point between the two. But after that has happened, it must once again reach the middle between the previous point of reference and the target and so on indefinitely, leading to an infinite regression of motions that the arrow must execute. Of course, this paradox was resolved by the introduction of convergent series - a sum of an infinite number of ever decreasing terms can lead to a finite answer. Unfortunately, not all infinities that have appeared in theories of nature since have been so innocuous. There is a hint already in the work of Newton for the complexities that were to come; a point particle is an inherently singular object - its density is infinite and the force of gravitation diverges as two such particles come together. Nevertheless, this posed no problem in the theory of Newton, since it was always assumed that a point particle is a kind of Platonic abstraction, something that should not be taken too seriously in the limit where calculations produce unbounded results.

The paradigm that the singularities produced by mathematical models of physical systems can simply be ignored suffered a massive blow with the advent of quantum field theory. A great deal of confusion was generated once physicists realized that quantum corrections produce seemingly infinite corrections to classical results - a clearly ridiculous conclusion that by then already contradicted a large swathe of experiments. The following decades lead to the theory of renormalization, which at a fundamental level turned out to be an admission of facts very similar to the ones that point particles in classical mechanics already forced us towards. The theory we are working with breaks down at small distances, hence we must impose some cutoff and simply admit we do not know how nature behaves on smaller scales (historically the hard part was showing that all observables can be expressed in terms of a finite number of variables characterizing low energy experiments and the specific value of where you put the cutoff is irrelevant).

Time and time again we see that infinities in physical theories are nothing to be feared of and instead provide valuable information about the range of applicability for the theory. This dissertation looks at the interplay of two ideas - resurgence and modularity - each of which plays an interesting rôle in characterizing infinities present in candidates for fundamental theories of nature (and far beyond)! Resurgence is deeply linked with perturbation theory - the most standard tool for the working physicist that is nearly always the first line of attack when a new problem emerges. Just like the arrow of Zeno, perturbation theory replaces the observable of interest with an infinite series, but, unlike in the thought experiment of the Greek philosopher, the series is actually divergent. So how should we understand this divergence? Is it telling us something about the range of applicability for the theory we are using, like renormalization in quantum field theory or the singularity theorems of Penrose and Hawking in general relativity? The answer turns out to be no - instead the series diverges because we are missing non-perturbative information; because perturbation theory is too crude a method. If you only ever calculate Feynman diagrams in quantum chromodynamics, you will miss the fact that the quarks and gluons that make up the edges of those graphs are actually not present in the world described by the theory and instead become confined. If you only analyse small waves on top of a laminar flow of water, you necessarily miss the fact that once those waves become large enough turbulence kicks in. The divergence of perturbative series is not a fault of any particular physical theory, instead it's a fault of the mathematical methods used to study those theories. Resurgence is a framework that turns this seeming weakness into a strength - in a wide range of situations it turns out that an analysis of the specific way how the perturbative series diverges actually contains information about the non-perturbative effects. Not only can one cure the infinities plaguing perturbation theory, but insight about physics far beyond is revealed in the process. Those tiny waves on top of a calm stream did always know about turbulence!

Modularity is a significantly more abstract idea. It finds its origins in number theory and seems to produce a never ending list of connections between areas of pure mathematics. The rôle it plays in physics is still not fully clear, but models that enjoy it as a symmetry have desirable properties and give us significantly more tools to solve for observables in them exactly. This is a symmetry that is present in many string theories as well as some particularly nice quantum field theories (for example, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 4 dimensions). The function of modularity in string theory relates particularly well to the story about the presence of infinities in theories of nature. In quantum field theories there were degrees of freedom present on very small distance scales whose contribution forced many quantities to diverge. Modularity (among other dualities) in string theory identifies different parts of the moduli space and provides a natural cutoff in this way leading to a UV-finite theory - no renormalization required.

In this thesis we explore how modularity and resurgence interact with each other within a string theory context. They both describe intricate interrelations between perturbative and non-perturbative phenomena and therefore provide stringent constraints on observables of interest. In chapters 2 and 3 we introduce respectively the mathematics of resurgence and modularity. In chapter 4, based on [1], we discuss generalised Eisenstein series in string perturbation theory and apply Cheshire cat resurgence to their Fourier zero-modes. In chapter 5, based on [2], we continue our analysis of generalised Eisenstein series, but now in the context of higher derivative corrections to type IIB supergravity; we use Poincaré series and spectral methods to derive non-perturbative information about them. Finally, in chapter 6, based on [3], we discuss integrated correlators in $\mathcal{N} = 4$ supersymmetric Yang-Mills and formulate a modular invariant version of transseries and resummation, demonstrating the need to include a novel, non-perturbative modular completion.

CHAPTER 2

Theory of resurgence

The modern theory of resurgence finds its origins within the work of Écalle who developed the framework within the context of differential, difference and functional equations [4]. More modern introductions focusing on the physical side include [5–7] and a more mathematical perspective can be found in [8]. In this section we introduce the basic terminology needed for an understanding of resummation and resurgence, but don't touch upon the mathematical details and the more algebraic aspects of the theory.

As was already mentioned in the introduction, most problems in physics are too complicated to be solved exactly and we need to instead rely on a method of approximation - perturbation theory. In order to make use of this, one must start by identifying a small (in some appropriate sense) parameter x that observables in our theory depend upon and expand them in a power series

$$\Phi_P(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{2.1}$$

In this equation $\Phi_P(x)$ describes the perturbative part of an observable of our interest $\mathcal{O}(x)$ and is merely a formal expression, since generically the coefficients have factorial growth $a_k \sim k!$ and the series diverges for any non-zero value of x. Such divergent series were studied by Euler and other famous mathematicians, but their meaning was somewhat unclear and a general sense of distrust about their properties was common. Finally, a definition in terms of asymptotics

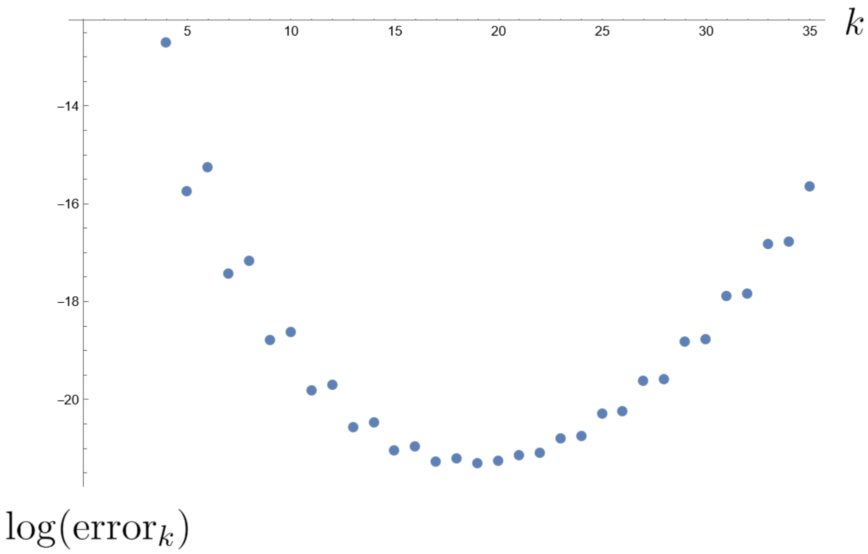

Figure 2.1: A graph showing the relative error in the Stirling approximation if the first k terms are added from the bracket in equation (2.2) for $n = 3$. We see that for this choice of n optimal truncation happens for $k = 19$.

was given by Poincaré who understood that the divergent series really encode the best possible approximation to the function of interest at a fixed order in the parameter x. A classic example is the Stirling approximation for the factorial function, which states

$$n! \sim n^n e^{-n}\sqrt{2\pi n}\Big(1 + \frac{1}{12n} + \frac{1}{288n^2} + ...\Big) \qquad \text{for } n \gg 1. \tag{2.2}$$

As can be seen, the coefficients in the series initially get smaller, but this pattern does not hold. In fact, it can be shown that for large values of the parameter the growth is factorial and the series diverges. Nevertheless, if the series is truncated at a finite order (the precise point of optimal truncation depends on the value of n), it gives a very accurate numerical estimate for the factorial. From figure 2.1 we see that even for the small value of $n = 3$ the agreement is exceptional. Applying this reasoning to the observable from before, we say that $\mathcal{O}(x)$ is *asymptotic* to $\Phi_P(x)$ and write

$$\mathcal{O}(x) \sim \Phi_P(x) = \sum_{k=0}^{\infty} a_k x^k \iff \lim_{x\to 0^+} x^{-n}\Big|\mathcal{O}(x) - \sum_{k=0}^{n} a_k x^k\Big| = 0 \quad \forall n \in \{0, 1, 2, ...\}, \tag{2.3}$$

which is exactly the notion that Poincaré devised. While asymptotic series are in some ways different to their convergent counterparts, they also share a plethora of properties: one may add them, multiply them or take their derivatives and under these operations the asymptotic relationship will be preserved. So we have found that the asymptotic series form a ring with a derivation

$$\Phi_P(x) \in \mathbb{C}[[x]] := \Big\{ \sum_{k=0}^{\infty} a_k x^k | a_k \in \mathbb{C} \Big\} \qquad \text{and} \qquad \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k, \tag{2.4}$$

where addition and multiplication of formal series is defined in the obvious way. Clearly we have made some progress in understanding the nature of perturbation theory - starting with an observable $\mathcal{O}(x)$ we generate a perturbative asymptotic series, which after truncation at a finite order gives us an approximate value for the exact observable. But this answer feels deeply unsatisfactory - had we put in the effort to calculate additional coefficients after the point of optimal truncation, adding them to our original observable would only make the answer worse. Could it really be that perturbation theory generates an infinite tail of meaningless information? Thankfully the answer is a resounding no, but we need to make some additional restrictions on the form of the asymptotic series. While in nearly all instances of physical relevance the series will be divergent, the rate at which the coefficients a_k grow is still bounded, and consequently the theory of resummation that we describe is also restricted to asymptotic series of a particular type. We say that a series $\sum_{k=0}^{\infty} a_k x^k$ has *Gevrey order-1* and belongs to $\mathbb{C}[[x]]_1 \subset \mathbb{C}[[x]]$ if the coefficients have at most factorial growth, more precisely

$$\mathbb{C}[[x]]_1 := \Big\{ \sum_{k=0}^{\infty} a_k x^k \big| |a_k| < BC^k k! \Big\}, \tag{2.5}$$

where $B, C > 0$ are fixed numbers and the inequality holds for all k.

With this restriction in mind, we may define a map that takes a formal Gevrey-1 series and returns a function that is holomorphic in the vicinity of the origin. This map is called the *Borel transform* and its purpose is, of course, to encode all that additional information which was hiding in the large order behaviour of the coefficients a_k. Throughout this thesis we will meet a variety of definitions for the Borel transform and the particular choice is really dictated by the application in mind, but for the simplest instance we use

$$\mathcal{B} : \mathbb{C}[[x]]_1 \to \mathbb{C}\{t\} \qquad \text{with} \qquad \mathcal{B}\Big[\sum_{k=0}^{\infty} a_k x^k\Big](t) := \sum_{k=0}^{\infty} \frac{a_{k+1}}{\Gamma(k+1)} t^k, \tag{2.6}$$

where $\mathbb{C}\{t\}$ is the set of holomorphic germs around the point $t = 0$ (these are simply all power series that converge in some open disc around the origin). Notice that the Borel transformation loses information about the constant coefficient a_0, but we can easily add it back in at a later step of the analysis. Since the original coefficients had a factorial growth and Borel transforming involved a division by $\Gamma(k+1)$, the new power series in t is indeed guaranteed to have a positive radius of convergence. The complex t-plane is often called the Borel plane and the function $\mathcal{B}[\Phi_P](t)$ will have singularities - poles and branch cuts - throughout it. Finally, we are at a stage where we can address our initial goal, the resumming of the asymptotic series $\Phi_P(x)$. We assume that $\mathcal{B}[\Phi_P](t)$ has no natural boundaries and can be endlessly continued along any curve that avoids a discrete set of singularities, in which case we may choose an angle $\theta = \arg(t)$ such that the Borel transform is regular in this direction. Then we define the *directional resummation* of the asymptotic series as

$$\mathcal{S}_\theta[\Phi_P](x) := a_0 + \int_0^{e^{i\theta}\infty} \mathcal{B}[\Phi_P](t) e^{-\frac{t}{x}} \mathrm{d}t, \tag{2.7}$$

which we easily recognise as a Laplace transform along the specified ray. Just to reiterate what we have done: we started with a formal power series $\Phi_P(x)$ that described the asymptotics of a function $\mathcal{O}(x)$ that we were interested in, then we performed a Borel transformation $\mathcal{B}[\Phi_P](t)$, which originally converged in a disc, but was assumed to have a nice analytic continuation and finally we constructed the resummation $\mathcal{S}_\theta[\Phi_P](x)$, which is an analytic function in the variable x defined on a half-plane $|\theta - \arg(x)| < \frac{\pi}{2}$. Furthermore, one can easily show the asymptotic relation

$$\mathcal{S}_\theta[\Phi_P](x) \sim \Phi_P(x) \tag{2.8}$$

holds for any allowed value of θ. It is clearly tempting to identify $\mathcal{O}(x)$ with the resummation $\mathcal{S}_\theta[\Phi_P](x)$, but there is a problem - different choices of θ generically give different functions. Whenever there is a singularity along $\arg(t) = \theta$, the two lateral resummations $\mathcal{S}_{\theta^+}$ above the singular direction and $\mathcal{S}_{\theta^-}$ below it will differ, therefore we have traded an originally divergent

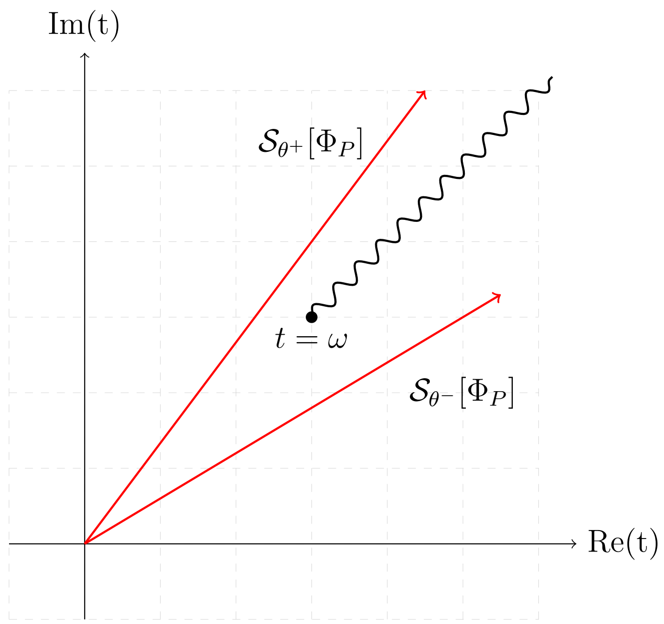

Figure 2.2: There are two possible lateral resummations around a singularity ω in the Borel plane and the discontinuity introduces an exponentially suppressed contribution.

answer $\Phi_P(x)$ for a new one that is ambiguous [1]. Clearly the singular directions are particularly important and deserve a name of their own, they are called *Stokes rays* after G. Stokes who was the first to observe this type of phenomenon in his study of the Airy function (to be discussed in 2.2).

In order to satisfy our desire for an unambiguous resummation scheme, we must analyse the singularity structure of the Borel transformation in more detail. We make a further assumption that the only singularities that appear are simple poles or logarithmic branch cuts so that if $t=\omega$ is a singular point we have the following expansion around it

$$\mathcal{B}[\Phi_P](t) = \frac{b_0}{2\pi(t-\omega)} + \frac{\log(t-\omega)}{2\pi}\mathcal{B}[\Phi_{NP}](t-\omega) + \text{reg}(t-\omega). \tag{2.9}$$

In this formula b_0 in an arbitrary number while $\mathcal{B}[\Phi_{NP}](t-\omega)$ and $\text{reg}(t-\omega)$ are are functions holomorphic in a neighborhood of $t=\omega$. With some hindsight, we have decided to write one of the functions as a Borel transformation of some other asymptotic series $\Phi_{NP}(x) = \sum_{k=0}^{\infty} b_k x^k$. If $\arg(\omega) = \theta$ and ω is the only singularity on the ray $\arg(t) = \theta$, we may calculate the

[1]It is quite amusing to observe the similarity with renormalization, where originally divergent integrals were traded for ones that depend on an arbitrary cut-off, although the analogy does not seem to extend further.

discontinuity in the resummation as we move across it

$$(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-})[\Phi_P](x) = -ie^{-\frac{\omega}{x}}\Big(b_0 + \int_0^{e^{i\theta}\infty} \mathcal{B}[\Phi_{NP}](t)e^{-\frac{t}{x}}\mathrm{d}t\Big) = -ie^{-\frac{\omega}{x}}\mathcal{S}_\theta[\Phi_{NP}](x). \tag{2.10}$$

We assumed that $\mathcal{B}[\Phi_{NP}](t)$ has no singularities along $\arg(t) = \theta$ so that the resummation $\mathcal{S}_\theta$ would be defined. The content of equation (2.10) is fairly clear - the discontinuity as we cross a Stokes ray can be captured by the resummation of another asymptotic series, although multiplied by the function $e^{-\frac{\omega}{x}}$. It is worthwhile to notice that this exponential prefactor vanishes perturbatively, since in the limit $x \to 0^+$ it decays faster than any polynomial, hence the resummation procedure has produced a term that was previously invisible! This motivates us to consider a larger space of formal power series that also includes terms which are exponentially suppressed in $\frac{1}{x}$; these are called *transseries* and capture aspects of asymptotics that go beyond a standard analysis. The original definition of asymptotic series (2.3) can be generalised to terms like $e^{-\frac{\omega}{x}}\Phi_{NP}(x)$, which encode corrections smaller than any power of x. In fact, if we construct the right kind of transseries, the problem of the resummation having an ambiguous value goes away. To see this, consider the one-parameter transseries defined by

$$\Phi(x;\sigma) = \Phi_P(x) + \sigma e^{-\frac{\omega}{x}}\Phi_{NP}(x), \tag{2.11}$$

where $\sigma \in \mathbb{C}$ is for now arbitrary. Since the Borel transform $\mathcal{B}[\Phi_P](t)$ has singularities on the Stokes ray $\arg(t) = \theta$, we still suffer from all of the previously identified problems, but notice that a simultaneous shift $\sigma \to \sigma + i$ as we go from below to above the cut cancels the discontinuity in (2.10) and makes the resummation continuous. So we learn that if we think of the parameter σ as piecewise constant on sectors separated by Stokes rays, but allowed to jump as a discontinuity is encountered, the resummation function can be arranged to give an unambiguous answer. We now give two examples that illustrate this idea and show how solutions to differential equations may be found by starting with asymptotic series.

2.1 Euler's equation

The first example is very simple yet captures some of the core ideas of resurgence. Consider the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = x - y, \tag{2.12}$$

which was first studied by Euler. Since this is a first order linear equation, its solution can be found by standard methods (like multiplying by the integrating factor), but we imagine that we are ignorant of the basic theory of differential equations and seek a solution in the form of a formal power series. A simple exercise establishes that

$$y_P(x) = \sum_{k=0}^{\infty} (-1)^k k! x^{k+1} \tag{2.13}$$

solves (2.12) and shows the expected Gevrey-1 behaviour, establishing that the series diverges for any non-zero x. Of course, this is not a solution in the form of a function $y : \mathbb{R} \to \mathbb{R}$, but we shall find the coefficients encode valuable information that allows us to reconstruct the actual solution to the differential equation. We start by computing the Borel transformation given by

$$\mathcal{B}[y_P](t) = \sum_{k=0}^{\infty} (-1)^k t^k = \frac{1}{1+t}, \tag{2.14}$$

which initially converged for $|t| < 1$, but has a natural extension to $\mathbb{C} \backslash \{-1\}$. It is clear that the only singularity is a simple pole at $t = -1$, so we may study the resurgent properties of the series by use of standard tools. Next we consider the resummation

$$\mathcal{S}_\theta[y_P](x) = \int_0^{e^{i\theta}\infty} \frac{e^{-\frac{t}{x}}}{1+t} \mathrm{d}t, \tag{2.15}$$

valid for any $\theta \neq \pi$, which is an analytic function of x in the half-plane $|\theta - \arg(x)| < \frac{\pi}{2}$. As we vary the angle θ, we obtain a function on the whole of $\mathbb{C} \backslash (-\infty, 0]$, and a simple calculation by differentiating under the integral sign shows that for $x \notin \mathbb{R}_{\leq 0}$ the function $\mathcal{S}_{\arg(x)}[y_P](x)$ solves the differential equation (2.12). Through an analysis of the divergent power series we have found a particular solution to the differential equation! But it is too early to celebrate, we are still missing the one dimensional vector space spanned by the solution to the homogeneous part $e^{\frac{1}{x}}$. From the discussion of resurgence in the first part of this chapter, we know that a transseries should be constructed in order to extract all of the information that the original asymptotic expansion contained. Since the singularity structure of $\mathcal{B}[y_P](t)$ is so simple, the corresponding non-perturbative completion just picks the residue at $t = -1$ of (2.15) and gives

$$y(x;\sigma) = y_P(x) + \sigma e^{\frac{1}{x}}. \tag{2.16}$$

The homogeneous solution has been recovered and the full solution to Euler's equation is finally given by $\mathcal{S}_\theta[y](x;\sigma)$. In this solution the transseries parameter σ plays a a double rôle: the expected free parameter describing the initial condition for a first order differential equation and also a counter to the discontinuity as we cross the Stokes line at $\theta = \pi$ (we need to shift $\sigma \to \sigma - 2\pi i$, if we cross the line anti-clockwise). We see that all of the analytic structure of the solution was encoded in the original series (2.13) and we have ended up with an infinitely-sheeted Riemann surface.

2.2 Airy equation

We now look at another classic although slightly more advanced application of resurgence theory. Indeed, many of the ideas that are central to the resurgence program (like the Stokes phenomenon) were first found through an exploration of the Airy function. Initially introduced by the physicist G.B. Airy who was studying the behaviour of wave optics, the function solves the differential equation

$$\frac{d^2y}{dx^2} = xy. \tag{2.17}$$

We see that compared to Euler's equation (2.12) the order of the equation has increased to two, but it is still linear. Unlike in the case of the Euler equation, the point $x = 0$ is regular and the solution can be extended to an entire function on the whole of $\mathbb{C}$, therefore it is impossible to apply resurgent methods to a series expansion as $x \to 0$. Nevertheless, we proceed by expanding the function around the essential singularity at $x = \infty$, which leads to exactly two possible formal solutions

$$y_\pm(x) = x^{\frac{5}{4}} e^{\pm\frac{2}{3}x^{3/2}} \Phi_\pm(x^{-\frac{3}{2}}) \qquad \text{with} \qquad \Phi_\pm(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k-\frac{1}{6})\Gamma(k-\frac{5}{6})}{\Gamma(k)} \Big(\pm\frac{3}{4}z\Big)^k. \tag{2.18}$$

Once again we find that a factorially divergent piece $\Phi_\pm(x^{-\frac{3}{2}})$ is present, hence we will need to define a resummation. In contrast to the Euler equation, both solutions have an exponential prefactor $e^{\pm\frac{2}{3}x^{3/2}}$ that multiplies the asymptotic part, so there is actually no perturbative sector. Nevertheless, the two terms may be assembled into a formal transseries solution to (2.17) as

$$y(x;\sigma_+,\sigma_-) = \sigma_+ y_+(x) + \sigma_- y_-(x). \tag{2.19}$$

Here the transseries parameters σ_+ and σ_- play the rôle of the expected two initial conditions to a second order differential equation. We next compute the Borel transform of the formal series, which is given by a hypergeometric function

$$\mathcal{B}[\Phi_\pm](t) = \pm\frac{3\pi}{2}\,{}_2F_1\Big(\frac{1}{6},\frac{5}{6};1\big|\pm\frac{3t}{4}\Big). \tag{2.20}$$

It is a well-known fact that the function ${}_2F_1(a,b;c|z)$ has a branch cut starting at $z=1$, hence we conclude that $\mathcal{B}[\Phi_\pm](t)$ have branch cuts starting respectively at $t=\pm\frac{4}{3}$ and consequently the directions $\theta=0,\pi$ are Stokes rays. From equation (2.18) we see that the constant term is missing, hence the directional resummation has the simple form

$$\mathcal{S}_\theta[\Phi_\pm](x) = \int_0^{e^{i\theta}\infty}\mathcal{B}[\Phi_\pm](t)e^{-tx^{3/2}}\mathrm{d}t \qquad \theta\neq 0,\pi; \qquad \big|\theta+\frac{3}{2}\arg(x)\big|<\frac{\pi}{2}. \tag{2.21}$$

Since in this example we have concrete expressions for the Borel transformations, we may explicitly calculate the discontinuity as one moves across a Stokes ray. From standard properties of the hypergeometric function (such as equation (4.43)), we find the connection formulae

$$\begin{aligned}(\mathcal{S}_{0^+}-\mathcal{S}_{0^-})[y_+](x) &= -i\mathcal{S}_0[y_-](x)\,,\\ (\mathcal{S}_{\pi^+}-\mathcal{S}_{\pi^-})[y_-](x) &= i\mathcal{S}_\pi[y_+](x)\,.\end{aligned} \tag{2.22}$$

We see that all of the analytic information that was encoded in the singularity structure of the Borel transformations (2.20) has been transformed into a set of algebraic equations (2.22) that are closed among themselves (since no other formal solutions are generated in addition to the $y_+(x)$ and $y_-(x)$ we began with). As in the case of the Euler equation, it is possible to check that $\mathcal{S}_\theta[y(\sigma_+,\sigma_-)](x)$ does in fact solve the Airy equation as long as the angle θ satisfies

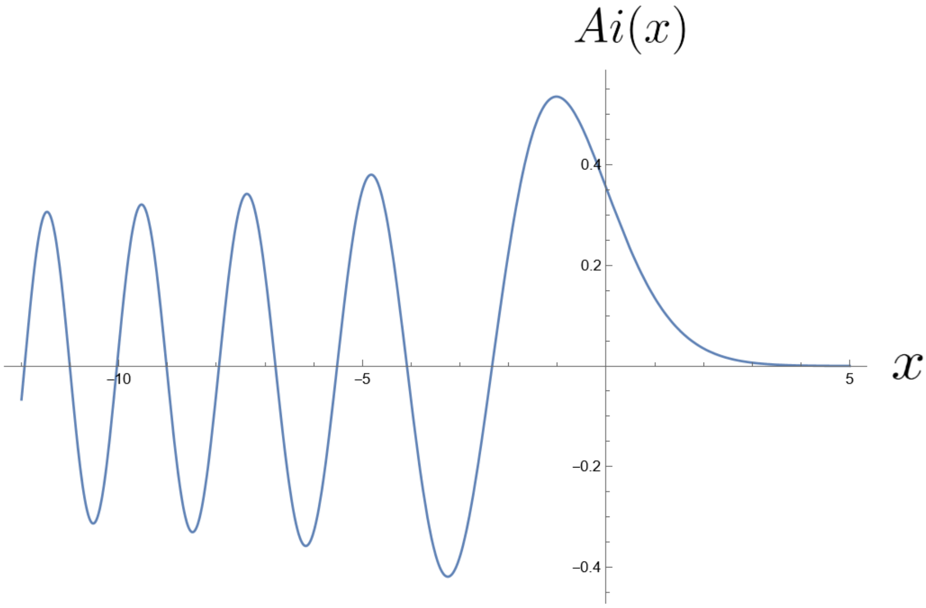

Figure 2.3: A plot of the Airy function clearly demonstrating the oscillatory behavior for $x < 0$ and exponential decay for $x > 0$. Both σ_+ and σ_- are turned on for negative values of x, while only σ_- plays a rôle in the asymptotics for positive x.

the constraint in (2.21) and additionally one shifts $\sigma_- \to \sigma_- + i\sigma_+$ whenever the Stokes ray along $\theta = 0$ is crossed and $\sigma_+ \to \sigma_+ - i\sigma_-$ whenever the Stokes ray at $\theta = \pi$ is crossed (in both cases anti-clockwise). This tells us that the appropriate asymptotics for the Airy function, which are encoded in the transseries ansatz, change as we vary the phase of the function $\arg(x)$ and additionally explains how a single analytic function can be oscillatory in one region of the complex plane, while exponentially decaying in another. The perplexing asymptotic behaviour, originally explained by Stokes using an early version of Picard-Lefschetz theory, can be seen in figure 2.3. Additionally, one may check that performing a full rotation on the resummation in the original x-plane via $x \to e^{2\pi i}x$ has trivial monodromy, if the transseries parameters are shifted appropriately (note that a 2π rotation in x will correspond to a 3π rotation in t). This is a manifestation of the fact that the Airy function is entire.

CHAPTER 3

The modular group

The modular group $\mathrm{SL}(2,\mathbb{Z})$ has played a very prominent rôle in mathematics since the 19$^{\text{th}}$ century, rearing its head in areas as disparate as number theory, complex analysis, finite group theory and representation theory. There are many classic texts to get acquainted with the theory from a mathematical side [9–11], but a recent set of lecture notes emphasizing the physics has also become available [12]. The purpose of this section is to give an overview of the theory of modular forms and functions in order to highlight the parts relevant for an understanding of the physics in this thesis.

The modular group consists of 2×2 matrices with integer entries and unit determinant

$$\mathrm{SL}(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| (a,b,c,d) \in \mathbb{Z}^4, ad-bc=1 \right\}. \tag{3.1}$$

While the group has infinite order, it is generated by just two elements

$$\mathrm{SL}(2,\mathbb{Z}) = \langle S,T | S^2 = (ST)^3 = -1 \rangle \qquad \text{with} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

By this we simply mean that any $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ can be written as $\gamma = \pm T^{a_1} S T^{a_2} S ... S T^{a_n}$ for some integral coefficients $a_i \in \mathbb{Z}$ and any three consecutive ST or TS may be omitted.

The most important observation about the modular group, from which nearly all of its applications arise, is that it possesses an action on the upper complex half-plane (also called the Poincaré plane). It is useful to start by observing that $\mathrm{SL}(2,\mathbb{Z})$ is a discrete subgroup of the Lie group $\mathrm{SL}(2,\mathbb{R})$, which has an action on the upper half-plane via Möbius transformations

$$\mathfrak{H} := \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\} = \mathrm{U}(1)\backslash \mathrm{SL}(2,\mathbb{R}), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \quad \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}. \tag{3.3}$$

It is easy to show that the action is transitive and every point has stabilizer $\mathrm{U}(1)$, hence the first identity follows from the orbit-stabilizer theorem. This way of characterising the upper half-plane has a natural extension to other Lie groups leading to the theory of automorphic forms, which have also found applications in physics [13] and play a central rôle in the famous Langlands program. Of course the modular group also inherits an action on $\mathfrak{H}$ from (3.3), which has the effect of tessellating the plane into a multitude of pieces - each serving as a fundamental domain of the action. To get a better understanding of how this comes about, we note the action of the two generators $S \cdot \tau = -\frac{1}{\tau}$ and $T \cdot \tau = \tau + 1$. Clearly the T generator is just implementing discrete translations along the real axis, but the action of S is more intricate. Irrespective of this, it is easy to find the fundamental domain $\mathcal{F}$ - the action of T allows us to restrict it to the strip $-\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2}$ and the action of S exchanges the interior and exterior of the unit circle, therefore we may pick

$$\mathcal{F} := \mathrm{SL}(2,\mathbb{Z})\backslash \mathfrak{H} = \big\{\tau \in \mathfrak{H} \big| -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}, |\tau| \geq 1\big\} \cup \{\tau \in \mathfrak{H} \big| |\tau| = 1, 0 \leq \mathrm{Re}(\tau) \leq \frac{1}{2}\}. \tag{3.4}$$

A depiction of $\mathcal{F}$ can be seen in Figure 3.1. It is easy to convince oneself that topologically $\mathcal{F}$ is a sphere with a missing point - the asymptotic point at $i\infty$ (in the literature often called the cusp).

Up until now we have focused on $\mathfrak{H}$ as a topological space, but it actually possesses a natural geometry described by an $\mathrm{SL}(2,\mathbb{R})$-invariant metric. If we parametrise $\tau = \tau_1 + i\tau_2$ with $\tau_2 > 0$, we may write this metric as $\mathrm{d}s^2 = \tau_2^{-2}(d\tau_1^2 + d\tau_2^2)$. This is actually the oldest non-Euclidean geometry - the hyperbolic plane, first described by Lobachevsky and Bolyai as a counterexample to the parallel postulate of Euclid's *Elements*. The metric establishes that the geodesics are either straight lines parallel to the imaginary axis or circles whose center lies on the real axis

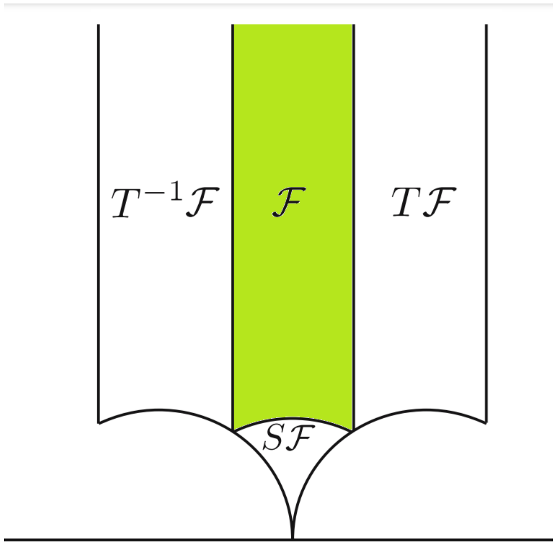

Figure 3.1: The fundamental domain $\mathcal{F}$ and some of its $\mathrm{SL}(2,\mathbb{Z})$ translates.

(observe that the boundary of $\mathcal{F}$ consists entirely of geodesics). One may also measure areas with respect to this metric and, while the fundamental domain is unbounded, it actually has a finite volume given by $\mathrm{Vol}(\mathcal{F}) = \frac{\pi}{3}$. It is also useful to introduce the Laplace-Beltrami operator

$$\Delta = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2), \tag{3.5}$$

which plays a major rôle in the study of modular functions and spectral theory - topics we describe later.

3.1 Modular functions and forms

We found that there is an action $\mathrm{SL}(2,\mathbb{Z}) : \mathfrak{H} \to \mathfrak{H}$, so one naturally wonders what type of functions have nice transformation properties under it? While it might seem simplest to consider function invariant under the action (and these will play a major rôle in what follows), it turns out to be useful to retain an automorphy factor and define a transformation property as follows

$$f(\gamma\cdot\tau) = (c\tau+d)^k f(\tau) \qquad \text{for} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \tag{3.6}$$

where the integer $k \in \mathbb{Z}$ is called the weight. If we choose $\gamma = -\mathbf{1}_2$, we instantly establish that a non-zero function requires k to be even. Similarly, by choosing the generator of translations

$\gamma = T$ and substituting into equation (3.6), we find that all functions of this type must be periodic $f(T \cdot \tau) = f(\tau + 1) = f(\tau)$. This implies that they must have a Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n(\tau_2) q^n \qquad \text{with} \qquad q = e^{2\pi i \tau}, \tag{3.7}$$

where the variable q is constrained to live in the interior of the unit circle. We did not assume that the function $f : \mathfrak{H} \to \mathbb{C}$ is holomorphic, which is reflected in the coefficients $a_n(\tau_2)$ depending on the imaginary part. Had we assumed holomorphicity, they would have to be constants. The coefficient functions $a_n(\tau_2)$ will generically have polynomial growth at infinity with subleading exponentially suppressed corrections. We are usually interested in functions that have controlled growth at the cusp $\tau \to i\infty$, which corresponds to the limit $q \to 0$, therefore we introduce the terminology

- f is a modular function if only a finite number of coefficients $a_n(\tau_2)$ with $n < 0$ are non-zero. If the function is holomorphic, this means that in the q-plane the function has a finite order pole at the origin;

- f is a modular form if all coefficients $a_n(\tau_2)$ vanish for $n < 0$. This means that the function approaches the coefficient $a_0(\tau_2)$ as $\tau \to i\infty$, which is independent of τ_1;

- f is a cusp form if it is a modular form and $a_0(\tau_2) = 0$, therefore cusp forms go to 0 at the cusp.

While it is possible to consider even more general modular objects, we shall find that for the topics covered in this thesis the definition given suffice.

3.1.1 Holomorphic modular forms

The simplest examples of modular forms are holomorphic ones at positive weight $k > 0$. A complete classification of such forms is known and is fully described by a class of functions called holomorphic Eisenstein series. For $k > 2$ they are defined by the sum

$$G_k(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m + n\tau)^k}, \tag{3.8}$$

which is a modular form of weight k and vanishes for k odd (as was expected, since there are no forms of odd weight). In the case $k = 2$ the sum is conditionally convergent and can be used to define a holomorphic function, but it is not a modular form (although it still has calculable transformation properties under modular transformations). The Eisenstein series have a well-known Fourier expansion given by

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{\Gamma(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{3.9}$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ is a divisor-sum function. We have now found an example of a modular form of every even weight $k > 2$, but are there any more? Notice that if we define $\mathcal{M}_k$ to be the space of holomorphic modular forms of weight k, then the space of all forms $\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k$ has a ring structure with $\mathcal{M}_k \mathcal{M}_l \subset \mathcal{M}_{k+l}$. Division in this space is not generically allowed, since if one divides by a cusp form the result will not be holomorphic at $i\infty$. For example, at weight 8 we find two modular forms $G_4(\tau)^2$ and $G_8(\tau)$, but they turn out to be constant multiples of each other and $\dim(\mathcal{M}_8) = 1$. In fact, it is possible to prove that the whole ring of holomorphic modular forms is freely generated by the Eisenstein series of weight 4 and 6 so that

$$\mathcal{M} \cong \mathbb{C}[G_4, G_6]. \tag{3.10}$$

By this we simply mean that every modular form can be written as a sum of terms $G_4(\tau)^a G_6(\tau)^b$ and $\dim(\mathcal{M}_\mathrm{k})$ equals the number of distinct ways one can write $k = 4a + 6b$ for non-negative integers a and b. This is a very powerful result, since, in particular, it implies $\dim(\mathcal{M}_\mathrm{k}) < \infty$ for any k. In turn that establishes we can prove that two modular forms are the same by simply checking a finite number of coefficients in their Fourier expansion. A comparison of the constant Fourier coefficient in (3.9) gives us the identity $G_8(\tau) = \frac{3}{7} G_4(\tau)^2$, while equating all of the other Fourier modes then gives the non-trivial arithmetic formula

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(n-m)\sigma_3(m). \tag{3.11}$$

We find that the constraints coming from modularity are so restrictive that surprising results are naturally derived through a simple paucity of the allowed objects. This is a theme that will be seen many more times throughout this thesis.

3.1.2 Maass forms

In the previous section we gave a complete classification of holomorphic modular forms and showed that the defining transformation property (3.6) is very restrictive. We now turn our attention to non-holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$, but impose additional constraints. Remember that the hyperbolic plane admits a natural Laplace operator $\Delta = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$, which in particular preserves the modular invariance property (such that if $f(\tau)$ satisfies the functional equation $f(\gamma \cdot \tau) = f(\tau)$ then so will $\Delta f(\tau)$). This motivates us to define a Maass form $f(\tau)$ as a modular-invariant function that additionally is an eigenvector of the Laplacian

$$\Delta f(\tau) = s(s-1)f(\tau). \tag{3.12}$$

Here $s \in \mathbb{C}$ is arbitrary and we also require that the function has at most polynomial growth at the cusp. The choice to label the eigenvalue as $s(s-1)$ might initially seem a little odd, but is standard in the literature. Clearly equation (3.12) is left invariant by the involution $s \to 1-s$, therefore there is some ambiguity in the value of s and, if one wishes, it is possible to restrict to the range $\mathrm{Re}(s) \geq \frac{1}{2}$. Much like in the case of holomorphic forms, a lot is known about the space of allowed Maass forms through an analysis of the modular transformation property. The simplest examples are once again called Eisenstein series, but this time they are non-holomorphic functions that are modular invariant

$$E^*(s;\tau) = \frac{1}{2}\Gamma(s) \sum_{(m,n)\neq(0,0)} \frac{(\tau_2/\pi)^s}{|m+n\tau|^{2s}}. \tag{3.13}$$

This definition converges absolutely for $\mathrm{Re}(s) > 1$, but a meromorphic continuation exists on the whole $\mathbb{C}$. To see this, we note the formula for the Fourier expansion of the non-holomorphic Eisenstein series

$$E^*(s;\tau) = \xi(2s)\tau_2^s + \xi(2s-1)\tau_2^{1-s} + 4\sum_{k=1}^{\infty} \cos(2\pi k\tau_1) \frac{\sigma_{2s-1}(k)}{k^{s-\frac{1}{2}}} \sqrt{\tau_2} K_{s-\frac{1}{2}}(2\pi k\tau_2), \tag{3.14}$$

where $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \xi(1-s)$ is the completed zeta function and $K_s(z)$ is a modified Bessel function of the second kind. The Fourier series converges for any $s \neq 0, 1$ and satisfies the reflection formula $E^*(s;\tau) = E^*(1-s;\tau)$. Furthermore, the poles at the points $s = 0, 1$ have τ-

independent residues given by $\text{res}_{s=1}E^*(s;\tau) = -\text{res}_{s=0}E^*(s;\tau) = \frac{1}{2}$. Of course, as required by the definition of a Maass form, the Eisenstein series also satisfy a Laplace eigenvalue equation

$$\Delta E^*(s;\tau) = s(s-1)E^*(s;\tau). \tag{3.15}$$

If we denote the space of Maass forms with eigenvalue $s(s-1)$ by $\mathcal{N}(s) = \mathcal{N}(1-s)$, we have found an element in this space for every s: when $s \neq 0,1$ these are simply the Eisenstein series $E^*(s;\tau)$, while for the special values $s = 0,1$ the eigenvalue vanishes and the constant functions satisfy the equation. It is natural to ask if the space really is 1-dimensional for every s, or have we missed some functions? At generic values of s there are indeed no more modular invariant solutions, but for $\text{Re}(s) = \frac{1}{2}$ there exists a discrete set of points for which $\mathcal{N}(s) = \mathbb{C}E^*(s;\tau) \bigoplus \mathcal{S}(s)$ with $\mathcal{S}(s)$ the space of non-holomorphic cusp forms. These are quite different beasts to their Eisenstein series cousins - similarly to them they satisfy an eigenvalue equation

$$\Delta\phi_n(\tau) = \mu_n\phi_n(\tau), \qquad \text{where} \qquad \mu_n = -\Big(\frac{1}{4} + t_n^2\Big), \quad 0 < t_1 < t_2 < \dots, \tag{3.16}$$

with the spectral parameters t_n, specifying the eigenvalue μ_n, forming an infinite and unbounded set of sporadic positive numbers. Although, as the name indicates, they have a vanishing Fourier zero-mode and admit the series expansion

$$\phi_n(\tau) = \sum_{k\neq 0} h_k^{(n)} \tau_2^{\frac{1}{2}} K_{it_n}(2\pi|k|\tau_2)e^{2\pi i k\tau_1}\,, \tag{3.17}$$

with the Fourier coefficients $h_k^{(n)}$ once more a set of sporadic real numbers. Given the outer automorphism of order two $\tau \to -\bar{\tau}$, we can divide the cusp forms into even ones, i.e. $\phi_n(\tau) = \phi_n(-\bar{\tau})$, and odd ones, i.e. $\phi_n(\tau) = -\phi_n(-\bar{\tau})$. These objects are quite mysterious and have links with quantum chaos [14]. While it is known that an infinite number of cusp forms exist (by using the Selberg trace formula), their exact distribution and other properties are currently unknown. Nevertheless, the interested reader can find both spectral parameters and Fourier coefficients for various even/odd Maass cusp forms on the L-functions and modular forms database (LMFDB) [15]. This concludes our discussion about the classification of Maass forms and we once again see how the modular group severely restricts the allowed space of functions.

3.2 Poincaré series representation

Now that we have given some examples of modular functions and forms, we turn to a more systematic analysis of their representations. If we wish to produce a function that is invariant under the action of some group, a simple method is to sum over the images of said action (for example, $\sum_{n\in\mathbb{Z}} f(x+n)$ has period 1 if the sum converges absolutely, i.e. it is invariant under the additive group $\mathbb{Z}$). In the case of the modular group and its action on $\mathfrak{H}$ this method is called constructing a Poincaré series and is very useful for calculations. We start by defining the Borel subgroup, which stabilises the cusp $\mathrm{B}(\mathbb{Z}) = \mathrm{stab}(\mathrm{i}\infty)$ and has the explicit representation

$$\mathrm{B}(\mathbb{Z}) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\} \subset \mathrm{SL}(2,\mathbb{Z}). \tag{3.18}$$

Next we choose a periodic function (usually called the seed function) $g : \mathfrak{H} \to \mathbb{C}$ such that $g(\tau) = g(\tau+1)$, therefore one has a well-defined action of the coset $\mathrm{B}(\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{Z})$ on it. Finally, we may construct a function by summing over the orbit of that coset

$$f(\tau) = \sum_{\gamma\in\mathrm{B}(\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{Z})} g(\gamma\cdot\tau), \tag{3.19}$$

which will have our desired modular invariance property as long as the sum converges absolutely. Poincaré series make modular properties manifest, but they also have drawbacks, for example, the Fourier series are usually quite hard to extract. Another benefit is that the seed function is usually simpler than the full modular function, for example, the non-holomorphic Eisenstein series have the fairly elementary Poincaré series

$$E(s;\tau) = \frac{E^*(s;\tau)}{\xi(2s)} = \sum_{\gamma\in\mathrm{B}(\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma\cdot\tau)^s, \qquad \mathrm{Re}(s) > 1, \tag{3.20}$$

where a new normalization for the Eisenstein series was introduced for later convenience. We see that in this case the seed function is simply the imaginary part of the modular parameter, which, as expected, is invariant under $\tau \to \tau + 1$. It is important to observe that the seed function is not unique - obviously one could shift it by the action of any element $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ or even take an infinite sum $\sum_{k=0}^{\infty} a_k g(\gamma_k\cdot\tau)$ with γ_k different matrices in the modular group and $\sum_{k=0}^{\infty} a_k = 1$. The final function might look nothing like the seed we began with even though the Poincaré series are identical.

To illustrate this, we note that yet another Poincaré seed for $E(s;\tau)$ was given in [16, Eq. (3.10)]:

$$\sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \left[\sqrt{|k|\tau_2} K_{s-\frac{1}{2}}(2\pi|k|\tau_2) e^{2\pi i k \tau_1}\right]_\gamma = \frac{\pi^{2s+1/2}\sigma_{2s-1}(k)E(s;\tau)}{2|k|^{s-1}\cos(\pi s)\Gamma(s+1/2)\zeta(2s-1)}\,, \tag{3.21}$$

where the notation $[\cdots]_\gamma$ means that γ acts on all occurrences of τ (and $\bar\tau$) inside the bracket using the fractional linear action (3.3). As we can easily see from (3.14), the seed appearing in the Poincaré sum is given by the generic Fourier non-zero mode of $E(s;\tau)$ and is therefore expected again to be proportional to $E(s;\tau)$.

Finally, we also observe that the method can be applied to modular forms of weight k (i.e. functions that are not quite invariant under modular transformations). If we are given $f_k(\tau)$, a modular form of weight k, we may construct a meromorphic differential form $\mathfrak{f}_k(\tau) = f_k(\tau)(\mathrm{d}\tau)^{\frac{k}{2}}$ that is modular-invariant. To connect this with the example of holomorphic Eisenstein series we saw before, define the differential k-form $\mathfrak{g}_{2k}(\tau) = G_{2k}(\tau)(\mathrm{d}\tau)^k$, which admits the simple Poincaré series

$$\mathfrak{g}_{2k} = 2\zeta(2k) \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{d}(\gamma\cdot\tau)^k. \tag{3.22}$$

3.3 Spectral theory

If we are interested in studying functions $f : \mathfrak{H} \to \mathbb{C}$ that are modular-invariant, a very powerful tool of analysis is spectral theory. The key idea behind spectral theory is to decompose any modular invariant function as a linear combination of "good" basis elements, i.e. normalisable eigenfunctions of the hyperbolic Laplace operator. Since the functions are invariant under modular transformations, we can really think of them as being defined on the fundamental domain $\mathcal{F}$. Of course, the fundamental domain inherits its geometry from that of $\mathfrak{H}$ and furthermore we may consider the Hilbert space of square-integrable functions $L^2(\mathcal{F})$ with respect to the Petersson inner product

$$(f,g) = \int_{\mathcal{F}} f(\tau)\,\overline{g(\tau)}\mathrm{d}\mu, \tag{3.23}$$

where the invariant Haar measure is $\mathrm{d}\mu = \tau_2^{-2}\mathrm{d}\tau_1\mathrm{d}\tau_2$.

Note that for a function f to be an element of $L^2(\mathcal{F})$, its growth at the cusp $\tau_2 \gg 1$ must be at most $|f(\tau)| = O(\tau_2^{\frac{1}{2}})$. In applications of our interest, we will often encounter modular invariant functions f violating such bound, i.e. non-$L^2(\mathcal{F})$ normalisable functions. Although this growth condition seems quite restrictive, spectral analysis methods can be extended from square-integrable functions to a broader class of functions that have moderate growth at the cusp. If a function f has cuspidal growth $|f(\tau)| = O(\tau_2^{\alpha})$ with $\mathrm{Re}(\alpha) > \frac{1}{2}$, we can find a coefficient β such that the new modular invariant combination $f(\tau) - \beta E(\alpha;\tau)$ has a tamer growth at the cusp. More generally, we will be discussing modular invariant functions whose asymptotic expansion at the cusp is controlled by finitely many non-integrable power-like terms $\tau_2^{\alpha_i}$ with $\mathrm{Re}(\alpha_i) > \frac{1}{2}$. Although such functions $f(\tau)$ are not elements of $L^2(\mathcal{F})$, we can find coefficients β_i for which the linear combination

$$f_{\text{new}}(\tau) = f(\tau) - \sum_i \beta_i E(\alpha_i;\tau) \in L^2(\mathcal{F}) \,, \tag{3.24}$$

is L^2-normalisable.

Modulo the caveat just mentioned, we now consider in more detail the Hilbert space $L^2(\mathcal{F})$ with inner product (3.23). One of the main benefits of working with a vector space is that we can always express a generic element in terms of a basis. Furthermore, since the space has a geometric structure and an associated Laplace-Beltrami operator $\Delta : L^2(\mathcal{F}) \to L^2(\mathcal{F})$, which is in particular self-adjoint with respect to the inner product (3.23), it is natural to use the Laplace eigenfunctions as a basis for $L^2(\mathcal{F})$.

The spectrum of the hyperbolic Laplacian is closely related to Maass forms discussed in 3.1.2 and decomposes into three distinct eigenspaces (we refer to [9, 10] for details):

- The constant function $f(\tau) = 1$ is clearly an eigenfunction of Δ with eigenvalue 0, and it is an element of $L^2(\mathcal{F})$, since $(1,1) = \mathrm{Vol}(\mathcal{F}) = \frac{\pi}{3}$ is the volume of the fundamental domain;

- The continuous part of the spectrum is spanned by $E(t;\tau)$ with $\mathrm{Re}(t) = \frac{1}{2}$ and eigenvalue $t(t-1)$ given (3.15);

- The discrete part of the spectrum is spanned by the Maass cusp forms, $\phi_n(\tau)$, with $n \in \mathbb{N}^{>0}$ and eigenvalue $\mu_n = -\left(\frac{1}{4} + t_n^2\right)$ given (3.16).

Once the basis of eigenfunctions for the Laplacian is understood, we are naturally led to consider the Roelcke-Selberg spectral decomposition:

$$f(\tau) = \langle f\rangle + \int_{\mathrm{Re}(t)=\frac{1}{2}} (f,\mathrm{E}_t)\, E(t;\tau)\frac{\mathrm{d}t}{4\pi i} + \sum_{n=1}^{\infty}(f,\phi_n)\phi_n(\tau)\,, \tag{3.25}$$

for a generic $f \in L^2(\mathcal{F})$ (inside the inner product we use the short-hand notation $\mathrm{E}_t := E(t;\tau)$).

The first term is simply $\langle f\rangle = \int_{\mathcal{F}} f(\tau)\,\mathrm{d}\mu$, which can be understood as the average of the function over the fundamental domain, or equivalently as the spectral overlap with the constant function $\langle f\rangle = (f,1)$. The remaining part of the decomposition can be understood as a "linear" combination of orthonormal basis elements whose coefficients are simply given by the inner product of the function $f(\tau)$ under consideration and the respective basis element.

In the case of a Poincaré series representation, an extraction of the Fourier zero-mode was an involved process, in contrast spectral representation gives easy access to this information. Notice that if we Fourier decompose $f \in L^2(\mathcal{F})$ as

$$f(\tau) = \sum_{k\in\mathbb{Z}} f_k(\tau_2)e^{2\pi i k\tau_1}\,,$$

the spectral decomposition (3.25) immediately provides for a nice contour integral representation for the Fourier zero-mode $f_0(\tau_2)$. Since from (3.17) we know that the Maass cusp forms have vanishing Fourier zero-mode, we conclude that only the Eisenstein series can contribute. Furthermore, from the Fourier decomposition (3.14) for $E(t;\tau)$, we know that the zero-mode of the Eisenstein series contains only two power-behaved terms, τ_2^t and τ_2^{1-t}. We can however combine the reflection property $\xi(2t)E(t;\tau) = \xi(2-2t)E(1-t;\tau)$, with a change of variables $t \to 1-t$ to show that both terms τ_2^t and τ_2^{1-t} give an equal contribution, arriving at

$$f_0(\tau_2) = \langle f\rangle + \int_{\mathrm{Re}(t)=\frac{1}{2}} (f,\mathrm{E}_t)\,\tau_2^t\frac{\mathrm{d}t}{2\pi i}\,. \tag{3.26}$$

This formula may appear rather useless since to extract the Fourier zero-mode $f_0(\tau_2)$ it would seem necessary to already know the full modular function $f(\tau)$ in order to compute its spectral overlap (f,E_t). However the use of Poincaré series leads to a neat idea called the *unfolding trick*, which sometimes makes the calculation possible. The trick has many guises (some of which we also describe later in the thesis), but here we look at a simple application that shows the overlap with Eisenstein series on the critical line $\mathrm{Re}(t) = \frac{1}{2}$ is given by the Mellin transform of the Fourier zero-mode

$$(f, \mathrm{E}_t) = \int_{\mathcal{F}} f(\tau)\overline{E(t;\tau)}\mathrm{d}\mu = \int_{\mathcal{F}} f(\tau) \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma\cdot\tau)^{\bar{t}}\mathrm{d}\mu = \int_{\mathrm{B}(\mathbb{Z})\backslash\mathfrak{H}} f(\tau)\tau_2^{1-t}\mathrm{d}\mu$$
$$= \int_0^\infty f_0(\tau_2)\tau_2^{-1-t}\mathrm{d}\tau_2 = \mathcal{M}[f_0](-t). \tag{3.27}$$

In the calculation we used the fact that on the critical line we have the identity $\bar{t} = 1-t$ and in the third equality we exchanged the sum and the integral, while simultaneously making a change of variable $\tau \to \gamma\cdot\tau$, which has the effect of moving the fundamental domain around in such a way that it tessellates the cylinder

$$\mathrm{B}(\mathbb{Z})\backslash\mathfrak{H} = \left\{\tau_1 + i\tau_2 \in \mathfrak{H} : |\tau_1| \le \frac{1}{2}\,,\ \tau_2 > 0\right\}\,. \tag{3.28}$$

The integral over τ_1 can then be trivially calculated and the final answer projects on the the zero-mode of the function. It is expressed using the Mellin transform, which for a function $g : [0,\infty) \to \mathbb{C}$ is defined by $\mathcal{M}[g](t) := \int_0^\infty g(x)x^{t-1}\mathrm{d}x$. Formula (3.26) is then recognised as nothing more than the Mellin inversion formula.

Finally, we notice that once the spectral overlap (f, E_t) is known, the integral representation (3.26) enables us to explore both the large $\tau_2 \to \infty$ asymptotics as well as the small $\tau_2 \to 0$ asymptotics by a suitable choice on how we close the t-contour of integration at infinity.

Applications to physics

Now that we have introduced the relevant mathematical techniques, we look at 3 applications to physics. While in each case the analysis concerns a particular space of functions that arises in the relevant physical context, the methods and ideas developed may be applied more generally. All of the examples are related to string theory, which is a framework to describe quantum gravity where particles are replaced by extended 1-dimensional strings. In this thesis we don't provide an introduction to string theory, but standard texts are [17–20]. Nevertheless, before we discuss the specifics, it's good to give a general overview of how modular symmetry appears in string theory. Broadly there are two different sources for the appearance of $\mathrm{SL}(2,\mathbb{Z})$: it serves as the mapping class of a torus and also as the S-duality group of type IIB string theory.

String perturbation theory involves a sum over all Riemann surfaces in order to calculate the scattering amplitude of oscillating strings (identified with particles) and the torus plays a particularly important rôle as the first quantum correction to the tree-level answer. By the mapping class of a surface, in this case the torus, we simply mean the group that describes large diffeomorphisms (those which are not connected to the identity), therefore in this case invariance under modular transformations simply comes about because our choice of coordinates on a torus should not matter.

The second occurrence of modular symmetry is somewhat more mysterious and traces its origin back to an observation of Montonen and Olive [21] in gauge theory. They found that the spectrum of the theory remains invariant if the coupling constant is inverted while simultaneously fundamental fields are exchanged with solitons. Even though the duality did not work out in the Georgi-Glashow model they had considered originally, it was established to be correct in the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills [22]. In supersymmetric theories the coupling constant g^2_{YM} and theta angle θ naturally combine into a complex parameter $\tau = \theta/(2\pi) + 4\pi i/g^2_{YM} \in \mathfrak{H}$. In this case the periodicity of the angle is implemented by $T : \tau \to \tau + 1$, while the Montonen-Olive duality is given by $S : \tau \to -\frac{1}{\tau}$, thereby enhancing the symmetry to the full $\mathrm{SL}(2, \mathbb{Z})$. Since $\mathcal{N} = 4$ SYM is a conformal theory and the coupling in it does not run, different choices of τ may be seen as parametrizing inequivalent theories, and, due to the duality discussed, the correct conformal manifold is the fundamental domain $\mathcal{F}$.

Another way how to view this duality is from a holographic perspective via the AdS/CFT correspondence. In an iconic paper [23] Maldacena proposed that a string theory in a negatively curved spacetime (one that asymptotically approaches the AdS solution) could be identified with a conformal field theory that lives on the boundary of that spacetime. With this correspondence a theory of quantum gravity is shown to be equivalent to another theory with no gravity at all! In the first and most widely studied implementation of the correspondence Type IIB string theory on a background $AdS_5 \times S^5$ in the presence of N $D3$-branes ought to be identified with $\mathcal{N} = 4$ SYM theory in 4 dimensional flat spacetime with gauge group $SU(N)$. By then it was already known that in Type IIB string theory one may also define a modular parameter $\tau_s = \chi + i/g_s$, where χ is the expectation value of the RR 0-form field (often also called the axion), while $g_s = e^{\Phi}$ is the string coupling (related to the expectation value of the dilaton field). The string theory was conjectured to be invariant under $\mathrm{SL}(2, \mathbb{Z})$ transformations on the parameter τ_s (with appropriate action on the other string and brane degrees of freedom as well), and this powerful non-perturbative invariance was called S-duality. Under AdS/CFT the two modular parameters are identified $\tau = \tau_s$ and the Montonen-Olive duality of gauge theory is revealed to be the same as S-duality of string theory.

CHAPTER 4

Superstring perturbation theory and modular graph functions

Identically to quantum field theory, the calculation of scattering amplitudes plays a major rôle in string theory. Indeed, a formulation for an all-order perturbative S-matrix of graviton scattering was one of the major early successes of string theory. In QFT perturbation theory is organised by the number of loops in the associated Feynman graph, while in string theory this notion is replaced by inequivalent Riemann surfaces (with genus of the surface playing the rôle of loops). Restricting our attention to the scattering of 4 gravitons in Type II theories for the moment, string theory gives us an asymptotic expansion in the string coupling g_s

$$\mathcal{A}(k_i, \epsilon_i) \sim \kappa_{10}^2 \mathcal{R}^4 \sum_{h=0}^{\infty} g_s^{2h-2} \mathcal{A}^{(h)}(s_{ij}). \tag{4.1}$$

In this formula κ_{10} is related to the 10 dimensional Newton's constant, $\mathcal{R}^4$ is a linearised contraction of 4 Riemann tensors that is constructed out of the graviton polarisation tensors ϵ_i and $s_{ij} = -\frac{\alpha'}{4} k_i \cdot k_j$ are Mandelstam invariants constructed from the momenta of the gravitons k_i. The form of the series (4.1) is severely constrained by $\mathcal{N} = 2$ supersymmetry in 10 dimensions so that only a single contraction of the polarisation tensors is allowed to appear (this also has implications for the low-energy effective action discussed later). The series is also known to be divergent [24] with the coefficient functions $\mathcal{A}^{(h)} \sim h!$, furthermore the Borel transformation of the series has singularities on the positive real axis indicating the presence of non-perturbative effects (which include D-instantons at order $\sim e^{-1/g_s}$ and gravitational instantons at order $\sim e^{-1/g_s^2}$). The genus-h amplitude $\mathcal{A}^{(h)}$ is given by an integral over the moduli space of genus-h Riemann surfaces of the expectation value of a product of 4 graviton vertex operators. For $h = 0$ the answer is easily calculable and reproduces the Virasoro-Shapiro amplitude, while the cases $h \geq 2$ are generically intractable using current techniques. In this chapter we focus on the sweet spot $h = 1$ corresponding to a toroidal worldsheet.

For closed-string amplitudes (describing gravitational interactions) at genus one the world-sheet is a torus $\mathfrak{T} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with complex modular parameter lying in the upper complex half-plane $\tau = \tau_1 + i\tau_2 \in \mathfrak{H}$. In order to calculate the amplitude of a scattering process, one introduces punctures $z_j \in \mathfrak{T}$ and integrates them over all inequivalent configurations. The amplitude for the 4 graviton case may then be written as

$$\mathcal{A}^{(1)}(s_{ij}) = \frac{\pi}{16} \int_{\mathcal{F}} \frac{|\mathrm{d}\tau|^2}{\tau_2^2} \mathcal{M}_4^{(1)}(s_{ij}|\tau), \tag{4.2}$$

with the fundamental domain $\mathcal{F}$ serving as the moduli-space of inequivalent tori. For further discussion it is useful to introduce a slightly generalised version of the integrand, which happens to be relevant for n graviton scattering and is given by

$$\mathcal{M}_n^{(1)}(s_{ij}|\tau) = \left(\prod_{j=2}^{n} \int_{\mathfrak{T}} \frac{\mathrm{d}^2 z_j}{\tau_2} \right) \exp\left(\sum_{1 \leq i < j}^{n} s_{ij} G(z_i - z_j, \tau) \right), \tag{4.3}$$

where translational invariance can be used to set z_1 to an arbitrary value. In this expression the Mandelstam invariants $s_{ij} \in \mathbb{C}$ are taken to be independent complex numbers, and $G(z, \tau)$ is the Green function on a torus given by

$$G(z, \tau) = \frac{\tau_2}{\pi} \sum_{(m,n) \neq (0,0)} \frac{e^{2\pi i(mv - nu)}}{|m\tau + n|^2}, \tag{4.4}$$

where $z = u\tau + v$ with $u, v \in [0, 1)$. This sum is only conditionally convergent and is understood using the Eisenstein summation convention [11].

When one Taylor expands the exponential of (4.3) in the s_{ij}, they are naturally led to a graphical scheme for organising the terms that emerge - these objects are called *modular graph functions* (MGFs) and were introduced in [25]. In order to construct a graph out of the terms in the series, we associate a vertex with each of the punctures $z_1, z_2, ..., z_n$ and an edge connecting vertices i and j with each occurrence of the propagator $G(z_i - z_j, \tau)$. In turn, every vacuum graph produced from a scalar field theory defined on a torus will also be associated to a modular graph function. We define the *weight* of an MGF as the number of edges in the corresponding graph (which is also the number of Green functions in the chosen monomial). It is important to note that weight as defined here is distinct from modular weight, which is vanishing for all MGFs.

In order to understand the structure of MGFs a little better, it is useful to parameterise the punctures as $z_j = u_j\tau + v_j$ with $u_j, v_j \in [0, 1)$ and $\frac{\mathrm{d}^2 z_j}{\tau_2} = \mathrm{d}u_j\,\mathrm{d}v_j$. In this case we use the lattice-sum representation of the Green function (4.4) to observe that each integral over a puncture simply enforces momentum conservation at the associated vertex. Since the torus is a compact space, the momenta are discrete and form a two-dimensional lattice (with origin removed)

$$p = m\tau + n \in \Lambda' , \qquad \Lambda' = (\mathbb{Z} + \tau\mathbb{Z}) \setminus \{0\}. \tag{4.5}$$

We are thus guaranteed that every one-particle reducible graph vanishes, since the momentum flowing through the reducible edge must be 0. As a result, the simplest non-trivial MGFs appear at one loop and are non-holomorphic Eisenstein series of weight $w > 1$

$$\mathcal{E}_w(\tau) = \left(\frac{\tau_2}{\pi}\right)^w \sum_{p\in\Lambda'} \frac{1}{|p|^{2w}} . \tag{4.6}$$

Here we introduced a third normalisation for this function (this is important to make some of their algebraic and transcendentality properties manifest). At two loops, every MGF associated to a connected graph can be expressed as a function $C_{a,b,c}$ of weight $w = a{+}b{+}c$:

$$C_{a,b,c}(\tau) = \left(\frac{\tau_2}{\pi}\right)^{a+b+c} \sum_{p_1,p_2,p_3\in\Lambda'} \frac{\delta(p_1{+}p_2{+}p_3)}{|p_1|^{2a}|p_2|^{2b}|p_3|^{2c}} . \tag{4.7}$$

The graphs corresponding to the MGFs $\mathcal{E}_w$ and $C_{a,b,c}$ are depicted in figure 4.1. There are obvious ways how one may construct MGFs at higher loop order [25] or even generalise to objects that carry non-zero modular weight, so called modular graph *forms* [26], but in this chapter we only analyse the two-loop, modular-invariant case.

MGFs have a variety of interesting connections to number theory. For example, when

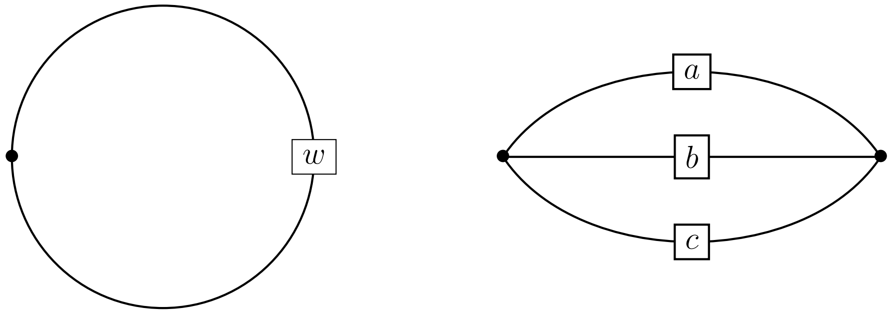

Figure 4.1: The graphs corresponding to the one-loop and two-loop modular graph functions $\mathcal{E}_w$ and $C_{a,b,c}$, where a link with a boxed number w indicates w concatenated Green functions.

computing the asymptotic expansion near the cusp $\tau \to i\infty$ of MGFs it is natural to encounter multiple zeta values (MZVs), i.e. generalisations of the Riemann zeta function defined by iterated (conical) sums. The weight of an MGF can be identified with the transcendental weight of the corresponding MZV[1] [27] appearing in the expansion at the cusp, however in general the loop order of an MGF is only an upper bound on the maximal depth of its possible MZVs.

Furthermore, one finds that there exists an intricate web of connections between different MGFs and $\mathbb{Q}$-linear combinations of multiple zeta values. Some easy examples were first discussed in [28]:

$$C_{1,1,1}(\tau) = \mathcal{E}_3(\tau) + \zeta_3\,, \qquad C_{2,2,1}(\tau) = \frac{2}{5}\mathcal{E}_5(\tau) + \frac{\zeta_5}{30}\,. \tag{4.8}$$

Observe that both sides of equations (4.8) are consistent with the defined weight assignments if furthermore the Riemann zeta[2] ζ_w is assigned weight w. This is a generic feature: algebraic relations between MGFs respect the weight grading but mix different loop orders (i.e. the loop order is only a filtration). In section 4.1 we introduce the notion of depth as a more useful alternative to loop order, at least for classifying algebraic relations.

Additionally to the algebraic relations discussed before, there are also differential equations relating different MGFs. Since they are modular functions, the equations they satisfy are with

[1]Although not of crucial importance here, MZVs are defined by the conical sum $\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r} k_1^{-n_1}k_2^{-n_2}\ldots k_r^{-n_r}$ with $n_i \in \mathbb{N}$ and $n_r \geq 2$. The transcendental weight of a MZV is given by $w = \sum_{1\leq i\leq r} n_i$ while its depth by r.

[2]Throughout this chapter, we shall write the Riemann zeta function either as $\zeta(s)$ or as ζ_s depending respectively on whether s is generic or fixed to some specific integer value.

respect to the $\mathrm{SL}(2,\mathbb{R})$ invariant Laplacian Δ defined before (3.5). At two loops it can be shown [28] that

$$\begin{aligned}\Delta C_{a,b,c} = {} & (a(a-1)+b(b-1)+c(c-1))C_{a,b,c} \\ & + ab(C_{a-1,b+1,c}+C_{a+1,b-1,c}+C_{a+1,b+1,c-2}-2C_{a,b+1,c-1}-2C_{a+1,b,c-1}) \\ & + bc(C_{a,b-1,c+1}+C_{a,b+1,c-1}+C_{a-2,b+1,c+1}-2C_{a-1,b,c+1}-2C_{a-1,b+1,c}) \\ & + ca(C_{a+1,b,c-1}+C_{a-1,b,c+1}+C_{a+1,b-2,c+1}-2C_{a+1,b-1,c}-2C_{a,b-1,c+1}),\end{aligned} \tag{4.9}$$

where one of the indices on the right hand side might get reduced to 0 or -1, in which case the two-loop function is replaced by

$$C_{w-\ell,\ell,0} = \mathcal{E}_\ell \mathcal{E}_{w-\ell} - \mathcal{E}_w\,, \qquad C_{w+1-\ell,\ell,-1} = \mathcal{E}_\ell \mathcal{E}_{w-\ell} + \mathcal{E}_{\ell-1}\mathcal{E}_{w-\ell+1}. \tag{4.10}$$

Formally in this procedure the divergent non-holomorphic Eisenstein series $\mathcal{E}_1$ can appear, but it always cancels out of the final answer. It was shown in [28] that the system of linear equations (4.9) can be diagonalised by the introduction of eigenfunctions $\mathcal{C}_{w;m;p}$, which are linear combinations of different $C_{a,b,c}$ with a fixed weight $w=a+b+c$. These eigenfunctions then satisfy a significantly more manageable differential equation

$$(\Delta-(w-2m)(w-2m-1))\mathcal{C}_{w;m;p} = t^{(0)}_{w;m;p}\mathcal{E}_w + \sum_{\ell=2}^{[w/2]} t^{(\ell)}_{w;m;p}\mathcal{E}_\ell\mathcal{E}_{w-\ell}, \tag{4.11}$$

where $t^{(0)}_{w;m;p}$ and $t^{(\ell)}_{w;m;p}$ are constants, m is a label for the eigenvalue of the differential equation, and p labels the degeneracy of the fixed eigenspace. The explicit coefficients connecting the two bases $C_{a,b,c}$ and $\mathcal{C}_{w;m;p}$ can be found in [29].

While the representation in terms of lattice sums is convenient for a graphical interpretation and establishing connections to MZVs, the sums are hard to manipulate and many identities are hidden. Moreover, we will be interested in finding the asymptotic behaviour of MGFs as $\tau\to i\infty$, which is a task of considerable difficulty from the perspective of lattice sums. Instead it is much more convenient to use the differential equations satisfied by MGFs such as (4.9) and (4.11) and solve them using a Poincaré series ansatz. We shall discuss said method in the following sections.

Due to the $\mathrm{SL}(2,\mathbb{Z})$ invariance all modular graph functions have period one and hence may be Fourier expanded in τ_1. This expansion contains a lot of information about the behaviour of the function as the modular parameter approaches the cusp $\tau \to i\infty$. To proceed, we introduce the following variables

$$y = \pi\tau_2\,, \qquad q = e^{2\pi i\tau}\,, \qquad \bar{q} = e^{-2\pi i\bar{\tau}}\,, \tag{4.12}$$

in which the non-holomorphic Eisenstein series (4.6) for positive integer weight $w > 1$ can be written as the Fourier series

$$\begin{aligned} \mathcal{E}_w(\tau) = (-1)^{w-1}\frac{\mathrm{B}_{2w}}{(2w)!}(4y)^w + \frac{4(2w-3)!\zeta_{2w-1}}{(w-2)!(w-1)!}(4y)^{1-w} \\ + \frac{2}{\Gamma(w)}\sum_{n=1}^{\infty} n^{w-1}\sigma_{1-2w}(n)\left[\sum_{a=0}^{w-1}(4ny)^{-a}\frac{\Gamma(w+a)}{a!\,\Gamma(w-a)}\right](q^n+\bar{q}^n), \end{aligned} \tag{4.13}$$

where $\sigma_s(n)$ is a divisor sum and we have introduced the Bernoulli numbers B_{2w} that are rational numbers related to even Riemann zeta values by

$$2\zeta_{2w} = (-1)^{w+1}\frac{(2\pi)^{2w}}{(2w)!}\mathrm{B}_{2w}\,, \qquad w = 1,2,3,\ldots\,. \tag{4.14}$$

The general Fourier expansion for a modular graph function is quite similar to equation (4.13) - one can show that MGFs grow at most polynomially at the cusp, and the expansion must be of the form $\sum_{M,N=0}^{\infty} L_{M,N}(y)q^M\bar{q}^N$ with $L_{M,N}(y)$ a Laurent polynomial [25]. The dominant behaviour at the cusp clearly comes from $L_{0,0}$. Some examples at two-loop level are as follows [28, 30]

$$\begin{aligned} C_{2,1,1}(\tau) &= \frac{2y^4}{14175} + \frac{\zeta_3 y}{45} + \frac{5\zeta_5}{12y} - \frac{\zeta_3^2}{4y^2} + \frac{9\zeta_7}{16y^3} + O(q,\bar{q})\,, \\ C_{2,2,2}(\tau) &= \frac{38y^6}{91216125} + \frac{\zeta_7}{24y} - \frac{7\zeta_9}{16y^3} + \frac{15\zeta_5^2}{16y^4} - \frac{81\zeta_{11}}{128y^5} + O(q,\bar{q})\,. \end{aligned} \tag{4.15}$$

Notice the recurrent appearance of odd zeta values in the expansion, as well as uniform transcendental weight $w = a + b + c$ for each term once an assignment of weight 1 is given to $y = \pi\tau_2$. Unlike the case of Eisenstein series, the zero Fourier mode gets additional, exponentially suppressed contributions from the terms $L_{N,N}(q\bar{q})^N$ with $N > 0$.

The focus of this chapter of the thesis is precisely to reconstruct the $(q\bar{q})^N$ non-perturbative terms to the zero Fourier mode from the purely perturbative Laurent polynomials, or rather a suitable deformation thereof, using resurgent analysis following similar methods to the ones developed in [31, 32]. The structure of the differential equation (4.11) fixes the functional form of these exponentially suppressed terms (see [33, Thm. 1.3]) in terms of incomplete Gamma functions and Laurent polynomials. We shall not rely on these results and arrive at fully explicit expression from resurgent analysis.

4.1 Depth-two Laplace systems

One of the key results of [34, 35] was to show that all two-loop modular graph functions can be written as rational linear combinations of modular invariant functions called *generalised Eisenstein series*, which are labelled by three parameters: s_1, s_2 called the weights (and closely related to the weight of the MGF) and λ characterising the eigenvalue. Generalised Eisenstein series are defined by the differential equation

$$[\Delta - \lambda(\lambda-1)]\, \mathcal{E}\left(\lambda; s_1, s_2\middle|\tau\right) = \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) \tag{4.16}$$

subject to the boundary condition that the term of order y^λ in the Laurent polynomial around the cusp $y \gg 1$ has vanishing coefficient. This boundary condition uniquely fixes the modular-invariant solution, since the ordinary Eisenstein series is the only modular-invariant solution with polynomial growth at the cusp to the corresponding homogeneous equation (3.15). The weights and the eigenvalue of the generalised Eisenstein series are fixed by the weight of the corresponding MGF. Indeed, every connected two-loop modular graph function $C_{a,b,c}(\tau)$ can be expressed as a rational linear sum of $\mathcal{E}\left(s; m, k\middle|\tau\right)$ with $w = k + m = a + b + c$ together with lower-depth objects (like non-holomorphic Eisenstein series and zeta values). The eigenvalue is additionally constrained to lie in the spectrum

$$s \in \mathrm{Spec}_1(k,m) := \{|k-m|+2,\ |k-m|+4,\ \dots, k+m-2\}\,, \qquad k,m \in \mathbb{N}^{\geq 2}\,. \tag{4.17}$$

4.2 Poincaré series approach

We already discussed the Poincaré series representation of a modular invariant function in section 3.2. Remember that the idea is to use a sum over $\mathrm{SL}(2,\mathbb{Z})$ images of a simpler (modular non-invariant) seed function to reconstruct the modular function of our interest. In this case we want to represent the generalised Eisenstein series $\mathcal{E}\left(s;m,k\middle|\tau\right)$ as a sum over images of some seed function $\mathrm{e}\big(s;m,k\big|\tau\big)$. In this sum it is more convenient to choose the seed function to be periodic in the real direction and quotient out by the Borel subgroup. As a result we can write

$$\mathcal{E}\left(s;m,k\middle|\tau\right) = \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{Z})} \mathrm{e}\big(s;m,k\big|\gamma\cdot\tau\big)\,, \tag{4.18}$$

where $\mathrm{e}\big(s;m,k\big|\tau+1\big) = \mathrm{e}\big(s;m,k\big|\tau\big)$. The advantage of choosing the seed to be periodic comes from observing that it can then be Fourier decomposed as

$$\mathrm{e}\big(s;m,k\big|\tau\big) = \sum_{n\in\mathbb{Z}} c_n(y) e^{2\pi i n \tau_1}\,, \tag{4.19}$$

for real coefficient functions that satisfy $c_n(y) = c_{-n}(y)$, since we want a real-analytic function that is even under the involution $\tau\to-\bar{\tau}$.

Upon substitution of (4.19) into (4.16) we can "fold" $\mathcal{E}_k$, i.e. use its known Poincaré sum representation (3.20), to arrive at a simpler equation for the seed function

$$(\Delta - s(s-1))\mathrm{e}\big(s;m,k\big|\tau\big) = (-1)^{k-1}\frac{\mathrm{B}_{2k}}{(2k)!}(4y)^k\mathcal{E}_m(\tau)\,. \tag{4.20}$$

From the known Fourier series of $\mathcal{E}_m$ we can find an expression for the Fourier coefficients $c_n(y)$ that were defined in (4.19). These are Laurent polynomials in y and matching corresponding powers on both sides of equation (4.20) gives an expression for $c_n(y)$. In [34, 35], it was shown that the solution to this Laplace equation is given by

$$\begin{aligned} c_0(y) &= (-1)^{k+m}\frac{\mathrm{B}_{2k}\mathrm{B}_{2m}(4y)^{k+m}}{(2k)!(2m)!(\mu_{k+m}-\mu_s)} - (-1)^k\frac{4\mathrm{B}_{2k}(2m-3)!\zeta_{2m-1}(4y)^{k+1-m}}{(2k)!(m-2)!(m-1)!(\mu_{k-m+1}-\mu_s)}\,,\\ c_n(y) &= (-1)^k\frac{2\mathrm{B}_{2k}}{(2k)!\Gamma(m)}\sigma_{1-2m}(|n|)|n|^{m-k-1}\sum_{\ell=k-m+1}^{k-1} g^{+}_{m,k,\ell,s}(4|n|y)^{\ell}e^{-2|n|y} \quad n\neq 0, \end{aligned} \tag{4.21}$$

where $\mu_s = s(s-1)$ and $g^+_{m,k,\ell,s}$ are the rational coefficients

$$g^+_{m,k,\ell,s} = \frac{\Gamma(\ell)}{\Gamma(\ell+s)} \sum_{i=\ell}^{k-1} \frac{(\ell+1-s)_{i-\ell}\Gamma(s+i)\Gamma(m+k-i-1)}{\Gamma(k-i)\Gamma(i+1)\Gamma(m-k+i+1)}, \tag{4.22}$$

with $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ the (ascending) Pochhammer symbol.

Equation (4.21) can be rewritten in a more suggestive way if we introduce one of the many flavours of iterated integrals at depth one:

$$\mathcal{E}_0(k, 0^p; \tau) = \frac{(2\pi i)^{p+1-k}}{p!} \int_\tau^{i\infty} (\tau-\tau_1)^p G^0_k(\tau_1) \mathrm{d}\tau_1 \,, \tag{4.23}$$

with even $k > 2$ and the notation 0^p is a short-hand of p successive zeros. Higher-depth versions, where the iterated integral structure becomes more evident, can be found in [36, 37].

The symbol G^0_k appearing in the integrand denotes the cuspidal part of the standard holomorphic Eisenstein series, $G_k(\tau)$, introduced in section 3.1.1:

$$\begin{aligned} \mathrm{G}_k(\tau) &= \sum_{p\in\Lambda'} \frac{1}{p^k} = 2\zeta_k + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \,, \qquad k \in \{4, 6, 8, ...\} \,, \\ \mathrm{G}^0_k(\tau) &= \mathrm{G}_k(\tau) - 2\zeta_k \end{aligned} \tag{4.24}$$

and it is convenient to define $\mathrm{G}^0_0 = -1$.

The integral in (4.23) converges for $p \geq 0$ and from the q-expansion of (4.24) one can easily obtain [32, 36]

$$\begin{aligned} \mathcal{E}_0(k, 0^p; \tau) &= -\frac{2}{(k-1)!} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{(mn)^{p+1}} q^{mn} = -\frac{2}{(k-1)!} \sum_{m=1}^{\infty} m^{k-p-2} \sigma_{1-k}(m) q^m \\ &= -\frac{2}{(k-1)!} \sum_{m=1}^{\infty} m^{-p-1} \sigma_{k-1}(m) q^m \,, \end{aligned} \tag{4.25}$$

which can also be considered formally for arbitrary $k, p \in \mathbb{C}$, providing an analytic continuation of this function in these parameters.

Going back to the Fourier modes (4.21) for the seed function $\mathrm{e}\big(s; m, k \big| \tau\big)$, we see that the general seed for all depth-two modular invariant functions can be written as

$$\mathrm{e}\big(s;m,k\big|\tau\big) = c_0(y) - (-1)^k \frac{2\mathrm{B}_{2k}\Gamma(2m)}{(2k)!\Gamma(m)} \sum_{\ell=k-m+1}^{k-1} g^{+}_{m,k,\ell,s}(4y)^{\ell}\mathrm{Re}[\mathcal{E}_0(2m,0^{k+m-\ell-1})]. \tag{4.26}$$

Noticeably, the use of Poincaré series has reduced the depth of the objects under consideration by one unit, thus making the problem more tractable. Furthermore, when $k > m$, the Poincaré seed just obtained gives rise to a convergent Poincaré sum.

Once the Poincaré seeds for the $\mathcal{E}\left(s;m,k\middle|\tau\right)$ are known, we are also able to derive similar expressions for all two-loop MGFs, for example [31],

$$C_{2,1,1}(\tau) = \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \left[\frac{2y^4}{14175} + \frac{y\zeta(3)}{90} + \frac{y}{90}\sum_{m=0}^{\infty}\sigma_{-3}(m)(q^m+\bar{q}^m)\right]_{\gamma}. \tag{4.27}$$

Again we note that a perk of using such a Poincaré series representation is that the depth of the MGF was reduced by one, since the sum in the brackets is related to the depth-1 object $\mathcal{E}_2$ through its Fourier series (4.13). Since lower depth objects are easier to study, equations like (4.27) open up new avenues of analysis.

4.3 Resurgent analysis for Poincaré series

The task at hand is now to start from the Poincaré-series representation (4.18) in terms of seed functions and extract the asymptotic expansion at the cusp of the modular-invariant function $\mathcal{E}\left(s;m,k\middle|\tau\right)$.

We can consider again the Eisenstein series as a warm-up exercise, and very standard results [9,13] tell us how to obtain the asymptotic expansion at the cusp (4.13) from its Poincaré sum representation (3.20). For more general Poincaré series the analysis is more involved, but in principle it is possible to rewrite each Fourier coefficient of a modular invariant function in terms of some convoluted integral transform of the Fourier coefficients of its seed function as well as involving complicated Kloosterman sums. We review this general procedure in appendix A.

In the present case we can see that the non-zero Fourier mode of the general seed (4.21) is of the form

$$c_n(y) = \sum_{\ell=k-m+1}^{k-1} \Big[(-1)^k \frac{\mathrm{B}_{2k}}{(2k)!\Gamma(m)} g^+_{m,k,\ell,s}\Big] \sigma_{2m-1}(|n|)|n|^{\ell-k-m}(4y)^\ell e^{-2|n|y}\,, \tag{4.28}$$

hence a finite and rational linear combination of seeds of the type

$$\sigma_a(n)|n|^b y^r e^{-2|n|y}\,. \tag{4.29}$$

Seeds of precisely this form were studied in [31,38], where it was shown how to use the procedure outlined in Appendix A to compute the Laurent polynomial of the associated Poincaré sum. To summarise the result, we consider the Poincaré sum

$$\Phi(\tau) = \sum_{\ell\in\mathbb{Z}} a_\ell(y) e^{2\pi i \ell \tau_1} = \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \varphi(\gamma\cdot\tau)\,, \tag{4.30}$$

with seed function given by terms of the form (4.29)

$$\begin{aligned} \varphi(\tau) &= \sum_{n\in\mathbb{Z}\backslash\{0\}} c_n(y) e^{2\pi i n\tau_1}\,, \\ c_n(y) &= \sigma_a(n)|n|^b y^r e^{-2|n|y}\,. \end{aligned} \tag{4.31}$$

Then the Laurent polynomial part of the asymptotic expansion at the cusp $y\to\infty$, for the zero-mode coefficient $a_0(y)$ is given by

$$\begin{aligned} a_0(y) \sim I(a,b,r|y) = \frac{2^{3-2r} y^{1+b-r}}{\Gamma(r)\pi^{2b-2r}} \Bigg[& \frac{y}{\pi^2} \frac{\Gamma(b+1)\Gamma(2r-b-2)}{\Gamma(r-b-1)} \frac{\zeta(2r-a-2b-2)\zeta(1-a)}{\zeta(2r-a-2b-1)} \\ &+ \Big(\frac{y}{\pi^2}\Big)^{a+1} \frac{\Gamma(a+b+1)\Gamma(2r-a-b-2)}{\Gamma(r-a-b-1)} \frac{\zeta(2r-a-2b-2)\zeta(a+1)}{\zeta(2r-a-2b-1)} \\ &+ \Big(\frac{\pi^2}{y}\Big)^b \sum_{n\geq 0} \Big(\frac{-\pi^2}{y}\Big)^n \frac{\Gamma(2r+n-1)}{n!\cdot\Gamma(r+n)} \\ &\qquad \times \frac{\zeta(-b-n)\zeta(-a-b-n)\zeta(2r-a-b+n-1)\zeta(2r-b+n-1)}{\zeta(2r+2n)\zeta(2r-a-2b-1)} \Bigg]\,. \end{aligned} \tag{4.32}$$

Using Ramanujan's identity

$$\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^s}=\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\,, \tag{4.33}$$

the last term can be rewritten in Dirichlet series form

$$\begin{aligned}&\frac{\zeta(-b-n)\zeta(-a-b-n)\zeta(2r-a-b+n-1)\zeta(2r-b+n-1)}{\zeta(2r+2n)}\\&=4\sin\left(\frac{\pi(b+n)}{2}\right)\sin\left(\frac{\pi(a+b+n)}{2}\right)\frac{\Gamma(1+b+n)\Gamma(1+a+b+n)}{(2\pi)^{a+2b+2n+2}}\sum_{m>0}\frac{\sigma_a(m)\sigma_{a+2b+2-2r}(m)}{m^{a+b+n+1}}\,.\end{aligned} \tag{4.34}$$

A few comments regarding the general expression (4.32) are in order.

- For generic a, b, r this asymptotic series is a Gevrey-1, factorially divergent formal power series. Shortly we will use Borel resummation in order to reconstruct the non-perturbative properties of $a_0(y)$ at the cusp $y\to\infty$. As usual, ambiguities in prescribing a unique resummation procedure will allow us to obtain the exponentially suppressed contributions, $(q\bar{q})^n$, which are hidden in the purely perturbative asymptotic result (4.32).

- For a, b integers with a odd (as for the case under consideration (4.28)), the series in (4.32) terminates after a finite number of terms. This can be easily understood by noticing that for n large enough either $\zeta(-b-n)$ or $\zeta(-a-b-n)$ will be a zeta value at a negative even integer, hence vanishing, while all other factors will be regular. For a, b integers with a odd we then have that the series in (4.32) does terminate for $n > n_{max} = \max(-b,-a-b)+1$. In particular for our choice of seeds (4.28), we have $a = 2m-1\geq 0$ while $-b\in\{m+1, m+2, ..., 2m, 2m+1\}$, hence for the case of interest (4.32) always truncates for $n>-b+1$.

- The parameter b serves the purpose of a regulator. When b is arbitrary our expression (4.32) is a formal asymptotic power series for which we can make use of resurgent analysis to reconstruct the exponentially suppressed terms in the zero-mode. At the end of the day, when we set b to its physical values appearing in (4.28), the asymptotic power series will truncate to the expected finite Laurent polynomial, while the non-perturbative terms will survive. This is an instance of Cheshire cat resurgence.

In [34] it was indeed shown that if we use the general expression (4.32) specialised to the seed $\mathrm{e}\big(s;m,k\big|\tau\big)$ from (4.26) then we obtain a truncating Laurent polynomial for the modular invariant functions $\mathcal{E}\left(s;m,k\big|\tau\right)$:

$$\mathcal{E}\left(s;m,k\big|\tau\right) = \mathrm{P}^{(s)}_{m,k}(y) + O(q,\bar{q})\,,$$

where the Laurent polynomial $\mathrm{P}^{(s)}_{m,k}(y)$ is given by

$$\begin{aligned}\mathrm{P}^{(s)}_{m,k}(y) =& \frac{(-4)^{k+m}\mathrm{B}_{2m}\mathrm{B}_{2k}}{(k{+}m{-}s)(k{+}m{+}s{-}1)(2m)!(2k)!}y^{k+m} - \frac{2(-1)^m4^{1+m-k}\mathrm{B}_{2m}\Gamma(2k{-}1)\zeta_{2k-1}}{\Gamma(k)\Gamma(k)(m{-}k{+}s)(m{-}k{-}s{+}1)(2m)!}y^{1+m-k}\\ &- \frac{2(-1)^k4^{1+k-m}\mathrm{B}_{2k}\Gamma(2m{-}1)\zeta_{2m-1}}{\Gamma(m)\Gamma(m)(k{-}m{+}s)(k{-}m{-}s{+}1)(2k)!}y^{1+k-m}\\ &+ \frac{4^{3-m-k}\Gamma(2m{-}1)\Gamma(2k{-}1)\zeta_{2m-1}\zeta_{2k-1}}{[\Gamma(m)\Gamma(k)]^2(k{+}m{-}s{-}1)(k{+}m{+}s{-}2)}y^{2-k-m} + c^{(s)}_{m,k}\zeta_{k+m+s-1}y^{1-s}\,,\end{aligned} \tag{4.35}$$

with the rational coefficient

$$c^{(s)}_{m,k} = \frac{4^{2-s}(-1)^{m+s+1}\mathrm{B}_{s+m-k}\mathrm{B}_{k+m-s}\mathrm{B}_{k+s-m}(2s)!}{(s{+}m{-}k)\Gamma(m)\Gamma(s)\mathrm{B}_{2s}(k{+}m{-}s)!(k{+}s{-}m)!}\sum_{\ell=k-m+1}^{\min(k-1,s)}(-1)^\ell g^+_{m,k,\ell,s}\frac{\Gamma(\ell{+}s{-}1)}{\Gamma(\ell)(s{-}\ell)!}\,, \tag{4.36}$$

expressed in terms of the rational numbers $g^+_{m,k,\ell,s}$ defined in (4.22).

The last term in (4.35) satisfies the homogeneous Laplace equation (4.16) and its coefficient can also be rewritten [39][3] as

$$c^{(s)}_{m,k} = -4\pi^{\frac{s-m-k-1}{2}}\frac{\Gamma\Big(\frac{m+k+s-1}{2}\Big)\xi(s+1-m-k)\xi(m+s-k)\xi(k+s-m)}{(2s-1)\Gamma(m)\Gamma(k)\,\xi(2s)}\,. \tag{4.37}$$

4.4 Resumming an evanescent tail

Since we are interested in exploiting the asymptotic nature of the general expression (4.32), we can simply focus on its last term which, for generic a,b,r, does indeed produce the factorially

[3] We also obtain this form for the coefficient directly from spectral theory in the next chapter, where it is given in equation (5.72)

divergent asymptotic tail

$$I_{asy}(a,b,r|y) = \frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-2}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)}\sum_{n>0}\sigma_a(n)\sigma_{a+2b+2-2r}(n) \tag{4.38}$$
$$\sum_{m\geq 0}\frac{\Gamma(m+a+b+1)}{(4ny)^{m+a+b+1}}\frac{\Gamma(2r+m-1)\Gamma(1+b+m)}{\Gamma(m+r)\Gamma(m+1)}\Big[(-1)^m\cos\Big(\frac{a\pi}{2}\Big)-\cos\Big(\frac{(a+2b)\pi}{2}\Big)\Big],$$

after making use of Ramanujan's identity as discussed above. We note that $I_{asy}(a,b,r|y)$ should be understood only as a formal power series in y^{-1} with zero radius of convergence.

The next step is to perform a standard Borel resummation for (4.38). Rewriting the integral representation of the gamma function as

$$\frac{\Gamma(m+a+b+1)}{(4ny)^{m+a+b+1}} = \int_0^\infty e^{-4nyt}t^{m+a+b}\mathrm{d}t\,, \tag{4.39}$$

we can then define the directional Borel resummation (see chapter 2 for an overview) of the formal power series $I_{asy}(a,b,r|y)$ as

$$\mathcal{S}_\theta\Big[I_{asy}(a,b,r)\Big](y) = \tag{4.40}$$
$$\frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-2}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)}\frac{\Gamma(2r-1)\Gamma(1+b)}{\Gamma(r)}\sum_{n>0}\sigma_a(n)\sigma_{a+2b+2-2r}(n)\int_0^{e^{i\theta}\infty}e^{-4nyt}B(t)\mathrm{d}t\,,$$

where the Borel transform in the case at hand is given by

$$\begin{aligned} B(t) &= \sum_{n\geq 0}t^{a+b+n}\frac{(2r-1)_n(1+b)_n}{(r)_n n!}\Big[(-1)^n\cos\Big(\frac{a\pi}{2}\Big)-\cos\Big(\frac{(a+2b)\pi}{2}\Big)\Big] \\ &= t^{a+b}\Big[{}_2F_1(2r-1,1+b;r|-t)\cos\Big(\frac{a\pi}{2}\Big)-{}_2F_1(2r-1,1+b;r|t)\cos\Big(\frac{(a+2b)\pi}{2}\Big)\Big]\,, \end{aligned} \tag{4.41}$$

with ${}_2F_1(a,b;c|z)$ denoting a standard hypergeometric function.

We see that for $\theta\in(0,\pi/2)$, the directional Borel resummation $\mathcal{S}_\theta\Big[I_{asy}(a,b,r)\Big](y)$ does indeed define an analytic function in the complex wedge $-\pi/2-\theta<\arg(y)<\pi/2-\theta$ whose asymptotic expansion near $y\to\infty$ is precisely given by (4.38). Furthermore, if we take two different directions $\theta_1,\theta_2\in(0,\pi/2)$, with $\theta_1<\theta_2$, it is simple to see that $\mathcal{S}_{\theta_1}\Big[I_{asy}(a,b,r)\Big](y)$ and $\mathcal{S}_{\theta_2}\Big[I_{asy}(a,b,r)\Big](y)$ are analytic continuations of one another since the integrand is regular in the complex wedge $\theta_1\leq\arg(t)\leq\theta_2$.

A similar story can be repeated for $\theta \in (-\pi/2, 0)$, however if we define the lateral Borel resummation as

$$\mathcal{S}_{\pm}\Big[I_{asy}(a,b,r)\Big](y) = \lim_{\theta\to 0^{\pm}} \mathcal{S}_{\theta}\Big[I_{asy}(a,b,r)\Big](y)\,, \tag{4.42}$$

we see that the two continuations $\mathcal{S}_{\pm}\big[I_{asy}(a,b,r)\big](y)$, belonging to the same germ of analytic functions, differ on the common domain of analyticity, since the integrand, and in particular ${}_2F_1(2r-1, 1+b; r|t)$, has a branch-cut singularity precisely along the direction $\arg(t) = 0$, which is a Stokes direction. Thus we have obtained two distinct continuations of the same formal power series (4.38) that differ precisely on the direction of interest, namely $y > 0$. This is generically a signal that we have to include non-perturbative, exponentially suppressed corrections [6].

From the properties of the hypergeometric series we can easily compute its discontinuity across the branch cut $t \in [1, \infty)$:

$$\begin{aligned} &\mathrm{Disc}_0\Big[{}_2F_1(a,b;c|t)\Big] = \lim_{\epsilon\to 0^+}\Big[{}_2F_1(a,b;c|t+i\epsilon) - {}_2F_1(a,b;c|t-i\epsilon)\Big] \\ &= \frac{2\pi i\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}(t-1)^{c-a-b}{}_2F_1(c-a, c-b; c-a-b+1|1-t)\,, \end{aligned} \tag{4.43}$$

valid for $t > 1$. We can then compute the difference between the two lateral resummations, related to the Stokes automorphism, and find

$$\begin{aligned} &(\mathcal{S}_+ - \mathcal{S}_-)\Big[I_{asy}(a,b,r)\Big](y) \\ &= \frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-2}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)}\frac{\Gamma(2r-1)\Gamma(1+b)}{\Gamma(r)}\sum_{n>0}\sigma_a(n)\sigma_{a+2b+2-2r}(n)\int_0^\infty e^{-4nyt}\mathrm{Disc}_0 B(t)\mathrm{d}t \\ &= -\frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-2}}{2^{a+2b}\Gamma(r)\zeta(2r-a-2b-1)}\sum_{n>0}\sigma_a(n)\sigma_{a+2b+2-2r}(n)2\pi i\cos\Big(\frac{(a+2b)\pi}{2}\Big)e^{-4ny} \\ &\times\int_0^\infty e^{-4nyt}\frac{(t+1)^{a+b}t^{-r-b}}{\Gamma(1-r-b)}{}_2F_1(1-r, r-b-1; 1-r-b|-t)\mathrm{d}t\,, \end{aligned} \tag{4.44}$$

where in the last step we substituted the discontinuity (4.43) and shifted the integration variable $t \to t+1$.

Notice that this discontinuity in resummation is purely non-perturbative in nature due to the presence of the exponentially suppressed term $(q\bar{q})^n = e^{-4ny}$. The present discussion is very similar to [31, 32, 40]: the starting asymptotic series (4.32) cannot be easily Borel resummed as it is, however by realising that the factorially growing coefficients are "dressed" by a suitable Dirichlet series we obtain (4.38), amenable to standard Borel resummation. The infinitely many exponentially suppressed corrections $(q\bar{q})^n = e^{-4ny}$ can be seen as arising from the unfolding of the Dirichlet series (4.34) combined with the shift $y \to 4ny$.

We can now define the median resummation of the asymptotic formal power series $I_{asy}(a, b, r|y)$,

$$\mathcal{S}_{med}\Big[I_{asy}(a, b, r)\Big](y) = \mathcal{S}_{\pm}\Big[I_{asy}(a, b, r)\Big](y) \mp i\,\mathrm{Im}[\sigma(a, b)]\,\mathrm{NP}(a, b, r|y)\,, \tag{4.45}$$

which is independent of our choice of sign, i.e. of direction of resummation, having defined the imaginary part of the transseries parameter

$$\mathrm{Im}\,[\sigma(a, b)] = \cos\Big(\frac{(a + 2b)\pi}{2}\Big)\,, \tag{4.46}$$

and the non-perturbative part $\mathrm{NP}(a, b, r|y)$ is given

$$\begin{aligned}\mathrm{NP}(a, b, r|y) = &-\frac{(4y)^{2+a+b-r}\pi^{2r-a-2b-1}}{2^{a+2b}\Gamma(r)\zeta(2r - a - 2b - 1)}\sum_{n>0}\sigma_a(n)\sigma_{a+2b+2-2r}(n)e^{-4ny} \\ &\times\int_0^\infty e^{-4nyt}(t + 1)^{a+b}t^{-r-b}{}_2\tilde{F}_1(1 - r, r - b - 1; 1 - r - b| - t)\mathrm{d}t\,,\end{aligned} \tag{4.47}$$

where ${}_2\tilde{F}_1(a, b; c|z) = {}_2F_1(a, b; c|z)/\Gamma(c)$ denotes the regularised hypergeometric function. In this context by a transseries we simply mean an expression that includes the perturbative series as well as all the exponentially suppressed non-perturbative terms.

We notice that the discontinuity (4.44) and in particular the Stokes constant $\cos[(a+2b)\pi/2]$, only fixes the imaginary part of the transseries parameter $\sigma(a, b)$, i.e. the overall piecewise constant (jumping only at Stokes directions) in front of the non-perturbative terms. Following [31, 32], we will make the assumption that the complete transseries parameter does in fact depend analytically on $(a + 2b)$, and the "minimal analytic completion" with non-trivial real part is simply

$$\sigma_{\pm}(a, b) = \exp\Big(\pm i\pi\frac{a + 2b - 1}{2}\Big) = \sin\Big(\frac{(a + 2b)\pi}{2}\Big) \mp i\cos\Big(\frac{(a + 2b)\pi}{2}\Big) \tag{4.48}$$

where once more we stress that the sign $\pm$ is correlated with the choice of resummation as in (4.45).

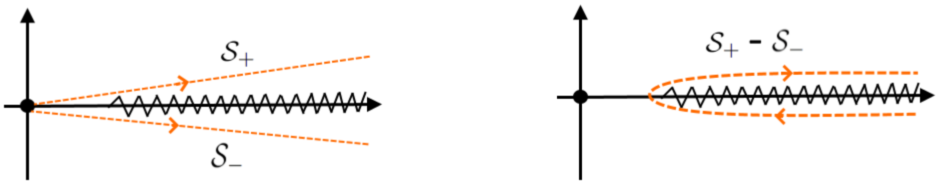

Figure 4.2: On the left diagram we show the two different lateral Borel resummations. On the right diagram, the difference between the two lateral Borel resummations is represented as an Hankel integral contour, used to evaluate the Stokes automorphism.

Usually when we look for transseries solutions to say non-linear ODEs, the imaginary part of the transseries parameter is fixed by the Stokes discontinuity, while its real part is determined via some initial condition. At the present time we do not have such an ODE construction for our problem and we are in a certain sense trying to bootstrap the full transseries entirely out of the perturbative data generated by the Poincaré sum of our seed (4.31) for $a, b \in \mathbb{C}$ generic, without having at our disposal any ODE or functional equation to guide us.

One of the key features of what is generally called "Cheshire cat resurgence" [41–44] is precisely that the Stokes constant vanishes for special values of the deformation parameter ($a + 2b$ in the present case or a supersymmetry breaking deformation in the aforementioned references) while non-perturbative corrections are expected to be present for all values of the deformation. This implies that the transseries parameter must have a non-vanishing real part as well. Our hypothesis (4.48) provides the minimal analytic completion to achieve this, and, as we will see later on, will produce the correct non-perturbative terms.

We then conclude that the non-perturbative resummation of (4.38) is given by

$$\mathcal{S}_{med}\Big[I_{asy}(a,b,r)\Big](y) = \mathcal{S}_{\pm}\Big[I_{asy}(a,b,r)\Big](y) + \sigma_{\pm}(a,b)\mathrm{NP}(a,b,r|y)\,. \tag{4.49}$$

Thanks to the discontinuity equation (4.44), we can easily see that the this is a well-defined analytic function providing a non-perturbative and unambiguous resummation for the formal asymptotic power series (4.38) which is also real (as one would have expected) for $y > 0$ with $a, b \in \mathbb{R}$ and continuous as $\arg(y) \to 0$.

4.5 Non-perturbative completion

We can now specialise the results of the previous section for the generic seed (4.31) to the case of interest for the seed functions $e(s;m,k)$ relevant for all two-loop MGFs. We can rewrite (4.26) as

$$
\mathrm{e}\big(s;m,k\big|\tau\big) = c_0(y) + (-1)^k \frac{2\mathrm{B}_{2k}}{(2k)!\Gamma(m)} \sum_{n\neq 0} \sum_{\ell=k-m+1}^{k-1} g^+_{m,k,\ell,s} (4y)^\ell |n|^{\ell-k-m} \sigma_{2m-1}(|n|) e^{-2|n|y} e^{2\pi i n \tau_1}\,, \tag{4.50}
$$

such that, manifestly, each seed is a finite combination of building blocks (4.31) with $a = 2m-1$, $b = \ell - k - m$, $r = \ell$ just analysed.

We start by decomposing $\mathcal{E}(s;m,k)$ in Fourier modes for $\tau_1 = \mathrm{Re}(\tau)$

$$
\mathcal{E}\left(s;m,k\big|\tau\right) = \sum_{n\in\mathbb{Z}} a_n(s;m,k|y) e^{2\pi i n\tau_1}\,, \tag{4.51}
$$

and focus on the asymptotic expansion for $y \to \infty$ of the zero-mode $a_0(s;m,k|y)$.

From the seed mode expansion (4.50) we can use the results of Appendix A to arrive at the Laurent polynomial part

$$
\begin{aligned}
&a_0(s;m,k|y) \sim \\
&(-1)^{k+m} \frac{\mathrm{B}_{2k}\mathrm{B}_{2m} 4^{k+m} I_0(k+m|y)}{(2k)!(2m)!(\mu_{k+m}-\mu_s)} - (-1)^k \frac{4\mathrm{B}_{2k}(2m-3)!\zeta_{2m-1} I_0(k+1-m|y)}{(2k)!(m-2)!(m-1)!(\mu_{k-m+1}-\mu_s)} \\
&+ (-1)^k \frac{2\mathrm{B}_{2k}}{(2k)!\Gamma(m)} \sum_{\ell=k-m+1}^{k-1} g^+_{m,k,\ell,s} 4^\ell I(2m-1,\ell-k-m,\ell|y)\,,
\end{aligned} \tag{4.52}
$$

where $I_0(r|y)$, defined in (A.8a), comes from the Poincaré sum of the seed function zero-mode $c_0(y)$ (4.21), while $I(a,b,r|y)$ (4.32) comes from the Poincaré sum of the non-zero modes.

As explained in [31, 34, 35], there are a few instances where the above expression has to be regulated. For example, it is fairly easy to see from (A.8a) that whenever $k = m$ the second contribution, naively proportional to $I_0(1|y)$, is divergent. The correct way to proceed is to regulate this expression by shifting $k \to k + \epsilon$, where the expression becomes regular for all $m, k \geq 2$. To render the whole expression regular, it is actually enough to consider the regulator $I_0(k+1-m|y) \to I_0(k+\epsilon+1-m|y)$ and $I(2m-1,\ell-k-m,\ell|y) \to I(2m-1,\ell-k-\epsilon-m,\ell|y)$.

The $I(a,b,r|y)$ contribution, coming from the Poincaré sum of all the non-zero Fourier modes of the seed function, gets specialised to the particular values $I(2m-1,\ell-k-m,\ell|y)$, hence the regulator $k\to k+\epsilon$ amounts to considering an analytic continuation in the b parameter, thus introducing an asymptotic tail of factorially growing terms just as discussed in the previous section.

As we send $\epsilon\to 0$ this tail will disappear and the precise combination of $I_0(r|y)$ and $I(a,b,r|y)$ contributions will give rise to the finite Laurent polynomial (4.35). However as argued above, the non-perturbative terms needed to provide an unambiguous resummation of the formal power series for $\epsilon\neq 0$ will survive in this limit, thus giving us the full, non-perturbative zero-mode contribution $a_0(s;m,k|y)$ to $\mathcal{E}\left(s;m,k\middle|\tau\right)$.

In more detail, we can consider the non-perturbative resummation (4.49) for the formal power series $I_{asy}(a,b,r)$ and specialise it to the current case (4.52) arriving at:

$$\begin{aligned}
&a_0(s;m,k|y) = \mathrm{P}^{(s)}_{m,k}(y)+ \qquad (4.53)\\
&\lim_{\epsilon\to 0}\Big\{(-1)^k\frac{2\mathrm{B}_{2k}}{(2k)!\Gamma(m)}\sum_{\ell=k-m+1}^{k-1} g^{+}_{m,k,\ell,s}4^l\mathcal{S}_\pm\big[I_{asy}(2m-1,\ell-k-\epsilon-m,\ell|y)\big]+\mathrm{NP}^{\epsilon}_{\pm}(s;m,k|y)\Big\},
\end{aligned}$$

where we collected in $\mathrm{P}^{(s)}_{m,k}(y)$ all the regular and finitely many perturbative terms arising from the limit for $\epsilon\to 0$ of (4.52) and which reproduce the Laurent polynomial (4.35). When ϵ is sent to zero we know, from our discussion above, that the resummation of the asymptotic tail I_{asy} vanishes identically, i.e. there is no asymptotic tail when $\epsilon=0$. Finally the non-perturbative terms, which will survive in the $\epsilon\to 0$ limit, are given by

$$\begin{aligned}
&\mathrm{NP}^{\epsilon}_{\pm}(s;m,k|y) = \qquad (4.54)\\
&\frac{2\times 4^{2+m-k}y^{m+1-k}}{\Gamma(m)}\sum_{n>0}\sigma_{1-2k}(n)\sigma_{2m-1}(n)e^{-4ny}\sum_{\ell=k-m+1}^{k-1}\frac{g^{+}_{m,k,\ell,s}e^{\pm i\pi(\ell-k-\epsilon)}}{\Gamma(\ell)}\\
&\times\int_0^\infty e^{-4nyt}(t+1)^{\ell+m-k-1}\,t^{k+m+\epsilon-2\ell}\,{}_2\tilde{F}_1(1-\ell,k+m-1;k+m+1+\epsilon-2\ell|-t)\mathrm{d}t\,,
\end{aligned}$$

The suffix $\pm$ is a reminder that we have already specialised the transseries parameter $\sigma_\pm(a,b)$ from (4.48) to the present case $a=2m-1, b=\ell-k-m-\epsilon$:

$$\sigma_\pm(2m-1,\ell-k-m-\epsilon) = e^{\pm i\pi(\ell-k-\epsilon-1)}\xrightarrow{\epsilon\to 0}(-1)^{\ell+k+1}\,. \qquad (4.55)$$

The main result of this chapter is given by equation (4.54), which contains all the exponentially suppressed $(q\bar{q})^n = e^{-4ny}$ terms in the zero-mode sector of all depth-two modular invariant functions $\mathcal{E}(s;m,k)$.

The key rôle of the parameter ϵ is to regulate the Borel transform integrand in equation (4.54). To make things clearer, let us analyse the various terms appearing in the integrand and see what the regulator ϵ does.

Firstly, we see from (4.55) that the transseries parameter is perfectly regular in this limit and it reduces to $(-1)^{\ell+k+1}$. Secondly, for the range of parameters considered here, the term $(t+1)^{\ell+m-k-1}$ is simply a polynomial in t of degree at most $m-2$. Similarly, the regularised hypergeometric function ${}_2\tilde{F}_1(1-\ell, k+m-1; k+m+1+\epsilon-2\ell|-t)$ is also a polynomial in t of degree $\ell-1$, since its first entry is a non-positive integer while the third entry is generic due to the presence of the regulator ϵ.

Hence we arrive at the conclusion that the integrand can be written as a polynomial in t multiplied by $t^{k+m+\epsilon-2\ell}$, a non-integer power of t, and the usual exponential damping factor. For generic ϵ, each monomial in t can be easily integrated to produce a gamma function multiplied by a power of $(4ny)$, i.e.

$$\int_0^\infty e^{-4nyt} t^{k+m+\epsilon-2\ell} t^n \mathrm{d}t = \frac{\Gamma(k+m+n+1+\epsilon-2\ell)}{(4ny)^{k+m+n+1+\epsilon-2\ell}}\,. \tag{4.56}$$

We need to distinguish two cases now:

- When $2\ell \leq k+m$, the regulating factor $t^{k+m+\epsilon-2\ell}$ is a positive power of t and the integral is regular in the limit $\epsilon \to 0$, hence we can directly compute:

  $$\int_0^\infty e^{-4nyt}(t+1)^{\ell+m-k-1}\, t^{k+m-2\ell}\, {}_2\tilde{F}_1(1-\ell, k+m-1; k+m+1-2\ell|-t)\mathrm{d}t\,, \tag{4.57}$$

  which is a polynomial of degree $2m-1$ in $(4ny)^{-1}$.

- When $2\ell \geq k+m+1$, the regulating factor $t^{k+m+\epsilon-2\ell}$ is a negative power of t making the integral ill-defined in the strict $\epsilon = 0$ limit. However, in this case, the regularised hypergeometric series ${}_2\tilde{F}_1(1-\ell, k+m-1; k+m+1+\epsilon-2\ell|-t)$ has a negative third entry. If we write the hypergeometric in terms of Gauss' series

$$ {}_2\tilde{F}_1(1-\ell, k+m-1; k+m+1+\epsilon-2\ell| -t) = \sum_{n=0}^{\infty} \frac{(1-\ell)_n (k+m-1)_n}{\Gamma(k+m+1+\epsilon+n-2\ell)} \frac{(-t)^n}{n!} \,, $$

it is now manifest that for $0 \le n \le 2\ell - k - m$ the coefficient of t^n vanishes in the $\epsilon \to 0$ limit, being proportional to $\Gamma(k+m+1+\epsilon+n-2\ell)^{-1}$. This vanishing behaviour exactly cancels the divergence that would originate from integrating, as in (4.56), any negative power of t generated from the factor $t^{k+m+\epsilon-2\ell}$ term. We find that again there is a well-defined limit $\epsilon \to 0$, that has to be taken *after* having performed the t-integral:

$$ \lim_{\epsilon \to 0} \Big[\int_0^\infty e^{-4nyt} (t+1)^{\ell+m-k-1}\, t^{k+m+\epsilon-2\ell}\, {}_2\tilde{F}_1(1-\ell, k+m-1; k+m+1+\epsilon-2\ell| -t) \mathrm{d}t \Big] \,, $$

which is again a polynomial of degree $2m-1$ in $(4ny)^{-1}$.

This concludes the proof that equation (4.54) is regular as $\epsilon \to 0$. This limit is interpreted as describing the non-perturbative corrections to the Fourier zero mode of $\mathcal{E}(s; m, k)$ through resurgent analysis and we have recovered the exact behaviour of the zero-mode at the cusp.

4.6 Some Examples

We list some of the results for the zero-mode $a_0(s; m, k|y)$ that follow from the previously derived calculations for a few small values of m, k, s.

In the $(2, k)$ sector, where there is a single eigenvalue $s = k$, we have:

$$ a_0(2; 2, 2|y) = \frac{y^4}{20250} - \frac{y\zeta_3}{45} - \frac{5\zeta_5}{12y} + \frac{\zeta_3^2}{4y^2} + \sum_{n=1}^{\infty} \frac{e^{-4ny}\sigma_{-3}(n)^2}{2y^2} \,, \tag{4.58} $$

$$ a_0(3; 2, 3|y) = \frac{y^5}{297675} - \frac{y^2\zeta_3}{1890} - \frac{\zeta_5}{360} - \frac{7\zeta_7}{64y^2} + \frac{\zeta_3\zeta_5}{8y^3} + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-5}(n)\sigma_{-3}(n) \Big[\frac{1}{4y^3} + \frac{n}{4y^2}\Big] \,, \tag{4.59} $$

$$ \begin{aligned} a_0(4; 2, 4|y) = {}& \frac{y^6}{3827250} - \frac{y^3\zeta_3}{28350} - \frac{\zeta_7}{720y} - \frac{25\zeta_9}{432y^3} + \frac{5\zeta_3\zeta_7}{64y^4} \\ & + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)\sigma_{-3}(n) \Big[\frac{5}{32y^4} + \frac{5n}{24y^3} + \frac{n^2}{12y^2}\Big] \,. \end{aligned} \tag{4.60} $$

Equation (4.58) is identical to the result of [33] for the exponentially suppressed terms of the

MGF $C_{2,1,1}$ once we use the fact that $\mathcal{E}(2;2,2) = -C_{2,1,1} + \frac{9}{10}\mathcal{E}_4$.

In the $(3,k)$ sector, with $k \geq 3$, we encounter two choices of eigenvalues $s \in \{k-1, k+1\}$:

$$a_0(2;3,3|y) = \frac{y^6}{6251175} - \frac{y\zeta_5}{630} - \frac{5\zeta_7}{288y} + \frac{\zeta_5^2}{32y^4} + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-5}(n)^2\Big[\frac{1}{16y^4} + \frac{n}{4y^3} + \frac{n^2}{8y^2}\Big], \quad (4.61)$$

$$a_0(4;3,3|y) = \frac{2y^6}{8037225} - \frac{y\zeta_5}{3780} - \frac{35\zeta_9}{1152y^3} + \frac{9\zeta_5^2}{128y^4} + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-5}(n)^2\Big[\frac{9}{64y^4} + \frac{n}{4y^3} + \frac{n^2}{8y^2}\Big], \quad (4.62)$$

$$\begin{aligned} a_0(3;3,4|y) &= \frac{y^7}{80372250} - \frac{y^2\zeta_5}{25200} - \frac{\zeta_7}{4536} - \frac{49\zeta_9}{11520y^2} + \frac{5\zeta_5\zeta_7}{256y^5} \\ &\quad + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)\sigma_{-5}(n)\Big[\frac{5}{128y^5} + \frac{5n}{32y^4} + \frac{7n^2}{48y^3} + \frac{n^3}{24y^2}\Big], \end{aligned} \quad (4.63)$$

$$\begin{aligned} a_0(5;3,4|y) &= \frac{y^7}{49116375} - \frac{y^2\zeta_5}{113400} - \frac{\zeta_7}{15120} - \frac{77\zeta_{11}}{4608y^4} + \frac{3\zeta_5\zeta_7}{64y^5} \\ &\quad + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)\sigma_{-5}(n)\Big[\frac{3}{32y^5} + \frac{37n}{192y^4} + \frac{7n^2}{48y^3} + \frac{n^3}{24y^2}\Big]. \end{aligned} \quad (4.64)$$

As a last example, in the $(4,k)$ sector with $k \geq 4$ and eigenvalues $s \in \{k-2, k, k+2\}$, we have:

$$\begin{aligned} a_0(2;4,4|y) &= \frac{y^8}{1205583750} - \frac{y\zeta_7}{7560} - \frac{5\zeta_9}{3888y} + \frac{5\zeta_7^2}{512y^6} \\ &\quad + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)^2\Big[\frac{5}{256y^6} + \frac{5n}{64y^5} + \frac{35n^2}{288y^4} + \frac{5n^3}{72y^3} + \frac{n^4}{72y^2}\Big], \end{aligned} \quad (4.65)$$

$$\begin{aligned} a_0(4;4,4|y) &= \frac{y^8}{982327500} - \frac{y\zeta_7}{45360} - \frac{7\zeta_{11}}{6912y^3} + \frac{5\zeta_7^2}{384y^6} \\ &\quad + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)^2\Big[\frac{5}{192y^6} + \frac{5n}{48y^5} + \frac{25n^2}{192y^4} + \frac{5n^3}{72y^3} + \frac{n^4}{72y^2}\Big], \end{aligned} \quad (4.66)$$

$$\begin{aligned} a_0(6;4,4|y) &= \frac{y^8}{580466250} - \frac{y\zeta_7}{113400} - \frac{5055\zeta_{13}}{530688y^5} + \frac{25\zeta_7^2}{768y^6} \\ &\quad + \sum_{n=1}^{\infty} e^{-4ny}\sigma_{-7}(n)^2\Big[\frac{25}{384y^6} + \frac{29n}{192y^5} + \frac{7n^2}{48y^4} + \frac{5n^3}{72y^3} + \frac{n^4}{72y^2}\Big]. \end{aligned} \quad (4.67)$$

One can check that these results are in agreement with the differential equation (4.16).

4.7 Exact results

From the general results derived previously, we know that the zero-mode (4.53) for the modular invariant function $\mathcal{E}(s;m,k)$ is given by

$$a_0(s;m,k|y) = \mathrm{P}^{(s)}_{m,k}(y) + \mathrm{NP}^{(s)}_{m,k}(y)\,, \tag{4.68}$$

with the perturbative terms given by the Laurent polynomials (4.35) and the non-perturbative terms

$$\mathrm{NP}^{(s)}_{m,k}(y) = \lim_{\epsilon\to 0} \mathrm{NP}^{\epsilon}_{\pm}(s;m,k|y)\,,$$

simply obtained from (4.54) by sending $\epsilon \to 0$.

We can use the general result (4.54) to write the non-perturbative terms as

$$\mathrm{NP}^{(s)}_{m,k}(y) = \sum_{n>0} e^{-4ny} \frac{n^{k+m-2}\sigma_{1-2m}(n)\sigma_{1-2k}(n)}{\Gamma(m)\Gamma(k)} \phi^{(s)}_{m,k}(4ny)\,, \tag{4.69}$$

where we used the divisor sum identity $\sigma_s(n) = n^s\sigma_{-s}(n)$, and defined $\phi^{(s)}_{m,k}(y)$ by

$$\begin{aligned}\phi^{(s)}_{m,k}(y) = \lim_{\epsilon\to 0} \Big[8\Gamma(k) y^{1+m-k} \sum_{\ell=k-m+1}^{k-1} \frac{g^{+}_{m,k,\ell,s}(-1)^{\ell+k}}{\Gamma(\ell)} \\ \times \int_0^\infty e^{-yt}(t+1)^{m+\ell-k-1}\, t^{k+m+\epsilon-2\ell}\, {}_2\tilde{F}_1(1-\ell,k+m-1;k+m+1+\epsilon-2\ell|-t)\,\mathrm{d}t \Big]\,.\end{aligned} \tag{4.70}$$

The non-perturbative terms in the zero Fourier mode could have also been obtained by using the ansatz (4.69) and substituting it into the inhomogeneous Laplace equation (4.16) satisfied by the $\mathcal{E}(s;m,k)$. From the Fourier mode expansion for the Eisenstein series (4.13) we can readily isolate the $(q\bar{q})^n$ contribution of the source term $\mathcal{E}_m\mathcal{E}_k$. This results in a second-order differential equation for $\phi^{(s)}_{m,k}(y)$ that could be solved using a Laurent series ansatz. The solution found in this way can be checked to agree with the results presently obtained via resurgent analysis. Additionally, in section 5.6 we obtain the same result from a spectral theory perspective, which also shows that the formula is not restricted to integer weights and eigenvalue (see eqn. (5.77)).

From the discussion below (4.56), we have that $\phi^{(s)}_{m,k}(y)$ is a polynomial of degree $k+m-2$

in y^{-1} with rational coefficients. We will now prove that

$$\phi^{(s)}_{m,k}(y) = \tag{4.71}$$

$$\frac{8}{y^2} + \frac{8[m(m-1)+k(k-1)-4]}{y^3} + 4\frac{\Big\{[m(m-1)+k(k-1)-7]^2 + 2s(s-1) - 13\Big\}}{y^4} + O(y^{-5}).$$

Note that the coefficients of higher corrections in y^{-1} will in general have a dependence on the eigenvalue s.

By using the integral transform (4.56), we observe that the leading contribution to $\phi^{(s)}_{m,k}(y)$ as $y \to \infty$ comes from the lowest power of t in the integrand of equation (4.70). To isolate this monomial, we start by noting that the lowest exponent for the factor $t^{k+m+\epsilon-2\ell}$ is clearly given by the highest value of the parameter $\ell = \ell_{max} = k-1$. In this case we have a simplification for the coefficients (4.22) $g^+_{m,k,\ell,s}$ appearing in (4.70), in that $g^+_{m,k,k-1,s} = \frac{1}{k-1}$ is independent of the eigenvalue s.

To obtain the lowest power of t in the integrand of (4.70), we similarly have to choose the constant term for both the hypergeometric series as well as the binomial when $\ell = k-1$:

$$\begin{aligned} {}_2\tilde{F}_1(2-k, k+m-1; m+3+\epsilon-k|-t) &= \frac{1}{\Gamma(m+3+\epsilon-k)} + \frac{(k-2)(k+m-1)}{\Gamma(m+4+\epsilon-k)} t + O\big(t^2\big)\,, \\ (t+1)^{m-2} &= 1 + (m-2)t + O\big(t^2\big)\,. \end{aligned} \tag{4.72}$$

We then arrive at the leading, large-y asymptotic for (4.70) given by

$$\phi^{(s)}_{m,k}(y) \sim \lim_{\epsilon\to 0}\Big[8\Gamma(k) y^{1+m-k}\frac{g^+_{m,k,k-1,s}}{\Gamma(k-1)}\int_0^\infty e^{-yt}\,\frac{t^{m+2+\epsilon-k}}{\Gamma(m+3+\epsilon-k)}\,\mathrm{d}t\Big] \sim \frac{8}{y^2} + \dots\,, \tag{4.73}$$

where we used the standard integral (4.56) and reproduced the leading order in (4.71).

For the sub-leading correction in (4.71) we need to investigate higher powers of t in the integrand of (4.70). Firstly we observe that decreasing $\ell \to \ell_{max} - 1 = k-2$ increases the power of t by 2 for the $t^{k+m+\epsilon-2\ell}$ term in the integrand. Hence we deduce that the next sub-leading correction comes again from $\ell = \ell_{max} = k-1$ where we consider instead the linear terms in t for the hypergeometric function and the binomial (4.72). As a result, since the coefficient $g^+_{m,k,k-1,s} = \frac{1}{k-1}$ does not depend on the eigenvalue s, we have that, just like for the leading term, the $\frac{1}{y^3}$ coefficient must once more be eigenvalue independent.

The calculation is very similar to the one presented above

$$\phi^{(s)}_{m,k}(y) \sim \frac{8}{y^2} + \lim_{\epsilon\to 0}\Big[8\,y^{1+m-k}\int_0^\infty e^{-yt}\,\frac{t^{m+3+\epsilon-k}}{\Gamma(m+3+\epsilon-k)}\Big[(m-2)+\frac{(k-2)(k+m-1)}{m+3+\epsilon-k}\Big]\,\mathrm{d}t\Big]$$
$$\sim \frac{8}{y^2} + \frac{8[m(m-1)+k(k-1)-4]}{y^3} + \dots\,, \tag{4.74}$$

and we reproduce, as anticipated, the sub-leading term of equation (4.71).

Getting analytic expressions for higher-order terms becomes slightly more complicated, since multiple values of ℓ in (4.70) start contributing and the coefficients $g^+_{m,k,\ell,s}$, see (4.22), are in general eigenvalue dependent, thus higher-order terms do depend on the eigenvalue s as well. For example, we can repeat a very similar discussion to the one above above for the $O(y^{-4})$ term, which receives two different contributions - one from $\ell = k-1$ and a second one from $\ell = k-2$. By using (4.22) to obtain the coefficient $g^+_{m,k,k-2,s}$

$$g^+_{m,k,k-2,s} = \frac{m(m-1)}{(k-2)} + \frac{(k-s-1)(k+s-2)}{(k-1)(k-2)}\,, \tag{4.75}$$

and then collect the appropriate powers of t in the integrand, we arrive at

$$\phi^{(s)}_{m,k}(y) \sim \tag{4.76}$$
$$\frac{8}{y^2} + \frac{8[m(m-1)+k(k-1)-4]}{y^3} + 4\frac{\Big\{[m(m-1)+k(k-1)-7]^2+2s(s-1)-13\Big\}}{y^4} + \dots\,.$$

All of the results here discussed can be checked for comparison with the examples given in section 4.6 and are consistent with the Laplace equation (4.16). In the next chapter we obtain another representation for the function $\phi^{(s)}_{m,k}(y)$ from the perspective of spectral theory and show that the large y asymptotics obtained here is more general and also works for non-integer m, k, s (see equation (5.78)).

4.8 Modularity and recovering the small-y behaviour

Up until now we have used the asymptotic nature of the large-y perturbative expansion to reconstruct the non-perturbative, exponentially suppressed $(q\bar{q})^n$ corrections via resurgent analysis. Now we want to understand a similar, yet conceptually different problem, namely is it possible to reconstruct the perturbative data, i.e. the Laurent polynomials (4.35), from the small-y expansion of the $(q\bar{q})^n$ terms? We will see that, complementary to resurgence, modularity will play a crucial rôle.

First of all, we recall here an important lemma proved in [45].

Lemma. *If* $\mathrm{F}(\tau)$ *is an* $\mathrm{SL}(2,\mathbb{Z})$ *invariant function on the upper half-plane such that at the cusp* $y\to\infty$*, with* $y=\pi\tau_2=\pi\,Im(\tau)$*, it satisfies the growth condition* $\mathrm{F}(\tau)=O(y^s)$ *with* $s>1$*, then each of its Fourier modes* $\mathrm{F}_n(y)=\int_0^1\mathrm{F}(\tau_1+iy/\pi)e^{-2\pi in\tau_1}\mathrm{d}\tau_1$ *satisfies the bound* $\mathrm{F}_n(y)=O(y^{1-s})$ *in the limit* $y\to0$*.*

Very roughly, the key idea behind this lemma is that a cuspidal growth of order y^s suggests that the modular invariant function $\mathrm{F}(\tau)$ must be bounded by $\mathcal{E}_s(\tau)$ on the whole upper half-plane and since for small y we have $\mathcal{E}_s(\tau)=O(y^{1-s})$, then the same bound must hold for $\mathrm{F}(\tau)$. Let us apply this lemma to our modular invariant functions $\mathcal{E}\,(s;m,k)$ and in particular let us try and understand the small-y behaviour of its zero-mode (4.68).

From the explicit Laurent polynomial (4.35) it is clear that $\mathcal{E}\left(s;m,k\middle|\tau\right)=O(y^{k+m})$ as $\tau\to i\infty$, hence from the lemma we deduce that for small y each Fourier mode of $\mathcal{E}\,(s;m,k)$ cannot be more singular than $O(y^{1-k-m})$. We can easily see from (4.35) that, for the spectrum of eigenvalues considered here, none of the perturbative terms is more singular than y^{1-k-m} and we conclude that the $(q\bar{q})^n$ terms (4.69), which were exponentially suppressed for large y, can at most diverge as y^{1-k-m} as $y\to0$.

We can run a more refined argument to analytically obtain part of the small-y limit of the $(q\bar{q})^n$ terms. To this end we can consider the modular invariant linear combination

$$\mathrm{F}(\tau)=\mathcal{E}\left(s;m,k\middle|\tau\right)+\alpha\,\mathcal{E}_{m+k}(\tau)\,, \tag{4.77}$$

where the constant α, given by

$$\alpha=\frac{\mathrm{B}_{2m}\mathrm{B}_{2k}(2m+2k)!}{\mathrm{B}_{2m+2k}(k{+}m{-}s)(k{+}m{+}s{-}1)(2m)!(2k)!}, \tag{4.78}$$

is chosen in a such a way (see (4.13) and (4.35)) that the coefficient of the leading term y^{k+m} of (4.77) is vanishing.

If we assume that $k>m$, we have thus obtained a new auxiliary modular invariant function $\mathrm{F}(\tau)$ with the tamer growth at the cusp $\mathrm{F}(\tau)=O(y^{1+k-m})$. Note that we have excluded the diagonal case, $k = m$, since $F(\tau)$ would grow at the cusp linearly as $O(y^1)$, hence the Lemma cannot be applied directly; we will however show a diagonal example where the results are consistent with the non-diagonal expectations and we can view the diagonal case as the limit $k \to m$.

By applying the lemma to $\mathrm{F}(\tau)$ we deduce that its small-y limit cannot be more singular than $O(y^{m-k})$. However, if we inspect all the powers appearing in the perturbative expansion (4.13) and (4.35) of the zero-mode, we find that the terms y^{1-s}, y^{2-k-m}, coming from $\mathcal{E}\left(s; m, k \middle| \tau\right)$, and the term y^{1-k-m}, coming from $\alpha\, \mathcal{E}_{k+m}(\tau)$, all violate the bound. Since the addition of $\alpha\, \mathrm{E}_{k+m}(\tau)$ does not modify the $(q\bar{q})^n$ sector, we must conclude that the small-y limit of the $(q\bar{q})^n$ terms (5.77) must exactly cancel against these singular terms. The small-y expansion of the $(q\bar{q})^n$ must then take the form:

$$\mathrm{NP}^{(s)}_{m,k}(y) = -c^{(s)}_{m,k}\zeta_{k+m+s-1}y^{1-s} - \frac{4^{3-m-k}\Gamma(2m-1)\Gamma(2k-1)\zeta_{2m-1}\zeta_{2k-1}}{[\Gamma(m)\Gamma(k)]^2(k+m-s-1)(k+m+s-2)}y^{2-k-m}$$
$$- \alpha\frac{4(2m+2k-3)!\zeta_{2m+2k-1}}{(m+k-2)!(m+k-1)!}(4y)^{1-m-k} + O(y^{m-k})\,. \tag{4.79}$$

Obtaining this expression directly from the small-y limit of (5.77) is not straightforward. A somewhat naive way to proceed is to expand the exponential factor $(q\bar{q})^n = e^{-4ny}$ for small-y and compute the sum over n term by term via its analytic continuation as a Dirichlet series using Ramanujan's identity (4.33).

To illustrate this, we first repeat the calculation, discussed in the previous section, to obtain the most singular term at small-y for $\phi^{(s)}_{m,k}(y)$

$$\phi^{(s)}_{m,k}(y) = \frac{8\Gamma(2m-1)\Gamma(2k-1)}{(k+m-s-1)(k+m+s-2)\Gamma(m)\Gamma(k)}y^{2-k-m} + O(y^{3-k-m})\,. \tag{4.80}$$

We can now consider its contribution in the small-y limit to $\mathrm{NP}^{(s)}_{m,k}(y)$ given by:

$$\begin{aligned}\mathrm{NP}^{(s)}_{m,k}(y) &= \sum_{n>0} e^{-4ny}\frac{n^{k+m-2}\sigma_{1-2m}(n)\sigma_{1-2k}(n)}{\Gamma(m)\Gamma(k)} \\ &\quad\times\Big[\frac{8\Gamma(2m-1)\Gamma(2k-1)}{(k+m-s-1)(k+m+s-2)\Gamma(m)\Gamma(k)}(4ny)^{2-k-m}+O(y^{3-k-m})\Big] \\ &\sim \frac{8\Gamma(2m-1)\Gamma(2k-1)}{(k+m-s-1)(k+m+s-2)[\Gamma(m)\Gamma(k)]^2}(4y)^{2-k-m}\sum_{n>0}\sigma_{1-2m}(n)\sigma_{1-2k}(n)+O(y^{3-k-m}) \\ &\sim -\frac{4^{3-m-k}\Gamma(2m-1)\Gamma(2k-1)\zeta_{2m-1}\zeta_{2k-1}}{[\Gamma(m)\Gamma(k)]^2(k+m-s-1)(k+m+s-2)}y^{2-k-m}+O(y^{3-k-m})\,,\end{aligned} \tag{4.81}$$

where we expanded the exponential term $e^{-4ny}=1+O(y)$ to leading order at small-y and used the analytic continuation at $s=0$ of Ramanujan's identity to resum

$$\sum_{n>0}\sigma_{1-2m}(n)\sigma_{1-2k}(n)\text{``}=\text{''}\zeta(0)\zeta(2m-1)\zeta(2k-1)\,. \tag{4.82}$$

This calculation reproduces precisely the expected y^{2-k-m} term in equation (4.79) .

Using the explicit examples (4.58), (4.61) and (4.65) presented before, it is possible to perform a similar argument to compute also the sub-leading corrections (4.79) by means of analytically continuing the sum over n as a Dirichlet series. We notice, however, that the most singular term in (4.79) is of order y^{1-m-k} and cannot possibly be obtained via this naïve analysis.

A more careful analysis of the small-y expansion of (5.77) can be derived from a Mellin transform argument. From the generic expression (5.77), it is easy to see that $\mathrm{NP}^{(s)}_{m,k}(y)$ is given by a finite linear combination of functions defined by

$$D_{a,b;c}(y)=\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^c}e^{-ny}, \tag{4.83}$$

with $a=1-2m, b=1-2k$ and $c\in\mathbb{Z}_{\leq 0}$.

In appendix B we derive the small-y behaviour (B.5) of the function $D_{a,b;c}(y)$ with $a,b,c\in\mathbb{C}$ generic. Using Mellin inversion formula, the asymptotic expansion at $y\to 0$ of $D_{a,b;c}(y)$ is related to the poles and residues of its Mellin transform $M_{a,b;c}(y)$.

We refer to appendix B for the general discussion and present here a few concrete examples. Let us consider the non-perturbative terms $\mathrm{NP}^{(3)}_{2,3}(y)$ for the zero Fourier mode of the depth-2 modular function $\mathcal{E}(3;2,3)$, which are given by (4.59)

$$\begin{aligned}\mathrm{NP}^{(3)}_{2,3}(y) &= \frac{1}{4}\sum_{n=1}^{\infty}\sigma_{-5}(n)\sigma_{-3}(n)e^{-4ny}\Big[\frac{1}{y^3}+\frac{n}{y^2}\Big]\\ &= \frac{1}{4y^3}D_{-5,-3;0}(4y)+\frac{1}{4y^2}D_{-5,-3;-1}(4y).\end{aligned}$$

The relevant Mellin transforms, see (B.3), are

$$M_{-5,-3;0}(t) = \int_0^\infty D_{-5,-3;0}(y)\,y^{t-1}\mathrm{d}y = \frac{\Gamma(t)\zeta(t)\zeta(3+t)\zeta(5+t)\zeta(8+t)}{\zeta(8+2t)}\,, \tag{4.84}$$

$$M_{-5,-3;-1}(t) = \int_0^\infty D_{-5,-3;-1}(y)\,y^{t-1}\mathrm{d}y = \frac{\Gamma(t)\zeta(-1+t)\zeta(2+t)\zeta(4+t)\zeta(7+t)}{\zeta(6+2t)}\,, \tag{4.85}$$

from which it is easy to see that $M_{-5,-3;0}(t)$ has simple poles at $t\in\mathbb{Z}$ in the range $-8\leq t\leq 1$, while $M_{-5,-3;-1}(t)$ has simple poles at $t\in\mathbb{Z}$ in the range $-7\leq t\leq 2$, excluding $t=1$. Both of the transforms also have poles coming from the zeta function in the denominator that are associated with non-trivial zeros of the Riemann zeta.

Referring to appendix B for the details, we can use the Mellin inversion formula (B.4) and, after the little exercise of computing the residues at these poles, we arrive at the small-y expansion for $\mathrm{NP}^{(3)}_{2,3}(y)$:

$$\begin{aligned}\mathrm{NP}^{(3)}_{2,3}(y) \sim{}& \frac{11\zeta_9}{128y^4}-\frac{\zeta_3\zeta_5}{8y^3}+\frac{7\zeta_7}{64y^2}-\frac{\zeta_3^2}{42y}+\frac{\zeta_5}{360}+\frac{\zeta_3y^2}{1890}-\frac{\zeta_7y^3}{3240\zeta_5}+\frac{\zeta_3\zeta_5y^4}{23625\zeta_7}-\frac{y^5}{297675}\\ &+\sum_{\rho_n}\frac{4^{1-t}(1+t)\Gamma(t-3)\zeta(t-3)\zeta(t)\zeta(t+2)\zeta(t+5)}{2\zeta'(\frac{1}{2}+i\rho_n)}y^{-t}\Big|_{t=-\frac{3}{4}+\frac{i\rho_n}{2}}\,,\end{aligned} \tag{4.86}$$

where the sum in the second line is over the non-trivial zeros of the zeta function $\frac{1}{2}+i\rho_n$. A comparison with (4.59) reveals that the small-y limit of the non-perturbative terms not only matches perfectly the expected behaviour (4.79) but it actually cancels exactly the full Laurent polynomial part:

$$\mathrm{NP}^{(3)}_{2,3}(y) \sim \frac{11\zeta_9}{128y^4}-P^{(3)}_{2,3}(y)-\frac{\zeta_3^2}{42y}-\frac{\zeta_7y^3}{3240\zeta_5}+\frac{\zeta_3\zeta_5y^4}{23625\zeta_7}+\sum_{\rho_n}\#y^{\frac{3}{4}+\frac{i\rho_n}{2}}\,, \tag{4.87}$$

with the sum over ρ_n having the same coefficients as in (4.86). The difference between the small-y limit of the non-perturbative sector and the Laurent series is given by the expected

y^{1-k-m} monomial of (4.79) and terms sub-leading as $y \to 0$. Although the polynomial piece presents novel types of coefficients, in the form of ratios of zeta values, they do respect uniform transcendentality with standard weight assignment. It is not clear what weight assignment (if any) should be given to the sum over the non-trivial zeros of the zeta function.

As a second example, we can analyse the small-y limit of the non-perturbative terms $\mathrm{NP}^{(s)}_{m,k}(y)$ in the diagonal sector $k = m$. For simplicity let us consider the non-perturbative terms $\mathrm{NP}^{(2)}_{2,2}(y)$ of the modular function $\mathcal{E}\,(2;2,2)$ that are given by (4.58)

$$\mathrm{NP}^{(2)}_{2,2}(y) = \sum_{n=1}^{\infty} \frac{e^{-4ny}\sigma_{-3}(n)^2}{2y^2} = \frac{1}{2y^2} D_{-3,-3;0}(4y). \tag{4.88}$$

The corresponding Mellin transform is given by

$$M_{-3,-3;0}(t) = \int_0^{\infty} D_{-3,-3;0}(y)\, y^{t-1} \mathrm{d}y = \frac{\Gamma(t)\zeta(t)\zeta(3+t)^2\zeta(6+t)}{\zeta(6+2t)}, \tag{4.89}$$

which has poles at $t \in \mathbb{Z}$ in the range $-6 \le t \le 1$. The key difference between the diagonal sector and the previous, non-diagonal example is the appearance of a double pole at $t = -2$, while all others are simple poles. Referring again to appendix B for all the details, in this case we have that a second order pole in the Mellin transform signals the presence of logarithmic corrections, $\log y$, in the asymptotic expansion as $y \to 0$ of $\mathrm{NP}^{(2)}_{2,2}(y)$.

The asymptotic expansion as $y \to 0$ of $\mathrm{NP}^{(2)}_{2,2}(y)$ is given by

$$\begin{aligned} \mathrm{NP}^{(2)}_{2,2}(y) \sim\ & \frac{7\zeta_7}{48y^3} - \frac{\zeta_3^2}{4y^2} + \frac{5\zeta_5}{12y} + \frac{\zeta_3}{15}\Big[\log\Big(\frac{8\pi y}{A^{24}}\Big) + \frac{\zeta_3'}{\zeta_3} - \frac{\zeta_4'}{\zeta_4}\Big] + \frac{\zeta_3 y}{45} - \frac{\zeta_5 y^2}{108\zeta_3} + \frac{2\zeta_3^2 y^3}{2835\zeta_5} - \frac{y^4}{20250} \\ & + \sum_{\rho_n} \frac{2^{2-2t}\Gamma(t-2)\zeta(t-2)\zeta(t+1)^2\zeta(t+4)}{\zeta'(\frac{1}{2}+i\rho_n)} y^{-t}\Big|_{t=-\frac{3}{4}+i\frac{\rho_n}{2}} \\ \sim\ & \frac{7\zeta_7}{48y^3} - P^{(2)}_{2,2}(y) + \frac{\zeta_3}{15}\Big[\log\Big(\frac{8\pi y}{A^{24}}\Big) + \frac{\zeta_3'}{\zeta_3} - \frac{\zeta_4'}{\zeta_4}\Big] - \frac{\zeta_5 y^2}{108\zeta_3} + \frac{2\zeta_3^2 y^3}{2835\zeta_5} + \sum_{\rho_n} \# y^{\frac{3}{4}+\frac{i\rho_n}{2}}\,, \end{aligned} \tag{4.90}$$

where A is the Glaisher–Kinkelin constant, $\log A = \frac{1}{12} - \zeta'(-1)$. If we assign transcendental weight 1 to $\log(8\pi y/A^{24}) + \zeta_3'/\zeta_3 - \zeta_4'/\zeta_4$, then the polynomial/logarithmic piece respects uniform transcendentality. This appears to be a variant of the transcendentality assignments in [46]. Note that again we reproduce the expected behaviour (4.79), furthermore, after comparison with the Laurent polynomial $P^{(2)}_{2,2}(y)$ (4.58), we have the stronger statement that the non-perturbative terms cancel exactly the perturbative ones as for the previous non-diagonal example.

For the general small-y expansion of $\mathrm{NP}^{(s)}_{m,k}(y)$, we see that (5.77) can be expressed as a finite linear combination of building blocks $D_{a,b;c}(y)$, defined in (4.83), with $a = 1-2m, b = 1-2k$ and $c \in \mathbb{Z}_{\leq 0}$. For this range of parameters, it is easy to see from the general formula (B.5), derived in appendix B, that the coefficients appearing in the polynomial piece of $\mathrm{NP}^{(s)}_{m,k}(y)$ in the small-y limit can be at most ratios of bilinears in odd zetas divided by a single odd zeta, or in the diagonal case $m = k$ contain at most one derivative of a Riemann zeta in the logarithmic part. Of course, one always has in addition the infinite sum over the non-trivial zeros of the zeta function.

Although quite different in spirit to the main message of this chapter, the small-y behaviour can also be retrieved by exploiting the spectral decomposition of the modular functions $\mathcal{E}(s;m,k)$ in terms of L^2-normalisable eigenfunctions of the $\mathrm{SL}(2,\mathbb{R})$ invariant Laplacian $\Delta = 4\mathrm{Im}(\tau)^2\partial_\tau\partial_{\bar\tau}$. This interesting interplay between resurgence and spectral theory, generalising the results of this chapter, is analysed in chapter 5 of the thesis. See also [47, 48] for the spectral decomposition of $\mathcal{E}(s;m,k)$ with $s,m,k \in \mathbb{C}$. In particular, amongst the L^2-normalisable eigenfunctions of the Laplacian, *non-holomorphic* cusp forms should play a special rôle in reconstructing the "instanton" sector, i.e. q^n terms in the Fourier decomposition of the modular functions $\mathcal{E}(s;m,k)$. Previous works [35] have shown from the different point of iterated integrals that *holomorphic* cusp forms do also play a rôle in the instantonic sector, however there is no obvious or straightforward connection between the holomorphic and non-holomorphic cusp functions.

As a concluding remark for this chapter we want to stress that if resurgent analysis allows us to retrieve the exponentially suppressed and non-perturbative corrections at large-y from the perturbative data, modularity dramatically intertwines the two and permits us to reconstruct the Laurent polynomial from the small-y behaviour of the infinite tower of $(q\bar{q})^n$ terms, no longer exponentially suppressed.

CHAPTER 5

S-duality in Type IIB string theory and generalised Eisenstein series

When discussing the low-energy expansion of type IIB superstring theory, the group $\mathrm{SL}(2,\mathbb{Z})$ is interpreted as the non-perturbative U-duality group of ten-dimensional type IIB string theory [49]. In the classical theory the vacuum expectation value of the axio-dilaton scalar field is given by

$$\tau = \chi + \frac{i}{g_s} = \mathrm{Re}(\tau) + i\,\mathrm{Im}(\tau)\,,$$

with g_s the string coupling constant, and parametrises the coset space $\mathfrak{H} = SL(2,\mathbb{R})/U(1)$, which is the upper half-plane that we are quite familiar with by now. However, quantum corrections [50] generate an anomaly in the U(1) R-symmetry thus breaking $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{Z})$. This U-duality symmetry group, $\mathrm{SL}(2,\mathbb{Z})$, acts on the axio-dilaton in the standard way

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \qquad \gamma\cdot\tau := \frac{a\tau+b}{c\tau+d}\,. \tag{5.1}$$

Note that although the transformation property is identical to the one is section 4, the physical interpretation is radically different. While in the case of string perturbation theory the parameter τ was identified as a modulus of a torus, now we instead think of it as a dynamical field in the theory. Since the axio-dilaton includes the string coupling, g_s, U-duality is an extremely powerful and non-perturbative symmetry, relating perturbative and non-perturbative effects.

The low-energy expansion of type IIB superstring theory is therefore expected to be invariant (or covariant) under $\mathrm{SL}(2,\mathbb{Z})$, where τ parameterises a fundamental domain that can be chosen to be the standard $\mathcal{F}$ defined in (3.4).

At low energy type IIB supergravity receives corrections coming from excited string states, which can be neatly assembled in an effective Lagrangian. Focusing for simplicity only on four-graviton interactions, we expect to find an effective Lagrangian containing the standard Einstein-Hilbert term, as well as an infinite tower of higher-derivative corrections schematically of the form $d^{2n}R^4$, where R^4 is a certain contraction of Riemann tensors and d is the covariant derivative. For $n \leq 3$ these terms are fixed by supersymmetry to be of the form (in the string frame)

$$\mathcal{L}_{\text{eff}} = (\alpha')^{-4} g_s^{-2} R + (\alpha')^{-1} g_s^{-\frac{1}{2}} \pi^{\frac{3}{2}} \mathcal{E}_{\frac{3}{2}}(\tau) R^4 + \alpha' g_s^{\frac{1}{2}} \pi^{\frac{5}{2}} \mathcal{E}_{\frac{5}{2}}(\tau) d^4 R^4 - (\alpha')^2 g_s \pi^3 \mathcal{E}\left(4; \tfrac{3}{2}, \tfrac{3}{2} \middle| \tau\right) d^6 R^4 + \dots, \tag{5.2}$$

where $\alpha' = \ell_s^2$ is the square of the string length scale.

As expected, the leading term when $\alpha' \to 0$ is simply given by the Einstein-Hilbert term (where R is the Ricci scalar). Although we only wrote the bosonic part, this term comes with its supersymmetric completion, involving other bosonic as well as fermionic fields, reproducing the type IIB supergravity lagrangian in ten dimensions.

For the higher-derivative corrections here reviewed, we have that maximal supersymmetry uniquely fixes the Lorentz contractions of the tensor indices and forbids the presence of R^2 and R^3 interactions. The first correction is proportional to R^4 [51,52] which is a 1/2-BPS operator, i.e. it preserves only 16 of the 32 supersymmetries associated with ten-dimensional maximal supersymmetry. Similarly, the higher-derivative term d^4R^4 is 1/4-BPS while d^6R^4 is 1/8-BPS and it is the last term to be protected by supersymmetry.

The ellipsis in (5.2) represents various supersymmetric completions, as well as higher-order terms and terms contributing to higher-point amplitudes. A particular class of interesting higher-point BPS amplitudes involve the scattering of four gravitons with certain massless fields of type IIB supergravity carrying specific U(1) charges [53, 54] and transforming covariantly under U-duality. The modular properties of these amplitudes have been analysed in [55], while their connection with the holographic dual picture of integrated correlators in $\mathcal{N} = 4$ SYM is presented in [56, 57]. We will not be discussing these corrections here.

Interestingly, the coefficients of the higher-derivative and BPS protected corrections displayed in (5.2) can be computed exactly and are expressible in terms of special modular invariant functions. In particular, we see that the coefficient of the R^4 [58–60] and the d^4R^4 [61] interactions involve non-holomorphic Eisenstein series, already introduced in (4.13) for integer weight, but which have the more general representation:

$$\begin{aligned}\mathcal{E}_s(\tau) &:= \sum_{(m,n)\neq(0,0)} \frac{(\tau_2/\pi)^s}{|m+n\tau|^{2s}} \\ &= \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\xi(2s-1)}{\Gamma(s)}\tau_2^{1-s} + \frac{4}{\Gamma(s)}\sum_{k\neq 0}|k|^{s-\frac{1}{2}}\sigma_{1-2s}(k)\tau_2^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|k|\tau_2)e^{2\pi ik\tau_1}\end{aligned} \tag{5.3}$$

with $\tau = \tau_1 + i\tau_2 \in \mathfrak{H}$ and $\mathrm{Re}(s) > 1$ for now. In this chapter another convention for the Eisenstein series, given in equation (3.20), will be useful, since it naturally appears when spectral theory is considered. As was mentioned before already, the Eisenstein series also satisfy reflection formulas that we note here for later convenience

$$\Gamma(s)\mathcal{E}_s(\tau) = \Gamma(1-s)\mathcal{E}_{1-s}(\tau)\,, \tag{5.4}$$

$$\xi(2s)E(s;\tau) = \xi(2-2s)E(1-s;\tau)\,. \tag{5.5}$$

We have already met the coefficient of d^6R^4 [62] as well, it is the generalised non-holomorphic Eisenstein series, defined in (4.16) via the differential equation

$$[\Delta - \lambda(\lambda-1)]\,\mathcal{E}\left(\lambda; s_1, s_2\middle|\tau\right) = \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau), \tag{5.6}$$

where as before we call the coefficients s_1, s_2 the weights. In chapter 4 we analysed the special case of integral weights and eigenvalue by using the fact that for these values Bessel functions may be expanded in terms of a finite sum of exponentials. The purpose of this chapter is to extend these methods to apply to the more general case relevant for corrections to Type IIB supergravity.

Beyond d^6R^4, higher derivative corrections are not supersymmetrically protected any longer, hence the same methods leading to the exact results presented above cannot be applied. However, novel results have been obtained by considering the holographic dual of type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, famously given by $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$.

Thanks to supersymmetric localisation [63], we can obtain various specific $\mathcal{N}=4$ four-point integrated correlators of superconformal primaries of the stress-energy tensor multiplet by taking different combinations of four derivatives of the partition function for the $\mathcal{N}=2^*$ theory (a massive deformation of $\mathcal{N}=4$) on a squashed S^4 with respect to different parameters (squashing, mass and complexified coupling).

In [64, 65] the authors exploited these supersymmetric localisation results to compute the large-N expansion of such integrated correlators while keeping fixed the modular parameter, τ, now denoting the Yang-Mills complexified coupling $\tau = \theta/2\pi + 4\pi i/g^2_{_{YM}}$. Using the AdS/CFT dictionary, we identify $g^2_{_{YM}} = 4\pi g_s$ and $(g^2_{_{YM}}N)^{\frac{1}{2}} = L^2/\alpha'$, where g_s is the string coupling constant and L is the scale of $\mathrm{AdS}_5\times \mathrm{S}^5$. Hence the large-$N$ limit of such integrated correlators can help us in understanding higher derivative corrections in type IIB superstring theory on $\mathrm{AdS}_5\times \mathrm{S}^5$ beyond d^6R^4 [64,65] as well as non-perturbative effects in α' [66,67]. These correlators and their resurgent properties are discussed in much more detail in chapter 6.

As a consequence of $\mathcal{N}=4$ Montonen–Olive duality (also known as S-duality), order by order at large-N we must have an expansion with coefficients that are non-holomorphic modular invariant functions of τ. From [64,65] we know that half-integer orders in $1/N$ produce only non-holomorphic Eisenstein series. However, for integer orders in $1/N$ this expansion is conjectured [68] to involve an infinite class of generalised Eisenstein series, $\mathcal{E}\left(\lambda; s_1, s_2\middle|\tau\right)$, with half-integer indices s_1, s_2 and spectrum of eigenvalues $\lambda \in \mathrm{Spec}_2(s_1, s_2)$ constrained by

$$\lambda \in \mathrm{Spec}_2(s_1,s_2) := \{s_1{+}s_2{+}1,\ s_1{+}s_2{+}3,\ s_1{+}s_2{+}5,\ ...\}\,, \qquad s_1,s_2 \in \mathbb{N}+\frac{1}{2}. \tag{5.7}$$

Remember that a first class of generalised Eisenstein series having spectrum $\mathrm{Spec}_1(s_1, s_2)$ was defined in (4.17) and it was relevant for the study of MGFs. Similarly, here we find that the coefficient of the d^6R^4 higher-derivative correction, $\mathcal{E}\left(4; \frac{3}{2}, \frac{3}{2}\middle|\tau\right)$, in (5.2) is simply the first instance of generalised Eisenstein series belonging to this second spectrum of functions (5.7). For future reference, we notice that within this second flavour of generalised Eisenstein series, $\mathcal{E}\left(\lambda; s_1, s_2\middle|\tau\right)$, relevant for higher derivative corrections and the large-N expansion of integrated correlators, the eigenvalue λ has always opposite even/odd parity when compared to the "weight" $w = s_1 + s_2$.

As a final comment, we stress again that from the gauge theory side we obtain exact expressions for four-point correlators which are *integrated* against different measures over the four insertion points. A difficult open problem is how to reconstruct from the dual IIB superstring side which higher-derivative corrections are responsible for a given generalised Eisenstein series in the large-N expansion. However, in [64, 65] the authors used the gauge theory results to reproduce exactly the known BPS corrections to the low-energy expansion of the four-graviton amplitude (5.2) in type IIB superstring theory in ten-dimensional flat-space. We expect the generalised Eisenstein series (5.7) to have important implications in our understanding of flat-space higher derivative corrections as well as for the structure of a similar expansion in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

5.1 A new Niebur-Poincaré series

One way of constructing a Poincaré series representation for the generalised Eisenstein series,

$$\mathcal{E}\left(\lambda; s_1, s_2 \middle| \tau\right) = \sum_{\gamma \in \mathrm{B}(\mathbb{Z}) \backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{e}\big(\lambda; s_1, s_2 \big| \gamma \cdot \tau\big) \ , \tag{5.8}$$

relies on rewriting the Laplace equation (5.6) after having replaced one of the Eisenstein series in the source term, say $\mathcal{E}_{s_1}(\tau)$, by its Poincaré series (5.3), usually dubbed as folding $\mathcal{E}_{s_1}(\tau)$. This leads us to consider an auxiliary Laplace equation for the candidate seed function $\mathrm{e}\big(\lambda; s_1, s_2 \big| \tau\big)$:

$$[\Delta - \lambda(\lambda - 1)]\, \mathrm{e}\big(\lambda; s_1, s_2 \big| \tau\big) = \frac{2\zeta(2s_1)}{\pi^{s_1}} \tau_2^{s_1} \mathcal{E}_{s_2}(\tau) \,. \tag{5.9}$$

We can first rewrite the source term as a Fourier series (5.3) with respect to $\tau_1 = \mathrm{Re}(\tau)$, and then find a particular solution for this Laplace equation mode by mode. For the Fourier zero-mode sector there is no issue in finding such a particular solution. However, for a Fourier non-zero mode it is rather difficult to find a particular solution to (5.9) which is expressible in terms of simple building-block seed functions for generic values of s_1, s_2 and λ .

In chapter 4 it was discussed how all two-loop MGFs, or more broadly all generalised Eisenstein series with spectrum given by (4.17), can be written as Poincaré series of finite linear combinations of the building-block seed functions introduced in [31]

$$\varphi(a, b, r|\tau) = \sum_{m \neq 0} \sigma_a(m)(4\pi|m|)^b \tau_2^r e^{-2\pi|m|\tau_2} e^{2\pi i m \tau_1} \,, \tag{5.10}$$

for different values of the parameters (a, b, r). It was nonetheless noticed in [31] that such seeds are rather ill-suited to describe generalised Eisenstein series relevant for higher-derivative corrections and integrated correlators, where the spectrum is (5.7). For these generalised Eisenstein series it is still possible to write a seed function in terms of building-blocks (5.10), but one requires an infinite sum over such simple seeds, thus making it quite hard to extract the asymptotic expansion at the cusp or other analytic properties from the corresponding Poincaré series. Other types of Poincaré series have been proposed in the literature [45, 69] for the diagonal elements, i.e. $s_1 = s_2$, in the first family (5.7), while in [70] other examples in this class are analysed directly from the differential equation point of view.

To construct a class of Poincaré seeds suited for discussing both (4.17)-(5.7) in a uniform manner, we have to re-examine the Laplace equation (5.9). From the Fourier decomposition of the Eisenstein series (5.3), we notice that the m^{th} Fourier mode, with $m \neq 0$, of the source term is schematically of the form

$$\sigma_a(m)|m|^{b-\frac{1}{2}}\tau_2^{r+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|\tau_2)e^{2\pi i m\tau_1}\,,$$

for some specific values of the parameters (a, b, r, s). Thanks to the recurrence relations satisfied by the modified Bessel function $K_s(y)$, for both spectra (4.17)-(5.7) it is always possible to find a finite linear combination over different values of the parameters[1] (a, b, r, s) of terms as above which is a solution to (5.9) in the m^{th} Fourier mode sector.

With this fact at hand, we can now introduce a novel space of Poincaré seeds and associated Poincaré series which is both general enough, in that every string theory generalised Eisenstein series (4.17)-(5.7) can be written as a Poincaré series of finite linear combinations of these novel seeds, and simple enough so that we can easily extract asymptotic data both at the cusp $\tau_2 \gg 1$ and at the origin $\tau_2 \to 0$.

[1]In this context the parameter a is rather special, since it is the index of the divisor sum function $\sigma_a(m)$. From the Laplace equation (5.9) and the Fourier mode decomposition (5.3) it is easy to see that $a = 1 - 2s_2$ for the present discussion.

We define the seed function

$$\begin{aligned}\upsilon\,(a,b,r,s|\tau) &= \sum_{m\neq 0} \upsilon_m(a,b,r,s|\tau_2) e^{2\pi i m \tau_1} \\ &:= \sum_{m\neq 0} \sigma_a(m)|m|^{b-\frac{1}{2}} \tau_2^{r+\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|\tau_2) e^{2\pi i m\tau_1}\,, \end{aligned} \tag{5.11}$$

which depends on four complex parameters (a,b,r,s). Given that the Bessel function $K_s(y)$ is exponentially suppressed for large values of its argument, we immediately have that the sum over the Fourier mode, m, is absolutely convergent for any values of the parameters (a,b,r,s). This property of the Bessel function implies as well that the seed function is exponentially suppressed for $\tau_2 \gg 1$, however, the limit $\tau_2 \to 0$ is more delicate to analyse.

Under the assumption that the Poincaré series of such a class of seed functions is well-defined, we can introduce a novel class of modular invariant functions which we denote by

$$\Upsilon\,(a,b,r,s|\tau) := \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \upsilon\,(a,b,r,s|\gamma\cdot\tau)\ . \tag{5.12}$$

The convergence of this Poincaré series is studied in appendix C, where we prove that absolute convergence is guaranteed when

$$\min\big(\mathrm{Re}(r+1-s),\, \mathrm{Re}(r+s),\, \mathrm{Re}(r-b),\, \mathrm{Re}(r-a-b)\big) > 1\,. \tag{5.13}$$

In what follows, we can relax the requirement of absolute convergence and consider if necessary the Poincaré series (5.12) in terms of its analytic continuation in some of its complex parameters (a,b,r,s), in direct analogy with the discussion below (3.21).

The keen-eyed reader will notice that the new seeds (5.11) are very reminiscent of the rather unconventional Poincaré series representation (3.21) for $\mathcal{E}_s(\tau)$. The reason is that, very much like (3.21), our expression (5.11) can be obtained as an infinite sum over all Fourier non-zero modes, $m\neq 0$, of the difference between two Niebur–Poincaré series [71, 72]. We will shortly prove that both string theory generalised Eisenstein series (4.17)-(5.7) can be obtained from finite linear combinations of these new Niebur–Poincaré series (5.12).

As already stressed, one of the perks of a Poincaré series representation is that in general it simplifies the complexity of the objects under consideration. In particular, from the seed function definition (5.11) we can already deduce various algebraic and differential identities satisfied by the modular objects $\Upsilon(a,b,r,s|\tau)$. Firstly, we note that the seed functions (5.11) are invariant under the reflection $s \to 1-s$,

$$\upsilon(a,b,r,1-s|\tau) = \sum_{m\neq 0} \sigma_a(m)|m|^{b-\frac{1}{2}}\tau_2^{r+\frac{1}{2}} K_{\frac{1}{2}-s}(2\pi|m|\tau_2)e^{2\pi i m\tau_1} = \upsilon(a,b,r,s|\tau) \ , \tag{5.14}$$

due to the Bessel function identity $K_s(y) = K_{-s}(y)$. Similarly, we have invariance under the transformation $(a,b) \to (-a, b+a)$,

$$\upsilon(-a,a+b,r,s|\tau) = \sum_{m\neq 0} \sigma_{-a}(m)|m|^{a+b-\frac{1}{2}}\tau_2^{r+\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|\tau_2)e^{2\pi i m\tau_1} = \upsilon(a,b,r,s|\tau) \ , \tag{5.15}$$

a straightforward consequence of the identity $\sigma_{-a}(m) = |m|^{-a}\sigma_a(m)$. From these two observations we deduce that the modular functions must also inherit these symmetries,

$$\Upsilon(a,b,r,s|\tau) = \Upsilon(a,b,r,1-s|\tau) \ , \tag{5.16}$$

$$\Upsilon(a,b,r,s|\tau) = \Upsilon(-a,b+a,r,s|\tau) \ . \tag{5.17}$$

More interestingly, given the well-known Bessel function recurrence relation

$$K_{s+1}(y) - K_{s-1}(y) = \frac{2s}{y} K_s(y) \, , \tag{5.18}$$

we can immediately derive the three-term recursion

$$\Upsilon(a,b,r,s+1|\tau) - \Upsilon(a,b,r,s-1|\tau) = \frac{2s-1}{2\pi}\Upsilon(a,b-1,r-1,s|\tau) \ . \tag{5.19}$$

Note that even if we consider a seed function whose parameters (a,b,r,s) satisfy the conditions (5.13) for absolute convergence of the Poincaré series, repeated applications of this recursion formula (5.19) will inevitably bring us outside of the domain (5.13) where the analytic continuation of (5.12) has to be discussed carefully.

Finally, given that our discussion started from the inhomogeneous Laplace equation (5.6), it is natural to consider the action of the Laplace operator on (5.12). By simply applying the

Laplacian to (5.11) and using the known identity for the derivative of the Bessel function,

$$K'_s(y) = -\frac{s}{y}K_s(y) - K_{s-1}(y)\,, \tag{5.20}$$

we arrive at

$$\big[\Delta - (r+1-s)(r-s)\big]\,\Upsilon\,(a,b,r,s|\tau) = -4\pi r \Upsilon\,(a,b+1,r+1,s-1|\tau)\;, \tag{5.21}$$

or equivalently making use of (5.19):

$$\big[\Delta - (r+s)(r+s-1)\big]\,\Upsilon\,(a,b,r,s|\tau) = -4\pi r \Upsilon\,(a,b+1,r+1,s+1|\tau)\;. \tag{5.22}$$

We have thus obtained that the functions $\Upsilon\,(a,b,r,s|\tau)$ satisfy a closed system of inhomogeneous Laplace eigenvalue equations where the source term is given by yet another function of the same type, but different parameters (a,b,r,s).

Both Laplace equations (5.21)-(5.22) simplify dramatically for $r=0$, where they reduce to

$$\big[\Delta - s(s-1)\big]\,\Upsilon\,(a,b,0,s|\tau) = 0\,, \tag{5.23}$$

and since the function $\Upsilon\,(a,b,0,s|\tau)$ is manifestly a modular invariant eigenfunction of Δ with eigenvalue $s(s-1)$ it must be proportional to $\mathcal{E}_s(\tau)$.

We will shortly prove that $\Upsilon\,(a,b,r,s|\tau)$ has polynomial growth at the cusp and compute explicitly its asymptotic expansion using the integral representation (3.26), thus easily fixing the coefficient of proportionality between $\Upsilon\,(a,b,0,s|\tau)$ and $\mathcal{E}_s(\tau)$. Alternatively, we can see from (5.11) that each summand with Fourier mode $m=k$ in the seed function $\upsilon\,(a,b,0,s|z)$ is proportional to the Poincaré seed (3.21) for $\mathcal{E}_s(\tau)$. The only difference with (3.21), is that the sum over m in (5.11) will simply produce a particular Dirichlet series which will contribute to the proportionality factor between $\Upsilon\,(a,b,0,s|\tau)$ and $\mathcal{E}_s(\tau)$.

As already mentioned, the novel seeds (5.11) are constructed precisely to provide for a broad enough basis of solutions to (5.9). Correspondingly, we will show that it is possible to produce finite linear combinations of $\Upsilon\,(a,b,r,s|\tau)$ which are solutions to the generalised Eisenstein series differential equation (5.6) relevant for string theory. A central part of this analysis is the observation that the space of functions $\Upsilon\left(a,b,r,s|\tau\right)$ contains all products of two Eisenstein series, i.e. all possible source terms of (5.6). The proof of this statement is very simple. If we consider the bilinear $\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau)$, we first fold $\mathcal{E}_{s_1}(\tau)$ and then re-express $\mathcal{E}_{s_2}(\tau)$ in Fourier modes arriving at

$$\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) = \frac{8\xi(2s_1)}{\Gamma(s_1)\Gamma(s_2)}\Upsilon\left(1-2s_2,s_2,s_1,s_2|\tau\right) + \frac{2\Gamma(s_1+s_2)\xi(2s_1)\xi(2s_2)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2))}\mathcal{E}_{s_1+s_2}(\tau) + \frac{2\Gamma(s_1+1-s_2)\xi(2s_1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+1-s_2))}\mathcal{E}_{s_1+1-s_2}(\tau)\,. \tag{5.24}$$

Alternatively we can use the reflection formula (5.5), combined with (5.16)-(5.17), to derive

$$\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) = \frac{8\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)}\Upsilon\left(1-2s_1,s_1,1-s_2,1-s_1|\tau\right) + \frac{2\Gamma(s_1{+}s_2{-}1)\xi(2s_1{-}1)\xi(2s_2{-}1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2){-}3)}\mathcal{E}_{s_1+s_2-1}(\tau) + \frac{2\Gamma(s_1{+}1{-}s_2)\xi(2s_1)\xi(2s_2{-}1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1{-}s_2){+}2)}\mathcal{E}_{s_1+1-s_2}(\tau)\,. \tag{5.25}$$

Note that by folding $\mathcal{E}_{s_1}(\tau)$ we break the symmetry between $s_1 \leftrightarrow s_2$. This comes at a notable price in the diagonal case $s_1 = s_2$ where (5.24)-(5.25) have to be regulated. For $s_1 = s_2$, the right-hand side of both equations contains the divergent Eisenstein series $\mathcal{E}_1(\tau)$. However, since the bilinear $\mathcal{E}_{s_1}(\tau)^2$ is perfectly regular for $s_1 \neq 1$, this implies that the modular functions $\Upsilon\left(1-2s_1,s_1,s_1,s_1\right)$ and $\Upsilon\left(1-2s_1,s_1,1-s_1,1-s_1\right)$ must diverge as well. A regularised versions of (5.24)-(5.25) for the case $s_1 = s_2$ is easily obtained by considering the continuous limit away from the diagonal $s_1 = s_2$ case:

$$\mathcal{E}_{s_1}(\tau)^2 = \lim_{\epsilon\to 0}\left[\mathcal{E}_{s_1+\epsilon}(\tau)\,\mathcal{E}_{s_1}(\tau)\right]. \tag{5.26}$$

When $\epsilon \neq 0$ we can safely write the right-hand side using (5.24)-(5.25). As $\epsilon \to 0$ our formulae (5.24)-(5.25) produce a divergent contribution coming from $\mathcal{E}_{1+\epsilon}(\tau)$ which cancels against the similarly singular Υ thus leaving us with a regular expression. The need for a regularisation of the diagonal case $s_1 = s_2$ is an ubiquitous phenomenon [31, 38] and it is independent from the particular seeds considered in the present work.

5.2 Asymptotic expansion at the cusp

Let us now derive the asymptotic expansion near the cusp $\tau \to i\infty$ for the modular invariant functions $\Upsilon\left(a, b, r, s|\tau\right)$. Firstly we perform a Fourier mode decomposition,

$$\Upsilon\left(a, b, r, s|\tau\right) = \sum_{k\in\mathbb{Z}} \Upsilon_k(a, b, r, s|\tau_2) e^{2\pi i k \tau_1} \,, \tag{5.27}$$

and focus on deriving the asymptotic expansion for large τ_2 of the Fourier zero-mode $\Upsilon_0(a, b, r, s|\tau_2)$.

In the previous chapter we have already reviewed how to retrieve the Fourier modes of a Poincaré series from an integral transform (A.3) of the Fourier modes for the corresponding seed function. In particular, if we focus on the Fourier zero-mode sector (A.5) for the specific seeds (5.11) under consideration, we have to compute:

$$\begin{aligned}&\Upsilon_0(a, b, r, s|\tau_2) \\ &= \sum_{d=1}^{\infty} \sum_{m\neq 0} S(m, 0; d) \int_{\mathbb{R}} e^{-2\pi i m \frac{\omega}{d^2(\omega^2+\tau_2^2)}} \sigma_a(m) |m|^{b-\frac{1}{2}} \Big(\frac{\tau_2}{d^2(\omega^2+\tau_2^2)}\Big)^{r+\frac{1}{2}} K_{s-\frac{1}{2}}\Big(\frac{2\pi|m|\tau_2}{d^2(\omega^2+\tau_2^2)}\Big) \mathrm{d}\omega.\end{aligned} \tag{5.28}$$

In appendix D we show how the above integral transform can be rewritten as a nicer Mellin-Barnes type of contour integral, thus making the task of extracting the asymptotic expansion at the cusp much more manageable.

Relegating the more technical details to the appendix, we present here the key result of our analysis: the integral representation (5.28) can be rewritten as the Mellin-Barnes integral

$$\Upsilon_0(a, b, r, s|\tau_2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} U(a, b, r, s|t)\, \tau_2^t \frac{\mathrm{d}t}{2\pi i} \,, \tag{5.29}$$

where we define

$$\begin{aligned}U(a, b, r, s|t) := &\frac{\Gamma\big(\frac{r+1-s-t}{2}\big)\Gamma\big(\frac{r+s-t}{2}\big)\Gamma\big(\frac{t+r-s}{2}\big)\Gamma\big(\frac{t+r+s-1}{2}\big)}{2\pi^r\,\Gamma(r)\xi(2-2t)} \\ &\times \frac{\zeta(r+1-b-t)\zeta(r+1-a-b-t)\zeta(t+r-b)\zeta(t+r-a-b)}{\zeta(2r+1-a-2b)} \,.\end{aligned} \tag{5.30}$$

For this section, unless otherwise specified, we restrict ourselves to the range of parameters (5.13) for which the Poincaré series is absolutely convergent. However, at the end of ap-

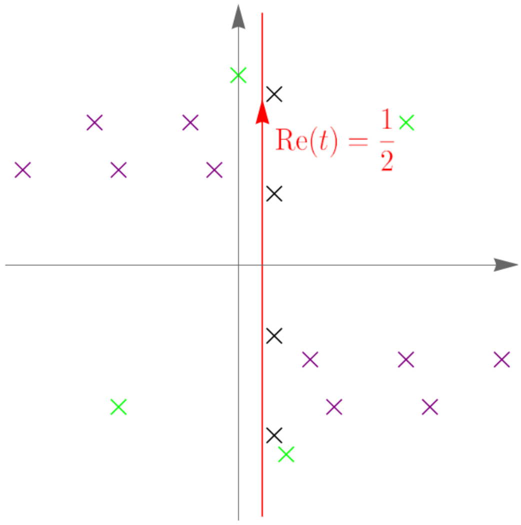

Figure 5.1: Schematic pole structure of $U(a, b, r, s|t)$. The infinite family of poles from the gamma functions is given in purple while the four poles from the zeta functions are given in green. In black, we have an infinite family of poles with $\text{Re}(t) = \frac{3}{4}$ (if Riemann hypothesis is correct) coming from the non-trivial zeroes of the Riemann zeta. The contour of integration, $\text{Re}(t) = \frac{1}{2}$, is indicated in red.

pendix D we explain that for parameters, (a, b, r, s), which do not produce convergent Poincaré series, the Mellin-Barnes representation (5.29) is still perfectly valid provided the contour of integration is modified from the straight line $\text{Re}(t) = \frac{1}{2}$ to a contour separating the two families of poles we are about to discuss.

It is now fairly straightforward to extract from the Mellin-Barnes integral (5.30) the asymptotic expansion of $\Upsilon_0(a, b, r, s|\tau_2)$ as $\tau_2 \gg 1$ by closing the contour of integration at negative infinity on the left semi-half plane $\text{Re}(t) < \frac{1}{2}$ and collecting residues from the different singular terms in (5.30). As we can see in Figure 5.1, when the parameters (a, b, r, s) define an absolutely convergent Poincaré series, i.e. when they satisfy (5.13), closing the contour at negative infinity in the half-plane $\text{Re}(t) < \frac{1}{2}$ selects two different types of poles:

- From the zeta functions $\zeta(t + r - b)$ and $\zeta(t + r - a - b)$ we have two poles located respectively at $t = b + 1 - r$ and $t = a + b + 1 - r$;
- From the gamma functions $\Gamma\left(\frac{t+r-s}{2}\right)$ and $\Gamma\left(\frac{t+r+s-1}{2}\right)$ we have two infinite families of poles located respectively at $t = s - r - 2n$ and $t = 1 - s - r - 2n$ with $n \in \mathbb{N}$.

It is easy to see that under the assumption (5.13) of an absolutely convergent Poincaré series, the above poles are all located in the half-plane $\text{Re}(t) < \frac{1}{2}$, while all remaining poles in (5.30) are located in the half-plane $\mathrm{Re}(t) > \frac{1}{2}$.

Computing the residues at said poles and summing over all of them produces the wanted asymptotic expansion for large-τ_2 of the Fourier zero-mode,

$$\Upsilon_0(a,b,r,s|\tau_2) \sim \frac{\zeta(2r-a-2b)}{2\Gamma(r)\zeta(2r-a-2b+1)}\Big[c^{(1)}(a,b,r,s)\tau_2^{b+1-r}+c^{(1)}(-a,a+b,r,s)\tau_2^{a+b+1-r}\Big]$$
$$+\sum_{n=0}^{\infty}\tau_2^{-r-2n}\big[\tau_2^{s}c_n^{(2)}(a,b,r,s)+\tau_2^{1-s}c_n^{(2)}(a,b,r,1-s)\big]\,, \tag{5.31}$$

where for convenience of presentation we defined the coefficients

$$c^{(1)}(a,b,r,s) = \frac{\Gamma\big(\frac{b+1-s}{2}\big)\Gamma\big(\frac{2r-b-s}{2}\big)\Gamma\big(\frac{b+s}{2}\big)\Gamma\big(\frac{2r+s-b-1}{2}\big)\zeta(1-a)}{\pi^b\,\Gamma(r-b)}\,, \tag{5.32}$$

$$c_n^{(2)}(a,b,r,s) = \frac{(-1)^n\pi^{2n+1-s}\Gamma\left(n+r\right)\Gamma\left(s-n-\frac{1}{2}\right)\Gamma\left(n+r-s+\frac{1}{2}\right)}{n!\Gamma\left(r\right)\Gamma(2n+r+1-s)}$$
$$\times\,\frac{\zeta(s-b-2n)\zeta(s-a-b-2n)\zeta(2n+2r+1-b-s)\zeta(2n+2r+1-a-b-s)}{\zeta(2r-a-2b+1)\zeta(4n+2r+2-2s)}\,. \tag{5.33}$$

Besides the first two terms τ_2^{b+1-r} and $\tau_2^{a+b+1-r}$, coming from the isolated poles of the two Riemann zeta functions, the remaining perturbative series is, for general parameters, (a,b,r,s), an asymptotic factorially divergent power series. From (5.33), the growth of the perturbative coefficients is $c_n^{(2)}(a,b,r,s) = O((2n)!)$ which combined with the power-like growth $(4\pi\tau_2)^{-2n}$ immediately suggests the presence of exponentially suppressed corrections $(q\bar{q}) = e^{-4\pi\tau_2}$.

While the modular functions $\Upsilon\left(a,b,r,s|\tau\right)$ provide for a natural extension of the generalised Eisenstein series, unlike the generalised Eisenstein series they have, for non-specific values of the parameters, non-terminating and factorially divergent formal power series expansions at the cusp $\tau_2 \gg 1$. Crucially, at non-generic and physically relevant points in parameter space, i.e. for special values of a,b,r,s corresponding to generalised Eisenstein series, the asymptotic tail of $\Upsilon\left(a,b,r,s|\tau\right)$ vanishes and (5.31) reduces to a sum of finitely many terms. In the next section, we show that this happens for $a\in\mathbb{Z}$ and (b,r,s) either all integers or all half-integers.

This dramatic change of the asymptotic series (5.31) from a factorially divergent formal power series to a finite sum can be understood quite easily from the contour integral representation given in (5.29). From the definition (5.30) of the function $U(a,b,r,s|t)$ we notice that the gamma functions generate two infinite families of poles in t on both side of the integration contour $\mathrm{Re}(t) = \frac{1}{2}$. At the same time, the four Riemann zeta functions present two pairs of identically spaced, infinite families of (trivial-)zeros in t again on both side of the integration contour $\mathrm{Re}(t) = \frac{1}{2}$. The truncation of the asymptotic series (5.31) to a finite sum happens precisely for special values of a, b, r, s for which these families of poles and zeroes start overlapping at some point. As we will show in the next section, the case of interest - the generalised Eisenstein series - neatly falls into this category.

The analytic continuation in (a, b, r, s) is crucial for fixing the exponentially suppressed $(q\bar{q})$-terms from the formal and factorially divergent perturbative expansion at the cusp, since for generic (a, b, r, s) the requirement of a well-defined Borel-Ecalle resummation of (5.31) allows for calculation of all $(q\bar{q})$-terms, similar to [31] and chapter 4.

Surprisingly, even when at special values of the parameters (a, b, r, s) the series (5.31) becomes a finite sum, such non-perturbative resurgent corrections do survive. This is the previously mentioned notion of Cheshire Cat resurgence [41–43] from the eponymous feline of Alice in Wonderland with a disappearing body but a lingering enigmatic grin. Since such a resurgence analysis is akin to the one carried out in chapter 4 for a different general class of seed functions (5.10), we will not repeat this calculation here. Later in the present chapter we will however revisit the calculation of exponentially suppressed terms from the spectral analysis point of view.

We conclude this section with a simpler "special" example, namely the case of the standard Eisenstein series. As previously remarked, since $\Upsilon\left(a, b, r = 0, s|\tau\right)$ is a modular solution to the Laplace equation (5.23) it must proportional to $E(s;\tau)$. Given the generic asymptotic expansion (5.31), we can now fix the constant of proportionality.

Firstly, it is a well-known result (5.3) that the asymptotic expansion at the cusp for $E(s;\tau)$ has only two power-behaved terms: τ_2^s and τ_2^{1-s}. These two terms are easily recognisable in (5.31) as regulated versions of the $n = 0$ terms $\tau_2^{s-r}c_0(a, b, r, s)+\tau_2^{1-s-r}c_0(a, b, r, \text{-}s)$, while all other terms vanish. More precisely, from the definition (5.30) we notice in the denominator the factor $\Gamma(r)$ is singular for $r = 0$, but easily regulated by considering $r = \epsilon$ and taking the limit $\epsilon \to 0$ at the very end. Only the poles of (5.30) located at $t = s + \epsilon$ and $t = 1 - s - \epsilon$ have a non-vanishing residue in the limit $\epsilon \to 0$ and produce precisely a multiple of the expected Eisenstein series Laurent polynomial (5.3). This allows us to fix the proportionality factor between $\Upsilon(a,b,r=0,s|\tau)$ and $E(s;\tau)$ as such

$$\Upsilon(a,b,0,s|\tau) = \frac{2\tan(\pi s)\Gamma(s)\zeta(1-b-s)\zeta(1-a-b-s)\zeta(s-b)\zeta(s-a-b)}{(2s-1)\pi^{s-1}\,\zeta(1-a-2b)\zeta(2-2s)}\,E(s;\tau)\,. \tag{5.34}$$

We stress again that we could have reached the same result from a direct comparison between each Fourier mode of the Poincaré seed (5.11) and the unusual Poincaré series (3.21) for $E(s;\tau)$. Applying (3.21) to each Fourier mode in (5.11), leaves us with a particular Dirichlet sum over the Fourier non-zero modes $m \in \mathbb{Z}\setminus\{0\}$ which, once evaluated, brings us back (5.34).

Lastly, an easy application of the recursion formula (5.19) shows that all of $\Upsilon(a,b,-n,s|\tau)$, with $n\in\mathbb{N}$, are also finite sums of Eisenstein series,

$$\Upsilon(a,b,-n,s|\tau) = \pi^n n!\sum_{k=0}^{n}\frac{(\text{-}1)^{k+1}(s+n-2k-\frac{1}{2})\Gamma(s-k-\frac{1}{2})}{k!\Gamma(n-k+1)\Gamma(n+s-k+\frac{1}{2})}\gamma(a,b+n,s+n-2k)E(s+n\text{-}2k;\tau)\,, \tag{5.35}$$

where the coefficient $\gamma(a,b,s)$ is the proportionality constant appearing in (5.34), i.e.

$$\gamma(a,b,s) = \frac{2\tan(\pi s)\Gamma(s)\zeta(1-b-s)\zeta(1-a-b-s)\zeta(s-b)\zeta(s-a-b)}{(2s-1)\pi^{s-1}\,\zeta(1-a-2b)\zeta(2-2s)}\,. \tag{5.36}$$

5.3 A ladder of inhomogeneous Laplace equations

We have just seen that this newly defined space (5.12) of modular invariant functions contains both single Eisenstein series (5.34) and products of two Eisenstein series (5.24)-(5.25). We now show that the functions $\Upsilon(a,b,r,s|\tau)$ are also closed under the action of the Laplace operator in τ. In particular, we describe a method of generating solutions to an infinite ladder of Laplace equations where the source term is a fixed function $\Upsilon(a,b,r,s|z)$ and the eigenvalues lie in the spectrum

$$\text{Spec}(r{+}s) = \{r{+}s{-}2,\ r{+}s{-}4,\ r{+}s{-}6,\ ...\}\,, \tag{5.37}$$

i.e. they take the form

$$\lambda_n(r+s) := r+s-2(n+1)\,, \tag{5.38}$$

with $n\in\mathbb{N}$. Once the source term $\Upsilon(a,b,r,s|\tau)$ is properly chosen, this spectrum reduces to the string theory spectra (4.17)-(5.7) and the constructed solution produces precisely a given generalised Eisenstein series expressed as a finite linear combination of novel Poincaré series

(5.12). Not to clutter the notation, in this section we will suppress the explicit τ-dependence.

The starting point of our analysis is the differential equation (5.22), rewritten here in a more convenient form

$$\big[\Delta - \lambda_0(r+s)(\lambda_0(r+s)-1)\big]\frac{\Upsilon\,(a,b-1,r-1,s-1)}{4\pi(1-r)} = \Upsilon\,(a,b,r,s)\ . \tag{5.39}$$

To construct this ladder of Laplace equations, we view this equation as the top element in a tower of similar equations with decreasing eigenvalues. We now look for linear combinations, $\mathrm{Y}_n(a,b,r,s)$, of functions $\Upsilon\,(a',b',r',s')$ with different parameters (a',b',r',s') and solutions to

$$\big[\Delta - \lambda_n(r+s)(\lambda_n(r+s)-1)\big]\mathrm{Y}_n(a,b,r,s) = \Upsilon\,(a,b,r,s)\ . \tag{5.40}$$

The starting Laplace equation (5.39) gives us the initial condition

$$\mathrm{Y}_0(a,b,r,s) = \frac{\Upsilon\,(a,b-1,r-1,s-1)}{4\pi(1-r)}\,, \tag{5.41}$$

while the rest of the ladder is generated from here by exploiting the crucial recursion relation (5.19) as we now show.

To simplify the discussion, we introduce a linear operator $\mathcal{D}$ that acts on the space of modular functions (5.12) as

$$\mathcal{D}\Upsilon\,(a,b,r,s) := \Upsilon\,(a,b,r,s-2) + \frac{2s-3}{2\pi}\Upsilon\,(a,b-1,r-1,s-1)\ , \tag{5.42}$$

for which the recursion relation (5.19) can then be written in the compact form

$$\mathcal{D}\Upsilon\,(a,b,r,s) = \Upsilon\,(a,b,r,s)\ .$$

One can easily check by induction that an n-fold application of this operator produces a sum of $n+1$ modular functions given by

$$\mathcal{D}^n\Upsilon\,(a,b,r,s) = \sum_{k=0}^{n}\binom{n}{k}\Big(\prod_{i=0}^{k-1}\frac{2(s+i-n)-1}{2\pi}\Big)\Upsilon\,(a,b-k,r-k,s+k-2n)\ . \tag{5.43}$$

While it is not immediately obvious how to use the Laplace equation (5.22) to invert (5.40) and find $\mathrm{Y}_n(a,b,r,s)$, we can use the recursion relation to rewrite (5.40) as

$$\begin{aligned}&\big[\Delta-\lambda_0(r+s-2n)(\lambda_0(r+s-2n)-1)\big]\mathrm{Y}_n(a,b,r,s)=\Upsilon\,(a,b,r,s)\\&=\mathcal{D}^n\Upsilon\,(a,b,r,s)=\sum_{k=0}^{n}\binom{n}{k}\Big(\prod_{i=0}^{k-1}\frac{2(s+i-n)-1}{2\pi}\Big)\Upsilon\,(a,b-k,r-k,s+k-2n)\ .\end{aligned}\tag{5.44}$$

Although $\mathcal{D}^n\Upsilon\,(a,b,r,s)$ is a linear combination of modular functions $\Upsilon\,(a,b',r',s')$ with different parameters (a,b',r',s'), we notice that the action of $\mathcal{D}^n$ produces a uniform shift on $r+s$, i.e. for every term in this linear combination we have $r'{+}s'=r{+}s{-}2n$. This means that if we consider the left-hand side of (5.44) term by term, we have reduced the problem to a collection of equations (5.39) for different values of parameters (a,b',r',s') but all satisfying $r'{+}s'=r{+}s{-}2n$. We can then use the inversion of the Laplacian (5.41) term by term to arrive at

$$\mathrm{Y}_n(a,b,r,s)=\sum_{k=0}^{n}\binom{n}{k}\Big(\prod_{i=0}^{k-1}\frac{2(s+i-n)-1}{2\pi}\Big)\frac{\Upsilon\,(a,b-k-1,r-k-1,s+k-2n-1)}{4\pi(k+1-r)}\,,\tag{5.45}$$

which is the sought-after solution to the ladder of Laplace equations (5.40) with eigenvalue $\lambda_n(r+s)=r{+}s{-}2(n+1)$ and source $\Upsilon\,(a,b,r,s)$.

Note that while in general this ladder does not terminate, whenever the parameter r is a strictly positive integer, which will be the relevant case for the MGFs spectrum (4.17), the ladder does in fact terminate after finitely many steps. This is easy to see from (5.45), let us assume that $r=n+1$ with $n\in\mathbb{N}$ for which (5.45) becomes ill-defined. In (5.45) the would-be $k=n$ term reduces to $\Upsilon\,(a,b-n-1,0,s-n-1)$ and according to the differential equation (5.44) the action of the Laplace eigenvalue operator on such a factor should produce the corresponding source proportional to $\Upsilon\,(a,b-n,1,s-n)$. However, this is not possible since $\Upsilon\,(a,b-n-1,0,s-n-1)$ is proportional (5.34) to the Eisenstein series $E(s-n-1;\tau)$, which is annihilated by (5.40) in the case $r=n+1$. We will come back to this point when discussing the ladder of equations for the first spectrum of generalised Eisenstein series.

In the current context, we are particularly interested in generating solutions to Laplace eigenvalue equations with sources given by products of two Eisenstein series. One of the perks of our approach is that the ladder of Laplace equations (5.40) just found precisely reduces to the desired inhomogeneous Laplace eigevalue equations when the source term $\Upsilon\left(a,b,r,s|\tau\right)$ is suitably chosen to reproduce the wanted bilinear in Eisenstein series as given in (5.24)-(5.25).

The first flavour of generalised Eisenstein series

Let us now use the ladder (5.40) just discussed to reconstruct the first string theory flavour of generalised Eisenstein series relevant for the study of two-loop MGFs. We consider integer indices $s_1, s_2 \geq 2$ and without loss of generality assume $s_1 \geq s_2$. We want to use the ladder equation to reproduce the finite spectrum of eigenvalues from (4.17)

$$\text{Spec}_1(s_1,s_2) = \{|s_1{-}s_2|{+}2,\ |s_1{-}s_2|{+}4,\ \dots,\ s_1{+}s_2{-}2\}\,.$$

Consequently, we specialise the ladder (5.40) to the case for which the source term $\Upsilon\left(a,b,r,s|\tau\right)$ produces the first representation we found for the product of two Eisenstein series (5.24), i.e. we specialise our ladder to

$$(a,b,r,s) = \Big(1-2s_2,\ s_2,\ s_1,\ s_2\Big)\,. \tag{5.46}$$

With this choice of parameters the ladder equation (5.40) reduces to

$$\Big[\Delta-\lambda_n^{(1)}(\lambda_n^{(1)}-1)\Big]\frac{8\xi(2s_1)}{\Gamma(s_1)\Gamma(s_2)}\Upsilon_n(1-2s_2,\ s_2,\ s_1,\ s_2|\tau) = \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) \tag{5.47}$$
$$-\frac{2\Gamma(s_1+s_2)\xi(2s_1)\xi(2s_2)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2))}\mathcal{E}_{s_1+s_2}(\tau)-\frac{2\Gamma(s_1+1-s_2)\xi(2s_1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+1-s_2))}\mathcal{E}_{s_1+1-s_2}(\tau)\,,$$

and the ladder eigenvalues, $\lambda_n^{(1)} = s_1+s_2-2(n+1)$, reproduce immediately the desired spectrum.

In this setup there is no issue with the large-τ_2 asymptotic behaviour for the solution $\Upsilon_n(1-2s_2, s_2, s_1, s_2|\tau)$: using the general expression (5.31) we can confirm that our ladder solution satisfies the desired boundary condition for which the coefficient of the homogeneous solution $y^{\lambda_n^{(1)}}$ vanishes. This means that for the specific parameters (5.46) the ladder solution (5.45) must reproduce (modulo single Eisenstein terms) the first flavour of generalised Eisenstein series, $\mathcal{E}(\lambda_n^{(1)}; s_1, s_2|\tau)$. Proceeding as we did before, we use (3.15) to invert the single Eisenstein source terms in (5.47) and arrive at

$$
\begin{aligned}
\mathcal{E}\left(\lambda_n^{(1)}; s_1, s_2\middle|\tau\right) =& \frac{8\xi(2s_1)}{\Gamma(s_1)\Gamma(s_2)} \Upsilon_n(1-2s_2,\, s_2,\, s_1,\, s_2|\tau) \\
&+ \frac{2\Gamma(s_1+s_2)\xi(2s_1)\xi(2s_2)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2))\mu(s_1+s_2,\lambda_n^{(1)})}\mathcal{E}_{s_1+s_2}(\tau) \\
&+ \frac{2\Gamma(s_1+1-s_2)\xi(2s_1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+1-s_2))\mu(s_2-s_1,\lambda_n^{(1)})}\mathcal{E}_{s_1+1-s_2}(\tau),
\end{aligned}
\tag{5.48}
$$

where we defined $\mu(s,\lambda) := s(s-1) - \lambda(\lambda-1)$. Unlike what happens in the case of the second spectrum, when the sources have integer indices, s_1, s_2, we notice that the spectrum of eigenvalues is bounded both from above and below. There is a maximal eigenvalue in the ladder which is given by $\lambda_0^{(1)}{=}s_1{+}s_2{-}2$ and agrees with the maximal eigenvalue obtained in the study of MGFs in the first spectrum (4.17). However the minimal eigenvalue in the ladder does not quite reproduce the minimal eigenvalue expected from (4.17).

As discussed below equation (5.45), in the case when the parameter $r = \tilde{n}+1$, with $\tilde{n} \in \mathbb{N}$, the ladder terminates after $\tilde{n}$ steps. In the present case (5.48), the parameter $r = s_1$ has precisely this property, hence the ladder terminates after $\tilde{n} = s_1 - 1$ steps, i.e. we have constructed generalised Eisenstein solutions (5.48) for $n = 0, ..., s_1 - 2$ and fixed sources. The minimal eigenvalue we obtain is then $\lambda_{s_1-2}^{(1)}{=}s_2{-}s_1{+}2$, in general lower than the minimal eigenvalue expected from the spectrum (4.17). These ladder solutions (5.48) with eigenvalues lower than the MGFs spectrum (4.17) correspond precisely to the modular objects discussed in section 7.3 of [35] and constructed from certain "overly-integrated seed functions".

The second flavour of generalised Eisenstein series

Now we repeat the analysis for the second flavour of generalised Eisenstein series (5.7) relevant for correction terms to type IIB supergravity. We hence consider half-integer indices $s_1, s_2 \in \mathbb{N} + \frac{1}{2}$ and want to reproduce the non-terminating spectrum of eigenvalues

$$
\mathrm{Spec}_2(s_1, s_2) = \{s_1{+}s_2{+}1,\ s_1{+}s_2{+}3,\ s_1{+}s_2{+}5,\ ...\}\,.
$$

To this end, we specialise the ladder (5.40) to the case for which the source term $\Upsilon(a, b, r, s|\tau)$ produces the second representation we found for the product of two Eisenstein series (5.25), i.e. we specialise our ladder to

$$(a,b,r,s) = \Big(1-2s_1,\, s_1,\, 1-s_2,\, 1-s_1\Big)\,, \tag{5.49}$$

and assume that s_1, s_2 are fixed half-integers, in which case (5.40) can be reduced to

$$\begin{aligned}&\Big[\Delta-\lambda_n^{(2)}(\lambda_n^{(2)}-1)\Big]\frac{8\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)}\mathrm{Y}_n(1-2s_1,\, s_1,\, 1-s_2,\, 1-s_1|\tau) = \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) \\ &-\frac{2\Gamma(s_1+s_2-1)\xi(2s_1-1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2)-3)}\mathcal{E}_{s_1+s_2-1}(\tau)-\frac{2\Gamma(s_1+1-s_2)\xi(2s_1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+1-s_2))}\mathcal{E}_{s_1+1-s_2}(\tau)\,.\end{aligned} \tag{5.50}$$

If we apply directly the ladder procedure with fixed parameters (5.49), we find that the ladder eigenvalues (dropping their explicit dependence from the fixed source indices s_1, s_2) are now $\tilde{\lambda}_n^{(2)} = -s_1-s_2-2n$, however, the exchange $\tilde{\lambda}_n^{(2)} \to \lambda_n^{(2)} = 1-\tilde{\lambda}_n^{(2)}$ leaves the equation invariant and produces the expected spectrum of eigenvalues

$$\lambda_n^{(2)} = s_1+s_2+2n+1\,.$$

This change is not without consequences: the constructed modular invariant solution, Y_n, does not quite land (modulo the previously discussed Eisenstein terms) on $\mathcal{E}(\lambda_n^{(2)}; s_1, s_2|\tau)$, the generalised Eisenstein series we are interested in, but rather on the reflected $\mathcal{E}(1-\lambda_n^{(2)}; s_1, s_2|\tau)$. We can use the general expression (5.31) to compute the asymptotic expansion of $\mathrm{Y}_n(1-2s_1, s_1, 1-s_2, 1-s_1|\tau)$ at large-τ_2 and confirm that the homogeneous solution $\tau_2^{1-\lambda_n^{(2)}}$ has vanishing coefficient, i.e. we land exactly on the opposite boundary condition compared to the wanted generalised Eisenstein series $\mathcal{E}(\lambda_n^{(2)}; s_1, s_2|\tau)$. This can be fixed by adding a suitable multiple of the modular invariant homogeneous solution, $\mathcal{E}_{\lambda_n^{(2)}}(\tau)$, such that the new solution satisfies the desired boundary condition of a vanishing coefficient for the homogeneous solution $\tau_2^{\lambda_n^{(2)}}$.

Lastly, with the help of the differential equation (3.15) we can easily invert the single Eisenstein source terms in (5.50). With all these considerations in mind, we arrive at the final expression

$$
\begin{aligned}
\mathcal{E}\left(\lambda_n^{(2)};s_1,s_2\middle|\tau\right)=&\frac{8\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)}\Upsilon_n(1-2s_1,\,s_1,\,1-s_2,\,1-s_1|\tau)\\
&-\frac{2\Gamma(\lambda_n^{(2)})\xi(2n{+}2)\xi(2s_1{+}2n{+}1)\xi(2s_2{+}2n{+}1)\xi(2(s_1{+}s_2{+}n))}{(2\lambda_n^{(2)}-1)\Gamma(s_1)\Gamma(s_2)\xi(2\lambda_n^{(2)}-1)\xi(2\lambda_n^{(2)})}\mathcal{E}_{\lambda_n^{(2)}}(\tau)\\
&+\frac{2\Gamma(s_1+s_2-1)\xi(2s_1-1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+s_2)-3)\mu(s_1+s_2-1,\lambda_n^{(2)})}\mathcal{E}_{s_1+s_2-1}(\tau)\\
&+\frac{2\Gamma(s_1+1-s_2)\xi(2s_1)\xi(2s_2-1)}{\Gamma(s_1)\Gamma(s_2)\xi(2(s_1+1-s_2))\mu(s_1+1-s_2,\lambda_n^{(2)})}\mathcal{E}_{s_1+1-s_2}(\tau)\,.
\end{aligned}
\tag{5.51}
$$

In summary, the ladder of Laplace equations (5.40) includes in a natural and uniform way the two string theory flavours of generalised Eisenstein series (4.17)-(5.7). In both cases (5.48)-(5.51), we expressed these generalised Eisenstein series as linear combination of finitely many novel Poincaré series (5.12). We now discuss some concrete examples for both flavours.

5.4 Examples

In this section we present some concrete and string theory relevant examples of our general construction. We begin with the generalised Eisenstein series $\mathcal{E}\left(4;\frac{3}{2},\frac{3}{2}\middle|\tau\right)$, coefficient of the higher derivative correction d^6R^4 in the effective low-energy action of type IIB superstring theory (5.2). For the given indices, $s_1=s_2=\frac{3}{2}$, the eigenvalue is $\lambda=\lambda_0^{(2)}=s_1+s_2+1=4$, hence $\mathcal{E}\left(4;\frac{3}{2},\frac{3}{2}\middle|\tau\right)$ is the function with smallest eigenvalue in the spectrum (5.7) for these sources.

Since this is a diagonal example where the indices s_1 and s_2 coincide, we need to use the regularisation scheme described in (5.26). Substituting the regularised parameters $s_1=\frac{3}{2}+\epsilon$, $s_2=\frac{3}{2}$ and $\lambda_0^{(2)}=4+\epsilon$ in the general expression (5.51) we derive

$$
\begin{aligned}
\mathcal{E}\left(4;\tfrac{3}{2},\tfrac{3}{2}\middle|\tau\right)=&\lim_{\epsilon\to0}\Big[\frac{4}{9\sqrt{\pi}\,\Gamma(\frac{3}{2}+\epsilon)}\Upsilon\left(\text{-}2(1+\epsilon),\tfrac{1}{2}+\epsilon,\text{-}\tfrac{3}{2},\text{-}\tfrac{3}{2}\text{ - }\epsilon|\tau\right)-\frac{\zeta(3+2\epsilon)}{9(2+\epsilon)\zeta(2+2\epsilon)}\mathcal{E}_{1+\epsilon}(\tau)\Big]\\
&-\frac{32\pi^6}{127\,575\zeta(7)}\mathcal{E}_4(\tau)-\frac{2\pi^2}{45\zeta(3)}\mathcal{E}_2(\tau)\,.
\end{aligned}
\tag{5.52}
$$

As previously stated, each term inside the limit is separately singular at $\epsilon=0$, however, this combination is such that the divergences in $1/\epsilon$ cancel out and produce a finite expression for

$\epsilon = 0$. We can substitute this regulated expression into the general formula (5.31) to recover the well-known asymptotic expansion [45] of the d^6R^4 correction

$$\mathcal{E}\left(4;\tfrac{3}{2},\tfrac{3}{2}\middle|\tau\right) \sim -\frac{2\zeta(3)^2\tau_2^3}{3\pi^3} - \frac{2\zeta(3)\tau_2}{9\pi} - \frac{2\pi}{45\tau_2} - \frac{4\pi^3}{25\,515\tau_2^3} \qquad \text{as} \quad \tau_2 \gg 1\,. \tag{5.53}$$

A second related example is the modular invariant function $\mathcal{E}\left(7;\tfrac{5}{2},\tfrac{3}{2}\middle|\tau\right)$ which arises at order $O(N^{-2})$ in the large-N expansion of the particular $\mathcal{N}=4$ SYM integrated correlator discussed in [65]. This case falls again into the spectrum (5.7), the indices are $s_1 = \frac{5}{2}$, $s_2 = \frac{3}{2}$ while the eigenvalue is $\lambda = \lambda_1^{(2)} = s_1 + s_2 + 3 = 7$ hence one step above the lowest eigenvalue in our Laplace tower (5.51). If we substitute these specific values for s_1, s_2 and $\lambda_1^{(2)}$ in (5.51) we obtain the Poincaré series representation

$$\begin{aligned}\mathcal{E}\left(7;\tfrac{5}{2},\tfrac{3}{2}\middle|\tau\right) = &-\frac{16}{15\pi^2}\Upsilon\left(-4,\tfrac{1}{2},-\tfrac{5}{2},-\tfrac{7}{2}|\tau\right) + \frac{16}{27\pi}\Upsilon\left(-4,\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{9}{2}|\tau\right) \\ &-\frac{4096\pi^{12}}{46\,414\,974\,375\zeta(13)}\mathcal{E}_7(\tau) - \frac{8\pi^4}{10\,935\zeta(5)}\mathcal{E}_3(\tau) - \frac{3\zeta(5)}{2\pi^4}\mathcal{E}_2(\tau)\,.\end{aligned} \tag{5.54}$$

Substituting this expression in the general formula (5.31) we obtain the asymptotic expansion

$$\mathcal{E}\left(7;\tfrac{3}{2},\tfrac{5}{2}\middle|\tau\right) \sim -\frac{2\zeta(3)\zeta(5)\tau_2^4}{15\pi^4} - \frac{\zeta(5)\tau_2^2}{30\pi^2} - \frac{4\zeta(3)}{2835} - \frac{2\pi^2}{3645\tau_2^2} - \frac{8\pi^6}{200\,930\,625\tau_2^6} \qquad \text{as} \quad \tau_2 \gg 1. \tag{5.55}$$

Finally, we discuss an example of generalised Eisenstein series belonging to the first spectrum (4.17). We consider the function $\mathcal{E}\left(3;3,2\middle|\tau\right)$ which captures the genuine depth-two part of the two-loop MGF $C_{3,1,1}(\tau)$ defined in (4.7),

$$C_{3,1,1}(\tau) = -4\,\mathcal{E}\left(3;3,2\middle|\tau\right) + \frac{43}{35}\mathcal{E}_5(\tau) - \frac{\zeta_5}{60}\,. \tag{5.56}$$

The indices are $s_1 = 3, s_2 = 2$ while the eigenvalue is $\lambda = \lambda_0^{(1)} = s_1 + s_2 - 2 = 3$ hence $\mathcal{E}\left(3;3,2\middle|\tau\right)$ is the function with largest eigenvalue in the first spectrum (4.17) for these sources. Substituting the specific values for s_1, s_2 and $\lambda_0^{(1)}$ in the general solution (5.48) we obtain

$$\mathcal{E}\left(3;3,2\middle|\tau\right) = -\frac{\pi^2}{945}\Upsilon\left(-3,1,2,1|\tau\right) + \frac{11}{70}\mathcal{E}_5(\tau) - \frac{\zeta(3)}{42}\mathcal{E}_2(\tau)\,. \tag{5.57}$$

It is interesting to compare the present Poincaré series representation (5.57) with a different one (finely tuned to represent all two-loop MGFs) considered in [31] for which we have

$$\mathcal{E}\left(3;3,2\middle|\tau\right) = \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \left[\frac{(\pi\tau_2)^5}{297\,675} - \frac{(\pi\tau_2)^2\zeta(3)}{1890} - \frac{(\pi\tau_2)^2}{1890}\sum_{m=1}^{\infty}\sigma_{-3}(m)(q^m+\bar{q}^m)\right]_\gamma . \quad (5.58)$$

Again thanks to the general expression (5.31), starting from (5.57) we can retrieve the known asymptotic expansion (see equation (4.59) derived before)

$$\mathcal{E}\left(3;3,2\middle|\tau\right) \sim \frac{\pi^5\tau_2^5}{297\,675} - \frac{\zeta(3)\pi^2\tau_2^2}{1890} - \frac{\zeta(5)}{360} - \frac{7\zeta(7)}{64\pi^2\tau_2^2} + \frac{\zeta(3)\zeta(5)}{8\pi^3\tau_2^3} \qquad \text{as} \quad \tau_2 \gg 1. \quad (5.59)$$

Compared to previous results in the literature, one novelty of the Poincaré series (5.12) is that all the examples here considered, and more broadly all generalised Eisenstein with spectra (4.17)-(5.7) can be expressed as linear combinations of finitely many $\Upsilon(a,b,r,s|\tau)$.

5.5 Spectral analysis point of view

The central ideas relating to spectral theory on $L^2(\mathcal{F})$ were already introduced in section 3.3; here we apply them to study generalised Eisenstein series. Let us briefly review how one can exploit the differential equation (5.6) to compute the spectral decomposition of the generalised Eisenstein series and in particular obtain a useful integral representation (3.26) for its Fourier zero-mode. Since the generalised Eisenstein series, $\mathcal{E}\left(\lambda;s_1,s_2\middle|\tau\right)$, is not an element of $L^2(\mathcal{F})$, one has to be a little careful in defining a proper regularised version for the spectral overlaps when dealing with functions not of rapid decay. This problem was addressed in a beautiful and classic paper by Don Zagier [73] from which we present a few key details; we also refer to [47] and appendix B of [48] for more details on the generalised Eisenstein series.

Firstly we want to understand the behaviour at the cusp $\tau_2 \gg 1$ of the generalised Eisenstein series by exploiting its differential equation (5.6), repeated here for convenience

$$[\Delta - \lambda(\lambda-1)]\mathcal{E}\left(\lambda;s_1,s_2\middle|\tau\right) = \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau).$$

As usual we perform the Fourier decomposition in $\tau_1 = \mathrm{Re}(\tau)$,

$$\mathcal{E}\left(\lambda;s_1,s_2\middle|\tau\right) = \sum_{k\in\mathbb{Z}} a_k(\lambda;s_1,s_2|\tau_2)e^{2\pi i k\tau_1}\,, \quad (5.60)$$

and thanks to linearity, we can solve the inhomogeneous Laplace equation mode by mode.

From (5.3) we easily extract the Fourier zero-mode contribution to the bilinear source term $\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau)$, comprised of power-behaved terms and exponentially suppressed terms $e^{-4\pi\tau_2}$. Thus we find a solution to the differential equation for the Fourier zero-mode $a_0(s_1,s_2;\lambda|\tau_2)$:

$$\begin{aligned}
a_0(\lambda;s_1,s_2|\tau_2) &= \frac{4\pi^{-s_1-s_2}\zeta(2s_1)\zeta(2s_2)}{(s_1+s_2-\lambda)(s_1+s_2+\lambda-1)}\tau_2^{s_1+s_2} \\
&+ \frac{4\pi^{-s_1}\xi(2s_2-1)\zeta(2s_1)}{(s_1+1-s_2-\lambda)(s_1-s_2+\lambda)\Gamma(s_2)}\tau_2^{s_1+1-s_2} + \frac{4\pi^{-s_2}\xi(2s_1-1)\zeta(2s_2)}{(s_2+1-s_1-\lambda)(s_2-s_1+\lambda)\Gamma(s_1)}\tau_2^{s_2+1-s_1} \\
&+ \frac{4\xi(2s_1-1)\xi(2s_2-1)}{(s_1+s_2-\lambda-1)(s_1+s_2+\lambda-2)\Gamma(s_1)\Gamma(s_2)}\tau_2^{2-s_1-s_2} + \alpha(\lambda;s_1,s_2)\tau_2^{1-\lambda} + O(e^{-4\pi\tau_2})\,.
\end{aligned} \tag{5.61}$$

The constant $\alpha(\lambda;s_1,s_2)$ parametrises the homogeneous solution, $\tau_2^{1-\lambda}$, and can not be determined by solely analysing the differential equation. However, the coefficient $\alpha(\lambda;s_1,s_2)$ will be promptly fixed by requiring modular invariance for the solution. Furthermore, since we are dealing with a second-order differential equation, we must have two linearly independent homogeneous solutions, which in the Fourier zero-mode sector are $\tau_2^{1-\lambda}$ and τ_2^{λ}. It is conventional to choose a vanishing coefficient for the second homogeneous solution, τ_2^{λ}. Once the modular invariant solution, $\mathcal{E}\left(\lambda;s_1,s_2\big|\tau\right)$, subject to this boundary condition has been found, we can always consider $\mathcal{E}\left(\lambda;s_1,s_2\big|\tau\right)+a\,\mathcal{E}_\lambda(\tau)$, with $a\neq 0$, which is a different modular invariant solution to the same Laplace system, but this time with a non-vanishing coefficient for τ_2^{λ}.

As anticipated, from the Fourier zero-mode analysis (5.61) we immediately deduce that the generalised Eisenstein series is not an element of $L^2(\mathcal{F})$. To simplify the discussion, we can assume that the eigenvalue λ is such that $\mathrm{Re}(\lambda)>\frac{1}{2}$, a condition that is satisfied by both spectra (4.17) and (5.7). With this assumption, from (5.61) we have full control over all power-behaved terms that might grow faster than $\tau_2^{\frac{1}{2}}$ at the cusp, and subsequently we can subtract suitable Eisenstein series in order to cancel all non-integrable terms thus obtaining a modular invariant and square-integrable function.

We are then led to consider the "regularised" linear combination

$$\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau) = \mathcal{E}\left(\lambda;s_1,s_2\big|\tau\right) - \sum_I \beta_I\mathcal{E}_I(\tau), \tag{5.62}$$

where $I\in\{s_1+s_2, s_1+1-s_2, s_2+1-s_1, 2-s_1-s_2\}$ and β_I are chosen such that the term of order τ_2^I in $\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau)$ has a vanishing coefficient if $\mathrm{Re}(I)>\frac{1}{2}$ and $\beta_I=0$ otherwise. By construction, we clearly have $\tilde{\mathcal{E}}(\lambda; s_1, s_2) \in L^2(\mathcal{F})$, hence its Fourier zero-mode can be given in terms of the contour integral representation (3.26).

Now that we have modified the generalised Eisenstein series to obtain a nice and square-integrable function, $\tilde{\mathcal{E}}(\lambda; s_1, s_2|\tau)$, we can combine the spectral methods described in the previous section with the Laplace equation (5.6). It is fairly easy to see from our definition (5.62) that the inhomogeneous Laplacian equation is modified to

$$[\Delta - \lambda(\lambda-1)]\tilde{\mathcal{E}}(\lambda; s_1, s_2|\tau) = \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) + \sum_I \big[\lambda(\lambda-1) - I(I-1)\big]\beta_I \mathcal{E}_I(\tau)\,. \tag{5.63}$$

Since both sides of this equation are in $L^2(\mathcal{F})$, we can now take the Petersson inner product against the constant function, the continuous part and the discrete part of the spectrum on both sides of (5.63) to obtain the spectral overlaps previously discussed. A slight complication arises from the fact that, although both sides of (5.63) are square-integrable, the source term is made of non-square integrable objects, hence a suitable regularisation is required to discuss the Petersson inner product for functions not of rapid decay.

To this end, we follow [73] and introduce a specific regularisation for the divergent integral

$$\mathcal{I}(s) = \int_0^\infty y^s \mathrm{d}y = \int_0^1 y^s \mathrm{d}y + \int_1^\infty y^s \mathrm{d}y = \mathcal{I}_1(s) + \mathcal{I}_2(s)\,. \tag{5.64}$$

Clearly the starting integral does not converge for any $s \in \mathbb{C}$, but the two parts it splits into do converge on disjoint regions. Namely for $\mathrm{Re}(s) > -1$ the integral $\mathcal{I}_1(s)$ is well-defined and we have $\mathcal{I}_1(s) = \frac{1}{s+1}$, while similarly for $\mathrm{Re}(s) < -1$ the second integral is well-defined and we have $\mathcal{I}_2(s) = -\frac{1}{s+1}$. Since both integrals admit an analytic continuation in $s \in \mathbb{C} \setminus \{-1\}$, we may define $\mathcal{I}(s) = \mathcal{I}_1(s) + \mathcal{I}_2(s) = 0$.

As a direct application of this formula, we compute the average $\langle \mathrm{E}_r \rangle = (\mathrm{E}_r, 1)$, i.e. the spectral overlap of an Eisenstein series with the constant function, as well as the spectral overlap $(\mathrm{E}_r, \mathrm{E}_t)$ for $r \neq t$:

$$\langle \mathrm{E}_r \rangle = \int_{\mathcal{F}} \Big[\sum_{\gamma \in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma\cdot\tau)^r \Big] \mathrm{d}\mu = \int_{\mathrm{B}(\mathbb{Z})\backslash \mathfrak{H}} \tau_2^r \frac{\mathrm{d}\tau_1\, \mathrm{d}\tau_2}{\tau_2^2} = \int_0^\infty \tau_2^{r-2} \mathrm{d}\tau_2 = 0\,, \tag{5.65}$$

$$
\begin{aligned}
(\mathrm{E}_r, \mathrm{E}_t) &= \int_{\mathcal{F}} E(r;\tau)\Big[\sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma\cdot\tau)^{\bar{t}}\Big]\mathrm{d}\mu = \int_{\mathrm{B}(\mathbb{Z})\backslash\mathfrak{H}} E(r;\tau)\tau_2^{\bar{t}}\,\frac{\mathrm{d}\tau_1\,\mathrm{d}\tau_2}{\tau_2^2} \\
&= \int_0^\infty \Big(\tau_2^{\bar{t}+r-2} + \frac{\xi(2r-1)\pi^r}{\Gamma(r)\zeta(2r)}\tau_2^{\bar{t}-r-1}\Big)\mathrm{d}\tau_2 = 0
\end{aligned}
\tag{5.66}
$$

In both calculations we make crucial use of the unfolding trick, which is described in section 3.3. Note that all of the above integrals are ill-defined and need to be regularised in the same way as the original integral $\mathcal{I}(s)$. We will shortly see more interesting examples where the unfolding procedure produces convergent integrals, which can nevertheless be treated via the same type of analytic continuation.

In particular, we can use the differential equation (5.63) to show the vanishing of the spectral overlap of $\tilde{\mathcal{E}}(s_1,s_2;\lambda|\tau)$ with the constant function,

$$
\begin{aligned}
&\langle\tilde{\mathcal{E}}(\lambda;s_1,s_2)\rangle = \int_{\mathcal{F}} \tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau)\,\mathrm{d}\mu \\
&= \frac{1}{\lambda(\lambda-1)}\int_{\mathcal{F}}\Big\{\Delta\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau) - \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) + \sum_I \big[I(I-1)-\lambda(\lambda-1)\big]\beta_I\mathcal{E}_I(\tau)\Big\}\,\mathrm{d}\mu = 0\,.
\end{aligned}
\tag{5.67}
$$

The first term vanishes since it is an integral of a total derivative over a closed surface, while the second and third term vanish due to the previously derived identities (5.65)-(5.66).

As a result, to derive a useful expression for the Fourier zero-mode integral representation (3.26) of $\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau)$, we only need considering the spectral overlap with the Eisenstein series $E(t;\tau)$ with $\mathrm{Re}(t)=\frac{1}{2}$:

$$
\begin{aligned}
(\tilde{\mathcal{E}}(\lambda;s_1,s_2),\mathrm{E}_t) &= \int_{\mathcal{F}}\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau)\,E(1-t;\tau)\mathrm{d}\mu = \int_{\mathcal{F}}\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau)\frac{\Delta E(1-t;\tau)}{t(t-1)}\mathrm{d}\mu \qquad (5.68)\\
&= \int_{\mathcal{F}}\Big\{\mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau) + \lambda(\lambda-1)\tilde{\mathcal{E}}(\lambda;s_1,s_2|\tau) + \sum_I[\lambda(\lambda-1)-I(I-1)]\beta_I\mathcal{E}_I(\tau)\Big\}\frac{E(1-t;\tau)}{t(t-1)}\mathrm{d}\mu\,,
\end{aligned}
$$

where in the Petersson inner product we used the fact that $\overline{E(t;\tau)} = E(\bar{t};\tau) = E(1-t;\tau)$ on the critical line $\mathrm{Re}(t)=\frac{1}{2}$ for which $\bar{t}=1-t$.

In the first line of (5.68) we used the differential equation satisfied by the Eisenstein series (3.15), while in the second line we integrated by parts and then used the inhomogeneous Laplace equation (5.63). Since we have already shown that the integral over the fundamental domain

$\mathcal{F}$ of a product of two Eisenstein series vanishes (5.66), the overlap $(\tilde{\mathcal{E}}(\lambda; s_1, s_2), \mathrm{E}_t)$ can be expressed simply as an integral of a triple product of Eisenstein series.

Once again this integral can be evaluated [73] via the unfolding trick by rewriting one of the Eisenstein series as a Poincaré series and then using the sum over images to unfold the fundamental domain $\mathcal{F}$ onto the strip $\mathrm{B}(\mathbb{Z})\backslash\mathfrak{H}$:

$$\begin{aligned}(\tilde{\mathcal{E}}(\lambda; s_1, s_2), \mathrm{E}_t) &= \frac{1}{(t-\lambda)(t+\lambda-1)} \int_{\mathcal{F}} \mathcal{E}_{s_1}(\tau)\mathcal{E}_{s_2}(\tau)E(1-t;\tau)\mathrm{d}\mu \\ &= \frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t-1)}.\end{aligned} \tag{5.69}$$

We can then write the spectral decomposition (3.25) for the generalised Eisenstein series

$$\begin{aligned}\mathcal{E}\left(\lambda; s_1, s_2 \middle| \tau\right) &= \int_{\mathrm{Re}(t)=\frac{1}{2}} \frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t-1)} E(t;\tau)\frac{\mathrm{d}t}{4\pi i} \\ &\quad + \sum_I \beta_I \mathcal{E}_I(\tau) + \sum_{n=1}^{\infty} (\tilde{\mathcal{E}}(\lambda; s_1, s_2), \phi_n)\phi_n(\tau),\end{aligned} \tag{5.70}$$

where the spectral overlap with the Maass cusp forms can be made more explicit, but it is of little concrete use given the poor analytic control over these objects.

We are now in the position of specialising the integral representation (3.26) to the case of $\mathcal{E}(s_1, s_2; \lambda)$ thus arriving at the useful expression for its Fourier zero-mode

$$\begin{aligned}a_0(\lambda; s_1, s_2|\tau_2) &= \sum_I \beta_I \Big[\frac{2\zeta(2I)}{\pi^I}\tau_2^I + \frac{2\xi(2I-1)}{\Gamma(I)}\tau_2^{1-I}\Big] \\ &\quad + \int_{\mathrm{Re}(t)=\frac{1}{2}} \frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t-1)}\tau_2^t \frac{\mathrm{d}t}{2\pi i},\end{aligned} \tag{5.71}$$

where again $I \in \{s_1+s_2, s_1+1-s_2, s_2+1-s_1, 2-s_1-s_2\}$ and β_I was defined in (5.62).

The integrand of (5.71) is a meromorphic function of t for which it is rather easy to understand the structure of singularities. Firstly, we note that the completed Riemann function $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is meromorphic with simple poles at $s=0$ and $s=1$, while it vanishes only at the non-trivial zeros of the Riemann zeta function, which, from the conjectural Riemann hypothesis, are of the form $s = \frac{1}{2} + i\rho_n$ with ρ_n real. We then deduce that the integrand of (5.71) has poles located at:

- $t = 1-I, I$ with $I \in \{s_1+s_2, s_1+1-s_2, s_2+1-s_1, 2-s_1-s_2\}$, for which one the completed Riemann zeta functions in the numerator has argument equal to 0 or 1 respectively;
- $t = \lambda, 1-\lambda$, coming from the two rational terms $[(t-\lambda)(t+\lambda-1)]^{-1}$;
- $t = \frac{3}{4} + i\frac{\rho_n}{2}$, coming from the non-trivial zeroes of $\xi(2t-1)$ present in the denominator.

We can now use (5.71) to distinguish between the different contributions arising in the asymptotic expansions of the Fourier zero-mode $a_0(\lambda; s_1, s_2|\tau_2)$ as $\tau_2 \gg 1$ or as $\tau_2 \to 0$. Focusing for the present time on the asymptotic expansion at the cusp $\tau_2 \gg 1$, we see that the integral contour in (5.71) can be closed in the left half-plane $\text{Re}(t) < 0$. In doing so, we pick up the residues for the poles located at $\text{Re}(t) < \frac{1}{2}$, which are:

(i) $t = I$ for $I \in \{s_1+s_2, s_1+1-s_2, s_2+1-s_1, 2-s_1-s_2\}$ with $\text{Re}(I) < \frac{1}{2}$;

(ii) $t = 1-I$ for $I \in \{s_1+s_2, s_1+1-s_2, s_2+1-s_1, 2-s_1-s_2\}$ with $\text{Re}(I) > \frac{1}{2}$;

(iii) $t = 1-\lambda$ under the original assumption $\text{Re}(\lambda) > \frac{1}{2}$.

The end result can be made more concrete by considering the case relevant for our spectra (4.17)-(5.7), where $s_1, s_2 \geq \frac{3}{2}$ and without loss of generality $s_1 \geq s_2$. Under these conditions and considering the non-diagonal case where $s_1 - s_2 \geq 1$, we simply collect the residues from the poles at $t \in \{s_2+1-s_1, s_2-s_1, 2-s_1-s_2, 1-s_1-s_2\}$ and $t = 1-\lambda$.

Note that, for this range of parameters, the square-integrable function $\tilde{\mathcal{E}}(\lambda; s_1, s_1)$ in (5.62) is obtained by removing suitable multiples of the Eisenstein series $\mathcal{E}_I(\tau)$ with $I \in \{s_1+s_2, s_1+1-s_2\}$. From the Fourier zero-mode (5.61), we see that this subtraction indeed removes the non-square integrable powers $\tau_2^{s_1+s_2}$ and $\tau_2^{s_1+1-s_2}$. However, since at the cusp $\mathcal{E}_I(\tau) \sim \#\tau_2^I + \#\tau_2^{1-I}$, we also introduce "unwanted" reflected powers $\tau_2^{1-s_1-s_2}$ and $\tau_2^{1-(s_1+1-s_2)} = \tau_2^{s_2-s_1}$. These unwanted terms are exactly cancelled by the residues coming from the above-mentioned poles located at $t \in \{s_2-s_1, 1-s_1-s_2\}$. The remaining poles at $t \in \{s_2+1-s_1, 2-s_1-s_2\}$ produce the remaining powers for the particular solution (5.61), while the pole at $t = 1-\lambda$ produces the homogeneous solution term.

The diagonal case, $s_1 = s_2$, requires some extra care since to define the square-integrable function $\tilde{\mathcal{E}}(\lambda; s_1, s_1)$ in (5.62) we need to subtract a regularised version for the divergent Eisenstein series $\mathcal{E}_1(\tau)$, see e.g. appendix B of [48]. At the same time, we see that the spectral overlap (5.69) develops a double pole at $t = 0$ and $t = 1$ precisely for $s_1 = s_2$. To avoid these complications, we can obtain the diagonal case as the off-diagonal limit $s_2 = s_1 - \epsilon$ with $\epsilon \to 0$.

We can directly use (5.71) to determine the previously unknown coefficient $\alpha(\lambda; s_1, s_2)$ multiplying the homogeneous solution $\tau_2^{1-\lambda}$. This coefficient was first computed in [39] with a similar method, and can now be calculated by simply picking up the pole of (5.71) at $t = 1-\lambda$, giving us

$$\alpha(\lambda; s_1, s_2) = -\frac{4\xi(s_1+s_2-\lambda)\xi(s_1-s_2+\lambda)\xi(s_2-s_1+\lambda)\xi(s_1+s_2+\lambda-1)}{(2\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2\lambda)}\,. \tag{5.72}$$

In the next section, we discuss the asymptotic expansion of (5.71) as $\tau_2 \to 0$, where the contour of integration has to be closed instead in the right half-plane $\mathrm{Re}(t) > 0$. This will select the "complementary" poles to the ones just discussed, and a new infinite family of poles coming from the non-trivial zeros of the Riemann zeta function will also play an essential rôle.

We conclude this section by analysing the spectral decomposition for the novel functions $\Upsilon(a,b,r,s|\tau)$. Firstly, from the previously determined asymptotic expansion at the cusp (5.31), we see that all $\Upsilon(a,b,r,s|\tau)$ are directly square integrable functions in the region of parameters a,b,r,s where the Poincaré series converges (5.13), i.e. we have immediately $\Upsilon(a,b,r,s) \in L^2(\mathcal{F})$ when (5.13) is satisfied.

As a consequence, we can directly compute the spectral overlaps without any need for subtracting Eisenstein series. We start by observing that the spectral overlap with the constant function vanishes

$$\langle \Upsilon(a,b,r,s)\rangle = \int_{\mathcal{F}} \Upsilon(a,b,r,s|\tau)\,\mathrm{d}\mu = 0\,,$$

since we can use the Poincaré series representation (5.12) for $\Upsilon(a,b,r,s|\tau)$ to unfold the integral from the fundamental domain $\mathcal{F}$ to the strip $\mathrm{B}(\mathbb{Z})\backslash\mathfrak{H}$, and we conclude that the integral over x vanishes since the seed function $\upsilon(a,b,r,s|\tau)$ does not have a Fourier zero-mode.

We proceed by computing the spectral overlap with the Eisenstein series. A calculation very similar to (5.69) yields

$$\begin{aligned}(\Upsilon(a,b,r,s)\,,\mathrm{E}_t) &= \int_{\mathcal{F}} \Upsilon(a,b,r,s|\tau)\,E(1-t;\tau)\mathrm{d}\mu = \int_{\mathrm{B}(\mathbb{Z})\backslash\mathfrak{H}} \upsilon(a,b,r,s|\tau)\,E(1-t;\tau)\frac{\mathrm{d}\tau_1\,\mathrm{d}\tau_2}{\tau_2^2}\\ &= \frac{\Gamma\big(\frac{r+1-s-t}{2}\big)\Gamma\big(\frac{r+s-t}{2}\big)\Gamma\big(\frac{t+r-s}{2}\big)\Gamma\big(\frac{t+r+s-1}{2}\big)}{2\pi^r\,\Gamma(r)\xi(2-2t)}\\ &\quad\times\frac{\zeta(r+1-b-t)\zeta(r+1-a-b-t)\zeta(t+r-b)\zeta(t+r-a-b)}{\zeta(2r+1-a-2b)}\,.\end{aligned} \tag{5.73}$$

The keen-eyed reader will notice that if we now plug the spectral overlap just derived into the integral representation formula for the Fourier zero-mode (3.26), we obtain exactly the same expression (5.29) previously derived from the Poincaré series representation. This is a significantly simpler derivation of (5.29) when compared to the Mellin-Barnes discussion presented in appendix D. However, we need to stress that without having already obtained the result (5.31), we could have not inferred immediately that the functions $\Upsilon(a, b, r, s|\tau)$ are in $L^2(\mathcal{F})$.

5.6 Non-perturbative terms and small-τ_2 behaviour

So far our analysis of the Fourier zero-mode (5.71) was only concerned with the power-behaved terms at the cusp $\tau_2 \gg 1$. In this section we show how the exponentially suppressed corrections $e^{-4\pi\tau_2}$ are encoded in (5.71) and clarify how the resurgent analysis carried out in chapter 4 nicely connects with the present discussion. In the limit $\tau_2 \to 0$, the non-perturbative terms stop being exponentially suppressed and instead produce an infinite sum of perturbative corrections related to the non-trivial zeros of the Riemann zeta function.

As discussed in the previous section, we can easily evaluate the perturbative expansion for the Fourier zero-mode integral representation (5.71) as $\tau_2 \gg 1$ by closing the contour of integration in the left half-plane $\mathrm{Re}(t) < 0$. Picking up various residues allows us to reproduce all power-behaved terms present in (5.61), however, the integral does not vanish when we push the contour of integration to infinity, but rather it produces the remaining exponentially suppressed corrections in the Fourier zero-mode sector.

We follow this procedure and push the contour of integration to the left half-plane $\mathrm{Re}(t) < 0$, while collecting the residues to arrive at

$$
\begin{aligned}
a_0(\lambda; s_1, s_2|\tau_2) &= \frac{4\pi^{-s_1-s_2}\zeta(2s_1)\zeta(2s_2)}{(s_1+s_2-\lambda)(s_1+s_2+\lambda-1)}\tau_2^{s_1+s_2} \qquad (5.74)\\
&+ \frac{4\pi^{-s_1}\xi(2s_2-1)\zeta(2s_1)}{(s_1+1-s_2-\lambda)(s_1-s_2+\lambda)\Gamma(s_2)}\tau_2^{s_1+1-s_2} + \frac{4\pi^{-s_2}\xi(2s_1-1)\zeta(2s_2)}{(s_2+1-s_1-\lambda)(s_2-s_1+\lambda)\Gamma(s_1)}\tau_2^{s_2+1-s_1}\\
&+ \frac{4\xi(2s_1-1)\xi(2s_2-1)}{(s_1+s_2-\lambda-1)(s_1+s_2+\lambda-2)\Gamma(s_1)\Gamma(s_2)}\tau_2^{2-s_1-s_2} + \alpha(\lambda; s_1, s_2)\tau_2^{1-\lambda}\\
&+ \int_{\mathrm{Re}(t)=\tilde{\gamma}} \frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t-1)}\tau_2^t \frac{\mathrm{d}t}{2\pi i}\,,
\end{aligned}
$$

where $\tilde{\gamma} < \min\big(\mathrm{Re}(I), \mathrm{Re}(1-I), \mathrm{Re}(\lambda), \mathrm{Re}(1-\lambda)\big)$. The integrand in (5.74) is manifestly analytic for t in the half-plane $\mathrm{Re}(t) \leq \tilde{\gamma}$ and we claim that the corresponding integral is exponentially suppressed at the cusp $\tau_2 \gg 1$ thus containing all of the non-perturbative, $e^{-4\pi\tau_2}$, terms.

For aesthetic reasons we perform the change of variables $t \to 1-t$ and use the reflection identity $\xi(s) = \xi(1-s)$ to rewrite the above integral as

$$\mathrm{NP}^{(\lambda)}_{s_1,s_2}(\tau_2) := \int_{\mathrm{Re}(t)=\gamma} \frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t)} \tau_2^{1-t} \frac{\mathrm{d}t}{2\pi i}\,, \tag{5.75}$$

where $\gamma > \max\big(\mathrm{Re}(I), \mathrm{Re}(1-I), \mathrm{Re}(\lambda), \mathrm{Re}(1-\lambda)\big)$ is arbitrary as long as it lies to the right of all the poles location. We can expand all the completed Riemann functions as $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ and use Ramanujan identity (D.12) in reverse to rewrite the particular combination of Riemann zeta functions appearing in (5.75) as a Dirichlet series for the product of two divisor functions, arriving at

$$\begin{aligned}\mathrm{NP}^{(\lambda)}_{s_1,s_2}(\tau_2) = &\sum_{n=1}^{\infty} \frac{4\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n) n^{s_1+s_2-1}\tau_2}{\Gamma(s_1)\Gamma(s_2)} \\ &\times \int_{\mathrm{Re}(t)=\gamma} \frac{\Gamma(\frac{t+s_1+s_2-1}{2})\Gamma(\frac{t+s_1-s_2}{2})\Gamma(\frac{t+s_2-s_1}{2})\Gamma(\frac{t+1-s_1-s_2}{2})}{(t-\lambda)(t+\lambda-1)\Gamma(t)} (\pi n \tau_2)^{-t} \frac{\mathrm{d}t}{2\pi i}\,.\end{aligned} \tag{5.76}$$

We have not managed to evaluate (5.76) in closed form, however, its asymptotic expansion as $\tau_2 \gg 1$ can be obtained via saddle point approximation. To proceed, we can use the Stirling approximation for the gamma functions to confirm that the integrand has a stationary point at $t = 4\pi n \tau_2$. Hence a simple steepest descent calculation produces the required asymptotic expansion,

$$\mathrm{NP}^{(\lambda)}_{s_1,s_2}(\tau_2) = \sum_{n=1}^{\infty} \frac{\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n) n^{s_1+s_2-2}}{\Gamma(s_1)\Gamma(s_2)} e^{-4\pi n \tau_2} \phi^{(\lambda)}_{s_1,s_2}(4\pi n \tau_2)\,, \tag{5.77}$$

where the first few perturbative corrections are given by (compare with (4.76))

$$\begin{aligned}&\phi^{(\lambda)}_{s_1,s_2}(y) = \\ &\frac{8}{y^2} + \frac{8[s_1(s_1-1)+s_2(s_2-1)-4]}{y^3} + 4\frac{\big\{[s_1(s_1-1)+s_2(s_2-1)-7]^2+2\lambda(\lambda-1)-13\big\}}{y^4} + O(y^{-5})\,.\end{aligned} \tag{5.78}$$

A few comments are in order. Firstly, although for general values of the parameters s_1, s_2 and λ the function $\phi^{(\lambda)}_{s_1,s_2}(y)$ contains infinitely many perturbative terms, we find that for some special cases, this series has only a finite number of terms. For example, if we fix $s_1 = 3, s_2 = 2$ and $\lambda = 2$, corresponding to the modular invariant function (5.57) belonging to the spectrum (4.17), the non-perturbative sector (5.76) simplifies to

$$\begin{aligned}\mathrm{NP}^{(3)}_{3,2}(\tau_2) &= \sum_{n=1}^{\infty} 16\pi\tau_2\, \sigma_{-3}(n)\sigma_{-5}(n)n^4 \int_{\mathrm{Re}(t)=5} t\,\Gamma(t-4)\,(4\pi n\tau_2)^{-t}\frac{\mathrm{d}t}{2\pi i}\\ &= \sum_{n=1}^{\infty} \frac{\sigma_{-3}(n)\sigma_{-5}(n)n^3}{2} e^{-4\pi n\tau_2}\Big[\frac{8}{(4\pi n\tau_2)^2}+\frac{32}{(4\pi n\tau_2)^3}\Big]\,,\end{aligned} \tag{5.79}$$

a result that was obtained using completely different techniques in (4.59).

Although we have not proven that for generic values of s_1, s_2 and λ the perturbative series $\phi^{(\lambda)}_{s_1,s_2}(y)$ contains infinitely many terms, it is easy to see that for the case associated with the spectrum (4.17) (corresponding to depth-two modular graph functions) the series $\phi^{(\lambda)}_{s_1,s_2}(y)$ is always a polynomial. This is expected from the Laplace equation (5.6) given that for the spectrum (4.17) the Eisenstein series appearing in the source term have integer index, hence the corresponding Bessel functions, which appear in the Fourier decomposition (5.3) and which are responsible for the non-perturbative terms, have half-integer index thus producing only finitely many perturbative terms in the non-perturbative sector.

A second comment we want to stress is that our expression (5.76) can be shown to be the exact solution to the Laplace equation (5.6) for the non-perturbative part of the Fourier zero-mode sector. Given the Laplace equation (5.6) and the Fourier decomposition (5.3) we must have

$$[\tau_2^2\partial_{\tau_2}^2 - \lambda(\lambda-1)]\mathrm{NP}^{(\lambda)}_{s_1,s_2}(\tau_2) = \sum_{n=1}^{\infty} \frac{32 n^{s_1+s_2-1}\sigma_{1-2s_1}(n)\sigma_{1-2s_2}(n)}{\Gamma(s_1)\Gamma(s_2)}\tau_2\, K_{s_1-\frac{1}{2}}(2\pi n\tau_2)K_{s_2-\frac{1}{2}}(2\pi n\tau_2)\,. \tag{5.80}$$

If we rewrite the source term using the Mellin-Barnes type integral representation for the product of two Bessel function

$$y\, K_{s_1-\frac{1}{2}}(2y)K_{s_2-\frac{1}{2}}(2y) = \int_{\mathrm{Re}(t)=\gamma} \frac{\Gamma(\frac{t+s_1+s_2-1}{2})\Gamma(\frac{t+s_1-s_2}{2})\Gamma(\frac{t+s_2-s_1}{2})\Gamma(\frac{t+1-s_1-s_2}{2})}{\Gamma(t)} y^{1-t}\frac{\mathrm{d}t}{16\pi i}\,, \tag{5.81}$$

where $\gamma > \max\big(\mathrm{Re}(I), \mathrm{Re}(1-I)\big)$, and then simply solve the differential equation for $\mathrm{NP}^{(\lambda)}_{s_1,s_2}(\tau_2)$

by inverting the differential operator as

$$\frac{1}{[\tau_2^2\partial_{\tau_2}^2-\lambda(\lambda-1)]}\tau_2^{1-t}=\frac{\tau_2^{1-t}}{(t-\lambda)(t+\lambda-1)}\,,$$

we find the exact integral representation (5.76).

Furthermore, we note that the formula for the perturbative series expansion (5.78) in the non-perturbative sector reproduces exactly the results obtained in chapter 4 for the modular invariant functions associated with the spectrum (4.17). We stress that in that case we started from the seed functions (5.10) and obtained the non-perturbative sector for the generalised Eisenstein series with spectrum (4.17) from a careful resummation of an evanescent, yet factorially divergent formal perturbative expansion in an example of Cheshire cat resurgence, very similar to our discussion below (5.33). We have now established that the results are actually more general than originally thought, and in particular (5.78) appears to be valid for all values of s_1, s_2 and λ and not just for the spectrum (4.17).

Finally, as discussed in chapter 4, it is easy to see that while for $\tau_2 \gg 1$ the Fourier zero-mode contribution (5.75) is non-perturbative and exponentially suppressed, its nature changes dramatically when $\tau_2 \to 0$. Rather than splitting the complete Fourier zero-mode $a_0(\lambda; s_1, s_2|\tau_2)$ in perturbative plus non-perturbative terms as in (5.71), we can analyse directly the integral representation (5.71) in the limit $\tau_2 \to 0$.

As previously anticipated just below (5.71), in the limit $\tau_2 \to 0$ we can close the contour of integration to the right half-plane $\mathrm{Re}(t) > 0$ and collect the residues from the various "complementary" poles plus the infinite set of completely novel poles located at $t = \frac{3}{4} + i\frac{\rho_n}{2}$ and coming from the non-trivial zeroes of $\xi(2t-1)$ present in the denominator of the integrand in (5.71). Once again, after we have pushed the contour of integration past all the poles, the remaining integral captures all exponentially suppressed contributions now of the form $e^{-\frac{4\pi}{\tau_2}}$ (similar behaviour was observed in [74] for the spectral decomposition of the partition function in certain 2-d conformal field theories). The asymptotic expansion of (5.71) as $\tau_2 \to 0$ is simply given by the sum over the residues of all the poles located at $\mathrm{Re}(t) > \frac{1}{2}$ plus a remaining contour integral,

$$
\begin{aligned}
a_0(\lambda;s_1,s_2|\tau_2) \sim\ & \frac{4\xi(2s_1-1)\xi(2s_2-1)\xi(2s_1+2s_2-2)}{(s_1+s_2-\lambda-1)(s_1+s_2+\lambda-2)\Gamma(s_1)\Gamma(s_2)\xi(2s_1+2s_2-3)}\tau_2^{s_1+s_2-1} \\
&+ \frac{4\xi(1-2s_1)\xi(2s_2-1)\xi(2s_2-2s_1)}{(s_1+1-s_2-\lambda)(s_1-s_2+\lambda)\Gamma(s_1)\Gamma(s_2)\xi(2s_2-2s_1-1)}\tau_2^{s_2-s_1} \\
&+ \frac{4\xi(1-2s_2)\xi(2s_1-1)\xi(2s_1-2s_2)}{(s_2+1-s_1-\lambda)(s_2-s_1+\lambda)\Gamma(s_1)\Gamma(s_2)\xi(2s_1-2s_2-1)}\tau_2^{s_1-s_2} \\
&+ \frac{4\xi(1-2s_1)\xi(1-2s_2)\xi(2-2s_1-2s_2)}{(s_1+s_2-\lambda)(s_1+s_2+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(1-2s_1-2s_2)}\tau_2^{1-s_1-s_2} \\
&- \alpha(1-\lambda;s_1,s_2)\tau_2^{\lambda} \\
&+ \sum_{\rho_n} \frac{2\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(1-t-\lambda)(t-\lambda)\Gamma(s_1)\Gamma(s_2)\pi^{\frac{1}{2}-t}\Gamma(t-\frac{1}{2})\zeta'(2t-1)}\tau_2^{t}\Big|_{t=\frac{3}{4}+i\frac{\rho_n}{2}} \\
&+ \widetilde{\mathrm{NP}}^{(\lambda)}_{s_1,s_2}(\tau_2)\,,
\end{aligned}
\tag{5.82}
$$

where the coefficient $\alpha(\lambda;s_1,s_2)$ is given in (5.72).

Similar to our large-τ_2 discussion, at small-τ_2 the non-perturbative terms, $\widetilde{\mathrm{NP}}^{(\lambda)}_{s_1,s_2}(\tau_2)$, come from having pushed the contour of integration past all the poles on the right t-half-plane,

$$
\widetilde{\mathrm{NP}}^{(\lambda)}_{s_1,s_2}(\tau_2) := \int_{\mathrm{Re}(t)=\gamma} \frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t-1)}\tau_2^{t}\frac{\mathrm{d}t}{2\pi i}\,,
\tag{5.83}
$$

where $\gamma > \max\big(\mathrm{Re}(I),\mathrm{Re}(1-I),\mathrm{Re}(\lambda),\mathrm{Re}(1-\lambda)\big)$. We proceed as before and expand all the completed Riemann functions as $\xi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, however, we notice that this time the ratio of Riemann zeta functions we obtain,

$$
\frac{\zeta(t+s_1+s_2-1)\zeta(t+s_1-s_2)\zeta(t+s_2-s_1)\zeta(t+1-s_1-s_2)}{\zeta(2t-1)}\,,
$$

cannot be written immediately as a Dirichlet series using Ramanujan identity (D.12). However, the present discussion is very similar to the spectral decomposition analysis considered in [74] for the study of certain partition functions in 2d CFTs. Building on [74], we can combine (D.12) with

$$
\frac{\zeta(2t)}{\zeta(2t-1)} = \sum_{n=1}^{\infty}\frac{\varphi^{-1}(n)}{n^{2t}}\,,
\tag{5.84}
$$

where $\varphi^{-1}(n)$ denotes the Dirichlet inverse[2] of Euler totient function, $\varphi(n)$, so that (5.83) can be rewritten in terms of a double Dirichlet series and an easier contour integral,

$$\begin{aligned}\widetilde{\mathrm{NP}}^{(\lambda)}_{s_1,s_2}(\tau_2) = & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4\sigma_{1-2s_1}(m)\sigma_{1-2s_2}(m)m^{s_1+s_2-1}\varphi^{-1}(n)}{\sqrt{\pi}\Gamma(s_1)\Gamma(s_2)} \\ & \times \int_{\mathrm{Re}(t)=\gamma} \frac{\Gamma(\frac{t+s_1+s_2-1}{2})\Gamma(\frac{t+s_1-s_2}{2})\Gamma(\frac{t+s_2-s_1}{2})\Gamma(\frac{t+1-s_1-s_2}{2})}{(t-\lambda)(t+\lambda-1)\Gamma(t-\frac{1}{2})} \Big(\frac{\tau_2}{\pi m n^2}\Big)^t \frac{\mathrm{d}t}{2\pi i}\,.\end{aligned} \tag{5.85}$$

We can now evaluate the asymptotic expansion as $\tau_2 \to 0$ of (5.85) via saddle point approximation. The integrand has a stationary point at $t = \frac{4\pi m n^2}{\tau_2}$ and a simple steepest descent calculation yields the required asymptotic expansion,

$$\widetilde{\mathrm{NP}}^{(\lambda)}_{s_1,s_2}(\tau_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sigma_{1-2s_1}(m)\sigma_{1-2s_2}(m)m^{s_1+s_2-\frac{3}{2}}\varphi^{-1}(n)}{\Gamma(s_1)\Gamma(s_2)\,n}\sqrt{4\tau_2}\,e^{-\frac{4\pi m n^2}{\tau_2}}\,\widetilde{\phi}^{(\lambda)}_{s_1,s_2}\Big(\frac{\tau_2}{4\pi m n^2}\Big)\,, \tag{5.86}$$

where the first few perturbative corrections are given by

$$\begin{aligned}&\widetilde{\phi}^{(\lambda)}_{s_1,s_2}(y) = \\ &8y^2 + 8\Big[s_1(s_1-1)+s_2(s_2-1)-\frac{11}{4}\Big]y^3 + 4\Big\{\Big[s_1(s_1-1)+s_2(s_2-1)-\frac{42}{8}\Big]^2 + 2\lambda(\lambda-1)-\frac{31}{4}\Big\}y^4 + O(y^5)\,.\end{aligned} \tag{5.87}$$

We note the striking similarity between the small-τ_2 exponentially suppressed terms (5.86)-(5.87) and the parallel large-τ_2 expressions (5.77)-(5.78). Equation (5.87) is directly analogous to the crossing equation (3.22) derived in [74].

Going back to the perturbative terms in (5.82), we see that the infinite series over ρ_n comes precisely from having collected the residues from the poles of $1/\xi(2t-1)$ in (5.71). Under the assumption that Riemann hypothesis is correct, these poles are associated with all non-trivial zeros of the Riemann zeta function $\zeta(s)$ located at $s=\frac{1}{2}+i\rho_n$ and $\rho_n \in \mathbb{R}$. Hence in the small-τ_2 limit, the last line in equation (5.82) behaves as the power $\tau_2^{\frac{3}{4}}$ modulated by oscillatory terms in τ_2 with frequencies determined by the ρ_n. A similar behaviour was already observed in [73] for the modular-invariant function $f(\tau) = \tau_2^{12}|\Delta(\tau)|^2$ with $\Delta(\tau)$ Ramanujan discriminant cusp

[2] The Dirichlet inverse, f^{-1}, of an arithmetic function, f, is defined such that the Dirichlet convolution of f with its inverse produces the multiplicative identity, i.e. $\sum_{d|n} f(d)f^{-1}(n/d) = \delta_{n,1}$. The Dirichlet series $L(f;s) := \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ has the property that $L(f^{-1};s) = (L(f;s))^{-1}$. The Dirichlet inverse, φ^{-1}, of Euler totient function, φ, is given by $\varphi^{-1}(n) = \sum_{d|n} d\,\mu(d)$ where μ is Möbius function.

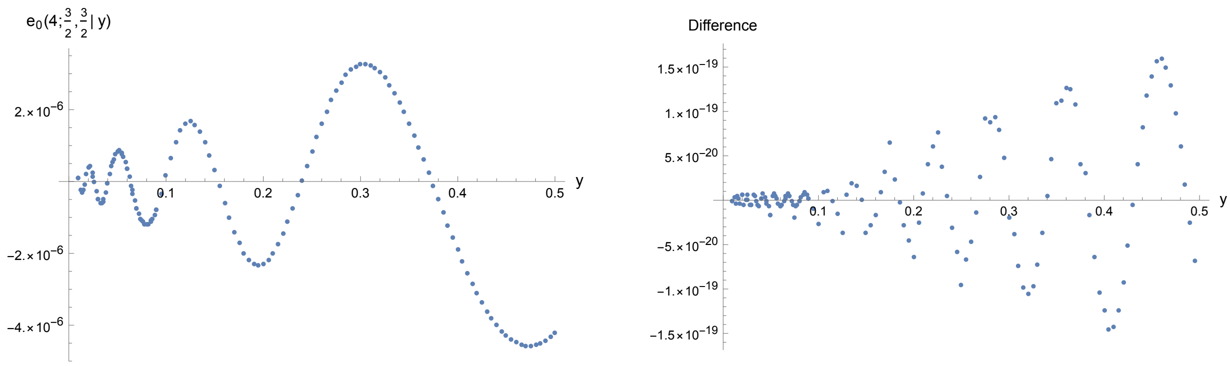

Figure 5.2: Comparison between the numerical evaluation of (5.71) and the small-τ_2 expansion (5.82). On the left, we plot $a_0(4; \frac{3}{2}, \frac{3}{2})$ after having subtracted all the terms in (5.82) but the Riemann zeta contributions. On the right, we plot the difference between the left data set and the predicted series of contributions in (5.82) from the first 10 non-trivial zeros of the Riemann zeta.

form. Similarly, in a different string theory context [75] and from a two-dimensional CFT context [74], the non-trivial zeros of the Riemann zeta function do appear from the asymptotic expansion for the spectral decomposition of different physical quantities.

In figure 5.2, we present numerical evidences for the small-τ_2 expansion (5.71) of the d^6R^4 case, $a_0(4; \frac{3}{2}, \frac{3}{2}|\tau_2)$. We have numerically evaluated to high precision the integral representation (5.71) for $a_0(4; \frac{3}{2}, \frac{3}{2}|\tau_2)$ at small τ_2 and subtracted from it all the terms in (5.82) but the Riemann zeta contributions. In figure 5.2 we first plot this quantity and then subtract from it the predicted series of contributions from the first 10 non-trivial zeros (5.82) of the Riemann zeta and plot this difference. As the second plot shows, our formula (5.82) is consistent with the numerical data within a 10^{-19} error over the whole range of τ_2 considered.

Although from a physical point of view, the limit $\tau_2 \to 0$ for the MGFs spectrum (4.17) corresponds simply to a particular degeneration limit of the worldsheet torus, for the generalised Eisenstein series associated with the spectrum (5.7), and in particular for the coefficient of the d^6R^4 in the low-energy expansion of type IIB superstring theory, this limit corresponds to the strong coupling regime $g_s \to \infty$. It would be extremely interesting, and equally difficult, to understand the string theory origins at strong coupling for the appearance of the non-trivial zeroes of the Riemann zeta.

5.7 The instanton sectors

So far we have focused our attention entirely on the Fourier zero-mode sector, while the spectral decomposition (3.25) in principle allows us to reconstruct all of the Fourier modes, in particular the Fourier non-zero modes which we will refer to as *instanton sectors*.

Given (5.70), we can extract the k-instanton sector $a_k(\lambda; s_1, s_2|\tau_2)$, i.e. the Fourier mode $e^{2\pi i k\tau_1}$, for the generalised Eisenstein series $\mathcal{E}\left(\lambda; s_1, s_2\middle|\tau\right)$:

$$\begin{aligned} a_k(\lambda; s_1, s_2|\tau_2) &= \sum_I \beta_I \frac{4}{\Gamma(I)} |k|^{I-\frac{1}{2}} \sigma_{1-2I}(k) \tau_2^{\frac{1}{2}} K_{I-\frac{1}{2}}(2\pi|k|\tau_2) \\ &+ \int_{\mathrm{Re}(t)=\frac{1}{2}} \Big[\frac{4\xi(t+s_1+s_2-1)\xi(t+s_1-s_2)\xi(t+s_2-s_1)\xi(t+1-s_1-s_2)}{(t-\lambda)(t+\lambda-1)\Gamma(s_1)\Gamma(s_2)\xi(2t-1)} \frac{4|k|^{t-\frac{1}{2}}\sigma_{1-2t}(k)}{\Gamma(t)} \\ &\quad \times \tau_2^{\frac{1}{2}} K_{t-\frac{1}{2}}(2\pi|k|\tau_2) \frac{\mathrm{d}t}{4\pi i} \Big] + \sum_{n=1}^{\infty} (\tilde{\mathcal{E}}(\lambda; s_1, s_2), \phi_n) h_k^{(n)} \tau_2^{\frac{1}{2}} K_{it_n}(2\pi|k|\tau_2)\,, \end{aligned} \tag{5.88}$$

where the coefficients $h_k^{(n)}$, associated with the Fourier expansion of Maass cusp forms, were defined in (3.17).

Although a complete analysis of the instanton sector is beyond the scope of this thesis, we note that a naive attempt at extracting the large-τ_2 behaviour of (5.88) would produce an incorrect result. At first glance we may try and expand directly the different Bessel functions for large argument, thus immediately obtaining the expected exponential suppression factor $e^{-2\pi|k|\tau_2}$, hallmark of the k-instanton sector. However, by doing so the perturbative expansion on top of the instanton factor q^k, for $k > 0$ with $q = e^{2\pi i\tau}$, or anti-instanton factor $\bar{q}^k$, for $k < 0$, would start at order τ_2^0 with sub-leading corrections $O(\tau_2^{-1})$, which turns out to be incorrect when compared with known results. Very likely a more careful analysis will show that the critical line integral is only conditionally convergent. This would explain why an expansion at large-τ_2 *before* having performed the integral would result in the wrong asymptotic behaviour.

In [34, 35], a representation for all generalised Eisenstein series with spectrum (4.17) was provided in terms of iterated integrals of holomorphic Eisenstein series. This representation is extremely convenient for extracting all of the instanton expansions and, by comparing with the examples discussed in [34, 35], we can clearly see that the above naive argument cannot possibly provide the correct answer for the generalised Eisenstein series with spectrum (4.17).

Furthermore, in the same references, the authors discovered that amongst the coefficients of the perturbative expansion in the instanton sector, $a_k(\lambda; s_1, s_2|\tau_2)$, besides rational numbers and odd-zeta values, a new class of numbers appears whenever the eigenvalue λ is such that the the vector space of holomorphic cusp forms of modular weight $w = 2\lambda$ has non-zero dimension. For these special eigenvalues the perturbative expansion at large-τ_2 of $a_k(\lambda; s_1, s_2|\tau_2)$ contains non-critical completed L-values of holomorphic cusp forms. Recently in [68, 76] a very similar (albeit so far completely different in nature) phenomenon was discovered for the generalised Eisenstein series with spectrum (5.7) for exactly the same eigenvalues.

It would be extremely interesting to extract the asymptotic expansion as $\tau_2 \gg 1$ of the k-instanton sector (5.88) and understand the origin of these completed L-values for holomorphic cusp forms from the spectral decomposition point of view (5.88). In particular, it is tantalising to conjecture some interplay between the non-holomorphic cusp forms and the appearance of holomorphic cusp forms.

CHAPTER 6

Integrated correlators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

We now switch gears and discuss a different, albeit related application of resurgence to the study of modular functions. $\mathrm{SL}(2,\mathbb{Z})$ plays a rôle not only in string theory and relatedly 2 dimensional CFT, but also in quantum field theories in higher dimensions. Here it usually serves as a duality group mapping the QFT to either itself or possibly a different theory, while simultanesouly inverting the coupling. In this section we study how different dualities (holographic and modular) interact to tell a beautiful story of interrelations between string theory and quantum fields.

Perhaps one of the most intriguing quantum field theories in four space-time dimensions is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with gauge group $SU(N)$. Amongst the many reasons for its undeniable appeal is that it provides for a moduli space of non-trivial superconformal theories parameterised by the Yang–Mills coupling $g_{_{YM}}$ and theta angle θ, conveniently packaged in the complex coupling $\tau := \theta/(2\pi) + 4\pi i/g^2_{_{YM}}$, upon which the Montonen–Olive [21] duality group $\mathrm{SL}(2,\mathbb{Z})$ acts.

Another important reason for our interest in $\mathcal{N} = 4$ SYM is that it provides a non-perturbative description of type IIB string theory in an $AdS_5 \times S^5$ background [23]. The gauge theory coupling τ is identified holographically with the type IIB string theory coupling $\tau_s := \chi + i/g_s$, i.e. $\tau = \tau_s$, while the string length scale ℓ_s is related to the number of colours

N of the dual $SU(N)$ gauge theory side by $(L/\ell_s)^4 = g_{_{YM}}^2 N$, with L the length scale of the $AdS_5 \times S^5$ background.

Thus, the string theory effective gravitational description at small ℓ_s corresponds on the gauge side to a large-N and finite τ regime. Although in this limit the bulk is weakly curved, as a consequence of $\ell_s/L \to 0$, the string theory remains strongly coupled due to τ_s being finite. Super-graviton scattering amplitudes on $AdS_5 \times S^5$ are consequently related to correlation functions of the stress tensor multiplet in $\mathcal{N} = 4$ SYM. Despite offering a non-perturbative definition of string theory through a well-defined CFT, practical applications of this duality aimed at exploring string theory in the gravity regime remain challenging since the CFT is still strongly coupled at large N. To advance our understanding of quantum gravity through AdS/CFT it is therefore necessary to analyse $\mathcal{N} = 4$ SYM non-perturbatively.

An extremely powerful method to extract non-perturbative properties of $\mathcal{N} = 4$ SYM for arbitrary coupling τ and number of colours N, is the use of supersymmetric localisation. It is precisely thanks to this tool that it was recently understood [63, 77] how to obtain certain integrals of the correlator of four superconformal primary operators, usually denoted[1] by $\mathcal{O}_2(x)$, in the $\mathcal{N} = 4$ stress tensor multiplet.

The particular integrated correlators of interest for the present chapter have been introduced in [63, 77] and are computed from derivatives of the S^4 partition function $Z_N(m, \tau)$ for $\mathcal{N} = 2^*$ SYM obtained by Pestun [78] in terms of an $SU(N)$ matrix model integral:

$$\mathcal{C}_N(\tau) \coloneqq \frac{1}{4}\Delta_\tau \partial_m^2 \log Z_N(m,\tau)\big|_{m=0}\,, \qquad \mathcal{H}_N(\tau) \coloneqq \partial_m^4 \log Z_N(m,\tau)\big|_{m=0}\,, \tag{6.1}$$

where $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ is the standard hyperbolic Laplacian and the complexified coupling constant $\tau = \tau_1 + i\tau_2 \in \mathfrak{H}$ parametrises the upper-half plane with the identification $\tau_1 = \theta/(2\pi)$ and $\tau_2 = 4\pi/g_{_{YM}}^2$.

When the mass parameter m is set to zero $\mathcal{N} = 2^*$ SYM reduces to $\mathcal{N} = 4$, hence the expressions just defined correspond to $\mathcal{N} = 4$ observables. More precisely, the quantities $\mathcal{C}_N(\tau)$ and $\mathcal{H}_N(\tau)$ are identified with integrals over the insertion points of four superconformal primary

[1] For simplicity we suppress R-symmetry indices.

operators of the form,

$$\mathcal{C}_N(\tau) = \int \langle \mathcal{O}_2(x_1) \cdots \mathcal{O}_2(x_4) \rangle \, \mathrm{d}\mu(\{x_i\}) \,, \qquad \mathcal{H}_N(\tau) = \int \langle \mathcal{O}_2(x_1) \cdots \mathcal{O}_2(x_4) \rangle \, \mathrm{d}\tilde{\mu}(\{x_i\}) \,. \tag{6.2}$$

We refer to [63, 77] for the precise relation between the supersymmetric localisation definitions (6.1) and the exact forms of the integrated correlators (6.2), and in particular for details on the integration measures $\mathrm{d}\mu(\{x_i\})$ and $\mathrm{d}\tilde{\mu}(\{x_i\})$ distinguishing $\mathcal{C}_N(\tau)$ and $\mathcal{H}_N(\tau)$.

Thanks to these results, it has finally become possible to perform holographic "precision-tests" [63–65, 77, 79], and reconstruct from the large-N expansion of the integrated correlators (6.1), the first few low-energy string theory corrections to the tree-level supergravity contribution to four-graviton scattering in $AdS_5 \times S^5$ as well as in flat-space.

Surprisingly, in a series of papers [80–82] an exact and modular covariant expression for finite τ was proven for a generalisation of the first integrated correlator $\mathcal{C}_N(\tau)$ to arbitrary classical gauge group, then extended to exceptional gauge groups in [83]. A key rôle in determining these astonishing results is played by the action of the Montonen–Olive duality group $\mathrm{SL}(2,\mathbb{Z})$ on the complex coupling τ, strongly constraining the space of modular invariant objects at play.

This led to a flourishing of exact results for other integrated correlators in $\mathcal{N} = 4$ SYM such as higher-point maximal $U(1)_Y$-violating correlators [56, 57], four-point functions of higher conformal dimensions operators [84–87] and giant gravitons [88], as well as integrated two-point functions of two superconformal primary operators in the presence of a half-BPS line defect [89, 90]. More recently these methods have also been applied to integrated correlators in less supersymmetric theories such as $\mathcal{N} = 2$ SYM [91–93]. We stress that these finite-N, finite-coupling results provide important data for numerical bootstrap studies, see e.g. [94, 95].

While novel studies [68] have shown intriguing conjectural relations between the two integrated correlators in (6.1), particularly in the large-N fixed-τ limit, we still lack an exact modular invariant expression for the second integrated correlator $\mathcal{H}_N(\tau)$ valid for arbitrary N and fixed τ. On the contrary, thanks to the pivotal results of [80, 81] we have an almost complete control over the first integrated correlator $\mathcal{C}_N(\tau)$. In particular, we know that for all N the quantity $\mathcal{C}_N(\tau)$ can be represented as a simple lattice sum integral whose systematic large-N expansion can be computed [82] starting from a lattice sum generating series over the number of colours N.

From the analysis of [82] it follows that the large-N, fixed-τ expansion of $\mathcal{C}_N(\tau)$ is an asymptotic factorially divergent formal series, which has to be completed by an infinite tower of modular invariant, non-perturbative exponentially suppressed terms at large-N, thus confirming the earlier impressive numerical studies of [66]. These non-perturbative corrections are extremely important and have the holographic interpretation of contributions from coincident (p, q)-string world-sheet instantons.

In this chapter we provide a resurgence analysis approach to the resummation of modular invariant large-N perturbative expansions akin to that for $\mathcal{C}_N(\tau)$. We show that it is possible to define a modified Borel resummation kernel with manifest modular invariance. By applying this resummation procedure to the formal perturbative large-N expansion of the integrated correlator $\mathcal{C}_N(\tau)$, we retrieve its complete exact transseries expansion previously only found via generating series methods. The modular invariant non-perturbative sectors of $\mathcal{C}_N(\tau)$ are amazingly encoded in its perturbative part. We also show that our approach is extremely useful in deriving novel non-perturbative results for a particular sector of the large-N expansion for the second integrated correlator $\mathcal{H}_N(\tau)$. The proposed modular invariant resurgent resummation is furthermore perfectly suited to perform a large-N 't Hooft limit expansion at large $\lambda := N g_{_{YM}}^2$, thus recovering the non-perturbative worldsheet instanton completions first obtained using resurgence analysis order by order in the genus expansion [66, 81].

The second goal we achieve is understanding how the large-N exact transseries expansion of the integrated correlator $\mathcal{C}_N(\tau)$ is encoded in an alternative and equivalent representation nicely found in [48] via $\mathrm{SL}(2, \mathbb{Z})$ spectral theory. The key ideas behind spectral representation are discussed in section 3.3 and we refer to it for further mathematical detail. We show that the large-N expansion of this spectral decomposition yields precisely the spectral decomposition of the large-N transseries expansion obtained via resurgence analysis.

From here we proceed by stating and deriving some results relating to the integrated correlator $\mathcal{C}_N(\tau)$. In particular, we present two equivalent representations valid for all complex coupling $\tau \in \mathfrak{H}$ and arbitrary number of colours N. Firstly, in section 6.1 we discuss the original exact expression for $\mathcal{C}_N(\tau)$ found in [80, 81] in terms of a lattice sum combined with a Borel-like integral transform. In section 6.2 we review how to construct from the lattice-sum representation a generating series over N and with it compute the exact large-N modular invariant transseries [67].

Secondly, in section 6.3 we discuss an equivalent expression for the integrated correlator, first presented in [48] and then extended in [86], in terms of an extremely simple spectral representation with respect to L^2-normalisable functions on the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$. From here we proceed and in section 6.4 formulate a method of resummation for modular invariant transseries and demonstrate the necessity of introducing a novel class of modular functions serving as a non-perturbative completion. Then in section 6.5 we apply these methods to the integrated correlators and demonstrate how resurgence recovers non-perturbative terms at large N. Finally, in section 6.8 we demonstrate that median resummation gives the exact answer for the observable of our interest starting from a spectral theory expression for the correlator.

6.1 Lattice sum representation

Even though the integrated correlator $\mathcal{C}_N(\tau)$ is defined in (6.1) by taking a suitable combination of derivatives of the S^4 partition function in the $\mathcal{N}=2^*$ mass deformed supersymmetric Yang–Mills theory, it was proven in [80,81] that this integrated correlator has the far more convenient lattice sum representation,

$$\mathcal{C}_N(\tau) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\,Y_{mn}(\tau)} B_N(t)\,\mathrm{d}t\,, \tag{6.3}$$

where we define the ubiquitous "lattice-sum coupling",

$$Y_{mn}(\tau) := \pi \frac{|n\tau+m|^2}{\tau_2}. \tag{6.4}$$

This extremely simple formula can be seen as a combination of the lattice sum over $(m,n)\in\mathbb{Z}^2$ and a Laplace integral of a "Borel transform" function $B_N(t)$, which is a rational function of t given by

$$B_N(t) := \frac{\mathcal{Q}_N(t)}{(t+1)^{2N+1}}\,. \tag{6.5}$$

The function $\mathcal{Q}_N(t)$ is a polynomial in the variable t of degree $2N-1$, which can be written for all $N\in\mathbb{N}$ as

$$\mathcal{Q}_N(t) := -\frac{1}{2}N(N-1)(1-t)^{N-1}(1+t)^{N+1} \tag{6.6}$$
$$\Big\{[3+(3t+8N-6)t]P_N^{(1,-2)}\Big(\frac{1+t^2}{1-t^2}\Big) + \frac{1}{1+t}(3t^2-8Nt-3)P_N^{(1,-1)}\Big(\frac{1+t^2}{1-t^2}\Big)\Big\}\,,$$

where $P_n^{(a,b)}(x)$ are Jacobi polynomials. We note that for any N the function $B_N(t)$ satisfies the inversion identity

$$t^{-1}B_N(t^{-1}) = B_N(t)\,. \tag{6.7}$$

The form of the function $B_N(t)$ given in (6.5) and (6.6) was conjectured in [81, 82] and then proved in [67] using matrix model methods.

Interestingly, in [82] it was shown that a more general version of the lattice sum expression (6.3), yields the integrated correlator of four superconformal primary operators (6.2) for $\mathcal{N}=4$ SYM with arbitrary classical gauge group $G = SO(N), USp(2N)$, then completed in [83] to the case of exceptional gauge groups. Goddard-Nuyts-Olive [96] electro-magnetic duality plays a fundamental rôle in dictating the particular lattice sum expressions appearing for different gauge groups. In this thesis, we focus our attention to the original discussion (6.3) of the integrated correlator in the $SU(N)$ theory.

As already noted, compared to the original expression (6.1), it is much easier to analyse the dependence of the integrated correlator from the parameters τ and N starting from the lattice sum expression (6.3). However, while the τ dependence has been basically trivialised, the dependence of (6.5) on the number of colours N is absolutely not transparent. This shortcoming was remedied in [67] where a generating function for the N-dependence was derived starting from (6.3), thus allowing for a direct calculation of the exact large-N, fixed-τ transseries expansion.

This generating series is defined as

$$\mathcal{C}_{SU}(z;\tau) := \sum_{N=1}^{\infty} \mathcal{C}_N(\tau)\, z^N\,, \tag{6.8}$$

with z an auxiliary complex variable. We can then invert (6.8) via

$$\mathcal{C}_N(\tau) = \oint_\gamma \frac{\mathcal{C}_{SU}(z;\tau)}{z^{N+1}} \frac{\mathrm{d}z}{2\pi i}\,, \tag{6.9}$$

where γ denotes a counter-clockwise contour circling the pole at $z=0$ of radius strictly less than one in order to avoid other singularities. From (6.3) we can equivalently define the generating series for the rational functions $B_N(t)$ which can be computed directly from (6.5):

$$B_{SU}(z;t) := \sum_{N=1}^{\infty} B_{SU(N)}(t) z^N = \frac{3tz^2\left[(t-3)(3t-1)(t+1)^2 - z(t+3)(3t+1)(t-1)^2\right]}{(1-z)^{\frac{3}{2}}\left[(t+1)^2-(t-1)^2 z\right]^{\frac{7}{2}}}\,, \tag{6.10}$$

leading to

$$\mathcal{C}_{SU}(z;\tau) := \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}\int_0^\infty e^{-t\,Y_{mn}(\tau)} B_{SU}(z;t)\,\mathrm{d}t\,. \tag{6.11}$$

This generating function satisfies several properties of note,

$$B_{SU}(z;t) = t^{-1} B_{SU}(z;t^{-1})\,, \qquad\qquad B_{SU}(z;t) = -B_{SU}(z^{-1};-t)\,,$$

as well as the integral identities,

$$\int_0^\infty \frac{B_{SU}(z;t)}{\sqrt{t}}\,\mathrm{d}t = 0\,, \qquad\qquad \int_0^\infty B_{SU}(z;t)\,\mathrm{d}t = \sum_{N=1}^{\infty}\frac{N(N-1)}{4} z^N\,. \tag{6.12}$$

The first of these equations, directly related to (6.7), is an inversion relation that follows automatically from the lattice sum definition of the integrated correlator (6.3), as was pointed out in [48] where the lattice sum is re-expressed in terms of a modular invariant spectral representation which will shortly be reviewed. The second equation in (6.12) is an inversion relation in the variable z, which relates the $SU(N)$ correlator with coupling $g^2_{_{YM}}$ to the $SU(-N)$ correlator with coupling $-g^2_{_{YM}}$, as previously discussed in [82].

6.2 Modular invariant large-N transseries

One of the main advantages of introducing a generating series such as $\mathcal{C}_{SU}(z;\tau)$ is that it has a much simpler form than $\mathcal{C}_N(\tau)$. This makes $\mathcal{C}_{SU}(z;\tau)$ extremely convenient for analysing the large-N properties of the integrated correlators. In particular, starting from (6.11) a key result of [67] was the derivation of the exact large-N transseries expansion for $\mathcal{C}_N(\tau)$ at fixed τ, which takes the form

$$\mathcal{C}_N(\tau) = \mathcal{C}_P(N;\tau) + \sigma\, \mathcal{C}_{NP}(N;\tau)\,. \tag{6.13}$$

In this expression $\mathcal{C}_P(N;\tau)$ contains the formal asymptotic perturbative expansion in $1/N$ for the integrated correlator $\mathcal{C}_N(\tau)$ given by

$$\mathcal{C}_P(N;\tau) = \frac{N^2}{4} + \sum_{\ell=0}^{\infty} N^{\frac{1}{2}-\ell} \sum_{m=0}^{\lfloor \ell/2 \rfloor} \tilde{b}_{\ell,m} E^*(\tfrac{3}{2}+\delta_\ell+2m;\tau)\,, \tag{6.14}$$

where $\delta_\ell = \ell\,(\mathrm{mod}\,2)$ and we denote with $\lfloor x \rfloor$ the floor of x. As discussed in [67], the constant coefficients $\tilde{b}_{\ell,m}$ can be easily computed starting from the generating series (6.11), but otherwise are not known in closed form for arbitrary ℓ and m. The first few coefficients $\tilde{b}_{\ell,m}$ for $\ell \leq 4$ where already computed in [64], while expressions for general ℓ and fixed m can be found in [81].

We stress that although this power-series in $1/N$ does not converge for any value of τ, it is nonetheless manifestly a modular invariant function of τ order by order in $1/N$. The only τ dependence in (6.14) appears through the non-holomorphic Eisenstein series, which have already appeared in this thesis multiple times and for this particular application are defined as

$$\begin{aligned} E^*(s;\tau) &:= \frac{\Gamma(s)}{2} \sum_{(m,n)\neq(0,0)} Y_{mn}(\tau)^{-s} = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \int_0^\infty e^{-t\,Y_{mn}(\tau)}\, t^{s-1}\,\mathrm{d}t \\ &= \xi(2s)\tau_2^s + \xi(2s-1)\,\tau_2^{1-s} + \sum_{k\neq 0} e^{2\pi i k\tau_1} 2\sqrt{\tau_2}\,|k|^{s-\frac{1}{2}}\sigma_{1-2s}(k)\,K_{s-\frac{1}{2}}(2\pi|k|\tau_2)\,. \end{aligned} \tag{6.15}$$

We additionally note that for this convention of normalisation the non-holomorphic Eisenstein series satisfy a particularly simple reflection formula $E^*(s;\tau) = E^*(1-s;\tau)$.

For the perturbative sector (6.14), the coefficient of each order in $1/N$ is given by a finite sum of non-holomorphic Eisenstein series, $E^*(s;\tau)$, of half-integer index s ranging from a maximal value $s = \frac{3}{2}+\ell$ to a minimal value $s = \frac{3}{2}$. From $\mathcal{C}_P(N;\tau)$ we can recover higher derivative corrections to the flat space-limit in the type IIB S-matrix of four gravitons at finite string coupling τ via the holographic dictionary [64]. Some of these terms were already mentioned in chapter 5 as corrections that are exactly derivable by use of S-duality in Type IIB string theory.

Importantly, the transseries expansion (6.13) of the full integrated correlator $\mathcal{C}_N(\tau)$ does also contain non-perturbative, exponentially suppressed terms at large-N, captured by $\mathcal{C}_{NP}(N;\tau)$ and given by the formal series

$$\mathcal{C}_{NP}(N;\tau) = \sum_{\ell=0}^{\infty} N^{2-\frac{\ell}{2}} \sum_{m=0}^{\ell} \tilde{d}_{\ell,m} D_N\big(\tfrac{\ell}{2}-2m;\tau\big)\,, \tag{6.16}$$

where the novel modular invariant function $D_N(s;\tau)$ is defined as[2]

$$D_N\,(s;\tau) := \sum_{(m,n)\neq(0,0)} \exp\Big(-4\sqrt{NY_{mn}(\tau)}\Big)\big(16Y_{mn}(\tau)\big)^{-s}\,. \tag{6.17}$$

Via the holographic dictionary, it was conjectured in [67] that these non-perturbative corrections capture the contribution of ℓ coincident (p,q)-string Euclidean world-sheet instantons wrapping a great two-sphere, S^2, on the equator of the five-sphere, S^5.

Finally, it was also argued that there is an ambiguity in resumming the large-N asymptotic perturbative expansion (6.14), which has to be compensated by a change in the non-perturbative sector captured by $\mathcal{C}_{NP}(N;\tau)$. This amounts to a jump in the transseries parameter σ, which is a piece-wise constant function of $\arg(N)$, taking values $\sigma = \pm\, i$ according to $\arg(N) > 0$ or < 0 respectively. In section 6.4, we show that the non-perturbative corrections (6.16), as well as the transseries parameter σ, can be fixed completely from a proper resurgence analysis of a modified modular invariant Borel resummation of the purely perturbative data (6.14). This explains the resurgent origins of the transseries (6.13), originally found solely via generating series methods.

6.3 Spectral representation

An alternative and equivalent representation to the lattice-sum integral expression (6.3) is obtained via $\mathrm{SL}(2,\mathbb{Z})$ spectral theory, a method of decomposing any (suitable) modular invariant function as a linear combination of "good" basis elements, i.e. L^2-normalisable eigenfunctions of the hyperbolic Laplace operator Δ.

[2]Note that compared to [67], where this class of functions was first introduced, we here use a slightly different and more convenient normalisation $D_N^{\text{there}}(s;\tau) = 2^{4s} D_N^{\text{here}}(s;\tau)$, which for the expansion (6.16), where a different indexing is implemented, in turns implies $d_{\ell,m}^{\text{there}} = 2^{6\ell-8m}\tilde{d}_{\ell,\ell-m}$.

For the present discussion we only highlight the fact that this integrated correlator possesses an astonishingly simple spectral representation, which only involves an integral over special normalisable eigenfunctions: the non-holomorphic Eisenstein series $E^*(s;\tau)$ with $\mathrm{Re}(s) = \frac{1}{2}$. This spectral approach to the integrated correlator (6.1) was introduced in [48], where the spectral overlap with Eisenstein series was first derived and it was also demonstrated that non-holomorphic cusp forms play no rôle for this observable.

Here we rederive the spectral representation for $\mathcal{C}_N(\tau)$ starting directly from the lattice-sum integral representation (6.3). We begin by considering the lattice sum in (6.3) and splitting it into the sum of the $(m,n) = (0,0)$ contribution and terms with $(m,n) \neq (0,0)$,

$$\mathcal{C}_N(\tau) = \frac{1}{2}\int_0^\infty B_N(t)\,\mathrm{d}t + \frac{1}{2}\sum_{(m,n)\neq(0,0)}\int_0^\infty e^{-t\,Y_{mn}(\tau)}\,B_N(t)\,\mathrm{d}t\,. \tag{6.18}$$

We now rewrite the second term in this expression in terms of the Mellin transform, $M_N(s)$, of the function $B_N(t)$ defined as

$$M_N(s) \coloneqq \int_0^\infty t^{s-1}\,B_N(t)\,\mathrm{d}t\,. \tag{6.19}$$

Given the expression (6.5) for $B_N(t)$, this Mellin integral can be shown to converge in the strip $-1 < \mathrm{Re}(s) < 2$ and has an analytic continuation to a meromorphic function of $s \in \mathbb{C}$. This transform can be inverted via Mellin inversion formula,

$$B_N(t) = \int_{\mathrm{Re}(s)=\alpha} t^{-s} M_N(s)\,\frac{\mathrm{d}s}{2\pi i}, \tag{6.20}$$

where the constant $\alpha \in \mathbb{R}$ is chosen in such a way that the original Mellin integral (6.19) converges for $\mathrm{Re}(s) = \alpha$. Crucially, we notice that the functional equation (6.7) translates immediately to the reflection formula,

$$M_N(1-s) = M_N(s)\,. \tag{6.21}$$

We now substitute Mellin inversion formula in (6.18) and perform a reflection $s \to 1-s$

while using (6.21) to arrive at,

$$\mathcal{C}_N(\tau) = \frac{1}{2}\int_0^\infty B_N(t)\,\mathrm{d}t + \int_{\mathrm{Re}(s)=1+\epsilon} M_N(s)\left(\frac{1}{2}\sum_{(m,n)\neq(0,0)}\int_0^\infty e^{-tY_{mn}(\tau)}t^{s-1}\mathrm{d}t\right)\frac{\mathrm{d}s}{2\pi i}\,. \quad (6.22)$$

The s-contour of integration has been shifted to $\mathrm{Re}(s) = 1+\epsilon$, with $\epsilon > 0$ sufficiently small, so that the t-integral and the lattice-sum are both convergent and we are allowed to use the integral representation (6.15) for the non-holomorphic Eisenstein series, thus arriving at the sought-after spectral representation for the integrated correlator

$$\mathcal{C}_N(\tau) = \langle \mathcal{C}_N \rangle + \int_{\mathrm{Re}(s)=\frac{1}{2}} M_N(s)E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}\,, \quad (6.23)$$

$$\langle \mathcal{C}_N \rangle := \int_0^\infty B_N(t)\,\mathrm{d}t = \lim_{s\to 1} M_N(s)\,, \quad (6.24)$$

where the additional factor 1/2 for the constant term originates from having moved the contour of integration back to $\mathrm{Re}(s) = \frac{1}{2}$ combined with the fact that $\mathrm{res}_{s=1}E^*(s;\tau) = \frac{1}{2}$.

While in the lattice-sum representation the N dependence is encoded entirely in the rational functions $B_N(t)$ given in (6.5), here this information is captured by the spectral overlap function, i.e. the Mellin transform $M_N(s)$. The function $M_N(s)$ can be obtained by exploiting an intriguing Laplace-difference equation found in [81] and satisfied by the integrated correlator:

$$\Delta\mathcal{C}_N(\tau) - (N^2-1)\Big(\mathcal{C}_{N+1}(\tau) - 2\mathcal{C}_N(\tau) + \mathcal{C}_{N-1}(\tau)\Big) - (N+1)\mathcal{C}_{N-1}(\tau) + (N-1)\mathcal{C}_{N+1}(\tau) = 0, \quad (6.25)$$

which fixes $\mathcal{C}_N(\tau)$ in terms of the initial data $\mathcal{C}_2(\tau)$ and $\mathcal{C}_1(\tau) = 0$.

By specialising (6.5) to the $SU(2)$ theory, i.e. by setting $N = 2$, we obtain the initial condition,

$$B_2(t) = \frac{9t - 30t^2 + 9t^3}{(t+1)^5}\,, \quad (6.26)$$

from which it is immediate to derive its Mellin transform,

$$M_2(s) = \frac{\pi s(1-s)(2s-1)^2}{2\sin(\pi s)}\,. \quad (6.27)$$

We can then combine (6.23) with (6.25) and the known Laplace equation

$$\Delta E^*(s;\tau) = s(s-1)E^*(s;\tau)\,, \tag{6.28}$$

to find a recurrence relation satisfied by the spectral overlaps, namely

$$N(N-1)M_{N+1}(s) = [s(s-1)+2(N^2-1)]M_N(s) - N(N+1)M_{N-1}(s)\,. \tag{6.29}$$

As show in [86], this recursion is solved by

$$M_N(s) = \frac{N(N-1)}{4}\frac{\pi s(1-s)(2s-1)^2}{\sin(\pi s)}{}_3F_2(2-N,s,1-s;3,2|1)\,, \tag{6.30}$$

thus implying from (6.24) that $\langle \mathcal{C}_N\rangle = N(N-1)/4$. The hypergeometric function in this equation is somewhat misleading, since the parameter $2-N$ is a non-positive integer for $N\geq 2$ and as a consequence, the hypergeometric function always reduces to a polynomial in $s(1-s)$ of degree $N-2$. In appendix E we find a more convenient expression given by

$$M_N(s) = \frac{2^{-2s}(2s-1)\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\,\Gamma(-s)}\int_0^1 x^{s-3}(1-x)^N{}_2F_1(s-1,s;2s|x)\mathrm{d}x + (s\leftrightarrow 1-s)\,, \tag{6.31}$$

where a certain regularisation is required when treating the x-integral near $x=0$, see in particular equation (E.10) and the detailed analysis presented in appendix E.

While at finite N it is straightforward to evaluate the Mellin transform (6.30) and obtain the spectral representation for the integrated correlator (6.23), it is absolutely not obvious how to deduce its large-N expansion. In section 6.8 we begin with the spectral overlap (6.31) and manifest how the resurgent structure of the integrated correlator is beautifully encoded in the spectral representation (6.23), recovering the entire modular transseries (6.13) from this perspective.

6.4 Resurgence of modular invariant transseries

In this section we want to show how the exact large-N transseries expansion at fixed τ for $\mathcal{C}_N(\tau)$ displayed in (6.13), so far only analysed numerically in [66] and derived in [67] via generating series methods, can be derived using resurgence analysis. In particular, we prove that it is possible to reconstruct the non-perturbative and modular invariant contributions (6.16) from a suitable resummation of the large-N formal, yet modular invariant perturbative sector (6.14).

We start by focusing our attention on the purely perturbative expansion of the integrated correlator (6.1), which at large-N and fixed τ has the formal asymptotic perturbative expansion (6.14), here rewritten for convenience

$$\mathcal{C}_N(\tau) \sim \mathcal{C}_P(N;\tau) = \frac{N^2}{4} + \sum_{\ell=0}^{\infty} N^{\frac{1}{2}-\ell} \sum_{m=0}^{\lfloor \ell/2 \rfloor} \tilde{b}_{\ell,m} E^*(\tfrac{3}{2}+\delta_\ell+2m;\tau)\,, \tag{6.32}$$

with $\delta_\ell = 0$ for even ℓ and $\delta_\ell = 1$ for odd ℓ.

As already noted previously, in the perturbative sector the coefficient of each order in $1/N$ is given by a finite sum of non-holomorphic Eisenstein series, $E^*(s;\tau)$, of half-integer index s ranging from the maximal value $s = \frac{3}{2}+\ell$ to the minimal one $s = \frac{3}{2}$. Following [68], we reorganise this formal power series in $1/N$ as

$$\mathcal{C}_P(N;\tau) = \frac{N^2}{4} + \sum_{r=0}^{\infty} N^{2-2r} \mathcal{C}_P^{(r)}(N;\tau)\,, \tag{6.33}$$

having defined

$$\mathcal{C}_P^{(r)}(N;\tau) \coloneqq \sum_{k=0}^{\infty} b_{r,k} N^{-\frac{3}{2}-k} E^*(\tfrac{3}{2}+k;\tau)\,. \tag{6.34}$$

Here we made the change of summation variables $\ell = 2r+k$ and $m = \lfloor \frac{k}{2} \rfloor$ and correspondingly denoted the rearranged coefficients by $b_{r,k} = \tilde{b}_{\ell,m}$. For fixed r, the formal power-series $\mathcal{C}_P^{(r)}(N;\tau)$ can be understood as collecting, order by order in $1/N$, the contributions to (6.32) coming from the "r-subleading index" non-holomorphic Eisenstein series, i.e. all $E^*(s;\tau)$ with index $s = \frac{3}{2}+\ell-2r$.

For example, we can focus on the contribution to (6.32) coming only from grouping all "leading-index" non-holomorphic Eisenstein series, i.e. all terms in (6.32) with $m = \lfloor \ell/2 \rfloor$ or equivalently specialising (6.34) to $k = \ell$ and $r = 0$:

$$\mathcal{C}_P^{(0)}(N;\tau) = \sum_{k=0}^{\infty} b_{0,k} N^{-\frac{3}{2}-k} E^*(\tfrac{3}{2}+k;\tau)\,, \tag{6.35}$$

where the coefficients $b_{0,k} = \tilde{b}_{k,\lfloor k/2 \rfloor}$ have been computed in [81] and are given by

$$b_{0,k} := \frac{(k+1)\Gamma(k-\frac{1}{2})\Gamma(k+\frac{5}{2})}{2^{2k+1}\,\pi^{3/2}\,\Gamma(k+1)}\,. \tag{6.36}$$

As manifest from this particular example (and other cases presented in [67, 81]), we notice that the coefficients $b_{r,k}$ appearing in the series (6.34) grow factorially with k for fixed r, i.e. $b_{r,k} \sim k!$, so that $\mathcal{C}_P^{(r)}(N;\tau)$ can be thought of as a formal asymptotic series with coefficients given by rational multiples of half-integer non-holomorphic Eisenstein series. This simple observation suggests immediately that a proper Borel-like resummation of the formal perturbative expansion (6.34), and hence of the whole perturbative sector (6.33), should by consistency require the introduction of the anticipated non-perturbative terms $\mathcal{C}_{NP}(N;\tau)$ presented in (6.16) and here rewritten for convenience,

$$\mathcal{C}_{NP}(N;\tau) = \sum_{\ell=0}^{\infty} N^{2-\frac{\ell}{2}} \sum_{m=0}^{\ell} \tilde{d}_{\ell,m} D_N\big(\tfrac{\ell}{2} - 2m;\tau\big)\,. \tag{6.37}$$

We stress once more that in [67] these terms have been recovered starting from the generating series (6.10), while presently we are in the process of describing how to retrieve them from a resurgence analysis approach to the resummation of the perturbative sector (6.34).

To this end, we proceed just like we did in the perturbative sector starting from (6.32) to arrive at (6.33), and rearrange the non-perturbative terms (6.37) as

$$\mathcal{C}_{NP}(N;\tau) = \sum_{r=0}^{\infty} N^{2-2r} \mathcal{C}_{NP}^{(r)}(N;\tau)\,, \tag{6.38}$$

$$\mathcal{C}_{NP}^{(r)}(N;\tau) = \sum_{k=-3r-1}^{\infty} d_{r,k}\, N^{-\frac{k+1}{2}} D_N\left(\tfrac{k+1}{2};\tau\right)\,, \tag{6.39}$$

where we made the change of summation variables $\ell = 4r+k+1$ and $m = r$ and correspondingly denoted $d_{r,k} = \tilde{d}_{\ell,m}$. Although using the methods of [67][3] it is possible to compute the coefficients $d_{r,k}$ for different values of r and k, no analytic expression similar to (6.36) has been found prior to this work. Using resurgence analysis we show how to derive the coefficients $d_{r,k}$

[3] We note again that, due to the change in normalisation (6.17) and in summation variables, to compare the non-perturbative coefficients $d_{r,k}$ with the results of [67] we must use $d_{r,k}^{\text{there}} = 2^{6r-8k} d_{r-k,4k-3r-1}$.

from the perturbative coefficients $b_{r,k}$ and manifest that at fixed value of r the numbers $d_{r,k}$ are once again factorially divergent as $k \to \infty$.

In what follows, we show that the large-N transseries representation (6.13) for the integrated correlator (6.1) can be recovered from the Borel-Écalle median resummation of the perturbative sectors (6.34), i.e.

$$\mathcal{C}_N(\tau) = \frac{N^2}{4} + \sum_{r=0}^{\infty} N^{2-2r} \mathcal{C}^{(r)}(N;\tau)\,, \tag{6.40}$$

$$\mathcal{C}^{(r)}(N;\tau) = \mathcal{C}_P^{(r)}(N;\tau) + \sigma\, \mathcal{C}_{NP}^{(r)}(N;\tau)\,. \tag{6.41}$$

As already mentioned previously, the median resummation contains the additional parameter called the transseries parameter $\sigma = \sigma(\arg(N))$, which is a piecewise constant function of $\arg(N)$. We will show that for the median resummation here considered the transseries parameter takes values $\sigma = \pm i$ according to whether $\arg(N) > 0$ or < 0, this will in turn be correlated with the how we perform the resummation of the perturbative sector $\mathcal{C}_P^{(r)}(N;\tau)$.

Thanks to our modular invariant resurgence analysis approach we find that:

(i) From the "r-subleading index" non-holomorphic Eisenstein series $E^*(s;\tau)$ with index $s = \frac{3}{2} + \ell - 2r$, grouped in $\mathcal{C}_P^{(r)}(N;\tau)$, we can retrieve all of the "r-subleading index" non-perturbative terms $D_N(s;\tau)$ with $s = \frac{\ell}{2} - 2r$, grouped in $\mathcal{C}_{NP}^{(r)}(N;\tau)$, see section 6.6;

(ii) As a consequence of modularity, in the 't Hooft limit where $\lambda = \sqrt{4\pi N/\tau_2}$ is kept fixed, the function $\mathcal{C}^{(r)}(N;\tau)$ reduces to the transseries expansion of the genus-r contribution to the integrated correlator, as well as the transseries expansion of the "dual 't Hooft-limit" genus-r contribution where $\tilde{\lambda} := (4\pi N)^2/\lambda$ is kept fixed, see section 6.7;

(iii) The sum over r in (6.40) is actually Borel summable and does not introduce any additional non-perturbative corrections. Furthermore, the large-N expansion of the spectral representation (6.23) leads directly to the spectral representation of $\mathcal{C}^{(r)}(N;\tau)$ whose spectral overlap encodes quite naturally both the perturbative, $\mathcal{C}_P^{(r)}(N;\tau)$, and non-perturbative, $\mathcal{C}_{NP}^{(r)}(N;\tau)$, sectors, see section 6.8.

6.5 Modular invariant resummation at large-N

Motivated from the case of present interest, namely the formal perturbative series (6.34), the main goal of this section is to define a modular invariant resummation for the formal but modular invariant series,

$$\Phi_P(N;\tau) \coloneqq \sum_{k=0}^{\infty} b_k\, N^{-\frac{3}{2}-k} E^*(\tfrac{3}{2}+k;\tau)\,, \tag{6.42}$$

where the coefficients b_k diverge factorially fast, i.e. $b_k \sim k!$.

Inspired by the particular exponential structure of the candidate non-perturbative terms, $D_N\,(s;\tau)$, defined in (6.17), we introduce a somewhat non-standard integral representation for the non-holomorphic Eisenstein series

$$N^{-s}E^*(s;\tau) = \int_0^\infty \mathcal{E}(\sqrt{N}t;\tau)\,\frac{2\Gamma(s)}{\Gamma(2s)}(4t)^{2s-1}\mathrm{d}t\,, \tag{6.43}$$

where we have defined a modular invariant modified Borel kernel

$$\mathcal{E}(t;\tau) \coloneqq D_{t^2}(0;\tau) = \sum_{(m,n)\neq(0,0)} e^{-4t\sqrt{Y_{mn}(\tau)}}\,, \tag{6.44}$$

which converges absolutely for all τ in the upper-half plane when $\mathrm{Re}(t) > 0$. Some of the properties of $\mathcal{E}(t;\tau)$ are presented in appendix F, in particular from (F.4) we see that (6.43) is convergent for $\mathrm{Re}(s) > 1$.

We are now in a position to define the Borel transform of the formal series (6.42) as

$$\mathcal{B}[\Phi_P](t) \coloneqq \sum_{k=0}^{\infty} b_k\,\frac{2\Gamma(k+\frac{3}{2})}{\Gamma(2k+3)}(4t)^{2k+2}\,, \tag{6.45}$$

which has a positive radius of convergence in the complex Borel t-plane under the assumption that $b_k \sim k!$, thus defining a germ of analytic functions at the origin.

Following standard resurgence analysis arguments, see chapter 2 for more information, we combine (6.45) with the integral representation (6.43) specialised to $s = k + \frac{3}{2}$, and define the directional Borel resummation of the original formal series (6.42) as

$$\mathcal{S}_\theta[\Phi_P](N;\tau) \coloneqq \int_0^{e^{i\theta}\infty} \mathcal{E}(\sqrt{N}t;\tau)\mathcal{B}[\Phi_P](t)\mathrm{d}t\,. \tag{6.46}$$

If the direction of integration $-\pi < \theta \leq \pi$ is such that the Borel transform $\mathcal{B}[\Phi_P](t)$ has no singularities, i.e. if $\arg(t) = \theta$ is *not* a Stokes direction, we have that the directional Borel resummation (6.46) is well-defined (under a moderate growth condition for the Borel transform) and it defines a modular invariant function of τ, which is analytic in N in the wedge $\mathrm{Re}(\sqrt{N}e^{i\theta}) > 0$ of the complex N-plane. From equation (6.43), we see that the asymptotic expansion of (6.46) at large-N reproduces the formal expansion (6.42) we started with, i.e. we have resummed (6.42) in a modular invariant way.

Furthermore, given two directions θ_1 and θ_2 with $\theta_1 < \theta_2$ such that $\mathcal{B}[\Phi_P](t)$ is regular in the wedge $\theta_1 \leq \arg(t) \leq \theta_2$, we find that $\mathcal{S}_{\theta_1}[\Phi_P](N;\tau) = \mathcal{S}_{\theta_2}[\Phi_P](N;\tau)$ on the common domain of analyticity. Hence $\mathcal{S}_{\theta_2}[\Phi_P](N;\tau)$ defines an analytic continuation of $\mathcal{S}_{\theta_1}[\Phi_P](N;\tau)$ to a wider wedge of the complex N-plane. However, if the direction $\theta = \theta_\star$ is a singular direction for $\mathcal{B}[\Phi_P](t)$, usually called a Stokes direction, we find instead that the analytic functions $\mathcal{S}_{\theta_\star-\epsilon}[\Phi_P](N;\tau)$ and $\mathcal{S}_{\theta_\star+\epsilon}[\Phi_P](N;\tau)$, with $\epsilon \to 0^+$, do not coincide on the common domain of analyticity and they crucially differ by non-perturbative terms. Near a Stokes direction $\theta_\star$, we are then naturally led to consider the lateral Borel resummations defined as

$$\mathcal{S}_{\theta_\star^\pm}[\Phi_P](N;\tau) \coloneqq \lim_{\epsilon\to 0^+} \mathcal{S}_{\theta_\star\pm\epsilon}[\Phi_P](N;\tau)\,. \tag{6.47}$$

In the case where N denotes the number of colours, we obviously want to define a resummation of (6.42) which is analytic in a neighbourhood of the positive real axis $\arg(N) = 0$. However, we will shortly see that for the cases of interest the direction $\theta = 0$ happens to be a Stokes ray. In particular, we need to consider the case where the Borel transform $\mathcal{B}[\Phi_P](t)$ has polar singularities at $t = 1$ plus a branch cut starting at $t = 1$ with an expansion of the form

$$\mathcal{B}[\Phi_P](t) \sim -\frac{1}{\pi}\sum_{k=1}^{M} \frac{d_{-k}(k-1)!}{(1-t)^k} + \Big(\sum_{k=0}^{\infty} \frac{d_k(t-1)^k}{k!}\Big)\frac{\log(1-t)}{\pi} + \mathrm{reg}(t-1), \tag{6.48}$$

with M a positive integer specifying the maximal order of the pole, while $\mathrm{reg}(t-1)$ denotes the analytic part at $t = 1$. Besides the polar part, we also have a logarithmic singularity multiplied by a new germ of analytic functions at the origin, which is specified by a series of factorially divergent coefficients d_k.

Starting from (6.48), we easily compute the difference between the two lateral resummations of the original series (6.42), related to the so-called Stokes automorphism, which takes the form

$$\Big(\mathcal{S}_+ - \mathcal{S}_-\Big)[\Phi_P](N;\tau) = -2i \sum_{k=-M}^{\infty} d_k \, N^{-\frac{k+1}{2}} D_N\left(\tfrac{1+k}{2};\tau\right) \,, \tag{6.49}$$

where for ease of notation we write $\mathcal{S}_\pm \coloneqq \mathcal{S}_{0^\pm}$ since $\theta = 0$ will be the only Stokes line we need considering. Notice that since the coefficients d_k are in general factorially divergent, as we show for the integrated correlator, the discontinuity equation (6.49) defines once again a formal series, which is however modular invariant and whose coefficients are no longer given by non-holormorphic Eisenstein series but rather they belong to the class of functions defined in (6.17).

We stress how general this result is: given a formal series of the form (6.42) whose Borel transform, $\mathcal{B}[\Phi_P](t)$, defines a germ of analytic function at the origin with a singular structure akin to (6.48), along any Stokes direction we must have a non-perturbative and modular invariant discontinuity in lateral resummations captured by an infinite sum of $D_N(s;\tau)$ functions.

To finally construct a complete modular invariant transseries starting from the purely perturbative sector (6.42), we additionally need to understand how to modify the lateral resummations $\mathcal{S}_\pm$ to take into account their discontinuity (6.49). Proceeding as we just did for the perturbative sector, we start from the non-perturbative modular invariant series in (6.49)

$$\Phi_{NP}(N;\tau) \coloneqq \sum_{k=-M}^{\infty} d_k \, N^{-\frac{k+1}{2}} D_N\left(\tfrac{1+k}{2};\tau\right) \,, \tag{6.50}$$

and consider the integral representation

$$N^{-\frac{k+1}{2}} D_N\left(\tfrac{k+1}{2};\tau\right) = \int_0^\infty \mathcal{E}(\sqrt{N}(t+1);\tau)\frac{t^k}{\Gamma(k+1)}\mathrm{d}t \,, \tag{6.51}$$

with $k \geq 0$. Just like the integral representation (6.43) leads to the perturbative Borel transform (6.45), we now use (6.51) to define the directional Borel resummation of the non-perturbative sector (6.50),

$$\widetilde{\mathcal{B}}[\Phi_{NP}](t) \coloneqq \sum_{k=0}^{\infty} \frac{d_k}{\Gamma(k+1)} t^k, \tag{6.52}$$

$$\mathcal{S}_\theta[\Phi_{NP}](N;\tau) \coloneqq \sum_{k=-M}^{-1} d_k\, N^{-\frac{k+1}{2}} D_N\left(\tfrac{1+k}{2};\tau\right) + \int_0^{e^{i\theta}\infty} \mathcal{E}(\sqrt{N}(t+1);\tau)\, \widetilde{\mathcal{B}}[\Phi_{NP}](t)\, \mathrm{d}t\,. \tag{6.53}$$

Note that the finitely many terms in (6.53) with a positive power of N have to be treated separately since they cannot be represented via (6.51), these terms correspond to the polar part of the singular behaviour of the Borel transform (6.48).

In standard applications of resurgence theory the Borel kernel is given by the usual Laplace measure $e^{-\sqrt{N}t}\mathrm{d}t$. In this case it is well appreciated that under a shift of integration variable $t \to t+1$, the measure naturally factors into the same integration kernel multiplied by the expected exponential suppression factor $^{-\sqrt{N}}$ which characterises the non-perturbative sectors. Due to the lattice sum nature (6.44) of the present modular invariant Borel integration kernel $\mathcal{E}(\sqrt{N}t;\tau)$, we do not have such property, i.e.

$$\mathcal{E}(\sqrt{N}(t+1);\tau)\mathrm{d}t \neq \mathcal{E}(\sqrt{N};\tau)\, \mathcal{E}(\sqrt{N}t;\tau)\mathrm{d}t\,.$$

This explains why we have to define a second Borel transform $\widetilde{\mathcal{B}}[\Phi_{NP}](t)$ in (6.53): while the building blocks of the perturbative sector are given by non-holomorphic Eisenstein series, they differ from those of the non-perturbative part, i.e. the functions $D_N(s;\tau)$. However, our modular invariant Borel kernel contains both objects:

$$N^{-s}E^*(s;\tau) = \int_0^\infty \mathcal{E}(\sqrt{N}t;\tau)\, \frac{2\Gamma(s)}{\Gamma(2s)}(4t)^{2s-1}\mathrm{d}t\,,$$

$$N^{-s}D_N(s;\tau) = \int_0^\infty \mathcal{E}(\sqrt{N}(t+1);\tau)\, \frac{t^{2s-1}}{\Gamma(2s)}\mathrm{d}t\,. \tag{6.54}$$

Given the discontinuity (6.49), it is now manifest that for the physically relevant domain $N > 0$, the two resummations $\mathcal{S}_\pm[\Phi_P](N;\tau)$ do differ and we have an ambiguity in how we resum the purely perturbative formal power series (6.42). Furthermore, while the original formal power series (6.42) is manifestly real for $N > 0$ and $\tau \in \mathfrak{H}$, neither of the two lateral resummations $\mathcal{S}_\pm[\Phi_P](N;\tau)$ is. To obtain a real and unambiguous resummation for $N > 0$, we have to consider an average between the two lateral resummations $\mathcal{S}_{\pm}[\Phi_P](N;\tau)$, usually referred to as median resummation [97],

$$\Phi(N;\tau) = \Phi_P(N;\tau) + \sigma\,\Phi_{NP}(N;\tau)\,. \tag{6.55}$$

The additional parameter σ, called the transseries parameter, is the piecewise constant function of $\arg(N)$ given by $\sigma = \pm i$ according to $\arg(N)\gtrless 0$, which in turn is correlated with the choice of lateral resummation

$$\mathcal{S}_{med}[\Phi_P](N;\tau) \coloneqq \begin{cases} \mathcal{S}_{+}[\Phi_P](N;\tau) + i\,\mathcal{S}_0[\Phi_{NP}](N;\tau)\,, & \arg(N) > 0\,, \\ \mathcal{S}_{-}[\Phi_P](N;\tau) - i\,\mathcal{S}_0[\Phi_{NP}](N;\tau)\,, & \arg(N) < 0\,. \end{cases} \tag{6.56}$$

We shortly show that the equality between the two seemingly different expressions comes from the discontinuity equation (6.49) combined with the fact that for the integrated correlator $\arg(N) = 0$ is not a Stokes direction for $\Phi_{NP}(N;\tau)$, which can then be resummed via (6.53) along $\theta = 0$, i.e. by considering $\mathcal{S}_0[\Phi_{NP}](N;\tau)$.

The median resummation produces an unambiguous resummation of (6.42) in a wedge of the complex-N plane which contains the physical domain $N > 0$. In particular, it is easy to show that (6.56) is real-analytic for $N > 0$ and $\tau \in \mathfrak{H}$, since it can be rewritten as

$$\begin{aligned} \mathcal{S}_{med}[\Phi_P](N;\tau) &= \frac{1}{2}(\mathcal{S}_{+} + \mathcal{S}_{-})[\Phi_P](N;\tau) = \int_{\mathcal{M}} \mathcal{E}(\sqrt{N}t;\tau)\mathcal{B}[\Phi_P](t)\mathrm{d}t\,, \\ &= \int_0^\infty \mathcal{E}(\sqrt{N}t;\tau)\mathrm{Re}\Big(\mathcal{B}[\Phi_P](t)\Big)\mathrm{d}t\,. \end{aligned} \tag{6.57}$$

Here we have defined for later convenience the notation for the median integration

$$\int_{\mathcal{M}} \coloneqq \frac{1}{2}\Big(\int_0^{\infty+i\epsilon} + \int_0^{\infty-i\epsilon}\Big)\,, \tag{6.58}$$

in the limit $\epsilon \to 0^+$. The particular integral representation (6.57) for the non-perturbative integrated correlator will be obtained directly from spectral theory in section 6.8.

6.6 Resurgence of the integrated correlators

Thanks to the analysis of the previous section, we have thus constructed an unambiguous resummation for a formal perturbative series in non-holomorphic Eisenstein series $E^*(s;\tau)$, schematically presented in (6.42), and have shown that it generically requires exponentially suppressed terms which involve the modular invariant function $D_N(s;\tau)$.

In this section we show how an application of this resummation method to the two integrated correlators (6.1) yields the complete large-N expansion of the first integrated correlator $\mathcal{C}_N(\tau)$, as well as a particular non-perturbative sector of the second integrated correlator $\mathcal{H}_N(\tau)$. This explains the appearance of such non-perturbative terms in the full transseries representation (6.40) of the integrated correlator $\mathcal{C}_N(\tau)$ and our analysis establishes a connection between the perturbative coefficients $\tilde{b}_{\ell,m}$ in (6.32), or alternatively the coefficients $b_{r,k}$ in (6.34), and the non-perturbative coefficients $\tilde{d}_{\ell,m}$ in (6.37), or alternatively the coefficients $d_{r,k}$ presented in (6.39), as we now show in more detail.

First integrated correlator

For concreteness we discuss the cases $r=0$ and $r=1$, although our analysis can be extended straightforwardly to arbitrary r. We begin by deriving the non-perturbative resummation of $\mathcal{C}_P^{(0)}(N;\tau)$ presented in (6.35), which contains all large-N perturbative contributions originating from non-holomorphic Eisenstein series with leading index.

Given the definition (6.45) we compute the associated Borel transform of this series, which takes the form

$$\mathcal{B}[\mathcal{C}_P^{(0)}](t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\Gamma(k-\frac{1}{2})\Gamma(k+\frac{5}{2})}{\Gamma(k+1)^2}t^{2k+2} = -6t^2\,{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};1|t^2). \tag{6.59}$$

As anticipated, we see that the Borel transform has two Stokes directions: one for $\theta=0$ and the other for $\theta=\pi$, both with logarithmic branch cuts starting respectively at $t=\pm1$ due to the hypergeometric function ${}_2F_1$. Since we are interested in obtaining a non-perturbative resummation for $\arg(N)=0$, we have to compute the singular behaviour of the Borel transform (6.59) near the point $t=1$. This can be obtained from the integral representation of the hypergeometric function thanks to which we find

$$\mathcal{B}[\mathcal{C}_P^{(0)}](t) \sim -\frac{(-2)}{\pi(1-t)} - 9\,t^2{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};2|1-t^2)\frac{\log(1-t)}{\pi} + \mathrm{reg}(t-1)\,, \tag{6.60}$$

where again $\mathrm{reg}(t-1)$ denotes the analytic part at $t=1$.

This structure is precisely of the form (6.48) previously analysed and, as a consequence, we have that the two lateral resummations of (6.59) do not coincide on the common domain of analyticity. We can compute the difference in lateral resummations (6.49) directly from the singular behaviour (6.60),

$$\begin{aligned}(\mathcal{S}_+ - \mathcal{S}_-)[\mathcal{C}_P^{(0)}](N;\tau) &= -2i\mathcal{S}_0[\mathcal{C}_{NP}^{(0)}](N;\tau) \\ &= 4iD_N(0;\tau) + 18i\int_0^\infty \mathcal{E}(\sqrt{N}(t+1);\tau)\,(1+t)^2{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};2|-t(2+t))\mathrm{d}t\,.\end{aligned} \tag{6.61}$$

Following our general discussion, we use this discontinuity and (6.53) to define the resummation of the non-perturbative sector which is then given by

$$\begin{aligned}&\mathcal{S}_0[\mathcal{C}_{NP}^{(0)}](N;\tau) \\ &= -2D_N(0;\tau) - 9\int_0^\infty \mathcal{E}(\sqrt{N}(t+1);\tau)(1+t)^2{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};2|-t(2+t))\mathrm{d}t\,.\end{aligned} \tag{6.62}$$

As already mentioned, the Borel transform of the non-perturbative sector is regular along the direction $\arg(t)=0$, hence (6.62) is precisely the Borel resummation of the formal series of non-perturbative corrections

$$\begin{aligned}\mathcal{C}_{NP}^{(0)}(N;\tau) &= \sum_{k=-1}^{\infty} d_{0,k}\,N^{-\frac{k+1}{2}}D_N\left(\tfrac{k+1}{2};\tau\right) \\ &= -2D_N(0;\tau) - 9N^{-\frac{1}{2}}D_N\left(\tfrac{1}{2};\tau\right) - \frac{117}{4}N^{-1}D_N(1;\tau) + \dots\,,\end{aligned} \tag{6.63}$$

with the coefficients $d_{0,k}$ given by

$$d_{0,-1} = -2\,,\qquad \widetilde{\mathcal{B}}[\mathcal{C}_{NP}^{(0)}](t) = -9(1+t)^2{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};2|-t(2+t)) = \sum_{k=0}^{\infty}\frac{d_{0,k}}{k!}t^k\,, \tag{6.64}$$

matching and extending the results of [67] (modulo the trivial change in normalisation in footnote 3) for the coefficients of the leading index non-perturbative terms, i.e. all $D_N(s;\tau)$ terms in (6.16) with $m=\ell$, presented in equation (3.26) of the same reference.

We can then construct the median resummation (6.56) and express it via the transseries

$$\mathcal{C}^{(0)}(N;\tau) = \mathcal{C}_P^{(0)}(N;\tau) + \sigma\, \mathcal{C}_{NP}^{(0)}(N;\tau)\,, \tag{6.65}$$

where the associated transseries parameter σ has value $\sigma = \pm i$ according to $\arg(N) \gtrless 0$, precisely matching the transseries (6.41) found in [67] by use of generating series methods. Furthermore, the median resummation of said transseries can be written as the average of the two lateral resummations presented in (6.57), taking the form

$$\begin{aligned}\mathcal{C}^{(0)}(N;\tau) = \mathcal{S}_{med}[\mathcal{C}_P^{(0)}](N;\tau) &= \int_{\mathcal{M}} \mathcal{E}(\sqrt{N}t;\tau)\Big[-6t^2{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};1|t^2)\Big]\mathrm{d}t \\ &= \int_0^\infty \mathcal{E}(\sqrt{N}t;\tau)\mathrm{Re}\Big(-6t^2{}_2F_1(-\tfrac{1}{2},\tfrac{5}{2};1|t^2)\Big)\mathrm{d}t\,,\end{aligned} \tag{6.66}$$

which will be retrieved from spectral methods later in section 6.8, see (6.134).

We can repeat this analysis for the contribution to (6.32) coming from all "sub-leading-index" non-holomorphic Eisenstein series, i.e. all terms in (6.32) with $m = \lfloor \ell/2 \rfloor - 1$ or equivalently consider (6.34) with $r=1$:

$$\mathcal{C}_P^{(1)}(N,\tau) = \sum_{k=0}^{\infty} b_{1,k} N^{-k-\frac{3}{2}} E^*(\tfrac{3}{2}+k;\tau)\,, \tag{6.67}$$

where the coefficients $b_{1,k} = \tilde{b}_{k+2,\lfloor k/2 \rfloor}$ have been computed in [81] and are given by

$$b_{1,k} \coloneqq -\frac{(k+1)^2(2k+13)\Gamma(k+\frac{5}{2})^2}{2^{2k+6}\,3\pi^{\frac{3}{2}}\Gamma(k+3)}\,. \tag{6.68}$$

The associated Borel transform (6.45) is then given by

$$\begin{aligned}\mathcal{B}[\mathcal{C}_P^{(1)}](t) &= -\frac{1}{24\pi}\sum_{k=0}^{\infty}\frac{(k+1)(2k+13)\Gamma(k+\frac{5}{2})^2}{\Gamma(k+1)\Gamma(k+3)}t^{2k+2} \\ &= -\frac{t^2}{8192}\Big[1248\,{}_2F_1(\tfrac{5}{2},\tfrac{5}{2};3|t^2) + 3400\,t^2\,{}_2F_1(\tfrac{7}{2},\tfrac{7}{2};4|t^2) + 1225\,t^4\,{}_2F_1(\tfrac{9}{2},\tfrac{9}{2};5|t^2)\Big]\,,\end{aligned} \tag{6.69}$$

where again, due to the presence of these particular hypergeometric functions, we find the two

Stokes directions $\arg(t) = 0$ and $\arg(t) = \pi$ with corresponding logarithmic branch cuts starting at $t = \pm 1$. The singularity structure of $\mathcal{B}[\mathcal{C}_P^{(1)}](t)$ near $t = 1$ is given by

$$\mathcal{B}[\mathcal{C}_P^{(1)}](t) \sim -\frac{1}{32\pi(1-t)^4} - \frac{3}{64\pi(1-t)^3} - \frac{(-77)}{512\pi(1-t)^2} - \frac{127}{1024\pi(1-t)} + \widetilde{\mathcal{B}}[\mathcal{C}_{NP}^{(1)}](t-1)\frac{\log(1-t)}{\pi} + \text{reg}(t-1)\,, \tag{6.70}$$

where, following the discussion around (6.49), we have already interpreted the germ of analytic functions multiplying the logarithm as the Borel resummation of the non-perturbative sector given by

$$\widetilde{\mathcal{B}}[\mathcal{C}_{NP}^{(1)}](t) = \tag{6.71}$$
$$-\frac{14[60t(t+2)+47]\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};5;-t(t+2)\right) + [113t(t+2)+269]\,{}_2F_1\left(\frac{1}{2},\frac{3}{2};5;-t(t+2)\right)}{8192(t+1)^4}\,.$$

We observe that, when compared to the Borel transform (6.60) for the case $r = 0$, the order of the pole at $t = 1$ has now increased to a fourth-order pole. Similarly to (6.62), from (6.71) we can now obtain the formal expansion of the sub-leading index non-perturbative sector

$$\mathcal{C}_{NP}^{(1)}(N;\tau) = \sum_{k=-3}^{\infty} d_{1,k}\, N^{-\frac{k+1}{2}} D_N\left(\tfrac{k+1}{2};\tau\right)\,, \tag{6.72}$$

with coefficients

$$d_{1,-4} = \frac{1}{192}\,,\quad d_{1,-3} = \frac{3}{128}\,,\quad d_{1,-2} = -\frac{77}{512}\,,\quad d_{1,-1} = \frac{127}{1024}\,, \tag{6.73}$$

$$\widetilde{\mathcal{B}}[\mathcal{C}_{NP}^{(1)}](t) = \sum_{k=0}^{\infty}\frac{d_{1,k}}{k!}t^k = -\frac{927}{8192} + \frac{3897}{16384}t - \frac{47217}{65536}\frac{t^2}{2!} + O(t^3)\,. \tag{6.74}$$

Once again these results extend those found in [67] for the sub-leading diagonal of non-perturbative terms presented in equation (3.26) of that reference, i.e. all $D_N(s;\tau)$ terms in (6.16) with $m = \ell - 1$.

In general, to reconstruct the non-perturbative completion of $\mathcal{C}_P^{(r)}(N;\tau)$, i.e. the non-perturbative sector completing the formal asymptotic expansion of "r-subleading index" non-holomorphic Eisenstein series (6.34), we start from the singular behaviour near $t = 1$ of the corresponding Borel transform

$$\mathcal{B}[\mathcal{C}_P^{(r)}](t) \sim -\frac{1}{\pi}\sum_{k=1}^{3r+1}\frac{d_{r,-k}(k-1)!}{(1-t)^k} + \widetilde{\mathcal{B}}[\mathcal{C}_{NP}^{(r)}](t-1)\frac{\log(1-t)}{\pi} + \text{reg}(t-1)\,, \tag{6.75}$$

with

$$\widetilde{\mathcal{B}}[\mathcal{C}_{NP}^{(r)}](t) = \sum_{k=0}^{\infty}\frac{d_{r,k}}{k!}t^k\,. \tag{6.76}$$

Equation (6.75) yields a difference in lateral Borel resummation of the form

$$(\mathcal{S}_+ - \mathcal{S}_-)[\mathcal{C}_P^{(r)}](N;\tau) = -2i\mathcal{S}_0[\mathcal{C}_{NP}^{(r)}](N;\tau)\,. \tag{6.77}$$

The coefficients $d_{r,k}$ of the non-perturbative sector are then entirely encoded in the discontinuity equation (6.77) of the Borel transform $\mathcal{B}[\mathcal{C}_P^{(r)}](t)$ along the Stokes line $t>0$, i.e.

$$\mathcal{S}_0[\mathcal{C}_{NP}^{(r)}](N;\tau) \sim \mathcal{C}_{NP}^{(r)}(N;\tau) = \sum_{k=-3r-1}^{\infty} d_{r,k}\, N^{-\frac{k+1}{2}} D_N\left(\tfrac{k+1}{2};\tau\right)\,. \tag{6.78}$$

Arguing as above, we must add to the lateral Borel resummation of the perturbative sector a suitable multiple of these non-perturbative terms to finally arrive at the modular invariant and unambiguous median transseries (6.41)

$$\begin{aligned}\mathcal{C}^{(r)}(N;\tau) &= \mathcal{C}_P^{(r)}(N;\tau) + \sigma\,\mathcal{C}_{NP}^{(r)}(N;\tau) = \mathcal{S}_{med}[\mathcal{C}_P^{(r)}](N;\tau)\\ &= \int_{\mathcal{M}}\mathcal{E}(\sqrt{N}t;\tau)\mathcal{B}[\mathcal{C}_P^{(r)}](t)\mathrm{d}t = \int_0^{\infty}\mathcal{E}(\sqrt{N}t;\tau)\mathrm{Re}\Big(\mathcal{B}[\mathcal{C}_P^{(r)}](t)\Big)\mathrm{d}t\,,\end{aligned} \tag{6.79}$$

where again the transseries parameter $\sigma = \pm i$ according to $\arg(N) > 0$ or < 0.

Second integrated correlator

We now apply the same resummation method to analyse a particular sector of the second integrated correlator $\mathcal{H}_N(\tau)$ presented in (6.1). The large-N expansion of $\mathcal{H}_N(\tau)$ was initiated in [65] and then conjectured in [68] to have the asymptotic perturbative form

$$\mathcal{H}_N(\tau) \sim 6N^2 + \mathcal{H}_N^h(\tau) + \mathcal{H}_N^i(\tau)\,, \tag{6.80}$$

where the two different perturbative sectors $\mathcal{H}^h_N(\tau)$ and $\mathcal{H}^i_N(\tau)$ are formal modular invariant power series in respectively half-integer powers and integer powers in $1/N$. In particular, the large-N expansion of $\mathcal{H}^h_N(\tau)$ is of the same form (6.33) for $\mathcal{C}_P(N;\tau)$ and only contains non-holomorphic Eisenstein series,

$$\mathcal{H}^h_N(\tau) = \sum_{r=0}^{\infty} N^{2-2r}\mathcal{H}^{(r)}_h(N;\tau)\,, \tag{6.81}$$

$$\mathcal{H}^{(r)}_h(N;\tau) \coloneqq \sum_{k=0}^{\infty} a_{r,k}N^{-\frac{3}{2}-k}E^*(\tfrac{3}{2}+k;\tau)\,. \tag{6.82}$$

For fixed value of r, the perturbative coefficients $a_{r,k}$ have been found in [68] exploiting an intriguing inhomogeneous Laplace difference equation relating this second integrated correlator $\mathcal{H}_N(\tau)$ to $\mathcal{C}_N(\tau)$. In particular, for the leading-index non-holomorphic Eisenstein series we have

$$a_{0,k} \coloneqq -\frac{(k+1)(k+3)\Gamma\left(k-\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{2^{2k-3}\pi^{3/2}\Gamma(k+1)}\,. \tag{6.83}$$

Focusing for concreteness on the $r=0$ case, it is then straightforward to compute the corresponding Borel transform (6.45),

$$\mathcal{B}[\mathcal{H}^{(0)}_h](t) = 192\,t^2{}_2F_1\left(-\tfrac{1}{2},\tfrac{3}{2};1|t^2\right) - 48\,t^4{}_2F_1\left(\tfrac{1}{2},\tfrac{5}{2};2|t^2\right) \tag{6.84}$$

from which we obtain the singular behaviour along the Stokes direction $\arg(t)=0$:

$$\mathcal{B}[\mathcal{H}^{(0)}_h](t) \sim -\frac{32}{\pi(1-t)} + \widetilde{\mathcal{B}}[\mathcal{H}^{(0)}_{h,NP}](t-1)\frac{\log(1-t)}{\pi} + \text{reg}(t-1)\,, \tag{6.85}$$

and interpret the germ of analytic functions multiplying the logarithm as the Borel resummation of the non-perturbative sector given by

$$\widetilde{\mathcal{B}}[\mathcal{H}^{(0)}_{h,NP}](t) = 48(t+1)^2\left[4\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{3}{2};1|-t(t+2)\right) + {}_2F_1\left(-\tfrac{1}{2},\tfrac{3}{2};2|-t(t+2)\right)\right]. \tag{6.86}$$

Following the same process as before, we extract from the singular behaviour (6.85) a novel formal series of non-perturbative corrections

$$\begin{aligned}\mathcal{H}^{(0)}_{h,NP}(N;\tau) &= \sum_{k=-1}^{\infty} h_{0,k}\, N^{-\frac{k+1}{2}} D_N\left(\tfrac{k+1}{2};\tau\right) \\ &= 32 D_N(0;\tau) + 240 N^{-\frac{1}{2}} D_N\left(\tfrac{1}{2};\tau\right) + 804 N^{-1} D_N(1;\tau) + \dots\,,\end{aligned} \tag{6.87}$$

with the coefficients $h_{0,k}$ given by

$$h_{0,-1} = 32\,, \qquad \widetilde{\mathcal{B}}[\mathcal{H}^{(0)}_{h,NP}](t) = \sum_{k=0}^{\infty} \frac{h_{0,k}}{k!} t^k = 240 + 804t + \frac{855}{2}\frac{t^2}{2!} + O(t^3)\,. \tag{6.88}$$

A similar analysis can be carried out for higher values of $r > 0$ to obtain the non-perturbative completion of the formal power series (6.81). However, we stress that this process does not define the full non-perturbative completion for the second integrated correlator $\mathcal{H}_N(\tau)$. The procedure here discussed can only resum the formal power series in half-integer powers of $1/N$ contained in (6.81). As discussed in [68], besides the sector just mentioned, the large-N expansion of the second integrated correlator (6.80) contains the formal modular invariant power series $\mathcal{H}^{(i)}_N(\tau)$ in integer powers of $1/N$.

Order by order in $1/N$, the coefficients of $\mathcal{H}^{(i)}_N(\tau)$ are given by finite linear combinations of a different class of modular functions - generalised Eisenstein series. They have already appeared in this thesis and were analysed extensively in chapter 5. Of course, they are also relevant for other string theory contexts, see e.g. [1, 31, 34], and display a more complicated structure of perturbative and non-perturbative corrections [2, 35, 76, 98]. In particular, we easily see that our resummation methods starting from the formal perturbative expansion (6.42) cannot be exploited to extract the non-perturbative completion to the sector $\mathcal{H}^{(i)}_N(\tau)$ of the second integrated correlator.

6.7 Large-N 't Hooft expansions

In this section we consider the standard large-N 't Hooft limit, where $\lambda := 4\pi N/\tau_2 = N g^2_{YM}$ is kept fixed as $N \to \infty$, starting from the transseries expansion (6.40). We show that in this limit the non-perturbative resummation $\mathcal{C}^{(r)}(N;\tau)$ naturally encodes the strong coupling genus-r 't Hooft transseries contribution, which includes an infinite tower of non-perturbative corrections of the form $e^{-2\ell\sqrt{\lambda}}$ with $\ell \in \mathbb{N}$, which can be interpreted as fundamental string world-sheet instantons. However, as a consequence of modularity we see that each $\mathcal{C}^{(r)}(N;\tau)$ also contains the strong coupling non-perturbative resummation of the "dual" genus-r 't Hooft expansion, where the dual 't Hooft coupling $\tilde{\lambda} = (4\pi N)^2/\lambda$ is kept fixed as $N \to \infty$. This resummation contains an infinite tower of non-perturbative corrections of the form $e^{-2\ell\sqrt{\tilde{\lambda}}}$ with $\ell \in \mathbb{N}$, which can be interpreted as an averaging of all dyonic-string world-sheet instantons.

Since in the 't Hooft limit we keep fixed $\lambda = 4\pi N/\tau_2 = N g^2_{_{YM}}$ as we send $N \to \infty$, we have that contributions from Yang-Mills instantons, of order $e^{-8\pi^2 N|k|/\lambda}$ with instanton number $k \neq 0$, are exponentially suppressed. In the Fourier mode expansion of $\mathcal{C}^{(r)}(N;\tau)$ with respect to $\tau_1 = \theta/(2\pi)$, such contributions can be idenfitied with the k^{th} Fourier mode. Hence in the 't Hooft limit we can restrict our attention to the analysis for the zero-mode sector of (6.79), which is obtained from

$$\mathcal{I}^{(r)}(N;\lambda) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{C}^{(r)}(N;\tau)\,\mathrm{d}\tau_1 = \int_{\mathcal{M}} \Big[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{E}(\sqrt{N}t;\tau)\,\mathrm{d}\tau_1\Big] \mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t\,. \tag{6.89}$$

To extract the 't Hooft expansion of this expression, we need to compute the zero Fourier mode of the modular invariant modified Borel kernel $\mathcal{E}(\sqrt{N}t;\tau)$. This calculation is presented in appendix F, where we derive an explicit formula in (F.8), here rewritten for convenience:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{E}(\sqrt{N}t;\tau) d\tau_1 = \frac{2}{e^{4t\sqrt{N\pi/\tau_2}} - 1} + \mathcal{U}(\sqrt{N}t;\tau_2)\,. \tag{6.90}$$

The function $\mathcal{U}(t;\tau_2)$ is given by either the contour integral representation (F.7) or as an infinite sum over Bessel functions (F.9)-(F.10). Substituting this expression for the zero-mode in (6.89) we arrive at

$$\mathcal{I}^{(r)}(N;\lambda) = I^{(r)}(\lambda) + N^{-1}\,\tilde{I}^{(r)}(\tilde{\lambda})\,, \tag{6.91}$$

$$I^{(r)}(\lambda) := \int_{\mathcal{M}} \frac{2}{e^{2t\sqrt{\lambda}} - 1}\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t\,, \tag{6.92}$$

$$\tilde{I}^{(r)}(\tilde{\lambda}) := \int_{\mathcal{M}} \mathcal{U}\Big(t;\frac{\tilde{\lambda}}{4\pi}\Big)\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t\,. \tag{6.93}$$

Here we have introduced the dual 't Hooft coupling $\tilde{\lambda} = (4\pi N)^2/\lambda$, and we have used the identity $\mathcal{U}(\sqrt{N}t;\frac{4\pi N}{\lambda}) = N^{-1}\mathcal{U}(t;\frac{\tilde{\lambda}}{4\pi})$, which follows easily from (F.7).

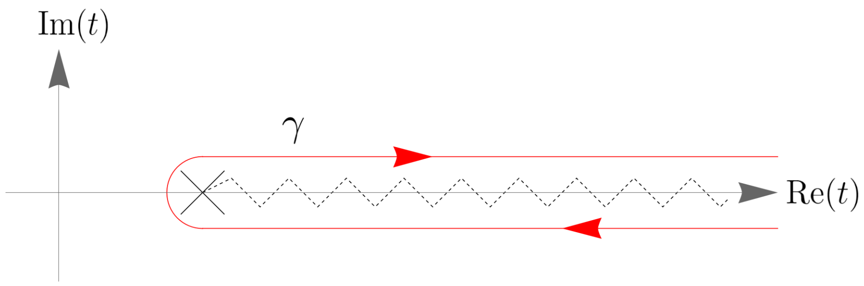

Figure 6.1: Hankel contour γ in the complex t-plane circling around the branch-cut singularity starting at $t = 1$.

If we plug this expression back into the full transseries representation (6.39), we obtain the complete 't Hooft limit expansion of the integrated correlator $\mathcal{C}_N(\tau)$:

$$\mathcal{I}(N;\lambda) \coloneqq \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{C}_N(\tau)\,\mathrm{d}\tau_1 = \sum_{r=0}^{\infty} N^{2-2r}\mathcal{I}^{(r)}(N;\lambda) \tag{6.94}$$
$$= \sum_{r=0}^{\infty} N^{2-2r} \int_{\mathcal{M}} \frac{2}{e^{2t\sqrt{\lambda}}-1}\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t + \sum_{r=0}^{\infty} N^{1-2r} \int_{\mathcal{M}} \mathcal{U}\Big(t;\frac{\tilde{\lambda}}{4\pi}\Big)\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t\,.$$

We stress that thanks to our careful rewriting of the complete transseries (6.40), the N-dependence has now been trivialised. Furthermore, as we will shortly demonstrate, the modular invariant median resummation (6.57) naturally leads to the median resummation of the genus-r large 't Hooft coupling expansion, given by the contribution $I^{(r)}(\lambda)$ in (6.91), plus the median resummation of the genus-r dual 't Hooft coupling expansion, encoded in the second term $\tilde{I}^{(r)}(\tilde{\lambda})$ in (6.91). As already appreciated in [67], we note that modular invariance unifies in the single expression (6.79), the seemingly different median 't Hooft-limit and dual 't Hooft-limit resummations studied in [81] and [66].

To clarify these statements, let us separately consider the two terms in (6.91). Starting with the analysis for $I^{(r)}(\lambda)$, we first rewrite the median resummation making use of (6.56),

$$I^{(r)}(\lambda) = \int_0^{\infty\pm i\epsilon} \frac{2}{e^{2t\sqrt{\lambda}}-1}\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t \mp \frac{1}{2}\int_{\gamma} \frac{2}{e^{2t\sqrt{\lambda}}-1}\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t\,, \tag{6.95}$$

where the Hankel contour γ is given in figure 6.1.

The first term in this expression can be easily computed by expanding the Borel transform

$\mathcal{B}[\mathcal{C}_P^{(r)}](t)$ using the definition (6.45) and then integrating term by term using the identity

$$\int_0^\infty \frac{2}{e^{2t\sqrt{\lambda}}-1} t^k \mathrm{d}t = 2^{-k}\lambda^{-\frac{k}{2}-\frac{1}{2}}\Gamma(k+1)\zeta(k+1)\,, \tag{6.96}$$

valid for $k \geq 0$. In this way we arrive at the formal asymptotic power series expansion

$$\int_0^{\infty\pm i\epsilon} \frac{2}{e^{2t\sqrt{\lambda}}-1}\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} b_{r,k}\,\xi(2k+3)\Big(\frac{\lambda}{4\pi}\Big)^{-k-\frac{3}{2}}\,, \tag{6.97}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s)$ is the completed zeta function as before. For example, substituting the $r = 0$ coefficients (6.36) or the $r = 1$ coefficients (6.68), one can check that (6.97) reduces respectively to the standard genus-0 and genus-1 large 't Hooft coupling expansion of the integrated correlator presented in [81], cf. equations (5.26) and (5.28) of the reference.

Alternatively, we note that

$$\frac{d}{dt}\left[\frac{2}{e^{2t\sqrt{\lambda}}-1}\right] = -\frac{4\sqrt{\lambda}}{4\sinh^2(t\sqrt{\lambda})}\,, \tag{6.98}$$

hence we can integrate (6.97) by parts[4] as

$$\int_0^{\infty\pm i\epsilon} \frac{2}{e^{2t\sqrt{\lambda}}-1}\,\mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t = \sqrt{\lambda}\int_0^{\infty\pm i\epsilon} \frac{1}{4\sinh^2(t\sqrt{\lambda})}\Big(-4\frac{d}{dt}\mathcal{B}[\mathcal{C}_P^{(r)}](t)\Big)\,\mathrm{d}t\,. \tag{6.99}$$

Again, substituting the Borel transform (6.59) for $r = 0$ or the $r = 1$ counterpart (6.69), it is easy to see that this expression reduces precisely to the corresponding modified Borel resummation for the 't Hooft genus expansion considered in [81].

An important difference between the current analysis and the complete genus-r large 't Hooft expansion is the absence in (6.97) of finitely many positive powers of λ which appear at any fixed genus. These powers will be retrieved by analysing the large-λ perturbative expansion of the "dual" 't Hooft contribution (6.93).

The second term in (6.95) can again be computed from our general analysis, starting from the singular behaviour (6.48) of the Borel transform $\mathcal{B}[\mathcal{C}_P^{(r)}](t)$, along the Stokes direction $t > 0$.

[4]It is easy to check from (6.59) and (6.69) that integration by parts does not produce any boundary contributions.

To compute the contribution coming from the polar part of $\mathcal{B}[\mathcal{C}_P^{(r)}](t)$, we simply need the polylogarithm, $\mathrm{Li}_k(x)$, identity

$$\oint_{|t|=1} \frac{2}{e^{2t\sqrt{\lambda}}-1} \frac{1}{(1-t)^k} \frac{\mathrm{d}t}{2\pi i} = -2 \frac{(4\lambda)^{\frac{k-1}{2}} \mathrm{Li}_{1-k}(e^{-2\sqrt{\lambda}})}{\Gamma(k)}\,, \tag{6.100}$$

valid for $k \in \mathbb{N}$, while to evaluate the contribution coming from the discontinuity of the logarithmic singularity we first shift the contour of integration $t \to t+1$ and then use

$$\int_0^\infty \frac{2}{e^{2\sqrt{\lambda}(t+1)}-1}\, t^k \, \mathrm{d}t = 2(4\lambda)^{-\frac{k+1}{2}} k! \mathrm{Li}_{k+1}(e^{-2\sqrt{\lambda}})\,, \tag{6.101}$$

valid for $\mathrm{Re}(k) > 0$.

We then express the second term in (6.95) as

$$\mp \frac{1}{2} \int_\gamma \frac{2}{e^{2t\sqrt{\lambda}}-1} \mathcal{B}[\mathcal{C}_P^{(r)}](t)\, \mathrm{d}t = \pm i \sum_{k=-3r-1}^{\infty} d_{r,k} (4\lambda)^{-\frac{k+1}{2}} \mathrm{Li}_{k+1}(e^{-2\sqrt{\lambda}})\,, \tag{6.102}$$

which, as anticipated, encodes all non-perturbative contributions from worldsheet instantons. It is easy to check that if we plug in the above expression the $r = 0$ non-perturbative coefficients (6.88), or similarly for the $r = 1$ (6.73), we retrieve the necessary non-perturbative completions to the formal large-λ perturbative expansion (6.97), which had been obtained previously in [66, 81] via resurgence analysis applied directly to (6.97).

An analogous analysis can be carried out for the second contribution $\tilde{I}^{(r)}(\tilde{\lambda})$. We again use the decomposition (6.56) to rewrite equation (6.93) as

$$\tilde{I}^{(r)}(\lambda) = \int_0^{\infty \pm i\epsilon} \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big) \mathcal{B}[\mathcal{C}_P^{(r)}](t)\, \mathrm{d}t \mp \frac{1}{2} \int_\gamma \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big) \mathcal{B}[\mathcal{C}_P^{(r)}](t)\, \mathrm{d}t\,, \tag{6.103}$$

with the same Hankel contour γ as shown in figure 6.1.

As before, the perturbative expansion at large-$\tilde{\lambda}$ is obtained from the first term in the above equation. Similarly to the analysis of (6.97), we expand the Borel transform $\mathcal{B}[\mathcal{C}_P^{(r)}](t)$ for t small and integrate order by order using

$$\int_0^\infty \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big)\, t^k \, \mathrm{d}t = \frac{\Gamma\Big(\frac{k}{2}+1\Big)\Gamma\Big(\frac{k}{2}\Big)\zeta(k)}{4\pi} \tilde{\lambda}^{\frac{1-k}{2}}\,, \tag{6.104}$$

valid for $\mathrm{Re}(k) > 1$ and proven in appendix F. Given the above identity and the definition (6.45) for the Borel transform, we arrive at the formal asymptotic power series expansion valid for large-$\tilde{\lambda}$:

$$\int_0^{\infty\pm i\epsilon} \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big)\, \mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t \sim \sum_{k=0}^{\infty} b_{r,k}\,\xi(2k+2)\Big(\frac{\tilde{\lambda}}{4\pi}\Big)^{-k-\frac{1}{2}}. \tag{6.105}$$

Using (F.10), we rewrite (6.105) as a Borel resummation of a modified Borel transform

$$\int_0^{\infty\pm i\epsilon} \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big)\, \mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t = \sum_{n=1}^{\infty} \frac{\tilde{\lambda}}{\pi} \int_0^{\infty\pm i\epsilon} n\, e^{-2nx\sqrt{\tilde{\lambda}}}\, \widehat{\mathcal{B}}[\mathcal{C}_P^{(r)}](x)\,\mathrm{d}x\,, \tag{6.106}$$

$$\widehat{\mathcal{B}}[\mathcal{C}_P^{(r)}](x) := \int_0^x \frac{x}{t\sqrt{x^2-t^2}} \mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t\,. \tag{6.107}$$

Substituting the $r = 0$ coefficients (6.36) or the $r = 1$ coefficients (6.68), one can check that (6.105) reduces respectively to the dual 't Hooft expansion at genus-0 or genus-1 of the integrated correlator presented in [48, 66]. In particular, the integral representation (6.106) in terms of a modified Borel transform[5] (6.107) of the original $\mathcal{B}[\mathcal{C}_P^{(r)}](t)$ is identical (modulo some integration by parts) to the Borel integral-representation presented in [66] for the dual 't Hooft genus expansion.

As in the previous decomposition (6.95), we find that second term in (6.103) encodes non-perturbative effects at large $\tilde{\lambda}$ and can be computed from our general analysis of the singular behaviour (6.48) of the Borel transform $\mathcal{B}[\mathcal{C}_P^{(r)}](t)$, along the Stokes direction $t > 0$. The polar and logarithmic parts can be rewritten as

$$\begin{aligned}
&\mp \frac{1}{2}\int_\gamma \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big)\, \mathcal{B}[\mathcal{C}_P^{(r)}](t)\,\mathrm{d}t \\
&= \mp i \sum_{k=1}^{3r+1} d_{r,-k}(k-1)! \oint_{|t|=1} \mathcal{U}\Big(t; \frac{\tilde{\lambda}}{4\pi}\Big) \frac{1}{(1-t)^k}\frac{\mathrm{d}t}{2\pi i} \pm i \sum_{k=0}^{\infty} \frac{d_{r,k}}{k!}\int_0^\infty \mathcal{U}\Big(t+1; \frac{\tilde{\lambda}}{4\pi}\Big)\, t^k\,\mathrm{d}t\,,
\end{aligned} \tag{6.108}$$

with γ the same Hankel contour in figure 6.1. While it is easy to compute the polar part directly from (F.9) or (F.10), we have not been able to find closed-form expressions akin to the 't Hooft limit analogues (6.100) and (6.101).

[5] We note that the net effect of this modified Borel transform is to simply multiply the Taylor coefficients of the Borel transform by a ratio of gamma functions, i.e. $\int_0^x \frac{x}{t\sqrt{x^2-t^2}} t^{2k+2}\,\mathrm{d}t = \frac{\sqrt{\pi}\Gamma(k+1)}{2\Gamma\left(k+\frac{3}{2}\right)} x^{2k+2}$.

However, it is a straightforward task to plug in the above expression the genus-0 non-perturbative coefficients (6.88), or similarly for genus-1 (6.73), and upon expanding at large-$\tilde{\lambda}$ retrieve the necessary non-perturbative completions to the formal perturbative expansion (6.97), obtained in [48,66] via resurgence analysis applied directly to (6.105). This is expected given that (6.106) relates the integral transform with kernel $\mathcal{U}(t;x)$ to a more standard Borel resummation for the modified Borel transform $\widehat{\mathcal{B}}[\mathcal{C}_P^{(r)}](x)$ given in (6.107). The non-perturbative terms in the dual 't Hooft genus expansion which are encoded in (6.108) can then be recovered directly from the directional Borel resummation (6.106), precisely as discussed in [66].

As previously stated, our analysis shows that the modular invariant median resummation (6.79) of the "r-subleading"-index non-holomorphic Eisenstein series (6.34), nicely encodes the median resummation of the genus-r large 't Hooft expansion and dual large 't Hooft expansion.

As a last comment, we notice that when we substitute the dual 't Hooft perturbative expansion (6.105) in the complete correlator $\mathcal{I}(\lambda)$, given in (6.94), this can rewritten as a series in positive powers of λ of the form:

$$\sum_{r'=0}^{\infty} N^{1-2r'} \sum_{k=0}^{\infty} b_{r',k}\, \xi(2k+2) \Big(\frac{\tilde{\lambda}}{4\pi}\Big)^{-k-\frac{1}{2}} = \sum_{r=0}^{\infty} N^{2-2r} \sum_{k=0}^{r-1} b_{r-k-1,k}\, \xi(2k+2) \left(\frac{\lambda}{4\pi}\right)^{k+\frac{1}{2}}, \qquad (6.109)$$

where we changed the summation variable $r' = r-k-1$, thus retrieving the "missing" powers of λ from the complete perturbative genus-r contribution in the 't Hooft limit, i.e. we have recovered the known perturbative genus expansion:

$$\mathcal{I}(N;\lambda) \sim \sum_{r=0}^{\infty} N^{2-2r} \mathcal{T}_P^{(r)}(\lambda)\,, \qquad (6.110)$$

$$\mathcal{T}_P^{(r)}(\lambda) := \sum_{k=0}^{r-1} b_{r-k-1,k}\, \xi(2k+2) \left(\frac{\lambda}{4\pi}\right)^{k+\frac{1}{2}} + \sum_{k=0}^{\infty} b_{r,k}\, \xi(2k+3) \Big(\frac{\lambda}{4\pi}\Big)^{-k-\frac{3}{2}}. \qquad (6.111)$$

This particular combination of positive and negative powers of λ is a direct consequence of having rearranged the perturbative large-N expansion (6.32), where each non-holomorphic Eisenstein series (6.15) at large τ_2 contributes $E^*(s;\tau) \sim \xi(2s)\tau_2^s + \xi(2s-1)\,\tau_2^{1-s}$.

If we extend the definition of $b_{r,k}$ for $r \in \mathbb{N}$ to negative values of k as

$$\hat{b}_{r,k} \coloneqq \begin{cases} b_{r+k+1,-k-2}\,, & k \in \mathbb{Z}\,, k \leq -2\,, \\ 0\,, & k = -1\,, \\ b_{r,k}\,, & k \in \mathbb{N}\,, \end{cases} \tag{6.112}$$

and make use of the functional equation $\xi(s) = \xi(1-s)$, we can rewrite (6.111) in the uniform manner

$$\mathcal{T}_P^{(r)}(\lambda) = \sum_{k=-r-1}^{\infty} \hat{b}_{r,k}\, \xi(2k+3) \Big(\frac{\lambda}{4\pi}\Big)^{-k-\frac{3}{2}}\,. \tag{6.113}$$

Note that the would-be singular term $\xi(1)$ does not appear in the above expression since it multiplies $\hat{b}_{r,-1} = 0$. Interestingly, the coefficients $\hat{b}_{r,k}$ are in fact identical to the continuation of $b_{r,k}$ to negative values of k.

The first non-trivial example of this fact appears at genus $r = 2$ for which the coefficients $b_{2,k}$ have been computed in [81] and are given by

$$b_{2,k} = \frac{(k+1)^2(20k^2+208k+219)\Gamma(k+\frac{5}{2})\Gamma(k+\frac{11}{2})}{2^{2k+12}\, 45\pi^{\frac{3}{2}}\Gamma(k+4)}\,. \tag{6.114}$$

Given the definition (6.112) and the explicit genus-0 and genus-1 coefficients (6.36)-(6.68) we obtain directly

$$\hat{b}_{2,-1} = 0\,, \qquad \hat{b}_{2,-2} = b_{1,0} = -\frac{39}{2048\sqrt{\pi}}\,, \qquad \hat{b}_{2,-3} = b_{0,1} = \frac{15}{32\sqrt{\pi}}\,, \tag{6.115}$$

while all other $\hat{b}_{2,k} = b_{k+3,-k-2} = 0$ for $k \leq -4$ since $b_{r,k}$ with $r < 0$ vanishes. Surprisingly, if we substitute in (6.114) negative values of k we find precisely these numbers, i.e. we have $\hat{b}_{2,k} = b_{2,k}$. We have confirmed that the equality $\hat{b}_{r,k} = b_{r,k}$ for all $k \in \mathbb{Z}$, in particular for $k < 0$, seems to persist at higher genus $r \geq 3$, however we do not have a proof of this statement, nor we understand the reasons behind it.

Since the analysis for the large-N 't Hooft expansion of the sector $\mathcal{H}_N^h(\tau)$ for the second integrated correlator is pretty much identical to the above discussion, we will not repeat it here. However, we want to highlight that the same analytic continuation to negative k of the perturbative coefficients $a_{r,k}$ for the "r-subleading index" non-Holomorphic Eisenstein series sector $\mathcal{H}_h^{(r)}(N;\tau)$ still seems to hold. That is if we take the analytic expressions for the coefficients

$a_{r,k}$, which following [68] can be computed recursively in r from the $b_{r,k}$, and continue them to negative values of k exactly as in (6.112), i.e. for all the values of r studied we find that $a_{r,-1} = 0$ and $a_{r,k} = a_{r+k+1,-k-2}$ for $k \in \mathbb{Z}\,, k \leq -2$ where again $a_{r,k} = 0$ for $r < 0$.

6.8 Transseries from spectral representation

In this section we show that the transseries (6.13) can also be neatly derived starting from the spectral representation (6.23) given in terms of the spectral overlap, $M_N(s)$, presented in (6.31). Firstly we show that the large-N expansion of (6.23) naturally leads to a Borel resummed version of (6.40), thus demonstrating that the series over $r \in \mathbb{N}$ of all sectors $N^{2-2r}\mathcal{C}^{(r)}(N;\tau)$ is indeed Borel summable as previously stated. From here we derive the spectral representation of $\mathcal{C}^{(r)}(N;\tau)$ for arbitrary r, thus reinterpreting the perturbative piece (6.34) as the polar contributions to the spectral integral, while the non-perturbative terms (6.37) arise as contributions from infinity. In this way, we produce a beautiful spectral-integral representation for the full transseries (6.40).

6.8.1 Spectral representation at large-N

We start by analysing the large-N expansion of the spectral representation (6.23) for the integrated correlator $\mathcal{C}_N(\tau)$. Given the functional identity $E^*(s;\tau) = E^*(1-s;\tau)$ of the non-holomorphic Eisenstein series (6.15) and the symmetry $M_N(s) = M_N(1-s)$ of the spectral overlap (6.31), at the price of losing manifest symmetry under $s \leftrightarrow 1-s$ we simply write

$$\mathcal{C}_N(\tau) = \frac{N(N-1)}{4} + \int_{\mathrm{Re}(s)=\frac{1}{2}} \widetilde{M}_N(s) E^*(s;\tau) \frac{\mathrm{d}s}{2\pi i}\,, \tag{6.116}$$

$$\widetilde{M}_N(s) := \frac{2^{1-2s}(2s-1)\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\,\Gamma(-s)} \int_0^1 x^{s-3}(1-x)^N {}_2F_1(s-1,s;2s|x)\mathrm{d}x\,, \tag{6.117}$$

where $M_N(s) = \frac{1}{2}(\widetilde{M}_N(s) + \widetilde{M}_N(1-s))$.

By changing integration variable in (6.117) to $x = 1 - e^{-\mu}$ with $\mu \in [0,\infty)$, the spectral overlap $\widetilde{M}_N(s)$ takes immediately the form of a standard Borel resummation in N,

$$\widetilde{M}_N(s) = \frac{2^{1-2s}(2s-1)\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\,\Gamma(-s)} \int_0^\infty e^{-\mu N} \mu^s B(s;\mu)\,\mathrm{d}\mu\,, \tag{6.118}$$

$$B(s;\mu) := \mu^{-3}e^{-\mu}\left(\frac{1-e^{-\mu}}{\mu}\right)^{s-3} {}_2F_1\left(s-1,s;2s|1-e^{-\mu}\right) , \tag{6.119}$$

where we notice for future reference that $\arg(\mu) = 0$ is not a Stokes direction for $B(s;\mu)$ and that $B(s;\mu) = -B(s;-\mu)$ as a consequence of the hypergometric function identity:

$$ {}_2F_1(a,b;c|x) = (1-x)^{-a}\, {}_2F_1(a,c-b;c|\frac{x}{x-1}) . \tag{6.120}$$

As one can see by direct calculation, the simple expression (6.118) solves identically the Laplace difference equation (6.29) after some integrations by parts.

The Borel transform $B(s;\mu)$ has an expansion for small μ of the form

$$B(s;\mu) = \mu^{-3} - \frac{s(s+5)}{24(2s+1)}\mu^{-1} + \frac{(s+2)(s+3)\left(5s^2+37s-12\right)}{5760(2s+1)(2s+3)}\mu + O(\mu^3) . \tag{6.121}$$

In particular, we notice the potentially singular behaviour of μ^{s-3} at the origin of the Borel μ-plane in (6.118). However, it is easy to check that the regularisation procedure discussed in (E.10), amounts to regarding the term μ^s in (6.118) as a regulator to perform the Borel integral term by term and only after integration taking s to lie on critical strip $\mathrm{Re}(s) = 1/2$.

We then compute the μ-integral in (5.7) by expanding the Borel transform for small μ and then integrate term by term arriving at the formal power series expansion

$$\widetilde{M}_N(s) = \sum_{r=0}^{2-2r} N^{2-2r-s}\mathcal{M}^{(r)}(s) , \tag{6.122}$$

where for the first two orders we have

$$\mathcal{M}^{(0)}(s) = \frac{2^{-2s}(2s-1)^2\Gamma(s)\Gamma(s+1)}{\sqrt{\pi}\Gamma\left(s+\frac{1}{2}\right)}\,\frac{\tan(\pi s)}{(s-1)(s-2)} , \tag{6.123}$$

$$\mathcal{M}^{(1)}(s) = -\frac{2^{-2s-1}s(s+5)(2s-1)^2\Gamma(s)\Gamma(s+1)}{24\sqrt{\pi}\Gamma\left(s+\frac{3}{2}\right)}\tan(\pi s) , \tag{6.124}$$

while for $r \geq 2$ we find the general form

$$\mathcal{M}^{(r)}(s) = \frac{2^{-2s-4r}(2s-1)^2\Gamma(s+2r)\Gamma(s+1)}{\sqrt{\pi}\,\Gamma\left(s+r+\frac{1}{2}\right)}P^{(r)}(s)\tan(\pi s) , \tag{6.125}$$

for some polynomials $P^{(r)}(s)$ of degree $2r-2$. For example we find

$$P^{(2)}(s) = \frac{5s^2+37s-12}{90}\,, \qquad P^{(3)}(s) = -\frac{35s^4+462s^3+1153s^2+750s-720}{5670}\,,$$
$$P^{(4)}(s) = \frac{175s^6+3745s^5+24579s^4+71327s^3+84086s^2+12648s-60480}{340200}\,. \tag{6.126}$$

A key observation about the functions $\mathcal{M}^{(r)}(s)$ is that they are all analytic functions in the strip $1/2 < \mathrm{Re}(s) < 3/2$ and they all have a simple zero at $s=1$, apart from the case $r=0$ for which we have

$$\lim_{s\to 1}\mathcal{M}^{(0)}(s) = -\frac{1}{2}\,. \tag{6.127}$$

This means that if we substitute the large-N expansion for the spectral overlap (6.122) in the integral representation (5.7), we can push the contour of integration to $\mathrm{Re}(s) = 1+\epsilon$ with $0<\epsilon<\frac{1}{2}$.

Only for the case $r=0$ we have to be careful, since by doing so we pick up the residue at $s=1$ coming from the non-zero limit $\lim_{s\to1}\mathcal{M}^{(0)}(s) = -\frac{1}{2}$ multiplying the simple pole of the non-holomorphic Eisenstein series (6.15), which in our normalisation has residue

$$\mathrm{res}_{s=1}E^*(s;\tau) = \frac{1}{2}\,. \tag{6.128}$$

This residue at $s=1$, coming from the case $r=0$, combines with the constant term $N(N-1)/4$ so that (5.7) can be rewritten as

$$\mathcal{C}_N(\tau) = \frac{N^2}{4} + \sum_{r=0}^{\infty} N^{2-2r}\int_{\mathrm{Re}(s)=1+\epsilon}\mathcal{M}^{(r)}(s)N^{-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}\,, \tag{6.129}$$

with $0<\epsilon<\frac{1}{2}$.

Few comments are in order at this point:

(i) The expansion in even powers of N (modulo the Mellin-like term N^{-s}) is a direct consequence of the previously noted fact that the Borel transform is an odd function of the Borel variable μ;

(ii) By comparing the expansion (6.129) with (6.40) it appears manifest, and it is proven in the next section, that $\mathcal{M}^{(r)}(s)N^{-s}$ must be the spectral overlap of $\mathcal{C}^{(r)}(N;\tau)$;

(iii) Once we identify (6.40) with the spectral representation for (6.129), we deduce, as previously stated, that the sum over r in (6.40) is Borel summable: the resummation over $r \in \mathbb{N}$ of the asymptotic series of spectral overlaps (6.122) is understood via the standard Borel transform (6.118).

We now move to show that (6.129) in fact provides the spectral representation of the modular invariant transseries (6.40).

6.8.2 Large-N transseries from a spectral perspective

By comparing the large-N spectral representation (6.129) with the previously analysed modular invariant transseries representation (6.40), we immediately see that the spectral representation for each $\mathcal{C}^{(r)}(N;\tau)$ must take the form

$$\mathcal{C}^{(r)}(N;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \mathcal{M}^{(r)}(s) N^{-s} E^*(s;\tau) \frac{\mathrm{d}s}{2\pi i}\,, \tag{6.130}$$

where the spectral overlaps, $\mathcal{M}^{(r)}(s)$, are precisely the ones we obtained from the large-N expansion (6.122).

To prove that (6.130) is in fact correct, we relate this expression directly with the median resummation formula (6.79), for example we show that when $r = 0$ the expression (6.130) reduces identically to (6.66). To this end we make use of the integral representation (6.43) for $N^{-s}E^*(s;\tau)$, presently appropriate since $\mathrm{Re}(s) = 1+\epsilon$, and we rewrite (6.130) as to obtain

$$\mathcal{C}^{(r)}(N;\tau) = \int_0^\infty \mathcal{E}(\sqrt{N}t;\tau)\Big[\int_{\mathrm{Re}(s)=1+\epsilon} (4t)^{2s-1}\frac{2\Gamma(s)\mathcal{M}^{(r)}(s)}{\Gamma(2s)}\frac{\mathrm{d}s}{2\pi i}\Big]\,\mathrm{d}t\,. \tag{6.131}$$

By comparing this expression with the median resummation formula (6.79), we conclude that for $t > 0$ we must have

$$\mathrm{Re}\Big(\mathcal{B}[\mathcal{C}_P^{(r)}](t)\Big) = \int_{\mathrm{Re}(s)=1+\epsilon} (4t)^{2s-1}\frac{2\Gamma(s)\mathcal{M}^{(r)}(s)}{\Gamma(2s)}\frac{\mathrm{d}s}{2\pi i}\,, \tag{6.132}$$

i.e. the Borel transform for the median resummation is related to the inverse Mellin transform of the spectral overlap $\mathcal{M}^{(r)}(s)$.

For concreteness, let us consider the explicit cases $r = 0$ presented in (6.123), for $r > 0$ the story is identical with just slightly different expressions (6.124)-(6.125). Starting from (6.123), we need to compute:

$$\int_{\mathrm{Re}(s)=1+\epsilon} (4t)^{2s-1} \frac{2\Gamma(s)\mathcal{M}^{(0)}(s)}{\Gamma(2s)} \frac{\mathrm{d}s}{2\pi i} = \int_{\mathrm{Re}(s)=1+\epsilon} t^{2s-1} \frac{4\Gamma(s-2)\Gamma(s+1)}{\Gamma\left(s-\frac{1}{2}\right)^2} \tan(\pi s) \frac{\mathrm{d}s}{2\pi i} \,. \tag{6.133}$$

These integrals can be understood as inverse Mellin transforms in the s variable and can be evaluated by closing the contour of integration in a suitable manner. For $0 < t < 1$ we must close the contour of integration to the right half-plane $\mathrm{Re}(s) > 1$, hence picking up the simple poles coming from $\tan(\pi s)$ and located at $s = n + 1/2$ with $n \in \mathbb{N}^{>0}$. Similarly, for $t > 1$ the contour of integration must be closed in the left half-plane $\mathrm{Re}(s) < 0$. The poles from $\tan(\pi s)$ are now compensated by the gamma functions at denominator, and we are left with the simple poles located at $s = -n$ with $n \in \mathbb{N}^{>0}$ coming from the double poles of the gamma functions at numerator combined with the simple zeroes of $\tan(\pi s)$. In both cases it is possible to show that the contribution at infinity vanishes.

Proceeding as just described, we see that (6.133) is given by

$$\int_{\mathrm{Re}(s)=1+\epsilon} (4t)^{2s-1} \frac{2\Gamma(s)\mathcal{M}^{(0)}(s)}{\Gamma(2s)} \frac{\mathrm{d}s}{2\pi i} = \begin{cases} -6t^2 \, {}_2F_1\left(-\frac{1}{2}, \frac{5}{2}; 1 \middle| \, t^2\right) \,, & 0 < t < 1 \,, \\ -\frac{3}{8t^3} \, {}_2F_1\left(\frac{5}{2}, \frac{5}{2}; 4 \, \middle| \, t^{-2}\right) \,, & t \geq 1 \,. \end{cases} \tag{6.134}$$

Since we want to show that the spectral representation (6.131) coincides with the median transseries resummation, then for $r = 0$ we must have that (6.134) equals (6.66). In particular, we see that (6.134) is identical to the Borel transform $\mathcal{B}[\mathcal{C}^{(0)}](t)$ given in (6.59) for $0 < t < 1$. Furthermore, as already discussed in detail we know that $\mathcal{B}[\mathcal{C}^{(0)}](t)$ has a branch-cut singularity starting at $t = 1$ with a discontinuity (6.60) which is purely imaginary for $t > 1$. From the integral representation for the hypergeometric function we can then derive that for $t \in [0, \infty)$ we have

$$\begin{aligned} \mathrm{Re}\Big(\mathcal{B}[\mathcal{C}^{(0)}](t)\Big) &= \frac{1}{2}(-6t^2) \lim_{\epsilon \to 0^+} \left[{}_2F_1\left(-\frac{1}{2}, \frac{5}{2}; 1 | (t+i\epsilon)^2\right) + {}_2F_1\left(-\frac{1}{2}, \frac{5}{2}; 1 | (t-i\epsilon)^2\right) \right] \\ &= \begin{cases} -6t^2 \, {}_2F_1\left(-\frac{1}{2}, \frac{5}{2}; 1 \middle| \, t^2\right) \,, & 0 < t < 1 \,, \\ -\dfrac{3}{8t^3} \, {}_2F_1\left(\frac{5}{2}, \frac{5}{2}; 4 \, \middle| \, t^{-2}\right) \,, & t \geq 1 \,, \end{cases} \end{aligned} \tag{6.135}$$

hence we conclude that indeed (6.134) is identical to $\mathrm{Re}\Big(\mathcal{B}[\mathcal{C}^{(0)}](t)\Big)$. For higher values of $r > 0$, we have checked that indeed the spectral representation (6.130) coincides precisely with the modular invariant median resummation (6.79) for $\mathcal{C}^{(r)}(N;\tau)$.

Before concluding, we also want to clarify how to extract directly from the spectral representation (6.130) the formal transseries expansion (6.41) in terms of the perturbative, non-holomorphic Eisenstein series part (6.34) and the non-perturbative sector (6.39). We again focus concretely on the case $r = 0$, although everything we say can be applied to arbitrary r. Substituting the expression (6.123) for $\mathcal{M}^{(0)}(s)$ in (6.130) and massaging some of the gamma functions we arrive at

$$\mathcal{C}^{(0)}(N;\tau) = \int_{\mathrm{Re}(s)=1+\epsilon} \frac{2^{2-4s}\Gamma(2s)}{\Gamma(s)} \frac{2\Gamma(s-2)\Gamma(s+1)}{\Gamma\left(s-\frac{1}{2}\right)^2} \tan(\pi s) N^{-s} E^*(s;\tau) \frac{\mathrm{d}s}{2\pi i}\,. \tag{6.136}$$

The perturbative part (6.35) is clearly obtained by formally closing the contour of integration to the right half-plane, $\mathrm{Re}(s) > 1$, and summing over minus (due to the orientation of the integration contour) the residues from the poles coming from $\tan(\pi s)$ and located at $s = k+3/2$ with $k \in \mathbb{N}$. A simple residue calculation immediately shows

$$-\mathrm{res}_{s=k+\frac{3}{2}}\left[\frac{2^{2-4s}\Gamma(2s)}{\Gamma(s)} \frac{2\Gamma(s-2)\Gamma(s+1)}{\Gamma\left(s-\frac{1}{2}\right)^2} \tan(\pi s) N^{-s} E^*(s;\tau)\right] = b_{0,k} N^{-\frac{3}{2}-k} E^*(k+\tfrac{3}{2};\tau), \tag{6.137}$$

for $k \in \mathbb{N}$ and where the coefficients $b_{0,k}$ are exactly the ones given in (6.36).

While the formal sum over all residues on the positive real s-axis reproduces the perturbative series $\mathcal{C}_P^{(0)}(N;\tau)$ in (6.35), the non-perturbative sector $\mathcal{C}_{NP}^{(0)}(N;\tau)$ in (6.87) is encoded in the formal contribution at infinity. This can be made explicit by first substituting in (6.136) the lattice sum representation (6.15) for the non-holomorphic Eisenstein series $E^*(s;\tau)$, and then evaluating the s-integral via a saddle-point analysis. One can easily see that at large-N the spectral representation (6.136) behaves as

$$\begin{aligned}&\frac{2^{2-4s}\Gamma(2s)\Gamma(s-2)\Gamma(s+1)}{\Gamma\left(s-\frac{1}{2}\right)^2}\left(NY_{mn}(\tau)\right)^{-s}\\ &\sim \exp\left[2s\Big(\log s - \log(2\sqrt{NY_{mn}(\tau)}) - 1\Big)\right]\frac{4\sqrt{\pi}}{\sqrt{s}}\Big(1+\frac{55}{24s}+O(s^{-2})\Big)\,.\end{aligned} \tag{6.138}$$

In the limit $N \gg 1$, the argument of the exponential function has a saddle point located at $s = s_\star$ with $|s_\star| \gg 1$:

$$s_\star := 2\sqrt{NY_{mn}(\tau)}\,. \tag{6.139}$$

By evaluating (6.138) at the saddle point $s = s_\star$, we find that the leading growth is given by $\exp(-4\sqrt{NY_{mn}(\tau)})$, i.e. precisely the exponential suppression factor of the non-perturbative $D_N(s;\tau)$ modular functions (6.17).

We notice furthermore, that the term $\tan(\pi s)$ in (6.136) is the realisation of the transseries parameter σ in (6.65). If we evaluate this factor at the saddle location (6.139), we see that it reduces to

$$\tan(\pi s_\star) \overset{|N|\gg 1}{\longrightarrow} \begin{cases} +i\,, & \arg(N) > 0\,, \\ -i\,, & \arg(N) < 0\,. \end{cases} \tag{6.140}$$

It is straightforward to expand around the saddle point by changing integration variables to

$$s = s_\star + i\,(NY_{mn})^{\frac{1}{4}}\delta s\,, \tag{6.141}$$

so that the integral representation (6.136) reduces at large-N to a gaussian integral in the fluctuations δs timed by an infinite formal series of perturbative corrections, which can be evaluated to arbitrary high order in $(NY_{mn})^{-1}$ thus recovering the non-perturbative sector $\mathcal{C}^{(0)}_{NP}(N;\tau)$ previously obtained in (6.87) via resurgent analysis arguments.

We conclude with a quicker and suggestive, albeit not completely rigorous way of showing that the non-perturbative coefficients $d_{r,k}$ of the non-perturbative sector (6.78) are encoded directly at the level of spectral overlap $\mathcal{M}^{(r)}(s)$. Focusing again on the showcase example where $r = 0$, we analyse the integrand of the spectral decomposition (6.136) and we expand it at large-s in the following manner,

$$\frac{2^{2-4s}\Gamma(2s)}{\Gamma(s)}\frac{2\Gamma(s-2)\Gamma(s+1)}{\Gamma\left(s-\frac{1}{2}\right)^2} = \frac{2^{2-4s}\Gamma(2s)}{\Gamma(s)}\left(\sum_{\ell=0}^{\infty}\delta_\ell\,(2s)^{-\ell}\right)\,, \tag{6.142}$$

where the first few δ_ℓ coefficients are given by

$$\delta_0 = 2\,,\qquad \delta_1 = 9\,,\qquad \delta_2 = \frac{153}{4}\,. \tag{6.143}$$

We find that the non-perturbative coefficients $d_{0,k}$ presented in (6.87) are in fact encoded entirely in the above expression via

$$d_{0,k} = \sum_{\ell=0}^{k+1} S_{k+1}^{(\ell)}\, \delta_\ell\,, \tag{6.144}$$

where $S_k^{(\ell)}$ denotes the Stirling number of the first kind. This identity follows from the properties of the Stirling numbers: the particular linear combination of the coefficients δ_ℓ defined in (6.144) allows us to rewrite (6.142) in the alternative large-s expansion

$$\begin{aligned}\frac{2^{2-4s}\Gamma(2s)}{\Gamma(s)}\left(\sum_{\ell=0}^{\infty}\delta_\ell\,(2s)^{-\ell}\right) &= \frac{2^{2-4s}\Gamma(2s)}{\Gamma(s)}\left(\sum_{k=-1}^{\infty} d_{0,k}\prod_{i=1}^{k+1}(2s-i)^{-1}\right)\\ &= \sum_{k=-1}^{\infty} d_{0,k}\frac{2^{2-4s}\Gamma(2s-k-1)}{\Gamma(s)}\,.\end{aligned} \tag{6.145}$$

As a last step, we notice that the particular factor in the summand,

$$\frac{2^{2-4s}\Gamma(2s-k-1)}{\Gamma(s)}\,,$$

is precisely the spectral overlap with the non-holomorphic Eisenstein series of the modular invariant function $N^{-\frac{k+1}{2}}D_N(\frac{k+1}{2};\tau)$, computed in [86] and for general index given by

$$D_N(p;\tau) = \langle D_N(p)\rangle + \int_{\mathrm{Re}(s)=\frac{1}{2}+\epsilon}\frac{2^{2-4s}\Gamma(2s-2p)}{\Gamma(s)}N^{p-s}E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i}\,. \tag{6.146}$$

A similar analysis can be carried out starting from $\mathcal{M}^{(r)}(s)$ with $r\geq 1$ to retrieve the non-perturbative coefficients $d_{r,k}$. While the expansion (6.145) is rather suggestive, we stress that this result does not quite show how to rigorously obtain the non-perturbative sector (6.78) from the spectral representation (6.136), unlike the previous two arguments exploiting either median resummation (6.132) or saddle point analysis.

We find it rather beautiful how the simple spectral representation (6.130) encodes in these various interesting ways the perturbative non-holomorphic Eisenstein series part (6.34), the non-perturbative $D_N(p;\tau)$ sector (6.39) and, as a matter of fact, the complete median resummation transseries representation (6.41).

CHAPTER 7

Conclusions

In this thesis we have discussed the interplay between modularity and resurgence within a string theory context. We have seen how the two ideas are complementary and provide information about non-perturbative physics in distinct yet interrelated ways. In chapters 2 and 3 we supplied the mathematical background needed to understand the thesis. The framework behind the resurgence program was introduced and elementary results from the theory of modular functions and forms were discussed.

In chapter 4 we discussed Modular Graph Functions and how a subclass of them could be expressed in terms of generalised Eisenstein series with integer weights. We applied Cheshire-cat resurgence to the Fourier zero-mode of this subclass of generalised Eisenstein series and demonstrated that median resummation gives the correct answer. Non-trivial cancellations in the limit $y \to 0$ were also observed - a clear manifestation of the modular origin for the function under study.

In chapter 5 we discussed generalised Eisenstein series in a more universal way that also includes applications to the low energy effective action of Type IIB string theory. We showed that the spectra relevant for both MGFs and also higher-derivative corrections may be embedded in the same space of modular functions, and additionally rederived many results from the perspective of $\mathrm{SL}(2,\mathbb{Z})$ spectral theory.

Finally, in chapter 6 we described a modular invariant construction of transseries relevant to a large N study of integrated correlators in $\mathcal{N} = 4$ SYM. The resummation in this framework introduces a novel class of modular functions that serve the rôle of non-perturbative completion - their exact form fixed by the divergence of the perturbative piece. This method also gives elementary access to the 't Hooft limit, of particular relevance to holographic applications.

Both modularity and resurgence are likely to continue playing an important rôle in high energy theory, since they allow us to see beyond the limitation of perturbation theory. Although we have not developed a general theory linking the the two ideas, this thesis provides ample evidence they are connected and a unified approach gives powerful tools to study systems of physical interest.

APPENDIX A

Fourier expansions of Poincaré series

Here, we collect some standard results (see e.g. [9]) on relating the Fourier series of a Poincaré series to that of its seed, following the notation of [31, 34]. We start from the relation

$$\Phi(\tau) = \sum_{\gamma \in \mathrm{B}(\mathbb{Z}) \backslash \mathrm{SL}(2,\mathbb{Z})} \varphi(\gamma \cdot \tau) \,, \tag{A.1}$$

between the Poincaré series $\Phi(\tau)$ and its seed $\varphi(\tau)$ that have the respective Fourier series

$$\Phi(\tau) = \sum_{\ell \in \mathbb{Z}} a_\ell(\tau_2) e^{2\pi i \ell \tau_1} \,, \qquad \varphi(\tau) = \sum_{\ell \in \mathbb{Z}} c_\ell(\tau_2) e^{2\pi i \ell \tau_1} \,, \tag{A.2}$$

with $\tau_1 = \mathrm{Re}(\tau)$ and $\tau_2 = \mathrm{Im}(\tau)$ as usual. The relation between the Fourier coefficients $a_\ell(\tau_2)$ and $c_\ell(\tau_2)$ is given by [9, 13]:

$$a_\ell(\tau_2) = c_\ell(\tau_2) + \sum_{d=1}^{\infty} \sum_{n \in \mathbb{Z}} S(n, \ell; d) \int_{\mathbb{R}} e^{-2\pi i \ell \omega - 2\pi i n \frac{\omega}{d^2(\tau_2^2 + \omega^2)}} c_n\Big(\frac{\tau_2}{d^2(\tau_2^2 + \omega^2)}\Big) \mathrm{d}\omega \,. \tag{A.3}$$

In the above formula, $S(n, \ell; d)$ denotes the Kloosterman sum

$$S(n, \ell; d) = \sum_{r \in (\mathbb{Z}/d\mathbb{Z})^\times} e^{2\pi i (nr + \ell r^{-1})/d} \,, \tag{A.4}$$

where $r \in (\mathbb{Z}/d\mathbb{Z})^\times$ denotes the finite sum over all $0 \le r < d$ that are coprime with d. If r is coprime with d it has a multiplicative inverse, denoted by r^{-1}, in $(\mathbb{Z}/d\mathbb{Z})^\times$.

The main focus for us is the Fourier zero mode $a_0(\tau_2)$ for which (A.3) specialises to

$$\begin{aligned} a_0(\tau_2) &= c_0(\tau_2) + \sum_{d=1}^{\infty}\sum_{n\in\mathbb{Z}}\sum_{r\in(\mathbb{Z}/d\mathbb{Z})^\times} e^{2\pi i n r/d}\int_{\mathbb{R}} e^{-2\pi i n\frac{\omega}{d^2(\tau_2^2+\omega^2)}} c_n\Big(\frac{\tau_2}{d^2(\tau_2^2+\omega^2)}\Big)\mathrm{d}\omega \\ &= I_0(\tau_2) + I(\tau_2)\,. \end{aligned} \tag{A.5}$$

As indicated in the second line it is useful to separate the contributions of c_0 from those of the c_n with $n \neq 0$, where we defined (changing also variables according to $\omega = \tau_2\, t$)

$$\begin{aligned} I_0(\tau_2) &= c_0(\tau_2) + \tau_2\sum_{d=1}^{\infty}\sum_{r\in(\mathbb{Z}/d\mathbb{Z})^\times}\int_{\mathbb{R}} c_0\Big(\frac{1}{\tau_2 d^2(1+t^2)}\Big)\mathrm{d}t\,, \\ I(\tau_2) &= \tau_2\sum_{d=1}^{\infty}\sum_{n\neq 0}\sum_{r\in(\mathbb{Z}/d\mathbb{Z})^\times} e^{2\pi i n r/d}\int_{\mathbb{R}} e^{-2\pi n\frac{it}{\tau_2 d^2(1+t^2)}} c_n\Big(\frac{1}{\tau_2 d^2(1+t^2)}\Big)\mathrm{d}t\,. \end{aligned} \tag{A.6}$$

In this appendix we only consider Poincaré seeds of a restricted functional form. More precisely, all seeds relevant for MGFs at two-loop order are given by (finite) linear combination of the basic objects

$$c_0(y) = (\pi\tau_2)^r = y^r\,, \tag{A.7a}$$

$$c_\ell(y) = \sigma_a(|\ell|)(4\pi|\ell|)^b\tau_2^r e^{-2\pi|\ell|\tau_2} = \sigma_a(|\ell|)(4\pi|\ell|)^b(y/\pi)^r e^{-2|\ell|y}\,, \tag{A.7b}$$

with $a, b, r \in \mathbb{C}$ and $y = \pi\tau_2$. Their contributions to the Laurent polynomial in $a_0(\tau)$ were found in [31] to be

$$I_0(r|y) = y^r + \frac{(-16)^{1-r}(2r)!(2r-3)!}{\mathrm{B}_{2r}(r-2)!(r-1)!}\zeta(2r-1)y^{1-r}\,, \tag{A.8a}$$

$$
\begin{aligned}
I(a,b,r|y) = \frac{2^{3-2r+2b}\pi}{\Gamma(r)}\Big(\frac{y}{\pi}\Big)^{1+b-r}\Bigg[&\frac{y}{\pi^2}\frac{\Gamma(b{+}1)\Gamma(2r{-}b{-}2)}{\Gamma(r{-}b{-}1)}\frac{\zeta(2r{-}a{-}2b{-}2)\zeta(1{-}a)}{\zeta(2r{-}a{-}2b{-}1)} \\
&+\Big(\frac{y}{\pi^2}\Big)^{a+1}\frac{\Gamma(a{+}b{+}1)\Gamma(2r{-}a{-}b{-}2)}{\Gamma(r{-}a{-}b{-}1)}\frac{\zeta(2r{-}a{-}2b{-}2)\zeta(a{+}1)}{\zeta(2r{-}a{-}2b{-}1)} \\
&+\left(\frac{\pi^2}{y}\right)^b\sum_{n\geq 0}\left(\frac{-\pi^2}{y}\right)^n\frac{\Gamma(2r{+}n{-}1)}{n!\cdot\Gamma(r{+}n)} \\
&\qquad\times\frac{\zeta(-b{-}n)\zeta(-a{-}b{-}n)\zeta(2r{-}a{-}b{+}n{-}1)\zeta(2r{-}b{+}n{-}1)}{\zeta(2r{+}2n)\zeta(2r{-}a{-}2b{-}1)}\Bigg],
\end{aligned}
\tag{A.8b}
$$

In view of (3.20), the result I_0 for a seed $c_0(y) = y^r$ is proportional to that of the standard non-holomorphic Eisenstein series $\mathcal{E}_r$.

The above results (A.8) were obtained originally in the range of parameters where the integrals and series converge. Nevertheless, we shall also require their values at analytically continued parameter values and refer the reader to [31, 34] for details on these analytic continuations.

APPENDIX B

A Mellin transform Lemma

In this appendix we will derive the asymptotic expansion near $y \to 0$ of the series:

$$D_{a,b;c}(y) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^c} e^{-ny}, \tag{B.1}$$

with $a, b, c \in \mathbb{C}$ while $y > 0$. Notice that this series is absolutely convergent for any $y > 0$ since $|\sigma_a(n)\sigma_b(n)n^{-c}| \leq n^{2+|a|+|b|+|c|}$. Furthermore, using $\sigma_a(n) = n^a \sigma_{-a}(n)$ we have

$$D_{a,b;c}(y) = D_{-a,b;c-a}(y) = D_{a,-b;c-b}(y) = D_{-a,-b;c-a-b}(y)\,.$$

To proceed, we wish to evaluate the Mellin transform

$$M_{a,b;c}(t) := \mathcal{M}[D_{a,b;c}](t) = \int_0^\infty D_{a,b;c}(y) y^{t-1} \mathrm{d}y. \tag{B.2}$$

Since the series (B.1) is exponentially suppressed as $y \to \infty$, we conclude that for sufficiently large $\mathrm{Re}(t) > t_0$ the integral converges absolutely. Hence, when $\mathrm{Re}(t) > t_0$, we can commute the sum with the integral and integrate term by term. After using the standard Ramanujan

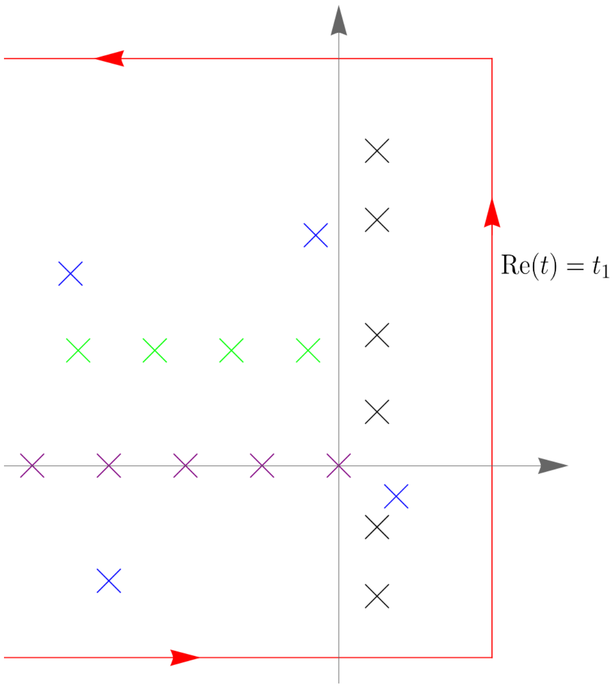

Figure B.1: Deformed t-integration contour for (B.4) and pole structure for the Mellin transform $M_{a,b;c}(t)$. For $a, b, c \in \mathbb{C}$ generic, there are three infinite families of poles: $t = -n\,,\ n \in \mathbb{Z}_{\geq 0}$, in purple, $t = -n - c + \frac{a+b}{2}\,,\ n \in \mathbb{Z}_{>0}$, in green and $t = \frac{1}{4} - c + \frac{a+b}{2} + \frac{i\rho_n}{2}$ in black, as well as four isolated poles for $t \in \{1-c, 1+a-c, 1+b-c, 1+a+b-c\}$ in blue.

identity (4.33) we obtain

$$M_{a,b;c}(t) = \frac{\Gamma(t)\zeta(t+c)\zeta(t+c-a)\zeta(t+c-b)\zeta(t+c-a-b)}{\zeta(2t+2c-a-b)}. \tag{B.3}$$

Although we derived this equation working in the wedge $\mathrm{Re}(t) > t_0$, we have that (B.3) is actually the unique meromorphic extension of $M_{a,b;c}(t)$ to the whole complex plane $t \in \mathbb{C}$.

The asymptotic expansion as $y \to 0$ of $D_{a,b;c}(y)$ is uniquely fixed by the singularities in t of its Mellin transform $M_{a,b;c}(t)$. To make this more precise we consider Mellin inversion formula

$$D_{a,b;c}(y) = \mathcal{M}^{-1}[M_{a,b;c}](y) = \frac{1}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty} M_{a,b;c}(t) y^{-t} \mathrm{d}t, \tag{B.4}$$

where $t_1 > t_0$ is arbitrary. The asymptotic expansion as $y \to 0$ of (B.4) can now be computed by closing the contour into a loop with $\mathrm{Re}(t) < 0$, as depicted in figure B.1, and evaluating it using Cauchy's residue theorem (and discarding exponentially suppressed corrections $e^{-4\pi^2/y}$).

Note that the Mellin transform (B.3) is a meromorphic function with an infinite number of poles for generic values of a, b, c. Hence from (B.4), we expect the expansion for $y \to 0$

of $D_{a,b;c}(y)$ to be a non-truncating asymptotic expansion. It is easy to see that for generic $a, b, c \in \mathbb{C}$ the Mellin transform (B.3) has only simple poles at locations:

- $t = -n\,,\ n \in \mathbb{Z}_{\geq 0}$, from the gamma function in the numerator;
- $t \in \{1 - c, 1 + a - c, 1 + b - c, 1 + a + b - c\}$, from the zeta functions in the numerator;
- $t = -n - c + \frac{a+b}{2}\,,\ n \in \mathbb{Z}_{>0}$, from the trivial zeros of the zeta function in the denominator.
- $t = \frac{1}{4} - c + \frac{a+b}{2} + \frac{i\rho_n}{2}$, from the non-trivial zeros of the zeta function in the denominator (the Riemann zeta function $\zeta(s)$ has non-trivial zeros at $s = \frac{1}{2} + i\rho_n$ with $\rho_n \in \mathbb{R}$ if the Riemann hypothesis is correct).

Nevertheless, for non-generic values of the parameters a, b, c, we can have that zeros at negative even integers of the zeta functions in the numerator, or the pole at one of the zeta function in the denominator, can cancel against the poles listed above thus leaving a smaller number of perturbative terms. Note that for non-generic values of a, b, c it is also possible to generate higher order poles, thus leading to logarithmic terms, $\log y$, in the asymptotic expansion as $y \to 0$ of $D_{a,b;c}(y)$.

Assuming generic $a, b, c \in \mathbb{C}$, we can compute (B.4) via residues calculus deforming the contour of integration as depicted in Figure B.1 and derive the asymptotic expansion as $y \to 0$ of $D_{a,b;c}(y)$ given by:

$$
\begin{aligned}
&D_{a,b;c}(y) \sim \qquad\qquad \text{(B.5)}\\
&y^{c-1}\frac{\Gamma(1-c)\zeta(1-a)\zeta(1-b)\zeta(1-a-b)}{\zeta(2-a-b)} + y^{c-a-1}\frac{\Gamma(1+a-c)\zeta(1+a)\zeta(1-b)\zeta(1+a-b)}{\zeta(2+a-b)}\\
&+ y^{c-b-1}\frac{\Gamma(1+b-c)\zeta(1-a)\zeta(1+b)\zeta(1-a+b)}{\zeta(2-a+b)}\\
&+ y^{c-a-b-1}\frac{\Gamma(1+a+b-c)\zeta(1+a)\zeta(1+b)\zeta(1+a+b)}{\zeta(2+a+b)}\\
&+ \sum_{\rho_n}\frac{\Gamma(t)\zeta(t+c)\zeta(t+c-a)\zeta(t+c-b)\zeta(t+c-a-b)}{2\zeta'(\frac{1}{2}+i\rho_n)}y^{-t}\Big|_{t=\frac{1}{4}-c+\frac{a+b}{2}+\frac{i\rho_n}{2}}\\
&+ \sum_{n=0}^{\infty}(-y)^n\frac{\zeta(c-n)\zeta(c-a-n)\zeta(c-b-n)\zeta(c-a-b-n)}{n!\zeta(2c-2n-a-b)}\\
&+ \sum_{n=1}^{\infty}y^{c-\frac{a+b}{2}}(-4\pi^2 y)^n\frac{\Gamma(\frac{a+b}{2}-c-n)\zeta(\frac{a+b}{2}-n)\zeta(\frac{b-a}{2}-n)\zeta(\frac{a-b}{2}-n)\zeta(-\frac{a+b}{2}-n)}{(2n)!\zeta(2n+1)}\,.
\end{aligned}
$$

This result is closely related to the small-τ_2 asymptotics of the Fourier zero-mode of the generalised Eisenstein series, shown in (5.82) yet derived using quite a different approach. Note that for generic $a, b, c \in \mathbb{C}$, the final two sums over n are asymptotic, factorially growing series. An interesting exercise would be to use resurgent analysis to derive the exponentially suppressed corrections $e^{-4\pi^2 k/y}$, with $k \in \mathbb{Z}_{>0}$, from a median resummation of such series. As a consistent, full-circle analysis the large-y expansion of such small-y non-perturbative corrections must reproduce back the starting large-y exponentially suppressed series (B.1). The presence of these terms was established using saddle point methods in chapter 5.

APPENDIX C

Convergence of the Poincaré series for $\Upsilon\,(a,b,r,s)$

In this appendix we discuss the region in parameter space, (a,b,r,s), for which the Poincaré series (5.12) converges absolutely. This will be achieved by constructing an auxiliary Poincaré series which has the same domain of absolute convergence but it is easier to analyse.

We start by observing that under a modular transformation $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ the magnitude of the seed functions $\upsilon\,(a,b,r,s|\tau)$ is bounded from above by an x-independent function

$$|\upsilon\,(a,b,r,s|\gamma\cdot\tau)\,| \leq \sum_{m\neq 0}\left|\left[\sigma_a(m)|m|^{b-\frac{1}{2}}\tau_2^{r+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|\tau_2)\right]_\gamma\right|,$$

simple consequence of triangle inequality combined with

$$\left|\left[e^{2\pi i\tau_1}\right]_\gamma\right| = 1\,.$$

Motivated by this observation, we define the auxiliary Poincaré series

$$\psi(a,b,r,s|\tau_2) := \sum_{m=1}^{\infty}\sigma_a(m)m^{b-\frac{1}{2}}\tau_2^{r+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi m\tau_2)\,, \tag{C.1}$$

$$\Psi(a,b,r,s|\tau) := \sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})}\left[\psi(a,b,r,s|\tau_2)\right]_\gamma\,, \tag{C.2}$$

and notice that the auxiliary Poincaré series (C.2) converges absolutely if and only if the original Poincaré series (5.12) does.

We continue by showing that (C.2) can be written in terms of a contour integral thus manifesting the convergence properties of the Poincaré series. Given a function $f(\tau_2)$ we define its Mellin transform as

$$\mathcal{M}[f](t) := \int_0^\infty f(\tau_2)\,\tau_2^t\,\frac{\mathrm{d}\tau_2}{\tau_2}\,, \tag{C.3}$$

and proceed to compute the Mellin transform of our new seed function

$$\begin{aligned}\tilde{\psi}(a,b,r,s|t) &:= \mathcal{M}[\psi(a,b,r,s)](t) = \int_0^\infty \psi(a,b,r,s|\tau_2)\,\tau_2^t\,\frac{\mathrm{d}\tau_2}{\tau_2} \\ &= \frac{1}{4\pi^{t+r+\frac{1}{2}}}\Gamma\Big(\frac{t+r+1-s}{2}\Big)\Gamma\Big(\frac{t+r+s}{2}\Big)\zeta(t+r+1-b)\zeta(t+r+1-a-b)\,,\end{aligned} \tag{C.4}$$

using the identities

$$\int_0^\infty K_s(y)y^b\mathrm{d}y = 2^{b-1}\Gamma\big(\frac{b+1-s}{2}\big)\Gamma\big(\frac{b+s+1}{2}\big)\,, \tag{C.5}$$

$$\sum_{m=1}^\infty \sigma_a(m)m^b = \zeta(-a-b)\zeta(-b)\,. \tag{C.6}$$

The Mellin transform (C.4) is well-defined in the strip

$$\mathrm{Re}(t) > \alpha = \max\big(\mathrm{Re}(s-r-1), \mathrm{Re}(-s-r), \mathrm{Re}(b-r), \mathrm{Re}(a+b-r)\big)\,. \tag{C.7}$$

We can now apply Mellin inversion formula to obtain the integral representation

$$\psi(a,b,r,s|\tau_2) = \mathcal{M}^{-1}[\tilde{\psi}(a,b,r,s)](\tau_2) = \int_{\beta-i\infty}^{\beta+i\infty} \tilde{\psi}(a,b,r,s|t)\,\tau_2^{-t}\frac{\mathrm{d}t}{2\pi i}\,, \tag{C.8}$$

where β is an arbitrary constant such that $\beta > \alpha$. The reason to derive (C.8) is that all of the explicit τ_2 dependence has now been reduced to the simple term τ_2^{-t}. At this point we can easily perform the Poincaré series (C.2) arriving at

$$\Psi(a,b,r,s|\tau) = \int_{\beta-i\infty}^{\beta+i\infty} \tilde{\psi}(a,b,r,s|t)\,E(-t;\tau)\frac{\mathrm{d}t}{2\pi i}\,. \tag{C.9}$$

The absolute convergence of the auxiliary Poincaré series (C.2), and hence of the original

Poincaré series (5.12), is then equivalent to understanding the conditions for which the Poincaré series of the integral representation (C.8) is absolutely convergent. This question is much easier to answer: with (C.8) the problem has been reduced to the convergence of the Poincaré series for Eisenstein series (3.20). We conclude that absolute convergence of (C.2) and (5.12) is guaranteed whenever

$$\mathrm{Re}(-t) = -\beta > 1 \qquad \Rightarrow \qquad \alpha < \beta < -1\,,$$

which, upon use of the condition (C.7) for a well-defined Mellin transform, reproduces precisely the domain in parameters space (5.13) stated in the main text

$$\min\big(\mathrm{Re}(r+1-s), \mathrm{Re}(r+s), \mathrm{Re}(r-b), \mathrm{Re}(r-a-b)\big) > 1\,.$$

It is interesting to note that the integral representation (C.9) implies that the spectral overlap $(\Psi(a,b,r,s), \phi_n)$ vanishes for all Maass cusp forms $\phi_n(z)$; such a result is not expected to hold for the more complicated $\Upsilon\,(a,b,r,s)$.

APPENDIX D

Mellin-Barnes representation

In this appendix we derive a Mellin-Barnes representation for the Fourier zero-mode $\Upsilon_0(a,b,r,s|y)$, starting from the general integral representation (A.5) specialised to the seed function (5.11) under consideration. Hence we start by considering

$$\begin{aligned}&\Upsilon_0(a,b,r,s|\tau_2) \\ &=\sum_{d=1}^{\infty}\sum_{m\neq 0} S(m,0;d)\int_{\mathbb{R}} e^{-2\pi i m\frac{\omega}{d^2(\omega^2+\tau_2^2)}}\sigma_a(m)|m|^{b-\frac{1}{2}}\Big(\frac{\tau_2}{d^2(\omega^2+\tau_2^2)}\Big)^{r+\frac{1}{2}}K_{s-\frac{1}{2}}\Big(\frac{2\pi|m|\tau_2}{d^2(\omega^2+\tau_2^2)}\Big)\mathrm{d}\omega.\end{aligned} \tag{D.1}$$

The Bessel function can now be substituted by its Mellin-Barnes integral representation

$$K_s(y)=\Big(\frac{y}{2}\Big)^s\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(t)\Gamma(t-s)\Big(\frac{y}{2}\Big)^{-2t}\frac{\mathrm{d}t}{4\pi i}, \tag{D.2}$$

where α is a real parameter such that $\alpha>\max\big(\mathrm{Re}(s),0\big)$. To perform the integral over ω we furthermore expand the exponential as

$$e^{-2\pi i m\frac{\omega}{d^2(\omega^2+\tau_2^2)}}=\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{-2\pi i m\omega}{d^2(\omega^2+\tau_2^2)}\Big)^k. \tag{D.3}$$

Substituting both the Mellin-Barnes representation for the Bessel function and the above convergent expansion in (D.1), we obtain

$$\Upsilon_0(a,b,r,s|\tau_2) = \sum_{d=1}^{\infty}\sum_{m\neq 0}\sum_{k=0}^{\infty}\int_{\mathbb{R}}\int_{\alpha-i\infty}^{\alpha+i\infty} S(m,0;d)\sigma_a(m)|m|^{b-\frac{1}{2}}\Big(\frac{\tau_2}{d^2(\omega^2+\tau_2^2)}\Big)^{r+\frac{1}{2}}$$
$$\Big(\frac{-2\pi i m\omega}{d^2(\omega^2+\tau_2^2)}\Big)^k\Big(\frac{\pi|m|\tau_2}{d^2(\omega^2+\tau_2^2)}\Big)^{s-2t-\frac{1}{2}}\frac{\Gamma(t)\Gamma(t-s+\frac{1}{2})}{k!}\frac{\mathrm{d}t\,\mathrm{d}\omega}{4\pi i}\,. \quad \text{(D.4)}$$

The integral over ω can be performed

$$\int_{\mathbb{R}}\frac{\omega^k}{(\omega^2+\tau_2^2)^{k+r+s-2t}}\,\mathrm{d}\omega = \frac{[1+(-1)^k]\tau_2^{4t+1-k-2r-2s}\,\Gamma(\frac{k+1}{2})\Gamma(\frac{k-1}{2}+r+s-2t)}{2\Gamma(k+r+s-2t)}, \quad \text{(D.5)}$$

provided that the integrand falls-off sufficiently fast as $\omega\to\pm\infty$, which in turns requires the parameter α to be bounded from above by $4\alpha < k+2\mathrm{Re}(r+s)-2$.

Under the conditions (5.13) for absolute convergence of the Poincaré series, we can easily see that for all $k\in\mathbb{N}$ the constraints on the parameter α:

$$\max\big(\mathrm{Re}(s),0\big) < \alpha < \frac{k+2\mathrm{Re}(r+s)-1}{4}\,,$$

always admit a non-vanishing strip of allowed integration contours in t.

At this point, we focus on the series in m, d and k. Firstly, given the explicit expression (A.4) for the Kloosterman sum $S(m,0;d)$ we use that $r\in(\mathbb{Z}/d\mathbb{Z})^\times$ implies $-r\in(\mathbb{Z}/d\mathbb{Z})^\times$ to derive $S(m,0;d)=S(-m,0;d)$. We can then replace the sum over all non-zero integers m by twice the sum over the positive integers $m>0$. Secondly, it is possible to evaluate explicitly the sum over d, which takes the form of a well-known Dirichlet series for the Ramanujan sum $S(m,0;d)$,

$$\sum_{d=1}^{\infty}\frac{S(m,0;d)}{d^{\tilde{s}}} = \frac{\sigma_{1-\tilde{s}}(m)}{\zeta(\tilde{s})}\,, \quad \text{(D.6)}$$

specialised to $\tilde{s}=2r+2k+2s-4t$. Finally, we note that the term $[1+(-1)^k]$ in the numerator of (D.5) restricts the sum over k to only run over even integers $2k$.

When the dust settles and after performing the change of variables $t\to\frac{t+r+s-1}{2}$, we are left with the expression

$$\Upsilon_0(a,b,r,s|\tau_2) \tag{D.7}$$
$$=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\int_{\frac12-i\infty}^{\frac12+i\infty}\frac{\sigma_a(m)\sigma_{2t-4k-1}(m)}{m^{t+r-2k-b}}\frac{(-1)^k\pi^{2k+1-r-t}\Gamma(k+\frac12-t)\Gamma\left(\frac{t+r-s}{2}\right)\Gamma\left(\frac{t+r+s-1}{2}\right)}{\Gamma(2k+1-t)\zeta(4k+2-2t)k!}\tau_2^{t-2k}\frac{\mathrm{d}t}{4\pi i}\,.$$

The next sum to evaluate is that over k. To this end, we begin by making the change of variable $t \to t' = t-2k$, thus shifting the contour of integration from $\mathrm{Re}(t) = \frac12$ to $\mathrm{Re}(t') = \frac12-2k$ and, after having changed the integration variable back to t, we are left with

$$\Upsilon_0(a,b,r,s|\tau_2) \tag{D.8}$$
$$=\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\int_{\frac12-2k-i\infty}^{\frac12-2k+i\infty}\frac{\sigma_a(m)\sigma_{2t-1}(m)}{m^{t+r-b}}\frac{(-1)^k\pi^{1-t-r}\Gamma(\frac12-k-t)\Gamma(\frac{t+2k+r-s}{2})\Gamma(\frac{t+2k+r+s-1}{2})}{\Gamma(1-t)\zeta(2-2t)k!}\tau_2^{t}\frac{\mathrm{d}t}{4\pi i}\,.$$

We would like to translate the shifted integration contour back to its original position at $\mathrm{Re}(t) = \frac12$, however, additional poles originating from $\Gamma(\frac12 - k - t)$ appear at $t = \frac12 - \ell$, with $\ell \in \mathbb{N}$ and $0 < \ell \leq k$. Although the shifted contour cannot be moved back immediately to its initial place, we can nevertheless rewrite it as a sum of two different contours: the original one along $\mathrm{Re}(t) = \frac12$ and a new contour encircling these new poles along the negative t-axis. As depicted in figure D.1, these two contours can be connected at infinity to form a single auxiliary contour of integration $\mathcal{C}$ which is independent from the summation variable k. We exchange the sum over k with the integral over $\mathcal{C}$ and perform the sum over k

$$\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\frac12-k-t)\Gamma(\frac{2k+r+t-s}{2})\Gamma(\frac{2k+r+s+t-1}{2})}{k!} \tag{D.9}$$
$$=\frac{\sin\left[\pi(r-t)\right]+\sin\left(\pi s\right)}{2\sin(\pi r)\cos(\pi t)}\frac{\Gamma\left(\frac{r+1-s-t}{2}\right)\Gamma\left(\frac{r+s-t}{2}\right)\Gamma\left(\frac{t+r-s}{2}\right)\Gamma\left(\frac{t+r+s-1}{2}\right)}{\Gamma(r)}\,.$$

We are then left with the expression

$$\Upsilon_0(a,b,r,s|\tau_2) = \sum_{m>0}\int_{\mathcal{C}}\Big(\frac{\sin\left[\pi(r-t)\right]+\sin\left(\pi s\right)}{2\sin(\pi r)\cos(\pi t)}\Big)\Big(\frac{\sigma_a(m)\sigma_{2t-1}(m)}{m^{t+r-b}}\Big) \tag{D.10}$$
$$\times\frac{\Gamma\left(\frac{r+1-s-t}{2}\right)\Gamma\left(\frac{r+s-t}{2}\right)\Gamma\left(\frac{t+r-s}{2}\right)\Gamma\left(\frac{t+r+s-1}{2}\right)}{\pi^r\Gamma(r)\xi(2-2t)}\tau_2^{t}\frac{\mathrm{d}t}{4\pi i}\,.$$

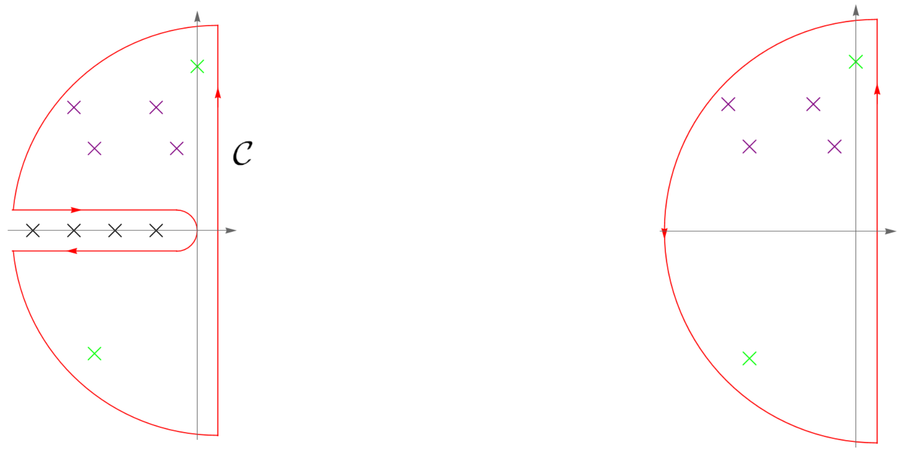

Figure D.1: On the left we show the auxiliary integration contour $\mathcal{C}$ and on the right the deformed contour. The poles from the gamma functions are depicted in purple, from the zeta functions in green and from the trigonometric functions in black.

Since the integration contour $\mathcal{C}$ is closed, the integral is uniquely fixed by the residues at the poles in the interior of $\mathcal{C}$. The only poles situated in the interior of the contour $\mathcal{C}$ are located at $t = -2n + s - r$ and $t = -2n + 1 - s - r$ for $n \in \mathbb{N}$ and come from the last two gamma functions at numerator in the above integrand. Furthermore, we notice that at these pole locations the ratio of trigonometric factors in (D.10) always evaluates to 1. We conclude that this ratio of trigonometric terms can be dropped from the contour integral (D.10) without changing the result

$$\Upsilon_0(a,b,r,s|\tau_2) = \sum_{m>0} \int_{\mathcal{C}} \Big(\frac{\sigma_a(m)\sigma_{2t-1}(m)}{m^{t+r-b}}\Big) \frac{\Gamma\big(\frac{r+1-s-t}{2}\big)\Gamma\big(\frac{r+s-t}{2}\big)\Gamma\big(\frac{t+r-s}{2}\big)\Gamma\big(\frac{t+r+s-1}{2}\big)}{\pi^r \Gamma(r)\xi(2-2t)} \tau_2^t \frac{\mathrm{d}t}{4\pi i} \,. \tag{D.11}$$

Since the trigonometric factors have been removed, we have that the previously mentioned poles which were located on the negative t-axis at $t = \frac{1}{2} - \ell$, with $\ell \in \mathbb{N}$ are no longer present in (D.11). As depicted in figure D.1, we are now free to deform the auxiliary contour of integration $\mathcal{C}$ to an infinite semi-circle. The contribution from the circle at infinity vanishes and the only non-trivial contribution to the integral comes from the line $\mathrm{Re}(t) = \frac{1}{2}$, hence we have managed to restore the original contour of integration.

Finally, we turn to the sum over m. We notice that at large-m the summand is bounded by

$$\Big|\frac{\sigma_a(m)\sigma_{2t-1}(m)}{m^{t+r-b}}\Big| = O\big(m^{-[\mathrm{Re}(r-b)-\max(\mathrm{Re}(a),0)]}\big) \,,$$

and, thanks to the conditions (5.13) for the absolute convergence of the Poincaré series, we can easily see that $\mathrm{Re}(r-b)-\max\big(\mathrm{Re}(a),0\big)>1$ for the range of parameters considered, hence this sum converges absolutely (note that for the convergence of this sum it is crucial we managed to reduce the contour $\mathcal{C}$ back to just the line $\mathrm{Re}(t)=\frac{1}{2}$). We can then use a well-known identity due to Ramanujan,

$$\sum_{m>0}\frac{\sigma_a(m)\sigma_b(m)}{m^s}=\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\,, \tag{D.12}$$

specialised to the case $b=2t-1\,,\ s=r+t-b$ and substitute it back in equation (D.10).

Our final result is the Mellin-Barnes integral representation for the Fourier zero-mode,

$$\Upsilon_0(a,b,r,s|\tau_2)=\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}U(a,b,r,s|t)\,\tau_2^t\frac{\mathrm{d}t}{2\pi i}, \tag{D.13}$$

where we define

$$\begin{aligned}U(a,b,r,s|t):=&\,\frac{\Gamma\big(\frac{r+1-s-t}{2}\big)\Gamma\big(\frac{r+s-t}{2}\big)\Gamma\big(\frac{t+r-s}{2}\big)\Gamma\big(\frac{t+r+s-1}{2}\big)}{2\pi^r\,\Gamma(r)\xi(2-2t)}\\&\times\frac{\zeta(r+1-b-t)\zeta(r+1-a-b-t)\zeta(t+r-b)\zeta(t+r-a-b)}{\zeta(2r+1-a-2b)}\,.\end{aligned} \tag{D.14}$$

The Mellin-Barnes integral representation (D.13) can be analytically continued to values of parameters, (a,b,r,s), for which the Poincaré series (5.12) is not absolutely convergent. In general, rather than the vertical line $\mathrm{Re}(t)=\frac{1}{2}$, the integration contour, γ, in (D.13) has to be chosen such that it separates two sets of poles of (D.14). The contour γ is such that the poles coming from

$$\Gamma\big(\frac{t+r-s}{2}\big)\Gamma\big(\frac{t+r+s-1}{2}\big)\zeta(t+r-b)\zeta(t+r-a-b)\,,$$

are located to the left of γ, while the remaining poles coming from

$$\frac{\Gamma\big(\frac{r+1-s-t}{2}\big)\Gamma\big(\frac{r+s-t}{2}\big)\zeta(r+1-b-t)\zeta(r+1-a-b-t)}{\xi(2-2t)}\,,$$

are located to the right of γ.

APPENDIX E

An alternative spectral overlap

In this appendix we derive an alternative expression for the spectral overlap (6.30) that will be fundamental in computing the large-N transseries expansion of the integrated correlator, $\mathcal{C}_N(\tau)$, starting from its spectral representation (6.23). Throughout this derivation we assume for simplicity that $N \geq 2$ is an integer (which is also the case of physical interest) and that the spectral parameter s lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Despite these assumptions, the regime of validity for the final result (E.5) will be more general and in particular it will provide an analytic continuation valid for $N \in \mathbb{C}$ with $\mathrm{Re}(N) > 0$.

We begin by rewriting the hypergeometric function appearing in the spectral overlap (6.30) via the integral representation,

$$
\begin{aligned}
&{}_3F_2(2-N, s, 1-s; 3, 2|1) \\
&= \frac{2(-1)^N \Gamma(N-1)}{\Gamma(1-s)\Gamma(s)} \oint_{\gamma_1} \frac{\Gamma(x+N-s-1)\Gamma(x+N+s-2)\Gamma(x)}{\Gamma(x+N-1)\Gamma(x+N)\Gamma(x+N+1)} \frac{\mathrm{d}x}{2\pi i}\,,
\end{aligned} \tag{E.1}
$$

where γ_1 is a contour around the poles coming from $\Gamma(x)/\Gamma(x+N-1)$ and located at $x \in \{0, -1, ..., -(N-2)\}$. It is convenient to make a change of integration variables $x \to x' = x+N$ which we then rename x again and, after using the reflection formula for the gamma functions,

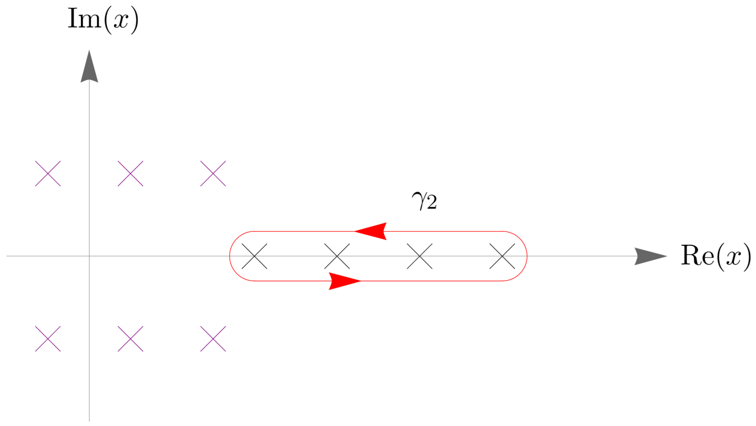

Figure E.1: The contour of integration γ_2 circles around the poles in the complex x-plane located at $x \in \{2, 3, ..., N\}$ while avoiding other singularities.

we reduce the integral to

$$
\begin{aligned}
&{}_3F_2(2-N, s, 1-s; 3, 2|1) \\
&= 2\Gamma(N-1)\sin(\pi s) \oint_{\gamma_2} \frac{\Gamma(x-s-1)\Gamma(x+s-2)}{\sin(\pi x)\Gamma(N+1-x)\Gamma(x-1)\Gamma(x)\Gamma(x+1)} \frac{\mathrm{d}x}{2\pi i}\,,
\end{aligned}
\tag{E.2}
$$

where γ_2 is a contour around the poles at $x \in \{2, 3, ..., N\}$ presented in figure E.1.

We now exploit the known asymptotic expansion of the gamma functions to find that the integrand of (E.2) behaves as $|\mathrm{Im}(x)|^{-3-N}$ as $|\mathrm{Im}(x)| \to \infty$. Since this bound is uniform throughout the x-plane, we have that the integrals coming from the horizontal contributions to γ_2 located at $|\mathrm{Im}(x)| = M$ vanish when we send $M \to \infty$. The non-vanishing contributions to the contour integral (E.2) can then be rewritten as

$$
\oint_{\gamma_2} = -\int_{\mathrm{Re}(x)=2-\epsilon} + \int_{\mathrm{Re}(x)=N+\epsilon}\,,
\tag{E.3}
$$

with $\epsilon > 0$ small enough.

Furthermore, given that the integrand of (E.2) does not have any poles in the domain $\mathrm{Re}(x) > N+\epsilon$, we can push the second contour of integration towards $\mathrm{Re}(x) = +\infty$ and show that it vanishes given the bound on the integrand discussed above. We then deduce that the original contour integral (E.2) reduces simply to an integral over $\mathrm{Re}(x) = 2-\epsilon$,

$$
\begin{aligned}
&{}_3F_2(2-N,s,1-s;3,2|1) \\
&= -2\Gamma(N-1)\sin(\pi s)\int_{\mathrm{Re}(x)=2-\epsilon} \frac{\Gamma(x-s-1)\Gamma(x+s-2)}{\sin(\pi x)\Gamma(N+1-x)\Gamma(x-1)\Gamma(x)\Gamma(x+1)}\frac{\mathrm{d}x}{2\pi i}\,.
\end{aligned}
\tag{E.4}
$$

We now push the contour of integration towards $\mathrm{Re}(x) \to -\infty$ and collect the residues from the poles originating from the two gamma functions at numerator, which are located at $x = -k+1+s$ and $x = -k+2-s$ for $k \in \mathbb{N}$, while the contribution at infinity vanishes thanks to a similar argument as above. Picking up these residues we arrive at the identity

$$
\begin{aligned}
&{}_3F_2(2-N,s,1-s;3,2|1) = 2\Gamma(N-1)\times \\
&\left[\frac{\Gamma(1-2s)\,{}_3F_2(s-2,s-1,s;2s,N+s-1|1)}{\Gamma(s+N-1)\Gamma(1-s)\Gamma(2-s)\Gamma(3-s)}+\frac{\Gamma(2s-1)\,{}_3F_2(-s-1,-s,1-s;2-2s,N-s|1)}{\Gamma(N-s)\Gamma(s)\Gamma(s+1)\Gamma(s+2)}\right],
\end{aligned}
\tag{E.5}
$$

where we notice that the two factors in this expression are related by the transformation $s \to 1-s$, as expected given that the equation we started with (E.1) had this symmetry.

Substituting (E.5) back in the spectral overlap (6.30), we derive the the symmetric expression

$$
\begin{aligned}
&M_N(s) = \\
&\frac{2^{-2s}\sqrt{\pi}(2s-1)\Gamma(1+N)\Gamma(\frac{3}{2}-s)\,{}_3F_2(s-2,s-1,s;2s,N+s-1|1)}{\sin(\pi s)\Gamma(3-s)\Gamma(-s)\Gamma(N+s-1)} + (s \leftrightarrow 1-s)\,.
\end{aligned}
\tag{E.6}
$$

The above equation can be simplified even further by using the Euler-like integral representation

$$
{}_3F_2(s-2,s-1,s;2s,N+s-1|1) = \frac{\Gamma(N+s-1)}{\Gamma(s)\Gamma(N-1)}\int_0^1 (1-x)^{N-2}x^{s-1}\,{}_2F_1(s-2,s-1;2s|x)\mathrm{d}x\,. \tag{E.7}
$$

Combining (E.6) with (E.7), we conclude that

$$
M_N(s) = \frac{2^{-2s}(2s-1)\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{\pi}\,\Gamma(-s)}\mathcal{I}_N(s) + (s \leftrightarrow 1-s)\,, \tag{E.8}
$$

$$
\mathcal{I}_N(s) \coloneqq \frac{N(N-1)}{(s-1)(s-2)}\int_0^1 (1-x)^{N-2}x^{s-1}\,{}_2F_1(s-2,s-1;2s|x)\mathrm{d}x\,. \tag{E.9}
$$

Finally, we rewrite the expression for $\mathcal{I}_N(s)$ by noticing that

$$\frac{d^2}{dx^2}(1-x)^N = N(N-1)(1-x)^{N-2}\,,$$

which can be substituted in (E.9) to perform twice an integration by parts. However, we need to be careful in doing so since the boundary terms diverge as $x \to 0^+$ and we must introduce a regulator as to cancel the singular terms. The end result is a formula for the spectral overlap (E.8) which is valid for $\mathrm{Re}(s) = \frac{1}{2}$ and $N \in \mathbb{C}$ with $\mathrm{Re}(N) > 0$

$$\mathcal{I}_N(s) = \lim_{\epsilon \to 0^+} \left[\frac{\epsilon^{s-2}}{s-2} + \frac{(s-2N-1)\epsilon^{s-1}}{2(s-1)} + \int_\epsilon^1 (1-x)^N x^{s-3} {}_2F_1(s-1, s; 2s|x) \mathrm{d}x \right]. \tag{E.10}$$

APPENDIX F

Properties of a modular invariant Borel kernel

In this appendix we review some known properties for the modular invariant Borel kernel $\mathcal{E}(t;\tau)$ as well as derive novel expressions which are of use in the main body of this work.

The family of modular invariant functions $D_N(s;\tau)$ defined in (6.17) was first introduced in [67], while a generalisation has also recently appeared in the study of torodial Casimir energy in 3-dimensional conformal field theories [99]. In this appendix we focus our attention to the special element (6.44) in this family, namely $\mathcal{E}(t;\tau) = D_{t^2}(0;\tau)$, which plays the rôle of a modular invariant kernel for our modified Borel resummation.

The Fourier mode decomposition of the lattice sum (6.44) can be analysed straightforwardly [67] starting from an integral representation valid for $\mathrm{Re}(t^2) > 0$

$$\mathcal{E}(t;\tau) = \sum_{(m,n)\neq(0,0)} \int_0^\infty e^{-xY_{mn}(\tau)-\frac{4t^2}{x}} \frac{2t}{\sqrt{\pi}x^{3/2}} \mathrm{d}x\,. \tag{F.1}$$

In particular, for our analysis of the large-N 't Hooft limit of section 6.7 we need an expression for the zero-mode sector of $\mathcal{E}(t;\tau)$, which can be easily extracted from (F.1) via standard Poisson resummation methods thus yielding

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{E}(t;\tau)\mathrm{d}\tau_1 = \frac{2}{e^{4t\sqrt{\pi/\tau_2}}-1} + \sum_{n=1}^{\infty} 4n\tau_2\, K_1(4nt\sqrt{\pi\tau_2})\,. \tag{F.2}$$

Alternatively, the lattice sum representation (6.44) can be written as a Poincaré series,

$$\mathcal{E}(t;\tau) = 2\sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Li}_0\Big(e^{-4t\sqrt{\pi/\tau_2}}\Big) = 2\sum_{\gamma\in \mathrm{B}(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \left[\frac{1}{\exp\left(4t\sqrt{\pi/\tau_2}\right)-1}\right]_\gamma . \tag{F.3}$$

As shown in [86], the spectral representation of $\mathcal{E}(t;\tau)$ can be obtained directly from the Poincaré series representation (F.3) and takes the form

$$\mathcal{E}(t;\tau) = \frac{1}{8t^2} + 4\int_{\mathrm{Re}(s)=\frac{1}{2}} \frac{(4t)^{-2s}\Gamma(2s)}{\Gamma(s)} E^*(s;\tau)\frac{\mathrm{d}s}{2\pi i} . \tag{F.4}$$

From the spectral representation we may also obtain a different expression for the Fourier zero-mode (F.2). We start by substituting in (F.4) the Fourier zero-mode (6.15) of the non-holomorphic Eisenstein series,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} E^*(s;\tau)\mathrm{d}\tau_1 = \xi(2s)\tau_2^s + \xi(2s-1)\tau_2^{1-s} . \tag{F.5}$$

To evaluate (F.4) we focus separately on the contribution coming from either of the two terms $\xi(2s)\tau_2^s$ and $\xi(2s-1)\tau_2^{1-s}$ in the above expression.

Starting with the term $\xi(2s)\tau_2^s$, we rewrite the completed Riemann zeta using the Dirichlet series representation of the zeta function $\zeta(2s) = \sum_{k\geq 1} k^{-2s}$ valid for $\mathrm{Re}(s) > 1/2$

$$\begin{aligned} 4\int_{\mathrm{Re}(s)=\frac{1}{2}+\epsilon} \frac{(4t)^{-2s}\Gamma(2s)}{\Gamma(s)}\xi(2s)\tau_2^s\frac{\mathrm{d}s}{2\pi i} &= \sum_{k=1}^{\infty} 4\int_{\mathrm{Re}(s)=\frac{1}{2}+\epsilon} \Gamma(2s)\Big(\frac{4t\sqrt{\pi}}{\sqrt{\tau_2}}k\Big)^{-2s}\frac{\mathrm{d}s}{2\pi i} \\ &= \frac{2}{e^{4t\sqrt{\pi/\tau_2}}-1} , \end{aligned} \tag{F.6}$$

where we simply evaluate the sum over all the residues coming from the poles of $\Gamma(2s)$ and subsequently perform the sum over k. Note that in doing so we have to shift the contour of integration to the right by an infinitesimal quantity, $\epsilon > 0$, since $\xi(2s)$ has a pole at $s = 1/2$. The contribution from this pole is cancelled by an equal and opposite pole coming from the second zero-mode term $\xi(2s-1)\tau_2^{1-s}$, since the complete non-holomorphic Eisenstein $E^*(s;\tau)$ is perfectly regular for $s=\frac{1}{2}$ and the integral expression (F.4) is unchanged if we move the contour of integration to $\mathrm{Re}(s) = 1/2+\epsilon$.

The remaining zero-mode contribution to $\mathcal{E}(t;\tau)$ comes from the leftover $1/(8t^2)$ term in

(F.4) and the non-holomorphic Eisenstein series factor $\xi(2s-1)\tau_2^{1-s}$ which can be combined as

$$\begin{aligned}\mathcal{U}(t;\tau_2) &\coloneqq \frac{1}{8t^2} + \int_{\mathrm{Re}(s)=\frac{1}{2}+\epsilon} \frac{4(4t)^{-2s}\Gamma(2s)}{\Gamma(s)}\xi(2s-1)\tau_2^{1-s}\frac{\mathrm{d}s}{2\pi i} \\ &= \int_{\mathrm{Re}(s)=1+\epsilon} \frac{\Gamma\left(s-\frac{1}{2}\right)\Gamma\left(s+\frac{1}{2}\right)\zeta(2s-1)}{2\pi t^2}\left(2t\sqrt{\pi\tau_2}\right)^{2-2s}\frac{\mathrm{d}s}{2\pi i}\,. \end{aligned} \tag{F.7}$$

We then conclude that the Fourier zero-mode of $\mathcal{E}(t;\tau)$ can also be expressed as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{E}(t;\tau)\,\mathrm{d}\tau_1 = \frac{2}{e^{4t\sqrt{\pi/\tau_2}}-1} + \mathcal{U}(t;\tau_2)\,, \tag{F.8}$$

where it is worth noting that for large values of t both terms are exponentially suppressed. Comparing with (F.2), we see that $\mathcal{U}(t;\tau_2)$ has the alternative representation

$$\mathcal{U}(t;\tau_2) = \sum_{n=1}^{\infty} 4n\tau_2\,K_1(4nt\sqrt{\pi\tau_2})\,. \tag{F.9}$$

Unfortunately $\mathcal{U}(t;\tau_2)$ does not seem to have a simpler expression in terms of elementary functions, however we can use the integral representation for the Bessel function to rewrite (F.9) in the useful Borel-like form

$$\mathcal{U}(t;\tau_2) = \sum_{n=1}^{\infty}\int_t^{\infty}(4n\tau_2)e^{-4nx\sqrt{\pi\tau_2}}\frac{x}{t\sqrt{x^2-t^2}}\,\mathrm{d}x\,. \tag{F.10}$$

To discuss the 't Hooft large-N limit in section 6.7, we need certain integrals involving the Fourier zero-mode just discussed. In particular, we need a formula for the moments t^α with respect to the measure $\mathcal{U}(t;\tau_2)\mathrm{d}t$. From the definition (F.7), we see that the t-dependence of such integrals is simply of the form $t^{\alpha-2s}$, we then consider the analytic continuations

$$\int_0^1 t^{\alpha-2s}\,\mathrm{d}t = \frac{1}{\alpha-2s+1}\,, \qquad \mathrm{Re}(s) < \frac{\alpha+1}{2}\,, \tag{F.11}$$

$$\int_1^{\infty} t^{\alpha-2s}\,\mathrm{d}t = -\frac{1}{\alpha-2s+1}\,, \qquad \mathrm{Re}(s) > \frac{\alpha+1}{2}\,. \tag{F.12}$$

After having performed the integral over t, we notice that the integrand of (F.7) acquires a single simple pole located at $s=\frac{\alpha+1}{2}$. Closing the contour of integration to the right half-plane

Re$(s) > 1$ picks up the residue at this pole without any boundary contribution, thus giving us

$$\int_0^\infty \mathcal{U}(t;\tau_2)\, t^\alpha \,\mathrm{d}t = \frac{\Gamma\Big(\frac{\alpha}{2}+1\Big)\Gamma\Big(\frac{\alpha}{2}\Big)\zeta(\alpha)}{4\pi}(2\sqrt{\pi\tau_2})^{1-\alpha}\,. \tag{F.13}$$

Alternatively, we may start from the expression in terms of Bessel functions (F.9) and compute

$$\int_0^\infty 4n\tau_2 K_1(4nt\sqrt{\pi\tau_2})\, t^\alpha\,\mathrm{d}t = \frac{\Gamma\Big(\frac{\alpha}{2}+1\Big)\Gamma\Big(\frac{\alpha}{2}\Big)}{4\pi}\, n^{-\alpha}\,(2\sqrt{\pi\tau_2})^{1-\alpha}\,, \tag{F.14}$$

valid for Re$(\alpha) > 0$, which can then be easily summed over n to reproduce (F.13).

In section 6.7 we also discuss the non-perturbative terms in the dual 't Hooft limit which require finding an expression for the moments t^α with respect to the measure $\mathcal{U}(t+1;\tau_2)\mathrm{d}t$. To this end, we start with the identity

$$\int_0^\infty \frac{t^\alpha}{(t+1)^{2s}}\,\mathrm{d}t = \frac{\Gamma(1+\alpha)\Gamma(2s-1-\alpha)}{\Gamma(2s)}\,, \tag{F.15}$$

which we then substitute in the defining formula (F.7) to derive

$$\int_0^\infty \mathcal{U}(t+1;\tau_2)\, t^\alpha\,\mathrm{d}t = \Gamma(1+\alpha)\int_{\mathrm{Re}(s)=\frac{1+\alpha}{2}+\epsilon} \frac{\Gamma(s-\frac{1}{2})\Gamma(2s-1-\alpha)\zeta(2s-1)}{4\sqrt{\pi}\Gamma(s)(4\sqrt{\pi\tau_2})^{2s-2}}\frac{\mathrm{d}s}{2\pi i}\,. \tag{F.16}$$

Although this expression does not quite yield a closed form such that (F.13), it still suffices for the discussion of section 6.7.

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