Systems Failure Diagnosis and Repair Kit using Survival Signature

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A Thesis presented for the degree of
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Dedicated to

To my parents

for their support and prayers

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Abstract

A pivotal aspect of studying systems involves diagnosing its failures, referred to in this thesis as identifying the components or types of components associated with system failure. System failure diagnosis serves various purposes, including facilitating maintenance activities and informing system design. This thesis delves into the study of system failure from two distinct perspectives: determining which types of components are most likely to lead to system failure and estimating the numbers of failed components of each type at the time of system failure. While Barlow and Proschan [7] introduced an importance measure that determines the probability of a component causing system failure based on the structure function, the complexity associated with the structure function may pose challenges in applying it to real complex systems. Therefore, for a general system structure containing multiple types of components, we use the concept of the survival signature introduced by Coolen and Coolen-Maturi [15] to derive the probability of a component of a specific type failing at the system failure time, ultimately leading to system failure.

Additionally, we derive probabilities of three events related to the number of failed components of multiple types at a future moment when the system fails, based on the survival signature. First, we determine the probability of the number of failed components at system failure, given that the system will fail at a specific time t and conditioning on the number of failed components prior to system failure. Second, the probability of the number of failed components at an unknown system failure, assuming the system is functioning at a certain time, is derived. We also consider

the probability of the number of failed components at system failure, assuming the system will fail in a specific future time interval. The results of the probabilities depend only on the distributions of failure times of component types and the survival signature of the system.

The results of these events are applied to various scenarios of the Repair Kit Problem (RKP). In particular, the probability of the number of failed components when the system fails is utilised to quantify the expected cost of a Return to Fit (RTF) visit, where a penalty cost is incurred if a repair kit cannot complete the intended job. The aim here is to determine the optimal repair kit under the cost model, that is, the repair kit with the minimum total expected cost, including holding cost and RTF cost. Three distinct scenarios for the RKP are considered. The first scenario involves determining an optimal repair kit intended to be provided with the system at the time of purchase, aiming to replace all failed components when the system fails. Second, an optimal repair kit designed to replace all failed components in preparation for a specific future time interval during which the system may fail is considered. The third scenario proposes an optimal minimal repair kit, which does not necessarily aim to replace all failed components at system failure, but rather to replace specific failed components that can restore the system to a functional status.

In cases where the distributions of failure times for some component types are unknown, we utilise the posterior predictive distribution for the probabilities of events related to the number of failed components and for the probability of a component of a specific type causing system failure. Based on these updated probabilities, the optimal repair kits are determined and compared to the scenario where the distributions of failure times for component types are assumed to be fully known.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

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Contents

Chapter 1

Introduction

1.1 Overview

One of the key aspects in the field of reliability is diagnosing system failures, which is valuable for various purposes, including maintenance activities and system design. Diagnosing system failures involves identifying the components or types of components most likely to lead to system failure and determining the numbers of failed components at the moment of system failure. This thesis concerns the derivation of probability distributions for these events. Having insight into these probabilities can serve different purposes. For example, components of the type with a higher probability of causing system failure can receive top priority for maintenance, inspection, and monitoring. Additionally, the probability distribution of the number of failed components offers various practical applications, such as optimising spare parts inventory by prioritising combinations of components with higher probabilities while minimising stock for those with lower probabilities, thereby preventing overstocking and reducing costs.

The literature offers various methods regarding these problems. Barlow and Proschan [7] developed an importance measure that determines the probability of a component causing system failure, primarily relying on the structure function. In terms of the numbers of failed components at system failure, various approaches are proposed in the literature to derive the probability distribution of the number of failed components based on different settings. Among these developments is the contribution by Eryilmaz [22], which derived the probability of the number of failed components at system failure for a system with exchangeable component failure times, utilising the system signature. Additionally, derivations for the same event were conducted for a k-out-of-m system [23] and for a series-parallel system [25], both composed of multiple types of components.

These studies have some limitations, including a primary dependency on the structure function, as for the Barlow and Proschan importance measure [7], and the assumption of the distribution of components failure times being identical, which can be informally regarded as components of a single type as in Eryilmaz [22]. If the latter assumption is relaxed, and components of multiple types are allowed, the investigation is often limited to specific system structures (e.g. [23], [25]). To address these challenges, we use the survival signature introduced by Coolen and Coolen-Maturi [15]. The survival signature does not allow individual components to be identified, so instead of finding the probability that a specific component caused system failure, we obtain an expression for the probability that system failure was caused by failure of a given component type. Indeed, this provides a new component type based importance measure which can be used in system design and deployment planning. Furthermore, the survival signature is used to derive the probability of various events related to the number of failed components of multiple types at system failure, considering different conditions, for a general system structure. These are regarded as the core contributions of this thesis.

1.2 Outline of thesis

The structure of the thesis is as follows. Chapter 2 provides an overview of the key reliability concepts discussed in the literature relevant to the thesis. This involves a review of main system structures and the definition of the concept of a coherent system. The chapter also explores various methods related to quantifying system reliability, such as structure function and path and cut sets. The chapter reviews the concept of the survival signature and how it is used to derive the reliability of a system with multiple types of components. Different methods for calculating or approximating the survival signature for complex systems, introduced in the literature, are outlined in this chapter.

In Chapter 3, the probabilities for events of interest related to the diagnosis of system failure in a general system with multiple types of components are derived using the survival signature. First, the chapter derives the probability of a component of a specific type failing at a certain time, given that the system fails at that time, indicating the contribution of a specific component type to system failure. We also derive probabilities for various events related to the numbers of failed components at the moment of system failure under different conditions. Initially, we derive the probability assuming system failure at a specific time t, while considering the number of failed components of each type prior to system failure. The probability of the numbers of failed components at system failure is also derived under two conditions: first, assuming that the system was functioning at a certain time t, and second, considering system failure at a specific future time interval. These probabilities have practical relevance, which will be further highlighted in this chapter. One notable application is the Repair Kit Problem (RKP), which will be thoroughly examined in Chapter 4. Some of the results from Chapter 3 were presented at the International Conference on Modelling in Industrial Maintenance and Reliability (held in July 2021) and at the International Workshop on Reliability Engineering and Computational Intelligence in the Netherlands (held in November 2022). Results of this chapter were also presented at a seminar held at Eindhoven University of Technology in February 2023.

In Chapter 4, we apply the probabilities derived in Chapter 3 to formulate the RKP under specific settings. The objective is to determine an optimal repair kit capable of completing a repair job under the cost model, that is to minimise the total expected cost. Three distinct scenarios for a repair kit, each motivated by specific needs, are formulated. First, we consider a repair kit provided with the system at the time of purchase, aiming to replace all failed components at system failure. We also consider a repair kit intended to replace all failed components in the event of a system failure at a specific future time interval. Finally, a minimal repair kit is introduced, which does not necessarily aim to replace all failures but rather to replace specific failures, aiming to restore the system to a functional status. Two closely related greedy heuristics are introduced to determine the optimal repair kit. Some results of this chapter were presented at the International Conference on Modelling in Industrial Maintenance and Reliability in July 2023.

Chapter 5 examines the probabilities introduced in Chapter 3 and the repair kits discussed in Chapter 4 from a Bayesian perspective. Specifically, posterior predictive distributions are employed for new failure times of certain component types when their failure time distributions are not fully known. Based on these predictive distributions, optimal repair kits are determined and compared to the scenarios where the distributions of component failure times are fully known.

Chapter 6 concludes this thesis by summarising the main contributions and offering some conclusions. In the final sections of Chapters 3, 4, and 5, interesting ideas for future research are suggested. Calculations in this thesis were conducted using the statistical software program R version 4.0.4. [52].

Chapter 2

Reliability theory foundation

2.1 Introduction

In this thesis, a *system* is defined as a set of linked components that performs a specific function. The term component describes any entity that is considered indivisible in the context of analysis, implying that the individual elements of the entity are not directly represented, only the complete entity is considered. For example, an automobile is a system and its constituent parts, such as its engine and gearbox, are its components. We should note that a component might be physically complicated. For instance, the engine itself can sometimes be a separate system. A system's reliability at a particular time point can be described as the probability that the system functions at that point in time.

This chapter provides an overview of some main reliability concepts. It covers the definition of state vector, structure function, path sets, and cut sets. By utilising these methods, one can determine the reliability of a system. However, the use of these methods may be limited when dealing with complex systems.

To overcome the problems associated with the use of these tools for complex systems and networks, the system signature [15] has been developed as an alternative method to quantify their reliability. However, the system signature requires all components to be of the same type, meaning that the components' failure time are exchangeable random quantities. Due to this strong assumption, the system signature approach is not feasible in the majority of real-world systems. Recently, the survival signature has been developed [15] to address this limitation when analysing the reliability of systems that contain multiple types of components. A brief overview of these methods is provided in this chapter.

2.2 System reliability

The reliability of a component can be interpreted in various ways. Generally speaking, it relates to its ability to accomplish a specified function correctly. As mentioned earlier, a system is made up of several components linked together to perform a specific function. Hence, the reliability of the system depends on the reliability of its components. To explain how a system and its components are related, we review the concepts of component state vector and structure function.

2.2.1 Structure function

For a system with m components, the *state vector* is a vector $\mathbf{x} = (x_1, ..., x_m) \in \{0,1\}^m$, representing the state of each component in the system, such that

$$x_i = \begin{cases} 1, & \text{if the } i\text{th component is functioning} \\ 0, & \text{if the } i\text{th component is not functioning.} \end{cases}$$

The component labelling is arbitrary but must be fixed to define $\mathbf{x}$. Thus, the state of a system can be determined by its state vector through a mathematical model [73]. A common approach in the literature to determine a system's state is based on the use of the structure function.

Definition 2.2.1 For a system with m components, let $\mathbf{x}$ be any possible state vector. The structure function is a map $\varphi(\mathbf{x}) : \{0,1\}^m \to \{0,1\}$ that relates the vectors in which the system works with a value of 1 and the vectors in which the system fails with a value of 0.

Calculating the structure function of a system requires an understanding of how it is structured (designed). A system's structure illustrates the interconnections

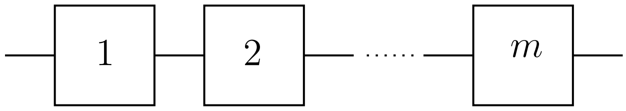


Figure 2.1: Series system with m components.

between its components. The way that the components are connected reveals how the condition of the components affects the condition of the system. Below are some examples of commonly known structure functions.

Example 2.2.1 (Series system) A series system refers to a system whose constituent components are linked to one another in series. Every component of a series system must function in order for the system to function as a whole. Figure 2.1 illustrates an m-component series system, and its structure function can be expressed using the equivalent expressions below,

$$\begin{aligned}\varphi(\mathbf{x}) &= \begin{cases} 1, & \text{if } x_i = 1 \text{ for all } i \\ 0, & \text{if } x_i = 0 \text{ for any } i \end{cases} \\ &= \prod_{i=1}^{m} x_i \\ &= \min(x_1, ..., x_m).\end{aligned}$$

Example 2.2.2 (Parallel system) Parallel systems are systems in which individual components are interconnected in parallel. It is necessary that at least one component of a parallel system functions in order for the system to function. Figure 2.2 shows a parallel system composed of m components, and the following equivalent expressions define the structure function of the system,

$$\begin{aligned}\varphi(\mathbf{x}) &= \begin{cases} 1, & \text{if } x_i = 1 \text{ for any } i \\ 0, & \text{if } x_i = 0 \text{ for all } i \end{cases} \\ &= 1 - \prod_{i=1}^{m} (1 - x_i) \\ &= \max(x_1, ..., x_m).\end{aligned}$$

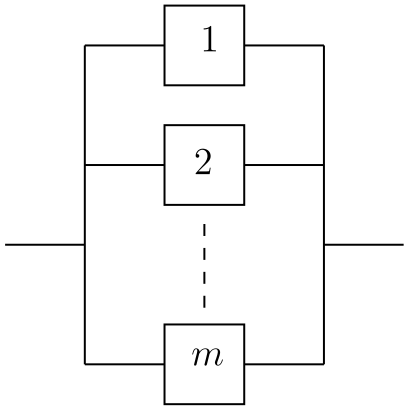


Figure 2.2: Parallel system with m components.

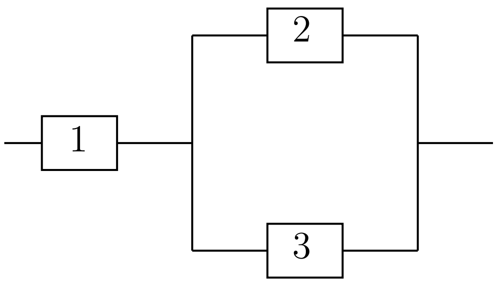


Figure 2.3: Series-parallel system of 3 components.

Example 2.2.3 (Hybrid system) A mixed system structure consists of two or more recognised sub-systems arranged in a series or parallel configuration to form an entire system. Figure 2.3 illustrates a hybrid system in which a series and parallel system are combined together. This system's structure function can be expressed as follows.

$$\begin{aligned}\varphi(\mathbf{x}) &= x_1\left(1-(1-x_2)(1-x_3)\right)\\ &= x_1(x_3+x_2-x_2x_3).\end{aligned}$$

Example 2.2.4 (k-out-of-m system) A system of m components functions (or is "good") if at least k of them function is known as a k-out-of-m:G system. A system of m components that fails if at least k of them fail is known as a k-out-of-m:F system. In general, the term k-out-of-m system refers to either a G or an F system [47]. It should be noted that series and parallel systems are equivalent to m-out-of-m:G (or 1-out-of-m:F) and 1-out-of-m:G (or m-out-of-m:F) systems, respectively. Figure 2.4 represents a k-out-of-m:G system with $k = 2$, $m = 3$. It

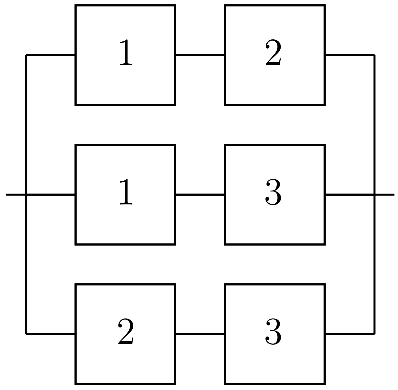


Figure 2.4: 2-out-of-3 system.

should be noted that when a component labeled i for $i \in \{1, 2, 3\}$, all components labeled i fail at once. So, the labels identify the physical components, not the types. The structure function for a k-out-of-m:G system can be given by the following equivalent formulas,

$$\begin{aligned}\varphi(\mathbf{x}) &= \begin{cases} 1, & \text{if} \quad \sum_{i=1}^{m} x_i \geqslant k \\ 0, & \text{if} \quad \sum_{i=1}^{m} x_i < k \end{cases} \\ &= \sum_j (\prod_{i \in B_j} x_i)[(\prod_{i \in B_j^c} 1 - x_i)].\end{aligned}$$

where B_j is any subset of $\{1, 2, ..., m\}$ with at least k elements, and the sum is taken over all the subsets. For example, for the 2-out-of-3:G, the structure function is

$$\begin{aligned}\varphi(\mathbf{x}) &= x_1 x_2[1 - x_3] + x_1 x_3[1 - x_2] + x_2 x_3[1 - x_1] + x_1 x_2 x_3 \\ &= x_1 x_2 + x_1 x_3 + x_2 x_3 - 2x_1 x_2 x_3.\end{aligned}$$

Using the definitions of a k-out-of-m:G system and a k-out-of-m:F system, a k-out-of-m:G system is essentially identical to a $(m - k + 1)$-out-of-m:F system. Accordingly, the structure function of a k-out-of-m:F can be expressed as follows.

$$\varphi(\mathbf{x}) = \begin{cases} 1, & \text{if} \quad \sum_{i=1}^{m} x_i \geqslant m - k + 1 \\ 0, & \text{if} \quad \sum_{i=1}^{m} x_i < m - k + 1. \end{cases}$$

2.2.2 Path and cut sets

It is often difficult or impossible to determine the structure functions of complex systems. The use of so-called path and cut sets can simplify derivation of structure functions. This subsection introduces the definition of these concepts followed by a useful theorem for extracting the structure function.

Definition 2.2.2 A set of system components, P, is called a *path set if* the system functions correctly whenever all components in P function. In particular, P is called a minimal path set if there is no proper subset of P that is a path set.

Definition 2.2.3 A set of system components, C, is called a *cut set if* the system fails whenever all components in C fail. Most importantly, C is called a minimal cut set if it does not contain a proper subset that is a cut set.

The following Theorem shows that either a minimal path or a minimal cut set can be used to represent the structure function [8].

Theorem 2.2.4 Consider a system with all minimal path sets $P_1, P_2, ..., P_k$ and all minimal cut sets $C_1, C_2, ..., C_l$. Then, the structure function of the system can be formulated in terms of the minimal path sets as

$$\varphi(\mathbf{x}) = 1 - \prod_{j=1}^{k} \left(1 - \prod_{i \in P_j} x_i \right)$$

or, in terms of the minimal cut sets as

$$\varphi(\mathbf{x}) = \prod_{j=1}^{l} \left(1 - \prod_{i \in C_j} (1 - x_i) \right).$$

Example 2.2.5 (Bridge system) Consider the bridge system illustrated in Figure 2.5. There are four minimal path sets and four minimal cut sets in the system. The minimal path sets are $P_1 = \{1,4\}, P_2 = \{2,5\}, P_3 = \{1,3,5\}$, and $P_4 = \{2,3,4\}$, while the minimal cut sets are $C_1 = \{1,2\}, C_2 = \{4,5\}, C_3 = \{1,3,5\}$, and $C_4 = \{2,3,4\}$. Using the minimal path sets, the structure function can be

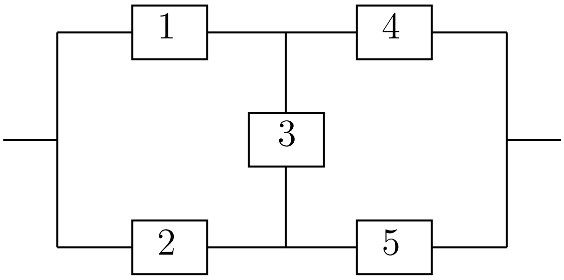


Figure 2.5: Bridge system with five components.

represented as follows.

$$\begin{aligned}\varphi(\mathbf{x}) &= 1 - (1 - x_1x_4)\,(1 - x_2x_5)(1 - x_1x_3x_5)(1 - x_2x_3x_4)\\ &= x_1x_4 + x_2x_5 + x_1x_3x_5 + x_2x_3x_4 - x_1x_3x_4x_5 - x_1x_2x_3x_5 - x_1x_2x_3x_4\\ &\quad - x_2x_3x_4x_5 - x_1x_2x_4x_5 + 2x_1x_2x_3x_4x_5\end{aligned}$$

Similarly, the structure function can be represented by the minimal cut sets.

$$\begin{aligned}\varphi(\mathbf{x}) = &\,(1 - (1 - x_1)(1 - x_2))\,(1 - (1 - x_4)(1 - x_5))\\ &\,(1 - (1 - x_1)(1 - x_3)(1 - x_5))\,(1 - (1 - x_2)(1 - x_3)(1 - x_4))\,.\end{aligned}$$

Both equations lead to the same structure function, and interested readers can verify the result.

2.2.3 Coherent systems

When designing a system, there are two fundamental prerequisites that one would naturally assume for a sensible system design. First, each component of a system must play a role in determining its functionality. Secondly, when the state of a component changes from 0 to 1, the system state should not be adversely affected. This implies that this change in the state of a component should not result in the system slipping from a functioning state to a failing state. This characteristic is called *monotonicity*. Mathematically, these two properties are described by the following two definitions.

Definition 2.2.5 Suppose that we have a system with m components such that $\mathbf{x} = (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_m)$ is a state vector of the system. Then, the component

x_i is called *irrelevant* if

$$\varphi(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_m) = \varphi(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_m)$$

for all possible state vectors $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_m) \in \{0, 1\}^{m-1}$.

Definition 2.2.6 A system with m components is considered to be a *monotone system if* $\mathbf{x} \leqslant \mathbf{y} \implies \varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$ where $\mathbf{x}, \mathbf{y} \in \{0, 1\}^m$ and $\mathbf{x} \leqslant \mathbf{y}$ is taken element-wise.

These two inherent properties result in the concept of coherent systems, a class of systems that are often investigated in the reliability literature. The definition of a coherent system is as follows.

Definition 2.2.7 A *coherent system* is one in which all its constituent components are relevant and its structure function is monotone [60].

These conditions imply that a coherent system functions if all the components are functioning and fails if none of them are functioning. This thesis assumes that all the systems being discussed are coherent systems.

2.3 System reliability computation

Consider a system with m independent components, and let X_i be a Bernoulli random variable representing the status of the ith component at some fixed time such that

$$X_i = \begin{cases} 1, & \text{if the } i\text{th component is functioning} \\ 0, & \text{if the } i\text{th component is not functioning.} \end{cases}$$

Let $p_i = P(X_i = 1)$ denote the probability of the event that component i functions and $\mathbf{p} = (p_1, ..., p_m)$. Using the structure function, the system reliability, denoted by S, can be computed based on the component reliability as follows [47].

$$S = P(\varphi(\mathbf{x}) = 1) = \varphi(\mathbf{p}).$$

In the examples below, we illustrate how the structure function can be used to determine the reliability of some systems.

Example 2.3.1 (Series and parallel systems) In order for a series system to function, all components must function. Therefore, if we assume that the random quantities X_i are independent, the reliability of a series system can be determined as follows.

$$\varphi(\mathbf{p}) = P(\varphi(\mathbf{x}) = 1) = P\left(\prod_{i=1}^{m} x_i = 1\right) = \prod_{i=1}^{m} P\left(x_i = 1\right) = \prod_{i=1}^{m} p_i.$$

Similarly, the reliability of a parallel system is found to be as:

$$\varphi(\mathbf{p}) = 1 - \prod_{i=1}^{m} \left(1 - p_i\right).$$

Example 2.3.2 (k-out-of-m system:G) If we assume that the random quantities X_i are independent and identical, then the reliability of the k-out-of-m:G system is equal to the probability that the number of functioning components is greater or equal to k. Consequently, the reliability of this system can be expressed by [47].

$$\varphi(\mathbf{p}) = \sum_{i=k}^{m} \binom{m}{i} (p)^i (1-p)^{m-i}.$$

where p is the reliability of a system component, which is the same for all components due to the *iid* assumption.

The earlier reliability calculations for the previous systems did not take into account time specifications and were treated as implicit. However, in numerous real-world situations, a specific time is often not predetermined or specified in advance. Therefore, a natural extension of the random variable X_i, the state of component i, is

$$X_i(t) = \begin{cases} 1, & \text{if the } i\text{th component is functioning at time } t \\ 0, & \text{if the } i\text{th component is not functioning at time } t. \end{cases}$$

When a component functions at $[0, t)$ and fails at $[t, \infty)$, then t is the component's *failure time* (also referred to as *lifetime*). Failure time can be considered to be a non negative real valued random variable, which will be denoted by T and can be described through probability distributions [17]. These distributions are generally known as lifetime distributions, and the most common include Exponential, Weibull, Gamma, Lognormal distributions.

Consider a system with m components and let $T_i \geqslant 0$ be the random failure time of component i. The cumulative distribution function (CDF) of T_i at time t, $F_{T_i}(t)$, represents the probability that component i fails before or at time t,

$$F_{T_i}(t) = P(T_i \leqslant t).$$

In a reliability framework, the focus is on the reliability function (also known as the survival function). The reliability of component i at time t is the probability that component i is functioning at time t, $S_{T_i}(t) = P(T_i > t)$. The reliability of component i at time t is

$$P(X_i(t) = 1) = P(T_i > t) = 1 - F_{T_i}(t).$$

2.4 Survival signature

Traditionally, system reliability quantification has relied on the structure function [2]. Samaniego [60] introduced the system signature as a tool for reliability assessment in systems composed of components with exchangeable failure time distributions, informally described as a single type of component. Samaniego's signature can be considered a summary representation of the structure function, sufficient for deriving the system reliability function when the failure times of all system components are exchangeable (i.e. when all components are of the same type). Due to the strong assumption of the components to be of a single type, the system signature approach might not be feasible in the majority of real-world systems. Coolen and Coolen-Maturi [15] introduced the survival signature as a method for quantifying system reliability for large systems with multiple types of components.

For a system consisting of m components of K types, let m_k be the components of type k, where $k \in \{1, ..., K\}$, such that $\sum_{k=1}^{K} m_k = m$, and assume that the failure times of components with the same type are independent and identical (*iid*) with full independence for failure times of components of different types. Let $\mathbf{x}^k = (x_1^k, ..., x_{m_k}^k) \in \{0, 1\}^{m_k}$ be the state vector of components with type k, then, the state vector for the whole system would be $\mathbf{x} = (\mathbf{x}^1, ..., \mathbf{x}^K) \in \{0,1\}^m$. The definition of the survival signature is as follows.

Definition 2.4.1 The survival signature for a system composed of m components of K types is denoted by $\Phi(l_1, ..., l_K)$, and it is defined as the probability that the system functions given that precisely l_k of its components of type k function for $l_k = 0, 1, ..., m_k$ for each $k \in \{1, 2, ..., K\}$.

In such a system, there are $\binom{m_k}{l_k}$ state vectors, $\mathbf{x}^k$, with precisely l_k components in each vector, $x_i^k = 1$, such that $\sum_{i=1}^{m_k} x_i^k = l_k$. Let S_l^k be the set of all these state vectors of components of type k. Moreover, let $S_{l_1,...,l_K}$ be the set of all state vectors for the whole system where $\sum_{i=1}^{m_k} x_i^k = l_k$ for $k = 1, ..., K$. Since it is assumed that the failure times of m_k components are *iid*, all state vectors $\mathbf{x}^k \in S_l^k$ are equally likely to occur. The survival signature of a system with multiple types of components can be calculated as

$$\Phi(l_1, ..., l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \times \sum_{\mathbf{x} \in S_{l_1,...,l_K}} \varphi(\mathbf{x}).$$

To derive the reliability of a multi-type system using the survival signature, let C_t^k denote the number of components of type k in the system that function at $t > 0$, and assuming independence between components of different types, then

$$P(T_s > t) = \sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \left[\Phi(l_1, ..., l_K) \prod_{k=1}^{K} P(C_t^k = l_k)\right].$$

Besides exchangeability of failure times of components of the same type, if we assume that they are independent and identically distributed with CDF $F_k(t)$, for components of type k, we get

$$P(T_s > t) = \sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \left[\Phi(l_1, ..., l_K) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k}\right)\right], \tag{2.4.1}$$

The survival signature has the main advantage of separating system structure from information about the components' failure times. When there exists only one type of component, so $K = 1$, the survival signature is equivalent to Samaniego's system signature [59][60]. Below, an example illustrating the calculation of the

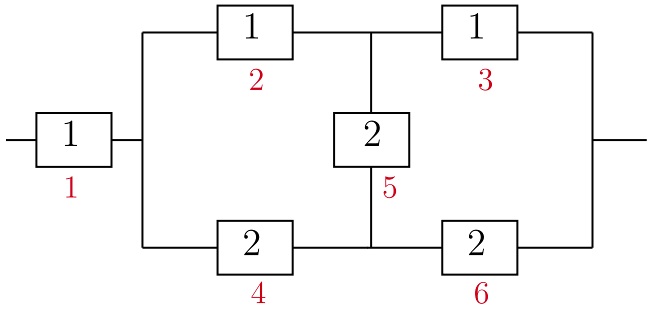


Figure 2.6: Bridge system with six components, numbered in red, of two types indicated in black.

survival signature is presented, followed by a brief overview of recent developments based on this concept.

Example 2.4.1 We demonstrate how the survival signature is calculated for the bridge system shown in Figure 2.6. In this system, there are two types of components: type 1 and type 2, as numbered within each component, with $m_1 = m_2 = 3$ being the number of components for both types. The survival signature must be identified for all $l_1, l_2 \in \{0, 1, 2, 3\}$, and this is shown in Table 2.1. In order to demonstrate how the survival signature is calculated, we will use the survival signature at $l_1 = 1$ and $l_2 = 2$, that is $\Phi(1, 2)$. The state vector for this system is $\mathbf{x} = (x^1_1, x^1_2, x^1_3, x^2_4, x^2_5, x^2_6)$. $\Phi(1, 2)$, is the probability that the system functions given precisely 1 component of type 1 functions and 2 components of type 2 function, so we consider all vectors $\mathbf{x}$ with $x^1_1 + x^1_2 + x^1_3 = 1$ and $x^2_4 + x^2_5 + x^2_6 = 2$. There are $\binom{3}{1}\binom{3}{2} = 9$ such vectors, but for only one of them the system functions, that is $\mathbf{x} = (1, 0, 0, 1, 0, 1)$. Given the assumption that the failure times of components of the same type are *iid*, and independence is also assumed for the failure times of components of different types, all the nine vectors have an equal probability of occurring. Therefore, $\Phi(1, 2) = 1/9$.

2.4.1 Developments based on survival signature

The emergence of survival signature has contributed to various facets of reliability analysis, including those involving relatively complex systems. In this subsection, we briefly highlight some of these developments. Coolen et al. [16] developed a non

l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_2	$\Phi(l_1, l_2)$
0	0	0	2	0	0
0	1	0	2	1	0
0	2	0	2	2	4/9
0	3	0	2	3	6/9
1	0	0	3	0	1
1	1	0	3	1	1
1	2	1/9	3	2	1
1	3	3/9	3	3	1

Table 2.1: Survival signature for bridge system shown in Figure 2.6.

parameteric predictive inference framework for system reliability based on survival signature. Patelli et al. [45][46] presented efficient simulation methods that relied on the survival signature to analyse the reliability of a large system. Feng et al. [26] presented a method based on the survival signature for coping with imprecision in the system resulting in upper and lower limits for its reliability function, and they also developed a new importance measure using the survival signature. Eryilmaz et al. [24] presented marginal and joint reliability importance measures for coherent systems with dependent and multiple component types, based on the survival signature.

From a Bayesian perspective, some studies have been developed that utilise the survival signature. Aslett et al. [6] demonstrate how knowledge and uncertainty regarding the reliability of components can be propagated to the system level. They assume that failure time data are available for tested components, possibly with right-censored data. The analysis encompasses both parametric and non-parametric approaches. Walter et al. [67] introduced a non parametric Bayesian approach that utilises sets of priors to analyse system reliability containing multiple types of components.

In Qin et al. [51] the reliability function is derived for multi-state systems with multi-binary-state components. Tavangar and Hashemi [64] employed the survival signature concept to evaluate the reliability of systems exposed to random external shocks. Additionally, they introduced maintenance policies for these systems based on representations of reliability derived from the survival signature. In recent studies, Hamdan et al. [31] have obtained the survival signature for a weighted k-out-of m system with multiple types of components and derived the system reliability function from this, and Zheng and Zhang [75] develop an analytical solution for the reliability of systems with dependent components based on the survival signature and copula theory.

2.4.2 Survival signature computation techniques

The survival signature offers a mechanism that aids in analysing relatively complex systems. Nonetheless, as with other classical approaches, the survival signature is highly influenced by the curse of dimensionality as the number of components and types increases. A number of efforts have been made in the literature with the aim of computing or approximating the survival signature of large systems. Aslett [5] [4] developed an R function that simplifies the calculation of the survival signature for systems with multiple component types. However, as the number of components in the system grows, the computational cost becomes more expensive. Reed [53] introduced an efficient method for exact computation of survival signature that is based on converting the fault tree diagram system into a binary decision diagram. The method performs well, as long as the system fault tree is already known. However, the challenge becomes more pronounced in the case of exceedingly large systems, necessitating the computation of the reliability structure. Xu et al. [70] proposed a new method for calculating the exact survival signature based on the extended universal generating function.

Several studies have also been conducted to approximate the survival signature. Behrensdorf et al. [9] propose a method for estimating the survival signature that relies on percolation theory and Monte Carlo simulation. The developed method is applied to a system containing 245 components of one type and 61 components of a different type. Di Maio et al. [20] proposed an approach to estimate the survival signature of complex infrastructures using entropy-based techniques within Monte Carlo simulations. Di Maioa et al. [21] address the problem of approximating the survival signature by framing it as a missing data problem. Their approach involves training an ensemble of Artificial Neural Networks (ANNs) using a dataset of survival signatures generated through Monte Carlo Simulation. Subsequently, this ensemble of trained ANNs is utilized to estimate the missing values of the survival signature.

2.5 Summary

This chapter reviewed some of the main reliability concepts introduced in the literature. It encompasses the definition of a state vector, structure function, path sets, and cut sets. Using these methods, the reliability of some simple systems was calculated. However, these methods may not be applicable when scaling up to large systems. The survival signature, introduced as an alternative method to address systems with multiple types of components, was also reviewed. Some methods for computing or approximating the survival signature, as well as developments using this signature, were also highlighted.

Chapter 3

Composition of component types at system failure

3.1 Introduction

A key aspect of studying systems is the diagnosis of the causes of system failure. We will refer to this as determining the components or types of components that are associated with system failure. System failure diagnosis is useful for several purposes, including maintenance activities and the design of a system. The purpose of this chapter is to examine system failure from two different viewpoints: determining which types of components are most likely to lead to system failure, and estimating the number of failed components of each type at the time of system failure.

To determine the components that cause system failure, some component importance measures have been discussed in the literature (e.g. [7], [38], [27]). However, these measures primarily depend on the structure function, which can be challenging for complex systems as seen in Chapter 2. Therefore, we devise a component probability for a component type leading to system failure at a particular time based on survival signature.

To determine the numbers of failed components at the moment of a system failure, a quantity particularly valuable for spare parts planning, only a few studies investigate such an event (e.g. [22], [23]). However, these studies have only considered systems with a single type of component. When systems with multiple types of components are considered, the investigations are limited to specific system structures. Moreover, the period of time during which the system was functioning prior to its failure, which affects the numbers of failed components at system failure, was not clearly considered. Therefore, for a general system, we derive probability distributions for the number of failed components of multiple types at the system failure time under different cases.

This chapter is organised as follows. Section 3.2 briefly reviews previous works introduced in the literature regarding components contributing to system failure and the numbers of failed components at system failure. A new measure of a component type causing system failure is developed in Section 3.3 based on the survival signature. Following that, three probability distributions are derived for the numbers of failed components of multiple types at the moment of system failure, each based on different conditions. Section 3.4 derives the distribution given exact system failure time and conditioning on the numbers of failed components of multiple types prior to system failure. The probability distribution of the number of failed components of multiple types at system failure, assuming the failure time is unknown, is derived in Section 3.5, taking into consideration the duration for which the system was functional. Section 3.6 derives the probability distribution of the number of failed components of multiple types at system failure, assuming system failure within a future time interval. Finally, Section 3.7 summarises key findings and suggests potential directions for future research.

3.2 Literature review

Determining the component or set of components related to the cause of system failure is not a new concept and has been studied in the literature. They are typically introduced as importance measures. The concept of importance measures was first introduced by Birnbaum in the 1960s [12] and defined as the probability that the functioning and failure of a specific component coincide with the system's functioning and failure, respectively. Since the introduction of Birnbaum's measure, various importance measures related to components causing system failure have been proposed.

Among these measures is the criticality importance measure for system failure, developed by Lambert in 1975 [38]. The measure is defined as the probability that a component is critical for the system at time t and fails by time t, assuming that the system has failed by time t. The definition of a critical component is as follows.

Definition 3.2.1 A component i is considered *critical* if its failure and functioning coincide with system failure and functioning, that is $\varphi(x_1, ..., x_{i-1}, x_i = 1, x_{i+1}, ..., x_m) = 1$ and $\varphi(x_1, ..., x_{i-1}, x_i = 0, x_{i+1}, ..., x_m) = 0$ [36] [37].

Another importance measure introduced in the literature is the Fussell-Vesely importance measure, developed by Fussell [27] and Vesely [66]. The measure comprises two types: the c-type, which evaluates whether a component participates in system failure through cut sets, and the p-type, which assesses whether a component contributes to system functioning through path sets. The c-type Fussell-Vesely measure for component i is defined as the probability that at least one minimal cut set containing component i fails by time t. The measure considers the possibility that a component i could play a role in system failure, when the component is not critical for the system.

One important measure closely related to a component being responsible for system failure is the Barlow and Proschan measure [7].

Definition 3.2.2 For a component i, Barlow and Proschan measure is defined as the probability that component i fails at time t given that the system fails at time t.

The measure depends primarily on the structure function of the system. Therefore, the use of this measure may be challenging with complex structural systems that are composed of a large number of components, as previously indicated in Chapter 2. For further details about these measures, the reader is referred to [36] and [69].

Besides finding the components or types of components that are most likely to trigger system failure, investigating the number of failed components in a broken (or functioning) system has recently received increased attention (e.g. [34], [25],

[18]). Understanding how this quantity is distributed can prove useful in many different applications. Implementing this event can assist systems administrators in decreasing expenses and increasing profits, and to utilise resources more effectively. In the context of spare parts planning, one can avoid unnecessary stocking of certain system components and keep the most critical ones.

Researchers have examined the distribution of the number failed components in a system that is still functioning. Asadi and Berred [3] considered the number of failed components in an operating $(m - k + 1)$-out-of-m system assuming that the failure times of all its components are independent and identically distributed, and they extended their results to any coherent system via system signature. For a k-out-of-m system that consists of multiple types of components, the distribution of the number of failed components is obtained by Eryilmaz [23] and used to find the optimal number of components of each type and optimal replacement time. Based on minimal path sets, Jasiński [34] has obtained the distribution for the number of failures for a general coherent system of multiple types of components.

The number of failed components at the moment of system failure has also been examined under a variety of conditions. For a coherent system that has exchangeable components' failure times having an absolutely continuous joint distribution, Eryilmaz [22] studied the distribution of the number of failed components when the system fails and concluded that it is equivalent to the system signature. Based on the same assumptions, the author also derived an expression for the number of failures that can occur after a certain point in time where the system was functioning until it fails, which he argued would be useful when calculating an extended warranty price.

When a system comprises multiple types of components, various studies have investigated the distribution of the number of failed components when the system fails. The distribution of the number of broken components of a specific component type in a failed k-out-of-m system constructed of different components types with its mean is considered by Eryilmaz [23]. For a series-parallel system when each subsystem has a specific type of distribution, Eryilmaz et al. [25] studied the joint distribution of the number of failures at the moment of system failure. The distribution of the same event has been considered by Eryilmaz et al. [19] for a series-parallel system consisting of multiple types of discrete failure time components. When the failure times of components are discretely distributed, Dembińska and Davies [18] obtained the distribution of failures when the system fails for a k-out-of-m system, with possibility of dependency between components and having multiple types of components. They considered the conditional distribution when the system fails at a specific point in time and also when the system was known to function at a certain time.

As noticed, the results of the discussed studies on the number of failed components when the system fails are limited to certain conditions. Among these limitations is the fact that some of the results were derived based on the concept of system signature, and hence are only applicable to systems with all components have exchangeable failure times. Another limitation arises when a system comprises multiple types of components, as the results are constrained to specific system structures, such as k-out-of-m or series-parallel configurations.

In the following sections, we first introduce a measure that utilises the survival signature in order to investigate the cause of the failure of the system. Then, we investigate the number of failed components in a coherent system composed of multiple types of components when it fails. Using the notion of survival signature, our objective is to determine the joint distribution of the numbers of failed components of each type upon system failure. The corresponding event is examined in three different settings. First, we consider the number of failed components of multiple types at a future moment at which the system fails, given the specific future failure time and the number of failed components of each type that occur prior to system failure. Secondly, this quantity is also studied when the system's failure time is unknown, assuming the system was functioning at some point in time. Thirdly, the number of failed components at the system failure time is considered assuming that the system will fail in a specific future time interval.

Throughout derivation of the probabilities of these events, we assume that the failure times of components of the same type are *iid* and independence is assumed for failure times of different types. Moreover, we assume that failure times of components are absolutely continuously distributed, that is, simultaneous failure of two or more components has probability zero.

3.3 Probability that a component type causes system failure

As explained in Section 3.2, Barlow and Proschan introduced a measure that finds the probability of a component causing system failure based on the structure function. In this section, we propose an alternative measure that is based on the survival signature. However, the survival signature does not allow individual components to be identified, so instead of finding the probability that a specific component caused system failure, we obtain an expression for the probability that system failure was caused by failure of a component of a specific type. As an illustration, having insight into this probability assists in arranging maintenance tasks. Components of the type with a high probability of causing system failure can receive top priority for maintenance, inspection, and monitoring.

Before proceeding to derive the probability of the event of interest, we introduce two lemmas that are helpful in this derivation.

Lemma 3.3.1 Consider a system with K types of components and m_k components of type k, where $k \in \{1, ..., K\}$. Let T_s denote the system failure time and T^k denote the failure time of a component of type k assuming it has an absolutely continuous distribution. Additionally, let $L_k(t)\big(L_k(t^-)\big)$ represent the number of functioning components of type k at time $t\big($at the time just before $t\big)$. Then the probability of having l_k functioning components by time t^- given that a component of type k fails exactly at t is

$$P(L_k(t^-) = l_k \mid T^k = t) = \frac{l_k h_k(t) P(L_k(t^-) = l_k)}{m_k f_k(t)},$$

where $h_k(t)$ and $f_k(t)$ denote the hazard rate and the probability density function of the failure time of components of type k at time t.

Proof: Let T_i^k denote the failure time of component i of type k and $Q_1, ..., Q_a$ are all possible subsets of $\{1, ..., m_k\}$ of size l_k, then

$$
\begin{aligned}
&P(L_k(t^-) = l_k \mid T^k = t) \\
&= P\left(\bigcup_{j=1}^{a}\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right) \mid T^k = t\right) \\
&= \sum_{j=1}^{a} P\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\} \mid T^k = t\right) \\
&= \sum_{j=1}^{a} P\left(T^k = t \mid \{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right) \\
&\quad \times P\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right)[P(T^k = t)]^{-1} \\
&= \sum_{j=1}^{a}\left(\sum_{i\in Q_j} P\left(T_i^k = t \mid T_i^k > t^-\right)\right) \\
&\quad \times P\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right)[P(T^k = t)]^{-1} \\
&\approx \sum_{j=1}^{a}\left(\sum_{i\in Q_j} P\left(t < T_i^k \leqslant t + \epsilon \mid T_i^k > t^-\right)\right) \\
&\quad \times P\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right)[P(t < T^k \leqslant t + \epsilon)]^{-1} \qquad \text{for small } \epsilon > 0.
\end{aligned}
$$

Multiplying the numerator and denominator by $\frac{1}{\epsilon}$ and taking the limit when $\epsilon \to 0$,

$$
\begin{aligned}
&P(L_k(t^-) = l_k \mid T^k = t) \\
&= \sum_{j=1}^{a}\left(\sum_{i\in Q_j} h_i^k(t)\right) P\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right)[m_k f_k(t)]^{-1} \\
&= l_k h_k(t)\left[\sum_{j=1}^{a} P\left(\{\bigcap_{i\in Q_j} T_i^k > t^-\} \cap \{\bigcap_{i\notin Q_j} T_i^k \leqslant t^-\}\right)\right][m_k f_k(t)]^{-1} \\
&= l_k h_k(t)\binom{m_k}{l_k}[1 - F_k(t^-)]^{l_k}[F_k(t^-)]^{m_k - l_k}[m_k f_k(t)]^{-1} \\
&= l_k h_k(t) P(L_k(t^-) = l_k)[m_k f_k(t)]^{-1}.
\end{aligned}
$$

□

Lemma 3.3.2 Consider a system with K types of components and m_k components of type k, where $k \in \{1, ..., K\}$. Let T_s denote the system failure time and T^k denote the failure time of a component of type k. Additionally, let $L_k(t)$ represent the number of functioning components of type k at time t. Assuming that the failure times of component types are absolutely continuously distributed, then

$$\begin{aligned}
&P(T_s = t \mid T^k = t, L_1(t) = l_1, \dots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \dots, L_K(t) = l_K, L_k(t^-) = l_k) \\
&= \Phi(l_1, \dots, l_{k-1}, l_k, l_{k+1}, \dots, l_K) - \Phi(l_1, \dots, l_{k-1}, l_k - 1, l_{k+1}, \dots, l_K)
\end{aligned}$$

Proof:

$$\begin{aligned}
&P(T_s = t \mid T^k = t, L_1(t) = l_1, \dots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \dots, L_K(t) = l_K, L_k(t^-) = l_k) \\
&= P(T_s = t \mid L_k(t) = l_k - 1, L_1(t) = l_1, \dots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \dots, L_K(t) = l_K, L_k(t^-) = l_k).
\end{aligned}$$

Since it is assumed that simultaneous failures of two or more components cannot occur, the number of functioning components not of type k at time t^- are the same as time t. Then,

$$\begin{aligned}
&P(T_s = t \mid L_k(t) = l_k - 1, L_1(t) = l_1, \dots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \dots, L_K(t) = l_K, L_k(t^-) = l_k) \\
&= P(T_s = t \mid L_k(t) = l_k - 1, L_1(t) = l_1, \dots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \dots, L_K(t) = l_K, L_1(t^-) = l_1, \dots, L_K(t^-) = l_K).
\end{aligned}$$

The probability of a system failing at time t given the number of the functioning components at times t^- and t equals the difference between the probability of the system functioning at time t^- given the number of functioning components at t^- and the probability of the system functioning at time t given the number of functioning components at t. That is the difference between the survival signatures at times t^- and t:

$$\Phi(l_1, \dots, l_{k-1}, l_k, l_{k+1}, \dots, l_K) - \Phi(l_1, \dots, l_{k-1}, l_k - 1, l_{k+1}, \dots, l_K).$$

$\square$

Definition 3.3.3 We define the following shorthand notation as this quantity will appear frequently in deriving the events of interest:

$$\Phi'_k(l_1,\ldots,l_K) := \Phi(l_1,\ldots,l_{k-1},l_k,l_{k+1},\ldots,l_K) - \Phi(l_1,\ldots,l_{k-1},l_k-1,l_{k+1},\ldots,l_K)$$

.

In the following, we consider the derivation of the probability of a component of a specific type causing system failure, followed by two examples. For a system comprising K types of components and m_k components of type k, where $k \in \{1, 2, ..., K\}$, we present the following theorem.

Theorem 3.3.4 Let T_s denote the system failure time and let T^k represent the failure time of a component of type $k \in \{1, 2, \ldots, K\}$, where the failure times of components are absolutely continuous distributed. We assume that the failure times of components of the same type are independent and identically distributed, and that the failure times of components of different types are independent. Denote by $L_k(t)$ the number of components of type k functioning at time t, and by $L_k(t^-)$ the number of components of type k functioning just before time t. If the probability distributions of failure times for all types of components are known, then the probability that a component of type k fails at time t, given that the system fails at time t,

$$\begin{aligned}
&P(T^k = t \mid T_s = t) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \left[\sum_{l_k=1}^{m_k} \Phi'_k(l_1,\ldots,l_K) l_k h_k(t) P(L_k(t^-) = l_k) \right] \\
&\quad \times P\left(\bigcap_{\substack{i=1 \\ i\neq k}}^{K} L_k(t) = l_k \right) \\
&\quad \times \left(\sum_{k=1}^{K} \left\{ \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \left[\sum_{l_k=1}^{m_k} \Phi'_k(l_1,\ldots,l_K) l_k h_k(t) P(L_k(t^-) = l_k) \right] \right.\right. \\
&\quad \left.\left. \times P\left(\bigcap_{\substack{i=1 \\ i\neq k}}^{K} L_k(t) = l_k \right) \right\} \right)^{-1}, \qquad (3.3.1)
\end{aligned}$$

where $h_k(t)$ denotes the hazard rate of the failure time of a component of type k, and $P(L_k(t) = l_k)$ is the probability of having l_k functioning components of type k at time t, which can be computed using the binomial distribution.

Proof:

$$P(T^k = t \mid T_s = t) = \frac{P(T_s = t \mid T^k = t)P(T^k = t)}{\sum_{k=1}^{K} P(T_s = t \mid T^k = t)P(T^k = t)},$$

where,

$$\begin{aligned}
&P(T_s = t \mid T^k = t) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \\
&\quad P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K) \\
&\quad \times P(L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K \mid T^k = t) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \\
&\quad P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K) \\
&\quad \times P(L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K)
\end{aligned}$$

(since all components not of type k are independent of type k components)

now,

$$\begin{aligned}
&P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K) \\
&= \sum_{l_k=0}^{m_k} P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K, L_k(t^-) = l_k) \\
&\quad \times P(L_k(t^-) = l_k \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l_k=0}^{m_k} P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K, L_k(t^-) = l_k) P(L_k(t^-) = l_k \mid T^k = t) \\
&= \sum_{l_k=0}^{m_k} P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, \\
&\quad L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K, L_k(t^-) = l_k) \\
&\quad \times \frac{l_k h_k(t) P(L_k(t^-) = l_k)}{m_k f_k(t)} \quad \text{(using Lemma 3.3.1)},
\end{aligned}
$$

thus,

$$
\begin{aligned}
&P(T_s = t \mid T^k = t) \\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \\
&\quad \left[\sum_{l_k=0}^{m_k} P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, \right. \\
&\quad L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K, L_k(t^-) = l_k) \\
&\quad \left. \times \frac{l_k h_k(t) P(L_k(t^-) = l_k)}{m_k f_k(t)} \right] \\
&\quad \times P(L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K).
\end{aligned}
$$

If $l_k = 0$, then

$$
\begin{gathered}
P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, \\
L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K, L_k(t^-) = l_k)
\end{gathered}
$$

becomes zero since there are no functioning components of type k cannot fail at $t > t^-$, therefore,

$$
\begin{aligned}
&P(T_s = t \mid T^k = t)\\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K}\\
&\quad \Bigg[\sum_{l_k=1}^{m_k} P(T_s = t \mid T^k = t, L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1},\\
&\quad L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K, L_k(t^-) = l_k)\\
&\quad \times \frac{l_k h_k(t) P(L_k(t^-) = l_k)}{m_k f_k(t)} \Bigg]\\
&\quad \times P(L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K).
\end{aligned}
$$

The first part can now be expressed using the survival signature as in Lemma 3.3.2, as follows

$$
\begin{aligned}
&P(T_s = t \mid T^k = t)\\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \left[\sum_{l_k=1}^{m_k} \Phi'_k(l_1, \ldots, l_K) \frac{l_k h_k(t) P(L_k(t^-) = l_k)}{m_k f_k(t)} \right]\\
&\quad \times P(L_1(t) = l_1, \ldots, L_{k-1}(t) = l_{k-1}, L_{k+1}(t) = l_{k+1}, \ldots, L_K(t) = l_K).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&P(T^k = t \mid T_s = t)\\
&= \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \left[\sum_{l_k=1}^{m_k} \Phi'_k(l_1, \ldots, l_K) l_k h_k(t) P(L_k(t^-) = l_k) \right]\\
&\quad \times P\left(\bigcap_{\substack{i=1\\ i\neq k}}^{K} L_k(t) = l_k \right)\\
&\quad \times \left(\sum_{k=1}^{K} \left\{ \sum_{l_1=0}^{m_1} \cdots \sum_{l_{k-1}=0}^{m_{k-1}} \sum_{l_{k+1}=0}^{m_{k+1}} \cdots \sum_{l_K=0}^{m_K} \left[\sum_{l_k=1}^{m_k} \Phi'_k(l_1, \ldots, l_K) l_k h_k(t) P(L_k(t^-) = l_k) \right] \right.\right.\\
&\quad \left.\left. \times P\left(\bigcap_{\substack{i=1\\ i\neq k}}^{K} L_k(t) = l_k \right) \right\} \right)^{-1}.
\end{aligned}
$$

$\square$

The primary difference between this measure and the Barlow-Proschan measure is that the former is concerned with a specific component, whereas the latter is concerned with a component of a certain type (i.e. the derived measure represents the sum of the probability of the same type). This measure will be utilised in Chapter 4 to formulate a minimal repair kit. To provide insight into the derived probability, we consider two systems that contain multiple types of components.

Example 3.3.1 (Series-parallel system) The first system, illustrated in Figure 3.1, consists of four components of two types, type 1 and type 2, as indicated within the components. The system functions if at least one component in each subsystem functions. This system's survival signature is presented in Table 3.1. We assume that all failure times of the components are Exponentially distributed with a failure rate of 0.025 for components of type 1 and 0.05 for components of type 2. Based on the failure time distributions assumed, components of type 1 will tend to fail after type 2 components fail. Therefore, we would expect for type 1 components to be more likely to trigger system failure as shown in Figure 3.2. The yellow line indicates the probability that the system will function at time t. Accordingly, it appears that the most relevant period to examine for determining the cause of system failure is before approximately $t = 75$. This is because there is a very small probability that the system will fail at any point after this time.

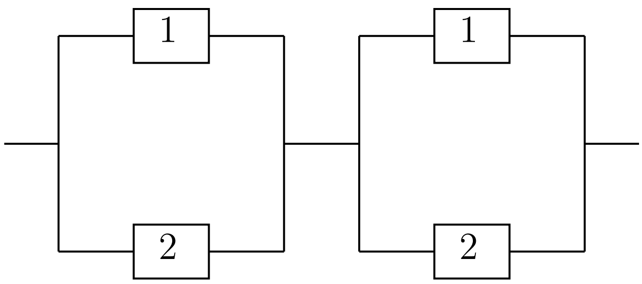


Figure 3.1: Series-parallel system with two types of components.

Example 3.3.2 (Hydro power plant system) This example illustrates how our measure is applied to a real-world hydro power plant system that consists of twelve components of six types. Figure 3.3 depicts the graphic design of the system. This system will be frequently used throughout the thesis, and in the following, we provide a description of its components. The first component in the system is the control gate (CG) situated within the dam, which manages the release and flow of water

l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_2	$\Phi(l_1, l_2)$
0	0	0	2	1	1
1	0	0	0	2	1
2	0	1	1	2	1
0	1	0	2	2	1
1	1	1/2			

Table 3.1: Survival signature of system in Figure 3.1

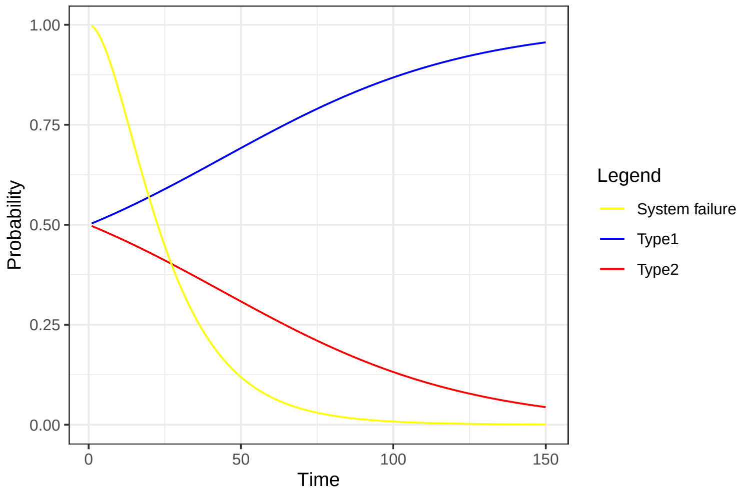


Figure 3.2: Probability of component types causing system failure for the system in Figure 3.1, with the yellow line indicating the probability of system failure exceeding time t.

from the reservoir. This water passes through two butterfly valves (BV1, BV2) that regulate its movement before reaching two turbines (T1, T2). These turbines, in turn, harness the kinetic energy of the flowing water, converting it into mechanical energy. To safeguard the system, three circuit breakers (CB1, CB2, CB3) are strategically placed to protect against electrical faults. The mechanical energy produced by the turbines is then converted into electrical energy by two generators (G1, G2), which produce alternating current. Finally, two transformers (TX1, TX2) within the powerhouse elevate the voltage of the alternating current for efficient transmission. The equivalent reliability diagram of the system is shown in Figure 3.5, where components are numbered according to type.

To apply the derived probability distribution, we assume that the failure times of all types of components follow a Weibull distribution, which has the following probability density function (PDF), with scale parameter α and shape parameter β, where where $\alpha, \beta > 0$

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}. \qquad \forall t \geqslant 0, 0 \text{ otherwise.}$$

We assume that the distributions of failure times of all types of components have the same scale $\alpha = 20$. However, each type has a different shape parameter β: 1.5 for type 1, 1.7 for type 2, 1.45 for type 3, 1.55 for type 4, 1.3 for type 5, and 1.6 for type 6. The system survival signature is presented in Table 3.2. Figure 3.4 shows the probability of each type triggering system failure at time t. The yellow line represents the probability that the system will function at time t, indicating that it will be unlikely to function after $t = 15$. The period up to $t = 15$ is, therefore, of most interest and should receive more attention when identifying the type that leads to system failure. If the system fails during this interval, it is highly probable that a type 5 component is the cause. This is likely because there are three components of type 5, and it has the lowest shape parameter, indicating a moderate decrease in the failure rate compared to the other types. Type 5 components are expected to fail last, making them the most likely responsible for causing the system failure. The component of type 1 is the second most likely to be the cause of system failure, with types 2, 3, and 4 having nearly identical probabilities of being the cause. Type 6, on the other hand, is the least likely to cause system failure.

3.4 Number of failed components given preceding failures

In this section, the number of failed components of multiple types at a future system failure time is considered, given that this time is assumed to be known. This event is also conditioned on the number of failures of different component types at various times before system failure. This probability distribution holds signifi-

l_1	l_2	l_3	l_4	l_5	l_6	$\Phi(l_1, l_2, l_3, l_4, l_5, l_6)$
1	1	1	1	2	[1,2]	1/12
1	1	1	2	2	[1,2]	1/6
1	1	2	1	2	[1,2]	1/6
1	2	1	1	2	[1,2]	1/6
1	1	1	1	3	[1,2]	1/4
1	1	2	2	2	[1,2]	1/3
1	2	1	2	2	[1,2]	1/3
1	2	2	1	2	[1,2]	1/3
1	1	1	2	3	[1,2]	1/2
1	1	2	1	3	[1,2]	1/2
1	2	1	1	3	[1,2]	1/2
1	2	2	2	2	[1,2]	2/3
1	1	2	2	3	[1,2]	1
1	2	1	2	3	[1,2]	1
1	2	2	[1,2]	3	[1,2]	1

Table 3.2: Survival signature of system in Figure 3.5 where $\Phi(l_1, l_2, l_3, l_4, l_5, l_6) = 0$ are excluded [26].

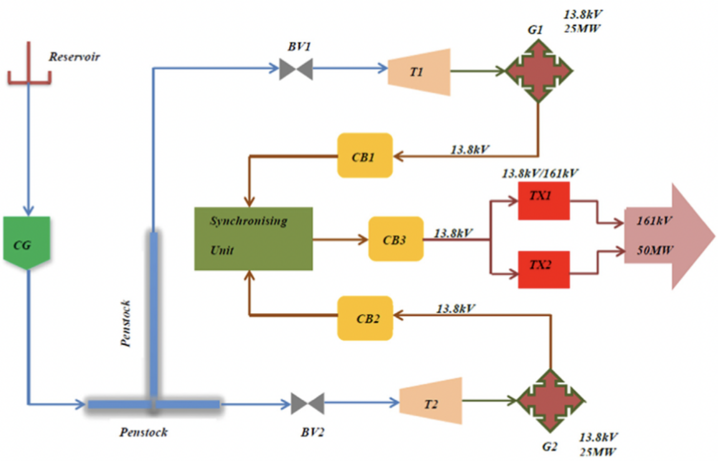


Figure 3.3: A graphic representation of the hydro power system [26].

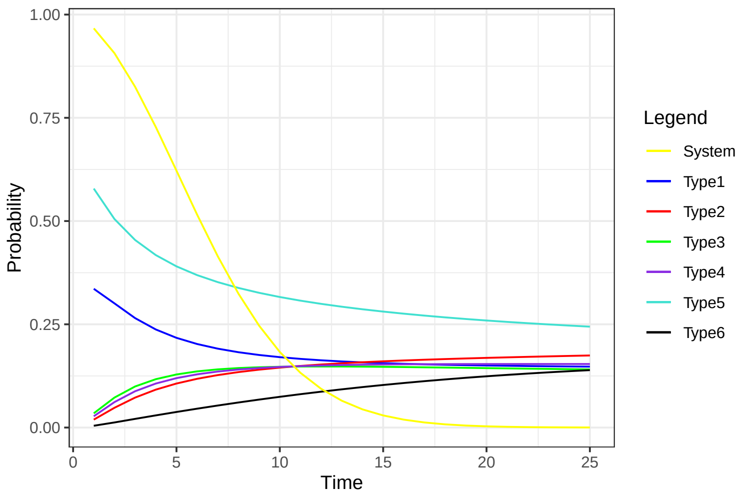


Figure 3.4: Probability of types leading to failure of the system shown in Figure 3.3, with the yellow line indicating the probability of system failure exceeding time t.

cant importance and offers various practical applications. It can aid in optimising spare parts inventory by prioritising combinations of components of multiple types with higher probabilities, while minimising stock for those with lower probabilities. This approach prevents overstocking and costs reduction. The specific information regarding the number of failures that occur before system failure is critical in determining if certain failure events tend to trigger subsequent failures.

For simplicity, we first consider a system containing two types of components and determine the probability distribution for the number of failed components of one type at the moment of system failure, given that there were some observed

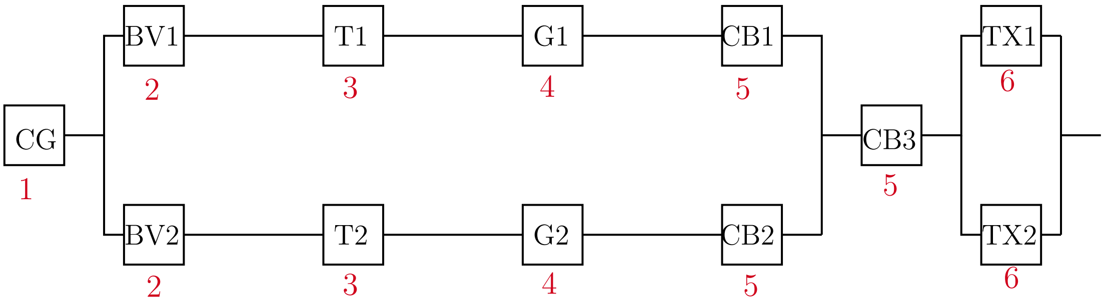


Figure 3.5: Hydro power plant system with 6 component types, indicated in red.

failed components of that type before system failure, and assuming system failure at a future time. The result can then be generalised to systems with multiple types of components. For a system that contains two types of components, with m_1 components of type 1 and m_2 components of type 2, respectively, the following theorem provides the distribution result along with its proof.

Theorem 3.4.1 Let T^k represent the failure time of a component of type $k \in \{1, 2\}$, where the failure times of components are absolutely continuous distributed. We assume that the failure times of components of the same type are independent and identically distributed, and that the failure times of components of different types are independent. We denote T_s as the random variable representing the system failure time, and $X_k(t)$ as the random variable representing the number of failed components of type k at time t. If we assume that the system will fail at a future time $t^\star$ and there were c_1 failed components of type 1 at time t_1 before system failure, then for $t^\star > t_1$ and $x_1 \geqslant c_1$, the probability of the event $X_1(t^\star) = x_1 \mid X_1(t_1) = c_1$,

$T_s > t^{\star-}$, $T_s = t^\star$, where $t^{\star-}$ is the time just before time $t^\star$, is given by

$$
\begin{aligned}
&P(X_1(t^\star) = x_1 \mid X_1(t_1) = c_1, T_s > t^{\star-}, T_s = t^\star) \\
&= \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)](m_1 - x_1 + 1)h_1(t^\star) \\
&\quad \times P(X_1(t^{\star-}) = x_1 - 1)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)](m_2 - x_2 + 1)h_2(t^\star) \\
&\quad \times P(X_2(t^{\star-}) = x_2 - 1)[P(X_2(t^\star) = x_2)]^{-1}\big] P(X_2(t^\star) = x_2)\Big] \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times \Big[\sum_{x_1=c_1}^{m_1} \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)](m_1 - x_1 + 1) \\
&\quad \times h_1(t^\star)P(X_1(t^{\star-}) = x_1 - 1)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)](m_2 - x_2 + 1)h_2(t^\star) \\
&\quad \times P(X_2(t^{\star-}) = x_2 - 1)[P(X_2(t^\star) = x_2)]^{-1}\big] P(X_2(t^\star) = x_2)\Big] \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\Big]^{-1},
\end{aligned}
\tag{3.4.2}
$$

where $h_k(t^\star)$ denotes the hazard rate of the failure time of a component of type k, and $P(X_k(t^\star) = x_k)$ is the probability of the number of failed components of type k at time $t^\star$, which can be computed using the binomial distribution.

Proof:

$$
\begin{aligned}
&P(X_1(t^\star) = x_1 \mid X_1(t_1) = c_1, T_s > t^{\star-}, T_s = t^\star) \\
&= P(T_s = t^\star, X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star-}) \big(P(X_1(t_1) = c_1, T_s > t^{\star-}, T_s = t^\star)\big)^{-1} \\
&= \big(P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star-}) \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star-})\big) \big(P(X_1(t_1) = c_1, T_s > t^{\star-}, T_s = t^\star)\big)^{-1} \\
&= \big(P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star-}) \\
&\quad \times P(T_s > t^{\star-} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1) P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\big) \\
&\quad \times \big(P(X_1(t_1) = c_1, T_s > t^{\star-}, T_s = t^\star)\big)^{-1}.
\end{aligned}
\tag{3.4.3}
$$

To determine the first probability in the numerator on the right hand side of Equation (3.4.3), we condition on all possible numbers of failed components of type 2, since they affect the probability of the system failing at time $t^\star$,

$$\begin{aligned}
&P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}) \\
&= \sum_{x_2=0}^{m_2} P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2) \\
&\qquad \times P(X_2(t^\star) = x_2 \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}).
\end{aligned} \tag{3.4.4}$$

Due to the assumption that two or more components, of the same or different types, fail simultaneously with probability zero, only the failure of one component coincides with system failure. System failure at time $t^\star$ can be due to failure of a component of type 1 or of type 2 at time $t^\star$. We derive the first term of (3.4.4) as follows.

$$\begin{aligned}
&P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2) \\
&= P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2, T^1 = t^\star) \\
&\quad \times P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2) \\
&\quad + P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2, T^2 = t^\star) \\
&\quad \times P(T^2 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)
\end{aligned}$$

$$\begin{aligned}
&= P(T_s = t^\star \mid X_1(t^\star) = x_1, T_s > t^{\star -}, X_2(t^\star) = x_2, T^1 = t^\star) \\
&\quad \times P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2) \\
&\quad + P(T_s = t^\star \mid X_1(t^\star) = x_1, T_s > t^{\star -}, X_2(t^\star) = x_2, T^2 = t^\star) \\
&\quad \times P(T^2 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)
\end{aligned}$$

$$\begin{aligned}
&= \frac{P(T_s = t^\star \mid X_1(t^\star) = x_1, X_2(t^\star) = x_2, T^1 = t^\star)}{P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_2(t^\star) = x_2, T^1 = t^\star)} \\
&\quad \times P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2) \\
&\quad + \frac{P(T_s = t^\star \mid X_1(t^\star) = x_1, X_2(t^\star) = x_2, T^2 = t^\star)}{P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_2(t^\star) = x_2, T^2 = t^\star)} \\
&\quad \times P(T^2 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2).
\end{aligned}$$

Using Lemma 3.3.2,

$$\begin{aligned}
&P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)\\
&= \frac{[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)]}{\Phi(m_1 - x_1 + 1, m_2 - x_2)}\\
&\quad \times P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)\\
&\quad + \frac{[\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)]}{\Phi(m_1 - x_1, m_2 - x_2 + 1)}\\
&\quad \times P(T^2 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)
\end{aligned} \tag{3.4.5}$$

where,

$$\begin{aligned}
&P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)\\
&= P(T_s > t^{\star -} \mid T^1 = t^\star, X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)\\
&\quad \times P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)\\
&\quad \times \left(P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)\right)^{-1}.
\end{aligned} \tag{3.4.6}$$

Now, $X_2(t^\star) = x_2$ in the second probability of the numerator of Equation (3.4.6) is omitted because we assume independence between failure times of components of type 1 and type 2. Thus,

$$\begin{aligned}
&P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)\\
&= \frac{\Phi(m_1 - x_1 + 1, m_2 - x_2) P(T^1 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1)}{P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)}\\
&= \frac{\Phi(m_1 - x_1 + 1, m_2 - x_2) P(T^1 = t^\star \mid X_1(t^\star) = x_1)}{P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)}.
\end{aligned} \tag{3.4.7}$$

Similar to the derivation of Equation (3.4.7),

$$\begin{aligned}
&P(T^2 = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2)\\
&= \frac{\Phi(m_1 - x_1, m_2 - x_2 + 1) P(T^2 = t^\star \mid X_2(t^\star) = x_2)}{P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)}.
\end{aligned} \tag{3.4.8}$$

Substituting the results of Equations (3.4.7) and (3.4.8) into Equation (3.4.5), we get

$$
\begin{aligned}
&P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}, X_2(t^\star) = x_2) \\
&= \big([\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)]P(T^1 = t^\star \mid X_1(t^\star) = x_1) \\
&\quad +[\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)]P(T^2 = t^\star \mid X_2(t^\star) = x_2)\big) \\
&\quad \times \big(P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, X_2(t^\star) = x_2)\big)^{-1}.
\end{aligned}
$$

The second term of Equation (3.4.4) is calculated as

$$
\begin{aligned}
&P(X_2(t^\star) = x_2 \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}) \\
&= P(T_s > t^{\star -} \mid X_2(t^\star) = x_2, X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times P(X_2(t^\star) = x_2, X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times \big(P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1)P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\big)^{-1} \\
&= P(T_s > t^{\star -} \mid X_2(t^\star) = x_2, X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times P(X_2(t^\star) = x_2 \mid X_1(t^\star) = x_1, X_1(t_1) = c_1)P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times \big(P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1)P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\big)^{-1} \\
&= P(T_s > t^{\star -} \mid X_2(t^\star) = x_2, X_1(t^\star) = x_1, X_1(t_1) = c_1)P(X_2(t^\star) = x_2) \\
&\quad \times \big(P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1)\big)^{-1}.
\end{aligned}
$$

Then, substituting these expressions into Equation (3.4.4) we find,

$$
\begin{aligned}
&P(T_s = t^\star \mid X_1(t^\star) = x_1, X_1(t_1) = c_1, T_s > t^{\star -}) \\
&= \sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)]P(T^1 = t^\star \mid X_1(t^\star) = x_1) \\
&\qquad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)]P(T^2 = t^\star \mid X_2(t^\star) = x_2)\big] \\
&\qquad \times P(X_2(t^\star) = x_2)[P(T_s > t^{\star -} \mid X_1(t^\star) = x_1, X_1(t_1) = c_1)]^{-1}.
\end{aligned}
$$

Substituting this into Equation (3.4.3),

$$
\begin{aligned}
&P(X_1(t^\star) = x_1 \mid X_1(t_1) = c_1, T_s > t^{\star -}, T_s = t^\star) \\
&= \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^1 = t^\star \mid X_1(t^\star) = x_1) \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^2 = t^\star \mid X_2(t^\star) = x_2)\big] \\
&\quad \times P(X_2(t^\star) = x_2)\Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times [P(X_1(t_1) = c_1, T_s > t^{\star -}, T_s = t^\star)]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^1 = t^\star \mid X_1(t^\star) = x_1) \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^2 = t^\star \mid X_2(t^\star) = x_2)\big] \\
&\quad \times P(X_2(t^\star) = x_2)\Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times \Bigg[\sum_{x_1=c_1}^{m_1} \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times P(T^1 = t^\star \mid X_1(t^\star) = x_1) \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times P(T^2 = t^\star \mid X_2(t^\star) = x_2)\big] P(X_2(t^\star) = x_2)\Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\Bigg]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^1 = t^\star) \\
&\quad \times P(X_1(t^\star) = x_1 \mid T^1 = t^\star)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^2 = t^\star) \\
&\quad \times P(X_2(t^\star) = x_2 \mid T^2 = t^\star)[P(X_2(t^\star) = x_2)]^{-1} \big] \\
&\quad \times P(X_2(t^\star) = x_2) \Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\times \Bigg[\sum_{x_1=c_1}^{m_1} \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^1 = t^\star) \\
&\quad \times P(X_1(t^\star) = x_1 \mid T^1 = t^\star)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^2 = t^\star) \\
&\quad \times P(X_2(t^\star) = x_2 \mid T^2 = t^\star)[P(X_2(t^\star) = x_2)]^{-1} \big] \\
&\quad \times P(X_2(t^\star) = x_2) \Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \Bigg]^{-1}.
\end{aligned}
$$

We consider the probability of a component failure time of type k being precisely equal to $t^\star$ as being negligible since, for continuous random variables, the probability of an exact value is zero. Instead, we approximate this probability by considering the component falling within a small interval around $t^\star$, denoted as $(t^\star, t^\star + \epsilon]$. The same applies for the probability of the system failure time being equal to $t^\star$. Then,

$$
\begin{aligned}
&P(X_1(t^\star) = x_1 \mid X_1(t_1) = c_1, T_s > t^{\star-}, T_s = t^\star) \\
&\approx \Bigg[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(t^\star < T^1 \leqslant t^\star + \epsilon) \\
&\quad \times P(X_1(t^\star) = x_1 \mid T^1 = t^\star)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(t^\star < T^2 \leqslant t^\star + \epsilon) \\
&\quad \times P(X_2(t^\star) = x_2 \mid T^2 = t^\star)[P(X_2(t^\star) = x_2)]^{-1}\big] P(X_2(t^\star) = x_2)\Bigg] \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times \Bigg[\sum_{x_1=c_1}^{m_1} \Bigg[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times P(t^\star < T^1 \leqslant t^\star + \epsilon) P(X_1(t^\star) = x_1 \mid T^1 = t^\star)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(t^\star < T^2 \leqslant t^\star + \epsilon) \\
&\quad \times P(X_2(t^\star) = x_2 \mid T^2 = t^\star)[P(X_2(t^\star) = x_2)]^{-1}\big] P(X_2(t^\star) = x_2)\Bigg] \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\Bigg]^{-1} \qquad \text{for small } \epsilon > 0.
\end{aligned}
$$

Multiplying by $\frac{1}{\epsilon}$ then taking the limit when $\epsilon \to 0$, $P(t^\star < T^k \leqslant t^\star + \epsilon)$ will result in $m_k f_k(t^\star)$ because there are m_k components of type k, then

$$\begin{aligned}
&P(X_1(t^\star) = x_1 \mid X_1(t_1) = c_1, T_s > t^{\star -}, T_s = t^\star) \\
&= \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] m_1 f_1(t^\star) \\
&\qquad \times P(X_1(t^\star) = x_1 \mid T^1 = t^\star)[P(X_1(t^\star) = x_1)]^{-1} \\
&\qquad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] m_2 f_2(t^\star) \\
&\qquad \times P(X_2(t^\star) = x_2 \mid T^2 = t^\star)[P(X_2(t^\star) = x_2)]^{-1}\big] \\
&\qquad \times P(X_2(t^\star) = x_2)\Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\quad \times \Bigg[\sum_{x_1=c_1}^{m_1} \Big[\sum_{x_2=0}^{m_2} \big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] m_1 f_1(t^\star) \\
&\qquad \times P(X_1(t^\star) = x_1 \mid T^1 = t^\star)[P(X_1(t^\star) = x_1)]^{-1} \\
&\qquad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] m_2 f_2(t^\star) \\
&\qquad \times P(X_2(t^\star) = x_2 \mid T^2 = t^\star)[P(X_2(t^\star) = x_2)]^{-1}\big] \\
&\qquad \times P(X_2(t^\star) = x_2)\Big] P(X_1(t^\star) = x_1, X_1(t_1) = c_1)\Bigg]^{-1}.
\end{aligned}$$

Using Lemma (3.3.1),

$$\begin{aligned}
&P(X_k(t^\star) = x_k \mid T^k = t^\star) \\
&= P(X_k(t^{\star -}) = x_k - 1 \mid T^k = t^\star) = \frac{(m_k - x_k + 1) h_k(t^\star) P(X_k(t^{\star -}) = x_k - 1)}{m_k f_k(t^\star)}.
\end{aligned}$$

Then,

$$
\begin{aligned}
&P(X_1(t^\star) = x_1 \mid X_1(t_1) = c_1, T_s > t^{\star -}, T_s = t^\star) \\
&= \Bigg[\sum_{x_2=0}^{m_2} \Big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)](m_1 - x_1 + 1) h_1(t^\star) \\
&\quad \times P(X_1(t^{\star -}) = x_1 - 1)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)](m_2 - x_2 + 1) h_2(t^\star) \\
&\quad \times P(X_2(t^{\star -}) = x_2 - 1)[P(X_2(t^\star) = x_2)]^{-1} \Big] P(X_2(t^\star) = x_2) \Bigg] \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \\
&\times \Bigg[\sum_{x_1=c_1}^{m_1} \Bigg[\sum_{x_2=0}^{m_2} \Big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)](m_1 - x_1 + 1) \\
&\quad \times h_1(t^\star) P(X_1(t^{\star -}) = x_1 - 1)[P(X_1(t^\star) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)](m_2 - x_2 + 1) h_2(t^\star) \\
&\quad \times P(X_2(t^{\star -}) = x_2 - 1)[P(X_2(t^\star) = x_2)]^{-1} \Big] P(X_2(t^\star) = x_2) \Bigg] \\
&\quad \times P(X_1(t^\star) = x_1, X_1(t_1) = c_1) \Bigg]^{-1}.
\end{aligned}
$$

□

This provides an expression for the probability of the number of failed components of one type for a system composed of two types of components. In the next subsection, we generalise the result to find the probability of the number of failed components of multiple types at system failure time for a system containing multiple types of components.

3.4.1 Case of multiple types of components

The probability distribution of the number of failed components at system failure, given the exact time of the failure and the number of failed components that occur prior to system failure, can be extended to systems with multiple types of components. For a system comprising K types of components and m_k components of type k, where $k \in \{1, 2, ..., K\}$, the following theorem provides the distribution result along with its proof.

Theorem 3.4.2 Let T^k denote the failure time of a component of type $k \in \{1, ..., K\}$, where the failure times of components are assumed to be absolutely continuous distributed. Further, assume that the failure times of components of the same type are independent and identically distributed, and that the failure times of components of different types are independent. Denote T_s as the random variable representing the system failure time, and $X_k(t)$ as the random variable representing the number of failed components of type k at time t. Suppose the system fails at a future time $t^\star$, and there were c_k failed components of type k at time t_k, for $k \in \{1, ..., K\}$, before system failure. Then, for $t^\star > t_1, t_2, ..., t_K$, $x_1 \geqslant c_1, x_2 \geqslant c_2, ..., x_K \geqslant c_K$, and $c_1, c_2, ..., c_K \neq x_1, x_2, ..., x_K$, the probability of the event $\bigcap_{k=1}^{K} X_k(t^\star) = x_k \mid \bigcap_{k=1}^{K} X_k(t_k) = c_k, T_s > t^{\star -}, T_s = t^\star$ is given by

$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K} X_k(t^\star) = x_k \mid \bigcap_{k=1}^{K} X_k(t_k) = c_k, T_s > t^{\star -}, T_s = t^\star\right) \\
&= \sum_{k=1}^{K}[\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
&\quad \times (m_k - x_k + 1)h_k(t^\star)P(X_k(t^{\star -}) = x_k - 1)[P(X_k(t^\star) = x_k)]^{-1} \\
&\quad \times \prod_{k=1}^{K} P\left(X_k(t^\star) = x_k, X_k(t_k) = c_k\right) \\
&\quad \times \left[\sum_{x_1=c_1}^{m_1} \cdots \sum_{x_K=c_K}^{m_K} \left[\sum_{k=1}^{K} \right.\right. \\
&\quad [\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
&\quad \times (m_k - x_k + 1)h_k(t^\star)P(X_k(t^{\star -}) = x_k - 1)[P(X_k(t^\star) = x_k)]^{-1} \\
&\quad \left.\left. \times \prod_{k=1}^{K} P\left(X_k(t^\star) = x_k, X_k(t_k) = c_k\right)\right]\right]^{-1},
\end{aligned} \tag{3.4.9}
$$

where, $h_k(t^\star)$ represents the hazard rate of component type k at time $t^\star$, while $P(X_k(t^\star) = x_k)$ denotes the probability of x_k failed components of type k at time $t^\star$, calculated through the binomial distribution.

Proof:

$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k \mid \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}, T_s=t^\star\right) \\
&=P\left(T_s=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}\right) \\
&\quad\times P\left(T_s>t^{\star-} \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k\right) \\
&\quad\times P\left(\bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k\right) \\
&\quad\times\left(P\left(\bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}, T_s=t^\star\right)\right)^{-1}.
\end{aligned} \tag{3.4.10}
$$

The first probability in the numerator of Equation 3.4.10 is calculated as

$$
\begin{aligned}
&P\left(T_s=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}\right) \\
&=\sum_{k=1}^{K} P\left(T_s=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}, T^k=t^\star\right) \\
&\quad\times P\left(T^k=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}\right) \\
&=\sum_{k=1}^{K} P\left(T_s=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, T_s>t^{\star-}, T^k=t^\star\right) \\
&\quad\times P\left(T^k=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right)=c_k, T_s>t^{\star-}\right).
\end{aligned}
$$

The first probability in the summation is determined as

$$
\begin{aligned}
&P\left(T_s=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, T_s>t^{\star-}, T^k=t^\star\right) \\
&=\frac{P\left(T_s=t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, T^k=t^\star\right)}{P\left(T_s>t^{\star-} \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right)=x_k, T^k=t^\star\right)}.
\end{aligned}
$$

Using Lemma 3.3.2 with some modifications to express it in terms of the number of failed components instead of functioning components,

$$
\begin{aligned}
&P\left(T_s = t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, T^k = t^\star\right)\\
&= [\Phi(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)\\
&\quad - \Phi(m_1 - x_1, \ldots, m_K - x_K)]\\
&= \Phi'_k(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&P\left(T_s = t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, T_s > t^{\star-}, T^k = t^\star\right)\\
&= \frac{\Phi'_k(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)}{\Phi(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, \ldots, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)}.
\end{aligned}
$$

The second probability in the summation is calculated as

$$
\begin{aligned}
&P\left(T^k = t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k, T_s > t^{\star-}\right)\\
&= P\left(T_s > t^{\star-} \mid T^k = t^\star, \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k\right)\\
&\quad \times P\left(T^k = t^\star \mid X_k\left(t^\star\right) = x_k\right)\\
&\quad \times \left(P\left(T_s > t^{\star-} \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k\right)\right)^{-1}\\
&= \Phi(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, \ldots, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)\\
&\quad \times P\left(T^k = t^\star \mid X_k\left(t^\star\right) = x_k\right)\\
&\quad \times \left(P\left(T_s > t^{\star-} \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k\right)\right)^{-1}.
\end{aligned}
$$

Substituting the results of these two probabilities back into the summation, we obtain

$$
\begin{aligned}
&P\left(T_s = t^\star \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k, T_s > t^{\star-}\right)\\
&= \sum_{k=1}^{K}[\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)]\\
&\quad \times P\left(T^k = t^\star \mid X_k\left(t^\star\right) = x_k\right)\\
&\quad \times \left(P\left(T_s > t^{\star-} \mid \bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k, \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k\right)\right)^{-1}.
\end{aligned}
$$

Substituting back into Equation 3.4.10, we get

$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k \mid \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k, T_s > t^{\star-}, T_s = t^\star\right)\\
&= \sum_{k=1}^{K}[\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)]\\
&\quad \times P\left(T^k = t^\star \mid X_k\left(t^\star\right) = x_k\right) \prod_{k=1}^{K} P\left(X_k\left(t^\star\right) = x_k, X_k\left(t_k\right) = c_k\right)\\
&\quad \times \left[\sum_{x_1=c_1}^{m_1} \cdots \sum_{x_K=c_K}^{m_K} \left[\sum_{k=1}^{K}\right.\right.\\
&[\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)]\\
&\quad \left.\left. \times P\left(T^k = t^\star \mid X_k\left(t^\star\right) = x_k\right) \prod_{k=1}^{K} P\left(X_k\left(t^\star\right) = x_k, X_k\left(t_k\right) = c_k\right)\right]\right]^{-1}.
\end{aligned}
$$

To find $P\left(T^k = t^\star \mid X_k\left(t^\star\right) = x_k\right)$, we use Lemma 3.3.1, so,

$$
\begin{aligned}
& P\left(\bigcap_{k=1}^{K} X_k\left(t^\star\right) = x_k \mid \bigcap_{k=1}^{K} X_k\left(t_k\right) = c_k, T_s > t^{\star-}, T_s = t^\star\right) \\
& = \sum_{k=1}^{K}[\Phi'_k(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
& \quad \times (m_k - x_k + 1)h_k(t^\star)P(X_k(t^{\star-}) = x_k - 1)[P(X_k(t^\star) = x_k)]^{-1} \\
& \quad \times \prod_{k=1}^{K} P\left(X_k\left(t^\star\right) = x_k, X_k\left(t_k\right) = c_k\right) \\
& \quad \times \left[\sum_{x_1=c_1}^{m_1} \cdots \sum_{x_K=c_K}^{m_K}\left[\sum_{k=1}^{K}\right.\right. \\
& \quad [\Phi'_k(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
& \quad \times (m_k - x_k + 1)h_k(t^\star)P(X_k(t^{\star-}) = x_k - 1)[P(X_k(t^\star) = x_k)]^{-1} \\
& \quad \left.\left. \times \prod_{k=1}^{K} P\left(X_k\left(t^\star\right) = x_k, X_k\left(t_k\right) = c_k\right)\right]\right]^{-1}.
\end{aligned}
$$

$\square$

Remark 3.4.1 Note that if $X_k(t^\star)$ denotes the numbers of failed components of type k at time $t^\star$ with a CDF $F_k(t^\star)$ for the failure time of components of type k, then for $x_k \in \{0, \ldots, m_k\}$,

$$
P(X_k(t^\star) = x_k) = \binom{m_k}{m_k - x_k}[F_k(t^\star)]^{x_k}[1 - F_k(t^\star)]^{m_k - x_k}.
$$

Also note that the probability $P\left(X_k(t^\star) = x_k, X_k(t_k) = c_k\right)$ can be calculated using a Multinomial distribution with c_k events $\in [0, t_k]$, $x_k - c_k \in [t_k, t^\star]$, and $m_k - x_k \in [t^\star, \infty]$, that is

$$
\begin{aligned}
& P\Big(X_k(t^\star) = x_k, X_k(t_k) = c_k\Big) \\
& = \binom{m_k}{c_k, x_k - c_k, m_k - x_k}[F_k(t_k)]^{c_k}[F_k(t^\star) - F_k(t_k)]^{x_k - c_k}[1 - F_k(t^\star)]^{m_k - x_k}.
\end{aligned}
$$

Example 3.4.1 (Series-parallel system) We show the probability of the number of failed components of multiple types with a simple example based on the system shown in Figure 3.6, which has two types of components. For both types, we assume that failure times are Exponentially distributed with a failure rate of 0.4 for type 1 and 0.2 for type 2. Figure 3.7 illustrates the probability of the number of failed components of type 1 and type 2 at different times of system failure, assuming one failure of type 1 occurred at time $t_1 = 10$ and one failure of type 2 at $t_2 = 5$ (Figure 3.7a), one failure of type 1 occurred at time $t_1 = 10$ and one failure of type 2 at $t_2 = 2$ (Figure 3.7b), and one failure of type 1 occurred at time $t_1 = 10$ and one failure of type 2 at $t_2 = 10$ (Figure 3.7c). The figure only includes the possible events of the number of failed components at the time of system failure, namely $X_1(t^*) = 2, X_2(t^*) = 1$ and $X_1(t^*) = 1, X_2(t^*) = 2$. In general, it appears that having two malfunctioning components of type 1 and one malfunctioning component of type 2 at the time of system failure is more likely to occur. If the failure of type 2 occurs before the failure of type 1, there is a higher probability of having two type 2 failures and one type 1 failure when the system fails early as shown in the first two figures. Figure 3.7b, with $t_1 = 10$ and $t_2 = 2$, shows that when the failure of type 2 happens very early compared to Figure 3.7a ($t_2 = 5$), the probability of having two type 2 failures and one type 1 failure increases for early system failures.

Figure 3.8 illustrates the probability of type 1 and type 2 component failures at the system failure time, assuming there were no prior failures of either type at different moments. Here, there are three possible combinations of the number of failed components of the two types at system failure. It is generally most likely to have two failures of type 1 and one failure of type 2 due to the high failure rate of type 1 components. If the zero failures of type 2 observed at earlier times than the zero failures of type 1, as shown in Figure 3.8a and 3.8b, the most likely scenario is to have two failures of type 2 and one failure of type 1 when the system fails early. Figure 3.8c illustrates that when the zero failures of the two types are observed at the same time ($t_1 = t_2 = 10$), and the system subsequently fails quickly, say at $t^* = 11$, it is most likely to have one failure of type 1 and one failure of type 2.

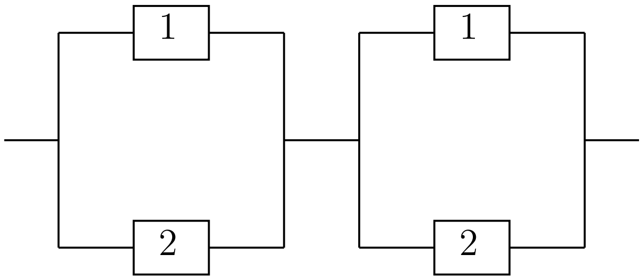
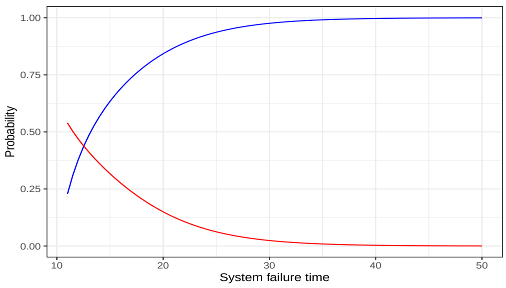


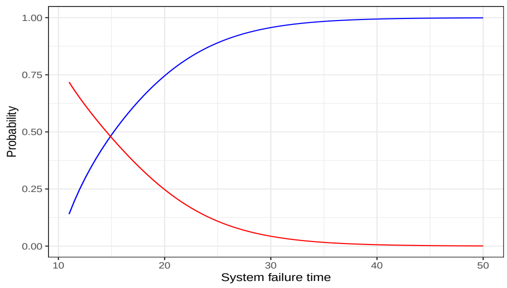
Figure 3.6: Series-parallel system with two types of components.

Example 3.4.2 (Hydro power system) The second example we present is based on the hydro power system, which consists of six different types of components, as illustrated in Figure 3.5. Figure 3.9 shows the probability of the number of failed components at system failure time, given that zero failures of type 1 and one failure of the types 2, 3, 4, 5, and 6 occurred prior to system failure at different moments. We assume that the failure times of components of types 1, 2, 3, 4, 5, and 6 have Exponential distributions with failure rates of 0.2, 0.5, 0.4, 0.3, 0.6, and 0.7, respectively. In this system, and given the earlier failures that occurred before system failure, there are only six possible combinations of failed components of multiple types at the time of system failure. Figure 3.9a shows the probability of the number of failures when the system fails, assuming that the previous failures that occurred before system failure were observed at the same time, with $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = 3$. In this scenario, it is most likely that an extra failure of type 5 will occur at the system failure moment. This may be attributed to both the high failure rate of type 5 components and their specific location within the system. Figure 3.9b illustrates a scenario where the failure of type 4, which occurred before system failure, was observed at $t_4 = 1$, while the failures of the other types were observed at $t_1 = t_2 = t_3 = t_5 = t_6 = 3$. Under this setting, it is highly probable to experience an additional failure of type 4 when the system fails. If we assume that the failure of type 2 occurred by $t_2 = 2$ instead of $t_2 = 3$, then the chance of having two failures of type 2 at system failure is increased as shown in Figure 3.9c.

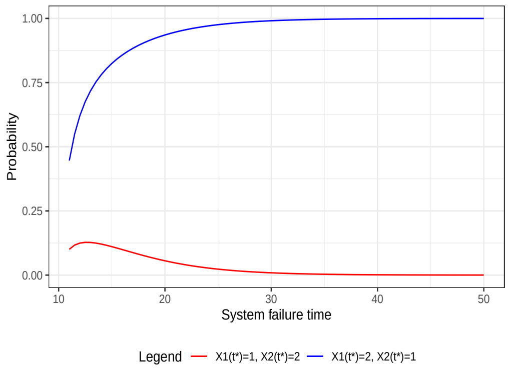
Figure 3.10 displays the probability distribution of the number of failed components at system failure. This distribution is conditioned on zero failures of type 1



(a) $X_1(10) = 1, X_2(5) = 1$



(b) $X_1(10) = 1, X_2(2) = 1$

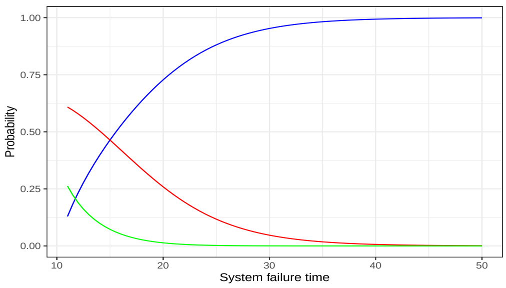


(c) $X_1(10) = 1, X_2(10) = 1$

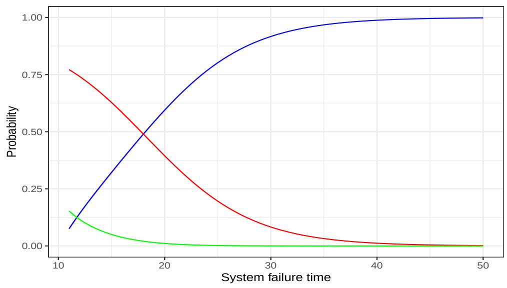
Figure 3.7: Probability of the number of failed components at system failure time for the System in Figure 3.6, considering one failure of type 1 and one failure of type 2 occurred before system failure at different times t_1, t_2.

and type 4 components, and one failure each of types 2, 3, 5, and 6 occurring prior to the system failure. Generally, it is observed that when assuming no failures of type 4 components occurred prior to system failure, the probability of the combination, which includes zero failures of type 1, one failure of types 2, 3, 5, and 6, and two failures of type 4, is less likely compared to the probability of the same combination when one failure of type 4 was observed at an earlier time before system failure, as shown in Figure 3.9.

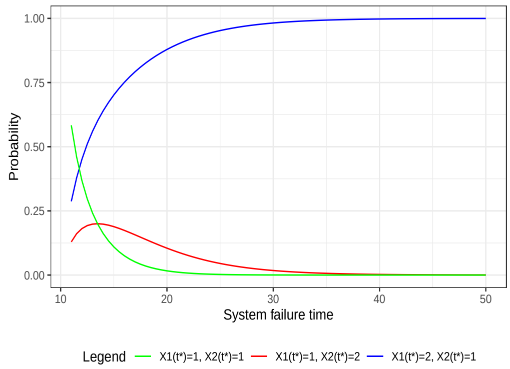
These two examples show that the number of failed components that observed prior to system failure and, moreover, the timing of these failures affect the number of failed components at the moment of system failure.



(a) $X_1(10) = 0, X_2(5) = 0$



(b) $X_1(10) = 0, X_2(2) = 0$

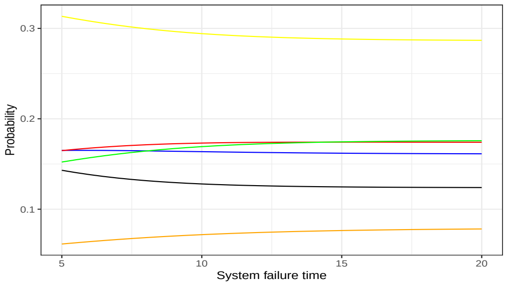


(c) $X_1(10) = 0, X_2(10) = 0$

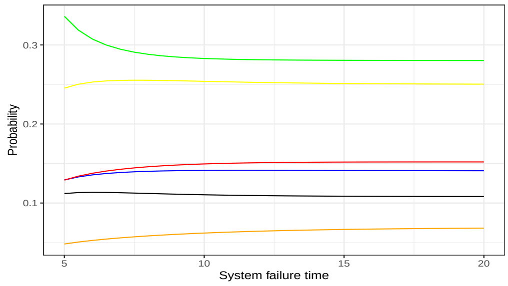
Figure 3.8: Probability of the number of failed components at system failure time for the System in Figure 3.6, considering zero failures of type 1 and zero failures of type 2 occurred before system failure at different times t_1, t_2.

3.5 Number of failed components at system failure given system age

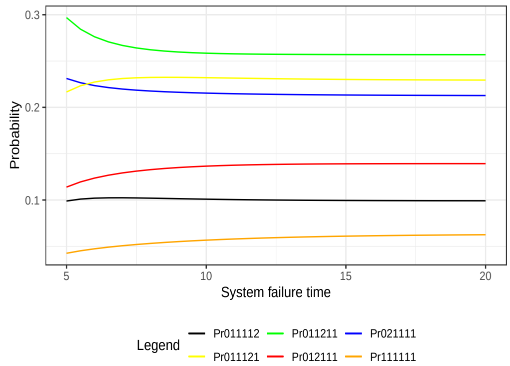
In this section, we examine the numbers of failed components of multiple types at a future unknown moment at which the system fails, given that the system was functioning at some point in time. The length of time the system was operational, which affects the probability of the number of failed components at the time of failure, is taken into account. Understanding the probability distribution of this event holds significant importance in various aspects. From a spare parts inventory perspective, this probability can mitigate the risk of parts unavailability and overstocking, which can be adjusted according to the system's operational period. This distribution can also be valuable for the warranty and insurance sector. Insurers can use the



(a) $X_1(3) = 0, X_2(3) = 1, X_3(3) = 1, X_4(3) = 1, X_5(3) = 1, X_6(3) = 1$



(b) $X_1(3) = 0, X_2(3) = 1, X_3(3) = 1, X_4(1) = 1, X_5(3) = 1, X_6(3) = 1$

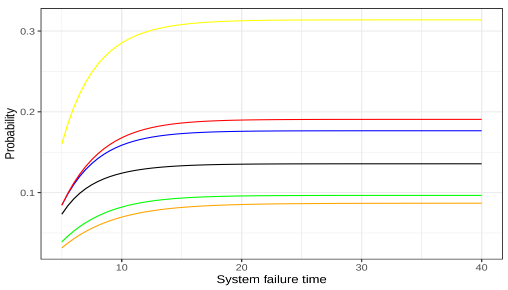


(c) $X_1(3) = 0, X_2(2) = 1, X_3(3) = 1, X_4(1) = 1, X_5(3) = 1, X_6(3) = 1$

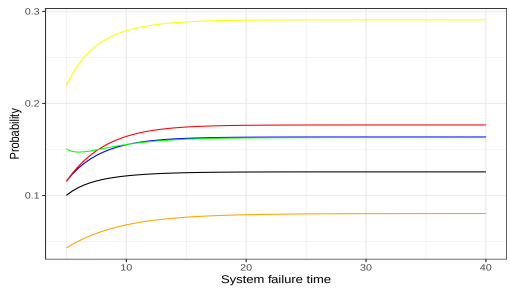
Figure 3.9: Probability of the number of failed components at system failure for the System in Figure 3.6, assuming zero failures of type 1 and one failure of the other types occurred before system failure at different times $t_1, \ldots, t_6$.

probability of the number of failed components, based on the system's age, to set insurance premiums based on expected failures and related costs.

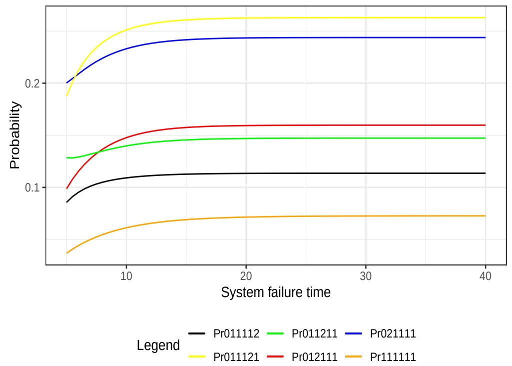
First, we consider a system composed of two types of components, and then the results are generalised to multiple types of components. For a system with two types of components, let $X_1(T_s)$ and $X_2(T_s)$ denote the number of failed components, respectively, of types 1 and 2, at time T_s of system failure. Our objective is to determine the probability of the event $X_1(T_s) = x_1, X_2(T_s) = x_2 \mid T_s > t$. The following theorem presents the probability distribution of the event, along with its proof.



(a) $X_1(3) = 0, X_2(3) = 1, X_3(3) = 1, X_4(3) = 0, X_5(3) = 1, X_6(3) = 1$



(b) $X_1(3) = 0, X_2(3) = 1, X_3(3) = 1, X_4(1) = 0, X_5(3) = 1, X_6(3) = 1$



(c) $X_1(3) = 0, X_2(2) = 1, X_3(3) = 1, X_4(1) = 0, X_5(3) = 1, X_6(3) = 1$

Figure 3.10: Probability of the number of failed components at system failure for the System in Figure 3.6, assuming zero failures of types 1 and 4, and one failure of types 2, 3, 5, and 6, occurred before system failure at different times $t_1, \ldots, t_6$.

Theorem 3.5.1 Let T^k denote the failure time of a component of type $k \in \{1, 2\}$, where the failure times of components are assumed to be absolutely continuous distributed. Further, assume that the failure times of components of the same type are independent and identically distributed, and that the failure times of components of different types are independent. Let T_s represent the random variable denoting the system failure time. If the probability distributions of the failure times of the component types are known, then the probability distribution of the event $X_1(T_s) =$

$x_1, X_2(T_s) = x_2 \mid T_s > t$ can be expressed as

$$\begin{aligned}
&P(X_1(T_s) = x_1, X_2(T_s) = x_2 \mid T_s > t) \\
&= \frac{1}{1 - F_{T_s}(t)} \int_t^{\infty} \Big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times (m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1)[P(X_2(u) = x_2)] \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times (m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1)[P(X_1(u) = x_1)] \Big] \, du, \qquad (3.5.11)
\end{aligned}$$

where, $F_{T_s}(t)$ denotes the cumulative distribution function of the system failure time T_s, while $h_k(u)$ represents the hazard rate of component type k at time u. The term $P(X_k(u) = x_k)$ represents the probability of x_k failed components of type k at time u, calculated through the binomial distribution, as noted in Remark 3.4.1.

Proof:

$$\begin{aligned}
&P(X_1(T_s) = x_1, X_2(T_s) = x_2 \mid T_s > t) \\
&= \int_t^{\infty} P(X_1(T_s) = x_1, X_2(T_s) = x_2 \mid T_s = u) f_{T_s}(u \mid T_s > t) \, du \\
&= \frac{1}{1 - F_{T_s}(t)} \int_t^{\infty} P(X_1(T_s) = x_1, X_2(T_s) = x_2 \mid T_s = u) \, f_{T_s}(u) \, du. \qquad (3.5.12)
\end{aligned}$$

For the first formula in the integral, the system failure time is now set at u, so

$$\begin{aligned}
&P(X_1(T_s) = x_1, X_2(T_s) = x_2 \mid T_s = u) = P(X_1(u) = x_1, X_2(u) = x_2 \mid T_s = u) \\
&= \frac{P(X_1(u) = x_1, X_2(u) = x_2) P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2)}{P(T_s = u)} \\
&= \frac{P(X_1(u) = x_1) P(X_2(u) = x_2) P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2)}{P(T_s = u)}. \qquad (3.5.13)
\end{aligned}$$

As we assume that two components cannot fail simultaneously, the system failure is either due to a failure of a component of type 1 or type 2. Thus,

$$\begin{aligned}
&P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2) \\
&= P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2, T^1 = u) P(T^1 = u \mid X_1(u) = x_1) \\
&\quad + P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2, T^2 = u) P(T^2 = u \mid X_2(u) = x_2).
\end{aligned}$$

If there are x_1 failures of components of type 1 by time u and one of these components fails exactly at u, then at the time just before u, u^-, we had $x_1 - 1$ failures. This means that by time u^- we would have $x_1 - 1$ failures of type 1 and x_2 failures of type 2, and by time u there will be x_1 failures of type 1 (increased by one) and x_2 failures of type 2 (as at time u^- since two failures cannot occur simultaneously). The same applies when we have x_2 failures of components of type 2 by time u and one of these fails exactly at u. Then,

$$\begin{aligned}
&P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2) \\
&= P(T_s = u \mid X_1(u^-) = x_1 - 1, X_2(u^-) = x_2, X_1(u) = x_1, X_2(u) = x_2) \\
&\quad \times P(T^1 = u \mid X_1(u) = x_1) \\
&\quad + P(T_s = u \mid X_1(u^-) = x_1, X_2(u^-) = x_2 - 1, X_1(u) = x_1, X_2(u) = x_2) \\
&\quad \times P(T^2 = u \mid X_2(u) = x_2) \\
&= [\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times P(T^1 = u) P(X_1(u) = x_1 \mid T^1 = u)[P(X_1(u) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times P(T^2 = u) P(X_2(u) = x_2 \mid T^2 = u)[P(X_2(u) = x_2)]^{-1}.
\end{aligned}$$

Using Lemma 3.3.1 for $P(X_k(u) = x_k \mid T^k = u)$,

$$\begin{aligned}
&P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2) \\
&= [\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^1 = u) \\
&\quad \times \frac{(m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1)}{m_1 f_1(u)} [P(X_1(u) = x_1)]^{-1} \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^2 = u) \\
&\quad \times \frac{(m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1)}{m_2 f_2(u)} [P(X_2(u) = x_2)]^{-1}.
\end{aligned} \tag{3.5.14}$$

By substituting the result of the probability $P(T_s = u \mid X_1(u) = x_1, X_2(u) = x_2)$, Equation (3.5.14), back into Equation (3.5.13), we obtain

$$
\begin{aligned}
&P(X_1(u) = x_1, X_2(u) = x_2 \mid T_s = u) \\
&= \Bigg[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^1 = u) \\
&\quad \times \frac{(m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1)}{m_1 f_1(u)} [P(X_2(u) = x_2)] \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(T^2 = u) \\
&\quad \times \frac{(m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1)}{m_2 f_2(u)} [P(X_1(u) = x_1)] \Bigg] \\
&\quad \times \Big[P(T_s = u) \Big]^{-1}.
\end{aligned}
$$

$$
\begin{aligned}
&\approx \Bigg[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] P(u < T^1 \leqslant u + \epsilon) \\
&\quad \times \frac{(m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1)}{m_1 f_1(u)} [P(X_2(u) = x_2)] \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] P(u < T^2 \leqslant u + \epsilon) \\
&\quad \times \frac{(m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1)}{m_2 f_2(u)} [P(X_1(u) = x_1)] \Bigg] \\
&\quad \times \Big[P(u < T_s \leqslant u + \epsilon) \Big]^{-1}, \qquad \text{for small } \epsilon > 0.
\end{aligned}
$$

Multiplying by $\frac{1}{\epsilon}$, then taking the limit when $\epsilon \to 0$,

$$
\begin{aligned}
&P(X_1(u) = x_1, X_2(u) = x_2 \mid T_s = u) \\
&= \Bigg[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] m_1 f_1(u) \\
&\quad \times \frac{(m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1)}{m_1 f_1(u)} [P(X_2(u) = x_2)] \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] m_2 f_2(u) \\
&\quad \times \frac{(m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1)}{m_2 f_2(u)} [P(X_1(u) = x_1)] \Bigg] \Big[f_{T_s}(u) \Big]^{-1} \\
&= \Bigg[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times (m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1) [P(X_2(u) = x_2)] \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times (m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1) [P(X_1(u) = x_1)] \Bigg] \Big[f_{T_s}(u) \Big]^{-1}. \qquad (3.5.15)
\end{aligned}
$$

Substituting the result of the probability $P(X_1(u) = x_1, X_2(u) = x_2 \mid T_s = u)$, Equation 3.5.15, into Equation 3.5.12,

$$\begin{aligned}
&P(X_1 = x_1, X_2 = x_2 \mid T_s > t) \\
&= \frac{1}{1 - F_{T_s}(t)} \int_t^{\infty} \Big[[\Phi(m_1 - x_1 + 1, m_2 - x_2) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times (m_1 - x_1 + 1) h_1(u) P(X_1(u^-) = x_1 - 1)[P(X_2(u) = x_2)] \\
&\quad + [\Phi(m_1 - x_1, m_2 - x_2 + 1) - \Phi(m_1 - x_1, m_2 - x_2)] \\
&\quad \times (m_2 - x_2 + 1) h_2(u) P(X_2(u^-) = x_2 - 1)[P(X_1(u) = x_1)] \Big] \, du,
\end{aligned}$$

□

This expression calculates the probability of the number of failed components of two types at the moment of system failure, assuming that the system was functioning at some point in time. The resulting expression depends solely on the failure time distributions of component types and the survival signature of the system. Moreover, the expression does not rely on the system failure time T_s where the f_{T_s} term is canceled in the derivation. The system reliability term outside the integral can be determined using the survival signature, as demonstrated in Chapter 2. The next subsection considers the case of multiple types of components.

3.5.1 Case of multiple types of components

The probability distribution of the numbers of failed components at system failure when it is assumed to be unknown, given that the system was functioning at some point in time, can be extended to multiple types of components. Consider a system with K types of components and m_k components of type k, for $k \in \{1, 2, ..., K\}$. The following theorem provides the probability distribution of the number of failed components of multiple types at the moment of system failure, assuming system functioning at time t, along with its proof.

Theorem 3.5.2 Let T^k denote the failure time of a component of type $k \in \{1, ..., K\}$, where the failure times of components are assumed to be absolutely continuous distributed. Additionally, suppose that the failure times of components of the same type are independent and identically distributed, and that the failure times of components of different types are independent. Let T_s represent the random variable denoting the system failure time. If the probability distributions of the failure times of the component types are known, then the probability distribution of the event $\bigcap_{k=1}^{K}\{X_k(T_s)=x_k\} \mid T_s>t$ can be expressed as

$$P\left(\bigcap_{k=1}^{K}\{X_k(T_s)=x_k\} \mid T_s>t\right)$$
$$=\frac{1}{1-F_{T_s}(t)}\int_t^{\infty}\sum_{k=1}^{K}$$
$$[\Phi_k'(m_1-x_1,\ldots,m_{k-1}-x_{k-1},m_k-x_k+1,m_{k+1}-x_{k+1},\ldots,m_K-x_K)]$$
$$\times\left(m_k-x_k+1\right)h_k\left(u\right)P\left(X_k\left(u^-\right)=x_k-1\right)\left(\prod_{\substack{i=1\\i\neq k}}^{K}P\left(X_i\left(u\right)=x_i\right)\right)du, \tag{3.5.16}$$

where, $F_{T_s}(t)$ denotes the cumulative distribution function of the system failure time T_s, and $h_k(u)$ represents the hazard rate of component type k at time u. The probability $P(X_k(u)=x_k)$ indicates the probability of x_k failed components of type k at time u, determined using the binomial distribution, as referenced in Remark 3.4.1

Proof:

$$P\left(\bigcap_{k=1}^{K}\{X_k(T_s)=x_k\} \mid T_s>t\right)$$
$$=\int_t^{\infty}P\left(\bigcap_{k=1}^{K}\{X_k\left(T_s\right)=x_k\} \mid T_s=u\right)f_{T_s}\left(u \mid T_s>t\right)\,du$$
$$=\frac{1}{1-F_{T_s}(t)}\int_t^{\infty}P\left(\bigcap_{k=1}^{K}\{X_k\left(T_s\right)=x_k\} \mid T_s=u\right)f_{T_s}\left(u\right)\,du. \tag{3.5.17}$$

For the first formula in the integral, the system failure time is now known at u, so

$$\begin{aligned}
&P\left(\bigcap_{k=1}^{K}\{X_k(T_s)=x_k\}\mid T_s=u\right)=P\left(\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\mid T_s=u\right)\\
&=\frac{P\left(\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\right)P\left(T_s=u\mid\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\right)}{P(T_s=u)}\\
&=\frac{\left[\prod_{k=1}^{K}P(X_k(u)=x_k)\right]P\left(T_s=u\mid\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\right)}{P(T_s=u)}.
\end{aligned}\tag{3.5.18}$$

As we assume that two or more components cannot fail simultaneously, the system failure time is due to only one type. Thus,

$$\begin{aligned}
&P\left(T_s=u\mid\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\right)\\
&=\sum_{k=1}^{K}P\left(T_s=u\mid\bigcap_{i=1}^{K}\{X_i(u)=x_i\},T^k=u\right)P\left(T^k=u\mid X_k(u)=x_k\right)\\
&=\sum_{k=1}^{K}P\left(T_s=u\mid\bigcap_{i=1}^{K}\{X_i(u)=x_i\},\bigcap_{\substack{i=1\\i\neq k}}^{K}\left\{X_i\left(u^-\right)=x_i\right\},X_k\left(u^-\right)=x_k-1\right)\\
&\quad\times P\left(T^k=u\mid X_k(u)=x_k\right)
\end{aligned}\tag{3.5.19}$$

Using Lemma 3.3.2 for the first term in Equation (3.5.19),

$$\begin{aligned}
&P\left(T_s=u\mid\bigcap_{i=1}^{K}\{X_i(u)=x_i\},\bigcap_{\substack{i=1\\i\neq k}}^{K}\left\{X_i\left(u^-\right)=x_i\right\},X_k\left(u^-\right)=x_k-1\right)\\
&=[\Phi(m_1-x_1,\ldots,m_{k-1}-x_{k-1},m_k-x_k+1,m_{k+1}-x_{k+1},\ldots,m_K-x_K)\\
&\quad-\Phi(m_1-x_1,\ldots,m_K-x_K)]\\
&\quad=\Phi_k'(m_1-x_1,\ldots,m_{k-1}-x_{k-1},m_k-x_k+1,m_{k+1}-x_{k+1},\ldots,m_K-x_K).
\end{aligned}$$

Thus,

$$\begin{aligned}
&P\left(T_s=u\mid\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\right)\\
&=\sum_{k=1}^{K}[\Phi_k'(m_1-x_1,\ldots,m_{k-1}-x_{k-1},m_k-x_k+1,m_{k+1}-x_{k+1},\ldots,m_K-x_K)]\\
&\quad\times P\left(T^k=u\right)P\left(T^k=u\mid X_k(u)=x_k\right)[P(X_k(u)=x_k)]^{-1}.
\end{aligned}$$

Using Lemma 3.3.1 for $P\left(T^k = u \mid X_k(u) = x_k\right)$,

$$
\begin{aligned}
&P\left(T_s = u \mid \bigcap_{k=1}^{K} \{X_k(u) = x_k\}\right) \\
&= \sum_{k=1}^{K} [\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
&\quad \times P\left(T^k = u\right) \frac{(m_k - x_k + 1)\, h_k(u)\, P\left(X_k(u^-) = x_k - 1\right)}{m_k f_k(u)} \left[P\left(X_k(u) = x_k\right)\right]^{-1}.
\end{aligned}
$$

By substituting the result of the probability $P\left(T_s = u \mid \bigcap_{k=1}^{K} \{X_k(u) = x_k\}\right)$ into Equation 3.5.18,

$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K} \{X_k(u) = x_k\} \mid T_s = u\right) \\
&= \sum_{k=1}^{K} [\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
&\quad \times P\left(T^k = u\right) \frac{(m_k - x_k + 1)\, h_k(u)\, P\left(X_k(u^-) = x_k - 1\right)}{m_k f_k(u)} \\
&\quad \times \left(\prod_{\substack{i=1 \\ i \neq k}}^{K} P\left(X_i(u) = x_i\right)\right) \left[P\left(T_s = u\right)\right]^{-1} \\
&\approx \sum_{k=1}^{K} [\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
&\quad \times P\left(u < T^k \leqslant u + \epsilon\right) \frac{(m_k - x_k + 1)\, h_k(u)\, P\left(X_k(u^-) = x_k - 1\right)}{m_k f_k(u)} \\
&\quad \times \left(\prod_{\substack{i=1 \\ i \neq k}}^{K} P\left(X_i(u) = x_i\right)\right) \left[P\left(u < T_s \leqslant u + \epsilon\right)\right]^{-1}.
\end{aligned}
$$

Multiplying by $\frac{1}{\epsilon}$, then taking the limit when $\epsilon \to 0$,

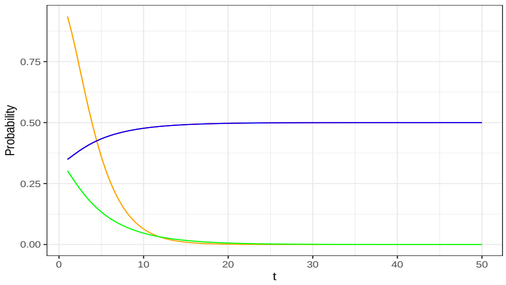
$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K} \{X_k(u) = x_k\} \mid T_s = u\right) \\
&= \sum_{k=1}^{K} [\Phi_k'(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)] \\
&\quad \times m_k f_k(u) \frac{(m_k - x_k + 1)\, h_k(u)\, P\left(X_k(u^-) = x_k - 1\right)}{m_k f_k(u)} \\
&\quad \times \left(\prod_{\substack{i=1 \\ i \neq k}}^{K} P\left(X_i(u) = x_i\right)\right) \left[f_{T_s}(u)\right]^{-1}.
\end{aligned}
$$

Substituting the result of probability $P\left(\bigcap_{k=1}^{K}\{X_k(u)=x_k\}\mid T_s=u\right)$ into Equation 3.5.17,

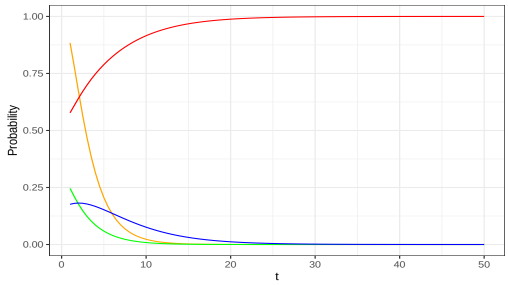
$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K}\{X_k(T_s)=x_k\}\mid T_s>t\right)\\
&=\frac{1}{1-F_{T_s}(t)}\int_t^{\infty}\sum_{k=1}^{K}\\
&\quad[\Phi_k'(m_1-x_1,\dots,m_{k-1}-x_{k-1},m_k-x_k+1,m_{k+1}-x_{k+1},\dots,m_K-x_K)]\\
&\quad\times(m_k-x_k+1)\,h_k(u)\,P\left(X_k\left(u^-\right)=x_k-1\right)\left(\prod_{\substack{i=1\\i\neq k}}^{K}P\left(X_i(u)=x_i\right)\right)du.
\end{aligned}
$$

□

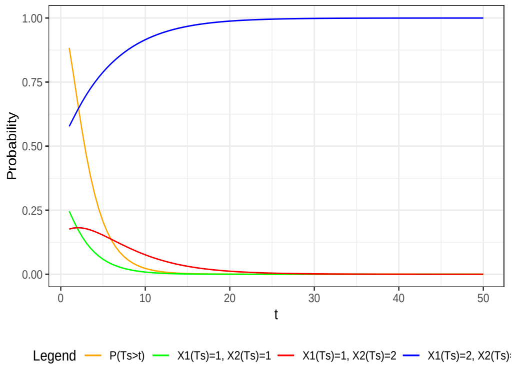
Example 3.5.1 (Series-parallel system) We illustrate our results on the probability of the number of failed components at system failure time, given that the system was functioning at time t, using a simple series-parallel system depicted in Figure 3.6. This system is composed of two types of components. For this system, there are only three different scenarios for the system to fail. In Figure 3.11, we show the probability of failures for both types, assuming they follow Exponential distributions, with the system functioning at time t. Figure 3.11a represents the scenario where both types have the same failure rate of 0.2. In Figure 3.11b, we depict the probability of failures when the failure rate of type 1 components is 0.2, and the failure rate of type 2 components is 0.4. Figure 3.11c illustrates a situation in which the failure rate of type 1 components is 0.4, while the failure rate of type 2 components is 0.2. The orange line in the figures indicates the probability of the system functioning at time t. Therefore, the time period that should receive more attention is the period before time $t = 10$. In the case where both types have the same failure rate, the probability of having two failures of type 1 and one failure of type 2, as well as having one failure of type 1 and two failures of type 2, is the same and is the most likely. Conversely, the probability of having one failure for both types is less likely. For the other two scenarios, the probability of failures depends on the failure rate of each type. When type 2 has a rate of 0.4, it is most likely to have two failures of type 2 and one failure of type 1. Conversely, when type 1 has a rate of 0.4, it is most likely to have two failures of type 1 and one failure of type 2.



(a) $T^1, T^2 \sim \text{Exp}(0.2)$



(b) $T^1 \sim \text{Exp}(0.2), T^2 \sim \text{Exp}(0.4)$



(c) $T^1 \sim \text{Exp}(0.4), T^2 \sim \text{Exp}(0.2)$

Figure 3.11: Probability of the number of failed components at system failure time for System in Figure 3.6 given that the system was functioning at time t. Note that the blue and red lines are overlapped in Figure 3.11a.

Example 3.5.2 (Hydro power system) In this example, we present our findings using the hydro power system illustrated in Figure 3.5, which comprises six types of components. Given the assumption that two or more components cannot fail simultaneously, there are a total of 136 potential combinations of failures for these six types that can occur at the moment of system failure. Figure 3.12 illustrates the probabilities associated with some of the most likely specific combinations of failed components at the time of system failure, assuming the system was in operation at time t, under various scenarios. In the first figure, we illustrate the probability of combinations of failures when the failure times of components follow the distributions outlined in Table 3.3. Based on these distributions, the most likely combination when the system fails, assuming the system was last known to be functioning at a very early time, is two failures of type 2 and no failures of the other types. When the system is still functioning at, say, time $t = 25$, then the combination of zero failures of types 1 and 3, and one failure for types 4, 5, and 6, and two failures for type 2 is the most likely one to occur when the system fails. If the system has been operational for an extended period, it is most likely to experience two failures of type 2, zero failures of type 1, and one failure of the other types at the time of failure.

When the shape parameters of components for type 1 and type 2 were adjusted to 1.1 and 0.8, respectively, the probabilities of most of these combinations significantly decreased, as depicted in Figure 3.12b. The only exception is in the probability of having one failed component of type 1 and one failed component of type 2, with no failures for the other types (green combination), which increased when assuming that the system was last known to function at earlier times. Under these modified parameters, Figure 3.12c illustrates some of the most likely combinations of failed components at the moment of system failure. In this scenario, when the system was last known to be functioning during its early stages, the most likely combination upon failure is having one failed component of type 1 and no failures of the other types. When the system continues to function at $t = 25$, then having one failed component of types 1, 2, 4, 5, and 6, with no failure of type 3, is most likely to occur, followed by the combination of having one failed component of types 1, 2, 5, and 6, with no failures for types 3 and 4. As the system continues to function, it becomes increasingly likely to experience an additional failed component at the time of system failure, ultimately resulting in one failure for each type.

As shown in these two examples, the probability of the numbers of failed components is highly affected by the period during which the system was in operation. This aspect was not clearly considered in similar events discussed in the literature.

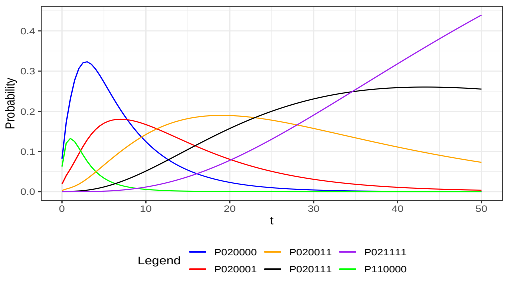
Component type	Distribution	Parameters
1	Weibull	$\alpha_1 = 1$, $\beta_1 = 0.5$
2	Weibull	$\alpha_2 = 2.5$, $\beta_2 = 2$
3	Exponential	failure rate = 0.02
4	Exponential	failure rate = 0.03
5	Exponential	failure rate = 0.06
6	Exponential	failure rate = 0.05

Table 3.3: Failure times distribution of the types of system components.

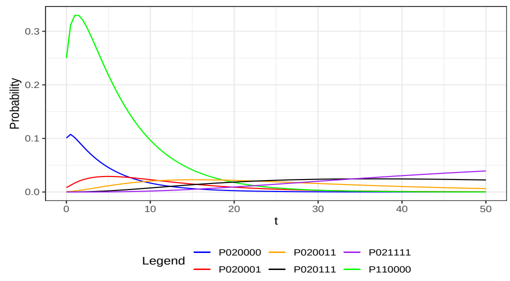
3.6 Number of failed components given system failure within a future time interval

This section studies the probability of the number of failures at the moment of system failure, given that the time of the system failure will occur within a future time interval. An event of this nature may be useful for on-site maintenance operations, for example, during periods when emergency supplies of spare parts are not available, such as on support vessels for wind farms [43]. This event is a natural extension of the event introduced in Section 3.5. If we assume that the system will fail within a future time interval $[t_1^\star, t_2^\star]$, then the probability distribution of the number of failed components of each type within a given system at the moment of failure can be calculated as follows.

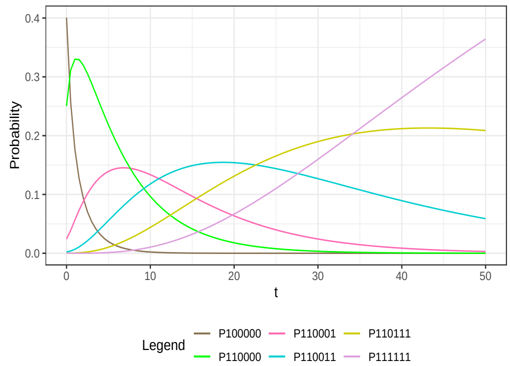
$$
\begin{aligned}
&P\left(\bigcap_{k=1}^{K} X_k\left(T_s\right) = x_k \mid T_s \in [t_1^\star, t_2^\star]\right) \\
&= \int_{t_1^\star}^{t_2^\star} P\left(\bigcap_{k=1}^{K} X_k\left(T_s\right) = x_k \mid T_s = u\right) f_{T_s}\left(u \mid T_s \in [t_1^\star, t_2^\star]\right) du \\
&= \int_{t_1^\star}^{t_2^\star} P\left(\bigcap_{k=1}^{K} X_k(u) = x_k \mid T_s = u\right) \frac{f_{T_s}(u)}{F_{T_s}(t_2^\star) - F_{T_s}(t_1^\star)} du \\
&= \frac{1}{F_{T_s}(t_2^\star) - F_{T_s}(t_1^\star)} \int_{t_1^\star}^{t_2^\star} P(\bigcap_{k=1}^{K} X_k(u) = x_k \mid T_s = u) f_{T_s}(u)\, du.
\end{aligned}
$$



(a) Probability of failures when distributions according to Table 3.3



(b) Probability of same combinations in 3.12a when $\beta_1 = 1.1$ and $\beta_2 = 0.8$



(c) Probability of some of most likely failures combinations when $\beta_1 = 1.1$ and $\beta_2 = 0.8$

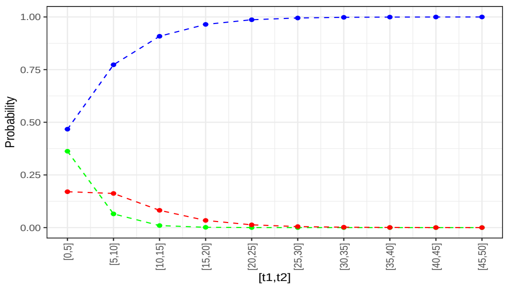
Figure 3.12: Probability of the number of failed components at system failure time for System in Figure 3.5 given that the system was functioning at time t.

The probability and density in the integration can be treated as in Subsection 3.5.1, thus,

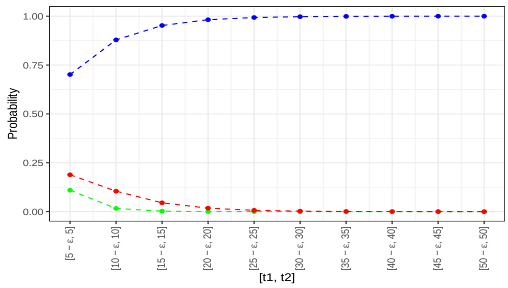
$$P\left(\bigcap_{k=1}^{K} X_k\left(T_s\right) = x_k \mid T_s \in [t_1^\star, t_2^\star]\right)$$
$$= \frac{1}{F_{T_s}(t_2^\star) - F_{T_s}(t_1^\star)} \int_{t_1^\star}^{t_2^\star} \sum_{k=1}^{K}$$
$$[\Phi(m_1 - x_1, \ldots, m_{k-1} - x_{k-1}, m_k - x_k + 1, m_{k+1} - x_{k+1}, \ldots, m_K - x_K)$$
$$- \Phi(m_1 - x_1, \ldots, m_K - x_K)]\,(m_k - x_k + 1)\, h_k\,(u)\, P\left(X_k\left(u^-\right) = x_k - 1\right)$$
$$\times \left(\prod_{\substack{i=1 \\ i \neq k}}^{K} P\left(X_i\left(u\right) = x_i\right)\right) du. \tag{3.6.20}$$

Example 3.6.1 (Series-parallel system) In this example, we calculate the probability of the number of failed components at the moment of system failure, considering that the system will fail within a future time interval. This calculation is performed for the system illustrated in Figure 3.6. The system's two types are assumed to follow Exponential distribution, with a failure rate of 0.4 for type 1 and 0.2 for type 2. Figure 3.13 shows the probability of failures for the two types when the system fails under various scenarios. In these scenarios, the most probable situation at the moment of system failure is having two failed components of type 1 and one failure of type 2. Figure 3.13a illustrates the probability of the number of failures if the system were to fail at different specified intervals. It shows that the most likely outcome, particularly when the system fails at a very early interval, is to have two failures of type 1, one failure of type 2, followed by, one failure of both types. Figure 3.13b illustrates the scenario where time $t_1^\star$ approaches $t_2^\star$ (so, $T_s \in [t_2^\star - \epsilon, t_2^\star]$, for small ϵ). This scenario corresponds to a case introduced in Section 3.4, where the system failure time is assumed to be known, but no prior knowledge about earlier failures is available. The probabilities in this figure can be compared to the probabilities in Figure 3.8, where differences become evident. This comparison confirms that having information about earlier failures affects the probability of failures at the moment of system failure. In Figure 3.13c, the probability of failures at system failure is presented when $t_2^\star$ approaches infinity leading to $T_s \geqslant t_1^\star$, which can be seen as the probability of the event introduced in Section 3.5. Therefore, this figure presents the same probabilities shown in Figure 3.11c.

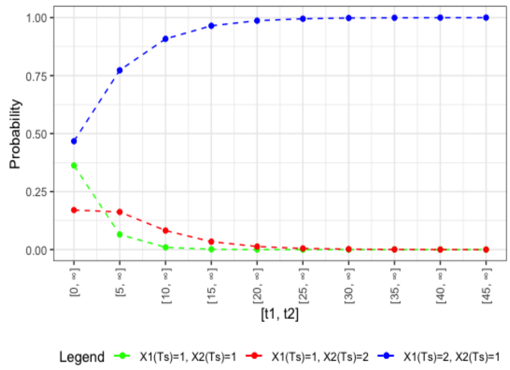
Example 3.6.2 (Hydro power system) Consider the hydro power system depicted in Figure 3.5. Let's assume that this system will fail within a future interval $[t_1^\star, t_2^\star]$, and the failure times of its components follow the distributions outlined in Table 3.3. In Figure 3.14, we present the probability of failures for some of the most likely combinations at the moment of system failure under various scenarios. Figure 3.14a illustrates these probabilities with the component failure time distributions as per Table 3.3. From the figure, we observe that if the system is assumed to fail at very early times, such as within the interval $[1, 5]$, then the most likely scenario at the moment of system failure is having two failed components of type 2 and zero



(a) Probability of failures given system failure at $[t_1^\star, t_2^\star]$.



(b) Probability of failures at $[t_1^\star, t_2^\star]$ when $t_1^\star$ approaches to $t_2^\star$.



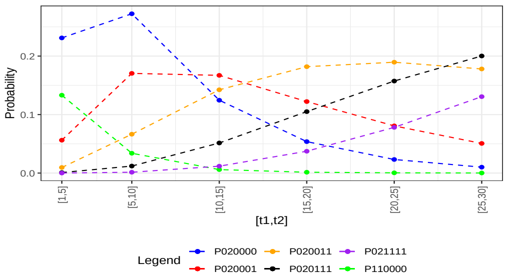
(c) Probability of failures at $[t_1^\star, t_2^\star]$ when $t_2^\star$ approaches to infinity.

Figure 3.13: Probability of the number of failed components at system failure time for System in Figure 3.6 given system failure within time interval $[t_1^\star, t_2^\star]$. Note that the dashed line is used to connect the points and does not represent a probability.

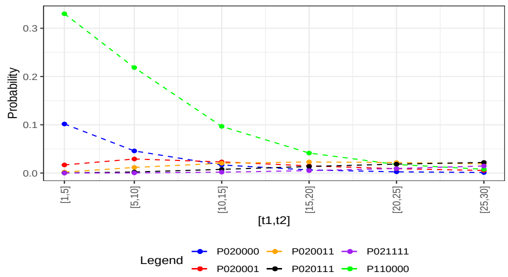
failed components of the other types. The second most likely combination of failed components is having one failed component each of types 1 and 2, with zero failures for the other types (green combination). Following this, the third most likely combination is having two failed components of type 2 and one failed component of type 6, with zero failures for the other types (red combination). The probabilities of other combinations of the number of failed components are similar when the system fails at very early moments.

If the system is assumed to fail within the time interval $[5, 10]$, then the probabilities for some of these combinations change. For example, the probability of the green combination clearly decreases, while the probability of the red combination increases. However, if we assume that the system will still be functioning and fail at a later time interval, $t = [25, 30]$, then the most likely scenario to occur when the system fails is having two failed components of type 2, one failed component of types 5 and 5, and zero failed components of the other types (orange combination).

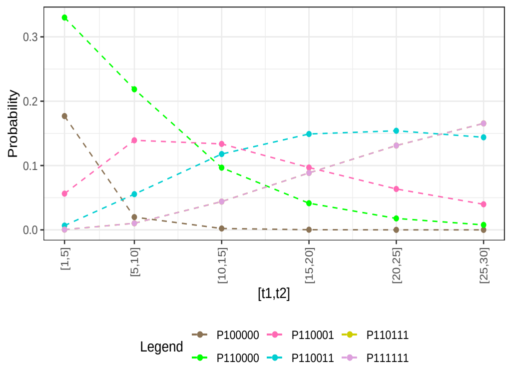
If we adjust the shapes of components of types 1 and 2 to 1.1 and 0.8, respectively, the probabilities change significantly, as depicted in Figure 3.14b. These changes are particularly noticeable. Figure 3.14c provides further insights into the probabilities of some of the most likely combinations of failed components under these updated parameter values.



(a) Probability of number of failed components assuming system failure at $[t_1^\star, t_2^\star]$.



(b) Probability of failure combinations in Figure 3.14a at $[t_1^\star, t_2^\star]$ when $\beta_1 = 1.1$ and $\beta_2 = 0.8$



(c) Probability of some of most likely failures combinations given system failure at $[t_1^\star, t_2^\star]$, when $\beta_1 = 1.1$ and $\beta_2 = 0.8$.

Figure 3.14: Probability of the number of failed components at system failure time for System in Figure 3.5 given system failure within time interval $[t_1^\star, t_2^\star]$. Note that the dashed line is used to connect the points and does not represent a probability.

3.7 Conclusions and future work

This chapter introduced a probability expression based on the survival signature that determines the probability of a specific component type failing simultaneously with the system, essentially identifying the probability of a component type causing the system failure. This probability provides valuable insights in various ways. Components with a high probability of causing system failure can be prioritised for maintenance, inspection, and monitoring. In Chapter 4, this probability will be utilised to determine the optimal minimal repair kit capable of restoring the system to a functional state without replacing all failed components.

The joint probability distribution for the numbers of failed components of multiple types at the moment of system failure was considered and derived using the survival signature under three different settings. Each scenario has its own potential applications. In the first scenario, we derive the probability of the number of failed components at the time of system failure, conditioning on both the exact system failure time and the number of failed components of each type that occurred before system failure, considering that these failures can occur at different times before system failure. This probability distribution has substantial relevance and can be applied in various practical situations. It can assist in optimising spare parts inventory planning by giving priority to combinations of components from different types with higher probabilities, while reducing the stock for those with lower probabilities.

Secondly, the probability distribution of the number of failed components of multiple types at system failure time is derived given that the system was functioning at some point in time, assuming that the system failure time is not known. This probability distribution has valuable applications in different domains. For instance, it can be employed in the insurance and warranty sector. Insurers can utilise the probability of component failures, taking into account the system's age, to determine insurance premiums based on probable failures and associated expenses.

Thirdly, the probability of the number of failed components of multiple types at system failure time is obtained assuming that the system will fail within a future time interval. Such an event could prove beneficial for on-site maintenance procedures, especially in situations where there is a lack of immediate access to emergency spare parts, as for example with the case of support vessels for wind farms [43]. In Chapter 4, these distributions of the number of failed components at the time of system failure will be used to determine the optimal repair kit for replacing all failed components when a system failure occurs, considering cost factors.

The derivation of the aforementioned probabilities was based on the assumption that two or more components cannot fail simultaneously. However, some system failure modes produce simultaneous failures of multiple components. An example of this is the Fukushima Daiichi nuclear disaster [72], caused by the loss of power, which specifically resulted from the simultaneous failures of emergency diesel generators, failing to provide backup power to the system. Therefore, it is important to derive the probabilities of the relevant events when system components can fail simultaneously.

For further future research, determining the optimal number of components of multiple types during the system design phase, aiming to minimise the expected cost of system failure, using the event derived in Section 3.5 for the number of failed components of multiple types at system failure, could be of significant interest.

Chapter 4

Repair kit problem

4.1 Introduction

In the highly competitive market, after-sales service has become an increasingly important factor for manufacturers and represents a strategic opportunity for financial gain. According to Amini et al. [1], $40-50\%$ of a manufacturer's profit is derived from complementary services they offer. Gebaue et al. [28] noted that manufacturers make a considerable investment in their after-sales services, including field service teams and parts inventories. Caterpillar is an example of a manufacturer that provides significant after-sales services. Some of their well-known services include guaranteed parts delivery and planned maintenance kits. The PC sector, including companies such as Dell Computer, is also well-known for its after-sales services, particularly rapid repair services, as noted by Cohen and Whang [14]. Medical products are another category of products that have a significant demand for after-sale services, particularly repair services. Cohen and Whang [14] pointed to an analysis by Blumberg [13] that has shown that the segment with the highest annual growth rate in demand for repair services is medical electronics/diagnostics, with a growth rate of 23.2%.

A key after-sales service offered by manufacturers is repair services at the customer's location. Blumberg [13] indicates that repair services are highly demanded in the US and Europe. Local repair services involve contacting the manufacturer or service provider when equipment malfunctions, and they, in turn, dispatch specialised engineers based on their experience and technical skills to diagnose and repair the problem. These engineers receive work assignments specifying the locations they need to visit, and they travel to these locations to perform the necessary repairs. When receiving their tasks, engineers must decide which spare parts to bring and in which quantities. This collection of spare parts is referred to as a repair kit. The successful completion of any repair job hinges on having all the necessary replacement parts available in the repair kit. Failure to bring the required parts may result in a return visit for replenishing the repair kit, known as a return to fit (RTF) visit. Manufacturers face a logistical decision regarding the contents of the repair kit to avoid the need for additional visits. This type of decision problem is known as the repair kit problem (RKP). It's important to note that second visits are not considered RKPs since the required parts are already known during the first visit.

As stated by Bijvank et al. [11], manufacturers usually decide which repair kit they will put in their technicians' vehicles based on their experience and limitations. These restrictions may pertain to the amount of space available in the vehicle or the availability of certain parts. It is therefore necessary to establish a mechanism for determining the parts of the repair kit that engineers should carry in their vehicles. This mechanism may be based on costs or on a promise of a specific level of service to the client.

Two types of models for the RKP are introduced in the literature; the cost model introduced by Smith et al. [62] and the service model introduced by Graves [29]. The objective of the cost model is to obtain a repair kit that minimises the total expected cost of the repair kit, which is usually comprised of both the holding cost and the cost of a possible RTF visit. This means that a trade-off must be made between the holding cost of the repair kit and the cost of a RTF visit. The service model aims to find the appropriate repair kit that minimises the holding cost based on a certain level of service, typically agreed with the customer. In general, the holding cost refers to the cost of each spare part included in the kit, while the RTF cost refers to the costs incurred when the job cannot be completed on the first attempt. This cost may include a loss of goodwill in addition to other costs such as staffing and labour costs. The RTF visit cost is commonly expressed in terms of some penalty costs. The service model is commonly preferred in practice over the cost model because, in the service model, customers can specify a minimum level of service [11]. The main challenge is quantifying the probability of an RTF visit occurring. In other words, it represents the probability that there are not enough spare parts available in the repair kit to complete the repair.

The so-called job fill rate, or job completion rate, is one of the most commonly used measures in the literature for formulating the RKP [50]. It is defined as the probability of no failures among components not currently stocked [29], or simply, the probability of no shortage in the repair kit. In the cost model, it is utilised to estimate the cost of the RTF visit, and in the service model, it serves as a constraint for successfully completing the job with the minimum holding cost. The specific derivation of the job fill rate depends on the settings and assumptions of the problem, such as whether it involves a single job or multiple jobs, the number of components needed per type (single or multiple), etc.

Since the RKP pertains to determining the number of spare parts needed to replace failed components, the probability of events related to the number of failed components at system failure and the component type that causes system failure, as introduced in Chapter 3, is well-suited for this context. Therefore, depending on the scenario, the probability of these events can serve as a measure for the service level required by the customer in the service model, while in the cost model, it can be used to quantify the cost of the RTF visit, which includes penalty costs.

The remainder of this chapter is organised as follows. An overview of previous work on the RKP is provided in Section 4.2. Section 4.3 demonstrates how to select an optimal repair kit that could be provided with the system at its purchase time, which aims to replace all failed components when the system fails. Using the probability derived in Section 3.5, various scenarios for calculating the RTF visit cost are considered to determine the optimal repair kit in this section. This section introduces two greedy algorithms which aim to identify the optimal repair kit. Section 4.5 provides a repair kit intended to replace all failed components when the system fails within a specific future time interval. Section 4.6 proposes an optimal repair kit that does not necessarily replace all failed components but instead aims to replace specific components, bringing the system back to a functional state. Finally, Section 4.7 concludes the chapter and provides interesting topics for future research.

4.2 Literature review

The most notable results for the RKP were observed during the 1980s and 90s. Only a few papers addressed the problem in the 00s, but it seems to have received more attention afterwards. Smith et al. [62] was the first known work to study the RKP. They sought to determine the optimal repair kit that minimises the expected total costs of the kit holdings and the penalties incurred due to incomplete jobs caused by insufficient parts. Graves [29] addresses the same problem and determines the optimal repair kit that guarantees a specified job fill rate with the minimum holding cost, and transformed the problem into binary Knapsack problem [58]. Both studies assume that components fail independently and at most one component of each type can be used for a repair job. Mamer and Smith [40] relax the independence assumption and allows more than one component of any type to be used. In all of these studies, the repair kit is assumed to be refilled after each job. In some cases, this assumption may be appropriate, particularly when dealing with complex repairs that require a prolonged period of time. In the literature, this is usually referred to as a single repair job.

As for repairs that do not require extensive time, such as office equipment or household appliances, the engineer usually visits multiple sites without restocking the repair kit. This general setting of visiting multiple jobs without restocking the kit was first considered by Heeremans and Gelders [33] and is referred to as a tour of repairs. They utilise tour fill rate (the proportion of tours without an RTF visit) in their formulation instead of job fill rate. Teunter [65] discussed a multiple-job RKP, which permits the use of multiple components per type in a job. In the context of multiple jobs, Teunter [65] was the first to introduce dependency between failures. The author also assumed that the required spare parts will be left at the client location, even if the job is not completed. As noted by Bijvank et al. [11], this assumption is inconsistent with practice as these spares can be used in the next job of the tour. Consequently, they examined the RKP for multiple jobs with multiple spares of the same type that can be used on a job, assuming that spares are only taken out of the repair kit if the job can be completed. The only assumption they make is that they assume that failures of components of different types are independent. It should be noted that in the literature of the RKP, the term “type” refers to a category of items that share similar physical characteristics. Therefore, in these discussed works, parts of the same type do not necessarily mean they have exchangeable failure times as assumed in Chapter 3.

In recent literature, various factors have been considered when formulating RKP models. Saccani et al. [57] examine the cost model for a tour RKP, in which multiple spares of the same type can be utilised on a single job. They consider two aspects: a financial constraint on the repair kit in which it does not exceed a specific value and the costs associated with replenishing the repair kit. Implementing a financial constraint on the repair kit can be beneficial, particularly in reducing losses related to theft. A case study by Bijvank et al. [11] showed that the solution they proposed to the company did not meet the criteria, as the company aimed to reduce the value of the kit due to theft concerns.

In the RKP literature, most works assume zero lead times for all spare parts. A lead time represents the time lag between the start and finish of a procedure. A zero lead time means that all parts required for the kit are always available prior to the start of the job(s), including high-value and heavy parts. This may not be the case in practice, as for instance for daily tours when restocking of the kit takes place overnight, some parts may have to be delivered from central warehouses to the engineering field site. Prak et al. [50] propose a multi-job multi-part service model with a positive lead time for all spare parts. A further factor that contributes to the development of the RKP is the use of information about the condition of the failed system before to the initial visit to the customer. Rippe and Kiesmüller [55] investigate a tour RKP and use the failure codes generated by sensors fitted in modern appliances as imperfect advance demand information for spare parts.

In an attempt to solve the RKP and obtain the optimal solution, several methods have been proposed in the literature. Teunter [65] introduced a greedy heuristic that starts with an empty repair kit and adds one component at each iteration until a full repair kit is achieved. The heuristic selects the repair kit with the minimum total expected cost. Similar to Teunter [65], Bijvank et al. [11] introduced a heuristic for selecting the optimal repair kit in which it can add more than one component of the same type at a time to fulfill multiple-component jobs. Saccani et al. [57] solve an integer linear model to determine the optimal repair kit given historical demand data and evaluate performance through simulation. Rippe and Kiesmüller [55] introduced two heuristics to address the RKP: a greedy heuristic treating the problem as a single job problem, and a part heuristic that decomposes the original Markov decision process into significantly smaller ones for each part.

In previous works discussed in the literature, the formulation of the RKP in both models was based on the job or tour fill rate. The calculation of the job or tour fill rate depends on the specific problem's settings. However, to the best of our knowledge, the derivation of these measures does not consider important factors such as the system's structure, system failure time, and component failure time. This crucial information, as demonstrated in Chapter 3, could significantly impact the determination of the number of failed components at system failure. Therefore, it is anticipated that the optimal repair kit may vary depending on this information.

In this chapter, we utilise the probabilities derived in Chapter 3, which are based on the system and component failure times, and take into account the system's structure via the survival signature. These probabilities are used to estimate the cost of RTF visits. The chapter focuses on the cost model, aiming to obtain optimal repair kits that minimise the expected total cost under different scenarios. The service model is left for future works. The setting of the RKP discussed in this chapter involves a single repair job that allows for the inclusion of multiple components of different types in the repair kit.

4.3 Repair kit for a new system

In a situation where a client or company has local technicians capable of replacing failed components, providing a kit of spare parts at the time of system purchase may be beneficial. Having a repair kit available on-site can help reduce system downtime and save costs associated with transporting the repair kit and personnel to the system site, in which they can be in a more complex job. This section aims to determine the optimal repair kit that is provided with the system so that it will be able to repair all of the failed parts, regardless of whether the system might function with some failed components after maintenance action. We determine the optimal repair kit in the cost model, that is the repair kit with the minimum total expected cost, which comprises the holding cost and the RTF visit cost. We assume that multiple parts of the same type may be required to complete the repair (i.e. a multi-part job). Let $S = (r_1, ..., r_K)$ denote a repair kit with r_k spare parts of type k, for $k \in \{1, ..., K\}$. The holding cost of the kit S is denoted by $C_h(S)$ and given by $C_h(S) = \sum_{k=1}^{K} r_k c_k$, where c_k represents the cost of a part of type k. We denote the total cost of the repair kit S by $C_t(S)$ and it is formulated as

$$C_t(S) = C_{RTF}(S) + C_h(S), \tag{4.3.1}$$

where $C_{RTF}(S)$ denotes the cost of a return-to-fit visit.

In this section, we present multiple scenarios in which the cost of the RTF visit is quantified. The first scenario involves a fixed penalty cost for not having sufficient spare parts to replace all failures. This penalty cost remains the same regardless of whether there is a shortage of one or more spares of any type.

The second scenario considers a penalty cost that varies from type to type and depends on the number of shortages for each type. The third scenario involves a penalty cost that varies depending on the type, regardless of the number of shortages of that type. This scenario can be seen as a special case of scenario two.

The last scenario is a general one, where there is a penalty cost depending on the type and the number of shortages of that type. Additionally, there is a fixed penalty if the job cannot be completed due to a shortage. This case combines elements of scenarios one and two.

4.3.1 Fixed penalty cost

This subsection quantifies the RTF visit cost, assuming there is a fixed penalty cost if one or more failed components cannot be replaced due to a shortage in the repair kit. Let C_p represent the penalty cost that must be paid in the event that the repair kit is lacking either one or more spares. Using the probability that the number of spare parts in the kit is greater than the number of failures when the system fails, the RTF cost under this scenario, denoted as $C_{RTF_{N_1}}(S)$, can be calculated as follows

$$C_{RTF_{N_1}}(S) = (1 - P(X_1(T_s) \leqslant r_1, ..., X_K(T_s) \leqslant r_K \mid T_s > 0))C_p, \tag{4.3.2}$$

where $X_k(T_s)$ represents the number of failures of type k at the system failure time T_s.

The probability in Equation (4.3.2) represents the probability that the number of failed components of each type in the system at the time of system failure is less than or equal to the number of spare parts in the repair kit. The condition on the system functioning at time 0 is intended to indicate that the system is new, as we plan to provide a kit at the time of purchase. The derivation of this probability was given in Section 3.5.1, leading to Equation 3.5.16. By substituting this expected RTF cost into $C_{RTF}(S)$ in Equation (4.3.1), we obtain the total expected cost of the repair kit. To illustrate the selection of the optimal repair kit, we provide an example of the hydro power plant system.

Example 4.3.1 (Hydro power system) Suppose that we have the system shown in Figure 4.1 that is composed of 12 components of 6 types. The distribution of failure times for components of each type is given in Table 4.1. The probability density function (PDF) and reliability function for component's failure time that is Weibull distributed with scale parameter α and shape parameter β, where $\alpha, \beta > 0$, are

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp - \left(\frac{t}{\alpha}\right)^{\beta}$$
$$R(t) = \exp - \left(\frac{t}{\alpha}\right)^{\beta}.$$

For a component's failure time that is Exponentially distributed with failure rate parameter $\lambda > 0$, the PDF and reliability function are

$$f(t) = \lambda \exp(-\lambda t)$$
$$R(t) = \exp(-\lambda t).$$

The aim is to find the optimal repair kit in order to replace all failed components when the system fails. For this system, there are $\prod_{k=1}^{6} m_k + 1 = 648$ combinations of repair kits including an empty kit and a full kit. Table 4.2 illustrates different optimal repair kits based on various holding costs for parts and fixed penalty costs which apply if not all failed components can be replaced. For holding costs $(4, 3, 2, 2, 1, 4)$ and $(8, 6, 4, 4, 2, 8)$, we observe that the total number of spare components in the optimal kits increases as the penalty cost increases, which is understandable given that the holding costs are relatively low. In contrast, in the last case, where holding costs are high compared to penalty costs, optimal kits tend to have only one spare component.

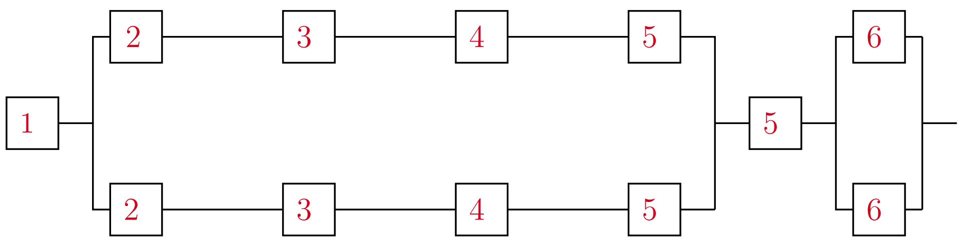


Figure 4.1: Hydro power plant system with 6 types of components, indicated in red.

If the shape parameters of distributions of failure times of components of types 1 and 2 change to $\beta_1 = 1.1$ and $\beta_2 = 0.8$, some of the optimal repair kits are adjusted accordingly. Table 4.3 presents the updated optimal repair kits under these new shape parameters. These adjustments arise from the change in the probability of the number of failed components being less than or equal to the number of spare

Component type	Distribution	Parameters
1	Weibull	$\alpha_1 = 1$, $\beta_1 = 0.5$
2	Weibull	$\alpha_2 = 2.5$, $\beta_2 = 2$
3	Exponential	$\lambda_3 = 0.02$
4	Exponential	$\lambda_4 = 0.03$
5	Exponential	$\lambda_5 = 0.06$
6	Exponential	$\lambda_6 = 0.05$

Table 4.1: Failure times distribution of the types of system components.

components in the repair kit, which in turn influences the RTF visit cost and, therefore, the total expected cost. For example, for the first optimal repair kit, $S = (1, 1, 0, 0, 1, 0)$, the probability $P(X_1(T_s) \leqslant 1, X_2(T_s) \leqslant 1, X_3(T_s) \leqslant 0, X_4(T_s) \leqslant 0, X_5(T_s) \leqslant 1, X_6(T_s) \leqslant 0 \mid T_s > 0)$ is 0.75 with a total expected cost of 12.954, while for the kit, $S = (1, 0, 0, 0, 1, 0)$, which was optimal under the same holding and penalty cost when $\beta_1 = 0.5$ and $\beta_2 = 2$, the probability $P(X_1(T_s) \leqslant 1, X_2(T_s) \leqslant 0, X_3(T_s) \leqslant 0, X_4(T_s) \leqslant 0, X_5(T_s) \leqslant 1, X_6(T_s) \leqslant 0 \mid T_s > 0)$ is 0.45 with a total expected cost of 16.009.

4.3.2 Penalty cost based on type and number of shortages

In this subsection, we quantify the RTF visit cost assuming that the penalty cost of shortages in the kit is different from one component type to another, and depends on the number of shortages for each type. This is particularly useful in situations where accessing a specific type is challenging, leading to a higher penalty cost, while others could be easily available, resulting in a lower penalty cost. Let C_p^k denote the penalty cost of a shortage of a component of type k, then the RTF cost of the repair kit S under this scenario, denoted as $C_{RTF_{N_2}}(S)$, is given by

$$C_{RTF_{N_2}}(S) = \sum_{x_1=0}^{m_1} \dots \sum_{x_K=0}^{m_K} \Bigg\{ P\left(X_1\left(T_s\right) = x_1, \dots, X_K\left(T_s\right) = x_K \mid T_s > 0\right) \times \left(\sum_{k=1}^{K} \max\left(x_k - r_k, 0\right) C_p^k \right) \Bigg\}. \tag{4.3.3}$$

Penalty cost C_p	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
20		$S = (1, 0, 0, 0, 1, 0)$
40	$(4, 3, 2, 2, 1, 4)$	$S = (1, 2, 0, 0, 1, 0)$
60		$S = (1, 2, 0, 1, 1, 1)$
20		$S = (1, 0, 0, 0, 0, 0)$
40	$(8, 6, 4, 4, 2, 8)$	$S = (1, 0, 0, 0, 1, 0)$
60		$S = (1, 1, 0, 0, 1, 0)$
20		$S = (0, 0, 0, 0, 1, 0)$
40	$(16, 12, 8, 8, 4, 16)$	$S = (1, 0, 0, 0, 0, 0)$
60		$S = (1, 0, 0, 0, 0, 0)$

Table 4.2: Optimal repair kits for the hydro power system under different fixed penalty costs and component holding costs.

The calculation of the RTF visit cost relies on the probability of the number of failed components at system failure, as derived in Section 3.5.1. For each combination of the number of failed components of multiple types, the probability that it is occurred is multiplied by the number of shortages for each component type, determined by the spare components in the repair kit S (if any), or by zero, and then by the penalty cost associated with that type.

Example 4.3.2 Based on the same system shown in Figure 4.1 and the failure time distributions in Table 4.1, Table 4.4 presents the optimal repair kits under different holding costs when the penalty costs of shortages differ for components of various types. When the penalty costs are high and the holding costs are low, the optimal repair kit contains a higher number of parts. For example, with penalty costs of $(50, 60, 70, 80, 90, 100)$ and holding costs of $(4, 3, 2, 2, 1, 4)$, the optimal repair kit has the highest number of parts compared to the other kits. When the penalty cost is low and the holding cost is relatively expensive, the optimal kit has fewer spare components. Type 5 is always included in all optimal repair kits since it has a relatively high probability of having failed when the system fails, as well as a high penalty cost in addition to having the lowest holding cost.

Penalty cost C_p	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
20		$S = (1, 1, 0, 0, 1, 0)$
40	$(4, 3, 2, 2, 1, 4)$	$S = (1, 2, 0, 0, 1, 0)$
60		$S = (1, 2, 0, 1, 1, 0)$
20		$S = (1, 0, 0, 0, 0, 0)$
40	$(8, 6, 4, 4, 2, 8)$	$S = (1, 1, 0, 0, 1, 0)$
60		$S = (1, 2, 0, 0, 1, 0)$
20		$S = (0, 0, 0, 0, 1, 0)$
40	$(16, 12, 8, 8, 4, 16)$	$S = (1, 0, 0, 0, 0, 0)$
60		$S = (1, 1, 0, 0, 1, 0)$

Table 4.3: Optimal repair kits for the hydro power system under different fixed penalty costs when $\beta_1 = 1.1$ and $\beta_2 = 0.8$.

If only the failure rate of components of type 6 changes to 0.7, all optimal repair kits change, as shown in Table 4.5. This change occurs due to the increase in the failure rate from 0.05 to a higher rate of 0.7, making it more likely to have two failures of components of type 6 when the system fails. This change is also influenced by the fact that type 6 components have the highest penalty cost.

4.3.3 Penalty cost based only on component type

In certain cases, a repair person may be faced with shortages of parts, in which case the penalty will be determined based on the type of part only, regardless of the number of shortages of that type. Therefore, if one part of type k is missing, the penalty is the same as if there were more parts of type k missing, but the penalty costs are assumed to differ per type. An instance of this might occur if some parts of a particular type could be obtained at short notice, so the cost would not be as high, but when parts of other types are not available, the system could be out of commission for a considerable period of time leading to a high cost. The formulation

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$		$S = (1, 1, 0, 0, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	$(4, 3, 2, 2, 1, 4)$	$S = (1, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 1, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	$(8, 6, 4, 4, 2, 8)$	$S = (1, 1, 0, 0, 1, 0)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 0, 0, 1, 0)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	$(16, 12, 8, 8, 4, 16)$	$S = (0, 0, 0, 0, 1, 0)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 1, 0, 0, 1, 0)$

Table 4.4: Optimal repair kits for the hydro power system under different penalty costs per type and component holding costs.

of the cost of a RTF visit in this setting, denoted as $C_{RTF_{N_3}}(S)$, is given as follows.

$$C_{RTF_{N_3}}(S) = \sum_{x_1=0}^{m_1} \dots \sum_{x_K=0}^{m_K} \sum_{k=1}^{K} P(X_1(T_s) = x_1, \dots, X_K(T_s) = x_K \mid T_s > 0) \times \left[\mathbf{I}\{x_k > r_k\}\tilde{C}_p^k\right], \tag{4.3.4}$$

where $\tilde{C}_p^k$ represents the penalty cost if there is a shortage of a component of type k.

Example 4.3.3 For this type of RTF visit cost, Table 4.6 provides the optimal repair kits for the system in Figure 4.1 based on different penalty and holding costs, where the failure times of parts types are distributed as shown in Table 4.1. The optimal repair kits in this example are the same as when the RTF visit cost is also influenced by the number of shortages per type (Table 4.4), except for two repair kits. In the first kit, the penalty is $(20, 30, 40, 50, 60, 70)$ and the holding cost is $(8, 6, 4, 4, 2, 8)$ which results in the optimal kit of $S = (1, 0, 0, 0, 1, 0)$, whereas if the number of shortages are taken into account, it is $S = (1, 1, 0, 0, 1, 0)$. The second kit is when the penalty cost per type is $(50, 60, 70, 80, 90, 100)$ and the holding cost is

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$		$S = (1, 1, 0, 0, 1, 2)$
$(20, 30, 40, 50, 60, 70)$	$(4, 3, 2, 2, 1, 4)$	$S = (1, 1, 0, 0, 1, 2)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 0, 1, 1, 2)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 2)$
$(20, 30, 40, 50, 60, 70)$	$(8, 6, 4, 4, 2, 8)$	$S = (1, 0, 0, 0, 1, 2)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 1, 0, 0, 1, 2)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	$(16, 12, 8, 8, 4, 16)$	$S = (0, 0, 0, 0, 1, 1)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 0, 0, 0, 1, 2)$

Table 4.5: Optimal repair kits for the hydro power system under different penalty costs per type and component holding costs when $\lambda_6 = 0.7$.

$(16, 12, 8, 8, 4, 16)$ which results in the optimal kit of $S = (1, 0, 0, 0, 1, 0)$ where it was $S = (1, 1, 0, 0, 1, 0)$ if the number of shortages were taken into account. This due to the fact that the penalty cost of missing a component of type 2 is relatively low compared to the other types. Additionally, adding one component of type 2 will not result in a considerable increase in the probability of that combination, especially considering that type 2 has a high holding cost.

Table 4.7 presents the optimal repair kits when the failure rate of components of type 5 decreases from 0.06 to 0.01. Comparing to the optimal repair kits in Table 4.6, when $\lambda_5 = 0.06$, all repair kits changed except for two repair kits (the second and third kit). Due to the significant decrease in the failure rate of type 5 components, it is noticeable that some of the optimal repair kits no longer include any components of type 5, whereas it was always present in the optimal repair kits when the failure rate was 0.06. This change occurred despite the high penalty cost associated with missing a component (or more) of type 5.

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$		$S = (1, 1, 0, 0, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	$(4, 3, 2, 2, 1, 4)$	$S = (1, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 1, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	$(8, 6, 4, 4, 2, 8)$	$S = (1, 0, 0, 0, 1, 0)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 0, 0, 1, 0)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	$(16, 12, 8, 8, 4, 16)$	$S = (0, 0, 0, 0, 1, 0)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 0, 0, 0, 1, 0)$

Table 4.6: Optimal repair kits for the hydro power system under different penalty costs per component type for any missing component, regardless of the number of missing components, and based on different component holding costs.

4.3.4 Penalty cost based on component type and fixed costs

In this subsection, the cost of a RTF visit is described in a more general context, where the total cost of RTF includes two related costs. The first cost quantifies the cost per component short, where each component type has its own penalty cost, while the second cost is an additional fixed cost if one or more components of any type cannot be replaced. The latter refers to the costs associated with the second visit, such as labour, fuel, or other expenses, and the former refers to the cost per part. The expected RTF visit cost in this setting is

$$\begin{aligned} C_{RTF_{N_4}}(S) = & \sum_{x_1=0}^{m_1} \dots \sum_{x_K=0}^{m_K} \{\, P\left(X_1\left(T_s\right) = x_1, \dots, X_K\left(T_s\right) = x_K \mid T_s > 0\right) \\ & \times \left(\sum_{k=1}^{K} \max\left(x_k - r_k, 0\right) C_p^k \right) \} \\ & + \left(1 - P\left(X_1\left(T_s\right) \leqslant r_1, \dots, X_K\left(T_s\right) \leqslant r_K \mid T_s > 0\right)\right) C_p. \end{aligned} \tag{4.3.5}$$

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$		$S = (1, 2, 0, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	$(4, 3, 2, 2, 1, 4)$	$S = (1, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 1, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$		$S = (1, 0, 0, 0, 0, 0)$
$(20, 30, 40, 50, 60, 70)$	$(8, 6, 4, 4, 2, 8)$	$S = (1, 0, 0, 0, 0, 0)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 2, 0, 1, 1, 0)$
$(10, 20, 30, 40, 50, 60)$		$S = (0, 0, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	$(16, 12, 8, 8, 4, 16)$	$S = (1, 0, 0, 0, 0, 0)$
$(50, 60, 70, 80, 90, 100)$		$S = (1, 0, 0, 0, 0, 0)$

Table 4.7: Optimal repair kits for the hydro power system under different penalty costs per component type for any missing component, regardless of the number of missing components, and based on different component holding costs when $\lambda_5 = 0.01$.

It should be noted that the cost of the RTF visit here combines the costs defined in Subsection 4.3.1 and 4.3.2.

Example 4.3.4 Table 4.8 presents the optimal repair kit for the system in Figure 4.1 based on various penalty costs per component type, an additional fixed penalty for missing one or more components, and holding costs. The distribution of failure times of the components are shown in Table 4.1. The number of parts in the optimal repair kit appears to be high when holding costs are low, and decreases when holding costs increase, when the penalty costs are fixed. Furthermore, we observe that type 1 always appears in the optimal repair kit despite having the highest holding cost. This is due to both its critical location and relatively high failure rate compared to other types. The same applies to type 5, where there is always a part of this type in optimal repair kit. Despite type 5's low failure rate, this occurs due to its low holding costs and location.

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Penalty cost C_p	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$			$S = (1, 2, 1, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	40	$(4, 3, 2, 2, 1, 4)$	$S = (1, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (1, 2, 1, 1, 2, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (1, 2, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	40	$(8, 6, 4, 4, 2, 8)$	$S = (1, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (1, 2, 0, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (1, 0, 0, 0, 1, 0)$
$(20, 30, 40, 50, 60, 70)$	40	$(16, 12, 8, 8, 4, 16)$	$S = (1, 1, 0, 0, 1, 0)$
$(50, 60, 70, 80, 90, 100)$			$S = (1, 1, 0, 0, 1, 0)$

Table 4.8: Optimal repair kits for the hydro power system taking into account different penalty costs per type, fixed penalty costs if a job is unsuccessful, and holding costs.

4.4 Algorithms for determining optimal kit

In the previous examples, we were able to obtain the best repair kits in all cases of quantifying RTF costs using a full search approach. Clearly, the repair kit must contain at least zero spare parts and at most $\sum_{k=1}^{K} m_k$. The total number of possible repair kit combinations is $\prod_{k=1}^{K}(m_k + 1)$. Typically, real-world systems consist of hundreds of components of multiple types. Consequently, it is infeasible to achieve an optimal repair kit through a full search approach in such cases. It is therefore necessary to employ a heuristic process in order to determine the exact or approximate optimal kit.

In this section, we introduce two algorithms that add one spare part to the repair kit or remove one spare part from the kit, depending on its impact on the total cost of the kit. The first algorithm begins with an empty kit. In Step 2, a part of type k is added to the kit, and the difference in total cost between the two kits is calculated. This difference, denoted as D_k, is computed for all types $k \in \{1, ..., K\}$. Step 3 selects the kit with the type that yields the lowest difference D_k when one part of that type is added to the kit, resulting in an updated kit. If the full kit is not achieved yet, the algorithm returns to Step 2 and add another part of type k, then calculate the difference in total cost between the updated repair kit and the previous one for all $k \in \{1, ..., K\}$. When the full repair kit is achieved, the algorithm ends with Step 5, where the optimal repair kit with the lowest expected total cost is selected. The mathematical explanation of the algorithm is provided in Algorithm 1.

Algorithm 1: Greedy Algorithm for selecting the optimal repair kit

---

Step 1:

$i := 0$ (first kit).

$S_i := (r_1, ..., r_K) = (0, ..., 0)$ (empty kit).

$C_t(S_i) := 0 + C_{RTF}(S_i)$.

Step 2:

For all $k \in \{1, 2, ..., K\}$ for which $r_k < m_k$,

$D_k := C_t(r_1, ..., r_{k-1}, r_k + 1, r_{k+1}, ..., r_K) - C_t(r_1, ..., r_K)$

(increase in the total cost if one part of type k is added to the kit).

Step 3:

$k' := \text{argmin}_k D_k$ (part type with the lowest D_k).

$i \leftarrow i + 1$ (next kit).

$r_{k'} \leftarrow r_{k'} + 1$ (add one part of type k').

$S_i := (r_1, ..., r_K)$ (updated kit).

$C_t(S_i) := C_h(S_i) + C_{RTF}(S_i)$.

Step 4:

If $r_k = m_k$ for all types k, go to step 5. If not, proceed to Step 2.

Step 5:

$S_{opt} := \text{argmin}_{S_i}(C_t(S_i))$.

Example 4.4.1 Using the algorithm on the hydro power system example, we obtain the same optimal repair kits for two scenarios of penalty types as in full search. Table 4.9 and 4.10 show the repair kits generated by the algorithm for the fixed penalty cost (Table 4.2) and the type-based penalty cost (Table 4.4), respectively. Note that

Table 4.9 is for the case of the fixed penalty cost when it is 60 and the holding cost is $(4, 3, 2, 2, 1, 4)$, while Table 4.10 is for the case of part type-based penalty cost when it is $(20, 30, 40, 50, 60, 70)$ and the holding cost is $(8, 6, 4, 4, 2, 8)$. From Table 4.10, repair kit number 3 is the optimal repair kit, which is the same one that is produced in a full search as shown in Table 4.4. Similarly, for the fixed penalty type, the optimal repair kit generated by the algorithm is repair kit number 6 as shown in Table 4.9, which matches the one found by full search.

i	S_i	$C_h(S_i)$	$C_t(S_i)$	k'
0	$[0, 0, 0, 0, 0, 0]$	0	60.00000	1
1	$[1, 0, 0, 0, 0, 0]$	16	27.98705	5
2	$[1, 0, 0, 0, 1, 0]$	20	25.76644	2
3	$[1, 1, 0, 0, 1, 0]$	32	22.74279	2
4	$[1, 2, 0, 0, 1, 0]$	44	20.11330	4
5	$[1, 2, 0, 1, 1, 0]$	52	19.78003	6
6	$[1, 2, 0, 1, 1, 1]$	68	19.70060	3
7	$[1, 2, 1, 1, 1, 1]$	76	19.76410	5
8	$[1, 2, 1, 1, 2, 1]$	80	20.26718	5
9	$[1, 2, 1, 1, 3, 1]$	84	21.26718	4
10	$[1, 2, 1, 2, 3, 1]$	92	23.19720	3
11	$[1, 2, 2, 2, 3, 1]$	100	25.16872	6
11	$[1, 2, 2, 2, 3, 2]$	116	28.94147	6

Table 4.9: Repair kits generated by the algorithm when the fixed penalty cost is 60 and the holding cost per part type is $(4, 3, 2, 2, 1, 4)$.

In Example 4.4.1, the algorithm successfully detects the optimal repair kit. However, due to the myopic nature of the algorithm, there may be instances in which it fails to locate the optimal repair kit. An example of this occurs when a system is highly likely to fail if two components of type k fail, but if a single component of that type fails, it has no effect. Therefore, adding one part of type k does not affect the total expected cost compared to adding two parts of that type. Consequently,

i	S_i	$C_h(S_i)$	$C_t(S_i)$	k'
0	$[0,0,0,0,0,0]$	0	46.12582	1
1	$[1,0,0,0,0,0]$	8	39.10526	5
2	$[1,0,0,0,1,0]$	10	32.63642	2
3	$[1,1,0,0,1,0]$	16	30.02443	4
4	$[1,1,0,1,1,0]$	20	31.52581	5
5	$[1,1,0,1,2,0]$	22	33.02889	5
6	$[1,1,0,1,3,0]$	24	35.02889	2
7	$[1,2,0,1,3,0]$	30	37.11722	6
8	$[1,2,0,1,3,1]$	38	39.67055	3
9	$[1,2,1,1,3,1]$	42	42.34243	4
10	$[1,2,1,2,3,1]$	46	46.28411	3
11	$[1,2,2,2,3,1]$	50	50.26513	6
12	$[1,2,2,2,3,2]$	58	58.00000	6

Table 4.10: Repair kits generated by the algorithm when the penalty cost per type is $(20, 30, 40, 50, 60, 70)$ and the holding cost per part type is $(8, 6, 4, 4, 2, 8)$.

it is highly likely that the myopic algorithm will not select D_k as the lowest in Step 3 and will choose another type. The following example illustrates the disadvantages of the algorithm.

Example 4.4.2 (Complex system) Consider the system depicted in Figure 4.2, which consists of 14 components of 6 types. The aim is to find the optimal repair kit to replace all failed components when the system fails. Assume that the failure times of the system components follow Exponential distributions with failure rates: $\lambda_1 = 0.3$ for type 1, $\lambda_2 = 0.7$ for type 2, $\lambda_3 = 0.02$ for type 3, $\lambda_4 = 0.03$ for type 4, $\lambda_5 = 0.06$ for type 5, and $\lambda_6 = 0.05$ for type 6. Let the penalty cost of a shortage of one or more components be $C_p = 200$, and the holding cost of each component per type be $C_1 = 25, C_2 = 20, C_3 = 8, C_4 = 8, C_5 = 8, C_6 = 2$. Upon calculating the total expected cost of the all 720 repair kits, we determine that $S = (2, 4, 0, 0, 1, 1)$ is the optimal kit with the lowest total expected cost of 174.8914. The algorithm, however, was not able to obtain the optimal repair kit when applied to this example. Table 4.11 presents the list of repair kits generated by the algorithm. Based on the algorithm, the best repair kit generated is $S = (2, 4, 2, 1, 1, 1)$, which has a total expected cost of 185.8811, 10.9897 more than the one obtained through the full search.

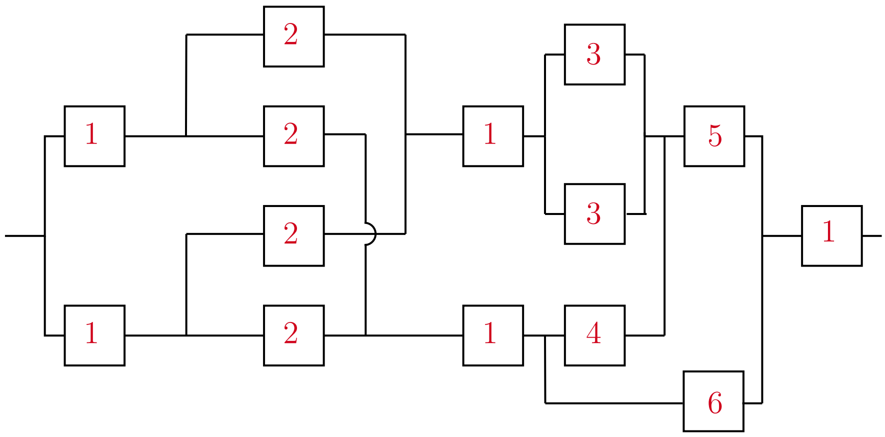


Figure 4.2: Reliability block diagram of the complex system.

This issue led us to consider a second algorithm that is actually similar to the previous one. Instead of starting with an empty kit, the updated algorithm starts with a kit containing the same number of components of each type as the system contains. Next, the part whose removal from the repair kit will result in the greatest reduction in the total expected cost of the repair kit is eliminated. The process is repeated until the repair kit becomes empty, and the repair kit with the lowest total expected cost is selected as the optimal kit. The algorithm's mathematical description is presented below in Algorithm 2. Using this algorithm on the complex system, the optimal kit $S = (2, 4, 0, 0, 1, 1)$ is now detected, which was not the case when applying Algorithm 1. A list of the repair kits that are generated by Algorithm 2 is presented in Table 4.12. These two algorithms provide a way to select the optimal repair kit, at least when a full search is not feasible. It should be noted that these algorithms parallel the greedy forward and backward step-wise search, a strategy commonly employed in other contexts such as variable selection.

Algorithm 2: Updated algorithm for selecting the optimal repair kit

Step 1:

$i := 0$ (first kit).

$S_i := (r_1, ..., r_K) = (m_1, ..., m_K)$ (full kit).

$C_t(S_i) := C_h(S_i) + C_{RTF}(S_i)$.

Step 2:

For all $k \in \{1, 2, ..., K\}$ for which $r_k > 0$,

$D_k := C_t(r_1, ..., r_K) - C_t(r_1, ..., r_{k-1}, r_k - 1, r_{k+1}, ..., r_K)$

(decrease in the total cost if one part of type k is removed from the kit).

Step 3:

$k' := \text{argmax}_k\, D_k$ (part type with the largest D_k).

$i \leftarrow i - 1$ (next kit).

$r_{k'} \leftarrow r_{k'} - 1$ (add one part of type k').

$S_i := (r_1, ..., r_K)$ (updated kit).

$C_t(S_i) := C_h(S_i) + C_{RTF}(S_i)$.

Step 4:

If $r_k = 0$ for all types k, go to Step 5. If not, proceed to Step 2.

Step 5:

$S_{opt} := \text{argmin}_{S_i}(C_t(S_i))$.

i	S_i	$C_h(S_i)$	$C_t(S_i)$	k'
0	$[0,0,0,0,0,0]$	0	200.0000	6
1	$[0,0,0,0,0,1]$	2	202.0000	5
2	$[0,0,0,0,1,1]$	10	209.9395	3
3	$[0,0,1,0,1,1]$	18	217.9384	4
4	$[0,0,1,1,1,1]$	26	225.9375	3
5	$[0,0,2,1,1,1]$	34	233.9375	1
6	$[1,0,2,1,1,1]$	59	244.5603	2
7	$[1,1,2,1,1,1]$	79	252.3695	2
8	$[1,2,2,1,1,1]$	99	248.7742	1
9	$[2,2,2,1,1,1]$	124	226.1510	2
10	$[2,3,2,1,1,1]$	144	192.5679	2
11	$[2,4,2,1,1,1]$	164	185.8811	1
12	$[3,4,2,1,1,1]$	189	189.0000	1
13	$[4,4,2,1,1,1]$	214	214.0000	1
14	$[5,4,2,1,1,1]$	239	239.0000	1

Table 4.11: Repair kits generated by Algorithm 1 for Example 4.4.2.

4.5 Repair kit for a system failure at a future time interval

The preceding section discussed a repair kit that is provided with a system upon purchase for use when the system fails. There is, however, a possibility that the optimal repair kit may vary depending on the time or length of a period during which the system may fail. In this section, we examine the optimal repair kit if it is assumed that the system fails within a certain period of time. In this context, such a repair kit can serve various purposes. As an example, when service operation vessels remain offshore for a period of time, usually 2 to 3 weeks [43], it is critical to determine in advance which repair kit is most suitable for that period during which problems may occur with, e.g., the turbine. Similar to the previous general kit, this

i	S_i	$C_h(S_i)$	$C_t(S_i)$	k'
0	$[5,4,2,1,1,1]$	239	239.0000	1
1	$[4,4,2,1,1,1]$	214	214.0000	1
2	$[3,4,2,1,1,1]$	189	189.0000	3
3	$[3,4,1,1,1,1]$	181	181.1491	1
4	$[2,4,1,1,1,1]$	156	178.0179	4
5	$[2,4,1,0,1,1]$	148	175.7883	3
6	$[2,4,0,0,1,1]$	140	174.8914	5
7	$[2,4,0,0,0,1]$	132	178.3849	2
8	$[2,3,0,0,0,1]$	112	181.2534	6
9	$[2,3,0,0,0,0]$	110	185.2563	2
10	$[2,2,0,0,0,0]$	90	207.3233	1
11	$[1,2,0,0,0,0]$	65	222.8129	2
12	$[1,1,0,0,0,0]$	45	221.6865	2
13	$[1,0,0,0,0,0]$	25	211.6071	1
14	$[0,0,0,0,0,0]$	0	200.0000	

Table 4.12: Repair kits generated by Algorithm 2 for Example 4.4.2.

setting is viewed from a cost perspective, considering different costs contributions to the total RTF expected cost.

The expected cost of a RTF visit for a repair kit that will be used in the event that a system fails in a future time interval is denoted by $C_{RTF_I}(S)$ and quantified using the probability 3.6.20 derived in Section 3.6, such that

$$\begin{aligned} C_{RTF_I}(S) = \sum_{x_1=0}^{m_1} \cdots \sum_{x_K=0}^{m_K} \Bigg\{ & P\left(X_1\left(T_s\right)=x_1, \ldots, X_K\left(T_s\right)=x_K \mid T_s \in [t_1^\star, t_2^\star]\right) \\ & \times \left(\sum_{k=1}^{K} \max\left(x_k - r_k, 0\right) C_p^k \right) \Bigg\} \\ & + \left(1 - P\left(X_1\left(T_s\right) \leqslant r_1, \ldots, X_K\left(T_s\right) \leqslant r_K \mid T_s \in [t_1^\star, t_2^\star]\right)\right) C_p. \end{aligned}$$

We provide some examples of optimal repair kits when a system fails in $[t_1^\star, t_2^\star]$ based on the different cases of quantifying RTF visit cost that were discussed in Section 4.3. The examples are based on the system illustrated in Figure 4.1 where the distribution of the failure times of the components is shown in Table 4.1. Table 4.13 presents an optimal repair kit assuming that the system will fail at some point within the period [5,10], considering different penalty and holding costs per type, assuming no additional costs will be incurred during the second visit (i.e., we set C_p to zero). Table 4.14 presents the optimal repair kits for various time intervals during which the system may fail.

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Extra cost C_p	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 1, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(4, 3, 2, 2, 1, 4)$	$S = (0, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (1, 2, 1, 1, 2, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 0, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(8, 6, 4, 4, 2, 8)$	$S = (0, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (0, 2, 1, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 0, 0, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(16, 12, 8, 8, 4, 16)$	$S = (0, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (0, 2, 0, 1, 1, 1)$

Table 4.13: Optimal repair kits when the system fails in $[5, 10]$, when there is a penalty cost per part type and no additional cost.

4.6 Minimal repair kit

There are instances where the focus is only on restoring the system to functional status regardless of whether some components remain unrepaired. A situation such as this may arise when there are some restrictions on the repair kit, such as its capacity, availability of some parts, or related expenses. Consequently, the purpose of this section is to determine the ideal minimal repair kit to be used in the event of a system failure. We define a minimal repair kit as a kit containing the least amount of spare parts that can be used to bring back a system to a functioning state with a degree of confidence. Taking into account that a system's failure is

Time interval $[t_1^\star, t_2^\star]$	Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$[0, 5]$	$(10, 20, 30, 40, 50, 60)$	$(4, 3, 2, 2, 1, 4)$	$S = (1, 1, 0, 0, 1, 1)$
		$(16, 12, 8, 8, 4, 16)$	$S = (0, 0, 0, 0, 1, 0)$
$[5, 10]$	$(10, 20, 30, 40, 50, 60)$	$(4, 3, 2, 2, 1, 4)$	$S = (0, 2, 1, 1, 1, 1)$
		$(16, 12, 8, 8, 4, 16)$	$S = (0, 2, 0, 0, 1, 1)$
$[10, 15]$	$(10, 20, 30, 40, 50, 60)$	$(4, 3, 2, 2, 1, 4)$	$S = (0, 2, 1, 1, 1, 1)$
		$(16, 12, 8, 8, 4, 16)$	$S = (0, 2, 0, 1, 1, 1)$
$[15, 20]$	$(10, 20, 30, 40, 50, 60)$	$(4, 3, 2, 2, 1, 4)$	$S = (0, 2, 1, 1, 1, 1)$
		$(16, 12, 8, 8, 4, 16)$	$S = (0, 2, 1, 1, 1, 1)$

Table 4.14: Optimal repair kits for different time intervals $[t_1^\star, t_2^\star]$, with penalty costs per part type and no additional costs.

due to a failure of only a single component, a kit containing one part of each type would be sufficient to restore the system. Another possibility for a minimal kit that guarantees system functionality is if it can be determined from the system structure in advance that providing some parts of the same type or different types will enable the system to be repaired. However, if only a few types are allowed to be carried in a repair kit due to certain limitations, a decision should be made based on which types of components are more likely to cause system failure. Thus, identifying the ideal minimal repair kit will require the use of the measure introduced in Section 3.3, i.e., the probability that a component of a particular type will result in the failure of the system. The selection of the optimal minimal repair kit will take into account both the holding cost and the return to fit cost as in the previous sections. To quantify the RTF visit cost of a minimal repair kit, let $S_m = (1^1, 1^2, ..., 1^K)$ denote a minimal repair kit, with 1^k denoting a component of type k. Based on the probability $P(T^k = t \mid T_s = t)$, given in Equation 3.3.1, where T^k represents the failure time of a component of type k, the RTF cost of the minimal repair kit S_m

can be expressed as follows.

$$C_{RTF_M}(S_m) = \sum_{\forall k \in S'_m} P\left(T^k = t \mid T_s = t\right) C_p^k + \sum_{\forall k \in S_m} \left(1 - P\left(T^k = t \mid T_s = t\right)\right) C_p. \tag{4.6.6}$$

The first part relates to the penalty cost associated with not having a part of type k included in the kit, while the second part relates closely to the general penalty cost associated with the second visit.

Example 4.6.1 Suppose that the system in Figure 4.1 fails at time $t = 10$, and the repair person is only permitted to have one part of only three types in the repair kit, owing to some limitations. Thus, we aim to determine the optimal minimal repair kit consisting of only three types that makes the trade-off between the types most likely to result in system failure and the associated penalties. At time $t = 10$, the probability of type 1 through type 6 causing system failure is 0.044, 0.900, 0.006, 0.008, 0.034, and 0.008 respectively. In Table 4.15, the optimal minimal repair kit is shown under varying penalty costs C_p and C_p^k assuming the failure times of components are distributed as shown in Table 4.1.

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Extra cost C_p	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal minimal kit S
$(10, 20, 30, 40, 50, 60)$			$S_m = (1^2, 1^4, 1^5)$
$(20, 30, 40, 50, 60, 70)$	45	$(4, 3, 2, 2, 1, 4)$	$S_m = (1^1, 1^2, 1^5)$
$(50, 60, 70, 80, 90, 100)$			$S_m = (1^1, 1^2, 1^5)$
$(10, 20, 30, 40, 50, 60)$			$S_m = (1^2, 1^4, 1^5)$
$(20, 30, 40, 50, 60, 70)$	45	$(8, 6, 4, 4, 2, 8)$	$S_m = (1^2, 1^4, 1^5)$
$(50, 60, 70, 80, 90, 100)$			$S_m = (1^2, 1^4, 1^5)$
$(10, 20, 30, 40, 50, 60)$			$S_m = (1^2, 1^4, 1^5)$
$(20, 30, 40, 50, 60, 70)$	45	$(16, 12, 8, 8, 4, 16)$	$S_m = (1^2, 1^4, 1^5)$
$(50, 60, 70, 80, 90, 100)$			$S_m = (1^2, 1^4, 1^5)$

Table 4.15: Optimal minimal repair kits for system failure at $t = 10$ under varying penalty costs per type, fixed penalty costs, and holding costs.

4.7 Conclusions and future work

This chapter discussed the repair kit problem (RKP), taking into account factors that have not received much attention in the literature, namely, system structure, system failure time and components lifetime. The chapter introduced three different scenarios for the RKP. Using the probabilities of events introduced in Chapter 3, the total expected cost of a repair kit is calculated under these scenarios.

The first scenario involves finding an optimal repair kit that could be provided with the system at the time of purchase that aims to replace all failed components when the system fails. Providing a repair kit upfront may contribute to minimising system downtime and reducing the associated costs of transporting personnel and equipment to the site. Under this scenario, the expected cost of the return to fit (RTF) visit, a second visit if a job cannot be completed successfully, is quantified from different perspectives, depending on the penalty cost that should be paid if a job cannot be completed. First, the expected RTF visit cost was quantified based on a fixed penalty cost if one or more components cannot be replaced. We also considered the case where the penalty cost depends solely on the component type, regardless of the number of shortages, and when the penalty depends on both the component type and the number of shortages. The expected cost of the RTF visit was also examined in a more general setting, where the penalty cost consists of both a fixed cost and a cost associated with the component type and the number of shortages of that type.

To determine the optimal repair kit for this scenario of the RKP, two closely related forward and backward greedy heuristics were introduced, aligning with those presented by Teunter [65] and Bijvank et al. [11]. For the hydro power example, the first heuristic was able to identify the optimal repair kit, which was the same as when a full search was applied. However, the heuristic failed to identify the optimal repair kit in another system. The total expected cost of the optimal repair kit selected by the algorithm was 6.28% higher compared to the total expected cost of the optimal repair kit detected in the full search. For this system, the second algorithm was able to identify the same optimal repair kit as the full search approach. Therefore, for other systems where conducting a full search is not possible, one might apply the two heuristics and select the optimal repair kit with the lowest total expected cost.

The second scenario of the RKP, considered in this chapter, focuses on determining an optimal repair kit that aims to replace all failed components when the system fails, assuming that system failure will occur within a specific future time interval. This scenario is motivated by the service operation vessels case study, where vessels remain offshore for a period of time [43]. Therefore, it is critical to determine in advance which repair kit is most suitable for that period. The probability of the number of failed components of multiple types at system failure, derived in Section 3.6 under the assumption that the failure will occur at a future time interval, is used to quantify the RTF visit cost.

The third scenario we considered in the RKP does not necessarily aim to replace all failures, as in the previous two scenarios, but rather focuses on finding the optimal repair kit that aims to restore the system to a functional status. Such a repair kit could be useful when there are restrictions on the repair kit, such as its capacity [57], the availability of certain components, or associated expenses [11]. The probability of a component of a specific type causing system failure, which is derived in Section 3.3, is used to quantify the expected cost of an RTF visit.

The minimal repair kit introduced in Section 4.6 does not necessarily replace all failures but aims to replace specific failures that can restore the system to a functional status. The minimal repair kit was introduced without considering the system's remaining functional time after repair. One potential avenue for further research involves expanding this minimal repair kit into a comprehensive solution that ensures system functionality with a high probability for a specific period of time. This extension could facilitate the preparation of major maintenance actions.

The identification of the optimal repair kit was based on a full search approach and on two closely related myopic algorithms. However, such algorithms do not consider the long-term consequences of adding or removing multiple components on the total expected cost of a repair kit. Therefore, exploring alternative computational algorithm methods for the discussed scenarios of the RKP becomes an interesting avenue.

This chapter focused on identifying the optimal repair kit within the cost model. Additionally, exploring similar scenarios for the RKP under the service model could be of interest, where customers can predefine a specific probability for job completion. The objective is to find a repair kit that fulfills this requirement with the minimum holding cost.

Chapter 5

Repair kit with Bayesian inference on component failure time

5.1 Introduction

The literature related to the investigation of components most likely to cause system failure or the determination of how many components fail at system failure time often assumes that components failure time distributions are fully known (e.g. [25], [7]). This assumption has also been applied in our work on these events in Chapter 3. Information of this nature, however, may not be directly available and may be difficult to obtain. Therefore, it is crucial to make inferences about the failure time of components.

Inferring component reliability poses significant challenges due to various complexities, such as censoring and a lack of data. Consider a scenario where we have a sample of systems that are observed until they experience failure. Following this, for each system, one of its components produces its failure time, while the failure times of the remaining components are censored. For instance, in the case of a series system failure, only one component will have a failure time that is directly observed, while the failure times of the remaining components are right-censored. Similarly, in a parallel system, one component's failure time is uncensored, whereas the failure times of the remaining components are left-censored. Moreover, life tests on the components of a system are often conducted during its initial development. However, once the system becomes operational, the available failure time data may only be accessible at the system level rather than for individual components [42]. Additionally, it is possible that, after deployment, the reliability characteristics of the components might change due to the working atmosphere of the system, where the focus is on reliability characteristics in the field [74]. Given the complexities mentioned, it is of interest to infer the distributions of component failure times.

The remainder of the chapter is outlined as follows. Section 5.2 provides a brief overview of some relevant literature on inferring component failure times from a Bayesian perspective. In Section 5.3, the posterior predictive distributions for new component failure times with Exponential or Weibull distributions are derived. The probabilities of the different events developed in Chapter 3 are then inferred based on these predictive distributions. Additionally, they are used to find optimal repair kits for some cases that were discussed in Chapter 4. A comparison is then made between these optimal repair kits and those produced when the distributions of component failure times are assumed to be known. In Section 5.4, we use a Markov Chain Monte Carlo method to generate posterior samples for the unknown parameters of the distributions of component failure times in situations where an analytical solution is not feasible. The generated samples are then used to estimate the posterior predictive distributions for new component failure times, which are subsequently used to find the optimal repair kits. Section 5.5 concludes the main ideas of the chapter and provides directions for future research.

5.2 Literature review

The literature on estimating reliability functions of system components is extensive, and various approaches have been proposed. Based on system failure time data, several studies have explored statistical inference techniques to make inference about component reliability. Bhattacharya and Samaniego [10] develop a non parametric maximum likelihood estimator for estimating component reliability. They utilise system signature and domination theory, assuming that component failure times are independent and identically distributed. Their study assumes that system failure times are based on a fixed system design. The study conducted by Ng et al. [44] explores the statistical inference of component failure time distribution by utilising failure time data of systems with identical structures, where the system signature is known. They make the assumption that component failure time distributions adhere to a general proportional hazard rate model and they estimate the proportionality parameter through the use of various estimation methods. Hall et al. [30] generalise Bhattacharya and Samaniego [10], by developing a non parametric estimator of component reliability based on system failure times with possibly different system structures. In Jin et al. [35], the same problem is addressed and non parametric estimation is developed when the sample of system failure times is obtained from arbitrary unknown designs. Based on complete and censored system failure times, Yang et al. [71] propose a stochastic expectation-maximization algorithm for determining the maximum likelihood estimates of model parameters. Based on progressive type-II censored failure times of systems and using system signature, Tavangar and Asadi [63] propose different methods for estimating component reliability.

Numerous contributions have also been made to infer about component failure time distributions from a Bayesian perspective. Using masked data (i.e. a component that caused system failure or the status of components at the time of system failure is unknown), Sarhan [61] obtained Bayesian estimates for component reliability for a series system, with components' failure times assumed to be independent and Exponentially distributed, with prior distributions belonging to a piecewise linear family. In Polpo et al. [48], a parametric Weibull model was used for failure time estimation of series and parallel systems. They chose Jeffrey's non informative prior distribution and utilised the Metropolis-Hasting simulation method to obtain the posterior distribution. In Polpo [49], non parametric Bayesian inference with the Dirichlet multivariate process as prior distribution is used to determine the reliability of components in series-parallel and parallel-series systems. Rodrigues et al. [56] consider Bayesian estimation for component reliability in coherent systems and assume a 3-parameter Weibull distribution for component failure times without requiring them to be identical. Their approach is not restricted to the common assumption that two or more components cannot fail at the same time. Their approach also considers left, right, and interval censored component failure times.

5.3 Bayesian inference with conjugate prior

The purpose of this section is to infer the failure time distributions of system components. Here, we assume that the failure times of some types of components follow Exponential distributions with unknown rates, while the failure times of components of other types are assumed to follow Weibull distributions with known shapes but unknown scales. For Exponential failure rates, we assume conjugate priors following the Gamma distribution. In the case of the Weibull scale, we adopt an Inverse Gamma prior distribution. These distributions were selected for their flexibility in describing a wide range of failure time data. Lawless [39] and Rinne [54] provide examples of various physical phenomena where the Weibull distribution is a suitable choice for modeling failure times.

To demonstrate the derivation of the posterior predictive distribution for the Weibull case, suppose that we have n_k test observations $\left(t_1^k, ..., t_{n_k}^k\right)$ for failure times of a component of type k. By denoting the unknown parameter (scale) as θ_k, the known shape parameter as β_k, and the Inverse Gamma prior parameters as a_k and b_k for shape and scale respectively, the posterior predictive distribution is given as follows.

$$
\begin{aligned}
f_{T^k_{n_k+1}}(t \mid \left(t_1^k, ..., t_{n_k}^k\right)) &= \int_0^\infty f\left(t \mid \theta_k\right) \pi\left(\theta_k \mid \left(t_1^k, ..., t_{n_k}^k\right)\right)\, d\theta_k \\
&= \int_0^\infty \frac{\beta_k}{\theta_k} t^{\beta_k - 1} \exp\left(-\frac{t^{\beta_k}}{\theta_k}\right) \\
&\quad \times \text{Inverse Gamma}\left(a_k + n_k, b_k + \sum_{i=1}^{n_k} \left(t_i^k\right)^{\beta_k}\right)\, d\theta_k \\
&= \beta_k t^{\beta_k - 1} \left(a_k + n_k\right) \left(b_k + \sum_{i=1}^{n_k} \left(t_i^k\right)^{\beta_k}\right)^{(a_k + n_k)} \\
&\quad \times \left(t^{\beta_k} + b_k + \sum_{i=1}^{n_k} \left(t_i^k\right)^{\beta_k}\right)^{-(a_k + n_k + 1)}.
\end{aligned}
\tag{5.3.1}
$$

The posterior predictive reliability function is

$$P\left(T^k_{n_k+1} > t \mid \left(t^k_1, ..., t^k_{n_k}\right)\right) = \left(\frac{b_k + \sum_{i=1}^{n_k}\left(t^k_i\right)^{\beta_k}}{t^{\beta_k} + b_k + \sum_{i=1}^{n_k}\left(t^k_i\right)^{\beta_k}}\right)^{a_k+n_k}. \tag{5.3.2}$$

In the following subsections, we use posterior predictive distributions for the probability that a component type will lead to system failure, and the probabilities of events related to the number of failed components at the time of system failure. Using these results, the optimal repair kit is determined under various scenarios and compared to the case in which the component failure times distribution is assumed to be fully known.

5.3.1 Optimal repair kits using predictive distributions

In this subsection, we examine the optimal repair kit for different scenarios of quantifying the expected RTF costs, introduced in Chapter 4, based on posterior predictive distributions. This investigation focuses on the hydropower system, which includes six types of components as shown in Figure 3.5. First, we assume that the distributions of failure times for components of types 1 and 2 are fully known and follow Weibull distributions with a shape parameter of 0.5 and a scale of 1, and a shape parameter of 2 and a scale of 2.5, respectively. For components of types 3, 4, 5, and 6, we assume Exponential distributions with unknown failure rates θ_k for $k \in \{3, 4, 5, 6\}$. The failure rate θ_k are assumed to have prior information represented by the following Gamma distribution with a shape parameter $\beta_k > 0$ and a rate parameter $\alpha_k > 0$:

$$f(\theta_k; \beta_k, \alpha_k) = \frac{\alpha_k^{\beta_k}}{\Gamma(\beta_k)}\theta_k^{\beta_k-1}e^{-\alpha_k\theta_k}.$$

The posterior predictive distribution for a new data $T^k_{n_k+1}$ is given as follows.

$$\begin{aligned}
f_{T^k_{n_k+1}}(t \mid (t^k_1, ..., t^k_{n_k})) &= \int_0^\infty f(t \mid \theta_k)\, \pi\left(\theta_k \mid (t^k_1, ..., t^k_{n_k})\right) d\theta_k \\
&= \int_0^\infty \theta_k e^{-\theta_k t} \frac{\left(\alpha_k + \sum_{i=1}^{n_k} t^k_i\right)^{\beta_k+n_k}}{\Gamma(\beta_k+n_k)} \\
&\quad \times \theta_k^{\beta_k+n_k-1} e^{-\left(\alpha_k+\sum_{i=1}^{n_k} t^k_i\right)\theta_k}\, d\theta_k \\
&= \frac{\left(\alpha_k + \sum_{i=1}^{n_k} t^k_i\right)^{\beta_k+n_k}}{\Gamma(\beta_k+n_k)} \\
&\quad \times \int_0^\infty \theta_k^{\beta_k+n_k} e^{-\left(\alpha_k+\sum_{i=1}^{n_k} t^k_i+t\right)\theta_k}\, d\theta_k \\
&= \frac{\left(\alpha_k + \sum_{i=1}^{n_k} t^k_i\right)^{\beta_k+n_k}}{\Gamma(\beta_k+n_k)} \\
&\quad \times \frac{\Gamma(\beta_k+n_k+1)}{\left(\alpha_k + \sum_{i=1}^{n_k} t^k_i + t\right)^{\beta_k+n_k+1}}.
\end{aligned}$$

We assume that the shape parameter $\beta_k = 1$ for all $k \in \{3, 4, 5, 6\}$, while for rate parameters, we take a rate of $\alpha_3 = 0.15$ for type 3, and $\alpha_4 = \alpha_5 = \alpha_6 = 0.09$ for types 4, 5, and 6.

Example 5.3.1 Assuming we have 20 observations as test data for types 3, 4, 5, and 6, we compute the posterior predictive distributions and use them to estimate the number of components of the six types that failed at the time of system failure given that the system was functioning at a certain point in time. The optimal repair kit is then determined based on these estimations. Using predictive distributions of types 3, 4, 5, and 6, Figure 5.1 shows the probability of different combinations for the numbers of failed components at the moment of system failure given that the system was functioning at time t. It should be noted that given the period of time during which the system has operated, the plotted combinations are generally the most likely to occur when the system fails. Table 5.1 shows different optimal repair kits for different holding and penalty costs when the RTF costs are computed according to Equation(4.3.2) where the probability term is based on the predictive distributions. We also take into account the uncertainty about the failure times of components when estimating the number of failures at the moment of system failure given that it fails at a future time interval $[t^\star_1, t^\star_2]$. Figure 5.2 presents the probability of the number of failed components of the six types when the system fails based on

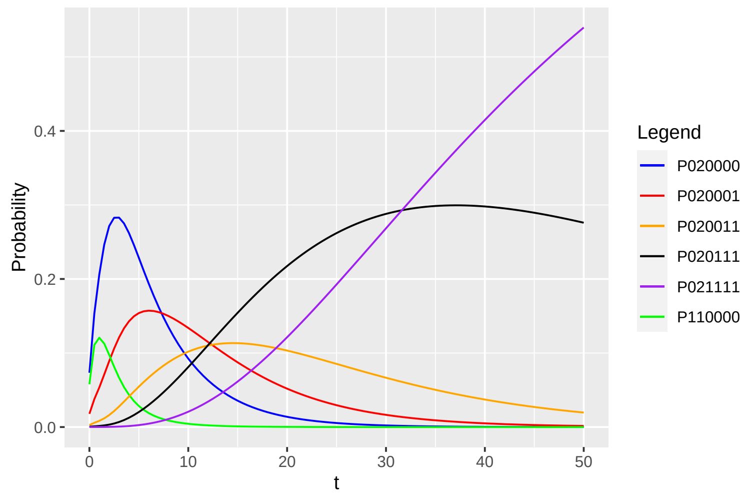


Figure 5.1: Probability of the number of failures at system failure time given system functioning at t, based on predictive distributions of types 3, 4, 5, and 6.

the predictive distributions of types 3, 4, 5, and 6, given the same assumptions as in the previous case regarding component failure times (i.e. distribution type, prior, size of test data, etc). For selecting the optimal repair kit, we consider the case in which the system fails at $[5, 10]$. Table 5.2 displays different optimal repair kits based on different penalties per part type shortage and assuming no extra penalty for the second visit. This calculation of the cost of the RTF visit is based on Equation (4.3.5), but with the condition $5 \leqslant T_s \leqslant 10$ instead of $T_s > 0$ and the second term is set to zero.

Secondly, we consider uncertainty about type 1 component, which are assumed to have a Weibull distribution with a known shape of 0.5 and an unknown scale in addition to uncertainty about types 3, 4, 5, and 6. This leaves type 2 as the only fully known type. As for the Weibull scale parameter, we assume a conjugate Inverse Gamma prior where the shape parameter and the scale parameter are both equal to 2. Equations (5.3.1) and (5.3.2) provide the posterior predictive probability density function and corresponding predictive reliability for a new failure time of type 1.

Example 5.3.2 Based on a set of 20 failure times as test data for type 1 components, the related posterior predictive distributions are calculated and applied to find the optimal repair kits. Using the predictive distributions for types 1, 3, 4,

Penalty cost C_p	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
20		$S = (1,0,0,0,0,0)$
40	$(4,3,2,2,1,4)$	$S = (1,2,0,1,1,0)$
60		$S = (1,2,1,1,1,1)$
20		$S = (1,0,0,0,0,0)$
40	$(8,6,4,4,2,8)$	$S = (1,0,0,0,0,0)$
60		$S = (1,0,0,0,1,0)$
20		$S = (0,0,0,0,0,0)$
40	$(16,12,8,8,4,16)$	$S = (1,0,0,0,0,0)$
60		$S = (1,0,0,0,0,0)$

Table 5.1: Optimal repair kits for the hydro power system using predictive distributions of types 3, 4, 5, and 6.

5, and 6, Figure 5.3 illustrates the probability of the number of failed components when the system fails. Table 5.3 displays optimal repair kits using the predictive distributions for a new failure time for types 1, 3, 4, 5 and 6. In addition, we take into consideration uncertainty about the failure time of the component of type 1, as well as the components of types 3, 4, 5, and 6, and use predictive distributions for those types to determine the probability of the number of failed components when the system fails at a future time interval when the system fails at $[t_1^\star, t_2^\star]$. Figure 5.4 illustrates the probability of failures at system failure time when the system fails at $[t_1^\star, t_2^\star]$ using predictive distributions of types 1, 3, 4, 5, and 6. In Table 5.4, we present optimal repair kits based on these predictive distributions if the system fails given that it failed at [1, 5].

Using predictive distributions, we also determine the optimal minimal repair kit that does not necessarily replace all failures, but aims to replace parts that restore the system to a functioning state. As discussed in Chapter 4 determining the minimal repair kit is related to the event of a component type causing system failures, that was addressed in Chapter 3. Assuming that the system will fail at a given time t, Figure 5.5 shows the probability of a component of a specific type

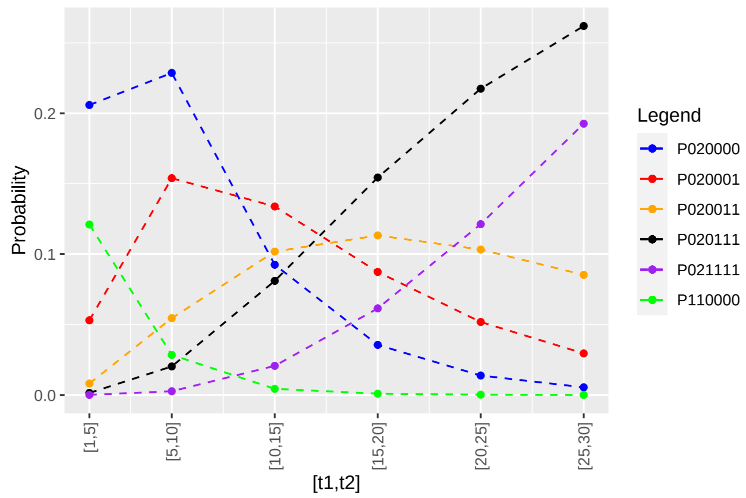


Figure 5.2: Probability of failures at system failure time given its failure at $[t_1^\star, t_2^\star]$ based on predictive distributions of types 3, 4, 5, and 6.

triggering a system failure. If we assume that the system will fail at $t = 10$, Table 5.5 presents different optimal minimal repair kits based on different parts holding costs and the given that only three parts are permitted.

5.3.2 Comparisons of repair kits under real and posterior predictive distributions

The previous subsection presented various scenarios for optimal repair kits and the associated probability events that are related to quantifying the expected cost of RTF visit for these repair kits based on predictive distributions. This was undertaken in two cases. In the first case, it is assumed that the probability distributions of the failure times of components of types 1 and 2 are known, while the predictive distribution is used for components of other types. In the second case, the assumption is made that only the probability distribution of failure times of components of type 2 is known. This subsection compares optimal repair kits when component failure times are assumed to be fully known versus when predictive distributions are used in both cases.

The first focus is on comparing between the optimal repair kits when assuming that the failure times of all component types are fully known and when only one of

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Extra cost C_p	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 1, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(4, 3, 2, 2, 1, 4)$	$S = (0, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (1, 2, 1, 1, 2, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 0, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(8, 6, 4, 4, 2, 8)$	$S = (0, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (0, 2, 1, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 0, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(16, 12, 8, 8, 4, 16)$	$S = (0, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (0, 2, 1, 1, 1, 1)$

Table 5.2: Optimal repair kits based on posterior predictive distributions of types 3, 4, 5, and 6, when the system fails at $[5, 10]$, when there is a penalty cost per part type shortage only.

them is known (type 2 which has Weibull distribution, as described earlier), while predictive distributions are used for the other types. For the repair kit which is intended to be placed nearby the system, as introduced in Section 4.3, Figures 3.12a and 5.3 present the probability of the number of failed components at system failure time when component failure times are all known as per Table 3.3, and when the distributions of failure times of components of types 1, 3, 4, 5, and 6 are not fully known. The settings for inferring about these types are the same as described in the Subsection 5.3.1. As regards this probability, we observe that there are some differences among the probabilities for different times when the system was last known to operate. The general probability pattern of the different combinations is, however, almost the same. In terms of optimal repair kits, we observe some differences between those developed using original component failure times (Table 4.2) and those developed using predictive distributions of types 1, 3, 4, 5, and 6 (Table 5.3). For example, in the case where the holding cost is (4, 3, 2, 2, 1, 4), all optimal kits have differed depending on the penalty cost. When the holding cost is (8, 6, 4, 4, 2, 8), only one kit remains the same, namely when the penalty cost is 20.

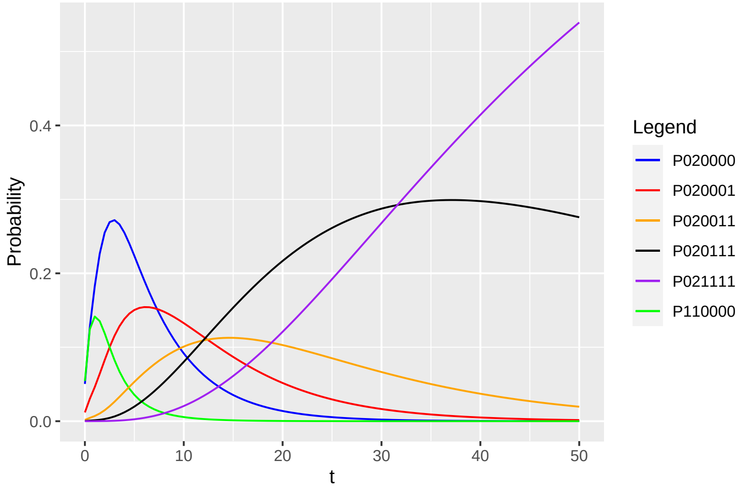


Figure 5.3: Probability of failures at the moment of system failure based on predictive distributions for components of types 1, 3, 4, 5, and 6.

The use of predictive distributions suggests that an empty kit is an optimal repair kit, when the holding cost is (16, 12, 8, 8, 4, 16) and the penalty is 20, whereas it is suggested to have one component of type 5 when real lifetime distributions are used. Moreover, in addition to type 2, if type 1 is also assumed to be known and has a Weibull distribution, and predictive distributions are used for types 3, 4, 5, and 6, then the optimal repair kits as shown in Table 5.1 do not improve to be the same or near when all types are known.

Considering optimal repair kits that aim at replacing all failures at a future time interval when the system fails, we also observe some differences between the optimal kits when the true component failure time are used and when predictive distributions are used. Figures 3.14a and 5.4 illustrate the probability of different combinations of failed components when the system fails, assuming the system will fail at $[t_1^\star, t_2^\star]$, based on known failure times distributions and based on predictive distributions of types 1, 3, 4, 5, and 6, respectively. For instance, if the system fails at a point in $[15, 20]$, then having two failures of type 2, one of types 5 and 6, and no failures of the other types (orange combination) is most likely with an approximate probability of 0.2, when using the true failure time distributions. When posterior predictive distributions are considered, having two failures of type 2, one failure of type 4, 5, and 6, and no failures of the other types is most likely (black

Penalty cost C_p	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
20		$S = (1, 0, 0, 0, 0, 0)$
40	$(4, 3, 2, 2, 1, 4)$	$S = (1, 1, 0, 0, 1, 0)$
60		$S = (1, 2, 0, 1, 1, 0)$
20		$S = (1, 0, 0, 0, 0, 0)$
40	$(8, 6, 4, 4, 2, 8)$	$S = (1, 0, 0, 0, 0, 0)$
60		$S = (1, 0, 0, 0, 1, 0)$
20		$S = (0, 0, 0, 0, 0, 0)$
40	$(16, 12, 8, 8, 4, 16)$	$S = (1, 0, 0, 0, 0, 0)$
60		$S = (1, 0, 0, 0, 0, 0)$

Table 5.3: Optimal repair kits for the hydro power system using predictive distributions of types 1, 3, 4, 5, and 6.

combination) with a probability of about 0.15. For comparison of optimal repair kits during system failure at $[t_1^\star, t_2^\star]$, we focus on the comparisons when the system fails at [5, 10]. We can see that there are some differences between optimal kits as determined by real failure time distributions (Table 4.13) and as determined by predictive distributions of types 1, 3, 4, 5, and 6 (Table 5.4). For kits resulting from real distributions, the optimal kit is $S = (0, 2, 0, 0, 1, 1)$ when the penalty cost per short type is $(10, 20, 30, 40, 50, 60)$, while when predictive distributions are used, the optimal kit includes one component of type 5. Similarly, when the penalty cost is $(50, 60, 70, 80, 90, 100)$, the optimal kit is $S = (0, 2, 0, 1, 1, 1)$, but it contains a component of type 3 when using predictive distributions. As for the other optimal kits, they remain the same. Furthermore, when the type 1 component is known as well as the components of type 2, the optimal repair kits at system failure time, when the system fails at [5, 10], are the same as the ones when the failure time of the type 1 component is unknown (not shown here).

For the optimal minimal repair kits, Figures 5.6 and 5.5 show the probability of a component of a specific type leading to system failure, based on true failure time distributions and predictive distributions of types 1, 3, 4, 5, and 6, respectively.

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Extra cost C_p	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 1, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(4, 3, 2, 2, 1, 4)$	$S = (0, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (1, 2, 1, 1, 2, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 0, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(8, 6, 4, 4, 2, 8)$	$S = (0, 2, 1, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (0, 2, 1, 1, 1, 1)$
$(10, 20, 30, 40, 50, 60)$			$S = (0, 2, 0, 1, 1, 1)$
$(20, 30, 40, 50, 60, 70)$	0	$(16, 12, 8, 8, 4, 16)$	$S = (0, 2, 0, 1, 1, 1)$
$(50, 60, 70, 80, 90, 100)$			$S = (0, 2, 1, 1, 1, 1)$

Table 5.4: Optimal repair kits based on posterior predictive distributions of types 1, 3, 4, 5, and 6, when the system fails at $[5, 10]$, when there is a penalty cost per part type shortage only.

Penalty cost per type $(C_p^1, C_p^2, C_p^3, C_p^4, C_p^5, C_p^6)$	Extra cost C_p	Holding cost $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal minimal kit S
$(10, 20, 30, 40, 50, 60)$			$S_m = (1^2, 1^4, 1^5)$
$(20, 30, 40, 50, 60, 70)$	45	$(4, 3, 2, 2, 1, 4)$	$S_m = (1^2, 1^4, 1^5)$
$(50, 60, 70, 80, 90, 100)$			$S_m = (1^1, 1^2, 1^5)$
$(10, 20, 30, 40, 50, 60)$			$S_m = (1^2, 1^4, 1^5)$
$(20, 30, 40, 50, 60, 70)$	45	$(8, 6, 4, 4, 2, 8)$	$S_m = (1^2, 1^4, 1^5)$
$(50, 60, 70, 80, 90, 100)$			$S_m = (1^2, 1^4, 1^5)$
$(10, 20, 30, 40, 50, 60)$			$S_m = (1^2, 1^4, 1^5)$
$(20, 30, 40, 50, 60, 70)$	45	$(16, 12, 8, 8, 4, 16)$	$S_m = (1^2, 1^4, 1^5)$
$(50, 60, 70, 80, 90, 100)$			$S_m = (1^2, 1^4, 1^5)$

Table 5.5: Optimal minimal repair kits when the system fails at $t = 10$ using predictive distributions of types 1, 3, 4, 5, and 6.

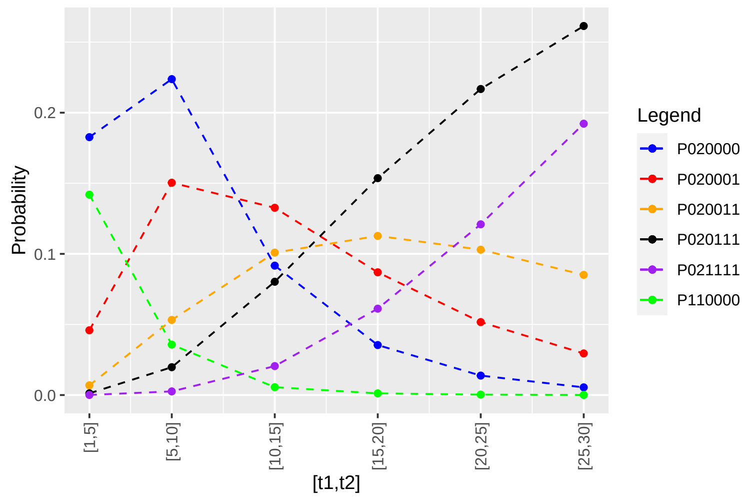


Figure 5.4: Probability of failures using predictive distributions of types 1, 3, 4, 5, and 6.

When the system fails at $t = 10$, most optimal minimal repair kits are the same, regardless of whether all failure time types are known or only type 2. The exception is the second optimal minimal repair kit, which is shown in Tables 4.15 and 5.5. When the failure time of the type 1 component is also known, then all optimal minimal repair kits become the same.

5.4 Bayesian inference with non-conjugate prior

There are certain cases where it is generally impossible to integrate out the parameters from a posterior distribution or determine the normalising constant of the posterior distribution. Such situations can occur when the prior knowledge about the parameters does not give rise to a conjugate posterior. With the development of Markov Chain Monte Carlo (MCMC) methods [41], this situation changed. The primary goal of these methods is to generate samples from the posterior distribution which are then used to make inferences about the likely values of the parameters.

This section considers one of the most popular MCMC methods for drawing samples from the posterior distribution, namely the Metropolis-Hastings (MH) algorithm [32]. Specifically, the algorithm will be used to generate samples from the posterior distribution of some unknown parameters related to the distribution of

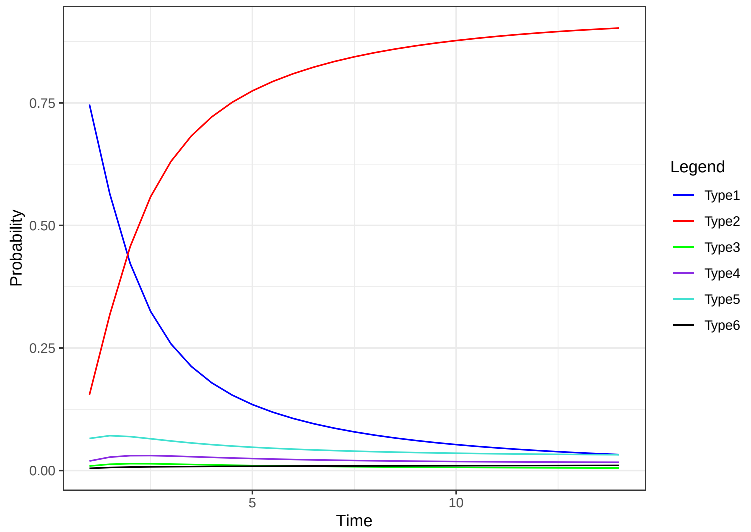


Figure 5.5: Probability of a component type causing system failure based on predictive distributions of types 1, 3, 4, 5, and 6.

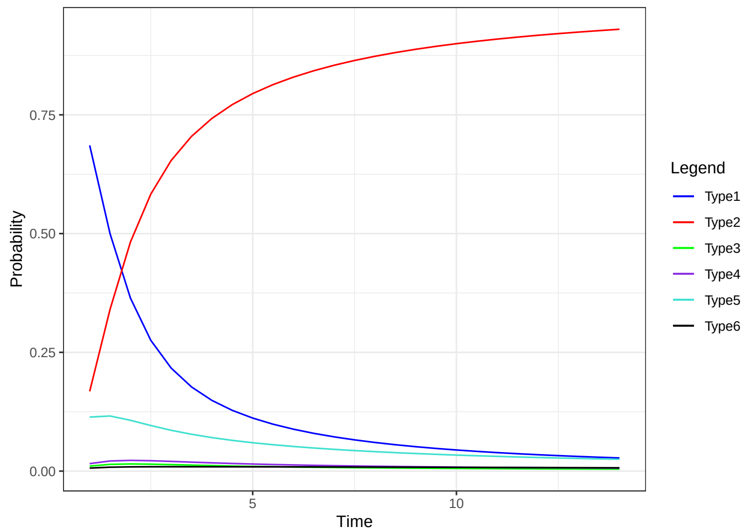


Figure 5.6: Probability of a component type causing system failure based on true lifetime distributions.

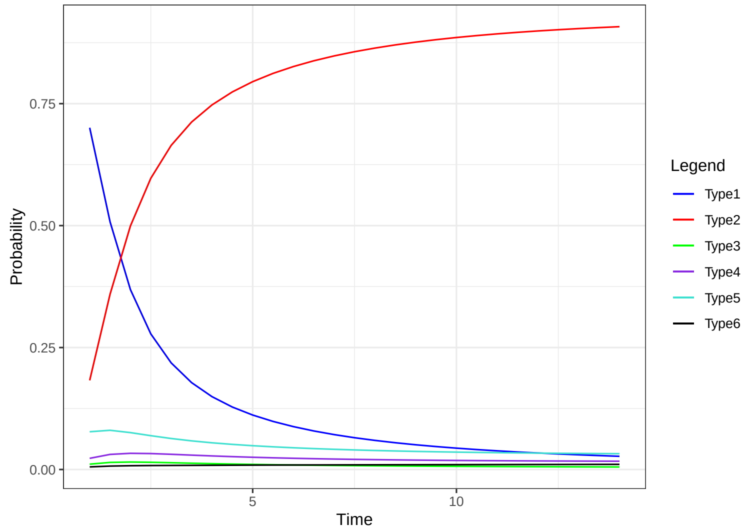


Figure 5.7: Probability of a component type causing system failure based on predictive distributions of types 3, 4, 5, and 6.

component failure times. These samples are then used to estimate the posterior predictive distribution of new component failure times, which will subsequently be used to determine the probability of events required to find the optimal repair kit.

The basic steps of the Metropolis-Hastings algorithm for sampling from an objective (target) distribution $\pi(\theta \mid (t_1, \ldots, t_n))$ are as follows. It begins by choosing an initial value for the unknown parameter $\theta^{(0)}$ and setting the number of iterations. At each iteration, a new candidate value θ' is generated from a proposal distribution $q(\theta' \mid \theta)$. An acceptance probability is then calculated which ensures exactly the correct proportion of samples are retained to ensure that the accepted samples are distributed according to the posterior. This process is repeated for the specified number of iterations, producing a sequence of values that represent samples from the posterior distribution. The mathematical steps of the algorithm are provided below in Algorithm 3.

In the following subsection, we provide some details on applying the algorithm to generate samples for the unknown rate parameter of the Exponential distribution, assuming the rate parameter has a prior represented by a uniform distribution. Based on the posterior samples, the optimal repair kits are determined for the hydro power system and compared to the case when the true distributions are fully known.

Algorithm 3: Metropolis-Hastings algorithm

Step 1: Initialize $\theta^{(0)}$.

Step 2: Repeat until the number of iterations is achieved:

1. Generate a uniform random number $u \sim \text{Unif}(0, 1)$.
2. Generate a candidate sample $\theta' \sim q(\theta' \mid \theta^{(i)})$.
3. Calculate the acceptance probability:
$$\alpha = \min\left(1, \frac{\pi(\theta' \mid t_1, \ldots, t_n)\, q(\theta^{(i)} \mid \theta')}{\pi(\theta^{(i)} \mid t_1, \ldots, t_n)\, q(\theta' \mid \theta^{(i)})}\right).$$
4. If $u \leqslant \alpha$, set $\theta^{(i+1)} = \theta'$; otherwise, set $\theta^{(i+1)} = \theta^{(i)}$.

5.4.1 Optimal repair kit using Metropolis–Hastings algorithm

In this subsection, we examine the optimal repair kit for a new system, as discussed in Section 4.3, and the related probability distribution developed in Section 3.5, where the posterior distributions of unknown parameters for some components' failure time distributions are obtained using the Metropolis-Hastings algorithm. The investigation is carried out for the hydro power system that consists of six types of components (Example 3.3.2). We assume that there are available data for some component types $(t_1^k, \ldots, t_{n_k}^k)$, which are assumed to be Exponentially distributed with unknown parameters θ_k. We further assume that these unknown parameters have uniform distributions with constants a_k and b_k. The posterior distribution of the unknown parameter θ_k is proportional to the product of the likelihood and the prior:

$$\pi(\theta_k \mid (t_1^k, \ldots, t_{n_k}^k)) \propto \theta_k^{n_k} e^{-\theta_k \sum_{i=1}^{n_k} t_i^k} \cdot \frac{1}{b_k - a_k} \quad \text{for } a_k \leqslant \theta_k \leqslant b_k.$$

To generate samples from the posterior distribution using the Metropolis-Hastings algorithm, we apply the following steps. First, an initial value $\theta_k^{(0)}$, a number of iteration N, and a proposal distribution $q(\theta_k' \mid \theta_k)$ are determined. A common choice for the proposal distribution which will be used here is a normal distribution centered at the current value, $q(\theta'_k \mid \theta_k) = \mathcal{N}(\theta_k,\, \sigma_k^2)$. For each iteration $i = 1, \ldots, N$, a candidate value θ'_k is generated from the proposal distribution $q(\theta'_k \mid \theta_k^{(i)})$. Then, the following acceptance probability α is calculated.

$$\alpha = \min\left(1, \frac{\pi(\theta'_k \mid t_1^k, \ldots, t_{n_k}^k)\, q(\theta^{(i)} \mid \theta'_k)}{\pi(\theta^{(i)} \mid t_1^k, \ldots, t_{n_k}^k)\, q(\theta'_k \mid \theta^{(i)})}\right).$$

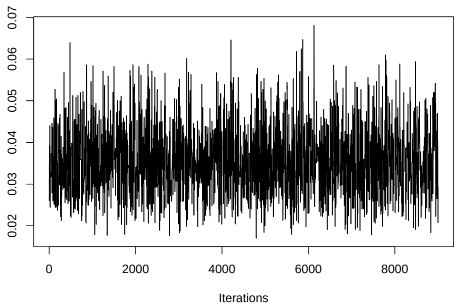
Since $q(\theta^{(i)} \mid \theta'_k)$ and $q(\theta'_k \mid \theta^{(i)})$ are symmetric, the acceptance probability is shortened to

$$\alpha = \min\left(1, \frac{\pi(\theta'_k \mid t_1^k, \ldots, t_{n_k}^k)}{\pi(\theta^{(i)} \mid t_1^k, \ldots, t_{n_k}^k)}\right).$$

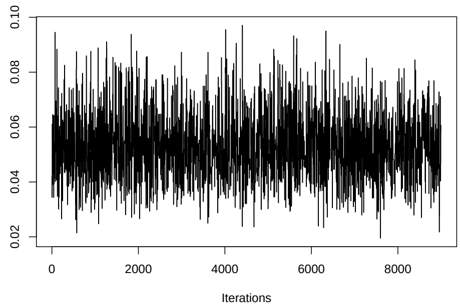
To accept or reject α, a uniform random number u $\sim$ Unif $(0, 1)$ is drawn. If $u < \alpha$ set $\theta_k^{(i+1)} = \theta'_k$, otherwise, set $\theta_k^{(i+1)} = \theta_k^{(i)}$. The sequence of $\theta_k^{(N)}$ values generated through this process will approximate the posterior distribution. In the following, we present an example of the probability distribution developed in Section 3.5, which is then used to obtain the optimal repair kit for a new system, with the posterior distributions determined based on the Metropolis-Hastings algorithm.

Example 5.4.1 For the hydro power system composed of six types of components, as shown in Figure 4.1, we assume that the distribution of the failure times for some component types are fully known, while others are not. It is assumed that the distribution of the failure times for components of types 1 and 2 are known and follow a Weibull distribution, with type 1 having a shape parameter of 0.5 and a scale parameter of 1, and type 2 having a shape parameter of 2 and a scale parameter of 2.5. For the components of types 3, 4, 5, and 6, it is assumed that there are some available data ($n_k = 20$) that are Exponentially distributed with unknown rates θ_k for $k \in \{3, 4, 5, 6\}$. We further assume that the parameters θ_k have uniform prior distributions with the hyperparameters $a_3 = 0.017$, $b_3 = 0.2$, $a_4 = 0.01$, $b_4 = 0.15$, $a_5 = 0.01$, $b_5 = 0.1$, $a_6 = 0.01$, and $b_6 = 0.83$.

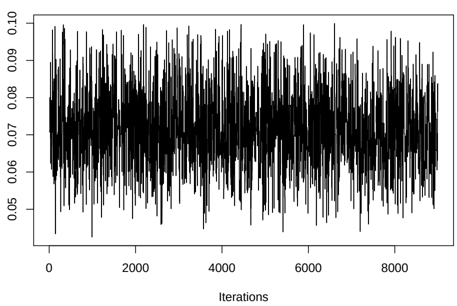
To generate samples from the posterior distributions for θ_k for $k \in \{3, 4, 5, 6\}$ using the Metropolis-Hastings algorithm, we set the initial value $\theta_k^{(0)} = \frac{a_k + b_k}{2}$ and define the number of iterations N = 10,000. For the proposal distributions, we choose normal distribution $N(\theta_k^{(i)}, \sigma_k^2)$ to achieve an acceptable acceptance rate. We use variances $\sigma_3^2 = 0.001$, $\sigma_4^2 = 0.002$, $\sigma_5^2 = 0.0015$, and $\sigma_6^2 = 0.005$. After applying the steps of the algorithm described earlier, the first 1,000 samples of the generated samples for each type $k \in \{3, 4, 5, 6\}$ are discarded as burn-in. This ensures that the remaining samples are representative of the posterior distribution and are not influenced by the initial values. Figure 5.8 presents the trace plots of the sampled values of the parameters of the failure time distributions for components of types 3, 4, 5, and 6, with an acceptance rate of around 0.3 for all four types. Then, these samples are utilised to compute the posterior predictive values, which are used for the determination of optimal repair kits.



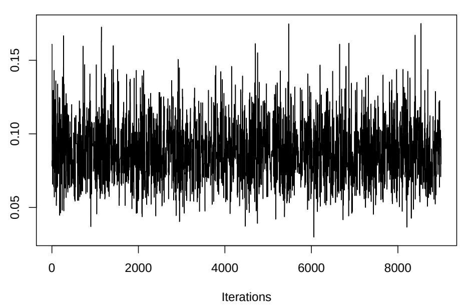
(a) Component type 3



(b) Component type 4



(c) Component type 5



(d) Component type 6

Figure 5.8: Trace plots of the sampled values of the parameters of failure time distributions for components of types 3, 4, 5, and 6.

Figure 5.9 presents the probability of the number of failed components of the six types when the system fails, given that the system is functioning at time t. If the system was last known to function at early times (e.g., from $t = 0$ to $t = 5$), the most

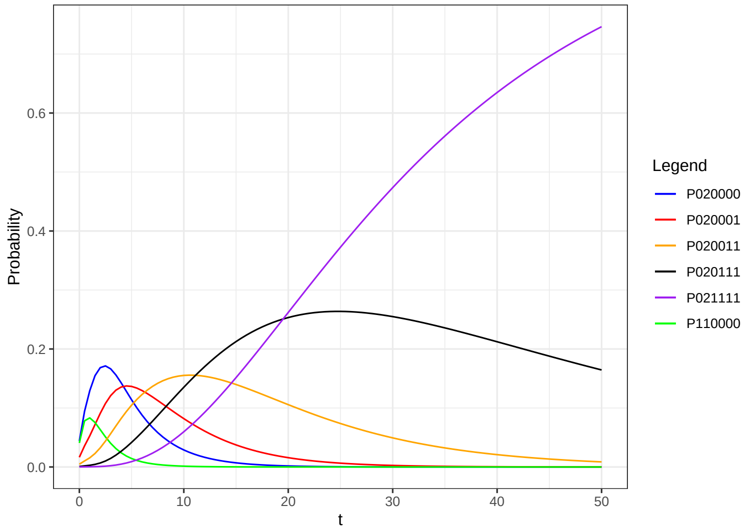


Figure 5.9: Probability of the number of failed components for the hydro power system at system failure using the MH algorithm for unknown parameters of components of types 3, 4, 5, and 6.

likely scenario is having two failed components of type 2 and zero failed components of the other types (blue combination). If the system continues to function between $t = 10$ and $t = 15$, the most likely scenario changes to having two failed components of type 2, zero failed components of types 1 and 3, and one failed component of the other types (black combination). When the system is still functioning at later stages, the most likely scenario becomes having two failed components of type 2, zero failed components of type 1, and one failed component of the other types (purple combination). Note that the plotted probabilities are for only the six most likely combinations. The probabilities of the remaining 642 combinations are not shown here.

Based on this probability distribution, Table 5.6 presents different optimal repair kits that are prepared for a new system, which can be used to replace all failed components when the system fails, given varying penalty costs and holding costs. For example, if the penalty cost for not completing the repair successfully is $C_p = 60$ and the holding cost per component type is $(4, 3, 2, 2, 1, 4)$, then the optimal repair kit is $S = (1, 2, 1, 1, 2, 1)$. If the holding cost increased to $(8, 6, 4, 4, 2, 8)$, the optimal repair kit would contain fewer spare parts: $S = (1, 0, 0, 0, 1, 0)$.

Penalty cost C_p	Holding cost per part type $(c_1, c_2, c_3, c_4, c_5, c_6)$	Optimal kit S
20		$S = (1, 0, 0, 0, 1, 0)$
40	$(4, 3, 2, 2, 1, 4)$	$S = (1, 1, 0, 1, 1, 0)$
60		$S = (1, 2, 1, 1, 2, 1)$
20		$S = (1, 0, 0, 0, 0, 0)$
40	$(8, 6, 4, 4, 2, 8)$	$S = (1, 0, 0, 0, 1, 0)$
60		$S = (1, 0, 0, 0, 1, 0)$
20		$S = (0, 0, 0, 0, 0, 0)$
40	$(16, 12, 8, 8, 4, 16)$	$S = (1, 0, 0, 0, 0, 0)$
60		$S = (1, 0, 0, 0, 1, 0)$

Table 5.6: Optimal repair kits for the hydro power system at system failure using the MH algorithm for unknown parameters of components of types 3, 4, 5, and 6.

When comparing these optimal repair kits, which are based on posterior samples generated using the MH algorithm, there are some differences when assuming the true failure time distributions of all component types are fully known (Table 4.2). For example, when the penalty cost is $C_p = 40$ and the holding cost is $(4, 3, 2, 2, 1, 4)$, the optimal repair kit is $S = (1, 2, 0, 0, 1, 0)$ when all failure time distributions are known, while it is $S = (1, 1, 0, 1, 1, 0)$ when the MH algorithm is used for posterior samples for component types 3, 4, 5, and 6. Similarly, when the penalty cost is $C_p = 60$ and the holding cost is $(16, 12, 8, 8, 4, 16)$, the optimal repair kit is $S = (1, 0, 0, 0, 0, 0)$ when all distributions are known, while it is $S = (1, 0, 0, 0, 1, 0)$ when the MH algorithm is used for types 3, 4, 5, and 6.

5.5 Conclusions and future work

The chapter examined the probability of a component of a specific type causing system failure and the probabilities of events related to the numbers of failed components of multiple types at system failure from a Bayesian perspective. The posterior predictive distributions for new failure times of certain component types are computed and used in calculating these probabilities. Optimal repair kits for various scenarios were determined based on these probabilities.

For the optimal repair kits intended to be provided with the system at its purchase time, differences are observed in the resulting optimal repair kits when assuming that the distributions of failure times for all component types are fully known versus assuming only one type has a fully known distribution for its failure time, while posterior predictive distributions are used for the other types. Additionally, when assuming that the failure times of two component types are fully known, the optimal repair kits do not improve to match those obtained when assuming the failure times of all component types are fully known.

When considering repair kits for a system that may fail within a specific time interval, differences are observed among the resulting optimal repair kits when assuming fully known failure times for all component types compared to when only the failure time distribution of one type is assumed to be fully known, with posterior predictive distributions used for the other types. Assuming that the distributions of the failure times for two types of components are fully known, did not yield any improvements. In fact, it resulted in the same optimal repair kits as when only one type distribution is known.

For the minimal repair kits, which are not necessarily aimed at replacing all failures but instead at restoring the system to a functional status, it is observed that the optimal minimal repair kits are identical when all distributions of all component types are fully known and when only the distribution of the failure time of one type is fully known, except for one minimal optimal repair kit. This particular repair kit became the same when assuming that the distribution of failure times of two types are known.

As for future research, it is of interest to further update the results regarding the probability of a component of a specific type leading to system failure and the probabilities related to the number of failed components at system failure using available information at the current time, as in Walter and Flapper [68]. Investigating how current information affects the results of optimal repair kits could provide valuable insights. Another direction for future research is to employ the joint posterior predictive density for the future failure times of components of the same type, given the data. This approach differs from using the marginal posterior predictive, as the future failure times will not be independent given the data, and the joint posterior predictive accounts for these dependencies. Another direction is to study a real-world system and including elicitation of expert judgements, especially in situations where data is limited or unavailable. It is also advantageous to use non-conjugate priors and modern Markov Chain Monte Carlo (MCMC) methods for computation. This approach allows for the implementation of more flexible model formulations.

Chapter 6

Conclusions

This thesis contributed novel methodology for system failure diagnosis. System failure was investigated from two perspectives. First, we introduced an importance component-type-based measure that calculated the probability of a component type failing at the moment of system failure, causing system failure. Secondly, three events related to the numbers of failed components of multiple types at system failure were derived under different conditions. First, the probability of the numbers of failed components of multiple types at system failure, given the exact failure time and conditioned on the numbers of failed components of multiple types that occur prior to system failure, was derived. Secondly, we derived the probability of the numbers of failed components of multiple types at system failure, assuming that the system was functioning at a certain point in time. The probability of the numbers of failed components of multiple types at the moment of system failure, assuming failure will occur within a future time interval, was also derived. The derivations of the probabilities for all these events were based on the use of the survival signature.

These probabilities can provide insights into several practical situations, including the Repair Kit Problem (RKP). The probabilities of some of these events were then utilised to formulate the RKP, specifically to calculate the expected cost of a possible Return to Fit (RTF) visit if a repair job cannot be completed successfully. We considered determining an optimal repair kit with the minimum total expected cost, comprising holding and RTF visit costs, under various scenarios. Initially, we examined a repair kit provided with a new system, aiming to replace all failed components in the event of a system failure. Two closely related heuristic approaches were proposed to determine the optimal repair kit. Another repair kit that was studied is prepared for a system failure within a future time interval, aiming to replace all failed components. Additionally, if the repair kit is restricted to a very limited number of components due to constraints such as space, we proposed a repair kit that does not necessarily aim to replace all failures but rather to restore the system to a functional status.

The determination of the probabilities of the numbers of failed components at system failure and the optimal repair kits was based on the assumption that all distributions of component failure times are fully known. Instead of making that assumption, we considered the situation in which some of the distributions may not be fully known. In this case, we derived posterior predictive distributions and applied them to the probability of the numbers of failed components at system failure, which is then used in determining the optimal repair kit. The optimal repair kits resulting from the use of posterior predictive distributions were then compared to those determined under the assumption of fully known distributions. The results indicated differences in all optimal repair kits when posterior predictive distributions were used compared to the case where all distributions of component failure times were assumed to be fully known. The only exception is for the optimal minimal repair kit, which suggests similar repair kits in both cases.

At the end of Chapters 3, 4, and 5, several ideas were provided for future research. In addition to these, we suggest another extension related to the repair kit problem. The repair kit problem discussed in this thesis was based on the assumption that the repair kit is determined to repair only one system (i.e. a single job). However, there are scenarios where the objective is to complete repairs for multiple systems at one or different locations (i.e. a tour job) (e.g. [65], [11]). Therefore, it is important to consider such scenarios, taking into account system structure, system and component failure times. This will require studying different events of interest, opening an interesting direction for further research. Another important future research is that all developments in this thesis focused on binary state systems (perfect functioning and complete failure). However, modern engineering systems often include systems and components with multiple states. Therefore, an obvious next step is to extend the probabilities for the numbers of failed components of multiple types at the moment of system failure and the RKP studied in the thesis to such systems. Qin and Coolen [51] introduced the survival signature for a multi-state system with multi-state components, which could provide a valuable basis for important generalisations.

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