Arithmetic and string theory

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Abstract

We study M-Theory solutions with G-flux on the Fermat sextic Calabi-Yau fourfold, focusing on the relationship between the number of stabilized complex structure moduli and the tadpole contribution of the flux. We emphasize first the point-of-view from Hodge theory by using Griffith residues to compute the length of the flux with respect to the dimension of the Zariski tangent space, and we propose an alternative approach to check that those are the only results by making use of elementary number theory.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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This work would not have been possible without the help from my advisor, Andreas, as well as Roberto and Daniel. Many thanks to them for having made the PhD experience enjoyable.

Dedication

Dedicated to my parents and my brothers for their unconditional support, as well as my friends, particularly Hadrien, Zakaria and Mehdi.

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CHAPTER 1

Introduction

In string theory, a large class of compactification models come from type IIB or F-theory [1]. In string compactification, the resulting four dimensional universe should have no massless scalar fields. Those massless fields come from the Kähler and complex structure moduli as well as the D-branes moduli, and one way to fix those complex structure moduli is to turn on appropriately quantized fluxes. A particularly nice description of those models is to consider the M-theory limit. This limit allows us to make statements on flux compactifications in a more mathematically amenable manner. In fact, this will allow us to study the following problem. Since we require fluxes to fix the various moduli appearing, those fluxes, in turn, contribute to various brane tadpoles which have to disappear in a compact manifold. The result is that we have to deal with tadpole cancellation conditions. In the usual picture, one should expect those conditions to be satisfied easily since there are as many equations to satisfy as there are moduli, and linear algebra would tell us that there is always a solution. However, in our case of study, the G_4 flux need not take value in a field, but rather over a half-integers or integers depending on the Calabi-Yau manifold. Furthermore its length has to be bounded above by some constant. Thus the usual intuition one has coming from linear algebra does not work and one needs to study those types of question more attentively.

This will be our main motivation to study flux compactifications, in particular the so-called G_4 flux in M-theory. On the one hand, flux quantization [2] imposes that G_4 must be integral or half-integral. The set of all G_4 fluxes obeying this condition shall be denoted as:

$$\Lambda_{phys} := \left\{ G \in \ H^{2,2}_{prim} | \ G + \frac{c_2}{2} \in \ H^4(X, \mathbb{Z}) \right\}. \tag{1.1}$$

We will be interested in the case where c_2 is odd, which means that despite appearances, Λ_{phys} is not a lattice when taking into consideration this constraint. We will explore the relationship between this constraint and lattices more thoroughly, ending up with a parametrization of a lattice resulting in Λ_{phys} for X being Fermat's sextic fourfold. On the other hand, the tadpole cancellation condition imposes that the self-intersection of G_4 must be below a certain bound. Furthermore, we require that the resulting physical theory must be supersymmetric Minkowski vacua, imposing further mathematical constraints on the type of differential form that G_4 must be.

We note those conditions by:

$$G_4 \in H^{2,2}_{prim} \tag{1.2}$$

$$G_4 \cdot G_4 \leq \frac{\chi}{24}, \tag{1.3}$$

where χ is the Euler characteristic of the manifold we consider. A priori the bound is quite tight and we look for obstructions to the existence of solutions G_4 satisfying all conditions above for a smooth manifold. In particular, we need to determine the length of a given G_4 flux and this will be done by using the theory of Griffith residues and computing their periods.

We explore in this text a set-up where all conditions are readily expressed and highlight the link between the geometric and arithmetic properties of such fluxes.

The first chapter introduces basic notions that will make sense of the above, and are used throughout the text. The goal is to provide a minimal introduction in order to be able to rely as little as possible on conjectures. We largely follow the exposition in [3]. The second chapter is devoted to a study of Griffith residues and their periods to make practical calculations, as well as a mathematical set up to express all conditions on G_4 easily. We finish the second chapter by making use of simple number theoretic theorems to highlight obstructions to the existence of solutions and collect evidence for an alternative unifying route to explore this problem. The third chapter is devoted to applying the second chapter results to the case of Fermat's sextic fourfold, and was taken from the work [4]. This will result in an explicit graph showing the tension between the aforementioned conditions as well as possible solutions to the problem. Lastly, in the fourth chapter, we highlight some situations where the previous propositions can be studied thoroughly.

In the first part of the present chapter, we introduce the basics of differential and algebraic geometry to make sense of the mathematical aspects of the problem and make sense of the quantization condition, and in the second part we introduce the problem as it appears in physics, making sense of the tadpole cancellation condition.

1.1 Mathematical setting

We start this section by making the link between algebraic and differential geometry appear, then introducing some useful tools and concluding with a bit of intersection theory.

1.1.1 Algebraic varieties

We are first interested in setting the stage of algebraic geometry. We will use this language heavily throughout the text, but we will make the link with differential geometry via a key result by Serre.

First let us define **complex projective space** as follows: $\mathbb{CP}^n$, is the space $\mathbb{C}^{n+1}\backslash\{0\}$ quotiented by the following equivalence relation:

$$(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n), \tag{1.4}$$

for any non-zero complex λ.

This means that any points that are linearly related in the complex plane are identified. All of those descriptions can be found in [3].

In our setting, we will simply understand **algebraic varieties** as defined as the zero-locus of some maximal ideal in some polynomial ring $\mathbb{CP}[x_1, ..., x_n]$ over complex projective space.

However, some further remarks are noteworthy :

1. $\mathbb{C}$ is algebraically closed.

2. We work with subvarieties in $\mathbb{CP}^n$, the complex projective space.

For example, a particularly interesting class of algebraic varieties are quadrics in $\mathbb{P}^n$, also known as quadratic forms in $n+1$ variables.

We make the distinction between affine varieties and projective varieties, but given our context we can simply understand them as being the vanishing locus of a polynomial or a homogeneous polynomial respectively.

$\mathbb{C}$ is algebraically closed but is also of characteristic 0. Furthermore, there is in principle no obstruction to formally study varieties over some other fields or rings, at the cost of perhaps introducing abstract varieties or schemes. This is mainly a first way to point out the fact that we will need to be thorough when tracking over which ring or field we are working with.

Perhaps more importantly, one has to give those objects some topology in order to be able to do some kind of geometry on them.

This topology is called the **Zariski topology**, and it differs from the usual topology induced by the Euclidean norm in $\mathbb{R}$. It is characterized by its closed sets, which are defined by the vanishing locus of some polynomial : its roots.

1.1.2 Complex manifolds

In parallel, we can construct complex manifolds, following [5]. One way to understand complex manifolds of complex dimension n is to define them by considering a real manifold of real dimension $2n$ and to further add an almost complex structure

$\mathfrak{I}$ such that locally:

$$\mathfrak{I}^2 = -I_n, \tag{1.5}$$

compatible with the metric meaning that for the metric g we have $\forall(u, v)$:

$$g(\mathfrak{I}u, \mathfrak{I}v) = g(u, v) \tag{1.6}$$

We can turn $\mathbb{CP}^n$ in a complex manifold by picking an appropriate coordinate system. The coordinate system can be chosen to be the open sets U_i with associated chart Φ_i defined by:

$$U_i := z_i \neq 0 \tag{1.7}$$

$$\Phi_i := (z_0/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_i). \tag{1.8}$$

The transition functions on the open sets $U_{ij} := U_i \cap U_j$ are $\Phi_{ij} := \Phi_i^{-1} \circ \Phi_j = \frac{z_i}{z_j}$ which are holomorphic, turning $\mathbb{CP}^n$ into a complex manifold.

Since this space is a manifold, we have implicitly used the usual Euclidean topology in this section. Perhaps noteworthy, the manifold we used in the example is called complex projective space, clearly a reference to the aforementioned projective varieties. How are the two constructions related ? The answer is given by Serre in [6], which relates the two a priori different topologies and constructions on them. For practical purposes, this means we can essentially use either framework of differential or algebraic geometry interchangeably. This is quite powerful since many results will be easier to state in one language or another.

A particularly important tool that is common to both languages is that of line bundles. Given a complex manifold X, a line bundle is given by the following data:

- A cover U_i.
- Trivilizations $U_i \times \mathbb{C}$.
- Analytical and nowhere zero transition functions.

In particular, the total space of a line bundle is also a manifold of dimension one more than the manifold X over which it is defined. Similarly, in algebraic geometry, one can define locally free invertible sheaves of rank 1, for which the definition ends up being the same via Serre's GAGA correspondance between analytic and algebraic geometry in the complex setting [6]. From this remark, one can see that hypersurfaces inside complex projective spaces, as manifolds, can be described by sections of a line bundle.

For notation, we will denote the hyperplane class by H, sections of the hyperplane line bundle by $\mathcal{O}(1)$, and more generally sections of line bundle that are described by a degree d polynomial by $\mathcal{O}(d)$.

1.1.3 Kähler manifolds : a special case

Now that we have introduced an almost complex structure $\mathfrak{J}$ to our manifold X with metric g, turning it into a complex manifold, it is also of interest to introduce a Kähler form J, which is a Hermitian form, meaning it satisfies :

$$J(x, y) = g(\mathfrak{J}x, y). \tag{1.9}$$

Furthermore we will ask that $dJ = 0$ so that in fact our manifold X is also a **Kähler manifold**. The reason for this will be given after introducing a few more mathematical notions.

There, the Kähler property plays an interesting role which facilitates the study of differential forms on complex manifolds immensely. Apart from Poincaré duality, which is an isomorphism between $H^k(X)$ and $H_{n-k}(X)$, but another duality of interest is Hodge duality, which is quite powerful to study Kähler manifolds.

For a manifold X of dimension n, we can define a **volume form** $Vol(X)$ which is a nowhere-zero n-form. Then, it is fairly natural to define the **Hodge operator** $\star$ acting on a differential form P as:

$$\star P =< Vol(X), P > \tag{1.10}$$

where $Vol(X)$ is the section of $\Lambda^n(X)$ the n-th exterior power of the cotangent bundle of X that is nowhere vanishing, which is the volume form induced by the metric $Vol(X) = \sqrt{det(g)}dx_0 \wedge \ldots \wedge dx_n$ for the metric g, and the inner form $<,>$ corresponds to the inner form of the exterior algebra of differential forms.

In fact, an interpretation of this formula is the following: it is fairly natural to ask to "complete" a k-form into an n-form by appending the "missing" $(n-k)$-forms. In fact, up to a sign due to orientability, taking the bidual recovers the k-form we started with:

$$\star \star P = (-1)^{k(n-k)}P. \tag{1.11}$$

Notably, the Hodge $\star$ operator depends on the metric. Furthermore, since we will work with Kähler manifolds, the volume form defined above is the same as one that could be built from using the symplectic structure.

Importantly, the example we gave of the **complex projective space, and any submanifold of it, as a complex manifold is also Kähler**. It carries a Kähler metric, known as the Fubini-Study metric. In fact the variety we will work with is also Kähler. We will make heavy use of the Kähler property and the so-called Kähler package in the following subsection in order to characterize some important invariants.

1.1.4 Differential forms of complex manifolds

We have seen that we can use algebraic and differential geometry interchangeably, and introduced the key example of Kähler manifolds. In this context, we introduce some key invariants and tools that we will use throughout the text. We assume some familiarity with Riemannian geometry, which can be found in [7]. The exposition when specialized to complex manifolds follows [3].

From the point of view of complex manifolds, the almost complex structure $\mathfrak{J}$ induces on the (co-)tangent bundles a decomposition in the following way. Complexify the (co-)tangent bundle of the underlying real manifold X by tensoring with $\mathbb{C}$ as $TX \otimes \mathbb{C} = T_{\mathbb{C}}X$. Since $\mathfrak{J}^2 = -1$ by definition, it has two eigenvalues as $\pm i$.

This induces a decomposition on the (co-)tangent bundle into a holomorphic (resp. anti-holomorphic) part, associated to the i (resp. $-i$) eigenspace which we denote by $T^{1,0}X$ (resp. $T^{0,1}X$), so that we have: $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$.

In light of this decomposition, we can understand in particular the space of differential forms. To this end, we note that the cotangent bundle and its exterior powers must be compatible with the decomposition induced by $\mathfrak{I}$. This means that we have for a given exterior power k:

$$\wedge^k T^*_{\mathbb{C}} X = \bigoplus_{j=0}^{k} \wedge^{j,k-j} X \tag{1.12}$$

with $\bigwedge^{p,q} X = \bigwedge^p T^{1,0} X \otimes \bigwedge^q T^{0,1} X$.

Differentials form belonging to $\bigwedge^{p,q}$ are said to be of p**-holomorphic parts and** q **anti-holormophic parts**, denoted as (p,q)**-forms**. We also need to know the decomposition of the exterior derivative due to the complex structure. It follows that:

$$d = \partial + \bar{\partial}, \tag{1.13}$$

where ∂ acts on a (p,q) form to give a $(p+1,q)$ form, and similarly $\bar{\partial}$ acts on a (p,q) form to give a $(p,q+1)$ form, and, crucially, $d^2 = 0$.

1.1.5 Cohomology theory and Hodge conjecture via Kähler package

Having introduced those notions, it is natural to introduce the notion of cohomology, following [5], which is necessary to study the problem of interest. The goal is to study the behaviour of differential forms under the exterior derivative. Furthermore, we tie things up by studying the special case of Kähler manifolds at the end.

So let $A_0, A_1, ...$ be abelian groups forming a cochain complex via homomorphisms $d_n : A_n \to A_{n+1}$ such that $d_{n+1} \circ d_n = 0$.

We use them to define the **cohomology groups** H^k as :

$$H^k = \frac{\text{Ker }(d_k)}{\text{Im }(d_{k-1})}. \tag{1.14}$$

For our purposes, the cohomology theories to consider are algebraic de Rham cohomology denoted by an index dR, for which the abelians groups are differential forms and the homomorphism is the exterior derivative and takes values in the field you consider your algebraic variety over, and Dolbeault cohomology, where the groups are given by (p, q) differential forms for a given p and the homomorphism is $\bar{\partial}$, denoted by an index $\bar{\partial}$ and takes values in $\mathbb{C}$ as such : $H^k_{\bar{\partial}}(X, \mathbb{C})$.

Note that usual de Rham cohomology for differentiable manifolds has coefficients in $\mathbb{R}$, but algebraic de Rham cohomology can have coefficients in the field of definition of the variety we consider.

As an example, we can mention the first non-trivial example which is the circle S^1. For the circle, $H^0_{dR}(S^1, \mathbb{R}) = \mathbb{R}$ and $H^0_{dR}(S^1, \mathbb{R}) = \mathbb{R} = [\phi] \cdot \mathbb{R}$ as well, where ϕ is the volume form on the circle. The reason it appears is precisely because it is not globally the differential of a function, meaning there are no functions f such that $\phi = df$ on the circle.

Since we are working with groups, it is key to emphasize that we need to pick coefficient rings over which those groups are defined. A given group H^k of a manifold X and coefficient ring R, is denoted by $H^k(X, R)$ when necessary to avoid confusion. We will denote $H^{p,q}$ instead of H^k in a complex setting.

Furthermore, in the setting we consider, to a cohomology theory $H^k(X, R)$ there is an associated homology $H_k(X, R)$ theory, and the two are related via Poincaré duality. This means that if X is of dimension n, we have $H^k(X, R) = H_{n-k}(X, R)$, under the condition that X is orientable.

Those two cohomology theories can be used to define some important topological invariants.

In particular the **Betti numbers** b_i are just the dimensions of $H^i(X, \mathbb{R})$, which can be used to defined the **Euler characteristic** $\chi(X) = \sum_i (-1)^i b_i$ of X.

In our previous example of the circle, $b_0 = 1$ and $b_1 = 1$ with the higher $b_i = 0$.

Dolbeault cohomology, which is the analog of de Rham cohomology for complex manifolds, can be used to define **Hodge numbers** $h^{p,q} = dim(H^{p,q}(X))$.

Those two are the main theories we will be working with, although many exists and are applicable in different contexts.

We now turn to the special case of Kähler manifolds as a special case where Dolbeault and de Rham cohomology are particularly well-behaved. The setting is that of Hodge theory.

Recall that a complex manifold being Kähler implies the following:

- The Laplacians given by either the exterior differential d of de Rham cohomology or the one given by $\bar{\partial}$ of Dolbeault cohomology coincide : $\Delta_d = 2\Delta_{\bar{\partial}}$.
- The decomposition of a d-harmonic form in terms of (p, q) holomorphic/anti-holomorphic forms is again d-harmonic.
- It is also compatible with complex conjugation, meaning that if some (p, q)-form is d-harmonic, then the conjugate (q, p)-form is d-harmonic as well.

This means that the notion of harmonic forms is quite nice for Kähler manifolds. In fact this can be understood to be a decomposition of de Rham cohomology in terms of Dolbeault cohomology, by making use of the fact that de Rham cohomology classes have a harmonic representation.

Naturally, we are interested in forms lying in H^{2p} whose decomposition lies in $H^{p,p}$, to which we can associate some submanifold in homology. Given that we have seen that differential geometry and algebraic geometry can be used interchangeably, and since we study in particular algebraic varieties, are those submanifolds also subvarieties, meaning they are the vanishing locus of some polynomial as well ?

Note that this conjecture depends on the cohomology group, and hence on the coefficient ring of said group. For example, in the case of Fermat's sextic, the rational Hodge conjecture, meaning with coefficients in $\mathbb{Q}$, is known to be true, but the integral Hodge conjecture, with coefficients in $\mathbb{Z}$ is not known to be true. The case of interest to us will be the case of the integral Hodge conjecture.

This is the question asked by the **Hodge conjecture**, and is still open. We can however notice one thing: if this conjecture holds, it is quite nice for us since polynomials are easier to work with than manifolds in general. Note however that for our purposes, it is neither necessary nor sufficient for the Hodge conjecture to be true to study the tadpole problem.

1.1.6 Chern classes, Calabi-Yau manifolds and invariants

The last and final tool we will recall is the theory of Chern classes, as well as how to compute them. Those are characteristic classes, meaning cohomological invariants, which can be used to tell the differences between vector bundles. The exposition here follows [3] in order to be able to readily perform computations.

A fairly straightforward way to define them is as follows. For a given curvature form R of a vector bundle, **Chern classes** c_k are the coefficients of the characteristic polynomial of said curvature form:

$$det\left(1+\frac{iR(t)}{2\pi}\right)=\sum_k c_k t^k. \tag{1.15}$$

Finally, the reason as to why we introduced those tools are the following : if one wants to have a supersymmetric Minkowski vacua after performing some compactification, one should use as a compactification space a Calabi-Yau manifold.

The definition of **Calabi-Yau manifolds** we will use is that they are smooth compact complex manifolds that have vanishing first Chern class.

Furthermore, we can describe such manifolds as some complex algebraic varieties, in which case we can use many of the tools we introduced to compute some relevant geometric properties.

For example, a condition for some N polynomials f_j of degree d_j in a product of l projective spaces $\mathbb{CP}$ of dimensions n_i to form a Calabi-Yau manifold can be expressed using Chern classes and the **adjunction formula** [3]:

$$\sum_{a=1}^{N} d_a^r = n_r + 1 \ (\forall\, r=1,...,l)\,. \tag{1.16}$$

The key observation we make for now is that in general, the degree of the polynomials and the dimensions of the projective spaces are linked via this formula, and hence we have a constraint that we will discuss in a later section.

In fact, in the context of algebraic geometry, the adjunction formula can be understood more generally. Consider the normal bundle short exact sequence for Y a submanifold of X:

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow N_{Y/X} \longrightarrow 0, \tag{1.17}$$

where T_Y is the cotangent bundle of Y, T_X the cotangent bundle of X and N designates the normal bundle, which is defined via this short exact sequence. In particular, for hypersurfaces Y embedded via i in an orientable manifold X, we have $i^*T_X = N \oplus T_Y$.

Furthermore, let $K_X := det(\bigwedge^n T^*_{\mathbb{C}}X) = det(T^*_{\mathbb{C}}X)$ designate the **canonical bundle** of X which is an invariant. We can dualize the previous sequence to get :

$$0 \longrightarrow N^*_{Y/X} \longrightarrow T_X|^*_Y \longrightarrow T^*_Y \longrightarrow 0. \tag{1.18}$$

We get :

$$det(T^*_X|_Y) \simeq det(N^*_{Y/X}) \otimes det(T^*_Y) = det(N_{Y/X})^* \otimes K_Y, \tag{1.19}$$

as well as :

$$det(T_X|^*_Y) \simeq det(T^*_X|_Y) \simeq det(T^*_X)|_Y \simeq K_X|_Y. \tag{1.20}$$

From which we get the **general adjunction formula** by tensoring with $det(N_{Y/X})$:

$$K_Y \simeq K_X|_Y \otimes det(N_{Y/X}). \tag{1.21}$$

The main point to emphasize is that for the specific case of Calabi-Yau manifolds, we do not know any metric. Thus it is necessary for practical purposes to work under the assumption of the Hodge conjecture, as this gives a characterization of the middle cohomology in terms of polynomials, which are easier to compute. In principle however, this assumption is not needed.

Furthermore, in order to compute invariants like the Hodge numbers $h^{p,q}$ or the Euler characteristic χ, we will need a few results from [8] for Calabi-Yau fourfolds.

In particular we need the following :

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}), \tag{1.22}$$

along with the following **arithmetic genus formulas** for Calabi-Yau fourfolds with vanishing first Chern class:

$$h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3} + h^{0,4} = \frac{1}{720}\int(-c_4 + 3c_2^2) \tag{1.23}$$

$$h^{1,0} - h^{1,1} + h^{1,2} - h^{1,3} + h^{1,4} = \frac{1}{180}\int(-31c_4 + 3c_2^2) \tag{1.24}$$

$$h^{2,0} - h^{2,1} + h^{2,2} - h^{2,3} + h^{2,4} = \frac{1}{120}\int(79c_4 + 3c_2^2) \tag{1.25}$$

where on the left hand-side it is the definition of the arithmetic genus for a projective complex manifold of dimension n, p_i, defined as:

$$p_i = \sum_{k=0}^{n}(-1)^k h^{i,n-k} \tag{1.26}$$

Furthermore, we have the constraint coming from the Euler characteristic χ :

$$\chi(X) = \int_X c_4(X) = \int_{\mathbb{P}^n} c_4(X) \wedge dH, \tag{1.27}$$

where dH is the Poincaré dual of X inside $\mathbb{P}^n$.

So that we can determine the Hodge numbers and the Euler characteristic if we know Chern classes. We can use the adjunction formula quite easily in the case of hypersurfaces of degree d in $\mathbb{P}^n$, and the computation of the **total Chern class** $c(X)$ is given by the expansion of :

$$c(X) = \frac{(1+H)^{n+1}}{(1+dH)}, \tag{1.28}$$

via the adjunction formula.

This makes the dependence of Hodge numbers and the Euler characteristic with respect to the dimension of the underlying projective space and the degree of the manifold explicit and makes the case of hypersurfaces quite simple because that dependence is rather straightforward.

Lastly, we need one more result to compute the relevant invariants. This is based on the observation that the arithmetic genus formula as well as the Euler characteristic fix only 2 of the remaining 3 Hodge numbers. In some special situation, we can use the following tool to fix the last one.

To this end, we will need the **Lefschetz hyperplane theorem**. Let X by an n-dimensional complex projective variety and Y an hyperplane section such that $X \backslash Y$ is smooth. The Lefschetz hyperplane theorem states that $H^k(X, \mathbb{Z}) \longrightarrow H^k(Y, \mathbb{Z})$ is an isomorphism for $k < n-1$ and injective for $k = n-1$.

In particular, this allows, along with the Euler characteristic and the arithmetic genus formula, the determination of the Hodge numbers for hypersurfaces in complex projective space.

1.1.7 Looking ahead: intersection theory

Since we have introduced many invariants for complex algebraic varieties, the natural question to ask is to classify possible varieties. For example, we have seen already a crucial classification problem: the Hodge conjecture.

Following that train of thought, a question that can be asked using the tools we have introduced is to characterized the subvarieties Y of a given variety X. This has been of historical importance, for example in the context of classical algebraic geometry, the classification of conics reduced to asking how many ways can we intersect a plane and a cone.

Here, we have noted that in any cohomology theory, we can associate a dual in homology under the condition that X is orientable.

Thus, computing the **length of a (real) differential form** G :

$$G \cdot G := \int_X G \wedge G, \tag{1.29}$$

is the same as computing the intersection number of the dual in homology and we formally identify both.

Naturally, this question has particular settings where it is remarkably rigid and relatively easy, such as the case for curves. Indeed, for curves the definition of **intersection multiplicity** which can be found in [9] fully determines the computation of intersection numbers.

However, we will be mainly interested in fourfolds. In particular we have seen that we are mainly interested in $H^{2,2} \cap H^4$. In this language, we are interested in computing intersection numbers of surfaces inside fourfolds.

A result in intersection theory we will use later on is that transverse intersections between holomorphic submanifolds always have an intersection number of $+1$.

This question is in general hard to answer, but for special cases we will introduce tools in chapter 2 which can be used to give answers.

1.2 Physics setting

In this section, we introduce the physical origins of the problem. We will first focus on some motivation for studying the so-called tadpole problem. We start by stating the physics setting from the point of view of type IIB string theory, and then translate everything to the point of view of M-theory which we will use in the rest of the text.

1.2.1 Type IIB supergravity action

Our starting point is the bosonic part of the low energy effective action of type IIB strings, which is given by, in the **democratic formulation** :

$$\begin{aligned} S_{IIB} =& 2\pi\Big(\int d^{10}x e^{-2\phi}\left(\sqrt{-g}R + 4\partial_M\phi\partial^M\phi\right) - \frac{1}{2}e^{-2\phi}\int H_3 \wedge \star H_3 \\ & - \frac{1}{4}\sum_{p=0}^{4}\int F_{2p+1}\wedge\star F_{2p+1} - \frac{1}{2}\int C_4 \wedge H_3 \wedge F_3\Big), \end{aligned} \tag{1.30}$$

following the conventions of [10]. Where we have introduced the following field strengths :

$$F_1 = dC_0 \tag{1.31}$$

$$F_3 = dC_2 - C_0 dB_2 \tag{1.32}$$

$$F_5 = dC_4 - \frac{1}{2}C_2 \wedge dB_2 + \frac{1}{2}B_2 \wedge dC_2 \tag{1.33}$$

$$F_5 = \star F_5 \tag{1.34}$$

$$F_9 = \star F_1 \tag{1.35}$$

$$F_7 = - \star F_3 \tag{1.36}$$

$$H_3 = dB_2 \tag{1.37}$$

The condition $F_5 = \star F_5$ must be supplemented in 1.30. Using this formulation, we can see that the C act as potential for the field strength F and that type IIB only has even potentials.

Some interesting objects in string theory are branes. For the particular action 1.30, the main topic of interest will be $D7$-branes, which are magnetic sources for the RR axion C_0.

1.2.2 D7-branes in type IIB supergravity

We will first explore the behaviour of 7-branes in the context of type IIB in order to showcase the subtleties that arise in flux compactifications.

In the context of type IIB strings, we can understand p-branes as source terms in the normal $n = 9 - p$ spatial directions. Typically, source fields can be thought of as fundamental solutions to Poisson's equation in n-dimensions :

$$\Delta\phi(x) = \delta^n(x)$$

As is well known, they behave as :

$$\phi(x) \propto \frac{1}{x^{n-2}}$$

for $n > 2$. This is the equation of motion for the C fields.

Crucially, this converges (except at 0) precisely in those cases $n > 2$. However, they scale logarithmically when $n = 2$, and the complex logarithm admits a branch-cut, showing the difficulties in this case. While this is heuristic, it highlights the main difficult in type IIB compactifications : $n = 2$ corresponds to $p = 7$, which motivates the study of D7 branes.

In the usual language, the field sourced by a 7-brane is denoted as C_8, which is dual to C_0. Note that C_0 appears in the above action as $F_1 = dC_0$. In the language of differential forms on a complex manifold, we can thus write our Poisson equation for C_8 in the presence of a D7 brane at some point z_{D7} as :

$$d \star F_9(z) = \delta^2(z - z_{D7})$$

By duality we also have that $F_9 = \star F_1$ which implies, in integral form :

$$\int \star F_9 = \oint_{S^1} dC_0 = 1 \tag{1.38}$$

in a neighborhood of the brane.

Defining formally for now the **axio-dilaton** τ as :

$$\tau = C_0 + ie^{-\phi} \tag{1.39}$$

where $< e^{\phi} >= g_s$[1] with g_s the string coupling constant. Importantly, τ is holomorphic from the equation of motion.

Paying attention to the axio-dilaton as defined in 1.39 we can write :

$$\tau(z) = \tau_0 + \frac{1}{2\pi i} ln(z - z_{D7}) + ... \tag{1.40}$$

Justifying our previous heuristics.

The logarithm branch-cut induces a monodromy $\tau \to \tau + 1$. Considering magnetic dualities between fields, recalling that 7-branes are electric sources for C_8 and

[1] $< . >$ denotes the vacuum expectation value

thus a magnetic source for the dual C_0, we can now see via 1.38 that τ is sourced by D7-branes.

1.2.3 $SL(2,\mathbb{Z})$ invariance and the axio-dilaton

In light of those results, we have reasons to look for a description of those non-pertubative effects. In fact, what those heuristics have shown is that the divergence in the axio-dilaton τ indicates a position of a 7-brane. We expose here some more properties of τ.

One can observe that the action 1.30 can be written to show manifest $SL(2,\mathbb{Z})$ invariance in the following manner [11] in the Einstein frame:

$$\tilde{S}_{IIB} = 2\pi \int d^{10}x\sqrt{-g}\left(R - \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{2(Im\,\tau)^2}\frac{1}{2}\frac{|G_3|^2}{Im\,\tau} - \frac{1}{4}|F_5|^2\right) + \frac{1}{4i}\int \frac{1}{Im\,\tau}C_4 + G_3 \wedge \bar{G}_3, \tag{1.41}$$

where we have introduced the G_3 **field strength** :

$$G_3 = F_3 - \tau H_3. \tag{1.42}$$

The main point of the type IIB action written this way is the manifest $SL(2,\mathbb{Z})$ invariance, given by the following transformations via

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

, as :

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{1.43}$$

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \tag{1.44}$$

$$C_4 \to C_4 \tag{1.45}$$

$$g_{\mu\nu} \to g_{\mu\nu}. \tag{1.46}$$

From our previous discussion resulting in 1.40 and showcasing a branch-cut, we can identify the presence of a $D7$ brane on the monodromy $\tau \to \tau+1$ by interpreting it as an $SL(2,\mathbb{Z})$ monodromy with :

$$M_{[1,0]} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This motivates the introduction of a notation to handle the presence of branes following [12]. Let us note a dyonic state with p and q coprime by $(p,q) = \Phi^a$, which naturally couples to the combination $pB_2 + qC_2$. We can then form them into a vector as in 1.43 to form an $SL(2,\mathbb{Z})$ charge vector Q^a.

By definition, a (p,q) string ends on a $[p,q]$ 7-brane. Hence, acting on Q^a by an $SL(2,\mathbb{Z})$ matrix $g_{[p,q]}$ such that :

$$g_{[p,q]} = \begin{pmatrix} p & r \\ -q & s \end{pmatrix}, \tag{1.47}$$

which induces a monodromy on $[p,q]$ 7-brane via :

$$M_{[p,q]} = g_{[p,q]} M_{[1,0]} g^{-1}_{[p,q]}. \tag{1.48}$$

The main point is the following [13]: we expect stacks of branes to give rise to a non-abelian gauge group. Usually, we only get $SU(N)$ as a gauge group when we consider only Chan-Paton factors. However, this dyonic system can admit several other non-abelian gauge groups, in particular the ADE groups.

In the first instance studying the expansion of the axio-dilaton, we understood that the divergences indicates the positions of 7-branes, but now we also understand that the monodromy of the axio-dilaton encodes the types of 7-branes appearing as well.

Furthermore, since, broadly speaking, branes lead to gauge groups in the compactificatified theory, this motivates the study of F-theory through the lens of this important $SL(2,\mathbb{Z})$ invariance. In fact, the idea of F-theory is to identify the $SL(2,\mathbb{Z})$ invariance with the data of an elliptic fibration to keep track of τ.

Lastly, to give some further informal motivation, we recall as stated earlier that branes act as sources for some fields. They also are dynamical and hence there is some back-reaction induced by the presence and position of branes. Hence F-theory is one way to study those backreactions.

1.2.4 Singularities and gauge groups

The main insight from the $SL(2,\mathbb{Z})$ invariance is that we can study it via elliptic fibrations as follows, first described in [14]. An introduction of the mathematics behind elliptic fibrations can be found in [10].

We admit that an elliptically fibred Calabi-Yau fourfold can be described using a **Weierstrass model** :

$$y^2 - (x^3 + fxz^4 + gz^6) = 0, \tag{1.49}$$

where f and g are two polynomials, depending on the base of the fibration.

We can introduce the discriminant of this elliptic fibration Δ:

$$\Delta = 4f^3 + 27g^2, \tag{1.50}$$

if $\Delta = 0$, the fibre is not smooth. In fact, [15] [16] provide a classification of co-dimension one singularities.

Furthermore, elliptic fibrations admit an invariant called the j-invariant defined as follows.

Let E be an elliptic fibration, and j **its j-invariant** :

$$j(E) := \frac{c_4^3(E)}{\Delta}. \tag{1.51}$$

In the context of F-theory we can rewrite the above formula as :

$$j(\tau) := \frac{4(24f)^3}{4f^3 + 27g^2}, \tag{1.52}$$

where τ is the axio-dilaton.

This means that the poles of the j-invariant corresponds to the monodromies, which can be identified with those of the branes, and in turn their location at those singularities.

This classification associates to each type of singularity an ADE Lie algebra and appears to be for purely combinatorial reasons from the mathematical point of view. From the physics perspective, this translates to the gauge group associated with a stack of 7-branes.

In short, there is a correspondence between the singular locus of the elliptic fibration and the union of loci of branes. Since loci of branes induce gauge groups in the compactified theory [17], we have a correspondence between singularities of the elliptic fibration and gauge groups.

Thus singularities are quite important and this leads to the following interpretation from physics. Since singularities encode the gauge algebras of the theory, we can hope that the gauge groups appearing in, for example, the Standard Model single out the geometry by forcing certain types of singularities. On the other hand, the appearance of those singularities provide a geometric explanation for the existence of gauge groups, meaning the choice is no longer arbitrary.

1.2.5 M-theory point-of-view

Due to string dualities, there is yet another point of view to study how elliptic fibrations arise, which is the point of view of M-theory. In fact, there is a duality between type IIB compactified on a circle and M-theory compactified on a torus. We derive this here following [10] and [8], mainly in order to introduce some notation and the object of interest, the G_4 field strength.

Start from the bosonic part of the $11d$ **supergravity action** :

$$S = 2\pi \left(\int_{\mathbb{R}^{1,10}} \sqrt{-g} R - \frac{1}{2} \int_{\mathbb{R}^{1,10}} dC_3 \wedge \star dC_3 - \frac{1}{6} \int_{\mathbb{R}^{1,10}} C_3 \wedge G_4 \wedge G_4 + \int_{\mathbb{R}^{1,10}} C_3 \wedge I_8 \right), \tag{1.53}$$

where crucially we have introduced the G_4 **field strength** which is :

$$G_4 = dC_3, \tag{1.54}$$

and some topological higher-order curvature term I_8, which will be of importance when discussing G_4.

For our purpose, we will skip most of the details, which can be found in [18].

The key result we will use is that M-theory compactified on an elliptically fibred fourfold X with base B and fiber volume V is dual to the circle reduction of radius R of type IIB on B via $R \sim \frac{1}{V}$.

From now on, we will take the point of view of M-theory and study the G_4 field strength introduced above. The main reason is that this point-of-view allows us to highlight issues with respect to flux compactifications.

Crucially, the G_4 field strength backreacts on the metric, but it stays Calabi-Yau up to warping, motivating further this point-of-view. A priori this is non-trivial since you expect the Calabi-Yau property to be lost upon backreaction.

To motivate further this perspective and introduce some basic mathematical tools we will use later on, it is a good exercise to study the G_4 field strength from a physics perspective (i.e. through string dualities and physical objects) and a mathematical one as done in [18].

G_4 field strengths are by definition 4-form living on some orientable Calabi-Yau manifold X of (real) dimension 8, and thus there is a canonical pairing with respect to pair G_4 with a 4-cycle in homology via Poincaré duality.

Since they lie in some cohomology class, to understand them we need to understand the (co-)homology of the underlying manifold.

In the general physics picture, the homology class represents the flux line corresponding to the field strength, which emanates from charged objects. Essentially a higher-dimensional analog to the usual electromagnetic theory, which is something we already mentioned in the type IIB picture.

From the M-theory picture over $\mathbb{R}^{1,2} \times X$ where X is an elliptically fibred fourfold over a base B, we can supplement 1.53 with $M2$-brane sources.

In this picture, **the equation of motion for** G_4 are, from this point of view, the following :

$$d \star G_4 = \frac{1}{2} G_4 \wedge G_4 - I_8 + \sum_{i=1}^{N_{M2}} [M2]_i \tag{1.55}$$

Where the brackets indicate the positions of the i $M2$-branes and N_{M2} the total charge (total number of branes). We can integrate the I_8 term over X to give the following [18] [19]:

$$\int_X I_8 = \frac{\chi(X)}{24}. \tag{1.56}$$

where I_8 is as noted before topological.

Now finally we can arrive at the tadpole cancellation condition in the M-theory picture as :

$$N_{M2} + \frac{1}{2} \int_X G_4 \wedge G_4 = \frac{\chi(X)}{24} \tag{1.57}$$

1.2.6 Moduli stabilization and G_4

In the bigger physics picture, we need this field strength to give some potential to the complex structure moduli in the compactified theory, in order to make them massive which may allow for stabilization. This is a general feature of Kaluza-Klein theory : compactification induces the creation of moduli, and those moduli need to be stabilized, which fluxes allow for in the setting of string theory.

Indeed, Kaluza-Klein theories, upon compactification, typically lead to a tower of massless fields. The goal of turning on fluxes is to act as a potential for those massless fields and in turn stabilize them. Since we are interested in vacuum expectation values in quantum field theory, we want to avoid the appearance of such massless fields. Hence stabilizing really means giving them a mass, and we want to stabilize all of them.

This stabilization process crucially depends on the Hodge operator as follows. Let us write the action S of a field strength F in a very formal manner and focus on the kinetic part :

$$S \propto \int F \wedge \star F.$$

The key observation here is the presence of the Hodge star operator which explicitly depends on the metric. In the context of Kaluza-Klein theory, it generically translates to an effective potential for the related moduli, leading to the idea that it may stabilize them.

This is the Lagrangian for the G_4 flux we are interested in. Furthermore, upon compactification, we can specify several properties of this G_4 depending on the physics we impose as follows.

Introducing Ω as the holomorphic top-form of our elliptically-fibred Calabi-Yau fourfold X and J as its Kähler form, we can write the minimas of the Gukov Vafa Witten superpotential, or **GVW superpotential** that arises when compactifying to 3 dimensions from the G_4-field strength as [20] [21]:

$$W = \int_X G_4 \wedge \Omega \tag{1.58}$$

$$\tilde{W} = \int_X G_4 \wedge J \wedge J, \tag{1.59}$$

where W is the chiral GVW superpotential while $\tilde{W}$ is the real version of this superpotential [22].

Adding the requirement that the vacua must be supersymmetric and Minkowski leads to :

$$D_i W = 0 \tag{1.60}$$

$$\partial_k \tilde{W} = 0 \tag{1.61}$$

$$W = 0, \tag{1.62}$$

where $i = 1, 2, ..., h^{1,3}$, $k = 1, ..., h^{1,1}$, $D_i = \partial_i + \partial_i K$ and K is the Kähler potential.

Since G_4 is a 4-form, we can prove some further properties of the G_4 flux from such equations 1.60. Since G_4 is real, we can infer from $W = 0 = \int_X G_4 \wedge \Omega$ that **its**

component of type $(0,4)$ **(and thus** $(4,0)$ **because it is real) vanishes** due to Ω being $(4,0)$.

Furthermore since $D_i\Omega$ form a basis of $H^{3,1}$ forms, **the** $(1,3)$ **components vanish and so do the** $(3,1)$ **components**.

Lastly, expanding J in a basis of $H^{1,1}$ forms j as $J = a_k j^k$, we can also see that :

$$\partial_k \tilde{W} = 0 \iff \int_X G_4 \wedge J \wedge j^k = 0 \tag{1.63}$$

implies that $G_4 \wedge J = 0$ and thus G_4 **is a primitive form**.

The requirement that it should be harmonic then, assuming the Hodge conjecture, translates to the form being of Hodge type $(2,2)$.

The reason it is a bit naive is that the general idea that it suffices to count the number of constraints to be able to tell if a given system admits a solution is only true over a field in which case systems form vector spaces, as in linear algebra. However, G_4 is crucially part of a $\mathbb{Z}$-module, and not a vector space.

Indeed, if it were in fact a vector space, we could easily normalize such a G_4 flux due to the presence of inverses for scalars. This is not the case here.

We can make a few comments and rewrite the conditions that must be obeyed by G_4 to be easier to manipulate. First of all, the second Chern class of X may not be divisible by two. In fact, in the example we will study it is not divisible by 2. However this requirement can also be understood, thanks to Poincaré duality in the following way:

$$\left(G_4 + \frac{c_2(X)}{2}\right) \cdot \omega = n \in \mathbb{Z}, \tag{1.64}$$

for all ω, where $\omega \in H^4(X,\mathbb{Z})$, and where the dot is just the intersection pairing in cohomology. This intersection product is $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ valued. From this we distinguish between two cases : either $c_2(X)$ is even in which case there is a priori no problem, or $c_2(X)$ is odd. In the latter, this means that G_4 is also in $\mathbb{Z}+\frac{1}{2}$.

We will introduce some notation to make the distinction between integral and half-integral cases more explicit. Let P be the inner form in $H^{2,2} \cap H^4(X,\mathbb{Z})_{prim}$. We notice first that a half integer cannot be a multiple of two, and that every half integer can be written in the form $\frac{n}{2}$ where n is an odd integer.

We define Q the **rescaled inner form** as the quadratic form such that, from the inner form P in $H^{2,2} \cap H^4(X, \mathbb{Z})_{prim}$ that is typically rational, we get an integral form, by multiplication of the LCM of the denominators. This is typically needed in the case where c_2 is odd or the integral Hodge conjecture is false. Note that this a priori only takes care of the length condition, and some further refinement may be imposed depending on the chosen variety.

Working under the assumptions that there are no anti-$M2$ branes present and thus $N_{M2} \geq 0$, the second condition can be put in the form of an inequality :

$$\frac{\chi(X)}{24} \geq \frac{1}{2} G_4 \cdot G_4 \tag{1.65}$$

Coincidentally, the absence of anti-$M2$ branes corresponds to $N = 1$ supersymmetric Minkowski vacua.

In short we have:

$$G + c_2/2 \in H^4(X, \mathbb{Z}) \tag{1.66}$$

$$\frac{\chi(X)}{24} \geq \frac{1}{2} G_4 \cdot G_4. \tag{1.67}$$

In mathematical language, it translates to finding a $\mathbb{Z}$-linear combinations of harmonic forms that are below a certain bound, possibly working with a certain parametrization that allows only certain $\mathbb{Z}$-linear combinations.

The main thing to note is that in general, meaning without working in a basis of harmonic forms, and assuming the Hodge conjecture, we can make the argument those two conditions are non-trivial working over $\mathbb{Z}$ a bit more precise as follows :

1. $H^{2,2}_{prim} \cap H^4(X, \mathbb{Z})$ is a $\mathbb{Z}$-module, which admits many bases.

2. The intersection pairing in the middle cohomology of an even dimensional manifold is a quadratic form.

3. Since the basis of our $\mathbb{Z}$-module may not have any particular property with respect to the Hodge structure, a generic solution written in this basis needs

to typically have many non-zero coefficients in its $\mathbb{Z}$-linear combination of generators, which is a basis-dependent statement.

4. Thus, in terms of intersection pairing, the result can be well below the bound imposed by physics.

To discuss those points, we will illustrate the tension by studying some G_4 field strength living in Fermat's sextic fourfold at the Hodge locus from different point of views. Despite this fourfold not being elliptically fibred, we will showcase some recent mathematical results that allow some computations to be carried out.

In light of the results we will get, we will discuss some possibilities for solutions to tadpole issues to be found from different point of views, as well as discuss some inputs from physics that may prove to be crucial and need to be the object of further studies.

One can point out that in [23], the authors state some implications of the non-existence of such fluxes and accordingly provide some examples of fourfolds where there is no flux satisfying both criterias 1.66.

Our goal here will be to provide another point-of-view on this problem via the example of the sextic, and state some conjectures and arguments supporting them, such that one may be able to construct examples of fourfolds which do admit a solution or some mathematical tension that need to be clarified.

We lastly note we can split the problem of finding G_4 fluxes according to those conditions in the following way. Let us introduce the **set of general Hodge cycles**, that respect 1.64 as :

$$S_H(X) := \{G_4 |\, G_4 + \frac{c_2}{2} \in H^4(X,\mathbb{Z}),\, G_4 \text{ is primitive of general Hodge type (2,2)}\},$$

where a general Hodge cycle here is an $H^{2,2} \cap H^4(X)$ cycle that stays $H^{2,2}$ as we move through moduli space.

This definition of $S_H(X)$, using that G_4 is of general Hodge type $(2,2)$, implies that all moduli are stabilized if appropriately quantized.

Naturally the condition on G_4 being general Hodge is quite difficult, and we will specify a way to check for this condition later on.

Let us further introduce the **set of field strength below the tadpole bound**:

$$S(Q,T) := \{G \in \mathbb{Z}^n \mid \exists\, k \in \mathbb{Z},\ Q(G) = k \leq T\} \tag{1.68}$$

which is the set of all G, coordinates in a chosen basis of $H^{2,2}_{prim}(X,\mathbb{Z})$ where Q is the rescaled inner form of $H^{2,2}_{prim} \cap H^4(X,\mathbb{Z})$ in a chosen basis, such that the associated G_4 flux is below the tadpole bound T with a length of k.

In essence, the conjectures in [23] translates to the following.

We expect that for most smooth Calabi-Yau fourfolds X we have :

$$S_H(X) \cap S(Q,T) = \emptyset \tag{1.69}$$

Where we do not exclude potential edge-cases that need to be characterized.

This translates to classifying fourfolds according to this physics problems. A very important remark is that the set of $S_H(X)$ can be quite big, and in fact infinite since any integral multiple of a general Hodge cycle will be again a general Hodge cycle, whereas the set $S(Q,T)$ is finite, which suggests an approach we will detail in the following section. Naturally, the assumption on smoothness is quite strong and we will discuss this assumption studying the Fermat's sextic.

CHAPTER 2

Length and quantization condition : possible approaches

We present here two approaches to the tadpole problem. The aim is to provide a review of results in as much generality as possible, while providing some references and commenting the way we will interpret and use those results.

The first section presents some tools in Hodge theory and introduces the relevant tools to study the tadpole conjecture. It is followed by section two on Fermat varieties, where we specialize the discussion of section one to Fermat's varieties which offer a nice example where the computations are feasible, and allows us to comment on the various assumptions behind the tadpole conjecture, such as smoothness. This culminates in stating the general approach we will take for Fermat's sextic in chapter 3.

The second section presents some tools in lattices-theory and number theory. The goal is twofold : review the computational tools used in the case of the sextic, and showcase the underlying arithmetic structure of the problem. We try to provide some alternative approaches to the one mentioned in section 2.3 at the end of this chapter.

2.1 First approach : Hodge theory, residues and algebraic cycles

As introduced previously, we want to study some self-dual 4-form of Hodge type (2,2) that obeys 1.66. This requires not only knowing what the cohomology groups $H^{2,2}$ and H^4 are but also seeing them as modules and finding points inside those modules.

The modules of interest in this context are finitely generated and possess some generators. In fact, in parallel with the theory of vector spaces (which are just free modules but over some field), it has many different bases.

Furthermore, since we do have Poincaré duality, we can study the problem in terms of homology or in terms of cohomology : because we only care about the middle cohomology of an even-dimensional manifold of complex dimension 4, Poincaré duality states that H^4 is isomorphic to H_4 in that case.

So, we need to provide results that allow us to :

1. **Compute a basis of the middle (co-)homology module.**
2. **Compute elements inside that module.**
3. **Check that they are Hodge and below the bound 1.66.**

We can observe right away that there are in principle many different bases for a given module. Furthermore, depending on the dimension of the module and the bound, enumerating all elements that might satisfy 1.66 is computationally intensive.

An important observation is that there is in fact one basis for which the problem is quite simple. For example, working in homology, the basis of Hodge cycles is quite appropriate since we can form linear combination of generators, guaranteeing we are Hodge, and the constraint in that case essentially comes down to finding short elements.

2.1.1 Algebraic subvarieties and algebraic cycles

Since we now talk about Calabi-Yau varieties, rather than manifolds, it is natural to ask about subvarities. Indeed, since we are interest in homology and we will make heavy use of Poincaré duality, some formalism need to be introduced there.

While the definition of a subvariety follows from the definition of a variety, we need to introduce the related notion of algebraic cycles.

First of all, we can classify subvarieties by either their dimension, or by their codimension. Here we will use the dimension.

Then an **algebraic cycle** C is a formal linear combination of subvarieties S of some variety V :

$$C := \sum_i a_i S_i \ \in V \tag{2.1}$$

For example, if our variety V is not irreducible, then an algebraic cycle of maximal dimension is merely a linear combination of irreducible components.

In terms of homology, we can naturally associate a homology class to some cycle C to the sum of the classes of its components, formally :

$$[C] := \sum_i a_i [S_i]. \tag{2.2}$$

For example, quadrics and quadratic forms have a codimension that can vary widely. If the quadratic form is defined by a matrix $A(x) \cdot y$ for vectors x, y, the codimension can vary widely depending on the entries of A. If the resulting variety is for example smooth of dimension 2, an example of algebraic cycle would be formal linear combination of curves (dimension 1 subvarieties) sitting inside the variety defined by the quadratic form. This is analogous to the historical construction of conics.

In the set up we will consider in the following, we can make this definition less abstract and consider algebraic cycles as follows.

A complete intersection algebraic cycle stems from a factorization of the equation of the variety X. Namely, for polynomials f_i of degree d_i an algebraic cycle of type (d_1, d_2, d_3) will be those that form factor with the defining equation of X as:

$$F = f_1 P_1 + f_2 P_2 + f_3 P_3,$$

where we only consider those algebraic cycles that are complete intersections defined by $f_1 = 0, f_2 = 0, f_3 = 0$.

Naturally, we have to fix some ring R for the homology groups. We are mostly interested in the case where $R = \mathbb{Q}$ or $R = \mathbb{Z}$.

Since we have made some assumption about the varieties we work with, we can state the Hodge conjecture:

If V is a projective complex manifold, then every Hodge class on V is algebraic.

Naturally we need to emphasize the difference between rational or integral Hodge conjecture. Note that the ring of definition of the coefficients must be obviously the same in both cohomology and homology, meaning when we work in some Hodge cohomology class $H^{p,p} \cap H^{2p}(V, R)$ we also need to have the coefficients of the linear combination defining $[C]$ to be in R. The current state of this conjecture depends on the variety studied.

However, we do note that, in the cases where it is not a conjecture, it is quite a powerful tool since we have an algebraic way to described our basis of choice for the tadpole problem.

2.1.2 Introducing residues

For now, we will focus on the tools we can use to find generators of those modules. For our purposes, the most appropriate tool to solve this issue is the usage of the theory of residues and their periods. In particular, we will restrict ourselves to study hypersurfaces since in that case, it will be fairly straightforward to showcase some interesting results and point out the subtleties that arise when studying the tadpole problem.

As per usual, let X be a smooth algebraic variety of degree d defined by $f = 0$, an hypersurface of $\mathbb{CP}^{n+1}$. Furthermore, let us assume the dimension n of X is even, as is the case in the M-theory picture introduced in the introduction. Let P be some monomial $z_0, z_1, ..., z_{n+1}$ and let us denote by β the tuple of the degree of each variable such that $|\beta| = (k+1)d - n - 2$ and where k is an integer describing which part of the cohomology we work with. For example, $k = 1$ for Fermat's sextic fourfold corresponds to $h^{1,3}$.

Let us, with that notation, define **residues forms** ω_β as the following :

$$\omega_\beta = \mathrm{Res}\left(\frac{P_\beta \Omega}{f^{k+1}}\right) \in H^k_{dR}(X)_{prim}, \tag{2.3}$$

with $\beta = (\beta[0], \beta[1], ...)$ and $P_\beta = x_0^{\beta[0]} x_1^{\beta[1]} ...$ To understand this we need to introduce the residue map. To define this, consider a cycle $C \in H_{n-1}(X, \mathbb{C})$ and take a tube $T(C)$ around it that lies within the complement of X. Since this is an n-cycle, we can integrate a rational n-form Q around it to get a number by computing the integral :

$$\int_{T(C)} Q = a, \tag{2.4}$$

where a is some number.

The parallel here should be drawn between this formula and Cauchy's residue formula. In Cauchy's case we have :

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = 1, \tag{2.5}$$

where γ is a contour encircling the origin, while in this generalization to arbitrary differential forms α we have:

$$\frac{1}{2\pi i}\int_\gamma \frac{dz \wedge \alpha}{z} = \alpha \tag{2.6}$$

Due to the duality between homology and cohomology, this also implies a cohomology class which is the residue, given by the **residue map**. In summary this is a map :

$$Res : H^n(\mathbb{P}^n \backslash X, \mathbb{C}) \longrightarrow H^{n-1}(X, \mathbb{C}) \tag{2.7}$$

We will not study this map in great details and mostly use the results in [24] [25]. The important lesson here is that it allows us to describe the modules we are interested in by computing integrals. Those integrals can be called periods of the cycle defined by the domain of integration.

Furthermore, notice that in general there is no guarantee that the resulting number is an integer. However, we do know that it is primitive, which was one of the requirement we needed to study the tadpole problem in the M-theory picture.

Lastly an important comment : residue forms are not necessarily exactly dual to algebraic cycles. That would be the case if the Hodge conjecture is true, showing once again the importance of this conjecture, but also that residue forms are a rather convenient tool regardless of the state of this conjecture.

2.1.3 Vanishing cycles

Now that we have an appropriate tool to make computation we need the right setting, which is to have some basis of the (co-)homology modules with respect to integral coefficients.

To this end we need to introduce **vanishing cycles**, which we denote by $\delta_{\beta'}$. Rather than opting to state the definition in all its abstraction, and since we are just here stating result that we will use, we will simply mention here that they form generators of the integral primtive middle homology, and we can compute their periods.

The main point is that, as we move around in moduli space, we can use them to describe the middle cohomology of our family of varieties. An example of vanishing cycle is given in 3.29.

Let us give an example of how vanishing cycles work by considering the example of projective curves. Let us consider a family of curves X_t degenerating into a singular curve X_s. By considering a basis α_t, β_t of $H^1(X_t)$ so that in the limit $t \to 0$ we have $\beta_0 \in H^1(X_s)$ and $\alpha_t \to 0$ as $t \to 0$. This gives a new basis of $H^1(X_s)$ that is related to the old basis via Picard-Lefschetz.

This is equivalent to studying the singular curve at different affine patches and seeing how they glue together, and noticing that the α are disappearing, hence the name "vanishing".

An important note for the example we study, Fermat's sextic, is that to have a proper basis you also need to add a linear algebraic cycle as the vanishing cycles do not suffice. In general this can be seen from the **Leray-Thom-Gysin sequence**

in homology [26, §4.6].

From this result, which is again some number in general, we can infer that by rescaling the residue forms ω_β we can hope that this number is in fact an integer, which means we in principle have found a basis over $\mathbb{Z}$ of the primitive middle cohomology in terms of rescaled residue forms.

As with residue forms, we want to highlight that the Hodge conjecture does not necessarily need to be true for vanishing cycles to exist and be used to study our problem.

2.1.4 Intersection pairing formula

Lastly, we can also formally compute :

$$\omega_{\beta_1} \cdot \omega_{\beta_2} := \int_X \omega_{\beta_1} \wedge \omega_{\beta_2} \tag{2.8}$$

as well, which essentially means we have in principle "solved" the issue of computing the length of the G_4 flux, as we have a basis in cohomology, and a way to compute intersections.

We need an explicit formula for the intersection pairing 2.8. In the context of smooth hypersurfaces X of degree d, complex even dimension n and defining equation $F = 0$, we can compute **the intersection pairing** using [25] :

$$\int_X \omega_{\beta 1} \wedge \omega_{\beta 2} = \frac{-(2\pi i)^n}{\frac{n!}{2}} \cdot c \cdot (d-1)^{n+2} d, \tag{2.9}$$

where $c \in \mathbb{C}$ is the unique number such that :

$$P_{\beta 1} P_{\beta 2} = c \cdot det(H(F)) \bmod(J(F)), \tag{2.10}$$

where J and H are respectively the Hessian of F and the Jacobian ideal of F.

The **Jacobian ideal** is defined, for R a ring, as :

$$J(F) = R[x, y, z, \ldots]/(F, \partial_x F, \partial_y F, \ldots) \tag{2.11}$$

meaning it's the ideal generated by partial derivatives of F, while the Hessian here is the classical Hessian from calculus.

This formula 2.9 contains the parameters you expect it to contain in the form of the degree d and the dimension of the ambient space n, since it counts intersections.

Furthermore for our purpose of studying Hodge cycles, we notice the importance of modding out by the Jacobian ideal of F, which in turns determines how we can pair the polynomials $P_{\beta 1}, P_{\beta 2}$.

Of course when computing the constant c, there is an implicit dependence on the degree of the Fermat variety considered since it will typically be inversely proportional to $d^{n+2}(d-1)^{n+2}$.

2.1.5 Hodge locus

This means we now need to know if a given G_4 flux in this basis is also of Hodge type (2,2) or not. To this end we need to define what is a Hodge locus.

Let us then consider degree $2p$ Hodge classes and consider the subset of $H^{p,p} \cap H^{2p}(X_0)$ for some variety X_0 inside a family of varieties X_s. As s moves, typically forms in $H^{p,p} \cap H^{2p}(X_0)$ will not be $H^{p,p}$ anymore. We are here interested in forms that do stay invariant and if this subset is of maximal dimension, we call it the **Hodge locus**.

This means in pratical purposes that if we find a G flux such that it has a Hodge locus of maximal dimension, then we know that it will be of general Hodge type $(2,2)$ (for the case of fourfolds).

Perhaps a few general comments are needed there. First of all, despite the problem being reduced to some simpler questions, the calculations are still computationally expensive. Moreover, we do not in general look for integers since $c_2(X)$ might be odd as discussed earlier, and we need to look for half-integers, a condition which needs to be taken care of when doing any computations.

However the important observation is that the Hodge conjecture is just a tool, and while the intersection with the tadpole problem is non-zero, it is a priori neither necessary nor sufficient to solve the Hodge conjecture to understand the tadpole problem and solve it.

Furthermore we can now make the arguments introduced in the introduction more precise : when working with the basis of residues, for G_4 to be in the Hodge locus, it requires in general to have a G_4 which has many non-zero coefficients. In turn, the length of the flux is given by a quadratic form because X is even dimensional, turning on many non-zero coefficients tends to increase the length. This is the expected feature of basis that is in fact not Hodge.

2.2 Fermat varieties as an example

Let us in this section emphasize that the tools introduced before are valid regardless of the fourfold taken. With that in mind, we can still pick a slightly more general setting that the one imposed by physics to make some observations that are not otherwise obvious.

In this section we define Fermat varieties X as degree d hypersurfaces of dimension n in $\mathbb{CP}^{n+1}$:

$$X := \sum_{i=0}^{n+1} x_i^d = 0. \tag{2.12}$$

2.2.1 Vanishing cycles and Hodge locus for Fermat

We specialize the discussion of the previous section to the case of Fermat varieties. In this context, the family of varieties to consider will be the affine patches obtained from setting one coordinate to 1.

For **vanishing cycles** $\delta_{\beta'}$ and residue forms ω_β, we have the following formula from [27]:

$$\int_{\delta_{\beta'}} \omega_\beta = \frac{1}{d^{n+1}\frac{n}{2}!(2\pi i)} \prod_{i=0}^{n+1} (\zeta_d^{(\beta_i+1)(\beta'_i+1)} - \zeta_d^{(\beta_i+1)\beta'_i})\Gamma(\frac{\beta_i+1}{d}), \tag{2.13}$$

for primitive $h^{2,2}$ forms and we have used the following definitions:

- For vanishing cycles, for every $\beta'_i \in \{0, 1, 2, 3, 4, \ldots\}^{n+1}$ consider the homological cycle

$$\delta_{\beta'} := \sum_{a\in\{0,1\}^5} (-1)^{\sum_{i=1}^5 (1-a_i)} \Delta_{\beta'+a}$$

where $\Delta_{\beta'+a} : \left[\Delta^4 := \{(t_1, t_2, t_3, t_4, t_5, \ldots) \in \mathbb{R}^{n+1} : t_i \geq 0 \ , \ \sum_{i=1}^{n+1} t_i = 1\}\right] \to U_0$. An affine patch is given by

$$\Delta_{\beta'+a}(t) := \left(\zeta_{2d}^{2(\beta_1'+a_1)-1} t_1^{\frac{1}{d}}, \zeta_{2d}^{2(\beta_2'+a_2)-1} t_2^{\frac{1}{d}}, \zeta_{2d}^{2(\beta_3'+a_3)-1} t_3^{\frac{1}{d}}, \zeta_{2d}^{2(\beta_4'+a_4)-1} t_4^{\frac{1}{d}}, \zeta_{2d}^{2(\beta_5'+a_5)-1} t_5^{\frac{1}{d}}, \ldots\right).$$

- For residue forms, we have :

$$\omega_\beta := Res\left(\frac{x^\beta \Omega_0}{Q(x)^{k+1}}\right),$$

where

$$\Omega_0 = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_5 \wedge \ldots,$$

where we omit the term dx_i and where Ω_0 is the standard degree d top form of $\mathbb{P}^{n+1}$, x^β is the monomial

$$x^\beta = x_0^{\beta_0} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} x_4^{\beta_4} x_5^{\beta_5} \ldots,$$

with $|\beta| := \frac{1}{d}\sum \beta_i = k \in \mathbb{Z}$, $0 \leq \beta_i \leq n$, and $0 \leq k \leq n$ determines the Hodge type:

$$\omega_\beta \in H^{n-k,k}(X)\,.$$

The **Hodge locus** will be described via a matrix ρ_{IJ} defined as

$$\rho_{IJ} = \omega_{\beta_I+\beta_J} \cdot G_4, \tag{2.14}$$

where $|\beta_I| = |\beta_J|$ is appropriately chosen, eg 1 for Fermat's sextic.

A G_4 flux will be general Hodge if the associated matrix ρ_{IJ} is of maximal rank. Indeed, denoting the associated square matrix by $\rho(G) := (\{\rho_{IJ}(G)\}_{I,J})$, it follows that, for a family of hypersurfaces T parametrized by t, with special point t_0:

$$\text{rank } \rho(G) = \text{Codim}(T_{t_0} V_G \subseteq T_{t_0}\mathsf{T}) \,, \tag{2.15}$$

where T_{t_0} is the Zariski tangent space, and V_G is defined by :

$$V_G := \{t \in (\mathsf{T}, t_0) | G_t \in H^{p,p}(X_t) \cap H^{2p}(X_t, \mathbb{Z})\} \,. \tag{2.16}$$

All this is saying is that for a cycle to be Hodge, it must stay in $H^{p,p} \cap H^{2,p}$ as we move around in moduli space.

We say that G is a general Hodge cycle if

$$\text{Codim}(V_G \subseteq \mathsf{T}) = h^{3,1}(X). \tag{2.17}$$

An alternative way to see the appearance of this matrix from the point of view of physics is the following. Consider the equation:

$$W(t) = \int_{X_t} G_4 \wedge \Omega_t, \tag{2.18}$$

where t parametrizes where we are in moduli space and Ω_t is the holomorphic top form associated to the variety X_t. All we have done so far is rewrite the equation of the GVW superpotential at a different point in moduli space. Fix $t = 0$ to be the point of interest.

Then a flat direction of the potential is given by a curse $t(s)$ such that $D_I(W(t(s))) = 0$ where the derivative is taken with respect to complex structure moduli. Since $D_I(W_t)|_{t=0} = 0$ by our assumptions, a first order expansion leads to:

$$\partial_s t(0) \partial_J D_I W(0) = 0. \tag{2.19}$$

But because $D_I(W_t)|_{t=0} = 0$ this is equal to:

$$D_J D_I W(0) = 0, \tag{2.20}$$

so that we recover that $\rho_{IJ} := D_I D_J W$ needs to have maximal rank purely from physical consideration.

2.2.2 A result by Deligne

We have decided to use formula 2.13, as specialized in the case of Fermat varieties, to be able to perform computations.

At a first glance, it is not clear why this formula is relevant when trying to solve the problem for Fermat varieties, since the result seems to lie in some $\mathbb{Q}[\zeta_d]$ and has even some transcendental components, far from the goal we have set to get something in $\mathbb{Z}$.

This fact holds in general as proved by Deligne in [28] : when computing periods such as this one, the result of the computation will depend on the **field of definition of the variety as well as the one of the relevant algebraic cycles**.

When wanting to explore the integral middle cohomology, we need to go from this field extension $\mathbb{Q}[\zeta_d]$ and handle the associated transcendental functions Γ to be able to recover an integral result. This suggests some further simplification of the formula and some notation:

$$\int_{\delta_{\beta'}} \omega_\beta = \frac{1}{d^{n+1}\frac{n}{2}!(2\pi i)} \prod_{i=0}^{n+1} (\zeta_d^{(\beta_i+1)(\beta'_i+1)} - \zeta_d^{(\beta_i+1)\beta'_i}) \Gamma(\frac{\beta_i+1}{d}) \tag{2.21}$$

$$= \frac{1}{d^{n+1}\frac{n}{2}!(2\pi i)} \prod_{i=0}^{n+1} (\zeta_d^{(1+\beta_i)\beta'_i}) \cdot (\zeta_d^{1+\beta_i} - 1) \cdot \Gamma(\frac{\beta_i+1}{d}) \tag{2.22}$$

$$= z_u z_\beta Z(\beta, \beta'), \tag{2.23}$$

where we made the identifications:

$$z_u = \frac{1}{d^{n+1}\frac{n}{2}!(2\pi i)} \tag{2.24}$$

$$z_\beta = \prod_{i=0}^{n+1} (\zeta_d^{1+\beta_i} - 1) \cdot \Gamma(\frac{\beta_i+1}{d}) \tag{2.25}$$

$$Z(\beta, \beta') = \prod_{i=0}^{n+1} \zeta_d^{(1+\beta_i)\beta'_i}. \tag{2.26}$$

We note that z_u is a universal constant, z_β only depends on the residue form ω_β and is in fact a constant, while $Z(\beta, \beta')$ depends on both ω_β and the vanishing cycle considered.

We will thus need to normalize with respect to all these terms if we want to find a suitable G_4 flux.

2.2.3 Shioda results for Fermat varieties

While periods can be computed in general under some mild assumptions, they are a powerful tool when it comes to Fermat varieties.

The starting point is to observe that Fermat varieties of dimension n and degree d have a large group of automorphisms [29]. This group, which we denote by $Aut(X)$, is:

$$Aut(X) = \mathfrak{S}_{n+2} \ltimes (\mathbb{Z}/d\mathbb{Z})^{n+1}, \tag{2.27}$$

where $\mathfrak{S}_{n+2}$ is the group of permutations of $n+2$ variables, acting in the obvious way, while $(\mathbb{Z}/d\mathbb{Z})^{n+1}$ acts on the variables via multiplication by d-roots of unity.

A noteworthy fact is that this doesn't take into account the $\mathbb{C}^*$ action coming from working in complex projective space.

However, **this automorphism group $Aut(X)$ also induces, in the case of Fermat variety, a decomposition in** $H^4(X)$. Indeed consider the **character group** A of $Aut(X)$:

$$= A := \{\mathbf{a} = (a[0], a[1], ..., a[n]) | a_i \in \mathbb{Z}_d \text{ and } \sum_i a[i] = 0 \bmod d\}.$$

The elements of A are maps that associate to an element g of $Aut(X)$ the phase :

$$\mathbf{a}(g) = \prod_i \zeta_d^{a[i]}.$$

Let $V(\mathbf{a})$ denote the subspace of $H^4(X)$ such that for a class $\eta \in V(\mathbf{a})$ we have:

$$g^*\eta = \mathbf{a}(g)\eta.$$

From [30] we have the following theorem:

The spaces $V(\mathbf{a})$ and their Hodge type are characterized by

a) $\dim_{\mathbb{C}}(V(\mathbf{a})) = 1$ if and only if $a[i] \neq 0$ for all i. Otherwise $\dim_{\mathbb{C}}(V(\mathbf{a})) = 0$.

b) The hodge type of forms in $V(\mathbf{a})$ are characterized by the $\mathbf{a}$.

This is to be interpreted in the following way : **the decomposition on $H^4(X)$ induced by the automorphism group $Aut(X)$ is finer than the Hodge decomposition**, and hence we can use this decomposition to study the Hodge structure.

2.2.4 Automorphisms, residues and algebraic cycles

From the previous theorem by Shioda, we can look back at our residue forms, and in particular 2.13, to see what this decomposition implies.

Let us first define some terms related to the decomposition coming from $Aut(X)$. Recall that we have characterized residues forms by some polynomial P_β depending on the tuple β.

We will refer to the tuple $\mathbf{a}$ as n-decomposable if it can be written maximally as a sum of n-pairs $a[i] + a[j] = 0 \; mod(d)$. That is, we take into account permutations of i and j.

By considering again $V(\mathbf{a})$, we can use this notion of decomposability again at the level of residue forms by saying a residue form ω_β is n-decomposable if the tuple β is n-decomposable in the same sense as $\mathbf{a}$.

Naturally this gives us also another way to understand complete intersection algebraic cycles as factorizations. We have specialized the formal definition of algebraic cycles to the case of hypersurfaces by saying they are just the vanishing locus of polynomials $f_i = 0$.

For example, let us consider the case of f_i having degree one, the so-called linear cycles. In the case of Fermat varieties, those linear cycles will take the form :

$$f_i := x_j - \zeta_{2d}^{l_i} x_k = 0 \tag{2.28}$$

for some parameter l_i and for some permutation of the coordinates σ that results in x_j and x_k.

Such that each of them can be **characterized by the l_i specifying which ζ_{2d} we pick, and naturally we write the associated linear cycle by** $C^{l_0,l_1,l_2,\cdots}$.

Considering the action of $Aut(X)$ on homology, we can express elements ν_a of $V(\mathbf{a})$ in terms of algebraic cycles. For example, for linear algebraic cycles inside a Fermat variety we have :

$$\nu_a \sim \sum_{l_0,l_1,l_2,\ldots} \zeta_d^{a[0]l_0+a[1]l_1+a[2]l_2+\ldots} C_\sigma^{l_0,l_1,l_2,\ldots} \tag{2.29}$$

where we also **need** σ as well as the l_i to fully specify the algebraic cycle.

Here the permutations just swap the coordinates. However, when it comes to picking a basis, permutations do play a role.

2.2.5 Aut(X) and intersection pairing

Let us now focus on computing intersections of two residues forms and specialize to the case of Fermat varieties in light of the results on $Aut(X)$ and study its impact.

Since we have seen that the result of 2.13 is in particular in $\mathbb{Q}[\zeta_d]$ and we want an integral result for some $\mathbb{Z}$-linear combination of residue forms, we need to understand how the intersection pairing works for residues.

In particular the intersection will be 0 if the product of the polynomials $P_{\beta 1}$ and $P_{\beta 2}$ falls in the Jacobian ideal associated with the Fermat variety given by $F = 0$.

In particular, if F is a polynomial of degree d, then we see that if one of the $\beta_1[i] + \beta_2[i] \geq 5$, **then it will result in a** 0 **intersection**. Since furthermore we are interested in the middle cohomology, this forces the residues form to only intersect in pairs that are defined only in terms of the associated β tuples. This can be seen from the results of Shioda by considering the way $Aut(X)$ acts on elements of cohomology from the previous section.

Consider the intersection of two residues $\omega_{\beta 1}$ and $\omega_{\beta 2}$ and compute the action of $Aut(X)$ on the result :

$$\int_X \omega_{\beta 1} \wedge \omega_{\beta 2} \longrightarrow \int_X \omega_{\beta 1} \wedge \omega_{\beta 2} \prod_i \zeta_d^{a1[i]+a2[i]} \tag{2.30}$$

Since this result must be invariant (recall that it is just a number), it automatically results in $a1 = -a2$ for the integral to be non-zero : residues are compatible with the decomposition induced by $Aut(X)$.

We will call such a pair a **pair of complex conjugate residue forms, or a self-conjugate residue form if the two elements of the pair are the same**. The justification of this notation can be seen by considering 2.13.

2.2.6 Complex conjugation for Fermat

Indeed, knowing that the residue forms are paired up, and given the constraints on the tuples β coming from working in middle cohomology which constraint their length, we can observe the following when we consider a pair of complex conjugate ω_β and $\bar{\omega_\beta}$.

Set a_β to be the constant of proportionality such that :

$$\bar{\omega}_\beta = a_\beta \omega_{\bar{\beta}} \, . \tag{2.31}$$

This already implies that

$$\omega_\beta = \bar{a}_\beta \bar{\omega}_{\bar{\beta}} \, , \tag{2.32}$$

so that

$$a_{\bar{\beta}} = \frac{1}{\bar{a}_\beta}. \tag{2.33}$$

This in particular implies that

$$\bar{\omega}_\beta \cdot \bar{\omega}_{\bar{\beta}} = \frac{a_\beta}{\bar{a}_\beta} \omega_\beta \cdot \omega_{\bar{\beta}}. \tag{2.34}$$

Since we need the result to hold for every vanishing cycles $\delta_{\beta'}$. As the result must be integral and in particular real, we have:

$$\int_{\delta_{\beta'}} \bar{\omega}_\beta = \overline{\int_{\delta_{\beta'}} \omega_\beta} \,. \tag{2.35}$$

We can then find a_β by computing:

$$a_\beta = \frac{\overline{\int_{\delta_{\beta'}} \omega_\beta}}{\int_{\delta_{\beta'}} \omega_\beta}, \tag{2.36}$$

which must hold for every vanishing cycle.

In particular, this **constant of proportionnality only depends on a single β for every pair of complex conjugates**.

2.2.7 Integrality and $Z(\beta, \beta')$

One more implication is that now that we know that every element in a pair of complex conjugate residue forms is proportional to the other, we can look back at 2.13 and handle the cyclotomic part $Z(\beta, \beta')$ which in general depends on the vanishing cycle. As a reminder, $Z(\beta, \beta')$ is given by:

$$Z(\beta, \beta') = \prod_{i=0}^{n+1} \zeta_d^{(1+\beta_i)\beta'_i}. \tag{2.37}$$

Consider the integrality condition on a sum of pairs of complex conjugate residue forms ω_β and $\bar{\omega}_\beta$. We have the following simplifcation:

$$(\omega_\beta + \bar{\omega}_\beta) \cdot \delta_{\beta'} = \nu_\beta Z(\beta, \beta') + \bar{\nu}_\beta \bar{Z}(\beta, \beta') \in \mathbb{Z} \ \forall \beta', \tag{2.38}$$

where we introduced the normalization we are looking for ν_β, which of course depends on the previous results (namely a_β, z_u and z_β).

We see from this formula, up to redefinition of ν_β by taking the complex conjugate, that the following **inner form** appears:

$$< x, y >= x\bar{y} + y\bar{x}. \tag{2.39}$$

Noticing that the roots of unity ζ_d form a group isomorphic to $\mathbb{Z}_d$, we have there found a way to make the result integral : we need to pick ν_β to be a $\mathbb{Z}$-linear combination of generators of the **dual of the lattice spanned by** $Z(\beta, \beta')$.

Crucially, this simple description is specific to Fermat hypersurfaces. In general, we expect that imposing this integrality condition will be less straightforward.

2.2.8 Linear cycles and $c_2(X)$

The flux quantization condition imposes, for some flux G, to have :

$$G + \frac{c_2(X)}{2} \in \ H^4(X, \mathbb{Z}). \tag{2.40}$$

We will focus on this condition and relate it to the decomposition induced by $Aut(X)$ and linear algebraic cycles.

In fact, in the case of interest of Fermat's sextic, a basis for $H^4(X, \mathbb{Z})$ is given by not only appropriate vanishing cycles but also a linear cycle, and $c_2(X) = 15$ which is odd. Naturally, this implies that G must also belong in $\mathbb{Z} + \frac{1}{2}$.

The way we enforce a flux G to be half-integral is through the linear cycle. Indeed the condition of half-integrality should be understood as the decomposition between the primitive part and the non-primitive part of $H^{2,2} \cap H^4$: the cohomology class of c_2 has a non-primitive part.

It follows that this non-primitive part intersects with linear cycles, and hence the constraints for G come from the linear cycle we pick to complete the basis of H^4.

To this end, we use [24] and the following formula for **the period of a residue form along a linear cycle** in a Fermat variety of even dimension n and degree d, with some linear algebraic cycle specified by a tuple l and a permutation σ :

$$\int_{C_\sigma^l} \omega_\beta = \begin{cases} sign(\sigma)\frac{(2i\pi)^n}{d^{n+1}\frac{n}{2}!}\zeta_{2d}^{\sum_{e=0}^{n/2}(\beta[i_{\sigma(2e)}]+1)\cdot(1+2l[i_{\sigma(2e+1)}])} & \text{if } \beta[i_{\sigma(2e)-2}]+\beta[i_{\sigma(2e)-1}] = d-2 \; \forall \, e \\ 0 & \text{otherwise.} \end{cases} \tag{2.41}$$

This should be interpreted in the light of Aut(X): the entries of the tuple β need to have their entries paired up *according to the permutation chosen*.

In this case thus we not only have a dependence on the $\mathbb{Z}_d^{n+1}$ part of the semidirect product of Aut(X), but also on the permutation part $\mathfrak{S}_{n+2}$.

In particular, since we need this result to be half-integral, we see right away that any rescaling we want to do to respect 2.13 will be modified and parametrized to respect the condition of half-integrality.

2.2.9 Quick comment on Calabi-Yau criteria

As discussed before, the result of 2.13 lies in some cyclotomic field $\mathbb{Q}[\zeta_d]$. Furthermore, we can use previous results to highlight the constraint coming from the fact that we study Calabi-Yau manifolds.

Indeed, let us relax this assumption and focus on a similar problem but for all Fermat varieties, such that we do not have a relationship between degree and dimension (via adjunction). Let us fix the dimension n to be 4 such that the dimension of the ambient projective space is 5, and keep the degree moving. For example, we can consider the quintic in $\mathbb{P}^5$ instead of the sextic in $\mathbb{P}^5$.

Now introduce some big $\mathcal{O}$ notation this time to refer to asymptotics, since we notice that typically the norm of terms like z_c and $Z(\beta, \beta')$ does not depend on the degree in the same way z_a does.

The contribution of normalizing some ω from z_u will typically be $\mathcal{O}(d^{-(n+1)})$. So let us pair two such normalized forms and use formula 2.9 and again focus on the dependency of the degree. We find that:

$$\omega_{\beta 1} \cdot \omega_{\beta 2} \propto d^{-(n+1)}(d^{n+1}) \cdot (d^{n+1}) \propto d^{n+1}. \tag{2.42}$$

So that we see the influence of the Calabi-Yau requirement here because at fixed n, lowering the degree d would give lower intersection numbers with the caveat that it also modifies $h^{2,2}$ and $h^{3,1}$ (and hence the would-be tadpole bound) of course.

This is an illustration of the fact that we have in principle lost some free parameters due to working with Calabi-Yau manifolds, and is our first hint to be able to plan a strategy to study the tadpole problem.

2.2.10 General approach and a potential solution to the difficulties

From this section and having considered the problem, we can guess a general approach to study the problem:

- **Compute all residues**
- **Find a basis of the integral middle homology using the Leray-Thom-Gysin sequence and vanishing cycles**
- **Compute the periods of the residues along that basis in homology and normalize the residues**
- **Find elements of the set** $S(Q, T)$
- **Check if those elements also belong to the set** $S_H(\rho, X)$

Computationally this can be extremely costly depending on the value of $h^{3,1}$ and $h^{2,2}$. Given that those values are quite high typically, we can restrict ourselves in a physical manner by limiting ourselves to **symmetric forms under the action of a subgroup of Aut(X)**.

We have tools in the forms of computation of periods, residue forms and vanishing cycles that can work given any fourfold and check the list of requirements we have given in the beginning of the section.

The collection of those tools allow, in principle, practical calculations to be made. They also show that respecting both conditions imposed by physics for the G_4 flux is a very hard problem to study, because of the aforementioned tension.

Intuitively, we have two constraints in the form of (in)equations to obey : ρ_{IJ} of maximal rank and small $G_4 \cdot G_4$. The discussion in this section, in particular the difficulties mentioned, then suggests some heuristics to tackle the problem.

We do expect some tension, as the conjecture is that the intersection of $S_H(\rho, X)$ and $S(Q,T)$ is empty for most smooth fourfolds. But there is a potential way to circumvent this tension by taking quotients and thus not working with the assumption of smoothness, which ties into our seemingly arbitrary way to reduce the dimensions of the problem.

Indeed, suppose we have found some solutions $z_i \in S(Q,T)$ but none of them belonging in $S_H(\rho, X)$ due to the corresponding ρ not being of full rank. Typically such solutions will have flat directions, meaning some coordinates of the z_i being 0 and hence not contributing to the rank of ρ. Then we can hope that there exists a quotient, resulting in an orbifold, such that those flat directions get removed. This will result in a different ρ with a lower possible maximal rank, with this condition on the rank now possibly being satisfied.

We can see now that the case of Fermat varieties is interesting because of the large group of automorphisms which opens up the way to perform the computation of the periods, as well as providing a possible solution by taking quotient. Similar to the Hodge conjecture, this is an assumption that has significant overlap with the tadpole problem, but in principle is not needed, since most fourfolds don't enjoy this same property as Fermat varieties.

Similarly, it is not guaranteed that quotienting always helps with respect to the tadpole problem. A quotient that both removes all flat directions and results in an orbifold may not exist in the first place, and even then we still have to check that the rank is indeed maximal. This suggests fixing a fourfold X, looking at a set of solutions of given lengths below the tadpole bound, and looking at the symmetries of those solutions, to which we hope to associate a quotient. Then taking the quotient, removing more and more flat directions, the expectation is that the dimension of the Hodge locus gets closer to being maximal.

2.3 Second approach : using number theory

A first case of interest is, assuming c_2 is even, to use the theory of lattices. By a lattice here we will take the definition to be of a finitely generated free $\mathbb{Z}$-module over some ring R given a bilinear form $\langle\ ,\ \rangle$.

We will try to show in this section that while approaching the problem purely in terms of lattices might seem like a good idea, the tadpole problem is very specific and we will offer a different point of view. Namely, rather than the usual approach which is to try to find suitable points satisfying the tadpole conditions, we will show that it is more convenient to find obstructions to those points existing.

In a sense, the set $S(Q, T)$ is interesting, but we should rather focus on finding what should belong there but does not, and why it does not. To this end, the too general formalism of lattices is not sufficient.

2.3.1 Review of lattices

Working with the above assumptions, **we have the following data**[1] :

1. A lattice Λ, with bilinear form $\langle\ ,\ \rangle$, which in our case will be a positive definite inner form.

2. A basis of Λ, e_i, with i ranging from 0 to n, where $n+1$ is the dimension of the lattice

3. Given $\langle\ ,\ \rangle$ and e_i, those two elements reduce to computing a Gram matrix $G_{ij} = \langle e_i, e_j\rangle$, with an invariant signature (p,n) denoting the number of positive and negative eigenvalues respectively

4. The dual of this lattice, Λ^*, given by G^*_{ij} and dual basis e^*_i defined by:

$$< e^*_i, e_j >= \delta_{ij},$$

   where δ is Kronecker's delta symbol.

[1]Lattices can be defined more generally : Let R is an integral domain with field of fraction K. An R-submodule M of a K-vector space V is a lattice if M is finitely generated over R.

5. An invariant, the covolume of this G_{ij}, defined as $coVol(G) = |det(G)|$

6. Elements of Λ will be labelled with vectors $\ell = \sum_i \mu_i e^i$ with $\mu_i \in \mathbb{Z}$

Given this formulation, we simply recall that in principle we just need to compute the inner form $\langle \ell, \ell \rangle$ for any ℓ that is relevant and can potentially lead to a result in $S(Q,T)$. Naturally this raises the question of how to find such ℓ.

We recall that since we deal with integers, the set $S(Q,T)$ is finite, and hence there are finitely many ℓ to check for. The problem is that the data we have is typically high dimensional, meaning the n in our data is quite big, and there is some ambiguity in finding a basis for Λ since it is not unique. Setting those choices is in general quite difficult and ultimately leads to difficulties finding the candidates for ℓ.

2.3.2 Finding the short vectors in a lattice

To see the difficulties relevant to our problem we can first find the bounds on the entries μ_i of every ℓ that give a norm $\langle \ell, \ell \rangle \leq T$ below some bound T. We want to find $|\mu_i|$ such that we are guaranteed to enumerate all ℓ below the bound T, and to this end we will call the norm of ℓ as the inner form $\langle \sum_i \mu_i e_i, \sum_i \mu_i e_i \rangle$. We can then compute the following for any ℓ below the bound :

$$\mu_i^2 = \langle e_i^*, \ell \rangle^2, \tag{2.43}$$

which simply uses duality, and we have squared the usual result in order to simplify the argument later on.

From there we can work out, since the **inner form is positive definite** :

$$\mu_i^2 = \langle e_i^*, \ell \rangle^2 \leq \langle e_i^*, e_i^* \rangle \cdot \langle \ell, \ell \rangle = \langle e_i^*, e_i^* \rangle \cdot T \tag{2.44}$$

This is just the usual Cauchy-Schwarz inequality but squared, as it is usually stated using the norm $||\,.\,|| = \sqrt{\langle\ ,\ \rangle}$ associated to the inner form.

For some μ_i, the result depends on the length of the dual basis e_i^* as well as the bound T, showing the inherent computational complexity as the dimension increases.

Naturally since we work with $\mathbb{Z}$-modules, we expect e_i^* to be rational in general, unless Λ is self-dual, and thus this inequality may or may not be so bad at first sight, despite the dependence on T.

However, the main point is that this also implicitly depends on the dimension of Λ. Even if it turns out that every μ_i is contained in a short range, say $[-2, 2]$ for example, the number of points to check will be exponential in the dimension. In this example, there are 5 integers in $[-2, 2]$, the total amount of points to check will be 5^{n+1}. For any high enough $n + 1$ this quickly becomes out of reach for practical purposes.

2.3.3 An improvement : basis reduction

We have encountered the first difficulty with using lattices : the computational complexity. We have seen that it depends on the length of the (dual) basis e_i^*. However there is a priori no obstruction to picking a different basis that will generate the same (dual) lattice.

In fact, had we worked over a vector space rather than a module, this is what we do all the time : given a certain matrix, we tend to immediately change basis to put it in better form, diagonal, triangular, block, etc... This is not possible for a $\mathbb{Z}$-module. However the idea should stay the same.

The question becomes : can we find some alternative basis that is shorter than the original e_i^* ? The answer in general is yes. In fact, in two dimensions, the **Lagrange-Gauss algorithm** provides a definite basis that is the shortest possible for a given lattice. The result of this algorithm in two dimensions is that the two resulting vectors will be the shortest linearly independent vectors in the lattice. Let L be the lattice generated by the following vectors, taken from [31] :

$$e_1 = [66586820, 65354729] \tag{2.45}$$

$$e_2 = [6513996, 6393464] \tag{2.46}$$

Those vectors are arbitrarily long, but we can shorten them via the Lagrange-Gauss algorithm as follows. First notice that $||e_2|| \leq ||e_1||$, otherwise we should swap them. While $||e_2|| \leq ||e_1||$:

- Round $q := \lfloor \langle e_1, \frac{e_2}{||e_2||} \rangle \rceil$ to nearest integer
- $r := e_1 - q \cdot e_2$
- $e_1 := e_2$
- $e_2 := r$

We can see the similarities between this algorithm and Euclidean division.

Applied on the previous vectors, this leads to a reduced basis of :

$$r_1 = [2280, -1001] \tag{2.47}$$

$$r_2 = [-1324, -2376] \tag{2.48}$$

While the norm of the second basis vector has not changed, the norm of the first one has considerably diminshed, by about 10^3.

The **LLL** [32] algorithm can be understood as a generalization of this algorithm to higher dimensions, or as a way to make the Gram-Schmidt process work over $\mathbb{Z}$ rather than just over fields. This is the main algorithm used when one performs basis reductions, although several other algorithms exists.

In fact, we can look back at the situation of vector spaces to draw inspiration from, since this is the ideal -and unachievable- case. In that case, we can understand **the problem of finding short vectors also as being tied to how orthogonal they are from one another**.

Note that the LLL algorithm guarantees some form of optimality of the reduction in polynomial time, as opposed to most algorithm which are of higher time complexity. Generally, finding short vectors in a lattice is a difficult problem that is considered NP-hard.

In the context of lattices, this is quite intuitive : if we have a bunch of points in a plane, picking any two of them that are non-linearly dependent will lead to a basis. In particular, we can pick very long. The consequences being of course that it will also imply they won't be very orthogonal.

The reason is that once we have described the lattice in terms of its Gram matrix G obtained from taking the inner form between the basis vectors e_i, any change of basis is necessarily with integral coefficients and of determinant ± 1. This already can be seen with our invariant the covolume. Let P be a change of basis for G. Since the covolume is an invariant we have :

$$coVol(P^t \cdot G \cdot P) = |det(P^t \cdot G \cdot P)| = |det(P)^2 \cdot det(G)| := coVol(G) \tag{2.49}$$

which already imposes $det(P) = \pm 1$.

Thus a hand-wavy argument that can allow us to gauge how reduced our basis is is to look at the number of 0's appearing in the Gram matrix, as they reflect orthogonal directions. Note that this is very much hand-wavy, and there's no guarantee of optimality. In fact, lattice problems such as the shortest vector problems, which is of course closely related to what we are doing, are NP-hard and no known solution exists in general.

2.3.4 A finite number of solutions and Fincke-Pohst algorithm

Since we have spent some time introducing the notion of volumes in lattices, as well as the invariant covolume, we want to show some important observation with respect to the tadpole problem.

Let G_4 be a flux whose self intersection is below some bound T, without specifically requiring this bound to be related to physics, as long as it is a bound. We essentially have that :

$$0 \leq Q(G_4) \leq T \tag{2.50}$$

This set is **closed and has finite volume** because Q is positive definite. Since we are working with a lattice, we can simply rely on the fact that there are finitely many lattice points inside a finite volume. Recall we have introduced the set $S(Q, T)$

in the introduction as :

$$S(Q,T) := \{G \mid \exists\, k \in \mathbb{Z},\ Q(G) = k \leq T\}. \tag{2.51}$$

We have simply stated in a lattice theoretic way that this set is finite, and for every k below T there are finitely many solutions, and hence finitely many G_4 fluxes below the tadpole bound.

Since it is finite, we need to find ways to characterize this set, and for this we will introduce a different vocabulary, although everything could be rephrased in terms of lattices up to some minor reformulations.

To make the transition smoother let us furthermore notice an important property that stems from noticing that G_4 must be positive semi-definite. When we introduced the LLL basis reduction we noticed the parallel with the Gram-Schmidt algorithm for vector spaces. Similarly, drawing inspiration from the Cholesky decomposition of positive definite matrices over some field, **the Fincke-Pohst algorithm** [33] (implemented in PARI/GP [34] and now used in SAGEMATH [35]) extends this concept over rings.

To illustrate the strength of this algorithm, we use the example of Cholesky decomposition of a symmetric positive definite matrix. Let A be the following matrix :

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 5 & 5 \\ 1 & 5 & 14 & 14 \\ 1 & 5 & 14 & 15 \end{pmatrix} \tag{2.52}$$

The algorithm is simply to verify the equality $A := LL^t$, hence to find a square root of A in the form of L, via Gaussian reduction. The result in that case is :

$$L := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 1 \end{pmatrix} \tag{2.53}$$

The point is the following : if we are trying to solve $x \cdot A \cdot x = n$ for some number n, then the search bounds are better by using L than using A since the norm of the rows/columns of A are longer than those of L. Hence, by using the fact that we deal with a specific problem with a certain amount of symmetries (here: the quadratic form is positive definite), we have considerably reduced the computational complexity.

There are two principal reasons as to why this algorithm is compelling. First of all, as we have discussed above, there is a priori no hope to achieve diagonalization of our Gram matrix, although that would make the problem of enumeration considerably easier. Furthermore, if we take a square matrix, it is clear that typically the number of points to enumerate is extremely large, and often those points are not very interesting as they are way above the bound we impose.

Thus, reducing the Gram matrix to a product of triangular matrices is a good way to answer those difficulties, reducing the search bounds and making the enumeration easier. In a sense, while basis reduction algorithm and in particular LLL allows one to pick a shorter and more orthogonal basis, the Fincke-Pohst reduces the size of the box/ellipsoid that contains the interesting points.

2.3.5 Quadratic forms and number theory

Moving away from the vocabulary of lattices, we will talk about quadratic forms, following Cassels [36], in place of Gram matrices. The main motivation is to clarify the points laying in $S(Q, T)$, as it is core to our strategy of enumerating every possible solution in this set, and of checking if they also lay in $S_H(\rho, X)$.

Part of the data included in the set $S(Q, T)$ is the following:

- Coefficients a_{ij} of the quadratic form Q.
- The numbers k that are integers such that $0 < k \leq T$.
- The finite number of solutions for each and every k, $Q(G) = k$.

In order to start this section with some fairly easy results from number theory, we will focus on the coefficients a_{ij} and k. First let us comment on the possible half-integrality condition coming from $G_4 + \frac{c_2}{2} \in H^4(X, \mathbb{Z})$.

On the geometric side of things, for the example we will study, this will be characterized with respect to linear cycles as we have seen previously. What can be said about this condition in other set ups? In general this condition translates to G_4 being itself half-integral, so that the coefficients of Q are typically not only integers but also half-integers.

Since a half-integer is written as $\frac{n}{2}$ with n odd, we might as well multiply the quadratic form Q by a factor of 2 and look for odd integers solutions. Upon division by two, we will recover $G \cdot G$. Here we propose a slightly different point of view of this operation.

In number theory, the **Chinese Remainder theorem (or fundamental theorem of arithmetic)** tells us that every integer can be written uniquely as a product of powers of primes. In particular that is the case for the coefficients of our quadratic form as well as the possible solutions. In that sense, we interpret the condition that $G \cdot G = \frac{n}{2}$ by rewriting $\frac{n}{2}$ as :

$$\frac{n}{2} = 2^{-1} 3^{a_3} 5^{a_5} \dots, \tag{2.54}$$

where n is odd.

This trivial change of notation is in fact very important for the following reason. We know that $S(Q, T)$ is finite and we seek to characterize it. Instead of trying to find the solutions G corresponding to some flux, we already know that the only possible integers k appearing are bounded above and below. However we can characterize this set by what is not part of it, rather than what belongs to it. Meaning, we will try to find the k for which there are no solutions G rather than finding those that admit solutions.

The reason is the following. Up to rescaling our quadratic form as before to make it integral, we know that the result must be an integer. In the language of quadratic forms, we therefore ask which integers are represented by our quadratic form Q.

A criteria is thus the following : let us assume the integer k is represented by Q, meaning we have a non-trivial integer solution G such that $Q(G) = k$. Then in particular, we can take the reduction modulo any integer and it should still be true because reduction modulo n is an equivalence relation.

Since we have seen every integer can be represented uniquely as a product of prime powers, we arrive at the following well-known result : **there is a non-trivial solution to the representation of an integer by an integral quadratic form if there are non-trivial solutions modulo every prime-power as well as the reals.**

Thus we are interested in the contrapositive of this proposition : we now know that if we do not have a solution for the representation of some integer k modulo some prime power, then we do not have an integral solution. So we have a characterization of elements of $S(Q, T)$ that eliminates possible solutions, and very importantly this does not rely on any conjecture. We thus speak of an obstruction if there are no solutions modulo p^k.

The converse to this equivalence relation, meaning asking to have a solution in the integers if you have a solution at every prime power is known as the Hasse principle, or local to global principle.

2.3.6 p-adic numbers and representation of integers

Our previous condition requires the reduction modulo every prime power. For practical purposes, is every prime appearing relevant ? The answer is no : the possible primes and prime powers we need to consider is finite since at some point we no longer perform any reduction, and thus every prime or prime power greater than this threshold will not induce any possible constraint on the representation of integers.

Naturally, this is at first quite scary since this condition can depend on very large prime powers, and while $\mathbb{Z}/(p^k\mathbb{Z})$ always has finitely many element, this at first does not seem to help out much when it comes to the computational problems previously managed in lattice-theoretic terms.

This motivates the introduction of the p**-adic numbers**. In this context, let us redefine the integers $\mathbb{Z}$ in the following way : we will see the integers $\mathbb{Z}$ as a subring of the ring of rationals $\mathbb{Q}$.

The field $\mathbb{Q}$ usually comes equipped with the usual absolute value defined as :

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if x} \geq 0 \end{cases}$$

The completion of $\mathbb{Q}$ with respect to this absolute value, meaning we ask for all Cauchy sequences to converge, leads to the reals $\mathbb{R}$. However, $\mathbb{Q}$ can come with other valuations, the p-adic valuations, denoted as $|.|_p$, defined in two steps. First introduce the valuation $v_p(n)$ for an integer n, as follows:

$$v_p(n) = \begin{cases} max(k) & \text{if } p^k \text{ divides } n \\ 0 & \text{if } p \text{ does not divide } n \end{cases}$$

Now you can introduce the p-adic valuation for the rationals $\mathbb{Q}$ by writing any rational as $\frac{n_1}{n_2}$ with n_1, n_2 coprime as follows :

$$v_p\left(\frac{n_1}{n_2}\right) = v_p(n_1) - v_p(n_2)$$

Asking for the completion of $\mathbb{Q}$ with respect to this valuation leads to the p-adic numbers $\mathbb{Q}_p$, which we can extract the p-adic integers $\mathbb{Z}_p$ from.

The point of this construction is to understand the following intuitive fact : if we ask for a quadratic form to represent an integer, in particular it must represent this integer in every possible base, not just base 10. This construction is analytic in nature, and to connect the dots we will expose the algebraic construction of the p-adics. However do note that the two are equivalent and we can see already a subtlety with this formalism : the p-adic integers are those p-adics numbers with a non-negative valuation. In particular, $\frac{1}{2}$ is a p-adic integer for every $p \neq 2$, which indicates to us that not only do we need to care about $\mathbb{Z}$ but also its field of fractions $\mathbb{Q}$ to find solutions.

The algebraic construction most relevant to us is to consider the p-adic integers Z_p as the inverse limit of the rings $\mathbb{Z}/(p^n\mathbb{Z})$ which is defined as the sequences $a_0, a_1, \ldots$ such that $a_i \in \mathbb{Z}/(p^i\mathbb{Z})$ and $a_i = a_{i+1} \mod p^i$ for all i, and $\mathbb{Q}_p$ is the field of fractions of this ring. Naturally both constructions are equivalent.

Following the usual conventions, we talk about a place for a given prime p, and the " infinity " place for the real numbers.

However this shows the main motivation behind the p-adics : it is a powerful tool to keep track of all possible modular reductions to **find obstructions using the fundamental theorem of arithmetic**. Thus we can restate the condition for obstruction as follows :

If there is a non-trivial integral representation z of some number n by an integral quadratic form Q, such that $Q(z) = n$, then we have a non-trivial solutions in $\mathbb{Z}_p$ for every prime p, including the reals.

2.3.7 Hensel's lemma and non-obstructions

Since we have a criteria for obstruction, it is good to give a criteria for non-obstructions.

Suppose we know that there are no obstruction modulo p, how do we know if there is no k such that there is an obstruction modulo p^k ?

The criteria is given by **Hensel's lemma** which is the following statement.

Let A be a complete commutative ring with respect to an ideal $\mathfrak{m}$[2]. Let $f := f_1, f_2, \ldots, f_n$ be a system of polynomials in $A[x_1, x_2, \ldots, x_n]$ and let J denote the Jacobian matrix of this system. Suppose a is a solution :

$$f_i(a) = 0 \mod (det(J)^2)\mathfrak{m} \tag{2.55}$$

then there is some b such that $f(b) = 0$ in A^n.

[2] in this context complete refers to the p-adic valuation

With one equation, it is quite easy to see what happens and the statement is simplified greatly.

If f is a multivariate polynomial, and some $a \in \mathbb{Z}/p\mathbb{Z}$ satisfies :

$$v_p(f(a)) < ||(\nabla f)(a)||_p^2 \tag{2.56}$$

then there is a solution in the p-adics $\mathbb{Z}_p$.

This statement is analogous to Newton algorithm, which relies on the usual Taylor expansion, but compatible with the p-adic valuation.

In particular, if $f = 0 \mod p$ and $(\nabla f)(a) \neq 0 \mod p$ then the solution is lifted to a solution in the p-adics.

Thus we have a criteria for a possible lifting of the solutions from modular arithmetic to the p-adics via an analogue of Newton's algorithm for approximation of roots, except replaced with the p-adic valuation instead of the usual Euclidean norm.

In practice, this means that in principle we have a way to check for finitely many primes if there are no-obstructions, instead of having to check for all prime powers.

2.3.8 Abstract varieties and related tools

Looking back when we first started looking at characteristics of the problem, we noted that we essentially were computing intersection numbers for special algebraic cycles of a fourfold described in complex projective space. From the point of view of the fundamental theorem of arithmetic, and in order to find obstructions, there is an alternative route, which is of course related to the p-adic numbers introduced above.

Indeed, since we are looking for computing some self-intersection over the integers $\mathbb{Z}$, we can also consider the same fourfold X but defined over the rationals $\mathbb{Q}$. Naturally, it is typically not algebraic closed, however we can consider extensions $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ which are closed. With this description, we also lose the rather strict and usual definition of an algebraic variety, and need to work with abstract varieties or schemes. Furthermore, many notions that made sense in the usual complex (and smooth) setting do not make much sense over $\mathbb{Q}$.

However, we can repeat the same constructions with the p-adics and ask to describe the variety over $\mathbb{Q}$, $\mathbb{Q}_p$, and define some cohomology theories associated with it and essentially perform the same computations, albeit with different tools.

The cohomology of interest in this setting is **étale cohomology**, presented in [37], notably containing the definitions and the very important comparison theorem between étale cohomology and the usual complex cohomology.

2.3.9 Relation to the tadpole problem

We have found tools in the previous sections to compute the length of the G_4 flux, as well as the dimension of the Hodge locus to check the type of a G_4 flux, and identified particular situations, such as Fermat varieties, where those tools can be applied readily. Furthermore, we have determined obstructions to the existence of such fluxes. For this we have introduced the p-adic numbers as a powerful tool to keep track of those obstructions, but in what situations can those tools be readily applied to study the tadpole problem ? And on the mathematical side, what are the results related to the tadpole problem from this point of view ?

First, in parallel with identifying the Hodge conjecture as having a significant overlap, but perhaps not complete, with the tadpole problem, we expose two mathematical conjectures related to the point of view taken from the fundamental theorem of arithmetic. We expect both conjectures to thus have overlap with the tadpole problem, but neither of them being necessary nor sufficient to fully solve the problem.

Directly linked with the fundamental theorem of arithmetic is the local to global principle, or Hasse principle. We have seen that if there is an integer solution to the problem, then there must be a non-trivial solution at every place $\mathbb{Z}_p$ as well as the reals. **The Hasse principle** asks the converse of this problem : if we have a local solution at every place p as well as the reals for the representation of some integer n, do we get an integral, or global, solution ?

This principle is in general of course **false**. In the particular context of quadratic forms we have the following results :

- Over the rationals $\mathbb{Q}$ and the corresponding local places $\mathbb{Q}_p$ this principle holds, a proof of which can be found in [38].
- Over the integers $\mathbb{Z}$ it is known to fail in general, with many counter-examples being found.

At first, the fact that it does not hold over the integers seems discouraging. However, recall that our situation is not generic : we typically want to represent numbers that are quite small -hence the tension in the first place- and our quadratic form comes from an algebraic variety. So typically the requirements for the Hasse principle to be interesting with respect to physics are both more specific, and also quite a bit smaller, since we do not care too much about integers above the tadpole bound, it would suffice for it to hold below this threshold. Moreover, some obstructions to the Hasse principle have a topological origin and can be explained (at least for certain curves and surfaces) by using étale cohomology and computing the Brauer-Manin obstruction, being covered in the lectures [39]. However one should point out that this obstruction to the Hasse principle is not the only one and does not account for all obstructions as the famous example of [40] shows.

Furthermore, it is important to keep track of the various spaces we are working with. We can work at the level of the fourfold X and study its (co-)homology classes, or once we have determined the quadratic form, we can work with this quadratic form on its own. For example, we can ask if the fourfold X we study has obstructions to having rational or integral points, or we can simply ask to find obstructions to the existence of rational or integral points for quadratic form in $H^{2,2}(X,\mathbb{Q})$ or $H^{2,2}(X,\mathbb{Z})$.

A less direct link is to look back at the overlap with the Hodge conjecture and ask if there is an p-adic analog to this conjecture. This analog is known as the Tate conjecture which we state here.

Let X be a smooth projective variety over a field k of characteristic p, finitely generated over its prime field (the unique minimal subfield of k). Let k_s be its separable closure and G the associated (absolute) Galois group of k, and $\bar{X}$ be the variety over this algebraically closed field. Let $l \neq p$ be a prime, and let algebraic cycles of codimension r be the elements of the free abelian group $Z^r(\bar{X})$. The **Tate**

conjecture states that the cycle class map, which associates an element of the étale cohomology to every algebraic cycle :

$$c_r : Z^r(\bar{X}) \otimes \mathbb{Q}_l \longrightarrow H^{2r}(\bar{X}, \mathbb{Q}_l(r)) \tag{2.57}$$

is surjective.

This is the analog of the integral Hodge conjecture, but for l-adic coefficients, and hence provides a possible framework to rephrase the tadpole conjecture in number-theoretic terms.

Naturally, one type of fourfold for which studying the number theoretic approach is particularly nice is the case of modular Calabi-Yau fourfolds. Since we are interested in computing the lack of existence of solutions to the tadpole problem by using the fundamental theorem of arithmetic, it is natural to consider the primes appearing, which has led us to consider the above conjectures.

However, for pratical purposes, this is quite difficult in general since there are possibly many computations to do, one for every prime (or prime power). However, modular fourfolds offer a nice setting to reduce this computational burden because they offer a global approach to the problem. For example, in the case of modular elliptic curves, being modular allows to compute their properties upon reduction in less time than one would by naively reduction modulo every suitable prime power. Note as well that considering the original physical requirements of an elliptic fibration, this hints as well to a deep link between modularity and the tadpole conjecture.

From this chapter we have exposed many links between the tadpole conjectures and various results or conjectures in mathematics. However, the only result that does not rely on conjectures is that there are typically obstructions to the existence of points below the tadpole bound. We have showcased different arguments and possible solutions to the tadpole conjecture by using various (conjectural) results in mathematics. While there is a clear relationship and overlap with the tadpole conjecture, it is not clear that they provide any criteria for the existence of solutions to the tadpole problem, but it is not clear that they are not needed either, as in principle the tadpole problem can be phrased without resorting to any sort of conjecture.

To summarize, from the point-of-view of p-adic numbers, there are two conjectures that are relevant to the tadpole problem.

- The Hasse principle: if you find solution to the quadratic equation modulo n for every integer n are you, in this context, guaranteed to have an integral solution ?

- The p-adic analogue of the Hodge conjecture : the Tate conjecture, which in its integral version, is still open for Fermat's sextic fourfold. Can we find an algebraic cycle to each form in $H^{2,2} \cap H^4(X, \mathbb{Z}_p)$?

We expect the Hasse principle to not hold, however, it would be interesting to see the link between the Tate conjecture and the tadpole problem in greater detail.

CHAPTER 3

G_4 fluxes and Fermat's sextic fourfold

In this chapter, we detail the various tools and results exposed in the previous chapter with more rigour and specialize to the case of Fermat's sextic fourfold. We begin by recalling a few facts about the sextic, and then move on to showcase some examples, commenting on possible links with the last chapter as well.

3.1 Mathematical facts about the sextic

3.1.1 Definition and properties of Fermat's sextic

Fermat's sextic fourfold is defined as the hypersurface

$$X \equiv x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 = 0 \tag{3.1}$$

in $\mathbb{P}^5$ with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3 : x_4 : x_5]$. Its large group of automorphism is given by :

$$\mathrm{Aut}(X) = \mathfrak{S}_6 \ltimes \mathbb{Z}_6^5$$

where we identify

$$\mathbb{Z}_6^5 \simeq \mathbb{Z}_6^6 / D$$

where D is the diagonal

$$D \equiv \mathrm{Im}(a \in \mathbb{Z}_6 \mapsto (a, a, a, a, a, a) \in \mathbb{Z}_6^6).$$

- $\mathfrak{S}_6$ is the group of permutations of six elements, which acts on X by permutation of coordinates
- The group $\mathbb{Z}_6^6/D$ acts on X as

  $$\ell : [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [\zeta^{\ell_0} x_0 : \zeta^{\ell_1} x_1 : \zeta^{\ell_2} x_2 : \zeta^{\ell_3} x_3 : \zeta^{\ell_4} x_4 : \zeta^{\ell_5} x_5],$$

  with $\zeta = e^{\frac{\pi i}{3}}$ the primitive sixth root of unity.

3.1.2 Chern classes, Euler characteristic and Hodge numbers

The total Chern class of X is by adjunction for hypersurfaces 1.28

$$c(X) = (1 + H)^6/(1 + 6H) = 1 + 15H^2 - 70H^3 + 435H^4\,, \tag{3.2}$$

where $H \in H^{1,1} \cap H^2(X, \mathbb{Z})$ is the hyperplane class.

Notably, the second chern class $c_2(X) = 15$ which means we will have to perform some rescaling if we want to use some number theory, as discussed previously.

From the adjunction formula, we can compute the Euler characteristic χ :

$$\chi(X) = \int_X c_4(X) = \int_{\mathbb{P}^5} c_4(X) \wedge dH = \int_{\mathbb{P}^5} 435H^4 \wedge 6H = 2610 \tag{3.3}$$

From the arithmetic genus formula, as well as noting that we can use Lefschetz's hyperplane theorem in the case of the sextic which results in $h^{1,1} = 1$, we can compute the hodge numbers :

$$h^{4,0} = h^{0,4} = 1 \tag{3.4}$$

$$h^{3,1} = h^{1,3} = 427 - 1 = 426 \tag{3.5}$$

$$h^{2,2} = 1752 \tag{3.6}$$

These numbers, in light of the previous chapter's discussion on lattices, are way too big to consider the whole problem and we will restrict our problem.

3.1.3 Jacobian and Hessian

We will make frequent use of the Jacobian ideal and the Hessian for our computations, and we list them here.

First we compute the Jacobian :

$$J(X) = \begin{pmatrix} 6x_0^5 & 6x_1^5 & 6x_2^5 & 6x_3^5 & 6x_4^5 & 6x_5^5 \end{pmatrix} \tag{3.7}$$

which has a global factor of 6, with associated Jacobian ideal $Jac(X)$:

$$Jac(X) =< F, 6x_0^5, 6x_1^5, 6x_2^5, 6x_3^5, 6x_4^5, 6x_5^5 > \tag{3.8}$$

where F is the defining equation of X.

The associated Jacobian ideal is thus spanned by monomials of degree 5, hence any monomial with a power of 5 or above will automatically lie in the Jacobian ideal.

As for the Hessian, we can simply compute it by taking another derivative :

$$H(X) = \begin{pmatrix} 30x_0^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30x_1^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 30x_2^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30x_3^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 30x_4^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30x_5^4 \end{pmatrix} \tag{3.9}$$

3.1.4 Residues for Fermat's sextic

Recall that in the case of fourfolds, the primitive part of the middle cohomology is defined as :

$$H^4(X)_{\text{prim}} = \{A \in H^4(X) | A \cdot H^2 = 0\} \, ,$$

and can be described by the residue map

$$Res : H^5(\mathbb{P}^5 \setminus X) \to H^4(X).$$

The residue mapping is surjective onto the primitive middle cohomology and a basis of $H^4(X)_{\text{prim}}$ is given by the forms [41]

$$\omega_\beta := Res\left(\frac{x^\beta \Omega_0}{Q(x)^{k+1}}\right)$$

where :

- $$\Omega_0 = \sum_{i=0}^{5} (-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_5$$

  is the standard degree 6 top form of $\mathbb{P}^5$.

- x^β is the monomial

  $$x^\beta = x_0^{\beta_0} x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} x_4^{\beta_4} x_5^{\beta_5}$$

  with $|\beta| := \frac{1}{6} \sum \beta_i = k \in \mathbb{Z}$, $0 \leq \beta_i \leq 4$, and $0 \leq k \leq 4$ determines the Hodge type:

  $$\omega_\beta \in H^{4-k,k}(X) \, .$$

In our approach we are interested in $k = 1, 2$ corresponding to $H^{1,3}$ and $H^{2,2}$ respectively, meaning we are interested in monomials of degree 6 and 12 respectively.

3.1.5 Intersection pairing

The inner form between two residues ω_P and ω_Q can be computed using [26] and specializing 2.9 to Fermat's sextic as follows. Let the monomials P and R be of degrees $6p$ and $6q$ such that $p+q=4$ we have :

$$\omega_P \cdot \omega_R = \int_X \omega_P \wedge \omega_R = c\ (-1)^{p+1}\ \frac{(2\pi i)^4}{p!q!} 5^6 6 \tag{3.10}$$

for $c \in \mathbb{C}$ the unique number such that

$$PR \equiv c \det(\mathrm{Hess}(X)) \pmod{\mathrm{Jac}(X)}, \tag{3.11}$$

For the Fermat sextic the determinant of the Hessian 3.9 is simply

$$\det(\mathrm{Hess}(Q)) = 30^6 \prod_i x_i^4\,, \tag{3.12}$$

and so for our monomial basis

$$\omega_\beta \cdot \omega_{\beta'} = 0 \tag{3.13}$$

except when $\beta_i = 4 - \beta_i'$ for all $i = 0, 1, 2, 3, 4, 5$.

We define the complex conjugate monomial as :

$$\bar{\beta}_i := 4 - \beta_i \tag{3.14}$$

which implies that

$$|\bar{\beta}| = \tfrac{1}{6} \sum_{i=0}^{5} (4 - \beta_i) = 4 - |\beta|. \tag{3.15}$$

Crucially, for $h^{2,2}$ forms we have $|\bar{\beta}| = |\beta| = 2$. Furthermore, since $x^\beta x^{\bar{\beta}} = \prod_i x_i^4$ for the intersection to be non-zero since any monomial with a term of degree $d_i > 4$ will lie in the Jacobian ideal 3.7, it follows that $c = 30^{-6}$ and :

$$\omega_\beta \cdot \omega_{\bar{\beta}} = (-1)^{|\beta|+1} \left(\frac{1}{30}\right)^6 \frac{(2\pi i)^4}{|\beta|!|\bar{\beta}|!} 5^6 \cdot 6 = (-1)^{|\beta|+1} \frac{(2\pi i)^4}{|\beta|!|\bar{\beta}|!} \frac{1}{6^5}. \tag{3.16}$$

3.2 Algebraic and Hodge cycles

Define classes in $H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ as Hodge cycles, and classes in $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ integral Hodge cycles. After the work of Shioda [30], we know that the residue forms generate the space of complexified primitive Hodge cycles

$$H^{2,2}(X)_{\text{prim}} = (H^{2,2}(X)_{\text{prim}} \cap H^4(X, \mathbb{Z})) \otimes \mathbb{C},$$

.

For $|\beta| = 2$, we have that $\omega_\beta \in H^{2,2}(X)_{\text{prim}}$. Such a form ω_β will be called:

- 3-decomposable if $\beta = (a, 4-a, b, 4-b, c, 4-c)$
- 1-decomposable if $\beta = (a, 4-a, 0, 2, 3, 3)$
- indecomposable if $\beta = (0, 0, 3, 3, 3, 3)$

up to permutations and for $0 \leq a, b, c \leq 4$. The 1751 classes in $H^{2,2}(X)_{\text{prim}}$ are thus organized into 1001 3-decomposable, 720 1-decomposable and 30 indecomposable cycles. Note that 2-decomposable cycles are automatically 3-decomposable from the constraint $|\beta| = 2$.

Over $\mathbb{Q}$ the Hodge conjecture is true, as proved in [30]. We introduce the algebraic cycles associated with the above residues[1] :

- Linear cycles: which can be obtained as the orbit of $\text{Aut}(X) = \mathfrak{S}_6 \ltimes \mathbb{Z}_6^5$ on

$$C := \{x_0 - \mu x_1 = x_2 - \mu x_3 = x_4 - \mu x_5 = 0\} \subseteq X, \tag{3.17}$$

where $\mu = e^{\frac{\pi i}{6}}$ is the primitive 12th root of unity. Given $\sigma \in \mathfrak{S}_6$ and $\ell \in \mathbb{Z}_6^6/D \simeq \mathbb{Z}_6^5$ we denote by C_σ^ℓ the linear cycle given by the equations:

$$x_{\sigma(0)} - \mu^{2(\ell_1-\ell_0)+1} x_{\sigma(1)} = x_{\sigma(2)} - \mu^{2(\ell_3-\ell_2)+1} x_{\sigma(3)} = x_{\sigma(4)} - \mu^{2(\ell_5-\ell_4)+1} x_{\sigma(5)} = 0. \tag{3.18}$$

[1] Those are the algebraic cycles we can compute the residues of, but in principle residue forms can be dual to other cycles.

- Aoki-Shioda cycles: which are obtained as the orbit of $\mathrm{Aut}(X)$ on

$$S := \{x_3^2 - \sqrt[3]{2}x_0x_1 = x_0^3 + x_1^3 + ix_2^3 = x_4 - \mu x_5 = 0\} \subseteq X. \tag{3.19}$$

Similarly we denote S_σ^ℓ for $\sigma \in \mathfrak{S}_6$ and $\ell := jklm$ with $j, k \in \mathbb{Z}/2\mathbb{Z}$, $m \in \mathbb{Z}/3\mathbb{Z}$ and $l \in \mathbb{Z}/6\mathbb{Z}$. They are given by:

$$\begin{aligned} S_\sigma^{jklm} := \{&x_{\sigma(0)}^3 + e^{i\pi k}x_{\sigma(1)}^3 + ie^{i\pi j}x_{\sigma(2)}^3 = 0 \\ &x_{\sigma(3)}^2 - 2^{1/3}e^{i\pi k}e^{\frac{2i\pi m}{3}}x_{\sigma(0)}x_{\sigma(1)} = 0 \\ &x_{\sigma(4)} - e^{i\pi/6}e^{i\pi l2/3}x_{\sigma(5)} = 0\} \end{aligned}$$

- Type 3 cycles: which are in the orbit of $\mathrm{Aut}(X)$ on

$$T := \{x_0^2 - \sqrt[3]{2}x_1x_2 = x_3^2 - \sqrt[3]{2}x_4x_5 = x_1^3 + x_2^3 + ix_4^3 + ix_5^3 = 0\} \subseteq X. \tag{3.20}$$

We denote them by T_σ^ℓ, where $\sigma \in \mathfrak{S}_6$ and $\ell \in \mathbb{Z}_6^6/D$. The explicit equations of T_σ^ℓ are:

$$\begin{gathered} \zeta^{2\ell_0}x_{\sigma(0)}^2 - \sqrt[3]{2}\zeta^{\ell_1+\ell_2}x_{\sigma(1)}x_{\sigma(2)} = 0, \\ \zeta^{2\ell_3}x_{\sigma(3)}^2 - \sqrt[3]{2}\zeta^{\ell_4+\ell_5}x_{\sigma(4)}x_{\sigma(5)} = 0, \\ (-1)^{\ell_1}x_{\sigma(1)}^3 + (-1)^{\ell_2}x_{\sigma(2)}^3 + i(-1)^{\ell_4}x_{\sigma(4)}^3 + i(-1)^{\ell_5}x_{\sigma(5)}^3 = 0. \end{gathered} \tag{3.21}$$

From [25] we can compute the periods of any residue forms over such cycles. The complete labelling of all such algebraic cycles is in the appendix B.1, B.2 and B.3.

For the particular case of linear cycles of the sextic, an explicit formula was obtained in [24] and is the following:

$$\int_{C_\sigma^\ell} \omega_\beta = \begin{cases} (2\pi i)^2 \frac{\mathrm{sgn}(\sigma)}{6^3 \cdot 2} \mu^{\sum_{e=0}^2 (\beta_{\sigma(2e)}+1)(2(\ell_{2e+1}-\ell_{2e})+1)} & \text{if } \beta_{\sigma(2e-2)} + \beta_{\sigma(2e-1)} = 4 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

3.2.1 Intersection numbers of complete intersection algebraic cycles

The above formula can be used to compute the intersection numbers of linear algebraic cycles depending on the parameters l and σ. We provide here an alternative, which can hold for complete intersection algebraic cycles, and do not rely on periods.

The intersection numbers for the linear cycles are given by :

$$\begin{aligned} C^2 &= 21 \\ C^{l_0 l_1 l_2} \cdot C^{l_0 l_1 l_2'} &= -4 \\ C^{l_0 l_1 l_2} \cdot C^{l_0 l_1' l_2'} &= 1 \\ C^{l_0 l_1 l_2} \cdot C^{l_0' l_1' l_2'} &= 0 \end{aligned}$$

Indeed, starting from the observation that :

$$H^2 \cdot H^2 = 6$$

And noticing, at the level of the factorization, that :

$$H^2 = \sum_{l_0=0}^{5} C^{l_0 l_1 l_2} \tag{3.23}$$

$$\implies H^2 \cdot C_\sigma^l = 1 \tag{3.24}$$

Which follows by symmetry, and the fact that the intersection is 0 if σ and l differ in all three components. Now we compute :

$$\sum_{l_1'=0,l_0'=0}^{5} C^{l_0l_1l_2} \cdot C^{l_0'l_1'l_2'} = 6$$

$$\implies \sum_{l_1'=0} (C^{l_0l_1l_2} \cdot C^{l_0l_1l_2'} + 5C^{l_0l_1l_2} \cdot C^{l_0'l_1l_2'}) = 6$$

$$\implies C^{l_0l_1l_2} \cdot C^{l_0l_1l_2'} + 5C^{l_0l_1l_2} \cdot C^{l_0l_1'l_2'} + 5(C^{l_0l_1l_2} \cdot C^{l_0'l_1l_2'} + 5C^{l_0'l_1l_2} \cdot C^{l_0l_1'l_2'}) = 6$$

$$\implies C^{l_0l_1l_2} \cdot C^{l_0l_1l_2'} + 5 + 5 + 25 \times 0 = 6$$

$$\implies C^{l_0l_1l_2} \cdot C^{l_0l_1l_2'} = -4$$

Following the same procedure, as a sanity check, we have indeed that :

$$C^2 = 21.$$

Which corresponds exactly to what we can find with the adjunction formula taking all $d_i = 1$. Alternatively, or to check that those formulas are correct, we can simply observe that linear cycles are just given by hyperplanes. Thus the intersection of 5 hyperplanes in $\mathbb{P}^5$ has intersection 1, while they do not intersect if there are more of them (6 here).

For Aoki-Shioda and type 3 cycles, one can work-out the self-intersection using adjunction which results in the self-intersections being :

$$S^2 = 66 \tag{3.25}$$

$$T^2 = 120 \tag{3.26}$$

Thus we can start computing the intersection numbers for Aoki-Shioda cycles at fixed permutation for example, knowing that again :

$$H^2.H^2 = 6$$

$$C^2 = 66,$$

and working backwards. For example, we have

$$
\begin{aligned}
12 &= \sum_{k'=0}^{1} C^{jklm} \cdot C^{jk'lm} \\
\implies 12 &= 66 + C^{jklm} \cdot C^{jk'lm} \\
\implies C^{jklm} \cdot C^{jk'lm} &= -54
\end{aligned}
$$

The 12 coming from the fact that you have a degree two and a degree one in the factorization, so that you end up, when summing over k, with a term $2H^2$. We find, after doing all the possible combinations :

$$
\begin{aligned}
S^{jklm} \cdot S^{k'j'm'l'} &= 12 & S^{jklm} \cdot S^{kj'm'l'} &= 0 \\
S^{jklm} \cdot S^{k'jm'l'} &= 0 & S^{jklm} \cdot S^{k'j'm'l} &= -24 \\
S^{jklm} \cdot S^{k'j'ml'} &= -6 & S^{jklm} \cdot S^{kj'ml'} &= 18 \\
S^{jklm} \cdot S^{kj'ml'} &= 18 & S^{jklm} \cdot S^{k'jml'} &= 18 \\
S^{jklm} \cdot S^{kj'm'l} &= 36 & S^{jklm} \cdot S^{k'jm'l} &= 36 \\
S^{jklm} \cdot S^{kjm'l'} &= 12 & S^{jklm} \cdot S^{k'j'ml} &= 66 \\
S^{jklm} \cdot S^{kj'ml} &= -54 & S^{jklm} \cdot S^{k'jml} &= -54 \\
S^{jklm} \cdot S^{kjml'} &= -6 & S^{jklm} \cdot S^{kjm'l} &= -24
\end{aligned}
$$

Where a prime indicates a different index than it's unprimed counterpart. Which, as a sanity check, leads to $C^2 = 66$ if working forward.

More generally, one can use the results in [27] to compute the homology class of a given cycle and then the intersection numbers using periods of residue forms.

3.2.2 Dimension of the Hodge locus

In order to know if a flux G is a general Hodge cycle we need to compute the rank of the matrix ρ_{IJ} defined as follows.

$$
\rho_{IJ}(G) := \omega_{\beta_I+\beta_J} \cdot G, \tag{3.27}
$$

for $|\beta_I| = |\beta_J| = 1$.

Since $|\beta_I| = |\beta_J| = 1$ corresponds to $h^{3,1}$ forms, this is a 426×426 matrix. The entries are then given by the coordinates of G in the basis of residue forms.

In order to see the relationship with respect to the Aoki-Shioda results, we can understand it as follows. Let $G := \sum_\beta \alpha_\beta \omega_\beta$ for some rescaling/constant α_β. Then ρ_{IJ} will be given by :

$$\rho_{IJ} = \sum_{|\beta|=2} \alpha_\beta \frac{-(2\pi i)^4}{2^2 6^5} \omega_{\beta_I+\beta_J} \cdot \omega_\beta = \frac{-(2\pi i)^4}{2^7 3^5} \alpha_{\overline{\beta_I+\beta_J}}\,, \tag{3.28}$$

with the universal constant $\frac{-(2\pi i)^4}{2^7 3^5}$.

3.2.3 Vanishing cycles

We have seen that vanishing cycles can be used to provide a basis for the middle homology. Let us specialize the discussion to Fermat's sextic and provide a proper definition in this case.

Let us fix the affine patch of the sextic to be :

$$U_0 := \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 |\ 1 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 = 0\}\,.$$

where we have fixed $x_0 = 1$.

Vanishing cycles are defined as follows : for every $\beta \in \{0,1,2,3,4\}^5$

$$\delta_\beta := \sum_{a \in \{0,1\}^5} (-1)^{\sum_{i=1}^5 (1-a_i)} \Delta_{\beta+a}, \tag{3.29}$$

where $\Delta_{\beta+a} : \Delta^4 := \{(t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}^5 : t_i \geq 0\ ,\ \sum_{i=1}^5 t_i = 1\} \to U_0$ is given by

$$\Delta_{\beta+a}(t) := \left(\zeta_{12}^{2(\beta_1+a_1)-1} t_1^{\frac{1}{6}}, \zeta_{12}^{2(\beta_2+a_2)-1} t_2^{\frac{1}{6}}, \zeta_{12}^{2(\beta_3+a_3)-1} t_3^{\frac{1}{6}}, \zeta_{12}^{2(\beta_4+a_4)-1} t_4^{\frac{1}{6}}, \zeta_{12}^{2(\beta_5+a_5)-1} t_5^{\frac{1}{6}}\right).$$

The periods of primitive classes can be explicitly computed [42] as follows:

$$\int_{\delta_{\beta'}} \omega_\beta = \frac{(-1)^{|\beta|}}{6^5} \frac{1}{|\beta|! 2\pi i} \prod_{i=0}^{5} \Gamma\left(\frac{\beta_i + 1}{6}\right) \left(\zeta^{(\beta'_i+1)(\beta_i+1)} - \zeta^{(\beta'_i)(\beta_i+1)}\right), \tag{3.30}$$

where $\beta'_0 := 0$.

This formula extended to other affine patches by setting $x_i = 1$ resulting in the same formula with $\beta'_i = 0$.

Thus for practical purposes we can simply represent vanishing cycles as tuples

$$\delta_{\beta'} = [\beta'_0, \beta'_1, \beta'_2, \beta'_3, \beta'_4, \beta'_5],$$

with each β' ranging from 0 to 4 with at least one of them being 0.

3.2.4 Basis over $\mathbb{Z}$

We have so far worked implicitly over $\mathbb{Q}$. However, vanishing cycles are not enough to provide a basis over $\mathbb{Z}$ and we need to supplement them with an extra linear algebraic cycle. This can be seen from the Leray-Thom-Gysin sequence in homology [26, §4.6]

$$0 \to H_4(X, \mathbb{Z})_{\mathrm{prim}} \to H_4(X, \mathbb{Z}) \xrightarrow{f} H_2(X_\infty, \mathbb{Z}) \to 0,$$

where $H_4(X, \mathbb{Z})_{\mathrm{prim}} := \mathrm{Im}(H_4(U_0, \mathbb{Z}) \to H_4(X, \mathbb{Z}))$, f is the intersection map and

$$X_\infty = X \cap \{x_0 = 0\} = \{x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 = 0\},$$

is the Fermat sextic threefold at infinity. Since

$$H_2(X_\infty, \mathbb{Z}) = \mathbb{Z} \cdot [L],$$

for some line $L \subseteq X_\infty$, we get the following decomposition

$$H_4(X, \mathbb{Z}) = H_4(X, \mathbb{Z})_{\mathrm{prim}} \oplus \mathbb{Z} \cdot [C], \tag{3.31}$$

for any linear algebraic cycle $C \subseteq X$.

Notice that in general the primitive part of an integral cycle is not integral. However, in our case, the above result means that for Aoki-Shioda and Type 3 cycles (corresponding to 1-decomposable and indecomposable cycles respectively) this will result in a 0 intersection with any linear cycles. Thus, this condition on the basis needs to be studied for the specific case of linear (or 3-decomposable) cycles.

3.2.5 Complex Conjugation

Since $G \in H^4(X)$, G is in particular real, and we need to study the action of complex conjugation. Set :

$$\bar{\omega}_\beta = c_\beta \omega_{\bar{\beta}} \,. \tag{3.32}$$

where c_β is a constant of proportionality to be determined.

Pick any vanishing cycle $\delta_{\beta'}$ and integrate $\bar{\omega}_\beta + \omega_\beta$.

$$\int_{\delta_{\beta'}} \bar{\omega}_\beta = \overline{\int_{\delta_{\beta'}} \omega_\beta} \,. \tag{3.33}$$

Using $\beta' = (0^6)$ we find for $|\beta| = 2$ that

$$c_\beta = \frac{\overline{\int_{\delta_{\beta'}} \omega_\beta}}{\int_{\delta_{\beta'}} \omega_{\bar{\beta}}} = -\prod_{i=0}^{5} \frac{\Gamma\left(\frac{\beta_i+1}{6}\right)}{\Gamma\left(\frac{5-\beta_i}{6}\right)} \frac{\overline{\zeta_6^{\beta_i+1} - 1}}{\zeta_6^{5-\beta_i} - 1} = -\prod_{i=0}^{5} \frac{\Gamma\left(\frac{\beta_i+1}{6}\right)}{\Gamma\left(\frac{5-\beta_i}{6}\right)} \,. \tag{3.34}$$

From the n-decomposability of the residues we have the following :

- When β is 3-decomposable we have that (up to permutation) $\beta_{2i} = 4 - \beta_{2i+1}$, so that $c_\beta = -1$.

- When β is 1-decomposable we have $\beta = (\beta_0, 4 - \beta_0, 0, 2, 3, 3)$ and hence

$$c_\beta = -\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)^2} = -\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)^2} = -\frac{\sqrt{\pi}2^{1-1/3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\sqrt{\pi}2^{1-2/3}} = -2^{1/3} \,. \tag{3.35}$$

- When β is indecomposable we can set $\beta = (4, 4, 1, 1, 1, 1)$ and hence

$$c_\beta = -\frac{\Gamma\left(\frac{5}{6}\right)^2\Gamma\left(\frac{1}{3}\right)^4}{\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{2}{3}\right)^4} = -\frac{\Gamma\left(\frac{2}{3}\right)^2\left(2^{1-2/3}\sqrt{\pi}\right)^2 2\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)^2\left(2^{1-2/6}\sqrt{\pi}\right)^2 2\Gamma\left(\frac{1}{3}\right)^2} = -\frac{2^{2/3}}{2^{4/3}} = -2^{-2/3} \,. \tag{3.36}$$

Where we have used the Euler reflection formulas repeatedly:

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)} \\ \Gamma(z)\Gamma(z+\tfrac{1}{2}) &= 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)\,.\end{aligned} \tag{3.37}$$

3.2.6 Integrality with respect to vanishing cycles

Let us first fix some notation to study the integrality condition with respect to vanishing cycles. For some residue form ω_β with $|\beta| = 2$:

$$\begin{aligned}\int_{\delta_{\beta'}} \omega_\beta &= \frac{1}{6^5}\frac{1}{4\pi i}\prod_i \Gamma\left(\frac{\beta_i+1}{6}\right)\left(\zeta_6^{(\beta'_i+1)(\beta_i+1)} - \zeta_6^{(\beta'_i)(\beta_i+1)}\right) \\ &= \frac{1}{6^5}\frac{1}{4\pi i}\prod_i \Gamma\left(\frac{\beta_i+1}{6}\right)\left(\zeta_6^{(\beta_i+1)} - 1\right)\zeta_6^{\beta'_i(\beta_i+1)} \\ &= z_u z_\beta Z(\beta,\beta')\end{aligned} \tag{3.38}$$

with

$$\begin{aligned}z_u &= \frac{1}{6^5}\frac{1}{4\pi i}, \\ z_\beta &= \prod_i \Gamma\left(\frac{\beta_i+1}{6}\right)\left(\zeta_6^{(\beta_i+1)} - 1\right), \\ Z(\beta,\beta') &= \zeta_6^{\sum_i \beta'_i(\beta_i+1)}.\end{aligned} \tag{3.39}$$

Note that z_u is a universal constant, z_β depends only on the choice of residue, whereas $Z(\beta,\beta')$ depends on the vanishing cycles as well as the residue.

From the definition of $Z(\beta,\beta')$ we get :

$$Z(\bar\beta,\beta') = \zeta_6^{\sum_i \beta'_i(\bar\beta_i+1)} = \zeta_6^{\sum_i \beta'_i(5-\beta_i)} = \overline{\zeta_6^{\sum_i \beta'_i(\beta_i+1)}} = \overline{Z(\beta,\beta')} \tag{3.40}$$

The ansatz for G is defined as :

$$G = \sum_{\beta\in\mathcal{I}} \frac{\nu_\beta}{z_u z_\beta}\,\omega_\beta + \frac{\bar\nu_\beta}{\bar z_u \bar z_\beta}\,\bar\omega_\beta\,. \tag{3.41}$$

Here, $\mathcal{I}$ is a subset of βs with $|\beta| = 2$ which contains $\beta = (2^6)$ and exactly one member from each pair $\beta, \bar\beta$. The only non-trivial data once the ω_β are picked are the ν_β. Importantly the maximally symmetric residue form with $\beta = [2,2,2,2,2,2]$

is manifestly real and as such is the only one who doesn't pair up with another residue form.

In order to determine the ν_β, we can use this ansatz in condition 3.38:

$$G \cdot \delta_{\beta'} = \sum_{\beta \in \mathcal{I}} \nu_\beta Z(\beta, \beta') + \bar{\nu}_\beta \bar{Z}(\beta, \beta') \in \mathbb{Z} \ \forall \beta' \tag{3.42}$$

Defining the inner form

$$\langle a, b \rangle \equiv a\bar{b} + \bar{a}b \tag{3.43}$$

on $\mathbb{C}$, the above relations can now be understood as the condition that the ν_β are contained in the lattice $\mathcal{Z}^*$ dual to the lattice $\mathcal{Z}$ spanned by the $Z(\beta, \beta')$.

Since with the above inner form, the sixth roots of unity span the A_2 lattice (or A_1 in the case of the $[2,2,2,2,2,2]$ form), we have in general that the lattice spanned by $Z(\beta, \beta')$ is included in the lattice $A_2^{875} \oplus A_1$. The inclusion is strict, as we shall see in the examples.

3.2.7 Physical constraint: integrality with respect to a linear cycle

To respect the half integrality condition we need to consider integrality with respect to a linear cycle 3.31. This will single out points in the lattice defined by $Z(\beta, \beta')$, which are the properly quantized G fluxes. Let us call this set Λ_{phys}.

In order to find elements of this set, the following algorithm can be used :

- Quantize some G with respect to the vanishing cycle condition 3.38, resulting in the lattice of ν_β.
- Write a general G ansatz with respect to the above lattice.
- Plug this ansatz in 3.18 for a linear cycle C^l_σ.
- Only finitely many 3-decomposable forms will have non-zero intersection with C^l_σ. Impose half-integrality of those intersections.

This will result in some parametric equations determining which integers can be both in the lattice of ν_β and Λ_{phys}.

3.2.8 Checking the normalization

We aime to prove that the normalization in the formula 2.13 is correct.

Since both residues and vanishing cycles form a basis of the same vector space, we can compute in either one of those basis. Using a basis of $H^4(X, \mathbb{C})_{prim}$ composed of normalized residues η_β we have

$$\delta_{\beta'} = \sum_\beta I_{\beta'\beta}\eta_\beta \tag{3.44}$$

where the sum runs over all β that are needed to generate $H^4(X, \mathbb{C})_{prim}$, i.e. $\sum_i \beta_i = 6k$ for $k = 0, 1, 2, 3, 4$ and $\beta_i \leq 4$. The coefficients in this sum are

$$I_{\beta'\beta} = \delta_{\beta'} \cdot \eta_{\bar{\beta}} = c_\beta \; c_{\beta'\,\beta}, \tag{3.45}$$

(no summation here) where the $c_{\beta'\,\beta}$ are the constant of normalization of the vanishing cycles.

Choosing $\beta' = (0^6)$ we find that

$$\begin{aligned} 2 = \delta_{\beta'} \cdot \delta_{\beta'} &= \left(\sum_\beta I_{\beta'\beta}\eta_\beta\right) \cdot \left(\sum_{\beta''} I_{\beta'\beta''}\eta_{\beta''}\right) = \sum_\beta I_{\beta'\beta} I_{\beta'\bar{\beta}} \\ &= \left(\frac{1}{6^5}\frac{1}{2\pi i}\right)^2 \left(\sum_\beta \frac{1}{|\beta|!(4-|\beta|)!} c_\beta c_{\bar{\beta}} \prod_i \left(\zeta^{\beta_i+1} - 1\right)\left(\zeta^{5-\beta_i} - 1\right) \Gamma\left(\frac{\beta_i + 1}{6}\right) \Gamma\left(\frac{5 - \beta_i}{6}\right)\right) \end{aligned} \tag{3.46}$$

We can simplify this as follows. First note that

$$\left(\zeta^{\beta_i+1} - 1\right)\left(\zeta^{5-\beta_i} - 1\right) = 2 - 2\cos\left(2\pi\,\frac{\beta_i + 1}{6}\right) = 4\,\sin^2\left(\pi\,\frac{\beta_i + 1}{6}\right) \tag{3.47}$$

(the same result is in fact found for the corresponding factors for any β') as well as the reflection formula

$$\Gamma\left(\frac{\beta_i + 1}{6}\right)\Gamma\left(\frac{5 - \beta_i}{6}\right) = \frac{\pi}{\sin\left(\pi\frac{\beta_i+1}{6}\right)}\,. \tag{3.48}$$

Hence we find

$$
\begin{aligned}
2 &= \left(\frac{1}{6^5}\frac{1}{2\pi i}\right)^2 (4\pi)^6 \left(\sum_\beta \frac{1}{|\beta|!(4-|\beta|)!} c_\beta^2 \prod_i \sin\left(\pi\frac{\beta_i+1}{6}\right)\right) \\
&= \left(\frac{1}{6^5}\frac{1}{2\pi i}\right)^2 (4\pi)^6 \frac{6^5}{(2Zi)^4} \left(\sum_\beta (-1)^{|\beta|+1} \prod_i \sin\left(\pi\frac{\beta_i+1}{6}\right)\right) \\
&= \frac{2}{3^5}\sum_\beta (-1)^{|\beta|} \prod_i \sin\left(\pi\frac{\beta_i+1}{6}\right)
\end{aligned}
\tag{3.49}
$$

We can work out the sum by brute force on a computer with the result

$$
243 = 3^5 \tag{3.50}
$$

This confirms that the inner form between vanishing cycles and residue forms is the correct one.

3.2.9 Restriction to symmetric forms

Since the lattices involved have a high dimension, we need to restrict ourselves to some symmetric forms to be able to perform computations. Furthermore, restricting ourselves to such symmetric forms reminds us that quotienting might be a solution to find suitable G fluxes so there is an additional physical motivation to do so.

We will only consider symmetries with respect to $(\mathbb{Z}/6\mathbb{Z})^5$ and work at fixed permutation $\mathfrak{S} = I_6$, the identity.

We will consider symmetries that act on the homogeneous coordinates x_i as

$$
g : (x_0, x_1, x_2, x_3, x_4, x_5) \ \to \ (\zeta^{g_0}x_0, \zeta^{g_1}x_1, \zeta^{g_2}x_2, \zeta^{g_3}x_3, \zeta^{g_4}x_4, \zeta^{g_5}x_5)\,, \tag{3.51}
$$

with ζ a primitive 6th root of unity and $g_i \in [0, 1, 2, 3, 4, 5]$.

Note that the homogeneous coordinates are only defined modulo the $\mathbb{C}^*$ action of $\mathbb{P}^5$, so that we can identify

$$
(g_0, g_1, g_2, g_3, g_4, g_5) \simeq (g_0+1, g_1+1, g_2+1, g_3+1, g_4+1, g_5+1)\,. \tag{3.52}
$$

Furthermore, we will only consider groups Γ which preserve the holomorphic top-form Ω, so that we need

$$\sum_i g_i = 0 \bmod 6 \;\; \forall\, g \in \Gamma\,. \tag{3.53}$$

The action of some $g \in \Gamma$ on ω_β is

$$\omega_\beta \to \omega_\beta \zeta^{\sum_i g_i(\beta_i+1)} \tag{3.54}$$

so that the invariant subspace in the middle cohomology is spanned by H^2 together with those residues for which

$$\sum_i g_i(\beta_i + 1) = 0 \bmod 6 \;\; \forall\, g \in \Gamma \tag{3.55}$$

3.2.10 Quadratic form

From the vanishing cycle condition, we can build a quadratic form as follows:

- Build the lattice of ν_β.
- Write a general ansatz for G with $\nu_\beta = [n_0, n_1, \ldots]$ a general lattice point.
- Compute $Q(G)$ formally with the above ansatz. This will result in an homogeneous polynomial of degree 2 in the n_i.
- Put this polynomial in matrix form and get Q. Only keep the odd integers represented by Q, corresponding to points in Λ_{phys}. Alternatively, build a parametrization with respect to the linear cycle condition and keep only points which obey this parametrization.

In this form, this will allow us to perform some number theory and in particular apply the Fincke-Pohst algorithm for enumeration of all vectors below a certain length. Then we are simply left to check the rank of ρ corresponding to each of those vectors.

Crucially, note that the n_i corresponds to different rescaling of a single component of a pair of complex conjugate residues, it does not correspond to residue forms themselves. Hence, while it makes computations easier, it also makes the geometric intuition a bit hidden.

3.3 Examples

3.3.1 Symmetric under $(\mathbb{Z}/6\mathbb{Z})^4$

The group $(\mathbb{Z}/6\mathbb{Z})^4$ acts on residues as follows :

$$\begin{array}{cccccc} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array} . \tag{3.56}$$

For a form ω_β to be invariant we need that $\beta_i = \beta_j$ for all i, j. The only invariant residues are hence

$$\begin{array}{c|c|c} & \beta & |\beta| \\ \hline \omega_{(0^6)} & (0,0,0,0,0,0) & 0 \\ \omega_{(1^6)} & (1,1,1,1,1,1) & 1 \\ \omega_{(2^6)} & (2,2,2,2,2,2) & 2 \\ \omega_{(3^6)} & (3,3,3,3,3,3) & 3 \\ \omega_{(4^6)} & (4,4,4,4,4,4) & 4 \end{array} \tag{3.57}$$

In particular, there is now only a single term in $G\ (\omega_{(2^6)})$ that is non-zero and the matrix ρ is one-dimensional.

Since $\omega_{(2^6)}$ is the only real residue, which we known from the way complex conjugation acts, we can impose that $G \cdot \delta_{\beta'} \in \mathbb{Z}$, resulting in $\bar{\nu}_{(2^6)} \in A_1^*$ which implies that $\nu_{(2^6)} = \frac{n}{2}$ for $n \in \mathbb{Z}$ so that

$$G = n\frac{\omega_{(2^6)}}{z_u z_{(2^6)}} = n\frac{2 \cdot 3^5\, i}{\pi^2}\ \omega_{(2^6)} \tag{3.58}$$

Next we impose the linear cycle condition :

$$G \cdot C_\ell \in \mathbb{Z} + \frac{1}{2}\,, \tag{3.59}$$

which for $C_{0,0,0}$ reads

$$\int_{C_{id}^{0,0,0}} G = n \frac{2\,\cdot\,3^5\,i}{\pi^2}\;(2\pi i)^2 \frac{1}{2\cdot 6^3} \zeta_{12}^{\sum_{e=1}^{3}(1+2)1} = -n\frac{3^2}{2} \in \mathbb{Z} + \frac{1}{2} \tag{3.60}$$

so that n must be odd, $n = 2m+1$ for $m \in \mathbb{Z}$.

We hence find that Λ_{phys} is described as

$$\Lambda_{phys} = \left\{ G_m = (2m+1)\frac{2\,\cdot\,3^5\,i}{\pi^2}\;\omega_{(2^6)} | m \in \mathbb{Z} \right\}\,. \tag{3.61}$$

The shortest choices of G are $m = 0$ and $m = -1$ for which

$$G_0^2 = G_{-1}^2 = \frac{3^5}{2}\,. \tag{3.62}$$

Furthermore, the matrix ρ is just a number in this case and it is non-zero, i.e. has full rank, whenever $G \neq 0$, which is true for any m.

Note that

$$\frac{\chi(X)}{24} - \frac{1}{2}G_0^2 = 48 > 0\,, \tag{3.63}$$

so that this flux is a perfectly viable solution.

However, choosing the next to shortest flux for $m = 1$ results in

$$\frac{\chi(X)}{24} - \frac{1}{2}G_1^2 = -438 < 0, \tag{3.64}$$

and does hence not give a consistent solution.

3.3.2 Symmetric under $(\mathbb{Z}/6\mathbb{Z})^3 \times (\mathbb{Z}/3\mathbb{Z})$

Here, the generators of $(\mathbb{Z}/6\mathbb{Z})^3 \times (\mathbb{Z}/3\mathbb{Z})$ are acting on residues by :

$$\begin{array}{cccccc} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{array} \tag{3.65}$$

so that the relevant invariant residue forms are

$$\begin{array}{c|cc} & \beta & |\beta| \\ \hline \omega_{(1^6)} & (1,1,1,1,1,1) & 1 \\ \omega_{(0^4 3^2)} & (0,0,0,0,3,3) & 1 \\ \omega_{(2^6)} & (2,2,2,2,2,2) & 2 \\ \omega_{(1^4 4^2)} & (1,1,1,1,4,4) & 2 \\ \omega_{(3^4 0^2)} & (3,3,3,3,0,0) & 2 \end{array} \tag{3.66}$$

In this case we have

$$\mathcal{Z}_{6^3 3} = Span_{\mathbb{Z}} \left\{ Z((1^4 4^2), \beta') \oplus Z((2^6), \beta') \right\} \tag{3.67}$$

By working these out for all β', one finds that the generators of $\mathcal{Z}_{6^3 3}$ are $(1,0)$, $(\zeta_6^2, 0)$, $(0,1)$, so that $\mathcal{Z}_{6^3 3} = A_2 \oplus A_1$ and $\mathcal{Z}^*_{6^3 3} = A_2^* \oplus A_1^*$.

Hence any $\mathbb{Z}$-linear combination of those generators will give an integral result with respect to vanishing cycles. We furthermore note that ζ_6^2 is the shortest vector in the A_2^* lattice.*

We now impose the linear cycle condition using $C_{0,0,0}$. As $C_{0,0,0} \cdot \omega_{(1^4 4^2)} = 0$, as is expected from the fact that it is not 3-decomposable, this only constrains the A_1^* summand such that :

$$\int_{C_{id}^{0,0,0}} G = \frac{2\nu_{(2^6)}}{z_u z_{(2^6)}} \int_{C_{id}^{0,0,0}} \omega_{(2^6)} = -3^2 \nu_{(2^6)} \in \mathbb{Z} + \frac{1}{2}. \tag{3.68}$$

As for the first example, we hence find

$$\nu_{(2^6)} = (2n_1 + 1)/2, \tag{3.69}$$

together with $\bar{\nu}_{(1^4 4^2)} \in A_2^*$, i.e

$$\bar{\nu}_{(1^4 4^2)} = n_2 + n_3 \frac{\zeta_{12}^{-1}}{\sqrt{3}}\,. \tag{3.70}$$

The matrix $\widetilde{\rho_{IJ6^33}}$ is:

$$\widetilde{\rho_{IJ6^33}} = \begin{pmatrix} \alpha_{(2^6)} & \alpha_{(3^4 0^2)} \\ \alpha_{(3^4 0^2)} & 0 \end{pmatrix} \tag{3.71}$$

The shortest $G \in \Lambda_{phys}$ for which ρ has rank one is found for $n_1 = n_2 = n_3 = 0$ in which case we have

$$\frac{\chi(X)}{24} - \frac{1}{2} G_0^2 = 48 > 0\,, \tag{3.72}$$

However, this vector does not correspond to a matrix ρ of full rank since only $\alpha_{(2^6)}$ is non-zero in this case.

The shortest G with ρ of full rank is found for $n_1 = n_2 = 0$ and $n_3 = 1$, so that

$$\frac{\chi(X)}{24} - \frac{1}{2} G_0^2 = \frac{2610}{24} - \frac{1}{2}\left(\frac{3^5}{2} + 3^2 2^6\right) = -240 < 0, \tag{3.73}$$

where we have used $S_{(1^4 4^2)} = \frac{3^2}{2^6}$.

Hence, there are no solutions with ρ of maximal rank within the tadpole bound here.

3.3.3 Symmetric under $(\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$

Here the action of the group $(\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ is given by

$$\begin{array}{cccccc} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{array} \tag{3.74}$$

and the invariant residue forms are:

$$\begin{array}{ccc} & \beta & |\beta| \\ \hline \omega_{(0^4 3^2)} & (0,0,0,0,3,3) & 1 \\ \omega_{(0^3 2^3)} & (0,0,0,2,2,2) & 1 \\ \omega_{(0^3 4 1^2)} & (0,0,0,4,1,1) & 1 \\ \omega_{(1^6)} & (1,1,1,1,1,1) & 1 \\ \omega_{(1^3 3 0^2)} & (1,1,1,3,0,0) & 1 \\ \omega_{(2^3 0^3)} & (2,2,2,0,0,0) & 1 \\ \omega_{(0^3 4^3)} & (0,0,0,4,4,4) & 2 \\ \omega_{(1^4 4^2)} & (1,1,1,1,4,4) & 2 \\ \omega_{(1^3 3^3)} & (1,1,1,3,3,3) & 2 \\ \omega_{(2^3 0 3^2)} & (2,2,2,0,3,3) & 2 \\ \omega_{(2^6)} & (2,2,2,2,2,2) & 2 \\ \omega_{(2^3 4 1^2)} & (2,2,2,4,1,1) & 2 \\ \omega_{(3^3 1^3)} & (3,3,3,1,1,1) & 2 \\ \omega_{(3^4 0^2)} & (3,3,3,3,0,0) & 2 \\ \omega_{(4^3 0^3)} & (4,4,4,0,0,0) & 2 \end{array} \tag{3.75}$$

The fundamental difference compared to previous examples is that the lattice $\mathcal{Z}$ is not a direct sum of A_2 and A_1 and the inclusion is strict, meaning :

$$\mathcal{Z}_{6^2 23} = Span_{\mathbb{Z}} \left\{ Z(\omega_{(0^3 4^3)}, \beta'), Z(\omega_{(1^4 4^2)}, \beta'), Z(\omega_{(1^3 3^3)}, \beta'), Z(\omega_{(2^3 0 3^2)}, \beta'), Z(\omega_{(2^6)}, \beta') \right\}$$

is not equal to $A_1 \oplus A_2^4$.

A basis of the dual lattice $\mathcal{Z}^*_{6^2 23}$ is given in matrix form by :

$$P_{6^2 23} = \begin{pmatrix} \frac{i}{6\sqrt{3}} & -\frac{1}{4}+\frac{i}{12\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}+\frac{i}{2\sqrt{3}} & -\frac{3}{2}+\frac{i}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4}+\frac{i}{4\sqrt{3}} & -1+\frac{i}{2\sqrt{3}} & 0 & 0 & \frac{e^{i\pi\frac{5}{6}}}{\sqrt{3}} & -\frac{3}{2}+\frac{i}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^{i\pi\frac{5}{6}}}{\sqrt{3}} & -\frac{3}{2}+\frac{i}{2\sqrt{3}} & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and we can write a general element of $\mathcal{Z}^*_{6^2 23}$ as

$$\left(\nu_{(0^3 4^3)}, \nu_{(1^4 4^2)}, \nu_{(1^3 3^3)}, \nu_{(2^3 0 3^2)}, \nu_{(2^6)}\right) = P_{6^2 23} \cdot \mu, \tag{3.76}$$

with $\mu \in \mathbb{Z}^9$.

The Gram matrix of the lattice $\mathcal{Z}^*_{6^2 23}$ is given by

$$G_{6^2 23} = \begin{pmatrix} 987/2 & 555 & 0 & 0 & 240 & 672 & 0 & 0 & 81/2 \\ 555 & 1686 & 0 & 0 & 336 & 912 & 0 & 0 & 81 \\ 0 & 0 & 576 & 1440 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1440 & 4032 & 0 & 0 & 0 & 0 & 0 \\ 240 & 336 & 0 & 0 & 192 & 480 & 0 & 0 & 0 \\ 672 & 912 & 0 & 0 & 480 & 1344 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 216 & 540 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 540 & 1512 & 0 \\ 81/2 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 243/2 \end{pmatrix} \tag{3.77}$$

This is not an integral matrix as $\mathbb{Z}^*$ is not an integral lattice.

Finally, we have to impose G to lie in Λ_{phys}, we need to take care of the intersection of G with respect to a linear cycle. An appropriate choice here is again $C^{0,0,0}_{id}$, as before. While we have in total three 3-decomposable residues, namely $\omega_{(0^3 4^3)}, \omega_{(1^3 3^3)}, \omega_{(2^6)}$, we have fixed the permutation to be the identity, and thus only $\omega_{(2^6)}$ has a non-zero period with respect to this linear cycle since it is the only cycle which is 3-decomposable with respect to the identity permutation in this list.

Thus, to impose $G \in \Lambda_{phys}$, we are reduced to make sure that the period of the rescaled $\tilde{\omega_{(2^6)}}$ is half integral with respect to $C_{id}^{0,0,0}$. For any given μ, this condition becomes :

$$\frac{2 \cdot (\frac{n_1}{6} + \frac{n_2}{3} + \frac{n_9}{2})}{z_u z_{(2^6)}} \int_{C_{id}^{0,0,0}} \omega_{(2^6)} = -3^2 \left(\frac{n_1}{6} + \frac{n_2}{3} + \frac{n_9}{2} \right) \in \mathbb{Z} + \frac{1}{2}. \tag{3.78}$$

Note that n_2 has no impact on this condition, so we can freely choose it. The resulting constraint on n_1 and n_9 implies that $n_1 + 3n_9$ is an odd integer, so that we can write

$$n_1 = 2k + 1 - 3n_9, \tag{3.79}$$

for $k \in \mathbb{Z}$.

We are now ready to find all flux solutions for this model by generating all vectors in $\mathcal{Z}^*_{6^2 23}$ up to some given length by computer using the Fincke-Pohst algorithm, and then checking for each one if it is contained in Λ_{phys}. All lengths below 500 appearing in Λ_{phys} and the associated numbers of solutions are

Length	**Number of solutions**
243/2	2
411/2	4
603/2	4
627/2	12
675/2	12
843/2	24
987/2	8

Table 3.1: Lengths in Λ_{phys} below 500.

Rank	Minimum
2	243/2
4	411/2
6	843/2

Table 3.2: Minimum for each rank

We can now work out

$$\widetilde{\rho_{IJ6^223}} = \begin{pmatrix} 0 & 0 & 0 & \alpha_{(3^40^2)} & 0 & \alpha_{(2^341^2)} \\ 0 & \alpha_{(4^30^3)} & \alpha_{(2^303^2)} & \alpha_{(3^31^3)} & 0 & \alpha_{(2^6)} \\ 0 & \alpha_{(2^303^2)} & 0 & 0 & 0 & 0 \\ \alpha_{(3^40^2)} & \alpha_{(3^31^3)} & 0 & 0 & \alpha_{(2^303^2)} & \alpha_{(1^33^3)} \\ 0 & 0 & 0 & \alpha_{(2^303^2)} & 0 & \alpha_{(1^44^2)} \\ \alpha_{(2^341^2)} & \alpha_{(2^6)} & 0 & \alpha_{(1^33^3)} & \alpha_{(1^44^2)} & \alpha_{(0^34^3)} \end{pmatrix} \tag{3.80}$$

for any of these solutions, and examine the relationship between its rank and G^2.

A first observation is that generically the same length can be associated to different ranks. Of course this can happen trivially if we rescale G, but also happens in a different manner here. For example, the length $G \cdot G = \frac{4083}{2}$ [2] can correspond to $\widetilde{\rho_{IJ6^223}}$ having rank 4 or rank 6, via for example the following solutions :

$$\mu = \begin{pmatrix} -1, & -1, & -1, & 0, & 0, & 1, & 0, & 0, & 0 \end{pmatrix}$$
$$\mu = \begin{pmatrix} 0, & -1, & 0, & 0, & -1, & 1, & -1, & 1, & -1 \end{pmatrix}$$
$$.$$

We have performed a scan over all lengths up to 1500 and computed the associated rank of ρ for all these solutions. This allows us to find the minimal length of G for each rank of ρ. The result is shown in Table 3.2 and Figure 3.1.

This is similar to the previous example, where the solutions found were only below the tadpole if the rank of the matrix was not full. The plot in Figure 3.1 shows the

[2] We obtained this point by checking for all points with coordinates $-1, 0, 1$ to check results of the Fincke-Pohst algorithm

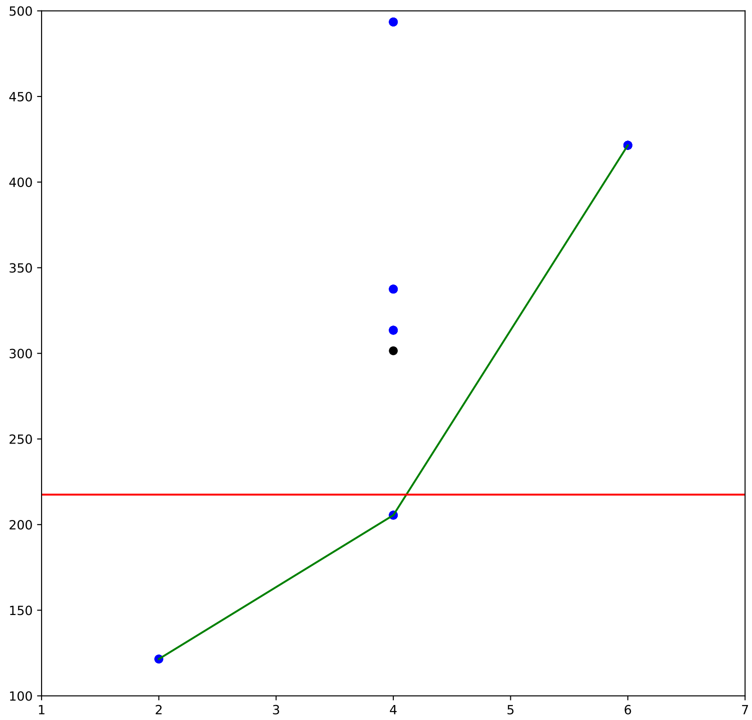

Figure 3.1: A plot of the minimal lengths found for each rank of ρ. The horizontal axis shows the rank of ρ and the vertical axis the tadpole contribution of the solutions. The red horizontal line shows the tadpole bound.

minimum lengths associated to every rank, as well as some further lattice points corresponding to rank four with non-minimal length. Interpolating the growth of G^2 with the rank of ρ shows that the tadpole bound is crossed well before a maximal rank of ρ is reached.

All of the solutions shown are quite simple in that at least one solution has $\mu_i \in \{-1, 0, 1\} \forall i$, except for length $\frac{603}{2}$. This is quite remarkable and indicates that constructing a basis of integral Hodge cycles using residues appears to be very efficient, at least when choosing appropriate linear combinations such as the ones in (3.76).

3.3.4 Quotients by $(\mathbb{Z}/6\mathbb{Z})^4$

The approach we have taken in this section naturally lends itself to study flux solutions on quotients of the Fermat sextic by the groups of symmetries considered. This appears to be a promising avenue to generate general Hodge cycles within the tadpole bound. On the one hand, the tadpole contribution should be significantly smaller, as we expect the self-intersection number of a symmetric flux to be divided by the order of the group for the quotient. On the other hand, the tadpole contribution of the geometry should be equal to the Euler characteristic of a crepant resolution of the quotient, which is typically of a similar magnitude than the original fourfold.

Let us exemplify this for the simple case of the Fermat sextic and $\Gamma_{6^4} = (\mathbb{Z}/6\mathbb{Z})^4$, where we can give a description using toric geometry. We first work out the Euler characteristic of a resolution. The family of sextic Calabi-Yau fourfolds is described as toric hypersurfaces by a pair of reflexive polytopes Δ, Δ^* with vertices

$$\Delta^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 5 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \end{pmatrix}. \tag{3.81}$$

Here Δ^* is the N-lattice polytope and Δ is the M-lattice polytope of the sextic fourfold X, and the mirror $X^\vee$ is found by reversing the roles of the two polytopes. Crucially, $X^\vee$ can also be found along the lines of [43] by taking (as resolution of) the quotient of X/Γ_{6^4}, and this is reflected in the face fan of Δ giving rise to the toric variety $\mathbb{P}^5/\Gamma_{6^4}$. It can be shown that Δ admits a fine and regular triangulation resulting in a projective crepant resolution $\widetilde{X_{\Gamma_{6^4}}}$ of X/Γ_{6^4} with

$$h^{1,1}(\widetilde{X_{\Gamma_{6^4}}}) = 426 \qquad h^{3,1}(\widetilde{X_{\Gamma_{6^4}}}) = 1 \qquad h^{2,1}(\widetilde{X_{\Gamma_{6^4}}}) = 0 \tag{3.82}$$

so that $\chi(\widetilde{X_{\Gamma_{6^4}}}) = 2610 = \chi(X)$ as expected for a mirror pair of Calabi-Yau fourfolds.

We now work out the fate of the tadpole contribution of the flux. As $\omega_{(2^6)}$ is invariant under Γ_{6^4} we will use the same notation to denote the image of this residue on the quotient. Following [44], we have that

$$\frac{\int_X \omega_{(2^6)} \wedge \omega_{(2^6)}}{\int_{X/\Gamma_{6^4}} \omega_{(2^6)} \wedge \omega_{(2^6)}} = \frac{\mathrm{Vol}(\Delta)}{\mathrm{Vol}(\Delta^*)} = 6^4 \,. \tag{3.83}$$

where Vol() is the lattice volume of the respective polytopes. The ratio here follows from the simple fact that the vertices of Δ^* span N, whereas the vertices of Δ span $N' \subset N$ with $N/N' = \Gamma_{6^4}$. The result above fits with the naive expectation that integrating an invariant form over a quotient is equal to the integral over the covering space divided by the order of the group.

Similar results can be obtained for other groups Γ as well. Here, the N-lattice polytope describing the quotient is given by a polytope which is a simplex with vertices v_i satisfying $\sum v_i = 0$ such that $N/N' = \Gamma$, where N' is again the sublattice of the N lattice spanned by the v_i. It hence follows from the same argument as above that the tadpole contribution of the flux is reduced by $|\Gamma|$.

Given a flux symmetric under a finite group of symmetries, taking the quotient hence leads to a significant reduction of the tadpole contribution of the flux. This comes with another feature, however: the fourfolds X/Γ are singular and the flux we have constructed is in general only defined on the singular fourfold X/Γ, i.e. these fluxes do not exist as properly quantized fluxes on a resolution of X/Γ. This is already indicated by the tadpole contribution of the flux being fractional in a way that does not originate from $c_2/2$.

3.3.5 Arithmetic obstructions - $(\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$

Given a point in the complex structure moduli space of a Calabi-Yau fourfold X, the intersection product of primitive Hodge cycles naturally corresponds to an integral quadratic form Q. The set of Hodge cycles below the tadpole bound is then

$$S(Q,T) := \{\mu \,|\, \exists\, k \in \mathbb{Z},\ Q(\mu) = k \leq T\},$$

with T the associated tadpole bound. While this set finite, performing an enumeration is computationally expensive and conceptually unsatisfactory, and we wish to find a necessary conditions for $S(Q,T)$ to be non-empty.

Recall that an integer m is called representable by Q if there exists integers $\mu \in \mathbb{Z}^n$ s.t. $Q(\mu) = m$. For an integer m to be representable, we have to have corresponding representations of the p-adic reductions m_p by G_p and μ_p for every prime p:

$$\mu \in \mathbb{Z}^n,\ m \in \mathbb{Z},\ G(\mu) = m \implies \exists\, \mu_p \in \mathbb{Z}_p^n \,|\, G_p(\mu_p) = m_p \in \mathbb{Z}_p. \tag{3.84}$$

Let us exemplify this point of view for the problem treated in this work. Let us study the existence of solutions for the gram matrix Q of residues symmetric under $(\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, (3.77), i.e. the set $S(Q, 1500)$. A direct observation from the results shown in Table 3.1 is that there are only 7 lengths in Λ_{phys} below 500, already well above the tadpole bound, indicating that the majority of integers below the tadpole bound cannot be represented.

In Table A.1, we have generated a list of all integers up to 1500 representable by Q which also includes their multiplicity. Note that this does not yet impose the physical quantization condition related to $c_2(X)/2$.

As we can see right away, there are very few lengths represented by this quadratic form to begin with. In fact, there are only 108 lengths represented up to 1500. Furthermore, as the length increases, so does the number of solutions.

Since it is valued in $\mathbb{Z} + \frac{1}{2}$ in general, we will multiply everything by a factor of 2. This allows us to recover the case of an integral quadratic form, and if we restrict ourselves to solutions that are odd, we recover Λ_{phys} for Q. This results in the following matrix :

$$\tilde{Q} = \begin{pmatrix} 987 & 1110 & 0 & 0 & 480 & 1344 & 0 & 0 & 81 \\ 1110 & 3372 & 0 & 0 & 672 & 1824 & 0 & 0 & 162 \\ 0 & 0 & 1152 & 2880 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2880 & 8064 & 0 & 0 & 0 & 0 & 0 \\ 480 & 672 & 0 & 0 & 384 & 960 & 0 & 0 & 0 \\ 1344 & 1824 & 0 & 0 & 960 & 2688 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 432 & 1080 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1080 & 3024 & 0 \\ 81 & 162 & 0 & 0 & 0 & 0 & 0 & 0 & 243 \end{pmatrix}$$

As an example, let us study whether or not 433 is represented by this matrix. Note that 433 is odd and thus in principle can lie in Λ_{phys}. We know that if there is a solution $\mu \in \mathbb{Z}$, then there must be solutions in the p-adics $\mu_p \in \mathbb{Z}_p$ for every prime p. So we can first perform a reduction mod 3 of $\tilde{Q}$, which is identically 0. Since $433 = 1 \pmod 3$, we have an obstruction : there is no non-trivial solution in the 3-adics.

Let us also work out the obstruction modulo 2 to see that it is indeed prime powers that matter, and not just the primes. For $n = 1$, the quadratic form $\tilde{Q}_2$ is:

$$\tilde{Q}_2 = n_1^2 + n_9^2. \tag{3.85}$$

Since $433 = 1 \pmod 2$ there are no obstructions. However, for $n = 2$, the quadratic form $\tilde{Q}_{2^2}$ is :

$$\tilde{Q}_{2^2} = 3n_1^2 + 2n_1 \cdot n_9 + 3n_9^2. \tag{3.86}$$

Since $433 = 1 \pmod 4$ there are in fact obstructions modulo 4, as $\tilde{Q}_{2^2}$ is always 0 or 3 modulo 4.

While the existence of obstructions to solutions holds for quadratic forms in general, here it is quite important to note the algebro-geometric origin of the quadratic forms we are considering. From the geometric and physical context, one can hope to a priori determine the obstructions, which leaves to determine the representation of integers for which there are no obstructions, thus severely constraining the set of physical solutions.

In fact one explanation possible for the fact that obstructions stems from the 2-adics and the 3-adics it that the Fermat sextic fourfold is known to be modular. Hence we expect the primes of bad reduction, the one that may induce obstructions, to be 2 and 3 as the Jacobian has a global factor of $6 = 2 \cdot 3$. Note that a priori non-trivial, since many coefficients of the quadratic form have prime factors other than 2 and 3, for example 3372 has prime factorization $2^2 \cdot 3 \cdot 281$.

It would be interesting to combine the observation about quotients made above with the number theoretic approach outlined here. Dividing a given quadratic form by prime powers can be described in this language as removing obstructions to the representation of integers by this quadratic form. It would be very interesting to systematically investigate this further.

CHAPTER 4

Conclusions and future research directions

4.1 Observations using Fermat's sextic

We introduced in chapter 2 various tools to be able to perform computations. Those were :

- Vanishing cycles
- Residues
- Algebraic cycles
- Quadratic forms

The first 3 lead to the first approach, while all four of them were combined to confirm the results.

The point-of-view adopted was to focus on forms that were below the tadpole bound, meaning those that lie in the set $S(Q, T)$, and then check if they were also part of the set $S(\rho, X)$.

Regarding the members of those sets we learned, in the cases studied for Fermat's sextic, that the set $S(Q, T)$ was mostly empty to begin with. This was explained using p-adic numbers for the quadratic form Q and the fundamental theorem of arithmetic. We expect the obstruction to come from the modularity of the sextic. In the general case, we expect modularity to play a role in the tadpole problem, in that modular varieties can offer models that can be studied analytically.

We have also seen some caveats to the fact that it was mostly empty : the number of variables of the quadratic forms obtained were not in general very high. With a higher number of variables, you expect more numbers to be represented, even if those numbers are, at the end of the day, points on surfaces[1]. We also have seen that the point-of-view of arithmetic allows us to apply algorithms that are much faster than naive enumeration, and it looks as a promising candidate for a unifying point-of-view on the problem.

Naturally, those observations were put in contrast with elements of the set $S(\rho, X)$. It turned out that for the most part, elements of $S(Q, T)$ did not belong to $S(\rho, X)$. In the most general case, we have seen the tension between the Hodge condition and the length condition exemplified by a curve : the minimal length for a general Hodge cycle of type $(2, 2)$ was above the tadpole bound by a far margin.

However, we have also seen a possible solution to the tadpole problem thanks to this. One of the working assumptions in the tadpole conjecture is to work with smooth fourfolds. However, when taking orbifold quotients, we did find a solution which coincidentally corresponds to the mirror sextic. This is natural from the point of view of toric geometry, as we expect the periods computed to depend on the order of the group of symmetry we quotient by. Then, quotienting just corresponds to removing flat directions and hence the odds of being general Hodge increase.

Finally, we have seen a major problem that is somewhat unavoidable with the tools introduced, which is that of computational complexity. Even if our basis seemed quite good, the Gram matrix having many off diagonal zeroes, and the algorithm used were fast considering that we were working with a case were said Gram matrix was positive definite, the computational complexity is still way too

[1]Note that in the two last cases studied for Fermat's sextic fourfold, the number of integers represented below the tadpole bound did not change despite the increase in the number of variables

high to be able to perform any meaningful computations in general. Indeed, for a complete intersection Calabi-Yay fourfold, the Hodge number $h^{2,2}$ is typically of the order of 100. Smaller than the sextic, but still out of reach as of today. This is one more argument that goes in the way of looking at this problem from the point-of-view of the p-adics.

Possible extension of this work involve studying the arithmetic properties and the interplay between the Hodge and length condition on other examples. In particular it would be interesting to study the link with mirror symmetry by taking quotients, as well as study the various arithmetic conjectures (Hasse principle, Tate conjecture, and importantly modularity) in this setting and see how big the overlap with the tadpole problem is. We hereby review those possible research directions.

4.2 A new perspective on $K3 \times K3$

4.2.1 Quotients of $K3 \times K3$

As restricting ourselves to symmetric forms on the sextic, it is quite natural to do the same analysis for product of $K3$ surfaces.

As a first remainder, let us quickly remember that thanks to Torelli's theorem for K3 surfaces, we have a correspondence between the classification problem for automorphism groups and lattices of $K3$ surfaces, due to Nikulin [45].

There we need to distinguish between symplectic group actions G_s and non-symplectic group actions G_n. In terms of taking quotients of $K3$ surfaces it is quite important since for a group G acting on K3 we have the following short exact sequence :

$$1 \to G_s \to G \to G_n \to 1 \tag{4.1}$$

Of course in this context, this is quite good, notably because the quotient of a $K3$ surface by a symplectic group action leads to ADE singularities.

However, this also tells us that we need to pay some attention when taking quotients. In fact, our desire is that after quotienting, we still get some (smooth and rational) $K3$ surface.

A first result by Nikulin [46], again, gives an answer in the form of the following result : for a $K3$ surface with a non-symplectic involution, the result depends on the fixed points of the involution. If there is no fixed point, the involution will be an Enriques surface, otherwise (fixed loci) it will be a $K3$ surface.

As for the non-symplectic part of some group action, [47] gives a proof that if the non-symplectic part G_n has order at least 3, then the quotient is again rational.

A further desirable property would be to have the same effect as taking orbifold quotients, meaning we essentially restrict ourselves to some symmetric forms.

An example of this would be in the case of the sextic and its mirror : taking the quotient has just reduced the dimension of $H^{2,2}$, so that the associated quadratic form is lower dimensional, but has not changed the actual value of the coefficients.

Noting furthermore that the intersection matrix for $K3$ surfaces is well known, and we can restrict ourselves to the case where the only obstruction comes from the fact that this is an even lattice and hence no odd integers can be represented, this offers a nice toy model.

So we understand that the main obstruction to finding suitable fluxes in the case of $K3 \times K3$ comes from the Hodge condition rather than the representation of integers, which makes it a good candidate to study the behaviour of this Hodge condition.

So we get some idea, using those results and the observations made in the case of the sextic, on how to build some examples of fluxes.

The question is the following :

let us fix some integer n below the tadpole bound for $K3 \times K3$. There is a finite amount of solutions to the representation of n by the intersection matrix in middle cohomology, due to it being an integral quadratic form. The task is to classify those solutions.

Pick the solutions with the most non-zero coefficients. As we noted, we want to "get rid" of the directions corresponding to zero coefficients in the solutions by quotienting. Can we build, using the work of Nikulin and Xiao, a quotient of $K3 \times K3$ corresponding to this solution ?

If the answer is yes, then we have reduced the problem. Then we are left to check the behaviour of the Hodge condition, and we have a good candidate for doing so, since we have many non-zero coefficients for our flux.

Perhaps there is still no flux that is Hodge, even if we have guaranteed that it is below the tadpole bound. However, this set up of $K3 \times K3$ is still interesting because it allows precisely to know the behaviour of the Hodge condition, which is very difficult to handle in general even without adding the requirement on the length of the flux.

Another incentive is of course that the results by Nikulin and Xiao allow the problem of the Hodge condition in this set up to be readily translated in terms of group theory (in particular, automorphism groups of $K3$). Not only is this more approachable computationally, but one can hope to extend this approach to other fourfolds if it proves fruitful in either providing some criterias to study the Hodge condition, or obstructions to this condition in case of a negative result.

To examplify this discussion, we notice we could have done the same analysis for the Fermat's sextic fourfold, for which we would have gotten the correspondance between the rank of the group we quotient by, the rank of ρ and the relative number of moduli stabilized by a flux below the tadpole bound, respectively as:

order of the group	rank of ρ	relative percentage of moduli stabilized
6^4	1	1
$6^3 \cdot 3$	1	$\frac{1}{2}$
$6^2 \cdot 3 \cdot 2$	4	$\frac{2}{3}$

At fixed permutation, there are only finitely many admissible groups, represented in table C and it would certainly be interesting to confirm that the mirror sextic is indeed the only possible solution. Note that the above table 4.2.1 is dubious in that each entry represents the maximal rank of ρ below the tadpole bound, but it is not at fixed length. Hence the interest to first carry over to $K3 \times K3$.

Note that similar results were already performed in [48] and [49] for the special case of $\mathbb{Z}_2$ orbifolds. In particular, the classification of gauge groups obtained was made. It would be particularly nice to put emphasis on the Hodge condition with respect to the length by building on those results.

One possible argument to motivate the study of mirror symmetry with respect to the tadpole conjecture is the following.

We have seen that the self-intersection depends on the degree in the case of the sextic, because it is Calabi-Yau, up to some factor. This factor seems to typically depend on the order of the maximal subgroup of symmetry under which the residue form is invariant, eg $(\mathbb{Z}/6\mathbb{Z})^4$ for the residue form with $\beta = (2^6)$.

In the case of hypersurfaces, mirror symmetry is quite powerful since swapping $h^{3,1}$ and $h^{1,1}$ results in a lower dimension mirror $h^{\tilde{3},1} = h^{1,1}$ thus reducing the requirement to find the flux to be general Hodge.

Then typically, since the mirror corresponds to such maximal subgroups, we expect that mirror symmetry offer solutions to the tadpole problem, at the cost of smoothness.

4.2.2 p-adics, K3, and the sextic

We have seen that there are typically obstructions to the representation of integers by quadratic forms. We have however to remember the origin of these integers : they count integral points on the intersection of surfaces on a Calabi-Yau fourfold.

This is a problem when considering the sextic. As we have seen before, there seems to be obstructions to the representation of integers in $H^{2,2}_{prim}$ on the Fermat's sextic fourfold. Thus, the analysis for this particular fourfold is quite difficult since we also have to worry about the Hodge type of a given cycle.

This is however not the case for the $K3$ lattice, which is $K3 = -E8 \oplus -E8 \oplus U \oplus U \oplus U$ with $E8$ the $E8$ root lattice and U defined as :

$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.2}$$

Notably, we know that $E8$ represents all even integers that are quite small, especially under 24. So far we have hoped that this would let us get rid of the criteria on the representation of integers to focus on the Hodge type criteria. However, we expose here a caveat to this assumption.

We study this via the Hasse principle : if we have a solution in $\mathbb{Z}_p$ for all primes p, then we should have a solution in $\mathbb{Z}$. This is not true in general, and we propose here to contrast the previous discussion about quotients with a very interesting example following Hasse's principle.

Remember that for a product of $K3$ surfaces X, the middle cohomology $H^4(X,\mathbb{Z})$ is determined (possibly up to torsion) by the cohomology of the K3 surfaces via Künneth formula :

$$H^n(X,\mathbb{Z}) \simeq \sum_{i+j=n} H^i(K3,\mathbb{Z}) \otimes H^j(\tilde{K3},\mathbb{Z}). \tag{4.3}$$

Let us fix for the sake of this argument the case $i = j = 2$. This will result in the flux being in both K3 surfaces.

We have seen earlier that generically the representation of integers by the middle cohomology of a $K3$ surface may not be an issue. However, there is a very interesting counter-example to this intuitive statement from [50], coming from the second étale cohomology group, which is known as the Brauer-Manin obstruction. Note that the Brauer-Manin obstruction is not a priori the only obstruction that can appear, however for smooth projective rationally connected varieties it is conjectured to be so [51].

In this paper, the example of a family of sextic $K3$ surfaces in $\mathbb{P}^3_{[1,1,1,3]}$ for which there are points in the p-adics $\mathbb{Q}_p$ for every prime p but no rational points are exhibited.

There are two comments regarding this result. The first one is that étale cohomology plays an important role there, and it does so when computing l-adic cohomology as well, especially when considering modularity as in [52]. This reinforce the importance of p-adic considerations when studying the tadpole problem.

Note that modularity is important when studying representation of integers because it allows one to make global estimation out of local considerations. For example, in the case of elliptic curves, it is easy to determine which primes will pose an obstruction, and those are exactly the primes of bad reduction for modular elliptic curves.

Perhaps less obviously, this also ties up into the possibility that quotienting offers a solution to the tadpole problem since quotienting might remove the points below the tadpole bound. We do note however that in general we are interested in points on the quadratic form defined by the intersection pairing on $H^{2,2}_{prim}$, which is different from simply the obstruction of rational points on a variety a priori. Thus we would need in principle that the Hasse principle applies in cohomology rather than on the variety.

Thus while we have seen the relative importance of the Hodge conjecture, we see here exemplified the importance of the other conjectures mentioned in chapter 2, namely the Tate conjecture and the Hasse principle. $K3 \times K3$ offers some nice toy models to explore those possibilities. However, we have to highlight the difficulties that may arise when taking proper fourfolds, as is the case for the sextic.

With that in mind, we make the following statement, up to the sextic obeying some modularity theorem : the only p-adics of obstruction for the representation of integers in the sextic fourfold are Z_2 and Z_3. Assuming some form of modularity theorem for the sextic, we can simply use Hensel's lemma to find non-obstructions.

We have seen that the Jacobian of the sextic X is :

$$J(X) = (6x_0^5, 6x_1^5, 6x_2^5, 6x_3^5, 6x_4^5, 6x_5^5). \tag{4.4}$$

$$\tag{4.5}$$

For every $p \neq 2, 3$ we can find solutions a such that $J(a) \neq 0$ and $X(a) = 0$. Indeed, for every prime p not equal to 2,3 we have to find $k := (k_0, k_1, k_2, k_3, k_4, k_5) \neq (0, 0, 0, 0, 0, 0)$ and $a := (a_0, a_1, a_2, a_3, a_4, a_5)$:

$$a_0^6 + a_1^6 + a_2^6 + a_3^6 + a_4^6 + a_5^6 = 0 \mod (p) \tag{4.6}$$

$$\iff -(a_1^6 + a_2^6 + a_3^6 + a_4^6 + a_5^6) = a_0^6 \mod (p) \tag{4.7}$$

and :

$$v_p(6a_0^5) = k_0 \tag{4.8}$$
$$v_p(6a_1^5) = k_1 \tag{4.9}$$
$$v_p(6a_2^5) = k_2 \tag{4.10}$$
$$v_p(6a_3^5) = k_3 \tag{4.11}$$
$$v_p(6a_4^5) = k_4 \tag{4.12}$$
$$v_p(6a_5^5) = k_5 \tag{4.13}$$

$$6a_0^5 = k_0 \mod (p) \tag{4.14}$$
$$6a_1^5 = k_1 \mod (p) \tag{4.15}$$
$$6a_2^5 = k_2 \mod (p) \tag{4.16}$$
$$6a_3^5 = k_3 \mod (p) \tag{4.17}$$
$$6a_4^5 = k_4 \mod (p) \tag{4.18}$$
$$6a_5^5 = k_5 \mod (p). \tag{4.19}$$

This immediately imposes that $a \neq (0,0,0,0,0,0)$. Furthermore, we have :

$$6a_0^6 = k_0 a_0 \mod (p). \tag{4.20}$$

Continuing the same pattern we have :

$$k_0a_0 + a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4 + a_5k_5 = 0 \mod (p). \tag{4.21}$$

This is a parametric linear diophantine equation and can be solved for any prime with all entries non-zero modulo p.

Note that in principle, one can compute any integer n represented by taking the appropriate cohomology theory. For example, in the case of interest, crystalline cohomology allows one to compute a basis of de Rham cohomology with p-adic coefficients to arbitrary precision since it suffices to compute $n \mod (p^k)$ for a sufficiently large k.

However it still leaves the question of obstructions open, and furthermore, to find an efficient and computationally realistic way to do it. One possible solution is to extend the results of [53] to other settings and study the properties of those Calabi-Yau modular forms.

Note that in particular the aforementioned paper relies heavily on mirror symmetry to compute the Yukawa coupling. However the hope is that similar methods can be developed to study the primitive middle cohomology of a fourfold from a global point of view by using modularity.

4.3 Far-fetched conjecture

We have extensively used the point-of-view of M-theory by using the G_4 flux until now. As mentioned in the introduction, everything can be said in the formalism of type IIB. In this section, we go back to that point-of-view from the lessons learned by using M-theory and make some conjectural statement.

Recall that in type IIB there are not only D7 branes, which are the main topic of study in this thesis, but also D3 branes. From [54], we see that those two types of branes can couple to form bound states. Furthermore, the number of moduli stabilized by the D7 branes is limited by the charge induced by the D3 branes, We have seen that to understand this problem, we really need to understand 4-cycles, and to that end, we could as well consider D3 branes wrapping around them, in which case [55] makes sense to consider to have a mapping between a geometric and number theoretic description.

On the mathematical side of things, this would imply studying the relationship between periods and L-functions in order to be able to count (appropriate) points on the variety we consider, which is an endeavour that was not explored in this thesis but is historically important.

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APPENDIX A

Short lattice points for residues symmetric under $\mathbb{Z}/6\mathbb{Z}^2 \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

This is a table of shortest lengths appearing in the lattice $\mathcal{Z}^*_{6^2 3 2}$ for the group $\Gamma = (\mathbb{Z}/6\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ s discussed in Section 3.3.3.

Table A.1: Lengths below 1500 and number of lattice points for $\mathcal{Z}^*_{6^2 3 2}$.

Length	**Number of solutions**	**Length**	**Number of solutions**
0	1	1971/2	12
243/2	2	990	24
192	6	1995/2	264
411/2	4	1008	36
216	6	2067/2	48
246	4	1038	264
603/2	4	1056	64
627/2	12	1062	24

Continued on next page

Table A.1 – continued from previous page

Length	Number of solutions	Length	Number of solutions
675/2	12	2139/2	112
342	4	1080	24
408	40	2187/2	146
843/2	24	2211/2	548
462	24	1110	88
486	2	2259/2	36
987/2	8	1134	156
504	4	2283/2	56
1035/2	24	1152	64
1059/2	76	2331/2	112
534	8	2355/2	96
558	24	1182	48
576	12	1200	264
1179/2	6	2403/2	24
624	24	1206	88
1251/2	4	1224	72
630	6	2475/2	42
648	6	1254	84
678	12	1272	112
696	8	2547/2	96
1395/2	24	1278	180
702	12	2571/2	396
1419/2	48	1296	144
720	24	2619/2	12
1491/2	24	2643/2	264
750	48	1326	456
768	42	1344	116
1539/2	12	2691/2	144

Continued on next page

Table A.1 – continued from previous page

Length	Number of solutions	Length	Number of solutions
1563/2	44	1350	12
792	78	2715/2	248
1611/2	36	1368	328
1635/2	8	2763/2	384
822	44	2787/2	88
840	36	1398	200
1683/2	24	1416	252
846	36	2835/2	144
1707/2	28	1422	384
864	6	2859/2	308
1755/2	24	1440	126
1779/2	84	2907/2	84
894	96	2931/2	168
912	48	1470	768
1827/2	150	1488	384
918	24	2979/2	520
1899/2	28	1494	48

APPENDIX B

Lists of Linearly Independent Algebraic Cycles

This appendix contains lists of linearly independent algebraic cycles of linear type, Aoki-Shioda type, and type 3. The tables list the powers of primitive roots of unity and permutation of homogeneous coordinates for each cycle.

Index	(ℓ_1, ℓ_3, ℓ_5)	Permutations of linear cycles
1,...,14	(0,0,0)	$\Sigma\backslash$(0 1 2 3 4 5)
15,...,299	(1,0,0),(2,0,0),(3,0,0),(4,0,0),(0,1,0) (1,1,0),(2,1,0),(3,1,0),(4,1,0),(0,2,0) (1,2,0),(2,2,0),(3,2,0),(4,2,0),(0,3,0) (1,3,0),(2,3,0),(3,3,0),(4,3,0)	Σ
300,...,305	(0,4,0)	(0 1 2 3 4 5), (0 1 2 4 3 5), (0 3 1 4 2 5), (0 1 2 5 3 4), (0 3 1 5 2 4), (0 4 1 5 2 3)
306,...,308	(1,4,0)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
309,...,311	(2,4,0)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
312,...,314	(3,4,0)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
315,...,317	(4,4,0)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
318,...,617	(0,0,1),(1,0,1),(2,0,1),(3,0,1),(4,0,1) (0,1,1),(1,1,1),(2,1,1),(3,1,1),(4,1,1) (0,2,1),(1,2,1),(2,2,1),(3,2,1),(4,2,1) (0,3,1),(1,3,1),(2,3,1),(3,3,1),(4,3,1)	Σ
618,...,623	(0,4,1)	(0 1 2 3 4 5), (0 1 2 4 3 5), (0 3 1 4 2 5), (0 1 2 5 3 4), (0 3 1 5 2 4), (0 4 1 5 2 3)
624,...,626	(1,4,1)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
627,...,629	(2,4,1)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
630,...,632	(3,4,1)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
633,...,635	(4,4,1)	(0 3 1 4 2 5), (0 3 1 5 2 4), (0 4 1 5 2 3)
636,...,800	(0,0,2),(1,0,2),(2,0,2),(3,0,2),(4,0,2) (0,1,2),(1,1,2),(2,1,2),(3,1,2),(4,1,2) (0,2,2)	Σ
801,...,814	(1,2,2)	$\Sigma\backslash$(0 1 2 5 3 4)
815,...,828	(2,2,2)	$\Sigma\backslash$(0 1 2 5 3 4)
829,...,842	(3,2,2)	$\Sigma\backslash$(0 1 2 5 3 4)
843,...,856	(4,2,2)	$\Sigma\backslash$(0 1 2 5 3 4)
857,...,871	(0,3,2)	Σ
872,...,886	(1,3,2)	Σ
887,...,901	(2,3,2)	Σ
902,...,914	(3,3,2)	$\Sigma\backslash$(0 1 2 4 3 5), (0 4 1 2 3 5)
915,...,927	(4,3,2)	$\Sigma\backslash$(0 1 2 4 3 5), (0 4 1 2 3 5)
928,...,933	(0,4,2)	(0 1 2 3 4 5), (0 1 2 4 3 5), (0 3 1 4 2 5), (0 1 2 5 3 4), (0 3 1 5 2 4), (0 4 1 5 2 3)
934	(1,4,2)	(0 3 1 4 2 5)
935	(2,4,2)	(0 3 1 4 2 5)
936	(3,4,2)	(0 3 1 4 2 5)
937	(4,4,2)	(0 3 1 4 2 5)
938,...,948	(0,0,3)	(0 1 2 3 4 5), (0 2 1 3 4 5), (0 3 1 2 4 5), (0 1 2 4 3 5), (0 2 1 4 3 5), (0 4 1 2 3 5), (0 3 1 4 2 5), (0 1 2 5 3 4), (0 2 1 5 3 4), (0 5 1 2 3 4), (0 4 1 5 2 3)
949,...,956	(1,0,3)	(0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 4 1 2 3 5), (0 3 1 4 2 5), (0 2 1 5 3 4), (0 5 1 2 3 4), (0 4 1 5 2 3)
957,...,964	(2,0,3)	(0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 4 1 2 3 5), (0 3 1 4 2 5), (0 2 1 5 3 4), (0 5 1 2 3 4), (0 4 1 5 2 3)
965,...,970	(3,0,3)	(0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 3 1 4 2 5), (0 2 1 5 3 4), (0 5 1 2 3 4)
971,...,976	(4,0,3)	(0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 3 1 4 2 5), (0 2 1 5 3 4), (0 5 1 2 3 4)
977,...,984	(0,1,3)	(0 1 2 3 4 5), (0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 4 1 2 3 5), (0 3 1 4 2 5), (0 2 1 5 3 4), (0 5 1 2 3 4)
985,...,989	(1,1,3)	(0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 4 1 2 3 5), (0 3 1 4 2 5)
990,...,994	(2,1,3)	(0 2 1 3 4 5), (0 3 1 2 4 5), (0 2 1 4 3 5), (0 4 1 2 3 5), (0 3 1 4 2 5)
995	(3,1,3)	(0 2 1 3 4 5)
996	(4,1,3)	(0 2 1 3 4 5)
997,998,999	(0,2,3)	(0 1 2 3 4 5) (0 2 1 3 4 5), (0 3 1 4 2 5)
1000,1001	(0,3,3)	(0 2 1 3 4 5), (0 3 1 2 4 5)

Table B.1: List of linearly independent linear cycles. Σ refers to the set of permutations.

Index	$(\ell_0, \ell_2, \ell_3, \ell_5)$	Permutations of Aoki-Shioda cycles
1002,...,1337	(0,1,0,0)	$\Sigma' \setminus$ (5 0 4 1 3 2), (5 1 4 0 3 2), (5 0 4 2 3 1), (5 2 4 0 3 1), (4 0 3 5 2 1), (5 0 3 4 2 1), (5 0 4 3 2 1), (4 3 5 0 2 1), (5 3 4 0 2 1), (5 1 4 2 3 0), (5 2 4 1 3 0), (4 1 3 5 2 0), (5 1 3 4 2 0), (5 1 4 3 2 0), (4 3 5 1 2 0), (5 3 4 1 2 0), (3 2 4 5 1 0), (4 2 3 5 1 0), (3 2 5 4 1 0), (5 2 3 4 1 0), (4 2 5 3 1 0), (5 2 4 3 1 0), (4 3 5 2 1 0), (5 3 4 2 1 0)
1338,...,1593	(1,1,0,0)	(0 1 2 3 4 5),(1 0 2 3 4 5),(2 0 1 3 4 5),(0 1 3 2 4 5),(1 0 3 2 4 5),(3 0 1 2 4 5), (0 2 3 1 4 5),(2 0 3 1 4 5),(3 0 2 1 4 5),(1 2 3 0 4 5),(2 1 3 0 4 5),(3 1 2 0 4 5), (0 1 2 4 3 5),(1 0 2 4 3 5),(2 0 1 4 3 5),(0 1 4 2 3 5),(1 0 4 2 3 5),(4 0 1 2 3 5), (0 2 4 1 3 5),(2 0 4 1 3 5),(4 0 2 1 3 5),(1 2 4 0 3 5),(2 1 4 0 3 5),(4 1 2 0 3 5), (0 1 3 4 2 5),(1 0 3 4 2 5),(3 0 1 4 2 5),(0 1 4 3 2 5),(1 0 4 3 2 5),(4 0 1 3 2 5), (0 3 4 1 2 5),(3 0 4 1 2 5),(4 0 3 1 2 5),(1 3 4 0 2 5),(3 1 4 0 2 5),(4 1 3 0 2 5), (0 2 3 4 1 5),(2 0 3 4 1 5),(3 0 2 4 1 5),(0 2 4 3 1 5),(2 0 4 3 1 5),(4 0 2 3 1 5), (0 3 4 2 1 5),(3 0 4 2 1 5),(4 0 3 2 1 5),(2 3 4 0 1 5),(3 2 4 0 1 5),(4 2 3 0 1 5), (1 2 3 4 0 5),(2 1 3 4 0 5),(3 1 2 4 0 5),(1 2 4 3 0 5),(2 1 4 3 0 5),(4 1 2 3 0 5), (1 3 4 2 0 5),(3 1 4 2 0 5),(4 1 3 2 0 5),(2 3 4 1 0 5),(3 2 4 1 0 5),(4 2 3 1 0 5), (0 1 2 3 5 4),(1 0 2 3 5 4),(2 0 1 3 5 4),(0 1 3 2 5 4),(1 0 3 2 5 4),(3 0 1 2 5 4), (0 2 3 1 5 4),(2 0 3 1 5 4),(3 0 2 1 5 4),(1 2 3 0 5 4),(2 1 3 0 5 4),(3 1 2 0 5 4), (0 1 2 5 3 4),(1 0 2 5 3 4),(2 0 1 5 3 4),(0 1 5 2 3 4),(1 0 5 2 3 4),(5 0 1 2 3 4), (0 2 5 1 3 4),(2 0 5 1 3 4),(5 0 2 1 3 4),(1 2 5 0 3 4),(2 1 5 0 3 4),(5 1 2 0 3 4), (0 1 3 5 2 4),(1 0 3 5 2 4),(3 0 1 5 2 4),(0 1 5 3 2 4),(1 0 5 3 2 4),(5 0 1 3 2 4), (0 3 5 1 2 4),(3 0 5 1 2 4),(5 0 3 1 2 4),(1 3 5 0 2 4),(3 1 5 0 2 4),(5 1 3 0 2 4), (0 2 3 5 1 4),(2 0 3 5 1 4),(3 0 2 5 1 4),(0 2 5 3 1 4),(2 0 5 3 1 4),(5 0 2 3 1 4), (0 3 5 2 1 4),(3 0 5 2 1 4),(5 0 3 2 1 4),(2 3 5 0 1 4),(3 2 5 0 1 4),(5 2 3 0 1 4), (1 2 3 5 0 4),(2 1 3 5 0 4),(3 1 2 5 0 4),(1 2 5 3 0 4),(2 1 5 3 0 4),(5 1 2 3 0 4), (1 3 5 2 0 4),(3 1 5 2 0 4),(5 1 3 2 0 4),(2 3 5 1 0 4),(5 2 3 1 0 4),(0 1 2 4 5 3), (1 0 2 4 5 3),(2 0 1 4 5 3),(0 1 4 2 5 3),(1 0 4 2 5 3),(4 0 1 2 5 3),(0 2 4 1 5 3), (2 0 4 1 5 3),(4 0 2 1 5 3),(1 2 4 0 5 3),(2 1 4 0 5 3),(4 1 2 0 5 3),(0 1 2 5 4 3), (1 0 2 5 4 3),(2 0 1 5 4 3),(0 1 5 2 4 3),(1 0 5 2 4 3),(5 0 1 2 4 3),(0 2 5 1 4 3), (2 0 5 1 4 3),(1 2 5 0 4 3),(0 1 4 5 2 3),(1 0 4 5 2 3),(4 0 1 5 2 3),(0 1 5 4 2 3), (1 0 5 4 2 3),(5 0 1 4 2 3),(0 4 5 1 2 3),(4 0 5 1 2 3),(5 0 4 1 2 3),(1 4 5 0 2 3), (4 1 5 0 2 3),(5 1 4 0 2 3),(0 2 4 5 1 3),(2 0 4 5 1 3),(4 0 2 5 1 3),(0 2 5 4 1 3), (2 0 5 4 1 3),(5 0 2 4 1 3),(0 4 5 2 1 3),(4 0 5 2 1 3),(2 4 5 0 1 3),(4 2 5 0 1 3), (5 2 4 0 1 3),(1 2 4 5 0 3),(2 1 4 5 0 3),(4 1 2 5 0 3),(1 2 5 4 0 3),(5 1 2 4 0 3), (1 4 5 2 0 3),(4 1 5 2 0 3),(2 4 5 1 0 3),(0 1 3 4 5 2),(1 0 3 4 5 2),(3 0 1 4 5 2), (0 1 4 3 5 2),(1 0 4 3 5 2),(4 0 1 3 5 2),(0 3 4 1 5 2),(3 0 4 1 5 2),(4 0 3 1 5 2), (1 3 4 0 5 2),(3 1 4 0 5 2),(4 1 3 0 5 2),(0 1 3 5 4 2),(1 0 3 5 4 2),(3 0 1 5 4 2), (0 1 5 3 4 2),(1 0 5 3 4 2),(5 0 1 3 4 2),(0 3 5 1 4 2),(3 0 5 1 4 2),(1 3 5 0 4 2), (0 1 4 5 3 2),(1 0 4 5 3 2),(4 0 1 5 3 2),(0 1 5 4 3 2),(4 0 5 1 3 2),(0 3 4 5 1 2), (3 0 4 5 1 2),(0 3 5 4 1 2),(3 0 5 4 1 2),(0 4 5 3 1 2),(4 0 5 3 1 2),(3 4 5 0 1 2), (4 3 5 0 1 2),(5 3 4 0 1 2),(1 3 4 5 0 2),(3 1 4 5 0 2),(1 3 5 4 0 2),(1 4 5 3 0 2), (3 4 5 1 0 2),(0 2 3 4 5 1),(2 0 3 4 5 1),(3 0 2 4 5 1),(0 2 4 3 5 1),(2 0 4 3 5 1), (4 0 2 3 5 1),(0 3 4 2 5 1),(3 0 4 2 5 1),(2 3 4 0 5 1),(3 2 4 0 5 1),(4 2 3 0 5 1), (0 2 3 5 4 1),(2 0 3 5 4 1),(0 2 5 3 4 1),(2 0 5 3 4 1),(5 0 2 3 4 1),(3 0 5 2 4 1), (2 3 5 0 4 1),(0 2 4 5 3 1),(2 0 4 5 3 1),(4 0 5 2 3 1),(0 3 4 5 2 1),(3 0 4 5 2 1), (2 3 4 5 0 1),(2 3 5 4 0 1),(2 4 5 3 0 1),(3 4 5 2 0 1),(1 2 3 4 5 0),(2 1 3 4 5 0), (3 1 2 4 5 0),(1 2 4 3 5 0),(2 1 4 3 5 0),(4 1 2 3 5 0),(1 3 4 2 5 0),(3 1 4 2 5 0), (2 3 4 1 5 0),(1 2 3 5 4 0),(1 2 5 3 4 0),(2 1 5 3 4 0),(5 1 2 3 4 0),(3 1 5 2 4 0), (1 2 4 5 3 0),(4 1 5 2 3 0),(1 3 4 5 2 0),(2 3 4 5 1 0)
1594,...,1645	(0,0,0,1)	(0 1 2 3 4 5),(1 0 2 3 4 5),(0 1 3 2 4 5),(1 0 3 2 4 5),(0 2 3 1 4 5),(2 0 3 1 4 5), (1 2 3 0 4 5),(2 1 3 0 4 5),(0 1 2 4 3 5),(1 0 2 4 3 5),(0 1 4 2 3 5),(1 0 4 2 3 5), (0 2 4 1 3 5),(2 0 4 1 3 5),(1 2 4 0 3 5),(0 1 3 4 2 5),(1 0 3 4 2 5),(0 1 4 3 2 5), (3 0 4 1 2 5),(2 0 3 4 1 5),(0 1 2 3 5 4),(1 0 2 3 5 4),(0 1 3 2 5 4),(1 0 3 2 5 4), (2 0 3 1 5 4),(0 1 2 5 3 4),(1 0 2 5 3 4),(0 1 5 2 3 4),(1 0 5 2 3 4),(0 2 5 1 3 4), (2 0 5 1 3 4),(1 2 5 0 3 4),(0 1 3 5 2 4),(1 0 3 5 2 4),(0 1 5 3 2 4),(3 0 5 1 2 4), (2 0 3 5 1 4),(0 1 2 4 5 3),(1 0 2 4 5 3),(0 1 4 2 5 3),(1 0 4 2 5 3),(2 0 4 1 5 3), (0 1 4 5 2 3),(1 0 4 5 2 3),(0 1 5 4 2 3),(4 0 5 1 2 3),(2 0 4 5 1 3),(0 1 3 4 5 2), (1 0 3 4 5 2),(3 0 4 1 5 2),(3 0 4 5 1 2),(2 0 3 4 5 1)
1646,..., 1687	(1,0,0,1)	(0 1 2 3 4 5),(1 0 2 3 4 5),(0 1 3 2 4 5),(1 0 3 2 4 5),(0 2 3 1 4 5),(2 0 3 1 4 5), (1 2 3 0 4 5),(2 1 3 0 4 5),(0 1 2 4 3 5),(1 0 2 4 3 5),(0 1 4 2 3 5),(1 0 4 2 3 5), (0 2 4 1 3 5),(2 0 4 1 3 5),(1 2 4 0 3 5),(0 1 3 4 2 5),(1 0 3 4 2 5),(0 1 4 3 2 5), (3 0 4 1 2 5),(2 0 3 4 1 5),(0 1 2 3 5 4),(1 0 2 3 5 4),(0 1 3 2 5 4),(0 1 2 5 3 4), (1 0 2 5 3 4),(0 1 5 2 3 4),(1 0 5 2 3 4),(0 2 5 1 3 4),(2 0 5 1 3 4),(1 2 5 0 3 4), (0 1 3 5 2 4),(0 1 5 3 2 4),(3 0 5 1 2 4),(0 1 2 4 5 3),(1 0 2 4 5 3),(0 1 4 2 5 3), (0 1 4 5 2 3),(0 1 5 4 2 3),(4 0 5 1 2 3),(0 1 3 4 5 2),(1 0 3 4 5 2),(2 0 3 4 5 1)
1688,...,1703	(0,1,0,1)	(0 1 2 3 4 5),(0 1 3 2 4 5),(0 2 3 1 4 5),(1 2 3 0 4 5),(0 1 2 4 3 5),(0 1 4 2 3 5), (0 1 3 4 2 5),(0 2 3 4 1 5),(1 2 3 4 0 5),(0 1 2 3 5 4),(0 1 2 5 3 4),(0 1 5 2 3 4), (0 1 2 4 5 3),(0 1 3 4 5 2),(0 2 3 4 5 1),(1 2 3 4 5 0)
1704,...,1719	(1,1,0,1)	(0 1 2 3 4 5),(0 1 3 2 4 5),(0 2 3 1 4 5),(1 2 3 0 4 5),(0 1 2 4 3 5),(0 1 4 2 3 5), (0 1 3 4 2 5),(0 2 3 4 1 5),(1 2 3 4 0 5),(0 1 2 3 5 4),(0 1 2 5 3 4),(0 1 5 2 3 4), (0 1 2 4 5 3),(0 1 3 4 5 2),(0 2 3 4 5 1),(1 2 3 4 5 0)
1720	(0,1,0,2)	(0,1,2,3,4,5)
1721	(1,1,0,2)	(0,1,2,3,4,5)

Table B.2: List of linearly independent Aoki-Shioda cycles. Σ' refers to the set of permutations with $\sigma(1) < \sigma(2)$.

Index	$(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$	Permutations of type 3 cycles
1722,...,1730	(0,0,0,0,0)	(0 1 2 3 4 5),(1 0 2 3 4 5),(0 1 3 2 4 5),(0 1 2 4 3 5),(1 0 2 4 3 5),(0 1 4 2 3 5),(0 1 3 4 2 5),(1 0 3 4 2 5),(2 0 3 4 1 5)
1731,...,1744	(1,0,0,0,0)	(0 1 2 3 4 5),(1 0 2 3 4 5),(2 0 1 3 4 5),(0 1 3 2 4 5),(0 1 2 4 3 5),(1 0 2 4 3 5),(2 0 1 4 3 5),(0 1 4 2 3 5),(0 1 3 4 2 5),(1 0 3 4 2 5),(3 0 1 4 2 5),(0 2 3 4 1 5),(2 0 3 4 1 5),(1 2 3 4 0 5)
1745,...,1749	(3,0,0,0,0)	(0 1 2 3 4 5),(0 1 2 4 3 5),(0 1 3 4 2 5),(0 2 3 4 1 5),(1 2 3 4 0 5)
1750	(1,0,0,1,0)	(0 1 2 3 4 5)
1751	(3,0,0,1,0)	(0 1 2 3 4 5)

Table B.3: List of linearly independent type 3 cycles.

APPENDIX C

Subgroups of $(\mathbb{Z}/6\mathbb{Z})^4$ acting on the Fermat Sextic

Below is a list of the occurring dimensions of invariant subspaces $H^{3,1}(X)_{inv} \subset H^{3,1}(X)$ and $H^{2,2}(X)_{inv} \subset H^{2,2}(X)$ for all subgroups of $(\mathbb{Z}/6\mathbb{Z})^4$. Note that a single entry potentially corresponds to genuinely different subgroups of $(\mathbb{Z}/6\mathbb{Z})^4$, i.e. subgroups which are not identified by permuting the x_i. As can be seen from the arrangement of the table, there is a matching between cases with $|\Gamma| = k$ and cases with $|\Gamma^\vee| = 6^4/k$, which is a consequence of mirror symmetry.

Table C.1: Orders and dimensions of invariant subspaces for all subgroups of $(\mathbb{Z}/6\mathbb{Z})^4$.

$h^{3,1}_{inv}(X)$	$h^{2,2}_{inv}(X)$	$\lvert\Gamma\rvert$	$h^{3,1}_{inv}(X)$	$h^{2,2}_{inv}(X)$	$\lvert\Gamma\rvert$
1	426	1751	1296	1	1
2	226	903	648	2	3
3	138	563	432	1	7
3	144	587	432	3	3
3	162	611	432	3	5
4	126	479	324	4	7
6	70	291	216	2	11

6	72	315	216	4	9
6	74	291	216	4	11
6	76	303	216	5	7
6	84	309	216	6	7
6	86	315	216	6	9
8	76	267	162	8	15
9	48	191	144	3	23
9	52	191	144	5	17
9	62	215	144	7	17
9	66	215	144	9	11
12	36	155	108	4	19
12	36	179	108	6	19
12	38	155	108	6	21
12	40	155	108	6	23
12	40	167	108	8	17
12	42	155	108	8	19
12	42	161	108	8	21
12	46	161	108	10	15
12	48	167	108	10	19
12	60	149	108	12	17
16	51	161	81	16	31
18	24	99	72	6	33
18	24	103	72	6	35
18	26	97	72	7	29
18	28	99	72	8	29
18	30	111	72	8	33
18	30	119	72	10	27
18	32	109	72	10	31
18	34	107	72	11	29
18	34	111	72	12	25

18	36	95	72	12	27
18	42	99	72	12	31
18	42	103	72	14	29
24	22	87	54	10	35
24	22	99	54	12	43
24	24	87	54	14	35
24	24	93	54	14	37
24	27	87	54	14	43
24	34	81	54	16	35
24	36	87	54	18	39
27	22	67	48	15	53
27	28	75	48	15	59
27	30	83	48	21	47
36	12	53	36	12	53
36	12	57	36	12	55
36	14	51	36	12	57
36	14	63	36	12	59
36	16	57	36	14	51
36	18	49	36	14	55
36	18	55	36	14	63
36	20	51	36	16	55
36	22	49	36	16	57
36	22	53	36	16	61
36	30	41	36	18	49