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Exceptional Mirror Symmetry and the String Worldsheet

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A thesis presented for the degree of
Doctor of Philosophy

Centre for Particle Theory
Department of Mathematical Sciences
Durham University
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Abstract

This thesis provides arguments to strengthen our understanding of mirror symmetry for manifolds with G_2 holonomy, by providing worldsheet arguments to demonstrate the physical equivalence of topologically distinct geometries. In particular we investigate the worldsheet superconformal field theories corresponding to manifolds with G_2 holonomy obtained by the quotient of the product of a Calabi-Yau threefold and a circle. The quotient acts on the Calabi-Yau as an antiholomorphic involution and on the circle by inversion. For such models, we argue that the Calabi-Yau mirror map implies a mirror map for the associated G_2 varieties by examining how antiholomorphic involutions behave under Calabi-Yau mirror symmetry. The mirror geometries identified by the worldsheet CFT are consistent with earlier proposals for twisted connected sum G_2 manifolds.

In order to be as self contained and pedagogical as possible, this thesis also provides a reasonably detailed review of Calabi-Yau manifolds and their associated CFTs. We also review details on the geometrical constructions of manifolds with G_2 holonomy, in order to explain the geometrical equivalence of our CFT results.

Declaration

The work in this thesis is based on research carried out in the Centre of Particle Theory group of the Department of Mathematical Sciences at Durham University, England. No part of this thesis has been submitted elsewhere for any degree or qualification.

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My deepest gratitude is owed to my siblings Kiran, Nikki and Kris along with the Shmees and my parents, in particular my mum. I don't think words can do justice to the level of appreciation I have for everything you have done for me and I simply hope that in the end you are proud of the work I have produced.

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To mama,

Now you have an answer when people ask "What is his PhD about?"

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0 Introduction

Over the past few decades string theory has been a huge talking point among physicists and mathematicians alike. String theory, at its core, aims to provide a solution to grand unification theory, by providing a quantum theory of gravity. In order to obtain a mathematically and physically consistent theory, string theories (including M-theory and F-theory) require extra dimensions to our observed 4-dimensional spacetime. In particular, superstrings exist in 10-dimensions and M-theory in 11-dimensions. Of course we need to address how one goes from a theory with these dimensions to the observed 4-dimensional reality we live in. The idea is simply to split the d-dimensional spacetime, $\mathcal{M}_d$, into a product

$$\mathcal{M}_d = \mathcal{M}_4 \times \mathcal{M}_{d-4},$$

where $\mathcal{M}_4$ represents 4-dimensional spacetime and $\mathcal{M}_{d-4}$ is the $(d-4)$-dimensional *internal* space. Letting strings propagate on such a geometry results in effective 4-dimensional physics at low energies, as in Kaluza-Klein compactifications [1]. That is, we compactify $\mathcal{M}_{d-4}$ down and study the result this has on the resulting 4-dimensional physics. The geometry is heavily restricted by requiring reasonable physics in 4D. By "reasonable physics", we mean [2]

1. The geometry of $\mathcal{M}_4 \times \mathcal{M}_{d-4}$ to be such that $\mathcal{M}_4$ is a maximally symmetric spacetime, and

2. We should have unbroken $\mathcal{N} = 1$ SUSY in 4D.

These conditions require the variations of the Fermi fields to vanish. The variation of the gravitino places two important constraints on the geometry. The first is that $\mathcal{M}_4$ is required to be Minkowski spacetime, which we denote by $\mathbb{M}_4$. The second condition is that for every covariantly constant spinor, $\nabla\xi = 0$, on the internal space, we get one copy of the $\mathcal{N} = 1$ SUSY algebra in 4D. The equations of motion can then be shown to imply that $\mathcal{M}_{d-4}$ is Ricci flat.

The existence of a covariantly constant spinor on $\mathcal{M}_{d-4}$ has important restrictions on the geometry. To understand this, we need the notion of *holonomy*, which we will shortly discuss. For now we just state that the result is that for the Heterotic string, we require the internal space to be a complex 3-dimensional Calabi-Yau manifold, while for M-theory we require the internal space to have holonomy given by the exceptional Lie group G_2.

It is no secret that string theory has received its fair share of criticisms over the years, particularly from those who question how physical the above conditions really are. However, it is fair to say that from a mathematical point of view, string theory is one of the most elegant and powerful theories developed in recent history. Part of the beauty of string theory is the dualities and symmetries it possesses. One of the most famous is the AdS/CFT correspondence [3], which has taken on a life as a whole area of research in itself.

One of the most elegant and profound symmetries in string theory comes from putting on more of a mathematical viewpoint hat. We can consider compactifying Type IIA/B strings on our Calabi-Yaus. This gives rise to a 4D theory with $\mathcal{N}=2$ SUSY, and so feels like a step in the wrong direction, however the beauty comes from the discovery of *mirror symmetry* [4–6], which is a deep connection between Calabi-Yaus: Calabi-Yaus come in pairs $(\mathcal{M}_{CY}, \mathcal{M}_{CY}^{\vee})$, known as a mirror pair, such that compactifying Type IIA on $\mathcal{M}_{CY}$ gives the same 4D physics as compactifying Type IIB on $\mathcal{M}_{CY}^{\vee}$. This is really meant as them having the same (isomorphic) worldsheet superconformal field theories (SCFTs). It is a necessary condition for two Calabi-Yaus to be mirror that the Hodge numbers $h^{2,1}$ and $h^{1,1}$ are swapped. The incredible thing is that the topologies of $\mathcal{M}_{CY}$ and $\mathcal{M}_{CY}^{\vee}$ are (in general) considerably different, and so it is not obvious, a priori, that they would stem from the same CFT.

Since its discovery, mirror symmetry for type II strings on Calabi-Yau manifolds has quickly evolved into a powerful tool [7] with intricate mathematical implications such as homological mirror symmetry [8].

This development has been driven by the wealth of examples that can be readily constructed and analyzed using techniques from toric geometry [9, 10], and a detailed understanding of the worldsheet CFT in which the equivalence for distinct target spaces could be proven directly [11–13]. Key technical advances in this development were Gepner models [14, 15], which give direct access to the worldsheet SCFT, as well as the detailed study of $\mathcal{N}=(2,2)$ models and in particular the correspondence between Calabi-Yau sigma models and Landau-Ginzburg models [16–20]. Extending the equivalence from the worldsheet theory to the full string theories, which includes BPS states associated with wrapped branes, not only vastly extended the scope of this duality, but also led to the geometric idea of mirror symmetry being T-duality along the fibres of a torus fibration (often referred to as the SYZ fibration) [21]. This picture becomes particularly clear for toroidal orbifolds [22].

As stated above, when considering M-theory compactifications, the role of the Calabi-Yau is replaced by that of a real 7-dimensional manifold with G_2 holonomy, $\mathcal{M}_{G_2}$. Again leaning on the more mathematical interest, one can ask similar questions about compactification of Type II strings on $\mathcal{M}_{G_2}$, which gives a 3D theory with $\mathcal{N} = 1$ SUSY. In particular, we can ask whether there is a notion of mirror symmetry here and if so how much can be said about it.

Indeed, a similar phenomenon in which topologically distinct $\mathcal{M}_{G_2}$ lead to isomorphic worldsheet SCFTs has been conjectured in [23], and has been dubbed 'G_2 mirror symmetry'. Whereas a necessary condition for a pair of Calabi-Yaus to be mirror is that their complex cohomologies are swapped, the corresponding condition for a pair of G_2 manifolds $\mathcal{M}_{G_2}$ and $\mathcal{M}^{\vee}_{G_2}$ is weaker and says merely that the sum of Betti numbers $b^2 + b^3$ is preserved.

There has been a significant development in the understanding of G_2 mirror symmetry in terms of the geometry [24–28]. While there has also been a significant development in the constructions of SCFTs corresponding to manifolds with G_2 holonomy [29–34], the understanding of G_2 mirror symmetry at the SCFT level is still not fully understood.

Organisation Of Thesis

The main aim of this thesis is to provide a detailed and pedagogical account of the existing material on both Calabi-Yau and G_2 mirror symmetry (explaining a lot of the results stated above) and to then strengthen the understanding of G_2 mirror symmetry from the SCFT point of view, and in particular to provide worldsheet arguments for some of the geometric mirror constructions that have appeared in the literature. Our approach is based on the work of [13], which showed how mirror symmetry for Calabi-Yau hypersurfaces in toric varieties can be demonstrated by using duality in $\mathcal{N} = (2, 2)$ gauged linear sigma models (GLSMs). In order to fully appreciate the significance of the results, a decent amount of time is spent going over existing results and highlighting important points as we progress. The layout of the thesis is as follows.

Chapter 1 is a short chapter that formally introduces the notion of holonomy, explains its significance to compactifications and highlights an important distinction between Calabi-Yaus and manifolds with G_2 holonomy. The main aim of this chapter, particularly Section 1.2, is to provide a background guiding principal for the following work.

Chapter 2 gives a detailed account of the geometry of Calabi-Yau manifolds while introducing a lot of concepts that will also be useful in the construction of manifolds with G_2 holonomy. The chapter first tackles the problem from a differential geometric point of view, then motivates and introduces the algebraic (toric) geometry construction. The relevant understandings of mirror symmetry are scattered throughout the chapter.

Chapter 3 starts with a general account of $\mathcal{N} = 2$ SCFTs and then moves on to provide a detailed study of the worldsheet SCFT for a Calabi-Yau target space. Again mirror symmetry is discussed throughout the chapter and related back to the geometrical understandings from the previous chapter. It is in this chapter that the results of [13] are presented. The important case of torodial orbifolds is not included in this chapter, but is instead presented as a detailed example in Appendix A.

Chapter 4 starts with a brief review of the group G_2 and manifolds with holonomy (contained in) G_2, before moving on to the explicit constructions. The torodial orbifolds due to Joyce are briefly discussed (again, these are discussed in detail in Appendix A), but the main focus is on quotients of Calabi-Yaus and circles and the twisted connected sum (TCS) construction, and how they are related. The chapter is concluded with a discussion of how toric geometry can be used in the TCS construction along with an understanding of constructing mirrors. Once more, G_2 mirror symmetry is discussed at relevant points, but questions are raised about the physical significance of these observations.

Chapter 5 then studies the SCFT of manifolds with G_2 holonomy. We first provide the general algebra of Shatashvili-Vafa and then describe how it can be reproduced in analogy to the geometrical constructions discussed in the previous chapter. Having laid significant ground work, we finally present the main result of this thesis, showing that what one expects to be a physically relevant G_2 mirror map, is indeed the case. In particular, we demonstrate that the results of [13] can be used to show that the construction two topologically different manifolds with G_2 holonomy have isomorphic SCFTs and so satisfy the criteria of a G_2 mirror pair, as per [23].

Chapter 6 then concludes the thesis, and provides suggestions for further work.

The thesis also contains three appendices. As mentioned above, Appendix A provides a detailed account of the example of torodial orbifolds and how they fit into the context of the thesis. Appendix B provides an account of rational forms and the Griffiths residue, and provides a link between a result for Gepner model states and the primitive cohomology of the target space. Appendix C presents a discussion of how one deals with the notion of twisted and untwisted states in Gepner models corresponding to quotients of Landau-Ginzburg orbifolds.[1]

[1] To the authors knowledge, the details of Appendices B and C haven't been presented formally in any literature to date, although it is believed a lot of researchers intuitively know them.

1 | Motivation

We start the main content of the thesis by formally introducing the notion of holonomy and its relation to the physical symmetry of SUSY. We also provide a motivational discussion about the similarities and differences between Calabi-Yaus and manifolds with G_2 holonomy. This discussion, particularly the fact that we can use Calabi-Yaus to make manifolds with G_2 holonomy, aims to provide a guiding principal for the work that follows, and should be kept in the back of the ones mind while reading the thesis.

1.1 Holonomy

The existence of a covariantly constant spinor on $\mathcal{M}_{d-4}$ has important restrictions on the geometry. To understand this, we need the notion of holonomy.

> **Definition.** [Holonomy] Let $\mathcal{M}$ be a smooth manifold equipped with some connection ∇ on the tangent bundle, and consider a $v \in T_p\mathcal{M}$. Let $n = \dim_{\mathbb{R}} \mathcal{M}$. Now consider a closed smooth loop $\gamma : [0,1] \to \mathcal{M}$ with $\gamma(0) = \gamma(1) = p$. Now consider parallel transporting v around γ, the result will be, in general, some other element $v' \in T_p\mathcal{M}$. As $T_p\mathcal{M}$ is an n-dimensional vector space, we know we can relate v and v' via some $GL(n,\mathbb{R})$ action, i.e. $P_\gamma \in GL(n,\mathbb{R})$ where P_γ denotes the parallel transport along γ. We then define the *holonomy group at* $p \in \mathcal{M}$ to be
>
> $$\mathrm{Hol}_p(\nabla) := \{P_\gamma \in GL(n,\mathbb{R}) \,|\, \gamma \text{ is a loop based at } p \in \mathcal{M}\}. \tag{1.1}$$
>
> This is a Lie group, where multiplication is given by composition and the inverse is given by running around the path in the opposite direction.

As the notation suggests, the holonomy is a property of the connection ∇, and so non-trivial changes of the connection can lead to non-trivial changes in the holonomy. This is not surprising, as it is the connection that defines what we mean by parallel transport. If we consider a (pseduo-)Riemannian manifold, then we know that there exists a unique connection: the Levi-Civita connection, which is defined by the requirement $\nabla^{LC} g = 0$ and it being torsion free. Here we have that lengths are preserved under parallel transport and so our holonomy group clearly restricts to $\text{Hol}_p(\nabla^{LC}) \subseteq O(n)$. If we further require our manifold to be orientable, then we get $SO(n)$. Unless otherwise specified, we shall work in this case going forward.

Now, as we have been careful to indicate, the holonomy seems to depend on the choice of base point $p \in \mathcal{M}$. Of course in general this is true, however if we have a connected manifold then any two points $p, q \in \mathcal{M}$ can be connected by some smooth path $\tau : [0,1] \to \mathcal{M}$ with $\tau(0) = p$ and $\tau(1) = q$, and so we can relate the holonomies at these two points, simply by

$$\text{Hol}_q(\nabla) = P_\tau \text{Hol}_p(\nabla) P_\tau^{-1}. \tag{1.2}$$

This provides an isomorphism between $\text{Hol}_p(\nabla)$ and $\text{Hol}_q(\nabla)$ and so it allows us to really speak about the holonomy of the manifold $\mathcal{M}$ itself. We denote this by $\text{Hol}(\mathcal{M})$.[1] The key thing to note here is that the holonomoy of a manifold is related to it's geometric properties, precisely because parallel transport measures curvature.

Manifolds who's holonomy is a proper subset of $SO(n)$ are known as *special holonomy* manifolds. The set of special holonomy manifolds was classified by Berger in [35]: for a simply-connected manifold $\mathcal{M}$ of real dimension n, that is neither locally a product nor symmetric, the only allowed special holonomy groups are

$$U\left(\frac{n}{2}\right), \quad SU\left(\frac{n}{2}\right), \quad Sp\left(\frac{n}{4}\right)\cdot Sp(1), \quad Sp\left(\frac{n}{4}\right), \quad G_2, \quad Spin(7), \quad \text{and} \quad Spin(9) \tag{1.3}$$

The last three cases are known as the *exceptional* holonomies. The cases that will be important to us are $U(n/2)$, $SU(n/2)$ and G_2. Manifolds with holonomy $U(n/2)$ and $SU(n/2)$ are called Kähler and Calabi-Yau, respectively. We note that these manifolds are necessarily even dimensional. In fact they are complex manifolds, with complex dimension $m = n/2$. The Lie group G_2 can be defined as the subgroup of $SO(7)$ that preserves the following 3-form

$$\Phi := dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}, \tag{1.4}$$

where we have used the short hand $dx_{ijk} := dx_i \wedge dx_j \wedge dx_k$. We therefore see that a manifold with $\text{Hol}(\mathcal{M}) = G_2$ is a real 7-dimensional manifold.

It follows from the above dimensional arguments, that Kähler and Calabi-Yau manifolds

[1] Here we have dropped reference to the connection, i.e. we should have written something like $\text{Hol}(\mathcal{M}, \nabla) \subseteq SO(n)$. However we will work soley with the Levi-Civita connection, and so drop the ∇. However, it is important to remember that the holonomy depends crucially on the connection.

$\mathcal{M}$	T^n		$\mathcal{M}_{CY_3}$		$\mathcal{M}_{G_2}$
$\dim_{\mathbb{R}}(X)$	n		6		7
$\mathrm{Hol}(X)$	id	$\subset$	$SU(3)$	$\subset$	G_2
Fraction SUSY Preserved	1	$>$	1/4	$>$	1/8

Table 1.1: *Relationship between the holonomy of the internal space $\mathcal{M}_{d-4} \in \{T^n, \mathcal{M}_{CY_3}, \mathcal{M}_{G_2}\}$ and the fraction of the SUSY preserved under compactification.*

with complex dimension $m = 3$ (i.e. $n = 6$) are possible internal spaces for superstring theories. Similarly G_2 is a possible internal space for M-theory. This argument followed purely from the dimensions, and further evidence is needed to substantiate these proposals.

1.1.1 Holonomy & SUSY

We now see the link between the the amount of SUSY that survives in 4D and holonomy of the internal space. The former is related to the number of covariantly constant spinors on $\mathcal{M}_{d-4}$, $\nabla\xi = 0$. However, this restricts the holonomy, i.e. the holonomy group has to be such that ξ is invariant. In particular, the more SUSY we have in 4D, the more restricted the holonomy. We summarise the important cases in Table 1.1. We note that this provides a nice connection between the geometrical symmetry of holonomy and the physical symmetry of SUSY.

Historically, Calabi-Yaus are seen as being physically important for the following reason: Heterotic string theory is a $10D$ theory with $\mathcal{N} = 1$, and so has 16 supercharges. This is equivalent to $\mathcal{N} = 4$ in $4D$ and so if we instead want $\mathcal{N} = 1$, we must only preserve 1/4 of the SUSY. This is exactly the case for compactifying on a manifold with $SU(3)$ holonomy, i.e. a 3-dimensional Calabi-Yau manifold. Similar arguments can be made for M-theory and G_2 holonomy.

One can also consider compactifying Type II strings on Calabi-Yaus. This gives rise to $\mathcal{N} = 2$ in $4D$, and so doesn't seem interesting from a physical perspective. However, as we will see, such a compactification process is very interesting mathematically, as it leads to the notion of mirror symmetry for Calabi-Yaus. It is this mathematical motivation that we will use in this work, and so we focus on compactifying Type II strings. Indeed we will also consider compactifying Type II strings down to $3D$ on a manifolds with G_2 holonomy, in order to look for and demonstrate mirror symmetry for these manifolds.

1.2 Calabi-Yau vs. G_2

This work will be focused almost entirely around Calabi-Yau manifolds and manifolds with G_2 holonomy. The following chapters will deal with each of these in far more detail, but here we present a brief account of their core features and how they are related. This is done in order to motivate the following work, and can be referred back to, in order to ground the overall picture.

Calabi-Yau manifolds contain two important complex differential forms on them. The first is the Kähler 2-form, J, and the second it the holomorphic 3-form, Ω. Manifolds with G_2 holonomy equally have an important real differential form, the associative 3-form, Φ, along with its dual coassociative 4-form, $\star\Phi$.

There is an important difference between Calabi-Yau manifolds and manifolds with G_2 holonomy. For the former there exists a theorem by Yau [36] (which proves a conjecture by Calabi [37]) that guarantees the existence of a Calabi-Yau metric under certain conditions, while no such theorem exists for manifolds with exceptional holonomy. Yau's theorem is incredibly powerful as typically looking for metrics is hard, but finding manifolds with special holonomy is typically much easier. Yau's theorem allows us to not worry about finding the explicit form of the metric and simply lean on the fact that we know one exists.

The absence of an equivalent theorem for manifolds with G_2 holonomy, provides a major hurdle in their study, and limits us to studying specific examples. An important example is that of Joyce orbifolds [38, 39].[2] The idea is to notice that a torus has trivial holonomy, but taking a quotient will restrict the holonomy. That is, we consider torodial orbifolds of the form T^n/Γ where Γ is a finite group. The two important examples to us are $T^6/\mathbb{Z}_2^2$ with $\mathbb{Z}_2^2 \subset SU(3)$, and $T^7/\mathbb{Z}_2^3$ with $\mathbb{Z}_2^3 \subset G_2$. The former defines a Calabi-Yau, and the $\mathbb{Z}_2^2$ acts on the coordinates $(x_1, ..., x_6)$ as

$$\begin{aligned}
\alpha &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (+x_1, +x_2, -x_3, a_4 - x_4, -x_5, a_6 - x_6) \\
\beta &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (-x_1, b_2 - x_2, +x_3, +x_4, b_5 - x_5, b_6 - x_6)
\end{aligned} \tag{1.5}$$

where a_4, a_6, b_2, b_5 and b_6 are each either 0 or $\frac{1}{2}$. The latter defines a G_2 manifold and the $\mathbb{Z}_2^3$ acts as

$$\begin{aligned}
\alpha &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (+x_1, +x_2, -x_3, a_4 - x_4, -x_5, a_6 - x_6, x_7) \\
\beta &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, b_2 - x_2, +x_3, +x_4, b_5 - x_5, b_6 - x_6, x_7) \\
\sigma &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6, -x_7),
\end{aligned} \tag{1.6}$$

[2] A detailed study of Joyce orbifolds and their relation to the work of this thesis is given in Appendix A.

where again a_i and b_i are either 0 or $\frac{1}{2}$. Noting that α, β act on the first six coordinates in the exactly the same way as in the Calabi-Yau case, leads us to think of

$$\frac{T^7}{\mathbb{Z}_2^3} = \frac{(T^6/\mathbb{Z}_2^2) \times S^1}{\mathbb{Z}_2}. \tag{1.7}$$

Indeed this provides insight into an important construction of manifolds with G_2 holonomy.

The key observation we need is that $SU(3) \subset G_2$, and so it is possible to consider embedding a Calabi-Yau inside a manifold with G_2 holonomy. Of course, from simple dimensional arguments, we need to supplement the Calabi-Yau with a 1-dimensional manifold if we hope to construct a manifold with G_2 holonomy. Given the Joyce example above, the natural candidate is S^1, and indeed it is well known that we can construct a manifold with G_2 holonomy as the resolution of the quotiented product

$$\mathcal{M}_\sigma = \frac{\mathcal{M}_{CY} \times S^1}{(\sigma, -)}, \tag{1.8}$$

where σ acts on the Calabi-Yau as an antiholomorphic involution, i.e.

$$\sigma : (J, \Omega) \mapsto (-J, \bar{\Omega}), \tag{1.9}$$

and $(-)$ is simply inversion on the S^1. Denoting the 1-form on S^1 by $d\theta$, we then have

$$\begin{aligned} \Phi &= J \wedge d\theta + \mathrm{Re}(\Omega) \\ \star\Phi &= \frac{1}{2} J \wedge J + \mathrm{Im}(\Omega) \wedge d\theta, \end{aligned} \tag{1.10}$$

which we note are indeed invariant under the quotient.

This relationship between Calabi-Yaus and manifolds with G_2 holonomy will be our central guiding point in a lot of what follows, and should be kept in the back of our minds as we develop the theory of Calabi-Yau manifolds.

2 Calabi-Yau: Geometry

This chapter is dedicated to reviewing, in some detail, the relevant theory of the geometry of Calabi-Yau manifolds. These are a particular example of a complex manifold, and so we first start with a summary of the relevant parts of complex manifold theory, and introduce the important example of a complex projective space. We then introduce Kähler manifolds and state the conditions that need to be satisfied in order to make a Calabi-Yau manifold. After introducing the concept of mirror symmetry for compactifications of type II strings on Calabi-Yaus, we move on to a detailed study of forming Calabi-Yaus as hypersurfaces in complex projective spaces.

Next, we introduce the relevant tools of algebraic geometry, in particular toric geometry, and show how these can be used to construct Calabi-Yaus in an almost trivial combinatoric manner. This not only simplifies constructions, but also allows us to deal with the issues of singularities in our Calabi-Yaus, as well as introduce a powerful method for constructing mirror Calabi-Yaus via the Batyrev construction.

The main sources for the material of this chapter are [40, 41] for the differential geometry, and [42, 43] for the algebraic geometry.

2.1 Complex Manifold Basics

We start with a discussion of the geometry of Calabi-Yau manifolds. As was mentioned in the last chapter, these are examples complex manifolds and so we start with a brief discussion of complex manifolds.

The definition of a complex manifold is exactly as we might expect.

> **Definition.** [Complex Manifold] A *complex manifold* is a manifold $\mathcal{M}$ of real dimension $2m$, but where our charts are now homeomorphic to $\mathbb{C}^m$, i.e. we have chart maps $\psi_i : U_i \to \mathbb{C}^m$, with $\{U_i\}$ being an open cover of $\mathcal{M}$. To get a smooth complex manifold, we further require that our chart transition maps $\psi_{ij} := \psi_i \circ \psi_j : \psi(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ are holomorphic maps from $\mathbb{C}^m$ to $\mathbb{C}^m$. We call $\mathcal{M}$ a complex manifold of dimension m.

An important object to define on a complex manifold is the complex structure.

Definition. [Complex Structure] Let $\mathcal{M}$ be a smooth manifold, then an *almost complex structure* is a tensor field which we view as a map $I : T\mathcal{M} \to T\mathcal{M}$ such that $I^2 = -1$. We call the pair $(\mathcal{M}, I)$ an *almost complex manifold.* If we introduce the complexified tangent bundle, $T_{\mathbb{C}}\mathcal{M} := T\mathcal{M} \otimes \mathbb{C}$, we can lift the action of I to $T_{\mathbb{C}}\mathcal{M}$ and induce a decomposition

$$T_{\mathbb{C}}\mathcal{M} = T\mathcal{M}^{(1,0)} \oplus T\mathcal{M}^{(0,1)}, \tag{2.1}$$

where $T\mathcal{M}^{(1,0)}$ is the holomorphic tangent bundle and $T\mathcal{M}^{(0,1)}$ the antiholomorphic tangent bundle. An element $v \in T\mathcal{M}^{(1,0)}$ obeys $Iv = +iv$, while $v \in T\mathcal{M}^{(0,1)}$ obeys $Iv = -iv$. A complex structure is called *integrable* if the Lie bracket of two holomorphic vector fields is again a holomorphic vector field.

We note here that the notion of an almost complex structure is defined at the level of the tangent bundle, and so can equally well be defined for a real manifold. However, the Nirnberg-Newlander theorem (see, e.g., [44] for a discussion) can be used to tell us that a manifold is a complex manifold if, and only if, the almost complex structure is integrable. As we will be working with complex manifolds only in this chapter, we always meet this condition and so we simply refer to an integrable almost complex structure as a complex structure.

Once we have a complex manifold and complex structure, we can apply the usual techniques of tensors and define holomorphic and antiholomorphic tensor fields. The ones we will be interested in are the complex differential forms.

Definition. [(p,q)-Form] Let $\mathcal{M}$ be a complex smooth manifold. We then define a (p,q)-*form* to be an element of

$$\Omega^{p,q}\mathcal{M} := \Gamma(\Lambda^{p,q}\mathcal{M}), \qquad \text{where} \qquad \Lambda^{p,q}\mathcal{M} = \Lambda^p T^*\mathcal{M}^{(1,0)} \otimes \Lambda^q T^*\mathcal{M}^{(0,1)}. \tag{2.2}$$

Just as in the case of deRham cohomology, we introduce the exterior derivative but now we have one for holomorphic, ∂, and one for antiholomorphic, $\bar{\partial}$. We define the Dolbeault cohomology with respect to either of these, we use $\bar{\partial}$:

$$H^{p,q}_{\bar{\partial}} = \frac{\ker(\bar{\partial} : \Omega^{p,q}(\mathcal{M}) \mapsto \Omega^{p,q+1}(\mathcal{M}))}{\mathrm{Im}(\bar{\partial} : \Omega^{p,q-1}(\mathcal{M}) \mapsto \Omega^{p,q}(\mathcal{M}))} \tag{2.3}$$

which lead us to the important definition of *Hodge numbers*:

$$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(\mathcal{M}). \tag{2.4}$$

Hodge numbers are just the complex equivalent of Betti numbers, and we can easily relate the two via

$$b^k = \sum_{j=0}^{k} h^{j,k-j}. \tag{2.5}$$

Recalling that the Euler characteristic is defined by

$$\chi := \sum_{k=0}^{\dim_{\mathbb{R}} \mathcal{M}} (-1)^k b^k, \tag{2.6}$$

we obtain an expression for it in terms of the Hodge numbers:

$$\chi = \sum_{k=0}^{\dim_{\mathbb{R}} \mathcal{M}} (-1)^k \sum_{j=0}^{k} h^{j,k-j}. \tag{2.7}$$

We often display Hodge numbers in a *Hodge Diamond* (where $\dim_{\mathbb{R}}(\mathcal{M}) = 2m$)

$$\begin{array}{ccccccc} & & & h^{m,m} & & & \\ & & h^{m,m-1} & \vdots & h^{m-1,m} & & \\ h^{m,0} & & \cdots & & \cdots & & h^{0,m} \\ & & h^{1,0} & \vdots & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} \tag{2.8}$$

This seems like a lot, however the $(m+1)^2$ Hodge numbers are not independent. The relations depend on the type of manifold we are considering and what structures it has, but we notice already that complex conjugation of the tangent spaces gives us $h^{p,q} = h^{q,p}$. The Hodge star operator (which acts as we might imagine, namely $\star : \Omega^{p,q} \to \Omega^{m-p,m-q}$) also tells us that $h^{p,q} = h^{m-p,m-q}$.

The notion of Hodge decomposition also carries over to the complex case. Introducing the codifferential $\bar{\partial}^\dagger := \mp \star \bar{\partial} \star$, we define the $\Delta = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. A *harmonic* (p,q)-form is then defined via $\Delta\omega = 0$. We denote the space of harmonic (p,q)-forms on $\mathcal{M}$ as $\mathcal{H}^{p,q}(\mathcal{M})$. Finally, as with the real case, there exists an isomorphism

$$\mathcal{H}^{p,q}(\mathcal{M}) \cong H^{p,q}_{\bar{\partial}}(\mathcal{M}), \tag{2.9}$$

i.e. every (p,q)-form can be represented by a harmonic (p,q)-form.

2.1.1 Chern Classes

So far we have just extended some of the structures/operators defined on real vector bundles to their complex versions defined on complex vector bundles. We now want to introduce something very important that doesn't have a real vector bundle equivalent.

Definition. [Chern Class] Let $(E, \pi, \mathcal{M})$ be a complex vector bundle,[1] and let A be the connection on E with associated curvature 2-form $F = dA + A \wedge A$. Then we define the *total Chern class* of E as

$$c(E) := \det\left(1 + \frac{i}{2\pi}F\right) \tag{2.10}$$

If E has complex rank k, then we can expand $c(E)$ in terms of the *Chern classes*:

$$c(E) = c_0(E) + c_1(E) + ... + c_k(E), \tag{2.11}$$

where the subscript denotes the power of F contained within the expression, namely:

$$\begin{aligned} c_0(E) &= [1], \\ c_1(E) &= \left[\frac{1}{2\pi i}\operatorname{Tr} F\right], \\ c_2(E) &= \left[\frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \big(\operatorname{Tr} F \wedge \operatorname{Tr} F - \operatorname{Tr}(F \wedge F)\big)\right] \\ &\vdots \\ c_k(E) &= \left[\left(\frac{i}{2\pi}\right)^k \det F\right]. \end{aligned} \tag{2.12}$$

The main Chern class that is of interest to us is the first Chern class of the tangent bundle. The curvature 2-form for $T^{(1,0)}\mathcal{M}$ is given by $F = -iR$, where R is the Ricci curvature. We therefore have

$$c_1(\mathcal{M}) := c_1\left(T^{(1,0)}\mathcal{M}\right) = \left[\frac{1}{2\pi}R\right], \tag{2.13}$$

where we have defined what we mean by the first Chern class of a complex manifold. In particular, note that a Ricci flat complex manifold has vanishing first Chern class.

The other Chern class that will be important to us is the top Chern class. This is a top form on E. If we again consider $E = T^{(1,0)}\mathcal{M}$, then we note that $\dim_{\mathbb{R}} T^{(1,0)}\mathcal{M} = \dim_{\mathbb{R}} \mathcal{M}$,

[1] Here we use notation $(E, \pi, \mathcal{M})$ for bundles, where E and $\mathcal{M}$ are the total space and base space, respectively, and $\pi : E \mapsto \mathcal{M}$ is the projection.

and so we can integrate the top Chern class over $\mathcal{M}$ itself.[2] It turns out this top form in $\mathcal{M}$ is actually what is known as the *Euler form*, and integrating it over $\mathcal{M}$ gives you the Euler characteristic. That is (if $\dim_{\mathbb{R}} \mathcal{M} = 2m$)

$$\chi = \int_{\mathcal{M}} c_m(\mathcal{M}). \tag{2.14}$$

Before moving on to study examples of complex manifolds we introduce one last important concept.

Definition. [Chern Character] Let E be a complex vector bundle or rank r, and express the total Chern class via $c(E) = \prod_{i=1}^r (1 + x_i)$. We define the *Chern character* to be

$$ch(E) := \sum_{i=1}^{r} e^{x_i}. \tag{2.15}$$

Now the Chern character seems like a strange thing to define, however we now note that it has the two nice properties that

$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2) \quad \text{and} \quad ch(E_1 \otimes E_2) = ch(E_1)ch(E_2). \tag{2.16}$$

Next we note that if we have a complex line bundle L, then L has rank 1 and so our Chern class, as defined above, is simply $c(L) = (1+x_1)$, but we can compare this to $c(L) = 1+c_1(L)$ and conclude that $x_1 = c_1(L)$. We therefore have that

$$ch(L) = e^{c_1(E)} = \sum_{\ell=0}^{\infty} \frac{c_1(L)^\ell}{\ell!} \tag{2.17}$$

Now, it follows from the expressions above that if E is given by the direct sum of r line bundles $\{L_1, ..., L_r\}$ then we have

$$ch(E) = ch(L_1 \oplus ... \oplus L_n) = ch(L_1) + ... + ch(L_n) = e^{c_1(L_1)} + ... + e^{c_1(L_r)}. \tag{2.18}$$

If we now compare this to the fact that $ch(E) = \sum_i e^{x_i}$ when $c(E) = \prod_i (1 + x_i)$ we see that

$$c(L_1 \oplus ... \oplus L_n) = \big(1 + c_1(L_1)\big)...\big(1 + c_1(L_r)\big), \tag{2.19}$$

[2]More technically we pullback the top Chern form on E to a top form on $\mathcal{M}$.

and in particular

$$c(L^{\oplus r}) = \big(1 + c_1(L)\big)^r \tag{2.20}$$

We also have a nice result for the tensor product of line bundles. Let $E = L_1 \otimes ... \otimes L_n$, then

$$ch(E) = ch(L_1 \otimes ... \otimes L_r) = ch(L_1)...ch(L_n) = e^{x_1}...e^{x_n} = e^{x_1+...+x_n}, \tag{2.21}$$

where $x_i = c_1(L_i)$. Now comes the interesting bit: this is still a line bundle, as $\dim(V \otimes W) = \dim V \times \dim W$, so we can compare it to $ch(L) = e^{c_1(L)}$ and conclude that

$$c_1(L_1 \otimes ... \otimes L_n) = c_1(L_1) + ... + c_1(L_n). \tag{2.22}$$

What will be of particular use to us when trying to construct Calabi-Yau manifolds later will be the specific case of this result

$$c\big(L^{\otimes d}\big) = 1 + dc_1(L), \tag{2.23}$$

where L is some line bundle.

2.2 Projective Space

Definition. [Complex Projective Space] Consider $\mathbb{C}^{n+1}$ with coordinates $(z_0, ..., z_n)$. Then we define the complex projective space as

$$\mathbb{CP}^n := \{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \backslash \{0\} \,|\, (z_0, ..., z_n) = (\lambda z_0, ..., \lambda z_n), \text{ for } \lambda \in \mathbb{C}^*\}. \tag{2.24}$$

Note that $\dim \mathbb{CP}^n = n$. We denote the coordinates on $\mathbb{CP}^n$ with square brackets and colons, $[z_0 : ... : z_n]$. The charts in $\mathbb{CP}^n$ are given by the open sets

$$U_i := \{[z_0 : ... : z_n] \,|\, z_i \neq 0\} \subset \mathbb{CP}^n. \tag{2.25}$$

It is clear that the set $\mathcal{U} := \{U_i \,|\, i = 0, ...n\}$ forms an open cover of $\mathbb{CP}^n$. We call the coordinates $(z_0, ..., z_n)$ the *homogeneous coordinates* of $\mathbb{CP}^n$. In what follows we will often use the shorthand $z = z_0, ..., z_n$.

We now want to construct two very important types of line bundle, defined on projective spaces. We give the definitions in a wordy manor (to avoid being too abstract) but of course they can be written down very concretely.

Definition. [Tautological & Hyperplane Line Bundles] Consider the complex projective space $\mathbb{CP}^n$. There is a "natural" line bundle we can construct over this: namely attach to each point $[z_1 : ... : z_{n+1}] \in \mathbb{CP}^n$ the line "projected away", i.e. the line given by $\pi^{-1}([z_0 : ... : z_n]) = \{(\lambda z_0, ..., \lambda z_n) \,|\, \lambda \in \mathbb{C}^*\} \subset \mathbb{C}^{n+1}$. This is known as the *tautological* (or *canonical*) *line bundle*, and we denote it by $\mathcal{O}_{\mathbb{CP}^n}(-1)$. The dual line bundle is called the *hyperplane line bundle* and we denote it $\mathcal{O}_{\mathbb{CP}^n}(1)$. The transition functions for the hyperplane line bundle are given by $g_{ij} : U_i \cap U_j \to z_i/z_j$, where $U_i, U_j \in \mathcal{U}$. That is $g_{ij}([z]) = \frac{z_i}{z_j}[z]$.

Remark 2.2.1. As a technical aside, we have been a little sloppy with notation above. We denoted the tautological/hyperplane line bundles themselves using the $\mathcal{O}(\pm 1)$ notation. Really we should just use L/L^{-1}, and then $\mathcal{O}(\pm 1)$ denotes the sheaf of holomorphic sections $\Gamma(L)/\Gamma(L^{-1})$. However this is standard notation, and we shall use $\mathcal{O}(\pm 1)$ to denote both the bundle itself and sections of the bundle, with the understanding following from context.

Recalling that the product of line bundles is again a line bundle, we introduce the notation

$$\mathcal{O}_{\mathbb{CP}^n}(d) := \otimes^d \mathcal{O}_{\mathbb{CP}^n}(1) \quad \text{and} \quad \mathcal{O}_{\mathbb{CP}^n}(-d) := \otimes^d \mathcal{O}_{\mathbb{CP}^n}(-1). \tag{2.26}$$

Now comes an important proposition that we will use later.

Proposition 2.2.2. *Any homogeneous polynomial of degree k in $\mathbb{CP}^n$ can be canonically identified with the holomorphic sections $\mathcal{O}_{\mathbb{CP}^n}(k)$.*

Proof. Consider a polynomial of degree k in the homogeneous coordinates $[z_0 : ... : z_n]$

$$P_k(z) = \sum_{|\nu|=k} a_\nu z_0^{\nu_0} ... z_n^{\nu_n}, \tag{2.27}$$

where $a_\nu \in \mathbb{C}$, and the sum is over the v_i, subject to the constraint $\nu_0 + ... + \nu_n = k$. Now this is *not* a polynomial in $\mathbb{CP}^n$ as it isn't scale invariant, i.e. $P_k(\lambda z) = \lambda^k P_k(z)$ but we want $P_k(\lambda z) = P_k(z)$. This is easily fixed by considering one of the charts $U_i \in \mathcal{U}$: we then simply divide by z_i^k, which we now write in a suggestive manner

$$s_i \equiv \frac{P_k(z)}{z_i^k} = \sum_{|\nu|=k} a_\nu \left(\frac{z_0}{z_i}\right)^{\nu_0} ... \left(\frac{z_n}{z_i}\right)^{\nu_n}. \tag{2.28}$$

Now this is only defined on U_i (as this is where we are guaranteed $z_i \neq 0$), but we get a globally defined polynomial by patching together the different s_i by multiplying by $(z_i/z_j)^k$

on the overlap $U_i \cap U_j$. However we now notice that this is simply k times the hyperplane line bundle's transition functions $g_{ij} :\in U_i \cap U_j \to z_i/z_j$, so

$$s_j = g_{ij}^{-k} s_i. \tag{2.29}$$

We can therefore think of the global polynomial as a section of $\mathcal{O}_{\mathbb{CP}^n}(k)$. This map is clearly bijective, as an element of $\mathcal{O}_{\mathbb{CP}^n}(k)$ is a linear functional from $\mathbb{C}^k \to \mathbb{C}$, but this is basically the definition of a polynomial of degree k in $\mathbb{CP}^n$, which proves the proposition. ∎

There is now an important Lemma associated to the proposition above.

Lemma 2.2.3. *The homogeneous coordinates of $\mathbb{CP}^n$ can be identified as sections of the hyperplane line bundle.*

2.2.1 Chern Classes

We now want to find the Chern classes of $\mathbb{CP}^n$, the question is how do we do this? We start by clarifying what a vector field in $T^{(1,0)}\mathbb{CP}^n$ is, and in particular what a zero vector is here.

Recall that $\mathbb{CP}^n$ is defined to be the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\lambda \in \mathbb{C}^*$. We can define this in terms of a projection $\pi(z) = [z]$, i.e. the fibres are given by the lines we project away. Now we can define a vector field in $T^{(1,0)}\mathbb{CP}^n$ by pushing down a vector $\widetilde{v} \in T(\mathbb{C}^{n+1} \setminus \{0\})$. That is, consider some open subset $U \in \mathbb{CP}^n$, then we have an open subset in $\mathbb{C}^{n+1} \setminus \{0\}$ given by $\pi^{-1}(U)$. We define our $\widetilde{v}$ vector field over $\pi^{-1}(U)$, and then get a vector field over U as $v([z]) := \pi_{*z}\widetilde{v}(z) = \pi_{*\lambda z}\widetilde{v}(\lambda z)$, where the second equality is our projective condition.

Now we want to ask the question of "what is a zero vector in $\mathbb{CP}^n$?" Well, it follows from above that $v([z]) = 0$ when $\pi_{*z}\widetilde{v}(z) = 0$, i.e. $\widetilde{v}$ is an element of the vertical subspace of the fibre, defined precisely as

$$V_z(\mathbb{C}^{n+1} \setminus \{0\}) := \ker \pi_{*z}, \tag{2.30}$$

The horizontal subspace, $H_z(\mathbb{C}^{n+1} \setminus \{0\})$, is the remaining orthogonal piece. Finally, recalling that the fibres are given by scaling the point $[z] \in \mathbb{CP}^n$, we see that our zero vectors are given by the push downs of

$$\widetilde{v} = \lambda\left(z_0 \frac{\partial}{\partial z_0} + ... + z_n \frac{\partial}{\partial z_n}\right) = \lambda V_E, \tag{2.31}$$

where we have defined the *Euler vector field* $V_E := z_i \partial_{z_i}$, and where $\lambda \in \mathbb{C}$.

Ok why is this useful to us? Well we note that we can span our holomorphic tangent bundle $T^{(1,0)}\mathbb{CP}^n$ by the push downs of the vectors $\{s_i(z)\frac{\partial}{\partial z_i}\}$, where $s_i(z)$ is a section in

$\mathcal{O}_{\mathbb{CP}^n}(1)$. In this way, we can define a surjective mapping

$$\varphi : \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \to T^{(1,0)}\mathbb{CP}^n, \tag{2.32}$$

where surjectivity is understood as we can produce a basis of $T^{(1,0)}\mathbb{CP}^n$. However we have just seen that the kernel of this map is the trivial line bundle $\mathbb{C}$ (i.e. the λ appearing in front of V_E), and we can embed this into $\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}$, giving us the short exact sequence[3]

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \xrightarrow{\varphi} T^{(1,0)}\mathbb{CP}^n \longrightarrow 0. \tag{2.33}$$

Now, given that a short exact sequence of complex vector bundles $0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2$, obeys the Chern class relation $c(E) = c(E_1)c(E_2)$, we conclude

$$c\big(\mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)}\big) = c(\mathbb{C}) \cdot c\big(T^{(1,0)}\mathbb{CP}^n\big). \tag{2.34}$$

Finally, using that trivially $c(\mathbb{C}) = 1$ and Equation (2.20) with $H = c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$, we conclude (using $c(\mathbb{CP}^n) \equiv c\big(T^{(1,0)}\mathbb{CP}^n\big)$)

$$c(\mathbb{CP}^n) = (1 + H)^{n+1}. \tag{2.35}$$

2.2.2 Sum Of $\mathbb{CP}^n$s

We can slightly generalise the result above, by now considering the whole thing again but now over a sum of complex projective spaces. In other words our base space becomes

$$\mathbb{CP}^{n_1} \oplus ... \oplus \mathbb{CP}^{n_\ell}. \tag{2.36}$$

Basically the whole thing is completely analogous, however now our middle term in the sequence is

$$\mathcal{O}_{\mathbb{CP}^{n_1}}(1)^{\oplus(n_1+1)} \oplus ... \oplus \mathcal{O}_{\mathbb{CP}^{n_\ell}}(1)^{\oplus(n_\ell+1)}, \tag{2.37}$$

and similarly the holomorphic tangent space term changes. However clearly the expression above is just a Whitney sum of line bundles[4] and so we have

$$c\Big(\mathcal{O}_{\mathbb{CP}^{n_1}}(1)^{\oplus(n_1+1)} \oplus ... \oplus \mathcal{O}_{\mathbb{CP}^{n_\ell}}(1)^{\oplus(n_\ell+1)}\Big) = \prod_{i=1}^{\ell}(1 + H_i)^{n_i+1}, \tag{2.38}$$

[3] This is an example of an *Euler sequence*.

[4] Given two vector bundles with the same base space $(E_1, \pi_1, \mathcal{M})$ and $(E_2, \pi_2, \mathcal{M})$, the Whitney sum is the vector bundle $(E_1 \oplus E_2, \pi_{12}, \mathcal{M})$, whose fibre over any $x \in \mathcal{M}$ is the direct sum of vector spaces of the fibres in E_1 and E_2.

where $H_i = c_1(\mathcal{O}_{\mathbb{CP}^{n_i}}(1))$, and so we conclude

$$c\big(\mathbb{CP}^{n_1} \oplus ... \oplus \mathbb{CP}^{n_\ell}\big) = \prod_{i=1}^{\ell}(1+H_i)^{n_i+1}. \tag{2.39}$$

2.2.3 Submanifolds Of Projective Spaces

Although we haven't proven it, it is hopefully clear that $\mathbb{CP}^n$ is indeed a complex manifold. It turns out it is also a compact manifold. This then leads into the following important theorem of Chow [45], which we word in a language useful to us.[5]

Theorem 2.2.4 (Chow)**.** *Any subspace of $\mathbb{CP}^n$ constructed by considering the zero locus of a finite number of homogeneous polynomial equations, is a compact complex submanifold.*

We do not prove this theorem, but just clarify that it seems reasonable: a homogeneous polynomial is a polynomial of the homogeneous coordinates $[z_0 : ... : z_n]$, and if we construct a polynomial out of them, and consider the zero locus (i.e. the points at which $P(z) = 0$) then we can use this condition to relate one of the coordinates to some of the others. In this way we reduce the dimension of the manifold we are considering by one. If we take two such polynomials and consider their mutual zero locus (i.e. the points when both $P_1(z)$ and $P_2(z)$ vanish), then we reduce the dimension by 2. This idea clearly generalises to saying that for every polynomial we introduce, we reduce the dimension by one. We call a manifold produced by the common zero locus of a finite collection of polynomials a *complete intersection*. Of course this does not prove that the resulting space is a compact, complex manifold, but we accept that as true and move on.

Given Chow's theorem, we can ask the question "what are the Chern classes of the resulting complex submanifolds?" The answer to this question will prove immensely useful to us later, but we shall answer it now.

Let $\mathcal{S} \subset \mathbb{CP}^n$ be a smooth hypersurface submanifold given by the zero locus of a homogeneous polynomial of degree d, $P(z)$, which we recall can be identified with a section of $\mathcal{O}_{\mathbb{CP}^n}(d)$. We now define the *normal bundle* of $\mathcal{S}$ to be

$$N_{\mathcal{S}} := \frac{T^{(1,0)}\mathbb{CP}^n|_{\mathcal{S}}}{T^{(1,0)}\mathcal{S}}. \tag{2.40}$$

That is, we consider the holomorphic vectors in $\mathbb{CP}^n$, restricted to $\mathcal{S}$, and quotient by vectors

[5] The technical content of Chow's theorem is: *an analytic subspace of a complex projective space, which is given by a closed subset, is an algebraic subvariety.* This language will be more meaningful to us later, but for now we just claim it implies what we write.

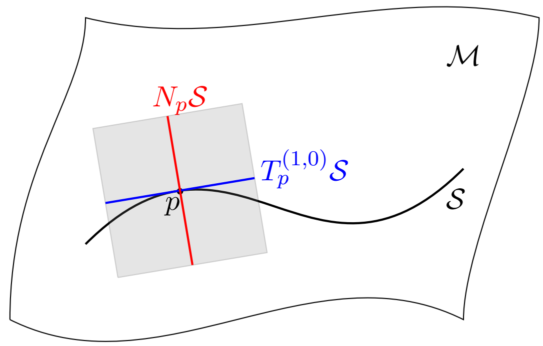


Figure 2.1: *A pictorial explanation of the normal bundle in two-dimensions. $T_p^{(1,0)}\mathcal{S}$ (blue) and $N_p\mathcal{S}$ (red) are holomorphic tangent and normal planes over $\mathcal{S} \subset \mathcal{M}$, at a point $p \in \mathcal{S}$. The shaded region is then $T_p^{(1,0)}\mathcal{M} \in T^{(1,0)}\mathcal{M}|_{\mathcal{S}}$, i.e. the holomorphic tangent plane at $p \in \mathcal{M}$. It is given by the span of both $N_p\mathcal{S}$ and $T_p^{(1,0)}\mathcal{S}$. The normal bundle, $N_{\mathcal{S}}$, is then formed in the usual way: the fibres are the $N_p\mathcal{S}$. We then get $T^{(1,0)}\mathcal{M}|_{\mathcal{S}} = T^{(1,0)}\mathcal{S} \oplus N_{\mathcal{S}}$, which can be rearranged to give a definition of the normal bundle.*

that are themselves tangent to $\mathcal{S}$. This clearly only leaves vectors normal to $\mathcal{S}$, hence the name. We demonstrate this pictorially for an abstract 2-dimensional manifold in Figure 2.1.

Now comes the crucial point: as we mentioned already, we can view $\mathcal{S}$ as the zero locus of our polynomial $P(z)$. However recall that Proposition 2.2.2 told us that a polynomial of degree d can be identified with a section of $\mathcal{O}_{\mathbb{CP}^n}(d)$, from which we conclude that $\mathcal{S}$ should be identified with the zeros in the fibres of $\mathcal{O}_{\mathbb{CP}^n}(d)$. In fact the normal bundle $N_{\mathcal{S}}$ of $\mathcal{S}$ is actually just given by $\mathcal{O}_{\mathbb{CP}^n}(d)|_{\mathcal{S}}$.[6] Finally, noting that essentially what we said above about the splitting of $T^{(1,0)}\mathbb{CP}^n|_{\mathcal{S}}$ into the normal bundle and $T^{(1,0)}\mathcal{S}$ is just the statement that

$$T^{(1,0)}\mathbb{CP}^n|_{\mathcal{S}} = T^{(1,0)}\mathcal{S} \oplus N_{\mathcal{S}} = T^{(1,0)}\mathcal{S} \oplus \mathcal{O}_{\mathbb{CP}^n}(d)|_{\mathcal{S}}, \tag{2.41}$$

we have the (split) short exact sequence

$$0 \longrightarrow T^{(1,0)}\mathcal{S} \longrightarrow T^{(1,0)}\mathbb{CP}^n|_{\mathcal{S}} \longrightarrow \mathcal{O}_{\mathbb{CP}^n}(d)|_{\mathcal{S}} \longrightarrow 0. \tag{2.42}$$

If we then use the result for the Chern classes of a short exact sequence, we can compute the

[6]This is linked to the *adjunction formulas*. See, e.g., [46] for more details.

total Chern class of $\mathcal{S}$ as

$$c(\mathcal{S}) = \frac{c(T^{(1,0)}\mathbb{CP}|_{\mathcal{S}})}{c(\mathcal{O}_{\mathbb{CP}^n}(d)|_{\mathcal{S}})} = \frac{c(\mathbb{CP}^n)}{c(\mathcal{O}_{\mathbb{CP}^n}(d))}, \tag{2.43}$$

where we have used that the total Chern class doesn't depend on whether we restrict to $\mathcal{S}$ or not. So finally recalling Equation (2.35) and Equation (2.23) (which tells us that $c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + dH$) we finally conclude

$$c(\mathcal{S}) = \frac{(1+H)^{n+1}}{1+dH}. \tag{2.44}$$

where as always $H = c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$.

Generalising

We can generalise this result to the cases when we consider a complete intersection manifold, i.e. our submanifold is now given by the common zero locus of multiple homogeneous polynomials. Let's say there are k polynomials of degrees d_i, $i \in \{1, ..., k\}$. Then it is hopefully intuitively clear that in this case we have that the result $N_{\mathcal{S}} = \mathcal{O}_{\mathbb{CP}^n}(d)|_{\mathcal{S}}$ generalises to

$$N_{\mathcal{S}} = \mathcal{O}_{\mathbb{CP}^n}(d_1)|_{\mathcal{S}} \oplus ... \oplus \mathcal{O}_{\mathbb{CP}^n}(d_k)|_{\mathcal{S}}, \tag{2.45}$$

i.e. each $\mathcal{O}_{\mathbb{CP}^n}(d_i)$ term represents the polynomial of degree d_i, and the direct sum the fact that we must satisfy all of them. Now recalling that each $\mathcal{O}_{\mathbb{CP}^n}(d_i)$ is itself a line bundle, we can use Equation (2.20) to obtain

$$c(N_{\mathcal{S}}) = \prod_{i=1}^{k}(1 + d_i H), \tag{2.46}$$

which gives us

$$c(\mathcal{S}) = \frac{(1+H)^{n+1}}{\prod_{i=1}^{k}(1+d_i H)}. \tag{2.47}$$

We can generalise this result further by allowing our base space to be a sum of complex projective spaces. However we need to be a bit more careful then simply plugging Equation (2.39) into the numerator of the above expression. The reason is that our polynomials could have different degrees in the different $\mathbb{CP}^{n_i}$s. For example, if we had $\mathbb{CP}^2 \oplus \mathbb{CP}^3$, which is a complex 5-dimensional manifold, we can produce a complex 2-dimensional manifold by introducing 3 polynomials. These polynomials can be of different degrees to each other, but we also have to take into account how the degree of each polynomial is distributed across the $\mathbb{CP}^2$ and $\mathbb{CP}^3$. We summarise this information in a *configuration matrix*. Say, for example, our polynomials had degrees $(1,3)$, $(4,2)$ and $(5,0)$, where (i,j) means degree i in the homogeneous coordinates of $\mathbb{CP}^2$ and degree j in the homogeneous coordinates of $\mathbb{CP}^3$, then our configuration matrix would be

$$\begin{array}{c|ccc|} \mathbb{CP}^2 & 1 & 4 & 5 \\ \mathbb{CP}^3 & 3 & 2 & 0 \end{array}_{\chi}, \tag{2.48}$$

where we have also indicated that we normally include the Euler characteristic in the bottom right, as this is a topological invariant of the space. To be completely clear on what the polynomials above are, if we denote the homogeneous coordinates of $\mathbb{CP}^2$ by $[z_0 : z_1 : z_3]$ and those of $\mathbb{CP}^3$ by $[w_0 : w_1 : w_2 : w_3]$, then a *particular* example would be

$$\begin{aligned} P_1(z,w) &= z_0 w_0^2 w_1 + z_2 w_0 w_2 w_3, \\ P_2(z,w) &= z_1^3 z_2 w_1 w_3, \\ P_3(z,w) &= z_0^2 z_1^2 z_3 + z_0^4 z_3. \end{aligned} \tag{2.49}$$

Luckily, the result for the total Chern class is relatively simple, given what we already know: the polynomials above are simply sections in $\mathcal{O}_{\mathbb{CP}^2}(1) \otimes \mathcal{O}_{\mathbb{CP}^3}(3)$, $\mathcal{O}_{\mathbb{CP}^2}(4) \otimes \mathcal{O}_{\mathbb{CP}^3}(2)$ and $\mathcal{O}_{\mathbb{CP}^2}(5)$, respectively. To write down the final result we want, we now consider the completely general configuration matrix

$$\begin{array}{c|ccc|} \mathbb{CP}^{n_1} & d_1^1 & \dots & d_k \\ \vdots & \vdots & & \vdots \\ \mathbb{CP}^{n_\ell} & d_1^\ell & \dots & d_k^\ell \end{array}_{\chi}, \tag{2.50}$$

we get that the total Chern class of $\mathcal{S}$ is given by

$$c(\mathcal{S}) = \frac{\prod_{i=1}^{\ell}(1+H_i)^{n_i+1}}{\prod_{r=1}^{k}(1+\sum_{s=1}^{\ell} d_r^s H_s)}, \tag{2.51}$$

where $H_i = c_1(\mathcal{O}_{\mathbb{CP}^{n_i}}(1))$.

2.3 Kähler Geometry

The final stepping stone needed to introduce Calabi-Yau geometry is Kähler geometry, which we do briefly now.

Definition. [Hermitian Metric/Manifold/Form] Let $(\mathcal{M}, I, g)$ be a complex manifold with Riemannian metric g and complex structure I. Then we call g *Hermitian* if $g(v,u) = g(Iv, Iu)$ for all $v, u \in T\mathcal{M}$. $\mathcal{M}$ is then called a Hermitian manifold. The Hermitian metric can then be used to define the *Hermitian form*:

$$J(v,u) = g(Iv,u), \tag{2.52}$$

for all vector fields u, v. The Hermitian form is a $(1,1)$-form, i.e. $J \in \Omega^{(1,1)}\mathcal{M}$.

Definition. [Kähler Form/Metric/Manifold] Let $(\mathcal{M}, I, g)$ be a Hermitian manifold with Hermitian form J. Then if J is closed we call it a Kähler form. The metric and manifold are then also called Kähler.

Proposition 2.3.1. *The following three conditions are equivalent (here ∇ is the Levi-Civita connection):*

(i) $\nabla J = 0$ *(i.e.* $\nabla_v J = 0$ *for arbitrary vector field* v*),*

(ii) $\nabla I = 0$*, and*

(iii) $dJ = 0$.

In particular, the existence of a Kähler form is equivalent to the complex structure being covariantly conserved.

This proposition is important as it tells us about the holonomy of Kähler manifolds. Indeed, the fact that the complex structure is covariantly conserved tells us that parallel transport must respect the decomposition of the tangent bundle into its holomorphic and antiholomorphic sectors. In other words, the holonomy group must preserve Hermiticity, which gives

$$\mathrm{Hol}(\mathcal{M}) \subseteq U(\dim_{\mathbb{C}} \mathcal{M}) \tag{2.53}$$

for a Kähler manifold $\mathcal{M}$.

Claim 2.3.2. Complex projective spaces are Kähler manifolds. The metric is given by the so-called Fubini-Study metric.

We do not prove this claim here, as the details are not directly relevant, but they can be found in, e.g., [40]. The importance of this claim comes from the following proposition.

Proposition 2.3.3. *Any complex submanifold of a Kähler manifold is itself a Kähler manifold.*

Proof. We do not prove this in detail, but simply point out that it is reasonable: the Kähler form is globally defined and closed, so if we restrict it to some submanifold, we will again get a closed $(1,1)$-form defined over all of our submanifold. A bit more technically, this is seen by the fact that the exterior derivative commutes with the pullback, and we can pull the Kähler form back from $\mathcal{M}$ onto the submanifold, and so $d(\varphi^* J) = \varphi^*(dJ) = 0$, and so the induced form is closed. ■

2.3.1 HyperKähler Geometry

Before moving on to the definition of a Calabi-Yau manifold, we quickly introduce hyperKähler manifolds.

> **Definition.** [HyperKähler Manifold] Let $(\mathcal{M}, g)$ be a Riemannian manifold and let (I, J, K) be a triple of complex structures. We call $(\mathcal{M}, g, I, J, K)$ a *hyperKähler manifold* if (I, J, K) are all Kähler w.r.t. the metric (i.e. define a Kähler manifold) and obey the quaternionic relations: $I^2 = J^2 = K^2 = IJK = -1$. HyperKähler manifolds of real dimension n have holonomy $\mathrm{Hol}(\mathcal{M}) \subset Sp(n/4)$, where $Sp(\cdot)$ is the compact symplectic group.

Proposition 2.3.4. *There is an S^2 worth of metric compatible complex structures on a hyperKähler manifold.*

Proof. If (I, J, K) are the complex structures of our hyperKähler manifold, then it is easy to check that

$$aI + bJ + cK \quad \text{where} \quad a^2 + b^2 + c^2 = 1 \tag{2.54}$$

is also a valid complex structure, and is Kähler w.r.t. the metric. ■

2.4 Calabi-Yau Geometry

We are now in a position to formally introduce Calabi-Yau manifolds. These are a particular kind of Kähler manifold, and as we argued in the last chapter play an important role in string theory compactifications. We start with a theorem due to Yau [36].

Theorem 2.4.1 (Yau)**.** *Let $(\mathcal{M}, I, g)$ be a compact Kähler manifold with associated Kähler form J. Further let R be $(1,1)$-form which represents the first Chern class of $\mathcal{M}$, i.e. $[R] \propto c_1(\mathcal{M})$. Then there exists a* ***unique*** *Kähler metric $\widetilde{g}$ on $\mathcal{M}$ with associated Kähler form $\widetilde{J}$ such that $[\widetilde{J}] = [J] \in H^2_{dR}(\mathcal{M}; \mathbb{R})$ and the Ricci form associated to $\widetilde{g}$ is R.*

This theorem is not easy to prove however, recalling that $c_1(\mathcal{M}) \sim [R]$, we get the following important corollary.

Corollary 2.4.2. *Let $(\mathcal{M}, I, g)$ be compact Kähler manifold with Kähler form J. Then, if $c_1(\mathcal{M}) = 0$ there exists a unique equivalent Kähler form, $[\widetilde{J}] = [J]$, such that $\widetilde{g}$ is Ricci flat.*

The resulting Ricci flat Kähler manifold is called a Calabi-Yau manifold. There are several, equivalent, definitions of a Calabi-Yau manifold, and we list the ones important to us in the following definition.[7]

Definition. [Calabi-Yau Manifold] Let $(\mathcal{M}, I, g)$ be a Kähler manifold of real dimension $2m$. Then we call it a *Calabi-Yau m-fold* if any of the following hold:

(i) $\mathcal{M}$ is Ricci flat, $R = 0$;

(ii) The first Chen class vanishes, $c_1(\mathcal{M}) = 0$;

(iii) The holonomy group is restricted to $\mathrm{Hol}(\mathcal{M}) \subseteq SU(m)$;

(iv) The canonical bundle is trivial (i.e. admits a global, non-vanishing section);

(v) $\mathcal{M}$ admits a globally defined, nowhere vanishing holomorphic m-form. Will we typically denote this as $\Omega \in H^{m,0}$.

2.4.1 Hodge Numbers

An important property of Calabi-Yau manifolds are their Hodge numbers. We already saw that Hodge numbers for a generic complex manifold obey:

- Complex conjugation: $h^{p,q} = h^{q,p}$, and
- Hodge star duality: $h^{p,q} = h^{m-p,m-q}$, where $m = \dim_{\mathbb{C}} \mathcal{M}$

Calabi-Yau manifolds have further restrictions on their Hodge numbers:

- It follows from condition (v) of the definition that $h^{m,0} = 1$. That is, Ω is a top dimensional holomorphic form, and therefore any other $(m,0)$-form can be expressed as $\alpha = f\Omega$, for some holomorphic function f.
- Given a $[\alpha] \in H^{0,q}(\mathcal{M})$, we have a unique $[\beta] \in H^{0,m-q}(\mathcal{M})$ such that

$$\int_{\mathcal{M}} \alpha \wedge \beta \wedge \Omega = 1. \tag{2.55}$$

[7] A nice discussion on how these different conditions agree can be found in [40].

This follows simply from the fact that the integrand is a (m, m)-form and so is (proportional to) the unique volume form. This implies that $h^{0,q} = h^{0,m-q}$, which can be paired with complex conjugation to give $h^{p,0} = h^{m-p,0}$. This is known as the *holomorphic duality*.

- It can be shown (see, e.g. [47]) that there are no 1-forms $h^{1,0} = 0$.[8]

Calabi-Yau 3-Folds

As we have explained, the case of interest are Calabi-Yau 3-folds, i.e. $m = 3$. The above conditions can then be used to simplify the Hodge diamond significantly. In particular we see that the only undetermined Hodge numbers are $h^{1,1}$ and $h^{2,1}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.56}$$

We can use this, along with the expression for the Euler characteristic, Equation (2.7), to obtain

$$\chi = 2(h^{1,1} - h^{2,1}) \tag{2.57}$$

for a Calabi-Yau 3-fold. The number $h^{1,1}$ classifies the infinitesimal deformations of the Kähler structure, while $h^{2,1}$ classifies the infinitesimal deformations of the complex structure. As we will see, the latter is generally not too difficult to compute, but the former can be a lot more challenging. However we are saved by recalling Equation (2.14), i.e. that we can express the Euler characteristic in terms of the top Chern class. Using this with the above, we then get the simple relation

$$\int_{\mathcal{M}} c_3(\mathcal{M}) = 2(h^{1,1} - h^{2,1}) = \chi \tag{2.58}$$

for a Calabi-Yau 3-fold.

[8] Here we have assumed that our complex dimension is at least $m = 2$. Otherwise this would clearly contradict $h^{m,0} = 1$ when $m = 1$.

$K3$ Surfaces

As will become clear later, we will also be interested in the case of Calabi-Yau 2-folds. These are also known as $K3$ surfaces, and we will use this name going forward in order to avoid confusion with our Calabi-Yau 3-folds. At first glance, there is one undetermined Hodge number, $h^{1,1}$. However, it turns out that the Euler characteristic for *every* $K3$ surface is the same: $\chi = 24$. In therms of the Hodge numbers we have

$$\chi = \sum_{k=0}^{4} (-1)^k b^k = 4 + h^{1,1}, \tag{2.59}$$

and so we see that every $K3$ surface actually has $h^{1,1} = 20$ and $b^2 = 22$. This gives insight into the fact that all $K3$ surfaces are in fact diffeomorphic as real manifolds to each other [48].

2.4.2 $K3$ Surfaces

The majority of the work that follows will be concerned with Calabi-Yau 3-folds, however a good understanding of $K3$ surfaces will be important. We therefore spend a little bit of time here going over some of the important properties of $K3$ surfaces. In what follows, we shall denote a $K3$ surface by the letter S.

Moduli Space

The main thing that will be important to us are notions related to the moduli spaces of complex structures and Ricci flat metrics (which are related to the Kähler form) for $K3$ surfaces. The material is based on Sections 2.3 and 2.4 of the great review of Aspinwall [49].

In order to measure the complex structure we introduce *periods*, which are defined as integrals of the holomorphic $(2,0)$-form Ω over integral 2-cycles in S. We start by defining an inner product on $H_2(S, \mathbb{Z})$ using the oriented intersection number of homology:

$$\alpha_1 \cdot \alpha_2 = \#(\alpha_1 \cap \alpha_2) \tag{2.60}$$

for $\alpha_1, \alpha_2 \in H_2(S, \mathbb{Z})$. This inner product gives $H_2(S, \mathbb{Z})$ the structure of a 22-dimensional lattice, and it can be shown that the signature of this lattice is $(3, 19)$. Let's denote the basis of this lattice by $\{e_i\}$. Next, we can use Poincaré duality to obtain another basis $\{e_i^*\}$ such that

$$e_i \cdot e_j^* = \delta_{ij}, \tag{2.61}$$

so that our lattice is self-dual. Lastly, the lattice can be shown to be even:

$$e \cdot e \in 2\mathbb{Z} \tag{2.62}$$

for all 2-cycles $e \in H_2(S, \mathbb{Z})$.

In total, then, we have that $H_2(S, \mathbb{Z})$ is isomorphic to a 22-dimensional even, self-dual lattice, with signature $(3, 19)$. We shall denote this lattice by $\Gamma^{3,19}$. It is known (see [50] for details) that these conditions completely fix $\Gamma^{3,19}$ to be of the form $\mathrm{diag}(-E_8, -E_8, U, U, U)$, where E_8 is the Cartan matrix of E_8 and

$$U \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.63}$$

is the hyperbolic lattice. In what follows, we shall simply write

$$\Gamma^{3,19} = (-E_8)^{\oplus 2} \oplus U^{\oplus 3} \tag{2.64}$$

We then have the following result, which we don't prove (see [49] for details).

Proposition 2.4.3. *Let S be a $K3$ surface with associated lattice $\Gamma^{3,19}$ as above. By considering $H^2(S, \mathbb{R}) = H^2(S, \mathbb{Z}) \otimes \mathbb{R}$, we get $\Gamma^{3,19} \subset \mathbb{R}^{3,19}$. The choice of a complex structure on S is then given by an oriented 2-plane $\Omega = x + iy$ for $x, y \in H^2(S, \mathbb{R})$. Changing the complex structure of S rotates Ω w.r.t. $\Gamma^{3,19}$.*

Next we consider the set of 2-forms on S. It follows from Poincaré duality that the lattice of integral cohomology is isomorphic to the lattice of integral homology, i.e. $H^2(S, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$. This tells us that the decomposition of $H^2(S, \mathbb{Z})$ into self-dual ($\mathcal{H}^+$) and anti-self-dual, ($\mathcal{H}^-$) forms,

$$H^2(S, \mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-, \tag{2.65}$$

must obey

$$\dim \mathcal{H}^+ = 3 \quad \text{and} \quad \dim \mathcal{H}^- = 19. \tag{2.66}$$

Let $\{s_1, s_2, s_3\} \subset H^2(S, \mathbb{R})$ be a valid basis for $\mathcal{H}^+$ that can be expressed in terms of the orthonormal frame of the cotangent bundle, $\{e_1, e_2, e_3, e_4\}$ as

$$\begin{aligned} s_1 &= e_1 \wedge e_2 + e_3 \wedge e_4 \\ s_2 &= e_1 \wedge e_3 + e_4 \wedge e_2 \\ s_1 &= e_1 \wedge e_4 + e_2 \wedge e_3. \end{aligned} \tag{2.67}$$

Then defining $dz_1 = e_1 + ie_2$ and $dz_2 = e_3 + ie_4$ we obtain the Kähler form as $J = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 = s_1$ and the holomorphic $(2,0)$-form as $\Omega^{(2,0)} = dz_1 \wedge dz_2 = s_2 + is_3$. The key thing is that $s_2 + is_3$ takes the form of our Ω above. So, if we denote by Σ the subspace $\mathcal{H}^+ \subset H^2(S, \mathbb{R})$, then we have that Σ is spanned by a choice of complex structure (i.e. an Ω) along with a direction in $H^2(S, \mathbb{R})$ specified by the Kähler form on S.

In particular we notice that rotations of Σ within $\Gamma^{3,19}$ can affect what we call the Kähler form and what we call the complex structure. This tradeable nature of the Kähler form and complex structure is particular property of $K3$ surfaces and does not apply to their 3-fold counterparts.

HyperKähler Structure

The other piece of information we will need later is that $K3$ surfaces admit a hyperKähler structure. This comes simply from the fact that $SU(2) \cong Sp(1)$. Proposition 2.3.4 tells us that hyperKähler manifolds have a S^2 worth of possible complex structures, and above this is exactly the freedom to rotate Σ. A rotation within this S^2 is referred to as a *hyperKähler rotation* of the $K3$, and will be incredibly useful later on.

2.4.3 Mirror Pairs

We can now consider compactifying string theories on Calabi-Yau manifolds. As explained in the last chapter, the cases of interest to us are the compactification of Type II strings on Calabi-Yau 3-folds. It can be shown (see, e.g., [51]) that the compactification of Type IIA strings on a Calabi-Yau 3-fold results in a 4-dimensional theory that contains $h^{1,1}$ abelian vector multiplets and $(h^{2,1}+1)$ hypermultiplets. Similarly Type IIB results in a 4-dimensional theory with $h^{2,1}$ abelian vector multiplets and $(h^{1,1}+1)$ hypermultiplets. The key observation is that these two results are symmetric under the exchange of $h^{1,1}$ and $h^{2,1}$. We call such a symmetry of the 4D physics *mirror symmetry*.

This observation becomes more interesting when noticing that such an exchange would simply result in mirroring the Hodge diamond along the diagonal. In particular, the resulting Hodge diamond is a valid Hodge diamond for a new Calabi-Yau with $h^{1,1}_{\text{new}} = h^{2,1}_{\text{old}}$ and $h^{2,1}_{\text{new}} = h^{1,1}_{\text{old}}$. This new Calabi-Yau would also have reversed Euler characteristic, $\chi_{\text{new}} = -\chi_{\text{old}}$. We call such a pair of Calabi-Yaus a *mirror* pair. Of course, we are not in general guaranteed that such mirror pairs exist. However, in [52] Kreuzer & Skarke obtained a list of the Hodge data of almost half a billion Calabi-Yau 3-folds. This list contains 30108 unique values for $(h^{1,1}, h^{2,1})$, which can be used to plot Figure 2.2 which gives strong support of the existence of Calabi-Yau mirror pairs.[9]

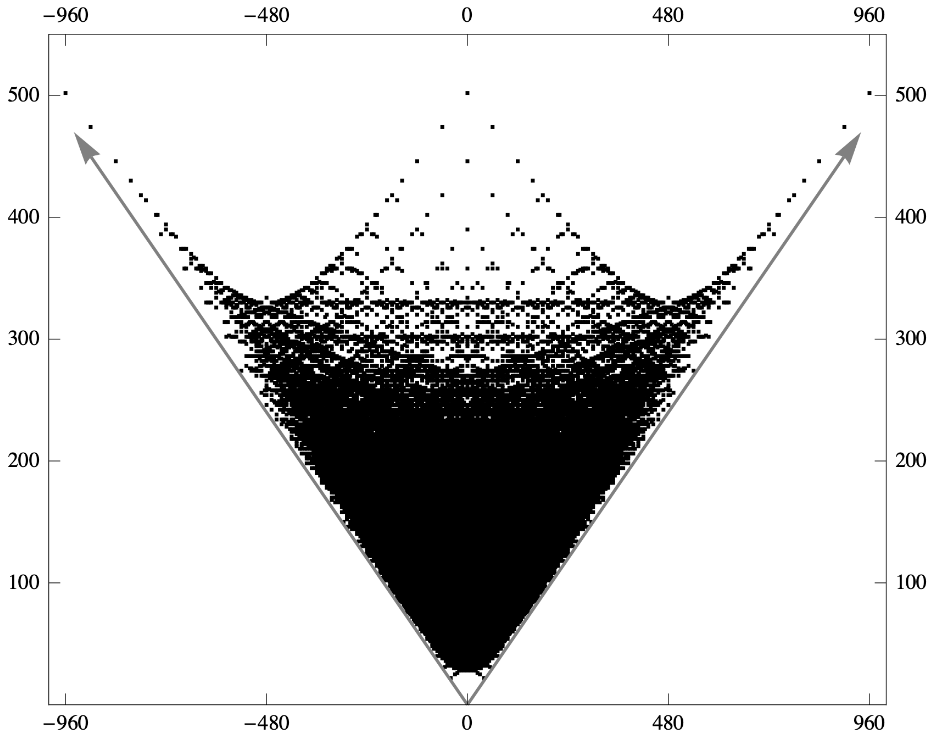


Figure 2.2: *A plot of $h^{1,1} + h^{2,1}$ against $\chi = 2(h^{1,1} - h^{2,1})$ for the Calabi-Yau 3-folds in Kreuzer-Skarke's list. We note that there is a close to perfect symmetry in the diagram, which gives strong support for the existence of mirror pairs of Calabi-Yau manifolds. Figure taken from [53].*

The majority of the rest of this chapter is dedicated to proving the existence of mirror Calabi-Yau pairs. We stress at this point that the two Calabi-Yaus have vastly different geometry and it is not a priori obvious that they would be so intimately related.

2.4.4 Lagrangian Submanifolds & The Antiholomorphic Involution

Before going on to how to construct of Calabi-Yaus using complex projective spaces, we first mention something that will be useful for us when discussing manifolds with G_2 holonomy.

Definition. [Lagrangian Submanifold] Let $\mathcal{M}$ be a symplectic manifold of dimension $2n$ with symplectic form ω. Then a submanifold $L \subset \mathcal{M}$ is called a *Lagrangian submanifold* if

(i) $\omega|_L = 0$, and

[9]Historically speaking, the Kreuzer-Skarke list was obtained after significant understanding of mirror symmetry was developed. However, a similar plot with less data points (2339 pairs of Hodge numbers) was given in [6] before the mathematical development of mirror symmetry.

(ii) $\dim L = \frac{1}{2} \dim \mathcal{M} = n$.

If we view the symplectic manifold as a complex manifold, and denote the volume form by Ω, then a Lagrangian submanifold is called *special* if, on top of the above two conditions, we further have $\mathrm{Im}(\Omega)|_L = 0$.

Langrangian submanifolds will be of interest to us in the context of fixed points of quotients of Calabi-Yaus. We recall from Section 1.2 that the important case will be an antiholomorphic involution, which we define now.

Definition. [Antiholomorphic Involution] Let $(\mathcal{M}, I, g)$ be a Calabi-Yau n-fold. Then a diffeomorphism $\sigma : \mathcal{M} \to \mathcal{M}$ is called an *antiholomorphic involution* if it satisfies:

(i) It is an involution; $\sigma^2 = \mathrm{id}$,

(ii) It is an isometry; $\sigma^*(g) = g$, and

(iii) It is antiholomorphic; $\sigma^*(I) = -I$.

An important consequence is that $\sigma^*(J) = -J$ and $\sigma^*(\Omega) = \bar{\Omega}$.

Corollary 2.4.4. *The fixed point locus of an antiholomorphic involution on a Calabi-Yau n-fold is a special Lagrangian submanifold.*

Proof. This follows simply from the fact that J is a symplectic form and Ω is used for the volume form. In the fixed point locus L_σ we have $J = 0$ and $\mathrm{Im}(\Omega) = 0$. ■

2.5 Constructing Calabi-Yaus In $\mathbb{CP}^n$

Recall that $\mathbb{CP}^n$ is a Kähler manifold and that any submanifold of a Kähler manifold is again Kähler. We can therefore ask the question of whether we can use complex projective spaces to form Calabi-Yaus. The answer is yes, and it is one of the most useful techniques to constructing them.

The procedure is straight forward: we define a Calabi-Yau manifold as a Kähler manifold with vanishing first Chern class, and we have a formula for the total Chern class for a hypersurface in $\mathbb{CP}^n$ given by the zero locus of a quasihomogenous polynomial, Equation (2.44). We can use this to find the degree of the polynomial needed to get vanishing first Chern class. We start with the general result and then work through some examples. We will then discuss the case of weighted projective spaces at the end.

2.5.1 General Result

Let $\mathcal{S} \subset \mathbb{CP}^{n_1} \oplus ... \oplus \mathbb{CP}^{n_\ell}$ be a given by the complete intersection of k polynomials with degrees d_r^s. Then the configuration matrix is

$$\begin{array}{c|ccc|} \mathbb{CP}^{n_1} & d_1^1 & \dots & d_k \\ \vdots & \vdots & & \vdots \\ \mathbb{CP}^{n_\ell} & d_1^\ell & \dots & d_k^\ell \end{array}_{\chi} , \tag{2.68}$$

and the total Chern class of $\mathcal{S}$ is given by

$$c(\mathcal{S}) = \frac{\prod_{i=1}^{\ell}(1+H_i)^{n_i+1}}{\prod_{r=1}^{k}(1+\sum_{s=1}^{\ell} d_r^s H_s)}. \tag{2.69}$$

We can expand the numerator and denominator in powers of H, and read the first Chern class off as the term linear in H_i:

$$c_1(\mathcal{S}) = \sum_{i=1}^{\ell} \left(n_i + 1 - \sum_{r=1}^{k} d_r^i \right) H_i. \tag{2.70}$$

Therefore, we get a Calabi-Yau manifold of dimension $(\sum_{i=1}^{\ell} n_i - k)$ when

$$\sum_{r=1}^{k} d_r^i = n_i + 1 \qquad \forall i \in \{1, ..., \ell\}. \tag{2.71}$$

If we continue the expansion of our total Chern class, we can find the top Chern class, which if we then integrate over $\mathcal{S}$ gives us our Euler characteristic. This is, of course, technically correct although it is often quite hard to compute in practice. However integrating on $\mathbb{CP}^n$ is *much* simpler, so we ask the question "is there any way we can get the result of $\int_{\mathcal{S}} c_{\text{top}}(\mathcal{S})$ as an integral over $\mathbb{CP}^n$?" The answer is yes, and it comes from using a modified version of Poincaré duality, which we state in the next theorem (see, e.g., [54] for more details).

Theorem 2.5.1. *Let $\mathcal{M}$ be an n-dimensional manifold, and let $\mathcal{S} \subset \mathcal{M}$ be some closed, k-dimensional submanifold. Then for any closed k-form $[\tau] \in H^k_{dR}(\mathcal{M};\mathbb{R})$ there exists a closed $(n-k)$-form $[\eta_{\mathcal{S}}] \in H^{n-k}_{dR}(\mathcal{M};\mathbb{R})$ such that*

$$\int_{\mathcal{S}} \tau = \int_{\mathcal{M}} \tau \wedge \eta_{\mathcal{S}}. \tag{2.72}$$

We call $\eta_{\mathcal{S}}$ the Poincaré dual class to $\mathcal{S}$.

The Poincaré class can be thought of as a delta function which restricts us to $\mathcal{S} \subset \mathcal{M}$. The true use of this comes from considering cases where the normal bundle to $\mathcal{S}$ is given by the restriction of a bundle E over $\mathcal{M}$, as in this case we simply have $\eta_{\mathcal{S}} = c_r(E)$ where r is the rank of E. This is exactly the case we are considering, e.g. $E = \mathcal{O}_{\mathbb{CP}^n}(d)$ and $N_{\mathcal{S}} = \mathcal{O}_{\mathbb{CP}^n}(d)|_{\mathcal{S}}$. So finally using that for us $\dim \mathcal{S} = \sum_{i=1}^{\ell} n_i$ and $\dim N_{\mathcal{S}} = k$ (that is each polynomial increases the dimension of the normal bundle by 1) we have

$$\chi = \int_{\mathbb{CP}^{n_1}\oplus\ldots\oplus\mathbb{CP}^{n_\ell}} c_{\sum_{i=1}^{\ell} n_i}(\mathcal{S}) \wedge c_k(E), \tag{2.73}$$

where

$$E = \bigoplus_{r=1}^{k} \left[\bigotimes_{s=1}^{\ell} \mathcal{O}_{\mathbb{CP}^{n_s}}(d_r^s) \right]. \tag{2.74}$$

2.5.2 Determining The Hodge Numbers

As we have seen, the Hodge numbers of a Calabi-Yau play an important role. We therefore want some method for computing the Hodge numbers for a Calabi-Yau defined by a hypersurface in some projective space. For Calabi-Yau 3-folds we have an incredibly useful method: $h^{2,1}$ counts the number of allowed monomials in the defining polynomial. This can be seen using the notion of a rational form and the Griffiths residue,[10] which we discuss in Appendix B.

The remaining undetermined Hodge number, $h^{1,1}$, is then computed via the Euler characteristic. That is, we use Equation (2.73) to obtain χ, and then Equation (2.57) to get $h^{1,1}$.

2.5.3 Quintic In $\mathbb{CP}^4$

Let's start with the simplest case: a single polynomial in $\mathbb{CP}^n$. Here we have $\ell = 1$ and $k = 1$ and so Equations (2.70) and (2.71) become

$$c_1(\mathcal{S}) = (n+1-d)H \qquad \Longrightarrow \qquad (n-1)\text{-dimensional Calabi-Yau if} \quad d = n+1. \tag{2.75}$$

So if we want to construct a Calabi-Yau 3-fold we have to consider a *quintic in* $\mathbb{CP}^4$. This is a very important example of a Calabi-Yau manifold and we now explore it in a bit more detail.

[10] We note here that for other Calabi-Yaus the monomials only count the *primitive* forms. In such a case, one usually uses other methods to compute the Hodge numbers, e.g. the Lefschetz hyperplane theorem.

Denoting the Calabi-Yau by Q and $H = \mathcal{O}_{\mathbb{CP}^4}(1)$, we have that the total Chern class is

$$c(Q) = \frac{(1+H)^5}{1+5H} = 1 + 10H^2 - 40H^3, \tag{2.76}$$

where the second line follows from expanding and truncating at H^3 as $\dim_{\mathbb{C}}(Q) = 3$. So we see that $c_3(Q) = -40H^3$. Next we have that our normal bundle $N_Q = \mathcal{O}_{\mathbb{CP}^4}(5)|_Q$ is a line bundle and so

$$\eta_Q = c_1\big(\mathcal{O}_{\mathbb{CP}^4}(5)\big) = 5H, \tag{2.77}$$

where we have used $c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + dH$. So we can compute our Euler characteristic via Equation (2.73)

$$\chi(Q) = \int_{\mathbb{CP}^4} (-40H^3) \wedge (5H) = -200, \tag{2.78}$$

where we have made use of

$$\int_{\mathbb{CP}^n} H^n = 1. \tag{2.79}$$

We can therefore summarise the Calabi-Yau manifold coming from the quintic in $\mathbb{CP}^4$ via the following configuration matrix

$$Q = \mathbb{CP}^4|5|_{-200}. \tag{2.80}$$

We now want to compute the Hodge numbers of the quintic. As explained above, we get $h^{2,1}$ by considering the number of allowed monomials in the defining polynomial. The number of independent degree d monomials in $(n+1)$ variables is given by the binomial coefficient $\begin{pmatrix} d+n \\ n \end{pmatrix}$, and so our quintic polynomial starts off with $\begin{pmatrix} 9 \\ 4 \end{pmatrix} = 126$ parameters. However we need to account for coordinate transformations (i.e. homogeneous linear change of variables) as well as the scaling. These collectively add up to[11] $(n+1)^2$ which for us is $5^2 = 25$, which finally leaves us with $h^{2,1} = 126 - 25 = 101$. We can then use this with our Euler characteristic to get $h^{1,1}$:

$$-200 = 2(h^{1,1} - 101) \qquad \Longrightarrow \qquad h^{1,1} = 1. \tag{2.81}$$

[11] Basically the homogeneous linear transformations of $(n+1)$ variables are given by the group $PGL(n+1, \mathbb{C})$, which is defined to be $GL(n+1, \mathbb{C})$ modded out by our scaling, so it has dimension $(n+1)^2 - 1$, with the -1 corresponding exactly to our scaling, so when we add this back in we're just left with $(n+1)^2$.

We can summarise this using the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.82}$$

2.5.4 Complete Intersections

We now go to the slightly more complicated case where we still only have a single $\mathbb{CP}^n$ but now we can use multiple polynomials. Here we have $\ell = 1$ and $k = n-3$, i.e. we need to reduce down to an 3-dimensional manifold. Our Calabi-Yau condition is simply

$$\sum_{r=1}^{\ell-3} d_r = n+1. \tag{2.83}$$

We now note that we actually require that $d_r \geq 2$ for all r. Why? Well imagine we have some $\mathbb{CP}^n$ and one of our polynomials has degree 1. We can always use a coordinate transformation such that this polynomial simply sets one of the homogeneous coordinates to zero, but then this just leaves us with $(k-1)$ polynomials in $\mathbb{CP}^{n-1}$.

Given this, we can show that there are actually only five solutions. We display them via their configuration matrices below (without their Euler characteristics)

$$\mathbb{CP}^4|5|, \qquad \mathbb{CP}^5|3\,3|, \qquad \mathbb{CP}^5|2\,4|, \qquad \mathbb{CP}^6|2\,2\,3| \quad \text{and} \quad \mathbb{CP}^7|2\,2\,2\,2|. \tag{2.84}$$

For clarity on why we can't have anymore, let's imagine we considered $\mathbb{CP}^8$. To get a 3-fold, we would need to consider 5 polynomials who's degrees sum up to 9. However we cannot do this if we also require that $d_r \geq 2$ for all r. The same idea applies to higher n. Of course we can generate other Calabi-Yau 3-folds by allowing our ambient space to be given by a direct sum of $\mathbb{CP}^n$s.

2.6 Weighted Projective Spaces

We now want to discuss *weighted* projective spaces and how the Calabi-Yau hypersurface story carries over. These are basically exactly the same as "regular" projective spaces, but now each homogeneous coordinate has its own weight under scaling. That is, the weighted projective space $\mathbb{WCP}^{(k_0,...,k_n)}$ is defined the same as a projective space but now with equivalence relation

$$[z_0 : ... : z_n] = [\lambda^{k_0} z_0 : ... : \lambda^{k_n} z_n]. \tag{2.85}$$

It is common to write a weighted projective space as $\mathbb{WCP}^n$ and then stating the weights as an $(n+1)$-tuple, i.e. we write "$\mathbb{WCP}^n$ with weights $(k_0, ..., k_n)$". We sometimes also use the notation $\mathbb{WCP}^n_{k_0,...,k_n}$. We will likely use a combination of all of these.

As we might expect, $\mathbb{WCP}^n_{k_0,...,k_n}$ and $\mathbb{CP}^n$ have a lot in common, however stuff is more subtle in the former. For example, let's consider trying to define a polynomial of degree d in $\mathbb{WCP}^n_{k_0,...,k_n}$. Let's illustrate some of the subtleties with an example.

Example 2.6.1. Consider $\mathbb{WCP}^2_{1,2}$. Let's define the polynomial

$$P(z_0, z_1) = z_0^2 z_1 + z_0^3, \tag{2.86}$$

this would be a polynomial of degree 3 in $\mathbb{CP}^2$, but for $\mathbb{WCP}^2_{1,2}$ we have

$$P(\lambda z_0, \lambda z_1) = (\lambda z_0)^2(\lambda^2 z_1) + (\lambda z_0)^3 = \lambda^4 z_0^2 z_1 + \lambda^3 z_0^3 \neq \lambda^d P(z_0, z_1). \tag{2.87}$$

▲

Definition. [Quasihomogeneous Polynomial] We call a polynomial in $\mathbb{WCP}^n_{k_0,...,k_n}$ *quasi-homogeneous of degree d* if

$$P(\lambda z_0, ..., \lambda z_n) = \lambda^d P(z_0, ..., z_n) \tag{2.88}$$

for some $d \in \mathbb{N}$.

Now, we can still define the tautological line bundle over $\mathbb{WCP}^n_{k_0,...,k_n}$ as the line bundle with fibres

$$\pi^{-1}[z_0 : ... : z_n] = (\lambda^{k_0} z_0, ..., \lambda^{k_n} z_n). \tag{2.89}$$

We denote the space of holomorphic sections of this space by $\mathcal{O}_{\mathbb{WCP}^n_{k_0,...,k_n}}(-1)$. We then similarly have the hyperplane line bundle $\mathcal{O}_{\mathbb{WCP}^n_{k_0,...,k_n}}(1)$, given by the dual of the above.

Note that the transition functions themselves are the same as before, i.e. $g_{ij} : U_i \cap U_j \to z_i/z_j$. The change comes by adapting Proposition 2.2.2:

Proposition 2.6.2. *Any quasihomogeneous polynomial of degree d in $\mathbb{WCP}^n_{k_0,\dots,k_n}$ can be canonically identified with the holomorphic sections $\mathcal{O}_{\mathbb{WCP}^n_{k_0,\dots,k_n}}(d)$.*

The proof follows completely analogously to that of Proposition 2.2.2, however now Lemma 2.2.3 changes to

Lemma 2.6.3. *The homogeneous coordinate z_i of $\mathbb{WCP}^n_{k_0,\dots,k_n}$ can be identified as sections of $\mathcal{O}_{\mathbb{WCP}^n_{k_0,\dots,k_n}}(k_i)$.*

This is easily understood as $P_i([z]) = z_i$ is a quasihomogeneous polynomial of degree k_i. Indeed we can understand Proposition 2.2.2 and Lemma 2.2.3 simply as specialisations of the above with $k_1 = ... = k_n = 1$.

We now proceed as before, and we arrive at an Euler sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{\mathbb{WCP}^n_{k_0,\dots,k_n}}(k_i)^{\oplus(\sum i)} \longrightarrow T^{(1,0)}\mathbb{WCP}^n_{k_0,\dots,kn} \longrightarrow 0, \tag{2.90}$$

from which, recalling $c(\mathcal{O}_{\mathbb{WCP}^n_{k_0,\dots,k_n}}(k_i)) = (1 + k_i H_{k_0,\dots,k_n})$, we conclude that

$$c\big(\mathbb{WCP}^n_{k_0,\dots,k_n}\big) = \prod_i (1 + k_i H_{k_0,\dots,k_n}) \tag{2.91}$$

where, of course, $H_{k_0,\dots,k_n} = c_1(\mathcal{O}_{\mathbb{WCP}^n_{k_0,\dots,k_n}}(1))$.

There is a very important difference between standard projective spaces and weighted projective spaces: the former are smooth manifolds, while the latter generically contain singularities. This singularity structure carries over to the Calabi-Yau hypersurface construction. Let's see this with an example.

Example 2.6.4. Consider the specific case of a single polynomial in $\mathbb{WCP}^4_{1,1,1,1,4}$, which we coordinatise using $[x_1 : x_2 : x_3 : x_4 : y]$. We have from Equation (2.91) that (using A to denote our ambient space, i.e. the $\mathbb{WCP}^4_{11114}$)

$$c_1(A) = 8H. \tag{2.92}$$

So if we want to have some subspace $\mathcal{S}$ that is Calabi-Yau, we are going to need a quasihomogenous polynomial of degree 8.

Let's consider the case $y^2 = P_8(x_i)$, where $P_8(x_i)$ is some degree 8 polynomial in the x_is. Now say we pick the patch $y = 1$, that is consider the chart where $y \neq 0$ and use the scaling to set $y = 1$. We now note that this scaling is *not* unique. From the equivalence relation, it follows that we can change our scaling by any $\lambda \in \mathbb{C}$ such that $\lambda^4 = 1$. However, different λ will affect the x_i values differently. In other words, we conclude that the x_is are not free complex numbers but in fact

$$x_i \in \frac{\mathbb{C}}{\mathbb{Z}_4}. \tag{2.93}$$

We see that this problem goes away when $x_i = 0$, and we call this point a *fixed point*. This type of singularity is known as an *orbifold* singularity.

We arrive at a similar issue in the context of the hyperplane class. Consider the smooth point $p = [1:0:0:0:0]$, which is defined by the 4 polynomials $y = x_2 = x_3 = x_4 = 0$, then we have

$$\int_p 4H \cdot H \cdot H \cdot H = 1 \qquad \Longleftrightarrow \qquad H^4 = \frac{1}{4}, \tag{2.94}$$

where the $4H$ and H factors are just the first Chern classes of the polynomials, and the 1 comes from simply integrating over p. In this way we see that our weighted projective space has *fractional* hyperplane class.

It is important to note that this singularity is a property of the ambient space $A = \mathbb{WCP}^4_{11114}$ itself, not the Calabi-Yau defined inside it. Indeed it is possible to obtain a smooth Calabi-Yau inside such a space. This is seen simply from the fact that $p = [1:0:0:0:0]$ is not a solution to our defining polynomial. ▲

The issues of orbifold singularities are not as scary as they seem, and are easily dealt with using the techniques of toric geometry, which we now discuss.

2.7 Toric Geometry

The above discussion has taken place in the language of *differential* geometry. From a pedagogical stand point, this is often a useful approach as it allows us to think about the structures pictorially and intuitively. However, as we have seen, this sometimes leads us down a path of reasonably complicated and/or fiddly computations. In particular we have just seen that for weighted projective spaces the hyperplane class is fractional, and we have orbifold singularities that we need to deal with. In order to address these points we turn to the closely related topic of *algebraic* geometry.

We note that Yau's theorem plays an important role in this switch: algebraic geometry will not give us a nice way to compute the metric of our space, but it will give us simple ways to construct spaces with vanishing first Chern class. It then follows from Yau's theorem that such a space has a unique metric, and so it doesn't matter if we don't find it.

2.7.1 Algebraic Varieties & Divisors

Definition. [Algebraically Closed Field] Let K be a field, and denote the ring of polynomials in n-variables by $K[x_1, ..., x_n]$, i.e. $f \in K[x_1, ..., x_n]$ means $f(x_1, ..., x_n) \in K$. Then we call K *algebraically closed* if any non-constant polynomial over K has a root in K, i.e. $f(x_1, ..., x_n) = 0$ has $x_1, ..., x_n \in K$.

Example 2.7.1. The ring of real numbers is *not* algebraically closed as $f(x) = x^2 + 1$ has root $x = \pm i \notin \mathbb{R}$. However the ring of complex numbers is algebraically closed. In what follows we shall assume we are using the complex numbers everywhere. ▲

Definition. [(Complex) Algebraic Variety] Consider the space $\mathbb{C}^n$,[12] and consider the algebraically closed field $\mathbb{C}$. Labelling the coordinates of $\mathbb{C}^n$ by $(z_1, ..., z_n)$, we have $f \in \mathbb{C}[z_1, ..., z_n]$ being a $\mathbb{C}$ valued function over $\mathbb{C}^n$. Now consider some set of polynomials $S \subset \mathbb{C}[z_1, ..., z_n]$ and define for their common zero locus

$$Z(S) := \{z \in \mathbb{C}^n \,|\, f(z) = 0, \ \forall f \in S\}. \tag{2.95}$$

Then a subspace $X \subset \mathbb{C}^n$ is called a *(complex) algebraic set* if $X = Z(S)$ for some S. If we can write X as the union of two proper algebraic sets then we say that X is *reducible*. If it is not reducible (and non-empty) it is irreducible. Finally, an irreducible algebraic set is called a *algebraic variety*. We can turn X into a topological space by defining our closed sets to be the algebraic sets. This is known as the *Zariski topology*. A subspace of X that is also an algebraic variety is called an *algebraic subvariety*. We shall simply say "(sub)variety" to mean "complex (sub)algebraic variety".

Definition. [Birationally Equivalent] Let X and Y be varieties, and assume X is irreducible. Then a *rational map* is a morphism from a non-empty open subset $U \subseteq X$ into Y, denoted $f : X \dashrightarrow Y$. A rational map is called *birational* if there exists a inverse of f that is rational, $f^{-1} : Y \dashrightarrow X$. We say that X and Y are *birationally equivalent*. A birational map is essentially a isomorphism between open subsets of X and Y.

Now note that the dimension of the space X is related to the cardinality of the set $S \subset \mathbb{C}[z_1, ..., z_n]$. This is simply the statement that each zero condition allows us to relate one of the z_i to some of the others, and so reduces the dimension by one. We therefore see

[12] We define algebraic varieties more generally in terms of *affine spaces*, which $\mathbb{C}^n$ is an example of.

that a codimesion 1 subvariety is simply a hypersurface in X given by the zero locus of some polynomial. This lends itself directly to our construction of Calabi-Yaus as hypersurfaces in complex projective spaces. Indeed it turns out that for complex manifolds all hypersurfaces arise in this way, which we say again the following definition.

> **Definition.** [Hypersurface] Given a complex space X, a *hypersurface*, Y, is a (sub)variety of codimension 1, i.e. $Y \subset X$ and $\dim Y = \dim X - 1$. A hypersurface is said to be *irreducible* if it corresponds to an irreducible (sub)variety. A general hypersurface is given by the union of its irreducible components, i.e. $Y = \cup Y_i$ where Y_i are the irreducible hypersurfaces. If X is compact then any hypersurface has only finitely many irreducible components.

Divisors

Hypersurfaces in algebraic geometry are best discussed in the language of divisors, which we now breifly discuss.

> **Definition.** [(Weil) Divisor] A *(Weil) divisor*, D, on X is a formal linear combination of irreduicible hypersurfaces, i.e.
>
> $$D = \sum_i a_i [Y_i], \qquad \text{where} \qquad a_i \in \mathbb{Z}. \tag{2.96}$$
>
> The coefficients a_i give the order to which the defining polynomial vanishes, with negative values corresponding to poles. It is hopefully clear that we can turn this into a group in the natural way (i.e. by our addition), in this way we define the *divisor group* of X, $\text{Div}(X)$.

Remark 2.7.2. We should clarify a bit the formal addition defined for Equation (2.96). This is simply defined in terms of the weightings of the defining polynomials and does not somehow correspond to "adding" two hypersurfaces together to get a new hypersurface. We can constrast this to the addition in homology, by recalling that hypersurfaces can be thought of in terms of homology (i.e. we can triangulate the hypersurface using simplicies). Homology has a well defined addition given in the expected way, namely $[Y_1]_{\text{homol}} + [Y_2]_{\text{homol}} = [Y_1 + Y_2]_{\text{homol}}$. Now, let's imagine that Y_1 and Y_2 are two different hypersurfaces, but suppose that the corresponding elements in homology are in the same class, i.e. $[Y_1]_{\text{homol}} = [Y_2]_{\text{homol}}$. If we then consider their difference then we get a vanishing result in homology, but the *divisor* $D = a_1[Y_1] - a_2[Y_2]$ is non-vanishing. In this sense a divisor is a finer notion than a homology class. In this way we see that the square bracket notation in Equation (2.96) does *not* mean the corresponding homology class, however it is standard notation and so we keep it.

Definition. [Exceptional Divisor] Let $f : X \dashrightarrow Y$ be a birational map. Then a divisor $D \in \text{Div}(X)$, which corresponds to the codimension-1 subvariety $Z \subset X$, is called *exceptional* if $f(Z)$ has at least codimension-2 in Y.

Example 2.7.3. The most important example of a exceptional divisor to us is a *blowup* (see [46] for a more detailed discussion). Consider $\mathbb{C}^2/\mathbb{Z}_2$ and embed it algebraically in $\mathbb{C}^3$ via

$$g(z_0, z_1, z_2) = z_0^2 + z_1^2 + z_2^2 = 0, \tag{2.97}$$

where (z_0, z_1, z_2) are the coordinates of our $\mathbb{C}^3$. This has an isolated singularity at the origin.[13] Let's denote this space by $A \subset \mathbb{C}^3$. Now consider $\mathbb{CP}^2$ with coordinates $[\xi_0 : \xi_1 : \xi_2]$. Next consider the subspace

$$\{(z_0, z_1, z_2), [\xi_0 : \xi_1 : \xi_2] \in \mathbb{C}^3 \times \mathbb{CP}^2 \,|\, \xi_i z_j = \xi_j z_i, \,\forall\, i, j\} \subset \mathbb{C}^3 \times \mathbb{CP}^2 \tag{2.98}$$

Away from the origin in $\mathbb{C}^3$, this condition is solved by $\xi_i = z_i$. We can therefore take our space A, excise the origin, replace $z_i \mapsto \xi_i$, and then take the closure of this. This gives the space

$$\xi_0^2 + \xi_1^2 + \xi_2^2 = 0 \tag{2.99}$$

as a subset in $\mathbb{CP}^2$. This defines a $\mathbb{CP}^1$, and we define $\widetilde{A} = A \backslash \{(0,0,0)\} \cup \mathbb{CP}^1$, where the $\mathbb{CP}^1$ replaces the origin. We can therefore define a birational map from $f : \widetilde{A} \dashrightarrow A$ that "blows down" the $\mathbb{CP}^1$ back to the origin. The $\mathbb{CP}^1$ is the exceptional divisor, and is often referred to as a blowup.[14] In this work the word "blowup" shall be used in this sense. ▲

As the definition makes clear, a Weil divisor is defined for *any* polynomial defined on our space, however not all polynomials correspond to *functions* on a space. More technically, they don't correspond to sections of the constant sheaf $\mathcal{O}_X := \underline{\mathbb{C}}$. As an example, if our ambient space (i.e. the space we start with) is $\mathbb{CP}^n$, we can define a hypersurface, and therefore a divisor, by $z_0 = 0$. However this equation is not projectively well defined, and so is not a function on $\mathbb{CP}^n$. This leads into the notion of a principal divisor.

Definition. [Principal Divisor] Let f be a meromorphic function on X. Then the divisor associated to f is

$$(f) := \sum \text{ord}_Y(f)[Y], \tag{2.100}$$

[13] A hypersurface $g(z_0, ..., z_k) = 0$ is smooth if and only if $\partial g/\partial z_0 = ... = \partial g/\partial z_k = g = 0$ has no solution.

[14] We note that one needs not replace a singular point in order to define a blow up. For example (see [49] for details), we can equally blowup the perfectly smooth $\mathbb{C}^3$ at a point $p \in \mathbb{C}^3$ by replacing it with a $\mathbb{CP}^2$.

where $\text{ord}_Y(f)$ is the order of f on Y, and the sum is done over all irreducible hypersurfaces of X. We call a divisor of this form *principal*. We denote the group of principal divisors $\text{Div}_0(X)$.

Principal divisors are useful to us in the context of comparing divisors, via the following definition.

Definition. [Linearly Equivalent Divisor] Let $D, D' \in \text{Div}(X)$. We call them *linearly equivalent*, denoted $D \sim D'$, if $D - D'$ is a principal divisor.

Definition. [Picard Group] The *Picard group* of X is given by the isomorphism classes of line bundles on X, with group operation being the tensor product. We denote the Picard group of X by $\text{Pic}(X)$.

Proposition 2.7.4. *There is a group homomorphism given by*

$$\begin{aligned} Div(X) &\to Pic(X) \\ D &\mapsto \mathcal{O}(D), \end{aligned} \tag{2.101}$$

where $\mathcal{O}(D)$ is a line bundle associated to the divisor D. Principal divisors are mapped to the identity element of $Pic(X)$, and so two linearly equivalent divisors $D \sim D'$ give rise to the same element in $Pic(X)$.

The proof of this proposition requires delving into the world of sheaf cohomology and so is omitted (see, e.g., [43]). However we note that we saw previously that to a hyperplane we can define the hyperplane line bundle H. This proposition is simply the algebraic geometry equivalent of this statement. In particular the $\mathcal{O}_{\mathbb{CP}^n}(1)$ we defined before really should have been written $\mathcal{O}_{\mathbb{CP}^n}(H)$, where $[H] = \{z_0 = 0\}$ is the hypersurface correspinding to the divisor $H = 1 \cdot [H]$.[15]

Corollary 2.7.5. *Let $Cl(X) := Div(X)/Div_0(X)$, known as the Weil divisor class. Then our group homomorphism of Proposition 2.7.4 provides an injection*

$$\phi : Cl(X) \to Pic(X). \tag{2.102}$$

This corollary is important because of the following proposition.

[15]Note that we could have defined $[H'] = \{z_1 = 0\}$, but in $\mathbb{CP}^n$ these two hypersurfaces differ by a coordinate redefinition, i.e. by a meromorphic function. It thus follows that $H \sim H'$.

Proposition 2.7.6. *If a line bundle $\mathcal{L} \in Pic(X)$ admits a global section, it is contained in the image of the above injection, i.e. there is a nontrivial Weil divisor $D \in Cl(X)$ associated to it.*

Proof. The key thing to note is that a global section in a line bundle is a hyperplane of the line bundle, and so corresponds to some divisor. Let s be such a non-zero global section and denote by D_s the associated divisor. This establishes a link between the section of a line bundle and a divisor, what we want is a link between the line bundle itself and D_s. Well, any line bundle can be defined by its sections, and any two sections are related by a meromorphic function, which is itself an element in $\text{Pic}(X)$ (it's a section in the sheaf of meromorphic functions). So if we consider a $\widetilde{s} = f \otimes s$, then we have, recalling the group structure on each space,

$$\phi([D_s]) = \phi([D_f + D_s]) = f \otimes s = \widetilde{s} = \phi([D_{\widetilde{s}}]), \tag{2.103}$$

where we have used that $D_f \sim 0$ as viewed as an element in $Cl(X)$. Then using that our map is injective, we have $D_s \sim D_{\widetilde{s}}$, and so we really are talking about the whole line bundle. ■

2.7.2 Toric Varieties, Cones & Fans

We are now in a position to start discussing toric geometry and how it can be used to construct toric varieties. There are two approaches to toric geometry: the spectrum approach and the coordinate approach. The former deals with a lot more algebraic geometry directly, while the latter is probably more intuitive, especially for a first time approach to the subject. For that reason, we shall focus almost entirely on the latter approach. We start by giving the important definitions.

> **Definition.** [Algebraic Torus] An *algebraic n-torus T* is given by the n-fold product of $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. That is $T = (\mathbb{C}^*)^n$, and we regard this as an abelian group.

> **Definition.** [Toric Variety] Let X be a $\mathbb{C}$ variety. Then we call X a *toric variety* if it contains an n-torus T as a dense open subset, such that the natural action of the torus on itself (i.e. simply multiplication in $(\mathbb{C}^*)^n$) extends to an action of T on the whole of X.

Example 2.7.7. Perhaps the most important example of a toric variety for us will be $\mathbb{CP}^n$ and $\mathbb{WCP}^n$. We show here that the former is indeed a toric variety. This is actually very straight forward: let's denote the homogeneous coordinates of $\mathbb{CP}^n$ by $[z_0 : ... : z_n]$. Then note that the open subset

$$T = \{[z] \,|\, z_i \neq 0 \,\forall i\} = (\mathbb{C}^*)^{n+1}/\mathbb{C}^* \subset \mathbb{CP}^n \tag{2.104}$$

where the quotient $\mathbb{C}^*$ is embedded diagonally into $(\mathbb{C}^*)^{n+1}$, is dense and is clearly isomorphic to $(\mathbb{C}^*)^n$, and so is an algebraic torus. We can have this act on $\mathbb{CP}^n$ simply by coordinatewise multiplication, so we see that $\mathbb{CP}^n$ is a toric variety. We similarly have that $\mathbb{WCP}^n$ is a toric variety. ▲

Definition. [Cone] Let N be a rank r lattice, and define $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Then a *(strongly convex, polyhedral) cone* $\sigma \in N_{\mathbb{R}}$ is a set

$$\sigma := \{a_1 v_1 + a_2 v_2 + ... + a_k v_k \,|\, a_i \geq 0 \,\forall i\} \qquad \text{such that} \qquad \sigma \cap (-\sigma) = \{0\}, \tag{2.105}$$

where $\{v_1, ..., v_k\} \subset N_{\mathbb{R}}$ is a finite set of vectors called the *generators* of σ. The dimension of a cone is given by the number of generators with non-zero coefficients. We call the boundary of a cone a *face*, and similarly we call a 1D cone an *edge* or a *ray*. A cone is called *rational* if $\{v_1, ..., v_k\} \subset N$, i.e. they are lattice points.

Definition. [Fan] A collection Σ of cones in $N_{\mathbb{R}}$ is called a *fan* if:

(i) each face of a cone in Σ is also a cone in Σ; and

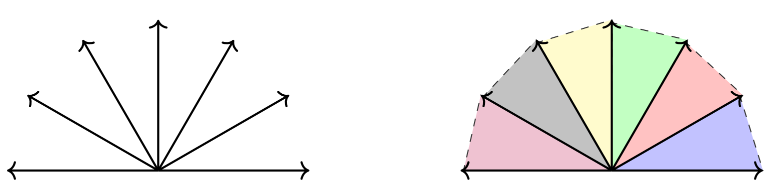
(ii) the intersection of any two cones in Σ is a face in each of the cones.

We denote the set of d-dimensional cones in Σ by $\Sigma(d)$. In particular, $\Sigma(1)$ denotes the set of edges in a fan. A *rational* fan is one whose cones are all themselves rational. A rational fan $\Sigma \subset \mathbb{R}^n$ can be identified with a set of points in $\mathbb{Z}^n$ (i.e. the lattice points).

Definition. [Simplical Cone/Fan] A cone σ is called *simplical* if it can be generated by a set of vectors $\{v_1, ..., v_k\}$ which form a basis for the vector space they span. A fan Σ is called simplical if all $\sigma \in \Sigma$ are simplical.

Unless otherwise specified, we will only consider rational and simplical cones and fans in this work.

Example 2.7.8. An example of a fan in 2D with 7 generating vectors is the following



The diagram on the left just shows the generating vectors, while the right-hand diagram also shows the 2D cones, indicated by the shaded regions. This fan contains a total of 14 cones: the 6 triangular faces, the 7 edges and the origin. ▲

As it will be important to us later, we also now introduce the dual lattice.

Definition. [Dual Lattice] Given our N lattice, we define the dual lattice $M := \hom(N, \mathbb{Z})$, and we denote their inner product by $\langle \, , \rangle : M \times N \to \mathbb{Z}$. We also define $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

2.7.3 Constructing Toric Varieties Using Fans

At first glance, it seems like toric varieties and fans are completely independent objects. However, as we will now see, they are related in a very elegant way. Before doing so, we stress that it is this link between the borderline trivial combinatorics game of picking points on a lattice and toric varieties, that gives us immense power when constructing Calabi-Yau manifolds.

As mentioned at the beginning of the section, we will use the coordinate approach to toric geometry. The key thing here is the following: given a fan Σ with n generating vectors $\{v_1, ..., v_n\}$, we have n associated rays $\rho_i = a_i v_i$ with $a_i > 0$, and to each of these rays we define a homogeneous coordinate $z_i \in \mathbb{C}$. A fan with $|\Sigma(1)| = n$ then has n corresponding homogeneous coordinates, and we can use these to define our toric variety.

Definition. [Exceptional Set] Let $S \subseteq \Sigma(1)$ denote any subset that does *not* span a cone in Σ. That is $\{\rho_1, \rho_2\} \in S$ if we do not have a 2D cone given by the face joining ρ_1 and ρ_2. Then we define $V(S) \subset \mathbb{C}^n$ to be the linear subspace defined by setting $z_\rho = 0$ for all $\rho \in S$. Finally we define the *exceptional set of* Σ to be

$$Z(\Sigma) = \bigcup V(S). \tag{2.106}$$

The final piece we need in order to get a toric variety from our fan is the following map.

$$\phi : \hom(\Sigma(1), \mathbb{C}^*) \to \hom(M, \mathbb{C}^*), \tag{2.107}$$

where M is the dual lattice to N. This is a map of maps, and is defined by

$$\phi : \big(f : \Sigma(1) \to \mathbb{C}^*\big) \mapsto \left(m \mapsto \prod_{\rho \in \Sigma(1)} f(v_\rho)^{\langle m, v_\rho \rangle}\right). \tag{2.108}$$

If we work in terms of coordinates, so that $v_j = (v_{j1}, ..., v_{jr})$ we can write ϕ explicitly as

$$\begin{aligned} \phi : (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^r \\ (t_1, ..., t_n) &\mapsto \bigg(\prod_{j=1}^n t_j^{v_{j1}}, ..., \prod_{j=1}^n t_j^{v_{jr}} \bigg), \end{aligned} \tag{2.109}$$

where the dimensions follow from the fact that $n = |\Sigma(1)|$ and N/M are rank r lattices, and the notation follows from the fact that $(\mathbb{C}^*)^k$ is a torus. We then define a quotienting group by

$$G := \ker \big(\hom(\Sigma(1), \mathbb{C}^*) \xrightarrow{\phi} \hom(M, \mathbb{C}^*) \big). \tag{2.110}$$

We can define an action of G on $\mathbb{C}^n \setminus Z(\Sigma)$ as follows: from the definition we have $G \subset \hom(\Sigma(1), \mathbb{C}^*)$, and so given a $g \in G$ and a $\rho \in \Sigma(1)$ we can define $g(v_\rho) \in \mathbb{C}^*$. We then use this to define an action of G on $\mathbb{C}^n$ simply by

$$g(z_1, ..., z_n) = \big(g(v_1)z_1, ..., g(v_n)z_n\big). \tag{2.111}$$

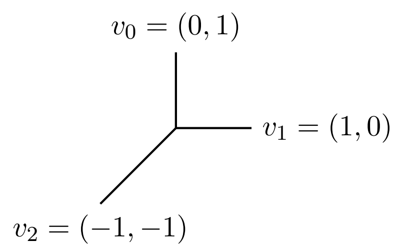
Recalling the definition Equation (2.106), it is clear that this action is closed in $\mathbb{C}^n \setminus Z(\Sigma)$; that is, $Z(\Sigma)$ just makes it so that certain elements can't vanish together, but $g(v_\rho) \neq 0$, and so this won't change this behaviour. We now finally arive at the definition of a toric variety associated to a fan.

> **Definition.** [Toric Variety From Fan] Let Σ be some fan with $n = |\Sigma(1)|$, and define $Z(\Sigma)$ and G via Equations (2.106) and (2.110), then
>
> $$X_\Sigma := \frac{\mathbb{C}^n \setminus Z(\Sigma)}{G} \tag{2.112}$$
>
> is a toric variety. The dense open torus is simply given by $T := (\mathbb{C}^*)^n/G \subset X_\Sigma$, and it acts on X_Σ by coordinatewise multiplication. It follows from Equation (2.109) that T has rank r and X_Σ is an r-dimensional toric variety.

It is important to note that the definition Equation (2.112) really is a property of Σ *not* just the rays used to generate it. This enters into the fact that the exceptional set $Z(\Sigma)$ depends explicitly on all the cones in Σ, not just the edges.

Remark 2.7.9. We note here that it is also possible to construct a fan from a toric variety. We do not discuss the details of how this is done here as we will only work in the fan to variety direction. Details of how this is done can be found in, e.g., [42].

Example 2.7.10. Let's start with something we already know is a toric variety: $\mathbb{CP}^n$. For ease of drawing, we consider $\mathbb{CP}^2$. This corresponds to a 2-dimensional lattice with 3 generating vectors. We claim that this corresponds to the following fan



where we have only drawn the edges, the rest of the fan is given by the 3 faces given by pairing two of the edges. Let's now verify that this is indeed $\mathbb{CP}^2$.

Firstly we note that the exceptional set is just point $\{0, 0, 0\}$, as the only combination of edges which doesn't span a cone is $S = \{(0, 1), (1, 0), (-1, -1)\}$. For clarity, this is *not* a cone in Σ for two reasons: firstly if we defined $\sigma_{012} = av_0 + bv_1 + cv_2$ then we would fail to satisfy $\sigma_{012} \cap (-\sigma_{012}) = \{(0, 0)\}$; secondly the intersection of σ_{012} and σ_{01}[16] is σ_{01}, but this is clearly not a face in σ_{012}. So we have $\mathbb{C}^3 \setminus Z(\Sigma_{\mathbb{CP}^2}) = \mathbb{C}^3 \setminus \{(0, 0, 0)\}$.

Next we need to find the group G. From Equation (2.109) we have

$$\begin{aligned} \phi : (\mathbb{C}^*)^3 &\to (\mathbb{C}^*)^2 \\ (t_0, t_1, t_2) &\mapsto (t_2^{-1} t_1, t_2^{-1} t_0), \end{aligned} \tag{2.113}$$

which follows from $v_0 = (v_{01}, v_{02}) = (0, 1)$ etc. Then G is defined as the kernel of this map, i.e. we want the right-hand element to be $(1, 1)$, which clearly requires $t_0 = t_1 = t_2$. In other words $G = \{(t, t, t) \,|\, t \in \mathbb{C}^*\}$ which is clearly isomorphic to $\mathbb{C}^*$, and so $G \cong \mathbb{C}^*$. We therefore arrive at

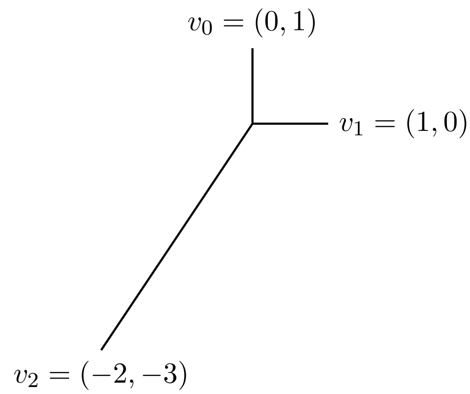
$$X_{\Sigma_{\mathbb{CP}^2}} = \frac{\mathbb{C}^3 \setminus \{(0, 0, 0)\}}{\mathbb{C}^*} = \mathbb{CP}^2. \tag{2.114}$$

Finally note that $T = (\mathbb{C}^*)^3 / G = (\mathbb{C}^*)^3 / \mathbb{C}^*$, where the $\mathbb{C}^*$ is embedded diagonally into $\mathbb{C}^*$, which is exactly what we had in Example 2.7.7. ▲

Example 2.7.11. Next let's look at a weighted projective space. Again for ease of drawing we consider $\mathbb{WCP}^2_{3,2,1}$. This again corresponds to a 2D lattice with 3 generating vectors. If we look through the details of Example 2.7.10, we see that the weightings of the coordinates enters in by the mapping ϕ, which is directly related to the entries of the vectors $\{v_0, v_1, v_2\}$.

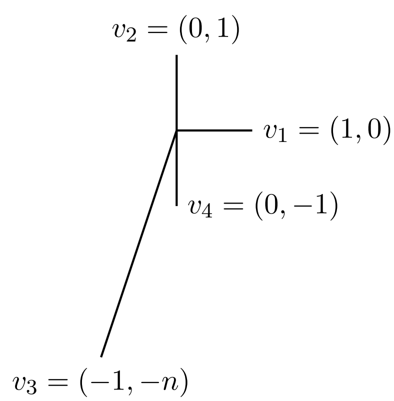
[16] Hopefully this notation is clear, but as we will use it a lot going forward, we explain it once: this simply means the cone with generating vectors v_0 and v_1.

We therefore just want to make it so that v_2 is three times v_0 in one entry and twice v_1 in the other. That is, we consider the diagram

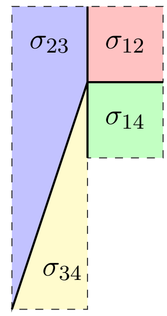


where again we have only drawn the edges. We leave the rest of this calculation as a nice exercise. ▲

Example 2.7.12. Let's now consider a new space, that will prove very useful to us going forward. Consider the 2D fan given by $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,-n)$ and $v_4 = (0,-1)$ with $n > 0$. The drawing of the edges is as follows



Now things are little more subtle when we ask the question "what are the 2D cones in this fan?", as we can*not* just take any cone spanned by two edges. Firstly we note that σ_{24} isn't a cone as it doesn't obey $\sigma_{24} \cap (-\sigma_{24}) = \{(0,0)\}$. Besides that, note that if we take both σ_{13} and σ_{14}, which are both well defined cones, their intersection would be σ_{14}, which is not a face in σ_{13}. Similarly we can't have σ_{13} and σ_{34}. Clearly there are different fans we can construct from these vectors, but here we want to consider the fan which contains cones $\{\sigma_{12}, \sigma_{14}, \sigma_{34}, \sigma_{23}\}$, which we try to depict in the following diagram



Let's now find the toric variety associated to this fan.

First, from the arguments above, we have that the exceptional set is given by

$$\begin{aligned} Z(\Sigma) &= V(S_{24}) \cup V(S_{13}) \cup V(S_{1234}) \\ &= (z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4) \cup (0,0,0,0) \\ &= (z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4) \end{aligned} \tag{2.115}$$

where the last line follows from $(0,0,0,0) \in (z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4)$, and the notation on the first line is hopefully clear.

Now we just need to find the group. We have the mapping

$$\phi : (t_1, t_2, t_3, t_4) \mapsto \big(t_1 t_3^{-1}, t_2 t_3^{-n} t_4^{-1}\big), \tag{2.116}$$

and so G is given by $t_1 = t_3$ and $t_2 = t_3^n t_4$, or in other words $G \cong (\mathbb{C}^*)^2$, which we can view as the embedding

$$\begin{aligned} (\mathbb{C}^*)^2 &\hookrightarrow (\mathbb{C}^*)^4 \\ (t, s) &\mapsto (t, t^n s, t, s). \end{aligned} \tag{2.117}$$

So our toric variety is given by

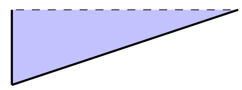
$$X_\Sigma = \frac{\mathbb{C}^4 \setminus \big((z_1, 0, z_3, 0) \cup (0, z_2, 0, z_4)\big)}{(\mathbb{C}^*)^2}. \tag{2.118}$$

Finally, the dense torus is given exactly by $T = (\mathbb{C}^*)^4/(\mathbb{C}^*)^2$, where the $(\mathbb{C}^*)^2$ is embedded as written above.

This surface, denoted F_n, is known as the *n-th Hirzebruch surface*. ▲

Example 2.7.13. As a final example consider the fan with edges $(0, 1)$ and $(n, 1)$.

$v_1 = (0,1)$ $v_2 = (n,1)$



The exceptional set for this fan is empty, as the only options for S are $\{v_1, v_2, \{v_1, v_2\}\}$, but all of these correspond to cones in Σ. To be clear, this is different to the cases for $\mathbb{CP}^2$ and $\mathbb{WCP}^2_{321}$ which had $Z(\Sigma) = \{(0,0,0)\}$. The group action is then found from the map

$$\phi : (t_1, t_2) \mapsto (t_2^n, t_1 t_2) \tag{2.119}$$

and so G requires $t_2^n = 1$ and $t_1 = t_2^{-1}$. This is just the group $\mathbb{Z}_n$, and so we have

$$X_\Sigma = \frac{\mathbb{C}^2}{\mathbb{Z}_n}. \tag{2.120}$$

The torus is given by $T = (\mathbb{C}^*)^2/\mathbb{Z}_n$. ▲

These final two examples are going to prove very useful for us going forward, especially when discussing fibrations and singularity blow ups. For this reason, it is important that these two examples are well understood at this point.

2.7.4 Weightings & Compactness

With the above examples helping ground the definitions that appeared before, we can now go on to discuss how powerful these fan diagrams actually are.

Weightings

The first thing we want to notice is something that was hopefully made suggestively clear from the examples: the group G gives us a quoienting corresponding to scaling(s) of the coordinates directly related to the entries of the vectors. In particular we have

$$[z_1, ..., z_n] \sim \left[\left(\lambda_1^{Q_1^1} ... \lambda_\ell^{Q_\ell^1}\right) z_1, ..., \left(\lambda_1^{Q_1^n} ... \lambda_\ell^{Q_\ell^n}\right) z_n\right], \tag{2.121}$$

where $\lambda_\alpha \in \mathbb{C}^*$ and $\sum_{i=1}^n Q_\alpha^i v_i = 0$ for all $\alpha = 1, ..., \ell$. In particular, for $\mathbb{CP}^2$ we have $\ell = 1$ with $\lambda = t$ and $Q_0 = Q_1 = Q_2 = 1$, i.e. $1 \cdot (0,1) + 1 \cdot (1,0) + 1 \cdot (-1,-1) = 0$. Similarly for $\mathbb{WCP}^2_{321}$ we have $\ell = 1$ and $Q_0 = 3$, $Q_1 = 2$ and $Q_2 = 1$. Then for F_n we have $\ell = 2$ with $\lambda_1 = t$, $\lambda_2 = s$ and $Q_t^1 = 1$, $Q_t^2 = n$, $Q_t^3 = 1$, $Q_t^4 = 0, Q_s^1 = 0, Q_s^2 = 1, Q_s^3 = 0$, and $Q_s^4 = 1$.

It is notationally convenient to display all this information in the form of a *weight system*, which we draw as

$$\begin{array}{ccc} z_1 & \cdots & z_n \\ \hline Q_1^1 & \cdots & Q_1^n \\ & \vdots & \\ Q_\ell^1 & \cdots & Q_\ell^n \end{array}$$

So for the $\mathbb{CP}^2$, $\mathbb{WCP}^2_{321}$ and F_n we have

$$\begin{array}{ccc} z_0 & z_1 & z_2 \\ \hline 1 & 1 & 1 \end{array} \qquad \begin{array}{ccc} z_0 & z_1 & z_2 \\ \hline 3 & 2 & 1 \end{array} \qquad \text{and} \qquad \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \\ \hline 1 & n & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \tag{2.122}$$

We haven't said anything about Example 2.7.13, as this doesn't have a weight system as there is no way to get the v_1 and v_2 to cancel with non-zero Q_i. In a way its weight system vanishes, and so we don't write anything.

These weight system diagrams are actually incredibly useful as they encode a lot of information. We will add more to them later, but for now notice that from the weight system we not only get the scaling weights, we can use them to reconstruct the generating vectors via $\sum_{i=1}^n Q_a^i v_i = 0$. Further we can immediately read off the dimension of the space – it is simply the number of columns minus the number of rows (excluding the row containing the z_is). This is not hard to see: the number of columns corresponds exactly to the number of coordinates, i.e. the power n factor appearing in the numerator of Equation (2.112), while the number of rows corresponds to how many different scalings we have, which corresponds exactly to the dimension of the group G in the denominator of Equation (2.112). For example, we see straight away that F_n is 2-dimensional from $4-2=2$. The fact that we can read off the dimension will prove additionally useful later when discussing so-called toric divisors and their linear relations.

Compactness

Next we want to ask the question "is it possible to read off whether the resulting toric variety is compact or not from the fan diagram?" The answer is yes, and is the content of the next proposition.

Proposition 2.7.14. *Let X_Σ be a toric variety associated to a fan Σ. Then X_Σ is compact iff the fan Σ fills $N_\mathbb{R}$.*

The proof of this proposition is easier to see when considering constructing a fan from a toric variety, and so we do not present here. We simply note that in the examples above, only

Example 2.7.13 is non-compact.

2.7.5 T-Invariant Subvarieties & Toric Divisors

So far we have been able to use fans to construct toric varieties, but we are yet to see how to construct subvarities and divisors within these spaces. This is something we will clearly need if we want to construct Calabi-Yaus as hypersurfaces in projective spaces.

First let's look at our T-invariant subvarieties, where T is the algebraic torus. These are particularly easy to describe in terms of our homogeneous coordinate description. Let Σ be a fan and X_Σ the associated toric variety. Then consider some $\sigma \in \Sigma$ which has generating vectors $\{v_1, ..., v_k\}$. We can associate a codimension k subvariety of X_Σ to this cone via

$$Z_\sigma := \{z \in X_\Sigma \,|\, z_1 = ... = z_k = 0\}, \tag{2.123}$$

where we see that it is codimension k from the fact that we have k conditions. Now as T acts on X_Σ by multiplication of non-vanishing complex numbers, this subvariety is clearly T-invariant. Then note that if we have two cones $\sigma, \widetilde{\sigma} \in \Sigma$ where the generating vectors of $\widetilde{\sigma}$ are contained within those for σ (i.e. $\widetilde{\sigma}$ is a face of σ), then the order of inclusion is flipped for the T-invariant subvarieties, i.e. $Z_\sigma \subset Z_{\widetilde{\sigma}}$. The claim is that these are the *only* types of T-invariant subvarieities. Putting this together with the fact that if the cone is *not* in the fan then Z_σ would correspond to an element of the exceptional set $Z(\Sigma)$, and so the subvariety would be empty, we have the following Lemma.

Lemma 2.7.15. *There is a one-to-one correspondance between non-empty T-invariant subvarieites and cones in fan, given by the ordering reversing mapping $\sigma \mapsto Z_\sigma$.*

It is interesting to note that each Z_σ is in fact a toric variety, and we can construct the lattice and fan from the lattice N and fan Σ for X_Σ: simply take the quotient of N by the sublattice $\sigma \cap N$, and then project every cone in Σ which contains σ as a face onto $\widetilde{N} = N/(\sigma \cap N)$.

Example 2.7.16. As an example, we can construct the T-invariant subvarieties of $\mathbb{CP}^2$ given in Example 2.7.10. We list them below

$$\begin{array}{cc} \sigma & Z_\sigma \\ \{0\} & \mathbb{CP}^2 \\ \{(0,1)\} & z_0 = 0 \\ \{(1,0)\} & z_1 = 0 \\ \{(-1,-1)\} & z_2 = 0 \\ \{(1,0),(-1,-1)\} & [1:0:0] \\ \{(0,1),(-1,-1)\} & [0:1:0] \\ \{(1,0),(0,1)\} & [0:0:1] \end{array} \tag{2.124}$$

which we can see obeys the order reversing inclusion, e.g. $\{(0,1)\} \subset \{(0,1),(-1,-1)\}$ and $[0:1:0] \subset z_0 = 0$. ▲

The important case of Lemma 2.7.15 for us is that each one-dimensional cone corresponds to a hypersurface in X_Σ. That is: we have a one-to-one correspondance between edges and toric divisors. In what follows we shall denote the toric divisor corresponding to z_i as D_i.

Now recall that Proposition 2.7.4 tells us that to each divisor we can associate some form of line bundle. For our toric divisors, these correspond to the hyperplane line bundles $\mathcal{O}(D_i)$. We can, of course, take a formal sum of our toric divisors to form some new divisor, i.e.

$$D = \sum_{i=1}^{n} a_i D_i. \tag{2.125}$$

Let's see what happens when we consider the case $a_i = \langle v_i, m\rangle$ for some $m \in M$, with M being the dual lattice to N. Then consider the monomial $z_1^{a_1}...z_n^{a_n}$, which is a section of $\mathcal{O}(\sum_i a_i D_i)$. Our equivalence relation Equation (2.121) then says that (just considering one α value for simplicity) our monomial is equivalent to

$$\left(\lambda^{Q^1} z_1\right)^{\langle v_1, m\rangle} ... \left(\lambda^{Q^n} z_n\right)^{\langle v_n, m\rangle} = \lambda^{\langle \sum_{i=1}^n Q^i v_i, m\rangle} z_1^{\langle v_1, m\rangle} ... z_n^{\langle v_n, m\rangle}. \tag{2.126}$$

Then recalling that $\sum_{i=1}^n Q^i v_i = 0$, we see that this monomial is completely invariant under this scaling. This tells us that the monomial is globally well defined on X_Σ, and so corresponds to a *globally* defined meromorphic section, and so it must correspond to a section in a trivial line bundle. That is we *must* have

$$\sum_{i=1}^{n} \langle v_i, m\rangle D_i \sim 0 \qquad \forall m \in M \tag{2.127}$$

This gives us a set of linear relations between the divisors. It follows from the fact that $\dim M = \dim N$, that we have $\dim N$ such linear relations between our toric divisors, i.e. m has coordinates $m = (m_1, ..., m_{\dim N})$ and we can consider the linear relations given by $m_1 = (1, 0, ..., 0)$, $m_2 = (0, 1, 0, ..., 0)$ etc. This gives us exactly $\dim N$ expressions. So in total we have $|\Sigma(1)| - \dim N$ linearly independent toric divisors.

2.7.6 Singularities & Blowups

We now discuss something that is really important for us: the presence of singularities. We note, from the definition Equation (2.112), our toric varieties are orbifolds with potential singularities, depending on what the group G is. We want to see how we can read off whether a toric variety is singular or not, given its associated fan.

The key point is to note the following: consider some fan Σ and form the toric variety X_Σ. Now consider some cone $\sigma \in \Sigma$ and form the toric variety $X_\sigma \subset X_\Sigma$, which we can *define* as the subset obtained by setting $z_\rho = 1$ for all $\rho \in \Sigma(1) \setminus \{\text{edges of } \sigma\}$. We can then patch these X_σ together to give $X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma$. This is just the statement that a fan is given by the union of its cones, and so the associated toric variety is given by the union of the subvarieties.

Now we define $\Sigma_\sigma \subset \Sigma$ to be the fan given by σ and all of its faces. From the explanation above, we have that $X_{\Sigma_\sigma} = \cup_{\widetilde{\sigma} \in \Sigma_\sigma} X_{\widetilde{\sigma}}$. Putting this together with the fact that there is clearly an injective embedding of $\widetilde{\sigma} \in \Sigma_\sigma$ into σ, simply by definition of Σ_σ, we have that $\cup_{\widetilde{\sigma} \in \Sigma_\sigma} X_{\widetilde{\sigma}} \cong X_\sigma$. We can therefore conclude that $X_{\Sigma_\sigma} \cong X_\sigma$. Finally note that $Z(\Sigma_\sigma) = \emptyset$, simply by the definition of Σ_σ: it contains all the possible cones. We then have the following proposition.

Proposition 2.7.17. *Let Σ be a fan and X_Σ be the associated toric variety. Then X_Σ is smooth (i.e. non-singular) iff every cone $\sigma \in \Sigma$ is generated by vectors which form a $\mathbb{Z}$-basis for $\sigma \cap N$.*

Proof. We show that the basis condition implies smoothness. Consider any top-dimensional cone $\sigma \in \Sigma$, by assumption this is generated by r linearly independent vectors, and so the group G acts trivially on X_{Σ_σ}. This is easiest to see by considering the weight system: there is no way to have these vectors cancel each other and so all $Q^i_\alpha = 0$. Putting this together with $Z(\Sigma_\sigma) = \emptyset$ we have that $X_{\Sigma_\sigma} = \mathbb{C}^r$, and so $X_\sigma \cong \mathbb{C}^r$ which is smooth. Finally putting this together with $X_\Sigma = \cup_{\sigma \in \Sigma} X_\sigma$, we conclude that X_σ is the union of smooth varieties, and so is smooth itself.

The reverse direction, that smoothness implies the basis criteria, is most easily shown using the spectrum approach, which we do not discuss in this work. We will also only be interested in going from the fan to the variety, and so we omit the rest of the proof. ■

For the examples discussed above, we see that:

- $\mathbb{CP}^2$ is smooth, which we know to be true,
- $\mathbb{WCP}^2_{321}$ is singular, which again we know to be true,
- F_n is smooth only when $n = 1$, and
- Example 2.7.13 is smooth only when $n = 1$.

So we have a condition for when the toric variety is singular, the obvious question for us to ask is "when does this singularity correspond to an orbifold?"

Proposition 2.7.18. *Let Σ be a fan and X_Σ be the associated toric variety. Then X_Σ is an orbifold iff Σ is simplical.*

Proof. Again we only show the condition $\implies$ orbifold direction. Let $\sigma \in \Sigma$ be a r-dimensional cone, then by definition of a simplical cone it can be generated by r vectors $\{v_1, ..., v_r\}$ which form a basis for the vector space they span. There is therefore only a finite number of ways we can get them to cancel each other, and so G is finite. We therefore have that $X_\sigma \sim \mathbb{C}^r/G$ and so is an orbifold, which in turn makes X_Σ an orbifold. ■

Of course every fan we have considered thus far has been simplical, and so we are dealing with orbifold toric varieties.

Blowup

Recalling Example 2.7.3, we know that singularities can be dealt with by the procedure of blowups. We now want to combine this with the result that a toric variety is smooth iff all the cones are generated by vectors which form a basis for the intersection of σ with N. In order to do that we need a couple definitions.

> **Definition.** [Fan Morphism] Let $\Sigma \subset N_\mathbb{R}$ and $\widetilde{\Sigma} \subset \widetilde{N}_\mathbb{R}$ be fans. Then a *fan morphism* from Σ to $\widetilde{\Sigma}$ is a homomorphism $\psi : N \to \widetilde{N}$, such that for every cone $\sigma \in \Sigma$ the image under $\psi \otimes \mathbb{R}$ is contained in some cone of $\widetilde{\Sigma}$. We say that ψ is *compatible* with Σ and $\widetilde{\Sigma}$.

> **Definition.** [Toric Morphism] Let $\Sigma \subset N_\mathbb{R}$ and $\widetilde{\Sigma} \subset \widetilde{N}_\mathbb{R}$ be fans, and let X_Σ and $X_{\widetilde{\Sigma}}$ be the associated toric varieties. Then a morphism $\phi : X_\Sigma \to X_{\widetilde{\Sigma}}$ is called *toric* if $\phi(T) \subseteq \widetilde{T}$ (i.e. ϕ maps the torus in X_Σ into the torus in $X_{\widetilde{\Sigma}}$), and $\phi|_T$ is a group homomorphism.

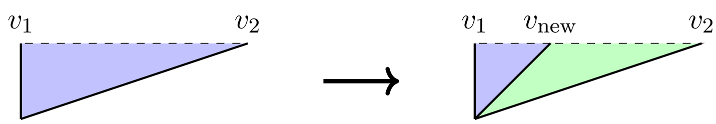
It can be shown that a fan morphism can be used to construct a toric morphism and vice versa (see Theorem 3.3.4 of [43]).

Definition. [Subdividing A Fan] Let $\Sigma \subset N_{\mathbb{R}}$ be a fan. Then another fan $\widetilde{\Sigma}$ *subdivides* Σ if

(i) $\Sigma(1) \subset \widetilde{\Sigma}(1)$, and

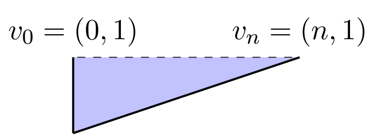
(ii) Each $\widetilde{\sigma} \in \widetilde{\Sigma}$ is contained in some $\sigma \in \Sigma$.

In terms of the toric diagrams, this is very straight forward: we can subdivide a fan Σ by introducing a new ray which "splits" an existing cone into two cones, as the following diagram is meant to indicate.

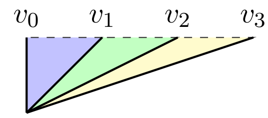


As we explained above, the idea is to take a singular toric variety and subdivide the fan such that we get a smooth result. In doing this, we introduce more toric divisors into $X_{\widetilde{\Sigma}}$. We then use the fact that the identity map on N if compatible with $\widetilde{\Sigma}$ and Σ and defines a birational toric morphism $f : X_{\widetilde{\Sigma}} \to X_\Sigma$. These divisors are then our exceptional divisors (c.f. Example 2.7.3), and so correspond to a blowup: we replace the orbifold singularities with a copy of $\mathbb{CP}^1$. The above diagram is clearly related to Example 2.7.13 and we show the full resolution in the next example.

Example 2.7.19. Recall that we have the fan



where we have suggestively renamed $v_1 \to v_0$ and $v_2 \to v_n$. We have shown that this is a singular space given by $X_\Sigma = \mathbb{C}^2/\mathbb{Z}_n$. To remove this singularity, we need to introduce new vectors so that every cone is generated by a basis of $\sigma \cap N$, where N is a 2D lattice. For $v_0 = (0, 1)$ the only edge that lies within σ_{0n} that meets this condition is $v_1 = (1, 1)$. However we then have the cone σ_{1n} which gives rise to a singularity if $n \neq 2$. However clearly all we have to do is include $v_2 = (2, n)$, and then continue this process until we reach $v_{n-1} = (n-1, 1)$. For $n = 3$ we get the following subdivided fan



which gives a smooth toric variety with weight system

$$\begin{array}{cccc} z_0 & z_1 & z_2 & z_3 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array} \tag{2.128}$$

This weight system generalises to the general n case, which follows from

$$v_{m-1} + v_{m+1} = (2m, 2) = 2v_m. \tag{2.129}$$

Each of the new rays gives a blowup and corresponds to a $\mathbb{CP}^1$, so we have a total of $(n-1)$ $\mathbb{CP}^1$s. Indeed we already showed in Example 2.7.3 that the blowup of $\mathbb{C}^2/\mathbb{Z}_2$ corresponds to replacing the origin with a copy of $\mathbb{CP}^1$. Here we are just looking at the more general case of $\mathbb{C}^2/\mathbb{Z}_n$, where a similar calculation to that of Example 2.7.3 can be used to show the blowup gives rise to $(n-1)$ exceptional $\mathbb{CP}^1$s. Details of this calculation can be found in [49].[17] ▲

2.7.7 Calabi-Yau Condition

Recall condition (iv) in the definition of a Calabi-Yau: it is a Kähler manifold with trivial canonical bundle. The canonical bundle is the line bundle given by the top-dimensional exterior power of the cotangent bundle. We then have the following proposition (see, e.g., [55]).

Proposition 2.7.20. *Let X be a nonsingular toric variety with $\{D_1, ..., D_d\}$ being the irreducible toric divisors. Then the canonical divisor is given by $D_K = -\sum_i D_i$. That is the canonical bundle is given by*

$$K_X = \mathcal{O}\left(-\sum_i D_i\right). \tag{2.130}$$

We therefore have that X is Calabi-Yau iff $D_K \sim 0$. This then gives us the following proposition.

Proposition 2.7.21. *Let X_Σ be a toric variety associated to some fan Σ. Then X_Σ is Calabi-Yau iff either of the following, equivalent, conditions apply*

[17] The $(1,-2,1)$ weight system gives rise exactly to the "(-2)-curves" discussed in this reference. Indeed the $(1,1)$ weight system is exactly the weight system of $\mathbb{CP}^1$, and the -2 weight is the curve.

(i) All the generating vectors end on the same affine hyperplane in $N_{\mathbb{R}}$; or

(ii) The weights Q^i_a obey $\sum_i Q^i_a = 0$ for all a.

Proof. (i) We can define a hyperplane in $N_{\mathbb{R}}$ precisely by the condition

$$H_N = \{w_i \in N_{\mathbb{R}} \,|\, \langle w_i, m\rangle = a\} \tag{2.131}$$

for some fixed $m \in M_{\mathbb{R}}$ and $a \in \mathbb{R}$. So the condition $\langle v_i, m\rangle = -1$, where $v_i \in \Sigma(1)$ defines a hyperplane in $N_{\mathbb{R}}$ that all the generating vectors end on.

(ii) This follows simply from $\sum_i \langle v_i, m\rangle Q^1_a = 0$ along with $\langle v_i, m\rangle = -1$.

■

Using Proposition 2.7.14 we then get the immediate corollary.

Corollary 2.7.22. *A toric Calabi-Yau manifold is non-compact.*

2.7.8 Updating The Weight System

So far we have seen that the weight system of a fan encodes a lot of information, however there is one important piece missing for the construction of a Calabi-Yau: the defining polynomial. We now work through how to update the weight system to encode this too.

Recall that we write our weight systems as

z_1	$\cdots$	z_n
Q^1_1	$\cdots$	Q^n_1
	$\vdots$	
Q^1_ℓ	$\cdots$	Q^n_ℓ

Now, to each coordinate z_i we have an associated toric divisor D_i, and so we can think of the columns as representing these toric divisors, i.e. we edit the weight system to look like

z_1	$\cdots$	z_n
Q^1_1	$\cdots$	Q^n_1
	$\vdots$	
Q^1_ℓ	$\cdots$	Q^n_ℓ
$\uparrow$		$\uparrow$
D_1	$\cdots$	D_n

Now, recall that the number of linearly independent toric divisors is given by $|\Sigma(1)| - \dim N$. Putting this together with the fact that $\dim N =$ (number of columns) $-$ (number of rows), and the fact that $|\Sigma(1)| =$ (number of columns), we immediately conclude that the number of linearly independent toric divisors is given by the number of rows. We can label these independent divisors H_j, and add them to our weight system as

$$\begin{array}{c|ccc} & z_1 & \dots & z_n \\ \hline H_1 \to & Q_1^1 & \dots & Q_1^n \\ & \vdots & \vdots & \\ H_\ell \to & Q_\ell^1 & \dots & Q_\ell^n \\ \hline & \uparrow & & \uparrow \\ & D_1 & \dots & D_n \end{array}$$

Indeed we can write the D_is in terms of the H_js using the weights, i.e.

$$D_i = \sum_{j=1}^{\ell} Q_j^i H_j. \tag{2.132}$$

Next we note that a hypersurface in our toric variety is given exactly by a divisor, i.e. we can express the defining polynomial, P, as a divisor. Whatever this divisor is, it can be related to our H_js by some given weights p_j. We can add this polynomial to our weight system too as

$$\begin{array}{c|ccc|c} & z_1 & \dots & z_n & \mathrm{P} \\ \hline H_1 \to & Q_1^1 & \dots & Q_1^n & p_1 \\ & \vdots & \vdots & & \vdots \\ H_\ell \to & Q_\ell^1 & \dots & Q_\ell^n & p_\ell \\ \hline & \uparrow & & \uparrow & \\ & D_1 & \dots & D_n & \end{array} \tag{2.133}$$

Finally, we recall that the total Chern class of a hypersurface space, $\mathcal{S}$, in some ambient space, A, is given by

$$c(\mathcal{S}) = \frac{c(A)}{c(P)}, \tag{2.134}$$

where P is the defining polynomial. Then, using

$$c(A) = \prod_{i=1}^{n}(1 + D_i) \quad \text{and} \quad c(P) = 1 + \sum_{j=1}^{\ell} p_j H_j, \tag{2.135}$$

we can quickly show that if we want a Calabi-Yau hypersurface we require

$$p_j = \sum_{i=1}^{n} Q_j^i. \tag{2.136}$$

2.7.9 Intersection Numbers & Fibration Structure

There are two more important pieces of information that we can understand from the fan of a toric variety. The first is the intersection number of subvarieties and the second is the idea of a fibration.

Intersection Numbers

We have shown that each homogeneous coordinate gives rise to a toric divisor, which is a hypersurface in X_Σ. We now want to ask the question "do these hypersurfaces intersect each other, and do they intersect themselves?" The answer to the latter comes from answering the former and then using Equation (2.127).

So how do we know if two toric divisors intersect each other? With some thought, it is clear that this happens only when the corresponding vectors form a cone in the fan. The most intuitive way to see this is probably just the fact that if they generate a cone in Σ, then by Lemma 2.7.15 they form a codimension 2 subvariety. This subvariety is formed exactly as the intersection of the 2 toric divisor hypersurfaces, which follows immediately from Equation (2.123). From here, we use Equation (2.127) to write the self intersection D_i^2 as $D_i \cdot (-a_j) D_j$ for $j \neq i$.

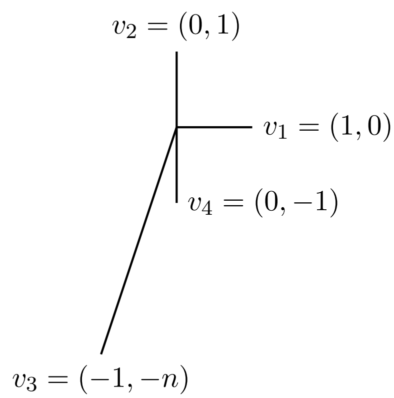
In order to be able to work out the self intersection numbers, we obviously need to know what the $D_i \cdot D_j$ with $j \neq i$ are. In fact the general k-point intersection $D_i \cdot D_j \cdot ... \cdot D_k$ plays an important role for us. Why? Well recall that when we wanted to compute the Euler characteristic we first found the top Chern class and integrated that over the space. Before we just quoted the result that, for $\mathbb{CP}^n$, $\int D^n = 1$, or if we were considering $\mathbb{CP}^{n_1} \oplus ... \oplus \mathbb{CP}^{n_m}$ that $\int D_1^{n_1} ... D_m^{n_m} = 1$. However we already saw that when you consider weighted projective spaces you need to be more careful as you get fractional results. So what is going on?

Well $D_i \cdot D_j$ is the intersection of the corresponding hyperplanes. So if $D_i \cdot D_j = 0$ they don't intersect, while if $D_i \cdot D_j = 1$ they intersect exactly once. We then generalise this to the case of n hypersurfaces intersecting. The key point is that if they only intersect once the resulting intersection space is smooth. So we conclude that $D_i \cdot D_j ... \cdot D_k = 1$ if the generating vectors $\{v_i, v_j, ..., v_k\}$ form a basis of a lattice, as per Proposition 2.7.17. This is precisely why we always took $\int D_i^n = 1$, i.e. $\mathbb{CP}^n$ is smooth. As Example 2.7.11 shows, weighted projective spaces are not smooth (i.e. (v_1, v_2) do not form a $\mathbb{Z}$ basis for $\sigma_{12} \cap N$) and so it follows that $D_i \cdot ... \cdot D_j \neq 1$.

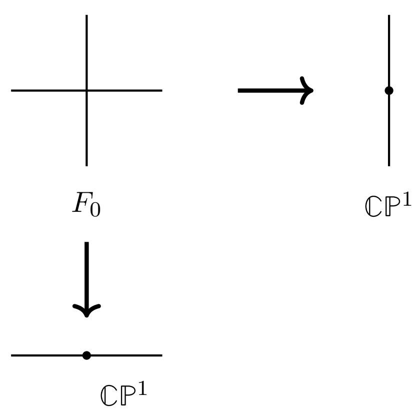
Now note that we also have that $D_i \cdot D_j ... \cdot D_k = 0$ if the span of the generating vectors don't span a cone in Σ. That is, if $\{v_i, v_j, ..., v_k\}$ don't span a cone in Σ then, by definition of the exceptional set, their common zero locus (which is exactly the intersection of the divisors) is removed from the toric variety X_Σ. So the intersection does not contribute to the integral over X_Σ, or any subspace, i.e. we require $D_i \cdot D_j \cdot ... \cdot D_k = 0$.

Fibration Structure

Recalling the fan for the n-th Hirzebruch surface F_n:



If we set $n = 0$ we would get a fan who's edges look like two perpendicular copies of the fan for $\mathbb{CP}^1$ (which has $v_1 = +1$ and $v_2 = -1$). That is, if we were to project the cones of F_0 horizontially or vertically, we would get copies of the $\mathbb{CP}^1$ fan.



From this perspective we start to see that the fans naturally encode some kind of fibration structure, i.e. $F_0 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$.

A similar argument can be made for F_n for $n > 0$. The key difference is when we ask "do the cones of F_n project into cones of $\mathbb{CP}^1$?" For F_0 the answer is yes: each quadrant is mapped to a half line. However for $F_{n>0}$ the answer is no: if we project the cone given by (v_2, v_3) horizontally, we are left with the full vertical line, but this is *not* a cone in $\mathbb{CP}^1$. The vertical projection, though, is fine as all cones project into half horizontal lines and so are cones in the $\mathbb{CP}^1$. This failure of σ_{23} to project into a cone of $\mathbb{CP}^1$ tells us that $F_{n>0}$ is a non-trivial $\mathbb{CP}^1$ fibration over $\mathbb{CP}^1$, with the horizontal $\mathbb{CP}^1$ being the base space. In other words the base space of our fibration must self intersect. We summaries this in the following table.

Edges Project Into Edges	Cones Project Into Cones	Contained Manner
✓	✓	Product (embedding)
✓	×	Twisted (self intersecting inclusion)

The fact that we have a non-trivial fibration can be seen by the fact that the divisors have non-vanishing self intersection. We have smooth cones generated by $\{v_1, v_2\}$ and $\{v_1, v_4\}$, so

$$D_1 \cdot D_2 = D_1 \cdot D_4 = 1, \tag{2.137}$$

Then from Equation (2.127), with $m = (1, 0)$ and $m = (0, 1)$, we have

$$D_1 - D_3 \sim 0 \quad \text{and} \quad D_2 - nD_3 - D_4 \sim 0. \tag{2.138}$$

It then follows from $D_1 \sim D_3$ and the above intersection relations that

$$D_3 \cdot D_2 = D_3 \cdot D_4 = 1. \tag{2.139}$$

All other non-self intersections vanish, as $\{v_1, v_3\}$ and $\{v_2, v_4\}$ don't span cones in Σ.

From here we get the self intersection numbers

$$D_1^2 = D_3^2 = 0, \qquad D_2^2 = n \quad \text{and} \quad D_4^2 = -n. \tag{2.140}$$

The important thing to note is the D_1 and D_3 don't self intersect: D_1 and D_3 correspond to setting $z_1 = 0$ and $z_3 = 0$ and so we are in the $[z_2 : z_4]$ line, it then follows from $D_1^2 = D_3^2 = 0$ that our fibres are given by $[z_1 : z_3] \cong \mathbb{CP}^1$. The fact that D_2 and D_4 self intersect proportionally to n encodes exactly the non-trivial nature of the $\mathbb{CP}^1$-bundle: the base space is self intersecting. Note that $n = 0$ returns that the base space is simply $\mathbb{CP}^1$ and we get $F_0 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$.

2.7.10 Polytopes

We have seen that in order to define a toric Calabi-Yau we need two peices of information $(\Sigma, \mathcal{L})$; Σ is a fan and $\mathcal{L}$ is the line bundle that encodes the defining polynomial. As per Proposition 2.7.6 there is corresponding a divisor $D_{\mathcal{L}} = \sum_i a_i D_i$ for integers a_i, where D_i are the divisors corresponding to the ray generators of Σ. The first Chern class of $\mathcal{L}$ is then given by

$$c_1(\mathcal{L}) = \sum_i a_i D_i. \tag{2.141}$$

Using Equation (2.135) we see that the first Chern class of the line bundle is given by

$$c_1(\mathcal{L}) = \sum_{j=1}^{\ell} p_j H_j. \tag{2.142}$$

We then put this together with Equations (2.132) and (2.136) and obtain

$$c_1(\mathcal{L}) = \sum_i D_i. \tag{2.143}$$

Therefore, in order to get a Calabi-Yau we require $a_i = 1$ for all i. There is an object in toric geometry which actually encodes both of these pieces of information together: a *polytope*.

Definition. [Polytope] Let $M_{\mathbb{R}}$ be some real vector space of dimension d. Consider some set of points $S \subset M_{\mathbb{R}}$. Then we can define a *polytope* by the convex hull of the set S, i.e.

$$\Delta = \mathrm{Conv}(S) := \left\{ \sum_i \lambda_i m_i \,\middle|\, \sum_i \lambda_i = 1,\ \forall m_i \in S,\ \text{and } \lambda_i \in \mathbb{R}_0^+ \right\} \subseteq M_{\mathbb{R}}. \tag{2.144}$$

The dimension of the polytope is equal to the dimension of the smallest affine subspace in $M_{\mathbb{R}}$ that contains Δ. We will focus on the cases where $M_{\mathbb{R}} = M \otimes \mathbb{R}$ for some lattice M. We then call a polytope $\Delta \subseteq M_{\mathbb{R}}$ a *lattice polytope* if the vertices of Δ are lattice points in M. We call the ordered set $\Delta \cap M = \{m_0, ..., m_k\}$ the *characters* of M. Lattice polytopes can always be made top-dimensional, i.e. $\dim \Delta = \dim M_{\mathbb{R}}$, simply by appropriately reducing the dimension of M.

An important object in a polytope are its faces.

Definition. [Polytope Face] Let $(N_{\mathbb{R}}, M_{\mathbb{R}})$ be a set of dual vector spaces and let $\Delta \subseteq M_{\mathbb{R}}$ be a polytope. Then given a non-zero vector $v \in N_{\mathbb{R}}$ and an $a \in \mathbb{R}$, we can define

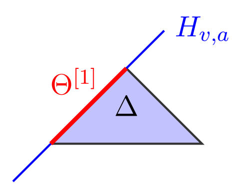
$$H_{v,a} := \{m \in M_{\mathbb{R}} \,|\, \langle m, v\rangle = a\} \quad \text{and} \quad H^+_{v,a} := \{m \in M_{\mathbb{R}} \,|\, \langle m, v\rangle \geq a\}. \tag{2.145}$$

$H_{v,a}$ is clearly a hypersurface in $M_{\mathbb{R}}$, and $H^+_{v,a}$ is the upper half plane associated to this hypersurface. We call a subset $\Theta \subseteq \Delta$ a *face* of Δ if there exists a $H_{v,a}$ and $H^+_{v,a}$ such that

$$\Theta = H_{v,a} \cap \Delta, \quad \text{and} \quad \Delta \subseteq H^+_{v,a}. \tag{2.146}$$

We will denote a dimension k face by $\Theta^{[k]}$.

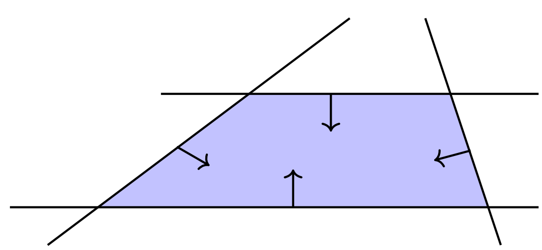
This definition is intuitively clear: consider some hypersurface in $M_{\mathbb{R}}$ that "touches" Δ, and then the intersection of this hypersurface with Δ is a face of Δ. We call a face of codimension-1 a *facet*, a face of dimension 1 an edge and a face of dimension 0 a vertex. Note we can think of a polytope as the convex hull of its vertices. We give a pictorial example of this for a 2D polytope corresponding to a triangle below.



Note that a polytope Δ is given precisely by the intersection of a finite number of half planes $H^+_{v_i,a_i}$, i.e.

$$\Delta = \bigcap_{i=1}^{\ell} H^+_{v_i,a_i} \tag{2.147}$$

is a polytope. It then follows from the definition of H_{v_i,a_i} that the vectors v_i are perpendicular to surfaces H_{v_i,a_i} and point into the intersection, as this is exactly what we need to ensure that $\langle m, v_i\rangle \geq a_i$ for all $m \in \Delta$. We give a pictorial example for a 2D polytope with $\ell = 4$ below.



We now note that for a lattice polytope, in which case $\dim \Delta = \dim M_{\mathbb{R}} = d$, each facet $\Theta^{[d-1]}$ has a unique supporting hyperplane. It follows from this that there is a unquie choice such that $v_{\Theta}^{[d-1]}$ is a primitive lattice point in N, which implies that $a_{\Theta}^{[d-1]}$ is an integer.

$$\begin{aligned} H_{v_{\Theta}^{[d-1]}, a_{\Theta}^{[d-1]}} &= \{m \in M_{\mathbb{R}} \,|\, \langle m, v_{\Theta}^{[d-1]} \rangle = -a_{\Theta}^{[d-1]}\}, \quad \text{and} \\ H^{+}_{\Theta^{[d-1]}} &= \{m \in M_{\mathbb{R}} \,|\, \langle m, v_{\Theta}^{[d-1]} \rangle \geq -a_{\Theta}^{[d-1]}\} \end{aligned} \tag{2.148}$$

and our polytope is given by

$$\Delta = \bigcap_{\{\Theta^{[d-1]}\}} H^{+}_{\Theta^{[d-1]}} = \{m \in M_{\mathbb{R}} \,|\, \langle m, v_{\Theta}^{[d-1]} \rangle \geq -a_{\Theta}^{[d-1]}, \, \forall \text{ facets } \Theta^{[d-1]} \subset \Delta\}. \tag{2.149}$$

The minus sign appearing in the above expressions is included for later convenience.

The polytope can then be used to define two different fans.

Definition. [Fan Over Faces] Let $\Delta \subseteq M_{\mathbb{R}}$ be a polytope with vertices $\{v_i\}$. We can define the *fan over the faces* to be the fan who's generating vectors are given are given by $\{v_i\}$. In other words, $\Delta \subseteq M_{\mathbb{R}}$ is given by the convex hull of the generating vectors of the fan.

We note that this fan is defined in $M_{\mathbb{R}}$. We can also construct a fan in $N_{\mathbb{R}}$ as follows.

Definition. [Normal Fan] Let $\Delta \subseteq M_{\mathbb{R}}$ be a polytope. To any face Θ of Δ, we can associate a cone

$$\check{\sigma}_{\Theta} := \bigcup_{r \geq 0, p_{\Delta} \in \Delta, p_{\Theta} \in \Theta} r \cdot (p_{\Delta} - p_{\Theta}), \tag{2.150}$$

and its dual $\sigma_{\Theta} \subseteq N_{\mathbb{R}}$, which obeys

$$\langle \check{\sigma}_{\Theta}, \sigma_{\Theta} \rangle \geq 0. \tag{2.151}$$

That is,

$$\sigma_{\Theta} := \{v \in N_{\mathbb{R}} \,|\, \langle p_{\Theta}, v \rangle \leq \langle p_{\Delta}, v \rangle, \, \forall \, p_{\Delta} \in \Delta \text{ and } p_{\Theta} \in \Theta\}. \tag{2.152}$$

Again the collection of such dual cones[18] for all faces gives us a fan $\Sigma_{\Delta} \subseteq N_{\mathbb{R}}$. The generating vectors of the fan are given by the vectors normal to the facets of Δ, and as such Σ_{Δ} is known as the normal fan of Δ.

An important note is that a k-dimensional face $\Theta^{[k]}$ of Δ is associated to a $(n-k)$-dimensional cone in Σ_{Δ}, where n is the dimension of Σ_{Δ}. In particular the facets in Δ

[18] It is customary to consider the full Δ being a face of itself, and the dual cone σ_{Δ} is then the zero-dimensional cone.

correspond to the rays in Σ_Δ. By an appropriate translation of Δ, the ray generators of the normal fan become equal to the lattice points $v_\Theta^{[d-1]}$. We can then use our polytope to define not only the normal fan Σ_Δ, but also the divisor class of a line bundle $\mathcal{L}$ on the associated toric variety X_{Σ_Δ}. That is, we identify the $a_\Theta^{[d-1]}$ in Equation (2.149) with the a_i in Equation (2.141).

Flipping this on its head: given a fan $\Sigma \subset N$ and line bundle $\mathcal{L}$ obeying Equation (2.141), we can define a polytope, known as the Newton polytope, as

$$\Delta_\Sigma = \{m \in M_\mathbb{R} \,|\, \langle m, \nu_i \rangle \geq -a_i, \, \forall \, \nu_i \in \Sigma(1)\}, \tag{2.153}$$

where $\Sigma(1)$ denotes the set of all rays in Σ. The group of holomorphic sections in $\mathcal{L}$ then has monomial basis

$$p(m) = \prod_i z_i^{\langle m, \nu_i \rangle + a_i}, \tag{2.154}$$

where the z_i are the homogeneous coordinates associated to the ray generators ν_i. In particular, we note that there is a one-to-one correspondence between the characters $\{m_i\} \in \Delta$ and the monomials in $\mathcal{L}$. We note that the holomorphicity of these sections is guarenteed by the condition $\langle m, \nu_i \rangle \geq -a_i$.

The case that is of interest to us is when the hypersurface defines a Calabi-Yau. This requires the defining polynomial to be a section of the anticanonical bundle of the ambient toric variety. This sets $a_i = 1$ for all i in Equation (2.141), and so the Newton polytope is simply defined by

$$\Delta_\Sigma = \{m \in M_\mathbb{R} \,|\, \langle m, \nu_i \rangle \geq -1, \, \forall \, \nu_i \in \Sigma(1)\}. \tag{2.155}$$

The Calabi-Yau is then given by the zero locus of

$$\sum_{m \in \Delta} \alpha_m p(m) = \sum_{m \in \Delta} \alpha_m \prod_{\nu_i \in \Sigma(1)} z_i^{\langle m, \nu_i \rangle + 1}, \tag{2.156}$$

where $\alpha_m \in \mathbb{C}$. If Δ is a lattice polytope (i.e. all its vertices are lattice points on M), then it follows that the ν_i are all lattice points on the dual lattice N. Taking their convex hull defines another polytope Δ°, known as the polar dual of Δ. The two lattices obey

$$\langle \Delta, \Delta^\circ \rangle \geq -1. \tag{2.157}$$

A lattice polytope whose polar dual is also a lattice polytope is called reflexive. It follows from this, and the fact that $(\Delta^\circ)^\circ = \Delta$, that (Δ, Δ°) are a pair of reflexive polytopes. A necessary condition of reflexivity is that the origin is the unique interior point of the polytope. The faces of Δ and Δ° are related to each other by

$$\langle \Theta^{[k]}, \Theta^{\circ[n-k-1]} \rangle = -1, \tag{2.158}$$

where n is the dimension of the polytopes. The important thing is that a k-dimensional face in Δ is related to a $(n-k-1)$-dimensional face in Δ°.

Of course we could have done this whole construction but now starting with a fan Σ° who's Newton polytope is Δ°. We would then have that Δ is given by the convex hull of the vertices of Σ°. Indeed in this way we see that the normal fan of one polytope corresponds to the fan over the faces of the other polytope.

If we now change notation $m_i \mapsto \nu_i^*$, then we see our pair of reflexive Newton polytopes are defined via

$$\begin{aligned} \Delta_\Sigma &= \{m \in M_\mathbb{R} \,|\, \langle m, \nu_i \rangle \geq -1, \, \forall \, \nu_i \in \Sigma(1)\} \\ (\Delta^\circ)_{\Sigma^\circ} &= \{n \in N_\mathbb{R} \,|\, \langle n, \nu_i^* \rangle \geq -1, \, \forall \, \nu_i^* \in \Sigma^\circ(1)\}, \end{aligned} \tag{2.159}$$

which define a pair of Calabi-Yaus $(X, X^\vee)$, with defining equations

$$\begin{aligned} G(z_i) &= \sum_{m \in \Delta} \alpha_m \prod_{\nu_i \in \Sigma(1)} z_i^{\langle m, \nu_i \rangle + 1} \\ G^\circ(z_i^\vee) &= \sum_{n \in \Delta^\circ} \alpha_n^\circ \prod_{\nu_i^* \in \Sigma^\circ(1)} z_i^{\langle n, \nu_i^* \rangle + 1} \end{aligned} \tag{2.160}$$

Batyrev Mirror Symmetry

We then have the following result due to Batyrev [9].

Proposition 2.7.23. *For a Calabi-Yau 3-fold defined via 4-dimensional reflexive Newton polytope Δ, the Hodge numbers are given by*

$$\begin{aligned} h^{1,1}(X_{\Delta,\Delta^\circ}) &= \ell(\Delta^\circ) - 5 - \sum_{\Theta^{\circ[3]}} \ell^*\left(\Theta^{\circ[3]}\right) + \sum_{(\Theta^{\circ[2]},\Theta^{[1]})} \ell^*\left(\Theta^{\circ[2]}\right) \ell^*\left(\Theta^{[1]}\right) \\ h^{2,1}(X_{\Delta,\Delta^\circ}) &= \ell(\Delta) - 5 - \sum_{\Theta^{[3]}} \ell^*\left(\Theta^{[3]}\right) + \sum_{(\Theta^{[2]},\Theta^{\circ[1]})} \ell^*\left(\Theta^{[2]}\right) \ell^*\left(\Theta^{\circ[1]}\right) \end{aligned} \tag{2.161}$$

where $\ell(...)$ denotes the number of lattice points of its argument and $\ell^*(...)$ only counts the lattice points in the relative interior of its argument.

We do not prove this result here but make the following comment. As detailed in Appendix B, the middle primitive Hodge numbers for Calabi-Yaus defined inside projective spaces are related to polynomails of set degree. In particular for a 3-fold, $h^{2,1}_{\text{Prim}} = h^{2,1}$ is given by the number of monomials of the same degree as the defining polynomial, minus the number of terms in the Jacobi ideal. The formula for $h^{2,1}$ above can be equally understood:

- $\ell(\Delta)$ corresponds exactly to the total number of monomials with the same degree as the defining polynomial.
- $(1+4)+\sum_{\Theta^{[3]}} \ell^*\left(\Theta^{[3]}\right)$ corresponds to the $\mathbb{C}^*$ scaling (the 1) as well as the automoprhism group of the toric varieity.[19]
- $\sum_{(\Theta^{[2]},\Theta^{\circ[1]})} \ell^*\left(\Theta^{[2]}\right) \ell^*\left(\Theta^{\circ[1]}\right)$ counts the non-primitive forms, that is it is the non-polynomial deformations.

Corollary 2.7.24. *The two Calabi-Yaus above form a mirror pair, and we use notation X_{Δ,Δ° and $X_{\Delta^\circ,\Delta}$ to denote the pair.*

Proof. This follows simply from Proposition 2.7.23 and swapping $\Delta \leftrightarrow \Delta^\circ$. ∎

This gives us an incredibly powerful method for constructing Calabi-Yau 3-folds and their mirror:

(i) Take a fan Σ corresponding to whatever ambient space we want.

(ii) Form the Newton polytope Δ_Σ along with its dual polytope Δ°.[20]

(iii) Consider $\Delta^\circ \to (\Delta^\circ)_{\Sigma^\circ}$, i.e. interpret Δ° as the Newton polytope of another fan Σ°.

(iv) The pair of Newton polytopes gives a pair of Calabi-Yau hypersurfaces $(X, X^\vee)$, which are mirror to each other.

Example 2.7.25. As an important example, we look at the quintic hypersurface $Q \subset \mathbb{CP}^4$ and find it's mirror. We start with the fan of $\mathbb{CP}^4$, which has weight system

$$\begin{array}{ccccc} z_0 & z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 & 1 \end{array} \tag{2.162}$$

[19] The automorphism group of a toric variety X has dimension $\text{rank}(N) + \dim \mathcal{R}$, where $\mathcal{R}$ is the set of Demazure roots, and corresponds exactly to the number of lattice points in the interior of the facets, here $\Theta^{[3]}$. See [56] for more details.

[20] We have to ensure that our polytopes are reflexive. Therefore if we end up in a situation where either are not, we simply go back and pick a different fan.

The generating vectors are

$$\begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.163}$$

The Newton polytope is then given by the convex hull of the vectors

$$\begin{pmatrix} \nu_0^* \\ \nu_1^* \\ \nu_2^* \\ \nu_3^* \\ \nu_4^* \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} \tag{2.164}$$

Our dual polytope Δ° is given by the convex hull of $(\nu_0, .., \nu_4)$.

Using

$$\ell(\Delta) = 125, \quad \ell(\Delta^\circ) = 6, \quad \ell^*(\Theta^{[3]}) = 19, \quad \text{and} \quad \ell^*(\Theta^{\circ[i]}) = 0 \tag{2.165}$$

for $i = 3, 2, 1$ we obtain $(h^{1,1}(Q), h^{2,1}(Q)) = (1, 101)$.

Finally we want to compute the toric variety associated to Σ°. This has generating vectors $(\nu_0^*, ..., \nu_4^*)$. The exceptional set is again just the origin. Our quotient map is then given as

$$\phi : \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} \mapsto \begin{pmatrix} t_0^{-1} t_1^4 t_2^{-1} t_3^{-1} t_4^{-1} \\ t_0^{-1} t_1^{-1} t_2^4 t_3^{-1} t_4^{-1} \\ t_0^{-1} t_1^{-1} t_2^{-1} t_3^4 t_4^{-1} \\ t_0^{-1} t_1^{-1} t_2^{-1} t_3^{-1} t_4^4 \end{pmatrix} \tag{2.166}$$

The kernel of this map is given by

$$t_i^5 = 1 \quad \text{and} \quad \prod_{i=1}^{5} t_i = 1. \tag{2.167}$$

In total this gives that the mirror quintic is given by a degree 5 polynomial in $\mathbb{CP}^4/\mathbb{Z}_5^3$. The monomial basis is then given by

$$p(n) = \prod_i z_i^{\langle n, \nu_i^* \rangle + 1}, \tag{2.168}$$

for $n \in \{\nu_0, ..., \nu_4, (0,0,0,0)\}$. The result is simply

$$z_0^5, \quad z_1^5, \quad z_2^5, \quad z_3^5, \quad z_4^5, \quad \text{and} \quad z_0 z_1 z_2 z_3 z_4. \tag{2.169}$$

We note that there is only one additional monomial compared to the Fermat terms z_i^5. This corresponds exactly to the fact that $h^{2,1}(Q^\vee) = 1$. ▲

2.7.11 Maximally Projective Crepant Partial (MPCP) Desingularisation

In the above we have skipped over an important detail: a priori we are not guaranteed that our fans do not contain singularities, and that these singularities don't hit our Calabi-Yau hypersurface. Indeed for a fan Σ constructed over the faces of a polytope Δ, we are not guaranteed that such a fan is even simplical: the toric variety contains bad (i.e. non-orbifold) singularities, as per Proposition 2.7.18. These bad singularities could be inherited into the Calabi-Yau and result in a singular space, which we do not want. Luckily, we have the following theorem for reflexive polytopes [9].

Theorem 2.7.26. *Let X be a toric variety associated to a reflexive polytope Δ. Then X admits at least one partial desingularisation (called a maximal projective crepant partial desingularisation in [9]) defined by a triangulation of Δ.*

The key thing is that this desingularisation is only partial, in general. However, it can be shown that if $\dim \Delta \leq 4$, then this desingularisation is a legitimate resolution and the toric variety is smooth. In particular, this means any Calabi-Yau defined as a hypersurface inside a toric variety stemming from a polytope with dimension 4 or less is smooth. This is the case for all Calabi-Yau 3-folds, and so we are guaranteed a smooth result. The X and $X^\vee$ appearing in Proposition 2.7.23 refer to the smooth Calabi-Yaus.

3 Calabi-Yau: Conformal Field Theory

In this chapter we will review the relevant material on the superconformal field theory (SCFT) realisations of Calabi-Yau manifolds. It has been shown that the existence of $\mathcal{N} = 1$ spacetime supersymmetry in heterotic compactifications actually requires the existence of $\mathcal{N} = (0, 2)$ SUSY on the worldsheet [57, 58]. As mentioned before, we are more interested in compactifications of Type II strings, where the left and right SUSYs are equal, so we require a $\mathcal{N} = (2, 2)$ field theory in $(1 + 1)$-dimensions. We therefore start with a review of some of the basic concepts and set notations of $\mathcal{N} = (2, 2)$ SCFTs. More detailed reviews can be found in [13, 59].

We then move towards nonlinear sigma models and SCFTs with Calabi-Yau target space. Here we will see that the states of the SCFT are deeply related to the cohomology of the Calabi-Yau. We then introduce mirror symmetry as an automorphism of the SCFT algebra, and relate it back to the result of the previous chapter.

Next we introduce the gauged linear sigma model and how it is related to the nonlinear sigma model of our Calabi-Yau. This leads to a discussion of the Landau-Ginzburg/Calabi-Yau correspondence, and establishes a link to the construction of Calabi-Yaus as hypersurfaces in complex projective spaces. We then meet the important construction of Gepner models and work through a few important examples.

Finally we study mirror symmetry in these models and review the construction of Hori & Vafa [13], and show how it is related to the Batyrev construction presented in the last chapter.

3.1 $\mathcal{N} = 2$ SCFTs

We have $\mathcal{N} = 2$ SUSY for both the left and right moving sectors, but for now we will just focus on one side, and assume the other is understood implicitly. We will return to how the left and right quantum numbers are related later.

The $\mathcal{N}=2$ Virasoro algebra contains four generators: the energy-momentum tensor, two supersymmetry currents and a $U(1)$ current (T, G^0, G^3, J), respectively. It is common to work with $G^{\pm} = \frac{1}{\sqrt{2}}(G^0 \pm iG^3)$ instead of G^0 and G^3, as these have definite charge under the R-symmetry of J, and we do so here. It can be shown [60] that the $U(1)$ current and one of the supersymmetry currents stem from the Kähler form in the geometry.[1] The algebra is defined via the mode relations

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2} - r\right)G_{m+r}^{\pm} \\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm} \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \\
\{G_r^{\pm}, G_s^{\pm}\} &= 0,
\end{aligned} \tag{3.1}$$

where c is the Virasoro central charge. The indices on the supersymmetry operators dictate whether we are in the Neveu-Schwarz (NS) or Ramond (R) sector, with $r, s \in \mathbb{Z}$ being R and $r, s \in \mathbb{Z} + \frac{1}{2}$ being NS.

We have two copies of this algebra, a left- and right-moving sector. We will use notation where unbarred objects correspond to the left-moving algebra and barred objects correspond to the right-moving sector.

3.1.1 Chiral & Antichiral States

An important thing to note at this point is that L_0 and J_0 commute, and so they can be simultaneously diagonalised. In this way, states in our Hilbert space actually have two labels $|h,q\rangle$, where h is the conformal weight while q denotes the J_0 charge, that is

$$L_0 |h,q\rangle = h |h,q\rangle \quad \text{and} \quad J_0 |h,q\rangle = q |h,q\rangle \, . \tag{3.2}$$

[1] The general statement is that the existence of a covariantly constant p-form on the target manifold gives rise to conformal dimension $\frac{p}{2}$ and $\frac{p+1}{2}$ currents in the algebra, the latter being the superpartner of the former. Here the Kähler form is a 2-form and so gives rise to a dimension 1 and dimension $\frac{3}{2}$ current, which are the J and G^3, respectively.

Then, using our anticommutation relation above, we see that for a general state in our $\mathcal{N}=2$ SCFT Hilbert space, we have (assuming unitarity)

$$0 \leq \langle h,q| \{G^{\pm}_{1/2}, G^{\mp}_{-1/2}\} |h,q\rangle = \langle h,q| (2L_0 \pm J_0) |h,q\rangle \,, \tag{3.3}$$

which gives us the relation

$$h \geq \frac{1}{2}|q|. \tag{3.4}$$

> **Definition.** [Superprimary States] Given a $\mathcal{N}=2$ SCFT, a state $|h,q\rangle$ is a superprimary if it obeys
>
> $$L_n |h,q\rangle = J_n |h,q\rangle = G^{\pm}_r |h,q\rangle = 0 \qquad \forall n,r>0. \tag{3.5}$$

> **Definition.** [Chiral State] In the Neveu-Schwarz sector of an $\mathcal{N}=2$ SCFT, a state $|h,q\rangle$ is called *left-chiral* if it obeys
>
> $$G^{+}_{-1/2} |h,q\rangle = 0. \tag{3.6}$$
>
> Similarly it is called *left-anti-chiral* if
>
> $$G^{-}_{-1/2} |h,q\rangle = 0. \tag{3.7}$$
>
> We similarly define a right-(anti-)chiral state by replacing $G^{\pm}_{-1/2}$ with $\overline{G}^{\pm}_{-1/2}$.

Proposition 3.1.1. *A state $|h,q\rangle$ in a unitary $\mathcal{N}=2$ SCFT has $h=\frac{q}{2}$ if, and only if, it is a chiral primary field. Similarly if $h=-\frac{q}{2}$ it is an anti-chiral primary.*

Proof. Given a chiral primary, following the steps leading to Equation (3.4), we get the result $h=\pm\frac{q}{2}$ easily.

Now assume that we have a state $|h,q\rangle = \left|\frac{q}{2},q\right\rangle$. Now we compute

$$0 = \langle h,q| \{G^{+}_{-1/2}, G^{-}_{+1/2}\} |h,q\rangle = \left|G^{-}_{+1/2}|h,q\rangle\right|^2 + \left|G^{+}_{-1/2}|h,q\rangle\right|^2, \tag{3.8}$$

where we have used the fact that the result vanishes, as per the derivation of Equation (3.4), along with $(G^{+}_{-1/2})^{\dagger} = G^{-}_{+1/2}$. So if we are in a unitary theory, we have positive norms, and so each term on the right must vanish independently, which, in particular, gives us the chiral condition $G^{+}_{-1/2}|h,q\rangle = 0$.

We also have the first step of the primary condition, $G^{-}_{+1/2}|h,q\rangle = 0$. To prove the rest, i.e. $G^{-}_{n-1/2}|h,q\rangle = 0$ $n>0$, we show that if such a state didn't vanish, then it would violate the general condition Equation (3.4): using the commutation relations, we start by noting that

$$L_0 J_n \left|\frac{q}{2}, q\right\rangle = \left(\frac{q}{2} - n\right) J_n \left|\frac{q}{2}, q\right\rangle \tag{3.9}$$

and so $J_n \left|\frac{q}{2}, q\right\rangle = 0$ for all $n > 0$, as otherwise we violate our condition $h \geq \frac{|q|}{2}$. Then we finally note that

$$[J_n, G^+_{-1/2}] \left|h, q\right\rangle = G^+_{n-1/2} \left|q, n\right\rangle, \tag{3.10}$$

but the left-hand side vanishes for our state $\left|\frac{q}{2}, q\right\rangle$, so we have a primary field.

The anti-chiral result follows along analogous lines. ■

Corollary 3.1.2. *The conditions* $G^-_{+1/2} \left|h, q\right\rangle = G^+_{-1/2} \left|h, q\right\rangle = 0$ *are sufficient to prove that* $\left|h, q\right\rangle$ *is a chiral primary. Similarly* $G^+_{+1/2} \left|h, q\right\rangle = G^-_{-1/2} \left|h, q\right\rangle = 0$ *gives an anti-chiral primary.*

Proof. This follows simply from the fact that these conditions impose $h = \frac{|q|}{2}$, and so by the above proposition, we have a (anti-)chiral primary. ■

Proposition 3.1.3. *A chiral primary field in a* $\mathcal{N} = 2$ *SCFT must have conformal weight* $h \leq \frac{c}{6}$*. In particular the bound* $h = \frac{c}{6}$ *is satisfied if, and only if,* $G^+_{-3/2} \left|h, q\right\rangle = 0$*. A similar result holds for anti-chiral primaries with* $G^+_{-3/2} \to G^-_{-3/2}$*.*

Proof. This follows simply by computing

$$\left\langle h, q\right| \{G^+_{-3/2}, G^-_{+3/2}\} \left|h, q\right\rangle \geq 0, \tag{3.11}$$

as the left-hand side gives

$$2h - 3q + \frac{2c}{3} = -4h + \frac{2c}{3}, \tag{3.12}$$

where we have used $q = 2h$ for a chiral primary. Again the equality is satisfied only when $G^+_{-3/2} \left|h, q\right\rangle = 0$. ■

We will see shortly that there is actually a *unique* chiral primary state that saturates the bound $h = \frac{c}{6}$, an important consequence of this is that when we have a non-degenerate theory (so the L_0 spectrum is discrete) there are only a finite number of chiral primary operators. There is also a unique antichiral primary state with $h = \frac{c}{6}$.

Chiral Ring

The fields associated to chiral states play an important role. Consider the OPE between two chiral fields:

$$\phi_i(z)\phi_j(w) = \sum_k C_{ij}^k \frac{\Phi_k(w)}{(z-w)^{h_i+h_j-h_k}} \tag{3.13}$$

for general fields $\Phi_k(w)$. Now, we impose the conservation of our $U(1)$ charge, i.e. of the q values: $q_i + q_j = q_k$. Then, using Equation (3.4), which holds for a general field, we have

$$h_k \geq \frac{q_k}{2} = \frac{q_i+q_j}{2} = h_i + h_j, \tag{3.14}$$

where we have used that ϕ_i and ϕ_j are chiral primaries. However, if $h_k > h_i + h_j$, then the OPE only contains positive powers of $(z-w)$, so in the limit $z \to w$, the right-hand side vanishes. The exception is precisely when $h_k = h_i + h_j = \frac{q_k}{2}$, but this makes Φ_k a chiral primary, as per Proposition 3.1.1. This allows us to define a closed product between chiral primaries as

$$(\phi_i \cdot \phi_j)(w) := \lim_{z\to w} \phi_i(z)\phi_j(w) = \sum_k C_{ij}^k \phi_k(w). \tag{3.15}$$

This product turns the set of chiral primaries into a *ring*,[2] known as the chiral ring. Obviously the same story follows for the anti-chiral ring, and equally for the right-(anti-)chiral rings. So in total a $\mathcal{N}=(2,2)$ SCFT has four chiral rings. We shall denote these four chiral rings by $\mathfrak{R}_{(c,c)}, \mathfrak{R}_{(c,a)}, \mathfrak{R}_{(a,c)}, \mathfrak{R}_{(a,a)}$, where c/a stand for chiral/antichiral. Note also that (c,c) and (a,a) are conjugate to each other, as are (c,a) and (a,c), and so we actually only have two independant chiral rings, say $\mathfrak{R}_{(c,c)}$ and $\mathfrak{R}_{(a,c)}$. We will denote the union of these two rings as $\mathfrak{R} = \mathfrak{R}_{(c,c)} \cup \mathfrak{R}_{(a,c)}$.

3.1.2 Spectral Flow

The $\mathcal{N}=2$ algebra contains an incredibly important property, known as spectral flow. We now show where this comes from and explain it's significance.

Proposition 3.1.4. *The $\mathcal{N}=2$ superVirasoro algebra contains a continuous automorphism*

[2] Note it really is a ring and not a group, as a chiral primary with $h \neq 0$ does not have an inverse, as this would require $h < 0$ for some fields, but this is not allowed in a unitary theory.

given by

$$\begin{aligned} L_n &\mapsto L_n^\eta = L_n + \eta J_n + \frac{\eta^2}{6} c\delta_{n,0} \\ J_n &\mapsto J_n^\eta = J_n + \frac{\eta}{3} c\delta_{n,0} \\ G_r^\pm &\mapsto (G^\eta)_r^\pm = G_{r\pm\eta}^\pm \end{aligned} \tag{3.16}$$

for continuous parameter η. The important case is when $\eta \in \mathbb{Z} + \frac{1}{2}$, as this provides a map between Ramond and Neveu-Schwarz sectors.

Proof. This follows by simply checking that $(L_n^\eta, J_\eta^n, G_\eta^\pm)$ satisfy the algebra mode relations. The details are tedious and so are omitted. ■

As Equation (3.16) deforms the generators of our SCFT, it necessarily deforms the Hilbert space. In other words, η parameterises a *flow* of the *spectrum* of our SCFT, hence the name spectral flow. We should note that the terminology *twist* is also used to describe this process. More concretely, denoting the starting Hilbert space by $\mathcal{H}_0$ and the resulting Hilbert space by $\mathcal{H}_\eta$, we define a unitary operator $U_\eta : \mathcal{H}_0 \to \mathcal{H}_\eta$ via

$$L_m^\eta = U_\eta L_m U_\eta^\dagger, \qquad J_m^\eta = U_\eta J_m U_\eta^\dagger, \quad \text{and} \quad |\phi_\eta\rangle = U_\eta |\phi\rangle . \tag{3.17}$$

This allow us to see that we really do have an automorphism on our theory quite easily: consider a state $|\phi\rangle = |h, q\rangle$ in our original theory, then

$$L_0^\eta |\phi_\eta\rangle = U_\eta L_0 U_\eta^\dagger U_\eta |\phi\rangle = U_\eta L_0 |\phi\rangle = h |\phi_\eta\rangle , \tag{3.18}$$

and similarly $J_0^\eta |\phi_\eta\rangle = q |\phi_\eta\rangle$, and so the physical results, the conformal weight and J_0 charge, are invariant. We are therefore dealing with the same theory, just in a different labelling. In fact, we could have defined our superVirasoro algebra to include a parameter η in it to start of with and built the theory up from there.

What we are more interested in, though, is how the conformal weight and J_0 charge change. That is, we want to calculate the actions[3]

$$L_0 |\phi_\eta\rangle = h_\eta |\phi_\eta\rangle \quad \text{and} \quad J_0 |\phi_\eta\rangle = q_\eta |\phi_\eta\rangle \tag{3.19}$$

[3]We might question acting on the states $|\phi_\eta\rangle$ with the "non-η" operators L_0 and J_0. That is, L_0 and J_0 are defined to act on our original Hilbert space, $\mathcal{H}_0$, and although $\mathcal{H}_\eta \cong \mathcal{H}_0$, that does not mean that L_0 and J_0 should be able to act on $\mathcal{H}_\eta$. The fact that we can do this is related to how η effects the modes of our operators. A nice short discussion of this can be found on pages 45 & 46 of [61].

and compare h_η/q_η to h/q. This is easily done:

$$L_0^\eta \left|\phi_\eta\right\rangle = \left(L_0 + \eta J_0 + \frac{\eta^2}{6}c\right)\left|\phi_\eta\right\rangle = \left(h_\eta + \eta q_\eta + \frac{\eta^2}{6}c\right)\left|\phi_\eta\right\rangle \tag{3.20}$$

and, similarly,

$$J_0^\eta \left|\phi\right\rangle = \left(q_\eta + \frac{\eta}{3}c\right)\left|\phi_\eta\right\rangle, \tag{3.21}$$

allowing us to conclude

$$(h,q) \mapsto \left(h - \eta q + \frac{\eta^2}{6}c, q - \frac{\eta}{3}c\right), \tag{3.22}$$

where c is the central charge and $\eta \in \mathbb{R}$.

As we already said, the case of most interest is $\eta = \pm 1/2$ as this provides a map between NS and R. The important case to consider is the flow of a chiral primary state:

$$\left|h_0 = \frac{q_0}{2}, q_0\right\rangle_{NS} \overset{\eta=\frac{1}{2}}{\longrightarrow} \left|h_{\frac{1}{2}} = \frac{c}{24}, q_{\frac{1}{2}} = q_0 - \frac{c}{6}\right\rangle_R, \tag{3.23}$$

and we note that the conformal weight in the Ramond sector is completely independent of which Neveu-Schwarz chiral primary we start from: it is *always* $h_{\frac{1}{2}} = \frac{c}{24}$. In this way we see this Ramond state is degenerate, as it can have multiple J_0 charges. Further to this, we claim that these Ramond states are, in fact, the ground states. This is shown as follows: consider a general Ramond state $\left|h,q\right\rangle_R$, and compute

$$\begin{aligned} 0 \le \left|G_0^+\left|h,q\right\rangle_R\right|^2 &= {}_R\left\langle h,q\right| G_0^- G_0^+ \left|h,q\right\rangle_R \\ &= {}_R\left\langle h,q\right| \{G_0^- G_0^+\} \left|h,q\right\rangle_R - {}_R\left\langle h,q\right| G_0^+ G_0^- \left|h,q\right\rangle_R \\ &= 2h - \frac{c}{12} - \left|G_0^-\left|h,q\right\rangle_R\right|^2 \end{aligned} \tag{3.24}$$

where we have made use of our algebra relations. Finally we use that $G_0^-\left|h,q\right\rangle_R = 0$ is possible to note that the smallest h value we can have is $h = \frac{c}{24}$, and so these must be our ground states, i.e. annihilated by all lowering operators. This gives us the Ramond sector condition $h \ge \frac{c}{24}$, which is just the equivalent of Proposition 3.1.3, which holds in the Neveu-Schwarz sector.

What we have just shown is that there is a one-to-one correspondence between chiral primaries and Ramond ground states. As there is a finite number of the latter in a non-degenerate theory, we have just provided an alternative proof that we only have a finite number of chiral primaries.

We can similarly consider setting $\eta = 1$, which provides a flow from NS to NS. For a chiral primary this acts as

$$\left|h_0 = \frac{q_0}{2}, q_0\right\rangle_{NS} \xrightarrow{\eta=1} \left|h_1 = -\frac{q_1}{2}, q_1 = q_0 - \frac{c}{3}\right\rangle_{NS}, \tag{3.25}$$

where we note that we obtain a antichiral primary on the right-hand side. Note that the vacuum $|0,0\rangle$ is both a chiral and anti-chiral primary, while the flowed to state is only antichiral with $q_1 = -\frac{c}{3}$. By similar calculation, if we set $\eta = -1$ then we will obtain a chiral primary with maximal conformal weight and $q = +\frac{c}{3}$. As the Neveu-Schwarz vacuum $|0,0\rangle$ is unique, we have just shown that there is a unique chiral primary operator with maximal conformal weight, which we claimed earlier was true.

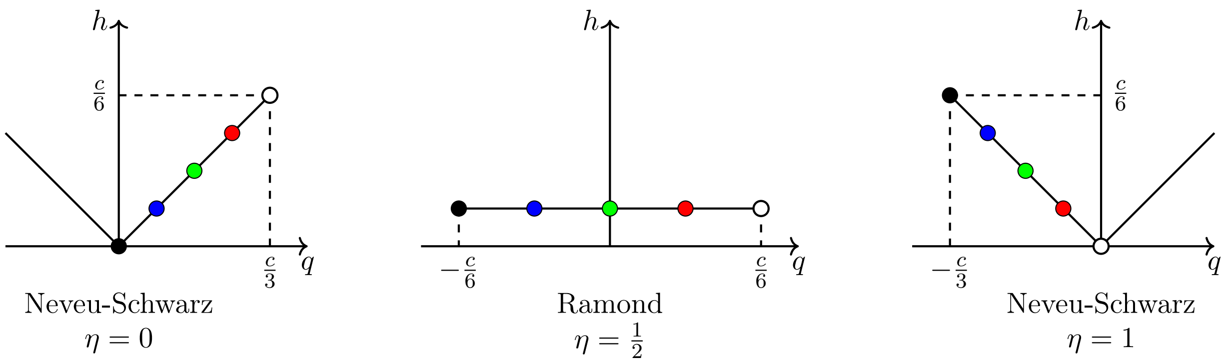


Figure 3.1: *Depiction of the spectral flow of chiral primary states. The initial Neveu-Schwarz chiral primaries (left) flow under $\eta = \frac{1}{2}$ to Ramond states with $h = \frac{c}{24}$ (middle). Upon a further $\eta = \frac{1}{2}$ (so $\eta = 1$ from the initial states) takes us back to the NS sector (right) where now all our states are anti-chiral. The colours on the circles are meant to depict how individual states flow.*

3.1.3 Minimal Models

The simplest class of $\mathcal{N} = 2$ SCFTs are the $\mathcal{N} = 2$ minimal models. These are rational conformal field theories and so contain a finite number of primary fields. They are uniquely defined by their central charge

$$c = \frac{3k}{k+2}, \tag{3.26}$$

where k is an integer known as the *level*. Note that $0 < c < 3$. We will denote the level k minimal model as MM_k. The superconformal primaries in MM_k are given by a triple (l, m, s) where

$$0 \leq l \leq k, \qquad s \sim s+4 \quad \text{and} \quad m \sim m + 2(k+2). \tag{3.27}$$

We identify $s = 0, 2$ as the NS states while $s = \pm 1$ are the R states. The conformal weights and $U(1)$ charges are given by

$$h^l_{m,s} = \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \quad \text{and} \quad q^l_{m,s} = -\frac{m}{k+2} + \frac{s}{2}. \tag{3.28}$$

Note that the chiral and antichiral conditions are given by $(l, m, s) = (l, \mp l, 0)$, respectively. Spectral flow maps $(l, m, s) \mapsto (l, m-1, s-1)$, and so our R ground states are given by $(l, m, s) = (l, \mp l - 1, -1)$ (see [59] for a nice review).

3.2 N=2 SCFTs with Calabi-Yau target

So far we have discussed $\mathcal{N} = 2$ SCFTs in general. We now want to specialise to the worldsheet of Type II strings compactified on a Calabi-Yau manifold, which is described by a nonlinear sigma model (NLSM) in $2D$.

3.2.1 Nonlinear Sigma Model

A NLSM is a scalar field theory, in which the fields are identified with the coordinates of the spacetime manifold. For string theory purposes, we focus on NLSMs in $2D$. We promote the whole construction to a supersymmetric theory and look to obtain a SCFT. We usually talk about a NLSM *on* a target manifold $\mathcal{M}$.

The NLSM on a Calabi-Yau manifold is, in particular, a NLSM on a Kähler manifold. We have already seen that Kähler implies $\mathcal{N} = (2, 2)$ on the worldsheet, and so this NLSM is a $\mathcal{N} = (2, 2)$ field theory in $(1+1)$-dimensions. Just as with $\mathcal{N} = 1$ in $(3+1)$-dimensions, there is a notion of chiral and antichiral superfields, i.e. fields which obey, respectively,

$$\bar{D}_{\dot{\alpha}} \Phi = 0 \quad \text{and} \quad D_\alpha \bar{\Phi} = 0, \tag{3.29}$$

for supercovariant derivatives $\bar{D}$ and D. Chiral and antichiral superfields are related by complex conjugation. We label the two coordinates via $\dot{\alpha}, \alpha = \pm$.

The Lagrangian (density) for the NLSM on a Kähler manifold is given by the D-term

$$\mathcal{L}_{\text{kin}} = \int d^4\theta K\big(\Phi_i, \bar{\Phi}_{\bar{i}}\big). \tag{3.30}$$

$K(\Phi_i, \bar{\Phi}_{\bar{i}})$ is a real function of chiral and antichiral superfields, and defines the Kähler metric via

$$g_{i\bar{j}} := \partial_i \partial_{\bar{j}} K\big(\Phi_i, \bar{\Phi}_{\bar{i}}\big). \tag{3.31}$$

$2D$ field theories with $\mathcal{N} = (2,2)$ contain two $U(1)$ R-symmetries: $U(1)_V$ and $U(1)_A$, where V stands for vector and A for axial.[4] In what follows we shall often work in a different charge basis, defined by

$$U(1)_L = \frac{U(1)_V + U(1)_A}{2} \quad \text{and} \quad U(1)_R = \frac{U(1)_V - U(1)_A}{2}, \tag{3.32}$$

where L/R stands for left/right, respectively. If our NLSM is superconformal, the $U(1)_L$ and $U(1)_R$ are generated by J and $\bar{J}$, respectively.

For these to be symmetries of our theory, we need to show that the action is invariant under their action. It is easy to show that the kinetic term is invariant provided $K(\Phi_i, \bar{\Phi}_{\bar{i}})$ has $(q_V, q_A) = (0,0)$. We note that if $K(\Phi_i, \bar{\Phi}_{\bar{i}}) = K(|\Phi_i|^2)$, then we can assign *any* charges to the individual chiral superfields (the antichiral superfields then have opposite charge).

However this only guarantees classical invariance and we need to check for the existence of anomalies. It can be shown [42] that $U(1)_V$ is not anomalous but that $U(1)_A$ can be, depending on the value of the first Chern class of the target manifold $\mathcal{M}$. In particular it can be shown that $U(1)_A$ is broken to $\mathbb{Z}_{2\kappa}$ where

$$\kappa = \langle c_1(\mathcal{M}), \varphi_*[\Sigma_{WS}] \rangle, \tag{3.33}$$

where $\varphi : \Sigma_{WS} \to \mathcal{M}$ is a map from the worldsheet to the target spacetime, and $[\Sigma_{WS}]$ is the homology class. From here we have that $U(1)_A$ is anomaly free when $c_1(\mathcal{M}) = 0$, i.e. it is a Calabi-Yau. In fact it turns out that $U(1)_A$ is anomaly free if, and only if, the target spacetime is Calabi-Yau. Putting this together with the fact that a $\mathcal{N} = (2,2)$ superconformal field theory requires both $U(1)_V$ and $U(1)_A$ to be non-anomalous, we see that the NLSM with Calabi-Yau target is expected to flow to a $\mathcal{N} = (2,2)$ SCFT.

3.2.2 Ramond Ground States

For a general quantum field theory, the Witten index at inverse temperature β is a modified partition function:

$$\text{Tr}\left[(-1)^F e^{-\beta H}\right], \tag{3.34}$$

where F is the fermion number operator, H is the Hamiltonian of the system and the trace is taken over the Hilbert space of states. For supersymmetric theories we have that every non-zero eigenvalue of H (i.e. non-zero energy) contains an equal number of fermions and

[4] It turns out that a $\mathcal{N} = (2,2)$ theory in $(1+1)$ dimensions can be obtained by the dimensional reduction of $\mathcal{N} = 1$ SUSY in $(3+1)$-dimensions. The $U(1)_V$ is the R-symmetry of the $4D$ theory and $U(1)_A$ corresponds to rotations in the compactified directions.

bosons and so give vanishing contributions. In this case, the Witten index simply becomes

$$\chi := \text{Tr}(-1)^F. \tag{3.35}$$

This was evaluated in [62] for a $\mathcal{N} = 1$ supersymmetric NLSM, and we briefly summarise the result here.

The Witten index only receives contributions from the R ground states, as these are the zero-energy eigenvalues. So, after ignoring modes with non-zero momentum, we are left with a supersymmetric quantum mechanics problem with a finite number of degrees of freedom. Our Majorana fermions, in the convienent basis, take the form $(\psi, \psi^*)^T$, with the two components related by complex conjugation. These components satisfy the Clifford algebra relations

$$\{\psi_i, \psi_j\} = \{\psi_i^* \psi_j^*\} = 0 \quad \text{and} \quad \{\psi_i, \psi_j^*\} = g_{ij}, \tag{3.36}$$

where g_{ij} are the components of the metric. We therefore identify ψ_i^* and ψ_i as the ith creation and annihilation operators, respectively. The Hilbert space of states here is rather straight forward to compute. We start by considering all states that are annihilated by the ψ_is. The wavefunction of a state is just an arbitrary function of the bosonic fields in our theory, which we denote by $A(\phi^k)$. We can now excite these states with our creation operators, to generate wavefunctions $A_{ij\ldots\ell}(\phi^k)$. Due to the fermionic nature of these creation operators, we must make sure we antisymmetrise the fermionic indices. From here it is easy to make the connection between these states and the differential forms of the manifold: simply identify ψ_i^* with wedging by dx^i and ψ_i by removing the form. This gives us the important result: *there is a one-to-one correspondence between Ramond ground states and differential forms.* We clarify at this point that really all we can say is that there is the same *number* of R ground states as there are differential forms, the fact that we have been able to pair them so nicely here is an artifact of our choice of basis.

In our basis, the supersymmetry charges take the neat form

$$Q = i \sum \psi_i^* p_i \quad \text{and} \quad Q^* = -i \sum \psi_i p_i, \tag{3.37}$$

where p_i is the covariant derivative for the boson associated to the fermion. We then see that Q adds a fermion index to our state while Q^* removes an index.

By then considering the action of Q on a generic wavefunction, $A_{ij\ldots\ell}(\phi^k)$, we can show that Q effectively acts as the exterior derivative on the differential forms. Equally Q^* acts as the adjoint operator d^*. The Hamiltonian in our SUSY theory is given simply by

$$H = QQ^* + Q^*Q \cong dd^* + d^*d, \tag{3.38}$$

and so we see ground states, i.e. those that contribute to the Witten index, are related to *harmonic* forms on the manifold. Finally, $(-1)^F$ is $+1$ for an even number of fermion excitations and -1 for an odd number, we arrive at the result

$$\text{Tr}(-1)^F = \sum_{p=1}^{\dim \mathcal{M}} (-1)^p b_p = \chi, \tag{3.39}$$

where b_p is the b-th Betti number of $\mathcal{M}$. Here χ is the Euler characteristic of the target manifold, and so motivates the notational choice of χ in Equation (3.35).

3.2.3 Chiral Rings

It is important to note that, at this level, we cannot identify individual Betti numbers of our supersymmetric sigma model: all we can say is that there is the same number of Ramond ground states as there are harmonic forms. If we are to introduce more structure to our target space, then it is possible to obtain further information and potentially get further topological relationships. We now look at how this happens for Calabi-Yau targets.

We now want to take the above discussion and look at it in situations where we have $(2,2)$ SUSY. The key thing here is that we have access to spectral flow, which allows us to map our R ground states to states in the NS sector. We will restrict the discussion to theories which satisfy the relation

$$q_L - q_R \in \mathbb{Z}. \tag{3.40}$$

As we will see, every state in our Calabi-Yau theory will actually already obey this condition, and so does not limit the results.

The condition for a R state to be a ground state, and so to contribute to the Witten index, is that it is annihilated by both $G_0^\pm$. This condition is mapped under spectral flow to the states being chiral or antichiral. In other words, spectral flow maps our R ground states to elements of our ring $\mathfrak{R}$. As the former are related to the differential forms of $\mathcal{M}$, we also obtain a relationship between $\mathfrak{R}$ and the forms. Using the fact that spectral flow from R to NS changes the charges by $-c/6$, we obtain the following [5]

$$\text{Tr}_R \left[t^{J_0} \bar{t}^{\bar{J}_0} \right] \big|_{G_0^\pm = \bar{G}_0^\pm = 0} = (t\bar{t})^{-c/6} \, \text{Tr}_{\mathfrak{R}} \left[t^{J_0} \bar{t}^{\bar{J}_0} \right]. \tag{3.41}$$

We note that it makes sense to include $J_0/\bar{J}_0$ in this trace: the trace is meant to be over Ramond ground states, which have zero eigenvalue with the Hamiltonian. It follows from the fact that both J_0 and $\bar{J}_0$ commute with the Hamiltonian, that we can include these into the argument of the trace without changing this condition.

From here we define

$$P_{\mathfrak{R}}(t,\bar{t}) := \mathrm{Tr}_{\mathfrak{R}}\left[t^{J_0}\bar{t}^{\bar{J}_0}\right], \tag{3.42}$$

which was called the *Poincaré polynomial* for the CFT in [5]. If we denote the number of states with charges (p,q) by $h^{p,q}$, we can write the Poincaré polynomial of the CFT as the sum

$$P_{\mathfrak{R}}(t,\bar{t}) = \sum_{p,q} h^{p,q} t^p \bar{t}^q. \tag{3.43}$$

We can equate this with the Poincaré polynomial of the target manifold $\mathcal{M}$:

$$P(t,\bar{t}) = \sum_{p,q=0}^{\dim\mathcal{M}} h^{\dim\mathcal{M}-p,q} t^p \bar{t}^q, \tag{3.44}$$

where $h^{m,n}$ are the Hodge numbers of $\mathcal{M}$. We then have the immediate Corollary.

Corollary 3.2.1. *There is* ***the same number*** *of states of charge* (p,q) *in our ring* $\mathfrak{R}$ *as there are* $(\dim\mathcal{M}-p,q)$*-forms on* $\mathcal{M}$.[5]

This result is clearly related to the result of the previous section, however we now note that we can look at individual Hodge numbers. The fact that we can do this is related directly to the fact that our algebra has two $U(1)$ charges (i.e. two generators J_0 and $\bar{J}_0$). It is crucial that we have both of these charges. Indeed it can be shown for a theory with only $U(1)_V$ (i.e. a Kähler target) that we would only be able to compute the Hodge numbers $h^{p,q}$ up to a set value of $p-q$. The existence of the $U(1)_A$ symmetry allows us to compute Hodge numbers up to set value of $p+q-\dim\mathcal{M}$, and so it is the combination of these two that allowed us to get the above result. This observation leads to the following important result: we can only determine the Hodge numbers of the target space up to the ambiguity

$$h^{p,q} \leftrightarrow h^{\dim\mathcal{M}-p,q}. \tag{3.45}$$

This is exactly the requirement of mirror symmetry we met before! This shows us that there is an intimate relationship between the charges of the states and the mirror map, which we will see more clearly going forward.

[5]We have changed convention compared to [5], which identifies (p,q) charge with (p,q)-forms. This relation is given a redefinition of t. We pick this convention for later convenience.

Of course in order to obtain Corollary 3.2.1, we have also used spectral flow to map the R ground states to elements in $\mathfrak{R}$. As explained above, we actually have two independent rings in general $\mathfrak{R}_{(c,c)}$ and $\mathfrak{R}_{(a,c)}$. We can therefore define two such Poincaré polynomials, one for each ring:

$$P_{(c,c)}(t,\bar{t}) := \mathrm{Tr}_{\mathfrak{R}_{(c,c)}}\left[t^{J_0}\bar{t}^{\bar{J}_0}\right] \quad \text{and} \quad P_{(a,c)}(t,\bar{t}) := \mathrm{Tr}_{\mathfrak{R}_{(a,c)}}\left[t^{J_0}\bar{t}^{\bar{J}_0}\right]. \tag{3.46}$$

States in $\mathfrak{R}_{(c,c)}$ necessarily have the same sign for the left and right charges, while states in $\mathfrak{R}_{(a,c)}$ have charges with different signs.

3.2.4 Odake Algbera

The above discussion holds for any $\mathcal{N}=(2,2)$ NLSM. Here we want to specialise to the case where our target manifold is a Calabi-Yau. Firstly, we note that, by central charge arguments, the CFT for our Calabi-Yau must have $c=9$: each spacetime dimension contributes $3/2$ to the central charge, so we have a total of $c=15$ but the 4-dimensional spacetime takes up $c_{ST}=6$ of these.

As a Calabi-Yau is, in particular, a Kähler manifold, the $\mathcal{N}=(2,2)$ SCFT is a good starting point: the $U(1)$ current gives us the Kähler form. However, it is not sufficient: we still need the holomorphic $(3,0)$-form, which we denote Ω. We account for Ω in the CFT by extending the $\mathcal{N}=2$ Virasoro by a field with quantum numbers $(h,q)_{NS}=(3/2,3)$, where the subscript indicates that the field lives in the NS sector. The resulting algebra is called an *Odake* algebra, and was first written down in [63], and it has the correct central charge.[6] We denote the field in our SCFT also by Ω. It is a complex field, and we write its decomposition as

$$\Omega = A + iB \tag{3.47}$$

The complex conjugate of this field (which is the $(0,3)$-form) is denoted $\Omega^* = A - iB$, and it has $(h,q)_{NS} = (3/2,-3)$. Note that the uniqueness of Ω and Ω^* in the SCFT follows from the fact that they are chiral/antichiral primaries that saturate the bound $h \leq \frac{c}{6}$. This is the statement that $h^{3,0}=h^{0,3}=1$.

The superpartner of Ω field is denoted Υ and is decomposed as $\Upsilon = \frac{1}{\sqrt{2}}(C+iD)$, such that (A,C) and (B,D) are superpartner pairs. So, in total, the generators of our Odake algebra are (T,G^0,J,G^3,A,B,C,D). The OPEs of these generators can be found in [29, 34]. We note here that associativity of the algebra only holds modulo an ideal generated by [64] (see

[6] A generic Odake algebra corresponds to an extension of the $\mathcal{N}=2$ Virasoro algebra by a $(n/2,n)$ field, and the central charge is $c=3n$. Here we are just considering $n=3$ as this is the relevant value for 3-folds.

also [29, 34])

$$N^1 = \partial A - (JB) \quad \text{and} \quad N^2 = \partial B + (JA), \tag{3.48}$$

where (...) stands for normal ordering.

As detailed in the original paper, these theories only admit a finite number of irreducible, unitary highest weight representations. The key thing for us will be the allowed massless reps, of which there are three for NS and three for R. As mentioned before, these representations are linked by spectral flow so that we only need to consider one set. The allowed values in the NS sector are[7]

$$(h, q)_{NS} = (3/2, -3), \quad (1/2, 1), \quad (1/2, -1) \quad \text{and} \quad (3/2, 3) \tag{3.49}$$

which have corresponding R values

$$(h, q)_R = (3/8, 3/2), \quad (3/8, -1/2), \quad (3/8, 1/2) \quad \text{and} \quad (3/8, -3/2), \tag{3.50}$$

respectively. We note at this point that every R ground state has $h = 3/8$, and so the R ground states are specified simply by their q values. This result is consistent with a general result for superconformal field theories that the conformal weight of Ramond ground states is $d/16$, where d is the number of Ramond Fermions (i.e. the dimension of the target spacetime).

Here we have only written down the quantum numbers for one side (say the left side) of our SCFT. The discussion is completely identical for the right hand side, and a general state is given by a product of two of the above states. As we will see, all the models we consider actually have $q_L = \pm q_R$. This restriction gives us an important proposition.

Proposition 3.2.2. *The (c, c) ring corresponds to the middle cohomology while the (c, a) ring gives the diagonal forms.*[8]

Proof. For our Odake algebra, we have that $q_{L,R} \in \{0, 1, 2, 3\}$.[9] So, if $q_L = q_R = q$ then

$$(q, q) \cong \alpha \in h^{3-q,q}, \tag{3.51}$$

i.e. there is a degree $(3 - q, q)$ form, as per Corollary 3.2.1. Similarly if $q_L = -q_R = q$, then

$$(q, -q) \overset{\eta=-1}{\longrightarrow} (q, 3 - q) \cong \beta \in h^{3-q,3-q}, \tag{3.52}$$

[7] The $(3/2, \pm 3)$ states are actually related to a single state, $(0, 0)$, by spectral flow with $\eta = \pm 1$. However, for future simplicity we treat them as their own fields here.

[8] By diagonal form, we simply mean an element in $h^{p,p}$ for $0 \leq p \leq \dim_{\mathbb{C}} \mathcal{M}$.

[9] The $0, 2$ cases are obtained by spectral flow of the $-3, -1$ cases, respectively.

where we have used spectral flow in order to ensure our differential form has positive degree. ■

Corollary 3.2.3. *An important consequence of Proposition 3.2.2 is that we require both* $\mathfrak{R}_{(c,c)}$ *and* $\mathfrak{R}_{(a,c)}$ *to be non-trivial in our theory, as otherwise some of the required Hodge numbers would vanish.*

We finally note that we can also relate the Hodge numbers to the charges in the R sector, by spectral flow. We simply use Equation (3.22) with $\eta = \pm 1/2$ and $c = 9$ so that $q_{NS} \mapsto q_R = q_{NS} \mp \frac{3}{2}$. From here we can say that to every state with charges $(q_L, q_R)_R$ there is a (m, n)-form, where these numbers obey

$$(q_L, q_R)_R = \left(\frac{3}{2} - m, n - \frac{3}{2}\right). \tag{3.53}$$

We emphasise here that we can only equate the *number* of these things. That is, if V_{q_L, q_R} denotes the vector space of states with charges $(q_L, q_R)_R$, then

$$\dim\left(V_{\frac{3}{2}-m, n-\frac{3}{2}}\right) = h^{m,n}. \tag{3.54}$$

It is generally true that a differential form can be represented by some state in the CFT, however we are not guaranteed that such a state will have definite charge.

3.2.5 Mirror Symmetry

An $\mathcal{N} = 2$ sigma-model with Calabi-Yau target has the following automorphism

$$\mathcal{M}_{\text{CY}} : (T, G^0, J, G^3, A, B, C, D) \mapsto (T, G^0, -J, -G^3, A, -B, C, -D). \tag{3.55}$$

Note, in particular, that it flips the sign of any state, $q \mapsto -q$. We are dealing with two copies of the algebra, and we have seen that the charges of the states are related to the degrees of the forms on the target manifold. In this context, mirror symmetry is understood as applying Equation (3.55) to one side, say the right side: $(q_L, q_R) \mapsto (q_L, -q_R)$. Note that this maps an element in (c, c) to an element of (a, c), and vice versa. From our relation to differential forms, this recovers the well known $h^{p,q} \mapsto h^{p,3-q}$.

Mirror symmetry thus suggests that two Calabi-Yau manifolds that have widely different geometry are actually deeply connected and give rise to the same SCFT (up to isomorphism). Indeed it is a well known fact that we can obtain the same SUSY multiplets in $4D$ from either compactifying IIA string theory on one Calabi-Yau or IIB on the mirror Calabi-Yau.

Although the argument to this point is promising, it is not entirely convincing, and in order to really appreciate the connection between mirror theories, we need to go to a more concrete construction.

3.3 Sigma Models

As explained at the beginning of the last section, the SCFT is linked to the spacetime geometry through the NLSM. The NLSM identifies the fields with coordinates on the target spacetime. This is done patch by patch, and so this construction is manifestly non-global. As we saw, the construction only defined the Calabi-Yau as a Kähler manifold with vanishing first Chern class. Such a definition is very precise, but it doesn't lend itself well to explicit constructions of Calabi-Yaus. A more practically useful geometrical construction is that of a hypersurface in some toric space (e.g. a weighted projective space). We can address both of these issues by looking at the $2D$ $\mathcal{N} = (2,2)$ theories in more detail, in particular at *gauged linear sigma models* (GLSM). Our discussion will largely follow [20, 42].

3.3.1 Gauged Linear Sigma Model

A GLSM is a $\mathcal{N} = (2,2)$ field theory, written in superspace, with a collection of n chiral superfields $\{\Phi_i\}$ along with a $U(1)$ gauge group.[10] We introduce a vector superfield $V \mapsto V + i(\Lambda - \bar{\Lambda})$, where Λ is a chiral superfield that labels the $U(1)$ action.

The Lagrangian of the GLSM contains four pieces:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_W + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{FI,\theta}. \tag{3.56}$$

which are given by

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= \int d^4\theta \sum_i \bar{\Phi}_i e^{2Q_i V} \Phi_i \\ \mathcal{L}_W &= \int d^2\theta W(\Phi_i) + c.c. \\ \mathcal{L}_{\text{gauge}} &= -\frac{1}{2e^2} \int d^4\theta \bar{\Sigma}\Sigma \\ \mathcal{L}_{FI,\theta} &= \frac{1}{2}\Big(- \int d\bar{\theta}^- d\theta^+ t\Sigma + c.c.\Big). \end{aligned} \tag{3.57}$$

where $W(\Phi_i)$ is the superpotential, e is the gauge coupling constant and $t = r - i\theta$. Here r is the FI parameter and θ the theta angle. The gauge field Σ is the field strength and is an

[10] The generalisation to $U(1)^s$ is straight forward.

example of a *twisted* chiral superfield, i.e. a field which obeys $\bar{D}_+\Sigma = D_-\Sigma = 0$. Viewing $\mathcal{L}_{\text{gauge}}$ as the twisted equivalent of $\mathcal{L}_{\text{kin}}$, we can then use $\mathcal{L}_{FI,\theta}$ to define a linear twisted superpotential, $\widetilde{W}(\Sigma) = -t\Sigma$. Explicit expressions for the component expansions of these Lagrangians can be found in [20].

We again need to ask about the $U(1)_V \times U(1)_A$ symmetries and anomaly conditions. The invariance of $\mathcal{L}_{\text{kin}}$ is of course the same as the NLSM discussion and gives the same result. The F-term, $\mathcal{L}_W$, tells us that the superpotential is required to have $(q_V, q_A) = (2, 0)$. This constrains the form it can take, namely we require it to be quasi-homogeneous:

$$W\big(\lambda^{q_V^i}\Phi_i\big) = \lambda^2 W\big(\Phi_i\big). \tag{3.58}$$

The twisted F-term, $\mathcal{L}_{FI,\theta}$, tells us that Σ must have $(q_V, q_A) = (0, 2)$. The anomaly conditions again carry over, along with the requirement that the $U(1)$ gauge group charges cancel [13, 42], i.e.

$$\sum_i Q_i = 0. \tag{3.59}$$

This is required to ensure $U(1)_A$ is non-anomalous. In particular if $\sum_i Q_i = p$ then $U(1)_A$ is broken to $\mathbb{Z}_{2p}$.

Remark 3.3.1. We note that Equation (3.59) is required in order for our target space to be Calabi-Yau. Indeed, note that Equation (3.59) takes the same form as condition (ii) of Proposition 2.7.21. This connection can be made much deeper, but we postpone this discussion until we have discussed mirror symmetry for this construction.

3.3.2 Connection to NLSM & The LG/CY Correspondence

In order to see the connection between GLSMs and NLSMs, we solve the equations of motion for the auxiliary fields:

$$D = -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right) \quad \text{and} \quad F_i = \frac{\partial W}{\partial \phi_i}, \tag{3.60}$$

where the lower case indicates the lowest component of the superfield. Doing this leaves us with a dynamical theory for the fields (ϕ_i, σ), which has potential energy

$$U(\phi_i, \sigma) = \frac{1}{2e^2}D^2 + \sum_i |F_i|^2 + 2|\sigma|^2 \sum_i Q_i^2|\phi_i|^2 \tag{3.61}$$

Let $\mathcal{M}_{\text{Vac}}$ denote the vacuum manifold of the GLSM. That is, in the GLSM we identify the chiral superfields $\{\Phi_1, ..., \Phi_n\}$ with coordinates on $\mathbb{C}^n$ and then consider the surface defined via minimising the potential energy. We then have the following proposition

Proposition 3.3.2. *The IR limit of a GLSM is the NLSM on $\mathcal{M}_{Vac}$.*

We do not prove this here (see [42] for more details) but instead work through an important example.

Example 3.3.3. Consider a theory of n chiral superfields all with charge $Q_i = 1$[11] and vanishing superpotential, $W = 0$. Then we have

$$U(\phi_i, \sigma) = \sum_i |\sigma|^2|\phi_i|^2 + \frac{e^2}{2}\bigg(\sum_i |\phi_i|^2 - r\bigg)^2. \tag{3.62}$$

If $r > 0$ then $U = 0$ is given by $\sigma = 0$ and

$$\sum_{i=1}^{n} |\phi_i|^2 = r. \tag{3.63}$$

This defines a sphere S^{n-1}. However we now need to account for the $U(1)$ action, so that in total the vacuum manifold is

$$\mathbb{CP}^{n-1} = \frac{\big\{(\phi_1, ..., \phi_n)\big|\sum_i |\phi_i|^2 = r\big\}}{U(1)}. \tag{3.64}$$

By an identical calculation, assigning different charges to the fields will produce a weighted projective space. ▲

Example 3.3.4. Let's now consider a generalisation of the above example: $\{\Phi_1, ..., \Phi_n\}$ and gauge group $U(1)^k = \prod_{a=1}^{k} U(1)_a$. The charges of the fields are $Q_{i,a}$, with $i = 1, ..., N$. Again we take the superpotential to vanish, which leaves

$$U = \sum_{i=1}^{n} |Q_{i,a}\sigma_a|^2|\phi_i|^2 + \sum_{a=1}^{k} \frac{e_a^2}{2}\bigg(\sum_{i=1}^{n} Q_{i,a}|\phi_i|^2 - r_a\bigg)^2. \tag{3.65}$$

For the case of $r_a > 0$, the vacuum manifold is then given by

$$\mathcal{M}_{Vac} = \frac{\big\{(\phi_1, ..., \phi_n)\big|\sum_i Q_{i,a}|\phi_i|^2 = r_a,\ \forall a = 1, ..., k\big\}}{U(1)^k}. \tag{3.66}$$

[11] Note that this doesn't obey Equation (3.59), and so the NLSM is anomalous and therefore cannot correspond to a Calabi-Yau.

The interesting thing then comes from the following argument. Consider $(\phi_1, ..., \phi_n)$ to be coordinates of a copy of $\mathbb{C}^n$. Denote the complexification of the $U(1)^k$ gauge group by $(\mathbb{C}^*)^k$, and define $Z(\phi_i) \subseteq \mathbb{C}^n$ to be the subset such that the $(\mathbb{C}^*)^k$ orbit doesn't contain a solution to

$$\sum_{i=1}^{n} Q_{i,a}|\phi_i|^2 = r_a \tag{3.67}$$

for all $a = 1, ..., k$. It can then be shown (see [42] for a detailed discussion) that $\mathcal{M}_{vac}$ is diffeomorphic to the quotient

$$\frac{\mathbb{C}^n \setminus Z(\phi_i)}{(\mathbb{C}^*)^k}. \tag{3.68}$$

Additionally there is a $(\mathbb{C}^*)^{n-k}$ holomorphic automorphism that acts freely and transitively on an open, dense submanifold of the quotient. This is *exactly* the construction of our toric varieties from fans introduced before, Equation (2.112). We also note that $Z(\phi_i)$ depends on the values of r_a, in the same way that $Z(\Sigma)$ in Equation (2.112) depends on the fan Σ. This provides further support for Remark 3.3.1: there is a deep connection between the GLSM and the construction of Calabi-Yaus using toric geometry. ▲

We now need to account for F-terms, i.e. non-vanishing superpotential. We will now show that appropriately chosen superpotentials lead to hypersurfaces in the ambient toric spaces. We focus on the simplest case of a hypersurface in $\mathbb{CP}^{n-1}$, but more general results can be found in [20].

Example 3.3.5. Consider a GLSM with $n+1$ chiral superfields $\{P, \Phi_1, ..., \Phi_n\}$ with gauge group charges $q_i = 1$ and $q_P = -n$, and superpotential

$$W = P \cdot G\big(\Phi_1, ..., \Phi_n\big) \tag{3.69}$$

where $G(\Phi)$ is a homogeneous polynomial of degree n. We assume $G(\Phi)$ is generic, in the sense that

$$G = \frac{\partial G}{\partial \Phi_1} = \frac{\partial G}{\partial \Phi_2} = ... = \frac{\partial G}{\partial \Phi_n} = 0 \qquad \Longrightarrow \qquad \Phi_1 = \Phi_2 = ... = \Phi_n = 0. \tag{3.70}$$

The potential energy for this system is given by

$$U = \big|G(\phi_i)\big|^2 + |p|^2 \sum_i \left|\frac{\partial G}{\partial \phi_i}\right|^2 + \frac{1}{2e^2}D^2 + 2|\sigma|^2\bigg(\sum_i |\phi_i|^2 + n^2|p|^2\bigg) \tag{3.71}$$

where

$$D = -e^2\bigg(\sum_i |\phi_i|^2 - n|p|^2 - r\bigg). \tag{3.72}$$

The vacuum manifold of this theory is defined by $U = 0$ and is r dependent. The case $r >> 0$ requires at least one of the ϕ_i to be non-zero. From here, the $|\sigma|^2 \sum |\phi_i|^2$ term gives $\sigma = 0$, while the $|p|^2 \sum |\partial_i G|^2$ term (along with Equation (3.70)) tells us that $p = 0$. Finally we require $G = 0$. We are thus left in exactly the case as before, but now with the constraint $G = 0$. In other words, the GLSM flows to the NLSM on $\mathcal{S} \subset \mathbb{CP}^{n-1}$, defined by a degree n homogenous polynomial. These are precisely the conditions for a Calabi-Yau manifold, $\mathcal{S} = \mathcal{M}_{CY}$.

The case $r << 0$ can similarly be shown to require $p \neq 0$ and so the field P picks up a vev, and breaks the $U(1)$ gauge group to a $\mathbb{Z}_n$ subgroup: $\phi_i \mapsto e^{\frac{2\pi i}{n}} \phi_i$. This leads to a theory with superpotential $W' = \sqrt{-r} \cdot G(\phi_i)$ subject to this $\mathbb{Z}_n$ action. This defines a Landau-Ginzburg (LG) orbifold. This recovers the well known Calabi-Yau/Landau-Ginzburg correspondence: we can view them as two different phases of the same GLSM.

It is important that we have a LG *orbifold*, as it is known that a LG theory only has non-trivial $\mathfrak{R}_{(c,c)}$, while $\mathfrak{R}_{(a,c)}$ contains just the identity, but we need both to be non-trivial for strings on Calabi-Yaus. However, as demonstrated in [65], the twisted states in the orbifold theory give rise to elements in $\mathfrak{R}_{(a,c)}$ and so save the day.

We can alter the action of the gauge group on this system to account for hypersurfaces in weighted projective spaces. We can pick $Q_P = -H$ and $Q_i = w_i$, where

$$w_i = \frac{H}{k_i + 2} \quad \text{and} \quad H = \text{lcm}(k_i + 2), \tag{3.73}$$

and then the anomaly condition enforces

$$\sum_i \frac{1}{k_i + 2} = 1. \tag{3.74}$$

Then we set

$$G(\Phi_i) = \Phi_1^{k_1+2} + ... + \Phi_n^{k_n+2} \tag{3.75}$$

in the superpotential. Our Calabi-Yau is then defined by this degree H Fermat hypersurface in $\mathbb{CP}^{n-1}_{w_1,...,w_n}$. The LG orbifold is then given by $W' = \Phi_1^{k_1+2} + ... + \Phi_n^{k_n+2}$ with $\mathbb{Z}_H$ quotient

$$\Phi_i \mapsto e^{\frac{2\pi i \gamma}{k_i+2}} \Phi_i. \tag{3.76}$$

▲

3.3.3 Gepner Models

It is known [5] that the IR limit of a LG model with $W = \Phi^{k+2}$ is a $(2,2)$ SCFT with central charge

$$c = \frac{3k}{k+2}. \tag{3.77}$$

which is the level k $\mathcal{N}=2$ minimal model, MM_k. The idea is then that the worldsheet SCFT (i.e. the nonlinear sigma model) is isomorphic to the SCFT obtained by the IR limit of the LG orbifold. We can therefore use the minimal models to construct and study the worldsheet SCFT.

Gepner [15] proposed a method for constructing the CFT of a Calabi-Yau as a direct product of $\mathcal{N}=2$ minimal models. At the GLSM level each term in the product corresponds to a different $\Phi_i^{k_i+2}$ in $G(\Phi_i)$.

The idea is to note that the central charge adds under products, and so we could form a $c=9$, $\mathcal{N}=2$ theory out of a collection of MMs with different levels. That is

$$\big(\mathcal{N}=2\big)_{c=9} = \bigotimes_{i=1}^{r} \big(\mathcal{N}=2\big)^{MM}_{c_i} \qquad \text{with} \qquad \sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} = 9. \tag{3.78}$$

The remaining part of our CFT corresponds to the $4D$ spacetime. Working in lightcone gauge, this is a CFT with central charge $c=3$ and consists of two bosons and their accompanying fermions. The fermions are described by an $\mathfrak{so}(2)_1$ affine Lie algebra,[12] which has four representations $(O_2)_{h=0,q=0}$, $(V_2)_{h=1/2,q=1}$, $(S_2)_{h=1/8,q=1/2}$ and $(C_2)_{h=1/8,q=-1/2}$. The NS sectors are O_2 and V_2 while S_2 and C_2 are the R sectors. As we are focusing on the Gepner model part here, we drop the fermions for now but shall return to them later.

Recall that the superconformal primaries in MM_k are defined by the triple (l,m,s) where the conformal dimension and $U(1)$ charge is given by Equation (3.28). The conformal weights and charges add under products of different MM_k. Therefore all we need to do is account for the orbifold action. As detailed in [65], at the level of the CFT the orbifold acts as a projection on the charges via

$$g = e^{2\pi i J_0}. \tag{3.79}$$

[12]Given a finite dimensional Lie algebra $\mathfrak{g}$, we define the (infinite dimensional) affine Lie algebras (also known as Kac-Moody algebras) by the central extension of $\mathfrak{g}$ along with the introduction of a derivative operator. We denote the affine Lie algebra as $\mathfrak{g}_k$, where k is known as the level. Affine Lie algebras will appear in several places throughout the thesis, but the technical details behind them will not be vital to the understanding of the work. For this reason they are not discussed in detail, but the interested reader is instead directed to [66] for details.

We therefore require our states to have integer charge, and our Gepner model is defined via

$$(Gep) = \left[MM_{k_1}, MM_{k_2}, ..., MM_{k_r}\right]\Big|_{U(1)\text{-projection}}, \tag{3.80}$$

where the $U(1)$-projection enforces

$$\sum_{i=1}^{r} \left[\frac{l_i}{k_i+2}\right] = 0, 1, 2, 3, \tag{3.81}$$

The restriction on the right-hand side follows from Equation (3.74) along with $l_i \leq k_i$. The integrality of the charges is also required to ensure spacetime SUSY (see [11] and references therein). This result is actually not surprising: we have already seen that our Odake algebra limits the NS charges to be $q = \pm 3, \pm 1$ which are equivalent, via spectral flow, to $q = 0, 1, 2, 3$. The above equation is nothing other than the NS charges of our states.

As we are considering an orbifold, we obtain both untwisted and twisted sectors. Let's start with the untwisted sector. Here we have $q_L = q_R$ and the charges of a state are simply given by the sum of the individual MM_k charges. In the R sector (which is where we will predominantly work), we therefore have the untwisted charges

$$\sum_{i=1}^{5} \left(\frac{l_i+1}{k_i+2} - \frac{1}{2}\right) = \sum_{i=1}^{5} \left(\frac{l_i}{k_i+2}\right) - \frac{3}{2}, \tag{3.82}$$

where we made use of our anomaly condition, Equation (3.74). If we then impose the Gepner condition, Equation (3.81), we see that the R charges are restricted to $q = \pm\frac{3}{2}, \pm\frac{1}{2}$. This same result is, of course, obtained by applying spectral flow to the allowed NS charges.

We now note that a chiral field Φ is identified with $(l, m, s) = (1, -1, 0)$, and so the l_i value determines the power of Φ_i. Therefore, a state with

$$\sum_{i=1}^{5} \frac{l_i}{k_i+2} = 0, 1, 2, 3 \tag{3.83}$$

correspond, respectively, to degree $0, H, 2H$ and $3H$ polynomials, where $H = \text{lcm}(k_i + 2)$. In particular the state $|l_i\rangle$ is identified geometrically with $\Phi_i^{l_i}$. Note that $l_i \leq k_i$ and so we must set $\Phi_i^{k_i+1} = 0$.[13] Using that the right-hand side of the above corresponds to states with charges $-3/2$, $-1/2$, $1/2$ and $3/2$, respectively, along with Equation (3.53), we get that they correspond to $(3, 0), (2, 1), (1, 2)$ and $(0, 3)$ forms.

[13] This is the requirement that we don't consider terms in the Jacobian ring.

Remark 3.3.6. It is interesting to note that this result can be related to the notion of a Griffiths residue and the primitive cohomology of the Calabi-Yau manifold (see Appendix B).

Let's now discuss the twisted sectors of our Gepner model. As detailed in [65] these states have $q_L = -q_R$, and the charge of a state depends on which twisted sector we are in:

$$q_L^\nu = \sum_{i|\nu\notin(k_i+2)\mathbb{Z}} \left(\frac{\nu}{k_i+2} - \left[\frac{\nu}{k_i+2}\right] - \frac{1}{2}\right), \tag{3.84}$$

where [...] stands for the integer part of the argument, and $\nu = 1, ..., H-1$ labels the twisted sector.

States in the twisted sector can become untwisted when $\nu \in (k_i+2)\mathbb{Z}$, in which case their charge is computed simply using

$$q_i = \frac{l_i+1}{k_i+2} - \frac{1}{2}. \tag{3.85}$$

and $(q_i)_L = (q_i)_R$ for these factors.

So, in total, a charge of a generic state is given by

$$\begin{aligned} q_L^\nu &= \sum_{i|\nu\in(k_i+2)\mathbb{Z}} \left(\frac{l_i+1}{k_i+2} - \frac{1}{2}\right) + \sum_{i|\nu\notin(k_i+2)\mathbb{Z}} \left(\frac{\nu}{k_i+2} - \left[\frac{\nu}{k_i+2}\right] - \frac{1}{2}\right) \\ q_R^\nu &= \sum_{i|\nu\in(k_i+2)\mathbb{Z}} \left(\frac{l_i+1}{k_i+2} - \frac{1}{2}\right) - \sum_{i|\nu\notin(k_i+2)\mathbb{Z}} \left(\frac{\nu}{k_i+2} - \left[\frac{\nu}{k_i+2}\right] - \frac{1}{2}\right) \end{aligned} \tag{3.86}$$

where the fully untwisted sector is identified with $\nu = 0$. We can write this in a more symmetric manner by defining $l_i^{(\nu)} + 1 := \nu \mod (k_i+2)$, then the two sums above take the same form. An overall state is considered untwisted if $q_L = q_R$ and twisted if $q_L = -q_R$, despite what the individual $(q_i)_L$ and $(q_i)_R$ obey.

We note that $\nu = 1$ always gives

$$q_L = -q_R = \sum_{i=1}^{5} \left(\frac{1}{k_i+2} - \frac{1}{2}\right) = -\frac{3}{2}, \tag{3.87}$$

and so corresponds to the $(3,3)$ form, as per Equation (3.53). Similarly $\nu = H-1$ gives rise to the $(0,0)$ form. The other ν values will give us either a $(1,1)$ or a $(2,2)$ form, up to some exceptions. Suppose that H is even, then we can set $\nu = H/2 \mod (k_i+2)$. We now claim that if $w_i = \frac{H}{k_i+2}$ is even, then the twist is trivial, i.e. $\nu = 0$. Let's see this: for some $n \in \mathbb{Z}$, we can write $\nu = H/2$ as

$$\frac{H}{2} + n(k_i+2) = \left(\frac{H}{2(k_i+2)} + n\right)(k_i+2) = \left(\frac{w_i}{2} + n\right)(k_i+2), \tag{3.88}$$

but if w_i is even, then we can always pick $n = -\frac{w_i}{2}$, and so $\nu = 0$. When w_i is odd, the above calculation shows us that

$$l_i^{(\frac{H}{2})} = \frac{k_i}{2}, \tag{3.89}$$

in which case the $U(1)$ charges vanish, $q_L = q_R = 0$. We are then left with the untwisted states $|l_i\rangle_R$, which have $(q_L)_i = (q_R)_i = \frac{l_i+1}{k_i+2} - \frac{1}{2}$, and correspond to $(2,1)$ or $(1,2)$ forms.

Example 3.3.7. Consider the Gepner model with $\{k_i+2\} = (5,5,5,5,5)$, from which we see that $H = 5$, and we are instantly lead to the conclusion that this is a Calabi-Yau in $\mathbb{CP}^4$: we have $w_1 = w_2 = ... = w_5$. We first check that we satisfy our central charge condition:

$$\sum_{i=1}^{5} \frac{3k_i}{k_i+2} = \sum_{i=1}^{5} \frac{9}{5} = 9. \tag{3.90}$$

The untwisted ground states are then given by monomials of degree $0, 5, 10$ and 15, remembering that we must set $X_i^4 = 0$ for all i, i.e. $k_i = 3$ not 5. Clearly there is only one monomial of degree 0, and the monomial of degree 15 corresponds to $X_1^3X_2^3X_3^3X_4^3X_5^3$, which again occurs only once.

We have already computed the degree 5 result back in Section 2.5.3; it is given by

$$\left(\binom{5}{5}\right) = \binom{9}{5} = 126. \tag{3.91}$$

We now need to account for the fact that $X_i^4 = 0$ for all i. The terms we need to remove are X_i^5 and $X_i^4X_j$, where $j \neq i$. The former is 5 terms (each i value), while the latter is 20 terms (5×4, the number of i times number of j). So we are left with $h^{2,1} = 101$. As $H = 5$ is odd, we do not have any further contributions to $h^{2,1}$, and so we can conclude $h^{2,1} = 101$. We can similarly check the order 10 result, but here we just quote the result that $h^{1,2} = h^{2,1}$ and move on.

We now just need to compute the diagonal Hodge numbers. These come from our twisted states. We have $\nu = 1,2,3,4$, with $\nu = 1$ corresponding to $h^{3,3}$ and $\nu = 4$ to $h^{0,0}$. Let's consider $\nu = 2$: by direct substitution this gives

$$q_L = -q_R = -\frac{1}{2}, \tag{3.92}$$

so corresponds to a $(2,2)$ form. Similarly, $\nu = 3$ corresponds to a $(1,1)$ form. So in total we have the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{3.93}$$

which is exactly that of the quintic in $\mathbb{CP}^4$. ▲

Example 3.3.8. Here we look at an example with H being even. Consider the Gepner model with $\{k_i+2\} = (8,8,4,4,4)$ and $H = 8$. First we check this meets the central charge condition:

$$c = \sum_{i=1}^{5} \frac{3k_i}{k_i+2} = 2\left(\frac{3\cdot 6}{8}\right) + 3\left(\frac{3\cdot 2}{4}\right) = \frac{9}{2} + \frac{9}{2} = 9, \tag{3.94}$$

so we're good.

The untwisted ground states correspond to monomials of order $0, 8, 16$ and 24, subject to $X_{1,2}^7 = X_{3,4,5}^3 = 0$. To save space, we simply state that these have multiplicity $1, 83, 83$ and 1, respectively. We then have unique twisted states for $\nu = 1,2,3,5,6,7$, which give $h^{0,0} = h^{3,3} = 1$ and $h^{1,1} = h^{2,2} = 2$.

We note that $\nu = 4$ is missing, which is the one we want to study here: the $i = 3,4,5$ terms become untwisted, and the $i = 1,2$ terms have

$$l_1 = l_2 = \frac{k_{1,2}}{2} = \frac{6}{2} = 3, \tag{3.95}$$

and so

$$\sum_{i=1}^{5} \frac{l_i}{k_i+2} = 2\frac{3}{8} + \sum_{i=3}^{5} \frac{l_i}{4} = \frac{1}{4}\big(3 + l_3 + l_4 + l_5\big) \tag{3.96}$$

which we require to be $0, 1, 2$ or 3. Using $l_{3,4,5} = 0, 1, 2$, the only possible combinations are

$$l_3 + l_4 + l_5 = 1 \quad \text{and} \quad l_3 + l_4 + l_5 = 5. \tag{3.97}$$

There are 3 ways to produce each of these: both have multiplicity 3 choose 1. We finally see that these do indeed contribute to $h^{2,1}$ and $h^{1,2}$, respectively, as they correspond to

$$\sum_{i=1}^{5} \frac{l_i}{k_i+2} = 1, 2, \tag{3.98}$$

respectively. Then we use the fact that the only (non-vanishing) contributions to the $U(1)$ charge comes from the untwisted states, we have $q_L = q_R = \mp\frac{1}{2}$, and so they are $(2,1)$ and $(1,2)$ forms, respectively. So in total we have $h^{2,1} = h^{1,2} = 83 + 3 = 86$, and our Hodge diamond is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 86 & & 86 & & 1, \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{3.99}$$

which is the Hodge diamond of a Calabi-Yau 3-fold in $\mathbb{WCP}^4_{1,1,2,2,2}$. ▲

3.3.4 Mirror Symmetry

Gepner Models

In [11], Greene and Plesser took the observation made in [67], that the quotient of a Gepner model by its full symmetry group yields an isomorphic theory, and extended it to more general quotients. In particular they looked at the geometrical phase of such a duality. We summarise the relevant content here.

Consider the Gepner model obtained by the minimal model product $(k_1, ..., k_r)$. Let d denote the order of $g = e^{2\pi i J_0}$. Then, this model has discrete symmetry group

$$G = \left(\prod_{i=1}^{r} \mathbb{Z}_{k_i+2} \right) \Big/ \mathbb{Z}_n, \tag{3.100}$$

where n is the order of g^2: $n = d$ for d odd and $n = d/2$ for d even. The mirror model is given by quotienting by $H \subset G$ defined such that

$$\sum_{i=1}^{r} \frac{\gamma_i}{k_i+2} \in \mathbb{Z}, \tag{3.101}$$

where $\gamma_i \in \mathbb{Z}_{k_i+2}$ represents an element of G. The mirror theory is shown to be isomorphic to the original Gepner model, but with one of the $U(1)$ charges reversed.

In terms of the corresponding LG orbifold, the statement is: the LG orbifold with superpotential

$$W(\Phi_i) = \sum_{i=1}^{r} \Phi_i^{k_i+2} \tag{3.102}$$

with orbifold action $\mathbb{Z}_H$ has a mirror LG orbifold with the same form of the superpotential[14]

$$\widetilde{W}_F(\Phi_i^\vee) = \sum_{i=1}^{r} (\Phi_i^\vee)^{k_i+2}, \tag{3.103}$$

but now the quotient is by $\Gamma^\vee \subset \prod_{i=1}^{r} \mathbb{Z}_{k_i+2}$ acting on the fields as

$$\Phi_i^\vee \mapsto e^{\frac{2\pi i \gamma_i}{k_i+2}} \Phi_i^\vee \tag{3.104}$$

subject to Equation (3.101). The $(2,1)$-forms of this dual theory are then related to the deformations of this equation. We note that the product $\Phi_1^\vee \Phi_2^\vee ... \Phi_r^\vee$ is always present. In fact, as we shall see, the mirror theory is actually defined as the LG orbifold with superpotential

$$\widetilde{W}(\Phi_i^\vee) = \sum_{i=1}^{r} (\Phi_i^\vee)^{k_i+2} + e^{t/H} \prod_{i=1}^{r} \Phi_i^\vee, \tag{3.105}$$

where t is the (FI, θ) parameter, as before.

Recall that the geometric phase is given by the Calabi-Yau defined by a hypersurface in a toric manifold, with defining polynomial given by $W(z_i) = 0$. For example, for $(k_i + 2) = 5$ for all i (and $r = 5$), we recover the quintic and mirror quintic Calabi-Yaus.

We can actually see the generation of this dual superpotential by looking at the states in the Gepner model. We go to the case of interest, namely $r = 5$. States with $q_L = q_R$ (i.e. elements of $\mathfrak{R}_{(c,c)}$) are the untwisted states, while states with $q_L = -q_R$ (elements of $\mathfrak{R}_{(a,c)}$) are the twisted states. Therefore, mirror symmetry acts on the Gepner model by mapping the twisted and untwisted states to untiwsted and twisted states, respectively. The original twisted states should now be interpreted as the untwisted states in the mirror model and so, as per the previous discussion, should be interpreted as monomials of degrees $0, H, 2H$ and $3H$. We can indeed see that this is the case as follows: we are now essentially mapping $\nu \mapsto -\nu$. This follows from the fact that the twisted states come from quotienting by $g = e^{2\pi i \nu J_0}$, but if we send $J_0 \mapsto -J_0$ this is the same as sending $\nu \mapsto -\nu$ in g. From here we simply interpret

[14] The subscript F is to indicate "Fermat type".

the $l_i^{(-\nu)}$ as the powers of the corresponding mirror homogeneous coordinates, i.e.

$$\left| l_i^{(-\nu)} \right\rangle \cong (\Phi_i^\vee)^{l_i^{-\nu}}. \tag{3.106}$$

Indeed this ties in nicely with the mirror description in terms of LG models. Let's look at the allowed deformations of $W_F^\vee$. The monomial $\Phi_1^\vee ... \Phi_5^\vee$, which is always present (by construction), would correspond to a state with $l_i^{(-\nu)} = 1$ for all i, and it is indeed true that this state always appears. This is seen simply from

$$l_i^{(-\nu)} + 1 = -\nu \mod (k_i + 2) \qquad \Longrightarrow \qquad l_i^{(-H+2)} = 1 \qquad \forall i. \tag{3.107}$$

Also note that $\nu = H - 2$ always gives a $(1,1)$-form in the original theory. This follows simply from

$$l_i^{(H-2)} + 1 = H - 2 \mod (k_i + 2) \qquad \Longrightarrow \qquad l_i^{(H-2)} = k_i - 1, \tag{3.108}$$

which together with

$$\sum_{i=1}^{5} \frac{k_i}{k_i + 2} = 3 \quad \text{and} \quad \sum_{i=1}^{5} \frac{1}{k_i + 2} = 1 \tag{3.109}$$

gives $\sum_i \frac{l_i^{(H-2)}}{k_i+2} = 2$, which is $q_L = -q_R = \frac{1}{2}$ and is the criteria for a $(1,1)$-form.

For the original untwisted states, we simply take the l_i values and plug them into $l_i + 1 = -\nu \mod (k_i + 2)$, and use this. For example, $l_i = 0$ for all i is the unique state that always gives the $(3,0)$-form, which should be mirrored to the $(0,0)$-form. Under this mirror map, this would give $\nu = H - 1$ for all i, but we know that this is the unique twist that gives the $(0,0)$-form, as required.

We note that we defined a twisted contribution to a state as one in which $\nu \notin (k_i + 2)\mathbb{Z}$, but that simply mapping $\nu \mapsto -\nu$ wont change this condition. However, the mirror of a twisted state is meant to be untwisted. The key thing is that it is untwisted w.r.t. the mirror Gepner model, i.e. we have a $\nu^\vee$ and a twisted contribution to a state in the mirror Gepner model obeys $\nu^\vee \notin (k_i+2)\mathbb{Z}$. This $\nu^\vee$ must account for the orbifold group of the mirror Gepner model, and so it is not easy to write down a direct relationship between ν and $\nu^\vee$. However, it is in principal not too difficult to obtain the relationship for specific cases. We highlight how this is done for quotients of the quintic in Appendix C.

Let's look at the two examples above and compute their mirrors.

Example 3.3.9. Consider the quintic Gepner model, $k_i+2 = 5$ for all i. The original untwisted states will simply turn into our diagonal forms. What we want to check is the twisted states, and check they give the right degree monomials. We have $\nu = 1, 2, 3, 4$ and so, after we send

$\nu \mapsto -\nu$, it is easy to show that

$$l_i^{(-1)} = 3 \qquad l_i^{(-2)} = 2 \qquad l_i^{(-3)} = 1 \quad \text{and} \quad l_i^{(-4)} = 0 \qquad \forall i \tag{3.110}$$

which give the monomials

$$(\widetilde{X}_1\widetilde{X}_2\widetilde{X}_3\widetilde{X}_4\widetilde{X}_5)^{4-\nu}, \tag{3.111}$$

which is exactly what we expect for the quintic. In particular we only have the always present superpotential deformation $\widetilde{X}_1\widetilde{X}_2\widetilde{X}_3\widetilde{X}_4\widetilde{X}_5$, corresponding to $\nu = H - 2 = 3$. This returns exactly the result of Greene and Plesser in [11]. ▲

Example 3.3.10. Let's now look for the mirror of the Calabi-Yau inside $\mathbb{WCP}^4_{1,1,2,2,2}$, which had $\{k_i + 2\} = (8, 8, 4, 4, 4)$. Here we have $h^{1,1} = 2$ corresponding to $\nu = 3, 6$. $\nu = H - 2 = 6$ gives us our always present $\widetilde{X}_1\widetilde{X}_2\widetilde{X}_3\widetilde{X}_4\widetilde{X}_5$, while $\nu = 3$ gives

$$\begin{aligned} l_{1,2}^{(-3)} + 1 = -3 \mod 8 \qquad &\Longrightarrow \qquad l_{1,2}^{(-3)} = 4 \\ l_{3,4,5}^{(-3)} + 1 = -3 \mod 4 \qquad &\Longrightarrow \qquad l_{3,4,5}^{(-3)} = 0 \end{aligned} \tag{3.112}$$

and so we get the deformation $\widetilde{X}_1^4\widetilde{X}_2^4$. ▲

Sigma Models

As we have seen, Gepner models are a particular instance of a phase of a more general theory of GLSMs. The question is whether the concept of mirror symmetry tracks back up the ladder into the general construction. The answer is yes and it was demonstrated in [13] (see also [42]). We outline the results here, and refer the reader to the reference for a more detailed discussion.

Consider a theory of a vector superfield V, a set of real superfields B_i, and a set of twisted chiral superfields Y_i. The imaginary part of the Y_is are periodic in 2π: $\vartheta_i = \frac{1}{2}(Y_i - \bar{Y}_i)$. Now consider the Lagrangian

$$\mathcal{L}' = \int d^4\theta \sum_i \left(e^{2Q_iV + B_i} - \frac{1}{2}\left(Y_i + \bar{Y}_i\right) B_i \right), \tag{3.113}$$

where Q_i are positive integers. The idea is to integrate out the fields in different orders. Fristly consider integrating out Y_i: this constrains B_i to obey

$$\bar{D}_+ D_- B_i = D_+ \bar{D}_- B_i = 0 \qquad \Longrightarrow \qquad B_i = \Psi_i + \bar{\Psi}_i \tag{3.114}$$

for a chiral superfield Ψ_i. It follows from the periodicity of Y_i that the imaginary part of Ψ_i is also periodic. Defining $\Phi_i = e^{\Psi_i}$ and plugging this back into $\mathcal{L}'$ then gives

$$\mathcal{L}_1 = \int d^4\theta \sum_i \bar{\Phi}_i e^{2Q_i V} \Phi_i, \tag{3.115}$$

but this is simply the kinetic term for a set of chiral superfields of charges Q_i. Finally, noting that $\mathcal{L}_{\text{gauge}}$ and $\mathcal{L}_{FI,\theta}$ are independent of Y_i and B_i, we can simply add them to both $\mathcal{L}'$ and $\mathcal{L}_1$ without affecting any of the calculation. We therefore end up with the Langrangian

$$\mathcal{L}_\Phi = \int d^4\theta \left(\sum_i \bar{\Phi}_i e^{2Q_i V} \Phi_i - \frac{1}{2e^2} \bar{\Sigma}\Sigma \right) + \frac{1}{2} \left(- \int d\bar{\theta}^- d\theta^+ t\Sigma + c.c. \right). \tag{3.116}$$

which is nothing but the GLSM of a set of chiral superfields of charges Q_i, with vanishing superpotential. We have seen that this is related to the NLSM on a weighted projective space.

If we now first integrate B_i out of $\mathcal{L}'$, we obtain

$$B_i = -2Q_i V + \log\left(\frac{Y_i + \bar{Y}_i}{2} \right). \tag{3.117}$$

Plugging this back into $\mathcal{L}'$, and using the fact that Y_i is a twisted chiral superfield, results in

$$\mathcal{L}_2 = \int d^4\theta \sum_i \left(-\frac{1}{2} \left(Y_i + \bar{Y}_i \right) \log\left(Y_i + \bar{Y}_i \right) \right) + \frac{1}{2} \sum_i \left(\int d\bar{\theta}^- d\theta^+ Q_i Y_i \Sigma + c.c. \right), \tag{3.118}$$

where $\Sigma = \bar{D}_+ D_- V$. Again we can simply add $\mathcal{L}_{\text{gauge}}$ and $\mathcal{L}_{FI,\theta}$ and obtain

$$\mathcal{L}_2 = \int d^4\theta \sum_i \left(-\frac{1}{2e^2} \bar{\Sigma}\Sigma - \frac{1}{2} \left(Y_i + \bar{Y}_i \right) \log\left(Y_i + \bar{Y}_i \right) \right) + \left(\int d\bar{\theta}^- d\theta^+ \left(\frac{1}{2} \sum_i Q_i Y_i - t \right) \Sigma + c.c. \right). \tag{3.119}$$

We then define the twisted superpotential for this theory as

$$\widetilde{W} = \left(\sum_i Q_i Y_i - t \right) \Sigma. \tag{3.120}$$

The important part is to equate the two expressions for B_i, which results in a relationship between the chiral superfields Φ_i and the twisted chiral superfields Y_i:

$$Y_i + \bar{Y}_i = 2\bar{\Phi}_i e^{2Q_i V} \Phi_i. \tag{3.121}$$

We can also relate the imaginary part of Y_i to the phase of Φ_i. This is not easily done in terms of the superfields, but can be seen if we consider a component expansion of the fields. If the lowest component of Φ_i is $\phi_i = \rho_i e^{i\psi_i}$, then the result is

$$d\vartheta_i = \star d\psi_i. \tag{3.122}$$

This map between chiral and twisted chiral superfields is exactly our mirror map. We now explain how this is related to the notion of mirror Calabi-Yaus discussed previously.

Solving the equations of motion for the dynamical Σ results in the D-term constraint

$$\partial_\Sigma \widetilde{W} = 0 \qquad \Longrightarrow \qquad \sum_i Q_i Y_i = t. \tag{3.123}$$

Finally, defining

$$X_i = e^{-Y_i}, \tag{3.124}$$

we see that the twisted superpotential takes the form

$$\widetilde{W}(X_i) = \sum_i X_i \qquad \text{subject to} \qquad \prod X_i^{Q_i} = e^{-t}. \tag{3.125}$$

A theory of a superfield with (twisted) superpotential is a LG theory. So here we have a LG theory for the twisted chiral superfields Y_i (expressed in terms of X_i) with twisted superpotential $\widetilde{W}(X_i)$ as above.

Let's now modify this construction slightly by introducing another chiral superfield P to our set $\{\Phi_1, ..., \Phi_n\}$, and we set the charge of P to be negative, $Q_P = -H$. Let $\widetilde{P} = e^{-Y_P}$ be the dual field to P, then it follows from the constraint above that

$$\widetilde{P}^{-H} X_1^{Q_1} ... X_n^{Q_n} = e^{-t}. \tag{3.126}$$

Defining

$$\Phi_i^\vee = X_i^{Q_i/H}, \tag{3.127}$$

then results in the condition

$$\widetilde{P} = e^{t/H}(\Phi_1^\vee)...(\Phi_n^\vee). \tag{3.128}$$

The twisted superpotential then takes the form

$$\widetilde{W}(\Phi_i^\vee) = (\Phi_1^\vee)^{H/Q_1} + ... + (\Phi_n^\vee)^{H/Q_n} + e^{t/H} \prod_{i=1}^n \Phi_i^\vee. \tag{3.129}$$

Note that this twisted superpotential is subject to an orbifold action $\Gamma^\vee \subset \prod_i \mathbb{Z}_{H/Q_i}$. Specifically it acts on the fields as

$$\Phi_i^\vee \mapsto \exp\left(\frac{2\pi i \gamma_i Q_i}{H}\right)\Phi_i^\vee, \qquad \text{subject to} \qquad \sum_i \frac{\gamma_i Q_i}{H} \in \mathbb{Z}. \tag{3.130}$$

The constraint condition comes from the fact that $e^{t/H}\prod_i \Phi_i^\vee \in \widetilde{W}(\Phi_i^\vee)$.

We therefore arrive at the result that the mirror of the NLSM on a weighted projective space is a LG model with the above superpotential. If we allow for negative charges then we obtain a LG orbifold. This is not quite what we want: we want to show that the mirror of a Calabi-Yau is again Calabi-Yau. However we now simply apply the Calabi-Yau/Landau-Ginzburg correspondence to the LG orbifold side, which gives that the mirror of the NLSM is a Calabi-Yau.

The first thing we note is that our starting GLSM has no superpotential. However, in order to get a NLSM on a hypersurface in a weighted projective space, i.e. a Calabi-Yau, we need a superpotential. This issue is addressed in [13], where they explain that the resulting mirror LG orbifold is unchanged by introducing the superpotential. The difference between the two cases is actually encapsulated in what are considered to be the fundamental fields on the mirror side: for the case with a superpotential the fundamental fields are the X_i while in the absence of the superpotential the fundamental fields are the Y_i. However we have a very simple relation between the two, and the LG orbifold is changed.

We therefore arrive at mirror symmetry as a map between two Calabi-Yaus. Note that if we pick the charges as $Q_i = w_i = H/(k_i+2)$ and $Q_P = -H$, then we arrive at the result of Greene and Plesser. Namely, the mirror of a LG orbifold with Fermat type superpotential is again a (deformation of) a LG orbifold with the same Fermat type superpotential but now with a different quotient group. Indeed Equation (3.130) becomes exactly the result of [11].

3.3.5 Mirror Symmetry for Toric Hypersurfaces

The above geometrical mirror map can be generalised to hypersurfaces in a generic toric variety, akin to the construction of Batyrev [9] discussed in Section 2.7.10.

Consider a GLSM with $(h+1)$ chiral superfields $(\Phi_1, ...\Phi_h, P)$ and gauge group $U(1)^k$. For each $U(1)$ we have a field strength Σ_a and associated FI parameter t_a. Let $Q_{i,a}$ denote the charge of Φ_i under the a^{th} $U(1)$ factor. Set

$$d_a := \sum_{i=1}^{h} Q_{i,a}, \tag{3.131}$$

and define t_i via[15]

$$t_a = \sum_{i=1}^{h} Q_{i,a} t_i. \tag{3.132}$$

Note that in order to have a Calabi-Yau we must obey the anomaly condition, Equation (3.59). This implies that the charges of P under the a^{th} $U(1)$ is $-d_a$.

We can again dualise this theory in order to obtain a theory with $(h+1)$ twisted chiral superfields $(Y_1, ..., Y_h, Y_P)$ and then define

$$\widetilde{P} := e^{-Y_P} \quad \text{and} \quad X_i := e^{-Y_i}. \tag{3.133}$$

The D-term constraint, Equation (3.123), gives k relations:

$$\sum_{i=1}^{h} Q_{i,a} Y_i - d_a Y_P = t_a. \tag{3.134}$$

In terms of the new variables this is

$$\left(\prod_{i=1}^{h} X^{Q_{i,a}} \right) \widetilde{P}^{-d_a} = e^{-t_a} \tag{3.135}$$

It the follows from [13], that in our case the twisted superpotential is actually empty and the defining hypersurface of the mirror Calabi-Yau is given by the above constraint along with

$$\sum_{i=1}^{h} X_i + \widetilde{P} = 0. \tag{3.136}$$

To write the mirror hypersurface equation in terms of mirror fields we now proceed as follows. The fan of the toric variety underlying the GLSM has h ray generators n_i[16] sitting in the N lattice which obey the k relations

$$\sum_{i=1}^{h} n_i Q_{i,a} = 0\,. \tag{3.137}$$

Let the superpotential take the form $W(\Phi_i, P) = P \cdot G(\Phi_i)$, where $G(\Phi_i)$ is a homogeneous polynomial of degrees $\{d_1, ..., d_k\}$ with respect to the $U(1)^k$. Next, let M denote the dual

[15] The t_i are defined up to redefinition of the $Q_{i,a}$.

[16] We previously used ν_i and ν_i^* for the generators. Here we swap to the more convenient notation of n_i and m_i.

lattice to N, and define m_ℓ such that we can write

$$G(\Phi_i) = \sum_{\ell=1}^{h^\vee} \prod_{i=1}^{h} \Phi_i^{\langle m_\ell, n_i \rangle + 1}\,, \tag{3.138}$$

where necessarily $\langle m_\ell, n_i \rangle \geq -1$ for all ℓ and i.

The Calabi-Yau/Landau-Ginzburg story carries over and again we get two phases of the GLSM: $p = 0$ gives the nonlinear sigma model on a Calabi-Yau defined by the vacuum manifold; $p \neq 0$ gives a LG orbifold with superpotential $W = G(\Phi_i)$.

We can now introduce $\{\Phi_1^\vee, ..., \Phi_{h^\vee}^\vee\}$, and define

$$\widetilde{P} = \prod_{\ell=1}^{h^\vee} \Phi_\ell^\vee \quad \text{and} \quad X_i = e^{-t_i} \prod_{\ell=1}^{h^\vee} \left(\Phi_\ell^\vee\right)^{\langle m_\ell, n_i \rangle + 1}. \tag{3.139}$$

Using Equations (3.131), (3.132) and (3.137) we can easily show that these then solve Equation (3.135). Plugging this into Equation (3.136) then gives the hypersurface equation

$$\sum_{i=1}^{h} e^{-t_i} \prod_{\ell=1}^{h^\vee} \left(\Phi_\ell^\vee\right)^{\langle m_\ell, n_i \rangle + 1} + \prod_{\ell=1}^{h^\vee} \Phi_\ell^\vee = 0\,. \tag{3.140}$$

This is the family of Calabi-Yau hypersurfaces identified by Batryrev's construction. Note that the last term here came from the $\widetilde{P}$, and so is the equivalent of the origin of N in the Batyrev construction.

An important point to note is that $h \neq h^\vee$, in general, and so the number of homogeneous coordinates defining the mirror (i.e. $h^\vee$) need not be the same as the number of homogeneous coordinates we start with (i.e. h). In the case of hypersurfaces in weighted projective spaces, it turns out that $h = h^\vee$, and so we use notation like $\Phi_i^\vee$, as before.

4 G_2: Geometry

In this chapter we move on to a discussion of manifolds with G_2 holonomy. We start by reviewing the basic definitions and concepts of the Lie group G_2 and what it takes for a manifold to have holonomy G_2. We introduce the conjecture for G_2 mirror symmetry due to Shatasvilli and Vafa, and then move on providing ways to construct G_2s. The main methods we focus on are quotients of Calabi-Yaus and circles as well as the twisted connected sum (TCS) construction. Mirror symmetry is discussed for each of these cases as we go, and we finish with a discussion of toric geometry and how tops can be used to reproduce the TCS construction. This provides a concrete way to form mirror G_2s in analogy to Batyrevs construction for Calabi-Yaus.

The main references for this chapter are [68] for the background G_2 discussion, [69–73] for the constructions of manifolds with G_2 holonomy and how they are related, and [74] for the discussion of tops and their relationship to the TCS construction.

4.1 G_2 Basics

We start by recalling the definition of the Lie group G_2.

Definition. [G_2 Group] Let $\{x_1, ..., x_7\}$ be a set of coordinates on $\mathbb{R}^7$. Then define the 3-form[1]

$$\Phi_0 := dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}, \tag{4.1}$$

where we have used the short hand $dx_{ijk} := dx_i \wedge dx_j \wedge dx_k$. Then the group G_2 is the subgroup of $SO(7)$ that preserves Φ_0. G_2 also preserves the 4-form

$$\star\Phi_0 := dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247}. \tag{4.2}$$

Note that, as the notation suggests, Φ_0 and $\star\Phi_0$ are related by Hodge dual.

As it is a subgroup of $SO(7)$, it also preserves the Euclidean metric on $\mathbb{R}^7$, $g_0 = dx_1^2 + ... + dx_7^2$ and the orientation. G_2 is a compact, simply-connected, semisimple Lie group of dimension 14.

We now claim that there is a $SU(3)$ subgroup of G_2 which is defined by the subgroup that leaves one of the coordinates unchanged, e.g. dx^1. A technical account of this can be found in [75][2]. A more quickly accessibly argument is as follows: consider $\mathbb{R}^7$ as the product space $\mathbb{R} \times \mathbb{C}^3$. Then we can consider the action of $SU(3)$ on this whole space as acting trivially on the $\mathbb{R}$ and with the standard action on the $\mathbb{C}^3$. The claim is then this action is indeed trivial on our forms Φ_0 and $\star\Phi_0$. It then immediately follows from the fact that $SU(2) \subset SU(3)$ that G_2 also contains a $SU(2)$ subgroup. As the $SU(2)$ subgroup of $SU(3)$ is defined by "ignoring one $\mathbb{C}$ factor in $\mathbb{C}^3$", we see that this $SU(2)$ subgroup leaves three of the $\mathbb{R}^7 \cong \mathbb{R}^3 \times \mathbb{C}^2$ coordinates alone, while acts on the $\mathbb{C}^2$ in the standard way.

The fact that we have an $SU(3)$ subgroup in G_2 will be important going forward. It gives us a hint that we could somehow embed a Calabi-Yau 3-fold inside our manifold with holonomy G_2.

4.1.1 G_2 Structures

As we will see going forward, the forms Equations (4.1) and (4.2) will come up again and again, and will prove very useful to us. Now clearly the specific forms given in Equations (4.1) and (4.2) are not the only choices, and indeed we will, in general, have classes of isomorphic 3/4-forms. This gives us the content of the next definition.

Definition. [Associative & Coassociative Forms] Let $\mathcal{M}$ be an oriented 7-manifold. Then at each point $p \in \mathcal{M}$, we define $\mathcal{P}^3_p\mathcal{M} \subset \Lambda^3_p\mathcal{M}$ to be the subset of 3-forms Φ, such that there exists an oriented isomorphism between $T_p\mathcal{M}$ and $\mathbb{R}^7$ that identifies $\Phi|_p$ and $\Phi_0|_p$ of Equation (4.1). A section of $\Phi \in \Gamma(\mathcal{P}^3\mathcal{M})$ can therefore be identified with Φ_0 globally. We call such a 3-form an *associative form.*[3]

We similarly define $\mathcal{P}^4_p\mathcal{M} \subset \Lambda^4_p\mathcal{M}$ and define $\star\Phi \in \Gamma(\mathcal{P}^4\mathcal{M})$ to be a 4-form which can be identified with $\star\Phi_0$, referring to such forms as *coassociative forms.*

We then have the definition of a G_2 structure.

[1] We are working with the form convention in [68].
[2] Here an alternative definition of the G_2 group as the automorphism group of the octonions is used.
[3] [68] uses the terminology "positive" in the place of associative and coassociatvie.

Definition. [(Oriented) G_2 Structure] Let $\mathcal{M}$ be an oriented 7-manifold with associative 3-form $\Phi \in \mathcal{P}^3\mathcal{M}$. Then consider the frame bundle F over $\mathcal{M}$. By definition, this is the bundle who's fibre at $p \in \mathcal{M}$ is the isomorphisms between $T_p\mathcal{M}$ and $\mathbb{R}^7$, i.e. it is a principal $GL_+(7,\mathbb{R})$-bundle over $\mathcal{M}$. Then define $Q_\Phi \subset F$ to be the subset that identifies Φ with Φ_0, then Q_Φ is a principal G_2-bundle on $\mathcal{M}$. This follows simply from the fact that Φ_0 has G_2 invariance. We call Q_Φ a *(oriented) G_2 structure on $\mathcal{M}$*.

As the notation suggests, Q_Φ is dependent on which associative 3-form Φ we start with. With some thought we see that we can actually go in the opposite direction; given a G_2 structure Q on $\mathcal{M}$ we can define corresponding associative 3-form Φ_Q, coassociative 4-form $\star\Phi_Q$ and metric g_Q. This follows simply from the fact that, $\Phi_0, \star\Phi_0$ and g_0 are G_2 invariant.[4]

So we see that we have 1-1 correspondence between G_2 structures and associative 3-forms. In particular, we see that, given a $\Phi \in \Gamma\mathcal{P}^3\mathcal{M}$, we can define an associated $\star\Phi$ and g. That is, we use Φ to define Q_Φ which we then use to define $\star\Phi_{Q_\Phi}$ and g_{Q_Φ}. There is an important point to note here: the Hodge dual $\star$ depends on the metric, but we have just shown that our metric depends on our associative 3-form Φ. We therefore see that the map

$$\Theta : \Phi \mapsto \star\Phi \tag{4.3}$$

is *non-linear* in Φ.

Notation. From now on we shall refer to the double of an associative 3-form and corresponding metric, (Φ, g), as a G_2 structure. Of course it is actually the corresponding unique principal G_2-bundle which is the G_2 structure, but we save a lot of notation this way.

We note that Φ determines a Riemannian metric and so we have a Levi-Civita connection. This leads to the following definition.

Definition. [Torsion Free G_2 Structure] Let $\mathcal{M}$ be a oriented 7-manifold with G_2 structure (Φ, g). Also let ∇ be the Levi-Civita connection of g. Then we define the *torsion* of (Φ, g) to be $\nabla\Phi$, and we say the G_2 structure is *torsion free* if $\nabla\Phi = 0$.

4.1.2 G_2-Manifolds

We can now define a G_2-manifold.

[4] Technically, in order to get the metric condition we need our G_2 structure to be oriented. We will always assume to be the case, unless otherwise specified.

Definition. [G_2-Manifold] We call the triple $(\mathcal{M}, \Phi, g)$ a *G_2-manifold* if $\mathcal{M}$ is an oriented 7-manifold with torsion free G_2-structure (Φ, g), $\nabla\Phi = 0$.

We then have the following proposition [68].

Proposition 4.1.1. *Let $(\mathcal{M}, \Phi, g)$ be a 7-manifold with G_2-structure (Φ, g). Then the following conditions are equivalent:*

(i) (Φ, g) *is torsion free,*

(ii) Hol$(\mathcal{M}) \subseteq G_2$, *and Φ is the induced 3-form,*

(iii) $\nabla\Phi = 0$, *where ∇ is the Levi-Civita connection of g,*

(iv) $d\Phi = d \star \Phi = 0$, *and*

(v) $d\Phi = d\Theta(\Phi) = 0$.

It is not too hard to see that these are equivalent: clearly (i) and (iii) are related by the definition of torsion free, similarly (iv) and (v) are related by Equation (4.3). Then in condition (ii) by "induced 3-form" we simply mean the 3-form that is invariant under parrallel transport $\nabla\Phi = 0$, which is (iii). Then to get from (iii) to (iv) you need the differential geometry result that a torsion free connection obeys

$$(d\omega)_{\mu_1 \dots \mu_n} = (n+1)\nabla_{[\mu_1}\omega_{\mu_2 \dots \mu_n]}, \tag{4.4}$$

so if $\nabla\Phi = 0$ then $d\Phi = 0$, we similarly get $d \star \Phi = 0$.

This proposition gives us five different ways to define a G_2-manifold. The two that will be of most use to us is (ii) and (iv). The key thing we note about (ii) is that the holonomy only needs to be a *subgroup* of G_2.[5] The question becomes "what needs to happen for $\mathrm{Hol}(\mathcal{M}) = G_2$?", which gives the next proposition.

Proposition 4.1.2. *Let $(\mathcal{M}, \Phi, g)$ be a compact G_2-manifold. Then Hol$(\mathcal{M}) = G_2$ if, and only if, the first fundamental group, $\pi_1(\mathcal{M})$, is finite.*

Proof. See Proposition 10.2.2 of [68]. ∎

As it will be important later, we also introduce the following definition.

[5]This is why we have previously always written "a manifold with holonomy G_2", rather than "a G_2-manifold".

Definition. [G_2-Involution] Let $(\mathcal{M}, \Phi, g)$ be a G_2-manifold, and $\iota : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism on $\mathcal{M}$. We call ι a *G_2-involution* if the following two conditions hold:

(i) It is an involution; $\iota^2 = \text{id}$.

(ii) It preserves the G_2-structure; $\iota^*(\Phi) = \Phi$.

4.1.3 Moduli Space & Mirror Conjecture

The moduli space of torsion-free G_2 structures plays an important role when looking for a notion of mirror symmetry. From a geometrical perspective, one can derive that the dimension of this moduli space is $b^3(\mathcal{M})$ [68], however it turns out that this is not the full story. Indeed, as noted in [23], there can be additional contributions from the antisymmetric 2-form that have no geometrical analogue. The dimension of the physical moduli space is then given by $b^2 + b^3$.

We then have the conjecture of generalised mirror symmetry [23, 76]:

Conjecture. *The degree of ambiguity left by being unable to decipher all the topological aspects of the target manifold using the algebraic formulation of quantum field theories is precisely explained by having topologically inequivalent manifolds allowed by the ambiguity to lead to the same quantum field theory up to deformation in the moduli of the quantum field theory.*

This conjecture is to be understood at the level of the field theory, not just the geometry, however we use it here to make the following statement. For manifolds with G_2 holonomy it tells us that a necessary condition is that mirror manifolds have the same $b^2 + b^3$. That is, if $\mathcal{M}_{G_2}$ and $\mathcal{M}^\vee_{G_2}$ are mirror, then

$$b^2(\mathcal{M}_{G_2}) + b^3(\mathcal{M}_{G_2}) = b^2(\mathcal{M}^\vee_{G_2}) + b^3(\mathcal{M}^\vee_{G_2}) \tag{4.5}$$

which we shall refer to as the Shatashvili-Vafa relation in what follows.

We note that Calabi-Yau manifolds are examples of G_2-manifolds and so should also obey this constraint. This is indeed true:

$$b^2(\mathcal{M}_{CY}) = h^{1,1} \quad \text{and} \quad b^3(\mathcal{M}_{CY}) = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} = 2(1 + h^{2,1}), \tag{4.6}$$

and so $b^2 + b^3$ is preserved under the mirror map $h^{m,n} \to h^{3-m,n}$.

4.1.4 Calibrated Submanifolds

Our associative 3-form, Φ, and coassociative 4-form, $\star\Phi$, are examples of *calibrations* on our G_2-manifolds. We now want to clarify what this means.

Definition. [Calibration & Calibrated Submanifolds] Let $(\mathcal{M}, g)$ be an n-dimensional Riemannian manifold. Then, a k-form $\Phi \in \Lambda^k\mathcal{M}$ is called a *calbiration* if

(a) Φ is closed, $d\Phi = 0$, and

(b) for any $p \in \mathcal{M}$, and any k-dimensional orientated subspace $S \subset \mathcal{M}$, $\Phi|_{T_pS}$ is less than or equal to the volume form on S, i.e. there is an $\alpha \leq 1$ such that

$$\int_S \Phi = \int_S \Phi|_{T_pS} = \alpha \cdot \int_S \text{vol}(T_pS) = \alpha \text{Vol}(S), \tag{4.7}$$

where $\text{vol}(T_pS)$ is the volume form on S. We sometimes write this simply as $\Phi|_{T_pS} = \alpha \cdot \text{vol}(T_pS)$.

If $\alpha = 1$, then we call S a *calibrated submanifold w.r.t. the calibration* Φ.

Proposition 4.1.3. *Calibrated submanifolds are submanifolds of minimum volume in their homology class.*

Proof. Assume $S \subset \mathcal{M}$ is a calibrated submanifold w.r.t. calibration Φ. Then assume that $\widetilde{S}$ is another submanifold in the same homology class as S, $[S] = [\widetilde{S}]$. That is $\widetilde{S} = S + \partial S'$ for some $(k+1)$-dimensional submanifold S'. Then, using Stoke's theorem,

$$\int_{\partial X} \omega = \int_X d\omega, \tag{4.8}$$

along with the fact that $d\Phi = 0$, we have

$$\text{Vol}(S) = \int_S \Phi = \int_{\widetilde{S}} \Phi \leq \int_{\widetilde{S}} \text{vol}(T_p\widetilde{S}) = \text{Vol}(\widetilde{S}), \tag{4.9}$$

and so S has minimal volume in its homology class. ■

Now, recalling Equations (4.1) and (4.2), we see straight away that Φ_0 and $\star\Phi_0$ act as calibrations and have calibrated 3-manifolds and calibrated 4-manifolds, respectively. For example, for Φ_0, we see that the 3-dimensional submanifold

$$S_{123} = \{(x_1, x_2, x_3, 0, 0, 0, 0) \,|\, x_1, x_2, x_3 \in \mathbb{R}\} \subset \mathbb{R}^7 \tag{4.10}$$

is a calibrated submanifold of Φ_0, as $\text{vol}(S_{123}) = dx_{123} = \Phi_0|_{S_{123}}$.[6]

We then note that an oriented 3-fold, $S \subset \mathbb{R}^7$, obeys $\Phi_0|_S = \text{vol}(S)$ if, and only if, $S = \gamma S_{123}$ for some $\gamma \in G_2$. So we have a group of calibrated submanifolds given by γS_{123}. The same story applies to calibrated submanifolds of $\star\Phi_0$, but with $S_{123} \to S_{4567}$, defined in the obvious way.

> **Definition.** [Associative & Coassociative Submanifolds] We call calibrated submanifolds w.r.t the Φ_0 *associative* 3*-folds*, and we similarly define *coassociative* 4*-folds*.

Proposition 4.1.4. *Let $(\mathcal{M}, \Phi, g)$ be a G_2-manifold, and let $\sigma : \mathcal{M} \to \mathcal{M}$ be a non-trivial isometric involution (i.e. a diffeormophism such that $\sigma^*(g) = g$ and $\sigma^2 = id_{\mathcal{M}}$ but $\sigma \neq id_{\mathcal{M}}$) obeying $\sigma^*(\Phi) = \Phi$. Then the fixed point locus*

$$F_\sigma := \{p \in \mathcal{M} \,|\, \sigma(p) = p\} \tag{4.11}$$

is an associative 3*-fold in $\mathcal{M}$. This implies that given any non-trivial $\gamma \in G_2$ that squares to the identity, the fixed point locus of γ is a associative* 3*-fold.*

Proof. Firstly we note that, from the fact that $\sigma^*(\Phi) = \Phi$, that σ^* must be in an element in G_2. Now, recalling Equation (4.1),

$$\Phi_0 := dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}, \tag{4.12}$$

we see that there are only two elementally distinct choices for $\gamma \in G_2$ such that our conditions are met: from $\sigma^*(g) = g$ we require $x_i \mapsto \pm x_i$ only. Then it can be quickly checked that the only ways to preserve Φ_0 are either $x_i \mapsto x_i$ for all i, which is not allowed as it is trivial, or

$$\begin{aligned} x_{1,2,3} \mapsto x_{1,2,3} \quad &\text{and} \quad x_{4,5,6,7} \mapsto -x_{4,5,6,7}, \\ x_{3,4,7} \mapsto x_{3,4,7} \quad &\text{and} \quad x_{1,2,5,6} \mapsto -x_{1,2,5,6}, \\ x_{2,5,7} \mapsto x_{2,5,7} \quad &\text{and} \quad x_{1,3,4,6} \mapsto -x_{1,3,4,6}, \qquad \text{or} \\ x_{1,4,6} \mapsto x_{1,4,6} \quad &\text{and} \quad x_{2,3,5,7} \mapsto -x_{2,3,5,7}. \end{aligned} \tag{4.13}$$

Clearly all 4 non-trivial choices are isomorphic by simply relabelling, and so we see that σ is conjugate in G_2[7] to

$$x_{1,2,3} \mapsto x_{1,2,3} \quad \text{and} \quad x_{4,5,6,7} \mapsto -x_{4,5,6,7}, \tag{4.14}$$

[6]From now on we shall just drop the T_p part and just write the submanifold S_{123} etc.

[7]Two maps $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are said to *conjugate* if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(f(x)) = g(h(x))$ for all $x \in \mathbb{R}^n$. Here we have $f = (x_{1,2,3} \mapsto x_{1,2,3}, x_{4,5,6,7} \mapsto -x_{4,5,6,7})$, $g = \sigma$ and $h = \gamma$ for non-trivial γ.

the fixed point of which is exactly S_{123}, i.e. $F_\sigma = \gamma S_{123}$ for $\gamma \in G_2$. We have done this at a point $p \in \mathcal{M}$ (as we are using local coordinates), but this clearly holds for all p, and so we have an associative 3-fold.

In this manner, we have shown that for any $\gamma \in G_2$ such that $\gamma^2 = 1$ but $\gamma \neq 1$, the fixed point locus of γ is an associative 3-fold. ■

Corollary 4.1.5. *Let $(\mathcal{M}, \Phi)$ be a connected, compact G_2-manifold. Consider the non-trivial (i.e. not the identity map) G_2-involution $\iota : \mathcal{M} \to \mathcal{M}$ with non-empty fixed point locus F_ι. Then F_ι is an associative 3-fold. It is also smooth, orientable and compact.*

4.2 Constructing G_2s

We now recall one of the big differences between Calabi-Yau manifolds and manifolds with G_2 holonomy: there is no version of Yau's theorem for the latter. This means we do not have a set of criteria that, once satisfied, guarantee that a 7-dimensional manifold will admit a G_2 metric. This limits us to considering specific constructions of manifolds with G_2 holonomy, and we now briefly review the three relevant ones: Joyce orbifolds, Calabi-Yau quotients and the twisted connected sum construction.

4.2.1 Joyce Orbifolds

The first examples of compact manifolds with G_2 holonomy were obtained by Joyce in [38, 39]. This construction looks at the resolutions of orbifold singularities of the quotient of T^7 by a finite group Γ that preserves the G_2 structure. These are considered as a detailed example of the content of this thesis in Appendix A. Here we just give a quick idea.

Specifically, they consider $\Gamma = \mathbb{Z}_2^3$ with action on the T^7 coordinates as[8]

$$\begin{aligned} \alpha &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (+x_1, +x_2, -x_3, a_4 - x_4, -x_5, a_6 - x_6, x_7) \\ \beta &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, b_2 - x_2, +x_3, +x_4, b_5 - x_5, b_6 - x_6, x_7) \\ \sigma &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6, -x_7), \end{aligned} \tag{4.15}$$

where $a_i, b_i = 0, 1/2$. A particularly interesting case is that of $a_4 = b_6 = 1/2$ and all others vanishing. There are 9 topologically inequivalent smoothings $\mathcal{M}_l$ of this orbifold that have a Ricci flat G_2 metric. The interesting thing is that the Betti numbers are given by

$$b^2(\mathcal{M}_l) = 8 + l \quad \text{and} \quad b^3(\mathcal{M}_l) = 47 - l, \tag{4.16}$$

[8]We note that we have picked a different labelling of the coordinates compared to [38, 39]. This is done in order to make the comparison with the Calabi-Yau $T^6/\mathbb{Z}_2^2$ clearer in Appendix A.

where l is a parameter that keeps track of how one deals with the fixed points of the orbifold. It is important to note that the sum b^2+b^3 for all 9 of these smooth G_2-manifolds are the same. We note that these Betti numbers satisfy the Shatashvili-Vafa condition, Equation (4.5), and so suggests that these 9 manifolds are in some sense mirror to each other. Indeed these orbifolds were considered in the context of discrete torsion in [77], where they reproduced the Betti numbers from a free field theory analysis. Importantly, the paper demonstrates the existence of two types of mirror map:

$$\begin{aligned} \mathcal{T}_3 &: IIA/B \text{ on } \mathcal{M}_l \to IIB/A \text{ on } \mathcal{M}_{8-l} \\ \mathcal{T}_4 &: IIA/B \text{ on } \mathcal{M}_l \to IIA/B \text{ on } \mathcal{M}_{8-l} \end{aligned} \tag{4.17}$$

Details of this construction are given in Appendix A.

4.2.2 Calabi-Yau Quotients

The next construction we want to look at is that of the quotient of a Calabi-Yau and a circle. We already partially motivated back in Section 1.2 that we can form a manifold with G_2 holonomy as the resolution of

$$\mathcal{M}_\sigma = \frac{\mathcal{M}_{CY} \times S^1}{(\sigma, -1)}, \tag{4.18}$$

where $\sigma : \mathcal{M}_{CY} \to \mathcal{M}_{CY}$ is an antiholomorphic involution and -1 is inversion on the circle. We shall now flush out some of the details of this claim.

We start by identifying $\mathbb{R}^7 \cong \mathbb{C}^3 \oplus \mathbb{R}$ by

$$(x_1, ..., x_7) \cong (z_1, z_2, z_3, x_7) = \big((x_1 + ix_2), (x_3 + ix_4), (x_5 + ix_6), x_7\big). \tag{4.19}$$

We can use (z_1, z_2, z_3) to define a holonomorphic 3-form and a $(1,1)$-form as

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3 \quad \text{and} \quad J = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 \tag{4.20}$$

From here, we can define our associative 3-form and coassociative 4-form as

$$\Phi = J \wedge dx + \mathrm{Re}(\Omega) \quad \text{and} \quad \star\, \Phi = \frac{1}{2} J \wedge J + \mathrm{Im}(\Omega) \wedge dx. \tag{4.21}$$

Importantly, we can also write down a metric as $g = dx^2 + g_{CY}$, where g_{CY} is the Calabi-Yau metric.

We now define an $SU(3)$ action that preserves Ω and J, and acts trivially on x. This defines an inclusion $SU(3) \hookrightarrow G_2$. It then follows that if we define a Calabi-Yau $\mathcal{M}_{CY}$ inside the $\mathbb{C}^3$ who's $(3,0)$-form is Ω and Kähler form is J and a circle $S^1 \subset \mathbb{R}$ with coordinate x, then $\mathcal{M}_{CY} \times S^1$ is a G_2-manifold with associative and coassociative forms as above.

Importantly, this G_2-manifold has holonomy $SU(3) \subset G_2$. We want to produce a G_2-manifold with $\text{Hol}(\mathcal{M}) = G_2$. We now turn to Proposition 4.1.2, which tells us that we get $\text{Hol}(\mathcal{M}) = G_2$ iff the first fundamental group is finite. This leads us to considering quotients of $\mathcal{M}_{CY} \times S^1$.

Firstly we note that the involution in $\mathcal{M}_\sigma$ above is is in fact a G_2-involution (i.e. it leaves Φ invariant) and so, by Corollary 4.1.5, its fixed point locus is a smooth, orientable and compact associative 3-fold, provided it is not empty. Recalling Corollary 2.4.4, it follows that the fixed point locus of $\mathcal{M}_\sigma$ can only be one of two things:

(i) Empty, i.e. $\sigma : \mathcal{M}_{CY} \to \mathcal{M}_{CY}$ is a free involution.

(ii) Two copies of a Lagrangian submanifold of $L_\sigma \subset \mathcal{M}_{CY}$. We get two copies as $-1 : S^1 \to S^1$ as fixed points $x = 0, 1/2$.

For case (i) the holonomy of the smoothing is not all of G_2 but actually only $SU(3) \ltimes \mathbb{Z}_2$, and were given the name *barely* G_2-manifold in [24]. The Betti numbers are simply given by the differential forms on $\mathcal{M}_{CY} \times S^1$ that are invariant under the involution. Using that the one-form on the S^1 is odd, we obtain

$$b^2 = h^{1,1}_+ \quad \text{and} \quad b^3 = h^{1,1}_- + h^{2,1} + 1, \tag{4.22}$$

where $h^{1,1}_\pm$ denote the $(1,1)$-forms that are even and odd under the involution. In particular we note that

$$b^2 + b^3 = 1 + h^{1,1} + h^{2,1}, \tag{4.23}$$

which is invariant under the Calabi-Yau mirror map $h^{1,1} \leftrightarrow h^{2,1}$.

Case (ii) is more interesting, and it was shown in [69] that $\mathcal{M}_\sigma$ can be smoothed to a manifold with holonomy G_2, provided there exists a $\mathbb{Z}_2$ bundle $\mathcal{Z}$ on L_σ, along with a nowhere vanishing, harmonic (w.r.t. the Kähler metric on $\mathcal{M}_{CY}$) one-form λ, valued in $\mathcal{Z}$ on L_σ. The Betti numbers here are given by[9]

$$\begin{aligned} b^2(\mathcal{M}_{G_2}) &= b^2_+(\mathcal{M}_\sigma) + 2b^0(L_\sigma, \mathcal{Z}) \\ b^3(\mathcal{M}_{G_2}) &= b^2_-(\mathcal{M}_{CY}) + b^3_+(\mathcal{M}_\sigma) + 2b^1(L_\sigma, \mathcal{Z}), \end{aligned} \tag{4.24}$$

[9]The factors of 2 here is included as we have defined L_σ as the special Lagrangian submanifold in $\mathcal{M}_{CY}$, but the fixed point set of σ is two copies of this.

where $b^k_+(\mathcal{M}_\sigma)$ counts the cohomology classes of $\mathcal{M}_\sigma$ that are even under the involution, and $b^i(L_\sigma, \mathcal{Z})$ are the $\mathcal{Z}$-twisted Betti numbers. We note that the hard part of this construction is not finding a Calabi-Yau 3-fold and antiholomorphic involution with non-empty fixed point locus. The difficulty lies in showing the existence of the harmonic one-form λ.

Understanding the existence of a mirror for case (ii) is slightly less clear, and is the main result of this thesis. For now we note the following: $\mathcal{M}_\sigma$ is made using a Calabi-Yau manifold and a circle. For both of these spaces we have a notion of mirror symmetry (it is simply T-duality for the S^1). We could then use these in order to *define* a mirror $\mathcal{M}_\sigma^\vee$. In fact we have three options:

(i) Mirror just $\mathcal{M}_{CY}$ and leave S^1 alone,

(ii) Leave $\mathcal{M}_{CY}$ alone and mirror S^1, or

(iii) Mirror both.

There is an important detail that needs to be addressed, though: what happens to the involution $(\sigma, -1)$? For example, in the last case we could define

$$(\mathcal{M}_\sigma)^\vee = \left(\frac{\mathcal{M}_{CY} \times S^1}{(\sigma, -1)}\right)^\vee = \frac{\mathcal{M}_{CY}^\vee \times (S^1)^\vee}{\tau}, \tag{4.25}$$

where τ is the involution needed in order to give rise to a quotient with equivalent $b^2 + b^3$. The question becomes "does such a τ exist, and if so, what is it?" In particular, is it an antiholomorphic involution on $\mathcal{M}_{CY}^\vee$ again? There is also the question of whether there is a unique τ that does this.

As $\mathcal{M}_{CY}^\vee$ is a Calabi-Yau, we can define an antiholomorphic involution on it, and it is not too hard to construct examples such that the involution will give rise to a G_2-manifold with $b^2 + b^3$ conserved. However, it is not obvious that this is what the physics tells us we should do. That is, from a physical point of view, mirror symmetry is a deeper statement than observing the invariance of cohomology: the full physics should be invariant.

The main result of this thesis is to demonstrate, using sigma model arguments, that this is indeed the case. That is, the mirror of the antiholomorphic involution is again an antiholomorphic involution. For now we continue with the development of the geometry and return to answering this question in Section 5.2.

4.2.3 Twisted Connected Sums

The next important construction of compact manifolds with G_2 holonomy are the *twisted connected sum* constructions. These were first introduced by Kovalev in [70] and developed further in [72] (see [71] for more background). We now briefly review the construction, proofs of the statements made can be found in the references.

Asymptotically Cylindrical Calabi-Yau 3-Folds & Building Blocks

The logic behind this construction is similar to the quotient construction above:[10] constructing metrics with holonomy G_2 is difficult, but constructing metrics with holonomy $SU(3)$ is a lot easier. We therefore want to use Calabi-Yau 3-folds as a starting point in our construction.

> **Definition.** [Calabi-Yau Cylinder] Let (S, I_S, g_S) be a $K3$ surface (i.e. a Calabi-Yau 2-fold). Then $X_\infty = \mathbb{R}^+ \times S^1 \times S$ along with
>
> $$I_\infty = I_\mathbb{C} + I_S \quad \text{and} \quad g_\infty = dt^2 + d\theta^2 + g_S \tag{4.26}$$
>
> where (t, θ) are the coordinates of $\mathbb{R}^+$ and S^1, is known as a *Calabi-Yau cylinder*. The Kähler and holomorphic top forms are related by
>
> $$J_\infty = dt \wedge d\theta + J_S \quad \text{and} \quad \Omega_\infty = (d\theta - idt) \wedge \Omega_S. \tag{4.27}$$

We then have the following important definitions (see [72] for a more details).

> **Definition.** [Asymptotically Cylindrical 3-Fold] A Calabi-Yau 3-fold, (X, I, g), is called *asymptotically cylinderical* (ACyl) if it is diffeomorphic to a Calabi-Yau cylinder, $X_\infty = \mathbb{R}^+ \times S^1 \times S$, outside a compact submanifold $K \subset X$. We call X_∞ the *asymptotic end* of X and (S, I_S, g_S) the *asymptotic $K3$ surface* of X.

> **Definition.** [Building Block] Let Z be a Kähler 3-fold with projection $\pi : Z \to \mathbb{CP}^1$, where a generic fibre is a smooth $K3$ surface S. Let S_0 be a smooth and irreducible fibre, and consider the natural restriction map
>
> $$\rho : H^{1,1}(Z, \mathbb{Z}) \to H^{1,1}(S_0, \mathbb{Z}) \cong \Gamma^{3,19} = (-E_8^{\oplus 2}) \oplus U^{\oplus 3}, \tag{4.28}$$
>
> and let $N = \text{im}(\rho)$. Then, if:
>
> (i) The anticanonical class $-K_Z \in H^2(Z)$ is primitive[11] and obeys $[-K_Z] = [S]$ (i.e. equal to the class of the fibre),
>
> (ii) The inclusion $N \hookrightarrow \Gamma^{3,19}$ is primitive,[12] and

[10]Historically this construction actually came first, though.

(iii) The group $H^3(Z,\mathbb{Z})$ is torsion-free,

then we call Z a *building block.*

It can be shown that a building block Z has $h^{1,0} = h^{2,0} = 0$. However, we note that a building block is *not* a Calabi-Yau 3-fold. This is because $c_1(Z) = [-K_Z] = [S]$. Nevertheless, we can make a Calabi-Yau 3-fold by simply excising a smooth fibre. That is, consider the fibre S_0 over a point $p_0 \in \mathbb{CP}^1$, then

$$X = Z \setminus S_0 \tag{4.29}$$

is a Calabi-Yau 3-fold. It follows from Theorem 3.4 of [72] that X is in-fact an ACyl 3-fold.

Gluing Procedure

The idea of Kovalev [70] was to take a pair of ACyl 3-folds, $X_\pm$, along with a pair of circles, $S^1_\pm$, and glue them together in order to make a manifold with G_2 holonomy. In particular, we can consider the spaces

$$\mathcal{M}_\pm := S^1_\pm \times X_\pm \tag{4.30}$$

and equip them both with G_2-structures in the same manner as our quotient construction above. That is we have associative and coassociative forms

$$\Phi_\pm = d\xi_\pm \wedge J_\pm + \mathrm{Re}(\Omega_\pm) \quad \text{and} \quad \star\,\Phi_\pm = \frac{1}{2} J_\pm \wedge J_\pm + \mathrm{Im}(\Omega_\pm) \wedge d\xi_\pm, \tag{4.31}$$

where $\xi_\pm$ are the coordinates on the new $S^1_\pm$. As before, these spaces only have holonomy $SU(3)$. We now give brief details on how to get holonomy G_2.

The asymptotic regions of $\mathcal{M}_\pm$ are given by

$$X_{\infty,\pm} \times S^1 \cong S_\pm \times \mathbb{R}^+ \times S^1_{b,\pm} \times S^1_{e,\pm}, \tag{4.32}$$

where the subscript b and e stand for "base" and "external", respectively. That is $S^1_{b,\pm} \subset X_{\infty,\pm}$ and $S^1_{e,\pm}$ are the new circles we add with coordinates $\xi_\pm$. The idea is to truncate $\mathcal{M}_\pm$ in the asymptotic regions and glue the resulting manifolds with boundary together in such a way that the resulting space has holonomy G_2.

It follows from Proposition 4.1.2, that this gluing procedure must be done in such a way that we obtain a finite first fundamental group. This immediately tells us that we cannot

[11] See the discussion of Appendix B for a definition.

[12] Let N be a lattice. Then, a sublattice $A \subset N$ is called *primitive* if any basis of A extends to a basis of N.

simply glue the spaces together by gluing $S^1_{b,+}$ to $S^1_{b,-}$ and $S^1_{e,+}$ to $S^1_{e,-}$, as this would give a space with infinite fundamental group. Instead, we consider the gluing procedure that glues $S^1_{b,\pm}$ to $S^1_{e,\mp}$. Of course this gluing must also be done in a way that is compatible with the $K3$ surfaces $S_\pm$ and the $\mathbb{R}^+$ factors. In total we then define a diffeomorphism as follows: fix $T > 0$ large enough[13] and consider the region $t \in (T, T+1) \subset \mathbb{R}^+$, then

$$\begin{aligned} \varphi : S_+ \times \mathbb{R}^+ \times S^1_{b,+} \times S^1_{e,-} &\to S_- \times \mathbb{R}^+ \times S^1_{b,-} \times S^1_{e,-} \\ \big((z_1^+, z_2^+), t, \theta_+, \xi_+\big) &\mapsto \big((z_1^-, z_2^-), 2T+1-t, \xi_-, \theta_-\big) \end{aligned} \tag{4.33}$$

where $(z_1^-, z_2^-) = \mathfrak{r}(z_1^+, z_2^+)$ for hyperKähler rotation $\mathfrak{r} : S_+ \to S_-$, i.e.

$$\mathfrak{r}^*\big(\operatorname{Im}(\Omega_{S_-})\big) = -\operatorname{Im}(\Omega_{S_+}), \quad \mathfrak{r}^*\big(\operatorname{Re}(\Omega_{S_-})\big) = J_{S_+} \quad \text{and} \quad \mathfrak{r}^*\big(J_{S_-}\big) = \operatorname{Re}(\Omega_{S_+}). \tag{4.34}$$

The claim is that the resulting space does indeed have holonomy G_2, and we introduce the following definition.

Definition. [Twisted Connected Sum G_2] Let $X_\pm$ be ACyl Calabi-Yau 3-folds and $\mathcal{M}_\pm = S^1_\pm \times X_\pm$ as above. Then we truncate $\mathcal{M}_\pm$ at $t = T+1$ and glue the two spaces together using φ and obtain a manifold with holonomy G_2, known as a twisted connected sum (TCS) G_2. If $X_\pm$ come from building blocks, $Z_\pm$, we use the notation

$$\mathcal{M}(Z_+, Z_-) := \big(Z_+ \times S^1_{e,+}\big) \#_\varphi \big(Z_- \times S^1_{e,-}\big) \tag{4.35}$$

We define regions $II^\pm$ to be the asymptotic regions of $\mathcal{M}_\pm$, and regions $I^\pm$ to be the remaining region, see Figure 4.1.

Cohomology

We now want to look at the cohomology of a TCS G_2. If we have a TCS coming from a set of building blocks $Z_\pm$, each of the building blocks has a restriction map of the form Equation (4.28), i.e. we have $\rho_\pm : H^{1,1}(Z_\pm, \mathbb{Z}) \to H^{1,1}(S_{0,\pm}, \mathbb{Z})$. We define

$$\begin{aligned} N_\pm &:= \operatorname{im}(\rho_\pm), \\ T_\pm &:= N_\pm^\perp \in H^2(S_{0,\pm}, \mathbb{Z}), \\ K_\pm &:= \ker(\rho_\pm)/[S_{0,\pm}]. \end{aligned} \tag{4.36}$$

[13] This is meant in the sense of Theorem 3.12 of [72], and is required in order to make sure the resulting G_2-structure is torsion-free.

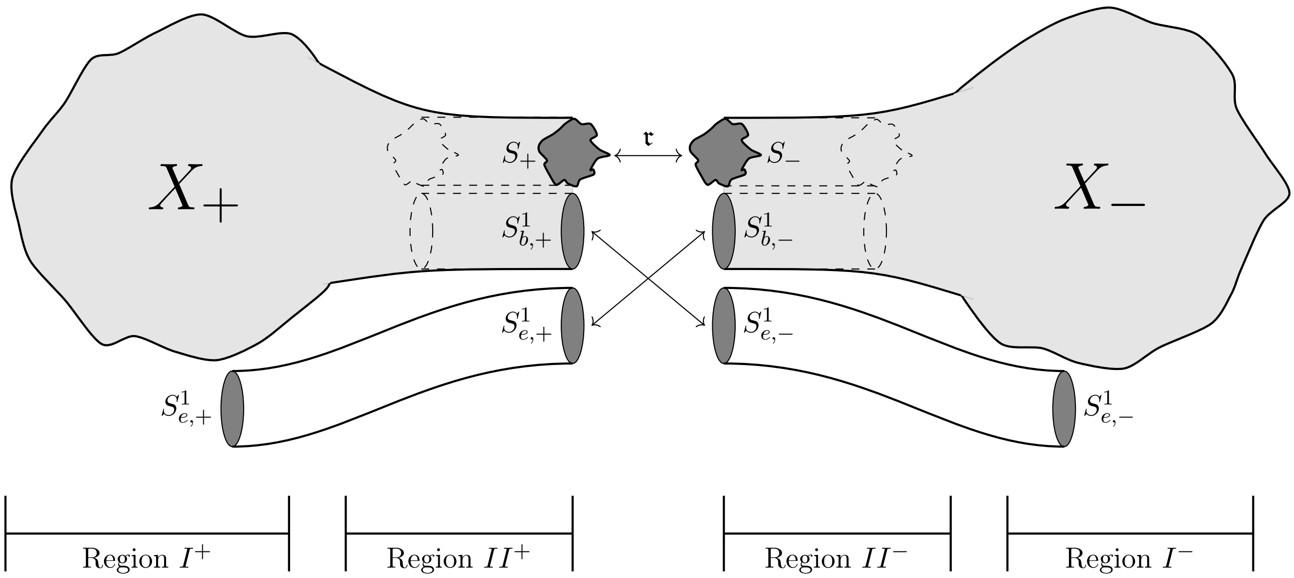


Figure 4.1: *The TCS construction of a manifold with G_2 holonomy. $X_\pm$ are ACyl Calabi-Yau 3-folds, who's asymptotic regions are $S_\pm \times \mathbb{R} \times S^1_{b,\pm}$ for $K3$ surfaces $S_\pm$. An additional circle is attached to each to define $\mathcal{M}_\pm = X_\pm \times S^1_{e,\pm}$. We glue $\mathcal{M}_\pm$ together by identitfying along the arrows, where $\mathfrak{r}$ is a hyperKähler rotation.*

The gluing diffeomorphism φ induces an isomorphism $H^2(S_{0,+}, \mathbb{Z}) \cong H^2(S_{0,-}, \mathbb{Z})$, and so it allows us to think of $N_\pm$ and $T_\pm$ as sitting inside the same lattice $\Gamma^{3,19} = (-E_8)^{\oplus 2} \oplus U^{\oplus}$. From here the integral cohomology groups of $\mathcal{M}(Z_+, Z_-)$ can be computed (see Theorem 4.9 of [72]). In this thesis we will work in a simplified case by imposing the *orthogonal gluing* condition,

$$N_\pm \otimes \mathbb{R} = (N_\pm \otimes \mathbb{R} \cap N_\mp \otimes \mathbb{R}) \oplus (N_\pm \otimes \mathbb{R} \cap T_\mp \otimes \mathbb{R}). \tag{4.37}$$

In this case we have

$$b^2 + b^3 = 23 + 2\big[|K_+| + |K_-| + h^{2,1}(Z_+) + h^{2,1}(Z_-)\big]. \tag{4.38}$$

Following the conjecture that $b^2 + b^3$ should be invariant under mirror G_2-manifolds, swapping one (or both) building blocks $Z_\pm$ for new building blocks $Z^\vee_\pm$ such that

$$h^{2,1}(Z^\vee_\pm) = |K_\pm| \quad \text{and} \quad |K^\vee_\pm| = h^{2,1}(Z_\pm) \tag{4.39}$$

would preserve $b^2 + b^3$ and constitute a potential mirror. We again note the interesting fact that we have three different mirror options:

$$\begin{aligned}(Z_+, Z_-) &\to (Z_+^\vee, Z_-)\\ (Z_+, Z_-) &\to (Z_+, Z_-^\vee)\\ (Z_+, Z_-) &\to (Z_+^\vee, Z_-^\vee).\end{aligned} \tag{4.40}$$

Mirror Gluing

To define a G_2 mirror we not only need to construct appropriate mirror building blocks, but furthermore need to find an isometry $\varphi^\vee$ to glue the asymptotically cylindrical Calabi-Yau threefolds $X_\pm^\vee$ to a TCS G_2 manifold. That such a 'mirror gluing' always exists was shown in [26, 27] by employing the following arguments. For type II strings on a G_2 variety $\mathcal{M}$ we not only need to specify the geometry of the target, but furthermore the B-field. If $\mathcal{M}$ is TCS, the B-field in general restricts non-trivially to $X_\pm$ and the asymptotic K3 fibres $S_{0\pm}$. Consistency of the gluing then implies that

$$B|_{S_{0-}} = B|_{S_{0+}}\,. \tag{4.41}$$

In the asymptotically cylindrical regions of $X_\pm$, mirror symmetry acting on $X_\pm$ implies that the $K3$ fibres $S_{0\pm}$ are mapped to their mirrors. Mirror symmery for a $K3$ surface S can be understood as a linear map acting on J_S, $\mathrm{Re}(\Omega_S^{2,0})$, $\mathrm{Im}(\Omega_S^{2,0})$, B_S that is specified by a choice of special Lagrangian fibration of S, and results in a mere reinterpretation of the same point in the CFT moduli space [78, 79].

Replacing both $Z_\pm$ by $Z_\pm^\vee$ then replaces $S_{0\pm}$ by $S_{0\pm}^\vee$, which in turn implies that $J_{S_{0\pm}^\vee}$, $\Omega^{2,0}_{S_{0\pm}^\vee}$, $B_{S_{0\pm}^\vee}$ satisfy the relations (4.34) and (4.41), so that the mirror symmetry canonically identifies a mirror gluing $\varphi^\vee$ that can be used to construct

$$[\mathcal{M}(Z_-, S_{0-}, Z_+, S_{0+}, \varphi)]^\vee := \mathcal{M}(Z_-^\vee, S_{0-}^\vee, Z_+^\vee, S_{0+}^\vee, \varphi^\vee)\,. \tag{4.42}$$

By using a similar logic as in the original SYZ argument, this mirror map is associated with performing 4 T-dualities along a coassociatve T^4 fibration of $\mathcal{M}$. Here, both $S_{e\pm}$ are contained in the coassociatve T^4.

Using a similar analysis one can show that there are gluings $\varphi^{\wedge\pm}$ which allow to construct

$$\begin{aligned}[\mathcal{M}(Z_-, S_{0-}, Z_+, S_{0+}, \varphi)]^{\wedge-} &:= \mathcal{M}(Z_-^\vee, S_{0-}^\vee, Z_+, S_{0+}, \varphi^{\wedge-})\\ [\mathcal{M}(Z_-, S_{0-}, Z_+, S_{0+}, \varphi)]^{\wedge+} &:= \mathcal{M}(Z_-, S_{0-}, Z_+^\vee, S_{0+}^\vee, \varphi^{\wedge+})\end{aligned} \tag{4.43}$$

and that these mirror maps are associated with associative T^3 fibrations. For $^{\wedge\pm}$, the SYZ picture implies that $S^1_{e\mp}$ are contained in the associative T^3 fibre, but $S^1_{e\pm}$ are not.

Besides sharing $b^2 + b^3$, the total integral cohomology

$$H^\bullet(M) = \bigoplus_k H^k(M, \mathbb{Z}) \tag{4.44}$$

satisfies the stronger condition that

$$H^\bullet(M) = H^\bullet(M^\vee) = H^\bullet(M^{\wedge\pm}) \tag{4.45}$$

for any type of gluing, not just orthogonal gluing. Note that this implies that $b^2+b^3+b^4+b^5 = 2(b^2 + b^3)$ is the same for all of these geometries.

Another interesting aspect of these mirror maps is that smooth G_2 manifolds can potentially have (geometrically) singular mirrors [27]. A TCS G_2 variety necessarily contains ADE singularities if there is a non-trivial ADE root lattice contained in $N_+ \cap N_-$, which is a possible realization of non-Higgsable clusters [80] in M-Theory [81]. Whereas a TCS G_2 manifold $\mathcal{M}$ might be such that $N_+ \cap N_-$ contains no roots, this does not necessarily hold for one of its mirrors. However, the presence of such singularities does not imply a non-abelian gauge group as there is necessarily a non-trivial B-field along the corresponding $\mathbb{CP}^1$s.

4.2.4 Comparing The Two

The quotient construction and the TCS constructions seem to share some similarities. In particular both consider taking a Calabi-Yau, attaching a circle, and then doing something in order to get finite fundamental group and so holonomy G_2. Of course the two constructions have big differences, but nevertheless it is reasonable to ask whether there is a way to link the two constructions.

This question was studied in [73], all be it from a slightly different angle. The paper uses the known results of a lift of the compactification of Type IIB strings on a Calabi-Yau orientifold to F-theory as motivation to study the lift of Type IIA orientifolds to M-theory on a G_2. In doing so, they demonstrate an elegant relationship between the quotient construction using $\mathcal{M}_\sigma$ and the TCS construction. Here we outline the details of their method.

Quotient

We start by noting that the Calabi-Yaus used in the TCS construction necessarily contain a $K3$ fibration. We therefore consider a $K3$ fibred Calabi-Yau, with base space $\mathbb{CP}^1$, i.e.

$S \hookrightarrow \mathcal{M}_{CY} \to \mathbb{CP}^1_b$, where b denotes "base". We denote the homogeneous coordinates of $\mathbb{CP}^1_b$ by $[z_1 : z_2]$. As $\sigma : \mathcal{M}_{CY} \to \mathcal{M}_{CY}$ is an antiholomorphic involution, we can pick it to act as an antiholomorphic involution w.r.t the $K3$ fibration. This in turn tells us that it acts on $\mathbb{CP}^1_b$ as $[z_1 : z_2] \mapsto [\bar{z}_1 : \bar{z}_2]$.

Claim 4.2.1. The fixed point locus of σ, restricted to the base space is a circle,

$$L_\sigma|_{\mathbb{CP}^1_b} \cong S^1, \tag{4.46}$$

and so cuts $\mathbb{CP}^1_b$ into two halves.

Proof. We have homogeneous coordinates $[z_1 : z_2]$ on $\mathbb{CP}^1$. We now swap to

$$z'_1 = z_1 + iz_2 \quad \text{and} \quad z'_2 = z_1 - iz_2. \tag{4.47}$$

The involution acts as $\sigma : [z'_1; z'_2] \mapsto [\bar{z}'_2, \bar{z}'_1]$. We then have that its action on the projectively well defined coordinate $z' = z'_1/z'_2$ is simply $z' \mapsto 1/\bar{z}'$. The fixed point of this is a circle with radius $|z'| = 1$. ∎

We now make the assumption that the $K3$ fibration over the S^1 of Equation (4.46) is trivial, i.e. that $\mathcal{M}_{CY}|_{S^1} = S^1 \times S_0$ for a smooth $K3$ surface S_0. The cutting of the $\mathbb{CP}^1$ into two halves splits the discriminant locus[14] of the fibration into two sets, and our assumption tells us that the product of the monodromies associated with the degeneration points is trivial in each of these sets. Our assumption also tells us that the fixed points locus of L_σ takes the form

$$L_\sigma = L_{S_0} \times S^1, \tag{4.48}$$

where S^1 is as per Equation (4.46), and L_{S_0} is the fixed point locus of the smooth $K3$ fibre S_0 over the S^1. See Figure 4.2.

As we have said, $\sigma : S_0 \to S_0$ acts as an antiholomorphic involution, that is:

$$\begin{aligned} \sigma^* &: J(S_0) \mapsto -J(S_0) \\ \sigma^* &: \Omega^{2,0}(S_0) \mapsto \overline{\Omega^{2,0}(S_0)}, \end{aligned} \tag{4.49}$$

where $J(S_0)$ and $\Omega^{2,0}(S_0)$ are the Kähler form and holomorphic $(2,0)$-form of S_0. The important part is that we can now use a hyperKähler rotation, ψ, to change complex structure

[14] Roughly speaking, the discriminant locus of a projection is the set of points such that the fibre is singular.

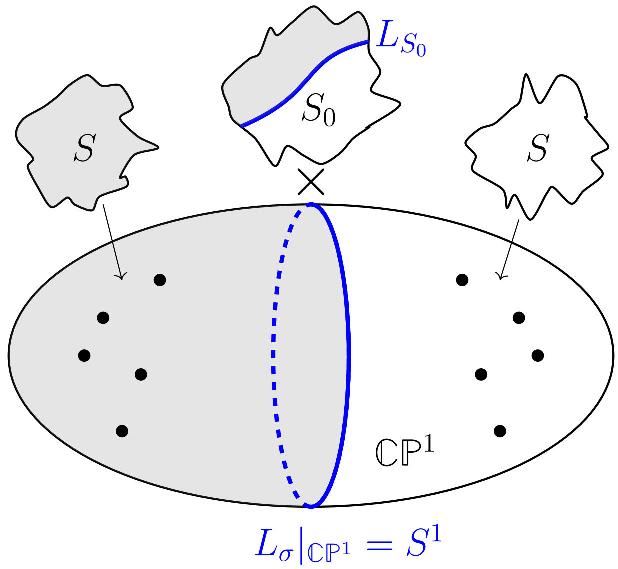


Figure 4.2: *Pictorial depiction of a $K3$ fibred Calabi-Yau 3-fold $S \hookrightarrow \mathcal{M}_{CY} \to_\pi \mathbb{CP}^1$ with antiholomorphic involution, σ, and its fixed points locus L_σ depicted in blue. The involution cuts the base $\mathbb{CP}^1$ in half with $L_\sigma|_{\mathbb{CP}^1} = S^1$. The fibration around this fixed circle is assumed to be trivial, with smooth fibre S_0. The involution acts in S_0 as an antiholomorphic involution that fixes a special Lagrangian submanifold L_{S_0}. The fixed point locus is thus $L_\sigma = S^1 \times L_{S_0}$. The shaded region is the fundamental region of the involution. The dots are the degeneration points, over which the $K3$ fibre S is singular. The product of the monodromies around these points in either half of the $\mathbb{CP}^1$ is necessarily trivial.*

on S_0 such that

$$J(S_0^\psi) = \operatorname{Re}\Omega^{2,0}(S_0) \quad \text{and} \quad \Omega^{2,0}(S_0^\psi) = J(S_0) - i\operatorname{Im}\Omega^{2,0}(S_0). \tag{4.50}$$

In this new complex structure, σ acts as

$$\begin{aligned} \sigma^* &: J(S_0^\psi) \mapsto J(S_0^\psi) \\ \sigma^* &: \Omega^{2,0}(S_0^\psi) \mapsto -\Omega^{2,0}(S_0^\psi). \end{aligned} \tag{4.51}$$

Such an involution is known as a non-symplectic involution, and they were classified by Nikulin [50, 82, 83]. The classification states that we can specify a non-symplectic involution by three integers (r, a, δ), constrained by $1 \leq r \leq 20$, $0 \leq a \leq 11$ and $\delta = 0, 1$. The allowed values all also obey $r - a \geq 0$.

The fixed point locus L_{S_0} is then given by a collection of $(f-1)$ disjoint $\mathbb{CP}^1$s along with a genus g surface. The triple (r, a, δ) determines f and g via[15]

$$g = \frac{20 - r - a}{2} + 1 \quad \text{and} \quad f = \frac{r - a}{2} + 1. \tag{4.52}$$

We are now in a position to compute the Betti numbers b^2 and b^3 for the resolved $\mathcal{M}_{G_2}$ coming from $\mathcal{M}_\sigma$. In particular we recall Equation (4.24) and use

$$b^0(L_\sigma) = f \quad \text{and} \quad b^1(L_\sigma) = b^1(L_{S_0}) + b^0(L_{S_0}) = 2g + f, \tag{4.53}$$

along with

$$b^2_+(\mathcal{M}_\sigma) = h^{1,1}_+(\mathcal{M}_{CY}) \quad \text{and} \quad b^3_+(\mathcal{M}_\sigma) = h^{1,1}_-(\mathcal{M}_{CY}) + h^{2,1}(\mathcal{M}_{CY}) + 1, \tag{4.54}$$

to obtain

$$b^2(\mathcal{M}_{G_2}) + b^3(\mathcal{M}_{G_2}) = 1 + h^{1,1}(\mathcal{M}_{CY}) + h^{2,1}(\mathcal{M}_{CY}) + 4f + 4g. \tag{4.55}$$

Recall that the resolution of $\mathcal{M}_\sigma$ requires the existence of a nowhere vanishing, harmonic one-form λ. The existence of it here follows from our previous assumption that the $K3$ surface doesn't vary over $S^1 = L_\sigma|_{\mathbb{CP}^1_b}$: λ is the volume form on this circle.

TCS

We now want to reproduce the above Betti numbers from a TCS construction. The idea of [73] was to use the antiholomorphic involution to divide the space into two and show that these spaces can be realised using building blocks.

The idea is to recall that σ splits the base $\mathbb{CP}^1_b$ in half, and so the fundamental region is topologically a bounded disc, $\mathcal{D}$. Letting R denote the radius of $\mathcal{D}$, we can use coordinates (r, ϕ) with $r \leq R$ and $\phi \in \{0, 2\pi\}$. By picking an appropriate region of the moduli space of $\mathcal{M}_{CY}$, we can put all of the singular $K3$ fibres in a small disc around the origin.

We then split $\mathcal{D}$ into two overlapping regions,

$$\begin{aligned} \mathcal{D}_- &= \left\{ (r, \phi) \,\middle|\, r < \frac{3}{4}R \right\} \\ \mathcal{D}_+ &= \left\{ (r, \phi) \,\middle|\, r > \frac{1}{4}R \right\}. \end{aligned} \tag{4.56}$$

If we can show that these two regions give ACyl 3-folds, then we are in business to find a

[15] There are two exceptional cases: $(r, a, \delta) = (10, 10, 0)$ and $(r, a, \delta) = (10, 8, 0)$. The former gives empty L_{S_0} while the latter is two tori.

TCS construction of our G_2. That is, we want to use $\mathcal{D}_\pm$ to define $\mathcal{M}_\pm$ that decomposes $\mathcal{M}_\sigma$. Importantly, the claim of the paper is that this decomposition is respected by the smoothing to $\mathcal{M}_{G_2}$, and so we get a decomposition of $\mathcal{M}_{G_2}$ as a TCS.

The first thing we notice is that the involution σ acts freely on the double cover of $\mathcal{D}$ everywhere except at $r = R$. Therefore its action for $\mathcal{M}_-$ is free and we simply have

$$\mathcal{M}_- = S^1_\theta \times X_-, \tag{4.57}$$

where X_- is the ACyl 3-fold that asymptotes at $r > \frac{1}{4}R$ to $S_0 \times S^1_\phi \times \mathbb{R}^+_r$. Here we have $S^1_{e,-} = S^1_\theta$, $S^1_{b,-} = S^1_\phi$ and $S_- = S_0$.

The region $\mathcal{M}_+$ is more difficult as it contains the fixed points $r = R$. However, we are saved by the crucial fact that there is a limit in the moduli space where the $K3$ fibre over $\mathcal{D}_+$ is trivial. We therefore have

$$\mathcal{M}_+ = \frac{(S_0 \times S^1_\phi \times \mathbb{R}_r) \times S^1_\theta}{(\sigma, -1)} = S^1_\phi \times \frac{(S_0 \times \mathbb{R}_r) \times S^1_\theta}{(\sigma, -1)} \tag{4.58}$$

where we have used the fact that σ doesn't change the ϕ coordinate on the disc. We note that, for $r < R$, the action is free and we simply get the product $S^1_\theta \times S^0 \times S^1_\phi \times \mathbb{R}$, which looks like an asymptotic region we want. Namely, we have $S^1_{e,+} = S^1_\phi$, $S^1_{b,+} = S^1_\theta$ and $S_+ = S_0$. However, there is an issure: σ acts on S_0 by an antiholomorphic involution which destroys the Calabi-Yau nature.

The solution to this problem is simple: we change complex structure on S_0 by using a hyperKähler rotation, $\mathfrak{r}_{r,a,\delta} : S_0 \mapsto S_0'$, such that σ acts holomorphically. We can then define X_+ to be the ACyl Calabi-Yau orbifold with asymptotic region $S_0' \times S^1_\theta \times \mathbb{R}^+_r$. The required hyperKähler rotation acts as

$$\mathfrak{r}^*_{r,a,\delta}\big(\operatorname{Im}(\Omega_{S'})\big) = -\operatorname{Im}(\Omega_S), \quad \mathfrak{r}^*_{r,a,\delta}\big(\operatorname{Re}(\Omega_{S'})\big) = J_S \quad \text{and} \quad \mathfrak{r}^*_{r,a,\delta}\big(J_{S'}\big) = \operatorname{Re}(\Omega_S). \tag{4.59}$$

We immediately note that this hyperKähler rotation is *exactly* the one needed for the gluing procedure in our TCS construction, Equation (4.34). Putting this together with the fact that the two S^1s are also swapped between $\mathcal{M}_\pm$, we arrive at a TCS construction of the singular $\mathcal{M}_\sigma$.

The next thing we need to do is look to resolve the space and obtain a TCS of $\mathcal{M}_{G_2}$. In order to do this, we need to find compact building blocks $Z_\pm$ with $c_1(Z_\pm) = [S_{0,\pm}]$ so that $X_\pm = Z_\pm \setminus S_{0,\pm}$.

The easier one of the two is Z_-. The idea is to notice that we could glue together two copies of X_- and get $\mathcal{M}_{CY}$ back. In this sense, we have

$$\mathcal{M}_{CY} = Z_- \# Z_-, \tag{4.60}$$

which establishes a relationship between the topologies of $\mathcal{M}_{CY}$ and Z_-. The relations that will be important to us are

$$h^{2,1}(Z_-) = \frac{1}{2}\Big(h^{2,1}(\mathcal{M}_{CY}) + h^{1,1}(\mathcal{M}_{CY})\Big) - 11 - |K_-|, \tag{4.61}$$

as well as

$$\begin{aligned} h^{1,1}_+(\mathcal{M}_{CY}) &= |K_-| + |N_+|, \\ h^{1,1}_-(\mathcal{M}_{CY}) &= |K_-| + |N_-| + 1. \end{aligned} \tag{4.62}$$

where K_- and $N_\pm$ are as per Equation (4.36).

The construction of Z_+ is a little more involved and requires introducing *Voisin-Borcea* Calabi-Yau 3-folds [84, 85]. These are Calabi-Yaus formed by the resolution of the quotient of a $K3$ surface and a 2-torus by a holomorphic involution:

$$Y_{r,a,\delta} = \left(\frac{\widetilde{S \times T^2}}{\eta}\right), \tag{4.63}$$

where η is a non-symplectic involution on S and acts as inversion on the complex coordinate of the T^2. As explained in [73], the idea is that $Y_{r,a,\delta}$ can be split itself into two 'Voisin-Borcea building blocks',

$$Y_{r,a,\delta} = \Upsilon_{r,a,\delta} \# \Upsilon_{r,a,\delta} \tag{4.64}$$

that have topology

$$h^{2,1}(\Upsilon_{r,a,\delta}) = 2g \quad \text{and} \quad |K(\Upsilon_{r,a,\delta})| = 2f \tag{4.65}$$

where g and f are as per Equation (4.52). The key thing is that $X_{r,a,\delta} = \Upsilon_{r,a,\delta} \setminus S_0$ is a non-compact ACyl 3-fold and so we can set $Z_+ = \Upsilon_{r,a,\delta}$.

Finally, plugging Equations (4.61) and (4.65) into Equation (4.38) returns

$$b^2(\mathcal{M}_{G_2}) + b^3(\mathcal{M}_{G_2}) = 1 + h^{1,1}(\mathcal{M}_{CY}) + h^{2,1}(\mathcal{M}_{CY}) + 4f + 4g, \tag{4.66}$$

which is exactly Equation (4.55), as required. We therefore get that the TCS construction is given by

$$\mathcal{M}_{G_2} = \big(\Upsilon_{r,a,\delta} \times S^1_\theta\big) \#_{\varphi_{r,a,\delta}} \big(Z_- \times S^1_\phi\big) \tag{4.67}$$

where $\varphi_{r,a,\delta}$ is the gluing diffeomorphism who's hyperKähler rotation is $\mathfrak{r}_{r,a,\delta}$ of Equation (4.59). As pointed out in [73], the nowhere vanishing, harmonic one-form λ needed in the quotient construction is given by $d\phi$ in the TCS description.

4.3 Toric Geometry: Tops

The construction of our mirror TCS G_2 required us to know the mirror building blocks $Z^{\vee}_{\pm}$. The obvious question is "is there a useful method to obtain these mirror building blocks?" We now recall that language of toric geometry provided a very powerful and borderline combinatoric method for constructing Calabi-Yaus and their mirrors. We now want to look to develop an equivalent construction for manifolds with G_2 holonomy.

The key lies in noting that we can form our G_2 using a $K3$ fibred Calabi-Yau. The Calabi-Yau (and its mirror) are defined by introducing a pair of reflexive polytopes (Δ, Δ°). We now recall that it is possible to probe for a fibration structure by projecting the fan into a sublattice and asking whether we get a fan again or not. We then consider the following set up [86–88], which we give in the form of a proposition.

Proposition 4.3.1. *Let $\Delta^{\circ} \subset N$ be a reflexive polytope in a lattice N, and $\Delta^{\circ}_F \subset \Delta^{\circ}$ be a subpolytope, i.e. there exists a sublattice $N_F \subset N$ such that $\Delta^{\circ}_F = \Delta^{\circ} \cap N_F$. If Δ°_F is reflexive, then the Calabi-Yau $X_{\Delta,\Delta^{\circ}}$ admits a fibration by $X_{\Delta_F,\Delta^{\circ}_F}$. The projection $N \to N/N_F$ gives rise to a projection of the hypersurfaces if there is an appropriate triangulation of Δ° that turns this into a toric morphism.*

The case that is of interest to us is when the fibration is of codimension 1, i.e. a $K3$ fibred Calabi-Yau 3-fold. We therefore want to look at the situations where N_F is codimension 1. Letting m_0 denote the primitive normal vector to N_F, i.e. $\langle m_0, N_F\rangle = 0$, allows us to see the fibration structure nicely [74]. We recall that the monomials for the defining equation are defined using the lattice points $m \in \Delta$:

$$G(z) = \sum_{m\in\Delta} \alpha_m \prod_{n_i\in\Delta^{\circ}} z_n^{\langle m,n\rangle+1}. \tag{4.68}$$

We therefore introduce an equivalence relation

$$m \sim m' \quad \text{if} \quad m - m' = k m_0, \tag{4.69}$$

for integer k. Let $\widetilde{M} = \{[M]\}$ denote the set of equivalence classes, then the defining equation can be rewritten as

$$
\begin{aligned}
G(z) &= \sum_{[M]\in\widetilde{M}} \sum_{m\in[M]} \alpha_m \prod_{n\in\Delta^\circ} z_n^{\langle m,n\rangle+1} \\
&= \sum_{[M]\in\widetilde{M}} \left(\prod_{n\in\Delta_F^\circ} z_n^{\langle m,n\rangle+1} \right) \left(\sum_{m\in[M]} \alpha_m \prod_{n\notin\Delta_F^\circ} z_n^{\langle m,n\rangle+1} \right) \\
&= \sum_{[M]\in\widetilde{M}} \alpha_m^F \prod_{n\in\Delta_F^\circ} z_n^{\langle m,n\rangle+1},
\end{aligned}
\tag{4.70}
$$

where we have defined

$$
\alpha_m^F := \sum_{m\in[M]} \alpha_m \prod_{n\notin\Delta_F^\circ} z_n^{\langle m,n\rangle+1}. \tag{4.71}
$$

This is, however, the defining equation for a Calabi-Yau $X_{\Delta_F,\Delta_F^\circ}$, where the coefficients of the monomials, α_m^F, depend on the remaining coordinates. This is exactly the set up for a fibration.

Next we note that Δ_F° separates Δ° into two halves:

$$
\begin{aligned}
\lozenge_1^\circ &:= \mathrm{Conv}\left(\{n\in\Delta^\circ | \langle m_0,n\rangle \geq 0\}\right) \\
\lozenge_2^\circ &:= \mathrm{Conv}\left(\{n\in\Delta^\circ | \langle m_0,n\rangle \leq 0\}\right)
\end{aligned}
\tag{4.72}
$$

so that

$$
\Delta^\circ = \lozenge_1^\circ \cup \lozenge_2^\circ \qquad \Delta_F^\circ = \lozenge_1^\circ \cap \lozenge_2^\circ\,. \tag{4.73}
$$

These two halves were named "top" and "bottom" in [89]. We then have the following definition.

Definition. [Top] Let $\Delta_F^\circ \subset \Delta^\circ$ be a pair of reflexive polytopes, as above, with $\dim\Delta_F^\circ = \dim\Delta^\circ - 1$. Let m_0 be the primitive normal vector to N_F, $\langle m_0, N_F\rangle = 0$. Then a *top* is defined as the lattice polytope

$$
\lozenge^\circ = \mathrm{Conv}\left(\{n\in\Delta^\circ | \langle m_0,n\rangle \geq 0\}\right). \tag{4.74}
$$

We will make the simple choice $m_0 = (0,0,0,1)$ by exploiting the $SL(4,\mathbb{Z})$ acting on N in the following.

Definition. [Projecting Top] A top $\lozenge^\circ$ is called *projecting* if the projection of $\lozenge^\circ$ to $N_F\otimes\mathbb{R}$ is contained in Δ_F°.

Projecting tops can be used to construct building blocks for TCS G_2 manifolds in analogy to Batyrev's construction of Calabi-Yau threefolds [74]. Given a projecting top, we can define its dual as

$$\langle \Diamond, \Diamond^\circ \rangle \geq -1 \qquad \langle \Diamond, n_0 \rangle \geq 0\,. \tag{4.75}$$

where $n_0 = (0,0,0,-1)$. Using $\Diamond \subseteq M_\mathbb{R}$, the normal fan $\Sigma_n(\Diamond)$ can be used in the construction of toric varities from polytopes discussed before, resulting in a compact hypersurface in the toric variety $X_{\Sigma_n(\Diamond)}$, which in general is not smooth. A fan refinement

$$\widetilde{\Sigma}_n(\Diamond) \to \Sigma_n(\Diamond)\,. \tag{4.76}$$

for which all rays introduced have generators which are lattice points on $\Diamond$ gives rise to a crepant partial desingularisation. The associated MPCP then defines a smooth hypersurface which we denote by $Z_{\Diamond,\Diamond^\circ}$. The defining equation of $Z_{\Diamond,\Diamond^\circ}$ is

$$F(z) = \sum_{m\in(\Diamond\cup(0,0,0,1))} \alpha_m z_0^{\langle m,n_0\rangle} \prod_{n\in\Diamond^\circ} z_n^{\langle m,n\rangle+1} = 0\,. \tag{4.77}$$

Here z_i are the homogeneous coordinates associated with the ray generators $n_i \in \Diamond^\circ$, note that n_0 is always a ray generator of $\Sigma_n(\Diamond)$.

The hypersurface $Z_{\Diamond,\Diamond^\circ}$ admits a K3 fibration with base $\mathbb{CP}^1$ such that

$$c_1(Z_{\Diamond,\Diamond^\circ}) = [S_0] \tag{4.78}$$

where $[S_0]$ is the cohomology class dual to the divisor class of a generic K3 fibre, i.e. $Z_{\Diamond,\Diamond^\circ}$ is a building block and $X_{\Diamond,\Diamond^\circ} = Z_{\Diamond,\Diamond^\circ} \setminus S_0$ is an asymptotically cylindrical Calabi-Yau threefold.

The topological data of $Z_{\Diamond,\Diamond^\circ}$ required for our purposes can be described by combinators [74].

Proposition 4.3.2. *Denoting k-dimensional faces of $\Diamond$ by $\Theta^{[k]}$, the Hodge numbers of $Z_{\Diamond,\Diamond^\circ}$ are $h^{i,0}(Z_{\Diamond,\Diamond^\circ}) = 0$ for all $i > 0$ and*

$$\begin{aligned} h^{1,1}(Z_{\Diamond,\Diamond^\circ}) &= -4 + \sum_{\Theta^{[3]}} 1 + \sum_{\Theta^{[2]}} \ell^*(\sigma_n(\Theta^{[2]})) + \sum_{\Theta^{[1]}} \ell^*(\Theta^{[1]})\ell^*(\sigma_n(\Theta^{[1]})) \\ h^{2,1}(Z_{\Diamond,\Diamond^\circ}) &= \ell(\Diamond) - \ell(\Delta_F) + \sum_{\Theta^{[2]}} \ell^*(\Theta^{[2]})\ell^*(\sigma_n(\Theta^{[2]})) - \sum_{\Theta^{[3]}} \ell^*(\Theta^{[3]}) \end{aligned} \tag{4.79}$$

where $\ell^*(\sigma_n(\Theta^{[k]}))$ counts lattice points on $\Diamond^\circ$ in the relative interior of the normal cone to $\Theta^{[k]}$, and ℓ and ℓ^* of polytopes/faces are defined as before. The ranks of the lattices N and K

defined for building blocks are given by

$$
\begin{aligned}
|N(Z_{\Diamond,\Diamond^\circ})| &= \ell^1(\Delta_F) - 3 + \sum_{ve\ \Theta_F^{\circ[1]}} \ell^*(\Theta_F^{\circ[1]})\ell^*(\Theta_F^{[1]}) \\
|K(Z_{\Diamond,\Diamond^\circ})| &= h^{1,1}(Z_{\Diamond,\Diamond^\circ}) - |N| - 1
\end{aligned}
\tag{4.80}
$$

where $\ell^1(...)$ counts points on the one-skeleton, and ve $\Theta_F^{[1]}$ denotes only those one-dimensional faces of Δ which are bounding a face of $\Diamond^\circ$ that is 'vertical', i.e. parallel to n_0.

As for reflexive polytopes, we can interchange the roles played by $\Diamond$ and $\Diamond^\circ$ resulting in another building block, $Z_{\Diamond^\circ,\Diamond}$, that satisfies [26]

$$
\begin{aligned}
h^{2,1}(Z_{\Diamond,\Diamond^\circ}) &= |K(Z_{\Diamond^\circ,\Diamond})| \\
h^{2,1}(Z_{\Diamond^\circ,\Diamond}) &= |K(Z_{\Diamond,\Diamond^\circ})| \,.
\end{aligned}
\tag{4.81}
$$

These are exactly the relations we needed to constitute a mirror in the TCS construction, Equation (4.39). Furthermore, the lattices $N(Z_{\Diamond,\Diamond^\circ})$ and $N(Z_{\Diamond^\circ,\Diamond})$ admit a primitive embedding

$$
N(Z_{\Diamond,\Diamond^\circ}) \oplus N(Z_{\Diamond^\circ,\Diamond}) \oplus U \hookrightarrow \Gamma^{3,19}. \tag{4.82}
$$

This implies that the K3 fibres of $Z_{\Diamond,\Diamond^\circ}$ and $Z_{\Diamond^\circ,\Diamond}$ are from algebraic mirror families [78]. The above relations play a crucial role in the construction of mirror G_2 manifolds of TCS type.

Any two projecting tops $\Diamond_1^\circ$ and $\Diamond_2^\circ$, for which $\Delta_{1F}^\circ = \Delta_{2F}^\circ$, can be joined to create a reflexive polytope Δ_{12}° [89]. A large fraction of the polytopes in the Kreuzer-Skarke list are of this type, and as their Hodge numbers can be understood from this decomposition as well, a number of patterns in the plot of Hodge numbers can be explained by this. Conversely, given a reflexive polytope Δ° that can be decomposed into two projecting tops, the Calabi-Yau threefold X_{Δ,Δ° admits a stable degeneration limit in which it becomes reducible into the two building blocks $Z_{\Diamond,\Diamond^\circ}$ and $Z_{\Diamond^\circ,\Diamond}$ [26]. This limit can be understood as stretching the base $\mathbb{CP}^1$ of the K3 fibration on X_{Δ,Δ°, separating the singular K3 fibres to its two ends. Cutting along the stretched base along the middle then decomposes X_{Δ,Δ° into the asymptotically cylindrical threefolds $X_{\Diamond_1,\Diamond_1^\circ}$ and $X_{\Diamond_2,\Diamond_2^\circ}$. This is, of course, simply the toric geometry description of the decomposition of a $K3$ fibred Calabi-Yau into a TCS considered above.

5 G_2: Conformal Field Theory

This chapter discusses the SCFT for G_2-manifolds, first presenting the general algebra due to Shatashvili and Vafa, before moving on to constructions using the Odake algebra. This allows us to make connections to the geometric constructions from the last chapter. As with the Calabi-Yau SCFT, we introduce the mirror map as an automorphism of the algebra. We then introduce the sigma model for a G_2-manifold, and ask questions about how we should think about the action of the antiholomorphic involution in the GLSM. This leads to simple explanation of how the mirror construction of Hori and Vafa can be nicely extended to the singular space $\mathcal{M}_\sigma$. In particular we observe that the conjectured geometrical mirror pair from the previous chapter, Equation (4.25), do indeed have isomorphic SCFTs and constitute genuine mirror pairs. We then end with a discussion on how the smoothing process that takes us from $\mathcal{M}_\sigma$ to $\mathcal{M}_{G_2}$ should be understood in the SCFT, and argue that we again obtain a genuine mirror map for $\mathcal{M}_{G_2}$.

The main references for this chapter are [23] for the general algebra and [29, 34] for the realisation of the algebra using the Odake algebra. The main result of the thesis, along with a discussion of how it fits into other known results, is then given from Section 5.2 onwards.

5.1 The Shatashvili-Vafa Algebra

Before discussing mirror symmetry for G_2-manifolds, we first want to construct the supersymmetric sigma model, i.e. we want to study the SCFT. We saw in Chapter 3 that for a Calabi-Yau manifold, the existence of spacetime SUSY required the SCFT to have $\mathcal{N} = (2, 2)$ SUSY. At the level of the SCFT this was obtained by extending the $\mathcal{N} = 1$ superVirasoro algebra by a $U(1)$ current, J. We now want to play a similar game for manifolds with G_2 holonomy and ask what extension (if any) of the $\mathcal{N} = 1$ superVirasoro we need.

First we make the observation noted in [23]. The $U(1)$ of the Odake algebra can be understood as follows: consider the sigma model with an n-dimensional Kähler target, so that we have $U(n)$ symmetry. In order to obtain a Calabi-Yau, we need to restrict to $SU(n)$ holonomy, and so the part of the $U(n)$ symmetry that remains unbroken is $U(1) = U(n)/SU(n)$. For manifolds with G_2 holonomy we can make a similar argument: we start with $SO(7)$ symmetry and have holonomy G_2. The quotient is not a group, however at the level of the SCFT it is simply a coset model with central charge $7/10$,[1] which is the infamous tricritical Ising model (see, e.g., [66] for a review).

The above observation suggests that we think of the role that the $U(1)$ plays in the Odake algebra is played by the trictrical Ising model for the G_2 SCFT. There is an important thing missing, though: the $U(1)$ gave us $\mathcal{N} = 2$ SCFT and so gave us access to spectral flow. It is spectral flow that allows us to map between NS and R states, and so gives rise to the spacetime SUSY. In order to make the above $U(1) \longrightarrow$ (tricitical Ising model) connection, we need to show that the latter also gives us a way to map NS and R states among each other.

In order to construct the SCFT of the G_2-manifold we start by recalling that [60] tells us that a covariantly constant p-form on the target manifold gives rise to a superpartner pair of currents with conformal dimensions $\frac{p}{2}$ and $\frac{p+1}{2}$. For the G_2-manifold we have two forms, the associative 3-form and its dual coassosiative 4-form. We therefore expect to add four currents to our algebra with conformal weights $(3/2, 2)$ and $(2, 5/2)$, which we denote (Φ, K) and (X, M), respectively.

The OPEs between the generators (T, G, Φ, K, X, M), where (T, G) are the generators of the $\mathcal{N} = 1$ superVirasoro algebra, are shown to close (see Appendix 1 of [23])[2], and so we can use them to define our G_2 SCFT. An important fact about this SCFT is that it possess two, non-commutative, $\mathcal{N} = 1$ algebras: we have the original one (T, G), and then a new one with generators $T_I = -\frac{1}{5}X$ and $G_I = \frac{i}{\sqrt{15}}\Phi$. The latter has central charge $c_I = \frac{7}{10}$, and gives us the tricritical Ising model that we expected.

The idea is then to split the stress energy tensor into two pieces: $T = T_I + T_r$, where T_r denotes the part that has vanishing OPE with the T_I (i.e. r means "remaining"). This allows us to give our states two conformal weights, h_I and h_r.

Recalling that the conformal weight of a Ramond ground state in a SCFT is $d/16$, where d is the dimension of the target space, tells us that our Ramond ground states must have $7/16$. The key thing is that the tricritical Ising model contains a Ramond ground state with $h_I = 7/16$, and so our G_2 SCFT contains a state

$$|h_I, h_r\rangle = \left|\frac{7}{16}, 0\right\rangle . \tag{5.1}$$

[1]Technically speaking the quotient is of the affine Lie algebra $\mathfrak{so}(7)_1$ and the $(\mathfrak{g}_2)_1$ algebras, which have central charges $7/2$ and $14/5$, respectively.

[2]It was noted in [29] that there is a typo in the $K(z)M(w)$ OPE.

The operator corresponding to this state can then be used to provide a map between R and NS states (see, e.g., [23]). This allows us to really conclude that the trictirical Ising model plays the role of the $U(1)$ in the Odake algebra.

We shall refer to the above algebra as the Shatashvili-Vafa algebra in what follows. As pointed out in [29], the algebra obtained in [23] was done in a free field representation, which guarantees the associativity of the OPEs. However, more abstractly in order to satisfy the Jacobi-like identities, we need to mod out by an ideal generated by

$$N = 4(GX) - 2(\Phi K) - 4\partial M - \partial^2 G. \tag{5.2}$$

5.1.1 Moduli Space

We now recall that the Witten index plays an important role in relating the states of the SCFT to the cohomology of the target spacetime. Here we are dealing with the extension of an $\mathcal{N} = 1$ algebra, and so we do not have the $U(1)_A$ and $U(1)_V$ charges that played such a crucial role in allowing us to determine individual Hodge numbers for Calabi-Yau targets. As explained in [23], here we are only able to compute either the sum of all even or all odd Betti numbers, we will use even for concreteness. This means that the SCFT only determines the target manifold up to manifolds that share the same $\sum_i b^{2i}$.

Luckily for manifolds with G_2 holonomy this is simplified slightly: the only undetermined Betti numbers are b^2, b^3, b^4 and b^5, but these are related by the Hodge star: $b^2 = b^5$ and $b^3 = b^4$. So, from the SCFT perspective, we can only compute the cohomology of the target space up to manifolds that have the same $b^2 + b^4 = b^2 + b^3$. This is *exactly* our Shatashvili-Vafa relationship, Equation (4.5), that we have been using to identify potential mirror G_2s. Indeed in [23] they show using the tricritical Ising decomposition above that they are only able to determine the value of $b^2 + b^3$ from the algebra. In other words, the moduli space of the Shatashvili-Vafa algebra has dimension $b^2 + b^3$

This tells us that, although the two manifolds could have widely different topology, they share the same SCFT and so correspond to physically equivalent theories. This is the context in which the generalised mirror conjecture above is understood.

We now move on to constructing the Shatashvili-Vafa algebra by thinking about the geometrical constructions of G_2s from before. We will, however, leave the discussion of mirror symmetry in these constructions until we have presented our general mirror arguments later.

5.1.2 From Odake

Now that we have both our Odake and Shatashvili-Vafa algebras, we can ask if we can link them in a similar fashion to the geometric constructions. We start by noting that the Ramond ground states of our Odake algebra all had $h = 3/8$, while the Shatashvili-Vafa algebra requires $h = 7/16$. Putting this together with the fact that the SCFT for a circle is given by a single boson-fermion pair, so that the R ground states have $h = 1/16$, we can generate the right conformal dimensions by attaching a copy of the S^1 SCFT to the Odake algebra.

Quotient Construction

We first look at the quotient construction

$$\mathcal{M}_\sigma = \frac{\mathcal{M}_{CY} \times S^1}{(\sigma, -)}, \tag{5.3}$$

where σ is an antiholomorphic involution on $\mathcal{M}_{CY}$ and $(-)$ the inversion on the S^1. This was done nicely in [29], and we summarise the construction here.

Denoting the Odake generators by $(T_{\mathrm{CY}}, G^0, J, G^3, A, B, C, D)$ and the boson-fermion generators by (j, ψ), we obtain the Shatashvili-Vafa generators via[3]

$$\begin{aligned} T &= T_{\mathrm{CY}} + T_{S^1} \\ G &= G^0 + G_{S^1} \\ \Phi &= A + (J\psi) \\ X &= (B\psi) + \frac{1}{2}(JJ) - \frac{1}{2}(\partial\psi\psi) \\ K &= C + (Jj) + (G^3\psi) \\ M &= (D\psi) - (Bj) + \frac{1}{2}(j\partial\psi) - \frac{1}{2}(\partial j\psi) + (JG^3) - \frac{1}{2}\partial G, \end{aligned} \tag{5.4}$$

where (...) stands for normal orderings, and

$$T_{S^1} = \frac{1}{2}(jj) + \frac{1}{2}(\partial\psi\psi) \quad \text{and} \quad G_{S^1} = (j\psi). \tag{5.5}$$

As before, we must take into consideration the ideal N of Equation (5.2). However, we note that N in fact belongs to the ideal generated by the Odake null fields N^1 and N^2 in

[3] The expression for M is corrected for a typo, as per [34].

Equation (3.48). We therefore obtain a realisation of the Shatashvili-Vafa algebra as

$$SV \hookrightarrow \frac{\mathrm{Od}^3 \times \mathrm{Free}_{S^1}}{\langle N^1, N^2 \rangle} \tag{5.6}$$

We are dealing with an $\mathcal{N} = (1,1)$ algebra, and so we have two copies of this: the left and right copies.

Following the geometrical construction of manifolds with G_2 holonomy from a quotiented product of a Calabi-Yau and a circle, we now want to ask how the involution acts on the generators of the SCFT. Geometrically, an antiholomorphic involution is defined via the action on the Kähler form, $J_K \mapsto -J_K$, and on the holomorphic $(3,0)$ form, $\Omega^{3,0} \mapsto \bar{\Omega}^{3,0}$. Recalling that the results of [60] tell us that the Kähler form gives rise to the (J, G^3) generators and that the imaginary parts of $\Omega^{3,0}$ gives rise to (B, D), we conclude that an antiholomorphic involution inverts the sign of these four generators. We can make similar arguments for the S^1 factor, where we see that the signs of both generators (j, ψ) are changed. So, in total, we see that an antiholomorphic involution acts on the generators as

$$(\sigma, -) : (T_{\mathrm{CY}}, G^0, J, G^3, A, B, C, D, j, \psi) \mapsto (T_{\mathrm{CY}}, G^0, -J, -G^3, A, -B, C, -D, -j, -\psi) \tag{5.7}$$

This acts on both the left and right algebras simultaneously. This is clearly an automorphism of the Shatashvili-Vafa algebra as the generators (T, G, Φ, X, K, M) are all invariant. Indeed this is one way to obtain the decomposition in Equation (5.4): they generate the subalgebra of $\mathrm{Od}^3 \times S^1$ fixed by σ.

We note an important point: the antiholomorphic involution does not have a unique geometrical interpretation. Here we have written its action at the level of the generators of the algebra, however this does *not* fix how it acts on individual elements of the Hilbert space. For example, an involution has the potential freedom to map states with the same quantum numbers, but the above is insensitive to this map. Geometrically this could correspond to the difference between sending $(z_1, z_2, ...,) \mapsto (\bar{z}_1, \bar{z}_2, ...)$ or $(z_1, z_2, ...,) \mapsto (\bar{z}_2, \bar{z}_1, ...)$, etc. In order to explicitly construct our mirror $\mathcal{M}_{G_2}$, we of course want to be more specific about which involution we are dealing with. We will be able to answer this question more clearly when we turn to the GLSM approach later.

TCS Construction

We also want to be able to see the construction of the Shatashvili-Vafa algebra akin to the TCS construction. This was done in [34] by considering the two regions I vs. II of the TCS construction in turn.

For regions $\mathrm{I}^{\pm}$ we are dealing with a product of an ACyl Calabi-Yau 3-fold and a circle. We therefore are in the realm of the above construction and we simply get Equation (5.6) again.

Regions $\mathrm{II}^{\pm}$ is slightly more complicated, but not too hard to see. Here we are geometrically looking at the product of a $K3$ surface, two S^1 factors and a copy of $\mathbb{R}^+$. The idea is to find a realisation of Od^3 in terms of the $K3$ surface, one of the S^1s and the $\mathbb{R}^+$, and then combine this with the result above to obtain a realisation of the Shatashvili-Vafa algebra. The result is rather straight forward: we can obtain Od^3 as

$$\mathrm{Od}^3 \hookrightarrow \frac{\mathrm{Od}^2 \times (\mathrm{Free})^2_{S^1\times\mathbb{R}^+}}{\langle N^1, N^2\rangle} \tag{5.8}$$

where Od^2 is the Odake algebra corresponding to the $K3$ surface. We then simply obtain

$$SV \hookrightarrow \frac{\mathrm{Od}^2 \times (\mathrm{Free})^3_{S^1\times\mathbb{R}^+\times S^1}}{\langle N^1, N^2\rangle}. \tag{5.9}$$

The final step of [34] is to check for compatibility of the two realisations at the junctions between regions: $\mathrm{I}^+\cap\mathrm{II}^+$, $\mathrm{II}^+\cap\mathrm{II}^-$ and $\mathrm{I}^-\cap\mathrm{II}^-$. We note that $\mathrm{I}^+\cap\mathrm{II}^+$ and $\mathrm{I}^-\cap\mathrm{II}^-$ will have the same form and so one needs to only check one. This is done by providing maps between the generators for each region, and checking that the ideals are unaffected. We do not present the details here, but they can be found in the reference.

5.1.3 Mirror Automorphism

Before looking at the sigma model for manifolds with G_2 holonomy, and the geometrical statements of mirror symmetry, we first observe mirror symmetry as an automorphism on the Shatashvili-Vafa algebra constructed using the Odake algebra as above.

Besides the antiholomorphic involution automorphism, Equation (5.7), the G_2 algebra contains three other interesting automorphisms $(\mathfrak{M}_{\mathrm{CY}}, T_{S^1}, \mathfrak{M}_{G_2})$, who's actions on the generators are given in Table 5.1. Contrary to the automorphism associated with antiholomorphic involutions, these automorphisms only act on *one side* of the $\mathcal{N}=(1,1)$ algebra, say the right side.

We note that the first automorphism is nothing other than our Calabi-Yau mirror map (Equation (3.55)), and the second is simply T-duality on the boson-fermion pair. The final map is just the composition of these two. Note that the latter acts on the Shatashvili-Vafa

	$T_{\rm CY}$	G^0	J	G^3	A	B	C	D	j	ψ
$\mathfrak{M}_{\rm CY}$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$+$
T_{S^1}	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$
$\mathfrak{M}_{G_2}$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$-$

Table 5.1: *Three automorphisms of the G_2 algebra formed via the product of the Calabi-Yau and circle algebras. The action is written via its action on the generators, with $(T_{CY}, G^0, J, G^3, A, B, C, D)$ corresponding to the Calabi-Yau and (j, ψ) the circle.*

generators trivially, i.e.

$$\mathfrak{M}_{G_2} : (T, G, \Phi, X, K, M) \mapsto (T, G, \Phi, X, K, M), \tag{5.10}$$

whereas the other two do not have easily defined action on these generators, e.g.

$$\mathfrak{M}_{\rm CY} : \Phi = A + (J\psi) \mapsto \Phi' = A - (J\psi). \tag{5.11}$$

Nevertheless, it is straightforward to show that both $\mathfrak{M}_{\rm CY}$ and T_{S^1} are automorphisms of the algebra. This follows simply from the fact that the Calabi-Yau subalgebra does not speak to the boson-fermion subalgebra, i.e. the OPEs between $(T_{\rm CY}, G^0, J, G^3, A, B, C, D)$ and (j, ψ) all vanish. Therefore if our map is an automorphism of the subalgebras, it must be an automorphism of the full algebra.

We note that the automorphism given in [77] takes the form

$$\mathfrak{M}_{GK} : (T, G, \Phi, X, K, M) \mapsto (T, G, -\Phi, X, -K, M). \tag{5.12}$$

As pointed out in [34], this automorphism is related to $\mathfrak{M}_{\rm CY}$ and T_{S^1} via a phase rotation on the Calabi-Yau generators:

$$Ph^{\pi} : (T_{\rm CY}, G^0, J, G^3, A, B, C, D) \mapsto (T_{\rm CY}, G^0, J, G^3, -A, -B, -C, -D). \tag{5.13}$$

We introduce these here as they will be important when discussing mirror symmetry in the TCS construction.

Mirror Involutions

We are now in a position to state an important observation: all three mirror automorphisms commute with the antiholomorphic involution automorphism. It follows from this that for every antiholomorphic involution on the original theory we obtain an antiholomorphic involution in the mirror theory. In other words, the mirror of antiholomorphic involutions are again antiholomorphic involutions.

We immediately note that this statement has been made simply at the level of the generators of the algebra. However, as we have said, at this level we are blind to *which* antiholomorphic involution we are doing. This situation is akin to the story of mirror symmetry in Calabi-Yau manifolds: one can note an automorphism of the algebra, but the story really becomes interesting once we have methods to dig deeper, in particular the GLSM and Gepner models.

TCS Construction

Before we look at the problem from the GLSM perspective, we quickly discuss the mirror map for the SCFT of the TCS construction. This was considered in [34] and gives a beautiful result. Recall that our TCS algebra can is decomposed into four regions, just as the geometry in the TCS construction is: regions $I^{\pm}$ and $II^{\pm}$. The idea is to start with a given G_2 mirror involution in region I^+ and then trace it through the gluing procedure and check that it defines a consistent mirror involution on the algebra in each region. In doing this they observe two different mirror maps. We summarise their results in Tables 5.2 and 5.3.

Region	I^+	II^+	II^-	I^-
Automorphism decomposition	$\mathfrak{M}_{CY} \circ T_\xi$	$\mathfrak{M}_S \circ Ph^\pi_S \circ T_\theta \circ T_\xi$	$Ph^\pi_S \circ \mathfrak{M}_S \circ T_\xi \circ T_\theta$	$\mathfrak{M}_{CY} \circ T_\theta$
SV automorphism	id	id	id	id

Table 5.2: *TCS mirror automorphism corresponding to Equation* (4.42). *That is, it is associated with performing* 4 *T-dualities along a coassociative* T^4 *fibration, with both* $S^1_{e,\pm}$ *being dualised.*

Region	I^+	II^+	II^-	I^-
Automorphism decomposition	$\mathfrak{M}_{CY} \circ Ph^\pi$	$\mathfrak{M}_S \circ T_\theta$	$Ph^\pi_S \circ T_\theta$	$Ph^\pi \circ T_\theta$
SV automorphism	$\mathfrak{M}_{GK}$	$\mathfrak{M}_{GK}$	$\mathfrak{M}_{GK}$	$\mathfrak{M}_{GK}$

Table 5.3: *TCS mirror automorphism corresponding to Equation* (4.43). *That is, it is associated with performing* 3 *T-dualities along an associative* T^3 *fibration, with only* $S^1_{e,-}$ *being dualised. We could also perform this same mirror map but with* $+ \leftrightarrow -$ *exchanged, which gives the other mirror map of Equation* (4.43).

The key thing to note is that the mirror map of Table 5.2 corresponds to mirroring *both* building blocks (as well as T-dualising the external circle), whereas the map of Table 5.3 only mirrors the Z_+ building block (without T-dualising $S^1_{e,+}$) and leaves Z_- unmirrored (but T-dualises $S^1_{e,-}$). Of course this latter type could also be done the other way around: mirror Z_- and leave Z_+ unmirrored. We therefore have 3 different mirroring options, corresponding exactly to Equation (4.40).

5.2 G_2 Sigma Models

We recall from our Calabi-Yau discussion that mirror symmetry is most powerfully understood from the sigma model perspective. In particular, the sigma model encapsulated the notions of mirror symmetry not only for Gepner models (which are a particular limit of the sigma model) but also reproduced the results of Greene-Plesser and also could be related to Batyrevs mirror maps for toric hypersurfaces.

For this reason, we now want to look at the sigma model for a manifold with G_2 holonomy and ask about what we can say about mirror symmetry in this context. We recall that the question we really want to probe is whether the following geometrical mirror proposal makes sense:

$$(\mathcal{M}_\sigma)^\vee = \left(\frac{\mathcal{M}_{CY} \times S^1}{(\sigma,-1)}\right)^\vee = \frac{\mathcal{M}_{CY}^\vee \times (S^1)^\vee}{\tau}, \tag{5.14}$$

where τ is an involution that gives rise to a quotient with equivalent $b^2 + b^3$. In particular we want to know whether the physics tells us whether such a τ exists, if it is unique and if τ should be an antiholomorphic involution on the mirror $\mathcal{M}_{CY}^\vee$.

The first thing we will need is the sigma model of the product of a Calabi-Yau and a circle. The key thing here is that this is a metric product, i.e. the metric for the theory is block diagonal

$$g_{CY\times S^1} = g_{CY} \oplus g_{S^1}. \tag{5.15}$$

From the sigma model perspective, this means that our sigma model splits into the sum of the Calabi-Yau and the circle models. The latter is simply the theory of a boson fermion pair, and the former we have discussed in detail above. It is known that the sigma model for a manifold with G_2 holonomy is a $(1,1)$ theory, but here we have a sum of a $(2,2)$ theory (the Calabi-Yau) and a $(1,1)$ theory (the circle). We generate the G_2 sigma model by taking the quotient of these sigma models by the antiholomorphic involution, we therefore need to know how the involution acts on these two models.

The S^1 sigma model is straight forward and it can be shown that the inversion simply acts as sign conjugation of the boson and fermion, i.e. if j is the boson and ψ the fermion, then the involution acts as $(j,\psi) \mapsto (-j,-\psi)$. The involution for the Calabi-Yau sigma model needs a little bit more thought, and we shall do this in Section 5.2.2 below. As a stepping stone, we first consider the limit of the GLSM corresponding to a G_2 Gepner model and study how the involution and mirror map work in this context.

5.2.1 G_2 Gepner models

The construction of G_2 Gepner models have been studied in [30–32]. Under Gepner's construction the full SCFT (in light-cone gauge) is given by a Gepner model and an $\mathfrak{so}(2)_1$ affine Lie algebra, which gives the two fermions in the non-compact directions. It can be shown using simple current arguments (see [59] for a review), that the NS vs R sectors of the two parts must agree, i.e. if we have a NS state in our Gepner model, we must take a NS state from our $\mathfrak{so}(2)_1$ model. Similarly we can show that the overall $U(1)$ charge of a state must be an odd integer.

Let's imagine we have a NS state in our Gepner model, and we want to add back in the $\mathfrak{so}(2)_1$ factor. We have two options: O_2 and V_2. These have $(h,q)_O = (0,0)$ and $(h,q)_V = (1/2,1)$. Now, we know that the NS states should have total charge being an odd integer, however we chose our spectral flow such that our Gepner models always had odd integer NS charge, and so we can only couple to the O_2 rep. This is all consistent: if we had taken a state of the form $q = q_{Gep} + q_{\mathfrak{so}(2)_1} = 2+1$, so that we were using the V_2 rep, we could use spectral flow to go to the state with $q_{Gep} = -1$. The spectral flow from NS to R in the $\mathfrak{so}(2)_1$ theory is given by either C_2 or S_2 (depending on which direction you flow). Either way, we are doing this same spectral flow twice (to go NS to R to NS) and so we are using $C_2^2 = S_2^2 = V_2$, which follows from the fusion rules of $\mathfrak{so}(2)_1$. Putting this together with $V_2 \times V_2 = O_2$, we see that our V_2 rep flows to an O_2 rep, as needed. All together, that is

$$(q_{Gep} = 2) \times V_2 \mapsto (q_{Gep} = -1) \times O_2. \tag{5.16}$$

Therefore we can always represent a state in the NS sector as a state in the Gepner model with odd integer charge along with the O_2 rep.

In order to construct our G_2 Gepner model, we need to split the $\mathfrak{so}(2)_1$ factor into two copies of $\mathfrak{so}(1)_1$. In other words, we want to treat the two fermions separatly, as one will remain a non-compact direction, whereas the other will be compactified on our S^1. There are three representations of $\mathfrak{so}(1)_1$: (O_1, V_1, S_1), which have conformal weights $(0, 1/2, 1/16)$, respectively. The thing we notice imediately is that the R representation S_1 has $h = 1/16$, which is *exactly* the conformal weight required in order to take the R ground states of a Calabi-Yau CFT and produce R ground states of a G_2 CFT, i.e. $h_{G_2} = 7/16 = 3/8 + 1/16 =$

$h_{Gep} + h_{S_1}$.

One can form the four reps of $\mathfrak{so}(2)_1$ out of the three reps of $\mathfrak{so}(1)_1$ as follows:

$$\begin{aligned} O_2 &= O_1 O_1 + V_1 V_1 \\ V_2 &= O_1 V_1 + V_1 O_1 \\ S_2 &= S_1 S_1 \\ C_2 &= S_1 S_1 \end{aligned} \tag{5.17}$$

The at-face-value equality of S_2 and C_2 is dealt with via arguments related to fixed points of simple current orbits in the $\mathfrak{so}(1)_1 \times \mathfrak{so}(1)_1$ theory (see, [30] for details). We can use this to write our generic NS $(Gep) \times \mathfrak{so}(2)_1$ state in terms of $\mathfrak{so}(1)_1$ reps, namely

$$\left(h = \frac{|q|}{2}, q \in \{\pm 3, \pm 1\}\right) \otimes \big(O_1 O_1 + V_1 V_1\big), \tag{5.18}$$

and similarly for the right states (i.e. tildes everywhere).

Anti-holomorphic Involution

The G_2 involution maps the $\mathfrak{so}(1)_1$ NS reps via $(O_1, V_1) \mapsto (O_1, -V_1)$.[4] As we have seen, it also maps states in our Gepner models by changing the sign of the $U(1)$ charges. In terms of the tuples (l_i, m_i, s_i) of the minimal model factors, the involution acts as

$$(l_i, m_i, s_i) \mapsto (l_i, -m_i, -s_i) \tag{5.19}$$

on the states in the highest weight representation. This actually gives the 'vanilla' involution (i.e. simply complex conjugation), but we can easily generalise this to involutions that swap homogeneous coordinates that have the same weight. At the Gepner level, this would be a map that swaps two minimal model factors that have the same level.

Using the general change of sign argument, we see that the states that existed in our Calabi-Yau Gepner model will split into one even and one odd piece under the involution. Namely, working in a basis of states with definite charge, our states are paired in their charge conjugates. We form the even and odd combinations in these pairs: the even ones couple with $O_1 O_1$ and survive while the odd ones couple with $V_1 V_1$ and survive.

Using the equivalence between the charges of the states in a Gepner model and the number of differential forms, along with identifying the presence of V_1 as wedging with $d\theta$ (the differ-

[4] It's action to S_1 is less easily written, but it acts via $S_2 \leftrightarrow C_2$.

ential form on the S^1), the above reproduces the geometrical argument that the differential forms that survive give Betti numbers

$$b^0 = b^7 = 1 \qquad b^2 = h^{1,1}_+ \quad \text{and} \quad b^3 = h^{1,1}_- + h^{2,1} + 1. \tag{5.20}$$

For clarity, $b^0 = 1$ corresponds to the $(0,0)$ form while $b^7 = 1$ to the $(3,3) \wedge d\theta$. The b^2 come from the even $(1,1)$ forms, $(1,1)_{\text{even}}$. Finally, b^3 forms come from $\{(3,0)+(0,3), (2,1)+(1,2), (1,1)_{\text{odd}} \wedge d\theta\}$. These are only the Betti numbers corresponding to the untwisted states under the involution σ as we have ignored the twisted sectors. Geometrically, this is the statement that we have the cohomology of $\mathcal{M}_\sigma$ but not of the resolved $\mathcal{M}_{G_2}$.

Mirror Involution

Next we want to look at the action of the mirror map on this construction and demonstrate that it gives rise to a mirror anti-holomorphic involution. Given the above arguments, this is straightforward; the key thing is that both maps act as a reversal of charges and commute. If we denote the mirror minimal model tuples as

$$(l_i, m_i, s_i) \mapsto (l_i^\vee, m_i^\vee, s_i^\vee), \tag{5.21}$$

then the charges of the mirror states are given in terms of $(m_i^\vee, s_i^\vee)$. As the mirror involution acts as a change of sign, it must act as

$$(l_i^\vee, m_i^\vee, s_i^\vee) \mapsto (l_i^\vee, -m_i^\vee, -s_i^\vee), \tag{5.22}$$

which is exactly equivalent to Equation (5.19). This tells us that the mirror involution has the same geometrical interpretation, namely it is an anti-holomorphic involution.

Note that it would be dangerous at this point to assume that states of definite charge have a one-to-one correspondence with differential forms by using Equation (3.53). Consider a state corresponding to a $(2,1)$-form in the original Gepner model: under the involution, this state is mapped to a state who's corresponding form is of Hodge type $(1,2)$. Acting with the mirror map on both of these states we find a $(1,1)$-form and a $(2,2)$-form. This now seems to imply that the involution σ on the mirror side has to map a $(1,1)$-form to a $(2,2)$-form, which cannot be achieved by an anti-holomorphic involution. One way to see the overly strong assumption in this argument is to observe that eigenstates of the charge operators do not need to correspond to forms of fixed degree. This can be made very explicit in orbifold models and we have treated one example in detail in Appendix A.

As we have said we expect our G_2 to have three different mirrors. In the quotient construction, geometrically these three mirrors correspond to: (i) mirroring the Calabi-Yau but leaving the circle factor alone, (ii) leaving the Calabi-Yau alone and doing T-duality on the circle, and (iii) doing both Calabi-Yau mirror and T-duality on the circle. For our Gepner model here we have only obtained one mirror map, corresponding to case (i). By studying the interplay of T-duality and the action of the involution on the $\mathfrak{so}(1)_1$ factor, one should be able to obtain similar results for the other two mirror maps. We do not do this calculation here, but claim that this construction exists, and provide evidence of this below.

5.2.2 G_2 GLSM

As we have already said, given a Calabi-Yau threefold $\mathcal{M}_{CY}$, a sigma model on the metric product $\mathcal{M}_{CY} \times S^1$ splits into the sum of a $(2,2)$ theory (the Calabi-Yau) and a $(1,1)$ theory (the circle). As manifolds with G_2 holonomy are not Kähler, their sigma model is $(1,1)$ theory, and we obtain this from the above by an quotienting by an involution. We are interested in how these G_2 involutions act on mirror pairs.

Anti-holomorphic Involution

The key observation which allows us to immediately write down anti-holomorphic involutions for GLSMs is that the chiral superfields Φ_i are identified with the homogeneous coordinates of the toric Calabi-Yau ambient space. Therefore the anti-holomorphic involution acts on these chiral superfields in *exactly* the same way that it acts on the coordinates. In particular the vanilla involution simply maps each chiral superfield to its anti-chiral partner. As the anti-holomorphic involution furthermore needs to map $G_\pm \mapsto G_\mp$ it follows that $\theta_\pm \mapsto \theta_\mp$ which implies that also the twisted chiral superfields Σ_a are sent to their complex conjugates.

The vanilla anti-holomorphic involution hence acts as

$$\sigma_v : \qquad \begin{aligned} \Phi_i &\mapsto \bar{\Phi}_i \\ \Sigma_a &\mapsto \bar{\Sigma}_a \end{aligned} \tag{5.23}$$

To have a symmetry of the GLSM, and not just the fields, we also need to make sure the anti-holomorphic involution is a symmetry of the action, Equation (3.56). This means that we need to restrict the complex parameters in the superpotential such that

$$W(\overline{\Phi}_i) = \overline{W(\Phi_i)}\,. \tag{5.24}$$

and furthermore we have to take the parameters t_a to be real.

We note that this condition requires that the coefficients in superpotential be real, but does not place any further constraints on them. We do not discuss this in detail here but simply note the following. The topology of the fixed point locus of the involution is, of course, dependent on what we pick. For example, if we consider the Calabi-Yau defined inside $\mathbb{WCP}^4_{1,1,1,1,4}$ then the defining equation (i.e. the superpotential) takes the form

$$\alpha_1\Phi_1^8 + \alpha_2\Phi_2^8 + \alpha_3\Phi_3^8 + \alpha_3\Phi_3^8 + \alpha_4\Phi_4^8 + \alpha_5\Phi_5^2 = 0, \tag{5.25}$$

where $\alpha_i \in \mathbb{C}$. By an appropriate choice of anti-holomorphic involution, the real equation is then given by

$$\pm\xi_1^8 \pm \xi_2^8 \pm \xi_3^8 \pm \xi_4^8 \pm \xi_5^2 = 0, \tag{5.26}$$

where ξ_i are real coordinates. The fixed point locus clearly depends on the choice of sign, in particular if we take all positive signs then it must be empty!

As the fixed point locus is dependent on the choice, it follows that the twisted sectors in the SCFT depend on the choice. In this thesis we do not study the twisted sectors, and so we do not expand further on this point, but simply point the interested reader to [33].

Mirror Involution

We can now trace this through the dualization procedure of [13] to find the action of σ on the mirror. Dualizing we find that

$$\sigma_v : \quad \begin{array}{rcrcl} \mathrm{Re}(Y_i) = & \bar{\Phi}_i e^{2Q_iV}\Phi_i \to & \bar{\Phi}_i e^{2Q_iV}\Phi_i & = & \mathrm{Re}(Y_i) \\ \mathrm{Im}(Y_i) = & \vartheta_i \to & -\vartheta_i & = & -\mathrm{Im}(Y_i) \end{array} \tag{5.27}$$

by using Equations (3.121) and (3.122).

For the case of weighted projective spaces we can directly track this action through to an action on the fields $\Phi_i^\vee$: We have $\Phi_i^\vee = X_i^{Q_i/H}$ with $X_i = e^{-Y_i/H}$, and so

$$\sigma_v : \Phi_i^\vee \mapsto \overline{\Phi}_i^\vee, \tag{5.28}$$

which is simply the vanilla anti-holomorphic involution on the mirror side again.

In the more general case of toric hypersurfaces (i.e. $h \le h^\vee$) we have that the fields in the dual theory are

$$\widetilde{P} = \prod_{\ell=1}^{h^\vee} \Phi_\ell^\vee \quad \text{and} \quad X_i = e^{-t_i} \prod_{\ell=1}^{h^\vee} \left(\Phi_\ell^\vee\right)^{\langle m_\ell, n_i\rangle + 1}. \tag{5.29}$$

where X_i are dual variable for the Φ_i and $\widetilde{P}$ is the dual variable of P. We can now simply change the roles of what is considered the starting point and what is considered the mirror. That is, we think of the vanilla involution σ_v as defined on $\mathcal{M}_{CY}^\vee$ instead of $\mathcal{M}_{CY}$, where it acts as $\sigma_v^\vee : \Phi_i^\vee \to \bar{\Phi}_i^\vee$. This then implies immediately that

$$\sigma_v : \qquad \begin{aligned} X_i &\mapsto \bar{X}_i \\ \widetilde{P} &\mapsto \overline{\widetilde{P}} \end{aligned} \tag{5.30}$$

and hence

$$\sigma_v : \qquad \begin{aligned} \Phi_i &\mapsto \bar{\Phi}_i \\ P &\mapsto \overline{P} \end{aligned} \tag{5.31}$$

We finally use that in the mirror theory the superpotential again obeys Equation (5.24), and hence recover the result that the vanilla involution in the GLSM is mapped to an involution of the same type for its mirror.

As the number $h^\vee$ of dual fields $\Phi_\ell^\vee$ can be larger than the number h of fields Φ_i, we cannot in general solve the above equations for $\Phi_\ell^\vee$ to directly show that complex conjugation of the Φ_i implies complex conjugation (and nothing else) of the $\Phi_\ell^\vee$. This does not prevent us from associating σ_v with $\sigma_v^\vee$. The action of an involution on an isomorphic theory must be unique up to automorphism, so that any freedom to associate σ_v with a different involution implies that this simply gives the vanilla involution in disguise.

A similar argument holds for the action of the involution of the circle part of the sigma model, where the action of T-duality identifies it with another involution inverting the coordinate on the circle. In summary, we hence have four isomorphic tuples

$$\begin{aligned} &\big(\mathcal{M}_{CY} \times S^1, (\sigma, -)\big) \\ \cong &\big(\mathcal{M}_{CY}^\vee \times (S^1)^\vee, (\sigma^\vee, -)\big) \\ \cong &\big(\mathcal{M}_{CY}^\vee \times (S^1), (\sigma^\vee, -)\big) \\ \cong &\big(\mathcal{M}_{CY} \times (S^1)^\vee, (\sigma, -)\big) \end{aligned} \tag{5.32}$$

As a Gepner model is a particular limit of the GLSM, our proof of the existence of the three mirror GLSMs also provides a proof of our claim above that there are three mirror maps for the G_2 Gepner model.

Even though we have focused the discussion on the vanilla involution σ_v, which always exists, it is clear that analogous results can be obtained for any other anti-holomorphic involution σ. By following through the same analysis investigating the dualisation procedure in the GLSM, such an involution σ will also have a mirror $\sigma^\vee$ which acts geometrically on $\mathcal{M}_{CY}^\vee$. An upshot of this realisation is that the set of anti-holomorphic involutions on $\mathcal{M}_{CY}$ is isomorphic to the set of anti-holomorphic involutions of $\mathcal{M}_{CY}^\vee$. We expect that this can be made precise for toric hypersurfaces by relating σ to automorphisms of Δ°.

5.2.3 G_2 Mirrors

Above, we have shown the equivalence of the vanilla anti-holomorphic involutions in the dual descriptions found after mirror symmetry and/or T-duality for both Gepner models and GLSMs. Of course, merely specifying the tuple $\left(\mathcal{M}_{CY} \times S^1, (\sigma, -)\right)$ does not yet define a G_2 model as we need to include an appropriate twisted sector, which is in general not unique.

For a given choice of superpotential obeying (5.24), and a choice of real parameters t_a, the mirror map identifies a corresponding superpotential and FI parameters on the mirror. This in particular means that the fixed loci L_σ and $L_{\sigma^\vee}$ are completely determined. However, there will in general be several inequivalent (partial) smoothings of the orbifold singularities of $\left(\mathcal{M}_{CY} \times S^1\right)/(\sigma_v, -)$ by choosing different bundles $\mathcal{Z}$ in the construction of [69]. In a TCS description, this freedom will appear as the freedom to resolve or deform the building block $\Upsilon_{r,a,\delta}$.

Given a point in the moduli space of the worldsheet SCFT of type II strings on $\mathcal{M}_{CY} \times S^1$, our analysis hence implies that pairwise isomorphic worldsheet CFTs must exists among the four isomorphic sets

$$\begin{aligned}
&\left\{(\mathcal{M}_{CY} \times S^1, (\sigma_v, -), L_\sigma^i)\right\} \\
\cong &\left\{(\mathcal{M}_{CY}^\vee \times (S^1)^\vee, (\sigma_v^\vee, -), L_\sigma^{\vee i})\right\} \\
\cong &\left\{(\mathcal{M}_{CY}^\vee \times S^1, (\sigma_v^\vee, -), L_\sigma^{\wedge - i})\right\} \\
\cong &\left\{(\mathcal{M}_{CY} \times (S^1)^\vee, (\sigma_v, -), L_\sigma^{\wedge + i})\right\}
\end{aligned} \tag{5.33}$$

where we have denoted different twisted sectors by $L_\sigma^i, L_\sigma^{\vee i}, L_\sigma^{\wedge - i}, L_\sigma^{\wedge + i}$.

It is beyond the scope of this thesis to investigate these sets and the precise identification between their elements. For specific models, some results can be found in [30, 32, 33].

It is intriguing to compare what we have found here with the mirror maps that were proposed in [26, 27] for twisted connected sum G_2 manifolds. For the three mirror maps found using the GLSM description, there are obvious candidates for a corresponding geometrical construction as a TCS, as indicated by the notation used (c.f. Equations (4.42) and (4.43)). Making this precise requires an in-depth analysis of twisted sectors, and an identification of the twisted sectors in the GLSM with different TCS realisations.

5.3 Example

We end with an example that collects and demonstrates the ideas presented in this thesis. The first thing we need is a $K3$ fibred Calabi-Yau threefold. Here we will consider the example of a Weierstrass elliptic fibration over a Hirzebruch surface. Locally this is a fibration of $\mathbb{WCP}^2_{321}$ over F_n. The weight system is given by

	y	x	w	z_1	z_2	z_3	z_4	P
$H_1 \rightarrow$	3	2	1	0	0	0	0	6
$H_2 \rightarrow$	$6+3n$	$4+2n$	0	1	n	1	0	$12+6n$
$H_3 \rightarrow$	6	4	0	0	1	0	1	12

where $[y : x : w]$ are the homogeneous coordinates of $\mathbb{WCP}^2_{321}$ and $[z_1 : z_2 : z_3 : z_4]$ are the coordinates of F_n. Recalling the weight systems for $\mathbb{WCP}^2_{321}$ and F_n, Equation (2.122), we can see the the fibration structure: the weights of y and x under H_2 and H_3 are what make this fibration non-trivial. A Weiserstrass elliptic fibration is given by the defining equation

$$y^2 = x^3 + f(z)xw^4 + g(z)w^6, \tag{5.34}$$

from which it follows that $f(z)$ has weights $(0, 8+4n, 8)$ and $g(z)$ has weights $(0, 12+6n, 12)$, where the three entries correspond to going down the rows in the weight diagram.

For simplicity we consider $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, and denote the coordinates for the two $\mathbb{CP}^1$s as $[u_1 : u_2]$ and $[z_1 : z_2]$. The defining equation for $\mathcal{M}_{CY}$ is then given by

$$y^2 = x^3 + f_{8,8}(u,z)xw^4 + g_{12,12}(u,z)w^6, \tag{5.35}$$

where the subscripts denote the degrees for (u, z). We shall take the base $\mathbb{CP}^1_b = \mathbb{CP}^1_{[z_0:z_1]}$. The fibration of $\mathbb{WCP}^2_{321}$ over $\mathbb{CP}^1_{[u_1:u_2]}$ is then a $K3$ surface, so in total we have a $K3$ fibration over $\mathbb{CP}^1_b$, as required.

Involution & Fixed Locus

We now want to look at how the antiholomorphic involution acts on $\mathcal{M}_{CY}$ and in particular look at the fixed point locus. As always we focus on the vanilla involution

$$\sigma : (y, x, w, u_1, u_2, z_1, z_2) \mapsto (\bar{y}, \bar{x}, \bar{w}, \bar{u}_1, \bar{u}_2, \bar{z}_1, \bar{z}_2). \tag{5.36}$$

As we have seen before, this action fixes a circle on each of the $\mathbb{CP}^1$ factors, so we have a fixed $T^2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. All that is left to do is look at how it acts on the elliptic fibration over $\mathbb{CP}^1_u \times \mathbb{CP}^1_z$. As explained in [90][5] (see also [73]), the fixed points of σ depend on the specific choice of $f_{8,8}(u,z)$ and $g_{12,12}(u,z)$. In particular one has to consider the discriminant locus $\Delta(u,z) = -(4f^3 + 27g^2)$. The values of (u,z) that give vanishing discriminant locus correspond to singular fibres. If we pick f and g such that these zeros do not lie in the fixed T^2 the fibration is trivial over this T^2, and then the fixed points locus of σ in the fibres is a disjoint union of two S^1s. So in total, the fixed point locus is $L_\sigma = T^3 \cup T^3$. As $L_\sigma|_{\mathbb{CP}^1_b} = S^1$, we thus have $L_{S_0} = T^2 \cup T^2$.

This fixed point locus contributes to the Betti numbers of the resolved G_2 as discussed above. In particular we have

$$b^0(L_\sigma) = 2 \quad \text{and} \quad b^1(L_\sigma) = 6. \tag{5.37}$$

Polytopes

Using the weight system above, we obtain the following generating vectors of the corresponding fan

$$\begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 3 & 0 & -1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \tag{5.38}$$

We immediately note that $\nu_w = (2,3,0,0)$ does not appear above, this is because it lies in the face connecting ν_{u_1} and ν_{u_2} (as well as for ν_{z_1} and ν_{z_2}). This is exactly the statement that we have a fibration structure: our fan is formed by taking the fan for $\mathbb{WCP}^2_{321}$ and adding generates above and below a set vertex.

[5] Note in this paper the base is a $\mathbb{CP}^2$ rather than $\mathbb{CP}^1 \times \mathbb{CP}^1$, but the logic is the same.

The dual vectors can then be computed as

$$\begin{pmatrix} \nu_0^* \\ \nu_1^* \\ \nu_2^* \\ \nu_3^* \\ \nu_4^* \\ \nu_5^* \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 1 & 6 & -6 \\ 1 & 1 & 6 & 6 \\ 1 & 1 & -6 & -6 \\ 1 & 1 & -6 & 6 \end{pmatrix}, \tag{5.39}$$

and our pair of reflexive polytopes (Δ, Δ°) are then given by the convex hull of the $\{\nu_i^*\}$ and $\{\nu_i\}$ vectors, respectively. We can use these polytopes to construct the Calabi-Yau threefold $\mathcal{M}_{\Delta,\Delta^\circ}$ and compute its Hodge numbers using the Batyrev result, Equation (2.161), to give $(h^{1,1}(\mathcal{M}_{\Delta,\Delta^\circ}), h^{2,1}(\mathcal{M}_{\Delta,\Delta^\circ})) = (3, 243)$.

As a consistency check, we can check that we can reproduce the Hodge number $h^{2,1} = 243$ by looking at the number of allowed monomials in the defining equation, Equation (5.35). These are determined by the allowed terms in $f_{8,8}(u,z)$ and $g_{12,12}(u,z)$. As both $[u_1 : u_2]$ and $[z_1 : z_2]$ are $\mathbb{CP}^1$s and they have the same degrees in f we can just work out the number of allowed monomials in $[u_1 : u_2]$ and then square that number to account for the $[z_1 : z_2]$. The same argument holds for g. For $f_8(u)$ we are thus looking at the number of degree 8 monomials in two variables. Recalling that the number of independent degree d monomials in $(n+1)$ variables is given by the binomial coefficient $\begin{pmatrix} d+n \\ n \end{pmatrix}$, this is simply $\begin{pmatrix} 9 \\ 1 \end{pmatrix} = 9$. So $f_{8,8}(u,z)$ has $9^2 = 81$ possible monomials. We similarly get that $g_{12,12}(u,z)$ has $13^2 = 169$ possible monomials. This gives a total of 250, which is bigger then the required 243. However, we need to account for the automorphisms and scalings. Each $\mathbb{CP}^1$ has an $SL(2,\mathbb{C})$ automoprhism group, which removes $3 + 3 = 6$, and the final one is removed by an overall scaling of Equation (5.35). So in total we have $h^{2,1} = 250 - 6 - 1 = 243$.

We now recall that the Betti numbers for the resolved $\mathcal{M}_{G_2}$ are given by

$$b^2(\mathcal{M}_{G_2}) = b^2_+(\mathcal{M}_\sigma) + 2b^0(L_\sigma) \quad \text{and} \quad b^3(\mathcal{M}_{G_2}) = b^2_-(\mathcal{M}_\sigma) + b^3_+(\mathcal{M}_\sigma) + 2b^1(L_\sigma). \tag{5.40}$$

Using the fact that σ is odd on all the $(1,1)$-forms on $\mathcal{M}_{CY}$, i.e. $b^2_+(\mathcal{M}_\sigma) = 0$ and $b^2_-(\mathcal{M}_\sigma) = 3$, and that $b^3_+(\mathcal{M}_\sigma) = h^{2,1}(\mathcal{M}_{CY}) + 1$, along with Equation (5.37), we get

$$\begin{aligned} b^2(\mathcal{M}_{G_2}) &= 0 + (2 \times 2) = 4 \\ b^3(\mathcal{M}_{G_2}) &= 3 + (243 + 1) + (2 \times 6) = 259 \end{aligned} \tag{5.41}$$

so that $b^2(\mathcal{M}_{G_2}) + b^3(\mathcal{M}_{G_2}) = 263$.

TCS

To see the TCS construction we need to find the building blocks $Z_\pm$. These are given in [73], and we summarise the result here.

We recall that

$$Z_+ = \Upsilon_{(r,a,\delta)} \tag{5.42}$$

where (r, a, δ) specify the Nikulin involution. This in turn is specified by L_{S_0}, which here is two disjoint T^2. This is given by the one of the exceptional cases and has $(r, a, \delta) = (10, 8, 0)$ and has corresponding $(f, g) = (2, 2)$. The relevant topological data of this building block is

$$|K_+| = 2f = 4 \quad \text{and} \quad h^{2,1}(Z_+) = 2g = 4. \tag{5.43}$$

For Z_- we need a $K3$ fibration over $\mathbb{CP}^1$ such that the first Chern class $c_1(Z_-)$ is given by the Poincaré dual of the homology class $[S_0] = [z_1] = [z_2]$. The defining equation is thus given by

$$y^2 = x^3 + f_{8,4}(u,z)xw^4 + g_{12,6}(u,z)w^6, \tag{5.44}$$

which is the same as for $\mathcal{M}_{CY}$ apart from the order of $[z_1 : z_2]$ in $f(u, z)$ and $g(u, z)$. The tops $\Diamond^\circ$ and $\Diamond$ then have vertices

$$\Diamond^\circ \sim \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 3 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Diamond \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 1 & 6 & -6 \\ 1 & 1 & 6 & 6 \\ 1 & 1 & -6 & 6 \end{pmatrix} \tag{5.45}$$

which we note are related to the polytope vertices above simply by removing $(2, 3, 0, -1)$ and $(1, 1, -6, -6)$, respectively.

The building block has $|K_-| = 0$, which we can plug into

$$h^{2,1}(Z_-) = \frac{1}{2}\left(h^{2,1}(\mathcal{M}_{CY}) + h^{1,1}(\mathcal{M}_{CY})\right) - 11 - |K_-| \tag{5.46}$$

which gives $h^{2,1}(Z_-) = 112$.

Of course these numbers can then be used in Equation (4.38) to reproduce $b^2(\mathcal{M}_{G_2}) + b^3(\mathcal{M}_{G_2}) = 263$, as per the derivation of Equation (4.67).

5.3.1 Mirror

We can now form the mirror threefold by swapping the roles of Δ and Δ°. We note immediately that we again have an elliptic fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1$: the $\mathbb{CP}^1$s are given by (ν_2^*, ν_3^*) and (ν_4^*, ν_5^*). We then note that we have the equivalent of ν_w above as $\nu_w^* = (1,1,0,0)$, and further the linear relation between $\nu_0^*, \nu_1^*, \nu_w^*$ is

$$3\nu_0^* + 2\nu_1^* + \nu_w^* = 0,$$

which is exactly the relation for $\mathbb{CP}^2_{321}$. We are thus dealing with a mirror who's defining equation is again of the form

$$\widetilde{y}^2 = \widetilde{x}^3 + \widetilde{f}_{8,8}(\widetilde{u},\widetilde{z})\widetilde{x}\widetilde{w}^4 + \widetilde{g}_{12,12}(\widetilde{u},\widetilde{z})\widetilde{w}^6. \tag{5.47}$$

where we have denoted the mirror coordinates with a tilde rather than a $\vee$ in order to lighten notation slightly. This is consistent with the general GLSM argument, which said that the mirror superpotential takes the same form. The key difference is that the mirror superpotential has a different quotient group, which restricts the allowed terms in the deformations of the defining equation. Here this will restrict the allowed terms in $\widetilde{f}_{8,8}$ and $\widetilde{g}_{12,12}$.

We can calculate this group using the polytopes (Δ, Δ°). A simple calculation[6] shows that the quotient group is $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ and that there is only one allowed term for $\widetilde{f}_{8,8}$,

$$\widetilde{u}_1^4\widetilde{u}_2^4\widetilde{z}_1^4\widetilde{z}_2^4 \tag{5.48}$$

and five terms for $\widetilde{g}_{12,12}$,

$$\widetilde{u}_1^{12}\widetilde{z}_1^{12}, \quad \widetilde{u}_2^{12}\widetilde{z}_1^{12}, \quad \widetilde{u}_1^{12}\widetilde{z}_2^{12}, \quad \widetilde{u}_2^{12}\widetilde{z}_2^{12}, \quad \text{and} \quad \widetilde{u}_1^6\widetilde{u}_2^6\widetilde{z}_1^6\widetilde{z}_2^6. \tag{5.49}$$

The first four of these are clearly related to each other by coordinate transformations and so only give one independent choice. We therefore get three possible terms, which corresponds exactly to $h^{2,1}(\mathcal{M}_{CY}^\vee) = h^{1,1}(\mathcal{M}_{CY}) = 3$.

Involution & Fixed Point Locus

We now want to look at the mirror involution and check it gives rise to an isomorphic G_2-manifold. The key thing is that our defining equation takes the same form and that, as per

[6] The idea is to consider the lattice spanned by the generating vectors $\{\nu_0^*, ..., \nu_5^*\}$ of Δ and take the quotient of the $\mathbb{Z}_4$ lattice by this sublattice. From here you can compute the quotient group. This was done using the Sage programming software.

our results, we are again considering the vanilla involution. We again get the same fixed point locus, as we are still fixing a T^2 in $\mathbb{CP}^1_u \times \mathbb{CP}^1_z$, and the zeros of the discriminant locus of the elliptic curve will still lie outside this fixed T^2. We therefore obtain $T^3 \cup T^3$ again and

$$b^0(L_\sigma^\vee) = 2 = b^0(L_\sigma) \quad \text{and} \quad b^1(L_\sigma^\vee) = 6 = b^1(L_\sigma). \tag{5.50}$$

We note that the invariance of the fixed point locus under the mirror map tells us that the $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ quotient group must act as an automorphism on the fixed point locus.

Mirror $\mathcal{M}_{G_2}$

We can now compute the Betti numbers for the mirror $\mathcal{M}_{G_2}^\vee$ and check that it obeys the Shatashvili-Vafa condition, i.e. $b^2 + b^3$ conserved. This is easily shown to be true:

$$\begin{aligned} b^2(\mathcal{M}_{G_2}^\vee) + b^3(\mathcal{M}_{G_2}^\vee) &= b^2(\mathcal{M}_\sigma^\vee) + b^3_+(\mathcal{M}_\sigma^\vee) + 2b^0(L_\sigma^\vee) + 2b^1(L_\sigma^\vee) \\ &= h^{1,1}(\mathcal{M}_\sigma^\vee) + h^{2,1}(\mathcal{M}_\sigma^\vee) + 1 + 2b^0(L_\sigma^\vee) + 2b^1(L_\sigma^\vee) \\ &= h^{2,1}(\mathcal{M}_\sigma) + h^{1,1}(\mathcal{M}_\sigma^\vee) + 1 + 2b^0(L_\sigma) + 2b^1(L_\sigma) \\ &= b^2(\mathcal{M}_{G_2}) + b^3(\mathcal{M}_{G_2}), \end{aligned} \tag{5.51}$$

and so these constitute mirror manifolds with G_2 holonomy. However we note that we really know that these are physically mirror, as we have demonstrated that they come from mirror GLSMs, and so come from the same SCFT.

TCS

We can confirm that this is indeed a topologically different $\mathcal{M}_{G_2}$ by computing the individual Betti number $b^2(\mathcal{M}_{G_2}^\vee)$ and $b^3(\mathcal{M}_{G_2}^\vee)$. This is easiest done using the TCS decomposition. We recall that the mirror is given by swapping the roles played by $\Diamond^\circ$ and $\Diamond$ and that this, in particular, results in Equation (4.39). The important point for us is that

$$|K_-^\vee| = h^{2,1}(Z_-) = 112. \tag{5.52}$$

Putting this, together with $N_- = 18$ we can use Equation (4.62) to obtain

$$h_-^{1,1}\left(\mathcal{M}_{CY}^\vee\right) = 112 + 18 + 1 = 131. \tag{5.53}$$

From this, and $h^{1,1}(\mathcal{M}_{CY}^{\vee}) = h^{2,1}(\mathcal{M}_{CY}) = 243$ we get

$$h_{+}^{1,1}\left(\mathcal{M}_{CY}^{\vee}\right) = 243 - 131 = 112. \tag{5.54}$$

Note that this tells us that our $K3$ fibred Calabi-Yau contains 224 reducible fibres. These fibres are swapped pairwise under the involution and taking even combinations is what gives rise to the above result.

We finally use

$$b_{\pm}^{2}(\mathcal{M}_{\sigma}^{\vee}) = h_{\pm}^{1,1}(\mathcal{M}_{CY}^{\vee}) \quad \text{and} \quad b_{+}^{3}(\mathcal{M}_{\sigma}^{\vee}) = h^{2,1}(\mathcal{M}_{CY}^{\vee}) + 1 \tag{5.55}$$

along with Equation (5.40) to obtain

$$\begin{aligned} b^{2}(\mathcal{M}_{G_2}^{\vee}) &= 112 + 4 = 116 \\ b^{3}(\mathcal{M}_{G_2}^{\vee}) &= 131 + 4 + 12 = 147 \end{aligned} \tag{5.56}$$

where we have also made use of Equation (5.50).

So we see that the individual Betti numbers are different, but importantly that the sum is invariant. This confirms that two topologically different manifolds with G_2 holonomy have isomorphic SCFTs and so constitute a mirror pair.

6 Conclusion & Discussion

This thesis presents a detailed review of mirror symmetry for Calabi-Yau manifolds and uses it to strengthen the understanding of mirror symmetry for G_2 manifolds, by providing worldsheet SCFT arguments, via the GLSM.

In order to be as self contained as possible, we have provided a detailed study of Calabi-Yau manifolds, from a geometrical perspective in Chapter 2 and from a SCFT perspective in Chapter 3. The relevant incantations of mirror symmetry for Type II strings on Calabi-Yau manifolds are presented throughout these chapters. The key construction for the arguments of this thesis is that of mirror symmetry of GLSMs due to Hori and Vafa [13].

In Chapter 4 we review what we mean by a G_2-manifold and the conditions required for a manifold to have G_2 holonomy. Due to the absence of a G_2 equivalent of Yau's theorem – i.e. there is no set of topological conditions that guarantees the existence of a Ricci flat metric with holonomy G_2 – we are restricted to specific constructions of manifolds with G_2 holonomy. The most important one for this thesis is that of a resolution of a quotient of a Calabi-Yau times a circle:

$$\mathcal{M}_{G_2} = \widetilde{\mathcal{M}_\sigma} \qquad \text{with} \qquad \mathcal{M}_\sigma = \frac{\mathcal{M}_{CY} \times S^1}{(\sigma, -)}$$

where σ acts on $\mathcal{M}_{CY}$ as an antiholomorphic involution and $(-)$ is inversion on the circle. Another important construction is that of a TCS G_2, and these are also reviewed, along with their connection to the quotient construction.

The main result this thesis sets out to demonstrate is that the mirror of $\mathcal{M}_{G_2}$ exists and that it is again given by an antiholomorphic quotient. This is meant not only at the level of meeting the Shatashvilli-Vafa condition – the conservation of the sum of Betti numbers $b^2 + b^3$ – but that the different geometries stem from the same (up to isomorphism) SCFT. This is done in Chapter 5, where we in fact show that there is a family of four different geometries that all have isomorphic SCFTs. Geometrically the three new geometries correspond to: mirror the Calabi-Yau and do T-duality on the circle; mirror only the Calabi-Yau and leave the circle alone; or leave the Calabi-Yau and do T-duality on the circle.

We now summarise the argument used to show that these distinct geometries have isomorphic worldsheet SCFTs.

The worldsheet CFT of Type II strings propagating on the covering space $\mathcal{M}_{CY} \times S^1$ enjoys Calabi-Yau mirror symmetry, as well as T-duality, giving rise to the three distinct duality maps. In particular, Calabi-Yau mirror symmetry can be shown using classic techniques such as Gepner models and GLSMs. It hence becomes possible to lift these duality maps to duality maps for the CFT describing the G_2 quotient as well. Given a pair of isomorphic CFTs and a pair of involutions that are identified under the isomorphism, one must find isomorphic theories after performing the quotient. What is not immediately obvious however, is if one can have a geometrical description in terms of a G_2 mirror pair on both sides. The involution which is used to form a G_2 variety from $\mathcal{M}_{CY} \times S^1$ must act geometrically as an anti-holomorphic involution on $\mathcal{M}_{CY}$. In the context of Calabi-Yau threefolds realized as toric hypersurfaces, the explicit description of mirror symmetry in terms of a GLSM made it possible for us to show that the mirror map precisely identifies pairs of anti-holomorphic involutions, so that Calabi-Yau mirror symmetry identifies pairs of dual quotients realised geometrically as G_2 varieties. This identification agrees with equivalences found using other techniques, such as a free-field description of toroidal orbifolds and Gepner models, where these are available.

Specifying an involution of a CFT is not enough to uniquely capture the quotient due to the presence of twisted sectors. Given a CFT and an involution, the set of possible twisted sectors must be uniquely determined however, so that our argument really identifies sets of models. For each possible twisted sector of the quotient CFT, there must be a possible twisted sector of its mirror. In the context of G_2 Gepner models, twisted sectors have received considerable attention in [30–33], but it remains an interesting question for future study how to understand the general picture.

The results presented in this thesis are consistent with earlier proposals of G_2 mirrors for twisted connected sum G_2 manifolds. Whenever quotients $\left(\mathcal{M}_{CY} \times S^1\right)/(\sigma, -)$ can be realised as twisted connected sums, this in particular gives us access to various smoothings by resolving and/or deforming the building blocks. It would be very interesting to work out in detail how this approach relates to twisted sectors both before and after mirror symmetry.

Additional Suggestions For Further Work

In this thesis, we have almost exclusively focused on the simplest case of an antiholomorphic involution: complex conjugation. We have referred to this as the *vanilla* involution. Although we have mentioned along the way how the arguments should hold for more generic involutions, it would be interesting to demonstrate this explicitly with examples.

Having described a detailed construction of G_2 mirror manifolds based on worldsheet arguments, it would be very interesting to start investigating enumerative problems based on the equality of effective superpotentials, e.g. along the lines of [24, 90, 91]. This is bound to be a hard problem with interesting mathematical ramifications [92].

For $Spin(7)$ mirror symmetry [23], we expect that results analogous to the ones presented here can be found. $Spin(7)$ manifolds can be realised as quotients of Calabi-Yau fourfolds by anti-holomorphic involutions [93], a construction which can be recast as a gluing of two simpler pieces analogous to TCS [94]. This was in turn used to propose a mirror construction in [95] which can be studied using similar techniques as used in this thesis. $Spin(7)$ mirror symmetry has been studied from the perspective of Gepner models in [96].

Appendices

A | Torodial Orbifold

Here we present a detailed analysis of the content of this thesis for the case of a torodial orbifold. Here we have a lot of control over the content of the theory and so it really helps to highlight how everything works together.

A.1 Calabi-Yau

Consider the Calabi-Yau formed as the orbifold $T^6/\mathbb{Z}_2^2$, where the $\mathbb{Z}_2^2$ acts via

$$\begin{aligned}\alpha &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (+x_1, +x_2, -x_3, a_4 - x_4, -x_5, a_6 - x_6)\\ \beta &: (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (-x_1, b_2 - x_2, +x_3, +x_4, b_5 - x_5, b_6 - x_6)\end{aligned} \tag{A.1}$$

where a_4, a_6, b_2, b_5 and b_6 are either 0 or $\frac{1}{2}$. As different values change the orbifold action, they have an effect on the twisted sector, i.e. the fixed points. Here we will focus solely on the untwisted sector as the notions we need are evident in the simpler untwisted sector. We include them here as it will allow a good connection with the Joyce orbifolds $T^7/\mathbb{Z}_2^3$ [38, 39].

As the values of the a_i and b_i won't affect our discussion, we set them all to zero. This orbifold was studied in [22], and then later generalised in [77], in the context of so-called discrete torsion and its role in mirror symmetry.

Before discussing the states in the CFT, we can use our orbifold action to compute the expected untwisted cohomology. We note that the values of a_i and b_i don't affect this, as they are simple shifts. Working with complex structure

$$z^j = x^{2j-1} + ix^{2j} \qquad \text{where} \qquad j = 1, 2, 3, \tag{A.2}$$

it is straight forward to check that the invariant forms are the $(0,0)$-form along with

$$\begin{gathered} dz_+^i d\bar{z}_+^i \\ dz_+^1 dz_+^2 dz_+^3, \quad dz_+^i dz_+^j d\bar{z}_+^k, \quad dz_+^i d\bar{z}_+^j d\bar{z}_+^k, \quad d\bar{z}_+^1 d\bar{z}_+^2 d\bar{z}_+^3 \\ dz_+^i d\bar{z}_+^i dz_+^j d\bar{z}_+^j, \\ dz_+^1 d\bar{z}_+^1 dz_+^2 d\bar{z}_+^2 dz_+^3 d\bar{z}_+^3, \end{gathered} \tag{A.3}$$

where $i,j,k \in \{1,2,3\}$ but $i \neq j \neq k$. From here we see that the non-zero, even Hodge numbers are $h_+^{0,0} = h_+^{3,0} = h_+^{0,3} = h_+^{3,3} = 1$ and $h_+^{1,1} = h_+^{2,2} = h_+^{2,1} = h_+^{1,2} = 3$.

Let's now turn to the CFT and ask "what are the RR ground states in our CFT?" For each coordinate x_j we have a left- and right-moving Majorana-Weyl spinor ψ^i and $\widetilde{\psi}^j$ respectively. Given we are working with the flat metric on T^6, the zero modes obey the anticommutation relations

$$\{\psi_0^i, \psi_0^j\} = \{\widetilde{\psi}_0^i, \widetilde{\psi}_0^j\} = 2\delta^{ij} \quad \text{and} \quad \{\psi_0^i, \widetilde{\psi}_0^j\} = 0. \tag{A.4}$$

We then define the complexified

$$\psi_\pm^j = \frac{1}{2}\big(\psi_0^j \pm i\widetilde{\psi}_0^j\big), \tag{A.5}$$

which can easily be checked to obey the standard creation and annihilation anticommutators

$$\{\psi_\pm^i, \psi_\mp^j\} = \delta^{ij} \quad \text{and} \quad \{\psi_\pm^i, \psi_\pm^j\} = 0. \tag{A.6}$$

We then adopt the convention that ψ_+^i are creation and ψ_-^i are annihilation operators. We note that these operators are left-right symmetric and so create left-right symmetric states.

Let's now look at the untwisted sector states, i.e. those states that are invariant under Equation (A.1). This action was defined on the coordinates x_i, but it acts on the fermions in the same way, as required by SUSY. The invariant states, i.e. the untwisted sector, are then easiest expressed using the $\psi_\pm^i$ algebra:

$$\begin{gathered} |0\rangle \\ |12\rangle, \quad |34\rangle, \quad |56\rangle \\ |135\rangle, \quad |136\rangle, \quad |145\rangle, \quad |146\rangle, \quad |235\rangle, \quad |236\rangle, \quad |245\rangle, \quad |246\rangle \\ |1234\rangle, \quad |1256\rangle, \quad |3456\rangle \\ |123456\rangle, \end{gathered} \tag{A.7}$$

where we have introduced the notation $|i...j\rangle := \psi_+^i ... \psi_+^j |0\rangle$. The twisted sector is straight forward to compute, but will not play a role here, instead the interested reader is directed to [77].

A.1.1 Link To Cohomology

We now want to find a relationship between the above RR states and the cohomology of the target Calabi-Yau manifold. The identification is very straight forward:

$$|ij...k\rangle \cong dx^i \wedge dx^j \wedge ... \wedge dx^k. \tag{A.8}$$

This is good, but really we want *complex* differential forms (e.g. the $(3,0)$-form Ω). For this reason we work in a different basis for our creation and annihilation operators. We define

$$\phi^i_\pm = \frac{1}{2}\big(\psi^{2i-1}_\pm + i\psi^{2i}_\pm\big) \quad \text{and} \quad \bar{\phi}^i_\pm = \frac{1}{2}\big(\psi^{2i-1}_\pm - i\psi^{2i}_\pm\big), \tag{A.9}$$

which obey

$$\{\phi^i_\pm, \bar{\phi}^j_\mp\} = \delta^{ij} \tag{A.10}$$

and all others vanishing. We identify the creation operators via the + subscript: i.e. ϕ^i_+ and $\bar{\phi}^i_+$. This set of operators will create states that are left-right symmetric and also form complex pairs. We then have

$$\phi^i_+ |0\rangle \cong dz^i \quad \text{and} \quad \bar{\phi}^i_+ |0\rangle \cong d\bar{z}^i. \tag{A.11}$$

We can now form the cohomology easily: we simply take products of the ϕ^i_+ and $\bar{\phi}^i_+$s and use the anticommutation properties. Of course we can only keep those that can be formed with the untwisted states listed above. It is not hard to verify that the only allowed combinations are

$$\begin{gathered} |0\rangle \\ \phi^i_+\bar{\phi}^i_+ |0\rangle \\ \phi^1_+\phi^2_+\phi^3_+ |0\rangle\,, \quad \phi^i_+\phi^j_+\bar{\phi}^k_+ |0\rangle\,, \quad \phi^i_+\bar{\phi}^j_+\bar{\phi}^k_+ |0\rangle\,, \quad \bar{\phi}^1_+\bar{\phi}^2_+\bar{\phi}^3_+ |0\rangle \\ \phi^i_+\bar{\phi}^i_+\phi^j_+\bar{\phi}^j_+ |0\rangle \\ \phi^1_+\bar{\phi}^1_+\phi^2_+\bar{\phi}^2_+\phi^3_+\bar{\phi}^3_+ |0\rangle\,, \end{gathered} \tag{A.12}$$

where $i,j,k \in \{1,2,3\}$ but $i \neq j \neq k$. These are, of course, the same results we arrived at in Equation (A.3).

The question we want to ask is: how do we write these forms in terms of our states $|ij...k\rangle$? The answer is to simply expand the ϕ^i_+ and $\bar{\phi}^i_+$ in terms of the ψ^i_+s. Let's first look at the

0, 2, 4 and 6-forms (i.e. the diagonal forms). These are very straight forward: consider, e.g.,

$$\begin{aligned} \phi^1_+\bar{\phi}^1_+ &= \frac{1}{4}(\psi^1_+ + i\psi^2_+)(\psi^1_+ - i\psi^2_+) \\ &= \frac{1}{2i}\psi^1_+\psi^2_+, \end{aligned} \tag{A.13}$$

so, up to a rescalling, this is simply $|12\rangle$. The same argument applies for all the other forms, and we get that the states with an even number of ψ^i_+s can simply be replaced with $\phi^i_+\bar{\phi}^i_+$. We shall call such states the *diagonal* states.

All that is left are the 3-forms. We compute these in the same manner, and obtain:

$$\begin{aligned} \Omega = \phi^1_+\phi^2_+\phi^3_+ &= |135\rangle - |245\rangle - |146\rangle - |236\rangle + i\big[\,|136\rangle - |246\rangle + |145\rangle + |235\rangle\big] \\ \bar{\Omega} = \bar{\phi}^1_+\bar{\phi}^2_+\bar{\phi}^3_+ &= |135\rangle - |245\rangle - |146\rangle - |236\rangle - i\big[\,|136\rangle - |246\rangle + |145\rangle + |235\rangle\big] \\ \omega_1 = \bar{\phi}^1_+\phi^2_+\phi^3_+ &= |135\rangle + |245\rangle - |146\rangle + |236\rangle + i\big[\,|136\rangle + |246\rangle + |145\rangle - |235\rangle\big] \\ \bar{\omega}_1 = \phi^1_+\bar{\phi}^2_+\bar{\phi}^3_+ &= |135\rangle + |245\rangle - |146\rangle + |236\rangle - i\big[\,|136\rangle + |246\rangle + |145\rangle - |235\rangle\big] \\ \omega_2 = \phi^1_+\bar{\phi}^2_+\phi^3_+ &= |135\rangle + |245\rangle + |146\rangle - |236\rangle + i\big[\,|136\rangle + |246\rangle - |145\rangle + |235\rangle\big] \\ \bar{\omega}_2 = \bar{\phi}^1_+\phi^2_+\bar{\phi}^3_+ &= |135\rangle + |245\rangle + |146\rangle - |236\rangle - i\big[\,|136\rangle + |246\rangle - |145\rangle + |235\rangle\big] \\ \omega_3 = \phi^1_+\phi^2_+\bar{\phi}^3_+ &= |135\rangle - |245\rangle + |146\rangle + |236\rangle + i\big[-|136\rangle + |246\rangle + |145\rangle + |235\rangle\big] \\ \bar{\omega}_3 = \bar{\phi}^1_+\bar{\phi}^2_+\phi^3_+ &= |135\rangle - |245\rangle + |146\rangle + |236\rangle - i\big[-|136\rangle + |246\rangle + |145\rangle + |235\rangle\big] \end{aligned} \tag{A.14}$$

We shall refer to this collection of states as the *non-diagonal* states from now on. We have named these for later convenience. We note that the $(3,0)$-form has decomposition $\Omega = A + iB$ with

$$A = |135\rangle - |245\rangle - |146\rangle - |236\rangle \quad \text{and} \quad B = |136\rangle - |246\rangle + |145\rangle + |235\rangle\,. \tag{A.15}$$

One can use the OPE between two fermions to then check that these expressions do indeed obey the OPEs required of A and B.

We immediately notice the difference between the diagonal and non-diagonal states: the former are given by single RR states, whereas the latter are given by a complex linear combination of all the RR states with three ψ^i_+s. This gives a first hint at a subtly: we know mirror symmetry is meant to map middle cohomology states to non-middle cohomology states, however we have just seen that these two classes of states take distinctly different forms. As we will see in Section A.1.3, the fix to this problem is that our 3-forms don't simply map to a single diagonal form, but to a complex linear combination of all the diagonal states. However, first it is instructive to compute the charges of our states under our $U(1)$ current.

A.1.2 Charges

In order to compute the charges of our states, we of course need to know the form the $U(1)$ current takes. For our theory of complex fermions, the left-moving $U(1)$ current takes the simple form

$$J = -\sum_{i=1}^{3} N\big(\phi^i \bar{\phi}^i\big) = \sum_{i=1}^{3} N\big(\psi^{2i-1}\psi^{2i}\big) \tag{A.16}$$

where the $N(...)$ stand for normal ordering. We have an analogous result for the right-moving current, but with tildes everywhere. Note that the current takes the form of a sum over the Kähler forms for the three T^2s that make up our T^6, i.e. $\omega_i \sim \psi^{2i-1}\psi^{2i}$. To compute the charges of our states, we need to find the zero mode in the expansion of J:

$$j_0 = -i \sum_{r\in\mathbb{Z}} \sum_{j=1}^{3} \psi_{-r}^{2j-1}\psi_r^{2j}. \tag{A.17}$$

We now make use of the following fact: the modes ψ_r^j with $r > 0$ will annihilate the vacuum, and because j_0 contains products of $\psi_{-r}^{2j-1}\psi_r^j$, along with the fact that we can anti-commute the different ψ^js and the fact that all our states are simply actions of ψ_0^js on the vacuum, means that we can effectively drop all the terms in j_0 that don't have $r = 0$. That is, we can instead simply consider the terms

$$\mathcal{J} = -i\big(\psi_0^1\psi_0^2 + \psi_0^3\psi_0^4 + \psi_0^5\psi_0^6\big) \in j_0. \tag{A.18}$$

The idea now is to express $\mathcal{J}$ in terms of the $\psi_\pm^i$s, as this will allow us to easily compute the charges of our states. Using Equation (A.5), we decompose $\mathcal{J}$ into two pieces $\mathcal{J} = \mathcal{J}_d + \mathcal{J}_{n-d}$ given by

$$\mathcal{J}_d = -i\big(\psi_+^1\psi_+^2 + \psi_-^1\psi_-^2 + \psi_+^3\psi_+^4 + \psi_-^3\psi_-^4 + \psi_+^5\psi_+^6 + \psi_-^5\psi_-^6\big). \tag{A.19}$$

and

$$\mathcal{J}_{n-d} = -i\big(\psi_+^1\psi_-^2 + \psi_-^1\psi_+^2 + \psi_+^3\psi_-^4 + \psi_-^3\psi_+^4 + \psi_+^5\psi_-^6 + \psi_-^5\psi_+^6\big). \tag{A.20}$$

The subscripts come from the fact that $\mathcal{J}_d$ is the only part of $\mathcal{J}$ that has any effect on the diagonal states and similarly $\mathcal{J}_{n-d}$ for the non-diagonal states.

Diagonal States

Let's start by looking at the diagonal states and using $\mathcal{J}_d$. It is then clear that, up to signs, this current is going to take our diagonal states and either add two ψ_+^is or take away two in the pairs $|12\rangle, |34\rangle$ or $|56\rangle$. Noting that when we remove the creation operators, we must first anticommute the $\psi_-^{2j-1}\psi_-^{2j}$ in $\mathcal{J}_d$, we see that these states come with a minus sign. For example

$$\begin{aligned}\psi_-^1\psi_-^2(\psi_+^1\psi_+^2|0\rangle) &= -\psi_-^2\psi_-^1\psi_+^1\psi_+^2|0\rangle \\ &= -\psi_-^2\psi_+^2|0\rangle + \psi_-^2\psi_+^1\psi_-^1\psi_+^2|0\rangle \\ &= -|0\rangle + \psi_+^2\psi_-^2|0\rangle - \psi_-^2\psi_+^1\psi_+^2\psi_-^1|0\rangle \\ &= -|0\rangle .\end{aligned} \tag{A.21}$$

where we have used $\{\psi_\pm^i, \psi_\mp^j\} = \delta^{ij}$ and $\{\psi_\pm^i, \psi_\pm^j\} = 0$. The same calculation holds for all other states – note that we can move bilinears in fermions freely, i.e. we can "jump" a $\psi_-^3\psi_-^4$ over the $\psi_+^1\psi_+^2$ in $|1234\rangle$ without the cost of any signs.

So we see our diagonal forms are mapped under $\mathcal{J}_d$ in the following way

$$\mathcal{J}_d: \quad \begin{aligned}|0\rangle &\mapsto -i\big[\,|12\rangle + |34\rangle + |56\rangle\,\big] \\ |12\rangle &\mapsto -i\big[-|0\rangle + |1234\rangle + |1256\rangle\,\big] \\ |34\rangle &\mapsto -i\big[-|0\rangle + |1234\rangle + |3456\rangle\,\big] \\ |56\rangle &\mapsto -i\big[-|0\rangle + |3456\rangle + |1256\rangle\,\big] \\ |1234\rangle &\mapsto -i\big[-|12\rangle - |34\rangle + |123456\rangle\,\big] \\ |1256\rangle &\mapsto -i\big[-|12\rangle - |56\rangle + |123456\rangle\,\big] \\ |3456\rangle &\mapsto -i\big[-|34\rangle - |56\rangle + |123456\rangle\,\big] \\ |123456\rangle &\mapsto -i\big[-|1234\rangle - |1256\rangle - |3456\rangle\,\big]\end{aligned} \tag{A.22}$$

The important thing to note is that *none* of these states are eigenstates of our current. We need to take a linear combination of states in order to get an eigenstate. By considering the (8×8) matrix defining the action of $\mathcal{J}_d$ on our diagonal states, we can compute the eigenvalues and eigenvectors. The results are presented in Table A.1.

We note at this point that the relative coefficients of these matches those of our non-diagonal states (Equation (A.14)), i.e. Σ and Ω etc have the same coefficients. We shall return to this in Section A.1.3.

Left-Charge	Eigenstate
$+3$	$\Sigma = -\big[\,\|0\rangle - \|1234\rangle - \|3456\rangle - \|1256\rangle\,\big] + i\big[\,\|56\rangle - \|123456\rangle + \|34\rangle + \|12\rangle\,\big]$
-3	$\bar{\Sigma} = -\big[\,\|0\rangle - \|1234\rangle - \|3456\rangle - \|1256\rangle\,\big] - i\big[\,\|56\rangle - \|123456\rangle + \|34\rangle + \|12\rangle\,\big]$
$+1$	$\sigma_1 = -\big[\,\|0\rangle + \|1234\rangle - \|3456\rangle + \|1256\rangle\,\big] + i\big[\,\|56\rangle + \|123456\rangle + \|34\rangle - \|12\rangle\,\big]$
-1	$\bar{\sigma}_1 = -\big[\,\|0\rangle + \|1234\rangle - \|3456\rangle + \|1256\rangle\,\big] - i\big[\,\|56\rangle + \|123456\rangle + \|34\rangle - \|12\rangle\,\big]$
$+1$	$\sigma_2 = -\big[\,\|0\rangle + \|1234\rangle + \|3456\rangle - \|1256\rangle\,\big] + i\big[\,\|56\rangle + \|123456\rangle - \|34\rangle + \|12\rangle\,\big]$
-1	$\bar{\sigma}_2 = -\big[\,\|0\rangle + \|1234\rangle + \|3456\rangle - \|1256\rangle\,\big] - i\big[\,\|56\rangle + \|123456\rangle - \|34\rangle + \|12\rangle\,\big]$
$+1$	$\sigma_3 = -\big[\,\|0\rangle - \|1234\rangle + \|3456\rangle + \|1256\rangle\,\big] + i\big[-\|56\rangle + \|123456\rangle + \|34\rangle + \|12\rangle\,\big]$
-1	$\bar{\sigma}_3 = -\big[\,\|0\rangle - \|1234\rangle + \|3456\rangle + \|1256\rangle\,\big] - i\big[-\|56\rangle + \|123456\rangle + \|34\rangle + \|12\rangle\,\big]$

Table A.1: *Eigenstates of the diagonal left $U(1)$ current $\mathcal{J}_d$, and their corresponding charges.*

Non-Diagonal States

We can proceed to compute how $\mathcal{J}_{n-d}$ effects the non-diagonal states in a similar fashion. Here we have the rule: if the state contains $\psi_+^{1,3,5}$ then it is replaced with $\psi_+^{2,4,6}$ and vice versa. In order to get the minus signs correct, we first write $\mathcal{J}_{n-d}$ with all annihilation operators to the right

$$\mathcal{J}_{n-d} = -i\big(\psi_+^1\psi_-^2 - \psi_+^2\psi_-^1 + \psi_+^3\psi_-^4 - \psi_+^4\psi_-^3 + \psi_+^5\psi_-^6 - \psi_+^6\psi_-^5\big), \tag{A.23}$$

so we must replace $\psi_+^{2,4,6} \mapsto \psi_+^{1,3,5}$ but $\psi_+^{1,3,5} \mapsto -\psi_+^{2,4,6}$. As before, we see how each of the individual $|ijk\rangle$ states are mapped under $\mathcal{J}_{n-d}$, and from there check that our non-diagonal states are eigenstates and compute the eigenvalues. We have:

$$\mathcal{J}_{n-d}: \quad \begin{aligned} |135\rangle &\mapsto -i\big[-|145\rangle - |145\rangle - |136\rangle\big] \\ |245\rangle &\mapsto -i\big[\,|145\rangle + |145\rangle - |246\rangle\big] \\ |146\rangle &\mapsto -i\big[-|246\rangle + |136\rangle + (156)\big] \\ |236\rangle &\mapsto -i\big[\,|136\rangle - |246\rangle + |145\rangle\big] \\ |136\rangle &\mapsto -i\big[-|236\rangle - |146\rangle + |135\rangle\big] \\ |246\rangle &\mapsto -i\big[\,|146\rangle + (256) + |245\rangle\big] \\ |145\rangle &\mapsto -i\big[-|245\rangle + |135\rangle - |146\rangle\big] \\ |235\rangle &\mapsto -i\big[\,|135\rangle - |245\rangle - |236\rangle\big] \end{aligned} \tag{A.24}$$

Recalling Equation (A.14) we therefore see that our non-diagonal states are our eigenstates with charges $\pm3, \pm1$, specifically:

$$q(\phi_+^1\phi_+^2\phi_+^3) = 3, \quad q(\bar{\phi}_+^1\bar{\phi}_+^2\bar{\phi}_+^3) = -3, \quad q(\bar{\phi}_+^i\phi_+^j\phi_+^k) = 1, \quad \text{and} \quad q(\phi_+^i\bar{\phi}_+^j\bar{\phi}_+^k) = -1. \tag{A.25}$$

In fact we have been a little careless here: the above charges are what we expect for the NS states, but here we are dealing with the R states. We go between these via spectral flow and this maps $j_0 \mapsto j_0 \pm \frac{3}{2}$, and so these charges should be shifted.

Right-Charge

We should also compute the right charge q_R. This comes from a similar derivation but now with tildes everywhere

$$\widetilde{\mathcal{J}} = -i\big(\widetilde{\psi}_0^1\widetilde{\psi}_0^2 + \widetilde{\psi}_0^3\widetilde{\psi}_0^4 + \widetilde{\psi}_0^5\widetilde{\psi}_0^6\big) \in \widetilde{j}_0. \tag{A.26}$$

However now we have

$$\widetilde{\psi}_0^i = -i\big(\psi_+^i - \psi_-^i\big), \tag{A.27}$$

and so we need to carry this factor of $-i$ through along with the relative sign between $\psi_\pm^i$. Everything in $\widetilde{\mathcal{J}}$ is bilinear, so we are really dealing with $(-i)^2 = -1$. We therefore get

$$\widetilde{\psi}_0^1\widetilde{\psi}_0^2 = -\big[\psi_+^1\psi_+^2 + \psi_-^1\psi_-^2 - \psi_+^1\psi_-^2 - \psi_-^1\psi_+^2\big] \tag{A.28}$$

etc. We therefore see that the signs on the non-diagonal forms cancel, i.e. we simply have

$$\widetilde{\mathcal{J}}_{n-d} = \mathcal{J}_{n-d} \tag{A.29}$$

however on the diagonal forms we get a relative sign

$$\widetilde{\mathcal{J}}_d = -\mathcal{J}_d \tag{A.30}$$

In other words, our non-diagonal states will have $q_L = q_R$ while the diagonal eigenstates have $q_L = -q_R$. This agrees with the result of Section 3.2.3: the non-diagonal states are elements of the (c, c) ring while the diagonal eigenstates are elements of the (a, c) ring.

A.1.3 Mirror Symmetry

We now want to look at how mirror symmetry acts on our Calabi-Yau. As explained in [77], in this context mirror symmetry is generated by three T-dualities along the coordinates

$$(j_1, j_2, j_3) \in \big\{(1,3,6), (1,4,5), (2,3,5), (2,4,6)\big\}, \tag{A.31}$$

which are exactly the combinations that appear in the imaginary parts of our 3-forms above.[1]

[1]We note that in [77] they also allow for T-dualising along $(1,3,5)$, $(1,4,6)$, $(2,3,6)$ and $(2,4,5)$, which correspond to the real parts of our 3-forms. Here we do not include these as they define the mirror map as one

We now want to ask how T-duality effects our Clifford algebra: it changes the sign of the right-moving fermion zero mode, and so it replaces the creation operator ψ_+^j with the annihilation operator ψ_-^j. This modifies the definition of the ground state to be in terms of the mapped operators, namely

$$\left(\psi_-^i\right)' |0\rangle' = 0, \tag{A.32}$$

which is equivalent to

$$\psi_-^i |0\rangle' = \psi_+^j |0\rangle' = 0 \tag{A.33}$$

where j labels the coordinates that are T-dualised and i labels all others. Using that $(\psi_+^j)^2 = 0$, we can then express our dual vacuum state in terms of the original one as

$$|0\rangle' = \psi_+^{j_1}\psi_+^{j_2}\psi_+^{j_3} |0\rangle \,, \tag{A.34}$$

with (j_1, j_2, j_3) the dualised indices.

We can now ask how states/forms in the T-dual picture relate to states/forms in the original picture. The idea is simple: we work with primes everywhere and then simply substitute in the relations at the end. The untwisted sector is then exactly the same as before, Equation (A.7), but with primes everywhere. For concreteness, let's work with T-dualising along $(1, 3, 6)$. In fact we are going to include a factor of i into our map, for a reason that will be explained shortly. Then we have

$$\left(\psi_\pm^{2,4,5}\right)' = \psi_\pm^{2,4,5}, \quad \left(\psi_\pm^{1,3,6}\right)' = \psi_\mp^{1,3,6} \quad \text{and} \quad |0\rangle' = i\,|136\rangle \,. \tag{A.35}$$

We now put this together with the anticommutators for the creation/annihilation operators

that maps the right-moving part of Ω to its complex conjugate, but these additional maps map $\Omega_R \mapsto -\Omega_R^*$. This additional minus sign can be accounted for by an additional automorphism of the algebra that introduces a phase: $\Omega_R \mapsto e^{i\phi}\Omega_R$, which corresponds exactly to Equation (5.13). In this appendix we will ignore this additional phase automorphism, and so we ignore these additional maps.

to obtain our relation. The result is the following

$$\mathfrak{M}: \quad \begin{aligned} &|135\rangle \mapsto -i\,|56\rangle \\ &|245\rangle \mapsto -i\,|123456\rangle \\ &|146\rangle \mapsto i\,|34\rangle \\ &|236\rangle \mapsto i\,|12\rangle \\ &|136\rangle \mapsto -i\,|0\rangle \\ &|246\rangle \mapsto -i\,|1234\rangle \\ &|145\rangle \mapsto i\,|3456\rangle \\ &|235\rangle \mapsto i\,|1256\rangle \end{aligned} \tag{A.36}$$

We can easily show from here that the states on the right hand side (i.e. the diagonal states) are mapped with the opposite sign behaviour. As we introduced the i factor we then get that $T^2 = \mathrm{id}$, which is required for it to be an involution.

The above allows us to ask the question of how our initial states are mapped. For example, the $(3,0)$-form Ω is mapped as

$$T(\Omega = \phi^1\phi^2\phi^3) = -\big[\,|0\rangle - |1234\rangle - |3456\rangle - |1256\rangle\,\big] + i\big[\,|56\rangle - |123456\rangle + |34\rangle + |12\rangle\,\big] = \Sigma, \tag{A.37}$$

where Σ is as defined in Table A.1. Again note that the factor of i we included is needed here, i.e. the real part of Ω is mapped to the imaginary part of Σ. A similar calculation will verify that the non-diagonal states in Equation (A.14) correspond, respectively, to the diagonal eigenstates in the table, i.e.

$$T(\bar{\Omega}) = T(\bar{\Sigma}), \qquad T(\omega_i) = \sigma_i \quad \text{and} \quad T(\bar{\omega}_i) = \bar{\sigma}_i \tag{A.38}$$

At the level of the charges, this implies that mirror symmetry maps

$$\mathfrak{M} : (q_L, q_R) \mapsto (q_L, -q_R). \tag{A.39}$$

So we see that mirror symmetry maps charge eigenstates to charge eigenstates. This is the result we expected: mirror symmetry maps $\mathfrak{R}_{(c,c)}$ to $\mathfrak{R}_{(a,c)}$ and vice versa. As we see, such a map takes a 3-form and maps it to a linear combination of all the diagonal forms.

A.2 G_2

Let's now look at the corresponding G_2. Joyce showed [38, 39] that one can construct a G_2 manifold via $T^7/\mathbb{Z}_2^3$, where the $\mathbb{Z}_2^3$ acts via

$$\begin{aligned} \alpha &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (+x_1, +x_2, -x_3, a_4 - x_4, -x_5, a_6 - x_6, x_7) \\ \beta &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, b_2 - x_2, +x_3, +x_4, b_5 - x_5, b_6 - x_6, x_7) \\ \sigma &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6, -x_7), \end{aligned} \tag{A.40}$$

where $a_i, b_i = 0, 1/2$. The α and β action here are simply the extension of Equation (A.1) to include the x_7 coordinate. We note that σ acts with a minus on $x_{2,4,6}$, which in the complex structure of our $T^6/\mathbb{Z}_2^2$ discussion, is nothing but complex conjugation. From here we see that we can identity[2]

$$\frac{T^7}{\mathbb{Z}_2^3} = \frac{\left(\frac{T^6}{(\alpha,\beta)}\right) \times S^1}{\sigma} = \frac{\mathcal{M}_{T^6} \times S^1}{\sigma}, \tag{A.41}$$

Hence we can apply our logic in order to check the involution carries through as we would like.

A.2.1 Antiholomorphic Involution

Before discussing the R ground states of our G_2 theory, we first want to ask how the anti-holomorphic involution acts on the states in our Calabi-Yau theory. This is particularly easy to do here: our complex structure is given by $z^i = x^{2i-1} + ix^{2i}$, and so complex conjugation simply acts via

$$\sigma : (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6). \tag{A.42}$$

This mapping is translated directly to the fermions, i.e. we map $\psi_+^{2,4,6} \mapsto -\psi_+^{2,4,6}$ and the others are left alone. From here we see that our diagonal and non-diagonal states have the desired behaviour, when compared to their differential forms: the $(0,0)$ and $(2,2)$ forms are invariant, the $(1,1)$ and $(3,3)$ forms are odd, and the 3-forms are mapped in pairs $(m, 3-m) \mapsto (3-m, m)$. In particular complex conjugation acts simply on the i appearing in our non-diagonal states, Equation (A.14).

However we note that the diagonal eigenstates in Table A.1 are not invariant but are also mapped via complex conjugation on the i factors. Note that this tells us that the real parts

[2]Strictly speaking we need to resolve the (α, β) action in T^6 to obtain the Calabi-Yau. We return to this shortly.

of diagonal states are the even forms while the imaginary parts are the odd forms. Putting this together with the non-diagonal states, we see that our charges are mapped via

$$\sigma : (q_L, q_R) \mapsto (-q_L, -q_R). \tag{A.43}$$

This is exactly the result we were expecting.

A.2.2 Untwisted Sector

We start by looking at the untwisted sector of this theory. Here the values of the a_i and b_i don't matter (as they only effect the fixed points), and so we can ignore them. It is straightforward to check that the invariant states are given by

$$|0\rangle\,, \qquad |ijk\rangle\,, \qquad |ijk\ell\rangle\,, \qquad |1234567\rangle \tag{A.44}$$

where

$$\begin{aligned} (ijk) &\in \big\{(127),(347),(567),(135),(146),(236),(245)\big\}, \\ (ijk\ell) &\in \big\{(1234),(3456),(1256),(1367),(1457),(2357),(2467)\big\}. \end{aligned} \tag{A.45}$$

We now note these take the exact form needed to be the extension of our states from our $T^6/\mathbb{Z}_2^2$ calculation: everything that was odd under the antiholomorphic involution is paired with a ψ_+^7 here. Geometrically, this is the statement that forms that are odd under the involution need to be wedged with $d\theta$. In particular, notice that for our 3-forms, Equation (A.14), we must now work with the real and imaginary parts. For example,

$$(\omega_1 + \bar{\omega}_1) \quad \text{and} \quad (\omega_1 - \bar{\omega}_1)\,|7\rangle \tag{A.46}$$

are the invariant states. This corresponds to taking the real and imaginary linear combinations of a $(2,1)$ and $(1,2)$ form and wedging the imaginary part with $d\theta$. Equally for the diagonal states, those corresponding to $(1,1)$ and $(3,3)$ forms come with a $|7\rangle$, while the $(0,0)$ and $(2,2)$ forms are invariant by themselves.

Additionally we note that the G_2 3-form and dual 4-form ($X := \star\Phi$) are expressed in the CFT as

$$\begin{aligned} \Phi &= |135\rangle - |245\rangle - |146\rangle - |236\rangle + |127\rangle + |347\rangle + |567\rangle \\ X &= |1457\rangle + |1367\rangle + |2357\rangle - |2467\rangle + |1234\rangle + |3456\rangle + |1256\rangle \end{aligned} \tag{A.47}$$

which matches the geometrical decomposition $\Phi = \text{Re}(\Omega) + J \wedge d\theta$ and $X = \text{Im}(\Omega) \wedge d\theta + \frac{1}{2} J \wedge J$.

A.2.3 Twisted Sector

We now want to investigate the twisted sector of our action, and ask how this changes the cohomology of our G_2. This problem has been studied from the perspective of the Joyce orbifold $T^7/\mathbb{Z}_2^3$ in [39] and then explained at the level of discrete torsion in [77]. Here we take a slighly different approach, and instead we want to consider

$$\mathcal{M}_\sigma = \frac{\widetilde{\left(\frac{T^6}{(\alpha,\beta)}\right)} \times S^1}{\sigma}, \tag{A.48}$$

where the tilde means we resolve the (α, β) action. This will, of course, give the same result as the references above.

Here the choice of the a_i, b_i in Equation (A.40) matters. We will work with $a_4 = b_6 = 1/2$ and others vanishing, i.e.

$$\begin{aligned} \alpha &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (+x_1, +x_2, -x_3, \frac{1}{2} - x_4, -x_5, -x_6, x_7) \\ \beta &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, -x_2, +x_3, +x_4, -x_5, \frac{1}{2} - x_6, x_7) \\ \sigma &: (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6, -x_7), \end{aligned} \tag{A.49}$$

First we want to consider the Calabi-Yau given by the resolution of $T^6/(\alpha, \beta)$. We know from our previous discussion that the untwisted sector gives contributions to the Hodge numbers

$$(h^{0,0}, h^{1,1}, h^{2,1}, h^{3,0}) = (1, 3, 3, 1), \tag{A.50}$$

along with their matching Hodge duals, $h^{m,n} = h^{3-m,3-n}$. The contribution from the twisted sector comes from considering the fixed points. Both α and β have 16 fixed points, however the action of the other on these fixed points leaves 8 in each case. We show this graphically in Figure A.1. Locally these fixed points are modelled by $T^2 \times \mathbb{C}^2/\{\pm 1\}$, and as standard we can choose to either blow up or deform the $\mathbb{C}^2/\{\pm 1\}$. In either case the Hodge numbers are the same, and each fixed point contributes one to both $h^{1,1}$ and $h^{2,1}$ (and their Hodge duals). So in total our Calabi-Yau has $(h^{1,1}, h^{2,1}) = (19, 19)$, and so is self-mirror. This is a special case and is known as the Schoen Calabi-Yau, and we shall denote it $\mathcal{M}_S$. In terms of Betti numbers, we have

$$b^2(\mathcal{M}_S) = \underset{T^2}{3} + \underset{\alpha}{8} + \underset{\beta}{8} = 19 \quad \text{and} \quad b^3(\mathcal{M}_S) = \underset{T^2}{8} + \underset{\alpha}{16} + \underset{\beta}{16} = 40. \tag{A.51}$$

We now want to consider the action of σ in $\mathcal{M}_\sigma = (\mathcal{M}_S \times S^1)/\sigma$, i.e. we want to compute $b^2_\pm(\mathcal{M}_S)$ and $b^3_\pm(\mathcal{M}_S)$ and use them to compute the Betti numbers for the smoothing $\mathcal{M}_{G_2} = \widetilde{\mathcal{M}_\sigma}$ via

$$b^2(\mathcal{M}_{G_2}) = b^2_+(\mathcal{M}_S) + e_2 \quad \text{and} \quad b^3(\mathcal{M}_{G_2}) = b^3_+(\mathcal{M}_S) + b^2_-(\mathcal{M}_S) + e_3, \tag{A.52}$$

where $e_{2,3}$ denotes the contributions from the fixed points of σ.

Let's start with $b^2(\mathcal{M}_S)$. It is clear that the 3 contribution from T^2 are all odd. The 8 that arises from the fixed points of α gives 4 even and 4 odd: the fixed points are identified in pairs (see Figure A.1) and so we can form one even and one odd combination. The 8 from the β fixed points are more subtle: here σ acts trivially (as its action is equivalent to βs action, but we have modded out by β), and so introduces discrete torsion. As detailed in [77], if we resolved the singularity in β via a blow up then σ will preserve the orientation of the exceptional divisor and so the ground state $(1,1)$-form is invariant. However, if we had deformed the β-singularity, then σ reverses the orientation and so our $(1,1)$-form is odd. Each of the 8 fixed points can be blown up or deformed independently, and so we have the above choice for each one. Therefore, if $\ell \in \{0, ..., 8\}$ denotes the number of blow ups, then we get a contribution of ℓ to $b^2_+(\mathcal{M}_S)$ and $(8-\ell)$ to $b^2_-(\mathcal{M}_S)$.

Now let's discuss $b^3_\pm(\mathcal{M}_S)$. The story is very similar to above: the $8+16=24$ that comes from T^2 and α are identified in pairs and so we have 12 even and 12 odd contributions. The 16 from β is dependent on the discrete torsion, and contributes $(8-\ell)$ to $b^3_+(\mathcal{M}_S)$ and ℓ to $b^3_-(\mathcal{M}_S)$.

Finally we need to add in the contributions from the σ fixed points, of which there are 4 independent ones. This is easily seen from the fact that in T^7/σ we have 16 fixed points, but these are reduced to 4 under (α, β), which are modded out to define $\mathcal{M}_S$. These four fixed points are T^3s and give contributions of 4 to $b^2(\mathcal{M})$ and 12 to $b^3(\mathcal{M})$. In total, then, our G_2 have Betti numbers

$$b^2(\mathcal{M}) = 8 + \ell \quad \text{and} \quad b^3(\mathcal{M}) = 47 - \ell, \tag{A.53}$$

which is in agreement with [39, 77].

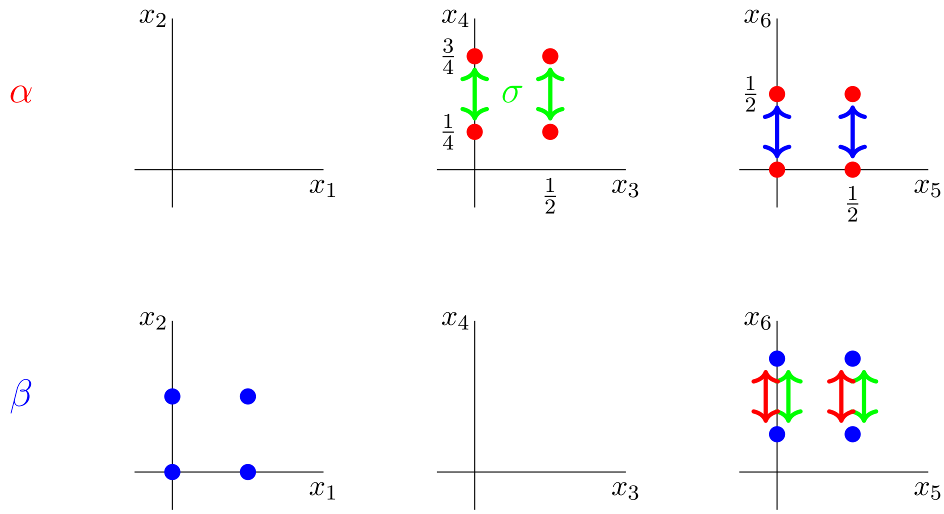


Figure A.1: *Fixed points in $T^6/(\alpha, \beta)$ and how they are mapped under (α, β, σ). All red objects correspond to α, blue to β and green to σ. The graphs indicate a decomposition of $T^6 = T^2 \times T^2 \times T^2$, and the number of fixed points understood multiplicitivly. The 16 fixed points of α are reduced to 4 under the action of (β, σ). However α and σ act on the fixed points of β in the same manner and so $\beta\sigma$ acts trivially. This leads to the introduction of discrete torsion in the G_2 manifold.*

A.2.4 Mirror Symmetry

As detailed in [77], here we have 4 notions of mirror symmetry. Just as with the Calabi-Yau torodial orbifold considered previously, these take the form of T-dualities:

$$\begin{aligned}
\mathcal{T}_3^+ &= \{(3,4,7),(2,4,5),(1,4,6)\} \\
\mathcal{T}_3^- &= \{(2,3,6),(5,6,7),(1,2,7),(1,3,5)\} \\
\mathcal{T}_4^+ &= \{(1,2,5,6),(1,3,6,7),(2,3,5,7)\} \\
\mathcal{T}_4^- &= \{(1,4,5,7),(1,2,3,4),(3,4,5,6),(2,4,6,7)\}.
\end{aligned} \tag{A.54}$$

We note that the combinations appearing in here are exactly the terms that appear in Φ and X above. The subscripts indicate how many T-dualities we do, and from chirality arguments we can see that $\mathcal{T}_3^\pm$ map compactifications on Type IIA/B to those on Type IIB/A, while $\mathcal{T}_4^\pm$ map Type IIA/B to Type IIA/B. The $\pm$ superscript indicates whether the discrete torsion signs are reversed or not, i.e. whether we blow up or deform the fixed points of β. This changes the topology of the resulting G_2 manifold, i.e. $\ell \mapsto (8-\ell)$ in Equation (A.53).

The key thing we want to notice is that within the $\mathcal{T}_4$ actions we have $(1,3,6,7)$, $(2,3,5,7)$, $(1,4,5,7)$ and $(2,4,6,7)$ which have the effect of mirroring the Calabi-Yau plus a T-duality in the additional S^1 direction. That is we can take our Calabi-Yau mirror maps in Equation (A.31) and add on a T-dual along the S^1 direction and generate a G_2 mirror map.[3]

Here we want to ask how the mirror maps effect the involution action σ. Namely we want to ask how $\sigma^\vee$ is related to σ. It is clear from the above calculation that

$$\left(\frac{\mathcal{M}_S \times S^1}{\sigma}\right)^\vee = \frac{\mathcal{M}_S^\vee \times (S^1)^T}{\sigma}, \tag{A.55}$$

and so we can set $\sigma^\vee = \sigma$. There are 9 independent G_2 manifolds we can form via the resolution of these spaces, labelled by the number of blow ups, ℓ. These blow ups appear in $\mathcal{M}_S$, and so we can label the 9 G_2s via

$$\mathcal{M}_{G_2}^\ell = \left(\frac{\widetilde{\mathcal{M}_S^\ell \times S^1}}{\sigma}\right). \tag{A.56}$$

Mirror symmetry maps $\mathcal{M}_{G_2}^\ell$ either back to itself or to $\mathcal{M}_{G_2}^{8-\ell}$, depending on whether we use $\mathcal{T}_4^+$ or $\mathcal{T}_4^-$, respectively.

The interesting thing in this case is that $\mathcal{M}_S$ is self mirror, and so even though the $\mathcal{M}_S^\ell$ look different, they are all diffeomorphic. This diffeomorphism alters the action of σ in the required way, namely we give σ an ℓ index and obtain

$$\mathcal{M}_{G_2}^\ell = \left(\frac{\widetilde{\mathcal{M}_S \times S^1}}{\sigma^\ell}\right) \tag{A.57}$$

We can therefore take two viewpoints on the situation: we either have a collection of different Calabi-Yaus, or we have a collection of different antiholomorphic involtuions.

[3]We note that, just as in the Calabi-Yau case, mirror symmetry does not simply map a 4-form to a 3-form, and vice versa. This we can see from the fact that our 4-form X contains exactly the terms that appear in $\mathcal{T}_4$, and so these terms will be mapped to the vacuum. For example, if we did the $(1,3,6,7)$ map, then $X \ni |1367\rangle \mapsto |0\rangle$, which geometrically is the 0-form. Similarly $\Phi \ni -|245\rangle \mapsto |1234567\rangle$. It is then easy to see that under any of the maps we actually exchange $\Phi + |0\rangle$ and $X + |1234567\rangle$. This is just the equivalent of the fact that a 3-form in the Calabi-Yau is mapped to a linear combination of the diagonal forms.

B Rational Forms & The Griffiths Residue

Here we discuss the notion of rational forms and residues in higher dimensions. The content is based on [97], and more details can be found there.

B.0.1 Rational Forms

Rational forms are essentially the higher degree equivalent to a rational function (which is just a $(0,0)$-form), i.e. they are forms that contain poles. We will be mostly interested in middle cohomology forms on $\mathbb{CP}^{n+1}$. It can be shown – see Theorem 2.9 of [97] for the general result – that any rational $(n+1)$-form on $\mathbb{CP}^{n+1}$ can be written in terms of a unique holomorphic $(n+1)$-form (the hatted notation means we omit that element)

$$\Omega_0 := \sum_{j=0}^{n+1} (-1)^j z^j dz^0 \wedge \ldots \wedge \widehat{dz^j} \wedge \ldots \wedge dz^{n+1}, \tag{B.1}$$

where $\{z^i\}$ are the coordinates of $\mathbb{CP}^{n+1}$. Our rational $(n+1)$-forms are then given by

$$\varphi = \frac{P(z)}{R(z)}\Omega_0, \tag{B.2}$$

where $P(z)$ and $R(z)$ are homogeneous polynomials with

$$\deg R = \deg P + (n+2). \tag{B.3}$$

This condition is simply needed to ensure that φ is projectively well defined. Let $V \subset \mathbb{CP}^{n+1}$ be the hypersurface defined by the zero locus $R(z) = 0$. We call V the *polar locus* of φ, and we similarly define the *pole order* of φ along V, in the obvious way. Our rational $(n+1)$-forms are clearly elements of $H^{n+1}(\mathbb{CP}^{n+1} \setminus V)$.

These rational forms also obey two important properties:

(i) For *any* rational $(n+1)$-form, φ, there exists an n-form, η, such that $\varphi + d\eta$ is an $(n+1)$-form with pole order $n+1$ along V.

(ii) If φ has pole order k along V, then there exists an n-form η with pole order $(k-1)$ such that $\varphi + d\eta$ has pole order $(k-1)$.

A detailed explanation of these results can be found in [97], but here we just give a general idea. We first look at condition (ii): If φ has pole order k along V, then its leading term is of the form

$$\varphi \sim \frac{1}{z^k}\Omega_0. \tag{B.4}$$

The idea is to pick a n-form η such that

$$d\eta \sim -\frac{1}{z^k}\Omega_0, \tag{B.5}$$

so that this leading order term cancels. Well we note that

$$d\left(\frac{1}{z^{k-1}}\right) = -(k-1)\frac{1}{z^k}dz, \tag{B.6}$$

so if we simply pick η to have leading term

$$\eta \sim \frac{1}{z^{k-1}}\Omega, \tag{B.7}$$

where Ω is an appropriately chosen n-form, then we can cancel our $1/z^k$ term in φ.

For condition (i), we note that it follows from Equation (B.3) that φ can have at most pole order $(n+2)$, i.e. $\deg P = 0$. However, we then use that we can reduce the pole order of φ by one, as per condition (ii), to see that the maximum pole order is $(n+1)$.

What is going to be of main interest to us is the case when we have a Calabi-Yau $\mathcal{M}_{CY} \subset \mathbb{CP}^{n+1}$ defined by the zero locus $Q(z) = 0$, with $\deg Q = n+2$. From now on we assume such a situation, although a lot of the results will hold for general polar locus V. A rational $(n+1)$-form with pole order k along $\mathcal{M}_{CY}$ then takes the form

$$\varphi = \frac{P(z)}{Q^k(z)}\Omega_0. \tag{B.8}$$

Here condition (ii) takes the form of the following proposition.[1]

[1]This proposition follows roughly from our calculation above, but a more explicit derivation/proof can be found in [97]: see the discussion leading up to Proposition 4.6.

Proposition B.0.1. *Let $\varphi = \frac{P(z)}{Q^k(z)}\Omega_0$ be a rational $(n+1)$-form with pole order k along $\mathcal{M}_{CY}$, then we can reduce the pole order to $(k-1)$ using an exact form $d\eta$, where η is a rational n-form of pole order $k-1$, if, and only if, P belongs to the Jacobi ideal, i.e. the ideal generated by $\{\frac{\partial Q}{\partial z^j}\}$.*

Note that in this situation, our degree condition, Equation (B.3), becomes

$$\begin{aligned} \deg P &= k \deg Q - (n+2) \\ &= (k-1)(n+2). \end{aligned} \tag{B.9}$$

B.0.2 Residue Map

We now want to employ our knowledge/intuition from complex analysis, in particular we want to look for the higher dimensional equivalent of the residue theorem

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = 1 \tag{B.10}$$

where γ is a closed path in $\mathbb{C}$ encircling the pole point, i.e. the origin here. We want to extended this idea to higher dimensions and write down something like

$$\frac{1}{2\pi i}\int_\gamma \frac{dz \wedge \alpha}{z} = \alpha, \tag{B.11}$$

where α is some smooth form.

In order to understand how we do this, let's recast Equation (B.10) in a more easily extendable language. Firstly we note that $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$, which is just the familiar result $\mathbb{CP}^1 \cong S^2$. The extension in what follows will be that $\mathbb{CP}^n$ can be identified as $\mathbb{C}^n$ with the hyperplane $\mathbb{CP}^{n-1}$ added at infinity. Now, in the language of homology, our closed curve $\gamma \subset \mathbb{C}$ is simply (homologous to) a circle around a 0-cycle, the point $z = 0$.

If we are working with our notational convention above of labelling our ambient space by $\mathbb{CP}^{n+1}$, we see for the case above, i.e. $n = 0$, our path is a circle over an n-cycle. We then generalise this to $n \geq 0$ by saying that we want to consider a circle bundle over an n-cycle. We denote the n-cycle by Γ and the circle bundle by $T(\Gamma)$, as it corresponds geometrically to a tube about Γ.[2]

The geometrical picture here is as follows: we consider our ambient $\mathbb{CP}^{n+1}$ and the polar

[2] In a more detailed derivation, we actually consider an $(n+1)$-cycle in $(\mathbb{CP}^{n+1} \setminus \mathcal{M}_{CY})$. However it turns out that all such $(n+1)$-cycles are homologous to a tube over an n-cycle $\Gamma \subset \mathcal{M}_{CY}$, see (3.3) of [97] for details.

locus $\mathcal{M}_{CY} \subset \mathbb{CP}^{n+1}$. Within this polar locus we consider an n-cycle $\Gamma \subset \mathcal{M}_{CY}$,[3] and we integrate our rational $(n+1)$-form, $\varphi \in H^{n+1}(\mathbb{CP}^{n+1} \setminus \mathcal{M}_{CY})$, over the tube $T(\Gamma)$. At each point $p \in \Gamma$, this picture simply reduces to integrating over a circle around a polar point, and so corresponds to the familiar case Equation (B.10). We have tried to depict this in Figure B.1 below. We are then left with a smooth n-form on $\Gamma \subset \mathcal{M}_{CY}$.

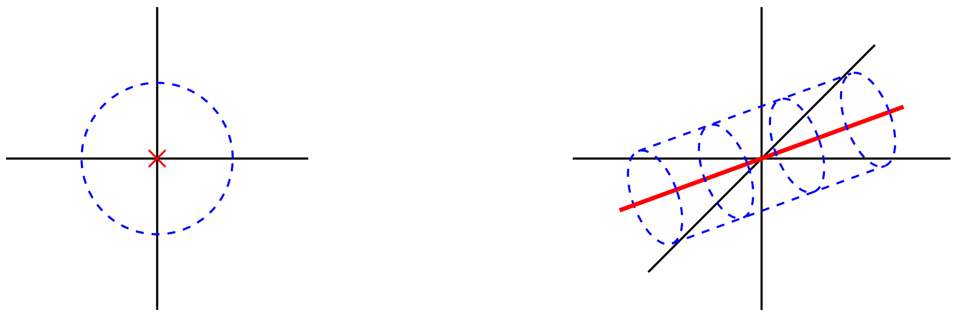


Figure B.1: *Pictorial depiction of the extension of the residue theorem to higher degree forms. Left: The familiar case of a pole on $\mathbb{C}$ (red cross) and a closed contour around it (blue dashed line). Right: the higher dimensional idea (where most dimensions are suppressed for obvious reasons); here the red line is meant to represent our n-cycle Γ and the dashed blue lines the tube $T(\Gamma)$. Every point along Γ has a circle around it, and can be thought of in the context of the left diagram.*

It turns out that the above procedure actually only gives us *primitive* n-forms on $\mathcal{M}_{CY}$, i.e. n-forms such that $\varphi \wedge \omega = 0$ where ω is the Kähler form.[4] We thus are left with the mapping

$$\text{Res} : H^{n+1}\big(\mathbb{CP}^{n+1} \setminus \mathcal{M}_{CY}\big) \to H^n_{\text{Prim.}}(\mathcal{M}_{CY}), \tag{B.12}$$

where

$$\int_\Gamma \text{Res}(\varphi) = \int_{T(\Gamma)} \varphi. \tag{B.13}$$

It can be shown that the residue map is in fact surjective.

B.0.3 Filtration

Next, we introduce a cohomology of rational forms. Let A^{n+1}_k denote the additive group of rational $(n+1)$-forms forms of pole order k along $\mathcal{M}_{CY}$. Then we define the cohomology

[3]We of course want Γ to be contained in $\mathcal{M}_{CY}$, as this is where we expect a non-vanishing residue.

[4]We can think of this in terms of homology by considering the dual: here ω becomes the hyperplane class and the wedge product the intersection number, so the vanishing result is the requirement that they don't intersect.

group

$$\mathcal{H}_k(\mathcal{M}_{CY}) := \frac{A_k^{n+1}(\mathcal{M}_{CY})}{dA_{k-1}^n(\mathcal{M}_{CY})}, \tag{B.14}$$

where we note that d changes both the form degree as well as the pole order. These groups obey a clear filtration:

$$\mathcal{H}_0(\mathcal{M}_{CY}) \subset \mathcal{H}_1(\mathcal{M}_{CY}) \subset ... \subset \mathcal{H}_{n+1}(\mathcal{M}_{CY}). \tag{B.15}$$

We can use our residue map to map this to a filtration of primitive cohomology. In particular, we have[5]

$$\text{Res}\big(\mathcal{H}_k(\mathcal{M}_{CY})\big) = \mathcal{F}^{n+1-k} H^n_{\text{Prim.}}(\mathcal{M}_{CY}) \tag{B.16}$$

where we have introduced

$$\mathcal{F}^{n+1-k} H^n_{\text{Prim.}}(\mathcal{M}_{CY}) = \bigoplus_{i \geq n+1-k} H^{i,n-i}_{\text{Prim.}}(\mathcal{M}_{CY}), \tag{B.17}$$

subject to the obvious constraint $0 \leq i \leq n$. This gives us the mapping

$$\begin{aligned}
\mathcal{H}_1(\mathcal{M}_{CY}) &\mapsto H^{n,0}_{\text{Prim.}}(\mathcal{M}_{CY}) \\
\mathcal{H}_2(\mathcal{M}_{CY}) &\mapsto H^{n,0}_{\text{Prim.}}(\mathcal{M}_{CY}) \oplus H^{n-1,1}_{\text{Prim.}}(\mathcal{M}_{CY}) \\
&\vdots \\
\mathcal{H}_{n+1}(\mathcal{M}_{CY}) &\mapsto H^{n,0}_{\text{Prim.}}(\mathcal{M}_{CY}) \oplus H^{n-1,1}_{\text{Prim.}}(\mathcal{M}_{CY}) \oplus ... \oplus H^{1,n-1}_{\text{Prim.}}(\mathcal{M}_{CY}) \oplus H^{0,n}_{\text{Prim.}}(\mathcal{M}_{CY}) \\
&= H^n_{\text{Prim.}}(\mathcal{M}_{CY})
\end{aligned} \tag{B.18}$$

We, of course, want to isolate a specific cohomology (e.g. $H^{2,1}_{\text{Prim.}}(\mathcal{M}_{CY})$), and the above filtration makes it clear that we can achieve this via forming the quotients

$$H^{p,n-p}_{\text{Prim.}}(\mathcal{M}_{CY}) = \frac{\mathcal{F}^p H^n_{\text{Prim.}}(\mathcal{M}_{CY})}{\mathcal{F}^{p+1} H^n_{\text{Prim.}}(\mathcal{M}_{CY})} = \frac{\mathcal{H}_{n+1-p}(\mathcal{M}_{CY})}{\mathcal{H}_{n-p}(\mathcal{M}_{CY})}, \qquad \text{with} \qquad p = 0, 1, ..., n. \tag{B.19}$$

Here p is related to the pole order k via $k = (n+1-p)$. It then follows from Equation (B.9) that

$$\deg P = (n-p)(n+2). \tag{B.20}$$

For Calabi-Yau 3-folds, $n = 3$ and so $\deg P = 5(3-p)$. In particular we notice that $p = 2$ gives

[5] The fact that we end up with a mixed degree form, i.e. we break holomorphicity, comes from iterating the reducing of the degree of our rational form. Details of this can be seen, e.g., in appendix A of [98].

$\deg P = 5 = \deg Q$. This is the statement that a primitive $(2,1)$-form has a corresponding polynomial of the same degree as the defining polynomial $Q(z)$, modulo the Jacobian, as per Proposition B.0.1. Here we have focused on a hypersurface in a single $\mathbb{CP}^n$, but the argument can be extended to more general hypersurfaces in projective spaces.

We note that the above result is related to restriction on allowed states in a Gepner model. Namely, recall Equation (3.83):

$$\sum_{i=1}^{5} \frac{l_i}{k_i+2} = 0, 1, 2, 3 \tag{B.21}$$

where l_i is the power of the chiral field Φ_i. In particular, the right-hand side corresponds to degree $0, H, 2H$ and $3H$ monomials, where H is the degree of the defining polynomial. We also had the Jacobi ideal restraint $\Phi_i^{k_i+1} = 0$. These are exactly the conditions for our primitive forms, and we could have written

$$\sum_{i=1}^{5} \frac{l_i}{k_i+2} = 3 - p, \tag{B.22}$$

where p is as above.

We now address the issue of $H^n_{\text{Prim.}}(\mathcal{M}_{CY}) \subseteq H^n(\mathcal{M}_{CY})$, i.e. there could be a form who's wedge with the Kähler form doesn't vanish. We call such forms "non-polynomial". Fortunately for us, this is not the case, which we shall now explain: we are dealing with Calabi-Yau 3-folds, so our middle cohomology are 3-forms. Taking the wedge with the Kähler form, which is a 2-form, would give us a 5-form. However the fifth cohomology group of a Calabi-Yau 3-fold is trivial. In the language of (r,s)-forms, we have, using that the Kähler form is a $(1,1)$-form,

$$\begin{aligned}
(3,0) &\mapsto (4,1) \\
(2,1) &\mapsto (3,2) \\
(1,2) &\mapsto (2,3) \\
(0,3) &\mapsto (1,4),
\end{aligned} \tag{B.23}$$

but there is no $(4,1)$ and $(1,4)$ cohomolgy groups for complex 3-dimensional spaces, and for a Calabi-Yau 3-fold we have $h^{3,2} = h^{2,3} = 0$. We therefore conclude that all middle cohmology forms are in fact primitive.

We note that this equality does *not* hold in general. For example, for a K3 surface, the middle cohomology are 2-forms, and the Kähler form itself is a 2-form, but is not primitive. Indeed, a general K3 surfaces always has $b^2 = h^{2,0} + h^{1,1} + h^{0,2} = 1 + 20 + 1 = 22$, while the residue calculation for the quartic in $\mathbb{CP}^3$ will give you 21 2-forms, the missing one is exactly the Kähler form. Note that we expect the primitive groups of $(2,0)$ and $(0,2)$-forms to saturate all such forms, as wedging with ω would give a $(3,1)$ or $(1,3)$-form. This general idea holds for all far left and far right (w.r.t. Hodge diamond) middle cohomology groups.

If were to consider the K3 surface in $\mathbb{CP}^2 \times \mathbb{CP}^1$, the residue would give 20 2-forms, the missing two corresponding to the hyplerplane divisors (i.e. the homology equivalent of our Kähler form) of the two factors in the product.

It is hopefully clear that a similar argument can be applied to any Calabi-Yau of even complex dimension, say $\dim_{\mathbb{C}} \mathcal{M}_{CY} = 2m$: the middle cohomology contains ω^m, where ω is the Kähler form, which is not primitive. For example, the Calabi-Yau 4-fold made from the sextic in $\mathbb{CP}^5$, it turns out that $h^{2,2} = 1752$, but the primitive subspace has dimension 1751, the missing term being exactly ω^2. As the second K3 examples above demonstrates, it is possible that the difference is more than just 1, however the key point is that if the Calabi-Yau has even complex dimension then the primitive cohomology is a proper subset.

C | Quintic Quotients

In this appendix, we look at quotients of Landau-Ginzburg (LG) orbifolds by a discrete group, namely the one that gives mirror symmetry. In particular, let's look at quotients of the orbifold LG quintic by a $G = \mathbb{Z}_5^m$ for $m = 0, 1, 2, 3$, i.e. look at

$$\big(LG(\text{Quintic})/\Gamma\big)/G, \tag{C.1}$$

where the Γ is the $U(1)$-projection on the LG model, in order to try reproduce Table 1 on page 78 of [61]. We look at this from the perspective of the Gepner model, in order to demonstrate how G changes the notion of untwisted and twisted states of the Gepner model.

C.1 General Picture

First let's look at the problem in a little more generality. We want to quotient our LG/Γ by a group G. This group is denoted in the notation of [61] as $\gamma = [n_1, ..., n_5]$, where the notation means $n_i \cong e^{2\pi i n_i J_0}$, subject to

$$\sum_i \frac{n_i}{k_i+2} \in \mathbb{Z}. \tag{C.2}$$

This condition is basically the requirement that the holomorphic 3-form is conserved, i.e. we still have a Calabi-Yau after quotient.

As we are taking two quotients of our original LG theory, we are going to generate a total of four sectors: untwisted-untwisted, untwisted-twisted, twisted-untwisted and twisted-twisted. We shall denote these as (u,u), (u,t), (t,u) and (t,t) from now on, with the first entry corresponding to our Γ action and the second the G action. What do these different sectors give us in terms of cohomology? We can make the educated guess that[1]

- (u,u): these will be the $(2,1)$-forms that survive both quotients, i.e. completely invariant monomials. In this sense they give the value of $h^{2,1}$ in the singular space before we deal

[1] Here we focus on the $(2,1)$-forms and $(1,1)$-forms, but of course it is really middle and diagonal cohomology they generate. However the point is still made clearly this way.

with the fixed points of G, i.e. they give $h^{2,1}_{\text{singular}}$.

- (t,u): these are the $(1,1)$-forms that are invariant under G, so correspond to $(1,1)$-forms that don't come from a blow up of the singularities of G, i.e. they give $h^{1,1}_{\text{singular}}$.
- (u,t): these are the $(2,1)$-forms we get by resolving the fixed point locus of G, so they give $h^{2,1}_{\text{res}} = h^{2,1} - h^{2,1}_{\text{singular}}$. These are the non-polynomial deformations of our defining equation.
- (t,t): these are the $(1,1)$-forms we get by blowing up the fixed points of G, so they give $h^{1,1}_{\text{blowup}} = h^{1,1} - h^{1,1}_{\text{singular}}$. These correspond to divisors that are reducible.

C.1.1 Singular Cohomology

So, how do we go about finding the twisted states of our G action? Well, we note that if $n_i = 1$ for all i, then we would just be considering exactly the quotient of the Gepner models discussed in Section 3.3.3, i.e. $\gamma = [g_0, ..., g_0]$ with $g_0 = e^{2\pi i J_0}$. We can use this to essentially "copy-paste" the previous derivation but now for different n_i values. The first two cases, (u,u) and (t,u), are easy to calculate: we simply work through the derivation of Equation (3.86), and then impose the requirement that

$$\sum_i \frac{n_i \ell_i^{(\nu)}}{k_i + 2} \in \mathbb{Z} \qquad \nu = 0, ..., H-1, \tag{C.3}$$

where $\nu = 0$ is the untwisted case. For the $(2,1)$-forms this is the statement that we are considering monomials that are γ invariant. Note that for the quintic *all* the $(1,1)$-forms will obey this as they have $\ell_1^{(\nu)} = ... = \ell_5^{(\nu)}$, so it follows from Equation (C.2) that our condition is met. This has the geometrical interpretation that we must preserve the Kähler form of our Calabi-Yau.

C.1.2 Non-Polynomial Deformations

Let's now deal with the remaining two cases in turn. Let's start with the non-polynomial deformations, (u,t). As we are untwisted w.r.t. our Γ action, this sector is actually quite easy to work out, once we make the observation above that we can simply consider the previous calculations as a special case of $n_i = 1$ for all i.

To start with let's work with the case that our G action only has one generator (e.g. for the quintic we have $\mathbb{Z}_5$). Here, we essentially replace ν with $n_i\tau$, where τ is our twist parameter. We then get that the twisted state is given by

$$(Q^{\tau}_{\text{tot}})_L = \sum_{\substack{i \\ n_i\tau\in(k_i+2)\mathbb{Z}}} \left(\frac{\ell_i+1}{k_i+2} - \frac{1}{2}\right) + \sum_{\substack{i \\ n_i\tau\notin(k_i+2)\mathbb{Z}}} \left(\frac{n_i\tau}{k_i+2} - \left[\frac{n_i\tau}{k_i+2}\right] - \frac{1}{2}\right), \tag{C.4}$$

subject to the usual half integral charge requirement. For the right charges, we simply include a sign in front of the second sum, i.e.

$$(Q^{\tau}_{\text{tot}})_R = \sum_{\substack{i \\ n_i\tau\in(k_i+2)\mathbb{Z}}} \left(\frac{\ell_i+1}{k_i+2} - \frac{1}{2}\right) - \sum_{\substack{i \\ n_i\tau\notin(k_i+2)\mathbb{Z}}} \left(\frac{n_i\tau}{k_i+2} - \left[\frac{n_i\tau}{k_i+2}\right] - \frac{1}{2}\right), \tag{C.5}$$

We shall return to the case when G has multiple generators (e.g. $\mathbb{Z}_5^2$ and $\mathbb{Z}_5^3$ for the quintic) in a moment.

C.1.3 Blow Ups

We now just need to account for the blow up terms, i.e. the (t,t) terms. These are also not too hard to account for: the idea is to again note that both the Γ and G actions go like g_0, and so here we simply consider twisting by $e^{2\pi i(\nu+n_i\tau)J_0}$. Therefore we take our result and replace $\nu \mapsto \nu + n_i\tau$ to give us

$$(Q^{\nu,\tau}_{\text{tot}})_L = \sum_{\substack{i \\ \nu+n_i\tau\in(k_i+2)\mathbb{Z}}} \left(\frac{\ell_i+1}{k_i+2} - \frac{1}{2}\right) + \sum_{\substack{i \\ \nu+n_i\tau\notin(k_i+2)\mathbb{Z}}} \left(\frac{\nu+n_i\tau}{k_i+2} - \left[\frac{\nu+n_i\tau}{k_i+2}\right] - \frac{1}{2}\right). \tag{C.6}$$

and

$$(Q^{\nu,\tau}_{\text{tot}})_R = \sum_{\substack{i \\ \nu+n_i\tau\in(k_i+2)\mathbb{Z}}} \left(\frac{\ell_i+1}{k_i+2} - \frac{1}{2}\right) - \sum_{\substack{i \\ \nu+n_i\tau\notin(k_i+2)\mathbb{Z}}} \left(\frac{\nu+n_i\tau}{k_i+2} - \left[\frac{\nu+n_i\tau}{k_i+2}\right] - \frac{1}{2}\right). \tag{C.7}$$

We note here that the change in sign of the second term between the above two expressions looks like it might be a problem (as we only want to consider states with $Q_L = \pm Q_R$), but we shall see that in all cases where both sums in Q_L are not identically zero (in the sense that *some* i in the sum is hit) that one of the sums actually gives gives a zero contribution.

These formulas can be adapted to give us the states in the (u,t) sector where G has multiple generators: if we had 2 generators just replace ν with $m_i\rho$ where m_i are the integers characterising the generator and ρ is the associated twist parameter. The extension to more generators should be clear.

C.1.4 The Master Formula

We can combine the above results to give a "master formula". We basically just take the results of the (t,t) sector but now allow for $\nu, \tau = 0$ corresponding to untwisted sectors. We generalise for G as well to have multiple generators as follows: we denote the different generators via $\gamma^{(\mu)} = [n_1^\mu, ..., n_5^\mu]$, and denote the associated twist parameters via τ^μ. We then get the overall formula

$$(Q_{\text{tot}}^{\nu,\vec{\tau}})_{L,R} = \sum_{\substack{i \\ \nu+n_i^\mu\tau^\mu \in (k_i+2)\mathbb{Z}}} \left(\frac{\ell_i+1}{k_i+2} - \frac{1}{2}\right) \pm \sum_{\substack{i \\ \nu+n_i^\mu\tau^\mu \notin (k_i+2)\mathbb{Z}}} \left(\frac{\nu+n_i^\mu\tau^\mu}{k_i+2} - \left[\frac{\nu+n_i^\mu\tau^\mu}{k_i+2}\right] - \frac{1}{2}\right). \tag{C.8}$$

where a sum is assumed in $n_i^\mu\tau^\mu$. For states that are in the fully untwisted sector, we impose the requirement that

$$\sum_i \frac{n_i^\mu \ell_i}{k_i+2} \in \mathbb{Z} \qquad \forall\mu. \tag{C.9}$$

As before, we can present the two sums above in a more symmetrical manner by defining $l_i^{(\nu,\vec{\tau})}$ as follows:

$$(Q_{\text{tot}}^{\nu,\vec{\tau}})_L = \sum_{i=1}^{5} \left(\frac{\ell_i^{(\nu,\vec{\tau})}+1}{k_i+2} - \frac{1}{2}\right) \qquad \text{where} \qquad \ell_i^{(\nu,\vec{\tau})}+1 := \nu + n_i^\mu\tau^\mu \mod (k_i+2). \tag{C.10}$$

The right charge is then give by changing the sign of any states with $\nu+n_i^\mu\tau^\mu \neq 0 \mod (k_i+2)$.

C.2 Quintic Calculation

We know that the orbifolded LG for the quintic is given by the product of minimal models $(k_i+2) = (5,5,5,5,5)$. The (2,1)-forms are given by finding the states such that

$$\sum_{i=1}^{5} \ell_i = 5 \qquad \text{subject to} \qquad \ell_i \leq 3 \quad \forall i. \tag{C.11}$$

Linking this to the fact that $|\ell_i\rangle \cong \Phi_i^{\ell_i}$, this is just the criteria that we are considering the monomials of degree 5, i.e. the possible deformations of the Fermat equation. The theory before we include G (i.e. the "uneffected Quintic") has the available states:

$$\begin{aligned}
(3,2,0,0,0) &\quad\Longrightarrow\quad 5\times 4 = 20 \text{ terms}\\
(3,1,1,0,0) &\quad\Longrightarrow\quad \frac{5\times 4\times 3}{2!} = 30 \text{ terms}\\
(2,2,1,0,0) &\quad\Longrightarrow\quad \frac{5\times 4\times 3}{2!} = 30 \text{ terms}\\
(2,1,1,1,0) &\quad\Longrightarrow\quad \frac{5\times 4\times 3\times 2}{3!} = 20 \text{ terms}\\
(1,1,1,1,1) &\quad\Longrightarrow\quad \frac{5\times 4\times 3\times 2\times 1}{5!} = 1 \text{ term}
\end{aligned}$$

for a total of 101 terms, as expected. The $(1,1)$-form is then given by the twisted state with $\nu = 2$, of which there is exactly 1 term.

We now want to account for the action of G. We know what the available actions are as they are presented in the table of [61]. Let's look at a few cases.

C.2.1 The $[0,1,2,3,4]$ Action

Let's start by looking at the $\mathbb{Z}_5$ action $[0,1,2,3,4] = [0,1,2,-2,-1]$, where equality follows from mod 5 in each n_i. This is the only example of a free action, so we expect no states to come from twisting by our G action.

First let's find the (u,u) states. The criteria Equation (C.9) becomes $\ell_2+2\ell_3-2\ell_4-\ell_5 = 0$ mod 5, which gives a a total of $h^{2,1}_{\text{singular}} = 21$, as shown in the Table C.1 below.

For the (t,u) states we simply get that our single form carries over to give $h^{1,1}_{\text{singular}} = 1$. From here we already see that we expect this action to be free as these results agree with those in the table of [61]. However we can check this is indeed the case.

Let's look for (u,t) states using Equations (C.4) and (C.5). For this involution we have

$$(Q^{\tau}_{\text{tot}})_L = \frac{\ell_1+1}{5} - \frac{1}{2} + 2\tau - 2 - \left[\frac{2\tau}{5}\right] - \left[\frac{3\tau}{5}\right] - \left[\frac{4\tau}{5}\right]. \tag{C.12}$$

We now note that

$$2\tau - 2 - \left[\frac{2\tau}{5}\right] - \left[\frac{3\tau}{5}\right] - \left[\frac{4\tau}{5}\right] = 2 \tag{C.13}$$

for $\tau = 1,2,3,4$, so we are left with

$$(Q^{\tau}_{\text{tot}})_L = \frac{\ell_1+1}{5} - \frac{1}{2} \overset{!}{=} -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \tag{C.14}$$

where the requirement is our usual half-integral charge requirement. Using finally that $\ell_i \leq 3$ for all i, we see that we cannot satisfy any of these, and so there are no twisted states of this

$(\ell_2, \ell_3, \ell_4, \ell_5)$	ℓ_1
$(3,0,1,1)$	0
$(1,3,1,0)$	0
$(0,1,3,1)$	0
$(1,1,0,3)$	0
$(2,2,0,1)$	0
$(1,0,2,2)$	0
$(2,1,2,0)$	0
$(0,2,1,2)$	0
$(3,1,0,0)$	1
$(0,3,0,1)$	1
$(1,0,3,0)$	1
$(0,0,1,3)$	1
$(2,0,0,2)$	1
$(0,2,2,0)$	1
$(1,1,1,1)$	1
$(2,0,1,0)$	2
$(1,2,0,0)$	2
$(0,0,2,1)$	2
$(0,1,0,2)$	2
$(1,0,0,1)$	3
$(0,1,1,0)$	3

Table C.1: *The possible untwisted $(2,1)$-forms for the quotient of the quintic LG orbifold by the $[0,1,2,3,4]$ action.*

form.

Finally we look for (t,t) states. These are a little tedious to compute by hand, but let's look at the case with $\nu = 1$. Then $\tau = 1$ results in $n_5 = 4$ becoming untwisted, $\tau = 2$ results in $n_3 = 2$ being untwisted, $\tau = 3$ results in $n_4 = 3$ being untwisted and $\tau = 4$ results in $n_2 = 1$ being untwisted. $n_1 = 0$ is always twisted. We therefore have

$$\begin{aligned}
(Q^{1,1}_{\text{tot}})_L &= \left(\frac{\ell_5+1}{5} - \frac{1}{2}\right) + \frac{6+4}{5} - 2 = \left(\frac{\ell_5+1}{5} - \frac{1}{2}\right) \\
(Q^{1,2}_{\text{tot}})_L &= \left(\frac{\ell_3+1}{5} - \frac{1}{2}\right) + \frac{16+4}{5} - 2 - \left[\frac{6+1}{5}\right] - \left[\frac{8+1}{5}\right] = \left(\frac{\ell_3+1}{5} - \frac{1}{2}\right) \\
(Q^{1,3}_{\text{tot}})_L &= \left(\frac{\ell_4+1}{5} - \frac{1}{2}\right) + \frac{21+4}{5} - 2 - \left[\frac{6+1}{5}\right] - \left[\frac{12+1}{5}\right] = \left(\frac{\ell_4+1}{5} - \frac{1}{2}\right) \\
(Q^{1,4}_{\text{tot}})_L &= \left(\frac{\ell_2+1}{5} - \frac{1}{2}\right) + \frac{36+4}{5} - 2 - \left[\frac{8+1}{5}\right] - \left[\frac{12+1}{5}\right] - \left[\frac{16+1}{5}\right] = \left(\frac{\ell_2+1}{5} - \frac{1}{2}\right)
\end{aligned} \tag{C.15}$$

which we have already shown has no allowed solutions. Similar results will follow for the other values of ν.

So in total we have $(h^{1,1}, h^{2,1})_{[0,1,2,3,4]} = (21, 1)$.

C.2.2 The $[0, 0, 0, 1, 4]$ Action

Let's also just look at the $[0, 0, 0, 1, 4]$ action. We can easily show that $h^{2,1}_{\text{singular}} = 25$ and $h^{1,1}_{\text{singular}} = 1$ via the above procedure. We now want to compute the (u,t) and (t,t) states.

For (u,t) we have

$$(Q^{\tau}_{\text{tot}})_L = \sum_{i=1}^{3} \left(\frac{\ell_i + 1}{5} - \frac{1}{2} \right) + \tau - 1 - \left[\frac{4\tau}{5} \right], \tag{C.16}$$

again we can easily compute that

$$\tau - \left[\frac{4\tau}{5} \right] = 1 \qquad \forall \tau = 1, 2, 3, 4. \tag{C.17}$$

We are then left with

$$(Q^{\tau}_{\text{tot}})_L = \sum_{i=1}^{3} \frac{\ell_i + 1}{5} - \frac{3}{2} \overset{!}{=} -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \tag{C.18}$$

which amounts to $\ell_1 + \ell_2 + \ell_3 = -3, 2, 7, 12$, respectively. Combining this with the fact that $0 \leq \ell_i \leq 3$ again, we can only meet the middle two conditions[2] as

$$\begin{aligned} \ell_1 + \ell_2 + \ell_3 = 2: &\qquad 3 \times (2, 0, 0) \quad \text{and} \quad 3 \times (1, 1, 0) \\ \ell_1 + \ell_2 + \ell_3 = 7: &\qquad 3 \times (3, 3, 1) \quad \text{and} \quad 3 \times (3, 2, 2). \end{aligned} \tag{C.19}$$

We now note that the twisted states actually make no contribution to the charge, as τ always comes with -1 here and we showed τs contribution was 1, and so we have $(Q^{\tau}_{\text{tot}})_R = (Q^{\tau}_{\text{tot}})_L$. So we see that, in total, we get $4 \times 6 = 24$ twisted $(2, 1)$ and $(1, 2)$ forms. This gives us a total of $25 + 24 = 49$, which is what is required as per the table in [61].

Finally we need to calculate the (t,t) states. For $\nu = 1$ we get that $n_4 = 1$ becomes untwisted when $\tau = 4$ and $n_5 = 4$ becomes untwisted when $\tau = 1$. Let's deal with the

[2] Note this makes sense: otherwise we would be getting additional $(3, 0)$ and $(0, 3)$ forms, which would break our Calabi-Yau-ness.

remaining cases first, $\tau = 2, 3$. Here we have

$$(Q_{\text{tot}}^{1,\tau=2,3})_L = \frac{5\tau+5}{5} - \left[\frac{4\tau+1}{5}\right] - \frac{5}{2} = -\frac{1}{2} \qquad \Longrightarrow \qquad (Q_{\text{tot}}^{1,\tau=2,3})_R = +\frac{1}{2} \tag{C.20}$$

so these are both $(2, 2)$-forms. Now let's look at the cases $\tau = 1, 4$:

$$\begin{aligned}(Q_{\text{tot}}^{1,1})_L &= \left(\frac{\ell_5+1}{5} - \frac{1}{2}\right) + \frac{1+4}{5} - 2 = \frac{\ell_5+1}{5} - \frac{3}{2} \\ (Q_{\text{tot}}^{1,4})_L &= \left(\frac{\ell_4+1}{5} - \frac{1}{2}\right) + \frac{16+4}{5} - \left[\frac{16+1}{5}\right] - 2 = \frac{\ell_4+1}{5} - \frac{3}{2}\end{aligned} \tag{C.21}$$

but this is required to be in $\{-3/2, -1/2, 1/2, 3/2\}$ but $\ell_5 \leq 3$ so we can't satisfy any of these.

Similar calculations can be done for the other ν values, and we just state the results here:

$$\begin{aligned}(Q_{\text{tot}}^{2,\tau=1,4})_L &= -(Q_{\text{tot}}^{2,\tau=1,4})_R = -\frac{1}{2} \\ (Q_{\text{tot}}^{3,\tau=1,4})_L &= -(Q_{\text{tot}}^{3,\tau=1,4})_R = +\frac{1}{2} \\ (Q_{\text{tot}}^{4,\tau=2,3})_L &= -(Q_{\text{tot}}^{4,\tau=2,3})_R = +\frac{1}{2}\end{aligned} \tag{C.22}$$

with all other terms not being allowed. So we get a total of $h^{1,1}_{\text{blowup}} = 4$ (and same for $(2, 2)$-forms).

So in total we have $(h^{1,1}, h^{2,1})_{[0,0,0,1,4]} = (49, 5)$, which is exactly what is required as per [61].

C.2.3 The $[0, 1, 1, 4, 4]$ Action

Finally let's consider the $[0, 1, 1, 4, 4]$ action. Once we have considered this, we will have worked out the Hodge numbers of one of the Calabi-Yaus in each mirror pair, so we can simply state the remaining results by mirror map.

Here we can show that the (u,u) sector gives $h^{2,1}_{\text{singular}} = 17$ and $h^{1,1}_{\text{singular}} = 1$.

For the (u,t) states we have

$$(Q_{\text{tot}}^{\tau})_L = \left(\frac{\ell_1+1}{5} - \frac{1}{2}\right) + 2\tau - 2\left[\frac{4\tau}{5}\right] - 2 = \left(\frac{\ell_1+1}{5} - \frac{1}{2}\right) \tag{C.23}$$

where the second equality follows from

$$2\tau - 2\left[\frac{4\tau}{5}\right] = 2, \qquad \tau = 1, 2, 3, 4. \tag{C.24}$$

So we see that there are no new twisted $(2,1)$-forms.

We just have to work out the (t,t) states now. For $\nu = 1$ we get that $n_{2,3} = 1$ become untwisted when $\tau = 4$ and $n_{4,5} = 4$ become untwisted when $\tau = 1$. If we consider the remaining cases, $\tau = 2, 3$, we can easily show that

$$(Q_{\text{tot}}^{1,\tau=2,3})_L = -(Q_{\text{tot}}^{1,\tau=2,3})_R = \frac{1}{2}. \tag{C.25}$$

Let's look at the case $\tau = 1$ now. We then have

$$\begin{aligned} (Q_{\text{tot}}^{1,1})_L &= \left(\frac{\ell_4+\ell_5+2}{5} - 1\right) + \frac{2+3}{5} - \frac{3}{2} \\ &= \frac{\ell_4+\ell_5+2}{5} - \frac{3}{2} \\ &\overset{!}{=} -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \\ \implies \qquad \ell_4+\ell_5 &\overset{!}{=} -2, 3, 8, 13. \end{aligned} \tag{C.26}$$

Finally using that $0 \leq \ell_4, \ell_5 \leq 3$, we see that we can only satisfy the 3 condition in 4 ways: $2 \times (3,0)$ and $2 \times (2,1)$. This corresponds to $Q_L = -1/2$. Now note that the untwisted contribution to this actually vanishes as

$$\frac{\ell_4+\ell_5+2}{5} - 1 = \frac{3+2}{5} - 1 = 0, \tag{C.27}$$

so we have $Q_R = -Q_L$, and so these states correspond to $(2,2)$-forms. We can similarly show that

$$(Q_{\text{tot}}^{1,4})_L = \frac{\ell_2+\ell_3+2}{5} - \frac{3}{2}, \tag{C.28}$$

so we get a further four twisted $(2,2)$-forms.

So in total the $\nu = 1$ sector gives us two twisted $(1,1)$-forms and eight twisted $(2,2)$-forms. By similar calculations we can show that the remaining sectors give

- $\nu = 2$: two twisted $(2,2)$-forms and eight twisted $(1,1)$-forms,
- $\nu = 3$: two twisted $(1,1)$-forms and eight twisted $(2,2)$-forms, and
- $\nu = 4$: two twisted $(2,2)$-forms and eight twisted $(1,1)$-forms.

so in total we have $2+2+8+8 = 20$ twisted $(1,1)$-forms (and same for twisted $(2,2)$-forms). Therefore in total we have $(h^{1,1}, h^{2,1})_{[0,1,1,4,4]} = (21, 17)$, which agrees with [61].

C.3 The Mirrors

In principal we could now apply the same logic in order to obtain the remaining quotients, i.e. those where G has more than one generator. This calculation is of course a little messy and so is not included here. We summarise the results (including the value of $h^{2,1}_{\text{untwisted}}$) in the following table.[3]

We see that only two cases contain twisted $(2,1)$-forms: the $(49,5)$ must have $49-25=24$ and the $(21,17)$ must have $21-5=16$ twisted states. From a geometrical point of view, we can account for these twisted states: the latter is considered on pages 28 & 29 of [11], the former we can describe now. The fixed point set of a $[1,0,-1,0,0]$ action is given by $[0 : z_2 : 0 : z_4 : z_5]$, which is simply a copy of $\mathbb{CP}^2$. Considering the intersection of this with our Calabi-Yau (i.e. the Fermat equation), we are left with considering the degree 5 polynomial $z_2^5 + z_4^5 + z_5^5 = 0$ inside $\mathbb{CP}^2$. Such a curve has genus

$$g = \frac{(5-1)(5-2)}{2} = 6, \tag{C.29}$$

which means that $h^{1,0} = 6$ on this curve. Locally the curve looks like $\mathbb{C}^2/\mathbb{Z}_5$, and such a space has, upon resolution, 4 $(1,1)$-forms, so in total this gives us $6 \times 4 = 24$ $(2,1)$-forms, as required. Note that this 6×4 decomposition appeared also in our CFT computation above. This suggests to a nice way to "see the geomtry" from the CFT.

[3]This table was generated using methods of toric geometry and the Sage programming software.

G-action	$(v_1, v_2, v_3, v_4, v_5)$	$(h^{2,1}, h^{1,1})$	$h^{2,1}_{\text{untwisted}}$
$[1, 0, -1, 0, 0]$	$\begin{matrix} 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$	(49,5)	25
$[0, 1, -1, 2, -2]$	$\begin{matrix} -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \\ 1 & 2 & 1 & -1 & -3 \\ 2 & 1 & 1 & -2 & -2 \end{matrix}$	(21,1)	21
$[1, -1, -1, 0, 1]$ $[1, -2, -1, 0, 2]$	$\begin{matrix} 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \\ 1 & 2 & 1 & -1 & -3 \end{matrix}$	(21,17)	5
$[1, 0, -1, -1, 1]$	$\begin{matrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 5 & -5 \end{matrix}$	(17,21)	17
$[-2, 1, 1, 0, 0]$ $[1, 1, 0, -2, 0]$	$\begin{matrix} 1 & 0 & 2 & 3 & -6 \\ 0 & 1 & 4 & 3 & -8 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 5 & -5 \end{matrix}$	(1,21)	1
$[0, 1, -1, 0, 0]$ $[0, -2, 0, 1, 1]$	$\begin{matrix} 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & -3 \\ 0 & 5 & 0 & 0 & -5 \\ 0 & 0 & 5 & 0 & -5 \end{matrix}$	(5,49)	5
$[1, 2, 0, -2, -1]$ $[1, 1, 0, -1, -1]$ $[0, 0, 0, 1, -1]$	$\begin{matrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{matrix}$	(1,101)	1

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