The Geometry of Hyperbolic Polynomials and an Application

HASSON, TRISTAN

The Geometry of Hyperbolic Polynomials and an Application

Tristan Hasson

A Thesis presented for the degree of
Doctor of Philosophy

Durham University

Department of Mathematics
Durham University
United Kingdom
June 2023

Abstract

This thesis aims to expand on the little known but rich mathematics of hyperbolic polynomials. Our two main results lie in rather different fields of mathematics. First we sit in algebraic geometry, looking at the space of hyperbolic polynomials themselves. We prove a result giving a class of operators on the space of polynomials preserving hyperbolicity. Our second result moves into differential geometry where we use Gårding's inequality for hyperbolic polynomials to prove a rigidity theorem for spacelike hypersurfaces in de Sitter space. To fix notation and give a feel for what hyperbolic polynomials are, we begin by giving an exposition of the theory introduced by Gårding in 1959.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematics, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

I would like to thank Dr. Wilhelm Klingenberg for his support over the years, I could not have asked for a better supervisor for me in this endeavour.

Contents

Abstract **ii**

Declaration **iii**

Acknowledgements **iv**

List of Figures **vii**

Dedication **viii**

0 Introduction **1**

0.1 Introducing Hyperbolic Polynomials 2

0.2 The Geometry of Hyperbolic Polynomials 3

0.3 Rigidity of Spacelike Hypersurfaces in de Sitter Space 4

1 Gårding's Inequality for Hyperbolic Polynomials **6**

1.1 Gårding Hyperbolic Polynomials 6

1.2 The Hyperbolicity Cone 7

1.3 Gårding's Inequality for Hyperbolic Polynomials 11

1.4 Edge and Lineality 15

2 The Geometry of Gårding Hyperbolic Polynomials **20**
2.1 The Space of Gårding Hyperbolic Polynomials 20
2.2 Univariate Hyperbolic Polynomials 27
2.3 Gårding–Nuij Sequences 31

3 A Rigidity Theorem for Spacelike Hypersurfaces in de Sitter Space **37**
3.1 de Sitter Space 38
3.1.1 The Manifold 38
3.1.2 Spacelike Hypersurfaces 40
3.2 Gårding's Inequality for Rigidity 41
3.2.1 The Equality Result 47
3.3 Integral Equations for Rigidity 49
3.3.1 Preliminaries 49
3.3.2 Non-holonomic frames 53
3.3.3 A Few Lemmas 54
3.3.4 The Integral Equations 57
3.3.5 Symmetry in Tilde 60
3.4 Rigidity in de Sitter Space 63

4 Conclusion **65**

A Python to Produce Images **69**

List of Figures

2.1 The hypersurface $P = 0$ of (2.3). 22

2.2 The hypersurfaces $P = 0$ of $F_s P$ for $s = 0.1$ and $s = 0.2$ respectively. 23

2.3 The hypersurface $P = 0$ of $T_{x,s} P$ at $s = 0.1$. 24

2.4 The hypersurfaces $P = 0$ of $T^4_{x,s} P$, $T^3_{x,s} T_{y,s} P$ and $T^2_{x,s} T^2_{y,s} P$. 25

Dedication

I dedicate this to my mother, father and brother who made me who I am.

CHAPTER 0

Introduction

Hyperbolic polynomials form a fairly self-contained little corner of mathematics. Originally arising from work in partial differential equations (see for example [16]), further study of the polynomials themselves has revealed interesting properties and usefulness in their own right. Gårding's paper *An inequality for hyperbolic polynomials* [9] published in 1959 gives, in his words, "a rather complete theory of homogeneous hyperbolic polynomials". While this paper gives a very solid grounding in the topic, since then further research has shed more light on the nature of hyperbolic polynomials and indeed applications to other areas of mathematics. An important paper of Nuij, *A note on hyperbolic polynomials* [22], published in 1969, gives important elementary results on the space of hyperbolic polynomials sitting in the space of polynomials. While the construction and properties of hyperbolic polynomials are well understood, there is no parameterisation of the space, so determining hyperbolicity in general is difficult (see for example [5] looking into testing for hyperbolicity).

While not well understood, hyperbolic polynomials lend themselves as objects with very useful applications. Hyperbolic polynomials come with associated hyperbolicity cones. These cones turn out to be of importance in modern optimisation, as fast and efficient algorithms can be applied to optimisation problems posed over hyperbolicity cones (see for example [12], [2]).

Another tool of modern optimisation is that of semidefinite programming. This looks at optimisation problems posed over sets of semidefinite matrices. In 1957, Lax published a paper [19] proposing a representation of hyperbolic polynomials on $\mathbb{R}^3$ as determinants of symmetric matrices. This was proved to be true by Lewis, Parrilo and Ramana [20] in 2005. In terms of optimisation, this meant that in $\mathbb{R}^3$ hyperbolic polynomials and semidefinite slices described the same class of cones for optimisation problems. For general dimension however, the question of semidefinite representation of hyperbolic polynomials remains opens as the so called "generalised Lax conjecture".

This thesis will look at the nature of hyperbolic polynomials as well as an application for a proof in differential geometry. The main results are: Theorem 2.3.2 of Chapter 2, which gives a condition for a class of operators on the space of polynomials to be hyperbolicity preserving; and Theorem 3.4.1 of Chapter 3 which proves that isometric spacelike hypersurfaces in de Sitter space must be the same hypersurface up to rigid transformation.

0.1 Introducing Hyperbolic Polynomials

As hyperbolic polynomials are a little obscure, notation and in fact definitions are a bit mismatched across the literature. We begin by giving a complete exposition of Gårding's theory, fixing the notation for the mathematical tools and objects we shall use throughout.

Hyperbolic polynomials are multivariate homogeneous polynomials (say P on $\mathbb{R}^n$), with the condition that there must be some direction $\mathbf{e} \in \mathbb{R}^n$ such that restricting P to any line $\mathbf{x} + t\mathbf{e}$ gives a polynomial in t with only real roots. If this is the case we say P is hyperbolic with respect to $\mathbf{e}$. While this definition is rather long winded, it turns out to give rise to some very interesting geometry. Essentially it means that the hypersurface in $\mathbb{R}^n$ given by $P = 0$ is "nice" in a particular way.

We go on to ask the question "if my polynomial P is hyperbolic with respect to

e, are there any other vectors in $\mathbb{R}^n$ that would satisfy the condition?" It turns out that for a given hyperbolic polynomial there is a collection of such vectors which we will call the hyperbolicity cone. We will give several definitions of this cone and prove that they are all equivalent.

These cones are of interest to the optimisation world, but for us they will be the setting for Gårding's inequality for hyperbolic polynomials, Theorem 1.3.4. This is an inequality between the values of the polynomial and its polar form over the hyperbolicity cone. Gårding also gives us a result for when we have equality, Theorem 1.4.4. We will use this inequality, and in particular the equality result, to prove our main result of Chapter 3.

0.2 The Geometry of Hyperbolic Polynomials

A few years after Gårding published his paper introducing hyperbolic polynomials [9], Nuij published a paper on the geometry of the space of hyperbolic polynomials [22]. In the first chapter we looked at hyperbolic polynomials themselves and their properties. Now we take a step back and look at how the collection of hyperbolic polynomials as a whole sits inside the space of all polynomials. We begin by summarising Nuij's results giving us properties such as openness and connectedness. In his proof, Nuij makes use of an operator to deform the polynomials and plot a path through the space of hyperbolic polynomials. For a polynomial P these operators take the form $P + s\frac{\partial P}{\partial x_n}$. Our main result of this chapter seeks to generalise Nuij's operators.

The idea for the way we will generalise these operators is based on work in a slightly different setting. We briefly look into the world of univariate hyperbolic polynomials (confusingly in the literature these are generally just termed "hyperbolic polynomials" so we introduce a new name to make the distinction clear). Univariate hyperbolic polynomials are simply polynomials in one variable with only real roots. A paper of Kurdyka and Paunescu [18] generalises Nuij's construction to higher derivatives, giving a large class of operators on univariate hyperbolic polynomials.

Our main result for the space of hyperbolic polynomials, Theorem 2.3.2, gives a similar generalisation to Kurdyka and Paunescu applied to Gårding's hyperbolic polynomials. These take the form

$$P_{a,s}(x_1, \dots, x_n) \coloneqq P(x_1, \dots, x_n) + \sum_{k=1}^{m} a_k s^k \frac{\partial^k P(x_1, \dots, x_n)}{\partial x_n^k}, \tag{1}$$

where the operator is defined by a given collection of polynomials a_k. We include results of Borcea and Brändén [3] on stable polynomials which we will use to prove our result.

0.3 Rigidity of Spacelike Hypersurfaces in de Sitter Space

Our final chapter moves away from the study of hyperbolic polynomials themselves and into the setting of differential geometry. We apply Gårding's inequality for hyperbolic polynomials to prove a rigidity result in de Sitter space. Our main result is based on a paper by Guan and Shen [11] looking at a rigidity result for hypersurfaces in Riemannian space forms. This was also the topic of the author's MSc thesis [14]. We now move away from the setting of Riemannian space forms and look into an equivalent rigidity theorem in a Lorentzian ambient manifold, namely de Sitter space. Our rigidity result, Theorem 3.4.1, states that if we have two spacelike hypersurfaces in de Sitter space (with a restriction on scalar curvature) with a local isometry between them, then this isometry is simply the restriction of some global isometry of de Sitter space. This is a rigidity theorem in the sense that simply stipulating a local isometry we get a global isometry. It is also a rigidity theorem in another sense as our global isometry is a rigid motion from one hypersurface to the other.

We begin by introducing de Sitter space, fixing notation, giving some idea of the geometry and defining what we mean by spacelike hypersurfaces. From here we look at the Weingarten maps of the hypersurfaces. We will prove that the second symmetric function of linear maps is hyperbolic with respect to the Weingarten maps of the two hypersurfaces. This allows us to apply Gårding's inequality, giving a relationship between this symmetric function and its polar form.

What we would like is to apply the equality result of Gårding's inequality which will prove our theorem. In order to do this we set up integral equations over the hypersurfaces and obtain our result. We include the definitions and results of Riemannian geometry that are needed. Unfortunately the result as stated in Guan and Shen's paper is incorrect, also in [14], an error is made in manipulating the intergrals. To fix this we work out a symmetry of the integrals so that all the right terms cancel when we combine them.

Finally, combining the intergral equations we are able to show that we have equality in Gårding's inequality. This will mean that the Weingarten maps of the two spacelike hypersurfaces are preserved under the local isometry. Now since this map preserves the first and second fundamental forms, it must be the restriction of some global isometry and our rigidity theorem is proved.

CHAPTER 1

Gårding's Inequality for Hyperbolic Polynomials

Beginning with basic definitions, this chapter will give the tools and necessary results to prove Gårding's inequality for hyperbolic polynomials. It is largely based in Gårding's original paper [9] and more recent work of Renegar [23]. We include extra proofs omitted by Gårding and some alternate notation and proofs have been used for clarity.

The property of hyperbolicity of a polynomial is defined in relation to a vector. It guarantees real roots when the polynomial is restricted to lines in the direction of this vector. We will see there is a collection of vectors for which the hyperbolic polynomial has this hyperbolic property, we will call this collection the hyperbolicity cone. Taking a selection of vectors from this cone, Gårding's inequality will then follow as a sort of analogue of the inequality for arithmetic and geometric means of real numbers.

1.1 Gårding Hyperbolic Polynomials

Definition 1.1.1. *Let $p(x_1, \ldots, x_n)$ be a degree m homogeneous polynomial on $\mathbb{R}^n$. If for some $\mathbf{e} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ the univariate polynomial $t \mapsto p(\mathbf{x} + t\mathbf{e})$ has only real* *roots for all* $\mathbf{x} \in \mathbb{R}^n$*, then we say* p *is* Gårding hyperbolic *with respect to* $\mathbf{e}$*.*

We will only consider hyperbolicity when $p(\mathbf{e}) > 0$ and replace p with $-p$ if required.

Throughout this thesis we will use $\mathfrak{L}_{\mathbf{x}}$ to denote the set of affine lines in $\mathbb{R}^n$ in the direction $\mathbf{x}$. Definition 1.1.1 for a polynomial p is then equivalent to every line $l \in \mathfrak{L}_{\mathbf{e}}$ intersecting the hypersurface $p = 0$ exactly m times.

As $t \mapsto p(\mathbf{x} + t\mathbf{e})$ is a polynomial in only one variable, we may factorise as

$$p(\mathbf{x} + t\mathbf{e}) = p(\mathbf{e}) \prod_{k=1}^{m} (t - \lambda_k(\mathbf{e}, \mathbf{x})). \tag{1.1}$$

Hyperbolicity with respect to a vector $\mathbf{e}$, as defined in Definition 1.1.1, is then equivalent to requiring that all $\lambda_k(\mathbf{e}, \mathbf{x})$ be real for all $\mathbf{x} \in \mathbb{R}^n$.

The following definition gives a larger class of polynomials containing the Gårding hyperbolic polynomials (however these will not necessarily be homogeneous). While we will focus on Gårding hyperbolic polynomials, it is interesting to note this alternate definition not depending on homogeneity.

Definition 1.1.2. *Let* $p \in \mathbb{C}[z_1, \ldots, z_n]$ *be a polynomial of degree* m *in* n *complex variables. Let* p_m *be the principal part of* p*, that is the terms of* p *with degree* m*. If, for some* $\mathbf{e} \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$*, we have* $p_m(\mathbf{e}) \neq 0$ *and* $p(\mathbf{x} + t\mathbf{e}) = 0$ *with* $\mathbf{x} \in \mathbb{R}^n$ *real, implies* $\mathrm{Im}(t) < t_0$ *where* t_0 *does not depend on* $\mathbf{x}$*, then we say* p *is* Nuij hyperbolic *with respect to* $\mathbf{e}$*.*

If a Nuij hyperbolic polynomial is homogeneous, then it is Gårding hyperbolic (see [22]). Unless otherwise stated, in sections 1.2, 1.3 and 1.4 the term hyperbolic will be used to mean only Gårding hyperbolic polynomials.

1.2 The Hyperbolicity Cone

Of course, if p is hyperbolic with respect to $\mathbf{e}$, there may well be other vectors that p is also hyperbolic with respect to. The main result of this chapter, Gårding's inequality, will be using collections of such vectors. As such, it will be important to understand the set of all directions that p is hyperbolic with respect to.

Definition 1.2.1. *Given a polynomial p on $\mathbb{R}^n$, define the* hyperbolicity cone*, $C(p)$, as the set of points $\mathbf{e} \in \mathbb{R}^n$ such that p is hyperbolic with respect to $\mathbf{e}$ with $p(\mathbf{e}) > 0$.*

If the polynomial p is hyperbolic with respect to $\mathbf{e}$, then we can define a different cone in relation to this initial $\mathbf{e}$ (compare definition of λ_{++} in [23]). We will then see in the next theorem that p is hyperbolic with respect to all vectors in this cone.

Definition 1.2.2. *Given p hyperbolic with respect to $\mathbf{e}$, define the* positive hyperbolicity cone *as the set*

$$C(p, \mathbf{e}) = \{\mathbf{b} \in \mathbb{R}^n \mid \text{the roots of } t \mapsto p(\mathbf{b} + t\mathbf{e}) \text{ are all negative}\}. \tag{1.2}$$

In other words, starting at the base point $\mathbf{b}$, the line in the direction $\mathbf{e}$ only crosses the hypersurface $p = 0$ in the negative $\mathbf{e}$ direction.

We now show $C(p, \mathbf{e}) \subset C(p)$ (see Theorem 3 in [23]).

Theorem 1.2.3. *Given p hyperbolic with respect to $\mathbf{e}$, we have $C(p, \mathbf{e}) \subset C(p)$. Furthermore, for any given $x \in C(p, \mathbf{e})$, we have $C(p, \mathbf{x}) = C(p, \mathbf{e})$.*

Proof. For any given $\mathbf{y} \in \mathbb{R}^n$, we must show that the polynomial $r \mapsto p(r\mathbf{x} + \mathbf{y})$ has only real roots.

Let $\alpha > 0$ be a real number. We begin by showing that the polynomial $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x})$ has only negative imaginary roots. Let r_0 be a root of said polynomial. Note that r_0 cannot be zero since that would imply $p(\alpha i\mathbf{e}) = 0$, implying $p(\mathbf{e}) = 0$, which cannot be true since $p(\mathbf{e}) > 0$. Now by homogeneity, we have

$$0 = p(\alpha i\mathbf{e} + r_0\mathbf{x}) = \frac{1}{r_0^m} p(\alpha i\mathbf{e} + r_0\mathbf{x}) = p(\frac{\alpha i}{r_0}\mathbf{e} + \mathbf{x}). \tag{1.3}$$

Since $\mathbf{x} \in C(p, \mathbf{e})$, we have that all roots of $t \mapsto p(\mathbf{b} + t\mathbf{e})$ are negative real. This means that $\frac{\alpha i}{r_0} = k$, for k some negative real number. This implies that r_0 must be a negative imaginary number, so all roots of $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x})$ are negative imaginary.

Now consider the polynomial $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x} + s\mathbf{y})$. We will show that for all $s \geq 0$ real, all of its roots have negative imaginary part. Note that we have just shown for $s = 0$ all roots are negative imaginary. Suppose there exists some

$s_0 > 0$ such that a root of $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x} + s\mathbf{y})$ has non-negative imaginary part. Then by continuity of the roots and the intermediate value theorem, there must exist an s' with $0 < s' \leq s_0$ such that a root of $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x} + s'\mathbf{y})$ has vanishing imaginary part. In other words, there must exist r', s' real numbers such that $p(\alpha i\mathbf{e} + r'\mathbf{x} + s'\mathbf{y}) = 0$. Setting $\mathbf{z} = r'\mathbf{x} + s'\mathbf{y} \in \mathbb{R}^n$ and $t_0 = \alpha i$, this implies there exists a root, t_0, of the polynomial $t \mapsto p(t\mathbf{e} + \mathbf{z})$ with t_0 imaginary. However, p is hyperbolic with respect to $\mathbf{e}$, so all such roots must be real. Hence we have a contradiction and so all roots of $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x} + s\mathbf{y})$ must have negative imaginary part.

Let $s = 1$, then we have in particular the roots of $r \mapsto p(\alpha i\mathbf{e} + r\mathbf{x} + \mathbf{y})$ have negative imaginary part. We set α to be any arbitrary number greater than zero. So if we let α tend to zero, then by the continuity of the roots (roots of a polynomial vary continuously as a function of the coefficients, see [13]), we have the roots of $r \mapsto p(r\mathbf{x} + \mathbf{y})$ have non-positive imaginary part. Now note that the polynomial $r \mapsto p(r\mathbf{x} + \mathbf{y})$ is real and so its complex roots must occur in conjugate pairs. Since we have shown that all roots must have non-positive imaginary part, this implies that all roots are real. Hence p is hyperbolic with respect to $\mathbf{x}$.

That $C(p, \mathbf{x}) = C(p, \mathbf{e})$ is a simple corollary of Theorem 1.2.5, since we will have that $C(p, \mathbf{x})$ and $C(p, \mathbf{e})$ are both the whole of the same connected component of $p \neq 0$. □

Note we have shown that p is hyperbolic with respect to all vectors in $C(p, \mathbf{e})$, however for certain polynomials there may be other vectors not in $C(p, \mathbf{e})$ that p is hyperbolic with respect to. Such a vector would give rise to a different positive hyperbolicity cone. Hence, when defining hyperbolicity, we really do need to give an initial hyperbolicity vector to fix this cone.

We define a third and final cone for a polynomial p, hyperbolic with respect to $\mathbf{e}$. We will show that it in fact defines the same cone $C(p, \mathbf{e})$ (see Proposition 1 in [23]), but can be more useful in proving certain results for hyperbolic polynomials. We consider the hypersurface $p = 0$, this divides up $\mathbb{R}^n$ into connected components where $p \neq 0$.

Definition 1.2.4. *Let p be hyperbolic with respect to $\mathbf{e}$, define $S(p, \mathbf{e})$ as the con-*

nected component of $\{\mathbf{x} \mid p(\mathbf{x}) \neq 0\}$ *containing* $\mathbf{e}$.

Theorem 1.2.5. *Let* p *be hyperbolic with respect to* $\mathbf{e}$, *then* $S(p, \mathbf{e}) = C(p, \mathbf{e})$.

Proof. We first prove the inclusion $S(p, \mathbf{e}) \subset C(p, \mathbf{e})$. Consider any $\mathbf{x} \in S(p, \mathbf{e})$ and suppose that the polynomial $t \mapsto p(\mathbf{x} + t\mathbf{e})$ has a positive root. Since $\mathbf{e} \in C(p, \mathbf{e})$, $t \mapsto p(\mathbf{e} + t\mathbf{e})$ must have all negative roots. So since the roots are continuous, we must have for any curve joining $\mathbf{e}$ and $\mathbf{x}$ that there exists a point $\mathbf{x}'$ on the curve such that $t \mapsto p(\mathbf{x}' + t\mathbf{e})$ has $t = 0$ as a root. However a point $\mathbf{x}'$ can only have $t = 0$ as a root if $p(\mathbf{x}') = 0$. This would imply that $\mathbf{x}$ was not in the connected component of $p \neq 0$ containing $\mathbf{e}$. Hence we have a contradiction, so $t \mapsto p(\mathbf{x} + t\mathbf{e})$ must have all negative roots and as such $\mathbf{x} \in C(p, \mathbf{e})$.

Now the inclusion $C(p, \mathbf{e}) \subset S(p, \mathbf{e})$. Consider any $\mathbf{x} \in C(p, \mathbf{e})$. Since $p(\mathbf{e}) \neq 0$, there must exist a ball centred at $\mathbf{e}$ that lies entirely inside $S(p, \mathbf{e})$. Now consider the line $t\mathbf{e}$. Since p is homogeneous meaning the hypersurface $p = 0$ is a cone, the maximal radius of a ball lying in $S(p, \mathbf{e})$ centred at $t\mathbf{e}$ increases linearly with t. This means that we can make this ball arbitrarily large for large enough t. Let ℓ be the straight line from $\mathbf{e}$ to $\mathbf{x}$, consider translating ℓ in the direction of $\mathbf{e}$. So any point $\mathbf{y} \in \ell$ translates to $\mathbf{y}' = \mathbf{y} + t\mathbf{e}$, in particular $\mathbf{e}$ translates to $\mathbf{e}' = (t + 1)\mathbf{e}$. The distance between $\mathbf{y}'$ and $\mathbf{e}'$ remains constant for all t, but as we know, the size of the ball around $\mathbf{e}'$ lying inside $S(p, \mathbf{e})$ can be made arbitrarily large for large enough t. So for large enough t, say t_0, the translated line segment ℓ' must lie entirely inside $S(p, \mathbf{e})$. Now, since $\mathbf{e}$ and $\mathbf{x}$ are in $C(p, \mathbf{e})$, all roots of the polynomials $t \mapsto p(\mathbf{e}+t\mathbf{e})$, $t \mapsto p(\mathbf{x}+t\mathbf{e})$ occur for negative values of t. In particular, for all $t \geq 0$ the lines $\mathbf{e}+t\mathbf{e}$, $\mathbf{x} + t\mathbf{e}$ do not cross the hypersurface $p = 0$, hence remain in the same connected component of $p \neq 0$. Putting this altogether, the lines $\mathbf{e}$ to $\mathbf{e}+t_0\mathbf{e}$, $\mathbf{e}+t_0\mathbf{e}$ to $\mathbf{x}+t_0\mathbf{e}$ and $\mathbf{x}+t_0\mathbf{e}$ to $\mathbf{x}$ do not cross the hypersurface $p = 0$. So $\mathbf{x}$ is in the same connected component of $p \neq 0$, hence $\mathbf{x} \in S(p, \mathbf{e})$. □

1.3 Gårding's Inequality for Hyperbolic Polynomials

This section will prove Gårding's inequality for hyperbolic polynomials. We will use this result to prove our rigidity theorem in Chapter 3. We have so far seen that given a polynomial hyperbolic with respect to some $\mathbf{e}$, we can find other vectors such that the same polynomial is also hyperbolic with respect to those vectors. We will now consider how we might manipulate a given hyperbolic polynomial and preserve hyperbolicity. In chapter 2 we will go into much greater detail of hyperbolicity preserving operators on the space of polynomials.

The following lemma (see Lemma 1 in [9]) gives one such construction, which will be of importance to us in proving the inequality.

Lemma 1.3.1. *Let p be a degree m polynomial on $\mathbb{R}^n$ that is hyperbolic with respect to $\mathbf{e} = (e_1, \dots, e_n)$. The polynomial*

$$q(x_1, \dots, x_n) = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k} p(x_1, \dots, x_n), \tag{1.4}$$

is hyperbolic with respect to $\mathbf{e}$.

Proof. We want to show that the univariate polynomial $t \mapsto q(\mathbf{x} + t\mathbf{e})$ has only real solutions, in this case $m - 1$ roots. Using the chain rule on the definition of q we have

$$\begin{aligned} q(\mathbf{x} + t\mathbf{e}) &= \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k} p(\mathbf{x} + t\mathbf{e}) \\ &= \sum_{k=1}^{n} \frac{d(x_k + te_k)}{dt} \frac{\partial}{\partial x_k} p(\mathbf{x} + t\mathbf{e}) \\ &= \frac{d}{dt} p(\mathbf{x} + t\mathbf{e}). \end{aligned} \tag{1.5}$$

Now by Rolle's Theorem we have that $t \mapsto q(\mathbf{x} + t\mathbf{e})$ has $m - 1$ real zeros, between the zeros of $t \mapsto p(\mathbf{x} + t\mathbf{e})$. □

Before we continue, we must define the polarisation of a homogeneous polynomial, which features in the inequality (see [9]).

Definition 1.3.2. *Let $p(\mathbf{x})$ be a homogeneous polynomial on $\mathbb{R}^n$ of degree m. We define the* polarized form of p *as the unique function $r(\mathbf{x}^1, \ldots, \mathbf{x}^m)$, $\mathbf{x}^k = \{x_j^k\}_{j=1}^n$, that is linear in each argument, invariant under permutations of the $\mathbf{x}^k$ and satisfies $r(\mathbf{x}, \ldots, \mathbf{x}) = p(\mathbf{x})$. It can be written explicitly as*

$$r(\mathbf{x}^1, \ldots, \mathbf{x}^m) = \frac{1}{m!}\Big(\prod_{k=1}^{m}\Big(\sum_{j=1}^{n} x_j^k \frac{\partial}{\partial x_j}\Big)\Big)\, p(\mathbf{x}). \tag{1.6}$$

Note that each term in the homogeneous polynomial is of degree m, and we have m different variables in the polynomial r. The terms in the polarised form amount to taking each term in p, duplicating it $m!$ times, and replacing each x_j with an x_j^k so that each $k \in \{1, \ldots, m\}$ appears once and all possible combinations of replacing the x_j with the x_j^k are included across the $m!$ duplicates.

The following theorem will be vital in proving the desired Gårding inequality for hyperbolic polynomials (see Theorem 4 in [9]).

Theorem 1.3.3. *Given p of degree $m > 1$, let p be hyperbolic $\mathbf{e}$. For $\mathbf{b} \in C(p, \mathbf{e})$, we have*

$$q(\mathbf{x}) = r(\mathbf{b}, \mathbf{x}, \ldots, \mathbf{x}), \tag{1.7}$$

is also hyperbolic $\mathbf{e}$. Furthermore we have $C(q, \mathbf{e}) \supset C(p, \mathbf{e})$.

Proof. To prove this, first consider what $r(\mathbf{b}, \mathbf{x}, \ldots, \mathbf{x})$ actually is. To construct r we took each term of p and repeated it to include all the ways of putting the $k = 1, \ldots m$ superscripts on the x_j. For q, we then say $\mathbf{x}^1$ is actually $\mathbf{b}$ and the rest of the $\mathbf{x}^k$ are all simply $\mathbf{x}$. So for each position the $\mathbf{x}^1 = \mathbf{b}$ term went in, there are $(m-1)!$ identical terms where the $(m-1)!$ permutations of the other $\mathbf{x}^2, \ldots, \mathbf{x}^m$ all became $\mathbf{x}$. Of course the completely polarised form r has the factor $\frac{1}{m!}$ to make $r(\mathbf{x}, \ldots, \mathbf{x}) = p(\mathbf{x})$. Without multiplicities, these terms are also obtained from $(\sum b_i \frac{\partial}{\partial x_i})p$. So we have

$$mq(\mathbf{x}) = (\sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i})p(\mathbf{x}). \tag{1.8}$$

Now, p is hyperbolic with respect to $\mathbf{b}$ and so, by Lemma 1.3.1, q must also be hyperbolic with respect to $\mathbf{b}$.

As we saw in the proof of Lemma 1.3.1, the zeros of

$$mq(\mathbf{x}+t\mathbf{b}) = \frac{d}{dt}p(\mathbf{x}+t\mathbf{b}), \tag{1.9}$$

separate the zeros of $p(\mathbf{x}+t\mathbf{b})$, so clearly $C(q,\mathbf{b}) \supset C(p,\mathbf{b})$. Of course $C(p,\mathbf{e}) = C(p,\mathbf{b})$ so $\mathbf{e} \in C(q,\mathbf{b})$, hence q is hyperbolic with respect to $\mathbf{e}$. Then $C(q,\mathbf{e}) = C(q,\mathbf{b})$, so we finally obtain $C(q,\mathbf{e}) \supset C(p,\mathbf{e})$. □

Now we have the tools to state and prove the main result of this section, the Gårding inequality for hyperbolic polynomials (see Theorem 5 in [9]).

Theorem 1.3.4 (Gårding Inequality for Hyperbolic Polynomials)**.** *Given p of degree $m > 1$, let p be hyperbolic with respect to $\mathbf{e}$ with $p(\mathbf{e}) > 0$. Let r be the polarised form of p, then for $\mathbf{x}^1, \dots, \mathbf{x}^m \in C(p,\mathbf{e})$, the following inequality holds*

$$r(\mathbf{x}^1, \dots, \mathbf{x}^m) \geq p(\mathbf{x}^1)^{\frac{1}{m}} \dots p(\mathbf{x}^m)^{\frac{1}{m}}. \tag{1.10}$$

Proof. We begin with a special case of the inequality. Let $\mathbf{b}, \mathbf{x} \in C(p,\mathbf{e})$. We will show that

$$q(\mathbf{x}) = r(\mathbf{b}, \mathbf{x}, \dots, \mathbf{x}) \geq p(\mathbf{b})^{\frac{1}{m}} p(\mathbf{x})^{\frac{m-1}{m}}. \tag{1.11}$$

First, without loss of generality, we may assume that $p(\mathbf{x}) = 1$, since p is homogeneous. Let t_i be the negative of the roots of $t \mapsto p(t\mathbf{x}+\mathbf{b})$, note that this means that all the t_i are positive. Then we have

$$\prod_{i=1}^{m}(t+t_i) = p(t\mathbf{x}+\mathbf{b}) = t^m + mqt^{m-1} + \dots + p(\mathbf{b}), \tag{1.12}$$

by definition. Now by expanding the left hand side and equating coefficients, we have

$$q(\mathbf{x}) = \frac{1}{m}\sum_{i=1}^{m} t_i, \tag{1.13}$$

and

$$p(\mathbf{b}) = \prod_{i=1}^{m} t_i. \tag{1.14}$$

Now (1.11) is simply a result of the theorem that the arithmetic mean is greater than the geometric mean.

With this special case, we now prove the theorem by induction. The base case for polynomials of degree $m = 2$ is already proved, as for degree 2 (1.10) and (1.11) are the same. Now assume the inequality holds for polynomials of degree $m-1$. We need to show that it must also hold for a polynomial p of degree m. Let r be the polarised form of p and let $q(\mathbf{x}) = r(\mathbf{b}, \mathbf{x}, \ldots, \mathbf{x})$ as before. Consider

$$\tilde{r}(\mathbf{x}^2, \ldots, \mathbf{x}^m) = r(\mathbf{b}, \mathbf{x}^2, \ldots, \mathbf{x}^m), \tag{1.15}$$

as r is the polarised form of p, it must be linear in each of the $\mathbf{x}^2, \ldots, \mathbf{x}^m$ and also invariant under permutations of the $\mathbf{x}^2, \ldots, \mathbf{x}^m$. Furthermore since $\tilde{r}(\mathbf{x}, \ldots, \mathbf{x}) = q(\mathbf{x})$, we conclude that $\tilde{r}$ is the polarised form of q. Since q is a degree $m-1$ polynomial and $C(q, \mathbf{e}) \supset C(p, \mathbf{e})$, we apply our assumption that the inequality holds for degree $m-1$ polynomials,

$$\tilde{r} \geq q(\mathbf{x}^2)^{\frac{1}{m-1}} \ldots q(\mathbf{x}^m)^{\frac{1}{m-1}}. \tag{1.16}$$

Now we can apply the inequality (1.11) from the special case to each of the $q(\mathbf{x}^i)$,

$$q(\mathbf{x}^i) \geq p(\mathbf{b})^{\frac{1}{m}} p(\mathbf{x}^i)^{\frac{m-1}{m}}. \tag{1.17}$$

Plugging all of these into (1.16) we obtain

$$r(\mathbf{b}, \mathbf{x}^2, \ldots, \mathbf{x}^m) = \tilde{r}(\mathbf{x}^2, \ldots, \mathbf{x}^m) \geq p(\mathbf{b})^{\frac{1}{m}} p(\mathbf{x}^2)^{\frac{1}{m}} \ldots p(\mathbf{x}^m)^{\frac{1}{m}}. \tag{1.18}$$

Since $\mathbf{b}$ was just any vector in $C(p, \mathbf{e})$, like the $\mathbf{x}^i$, we have (1.10) holds for degree m, hence the theorem is proved. $\square$

1.4 Edge and Lineality

We have now proven Gårding's inequality for hyperbolic polynomials. In the following section we will see a further result given by Gårding [9] for when we have equality in (1.10), Gårding's inequality. It is this equality result that will be vital for our rigidity theorem in Chapter 3.

In order to prove this, first we will define three spaces of vectors. As we will see in the following propositions, these will turn out to define the same space. We begin with a space associated to the cone $C(p, \mathbf{e})$.

Definition 1.4.1. *Given a hyperbolic polynomial p, we define the* edge *of the cone $C(p, \mathbf{e})$ as the set*

$$E_{C(p,\mathbf{e})} = \{\mathbf{x} \in \mathbb{R}^n \mid C(p, \mathbf{e}) + \mathbf{x} = C(p, \mathbf{e})\}. \tag{1.19}$$

We also define a space associated to the polynomial, p.

Definition 1.4.2. *Given a polynomial p hyperbolic with respect to $\mathbf{e}$, we define the* lineality *of p as the set*

$$L_p = \{\mathbf{x} \in \mathbb{R}^n \mid p(t\mathbf{x} + \mathbf{y}) = p(\mathbf{y}), \text{ for all } \mathbf{y}, t\}. \tag{1.20}$$

The final space we define gives the most useful characteristic of the constituent vectors.

Definition 1.4.3. *Given a polynomial hyperbolic with respect to $\mathbf{e}$, define the space*

$$X_{p,\mathbf{e}} = \{\mathbf{x} \mid \text{roots of } t \mapsto p(t\mathbf{e} + \mathbf{x}) \text{ are all zero}\}. \tag{1.21}$$

The following two propositions show that these three definitions all define the same space of vectors.

Proposition 1.4.1. $X_{p,\mathbf{e}} = E_{C(p,\mathbf{e})}$.

Proof. We begin by showing the inclusion $X_{p,\mathbf{e}} \subset E_{C(p,\mathbf{e})}$. Take $\mathbf{x} \in X_{p,\mathbf{e}}$. We need to show that for any $\mathbf{b} \in C(p, \mathbf{e})$ that $\mathbf{b} + \mathbf{x}, \mathbf{b} - \mathbf{x} \in C(p, \mathbf{e})$. The first step will be to show that the polynomial

$$(s,t) \mapsto p(s\mathbf{e} + t\mathbf{b} + \mathbf{x}), \tag{1.22}$$

has no roots when $t > 0$ and $s \geq h(\mathbf{e},\mathbf{x})$, where $h(\mathbf{e},\mathbf{x})$ is the greatest root of $s \mapsto p(s\mathbf{e}+\mathbf{x})$ (note that this is a general result for all $\mathbf{x}$, not just $\mathbf{x} \in X_{p,\mathbf{e}}$). When s and t get large, the principal part of $p(s\mathbf{e}+t\mathbf{b}+\mathbf{x})$ is

$$p(s\mathbf{e} + t\mathbf{b}) = p(\mathbf{e}) \prod_{k=1}^{m} (s - t\lambda_k(\mathbf{e},\mathbf{b})), \tag{1.23}$$

where the $\lambda_k(\mathbf{e},\mathbf{b})$ are the roots of $s \mapsto p(s\mathbf{e}+\mathbf{b})$. Note, as $\mathbf{b} \in C(p,\mathbf{e})$ all $\lambda_k(\mathbf{e},\mathbf{b})$ are strictly negative. This means as s tends to ∞, all the t_k tend to $-\infty$, where t_k are the roots of $t \mapsto p(s\mathbf{e} + t\mathbf{b})$. This means that as $s \to \infty$, the roots of $t \mapsto p(s\mathbf{e}+t\mathbf{b}+\mathbf{x})$ also tend to $-\infty$.

Now assume there exists some (s_0, t_0) with $t_0 > 0$, $s_0 \geq h(\mathbf{e},\mathbf{x})$ such that

$$p(s_0\mathbf{e} + t_0\mathbf{b} + \mathbf{x}) = 0. \tag{1.24}$$

Now consider the roots of $t \mapsto p(s\mathbf{e}+t\mathbf{b}+\mathbf{x})$ as s tends to infinity from s_0. Since one root is strictly positive at (s_0,t_0), but all eventually tend to $-\infty$, there must exist some $s' > s_0$ such that 0 is a root of $t \mapsto p(s'\mathbf{e}+t\mathbf{b}+\mathbf{x})$. This would mean that s' is a root of $s \mapsto p(s\mathbf{e}+\mathbf{x})$ with $s' > h(\mathbf{e},\mathbf{x})$. However $h(\mathbf{e},\mathbf{x})$ was defined to be the greatest root of $s \mapsto p(s\mathbf{e}+\mathbf{x})$. So we have a contradiction, so no such (s_0,t_0) can exist.

In particular, letting $t = 1$ then

$$s \mapsto p(s\mathbf{e} + \mathbf{b} + \mathbf{x}), \tag{1.25}$$

has no roots when $s \geq h(\mathbf{e},\mathbf{x})$. Now we use that $\mathbf{x} \in X_{p,\mathbf{e}}$, meaning all the roots of $s \mapsto p(s\mathbf{e}+\mathbf{x})$ are zero. Hence the polynomial $s \mapsto p(s\mathbf{e}+(\mathbf{b}+\mathbf{x}))$ only has negative roots, so $\mathbf{b}+\mathbf{x} \in C(p,\mathbf{e})$.

Note that $p(\lambda\mathbf{x}) = \lambda^m p(\mathbf{x})$ for all $\lambda \in \mathbb{R}$. This implies for any $\mathbf{x} \in X_{p,\mathbf{e}}$ we have $\lambda\mathbf{x} \in X_{p,\mathbf{e}}$, since if some $\mathbf{y} = \lambda_0\mathbf{x}$ has a non-zero root of $s \mapsto p(s\mathbf{e} + \mathbf{y})$ this would imply that $\mathbf{x} = \frac{1}{\lambda_0}\mathbf{y}$ also has such a non-zero root. Hence $-\mathbf{x} \in X_{p,\mathbf{e}}$, so if $\mathbf{b} \in C(p, \mathbf{e})$ then $\mathbf{b} - \mathbf{x} \in C(p, \mathbf{e})$. Altogether this proves $C(p, \mathbf{e}) - \mathbf{x} = C(p, \mathbf{e})$ and hence $\mathbf{x} \in E_{C(p,\mathbf{e})}$.

We now prove the other inclusion $E_{C(p,\mathbf{e})} \subset X_{p,\mathbf{e}}$. Let $\mathbf{x} \in E_{C(p,\mathbf{e})}$. Assume there esists some $s_0 > 0$ with $p(s_0\mathbf{e} + \mathbf{x}) = 0$. Then let s' be such that $0 < s' < s_0$. This implies $\mathbf{x} + s'\mathbf{e}$ is not in the cone $C(p, \mathbf{e})$ since the polynomial $s \mapsto p(s\mathbf{e} + (\mathbf{x} + s'\mathbf{e}))$ has a positive root $s_0 - s' > 0$. However, since $\mathbf{x} \in E_{C(p,\mathbf{e})}$ and preserves the cone under translation, this would imply $(\mathbf{x} + s'\mathbf{e}) - \mathbf{x}$ is not in $C(p, \mathbf{e})$. Clearly this is simply $s'\mathbf{e}$ and we have all positive scalar multiples of $\mathbf{e}$ are in $C(p, \mathbf{e})$. Hence we have a contradiction and $s \mapsto p(s\mathbf{e} + \mathbf{x})$ cannot have a positive root.

Now assume there exists some $s_0 < 0$ with $p(s_0\mathbf{e} + x) = 0$. As before take $s_0 < s' < 0$ meaning $\mathbf{x} + s'\mathbf{e}$ is not in $C(p, -\mathbf{e})$. Note that $C(p, \mathbf{e}) = -C(p, -\mathbf{e})$, so $-\mathbf{x} - s'\mathbf{e}$ must not be in $C(p, \mathbf{e})$. Since $\mathbf{x} \in X_{p,\mathbf{e}}$ this would imply $-s'\mathbf{e}$ is not in the cone $C(p, \mathbf{e})$. Clearly as before $-s'\mathbf{e}$ must be in $C(p, \mathbf{e})$, hence $s \mapsto p(s\mathbf{e} + x)$ cannot have a negative root, so the only root can be zero meaning $\mathbf{x} \in X_{p,\mathbf{e}}$. So we have proved both inclusions and $X_{p,\mathbf{e}} = E_{C(p,\mathbf{e})}$. □

Note that $E_{C(p,\mathbf{e})}$ is the set of vectors that preserve the cone $C(p, \mathbf{e})$ under translation. Since $C(p, \mathbf{e}) = C(p, \mathbf{b})$ for all $\mathbf{b} \in C(p\mathbf{e})$, we have $E_{C(p,\mathbf{e})} = E_{C(p,\mathbf{b})}$. This means

$$X_{p,\mathbf{e}} = E_{C(p,\mathbf{e})} = E_{C(p,\mathbf{b})} = X_{p,\mathbf{b}}. \tag{1.26}$$

Proposition 1.4.2. $X_{p,\mathbf{e}} = L_p$.

Proof. We first prove the inclusion $L_p \subset X_{p,\mathbf{e}}$. Let $\mathbf{x} \in L_p$, then

$$p(t\mathbf{e} + \mathbf{x}) = p(t\mathbf{e}) = t^m p(\mathbf{e}). \tag{1.27}$$

Therefore all the roots of the polynomial $t \mapsto p(t\mathbf{e} + \mathbf{x})$ are zero. Hence $\mathbf{x} \in X_{p,\mathbf{e}}$.

Now we prove the other inclusion $X_{p,\mathbf{e}} \subset L_p$. Let $\mathbf{x} \in X_{p,\mathbf{e}}$, we begin by showing $p(\mathbf{b}) = p(\mathbf{b} + \mathbf{x})$ for all $\mathbf{b} \in C(p, \mathbf{e})$. Since by Proposition 1.4.1 $\mathbf{x} \in E_{C(p,\mathbf{e})}$, the cone $C(p, \mathbf{e})$ is preserved under translation by $\mathbf{x}$. So we have $\mathbf{b} - \mathbf{x} \in C(p, \mathbf{e})$. Now consider the polynomials

$$\begin{aligned} &t \mapsto p(t\mathbf{b}) \\ &s \mapsto p(2\mathbf{x} + s(\mathbf{b} - \mathbf{x})). \end{aligned} \tag{1.28}$$

The first equation clearly has a root at zero with multiplicity m. Since $2\mathbf{x} \in X_{p,\mathbf{e}} = X_{p,(\mathbf{b}-\mathbf{x})}$, the second polynomial also has a root at zero with multiplicity m. Furthermore both polynomials take the value $p(2\mathbf{b})$ at $t = s = 2$. Since they are both degree m polynomials the information we have implies they are the same polynomial. So they must take the same value at $t = s = 1$. Therefore

$$p(\mathbf{b}) = p(\mathbf{b} + \mathbf{x}). \tag{1.29}$$

Now we need only show that this implies $p(\mathbf{y}) = p(\mathbf{y} + \mathbf{x})$. Consider the two polynomials

$$\begin{aligned} &t \mapsto p(\mathbf{y} + t\mathbf{e}) \\ &s \mapsto p((\mathbf{y} + \mathbf{x}) + s\mathbf{e}). \end{aligned} \tag{1.30}$$

Since p is hyperbolic with respect to $\mathbf{e}$, both $\mathbf{y} + t\mathbf{e}$ and $(\mathbf{y}+\mathbf{x}) + s\mathbf{e}$ eventually enter the cone $C(p, \mathbf{e})$ for some s_0, t_0. Then by (1.29), we have $p(\mathbf{y}+\lambda\mathbf{e}) = p((\mathbf{y}+\mathbf{x})+\lambda\mathbf{e})$ for all $\lambda > s_0, t_0$. Therefore they must be the same polynomial, so evaluating at $t = s = 0$ we have

$$p(\mathbf{y}) = p(\mathbf{y} + \mathbf{x}). \tag{1.31}$$

This shows $\mathbf{x} \in L_p$ and we have $X_{p,\mathbf{e}} = L_p$. □

We now give an extension to Theorem 1.3.4, with a result for equality in the Gårding inequality for hyperbolic polynomials.

Theorem 1.4.4. *In the setting of Theorem 1.3.4, we have the inequality*

$$r(\mathbf{x}^1, \ldots, \mathbf{x}^m) \geq p(\mathbf{x}^1)^{\frac{1}{m}} \ldots p(\mathbf{x}^m)^{\frac{1}{m}}. \tag{1.32}$$

Further to this, we have equality if and only if the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^m$ *are pairwise* *proportional modulo L_p.*

Proof. Let p, q and r be as defined in the proof of Theorem 1.3.4, let $\mathbf{b}, \mathbf{x} \in C(p, \mathbf{e})$ and as before let $\{t_i\}$ be the negative of the roots of $t \mapsto p(t\mathbf{x} + \mathbf{b})$. So we have

$$\prod_{i=1}^{m}(t + t_i) = p(t\mathbf{x} + \mathbf{b}), \tag{1.33}$$

and assuming $p(\mathbf{x}) = 1$, recall the inequality

$$q(\mathbf{x}) = r(\mathbf{b}, \mathbf{x}, \ldots, \mathbf{x}) \geq p(\mathbf{b})^{\frac{1}{m}} p(\mathbf{x})^{\frac{m-1}{m}}, \tag{1.34}$$

is simply a result of the arithmetic mean of the t_i being greater than the geometric mean. Now we know that the arithmetic mean is equal to the geometric mean if and only if $t_1 = \cdots = t_m$, so let us say $t_i = t_0$ for all i. Now from (1.33) we have

$$(t + t_0)^m = p(t\mathbf{x} + \mathbf{b}) = p((t + t_0)\mathbf{x} + \mathbf{b} - t_0\mathbf{x}). \tag{1.35}$$

Changing variables $\lambda = t + t_0$ gives

$$\lambda^m = p(\lambda\mathbf{x} + (\mathbf{b} - t_0\mathbf{x})). \tag{1.36}$$

Now since p is hyperbolic with respect to $\mathbf{x}$ and all the roots of $\lambda \mapsto p(\lambda\mathbf{x}+(\mathbf{b}-t_0\mathbf{x}))$ are zero, by Proposition 1.4.2 we have that $\mathbf{b}-t_0\mathbf{x} \in L_p$. Using this in the induction argument in the proof of Theorem 1.3.4, we obtain the result that

$$r(\mathbf{x}^1, \ldots, \mathbf{x}^m) = p(\mathbf{x}^1)^{\frac{1}{m}} \ldots p(\mathbf{x}^m)^{\frac{1}{m}}, \tag{1.37}$$

if and only if the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^m$ are pairwise proportional modulo L_p. $\square$

CHAPTER 2

The Geometry of Gårding Hyperbolic Polynomials

In the last chapter we explored properties of hyperbolic polynomials themselves. Now we take a step back and look at the space of hyperbolic polynomials. Nuij proved results in [22] on properties such as openness and connectedness. The main result of this chapter will combine ideas from Nuij on Gårding hyperbolic polynomials and from Kurdyka and Paunescu [18] on a related class of polynomial which we will call univariate hyperbolic. The literature can get confusing as the term hyperbolic polynomial is used to define two different types of polynomial, hence we had the names Gårding and univariate to distinguish between them. Further, in this chapter we denote Gårding and univariate hyperbolic polynomials with upper case and lower case letters respectively.

2.1 The Space of Gårding Hyperbolic Polynomials

In this section the term hyperbolic polynomial will be used to refer only to Gårding hyperbolic polynomials. We begin with a few results from [22] on the nature of the space of hyperbolic polynomials (see also [21]). Throughout we will use the topology defined by the Euclidean norm of the coefficients of the polynomial.

Theorem 2.1.1. *Every hyperbolic polynomial is the limit of strictly hyperbolic polynomials.*

This proof makes use of an important operator on the space of polynomials. Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $\mathbf{e}$, where $\mathbf{e}$ points in the x_1 direction. Given s real, for any $k > 1$ define the Nuij operator

$$T_{k,s}P(x) = P(x) + sx_k \frac{\partial P(x)}{\partial x_1}. \tag{2.1}$$

We will consider this construction later for the univariate case (see Lemma 2.2.2). As a simple corollary of Lemma 2.2.2, we have $T_{k,s}P(x)$ is hyperbolic with respect to $\mathbf{e}$. In particular, if $s \neq 0$ and $x_k \neq 0$ we have $T_{k,s}$ reduces the multiplicity of roots of any multiple roots of $P(x + t\mathbf{e})$ by one. Further, any roots of $T_{k,s}P(x + t\mathbf{e})$ not shared by $P(x + t\mathbf{e})$ are simple.

Proof of Theorem 2.1.1. Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be a degree m polynomial hyperbolic with respect to $\mathbf{e}$, where $\mathbf{e}$ points in the x_1 direction. Since $T_{k,s}$ reduces the multiplicity of roots of $P(x + t\mathbf{e})$ by one when $s \neq 0$ and $x_k \neq 0$, we have for $s \neq 0$

$$F_sP(x + t\mathbf{e}) = T^m_{2,s} \ldots T^m_{n,s}P(x + t\mathbf{e}) \tag{2.2}$$

has only simple roots except where $x_2 = \cdots = x_n = 0$. Hence F_sP is strictly hyperbolic for $s \neq 0$ and clearly tends to P as s approaches 0. This proves the theorem. □

We now give an example illustrating Theorem 2.1.1 giving insight into how exactly the operator F_s acts on a hyperbolic polynomial.

Example 2.1.1. *First we construct a simple hyperbolic polynomial that is not strictly hyperbolic. The polynomial $x^2+y^2-z^2$ has a basic round cone as its hyperbolicity cone. Multiplying two of these with non-empty intersection of their cones gives us a suitable non-strict hyperbolic polynomial. Figure 2.1 shows the hypersurface*

$P = 0$ for the polynomial

$$P(x, y, z) = x^4 + x^2y^2 - x^2z^2 + \frac{1}{4}y^4 - \frac{7}{2}y^2z^2 + \frac{1}{4}z^4 \tag{2.3}$$

which is hyperbolic with respect to the z coordinate direction. Now applying the

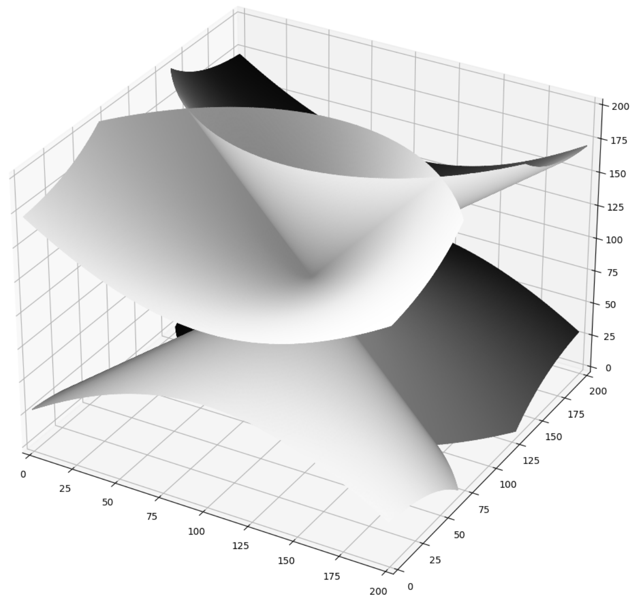

Figure 2.1: The hypersurface $P = 0$ of (2.3).

operator F_s we obtain

$$F_sP = P + 2s(x+y)\frac{\partial P}{\partial z} + s^2(x^2+4xy+y^2)\frac{\partial^2 P}{\partial z^2} + 2s^3(x^2y+xy^2)\frac{\partial^3 P}{\partial z^3} + s^4x^2y^2\frac{\partial^4 P}{\partial z^4} \tag{2.4}$$

explicitly

$$\begin{aligned} F_sP = {} & x^4 + x^2y^2 - x^2z^2 + \frac{1}{4}y^4 - \frac{7}{2}y^2z^2 + \frac{1}{4}z^4 \\ & + 2s(x+y)(-2x^2z - 7y^2z + z^3) \\ & + s^2(x^2+4xy+y^2)(-2x^2 - 7y^2 + 3z^2) \\ & + 12s^3z(x^2y + xy^2) \\ & + 6s^4x^2y^2. \end{aligned} \tag{2.5}$$

We can then see how F_s acts on P in Figure 2.2.

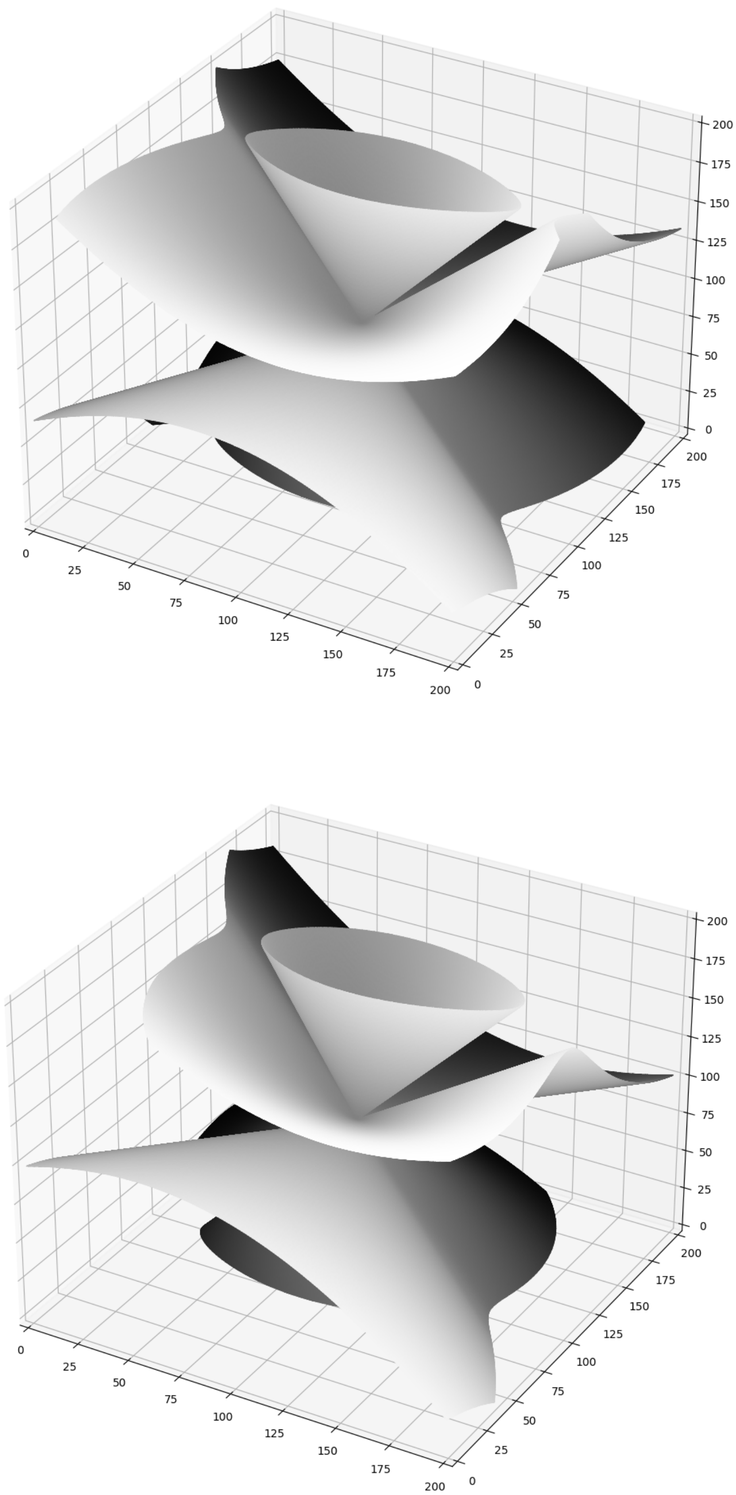

Figure 2.2: The hypersurfaces $P = 0$ of $F_s P$ for $s = 0.1$ and $s = 0.2$ respectively.

Note in the definition of Nuij's operator in (2.2) we repeat the operators $T_{k,s}$

m times. This is simply to form a general operator to deal with worst case scenarios of root multiplicity guaranteeing we land in the space of strictly hyperbolic polynomials. We obtain a variety of operators by taking different compositions of these $T_{k,s}$. We may end up in the space of strictly hyperbolic polynomials or on the boundary, depending on the operator and the initial polynomial. We follow with a few motivational examples of operators acting on $P = z^4$.

Example 2.1.2. *The hyperbolic polynomial $P = z^m$ is the worst case scenario for root multiplicity since every line in a hyperbolic direction has one root of multiplicity m. We can use it to illustrate how different combinations of the $T_{k,s}$ act to remove multiplicity. In this example we use $P = z^4$ and denote the operators by $T_{x,s}$ and $T_{y,s}$. We can see in Figure 2.3 the operator $T_{x,s}$ reduces the multiplicity of the root everywhere except for where $x = 0$.*

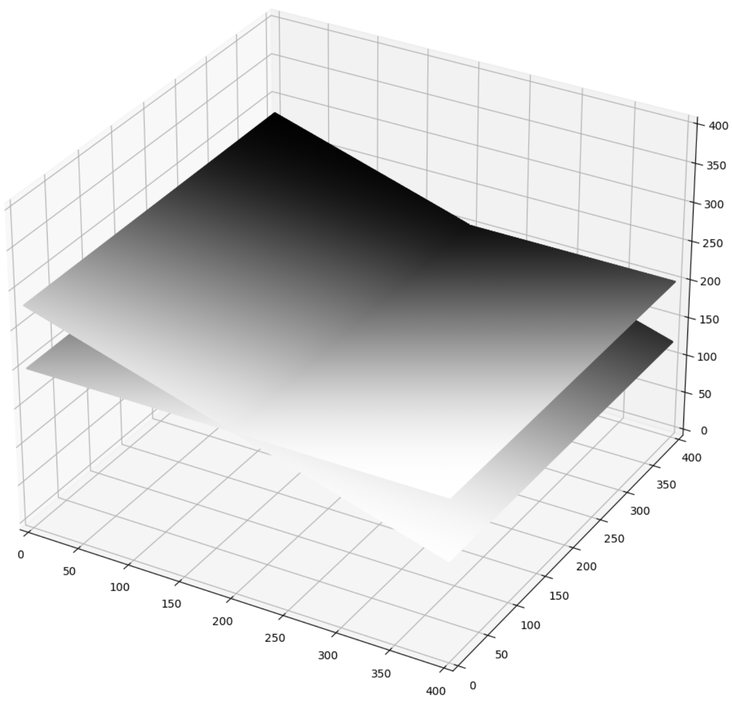

Figure 2.3: The hypersurface $P = 0$ of $T_{x,s}P$ at $s = 0.1$.

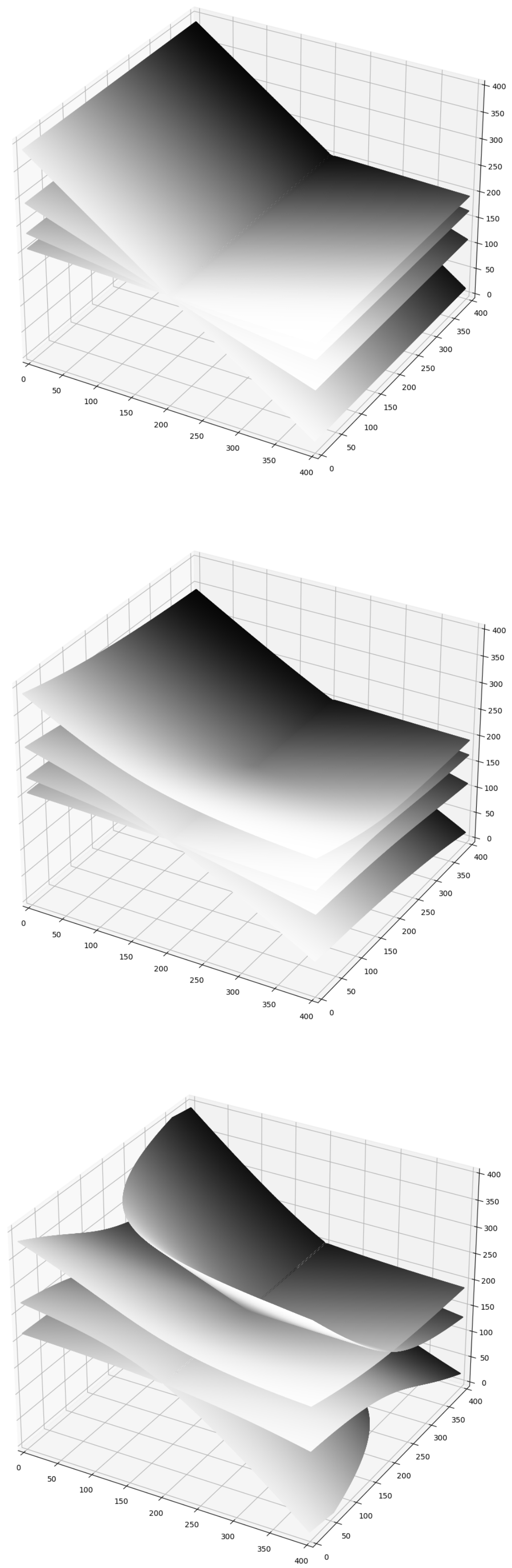

Figure 2.4: The hypersurfaces $P = 0$ of $T_{x,s}^4 P$, $T_{x,s}^3 T_{y,s} P$ and $T_{x,s}^2 T_{y,s}^2 P$.

Now taking combinations of the $T_{k,s}$ we see in Figure 2.4 $T_{x,s}$ and $T_{y,s}$ reducing the multiplicity by one everywhere except $x = 0$ and $y = 0$ respectively.

We saw in (2.4) the polynomial coefficients of the $\frac{\partial P}{\partial z}$, $\frac{\partial^2 P}{\partial z^2}$, etcetera arising purely from expanding the $T_{k,s}$ and not depending on the polynomial P. We will refer to these as homogenisation coefficients. We saw in Example 2.1.2 we can take a variety of combinations of the $T_{k,s}$ which give rise to different homogenisation coefficients.

This construction motivates work of Kurdyka and Paunescu in the univariate case [18] which we discuss in the next section. Further, this is the motivation for our analogous result for Gårding hyperbolic polynomials.

We now conclude this section with a couple more results of Nuij on the space of hyperbolic polynomials.

Theorem 2.1.2. *The space of strictly hyperbolic polynomials is open.*

Proof. Let $z' = (z_2, \ldots, z_n)$ and denote the zeros of $t \mapsto P(t, z')$ by $t_j(z')$. Define the number

$$d_P := \inf\{|t_j(z') - t_k(z')|; j \neq k, |z'| = 1\}. \tag{2.6}$$

Then since P is strictly hyperbolic we have $d_P > 0$ and depends continuously on the coefficients of the polynomial P. Hence the space of strictly hyperbolic polynomials is open. □

A polynomial P hyperbolic with respect to a vector $\mathbf{e}$ is said to be normalised if $P(\mathbf{e}) = 1$.

Theorem 2.1.3. *The space of normalised hyperbolic polynomials is connected and simply connected.*

Proof. In this proof for simplicity we will assume that $\mathbf{e}$ points in the z_1 direction and without loss of generality we assume $\mathbf{e} = (1, 0, \ldots, 0)$. Define a new operator G_t acting on the space of homogeneous polynomials by

$$G_t P(z_1, z') = P(z_1, tz'). \tag{2.7}$$

Note that $G_1 P = P$ and $G_0 P = P(\mathbf{e}) z_1^m$. Since P is normalised $P_(\mathbf{e}) = 1$, hence we can connect any normalised hyperbolic polynomial P to the fixed polynomial z_1^m

via the polynomials G_sP for $0 \leq s \leq 1$. This family of operators is equicontinuous on every bounded set of polynomials. Hence every closed curve in the space of normalised hyperbolic polynomials can be contracted to the point $G_0P = z_1^m$.

Note if P is strictly hyperbolic we can use the operator from (2.2) to connect P to the fixed polynomial F_1G_0P via $F_{1-s}G_sP$ for $0 \leq s \leq 1$ where the $F_{1-s}G_sP$ are strictly hyperbolic. Hence this theorem also holds for normalised strictly hyperbolic polynomials. □

2.2 Univariate Hyperbolic Polynomials

In this section we give results for univariate hyperbolic polynomials which motivate our result for Gårding hyperbolic polynomials. For more literature see for example [17] [18].

Definition 2.2.1 (Univariate Hyperbolic Polynomials). *Let $p \in \mathbb{R}[z]$ be a polynomial in one variable, we say p is* univariate hyperbolic *if it has only real roots.*

It is clear to see from the definitions that a polynomial P on $\mathbb{R}^n$ is Gårding hyperbolic with respect to $\mathbf{e}$ if for all lines $l \in \mathfrak{L}_{\mathbf{e}}$, restricting P to l gives a univariate hyperbolic polynomial.

The following is an important lemma of Nuij (see [22]).

Lemma 2.2.2. *If p is univariate hyperbolic, then $p + sp'$ is univariate hyperbolic for all $s \in \mathbb{R}$.*

Proof. Since p is univariate we may factorise

$$p(z) = A\prod_{j=1}^{k}(z - z_j)^{m_j}. \tag{2.8}$$

First we will assume that the zeros of p satisfy $\mathrm{Im}(z_j) < a$ and show $p + sp'$ satisfies the same. Indeed explicitly

$$p(z) + sp'(z) = p(z)\Big(1 + s\sum_{j=1}^{k} m_j(z - z_j)^{-1}\Big). \tag{2.9}$$

Now consider z for which $\mathrm{Im}(z) \geq a$. Then each term on the right hand side of (2.9) has strictly positive imaginary part, so clearly $p + sp'$ cannot be zero when $\mathrm{Im}(z) \geq a$. This is clearly also true when $\mathrm{Im}(z_j) > a$, so it follows that for p with real roots, the roots of $p + sp'$ are also real. □

Kurdyka and Paunescu generalise this construction [18] by giving a collection of linear maps preserving univariate hyperbolicity.

Definition 2.2.3. *Let $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. If for all univariate hyperbolic polynomials p of degree d we have*

$$p_a(z, s) := p(z) + \sum_{k=1}^{d} a_k s^k p^{(k)}(z), \tag{2.10}$$

is a univariate hyperbolic polynomial for all $s \in \mathbb{R}$, then we say a is a Nuij sequence.

Theorem 2.2.4. *A sequence $a = (a_1, \ldots, a_d) \in \mathbb{R}^n$ is a Nuij sequence if and only if the polynomial*

$$q_a(z) := z^d + \sum_{k=1}^{d} a_k (z^d)^{(k)}, \tag{2.11}$$

is hyperbolic. Note this is simply (2.10) with $p(z) = z^d$ and $s = 1$.

The proof of this theorem (and our result for Gårding hyperbolic polynomials) relies on a lemma of Borcea and Brändén [3] on another related type of polynomial called stable polynomials. For completeness we include a short exposition of the necessary results here.

Definition 2.2.5. *Let $p \in \mathbb{C}[z_1, \ldots, z_n]$ be a degree m polynomial. If for all $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $Im(z_i) > 0$ $1 \leq i \leq n$ we have $p(z_1, \ldots, z_n) \neq 0$ then we say p is* stable. *Denote the space of stable polynomials in m variables by $\mathcal{H}_m(\mathbb{C})$.*

The following are two general results of complex analysis which we state without proof. The first can be found for example in [1].

Theorem 2.2.6 (Hurwitz)**.** *Let $\{f_k\}$, f be functions holomorphic on some connected open set G such that the sequence $\{f_k\}$ converges uniformly to f on compact subsets of G. If f has a zero of order m at z_0, then for all sufficiently small $r > 0$ there exist* *sufficiently large $k \in \mathbb{N}$ such that f_k has precisely m zeros in the disk $|z - z_0| < r$ counting multiplicity.*

The second was introduced by Grace in 1902 [10].

Theorem 2.2.7 (Grace–Walsh–Szegö coincidence theorem)**.** *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be symmetric and multi-affine (of degree at most 1 in each variable). Given points $\zeta_1, \ldots, \zeta_n$ in a circular domain C, if the $\deg(f) = n$ or C is convex, then there exist a point $\zeta \in C$ such that*

$$f(\zeta_1, \ldots, \zeta_n) = f(\zeta, \ldots, \zeta). \tag{2.12}$$

The following final two results of Borcea and Brändén give us the tools sufficient to prove our theorem on linear operators preserving Gårding hyperbolicity.

Corollary 2.2.8. *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial with degree at most d in the variable z_1. Now denote the coefficents in the expansion in terms of z_1 by setting*

$$f(z_1, \ldots, z_n) = \sum_{k=0}^{d} Q_k(z_2, \ldots, z_n) z_1^k. \tag{2.13}$$

Let $e_k(x_1, \ldots, x_d)$ denote the elementary symmetric function of degree k in d variables. Then the polynomial f is stable if and only if the polynomial

$$\sum_{k=0}^{d} Q_k(z_2, \ldots, z_n) \frac{e_k(x_1, \ldots, x_d)}{\binom{d}{k}}, \tag{2.14}$$

is stable in all the variables $z_2, \ldots, z_n, x_1, \ldots, x_d$.

Proof. First, assume f is stable. Fix some $\zeta_2, \ldots, \zeta_n$ with $\text{Im}(z_i) > 0$, then

$$\sum_{k=0}^{d} Q_k(\zeta_2, \ldots, \zeta_n) \frac{e_k(x_1, \ldots, x_d)}{\binom{d}{k}}, \tag{2.15}$$

is symmetric and multi-affine. Now for any points $(\xi_1, \ldots, \xi_d)$ with $\text{Im}(\xi_k) > 0$, by the Grace-Walsh-Szegö Coincidence Theorem there is a point ζ_1 with $\text{Im}(\zeta_1) > 0$

such that

$$\sum_{k=0}^{d} Q_k(\zeta_2, \ldots, \zeta_n) \frac{e_k(\xi_1, \ldots, \xi_d)}{\binom{d}{k}} = \sum_{k=0}^{d} Q_k(\zeta_2, \ldots, \zeta_n) \frac{e_k(\zeta_1, \ldots, \zeta_1)}{\binom{d}{k}} \tag{2.16}$$

Now note

$$\frac{e_k(z_1, \ldots, z_1)}{\binom{d}{k}} = z_1^k, \tag{2.17}$$

combining with (2.16) we have

$$\sum_{k=0}^{d} Q_k(\zeta_2, \ldots, \zeta_n) \frac{e_k(\xi_1, \ldots, \xi_d)}{\binom{d}{k}} = f(\zeta_1, \ldots, \zeta_n) \neq 0, \tag{2.18}$$

hence (2.14) is stable.

Now assume (2.14) is stable. For any point $(z_2, \ldots, z_n, x_1, \ldots, x_d)$ with all variables with positive strictly positive imaginary part, we have (2.14) is not zero. Now using (2.17), for any point $(z_1, \ldots, z_n)$ with $\mathrm{Im}(z_k) > 0$ we have

$$f(z_1, \ldots, z_n) = \sum_{k=0}^{d} Q_k(z_2, \ldots, z_n) \frac{e_k(z_1, \ldots, z_1)}{\binom{d}{k}} \neq 0. \tag{2.19}$$

Hence f is stable which proves the corollary. □

Lemma 2.2.9. *Consider $T : \mathbb{C}_d[z] \to \mathbb{C}[z]$ a linear operator. We extend T to a linear operator on $\mathbb{C}[z, w]$ by setting $T(z^i w^j) = T(z^i) w^j$. Let T be such that $T[(z + w)^d] \in \mathcal{H}_2(\mathbb{C})$. If $f \in \mathcal{H}_1(\mathbb{C})$ with $\deg(f) \leq d$, then $T(f) \in \mathcal{H}_1(\mathbb{C}) \cup \{0\}$.*

Proof. First assume $f \in \mathcal{H}_1(\mathbb{C})$ is of degree exactly d in z. For $\epsilon > 0$ set

$$f_\epsilon(z) = f(z + \epsilon i). \tag{2.20}$$

We may factorise

$$f_\epsilon(z) = C(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_d) = C \sum_{k=0}^{d} (-1)^k e_k(\zeta_1, \ldots, \zeta_d) z^{n-k}, \tag{2.21}$$

where $C \neq 0$ and $\mathrm{Im}(\zeta_i) < 0$, since f is stable in z and $\epsilon > 0$. Now by assumption in the lemma

$$T[(z+w)^d] = \sum_{k=0}^{d} \binom{d}{k} T(z^{d-k}) w^k \in \mathcal{H}_2(\mathbb{C}), \tag{2.22}$$

so applying Hurwitz's theorem we have

$$\sum_{k=0}^{d} T(z^{d-k}) e_k(w_1, \ldots, w_d) \in \mathcal{H}_{d+1}(\mathbb{C}). \tag{2.23}$$

Since T is a linear operator

$$T(f_\epsilon)(z) = C \sum_{k=0}^{d} T(z^{d-k}) e_k(-\zeta_1, \ldots, -\zeta_d). \tag{2.24}$$

Now by (2.23) we have $T(f_\epsilon)(z) \in \mathcal{H}_1(\mathbb{C})$ since the ζ_i have strictly positive imaginary part (2.24) can not have a root when z has strictly positive imaginary part. Letting $\epsilon \to 0$ we have $T(f) \in \mathcal{H}_1(\mathbb{C}) \cup \{0\}$.

Finally for the case $f \in \mathcal{H}_1(\mathbb{C})$ has degree less than d, we consider the new function $f^\epsilon = (1-\epsilon i z)^{d-\deg(f)}$. Note $f^\epsilon \in \mathcal{H}_1(\mathbb{C})$, so by the above argument we have $T(f^\epsilon) \in \mathcal{H}_1(\mathbb{C})$ for all $\epsilon > 0$. Now since $f^\epsilon \to f$ as $\epsilon \to 0$, by Hurwitz' theorem we have $f \in \mathcal{H}_1(\mathbb{C}) \cup \{0\}$ which proves the lemma. □

2.3 Gårding–Nuij Sequences

Nuij makes important use of a collection of hyperbolicity preserving operators in his paper [22] to prove results on the space of hyperbolic polynomials. Assume that P is hyperbolic with respect to the x_n direction, then these operators are of the form:

$$T_{k,s}P \coloneqq P + s x_k \frac{\partial P}{\partial x_n} \tag{2.25}$$

where $s \in \mathbb{R}$ parameterises a collection of operators. In his paper Nuij combines these first order operators to form ones with higher order derivatives. For example in a 3-dimensional setting we could have

$$T_{x,s}T_{x,s}T_{y,s}T_{y,s} = P + s(2x+2y)\frac{\partial P}{\partial z} + s^2(x^2+4xy+y^2)\frac{\partial^2 P}{\partial z^2} + s^3(2x^2y+2xy^2)\frac{\partial^3 P}{\partial z^3} + s^4x^2y^2\frac{\partial^4 P}{\partial z^4} \quad (2.26)$$

Note this bears resemblance to the construction of Nuij sequences from Kurdyka and Paunescu's paper [18]. This motivates the following definition in the setting of Gårding's hyperbolic polynomials.

Definition 2.3.1. *A sequence $a = (a_1, \ldots, a_m)$ of polynomials, where $a_k \in \mathbb{R}_k[x_1, \ldots, x_{n-1}]$, is a* Gårding Nuij sequence *if, given any $P(x_1, \ldots, x_n)$, a polynomial of degree m hyperbolic with respect to the x_n direction, the polynomial*

$$P_{a,s}(x_1, \ldots, x_n) := P(x_1, \ldots, x_n) + \sum_{k=1}^{m} a_k s^k \frac{\partial^k P(x_1, \ldots, x_n)}{\partial x_n^k}, \quad (2.27)$$

is hyperbolic with respect to x_n for any $s \in \mathbb{R}$.

We now prove the main result of this chapter giving a condition for any sequence of homogenisation coefficients to give rise to a hyperbolicity preserving operator.

Theorem 2.3.2. *A sequence $a = (a_1, \ldots, a_m)$ where $a_k \in \mathbb{R}_k[x_1, \ldots, x_{n-1}]$ is a Gårding–Nuij sequence if and only if*

$$Q_a(x_1, \ldots, x_n) = x_n^m + \sum_{k=1}^{m} a_k \frac{\partial^k (x_n^m)}{\partial x_n^k}, \quad (2.28)$$

is a hyperbolic polynomial.

Proof. We begin with the easy direction. If a is a Gårding Nuij sequence then Q_a is a hyperbolic polynomial. Now Q_a is simply the Nuij sequence construction acting upon the polynomial x_n^m, so since $P = x_n^m$ is hyperbolic with respect to x_n, we have Q_a is hyperbolic.

This leaves us to prove: if Q_a is a hyperbolic polynomial, then a is a Gårding–

Nuij sequence. Define a map

$$T_a(P)(x_1, \ldots, x_n) \coloneqq P(x_1, \ldots, x_n) + \sum_{k=1}^{m} a_k \frac{\partial^k P(x_1, \ldots, x_n)}{\partial x_n^k}. \tag{2.29}$$

We need to show that if Q_a is hyperbolic then T_a preserves hyperbolicity.

Now, T_a preserves hyperbolicity if for all $v = (v_1, \ldots, v_n)$ fixed, the univariate polynomial given by restricting $T_a(P)$ to the line through v in the direction x_n has only real roots. Denote $T_a(P)$ restricted to this line by

$$T_{a,v}(P_v) = T(P)(v_1, \ldots, v_{n-1}, x_n), \tag{2.30}$$

so $T_{a,v} : \mathbb{C}_m[z] \to \mathbb{C}_m[z]$ is a linear map.

Explicitly we have

$$T_{a,v}(P_v)(x_n) = P_v(x_n) + \sum_{k=1}^{m} a_k(v_1, \ldots, v_{n-1}) \frac{dP_v(x_n)}{dx_n}, \tag{2.31}$$

and since the $v_1, \ldots, v_{n-1}$ are fixed, so is $a_k(v_1, \ldots, v_{n-1})$ which we will denote $a_{k,v}$.

To show T_a preserves hyperbolicity we will show $T_{a,v}((z+w)^m) = Q_{a,v}(z+w)$, to which we will be able to apply Lemma 2.2.9.

First consider $T_{a,v}$,

$$\begin{aligned} T_{a,v}((z+w)^m) &= T_{a,v}\Big(\sum_{i=0}^{m} \binom{m}{i} z^i w^{m-i}\Big) \\ &= \sum_{i=0}^{m} \binom{m}{i} w^{m-i} T_{a,v}(z^i). \end{aligned} \tag{2.32}$$

Now note that

$$T_{a,v}(z^i) = \sum_{j=0}^{i} a_{j,v} (z^i)^{(j)} = \sum_{j=0}^{i} a_{j,v} \frac{i!}{(i-j)!} z^{i-j}, \tag{2.33}$$

so plugging this into the above we obtain

$$\sum_{i=0}^{m}\binom{m}{i}w^{m-i}\Big(\sum_{j=0}^{i}a_{j,v}\frac{i!}{(i-j)!}z^{i-j}\Big)$$
$$=\sum_{i=0}^{m}\Big(\frac{m!}{(m-i)!i!}\Big)w^{m-i}\Big(\sum_{j=0}^{i}a_{j,v}\frac{i!}{(i-j)!}z^{i-j}\Big). \quad (2.34)$$

Now we can pick out the expression that comes with each $a_{j,v}$,

$$\sum_{i=j}^{m}\frac{m!}{(m-i)!i!}\frac{i!}{(i-j)!}z^{i-j}w^{m-i}, \quad (2.35)$$

and relabelling the index gives

$$\sum_{k=0}^{m-j}\frac{m!}{(m-j-k)!k!}z^{k}w^{m-j-k}. \quad (2.36)$$

Now we consider $Q_{a,v}(z+w)$. By definition we have

$$Q_{a,v}(z+w)=\sum_{i=0}^{m}\frac{m!}{(m-i)!}a_{i,v}(z+w)^{m-i}. \quad (2.37)$$

The coefficient of $a_{j,v}$ here is

$$\begin{aligned}\frac{m!}{(m-j)!}(z+w)^{m-j} &= \frac{m!}{(m-j)!}\sum_{k=0}^{m-j}\binom{m-j}{k}z^{k}w^{m-j-k}\\ &= \frac{m!}{(m-j)!}\sum_{k=0}^{m-j}\frac{(m-j)!}{(m-j-k)!k!}z^{k}w^{m-j-k} \quad (2.38)\\ &= \sum_{k=0}^{m-j}\frac{m!}{(m-j-k)!k!}z^{k}w^{m-j-k}.\end{aligned}$$

This is the same expression for the $a_{j,v}$ as in $T_{a,v}$ hence $T_{a,v}((z+w)^m)=Q_{a,v}(z+w)$.

Now since $Q_{a,v}(x_n)$ has only real roots, we have that $Q_{a,v}(x_n+w)$ is a stable polynomial in two variables. Hence since $T_{a,v}((x_n+w)^m)=Q_{a,v}(x_n+w)$, by Lemma 2.2.9 we have $T_{a,v}$ preserves stability. Restricting to real polynomials, $\mathbb{R}_m[x_1,\dots,x_n]$, we have that $T_{a,v}$ preserves real-rootedness. So for all $v\in\mathbb{R}^n$ we have if P restricted to the line through v in the direction x_n has only real roots, then $T(P)$ restricted to the same line also has only real roots. Hence T preserves Gårding hyperbolicity and $P_{a,1}(x_1, \ldots, x_n)$ is hyperbolic with respect to x_n.

It remains to prove hyperbolicity for general $s \in \mathbb{R}$. Choose $s \in \mathbb{R}^*$. Clearly if $Q_a(x_1, \ldots, x_n)$ is hyperbolic in the x_n direction, then so is $s^{-m}Q_a(x_1, \ldots, x_{n-1}, sx_n)$. Explicitly we have

$$\begin{aligned} s^m Q_a(x_1, \ldots, x_{n-1}, s^{-1}x_n) &= s^m \left(s^{-m} x_n^m + \sum_{k=1}^{m} a_k \frac{\partial^k \left((s^{-1}x_n)^m \right)}{\partial (s^{-1}x_n)^k} \right) \\ &= x_n^m + \sum_{k=1}^{m} \frac{m!}{(m-k)!} a_k s^k x_n^{m-k}. \end{aligned} \tag{2.39}$$

This is simply the construction $Q_{\tilde{a}}(x_1, \ldots, x_n)$ for a new set of homogenisation coefficients $\tilde{a} = (a_1 s, \ldots, a_k s^k, \ldots, a_m s^m)$. Using this $Q_{\tilde{a}}$ and the above argument we have that $P_{\tilde{a},1}(x_1, \ldots, x_n)$ is hyperbolic in the x_n direction. This is simply $P_{a,s}(x_1, \ldots, x_n)$, hence the theorem is proved. □

Discussion

We saw how Nuij used first order operators on polynomials to generate paths within the space of Gårding hyperbolic polynomials. With these he proved various properties of the space. The space however is still not well understood, we have no parameterisation and in general it is difficult to determine hyperbolicity for a given polynomial. This is rather a spanner in the works as hyperbolic polynomials are of great interest to the field of optimisation, lending their hyperbolicity cones to the posing of hyperbolic programming problems.

As such, it is of interest to further our understanding of this space, so we might better use these tools for our applications. The main result of this chapter has generalised the tools which Nuij used to prove results on the space. This gives us greater freedom in understanding how we might move within the space. Further it shows the importance of the trivial hyperbolic polynomial x_n^m.

It would now be of interest to consider operators acting on x_n^m and determine more explicitly our Gårding–Nuij sequences. Once these are determined we may use these operators to move more generally through the space of hyperbolic polynomials and try to shed light on some greater structure.

CHAPTER 3

A Rigidity Theorem for Spacelike Hypersurfaces in de Sitter Space

In this final chapter we move into the world of differential geometry and prove a rigidity theorem for spacelike hypersurfaces in de Sitter space. This proof will make vital use of hyperbolic polynomials, namely Gårding's inequality which we saw in Chapter 1.

The idea of this rigidity theorem is long standing and has been proved in many different settings. Essentially we wish to show that two locally isometric hypersurfaces (under the right conditions) must simply be the same hypersurface up to some rigid motion of the ambient space. In 1927 Cohn-Vossen proved this result for regular surfaces in $\mathbb{R}^3$ [4]. The theorem was proved for hypersurfaces in spherical space $\mathbb{S}^n$ by do Carmo and Warner in 1970 [8]. More recently the rigidity theorem was generalised to hypersurfaces in n-dimensional space forms by Guan and Shen [11]. Space forms being $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$, Riemannian manifolds of constant sectional curvature. See [14] for a detailed exposition of this paper.

In this chapter, we will now prove the rigidity result for spacelike hypersurfaces in de Sitter space.

3.1 de Sitter Space

3.1.1 The Manifold

De Sitter space is the Lorentzian manifold with constant sectional curvature $\bar{K} = 1$. It originally arose as a solution to the Einstein field equations in the study of general relativity (for an overview of de Sitter space from this viewpoint see for example [24]). It is most commonly realised as the set of points of unit distance from the origin in Minkowski space; it is the Lorentzian analogue of the Euclidean sphere. Minkowski space, which we will denote $(\mathbb{R}^{n+1}_1, g^M)$, is the set $\mathbb{R}^{n+2}$ equipped with the metric

$$g^M = ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2. \tag{3.1}$$

It will be more useful to us to use cylindrical polar coordinates on Minkowski space, (x_0, r, Z) for $Z \in \mathbb{S}^n$. The metric with respect to the coordinate basis $\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial r}, \frac{\partial}{\partial Z_i}\}_{i=1}^n$ is

$$g^M = ds^2 = -dx_0^2 + dr^2 + r^2\sigma, \tag{3.2}$$

where σ is the standard round metric on $\mathbb{S}^n$. This metric induces a quadratic form on $\mathbb{R}^{n+1}_1$:

$$||x|| = d(0, x) = -x_0^2 + r^2. \tag{3.3}$$

Now as mentioned before, de Sitter space, $(dS^{n+1}, \bar{g})$, is the set of points a unit distance from the origin, that is

$$dS^{n+1} := \{x \in \mathbb{R}^{n+1}_1 |\ ||x|| = 1\}. \tag{3.4}$$

This clearly simply gives a relation between x_0 and r, namely $r = \sqrt{x_0^2 + 1}$. If we then choose coordinates, on dS^{n+1}, (t, θ) for $\theta \in \mathbb{S}^n$, these map into $dS^{n+1} \subset \mathbb{R}^{n+1}_1$ by

$$(t, \theta) \to (t, \sqrt{t^2 + 1}, \theta). \tag{3.5}$$

Now let us see what the metric induced on dS^{n+1} from $\mathbb{R}^{n+1}_1$ looks like in terms of the coordinate basis $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta_i}\}_{i=1}^n$. First we express these basis vectors in terms of the coordinate basis on $\mathbb{R}^{n+1}_1$ by considering their action on a smooth function $f \in C^\infty(\mathbb{R}^{n+1}_1)$:

$$\begin{aligned} \frac{\partial}{\partial t} f(x_0, r, Z) &= \frac{\partial x_0}{\partial t}\frac{\partial}{\partial x_0} f + \frac{\partial r}{\partial t}\frac{\partial}{\partial r} f + \sum_j \frac{\partial Z_j}{\partial t}\frac{\partial}{\partial Z_j} f \\ &= \Big(\frac{\partial}{\partial x_0} + \frac{t}{\sqrt{t^2+1}}\frac{\partial}{\partial r}\Big) f, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \frac{\partial}{\partial \theta_i} f(x_0, r, Z) &= \frac{\partial x_0}{\partial \theta_i}\frac{\partial}{\partial x_0} f + \frac{\partial r}{\partial \theta_i}\frac{\partial}{\partial r} f + \sum_j \frac{\partial Z_j}{\partial \theta_i}\frac{\partial}{\partial Z_j} f \\ &= \frac{\partial}{\partial Z_i} f. \end{aligned} \tag{3.7}$$

Plugging these into the metric on $\mathbb{R}^{n+1}_1$ given in (3.2) we obtain

$$\begin{aligned} \bar{g} = ds^2 &= \Big(-1 + \frac{t^2}{t^2+1}\Big) dt^2 + (t^2+1)\sigma \\ &= \Big(\frac{-1}{t^2+1}\Big) dt^2 + (t^2+1)\sigma. \end{aligned} \tag{3.8}$$

where again σ is the round metric on $\mathbb{S}^n$. Now, t is simply the timelike coordinate of Minkowski space. We reparameterise this with an arc length parameter ρ along the t coordinate. Let $\gamma : \lambda \to (\lambda, \theta_0)$ be the t-coordinate curve for some constant θ_0, then let

$$\begin{aligned} \rho(t) &= \int_0^t ||\gamma'(\lambda)|| d\lambda \\ &= \int_0^t \sqrt{\Big\langle \frac{\partial}{\partial t}\Big|_\lambda, \frac{\partial}{\partial t}\Big|_\lambda \Big\rangle} d\lambda \\ &= \int_0^t \sqrt{\frac{1}{\lambda^2+1}} d\lambda \\ &= \operatorname{arsinh}(t). \end{aligned} \tag{3.9}$$

So we have $t = \sinh(\rho)$. Let's just go ahead and state that $dt = \cosh(\rho) d\rho$, and so the metric with respect to the basis $\{\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta_i}\}_{i=1}^n$ is

$$\bar{g} = ds^2 = -d\rho^2 + \cosh^2(\rho)\sigma. \tag{3.10}$$

3.1.2 Spacelike Hypersurfaces

We will be considering spacelike hypersurfaces in de Sitter space. The term *spacelike* refers to hypersurfaces whose normal is timelike, that is $\langle \nu, \nu \rangle_{dS} < 0$ for ν normal to $M \subset dS$. This will mean that the tangent vectors to M are indeed spacelike, and the metric positive definite. A hypersurface (M, g) will be represented as a graph over the unit sphere:

$$\begin{aligned} y : \mathbb{S}^n &\to dS^{n+1} \\ \zeta &\mapsto (y(\zeta), \zeta). \end{aligned} \tag{3.11}$$

So we have a natural coordinate system on M given by (ζ) for $\zeta \in \mathbb{S}^n$. Now let's see what the coordinate basis looks like. As before consider its action on a smooth function $C^\infty(dS)$:

$$\begin{aligned} \frac{\partial}{\partial \zeta_i} f(\rho(\zeta), \theta(\zeta)) &= \frac{\partial \rho}{\partial \zeta_i} \frac{\partial}{\partial \rho} f + \sum_j \frac{\partial \theta_j}{\partial \zeta_i} \frac{\partial}{\partial \theta_j} f \\ &= \Big(\frac{\partial y}{\partial \zeta_i} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \theta_i} \Big) f. \end{aligned} \tag{3.12}$$

Then the metric, g, induced from de Sitter space, with respect to this metric is

$$g = ds^2 = \sum_{i,j} \Big(- \frac{\partial y}{\partial \zeta_i} \frac{\partial y}{\partial \zeta_j} + \cosh^2(y(\zeta)) \sigma_{ij} \Big) d\zeta_i d\zeta_j, \tag{3.13}$$

for σ_{ij} the $d\zeta_i d\zeta_j$ component of the round metric on $\mathbb{S}^n$.

Now we find conditions on the graph y for the hypersurface M to be spacelike. First we need an expression for the normal, $\nu = \nu_0 \frac{\partial}{\partial \rho} + \sum_i \nu_i \frac{\partial}{\partial \theta_i}$. Using $\langle \nu, \frac{\partial}{\partial \zeta_i} \rangle_{dS} = 0$ for all i, we obtain

$$-\nu_0 \frac{\partial y}{\partial \zeta_i} + \cosh^2(y) \sigma \Big(\sum_j \nu_j \frac{\partial}{\partial \theta_j}, \frac{\partial}{\partial \theta_i} \Big) = 0. \tag{3.14}$$

The standard coordinates give a diagonal metric on the sphere, so we have

$$\nu_0 \frac{\partial y}{\partial \zeta_i} = \cosh^2(y) \sigma_{ii} \nu_i. \tag{3.15}$$

Since we are looking for spacelike hypersurfaces, we want ν to be timelike, so ν_0 better be non-zero. Let us just say $\nu_0 = \cosh^2(y)$, so

$$\nu = \cosh^2(y)\frac{\partial}{\partial\rho} + \sum_i \left(\frac{1}{\sigma_{ii}}\frac{\partial y}{\partial\zeta_i}\right)\frac{\partial}{\partial\theta_i}. \tag{3.16}$$

Now plugging this into $\bar{g}$,

$$\langle\nu,\nu\rangle_{dS} = -\cosh^4(y) + \cosh^2(y)\sum_i \sigma_{ii}\frac{\partial y}{\partial\zeta_i}\frac{\partial y}{\partial\zeta_i}. \tag{3.17}$$

So finally for M to be spacelike, we must have

$$\sum_i \frac{1}{\sigma_{ii}\cosh^2(y)}\left(\frac{\partial y}{\partial\zeta_i}\right)^2 < 1, \tag{3.18}$$

for all $\zeta \in \mathbb{S}^n$.

3.2 Gårding's Inequality for Rigidity

This section adapts a result of Guan and Shen in [11] to de Sitter space. We will be considering the Weingarten maps (or otherwise shape operators) for all points of a hypersurface, M, in de Sitter space, dS^{n+1}. These lie in the space of self-adjoint operators (with respect to the metric g on M) over T_pM for a point $p \in M$, which we will denote $\mathcal{SA}(T_pM)$ (or just $\mathcal{SA}(V)$ for an inner product space V).

We define the second symmetric function of a linear map W as

$$\sigma_2(W) = \sum_{i<j} \kappa_i\kappa_j, \tag{3.19}$$

where the κ_i are the eigenvalues of W.

Lemma 3.2.1. *For any given matrix W, we have*

$$\sigma_2(W) = \sum_{i<j} w_{ii}w_{jj} - w_{ij}w_{ji}. \tag{3.20}$$

Proof. First note that the trace of a linear operator does not depend on choice of basis. Therefore we have

$$tr(W) = \sum_i w_{ii} = \sum_i \kappa_{ii}. \tag{3.21}$$

Then

$$2\sigma_2(W) = 2\sum_{i<j} \kappa_i\kappa_j = \sum_{i,j} \kappa_i\kappa_j - \sum_i \kappa_i\kappa_i = tr(W)^2 - tr(W^2). \tag{3.22}$$

Now consider the two traces separately, for components w_{ij} of W under any given basis. We have

$$tr(W)^2 = \sum_{i,j} w_{ii}w_{jj} = 2\sum_{i<j} w_{ii}w_{jj} + \sum_i w_{ii}w_{ii}, \tag{3.23}$$

and

$$tr(W^2) = \sum_{i,j} w_{ij}w_{ji} = 2\sum_{i<j} w_{ij}w_{ji} + \sum_i w_{ii}w_{ii}. \tag{3.24}$$

Plugging (3.23) and (3.24) into (3.22) we obtain the result. □

We will now show that σ_2 is hyperbolic with respect to the identity matrix in $\mathcal{SA}(T_pM)$. We begin by showing the determinant map is hyperbolic with respect to the identity in $\mathcal{SA}(T_pM)$. Since the determinant is of degree n, we must show that the univariate polynomial

$$t \mapsto \det(M + tI) \tag{3.25}$$

has n real zeros for all $M \in \mathcal{SA}(T_pM)$. Since we only care if the roots are real, we may as well replace t with $-t$. Then this simply becomes the eigenvalue equation for M, but since M is self adjoint we know that all its eigenvalues are real. Therefore (3.25) has n real roots and the determinant is hyperbolic with respect to the identity.

Now we have a corollary to Lemma 1.3.1 (see [9]).

Corollary 3.2.2. *Given a degree $m > 1$ polynomial P, hyperbolic with respect to*

$\mathbf{a} \in \mathbb{R}^n$, *the polynomials* $\{P_i\}_{i=0}^m$, *defined by*

$$P(s\mathbf{a} + \mathbf{x}) = \sum_{i=0}^{m} s^i P_i(\mathbf{x}), \tag{3.26}$$

are also hyperbolic with respect to $\mathbf{a}$.

Proof. Consider evaluating this polynomial $P(s\mathbf{a} + \mathbf{x})$ at $s = 0$. We then have

$$P(\mathbf{x}) = P_0(\mathbf{x}), \tag{3.27}$$

and since P is hyperbolic $\mathbf{a}$, P_0 must also be hyperbolic $\mathbf{a}$. Now consider the polynomial

$$Q(s\mathbf{a} + \mathbf{x}) = \frac{d}{ds} P(s\mathbf{a} + \mathbf{x}) = \sum_{i=1}^{m} i s^{i-1} P_i(\mathbf{x}). \tag{3.28}$$

Evaluating this at $s = 0$, we obtain

$$Q(\mathbf{x}) = P_1(\mathbf{x}). \tag{3.29}$$

Note that this Q is exactly the same Q as defined in Lemma 1.3.1, so since P is hyperbolic $\mathbf{a}$, we have $Q = P_1$ is hyperbolic $\mathbf{a}$. Repeating this process shows that all the P_i are hyperbolic with respect to $\mathbf{a}$. □

Now note that the characteristic polynomial for a matrix satisfies

$$\begin{aligned} (-1)^n \det(M - tI) &= \sum_{i=0}^{n} (-1)^i \, t^{n-i} \text{ (sum of all combinations of } i \text{ eigenvalues)} \\ &= \sum_{i=0}^{n} (-1)^i \, t^{n-i} \sigma_i(M), \end{aligned} \tag{3.30}$$

where $\sigma_i(M)$ is the i^{th} symmetric function of the eigenvalues of M. Now since the determinant is hyperbolic with respect to the identity, we have $(-1)^n det(M)$ is hyperbolic with respect to $-I$. Then by Corollary 3.2.2, we have that the functions $(-1)^i \, \sigma_i$ are hyperbolic with respect to $-I$. So finally, the symmetric functions, in particular σ_2, are hyperbolic with respect to the identity.

Lemma 3.2.3. *Let K and $\bar{K}$ denote the scalar curvatures of M^n and dS^{n+1} respectively. We have yet another expression for σ_2:*

$$\sigma_2(W) = \frac{n(n-1)}{2}(K - \bar{K}). \tag{3.31}$$

As a corollary, since the scalar curvature is invariant under local isometries, we have $\sigma_2(W) = \sigma_2(\tilde{W})$.

Proof. For each point p, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of T_pM, made up of eigenvectors of $W(p)$. As a simple corollary to the Gauss equation we have

$$K(e_i, e_j) - \bar{K}(e_i, e_j) = \kappa_i \kappa_j. \tag{3.32}$$

By the definition of $\sigma_2(W)$, we have

$$\sigma_2(W) = \sum_{i<j} \kappa_i \kappa_j = \sum_{i<j} K(e_i, e_j) - \sum_{i<j} \bar{K}(e_i, e_j). \tag{3.33}$$

For the first term on the right hand side, we have

$$\begin{aligned} K &= \frac{1}{n(n-1)} \sum_{i,j} K(e_i, e_j) \\ &= \frac{1}{n(n-1)} \Big(\sum_{i<j} K(e_i, e_j) + \sum_{i=j} K(e_i, e_j) + \sum_{i>j} K(e_i, e_j) \Big). \end{aligned} \tag{3.34}$$

Since $K(e_i, e_j) = K(e_j, e_i)$ and $K(e_i, e_i) = 0$, this reduces to

$$K = \frac{2}{n(n-1)} \sum_{i<j} K(e_i, e_j). \tag{3.35}$$

Similarly for the second term on the right hand side of (3.33), we have

$$\frac{2}{n(n-1)} \sum_{i<j} \bar{K}(e_i, e_j) = \frac{1}{n(n-1)} \sum_{i,j} \bar{K}(e_i, e_j). \tag{3.36}$$

Of course the right hand side is not the definition of the scalar curvature of the ambient space, since we are not taking the average over a full orthonormal basis of the tangent space of the ambient manifold. However, since the ambient space

dS^{n+1} has constant sectional curvature, all of the $K(u,v)$ are the same for any two orthonormal vectors u and v. So it does not matter if we take the average over $n(n-1)$ pairs of orthonormal vectors or $n(n+1)$, we will still arrive at the scalar curvature. So we have

$$\frac{2}{n(n-1)}\sum_{i<j}\bar{K}(e_i,e_j)=\bar{K}. \tag{3.37}$$

Putting this all together, we obtain the result

$$\sigma_2(W)=\frac{n(n-1)}{2}(K-\bar{K}). \tag{3.38}$$

□

From Lemma 3.2.3 note that if $K>\bar{K}$ for all points on the hypersurface, then $\sigma_2(W)$ is positive for all points on the hypersurface. We now claim that this implies that the Weingarten maps $W(p)$ for every point $p\in M$ all lie in the same hyperbolicity cone. For any point $M\in\mathcal{SA}(T_pM)$, consider the affine line through M in the direction of I. Since σ_2 is hyperbolic with respect to the identity, this affine line must cross the hypersurface $\sigma_2=0$ exactly twice (with multiplicities). Let $t_1<t_2$ be the roots of $t\mapsto\sigma_2(M+tI)$. Since σ_2 is of degree two, if $t_1\neq t_2$ then $\sigma_2(M+tI)$ changes sign at t_1 and at t_2, and if $t_1=t_2$ then $\sigma_2(M+tI)$ has the same sign either side of the root. Importantly, note that in either case we have points $\sigma_2(M+tI)$ has the same sign for $t<t_1$ and $t>t_2$, and values $t_1<t<t_2$ have the opposite sign. Now by definition, the points $M+tI$ with $t>t_2$ are in $C(\sigma_2,I)$ and the points $M+tI$ with $t<t_1$ are in $C(\sigma_2,-I)$. Since $C(\sigma_2,I)=S(\sigma_2,I)$, for any affine line $M+tI$, we have $\operatorname{sgn}(\sigma_2(M+tI))=\operatorname{sgn}(\sigma_2(I))>0$ for all points with $t<t_1$ or $t>t_2$, and for any other $t_1<t<t_2$ we have $\operatorname{sgn}(\sigma_2(M+tI))<0$. This means σ_2 is hyperbolic with respect to any $W\in\mathcal{SA}(T_pM)$ with $\sigma_2(W)>0$. Since we stipulate that $K>\bar{K}$, by Lemma 3.2.3 σ_2 is hyperbolic with respect to $W(p)$ for all points $p\in M$. Furthermore, since $\sigma_2(W(p))$ is strictly greater than zero, it always stays in the same connected component of $\{M\in\mathcal{SA}(T_pM)\mid\sigma_2(M)\neq 0)\}$. So either $\{W(p)\mid p\in M\}\subset S(\sigma_2,I)$ or $\{W(p)\mid p\in M\}\subset S(\sigma_2,-I)$.

We will be considering two isometric hypersurfaces M and $\tilde{M}$ and their respective

Weingarten tensors W and $\tilde{W}$. We would like Theorem 1.3.4 to hold for $W(p)$ and $\tilde{W}(f(p))$ for all points p and $f(p)$ identified under the isometry. In order for this to hold, we need that $W(p)$ and $\tilde{W}(f(p))$ both lie in the same hyperbolicity cone, i.e. either $C(\sigma_2, I)$ or $C(\sigma_2, -I)$. Note that $S(\sigma_2, I) = -S(\sigma_2, -I)$.

We now show that by using a global isometry of de Sitter space, we can make sure all Weingarten maps for both hypersurfaces lie in $C(\sigma_2, I)$. Since the hypersurface is spacelike, it must be compact. Therefore, we can find a point p such that the ρ coordinate of p is greater than any other point on the hypersurface. As the hypersurface is described as the graph of a function $f : \mathbb{S}^n \to \mathbb{B}$, we can calculate the second fundamental form at this point as

$$B_{ij} = \Gamma^0_{ij} + \frac{\partial^2 f}{\partial \zeta_i \partial \zeta_j}, \tag{3.39}$$

where Γ^0_{ij} are Christoffel symbols on de Sitter space running over the $\{\zeta_i\}_{i=1}^n$ coordinates. We have used the fact that f has a global maximum at p, meaning that all first derivatives are zero. To find the Weingarten map, we simply contract with the induced inverse metric, g^{ij}, of the hypersurface at this point. Calculating the Christoffel symbols, we find

$$\sum_k g^{ik}\Gamma^0_{kj} = \cosh(\rho)\sinh(\rho)\ \delta^i_j, \tag{3.40}$$

where δ^i_j is the Kronecker delta symbol.

Now consider reflecting this hypersurface in the "equator" of de Sitter space, $\rho = 0$. This simply amounts to replacing the ρ coordinate of any point x in the hypersurface by $-\rho$. Plugging $-\rho$ in (3.40) we obtain

$$\cosh(-\rho)\sinh(-\rho)\ \delta^i_j = -\cosh(\rho)\sinh(\rho)\ \delta^i_j. \tag{3.41}$$

As the ρ coordinate of a point on the hypersurface is given by the function f, this reflection is obtained by replacing f with $-f$. So since

$$\frac{\partial^2(-f)}{\partial \zeta_i \partial \zeta_j} = -\frac{\partial^2(f)}{\partial \zeta_i \partial \zeta_j}, \tag{3.42}$$

we have the Weingarten map of the maximal point p, under the reflection, is given by

$$\hat{W}_{ij} = \sum_k g^{ik} \hat{B}_{kj} = -\cosh(\rho)\sinh(\rho)\, \delta^i_j - \sum_k g^{ik} \frac{\partial^2(f)}{\partial\zeta_k \partial\zeta_j} = -W_{ij}. \tag{3.43}$$

This shows that by reflecting about the "equator" we can move the Weingarten map of the maximum point from $C(\sigma_2, -I)$ to $C(\sigma_2, I)$. So up to global isometry, $W(p), \tilde{W}(f(p)) \in C(\sigma_2, I)$ for all points $p \in M$. Therefore the Gårding inequality for hyperbolic polynomials holds for all points on the hypersurfaces.

3.2.1 The Equality Result

Since $W(p), \tilde{W}(f(p)) \in C(\sigma_2, I)$ for all points $p \in M$, and further $\sigma_2(W) = \sigma_2(\tilde{W})$, we have the inequality

$$\sigma_{1,1}(W, \tilde{W}) \geq \sigma_2(W), \tag{3.44}$$

where $\sigma_{1,1}(W, \tilde{W})$ denotes the polarised form of $\sigma_2(W)$. By Theorem 1.4.4 we have if

$$\sigma_{1,1}(W, \tilde{W}) = \sigma_2(W), \tag{3.45}$$

then W and $\tilde{W}$ must be proportional modulo L_{σ_2}. As such, we would like to know what L_{σ_2} actually is, and it turn out that in fact it is trivial.

Proposition 3.2.1. $L_{\sigma_2} = \mathbf{0}$.

Proof. By Proposition 1.4.2, L_{σ_2} is the set of W such that the roots of the polynomial

$$t \mapsto \sigma_2(tI + W) \tag{3.46}$$

are all zero. Expanding this out, we have

$$\begin{aligned} \sigma_2(tI + W) &= \sum_{i<j} (t + w_{ii})(t + w_{jj}) - w_{ij} w_{ji} \\ &= \sum_{i<j} t^2 + (w_{ii} + w_{jj})t + w_{ii} w_{jj} - w_{ij} w_{ji}. \end{aligned} \tag{3.47}$$

For all the roots of this to be zero, we need

$$\sum_{i<j}(w_{ii} + w_{jj}) = (n-1)\sum_i w_{ii} = 0 \tag{3.48}$$

and

$$\sum_{i<j} w_{ii}w_{jj} - w_{ij}w_{ji} = 0. \tag{3.49}$$

Put simply, this means we need $\sigma_1(W) = \sigma_2(W) = 0$. Now using the definition of $\sigma_2(W)$ in terms of the eigenvalues κ_i of W, we have

$$\begin{aligned}
\sigma_2(W) &= \sum_{i<j} \kappa_i\kappa_j \\
&= \frac{1}{2}\sum_{i\neq j} \kappa_i\kappa_j \\
&= \frac{1}{2}\Big(\sum_{i,j} \kappa_i\kappa_j - \sum_i \kappa_i\kappa_i\Big) \\
&= \frac{1}{2}\Big(\sum_i \kappa_i\Big(\sum_j \kappa_j\Big) - \sum_i \kappa_i\kappa_i\Big).
\end{aligned} \tag{3.50}$$

Now since $\sigma_1(W) = \sum_i \kappa_i = 0$, we have

$$\sigma_2(W) = -\frac{1}{2}\sum_i \kappa_i\kappa_i. \tag{3.51}$$

Then clearly the only way for $\sigma_2(W) = 0$ is for all the $\kappa_i = 0$. Hence $L_{\sigma_2} = \mathbf{0}$. □

Since $\sigma_2(W) = \sigma_2(\tilde{W})$, if W and $\tilde{W}$ are proportional then we must have $W = \tilde{W}$. Hence we have

$$\sigma_{1,1}(W, \tilde{W}) = \sigma_2(W), \tag{3.52}$$

if and only if $W = \tilde{W}$.

3.3 Integral Equations for Rigidity

3.3.1 Preliminaries

Definition 3.3.1. *Given the usual metric on de Sitter space* $\bar{g} = -d\rho^2 + \bar{\phi}^2(\rho)\sigma$ *where* $\bar{\phi}(\rho) = \cosh(\rho)$*, define*

$$\bar{\Phi}(\rho) = \int_0^\rho \bar{\phi}(r)dr. \tag{3.53}$$

Throughout we will denote the metric form $g(X, Y)$ by $\langle X, Y\rangle_g$. Where there is no ambiguity, we will drop the "g" and simply write $\langle X, Y\rangle$.

Given isometric spacelike hypersurfaces $M, \tilde{M} \subset dS^{n+1}$, denote by Φ and $\tilde{\Phi}$ the restriction of $\bar{\Phi}$ to M and $\tilde{M}$ respectively. Similarly define ϕ and $\tilde{\phi}$ as the restriction of $\bar{\phi}$. Denote the isometry $f : M \to \tilde{M}$.

Definition 3.3.2. *Define the vector field* V *on de Sitter space as*

$$V = \bar{\phi}(\rho)\frac{\partial}{\partial\rho}. \tag{3.54}$$

Remark: It is clear by simple calculation that $\bar{\Phi}(\rho) = \sinh(\rho) = \bar{\phi}'(\rho)$.

We now give a few general definitions and results of Riemannian geometry (see for example [6]).

Definition 3.3.3. *Given a smooth function* f*, the* gradient $\mathrm{grad}(f)$ *at a point* p *is the unique vector such that for any vector* $X \in T_pM$ *we have*

$$\langle \mathrm{grad}(f), X\rangle = \mathrm{d}f(X), \tag{3.55}$$

where $\mathrm{d}f(X)$ *is the directional derivative of* f *in the direction of* X*.*

Proposition 3.3.1. *Let* M, g *be a hypersurface in* $\bar{M}, \bar{g}$*, with* g *the metric induced from* $\bar{g}$*. Let* $\bar{f}$ *be a smooth function on* $\bar{M}$ *and let* f *be the restriction of* $\bar{f}$ *to* M*. Then for any point* $p \in M$*, given a vector* $X \in T_pM \subset T_pdS$*, we have*

$$\langle \mathrm{grad}(\bar{f}), X\rangle_{dS} = \langle \mathrm{grad}(f), X\rangle_M. \tag{3.56}$$

Definition 3.3.4. *Let M, g be a Riemannian manifold, given smooth function f on M and vector fields X, Y, define the* Hessian

$$\mathrm{Hess}^{M,g}(f)(X,Y) \coloneqq \langle \nabla_X \mathrm{grad}(f), Y \rangle. \tag{3.57}$$

Proposition 3.3.2. *We also have the following expression for the Hessian*

$$\mathrm{Hess}^{M,g}(f)(X,Y) = \langle \mathrm{grad}(\langle \mathrm{grad}(f), Y \rangle), X \rangle - \langle \mathrm{grad}(f), \nabla_X Y \rangle. \tag{3.58}$$

Proof. Since the Levi-Civita connection is compatible with the metric, i.e.

$$\langle \mathrm{grad}(\langle Y, Z \rangle), X \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \tag{3.59}$$

it follows that

$$\langle \nabla_X \mathrm{grad}(f), Y \rangle = \langle \mathrm{grad}(\langle \mathrm{grad}(f), Y \rangle), X \rangle - \langle \mathrm{grad}(f), \nabla_X Y \rangle. \tag{3.60}$$

□

Definition 3.3.5 (Divergence)**.** *The* divergence *at $p \in M$ of a vector field X is defined as the trace of the covariant derivative,*

$$\mathrm{div}(X) \coloneqq \sum_i \langle \nabla_{e_i} X, e_i \rangle, \tag{3.61}$$

for some orthonormal basis $\{e_i\}$ of T_pM.

Lemma 3.3.6. *Given a function f and vector field X the following holds:*

$$\mathrm{div}(fX) = f\mathrm{div}(X) + \langle grad(f), X \rangle. \tag{3.62}$$

From this and the divergence theorem (see for example [7]) we obtain the following theorem.

Theorem 3.3.7 (Integration by Parts)**.** *Let M be a manifold (possibly with boundary), given a smooth function f and vector field X, the following (referred to as integration by parts) holds*

$$\int_M \langle \mathrm{grad}(f), X\rangle = -\int_M f \mathrm{div}(X) + \int_{\partial M} f\langle X, \nu\rangle. \tag{3.63}$$

We will only be dealing with manifolds with no boundary, in which case the last term will of course be zero.

The Weingarten map and σ_2

Let $H = \{h_{ij}\}$ be the second fundamental form of the hypersurface M^n inside $\bar{M}^{n+1}$, and let $W = \{w_{ij}\}$ be the Weingarten map. Note that W is the linear operator associated to the bilinear form H, characterised by $w_{ij} = g_{ik}h_{kj}$.

Remark: If w_{ij}, h_{ij}, g_{ij} are the components of W, H and g with respect to an orthonormal basis then we have $w_{ij} = \delta_{ik}h_{kj}$. Since h_{ij} is symmetric in i and j, this implies that, when with respect to an orthonormal basis, w_{ij} is also symmetric in i and j.

Recall the expression for $\sigma_2(W)$ from Lemma 3.2.1,

$$\sigma_2(W) = \sum_{i<j} w_{ii}w_{jj} - w_{ij}w_{ji}. \tag{3.64}$$

From this we can easily calculate the partial derivatives of σ_2 with respect to the components of W.

$$\frac{\partial \sigma_2(W)}{\partial w_{ij}} = \begin{cases} \sum\limits_{k\neq i} w_{kk} & \text{for } i = j \\ -w_{ji} & \text{for } i \neq j. \end{cases} \tag{3.65}$$

Definition 3.3.8. *Let b be a symmetric type $(0,2)$-tensor field, we say that b is a* Codazzi tensor *if*

$$(\nabla_X b)(Y, Z) = (\nabla_Y b)(X, Z). \tag{3.66}$$

Let B be the self-adjoint symmetric type $(1,1)$-tensor associated to b, that is

$$g(BX, Y) = b(X, Y). \tag{3.67}$$

Then we also call B a Codazzi tensor.

We have the following proposition from [11].

Proposition 3.3.3. *Assume that W is a Codazzi tensor and $\{e_1, \ldots, e_n\}$ is an orthonormal frame on M, then we have the following identity*

$$\sum_i \Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{w_{ij}}(W)\Big), e_i \Big\rangle = 0. \tag{3.68}$$

Proof. The proof is simply a calculation, for any given j,

$$\begin{aligned}
\sum_i \Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{w_{ij}}(W)\Big), e_i \Big\rangle &= \Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{w_{ii}}(W)\Big), e_i \Big\rangle + \sum_{i \neq j} \Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{w_{ij}}(W)\Big), e_i \Big\rangle \\
&= \sum_{l \neq i} \langle \mathrm{grad}(w_{ll}), e_i \rangle - \sum_{i \neq j} \langle \mathrm{grad}(w_{ji}), e_i \rangle \\
&= \sum_{l} \langle \mathrm{grad}(w_{ll}), e_i \rangle - \langle \mathrm{grad}(w_{ii}), e_i \rangle - \sum_{i \neq j} \langle \mathrm{grad}(w_{ii}), e_j \rangle \\
&= 0,
\end{aligned} \tag{3.69}$$

where from line one to two we have used (3.65) and from line two to three we have used that W is a Codazzi tensor. □

As σ_2 is degree two its polarised form has the following simple form (see [11]).

Proposition 3.3.4. *The polarised form of $\sigma_2(W)$ as defined in Definition 1.3.2 can be expressed as*

$$\sigma_{1,1}(W, \tilde{W}) = \frac{1}{2} \sum_{i,j} \frac{\partial \sigma_2(W)}{\partial w_{ij}} \tilde{w}_{ij}. \tag{3.70}$$

Proof. First note that the polarised form of σ_2 as defined in Definition 1.3.2 is

$$\sigma_{1,1}(W,\tilde{W}) = \sum_{i<j} \frac{1}{2}(w_{ii}\tilde{w}_{jj} + \tilde{w}_{ii}w_{jj}) + \frac{1}{2}(-w_{ij}\tilde{w}_{ji} - \tilde{w}_{ij}w_{ji}). \tag{3.71}$$

Now from (3.65)

$$\begin{aligned} \frac{1}{2}\sum_{i\neq j} \frac{\partial \sigma_2(W)}{\partial w_{ij}}\tilde{w}_{ij} &= \frac{1}{2}\sum_{i<j} -w_{ji}\tilde{w}_{ij} + \frac{1}{2}\sum_{i>j} -w_{ji}\tilde{w}_{ij} \\ &= \frac{1}{2}\sum_{i<j} -w_{ij}\tilde{w}_{ji} - \tilde{w}_{ij}w_{ji}, \end{aligned} \tag{3.72}$$

which is clearly the second term of (3.71). For $i = j$ terms we have

$$\begin{aligned} \frac{1}{2}\sum_{i=j} \frac{\partial \sigma_2(W)}{\partial w_{ij}}\tilde{w}_{ij} &= \frac{1}{2}\sum_{i} \Big(\sum_{k\neq i} w_{kk}\Big)\tilde{w}_{ii} \\ &= \frac{1}{2}\sum_{i<k} w_{kk}\tilde{w}_{ii} + \frac{1}{2}\sum_{i>k} w_{kk}\tilde{w}_{ii} \\ &= \frac{1}{2}\sum_{i<j} w_{ii}\tilde{w}_{jj} + \tilde{w}_{ii}w_{jj}, \end{aligned} \tag{3.73}$$

which is the second term of (3.71). Hence the proposition is proved. □

3.3.2 Non-holonomic frames

Let (M, g) be an n-dimensional Riemannian manifold and U be the domain of a coordinate chart on M. We can choose a collection of vector fields $\{e_i\}$ on $U \subset M$ such that

$$g(e_i, e_j) = \delta_{ij}, \tag{3.74}$$

for all points $p \in U$. This is called an orthonormal frame on U.

In general orthonormal frames do not arise as the coordinate basis of some coordinate chart on U.

Proposition 3.3.5 (Koszul Formula, see [6]). *For vector fields X, Y, Z on a manifold*

M, g with Levi-Civita connection ∇, we have the Koszul formula

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y)\Big). \quad (3.75)$$

Using $g(e_i, e_j) = \delta_{ij}$, this gives us a formula for the connection coefficients with respect to an orthonormal frame $\{e_i\}$:

$$\Gamma^k_{ij} \coloneqq \langle \nabla_{e_i} e_j, e_k \rangle = \frac{1}{2}\Big(\langle [e_i, e_j], e_k\rangle - \langle [e_j, e_k], e_i\rangle - \langle [e_i, e_k], e_j\rangle\Big). \quad (3.76)$$

3.3.3 A Few Lemmas

Let M be a spacelike hypersurface of dS^{n+1}. Let $\bar{g}$ and $\bar{\nabla}$ denote the metric and Levi-Civita connection on de Sitter space and let g and ∇ denote the induced metric and induced connection on M. The following lemmas (see [11]) give the integrands for the integral equations which follow. These will be vital in the main rigidity proof.

Lemma 3.3.9.

$$\langle \bar{\nabla}_{e_i} V, e_j \rangle_{dS} + \langle \bar{\nabla}_{e_j} V, e_i \rangle_{dS} = 2\phi' \bar{g}_{ij}. \quad (3.77)$$

Proof. We will prove this in two stages. First, we show that $(\mathcal{L}_V \bar{g})(e_i, e_j) = \langle \bar{\nabla}_{e_i} V, e_j \rangle_{dS} + \langle \bar{\nabla}_{e_j} V, e_i \rangle_{dS}$, and then that $(\mathcal{L}_V \bar{g})(e_i, e_j) = 2\phi'(\rho)\bar{g}_{ij}$.

So first, since the Lie derivative obeys Leibniz's rule, we have

$$\mathcal{L}_V(\bar{g}(e_i, e_j)) = (\mathcal{L}_V \bar{g})(e_i, e_j) + \bar{g}(\mathcal{L}_V e_i, e_j) + \bar{g}(e_i, \mathcal{L}_V e_j). \quad (3.78)$$

Using that the derivative of a function is the directional derivative and that $\mathcal{L}_X Y = [X, Y]$, we have

$$(\mathcal{L}_V \bar{g})(e_i, e_j) = \langle \mathrm{grad}(\langle e_i, e_j \rangle_{dS}), V \rangle_{dS} - \langle [V, e_i], e_j \rangle_{dS} - \langle e_i, [V, e_j] \rangle_{dS}, \quad (3.79)$$

and taking V into the inner product using compatibiliity with the metric, we obtain

$$(\mathcal{L}_V\bar{g})(e_i, e_j) = \langle\bar{\nabla}_V e_i, e_j\rangle_{dS} + \langle e_i, \bar{\nabla}_V e_j\rangle_{dS} - \langle[V, e_i], e_j\rangle_{dS} - \langle e_i, [V, e_j]\rangle_{dS}. \quad (3.80)$$

Now using the symmetry of the connection $\nabla_X Y - \nabla_Y X = [X, Y]$, we have

$$(\mathcal{L}_V\bar{g})(e_i, e_j) = \langle\bar{\nabla}_{e_i} V, e_j\rangle_{dS} + \langle\bar{\nabla}_{e_j} V, e_i\rangle_{dS}. \quad (3.81)$$

We now prove the second statement $(\mathcal{L}_V\bar{g})(e_i, e_j) = 2\phi'(\rho)\bar{g}_{ij}$, simply by calculation. We have the Cartan formula for the Lie derivative of a differential form

$$\mathcal{L}_X\omega = X\lrcorner(d\omega) + d(X\lrcorner\omega). \quad (3.82)$$

Applying this to $d\rho$, we get

$$\begin{aligned}\mathcal{L}_V d\rho &= V\lrcorner(dd\rho) + d(V\lrcorner d\rho)\\ &= d(\phi(\rho))\\ &= \phi'(\rho)d\rho\end{aligned} \quad (3.83)$$

and hence

$$\mathcal{L}_V d\rho^2 = 2\phi'(\rho)d\rho^2. \quad (3.84)$$

Now calculating $\mathcal{L}_V\phi^2(\rho)d\theta^2$ gives

$$\begin{aligned}\mathcal{L}_V\phi^2(\rho)d\theta^2 &= V\lrcorner(d(\phi^2(\rho)d\theta^2)) + d(V\lrcorner(\phi^2(\rho)d\theta^2))\\ &= V\lrcorner\Big(\frac{\partial(\phi^2(\rho))}{\partial\rho}d\rho\, d\theta^2\Big)\\ &= V\lrcorner(2\phi(\rho)\phi'(\rho)d\rho\, d\theta^2)\\ &= 2\phi^2(\rho)\phi'(\rho)d\theta^2.\end{aligned} \quad (3.85)$$

Putting this all together, we obtain

$$\begin{aligned}(\mathcal{L}_V\bar{g})(e_i, e_j) &= 2\phi'(\rho)(d\rho^2 + \phi^2(\rho)\theta^2)\\ &= 2\phi'(\rho)\bar{g}_{ij},\end{aligned} \quad (3.86)$$

Now combining (3.81) and (3.86), the lemma is proved. □

Lemma 3.3.10. *Let ν be the normal to M, then we have*

$$\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) = \phi' g_{ij} + h_{ij}\langle V, \nu\rangle_{dS}. \tag{3.87}$$

Proof. We start with the definitions of the Hessian for $\bar{\Phi}$ in de Sitter space and for Φ in M,

$$\mathrm{Hess}^{dS,\bar{g}}(\bar{\Phi})(e_i, e_j) = \langle \mathrm{grad}(\langle \mathrm{grad}(\bar{\Phi}), e_j\rangle_{dS}), e_i\rangle_{dS} - \langle \mathrm{grad}(\bar{\Phi}), \bar{\nabla}_{e_i} e_j\rangle_{dS}, \tag{3.88a}$$

$$\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) = \langle \mathrm{grad}(\langle \mathrm{grad}(\Phi), e_j\rangle_M), e_i\rangle_M - \langle \mathrm{grad}(\Phi), \nabla_{e_i} e_j\rangle_M. \tag{3.88b}$$

Subtracting (3.88a) from (3.88b), we obtain

$$\begin{aligned}
&\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) - \mathrm{Hess}^{dS,\bar{g}}(\bar{\Phi})(e_i, e_j) \\
&\qquad = \langle \mathrm{grad}(\langle \mathrm{grad}(\Phi), e_j\rangle_M), e_i\rangle_M - \langle \mathrm{grad}(\langle \mathrm{grad}(\bar{\Phi}), e_j\rangle_{dS}), e_i\rangle_{dS} \\
&\qquad\qquad - \langle \mathrm{grad}(\Phi), \nabla_{e_i} e_j\rangle_M + \langle \mathrm{grad}(\bar{\Phi}), \bar{\nabla}_{e_i} e_j\rangle_{dS}.
\end{aligned} \tag{3.89}$$

By Proposition 3.3.1, $\langle \mathrm{grad}(\langle \mathrm{grad}(\Phi), e_j\rangle_M), e_i\rangle_M = \langle \mathrm{grad}(\langle \mathrm{grad}(\bar{\Phi}), e_j\rangle_{dS}), e_i\rangle_{dS}$, and also $\langle \mathrm{grad}(\Phi), \nabla_{e_i} e_j\rangle_M = \langle \mathrm{grad}(\bar{\Phi}), \nabla_{e_i} e_j\rangle_{dS}$, so we have

$$\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) = \mathrm{Hess}^{dS,\bar{g}}(\bar{\Phi})(e_i, e_j) + \langle \mathrm{grad}(\bar{\Phi}), \bar{\nabla}_{e_i} e_j - \nabla_{e_i} e_j\rangle_{dS}. \tag{3.90}$$

Note that on de Sitter space, $\mathrm{grad}\bar{\Phi} = V$, so then from the definition of the Hessian, we obtain

$$\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) = \langle \bar{\nabla}_{e_i} V, e_j\rangle_{dS} + \langle V, \bar{\nabla}_{e_i} e_j - \nabla_{e_i} e_j\rangle_{dS}. \tag{3.91}$$

Note $\langle V, \bar{\nabla}_{e_i} e_j - \nabla_{e_i} e_j\rangle_{dS} = h_{ij}\langle V, \nu\rangle_{dS}$. Now take this equation and consider swapping e_i and e_j in each term

$$\mathrm{Hess}^{M,g}(\Phi)(e_j, e_i) = \langle \bar{\nabla}_{e_j} V, e_i\rangle_{dS} + h_{ji}\langle V, \nu\rangle_{dS}. \tag{3.92}$$

Adding this to (3.91), and using that both the Hessian and second fundamental form are symmetric in i and j, we obtain

$$2\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) = \langle \bar{\nabla}_{e_i} V, e_j \rangle_{dS} + \langle \bar{\nabla}_{e_j} V, e_i \rangle_{dS} + 2h_{ij} \langle V, \nu \rangle_{dS}. \tag{3.93}$$

Finally by Lemma 3.3.9 we obtain the result

$$\mathrm{Hess}^{M,g}(\Phi)(e_i, e_j) = \phi' g_{ij} + h_{ij} \langle V, \nu \rangle_{dS}. \tag{3.94}$$

□

3.3.4 The Integral Equations

By identifying points of M and $\tilde{M}$ under the isometry f, we can treat $\tilde{\phi}$ and $\tilde{\Phi}$ as functions on M, i.e. for $x \in M$, $\tilde{\phi}_M : x \mapsto \tilde{\phi}(f(x))$. We will abuse notation and use $\tilde{\phi}$ to denote $\tilde{\phi}_M$ in integrals over M. The following theorem is adapted from a lemma in [11].

Theorem 3.3.11. *Let M and $\tilde{M}$ be locally isometric spacelike hypersurfaces in de Sitter space. Given an orthonormal frame $\{e_i\}$ on M, which can be identified as an orthonormal frame on $\tilde{M}$ under the isometry, the following integral equations hold:*

$$\begin{aligned} &\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\, \tilde{\phi}'\, \mathrm{Hess}^{M,g}(\Phi)(e_i, e_j)\, d^n V \\ &\qquad = \int_M \Big[(n-1)\tilde{\phi}' \phi' \sigma_1(W) - 2\tilde{\phi}' \sigma_2(W) \langle V, \nu \rangle \Big]\, d^n V, \end{aligned} \tag{3.95a}$$

$$\begin{aligned} &\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\, \tilde{\phi}'\, \mathrm{Hess}^{M,g}(\Phi)(e_i, e_j)\, d^n V \\ &\qquad = \int_M \Big[(n-1)\tilde{\phi}' \phi' \sigma_1(\tilde{W}) - 2\tilde{\phi}' \sigma_{1,1}(W, \tilde{W}) \langle V, \nu \rangle \Big]\, d^n V, \end{aligned} \tag{3.95b}$$

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\ \phi'\ \mathrm{Hess}^{M,g}(\tilde{\Phi})(e_i,e_j)\ d^nV$$
$$= \int_M \Big[(n-1)\phi'\tilde{\phi}'\sigma_1(W) - 2\phi'\sigma_{1,1}(\tilde{W},W)\langle\tilde{V},\tilde{\nu}\rangle\Big]\ d^nV, \quad (3.95c)$$

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\ \phi'\ \mathrm{Hess}^{M,g}(\tilde{\Phi})(e_i,e_j)\ d^nV$$
$$= \int_M \Big[(n-1)\phi'\tilde{\phi}'\sigma_1(\tilde{W}) - 2\phi'\sigma_2(\tilde{W})\langle\tilde{V},\tilde{\nu}\rangle\Big]\ d^nV. \quad (3.95d)$$

Proof. Starting with Lemma 3.3.10 applied to M and $\tilde{M}$, we have

$$\mathrm{Hess}^{M,g}(\Phi)(e_i,e_j) = \phi' g_{ij} + h_{ij}\langle V,\nu\rangle_{dS} \quad (3.96a)$$

$$\mathrm{Hess}^{\tilde{M},\tilde{g}}(\tilde{\Phi})(e_i,e_j) = \tilde{\phi}'\tilde{g}_{ij} + \tilde{h}_{ij}\langle\tilde{V},\tilde{\nu}\rangle_{dS}. \quad (3.96b)$$

It is not hard to convince oneself that $\mathrm{Hess}^{\tilde{M},\tilde{g}}(\tilde{\Phi})(e_i,e_j) = \mathrm{Hess}^{M,g}(\tilde{\Phi})(e_i,e_j)$. Now we multiply (3.96a) by $\tilde{\phi}'$ and (3.96b) by ϕ'. Now take the four combinations of multiplying these two equations by either $\frac{\partial\sigma_2}{\partial w_{ij}}(W)$ or $\frac{\partial\sigma_2}{\partial w_{ij}}(\tilde{W})$, sum over the i and j and integrate over M to obtain

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}'\ \mathrm{Hess}^{M,g}(\Phi)(e_i,e_j)\ d^nV$$
$$= \int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\tilde{\phi}'\phi' g_{ij} + \frac{\partial \sigma_2}{\partial w_{ij}}(W)\tilde{\phi}' h_{ij}\langle V,\nu\rangle_{dS}\ d^nV, \quad (3.97a)$$

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\ \tilde{\phi}'\ \mathrm{Hess}^{M,g}(\Phi)(e_i,e_j)\ d^nV$$
$$= \int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\tilde{\phi}'\phi' g_{ij} + \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\tilde{\phi}' h_{ij}\langle V,\nu\rangle_{dS}\ d^nV, \quad (3.97b)$$

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\ \phi'\ \mathrm{Hess}^{M,g}(\tilde{\Phi})(e_i, e_j)\ d^nV$$
$$= \int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\phi'\tilde{\phi}'\tilde{g}_{ij} + \frac{\partial \sigma_2}{\partial w_{ij}}(W)\phi'\tilde{h}_{ij}\langle \tilde{V}, \tilde{\nu}\rangle_{dS}\ d^nV, \quad (3.97c)$$

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\ \phi'\ \mathrm{Hess}^{M,g}(\tilde{\Phi})(e_i, e_j)\ d^nV$$
$$= \int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\phi'\tilde{\phi}'\tilde{g}_{ij} + \frac{\partial \sigma_2}{\partial w_{ij}}(\tilde{W})\phi'\tilde{h}_{ij}\langle \tilde{V}, \tilde{\nu}\rangle_{dS}\ d^nV. \quad (3.97d)$$

We now focus on the right hand side of (3.97a). Since $\{e_i\}$ is an orthonormal basis, we have $g_{ij} = \delta_{ij}$, so for the first term we have

$$\sum_{i=j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\tilde{\phi}'\phi'. \tag{3.98}$$

Substituting in the expression for $\frac{\partial \sigma_2}{\partial w_{ij}}(W)$ for $i = j$ from (3.65) gives

$$\sum_i \sum_{k \neq i} w_{kk}\tilde{\phi}'\phi', \tag{3.99}$$

which is clearly equal to

$$(n-1)\tilde{\phi}'\phi'\sigma_1(W). \tag{3.100}$$

For the second term of the right hand side of (3.97a), note that since $g_{ij} = \delta_{ij}$, we have $h_{ij} = w_{ij}$. Therefore we have

$$\tilde{\phi}'\langle V, \nu\rangle \sum_{i,j} \frac{\partial \sigma_2(W)}{\partial w_{ij}} w_{ij}. \tag{3.101}$$

From the expression for $\sigma_{1,1}(W, \tilde{W})$ in Proposition 3.3.4, this is equal to

$$2\tilde{\phi}'\langle V, \nu\rangle \sigma_{1,1}(W, W), \tag{3.102}$$

but as the polarised form of σ_2, we have $\sigma_{1,1}(W, W) = \sigma_2(W)$. This together with (3.100) gives the right hand side of (3.95a). The remaining three equations are proved along the same lines, hence the theorem is proved. □

3.3.5 Symmetry in Tilde

There is a mistake in the paper of [11]. The working is omitted but in order to replicate their result we would need $\frac{\partial\sigma_2(W)}{\partial w_{ij}}$ terms to factor out of the divergence after integrating by parts which we cannot do. However we can use an extra step so that our main proof still works. To prove our rigidity result we will need certain terms of the integral equations above to cancel when subtracted from one another. The following theorem proves a symmetry which gives us this result.

Theorem 3.3.12. *The integral*

$$\int_M \sum_{i,j} \frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}'\ \mathrm{Hess}^{M,g}(\Phi)(e_i, e_j)\ d^nV \tag{3.103}$$

is invariant under switching $\tilde{\phi}'$ for ϕ' and Φ for $\tilde{\Phi}$ (henceforth referred to as symmetric in tilde), so that

$$\int_M \sum_{i,j} \frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \phi'\ \mathrm{Hess}^{M,g}(\tilde{\Phi})(e_i, e_j)\ d^nV - \int_M \sum_{i,j} \frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}'\ \mathrm{Hess}^{M,g}(\Phi)(e_i, e_j)\ d^nV = 0. \tag{3.104}$$

Proof. Using the expression for the Hessian in Proposition 3.3.2, we will start from the integral

$$\int_M \sum_{i,j} \frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}' \Big(\langle \mathrm{grad}(\langle \mathrm{grad}(\Phi), e_j\rangle), e_i\rangle - \langle \mathrm{grad}(\Phi), \nabla_{e_i} e_j\rangle \Big)\ d^nV. \tag{3.105}$$

Now taking the $\frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}'$ inside the inner products gives

$$\int_M \sum_{i,j} \Big\langle \mathrm{grad}(\langle \mathrm{grad}(\Phi), e_j\rangle), \frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}'\ e_i \Big\rangle - \Big\langle \mathrm{grad}(\Phi), \frac{\partial\sigma_2}{\partial w_{ij}}(W)\ \tilde{\phi}'\ \nabla_{e_i} e_j \Big\rangle\ d^nV, \tag{3.106}$$

and integration by parts on the first term gives

$$\int_M \sum_{i,j} -\langle \mathrm{grad}(\Phi), e_j\rangle \,\mathrm{div}\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\, e_i\Big) - \Big\langle \mathrm{grad}(\Phi), \frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\,\nabla_{e_i} e_j\Big\rangle\, d^n V. \tag{3.107}$$

Now by Theorem 3.3.6 we have

$$\int_M \sum_{i,j} -\langle \mathrm{grad}(\Phi), e_j\rangle \,\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\,\mathrm{div}(e_i) - \Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\Big), e_i\Big\rangle\Big) \\ - \Big\langle \mathrm{grad}(\Phi), \frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\,\nabla_{e_i} e_j\Big\rangle\, d^n V. \tag{3.108}$$

Rearranging gives

$$\int_M \sum_{i,j} \langle \mathrm{grad}(\Phi), e_j\rangle \,\Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\Big), e_i\Big\rangle \\ - \Big\langle \mathrm{grad}(\Phi), \frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\tilde{\phi}'\,\Big(\mathrm{div}(e_i) e_j + \nabla_{e_i} e_j\Big)\Big\rangle\, d^n V. \tag{3.109}$$

We will now examine the two terms in this integral separately. Starting with the first term, since grad satisfies Leibniz's rule, we have

$$\int_M \sum_{i,j} -\langle \mathrm{grad}(\Phi), e_j\rangle \,\Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\Big)\tilde{\phi}' + \frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\mathrm{grad}(\tilde{\phi}'), e_i\Big\rangle\, d^n V. \tag{3.110}$$

The linearity of the inner product and splitting the integral gives

$$\int_M \sum_{i,j} -\langle \mathrm{grad}(\Phi), e_j\rangle \,\Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\Big)\tilde{\phi}', e_i\Big\rangle\, d^n V \\ + \int_M \sum_{i,j} -\langle \mathrm{grad}(\Phi), e_j\rangle \,\Big\langle \frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\mathrm{grad}(\tilde{\phi}'), e_i\Big\rangle\, d^n V. \tag{3.111}$$

Factoring out $-\langle \mathrm{grad}(\Phi), e_j\rangle$ and $\tilde{\phi}'$ from the sum over i in the first integral gives

$$\int_M \sum_{j} -\Big(\tilde{\phi}'\,\langle \mathrm{grad}(\Phi), e_j\rangle \,\sum_i \Big\langle \mathrm{grad}\Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W)\Big), e_i\Big\rangle\Big)\, d^n V \\ + \int_M \sum_{i,j} -\langle \mathrm{grad}(\Phi), e_j\rangle \,\Big\langle \frac{\partial \sigma_2}{\partial w_{ij}}(W)\,\mathrm{grad}(\tilde{\phi}'), e_i\Big\rangle\, d^n V, \tag{3.112}$$

but then by Proposition 3.3.3 the first integral is zero. After factoring out $\frac{\partial \sigma_2}{\partial w_{ij}}(W)$ in the second integral, we see by Remark 3.3.1 that (3.112) is equal to

$$\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\langle \mathrm{grad}(\Phi), e_j\rangle\langle \mathrm{grad}(\tilde{\Phi}), e_i\rangle \; d^nV. \tag{3.113}$$

Now we will look at the second term of (3.109). First let

$$X = \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W) \Big(\mathrm{div}(e_i)e_j + \nabla_{e_i} e_j\Big), \tag{3.114}$$

then the second term of (3.109) is equal to

$$-\int_M \langle \mathrm{grad}(\Phi), \tilde{\phi}' X\rangle \; d^nV. \tag{3.115}$$

Trivially this is

$$-\int_M \frac{1}{2}\langle \mathrm{grad}(\Phi), \tilde{\phi}' X\rangle + \frac{1}{2}\langle \mathrm{grad}(\Phi), \tilde{\phi}' X\rangle \; d^nV. \tag{3.116}$$

Then integration by parts on the second term gives

$$-\int_M \frac{1}{2}\langle \mathrm{grad}(\Phi), \tilde{\phi}' X\rangle - \frac{1}{2}\Phi \mathrm{div}(\tilde{\phi}' X) \; d^nV. \tag{3.117}$$

By Theorem 3.3.6, this is

$$-\int_M \frac{1}{2}\langle \mathrm{grad}(\Phi), \tilde{\phi}' X\rangle - \frac{1}{2}\Phi\Big(\tilde{\phi}'\mathrm{div}(X) - \langle \mathrm{grad}(\tilde{\phi}'), X\rangle\Big) \; d^nV. \tag{3.118}$$

Note from Remark 3.3.1 we have $\tilde{\phi}' = \tilde{\Phi}$, so after expanding we have

$$-\int_M \frac{1}{2}\langle \mathrm{grad}(\Phi), \tilde{\Phi} X\rangle + \frac{1}{2}\langle \mathrm{grad}(\tilde{\Phi}), \Phi X\rangle - \frac{1}{2}\Phi\tilde{\Phi}\mathrm{div}(X) \; d^nV. \tag{3.119}$$

Plugging (3.113) and (3.119) into (3.109), we obtain

$$
\begin{aligned}
\int_M \sum_{i,j} \frac{\partial \sigma_2}{\partial w_{ij}}(W)\, \tilde{\phi}'\, \mathrm{Hess}^{M,g}(\Phi)(e_i, e_j)\, d^n V \\
= \int_M \sum_{i,j} \Big[& \frac{\partial \sigma_2}{\partial w_{ij}}(W) \langle \mathrm{grad}(\Phi), e_j \rangle \langle \mathrm{grad}(\tilde{\Phi}), e_i \rangle \\
& - \frac{1}{2} \Big\langle \mathrm{grad}(\Phi), \tilde{\Phi} \frac{\partial \sigma_2}{\partial w_{ij}}(W) \, \Big(\mathrm{div}(e_i) e_j + \nabla_{e_i} e_j \Big) \Big\rangle \\
& - \frac{1}{2} \Big\langle \mathrm{grad}(\tilde{\Phi}), \Phi \frac{\partial \sigma_2}{\partial w_{ij}}(W) \, \Big(\mathrm{div}(e_i) e_j + \nabla_{e_i} e_j \Big) \Big\rangle \\
& + \frac{1}{2} \Phi \tilde{\Phi} \mathrm{div} \Big(\frac{\partial \sigma_2}{\partial w_{ij}}(W) \, \Big(\mathrm{div}(e_i) e_j + \nabla_{e_i} e_j \Big) \Big) \Big] \, d^n V.
\end{aligned} \tag{3.120}
$$

Now from the explicit formula for $\frac{\partial \sigma_2}{\partial w_{ij}}(W)$ in (3.65) and that $W = w_{ij}$ is symmetric in i and j as in Remark 3.3.1, we have that the first term on the right hand side of (3.120) is symmetric in tilde. Clearly the middle two terms together are symmetric in tilde and the last term is as well. Therefore the whole integral is symmetric in tilde. □

3.4 Rigidity in de Sitter Space

Denote by $dS^{n+1,+}$ the region of de Sitter space for which ρ is positive. We can now prove the main rigidity result following the idea for the Riemannian case in [11].

Theorem 3.4.1. *Let M and $\tilde{M}$ be two spacelike hypersurfaces in $dS^{n+1,+}$ with $K > \bar{K}$ such that there exists a local isometry $f : M \to \tilde{M}$. Then f is the restriction of some global isometry F of de Sitter space.*

Proof. We start with the integral equations in Theorem 3.3.11, subtracting (3.95a) from (3.95c) and (3.95b) from (3.95d), by Theorem 3.3.12 the right hand sides will cancel and we obtain

$$
\int_M \tilde{\phi}' \sigma_2(W) \langle V, \nu \rangle \, d^n V = \int_M \phi' \sigma_{1,1}(\tilde{W}, W) \langle \tilde{V}, \tilde{\nu} \rangle \, d^n V, \tag{3.121a}
$$

$$
\int_M \phi' \sigma_2(\tilde{W}) \langle \tilde{V}, \tilde{\nu} \rangle \, d^n V = \int_M \tilde{\phi}' \sigma_{1,1}(W, \tilde{W}) \langle V, \nu \rangle \, d^n V. \tag{3.121b}
$$

Now note that $\sigma_2(W) = \sigma_2(\tilde{W})$ and $\sigma_{1,1}(W, \tilde{W}) = \sigma_{1,1}(\tilde{W}, W)$, so adding (3.121a) and (3.121b) and rearranging we obtain

$$\int_M \left(\tilde{\phi}'\langle V, \nu\rangle + \phi'\langle \tilde{V}, \tilde{\nu}\rangle\right)\left(\sigma_2(W) - \sigma_{1,1}(W, \tilde{W})\right)\, d^nV = 0. \tag{3.122}$$

Since $M, \tilde{M} \subset dS^{n+1,+}$, ϕ' and $\tilde{\phi}'$ are positive on all of M. Furthermore since M and $\tilde{M}$ are spacelike, $\langle V, \nu\rangle$ and $\langle \tilde{V}, \tilde{\nu}\rangle$ are strictly less than zero. So the quantity $\tilde{\phi}'\langle V, \nu\rangle + \phi'\langle \tilde{V}, \tilde{\nu}\rangle$ is strictly less than zero. Now from the Gårding inequality we have that $\sigma_2(W) - \sigma_{1,1}(W, \tilde{W})$ is less than or equal to zero. Therefore the only way the integral in (3.122) can be zero is if $\sigma_2(W) - \sigma_{1,1}(W, \tilde{W})$ is identically zero on all of M.

Now from the equality result of Gårding's inequality, since $\sigma_2(W) - \sigma_{1,1}(W, \tilde{W}) = 0$ is zero on all of M, we have that $W = \tilde{W}$. Since the first and second fundamental forms are preserved under the map f, it must be the restriction of some global isometry F (see for example [15]), hence the theorem is proved. □

Discussion

Rigidity of surfaces in various settings has made continual progress since Cohn-Vossen's paper on regular surfaces. Guan and Shen more recently gave a method to prove rigidity in more generality for hypersurfaces in Riemannian manifolds of constant sectional curvature. The work of this chapter continued this theme proving rigidity this time in a setting with indefinite metric. Further we give a rigorous exposition of the type of argument employed by Guan and Shen, correcting the gap.

This chapter also aims to show how hyperbolic polynomials may be applied in a somewhat unexpected setting. We are able to construct a function of the shape operator which is hyperbolic and then use the powerful Gårding inequality for hyperbolic polynomials as the jump pad to our rigidity result. These techniques may be of further interest to the differential geometry community where hyperbolicity may be employed on functions in other settings.

CHAPTER 4

Conclusion

This thesis has aimed to do three main things: to give a clear self-contained exposition of Gårding's theory of hyperbolic polynomials; to understand better the structure of the space of hyperbolic polynomials; and finally to demonstrate how hyperbolic polynomials can be applied to other areas.

The first of our main results is proved in Chapter 2. Following ideas of Nuij we incorporated work on univariate hyperbolic polynomials. This lead us to our result, defining and identifying a general condition for Gårding Nuij sequences. These giving rise to hyperbolicity preserving linear operators.

The third and final chapter contained our second main result, a theorem in differential geometry. This showed that two isometric spacelike hypersurfaces in de Sitter space must be the same hypersurface up to some global isometry. At the heart of this proof was Gårding's inequality as we constructed a hyperbolic polynomial acting on the second fundamental form of each hypersurface. Then using integral equations we were able to show in this case that the two sides of Gårding's inquality were equal. Thus we were able to apply the equality result of Gårding. The proof structure is based on a paper of Guan and Shen in the Riemannian case, however this chapter contains original work to translate to de Sitter space and patch errors.

In order to give these proofs we required a background on Gårding hyperbolic polynomials, in particular the full result of Gårding's ineqality for hyperbolic polynomial. In Chapter 1, we introduced new notation and included alternate proofs for simplicity and clarity in presenting Gårding's theory.

Looking to the future, following the work of this thesis, the main line of research will be to continue on from the results of Chapter 2. Namely, to gain a more rigorous understanding of what can be a Gårding Nuij sequence and more specific cases of them. By using the linear operators defined by the Gårding Nuij sequences, this will allow us to discern structures within the space of hyperbolic polynomials.

Bibliography

[1] L. V. Ahlfors, *Complex Analysis An Introduction to the Theory of Analytic Functions of One Complex Variable*, McGraw-Hill, Inc., 1979.

[2] H. H. Bauschke, O. Güler, A. S. Lewis, and H. S. Sendov, *Hyperbolic polynomials and convex analysis*, Canad. J. Math., 53 (3) (2001), pp. 470–488.

[3] J. Borcea and P. Brändén, *Pólya-Schur master theorems for circular domains and their boundaries*, Annals of Mathematics, 170 (2009), pp. 465–492.

[4] S. Cohn-Vossen, *Zwei Sätze über die Starrheit der Eiflachen*, Nachr. Ges. Wiss. Göttingen, (1927), pp. 125–134.

[5] P. Dey and D. Plaumann, *Testing hyperbolicity of real polynomials*. https://doi.org/10.48550/arXiv.1810.04055, 2018.

[6] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, USA, 1992.

[7] ——, *Differential Forms and Applications*, Springer-Verlag, Berlin Heidelberg, 1994.

[8] M. P. do Carmo and F. W. Warner, *Rigidity and convexity of hypersurfaces in spheres*, Journal of Differential Geometry, 4 (1970), pp. 65–76.

[9] L. Gårding, *An inequality for hyperbolic polynomials*, Journal of Mathematics and Mechanics, 8 (6) (1959), pp. 957–965.

[10] J. H. Grace, *The zeros of a polynomial*, Proceedings of the Cambridge Philosophical Society, 11 (1902), pp. 352–357.

[11] P. Guan and X. S. Shen, *A rigidity theorem for hypersurfaces in higher dimensional space forms*. https://doi.org/10.48550/arXiv.1306.1581, 2013.

[12] O. Güler, *Hyperbolic polynomials and interior point methods for convex programming*, Mathematics of Operations Research, 22 (1997), pp. 350–377.

[13] G. Harris and C. Martin, *Shorter notes: The roots of a polynomial vary continuously as a function of the coefficients*, Proceedings of the American Mathematical Society, 100 (1987), pp. 390–392.

[14] T. Hasson, *A rigidity theorem for hypersurfaces in space forms*, Master's thesis, 2018.

[15] H. Hopf, *Differential Geometry in the Large*, Springer, 1989.

[16] L. Hörmander, *Linear Partial Differential Operators*, Springer, 1963.

[17] V. Kostov, *Topics on Hyperbolic Polynomials in One Variable*, SMF, 2011.

[18] K. Kurdyka and L. Paunescu, *Nuij type pencils of hyperbolic polynomials*, Canadian Mathematical Bulletin, 60 (2017), pp. 561–570.

[19] P. D. Lax, *Differential equations, difference equations and matrix theory*, Communications on Pure and Applied Mathematics, 11 (1958), pp. 175–194.

[20] A. S. Lewis, P. A. Parrilo, and M. V. Ramana, *The Lax conjecture is true*, Proceedings of the American Mathematical Society, 133 (2005), pp. 2495–2499.

[21] A. Leykin and D. Plaumann, *Determinantal representations of hyperbolic curves via polynomial homotopy continuation*, 2016.

[22] W. Nuij, *A note on hyperbolic polynomials*, Mathematica Scandinavica, 23 (1968), pp. 69–72.

[23] J. Renegar, *Hyperbolic programs, and their derivative relaxations*, Foundations of Computational Mathematics, 1 (2006), pp. 59–79.

[24] A. Zee, *Einstein Gravity in a Nutshell*, Princeton University Press, 2013.

APPENDIX A

Python to Produce Images

In chapter 2 we looked at operators acting on hyperbolic polynomials. It is useful to see the hypersurface $P = 0$ for a given hyperbolic polynomial, especially to see how they are changed by the operators. We saw clearly how Nuij used operators to create paths through the space of hyperbolic polynomials from the boundary into the open set of strictly hyperbolic polynomials. The images were created using Python with the numpy, matplotlib and skimage libraries. The following code is an example demonstrating the process to produce the images in Figure 2.2.

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.art3d import Poly3DCollection

from skimage import measure
from skimage.draw import ellipsoid

spacing=0.01

x, y, z = np.mgrid[-1.0:1.0:spacing,
                   -1.0:1.0:spacing,
                   -1.0:1.0:spacing]
```

```

#p=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4
#dpdz=-2*x**2*z-7*y**2*z+z**3
#d2pdz2=-2*x**2-7*y**2+3*z**2
#d3pdz3=6*z
#d4pdz4=6

s=0.2
#cone = x**2+y**2-z**2
#cone = 0.25*(-2*z**2*(2*x**2 + 7*y**2) + (2*x**2 + y**2)**2 + z**4)
cone_1 = x**2 + 0.5*(y**2 - 2*3**0.5*y*z - z**2)
cone_2 = x**2 + 0.5*(y**2 + 2*3**0.5*y*z - z**2)

nuij = x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + 2*s*(x+y)*(-2*x**2*z-7*y**2*z+z**3) + s**2*(x**2+4*x*y+y**2)*(-2*x**2-7*y**2+3*z**2) + 12*s**3*z*(x**2*y+x*y**2) + 6*s**4*x**2*y**2

op_x=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + s*x*(-2*x**2*z-7*y**2*z+z**3)
op_y=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + s*y*(-2*x**2*z-7*y**2*z+z**3)
op_xx=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + 2*s*x*(-2*x**2*z-7*y**2*z+z**3) + s**2*x**2*(-2*x**2-7*y**2+3*z**2)
op_xy=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + s*(x+y)*(-2*x**2*z-7*y**2*z+z**3) + s**2*x*y*(-2*x**2-7*y**2+3*z**2)
op_yy=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + 2*s*y*(-2*x**2*z-7*y**2*z+z**3) + s**2*y**2*(-2*x**2-7*y**2+3*z**2)
op_xxx=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + 3*s*x*(-2*x**2*z-7*y**2*z+z**3) + 3*s**2*x**2*(-2*x**2-7*y**2+3*z**2) + s**3*x**3*(6*z)
op_xxy=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + s*(2*x+y)*(-2*x**2*z-7*y**2*z+z**3) + s**2*(2*x*y+x**2)*(-2*x
```

```
**2-7*y**2+3*z**2) + s**3*x**2*y*(6*z)
op_xyy=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + s*(x+2*y)*(-2*x**2*z-7*y**2*z+z**3) + s**2*(2*x*y+y**2)*(-2*x**2-7*y**2+3*z**2) + s**3*x*y**2*(6*z)
op_yyy=x**4+x**2*y**2-x**2*z**2+0.25*y**4-3.5*y**2*z**2+0.25*z**4 + 3*s*y*(-2*x**2*z-7*y**2*z+z**3) + 3*s**2*y**2*(-2*x**2-7*y**2+3*z**2) + s**3*y**3*(6*z)

# Plot the Nuij operator cone using marching cubes
verts_n, faces_n, normals_n, values_n = measure.marching_cubes(nuij, 0)
verts = verts_n
faces = faces_n

# Display resulting triangular mesh using Matplotlib.
fig = plt.figure(figsize=(10, 10))
ax = fig.add_subplot(111, projection='3d')

mesh = Poly3DCollection(verts[faces], alpha=1)

mesh.set_facecolor(np.array([[max(face)/len(verts),max(face)/len(verts),max(face)/len(verts)] for face in faces]))
mesh.set_3d_properties()
mesh.set_edgecolor('face')
ax.add_collection3d(mesh)

ax.set_xlim(0, 2/spacing)
ax.set_ylim(0, 2/spacing)
ax.set_zlim(0, 2/spacing)

plt.tight_layout()
plt.show()
```