L-functions for Classical Groups: The Integral Representations, Algebraicity and the p-adic Interpolations

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Abstract: The main theme of this thesis is the study of special values of L-functions through integral representations. We present an integral representation of the standard L-functions for classical groups via the doubling method. Our computations, comparing with the well known result for partial L-functions in [PR87], include all ramified local integrals with the explicit choice of local sections for Eisenstein series. When the classical group admits a Shimura variety, we have a well defined notion of algebraic modular forms. In this case, we calculate the Fourier expansion of Eisenstein series from which the properties of their special values can be easily read off. Utilizing our integral representations, we then prove the algebraicity of certain special L-values for modular forms on some classical groups. Furthermore, by our proper choice of the local sections for Eisenstein series, we construct the p-adic L-functions interpolating these special L-values.

Generalizing the classical doubling method, [CFGK19] presents an integral representation for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ by the twisted doubling method. In the final chapter of the thesis, we present another integral representation for the L-functions of $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ via a non-unique model and obtain some analytic results.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

The result in chapter 2-4 is contained in [Jin23] and some partial results are also obtained in [Jin22]. Some discussions in chapter 3 are taken from [BJar] which is joint with Thanasis Bouganis. Chapter 5 is completely taken from [JY23] which is joint with Pan Yan.

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I would like to express my gratitude to my supervisor Thanasis Bouganis for his guidance throughout my doctoral studies. Without his patience and encouragement, the results in the thesis may never be obtained. Some discussions in chapter 3 of this thesis are taken from the joint work with him, from which I learnt a lot about my research field. The other parts of the thesis (chapter 2-4) are also motivated by this work and I also thank him for various helpful discussions and suggestions. I would also like to thank my second supervisor Jack Shotton for many helpful conversations.

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The result in Chapter 5 is the joint work with Pan Yan. I would like to thank him for collaborating with me. Our collaboration and discussions also help me understand more mathematics beyond my current research field.

I thank my examiners Tobias Berger and Herbert Gangl and also the anonymous referees on my paper [BJar] for providing many helpful comments.

特别致谢

我想特别感谢我本科论文的导师，谢兵永老师。事实上，我大部分数论的基础理论都是向他学习的。没有他，我的数学生涯肯定会艰苦很多。当我在华东师范大学数学系就读时，他开展了一系列讨论班帮助我学习数论的知识。我们先后学习了代数数论和自守表示等做数论研究所需要的基础知识。自学这些内容是很困难的，我没有学明白的内容都需要依靠他在讨论班上的解释。特别地，也是在讨论班以及在完成本科论文的过程中，我知道了像朗兰兹纲领和BSD猜想这些数论中的中心问题。这让我对做数论研究有了一些感觉和自信。我同时要感谢我的朋友以及讨论班的伙伴谭钦匀。没有他我们的数论讨论班就无法开展。作为我的学长，我也在和他的讨论中向他学习到了许多知识。

当我在华东师范大学数学系就读时，还有许多老师给过我帮助。陆俊老师在我刚入学时给予了我不少学习数学的建议。他推荐了许多数学经典教材供我自学，还开展了一个微分几何的讨论班。我几乎每学期都参加周国栋老师的课程或讨论班，向他学习到了许多代数方面的知识。他给予了我许多学习上的指点和帮助，和他的诸多讨论让我在代数学习上轻松了许多。我同时感谢张通老师在我申请英国的研究生时提供了建议和帮助。我也要感谢郑璐予同学。在我读博士期间，她和我在网上讨论了几何表示论相关的一些内容。

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金雨波

Contents

Chapter 1

Introduction

One of the central problems in number theory is the study of special values of L-functions. The main object studied in this thesis is the standard L-function for automorphic forms on classical groups. In Chapter 2 and 3, we present an integral representation for these L-functions using the doubling method. Utilizing the integral representation, we prove the algebraicity of certain special L-values and construct p-adic L-functions interpolating these values in Chapter 4. These three chapters are mainly taken from [Jin23] improving some results obtained in [BJar] (joint with Thanasis Bouganis) and [Jin22]. Recently, [CFGK19] generalized the (classical) doubling method to obtain an integral representation for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$. The algebraicity result for such L-functions is far away from being proved due to the complicated Speh representations used as the inducing data for Eisenstein series. However, following [GS20], we derive a new integral representation for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ and obtain the analytic result in Chapter 5. This part is taken from [JY23] and is joint with Pan Yan.

We introduce our results and compare with works in the literature in the following three sections.

1.1 Integral representations

Let G be a classical group over a number field F defined as in (2.1.6) or (2.1.8). The first main theme of this thesis is an integral representation for standard L-functions of classical groups. One way to obtain such an integral representation is the doubling method originated in [Gar84b; PR87]. We briefly recall the setup for the doubling method in the following. More detailed expository on the doubling method will be given in Section 2.2.2.

Take a cuspidal representation π of $G(\mathbb{A})$ and a cusp form $\phi \in \pi$, where $\mathbb{A}$ is the adele ring of F. We consider a doubling embedding $G \times G \to H$ (2.1.11) into a bigger classical group H defined as in (2.1.10). We have a Siegel Eisenstein series $E(h; f_s)$ (2.2.9) defined on $H(\mathbb{A})$ associated to a section $f_s \in \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\chi|\nu(\cdot)|^s)$ (2.2.5) of the parabolic induction from a Siegel parabolic subgroup $P \subset H$. Here, the inducing data is a Hecke character χ (viewed as a character on $P(\mathbb{A})$) and a reduced norm ν. The main strategy of the doubling method is to pullback such an Eisenstein series $E(h; f_s)$ on $H(\mathbb{A})$ along the doubling embedding and pair with the cusp forms on $G(\mathbb{A})$. That is, we consider the global integral of the form

$$\begin{aligned}&\mathcal{Z}(s;\phi,f_s)\\&=\int_{(G\times G)(F)\backslash(G\times G)(\mathbb{A})} E((g_1,g_2);f_s)\overline{\phi_1(g_1)}\phi_2(g_2)\chi(\nu(g_2))^{-1}dg_1dg_2,\end{aligned} \tag{1.1.1}$$

where (g_1, g_2) is the image of $g_1 \times g_2 \in (G \times G)(\mathbb{A})$ in $H(\mathbb{A})$ and $\phi_1 := \pi(g_1)\phi, \phi_2 := \pi(g_2)\phi$ are certain translates of ϕ by some $g_1, g_2 \in G(\mathbb{A})$.

It is shown in [PR87] that this global integral has an Euler product expression (2.2.15) and thus reduces the study of (1.1.1) to the study of local integrals place by place. It is also well known that one can make a proper choice of the section f_s such that $\mathcal{Z}(s;\phi,f_s)$ represents the partial L-function of ϕ, i.e. all the ramified local L-factors are set to 1. For the study of arithmetic problems of special L-values and especially the construction of p-adic L-functions, the information at ramified places is indispensable.

The definition of local L-factors is indeed a fundamental problem in the study of automorphic representations (see also Remark 2.3.5). In [Lan70], Langlands conjectured that one can associate to any cuspidal representation $\pi = \otimes \pi_v$ a local L-factor $L_v(s, \pi_v)$ and an epsilon factor $\epsilon_v(s, \pi_v)$ such that the global L-function $L(s, \pi)$ satisfies a functional equation. In [Yam14], Yamana defined these local factors and proved the functional equation using the doubling method. His approach works for all irreducible automorphic representations of classical groups and is used in proving some analytic properties of L-functions. However, he did not construct the local sections of Eisenstein series and did not compute the local integrals explicitly so it is not clear how his computations can be used to study the algebraicity of special L-values or to construct p-adic L-functions.

We study the ramified local integrals in a different way which is inspired by [Shi95]. Fix an integral ideal $\mathfrak{n}$ of F. Assume ϕ is fixed by some open compact subgroup $K(\mathfrak{n})$ (2.3.23) and is an eigenform for a certain Hecke algebra $\mathcal{H}(K(\mathfrak{n}), \mathfrak{X})$ (defined in Section 2.3.4). For a Hecke character χ whose conductor divides $\mathfrak{n}$, we define the L-function $L(s, \phi \times \chi)$ to be a Dirichlet series of the Hecke eigenvalues of ϕ. This extends the definition of the L-function for symplectic groups in [Shi95] and is an analogue of the L-functions for classical modular forms defined by Dirichlet series of Fourier coefficients. In particular, the L-function $L(s, \phi \times \chi)$ has all bad Euler factors outside the conductor of χ. In [Shi95], Shimura constructs local sections of Eisenstein series explicitly at all places such that $\mathcal{Z}(s; \phi, f_s)$ represents $L(s, \phi \times \chi)$. Our main theorem on the integral representation, which extends his result to all classical groups, is stated as follows.

Theorem 1.1.1. *(Theorem 2.2.4, 3.4.2) There is a choice of f_s such that*

$$\mathcal{Z}(s; \phi, f_s) = C \cdot L\left(s + \frac{1}{2}, \phi \times \chi\right) \cdot \mathcal{Z}_\infty(s; \phi_\infty, f_s^\infty) \cdot \langle \phi', \phi \rangle, \tag{1.1.2}$$

where C is some nonzero constant depending on s, $\phi' := \pi(g')\phi$ is a translate of ϕ by some $g' \in G(\mathbb{A})$ and $\mathcal{Z}_\infty(s; \phi_\infty, f_s^\infty)$ a nonzero constant depending on the choice of the archimedean section $f_s^\infty := \prod_{v|\infty} f_{s,v}$ and $\phi_\infty = \prod_{v|\infty} \phi_v$. When the underlying *symmetric space of G is hermitian, and ϕ is a holomorphic cusp form as in Definition 3.2.3, we can further make a choice of f_s^∞ such that $\mathcal{Z}_\infty(s;\phi_\infty,f_s^\infty)$ is the constant given in Proposition 3.4.1.*

If G is a unitary group, we assume all places $v|\mathfrak{n}$ are nonsplit in the quadratic extension defining the group G. This is only for simplicity and also because the split case is well studied in [HLS06] and [EHLS20]. The main difficulty for extending the result of [Shi95] is to deal with the classical groups which are not totally isotropic (i.e. $r > 0$ in (2.1.7)). In this case, the doubling map and the image of the doubling embedding (2.1.13) are much more involved which complicates the computations. For the purpose of constructing p-adic L-functions, our local sections are also properly chosen such that the Eisenstein series has a nice Fourier expansion. This is indeed the core technical issue of this work. We do not study the archimedean integral in general in this work. For the special cases we are considering, the archimedean computations follow from [Shi97; Shi00] and [BJar]. For completeness, we will also present an integral representation for L-functions of Maass forms on general linear groups using the doubling method in the appendix.

Recently, Cai, Friedberg, Ginzburg and Kaplan [CFGK19] presented an integral representation for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ by the twisted doubling method generalizing the classical doubling method introduced above. In [Cai21], the unfolding of the global integral are also worked out for $G \times \mathrm{GL}_k$ with G any classical group. It will also be important to study the ramified integrals derived from the twisted doubling method. For example, one should expect that one can define the local L-factors and prove the functional equations for standard L-functions of $G \times \mathrm{GL}_k$ as in [Yam14]. In Chapter 5, we also derive a new integral representation for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ via a non-unique model (an independent introduction to this work will be given in Section 5.1). It is also an interesting question whether one can construct local sections and compute the ramified integral for $G \times \mathrm{GL}_k$ explicitly as we have done here for $G \times \mathrm{GL}_1$.

1.2 The algebraic result

The celebrated Deligne Conjecture [Del79] claims that the critical values of motivic L-functions, up to certain periods, are algebraic numbers. In this work, we study the automorphic counterpart of this conjecture. As the approach here relies heavily on the theory of Shimura varieties, we restrict ourselves to the classical group G whose underlying symmetric space is hermitian. Such groups (except some orthogonal groups) are listed at the beginning of Chapter 3. In all these cases, the notion of the algebraic modular forms is well defined. We refer the reader to the beginning of Section 3.3 for a summary of various characterizations of algebraic modular forms in the literature. All of them rely on the fact that the symmetric space of G is hermitian so that one can associate G to a Shimura variety.

We fix the following setup. Assume F is a totally real number field of degree $d(F)$ over $\mathbb{Q}$. Let $\boldsymbol{l} = (l, ..., l) \in \mathbb{Z}^{d(F)}$ be a parallel tuple satisfying

$$\begin{aligned} & l \geq \begin{cases} m+1 & \text{Case II}, \\ n+1 & \text{Case III, IV, V}. \end{cases} \quad \text{when } F \neq \mathbb{Q}, \\ & l \geq \begin{cases} m+1 & \text{Case II}, \\ n+r+1 & \text{Case III, IV, V}, \end{cases} \quad \text{when } F = \mathbb{Q}, \end{aligned} \tag{1.2.1}$$

with m, n, r as in (2.1.7). Fix a specific prime ideal $\boldsymbol{p}$ of F above an odd prime number p and an integral ideal $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2 = \prod_v \mathfrak{p}_v^{\mathfrak{c}_v}$ of F with $\mathfrak{n}_1, \mathfrak{n}_2, \boldsymbol{p}$ coprime. We make the following assumptions:

(1) $2 \in \mathcal{O}_v^\times$ and $\theta \in \mathrm{GL}_r(\mathcal{O}_v)$ for all $v|\mathfrak{n}\boldsymbol{p}$. Here, θ is the anisotropic part of G.

(2) $\boldsymbol{f} \in \mathcal{S}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}), \overline{\mathbb{Q}})$ is an algebraic eigenform for the Hecke algebra $\mathcal{H}(K(\mathfrak{n}\boldsymbol{p}), \mathfrak{X})$ as in Section 2.3.4.

(3) $\boldsymbol{f}$ is an eigenform for the $U(\boldsymbol{p})$ operator defined in (2.3.13) with eigenvalue $\alpha(\boldsymbol{p}) \neq 0$.

(4) $\chi = \chi_1\boldsymbol{\chi}$ where χ_1 has conductor $\mathfrak{n}_2$ and $\boldsymbol{\chi}$ has conductor $\boldsymbol{p}^{\boldsymbol{c}}$ for some integer $\boldsymbol{c} \geq 0$. We assume χ has infinity type $\boldsymbol{l}$. That is, $\chi_v(x) = x^l|x|^{-l}$ for all $v|\infty$.

(5) In the case when G is a unitary group, we further assume all places $v|\mathfrak{n}\boldsymbol{p}$ are

nonsplit in the imaginary quadratic field E defining the unitary group G.

We are interested in the special value of the L-function $L\left(s+\frac{1}{2}, \boldsymbol{f} \times \chi\right)$ at

$$s = s_0 := \begin{cases} l-\kappa & \text{Case II, III, IV,} \\ \frac{l}{2}-\kappa & \text{Case V.} \end{cases}, \tag{1.2.2}$$

with κ a constant depending on n given in (2.2.8).

Our main theorem on algebraicity is stated as follows.

Theorem 1.2.1. *(Theorem 4.2.1) For l, s_0 as above,*

$$\begin{aligned} &\frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}\Omega\cdot\langle \boldsymbol{f}, \boldsymbol{f}\rangle} \in \overline{\mathbb{Q}}, && \text{if } \boldsymbol{c}>0, \\ &\frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right) M\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}\Omega\cdot\langle \boldsymbol{f}, \boldsymbol{f}\rangle} \in \overline{\mathbb{Q}}, && \text{if } \boldsymbol{c}=0. \end{aligned} \tag{1.2.3}$$

where $d(F) = [F:\mathbb{Q}]$, $\boldsymbol{d}(\pi)$ is the constant given in (4.2.4), $\langle\cdot,\cdot\rangle$ is the Petersson inner product and $M\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)$ is the modification factor listed in Proposition 2.4.6. Here $\Omega = 1$ in Case II, III, IV and in Case V, Ω is the CM period (4.2.5) depending only on the group G.

When the group is totally isotropic (i.e. $r = 0$ in (2.1.7)), we also obtain a refined version of the above theorem. That is, we describe the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on special L-values in Theorem 4.2.2. The proof of this theorem uses the standard strategy in [BS00] and [Shi00]. That is, we derive the algebraicity from the integral representation (3.4.22), (3.4.23) reformulated from Theorem 1.1.1, 2.2.4, 3.4.2 and the algebraic properties of the Fourier coefficients of Eisenstein series in Corollary 4.1.11.

This kind of result is also obtained in [BS00; Shi95; Shi00] for symplectic and unitary groups and in [BJar] for quaternionic unitary groups. We explain what is new in our work. First of all, all these works, except [Shi95] for symplectic groups, only consider partial L-functions while the L-function considered here includes those ramified L-factors. Of course, there are only finitely many missing L-factors in the partial L-function and if these ramified L-factors are known to be algebraic one may manually add these factors to the algebraicity result of partial L-functions. But this only make sense at the special values beyond the absolutely convergence bound, i.e. $s = s_0 > \kappa$ with κ given in (2.2.11). Secondly, inspired by [BS00], our local sections of the Eisenstein series are chosen such that we only need information about the Fourier coefficients of rank greater or equal to $2m$ (where m is the Witt index of G). This allows us to get a better bound on l and discuss the special values below the absolutely convergence bound.

The orthogonal groups are not studied here for at least two reasons. Firstly, when the symmetric space of G is hermitian (i.e. $G(F_v)$ has Witt index 2 for any archimedean places v), the symmetric space of H is no longer hermitian so that one need to carefully define the meaning of algebraic modular forms on $H(\mathbb{A})$. Secondly, the archimedean computations for the Fourier coefficients of Eisenstein series will involve certain generalized Bessel functions studied in [Shi99a]. The analytic properties are studied there but there are no explicit formulas as for the confluent hypergeometric function in [Shi82] so that we do not know the algebraic properties of these functions so far.

1.3 The p-adic L-function

We keep the setup as in the previous section. In particular, we fix a prime ideal $\boldsymbol{p}$ of F above an odd prime number p. Once the algebraicity of special L-values is known, one can ask about the p-adic interpolation of these values. Our main theorems on p-adic L-functions (Theorem 4.2.5, Theorem 4.2.6, (4.2.17), (4.2.18), (4.2.20), (4.2.24)) are stated as follows.

Theorem 1.3.1. *Assume $\boldsymbol{f}$ is $\boldsymbol{p}$-ordinary in the sense that $\alpha(\boldsymbol{p}) \in \mathcal{O}_{\mathbb{C}_p}^{\times}$. Fix χ_1 to be a Hecke character of conductor $\mathfrak{n}_2$ and infinity type $\boldsymbol{l}$. That is $\chi_{1,v}(x) = x^l|x|^{-l}$ for any places $v|\infty$.*

(1) For unitary and quaternionic unitary groups, there exists a p-adic measure $\mu(\boldsymbol{f})$

on $\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)$ *such that for any finite order Hecke character* $\boldsymbol{\chi}$ *of conductor* $\boldsymbol{p}^{\boldsymbol{c}}$,

$$\begin{aligned}\int_{\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) = |\boldsymbol{\varpi}|^{\boldsymbol{c}\mathbf{d}_1\frac{m(m-1)}{2}} \pi^{d(F)\boldsymbol{d}(\pi)} \left(c_l(s_0) \prod_{i=0}^{n-1} \Gamma(\mathbf{d}_1(l-i)) \right)^{d(F)} \\ \times G^D(\chi)^{-m} M\left(s_0 + \frac{1}{2}, \boldsymbol{f} \times \chi\right) \cdot \frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\Omega \cdot \langle \boldsymbol{f}, \boldsymbol{f} \rangle}.\end{aligned} \tag{1.3.1}$$

(2) For symplectic groups we assume the Witt index m *has the same parity with the weight* l *of* $\boldsymbol{f}$, *i.e.* $l \equiv m \mod 2$. *For quaternionic orthogonal groups, we assume* $\boldsymbol{p}$ *splits in the quaternion algebra* D *if the group is not totally isotropic (i.e.* $r > 0$ *in* (5.2.6)*). Then in these two cases, there exists a* p*-adic measure* $\mu(\boldsymbol{f})$ *on* $\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)$ *such that for any finite order Hecke character* $\boldsymbol{\chi}$ *of conductor* $\boldsymbol{p}^{\boldsymbol{c}}$,

$$\begin{aligned}&\int_{\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) \\ =&|\boldsymbol{\varpi}|^{\boldsymbol{c}\mathbf{d}_1\frac{m(m-1)}{2}} N_{F/\mathbb{Q}}(\boldsymbol{p})^{\boldsymbol{c}\left(s_0-\frac{1}{2}\right)} G^D(\chi)^{-m} G^F(\chi)^{-1} \pi^{d(F)\boldsymbol{d}(\pi)} \\ &\times \left(c_l(s_0) \Gamma\left(s_0+\frac{1}{2}\right) \prod_{i=0}^{n\mathbf{d}_1-1} \Gamma\left(l-\frac{i}{2}\right) \right)^{d(F)} \cdot \frac{L_{\boldsymbol{p}}\left(s_0+\frac{1}{2},\chi\right)}{L_{\boldsymbol{p}}\left(\frac{1}{2}-s_0,\chi^{-1}\right)} \\ &\times M\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right) \cdot \frac{L\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)}{\langle \boldsymbol{f},\boldsymbol{f}\rangle}.\end{aligned} \tag{1.3.2}$$

Here:

(a) In the case of unitary groups, E *is the imaginary quadratic extension of* F *defining the unitary group. In other cases* $E = F$.

(b) $\Omega = 1$ *in the case of quaternionic unitary groups and* Ω *is the CM period* (4.2.5) *in the case of unitary groups.*

(c) $\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)$ *is the* p*-adic analytic group defined in* (4.2.12)*.*

(d) $d(F) = [F : \mathbb{Q}]$, $\mathbf{d}_1 = 2$ *for two quaternionic cases and* $\mathbf{d}_1 = 1$ *for symplectic and unitary groups.*

(e) For a division algebra D, $G^D(\chi)$ *is the Gauss sum of* χ *defined on* D.

(f) $c_l(s_0)$ *is given by Proposition 3.4.1 and* $\boldsymbol{d}(\pi)$ *is given in* (4.1.30)*.*

(g) $M(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi)$ *is given in Proposition 2.4.6 if* $\boldsymbol{c} = 0$ *and is understood as* 1 *if* $\boldsymbol{c} > 0$.

(h) $L_{\boldsymbol{p}}\left(s_0+\frac{1}{2},\chi\right)^{-1} = 1 - \chi(\boldsymbol{\varpi})|\boldsymbol{\varpi}|^{s_0+\frac{1}{2}}$ *if* $\boldsymbol{c} = 0$ *and is understood as* 1 *if* $\boldsymbol{c} > 0$.

The assumption for quaternionic orthogonal groups is technical and is necessary in our proof of Theorem 4.2.6. Again, the construction of p-adic L-functions relies on properly choosing the local sections of the Eisenstein series such that its Fourier coefficients have p-adic interpolations. We refer the reader to [Liu20] where it is carefully explained how these local sections should be chosen.

We compare our results with other works in the following.

For symplectic groups, p-adic L-functions have been constructed in [BS00] using the doubling method and in [CP04] using the Rankin-Selberg method. The p-adic L-functions for ordinary families are constructed in [Liu20]. We admit that the approach in our work is highly inspired by [BS00] and [Liu20]. Although all these works are concerning the base field $F = \mathbb{Q}$, there is no difficulty to generalize their work to any totally real field F as we have done here.

For unitary groups, p-adic L-functions are studied in [Eis21; EHLS20; HLS06; SU14; Wan15] for ordinary families. All these works assume that $\boldsymbol{p}$ is split in the imaginary quadratic extension E/F so that the local group $G(F_{\boldsymbol{p}})$ at $\boldsymbol{p}$ is a general linear group. Their local sections are always chosen as certain Godement-Jacquet sections, and we do not discuss this case in our work. When $\boldsymbol{p}$ is inert, the p-adic L-function is constructed in [Bou16] for totally isotropic groups (i.e. $r = 0$ in (2.1.7)). Our result for the general unitary group with $\boldsymbol{p}$ nonsplit is new.

The L-functions for the two quaternionic cases are less studied than symplectic and unitary groups. In our previous work [Jin22], we have constructed p-adic L-functions for these totally isotropic groups and restricted to the case when $\boldsymbol{p}$ splits in the quaternion algebra. This case is much simpler as the local group $G(F_{\boldsymbol{p}})$ will be either an orthogonal group or symplectic group and both are totally isotropic. We have removed these restrictions here.

We admit that the construction of p-adic L-functions for ordinary families is beyond the scope of this work. For p-adic families, one also needs to understand more about the geometry of Shimura varieties and p-adic modular forms. For example, in [Eis21; EHLS20; HLS06] the split assumption on $\boldsymbol{p}$ is also used to guarantee the nonvanishing of a certain ordinary locus in defining the p-adic modular forms (see also [Eis21, 5.3(2)]). In the two quaternionic cases, the geometry of Shimura varieties becomes more challenging as these Shimura varieties are not of PEL type.

Chapter 2

The Integral Representation of the Standard L-functions

In this chapter, we present an integral representation of the standard L-functions for classical groups. We use the doubling method originating from [Gar84b; PR87]. An integral representation for the partial L-functions is well known. Here we calculate all ramified local integrals to obtain integral representations for complete L-functions including ramified L-factors. The local sections of Eisenstein series are explicitly constructed and properly chosen for our later purpose of constructing p-adic L-functions.

This chapter is essentially taken from [Jin23, Section 2-4] and is organized as follows. In Section 2.1 we fix our setup for classical groups. We review the global integral from the doubling method and state our main results on integral representations in Section 2.2. The definition of the local L-factors and computation of the local integrals will be carried out in Section 2.3 and Section 2.4.

2.1 Classical groups

We review basics of hermitian forms and classical groups. The setup for classical groups is the following [Yam14, Section 2] and for generalities of hermitian forms the reader can refer to [Shi97, Chapter I]. We also give several examples regarding the quaternion algebra for which a comprehensive reference is [Voi21].

2.1.1 Algebras with involution

We start by fixing some general notations. For an associative ring R with identity, denote by $\mathrm{Mat}_{m,n}(R)$ the R-module of all $m \times n$ matrices with entries in R. Set $\mathrm{Mat}_n(R) = \mathrm{Mat}_{n,n}(R)$ and $\mathrm{GL}_n(R) = \mathrm{Mat}_n(R)^\times$. For $x \in \mathrm{Mat}_{m,n}(R)$, denote ${}^t x$ for its transpose. Denote by 1_n and 0_n, or even simply 1 and 0 if its size is clear from the context, for the identity matrix and zero matrix in $\mathrm{Mat}_n(R)$, respectively.

Let F be a local or global field and D an F-algebra with involution ρ whose center E contains F. The couple (D, ρ) considered in this thesis will belong to the following five types:

(a) $D = E = F$ and ρ is the identity map,

(b) D is a division quaternion algebra over $E = F$ and ρ is the main involution of D,

(c) D is a division algebra central over a quadratic extension E of F and ρ generates $\mathrm{Gal}(E/F)$,

(d) $D = \mathrm{Mat}_2(E), E = F$ and ρ is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\rho = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

(e) $D = \mathbf{D} \oplus \mathbf{D}^{\mathrm{op}}, E = F \oplus F$ and ρ is given by $(x, y)^\rho = (y, x)$, where $\mathbf{D}$ is a division algebra central over F and $\mathbf{D}^{\mathrm{op}}$ is its opposite algebra.

For $x = (x_{ij}) \in \mathrm{Mat}_{mn}(D)$, set $x^\rho = (x_{ij}^\rho)$ and $x^* = {}^t x^\rho, \hat{x} = (x^*)^{-1}$. For $x \in \mathrm{Mat}_n(D)$, $\nu(x) \in E, \tau(x) \in E$ stand for its reduced norm and reduced trace to the center E.

Example 2.1.1. (Quaternion algebras over $\mathbb{Q}$) Recall that a quaternion algebra over $\mathbb{Q}$ is a central simple algebra of dimension four. Picking up a basis, we can write it in the form

$$D = \mathbb{Q} \oplus \mathbb{Q}\zeta \oplus \mathbb{Q}\xi \oplus \mathbb{Q}\zeta\xi,$$

where

$$\zeta^2 = \alpha, \quad \xi^2 = \beta, \quad \zeta\xi = -\xi\zeta,$$

with α, β nonzero squarefree integers. We make the convention that $D \neq \mathrm{Mat}_2(\mathbb{Q})$ which is equivalent to $\alpha \neq 1, \beta \neq 1$. The involution of D is given by

$$\rho : D \to D : a + b\zeta + c\xi + d\zeta\xi \mapsto a - b\zeta - c\xi - d\zeta\xi.$$

Then (D, ρ) is an algebra with involution over $\mathbb{Q}$ of type (b) above.

Identify ζ, ξ with $\sqrt{\alpha}, \sqrt{\beta} \in \overline{\mathbb{Q}}$ and let $K = \mathbb{Q}(\xi)$ with involution ι the generator of $\mathrm{Gal}(K/\mathbb{Q})$. We can define an embedding

$$\mathfrak{i} : D \to \mathrm{Mat}_2(K), \quad a + b\zeta + c\xi + d\zeta\xi \mapsto \begin{bmatrix} a + c\xi & \alpha(b - d\xi) \\ b + d\xi & a - c\xi \end{bmatrix}.$$

One easily checks that for $x \in D$,

$$^t\mathfrak{i}(x)^\iota = I^{-1}\mathfrak{i}(x^*)I, \qquad I = \begin{bmatrix} -\alpha & 0 \\ 0 & 1 \end{bmatrix},$$

$$^t\mathfrak{i}(x) = J^{-1}\mathfrak{i}(x^*)J, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and $\mathfrak{i}$ induces the isomorphism

$$\mathfrak{i} : D \xrightarrow{\sim} \{x \in \mathrm{Mat}_2(K) : x^\iota IJ = IJx\}.$$

We extend this map to the embedding $\mathfrak{i} : \mathrm{Mat}_n(D) \to \mathrm{Mat}_{2n}(K)$ by sending $x = (x_{ij})$ to $(\mathfrak{i}(x_{ij}))$. Denote $I'_n = \mathrm{diag}[I, ..., I]$ and $J'_n = \mathrm{diag}[J, ..., J]$ with n copies. Then for $x \in \mathrm{Mat}_n(D)$,

$$^t\mathfrak{i}(x)^\iota = I_n'^{-1}\mathfrak{i}(x^*)I'_n, \qquad ^t\mathfrak{i}(x) = J_n'^{-1}\mathfrak{i}(x^*)J'_n,$$

and $\mathfrak{i}$ induces the isomorphism

$$\mathfrak{i} : \mathrm{Mat}_n(D) \xrightarrow{\sim} \{x \in \mathrm{Mat}_{2n}(K) : x^\iota I_n' J_n' = I_n' J_n' x\}.$$

The reduced norm and reduced trace are defined as

$$\nu(x) = \det(\mathfrak{i}(x)), \qquad \tau(x) = \mathrm{tr}(\mathfrak{i}(x)),$$

where det and tr are the usual determinant and trace of matrices. This definition is independent of the choice of K.

Example 2.1.2. (Quaternion algebras over $\mathbb{R}$) Let D be the quaternion algebra over $\mathbb{Q}$ as in Example 2.1.1. There are two possibilities for $D \otimes_{\mathbb{Q}} \mathbb{R}$.

On one hand, if both α, β are negative then

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} := \mathbb{R} \oplus \mathbb{R}\boldsymbol{i} \oplus \mathbb{R}\boldsymbol{j} \oplus \mathbb{R}\boldsymbol{ij},$$

with

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = -1, \qquad \boldsymbol{ij} = -\boldsymbol{ji}.$$

Here $\mathbb{H}$ is the Hamilton quaternion algebra with involution

$$\iota : a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{ij} \mapsto a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{ij},$$

and we can embed it into $\mathrm{Mat}_2(\mathbb{C})$ by the map $\mathfrak{i}$ defined as in Example 2.1.1. In this case $(\mathbb{H}, \iota)$ is again an algebra with involution over $\mathbb{R}$ of type (b).

On the other hand, if one of α, β is positive (without loss of generality we can assume $\beta = 1$) then the map $\mathfrak{i}$ similarly defined as in Example 2.1.1 gives an isomorphism between $(D \otimes_{\mathbb{Q}} \mathbb{R}, \rho)$ and $(\mathrm{Mat}_2(\mathbb{R}), \iota)$ where ι is the involution of $\mathrm{Mat}_2(\mathbb{R})$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\iota = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In this case $(\mathrm{Mat}_2(\mathbb{R}), \iota)$ is an algebra with involution over $\mathbb{R}$ of type (d).

Example 2.1.3. (Quaternion algebras over $\mathbb{Q}_p$) Let D be the quaternion algebra as in Example 2.1.1. There are two possibilities for $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ depending on whether the equation

$$\alpha x^2 + \beta y^2 = 1$$

has a solution (x, y) over $\mathbb{Q}_p$.

If it does not have a solution, we say that p ramifies and in this case $(D \otimes_{\mathbb{Q}} \mathbb{Q}_p, \rho)$ is again an algebra with involution over $\mathbb{Q}_p$ of type (b).

If it has a solution, we say that p splits and by changing basis we have

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D' := \mathbb{Q}_p \oplus \mathbb{Q}_p\zeta' \oplus \mathbb{Q}_p\xi' \oplus \mathbb{Q}_p\zeta'\xi',$$

with

$$\zeta'^2 = \alpha', \quad \xi'^2 = \beta' = 1, \quad \zeta'\xi' = -\xi'\zeta'.$$

In this case, by the map $\mathfrak{i}$ similarly defined as in Example 2.1.1, we can show that $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to an algebra with involution over $\mathbb{Q}_p$ of type (d). Note that, up to conjugation, $\mathrm{Mat}_2(\mathbb{Q}_p)$ has a unique maximal order $\mathrm{Mat}_2(\mathbb{Z}_p)$. If we fix a maximal order $\mathcal{O}$ of D, then we can and we shall always fix an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{Mat}_2(\mathbb{Q}_p)$ such that the image of $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is $\mathrm{Mat}_2(\mathbb{Z}_p)$.

2.1.2 Hermitian forms and classical groups

Let F be a local field or global field. Fix a triple (D, ρ, ϵ) with (D, ρ) an algebra D with involution ρ over F and $\epsilon = \pm 1$. Let W be a free left D-module of rank n. By an ϵ-hermitian space we mean a structure $\mathcal{W} = (W, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is an ϵ-hermitian form on W, that is, an F-bilinear map $\langle \cdot, \cdot \rangle : W \times W \to D$ such that

$$\langle x, y \rangle^\rho = \epsilon \langle y, x \rangle, \quad \langle ax, by \rangle = a \langle x, y \rangle b^\rho, \quad (a, b \in D,\ x, y \in W). \tag{2.1.1}$$

We always assume such a form to be non-degenerate, i.e. $\langle x, W \rangle = 0$ implies $x = 0$. We call $\mathcal{W}$ isotropic if $\langle x, x \rangle = 0$ for some $0 \neq x \in W$ and call $\mathcal{W}$ anisotropic if $\langle x, x \rangle = 0$ only for $x = 0$. The following is the fundamental theorem of the study of hermitian forms.

Proposition 2.1.4. *(Witt's Theorem) Let $\mathcal{W}$ be an ϵ-hermitian space of rank n. There exists $2m$ elements e_i, f_i with $1 \leq i \leq m \leq \frac{n}{2}$ such that*

$$W = \sum_{i=1}^{m} (De_i + Df_i) + Z,$$

$$\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0, \quad \langle e_i, f_j \rangle = \begin{cases} \epsilon & i = j, \\ 0 & i \neq j \end{cases} \qquad \text{for every } i, j, \tag{2.1.2}$$

$$Z = \{x \in W : \langle e_i, x \rangle = \langle f_i, x \rangle = 0 \text{ for all } i\}.$$

If $(Z, \langle \cdot, \cdot \rangle_Z)$, with $\langle \cdot, \cdot \rangle_Z$ the restriction to Z, is anisotropic then we call m the Witt index of $\mathcal{W}$.

We record some facts about the anisotropic ϵ-hermitian spaces in the following lemma.

Lemma 2.1.5. *(1) Let F be a non-archimedean local field and $\mathcal{W}$ an anisotropic ϵ-hermitian space of dimension r. Then*

$$\begin{aligned} r &= 0 && \text{if } (D, \rho) \text{ of type (a) and } \epsilon = -1, \\ r &\leq 4 && \text{if } (D, \rho) \text{ of type (a) and } \epsilon = 1, \\ r &\leq 3 && \text{if } (D, \rho) \text{ of type (b) and } \epsilon = -1, \\ r &\leq 1 && \text{if } (D, \rho) \text{ of type (b) and } \epsilon = 1, \\ r &\leq 2 && \text{if } (D, \rho) \text{ of type (c) and } \epsilon = -1. \end{aligned} \tag{2.1.3}$$

(2) Let F be an archimedean local field and $\mathcal{W}$ be an anisotropic hermitian space of dimension r. Then

$$\begin{aligned} r &= 0 && \text{if } (D, \rho) \text{ of type (a) and } \epsilon = -1, \\ r &\leq 1 && \text{if } (D, \rho) \text{ of type (b) and } \epsilon = -1, \\ r &= 0 && \text{if } (D, \rho) \text{ of type (b) and } \epsilon = 1, \end{aligned} \tag{2.1.4}$$

In other cases, r can be arbitrary non-negative integers.

(3) Let F be a number field and $\mathcal{W}$ an anisotropic ϵ-hermitian space of dimension r.

Then

$$\begin{aligned} r &= 0 && \text{if } (D,\rho) \text{ of type (a) and } \epsilon = -1, \\ r &\le 3 && \text{if } (D,\rho) \text{ of type (b) and } \epsilon = -1, \\ r &\le 1 && \text{if } (D,\rho) \text{ of type (b) and } \epsilon = 1, \end{aligned} \tag{2.1.5}$$

In other cases, r can be arbitrary non-negative integers.

Proof. For (1), the assertion for (D,ρ) of type (a), (c) is well known and recorded in [Shi97, Proposition 5.2]. The case (D,ρ) of type (b) with $\epsilon = 1$ can be proved similarly as there. The case (D,ρ) of type (b) with $\epsilon = -1$ is proved in [Tsu61]. The second part of the lemma can be checked directly and the third part follows by the Hasse principle. For the case (D,ρ) of type (b) with $\epsilon = 1$, one uses the well known Hasse principle for quadratic forms, and for the case (D,ρ) of type (b) with $\epsilon = -1$ one needs the Hasse principle for quaternionic skew-hermitian forms proved in [Hij63]. □

Denote the ring of all D-linear endomorphisms of W by $\mathrm{End}_D(W)$ and $\mathrm{GL}_D(W) = \mathrm{End}_D(W)^\times$. If we view elements of W as row vectors, then $\mathrm{GL}_D(W)$ acts on W from the right. The classical group of $\mathcal{W}$ is defined as

$$G := G(\mathcal{W}) := \{g \in \mathrm{GL}_D(W) : \langle xg, yg\rangle = \langle x, y\rangle \text{ for all } x, y \in W\}, \tag{2.1.6}$$

which is a (possibly disconnected) reductive algebraic group over F. By fixing a basis of W, we can identify $\mathrm{End}_D(W)$ with $\mathrm{Mat}_n(D)$ and $\mathrm{GL}_D(W)$ with $\mathrm{GL}_n(D)$. Then $\langle\cdot,\cdot\rangle$ can be expressed as a matrix of the form

$$\Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon\cdot 1_m & 0 & 0 \end{bmatrix} \text{ with } n = 2m + r,\ \theta^* = \epsilon\theta \in \mathrm{GL}_r(D) \tag{2.1.7}$$

and thus the classical group G can be realized as

$$G := G(W, \Phi) = \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\}. \tag{2.1.8}$$

We assume $m \geq 1$ throughout the thesis to avoid the discussion of definite classical groups.

Example 2.1.6. (Classical groups of type (d)) Let $\mathcal{W} = (W, \langle \cdot, \cdot \rangle)$ be an ϵ-hermitian space associated to (D, ρ) of type (d) with $\epsilon = \pm 1$ and $G = G(W, \Phi)$ the associated classical group defined as in (2.1.8). Identifying $\mathrm{GL}_n(D)$ with $\mathrm{GL}_{2n}(F)$, the group G is isomorphic to

$$\widetilde{G} := \{g \in \mathrm{GL}_n(D) : g\widetilde{\Phi}\,{}^t g = \widetilde{\Phi}\},$$

with

$$\widetilde{\Phi} = \begin{bmatrix} 0 & 0 & 1_{2m} \\ 0 & \widetilde{\theta} & 0 \\ -\epsilon \cdot 1_{2m} & 0 & 0 \end{bmatrix} \text{ with } n = 2m + r,\ {}^t\widetilde{\theta} = -\epsilon\widetilde{\theta} \in \mathrm{GL}_{2r}(F).$$

By the above example, the study of classical groups of type (d) is indeed covered by the study of classical groups of type (a). One can also show that in case (e) the classical group is the general linear group.

2.1.3 The doubling embedding

We keep the notation $G := G(\mathcal{W}) = G(W, \Phi)$ in (2.1.6) or (2.1.7). Doubling the underlying ϵ-hermitian space $\mathcal{W}$ we consider $\mathcal{V} = (W \oplus W, \langle\langle \cdot, \cdot \rangle\rangle)$ where

$$\langle\langle (x_1, x_2), (y_1, y_2) \rangle\rangle := \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle \text{ for } (x_1, x_2), (y_1, y_2) \in W \oplus W. \tag{2.1.9}$$

By fixing a basis of $\mathcal{V}$, the classical group $G(\mathcal{V})$ is isomorphic to

$$H = \{h \in \mathrm{GL}_{2n}(D) : gJ_n g^* = J_n\}, \quad J_n = \begin{bmatrix} 0 & 1_n \\ \epsilon \cdot 1_n & 0 \end{bmatrix}. \tag{2.1.10}$$

Note that

$$R \begin{bmatrix} \Phi & 0 \\ 0 & -\Phi \end{bmatrix} R^* = J_n,$$

with

$$R = \begin{bmatrix} 0 & \frac{\epsilon}{2}\cdot 1_r & 0 & 0 & \frac{\epsilon}{2}\cdot 1_r & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon\cdot 1_m \\ 1_m & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^{-1} & 0 & 0 & \theta^{-1} & 0 \\ 0 & 0 & 0 & 1_m & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \end{bmatrix}.$$

Then we define a doubling map

$$\begin{aligned} &G \times G \to H, \\ &(g_1, g_2) \mapsto R \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} R^{-1}. \end{aligned} \tag{2.1.11}$$

We thus view $G \times G$ as a subgroup of H and identify (g_1, g_2) with its image in H. More explicitly, if we write

$$g_1 = \begin{bmatrix} a_1 & f_1 & b_1 \\ h_1 & e_1 & j_1 \\ c_1 & k_1 & d_1 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} a_2 & f_2 & b_2 \\ h_2 & e_2 & j_2 \\ c_2 & k_2 & d_2 \end{bmatrix}, \tag{2.1.12}$$

with a_1, a_2, d_1, d_2 of size $m \times m$, e_1, e_2 of size $r \times r$, then

$$(g_1, g_2) = \begin{bmatrix} \frac{e_1+e_2}{2} & -\frac{j_2}{2} & \frac{\epsilon h_1}{2} & \frac{\epsilon(e_1-e_2)\theta}{4} & \frac{\epsilon h_2}{2} & \frac{\epsilon j_1}{2} \\ -k_2 & d_2 & 0 & \frac{\epsilon k_2 \theta}{2} & -\epsilon c_2 & 0 \\ \epsilon f_1 & 0 & a_1 & \frac{f_1\theta}{2} & 0 & b_1 \\ \epsilon\theta^{-1}(e_1 - e_2) & \epsilon\theta^{-1} j_2 & \theta^{-1} h_1 & \theta^{-1}\frac{e_1+e_2}{2}\theta & -\theta^{-1} h_2 & \theta^{-1} j_1 \\ \epsilon f_2 & -\epsilon b_2 & 0 & -\frac{f_2\theta}{2} & a_2 & 0 \\ \epsilon k_1 & 0 & c_1 & \frac{k_1\theta}{2} & 0 & d_1 \end{bmatrix}. \tag{2.1.13}$$

2.2 The global integral and the main result

We review the doubling method and summarize our main results on the integral representation of the standard L-functions for classical groups. The definition of the local L-factors and computation of the local integrals will be carried out in the next two sections.

2.2.1 The global groups

Let F be a number field with adele ring $\mathbb{A}$. We consider tuples (D, ρ, ϵ) of following five cases:

(Case I, Orthogonal)	(D, ρ) of type (a) with $\epsilon = 1$,
(Case II, Symplectic)	(D, ρ) of type (a) with $\epsilon = -1$,
(Case III, Quaternionic Orthogonal)	(D, ρ) of type (b) with $\epsilon = 1$,
(Case IV, Quaternionic Unitary)	(D, ρ) of type (b) with $\epsilon = -1$,
(Case V, Unitary)	(D, ρ) of type (c) with $D = E$ and $\epsilon = -1$.

The global groups we consider are

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\},\ \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.2.1}$$

with $n = 2m + r$ and $\theta^* = \epsilon\theta \in \mathrm{GL}_r(D)$ anisotropic (so the global Witt index of Φ is m) and

$$H := H(F) := \{h \in \mathrm{GL}_{2n}(D) : gJ_n g^* = J_n\}, \quad J_n = \begin{bmatrix} 0 & 1_n \\ \epsilon \cdot 1_n & 0 \end{bmatrix}, \tag{2.2.2}$$

together with a doubling embedding $G \times G \to H$ defined by (2.1.11).

We denote $\mathbb{A}_D = D \otimes_F \mathbb{A}$ for the adelization and $D_v = D \otimes_F F_v$ for the localization at a place v of F. For global groups, we will write $G(\mathbb{A}), H(\mathbb{A}), G(F_v), H(F_v)$ for its adelization and localization but simply write $G = G(F), H = H(F)$ for the rational points if its meaning is clear from the context.

Remark 2.2.1. In this thesis, we label our group as Case I-V for simplicity but we may also call the name of the group (i.e. orthogonal, symplectic, ...) so that one can easily compare the group here with the one in other papers. The notion of orthogonal, symplectic and unitary groups are well known and appear frequently in the literature while the groups of Case III, IV do not have a standard name. Here we call them quaternionic orthogonal or unitary depending on whether the form defining the group is hermitian (like the orthogonal group) or skew-hermitian (like the unitary). But indeed, both groups have been called 'quaternionic unitary' in the literature so the reader should be careful about its meaning. For example, the group studied in [Gar77] and [Shi99b] are the quaternionic orthogonal group here.

Remark 2.2.2. In this thesis, we shall use the term 'totally isotropic group' to indicate the group whose associated ϵ-hermitian form is totally isotropic (i.e. $r = 0$ in (2.1.7)) instead of using the term split group or quasi-split group. In the sequel, when discussing the local groups $G(F_v)$, we will distinguish the 'split' and 'nonsplit' case according to whether D_v is split or not.

2.2.2 The doubling method

Let $P \subset H$ be the Siegel parabolic subgroup whose Levi component is $\mathrm{GL}_n(D)$. More explicitly, $P = M \ltimes N$ with

$$M = \left\{ \begin{bmatrix} a & 0 \\ 0 & \hat{a} \end{bmatrix} : a \in \mathrm{GL}_n(D) \right\}, \quad N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in S_n(F) \right\}. \tag{2.2.3}$$

Here $S_n(F)$ is an additive algebraic group with

$$S_n(F) = \{ b \in \mathrm{Mat}_n(D) : \epsilon b + b^* = 0 \}. \tag{2.2.4}$$

Let $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ be a Hecke character and extend it to a character on $\mathrm{GL}_n(\mathbb{A}_D)$ (still denoted by χ) by taking the composite with the reduced norm $\nu : \mathrm{GL}_n(\mathbb{A}_D) \to \mathbb{A}_E^\times$. Consider the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\chi|\nu(\cdot)|^s) \tag{2.2.5}$$

consisting of functions $f_s : H(\mathbb{A}) \to \mathbb{C}$ such that

$$f_s(pg) = \chi(\nu(a))|\mathrm{N}_{E/F}(\nu(a))|^{s+\kappa} f_s(g), \tag{2.2.6}$$

for

$$p = \begin{bmatrix} a & b \\ 0 & \hat{a} \end{bmatrix} \in P(\mathbb{A}), \qquad a \in \mathrm{GL}_n(D), \tag{2.2.7}$$

where

$$\kappa = \begin{cases} \frac{n-1}{2} & \text{Case I,} \\ \frac{n+1}{2} & \text{Case II,} \\ \frac{2n+1}{2} & \text{Case III,} \\ \frac{2n-1}{2} & \text{Case IV,} \\ \frac{n}{2} & \text{Case V.} \end{cases} \tag{2.2.8}$$

We then form the Eisenstein series

$$E(h; f_s) = \sum_{\gamma \in P(F)\backslash H(F)} f_s(\gamma h), \qquad h \in H(\mathbb{A}), \tag{2.2.9}$$

on $H(\mathbb{A})$ associated to a standard section f_s.

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and $\phi_1, \phi_2 \in \pi$ be two cusp forms. The global integral we consider is

$$\begin{aligned} &\mathcal{Z}(s; \phi_1, \phi_2, f_s) \\ &= \int_{(G\times G)(F)\backslash(G\times G)(\mathbb{A})} E((g_1, g_2); f_s)\overline{\phi_1(g_1)}\phi_2(g_2)\chi(\nu(g_2))^{-1} dg_1 dg_2. \end{aligned} \tag{2.2.10}$$

Here we must take $\phi_1, \phi_2 \in \pi$ in the same representation space, otherwise the integral will be identically zero. The following basic identity is pivotal in the doubling method.

Proposition 2.2.3. *(Basic identity) Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and $\phi_1, \phi_2 \in \pi$ be two cusp forms. Then*

$$\mathcal{Z}(s; \phi_1, \phi_2, f_s) = \int_{G(\mathbb{A})} f_s(\delta(g, 1))\langle \pi(g)\phi_1, \phi_2 \rangle dg, \tag{2.2.11}$$

where

$$\langle \phi_1, \phi_2 \rangle = \int_{G(F)\backslash G(\mathbb{A})} \overline{\phi_1(g)}\phi_2(g) dg, \tag{2.2.12}$$

is the standard inner product on $G(\mathbb{A})$ *and*

$$\delta = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & -1_m & 0 & 1_m & 0 \\ 0 & \epsilon \cdot 1_m & 0 & 0 & 0 & 1_m \end{bmatrix}. \tag{2.2.13}$$

Proof. The proof is well known (see for example [PR87]). We sketch the idea here.

Unfolding the Eisenstein series, we obtain

$$\begin{aligned}&\mathcal{Z}(s;\phi_1,\phi_2,f_s)\\ =&\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})}\sum_{\gamma\in P(F)\backslash H(F)/G(F)\times G(F)}\sum_{\eta\in \mathrm{Stab}_\gamma\backslash G(F)\times G(F)}\\ &f_s(\gamma\eta(g_1,g_2))\overline{\phi_1(g_1)}\phi_2(g_2)\chi(\nu(g_2))^{-1}dg_1dg_2.\end{aligned}$$

Here $\mathrm{Stab}_\gamma = \gamma^{-1}P(F)\gamma\cap G(F)\times G(F)$ is the stabilizer of the orbit represented by γ. The representatives of $P(F)\backslash H(F)/G(F)\times G(F)$ can be chosen as

$$\delta_i = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & -e_i & 0 & 1_m & 0 \\ 0 & \epsilon \cdot e_i & 0 & 0 & 0 & 1_m \end{bmatrix} \text{ with } e_i = \begin{bmatrix} 1_i & 0 \\ 0 & 0 \end{bmatrix},\ 0\le i\le m,$$

and the stabilizer Stab_{δ_i} can be easily calculated. We can thus write

$$\begin{aligned}\mathcal{Z}(s;\phi_1,\phi_2,f_s) &= \sum_{0\le i\le m}\mathcal{Z}_i(s;\phi_1,\phi_2,f_s),\\ \mathcal{Z}_i(s;\phi_1,\phi_2,f_s) &= \int_{\mathrm{Stab}_{\delta_i}\backslash G(\mathbb{A})\times G(\mathbb{A})} f_s(\delta_i(g_1,g_2))\overline{\phi_1(g_1)}\phi_2(g_2)\chi(\nu(g_2))^{-1}dg_1dg_2.\end{aligned}$$

All the orbits represented by δ_i for $0\le i<m$ are negligible in the sense that they contain the unipotent radical of a proper parabolic subgroup of $G\times G$ as a normal

subgroup. Then using the cuspidality of π, we can show that

$$\mathcal{Z}_i(s;\phi_1,\phi_2,f_s)=0 \text{ for all } 0\le i<m.$$

It remains to calculate the contribution of the main orbit $\delta := \delta_m$. Note that $\delta(g,g)\delta^{-1}\in P$ for all $g\in G$, we calculate

$$\begin{aligned}\mathcal{Z}_m(s;\phi_1,\phi_2,f_s) &= \int_{G(\mathbb{A})}\int_{G(F)\backslash G(\mathbb{A})} f_s(\delta(g_2g_1,g_2))\overline{\phi_1(g_2g_1)}\phi_2(g_2)\chi(\nu(g_2))^{-1}dg_2dg_1\\ &= \int_{G(\mathbb{A})} f_s(\delta(g_1,1))\int_{G(F)\backslash G(\mathbb{A})}\overline{\phi_1(g_2g_1)}\phi_2(g_2)dg_2dg_1.\end{aligned}$$

□

Write $\pi=\otimes_v'\pi_v$ and assume $\phi_1=\otimes_v\phi_{1,v},\phi_2=\otimes_v\phi_{2,v}$ with $\phi_{1,v},\phi_{2,v}\in\pi_v$. Also choose the section f_s such that $f_s=\prod_v f_{s,v}$ is factorizable with local sections $f_{s,v}\in \mathrm{Ind}_{P(F_v)}^{H(F_v)}(\chi|\nu(\cdot)|^s)$. Due to the uniqueness of the pairing, $\langle\cdot,\cdot\rangle$ is factorizable in the sense that $\langle\phi_1,\phi_2\rangle=\prod_v\langle\phi_{1,v},\phi_{2,v}\rangle$, where

$$\langle\phi_{1,v},\phi_{2,v}\rangle=\int_{G(F_v)}\overline{\phi_{1,v}(g)}\phi_{2,v}(g)dg \tag{2.2.14}$$

is the local pairing. Then $\mathcal{Z}(s;\phi_1,\phi_2,f_s)$ has an Euler product expression

$$\begin{aligned}\mathcal{Z}(g_2;\phi_1,\phi_2,f_s)&=\prod_v\mathcal{Z}_v(s;\phi_{1,v},\phi_{2,v},f_{s,v}),\\ \mathcal{Z}_v(s;\phi_{1,v},\phi_{2,v},f_{s,v})&=\int_{G(F_v)}f_{s,v}(\delta(g,1))\langle\pi(g)\phi_{1,v},\phi_{2,v}\rangle dg.\end{aligned}\tag{2.2.15}$$

Hence, the global integral $\mathcal{Z}(s;\phi_1,\phi_2,f_s)$ can be studied locally place by place.

In some works of the doubling method (e.g. [BS00; Gar84b; Shi97; Shi00]), for $g_2\in G(\mathbb{A})$, the integral of the following form is considered

$$\mathcal{Z}'(g_2;\phi_1,f_s)=\int_{G(F)\backslash G(\mathbb{A})}E((g_1,g_2);f_s)\overline{\phi_1(g_1)}dg_1. \tag{2.2.16}$$

The computation of (2.2.16) is same as the one for (2.2.10). In particular, we have

$$\begin{aligned}\mathcal{Z}'(g_2;\phi_1,f_s) &= \chi(\nu(g_2))\int_{G(\mathbb{A})} f_s(\delta(g_1,1))\phi_1(g_2g_1)dg_1\\ &= \chi(\nu(g_2))\prod_v \mathcal{Z}'_v(g_2;\phi_{1,v},f_{s,v}), \qquad (2.2.17)\\ \mathcal{Z}'_v(g_2;\phi_{1,v},f_{s,v}) &= \int_{G(F_v)} f_{s,v}(\delta(g_1,1))\phi_{1,v}(g_2g_1)dg_1\end{aligned}$$

2.2.3 Main results on integral representations

The first main result of this thesis is an integral representation of standard L-functions. That is, we make the choice of f_s such that the global integral $\mathcal{Z}$ in (2.2.10) represents the L-function defined in Section 2.3. We summarize our result here.

Let $\mathfrak{o}$ be the ring of integers of F and $\mathcal{O}$ a maximal order of D. Denote by $\mathfrak{o}_v,\mathcal{O}_v$ their localizations and assume $D_v=\mathcal{O}_v\otimes_{\mathfrak{o}_v}F_v$. For a finite place v corresponds to a prime ideal $\mathfrak{p}_v$ of $\mathfrak{o}$, denote ϖ_v for the uniformizer of $\mathfrak{p}_v$ and set $q_v=|\varpi_v|_v^{-1}$. Fix an $\mathfrak{o}$-ideal $\mathfrak{n}=\mathfrak{n}_1\mathfrak{n}_2$ with $\mathfrak{n}_1,\mathfrak{n}_2$ coprime and write $\mathfrak{n}=\prod_v\mathfrak{p}_v^{\mathfrak{c}_v}$. Define the following open compact subgroup of $G(\mathfrak{o})$:

$$K(\mathfrak{n})=G(\mathfrak{o})\cap\begin{bmatrix}\mathrm{Mat}_m(\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O})\\ \mathrm{Mat}_{r,m}(\mathfrak{n}\mathcal{O}) & 1+\mathrm{Mat}_r(\mathfrak{n}'\mathcal{O}) & \mathrm{Mat}_{r,m}(\mathcal{O})\\ \mathrm{Mat}_m(\mathfrak{n}\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathfrak{n}\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O})\end{bmatrix}, \qquad (2.2.18)$$

where $\mathfrak{n}'=\prod_{v,\mathfrak{c}_v\geq 1}\mathfrak{p}_v$ is the support of $\mathfrak{n}$.

Let $\phi\in\pi$ be a factorizable cusp form with $\phi=\otimes'_v\phi_v$. Set S_∞ be the set of all archimedean places of F. Denote S_1 be the set consisting of places dividing $\mathfrak{n}_1$ and S_2 the set consisting of places dividing $\mathfrak{n}_2$. We make the following assumptions:

(1) $2\in\mathcal{O}_v^\times$ and $\theta\in\mathrm{GL}_r(\mathcal{O}_v)$ for all $v\in S_1\cup S_2$,

(2) ϕ is fixed by $K(\mathfrak{n})$ and is unramified outside S_1,S_2, i.e. fixed by $G(\mathfrak{o}_v)$ for $v\notin S_1\cup S_2\cup S_\infty$,

(3) ϕ is an eigenfunction for the Hecke algebra $\mathcal{H}(K(\mathfrak{n}),\mathfrak{X})$ as in Section 2.3.4,

(4) χ has conductor $\mathfrak{n}_2$,

(5) In Case V, all places $v \in S_1 \cup S_2$ are nonsplit in $\mathcal{O}$.

The standard L-function $L(s, \phi \times \chi)$ of ϕ twisted by χ is defined in Section 2.3.4. There is an Euler product expression

$$L(s, \phi \times \chi) = \prod_v L_v(s, \phi_v \times \chi_v). \tag{2.2.19}$$

When $v \notin S_1 \cup S_2$, $L_v(s, \phi_v \times \chi_v)$ is the unramified local L-factors defined with m in Section 2.3.1 replaced by the Witt index of $G(F_v)$. When $v \in S_1 \cup S_2$, $L_v(s, \phi_v \times \chi_v)$ are the ramified local L-factors and in particular $L_v(s, \phi_v \times \chi_v) = 1$ if $v \in S_2$. The integral representation for the partial L-function

$$L^{S_1 \cup S_2}(s, \phi \times \chi) := \prod_{v \notin S_1 \cup S_2} L_v(s, \phi \times \chi) \tag{2.2.20}$$

is well known. We make the choice of local sections $f_{s,v}$ properly for $v \in S_1 \cup S_2$ such that the global integral $\mathcal{Z}$ represents the complete L-function. Let

$$w = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & 1_r & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix} \tag{2.2.21}$$

be a Weyl element. Define $\eta_1 \in G(\mathbb{A})$ to be an element such that $(\eta_1)_v = w$ for $v \in S_1$ and $(\eta_1)_v = 1$ for $v \notin S_1$. Similarly set $\eta_2 \in G(\mathbb{A})$ to be an element such that $(\eta_2)_v = w$ for $v \in S_2$ and $(\eta_2)_v = 1$ for $v \notin S_2$.

Take $\phi_1 = \pi(\eta_1)\phi$, $\phi_2 = \pi(\eta_2)\phi$ and write

$$\mathcal{Z}(s; \phi, f_s) := \mathcal{Z}(s; \phi_1, \phi_2, f_s). \tag{2.2.22}$$

Theorem 2.2.4. *Keep the assumptions of ϕ, χ as above. Take the section f_s to be*

$$f_s = \prod_{v \notin S_1 \cup S_2 \cup S_\infty} f^0_{s,v} \cdot \prod_{v \in S_1} f^{\dagger, \mathfrak{c}_v}_{s,v} \cdot \prod_{v \in S_2} f^{\ddagger, \mathfrak{c}_v}_{s,v} \cdot \prod_{v \in S_\infty} f^{\infty}_{s,v}. \tag{2.2.23}$$

Then

$$\mathcal{Z}(s; \phi, f_s) = C \cdot L\left(s + \frac{1}{2}, \phi \times \chi\right) \cdot \mathcal{Z}_\infty(s; \phi_\infty, f^\infty_s) \cdot \prod_{v \nmid \infty} \langle \pi(\eta)\phi_v | U'(\mathfrak{n}_1), \phi_v \rangle. \tag{2.2.24}$$

Here:

(a) $f^0_{s,v}, f^{\dagger,\mathfrak{c}_v}_{s,v}, f^{\ddagger,\mathfrak{c}_v}_{s,v}$ *are local sections defined by* (2.4.6), (2.4.11), (2.4.15) *and* $f^\infty_{s,v}$ *can be chosen such that*

$$\mathcal{Z}_\infty(s;\phi_\infty, f^\infty_s) := \prod_v \mathcal{Z}_{v|\infty}(s;\phi_v, f^\infty_{s,v}) \neq 0, \tag{2.2.25}$$

(b) $U'(\mathfrak{n}_1) = \prod_{v|\mathfrak{n}_1} U'(\mathfrak{p}_v^{\mathfrak{c}_v})$ *is the Hecke operator defined by* (2.3.14) *and*

$$\eta = \prod_{v\in S_2} \begin{bmatrix} 0 & 0 & \varpi_v^{-\mathfrak{c}_v}\cdot 1_m \\ 0 & 1_r & 0 \\ \varpi_v^{\mathfrak{c}_v}\cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.2.26}$$

(c) C *is a constant given by*

$$C = \chi(\mathfrak{n}_1)^{m\mathbf{d}_1}|\mathfrak{n}_1|^{m\mathbf{d}_2(s+\kappa)}\mathrm{vol}(\mathrm{GL}_m(\mathcal{O})/\mathrm{GL}_m(\mathfrak{n}_2\mathcal{O})), \tag{2.2.27}$$

with

$$\mathbf{d}_1 = \begin{cases} 1 & \textit{Case I, II, V,} \\ 2 & \textit{Case III, IV,} \end{cases} \qquad \mathbf{d}_2 = \begin{cases} 1 & \textit{Case I, II,} \\ 2 & \textit{Case III, IV, V.} \end{cases} \tag{2.2.28}$$

Remark 2.2.5. This is proved by combining the local computations of Proposition 2.4.1, 2.4.2, 2.4.4. In Case III, IV, if D_v splits then the local computations follow from the one for Case I, II as the local group $G(F_v)$ is a symplectic group in Case III or an orthogonal group in Case IV (see also Section 2.3.3). For Case V, we do not cover the split case in this work for simplicity and also because this case is well studied in [HLS06] and [EHLS20]. Hence throughout the thesis we will assume all $v|\mathfrak{n}$ are nonsplit in $\mathcal{O}$ for Case V.

2.3 Hecke operators and local L-factors

In this and the next section, we fix the following local setup. Let F be a non-archimedean local field and $\mathfrak{o}$ its ring of integers with the maximal ideal $\mathfrak{p}$. Fix a uniformizer ϖ and the absolute value $|\cdot|$ on F normalized so that $|\varpi| = q^{-1}$ with

q the cardinality of the residue field. We consider tuples (D, ρ, ϵ) of following eight cases:

(Case I, Orthogonal)	(D, ρ) of type (a) with $\epsilon = 1$,
(Case II, Symplectic)	(D, ρ) of type (a) with $\epsilon = -1$,
(Case III, Quaternionic Orthogonal Nonsplit)	(D, ρ) of type (b) with $\epsilon = 1$,
(Case III', Quaternionic Orthogonal Split)	(D, ρ) of type (d) with $\epsilon = 1$,
(Case IV, Quaternionic Unitary Nonsplit)	(D, ρ) of type (b) with $\epsilon = -1$,
(Case IV', Quaternionic Unitary Split)	(D, ρ) of type (d) with $\epsilon = -1$,
(Case V, Unitary Nonsplit)	(D, ρ) of type (c) with $D = E$, $\epsilon = -1$,
(Case V', Unitary Split)	(D, ρ) of type (e) with $D = E$, $\epsilon = -1$.

We fix a maximal order $\mathcal{O}$ of D such that $D = \mathcal{O} \otimes_{\mathfrak{o}} F$. Let $\mathfrak{q}$ be a prime ideal in $\mathcal{O}$ above $\mathfrak{p}$ and fix $\widetilde{\varpi}$ a uniformizer of $\mathfrak{q}$.

2.3.1 Unramified local L-factors

In this and the next subsection, we do not consider three split cases (i.e Case III', IV', V'). Let

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\},\ \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.3.1}$$

with $n = 2m + r$ and $\theta^* = \epsilon\theta \in \mathrm{GL}_r(D)$ is anisotropic. Assume π is an unramified admissible representation of $G(F)$ and $\phi \in \pi$ a spherical vector. Also assume χ is an unramified character of $E^\times$.

Recall the Cartan decomposition

$$\begin{aligned} G(F) &= \coprod_{\substack{e_1,\dots,e_m \in \mathbb{Z} \\ 0 \le e_1 \le \dots \le e_m}} K_{e_1,\dots,e_m}, \\ K_{e_1,\dots,e_m} &= G(\mathfrak{o})\mathrm{diag}[\widetilde{\varpi}^{e_1}, \dots, \widetilde{\varpi}^{e_m}, 1_r, \widetilde{\varpi}^{-e_1}, \dots, \widetilde{\varpi}^{-e_m}]G(\mathfrak{o}). \end{aligned} \tag{2.3.2}$$

The local spherical Hecke algebra $\mathcal{H}$ is generated by all such double cosets $K_{e_1,\dots e_m}$.

The action of the Hecke operator associated to $[K_{e_1,\ldots,e_m}]$ on ϕ is given by

$$\phi|[K_{e_1,\ldots,e_m}] = \int_{K_{e_1,\ldots,e_m}} \pi(g)\phi dg. \tag{2.3.3}$$

Here, the measure dg is normalized such that $G(\mathfrak{o})$ has volume 1. Since the space of spherical vectors has dimension one, ϕ is an eigenvector under the action of Hecke operators, that is

$$\phi|[K_{e_1,\ldots,e_m}] = \lambda_{e_1,\ldots,e_m}(\phi)\phi, \tag{2.3.4}$$

for some scalar $\lambda_{e_1,\ldots,e_m}(\phi)$. We define the unramified local L-factors as

$$\begin{aligned} &L\left(s+\frac{1}{2}, \phi \times \chi\right) \\ =&b(s,\chi) \sum_{\substack{e_1,\ldots,e_m\in\mathbb{Z} \\ 0\le e_1\le\ldots\le e_m}} \lambda_{e_1,\ldots,e_m}(\phi) \left(\chi(\nu(\widetilde{\varpi}))|\mathrm{N}_{E/F}(\nu(\widetilde{\varpi}))|^{s+\kappa}\right)^{e_1+\ldots+e_m}. \end{aligned} \tag{2.3.5}$$

Here $b(s,\chi)$ is the normalizing factor given in the following list (taken from [Yam14, p.667] but Case III, IV should be calculated from [Shi99b, Proposition 3.5]).

(Case I, Orthogonal)

$$b(s,\chi) = \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} L(2s+n+1-2i, \chi^2),$$

(Case II, Symplectic)

$$b(s,\chi) = L\left(s+\frac{n+1}{2}, \chi\right) \prod_{i=1}^{\frac{n}{2}} L(2s-1+2i, \chi^2),$$

(Case III, Quaternionic Orthogonal Nonsplit)

$$b(s,\chi) = L\left(s+\frac{2n+1}{2}, \chi\right) \prod_{i=1}^{n} L(2s+2n+1-4i, \chi^2),$$

(Case IV, Quaternionic Unitary Nonsplit)

$$b(s,\chi) = \prod_{i=1}^{n} L(2s+2n+3-4i, \chi^2),$$

(Case V, Unitary) Set $\chi^0 = \chi|_{F^\times}$ and let $\chi_{E/F}$ be the quadratic character associated to E/F, then

$$b(s,\chi) = \prod_{i=1}^{n} L(2s+i, \chi^0\chi_{E/F}^{n+i}).$$

Here $L(s,\chi)$ means the local L-factor of Hecke L-functions.

Proposition 2.3.1. *Let $\alpha_i \in \mathbb{C}, 1 \le i \le m$ be the Satake parameters of ϕ. Then $L(s,\phi\times\chi)$ has an Euler product expansion with $L(s,\phi\times\chi)^{-1}$ given by the following list.*

(Case I, Orthogonal)

$$\prod_{i=1}^{\lfloor \frac{r}{2}\rfloor}\left(1-\chi(\varpi)^2 q^{2i-r-2s}\right)\times\prod_{i=1}^{m}\left(1-\chi(\varpi)\alpha_i q^{-1+\frac{r}{2}-s}\right)\left(1-\chi(\varpi)\alpha_i^{-1}q^{1-\frac{r}{2}-s}\right),$$

(Case II, Symplectic)

$$\left(1-\chi(\varpi)q^{-s}\right)\times\prod_{i=1}^{m}\left(1-\chi(\varpi)\alpha_i q^{-s}\right)\left(1-\chi(\varpi)\alpha_i^{-1}q^{-s}\right),$$

(Case III, Quaternionic Orthogonal Nonsplit)

$$\left(1-\chi(\varpi)q^{-r-s}\right)\times\prod_{i=1}^{m+r}\left(1-\chi(\varpi)^2 q^{4i-2r-2s}\right)$$
$$\times\prod_{i=1}^{m}\left(1-\chi(\varpi)\alpha_i q^{-1+r-s}\right)\left(1-\chi(\varpi)\alpha_i^{-1}q^{-r-s}\right),$$

(Case IV, Quaternionic Unitary Nonsplit)

$$\prod_{i=1}^{m+r}\left(1-\chi(\varpi)^2 q^{4i-2-2r-2s}\right)$$
$$\times\prod_{i=1}^{m}\left(1-\chi(\varpi)\alpha_i q^{-2+r-s}\right)\left(1-\chi(\varpi)\alpha_i^{-1}q^{1-r-s}\right),$$

(Case V, Unitary Inert) E/F is inert,

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\alpha_i q^{-1+r-s}\right)\left(1-\chi(\varpi)\alpha_i^{-1}q^{1-r-s}\right),$$

(Case V, Unitary Ramified) E/F is ramified,

$$\prod_{i=1}^{m}\left(1-\chi(\widetilde{\varpi})\alpha_i q^{\frac{r-1}{2}-s}\right)\left(1-\chi(\widetilde{\varpi})\alpha_i^{-1}q^{-\frac{r-1}{2}-s}\right),$$

Proof. The symplectic and unitary cases are given in [Shi00, Theorem 19.8]. The orthogonal case are given by [Shi04, Proposition 17.14] and the quaternionic orthogonal group are studied in [Shi99b, Theorem 3.12]. All can be computed using the method in [Shi97, Section 16]. For the quaternionic unitary groups, by the same manner, we calculate the following Dirichlet series,

$$\sum_{\substack{e_1,\ldots,e_m\in\mathbb{Z}\\ 0\le e_1\le\ldots\le e_m}} \lambda_{e_1,\ldots,e_m}(q^{-s})^{e_1+\ldots+e_m} = \alpha(s)\beta(2s-2m+1)\mathfrak{A}(s-n+1-r,s),$$

where

$$\alpha(s) = \prod_{i=1}^{m} \frac{1-q^{4i-4-2s}}{1-q^{2m+2i-3-2s}},$$
$$\beta(s) = \prod_{i=1}^{m} \frac{1-q^{2i-2-s}}{1-q^{2r+2i-2-s}},$$
$$\mathcal{A}(s',s) = \prod_{i=1}^{m} \frac{1-q^{2i-2-s-s'}}{(1-q^{-2-s'}\alpha_i)(1-q^{2m-s}\alpha_i^{-1})}.$$

Then

$$\sum_{\substack{e_1,\ldots,e_m\in\mathbb{Z}\\ 0\le e_1\le\ldots\le e_m}} \lambda_{e_1,\ldots,e_m}(q^{-s})^{e_1+\ldots+e_m} = \prod_{i=1}^{m} \frac{1-q^{4i-4-2s}}{(1-q^{n+r-3-s}\alpha_i)(1-q^{2m-s}\alpha_i^{-1})}.$$

Multiplying the normalizing factor $b(s,\chi)$ we obtain the result in the above list. □

2.3.2 Ramified local L-factors

Let

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\},\ \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon\cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.3.6}$$

with $n = 2m + r$ and $\theta^* = \epsilon\theta \in \mathrm{GL}_r(D)$ not necessarily anisotropic. In the ramified cases, we will always assume that 2 and θ are unramified, i.e. $2 \in \mathcal{O}^\times, \theta \in \mathrm{GL}_r(\mathcal{O})$. For an integer $\mathfrak{c} \ge 1$, we consider the following two open compact subgroups of $G(\mathfrak{o})$:

$$\begin{aligned} K(\mathfrak{p}^\mathfrak{c}) &= G(\mathfrak{o}) \cap \begin{bmatrix} \mathrm{Mat}_m(\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O}) \\ \mathrm{Mat}_{r,m}(\mathfrak{p}^\mathfrak{c}\mathcal{O}) & 1+\mathrm{Mat}_r(\mathfrak{p}\mathcal{O}) & \mathrm{Mat}_{r,m}(\mathcal{O}) \\ \mathrm{Mat}_m(\mathfrak{p}^\mathfrak{c}\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathfrak{p}^\mathfrak{c}\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O}) \end{bmatrix}, \\ K'(\mathfrak{p}^\mathfrak{c}) &= G(\mathfrak{o}) \cap \begin{bmatrix} \mathrm{Mat}_m(\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathfrak{p}^\mathfrak{c}\mathcal{O}) & \mathrm{Mat}_m(\mathfrak{p}^\mathfrak{c}\mathcal{O}) \\ \mathrm{Mat}_{r,m}(\mathcal{O}) & 1+\mathrm{Mat}_r(\mathfrak{p}\mathcal{O}) & \mathrm{Mat}_{r,m}(\mathfrak{p}^\mathfrak{c}\mathcal{O}) \\ \mathrm{Mat}_m(\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O}) \end{bmatrix}. \end{aligned} \tag{2.3.7}$$

Clearly, they are related by $K(\mathfrak{p}^{\mathfrak{c}}) = wK'(\mathfrak{p}^{\mathfrak{c}})w$ with w the Weyl element as (2.2.21). Let

$$\mathfrak{M} = \mathrm{GL}_m(D) \cap \mathrm{Mat}_m(\mathcal{O}), \mathfrak{Q} = \{\mathrm{diag}[u, 1_r, \hat{u}], u \in \mathfrak{M}\}, \mathfrak{X} = K(\mathfrak{p}^{\mathfrak{c}})\mathfrak{Q}K(\mathfrak{p}^{\mathfrak{c}}). \quad (2.3.8)$$

For $\xi = \mathrm{diag}[u, 1_r, \hat{u}] \in \mathfrak{Q}$, we define $\mathfrak{d}(\xi)$ be the integer such that $\nu(u) = \widetilde{\varpi}^{\mathfrak{d}(\xi)}$. The local Hecke algebra $\mathcal{H}(K(\mathfrak{p}^{\mathfrak{c}}), \mathfrak{X})$ associated to $K(\mathfrak{p}^{\mathfrak{c}})$ and $\mathfrak{X}$ is generated by double cosets $[K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})]$ with $\xi \in \mathfrak{Q}$. This kind of Hecke algebra generalizes the one in [Shi00, Section 19]. Let π be an admissible representation of $G(F)$. Assume $\phi \in \pi$ is a vector fixed by $K(\mathfrak{p}^{\mathfrak{c}})$, the Hecke operator $[K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})]$ acts on ϕ by

$$\phi|[K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})] = \int_{K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})} \pi(g)\phi dg. \quad (2.3.9)$$

If we assume the measure dg is normalized such that $K(\mathfrak{p}^{\mathfrak{c}})$ has volume 1, then the action can be written as a sum

$$\phi|[K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})] = \sum_{K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})/K(\mathfrak{p}^{\mathfrak{c}})} \pi(g)\phi. \quad (2.3.10)$$

The coset in the sum is characterized in the following lemma.

Lemma 2.3.2. *Let* $\xi = \mathrm{diag}[u, 1_r, \hat{u}]$ *with* $u \in \mathfrak{M}$ *then*

$$K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}}) = \coprod_{d,b,c} \begin{bmatrix} d & -b^*\theta^{-1} & c\hat{d} \\ 0 & 1 & b\hat{d} \\ 0 & 0 & \hat{d} \end{bmatrix} K(\mathfrak{p}^{\mathfrak{c}}), \quad (2.3.11)$$

where $d \in \mathrm{GL}_m(D)u\mathrm{GL}_m(D)/\mathrm{GL}_m(D)$, $b \in \mathrm{Mat}_{m,r}(\mathcal{O})/\mathrm{Mat}_{m,r}(\mathcal{O})d^*$ *and* $c \in \mathrm{Mat}_m(\mathcal{O})/d\mathrm{Mat}_m(\mathcal{O})d^*$ *satisfying* $\epsilon c + b^*\hat{\theta}b + c^* = 0$.

Proof. This is an analogue of [Shi00, Lemma 19.2] and can be verified in a straightforward way. $\square$

Assume $\phi \in \pi$ is an eigenvector for all $[K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})]$, that is there exists a scalar λ_ξ such that

$$\phi|[K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})] = \lambda_\xi(\phi)\phi. \quad (2.3.12)$$

For an integer $\mathfrak{n} \geq 1$, we consider a special Hecke operator

$$U(\mathfrak{p}^{\mathfrak{n}}) := [K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})] \text{ with } \xi = \mathrm{diag}[\varpi^{\mathfrak{n}} \cdot 1_m, 1_r, \varpi^{-\mathfrak{n}} \cdot 1_m]. \tag{2.3.13}$$

Denote the Hecke eigenvalue for operator $U(\mathfrak{p}^{\mathfrak{n}})$ as $\alpha(\mathfrak{p}^{\mathfrak{n}})$. Clearly by Lemma 2.3.2, one has $U(\mathfrak{p}^{\mathfrak{n}}) = U(\mathfrak{p})^{\mathfrak{n}}$ and $\alpha(\mathfrak{p}^{\mathfrak{n}}) = \alpha(\mathfrak{p})^{\mathfrak{n}}$. In later computations, we will also use another kind of Hecke operator

$$U'(\mathfrak{p}^{\mathfrak{n}}) := [K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})] \text{ with } \xi = \begin{bmatrix} 0 & 0 & \varpi^{-\mathfrak{n}} \cdot 1_m \\ 0 & 1_r & 0 \\ \epsilon\varpi^{\mathfrak{n}} \cdot 1_m & 0 & 0 \end{bmatrix}. \tag{2.3.14}$$

Its action on ϕ is defined similarly as above.

Assume χ is an unramified character, define the ramified local L-factors as

$$L\left(s+\frac{1}{2}, \phi \times \chi\right) = \sum_{\xi \in K(\mathfrak{p}^{\mathfrak{c}})\backslash \mathfrak{X}/K(\mathfrak{p}^{\mathfrak{c}})} \lambda_\xi(\phi) \left(\chi(\nu(\widetilde{\varpi}))|\mathrm{N}_{E/F}(\nu(\widetilde{\varpi}))|^{s+\kappa}\right)^{\mathfrak{d}(\xi)}. \tag{2.3.15}$$

If χ is ramified then we simply set

$$L\left(s+\frac{1}{2}, \phi \times \chi\right) = 1. \tag{2.3.16}$$

Proposition 2.3.3. *Let $\beta_i \in \mathbb{C}, 1 \leq i \leq m$ be the Satake parameters of ϕ and assume χ is unramified. Then $L(s, \phi \times \chi)$ has an Euler product expansion with $L(s, \phi \times \chi)^{-1}$ given by the following list.*

(Case I, Orthogonal)

$$\prod_{i=1}^{m} \left(1 - \chi(\varpi)\beta_i q^{-1+\frac{r}{2}-s}\right),$$

(Case II, Symplectic)

$$\prod_{i=1}^{m} \left(1 - \chi(\varpi)\beta_i q^{-s}\right),$$

(Case III, Quaternionic Orthogonal Nonsplit)

$$\prod_{i=1}^{m} \left(1 - \chi(\varpi)\beta_i q^{-1+r-s}\right),$$

(Case IV, Quaternionic Unitary Nonsplit)

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{-2+r-s}\right),$$

(Case V, Unitary Inert) E/F is inert,

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{-1+r-s}\right),$$

(Case V, Unitary Ramified) E/F is ramified,

$$\prod_{i=1}^{m}\left(1-\chi(\widetilde{\varpi})\beta_i q^{\frac{r-1}{2}-s}\right),$$

Proof. This is an analogue of [Shi00, Theorem 19.8] for symplectic and unitary cases. The proof for all the cases are the same so we only compute the orthogonal case as an example and omit the other cases.

The Satake map $\omega : \mathbb{T}(K(\mathfrak{p}^{\mathfrak{c}}),\mathfrak{X}) \to \mathbb{Q}[t_1,...,t_m]$ is defined as follows. Given a coset $d\mathrm{GL}_m(\mathfrak{o})$ for $d \in \mathrm{GL}_m(F)$, we can find a lower triangular matrix $g \in \mathrm{GL}_m(F)$ such that $d\mathrm{GL}_m(\mathfrak{o}) = g\mathrm{GL}_m(\mathfrak{o})$. Assume the diagonal elements of g are of the form $\varpi^{e_1},...,\varpi^{e_m}$ with $e_i \in \mathbb{Z}$ and set $\omega_0(d\mathrm{GL}_m(\mathfrak{o})) = \prod_{i=1}^{m}(q^{-i}t_i)^{e_i}$. For $K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}}) = \coprod_y yK(\mathfrak{p}^{\mathfrak{c}})$ with y as in Lemma 2.3.2, we then define

$$\omega([K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})]) = \sum_y \omega_0(y\mathrm{GL}_m(\mathfrak{o})).$$

Set $T = \chi(\varpi)q^{-s}$. By [Shi00, Lemma 19.9] we calculate the Dirichlet series

$$\sum_{\xi\in K(\mathfrak{p}^{\mathfrak{c}})\backslash\mathfrak{X}/K(\mathfrak{p}^{\mathfrak{c}})} \lambda_\xi(\phi)T^{\nu(d)} = \sum_{d\in \mathrm{GL}_m(F)/\mathrm{GL}_m(\mathfrak{o})} \mathrm{vol}(b,c)\omega_0(d\mathrm{GL}_m(\mathfrak{o}))T^{\nu(d)}.$$

Here $\mathrm{vol}(b,c)$ is the volume of $K(\mathfrak{p}^{\mathfrak{c}})\xi K(\mathfrak{p}^{\mathfrak{c}})/K(\mathfrak{p}^{\mathfrak{c}})$ with fixed d. Clearly

$$\mathrm{vol}\left(\mathrm{Mat}_{m,r}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})/\mathrm{Mat}_{m,r}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})d^*\right) = |\nu(d)|^{-r},$$

and $c' := cJ + \frac{1}{2}b^*\hat{\theta}b$ satisfies $c + \epsilon c'^* = 0$. Then by [Shi97, Lemma 13.2] we have

$$\begin{aligned}
&\sum_{\xi \in K(\mathfrak{p}^{\mathfrak{c}})\backslash \mathfrak{X}/K(\mathfrak{p}^{\mathfrak{c}})} \lambda_\xi(\phi)T^{\nu(d)} \\
=& \sum_{d \in \mathrm{GL}_m(F)/\mathrm{GL}_m(\mathfrak{o})} |\nu(d)|^{-r-m+1}\omega_0(d\mathrm{GL}_m(\mathfrak{o}))T^{\nu(d)} \\
=& \prod_{i=1}^{m}(1 - \chi(\varpi)\beta_i q^{m+r-2-s})^{-1}
\end{aligned}$$

Changing $s \mapsto s + \frac{n-1}{2} - \frac{1}{2}$ we obtain the result in above list. $\square$

2.3.3 The split case

Let

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\},\ \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.3.17}$$

with $n = 2m + r$ and $\theta^* = \epsilon\theta \in \mathrm{GL}_r(D)$ not necessarily anisotropic. In Case III', IV', this group is isomorphic to

$$\widetilde{G} := \widetilde{G}(F) := \{g \in \mathrm{GL}_{2n}(F) : g\widetilde{\Phi}\,{}^t g = \widetilde{\Phi}\},\ \widetilde{\Phi} = \begin{bmatrix} 0 & 0 & 1_{2m} \\ 0 & \widetilde{\theta} & 0 \\ -\epsilon \cdot 1_{2m} & 0 & 0 \end{bmatrix}, \tag{2.3.18}$$

with ${}^t\widetilde{\theta} = -\epsilon\widetilde{\theta} \in \mathrm{GL}_{2r}(F)$. In Case V', $G \cong \mathrm{GL}_n(F)$ is simply the general linear group. We omit the discussion of Case V' for simplicity as it is well studied in [Shi97; Shi00].

Unramified local L-factors

The group $\widetilde{G}$ is further isomorphic to

$$\widetilde{G}' := \widetilde{G}'(F) := \{g \in \mathrm{GL}_{2n}(F), g\widetilde{\Phi}'\,{}^t g = \widetilde{\Phi}'\},\ \widetilde{\Phi}' = \begin{bmatrix} 0 & 0 & 1_{m'} \\ 0 & \theta' & 0 \\ -\epsilon \cdot 1_{m'} & 0 & 0 \end{bmatrix}, \tag{2.3.19}$$

with $2n = 2m' + r'$ and ${}^t\theta' = -\epsilon\theta' \in \mathrm{GL}_{r'}(F)$ anisotropic. This is a group of Case I or II discussed in Section 2.3.1. Assume χ is an unramified character of $F^\times$, π an unramified admissible representation of $G(F)$ and $\phi \in \pi$ a spherical vector. Let π' be an unramified admissible representation of $\widetilde{G}'(F)$ and $\phi' \in \pi'$ a spherical vector obtained from π, ϕ under the isomorphism $G \cong \widetilde{G}'$. We thus define the local L-factor $L(s, \phi \times \chi) := L(s, \phi' \times \chi)$ as in Section 2.3.1. In particular, the normalizing factors are

(Case III', Quaternionic Orthogonal Split)

$$b(s,\chi) = L\left(s + \frac{2n+1}{2}, \chi\right) \prod_{i=1}^{n} L(2s - 1 + 2i, \chi^2),$$

(Case IV', Quaternionic Unitary Split)

$$b(s,\chi) = \prod_{i=1}^{n} L(2s + 2n + 1 - 2i, \chi^2).$$

Let α_i be the Satake parameters of ϕ then $L(s, \phi \times \chi)^{-1}$ are given by

(Case III', Quaternionic Orthogonal Split)

$$\left(1 - \chi(\varpi)q^{-s}\right) \times \prod_{i=1}^{n} \left(1 - \chi(\varpi)\alpha_i q^{-s}\right)\left(1 - \chi(\varpi)\alpha_i^{-1} q^{-s}\right),$$

(Case IV', Quaternionic Unitary Split)

$$\prod_{i=1}^{\lfloor \frac{r'}{2} \rfloor} \left(1 - \chi(\varpi)^2 q^{2i - r' - 2s}\right) \times \prod_{i=1}^{m'} \left(1 - \chi(\varpi)\alpha_i q^{-1 + \frac{r'}{2} - s}\right)\left(1 - \chi(\varpi)\alpha_i^{-1} q^{1 - \frac{r'}{2} - s}\right).$$

Ramified local L-factors

For an integer $\mathfrak{c} \geq 1$, we consider the open compact subgroup $K(\mathfrak{p}^{\mathfrak{c}})$ of $G(\mathfrak{o})$ as in Section 2.3.2. The isomorphism between G and $\widetilde{G}$ can be chosen such that the image of $G(\mathfrak{o})$ is $\widetilde{G}(\mathfrak{o})$. We will fix such an isomorphism throughout this thesis. In this case, the image of $K(\mathfrak{p}^{\mathfrak{c}})$ is

$$\widetilde{K}(\mathfrak{p}^{\mathfrak{c}}) = \widetilde{G}(\mathfrak{o}) \cap \begin{bmatrix} \mathrm{Mat}_{2m}(\mathfrak{o}) & \mathrm{Mat}_{2m,2r}(\mathfrak{o}) & \mathrm{Mat}_{2m}(\mathfrak{o}) \\ \mathrm{Mat}_{2r,2m}(\mathfrak{p}^{\mathfrak{c}}\mathfrak{o}) & 1 + \mathrm{Mat}_{2r}(\mathfrak{p}\mathfrak{o}) & \mathrm{Mat}_{2r,2m}(\mathfrak{o}) \\ \mathrm{Mat}_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathfrak{o}) & \mathrm{Mat}_{2m,2r}(\mathfrak{p}^{\mathfrak{c}}\mathfrak{o}) & \mathrm{Mat}_{2m}(\mathfrak{o}) \end{bmatrix} \qquad (2.3.20)$$

We can define the Hecke algebras $\mathcal{H}(K(\mathfrak{p}^{\mathfrak{c}}),\mathfrak{X})$ and $\mathcal{H}(\widetilde{K}(\mathfrak{p}^{\mathfrak{c}}),\mathfrak{X})$ similarly as Section 2.3.2. Let π be an admissible representation of $G(F)$. Assume $\phi\in\pi$ is a vector fixed by $K(\mathfrak{p}^{\mathfrak{c}})$ and is an eigenvector for the Hecke algebra $\mathcal{H}(K(\mathfrak{p}^{\mathfrak{c}}),\mathfrak{X})$. Let π' be an admissible representation of $\widetilde{G}(F)$ and $\phi'\in\pi$ a vector obtained from π,ϕ under the isomorphism $G\cong\widetilde{G}$. We thus define the local L-factor $L(s,\phi\times\chi):=L(s,\phi'\times\chi)$ as in (2.3.15), (2.3.16). In particular, $L(s,\phi\times\chi)^{-1}$ are given by

(Case III', Quaternionic Orthogonal Split)

$$\prod_{i=1}^{2m}\left(1-\chi(\varpi)\beta_i q^{-s}\right),$$

(Case IV', Quaternionic Unitary Split)

$$\prod_{i=1}^{2m}\left(1-\chi(\varpi)\beta_i q^{-1+r-s}\right).$$

The operator $U'(\mathfrak{p}^{\mathfrak{n}})$ is defined as in (2.3.14) for orthogonal and symplectic groups. In Case IV', the $U(\mathfrak{p}^{\mathfrak{n}})$ is also the one defined for Case I in (2.3.13). In Case III', we define $U(\mathfrak{p}^{\mathfrak{n}})$ as

$$U(\mathfrak{p}^{\mathfrak{n}}):=[\widetilde{K}(\mathfrak{p}^{\mathfrak{c}})\xi\widetilde{K}(\mathfrak{p}^{\mathfrak{c}})]\text{ with }\xi=\mathrm{diag}[\varpi^{\mathfrak{n}}\cdot 1_n,\varpi^{-\mathfrak{n}}\cdot 1_n]. \tag{2.3.21}$$

Remark 2.3.4. Note that when defining local L-factors, we always assume the group is chosen such that m is the Witt index in unramified cases which is not applied for ramified cases. In other words, in ramified cases our open compact subgroup $K(\mathfrak{p}^{\mathfrak{c}})$ is not chosen to be maximal. For example in above Case III', clearly the local group $\widetilde{G}$ can be further isomorphic to $\widetilde{G}'$ with $m'=n,r'=0$ as in the unramified computations. But we are still considering the open compact subgroup $\widetilde{K}(\mathfrak{p}^{\mathfrak{c}})$ rather than the bigger one

$$\widetilde{G}'(\mathfrak{o})\cap\begin{bmatrix}\mathrm{Mat}_n(\mathfrak{o}) & \mathrm{Mat}_n(\mathfrak{o})\\ \mathrm{Mat}_n(\mathfrak{p}^{\mathfrak{c}}\mathfrak{o}) & \mathrm{Mat}_n(\mathfrak{o})\end{bmatrix},$$

which causes our local L-factors to be of degree $2m$ rather than the expected n as in Case II. We make those restrictions because only these L-factors show up in our integral representations.

2.3.4 The global L-function

We summarize our definition for the standard L-function. Let F be a number field and

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\},\ \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.3.22}$$

be the global group as in Section 2.2.1. Let $\mathfrak{o}$ be the ring of integers of F and $\mathcal{O}$ a fixed maximal order of D such that $D = F \otimes_{\mathfrak{o}} \mathcal{O}$. Let $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2$ be an integral ideal of $\mathfrak{o}$ with $\mathfrak{n}_1, \mathfrak{n}_2$ coprime and $\chi : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ a Hecke character of conductor $\mathfrak{n}_2$. Consider the open compact subgroup

$$K(\mathfrak{n}) = G(\mathfrak{o}) \cap \begin{bmatrix} \mathrm{Mat}_m(\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O}) \\ \mathrm{Mat}_{r,m}(\mathfrak{n}\mathcal{O}) & 1 + \mathrm{Mat}_r(\mathfrak{n}'\mathcal{O}) & \mathrm{Mat}_{r,m}(\mathcal{O}) \\ \mathrm{Mat}_m(\mathfrak{n}\mathcal{O}) & \mathrm{Mat}_{m,r}(\mathfrak{n}\mathcal{O}) & \mathrm{Mat}_m(\mathcal{O}) \end{bmatrix}, \tag{2.3.23}$$

with $\mathfrak{n}'$ the support of $\mathfrak{n}$. For v a finite place corresponds to a prime ideal $\mathfrak{p}$ denote $\mathfrak{M}_v = \mathrm{GL}_m(D_v) \cap \mathrm{Mat}_m(\mathcal{O}_v)$ and set

$$\mathfrak{Q}_v = \begin{cases} G(\mathfrak{o}_v) & \mathfrak{p} \nmid \mathfrak{n}, \\ \mathrm{diag}[u, 1_r, \hat{u}] \text{ with } u \in \mathfrak{M}_v & \mathfrak{p} | \mathfrak{n}_1 \\ 1 & \mathfrak{p} | \mathfrak{n}_2. \end{cases} \tag{2.3.24}$$

Let $\mathfrak{Q} = \prod_v \mathfrak{Q}_v$ and $\mathfrak{X} = K(\mathfrak{n})\mathfrak{Q}K(\mathfrak{n})$. Define the global Hecke algebra $\mathcal{H} := \mathcal{H}(K(\mathfrak{n}), \mathfrak{X})$ associated to $K(\mathfrak{n})$ and $\mathfrak{X}$ be the one generated by double cosets $[K(\mathfrak{n})\xi K(\mathfrak{n})]$ with $\xi \in \mathfrak{Q}$. Its action on a cusp form $\phi \in \pi$ can be similarly defined as in the local case treated above. Assume $\phi \in \pi$ is fixed by $K(\mathfrak{n})$ and is an eigenfunction for $\mathcal{H}$. That is

$$\phi | [K(\mathfrak{n})\xi K(\mathfrak{n})] = \lambda_\xi(\phi)\phi. \tag{2.3.25}$$

The standard L-function of ϕ twisted by χ is defined as

$$L\left(s+\frac{1}{2}, \phi \times \chi\right) = b(s,\chi) \sum_{\substack{\xi \in K(\mathfrak{n})\backslash \mathfrak{X}/K(\mathfrak{n}) \\ \xi = \mathrm{diag}[u,1,\hat{u}]}} \lambda_\xi(\phi)\chi(\nu(u))|\mathrm{N}_{E/F}(\nu(u))|^{s+\kappa}. \tag{2.3.26}$$

Clearly, see for example [Shi00, Section 19], it has an Euler product expression

$$L\left(s+\frac{1}{2}, \phi \times \chi\right) = \prod_v L_v\left(s+\frac{1}{2}, \phi_v \times \chi_v\right), \tag{2.3.27}$$

with $L_v\left(s+\frac{1}{2}, \phi_v \times \chi_v\right)$ the local L-factors defined in the previous two subsections. Note that when $v \nmid \mathfrak{n}$ is unramified, $G(\mathfrak{o}_v)$ may have Witt index $m' \geq m$ and the unramified local L-factors is defined by replacing m with m' in Section 2.3.1.

Remark 2.3.5. We give several remarks on our L-functions.

(1) Here we define the L-function by a Dirichlet series associated to certain Hecke eigenvalues which can be viewed as an analogue of the L-function for classical (GL_2) modular forms. This kind of L-function is also studied in [Shi97; Shi00] for symplectic and unitary groups.

(2) As the reader may notice, we are writing $L(s, \phi \times \chi)$ to indicate its dependence on the cusp form ϕ. Unlike the GL_2 case, we do not have a clear correspondence between eigenforms and cuspidal representations. This is because, the subspace of π fixed by $K(\mathfrak{n})$ (take $\mathfrak{n}$ to be the conductor of π) may not be of dimension one.

(3) The unramified local L-factors defined here are really the Langlands L-function associated to the natural embedding of the L-group ${}^LG$ into a general linear group.

(4) We make no claim that our definition of ramified L-factors is 'correct'. Indeed, it is a conjecture of Langlands [Lan70] that for any cuspidal representation π one can associate to any place a local L-factor $L_v(s, \pi_v)$ and a local root number $\epsilon_v(s, \pi_v)$ such that the global L-function satisfies a functional equation of the form

$$L(s, \pi) = \epsilon(s, \pi)L(1-s, \pi).$$

Using the doubling method, Yamana [Yam14] gives a definition of local L-factors and proves the functional equation for classical groups. However, he does not define these factors explicitly as in our Proposition 2.3.1, 2.3.3 and it is not clear how his approach can be used to study algebraicity or p-adic properties which is of interest in this work. We have not compared our L-factors with his and we also do not know whether the L-function defined here can be proved to satisfy a functional equation.

2.4 The non-archimedean local integrals

We carry out the computations of the non-archimedean local integrals in this section. We keep the setting for non-archimedean local fields and tuples (D, ρ, ϵ) as in the beginning of Section 2.3. The unramified local integrals are well known but we will also review the computations for completeness. For ramified local integrals, it is also well known that one can choose a local section $f_{s,v}$ such that $\mathcal{Z}_v(s; \phi_{1,v}, \phi_{2,v}, f_{s,v}) \neq 0$. For our purpose, we explicitly construct certain local sections $f_s^\dagger, f_s^\ddagger, f_s^p$ such that $\mathcal{Z}$ represents the local L-factors defined in the last section or the p-adic modifications. These local sections will also be chosen such that the Eisenstein series has a nice Fourier expansion. To make our notations and computations consistent, we do not consider the split case (i.e. Case III', IV', V'). For Case III', IV', the local groups are isomorphic to the groups in Case I, II and our arguments can be directly extended to these two cases. The Case V' should be treated separately and we omit it for simplicity. We will assume in this thesis that Case V' does not occur in the ramified setting.

2.4.1 Setup for non-archimedean local integrals

Let

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\},\ \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}, \tag{2.4.1}$$

with $n = 2m + r$ and $\theta^* = \epsilon\theta \in \mathrm{GL}_r(D)$ not necessarily anisotropic. Let

$$H := H(F) := \{h \in \mathrm{GL}_{2n}(D) : hJ_nh^* = J_n\},\ J_n = \begin{bmatrix} 0 & 1_n \\ \epsilon \cdot 1_n & 0 \end{bmatrix}, \tag{2.4.2}$$

and define an embedding

$$\begin{aligned} &G \times G \to H, \\ &(g_1, g_2) \mapsto R \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} R^{-1}, \end{aligned} \tag{2.4.3}$$

with

$$R = \begin{bmatrix} 0 & \frac{\epsilon}{2} \cdot 1_r & 0 & 0 & \frac{\epsilon}{2} \cdot 1_r & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon \cdot 1_m \\ 1_m & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^{-1} & 0 & 0 & \theta^{-1} & 0 \\ 0 & 0 & 0 & 1_m & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \end{bmatrix}.$$

We identify (g_1, g_2) with its image in H. Let $P \subset H$ be the Siegel parabolic subgroup consisting elements of the form $\begin{bmatrix} * & * \\ 0_n & * \end{bmatrix}$, with Levi decomposition $P = M \ltimes N$ for

$$M = \left\{ \begin{bmatrix} a & 0 \\ 0 & \hat{a} \end{bmatrix} : a \in \mathrm{GL}_n \right\}, \quad N = \left\{ \begin{bmatrix} 1_n & b \\ 0 & 1_n \end{bmatrix} : b^* = -\epsilon b \in \mathrm{Mat}_n \right\}. \tag{2.4.4}$$

Consider the induced representation $\mathrm{Ind}_{P(F)}^{H(F)}(\chi|\nu(\cdot)|^s)$ for a character $\chi : E^\times \to \mathbb{C}^\times$. Let π be an admissible representation of $G(F)$ and $\phi_1, \phi_2 \in \pi$. For a section $f_s \in \mathrm{Ind}_{P(F)}^{H(F)}(\chi|\nu(\cdot)|^s)$ we consider the local integral

$$\mathcal{Z}(s; \phi_1, \phi_2, f_s) = \int_{G(F)} f_s(\delta(g, 1))\langle \pi(g)\phi_1, \phi_2 \rangle dg, \tag{2.4.5}$$

where

$$\delta = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & -1_m & 0 & 1_m & 0 \\ 0 & \epsilon \cdot 1_m & 0 & 0 & 0 & 1_m \end{bmatrix}.$$

2.4.2 The unramified local integrals

Assume χ is an unramified character of $E^\times$. Let π be an unramified admissible representation of $G(F)$ and $\phi_1 = \phi_2 = \phi \in \pi$ a spherical vector.

Take the local section $f_s^0 \in \mathrm{Ind}_{P(F)}^{H(F)}(\chi|\nu \cdot |^s)$ to be the spherical section normalized such that

$$f_s^0(pk) = \chi(\nu(a))|\mathrm{N}_{E/F}(\nu(a))|^{s+\kappa} b(s,\chi) \text{ with } p = \begin{bmatrix} a & b \\ 0 & \hat{a} \end{bmatrix}, k \in H(\mathfrak{o}). \tag{2.4.6}$$

Here $b(s,\chi)$ is the normalizing factor given in Section 2.3.1. Note that in (2.4.1), θ is not necessarily anisotropic. Let m' be the Witt index of G. That is G is isomorphic to the following F-group

$$G' := G'(\Phi') := \{g \in \mathrm{GL}_n(D) : g\Phi' g^* = \Phi'\} \tag{2.4.7}$$

with

$$\Phi' = \begin{bmatrix} 0 & 0 & 1_{m'} \\ 0 & \theta' & 0 \\ \epsilon 1_{m'} & 0 & 0 \end{bmatrix}, n = 2m' + r', \theta'^* = \epsilon\theta' \in \mathrm{GL}_{r'}(D) \text{ anisotropic }.$$

Then there exists a matrix $S \in \mathrm{GL}_r(D)$ with $S\Phi' S^* = \Phi$ and the isomorphism between G and G' are given by

$$\begin{aligned} G' &\xrightarrow{\sim} G, \\ g &\mapsto SgS^{-1}. \end{aligned} \tag{2.4.8}$$

Denote by π', ϕ' for the admissible representation and cusp form of $G'(F)$ obtained from π, ϕ under isomorphism (2.4.8). Then the local L-factor $L(s, \phi\times\chi) = L(s, \phi'\times\chi)$ is defined in Section 2.3.1 with m replaced by m'.

Proposition 2.4.1. *Let χ be an unramified character of $E^\times$ and π an unramified admissible representation of $G(F)$. For f_s^0 chosen as above and $\phi_1 = \phi_2 = \phi \in \pi$ a spherical vector, we have*

$$\mathcal{Z}(s; \phi, \phi, f_s^0) = L\left(s + \frac{1}{2}, \phi \times \chi\right) \langle \phi, \phi \rangle. \tag{2.4.9}$$

Proof. We have another doubling embedding

$$G' \times G' \to H,$$
$$(g_1, g_2) \mapsto R' \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} R'^{-1}, \qquad R' = R \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$$

which is compatible with (2.4.3),(2.4.8). Then the integral (2.4.5) is equivalent to the integral

$$\mathcal{Z}(s; \phi', \phi', f_s) = \int_{G'(F)} f_s(\delta(g, 1)) \langle \pi'(g)\phi', \phi' \rangle dg.$$

By Cartan decomposition, we have

$$\begin{aligned}
&\sum_{\substack{e_1, \dots, e_{m'} \in \mathbb{Z} \\ 0 \le e_1 \le \dots \le e_{m'}}} f_s^0(\delta \cdot (\mathrm{diag}[\widetilde{\varpi}^{e_1}, \dots, \widetilde{\varpi}^{e_{m'}}, 1_{r'}, \widetilde{\varpi}^{-e_1}, \dots, \widetilde{\varpi}^{-e_{m'}}], 1)) \\
&\times \int_{K_{e_1, \dots, e_{m'}}} \langle \pi'(g)\phi', \phi' \rangle dg \\
= &\sum_{\substack{e_1, \dots, e_{m'} \in \mathbb{Z} \\ 0 \le e_1 \le \dots \le e_{m'}}} \left(\chi(\nu(\widetilde{\varpi})) |\mathrm{N}_{E/F}(\nu(\widetilde{\varpi}))|^{s+\kappa}\right)^{e_1 + \dots + e_{m'}} b(s, \chi) \lambda_{e_1, \dots, e_{m'}}(\phi') \langle \phi', \phi' \rangle \\
= &L\left(s + \frac{1}{2}, \phi' \times \chi\right) \langle \phi', \phi' \rangle.
\end{aligned}$$

□

2.4.3 The local section $f_s^{\dagger, \mathfrak{c}}$

In this and the next two subsections, we consider the ramified local integrals. We will always assume $2 \in \mathcal{O}^\times$ and $\theta \in \mathrm{GL}_r(\mathcal{O})$ in the ramified cases. For an integer

$\mathfrak{c} \geq 1$ let $N'(\mathfrak{p}^{\mathfrak{c}})$ be the subgroup of $N(F)$ consisting the elements of the form

$$\begin{bmatrix} 1_r & 0 & x & y \\ 0 & 1_{2m} & \epsilon y^* & z \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 1_{2m} \end{bmatrix}, x \in S_r(\mathfrak{p}\mathcal{O}), y \in \mathrm{Mat}_{r,2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}), z \in S_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}). \tag{2.4.10}$$

Define $f_s^{\dagger,\mathfrak{c}} \in \mathrm{Ind}_{P(F)}^{H(F)}(\chi|\nu(\cdot)|^s)$ to be a local section supported on $P(F)J_nN'(\mathfrak{p}^{\mathfrak{c}})$ with

$$f_s^{\dagger,\mathfrak{c}}(pJ_nn) = \chi(\nu(a))|\mathrm{N}_{E/F}(\nu(a))|^{s+\kappa} \text{ for } p = \begin{bmatrix} a & b \\ 0 & \hat{a} \end{bmatrix} \in P(F), n \in N'(\mathfrak{p}^{\mathfrak{c}}). \tag{2.4.11}$$

Note that when pulling back along $G \times G \to H$, we have

$$f_s^{\dagger,\mathfrak{c}}((g_1k_1, g_2k_2)) = \chi(\nu(d_{k_1}))\chi(\nu(a_{k_2}))f_s^{\dagger,\mathfrak{c}}((g_1, g_2)) \tag{2.4.12}$$

for $k_1 = \begin{bmatrix} a_{k_1} & f_{k_1} & b_{k_1} \\ h_{k_1} & e_{k_1} & j_{k_1} \\ c_{k_1} & k_{k_1} & d_{k_1} \end{bmatrix} \in K'(\mathfrak{p}^{\mathfrak{c}})$ and $k_2 = \begin{bmatrix} a_{k_2} & f_{k_2} & b_{k_2} \\ h_{k_2} & e_{k_2} & j_{k_2} \\ c_{k_2} & k_{k_2} & d_{k_2} \end{bmatrix} \in K(\mathfrak{p}^{\mathfrak{c}})$.

The following proposition is an analogue of the computations in [Shi95, Section 4] for symplectic groups. We extend the arguments there to all classical groups. Especially, the main difficulty in the computations is to deal with the group with $r \neq 0$.

Proposition 2.4.2. *Assume $\phi \in \pi$ is fixed by $K(\mathfrak{p}^{\mathfrak{c}})$ and is an eigenvector of the Hecke algebra $\mathcal{H}(K(\mathfrak{p}^{\mathfrak{c}}), \mathfrak{X})$. Set $\phi_1 = \pi(w)\phi, \phi_2 = \phi$. Assume χ is an unramified character and $2 \in \mathcal{O}^\times, \theta \in \mathrm{GL}_r(\mathcal{O})$, then*

$$\begin{aligned} &\mathcal{Z}(s; \pi(w)\phi, \phi, f_s^{\dagger,\mathfrak{c}}) \\ =&\chi(\varpi)^{\mathfrak{c}m\mathbf{d}_1} q^{-\mathfrak{c}m\mathbf{d}_2(s+\kappa)} L\left(s + \frac{1}{2}, \phi \times \chi\right) \cdot \langle \phi | U'(\mathfrak{p}^{\mathfrak{c}}), \phi \rangle . \end{aligned} \tag{2.4.13}$$

Here

$$\mathbf{d}_1 = \begin{cases} 1 & \textit{Case I, II, V}, \\ 2 & \textit{Case III, IV}, \end{cases} \qquad \mathbf{d}_2 = \begin{cases} 1 & \textit{Case I, II}, \\ 2 & \textit{Case III, IV, V}, \end{cases} \tag{2.4.14}$$

Proof. Denote

$$\mathfrak{M}(\mathfrak{p}^{\mathfrak{c}}) = \mathrm{GL}_m(D) \cap \mathrm{Mat}_m(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}),\ \mathfrak{Q}(\mathfrak{p}^{\mathfrak{c}}) = \{\mathrm{diag}[\hat{u}, -1_r, u], a \in \mathfrak{M}(\mathfrak{p}^{\mathfrak{c}})\}.$$

We claim that

$$f_s^{\dagger}(\delta(g,1)) \neq 0 \text{ if and only if } g \in K(\mathfrak{p}^{\mathfrak{c}})\mathfrak{Q}(\mathfrak{p}^{\mathfrak{c}})K'(\mathfrak{p}^{\mathfrak{c}}).$$

Write $g = \begin{bmatrix} a & f & b \\ h & e & j \\ c & k & d \end{bmatrix}$ with a, d of size $m \times m$, e of size $r \times r$ and compute

$$\delta(g,1) = \begin{bmatrix} \frac{e+1}{2} & 0 & \frac{\epsilon h}{2} & \frac{\epsilon(e-1)\theta}{4} & 0 & \frac{\epsilon j}{2} \\ 0 & 1_m & 0 & 0 & 0 & 0 \\ \epsilon f & 0 & a & \frac{f\theta}{2} & 0 & b \\ \epsilon\theta^{-1}(e-1) & 0 & \theta^{-1}h & \theta^{-1}\frac{e+1}{2}\theta & 0 & \theta^{-1}j \\ -\epsilon f & 0 & -a & -\frac{f\theta}{2} & 1_m & -b \\ \epsilon k & \epsilon & c & \frac{k\theta}{2} & 0 & d \end{bmatrix}.$$

The elements in $P(F)J_nN'(\mathfrak{p}^{\mathfrak{c}})$ can be written as

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & D_1 & D_2 \\ 0 & 0 & D_3 & D_4 \end{bmatrix} J_n \begin{bmatrix} 1_r & 0 & x & y \\ 0 & 1_{2m} & \epsilon y^* & z \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 1_{2m} \end{bmatrix}$$
$$= \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ \epsilon D_1 & \epsilon D_2 & \epsilon D_1 x + D_2 y^* & \epsilon D_1 y + \epsilon D_2 z \\ \epsilon D_3 & \epsilon D_4 & \epsilon D_3 x + D_4 y^* & \epsilon D_3 y + \epsilon D_4 z \end{bmatrix},$$

with $x \in S_r(\mathfrak{p}\mathcal{O}), y \in \mathrm{Mat}_{r,2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}), z \in S_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})$. Assume $\tau(g,1)$ is of above form, then comparing two expressions we need

$$D_1 = \theta^{-1}(e-1),\ D_2 = \begin{bmatrix} 0 & \epsilon\theta^{-1}h \end{bmatrix},\ D_3 = \begin{bmatrix} -f \\ k \end{bmatrix},\ D_4 = \begin{bmatrix} 0 & -\epsilon a \\ 1 & \epsilon c \end{bmatrix},$$

and

$$\epsilon D_1 x + D_2 y^* = \theta^{-1}\frac{e+1}{2}\theta, \quad D_1 y + D_2 z = \begin{bmatrix} 0 & \epsilon\theta^{-1}j \end{bmatrix},$$

$$\epsilon D_3 x + D_4 y^* = \begin{bmatrix} -\frac{f\theta}{2} \\ \frac{k\theta}{2} \end{bmatrix}, \quad D_3 y + D_4 z = \begin{bmatrix} \epsilon & -\epsilon b \\ 0 & \epsilon d \end{bmatrix}.$$

First of all, write $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$, then

$$\begin{bmatrix} -\frac{f\theta}{2} \\ \frac{k\theta}{2} \end{bmatrix} = \epsilon D_3 x + D_4 y^* = \begin{bmatrix} -\epsilon f x - \epsilon a y_2^* \\ \epsilon k x + y_1^* + \epsilon c y_2^* \end{bmatrix}$$

implies

$$a y_2^* = -\epsilon f(\epsilon x - \frac{\theta}{2}), \qquad y_1^* + \epsilon c y_2^* = -k(\epsilon x - \frac{\theta}{2}).$$

Since by our assumption $\frac{\theta}{2} \in \mathrm{GL}_r(\mathcal{O})$ and the condition on x, the first equation forces a to be invertible and thus

$$y_2^* = -\epsilon a^{-1} f(\epsilon x - \frac{\theta}{2}), \qquad y_1^* = -(k - ca^{-1}f)(\epsilon x - \frac{\theta}{2}).$$

The condition on y then forces $a^{-1}f, k - ca^{-1}f \in \mathrm{Mat}_{m,r}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})$. Secondly, from

$$\theta^{-1}\frac{e+1}{2}\theta = \epsilon D_1 x + D_2 y^* = \epsilon\theta^{-1}(e-1)x + \epsilon\theta^{-1}h y_2^*,$$

we obtain

$$\epsilon(e - 1 + ha^{-1}f)x = \frac{e + 1 - ha^{-1}f}{2}\theta.$$

The condition on x then forces $e - ha^{-1}f + 1 \in \mathrm{Mat}_r(\mathfrak{p}\mathcal{O})$. Finally, comparing D_4 and $D_3 y + D_4 z$, the condition on z forces

$$\begin{bmatrix} 0 & -\epsilon a \\ 1 & \epsilon c \end{bmatrix}^{-1} \begin{bmatrix} \epsilon & -\epsilon b \\ 0 & \epsilon d \end{bmatrix} = \begin{bmatrix} \epsilon c a^{-1} & \epsilon d \\ -a^{-1} & a^{-1}b \end{bmatrix} \in \mathrm{Mat}_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})$$

and hence $a^{-1}, ca^{-1}, a^{-1}b \in \mathrm{Mat}_m(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})$. Since a is invertible, we can write

$$g = \begin{bmatrix} 1_m & 0 & 0 \\ ha^{-1} & 1_r & 0 \\ ca^{-1} & -\epsilon \hat{a} h^* \theta^{-1} & 1_m \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & e - ha^{-1} f & 0 \\ 0 & 0 & \hat{a} \end{bmatrix} \begin{bmatrix} 1_m & a^{-1} f & a^{-1} b \\ 0 & 1_r & -\epsilon\theta f^* \hat{a} \\ 0 & 0 & 1_m \end{bmatrix}$$

and our claim clearly follows.

By straightforward computations and the fact that (see also the proof of [Shi95, Lemma 6.2])

$$K(\mathfrak{p}^{\mathfrak{c}})\mathfrak{Q}(\mathfrak{p}^{\mathfrak{c}})K'(\mathfrak{p}^{\mathfrak{c}}) = K(\mathfrak{p}^{\mathfrak{c}})\mathfrak{Q}K(\mathfrak{p}^{\mathfrak{c}}) \cdot K(\mathfrak{p}^{\mathfrak{c}})\mathrm{diag}[\varpi^{-\mathfrak{c}} \cdot 1_m, 1_r, \varpi^{\mathfrak{c}} \cdot 1_m]K'(\mathfrak{p}^{\mathfrak{c}}),$$

we have

$$\begin{aligned} &\mathcal{Z}(s; \pi(w)\phi, \phi, f_s^{\dagger,\mathfrak{c}}) \\ =&\chi(\varpi)^{\mathfrak{c}m\mathbf{d}_1} q^{-\mathfrak{c}m\mathbf{d}_2(s+\kappa)} \sum_{\xi \in K(\mathfrak{p}^{\mathfrak{c}})\backslash \mathfrak{X}/K(\mathfrak{p}^{\mathfrak{c}})} \lambda_\xi(\phi)(\chi(\nu(\widetilde{\varpi}))|\mathrm{N}_{E/F}(\nu(\widetilde{\varpi}))|^{s+\kappa})^{\mathfrak{d}(\xi)} \\ \times& \int_{K(\mathfrak{p}^{\mathfrak{c}})\mathrm{diag}[\varpi^{-\mathfrak{c}}\cdot 1_m, -1_r, \varpi^{\mathfrak{c}}\cdot 1_m]K'(\mathfrak{p}^{\mathfrak{c}})} \langle \pi(gw)\phi, \phi \rangle \, dg \\ =&\chi(\varpi)^{\mathfrak{c}m\mathbf{d}_1} q^{-\mathfrak{c}m\mathbf{d}_2(s+\kappa)} L\left(s + \frac{1}{2}, \phi \times \chi\right) \langle \phi | U'(\mathfrak{p}^{\mathfrak{c}}), \phi \rangle . \end{aligned}$$

□

2.4.4 The local section $f_s^{\ddagger,\mathfrak{c}}$

If χ is a ramified character then $\mathcal{Z}(s; \phi_1, \phi_2, f_s^{\dagger,\mathfrak{c}})$ will be identically zero. In this subsection, we define a section $f_s^{\ddagger,\mathfrak{c}}$ as a twist of $f_s^{\dagger,0}$ such that $\mathcal{Z}(s; \phi_1, \phi_2, f_s^{\ddagger,\mathfrak{c}})$ is a non-zero constant. Assume χ has conductor $\mathfrak{p}^{\mathfrak{c}}$, we define

$$f_s^{\ddagger,\mathfrak{c}}(h) = \sum_{u \in \mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O})} \chi^{-1}(\nu(u)) \times f_s^{\dagger,0}\left(h \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & \frac{u}{\varpi^{\mathfrak{c}}} \\ 0 & 0 & 1_m & 0 & -\frac{\epsilon u^*}{\varpi^{\mathfrak{c}}} & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_m \end{bmatrix} \right). \tag{2.4.15}$$

The following lemma shows the reason for the twist.

Lemma 2.4.3. *When pulling back along $G \times G \to H$, we have*

$$f_s^{\ddagger,\mathfrak{c}}((g_1k_1, g_2k_2)) = \chi(\nu(k_2)) f_s^{\ddagger,\mathfrak{c}}((g_1, g_2)) \tag{2.4.16}$$

for $k_1 \in K(\mathfrak{p}^{2\mathfrak{c}}), k_2 \in K'(\mathfrak{p}^{2\mathfrak{c}})$.

Proof. For the notations of k_1, k_2 as before, we have

$$f_s^{\ddagger,\mathfrak{c}}((g_1k_1, g_2k_2)) = \chi(\nu(d_{k_1}))\chi(\nu(a_{k_2})) \sum_{u \in \mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O})} \chi^{-1}(\nu(u)) \times f_s^{\dagger,0}\left((g_1, g_2) \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & \frac{u'}{\varpi^{\mathfrak{c}}} \\ 0 & 0 & 1_m & 0 & -\frac{\epsilon u'^*}{\varpi^{\mathfrak{c}}} & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_m \end{bmatrix} \right)$$

with $u' = d_{k_2} u d_{k_1}^{-1}$. Then changing variables $u \to d_{k_2}^{-1} u d_{k_1}$ gives the desired result. $\square$

The following proposition is an analogue of the computations in [BS00] and [SU14, Proposition 11.16]. Again the main difficulty is to deal with the group with $r \neq 0$ and the arguments is similar to the proof of Proposition 2.4.2.

Proposition 2.4.4. *Assume $\phi \in \pi$ is fixed by $K(\mathfrak{p}^{\mathfrak{c}})$, χ is a character of conductor $\mathfrak{p}^{\mathfrak{c}}$ and $2 \in \mathcal{O}^{\times}, \theta \in \mathrm{GL}_r(\mathcal{O})$. Set $\phi_1 = \phi, \phi_2 = \pi(w)\phi$ and denote*

$$K_1(\mathfrak{p}^{\mathfrak{c}}) = G(\mathfrak{o}) \cap \begin{bmatrix} 1+\mathrm{Mat}_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) & \mathrm{Mat}_{2m,r}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) & \mathrm{Mat}_{2m}(\mathcal{O}) \\ \mathrm{Mat}_{r,2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) & 1+\mathrm{Mat}_{r}(\mathfrak{p}\mathcal{O}) & \mathrm{Mat}_{r,2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) \\ \mathrm{Mat}_{2m}(\mathfrak{p}^{2\mathfrak{c}}\mathcal{O}) & \mathrm{Mat}_{2m,r}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) & 1+\mathrm{Mat}_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) \end{bmatrix}. \tag{2.4.17}$$

Then

$$\begin{aligned} &\mathcal{Z}(s;\phi,\pi(w)\phi,f_s^{\ddagger,\mathfrak{c}}) \\ =&\mathrm{vol}(\mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O}))\left\langle \pi\left(\begin{bmatrix} 0 & 0 & \varpi^{-\mathfrak{c}}\cdot 1_m \\ 0 & 1_r & 0 \\ \varpi^{\mathfrak{c}}\cdot 1_m & 0 & 0\end{bmatrix}\right)\phi,\phi\right\rangle. \end{aligned} \tag{2.4.18}$$

Proof. Denote $d_u = \mathrm{diag}[1_{m+r}, \epsilon\varpi^{\mathfrak{c}}u^{-1}, 1_{m+r}, \epsilon\varpi^{-\mathfrak{c}}u^*]$, then

$$(w,w)^{-1}d_u\delta d_u^{-1}(w,w) = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & \frac{u}{\varpi^{\mathfrak{c}}} \\ 0 & 0 & 1_m & 0 & -\frac{\epsilon u^*}{\varpi^{\mathfrak{c}}} & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_m \end{bmatrix}.$$

Changing variables $g \mapsto w^{-1}g\begin{bmatrix} \epsilon\varpi^{-\mathfrak{c}}u & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & \epsilon\varpi^{\mathfrak{c}}\hat{u}\end{bmatrix}w$, we need to calculate

$$\begin{aligned} &\mathcal{Z}(s;\phi,\pi(w)\phi,f_s^{\ddagger,\mathfrak{c}}) \\ =&\int_{G(F)}\sum_{u\in\mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O})}\chi^{-1}(\nu(u))\left\langle\pi\left(g\begin{bmatrix} \epsilon\varpi^{-\mathfrak{c}}u & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & \epsilon\varpi^{\mathfrak{c}}\hat{u}\end{bmatrix}w\right)\phi,\phi\right\rangle \\ &\times f_s^{\dagger,0}\left(\delta(g,1)\tau d_u^{-1}(w,w)\right)dg. \end{aligned}$$

Write $g = \begin{bmatrix} a & f & b \\ h & e & j \\ c & k & d \end{bmatrix}$ with a, d of size $m \times m$, e of size $r \times r$ and compute that

$$\delta(g,1)\delta d_u^{-1}(w,w)$$
$$= \begin{bmatrix} \frac{e+1}{2} & 0 & \frac{\epsilon\varpi^{\mathfrak{c}} j\hat{u}}{2} & \frac{\epsilon(e-1)\theta}{4} & -\frac{j}{2} & \frac{\varpi^{-\mathfrak{c}} hu}{2} \\ 0 & 0 & 0 & 0 & -1_m & 0 \\ \epsilon f & 0 & \varpi^{\mathfrak{c}} b\hat{u} & \frac{f}{2} & -\epsilon b & \epsilon\varpi^{-\mathfrak{c}} au \\ \epsilon\theta^{-1}(e-1) & 0 & \varpi^{\mathfrak{c}}\theta^{-1} j\hat{u} & \theta^{-1}\frac{e+1}{2}\theta & -\epsilon\theta^{-1} j & \epsilon\varpi^{-\mathfrak{c}}\theta^{-1} hu \\ -\epsilon f & -\epsilon\cdot 1_m & -\varpi^{\mathfrak{c}} b\hat{u} & -\frac{f\theta}{2} & \epsilon b & -\epsilon\varpi^{-\mathfrak{c}}(a+1)u \\ \epsilon k & 0 & \varpi^{\mathfrak{c}} d\hat{u} & \frac{k\theta}{2} & -\epsilon(d+1) & \epsilon\varpi^{-\mathfrak{c}} cu \end{bmatrix}.$$

Suppose it is an element in $P(F)J_nN'(\mathfrak{p}^0)$, then as in the proof of Proposition 2.4.2 it can be written in the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ \epsilon D_1 & \epsilon D_2 & \epsilon D_1 x + D_2 y^* & \epsilon D_1 y + \epsilon D_2 z \\ \epsilon D_3 & \epsilon D_4 & \epsilon D_3 x + D_4 y^* & \epsilon D_3 y + \epsilon D_4 z \end{bmatrix}$$

with $x \in S_r(\mathfrak{p}\mathcal{O}), y \in \mathrm{Mat}_{r,2m}(\mathcal{O}), z \in S_{2m}(\mathcal{O})$. We need

$$D_1 = \theta^{-1}(e-1),\ D_2 = \begin{bmatrix} 0 & \epsilon\varpi^{\mathfrak{c}}\theta^{-1} j\hat{u} \end{bmatrix},\ D_3 = \begin{bmatrix} -f \\ k \end{bmatrix},\ D_4 = \begin{bmatrix} -1_m & -\epsilon\varpi^{\mathfrak{c}} b\hat{u} \\ 0 & \epsilon\varpi^{\mathfrak{c}} d\hat{u} \end{bmatrix},$$

and

$$\epsilon D_1 x + D_2 y^* = \theta^{-1}\frac{e+1}{2}\theta,\quad D_1 y + D_2 z = \begin{bmatrix} -\theta^{-1} j & \theta^{-1}\varpi^{-\mathfrak{c}} hu \end{bmatrix},$$
$$\epsilon D_3 x + D_4 y^* = \begin{bmatrix} -\frac{f\theta}{2} \\ \frac{k\theta}{2} \end{bmatrix},\quad D_3 y + D_4 z = \begin{bmatrix} b & -\varpi^{-\mathfrak{c}}(a+1)u \\ -(d+1) & \varpi^{-\mathfrak{c}} cu \end{bmatrix}.$$

By the same arguments as in the proof of Proposition 2.4.2, the conditions on x, y, z force

(1) d is invertible and $1 + d^{-1} \in \mathrm{Mat}_m(\mathfrak{p}^{\mathfrak{c}}\mathcal{O})$,

(2) $d^{-1}c$ have entries in $\mathfrak{p}^{2\mathfrak{c}}$, $d^{-1}k, jd^{-1}$ has entries in $\mathfrak{p}^{\mathfrak{c}}$ and bd^{-1} has entries in $\mathcal{O}$,

(3) $e - jd^{-1}k \in -1 + \mathrm{Mat}_r(\mathfrak{p}\mathcal{O})$.

These implies $-g \in K_1(\mathfrak{p}^{\mathfrak{c}})$ and we have

$$\begin{aligned}
&\mathcal{Z}(s;\phi,\pi(w)\phi,f_s^{\ddagger,\mathfrak{c}})\\
=&\int_{K_1(\mathfrak{p}^{\mathfrak{c}})}\sum_{u\in\mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O})}\chi^{-1}(\nu(u))\left\langle\pi\left(g\begin{bmatrix}\epsilon\varpi^{-\mathfrak{c}}u & 0 & 0\\ 0 & 1_r & 0\\ 0 & 0 & \epsilon\varpi^{\mathfrak{c}}\hat{u}\end{bmatrix}w\right)\phi,\phi\right\rangle dg\\
=&\mathrm{vol}(\mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O}))\left\langle\pi\left(\begin{bmatrix}0 & 0 & \varpi^{-\mathfrak{c}}\cdot 1_m\\ 0 & 1_r & 0\\ \varpi^{\mathfrak{c}}\cdot 1_m & 0 & 0\end{bmatrix}\right)\phi,\phi\right\rangle.
\end{aligned}$$

□

2.4.5 The p-adic section f_s^p

Assume χ is unramified, $\phi \in \pi$ is fixed by $K(\mathfrak{q}^2)$ and is an eigenvector for the Hecke algebra $\mathcal{H}(K(\mathfrak{q}^2),\mathfrak{X})$. We construct yet another section f_s^p as a twist of $f_s^{\dagger,0}$ which represent the p-adic modification factor in the construction of the p-adic L-functions. Again, here we are inspired by the idea of [BS00, p.1392 and p.1400].

For each $0 \le i \le m$, denote T_i for the Hecke operator given by the double coset

$$[K(\mathfrak{p})\xi_i K(\mathfrak{p})],\ \xi_i = \mathrm{diag}[u_i, 1_r, \hat{u}_i],\ u_i = \begin{bmatrix}1_{m-i} & 0\\ 0 & \widetilde{\varpi}\cdot 1_i\end{bmatrix}. \tag{2.4.19}$$

Suppose there is a double coset decomposition

$$\mathrm{GL}_m(\mathcal{O})u_i\mathrm{GL}_m(\mathcal{O}) = \coprod_j \delta_{ij}\mathrm{GL}_m(\mathcal{O}). \tag{2.4.20}$$

We define a local section $f_s^{p,i}$ by

$$f_s^{p,i}(h) = \sum_j \sum_{u \in \widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})\delta_{ij}^{-1}/\widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})} f_s^{\dagger,0}\left(h\begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & \frac{u}{\widetilde{\varpi}} \\ 0 & 0 & 1_m & 0 & -\frac{\epsilon u^*}{\widetilde{\varpi}} & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_m \end{bmatrix}\right). \tag{2.4.21}$$

Lemma 2.4.5. *Let λ_i be the eigenvalues of ϕ under T_i, i.e. $\phi|T_i = \lambda_i\phi$. Set $\phi_1 = \phi, \phi_2 = \pi(w)\phi$. Then for each $0 \le i \le m$,*

$$\begin{aligned} &\mathcal{Z}(s;\phi,\pi(w)\phi, f_s^{p,i}) \\ =&\chi(\widetilde{\varpi})^{\mathbf{d}_3(m-i)}q^{-(m-i)\mathbf{d}_4(s+\kappa)}\lambda_{m-i}L\left(s+\frac{1}{2},\phi\times\chi\right) \\ &\times\left\langle \pi\left(\begin{bmatrix} 0 & 0 & \widetilde{\varpi}^{-1}\cdot 1_m \\ 0 & 1_r & 0 \\ \widetilde{\varpi}\cdot 1_m & 0 & 0 \end{bmatrix}\right)\phi,\phi\right\rangle, \end{aligned} \tag{2.4.22}$$

with

$$\mathbf{d}_3 = \begin{cases} 1 & \textit{Case I, II, V Ramified,} \\ 2 & \textit{Case III, IV, V Inert.} \end{cases}, \qquad \mathbf{d}_4 = \begin{cases} 2 & \textit{Case V Inert,} \\ 1 & \textit{otherwise,} \end{cases} \tag{2.4.23}$$

Proof. Denote $d_{\widetilde{\varpi}} = \mathrm{diag}[1_{m+r}, \epsilon\widetilde{\varpi}, 1_{m+r}, \widetilde{\varpi}^{-1}]$ and

$$\delta_u = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & -u & 0 & 1_m & 0 \\ 0 & \epsilon u^* & 0 & 0 & 0 & 1_m \end{bmatrix}.$$

Then

$$(w,w)^{-1}d_{\widetilde{\varpi}}\delta_u d_{\widetilde{\varpi}}^{-1}(w,w) = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & \frac{u}{\widetilde{\varpi}} \\ 0 & 0 & 1_m & 0 & -\frac{\epsilon u^*}{\widetilde{\varpi}} & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_m \end{bmatrix}.$$

Changing variables $g \mapsto w^{-1}g \begin{bmatrix} \epsilon\widetilde{\varpi}^{-1}\cdot 1_m & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & \epsilon\widetilde{\varpi}\cdot 1_m \end{bmatrix} w$, we need to calculate

$$\begin{aligned}
&\mathcal{Z}(s;\phi_1,\phi_2,f_s^p) \\
&= \int_{G(F)} \sum_j \sum_{u\in\widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})\delta_{ij}^{-1}/\widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})} \left\langle \pi\left(g \begin{bmatrix} \epsilon\widetilde{\varpi}^{-1}\cdot 1_m & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & \epsilon\widetilde{\varpi}\cdot 1_m \end{bmatrix} w \right)\phi,\phi \right\rangle \\
&\times f_s^{\dagger,0}\left(\delta(g,1)\tau_x d_{\widetilde{\varpi}}^{-1}(w,w)\right) dg.
\end{aligned}$$

Write $g = \begin{bmatrix} a & f & b \\ h & e & j \\ c & k & d \end{bmatrix}$ and compute that

$$\delta(g,1)\delta_u d_{\widetilde{\varpi}}^{-1}(w,w)\begin{bmatrix} 1_r & 0 & 0 & 0 \\ 0 & \widetilde{\varpi}^{-1}\cdot 1_{2m} & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & \widetilde{\varpi}\cdot 1_{2m} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e+1}{2} & 0 & \frac{\widetilde{\varpi}\epsilon j}{2} & \frac{\epsilon(e-1)\theta}{4} & -\frac{j}{2} & \frac{\widetilde{\varpi}^{-1}h}{2} \\ 0 & 0 & 0 & 0 & -1_m & 0 \\ \epsilon f & 0 & \widetilde{\varpi}b & \frac{f}{2} & -\epsilon b & \epsilon\widetilde{\varpi}^{-1}a \\ \epsilon\theta^{-1}(e-1) & 0 & \theta^{-1}\widetilde{\varpi}j & \theta^{-1}\frac{e+1}{2}\theta & -\epsilon\theta^{-1}j & \epsilon\widetilde{\varpi}^{-1}\theta^{-1}h \\ -\epsilon f & -\epsilon\cdot 1_m & -\widetilde{\varpi}b & -\frac{f\theta}{2} & \epsilon b & -\epsilon\widetilde{\varpi}^{-1}(a+u) \\ \epsilon k & 0 & \widetilde{\varpi}d & \frac{k\theta}{2} & -\epsilon(du^*+1) & \epsilon\widetilde{\varpi}^{-1}c \end{bmatrix}.$$

By the same arguments as in the proof of Proposition 2.4.2, 2.4.4, this is an element in $P(F)J_nN'(\mathfrak{p}^0)$ if and only if

(1) d is invertible and $d^{-1}+u^*\in \mathrm{Mat}_m(\mathfrak{q}\mathcal{O})$,

(2) $d^{-1}c$ have entries in $\mathfrak{q}^2$, $d^{-1}k, jd^{-1}$ has entries in $\mathfrak{q}$ and bd^{-1} has entries in $\mathcal{O}$,

(3) $e-jd^{-1}k\in -1+\mathrm{Mat}_r(\mathfrak{q}\mathcal{O})$.

Since d is invertible, we can write

$$g=\begin{bmatrix} 1_m & -\hat{d}j^*\theta^{-1} & bd^{-1} \\ 0 & 1_r & jd^{-1} \\ 0 & 0 & 1_m \end{bmatrix}\begin{bmatrix} \hat{d} & 0 & 0 \\ 0 & e-jd^{-1}k & 0 \\ 0 & 0 & d \end{bmatrix}\begin{bmatrix} 1_m & 0 & 0 \\ -\theta k^*\hat{d} & 1_r & 0 \\ d^{-1}c & d^{-1}k & 1_m \end{bmatrix}.$$

Note that there is a permutation $j\mapsto j'$ such that $\hat{d}$ runs through $\mathrm{GL}_m(D)\cap \widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})\delta_{ij'}^{-1}$ for fixed j. Hence, when δ_{ij} running through the right coset

$$\mathrm{GL}_m(\mathcal{O})u_i\mathrm{GL}_m(\mathcal{O})/\mathrm{GL}_m(\mathcal{O}),$$

all such d run through

$$\mathrm{GL}_m(D)\cap\widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})u_i^{-1}.$$

Therefore,

$$\begin{aligned}&\mathcal{Z}(s;\phi,\pi(w)\phi,f_s^{p,i})\\ =&\sum_{\hat{d}\in \mathrm{GL}_m(D)\cap\widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})u_i^{-1}}\ \sum_{g\in K(\mathfrak{q}^2)\mathrm{diag}[\hat{d},1_r,d]K(\mathfrak{q}^2)}\chi(\nu(\widetilde{\varpi}d))|\mathrm{N}_{E/F}(\nu(\widetilde{\varpi}d))|^{s+\kappa}\\ &\times\left\langle\pi\left(g\begin{bmatrix}\epsilon\widetilde{\varpi}^{-1}\cdot 1_m & 0 & 0\\ 0 & 1_r & 0\\ 0 & 0 & \epsilon\widetilde{\varpi}\cdot 1_m\end{bmatrix}w\right)\phi,\phi\right\rangle.\end{aligned}$$

Note that when $\hat{d}$ runs through $\mathrm{GL}_m(D)\cap\varpi\mathrm{Mat}_m(\mathcal{O})u_i^{-1}$, we are taking a sum over

$$K(\mathfrak{q}^2)\mathfrak{Q}K(\mathfrak{q}^2)\cdot K(\mathfrak{q}^2)\mathrm{diag}[\widetilde{\varpi}^{-1}u_i,1_r,\widetilde{\varpi}u_i^{-1}]K(\mathfrak{q}^2).$$

We thus obtain

$$\begin{aligned}&\mathcal{Z}(s;\phi,\pi(w)\phi,f_s^{p,i})\\ =&\chi(\widetilde{\varpi})^{\mathbf{d}_3(m-i)}q^{-(m-i)\mathbf{d}_4(s+\kappa)}\lambda_{m-i}L\left(s+\frac{1}{2},\phi\times\chi\right)\\ &\times\left\langle\pi\left(\begin{bmatrix}0 & 0 & \widetilde{\varpi}^{-1}\cdot 1_m\\ 0 & 1_r & 0\\ \widetilde{\varpi}\cdot 1_m & 0 & 0\end{bmatrix}\right)\phi,\phi\right\rangle.\end{aligned}$$

as desired. □

Gluing all these $0\le i\le m$ together, we define the local section f_s^p by

$$f_s^p(h)=\sum_{i=0}^m(-1)^i q^{\mathbf{d}_3\left(\frac{i(i-1)}{2}-im\right)}f_s^{p,i}(h). \tag{2.4.24}$$

Proposition 2.4.6. *Assume χ is unramified, $\phi\in\pi$ is fixed by $K(\mathfrak{q}^2)$ and is an eigenvector for the Hecke algebra $\mathcal{H}(K(\mathfrak{q}^2),\mathfrak{X})$. Assume $2\in\mathcal{O}^\times,\theta\in\mathrm{GL}_r(\mathcal{O})$. Set $\phi_1=\phi,\phi_2=\pi(w)\phi$ and denote β_i for the Satake parameters of ϕ. Then*

$$\begin{aligned}\mathcal{Z}(s;\phi,\pi(w)\phi,f_s^p)=&(-1)^m q^{-\mathbf{d}_3\frac{m^2+m}{2}}L\left(s+\frac{1}{2},\phi\times\chi\right)M\left(s+\frac{1}{2},\phi\times\chi\right)\\ &\times\left\langle\pi\left(\begin{bmatrix}0 & 0 & \widetilde{\varpi}^{-1}\cdot 1_m\\ 0 & 1_r & 0\\ \widetilde{\varpi}\cdot 1_m & 0 & 0\end{bmatrix}\right)\phi,\phi\right\rangle,\end{aligned} \tag{2.4.25}$$

where $M(s,\phi\times\chi)$ is the modification factor given in the following list.

(Case I, Orthogonal)

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{\frac{r}{2}-s}\right),$$

(Case II, Symplectic)

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{-s+1}\right),$$

(Case III, Quaternionic Orthogonal Nonsplit)

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{1+r-s}\right),$$

(Case IV, Quaternionic Unitary Nonsplit)

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{r-s}\right),$$

(Case V, Unitary Inert) E/F is inert,

$$\prod_{i=1}^{m}\left(1-\chi(\varpi)\beta_i q^{1+r-s}\right),$$

(Case V, Unitary Ramified) E/F is ramified,

$$\prod_{i=1}^{m}\left(1-\chi(\widetilde{\varpi})\beta_i q^{\frac{r+1}{2}-s}\right).$$

Proof. By the above lemma, we have

$$\begin{aligned}\mathcal{Z}(s;\phi_1,\phi_2,f_s^p)=&\sum_{i=1}^{m}(-1)^i q^{\mathbf{d}_3\left(\frac{i(i-1)}{2}-im\right)}\chi(\widetilde{\varpi})^{\mathbf{d}_3(m-i)}q^{-(m-i)(s+\kappa)}\lambda_{m-i}\\ &\times L\left(s+\frac{1}{2},\phi\times\chi\right)\left\langle\pi\left(\begin{bmatrix}0&0&\widetilde{\varpi}^{-1}\cdot 1_m\\0&1_r&0\\ \widetilde{\varpi}\cdot 1_m&0&0\end{bmatrix}\right)\phi,\phi\right\rangle.\end{aligned}$$

It suffices to compute

$$\begin{aligned}&\sum_{i=1}^{m}(-1)^i q^{\mathbf{d}_3\left(\frac{i(i-1)}{2}-im\right)}\chi(\widetilde{\varpi})^{\mathbf{d}_3(m-i)}q^{-(m-i)(s+\kappa)}\lambda_{m-i}\\ =&(-1)^m q^{-\mathbf{d}_3\frac{m^2+m}{2}}\sum_{i=1}^{m}(-1)^i q^{\mathbf{d}_3\frac{i(i-1)}{2}}\chi(\widetilde{\varpi})^{\mathbf{d}_3 i}q^{-i(s+\kappa-\mathbf{d}_3)}\lambda_i.\end{aligned}$$

This equals to $M(s+\frac{1}{2},\phi\times\chi)$ in the above lists. Indeed, using [Shi00, Lemma 19.13] and the explicit description of the Satake map in the proof of Proposition 2.3.3, one can show that

$$\sum_{i=0}^{m}(-1)^i q^{\mathbf{d}_3 \frac{i(i-1)}{2}} \lambda_i \chi(\widetilde{\varpi})^{\mathbf{d}_3 i} q^{-i(s+\kappa-\frac{1}{2})}$$

is the Euler factor in Proposition 2.3.3 and the proposition easily follows. $\square$

Chapter 3

The Archimedean Theory and Algebraic Modular Forms

In this and the next chapter, we restrict ourselves to the following global setting. Let F be a totally real field of degree $[F : \mathbb{Q}] = d$ and consider tuples (D, ρ, ϵ) of following four cases:

(Case II, Symplectic)	(D, ρ) of type (a) with $\epsilon = -1$,
(Case III, Quaternionic Orthogonal)	(D, ρ) of type (b) with $\epsilon = 1$ and
	$D_v = \mathrm{Mat}_2(\mathbb{R})$ for any archimedean place v,
(Case IV, Quaternionic Unitary)	(D, ρ) of type (b) with $\epsilon = -1$,
	$D_v = \mathbb{H}$ for any archimedean place v,
(Case V, Unitary)	(D, ρ) of type (c) with $\epsilon = -1$,
	$D = E$ is an imaginary quadratic field.

Here $\mathbb{H}$ is the Hamilton quaternion algebra for which we fix an embedding into $\mathrm{Mat}_2(\mathbb{R})$. The global group G is defined as

$$G := G(F) := \{g \in \mathrm{GL}_n(D) : g\Phi g^* = \Phi\}, \quad \Phi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \theta & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}, \tag{3.0.1}$$

with $n = 2m + r$ and $\theta^* = \epsilon\theta \in \mathrm{GL}_r(D)$ is anisotropic (over F). We may also write it as $G_{m,r}$ to emphasize the index. In Case V, we assume $i\theta_v > 0$ for all archimedean place v for simplicity.

This chapter is expanded from [Jin23, Section 5, 6] and is organized as follow. We review the definition of symmetric spaces in Section 3.1. Both classical and adelic definition of modular forms are given in Section 3.2. In Section 3.3, we study the algebraic modular forms which will be used later. The algebraic modular forms for symplectic and unitary groups are already well studied in [Shi97; Shi00] and most discussions concerning quaternionic unitary groups in this chapter are taken from [BJar] which is joint with Thanasis Bouganis. In Section 3.4, we calculate the archimedean local integrals and summarize the integral representations.

3.1 Symmetric spaces

Let v be an archimedean place of F and $G_v = G(F_v)$ the localization of G at v. Fix a maximal compact subgroup K of G_v. Then by our assumption, G_v/K is a hermitian symmetric space. For a comprehensive study of hermitian symmetric spaces, the reader can refer to [Hel01; Sat80; Pya69] (see also [Hua63] and [Lanar, Section 3]). The symmetric spaces are well studied in [Shi97, Section 6, 7] and [Shi00, Section 3, 5] for symplectic and unitary groups. In this section, we discuss the realizations of symmetric spaces for Case III, IV. The Case IV is studied in [BJar] and the Case III is similar. We will start with a rather general and abstract setting in Section 3.1.1 to explain the idea and then give explicit realizations in the following subsections.

We remind the reader that the notation for our group

$$G \cong \{g \in \mathrm{GL}_n(D) : g^*\Phi^{-1}g = \Phi^{-1}\} \tag{3.1.1}$$

coincides with the notation in [BJar].

3.1.1 Abstract symmetric spaces

We only discuss Case IV in this subsection but the idea is the same for all cases. Let $\mathfrak{i}$ be any embedding $\mathrm{Mat}_n(\mathbb{H}) \to \mathrm{Mat}_{2n}(\mathbb{C})$. Then by the Skolem-Noether theorem ([Mil20, Theorem 2.10]) there exists $\alpha \in \mathrm{Mat}_{2n}(\mathbb{C})$ with $\alpha\alpha^* = 1$ such that ${}^t\mathfrak{i}(x) = \alpha\mathfrak{i}(x^*)\alpha^{-1}$. Let $\Psi \in \mathrm{GL}_n(D)$ be a skew-hermitian form similar to Φ^{-1} above, that is $\Psi = \gamma^*\Phi^{-1}\gamma$ for some $\gamma \in \mathrm{GL}_n(D)$. Then the group G_v is isomorphic to

$$\mathcal{G} = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^*Hg = H,\, {}^tgKg = K\}, \tag{3.1.2}$$

with $H = \mathfrak{i}(\Psi), K = \alpha^{-1}\mathfrak{i}(\Psi)$. We call it a realization of G_v. Suppose we are given two such data $(\mathfrak{i}_1, \Psi_1, H_1, K_1, \mathcal{G}_1)$ and $(\mathfrak{i}_2, \Psi_2, H_2, K_2, \mathcal{G}_2)$ with $\Psi_1 = S^*\Psi_2 S$. Again by Skolem-Noether there exists β with $\beta\beta^* = 1$ such that $\mathfrak{i}_1(x) = \beta^{-1}\mathfrak{i}_2(x)\beta$. Put $R = \mathfrak{i}_2(S)\beta$ then $H_1 = R^*H_2R, K_1 = {}^tRK_2R$. Therefore $g \mapsto RgR^{-1}$ gives isomorphism $\mathcal{G}_1 \cong \mathcal{G}_2$.

Following [Pya69], we will define the associated symmetric space via its Borel embedding into its compact dual symmetric space. In Case IV, the semisimple compact dual of our group is the group $\mathrm{SO}(2n)$ (with notations in [Hel01, page 330]), and the corresponding dual symmetric space is $\mathrm{SO}(2n)/\mathrm{U}(n)$. This space may be identified (see for example [Shi87, page 6]) with the space $V = L/\mathrm{GL}_n(\mathbb{C})$ where

$$L = \{U \in \mathrm{Mat}_{2n,n}(\mathbb{C}) : \quad {}^tUKU = 0\}. \tag{3.1.3}$$

We set

$$\Omega = \{U \in \mathrm{Mat}_{2n,n}(\mathbb{C}) : -iU^*HU > 0,\, {}^tUKU = 0\} \subset L, \tag{3.1.4}$$

with the action of $\mathrm{GL}_n(\mathbb{C})$ by right multiplication and $\mathcal{G}$ by left multiplication. The symmetric space $\mathcal{H}$ is defined as

$$\mathcal{H} := \mathcal{H}_{u_0} = \{z \in \mathrm{Mat}_n(\mathbb{C}) : U(z) \in \Omega\}, \quad U(z) := \begin{bmatrix} z \\ u_0 \end{bmatrix}, \tag{3.1.5}$$

for some fixed suitable u_0, which we make explicit later. The following lemma is a direct consequence of our definition for $\mathcal{H}$.

Lemma 3.1.1. *There is a bijection* $\mathcal{H} \times \mathrm{GL}_n(\mathbb{C}) \to \Omega$ *given by* $z \times \lambda = U(z)\lambda$.

By this lemma, it follows that for any element $\alpha \in \mathcal{G}$, we can find a $z' \in \mathcal{H}$ and $\lambda(\alpha, z) \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\alpha U(z) = U(z')\lambda(\alpha, z). \tag{3.1.6}$$

We then define the action of $\mathcal{G}$ on $\mathcal{H}$ by $\alpha.z := \alpha z := z'$ and $\lambda(\alpha, z)$ satisfies the cocycle relation

$$\lambda(\alpha_1\alpha_2, z) = \lambda(\alpha_1, \alpha_2 z)\lambda(\alpha_2, z) \text{ for } \alpha_1, \alpha_2 \in \mathcal{G}, z \in \mathcal{H}. \tag{3.1.7}$$

We set $j(\alpha, z) := \nu(\lambda(\alpha, z)) \in \mathbb{C}^\times$. We call $\lambda(\alpha, z)$ or $j(\alpha, z)$ automorphy factors. More explicitly, write $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, d \in \mathrm{Mat}_n(\mathbb{C})$, we have

$$\alpha U(z) = \begin{bmatrix} az + bu_0 \\ cz + du_0 \end{bmatrix} = \begin{bmatrix} (az + bu_0)(cz + du_0)^{-1}u_0 \\ u_0 \end{bmatrix} u_0^{-1}(cz + du_0). \tag{3.1.8}$$

That is,

$$\alpha z = (az + bu_0)(cz + du_0)^{-1}u_0, \qquad \lambda(\alpha, z) = u_0^{-1}(cz + du_0). \tag{3.1.9}$$

For $z_1, z_2 \in \mathcal{H}$, we set

$$\begin{aligned} \eta(z_1, z_2) &:= iU(z_1)^*HU(z_2), \qquad & \delta(z_1, z_2) &:= \nu(\eta(z_1, z_2)) \\ \eta(z) &:= \eta(z, z), & \delta(z) &:= \delta(z, z). \end{aligned} \tag{3.1.10}$$

We note that

$$U(z_1)^*HU(z_2) = \lambda(\alpha, z_1)^*U(\alpha z_1)^*HU(\alpha z_2)\lambda(\alpha, z_2), \tag{3.1.11}$$

and

$$
\begin{aligned}
iU(\alpha z_1)^* HU(\alpha z_2) &= \begin{bmatrix} \eta(\alpha z_1, \alpha z_2) & * \\ * & * \end{bmatrix}, \\
iU(z_1)^* HU(z_2) &= \begin{bmatrix} \eta(z_1, z_2) & * \\ * & * \end{bmatrix}.
\end{aligned}
\tag{3.1.12}
$$

We thus obtain that

$$\lambda(\alpha, z_1)^* \eta(\alpha z_1, \alpha z_2) \lambda(\alpha, z_2) = \eta(z_1, z_2), \tag{3.1.13}$$

and after taking the determinant, we have

$$\overline{j(\alpha, z_1)} \delta(\alpha z_1, \alpha z_2) j(\alpha, z_2) = \delta(z_1, z_2). \tag{3.1.14}$$

In particular,

$$\lambda(\alpha, z)^* \eta(\alpha z) \lambda(\alpha, z) = \eta(z), \qquad \delta(\alpha z) = |j(\alpha, z)|^{-2} \delta(z). \tag{3.1.15}$$

We now discuss the relation between different realizations of the symmetric space $\mathcal{H}$. Given H_1, K_1 and H_2, K_2 as above, we have seen at the beginning of this subsection that we can find an R such that $H_1 = R^* H_2 R, K_1 = {}^t R K_2 R$. We then have an isomorphism $\Omega_1 \cong \Omega_2$ given by $U \mapsto RU$ which induces an isomorphism $\rho : \mathcal{H}_1 \cong \mathcal{H}_2$. Indeed, for $z_1 \in \mathcal{H}_1$, there exists some $z_2 \in \mathcal{H}_2, \mu(z_1) \in \mathrm{GL}_n(\mathbb{C})$ such that

$$R \begin{bmatrix} z_1 \\ u_{01} \end{bmatrix} = \begin{bmatrix} z_2 \\ u_{02} \end{bmatrix} \mu(z_1), \tag{3.1.16}$$

for some $u_{01}, u_{02} \in \mathrm{Mat}_n(\mathbb{C})$ and the isomorphism can be given by $\rho(z_1) = z_2$.

In the following lemma we write ρ also for the isomorphism $\mathcal{G}_1 \to \mathcal{G}_2$ given by $\rho(g_1) := R g_1 R^{-1}$.

Lemma 3.1.2. *Let $\rho : \mathcal{G}_1 \to \mathcal{G}_2, \rho : \mathcal{H}_1 \to \mathcal{H}_2$ given as above. Then*

(1) $\rho(\alpha z) = \rho(\alpha)\rho(z)$ with $\alpha \in \mathcal{G}_1, z \in \mathcal{H}_1$;

(2) $\lambda(\rho(\alpha), \rho(z)) = \mu(\alpha z) \lambda(\alpha, z) \mu(z)^{-1}$;

(3) $\eta(\rho(z_1), \rho(z_2)) = \widehat{\mu(z_1)} \eta(z_1, z_2) \mu(z_2)^{-1}$ for $z_1, z_2 \in \mathcal{H}_1$.

Proof. (1) It suffices to prove that

$$\begin{bmatrix} \rho(\alpha z) \\ u_{02} \end{bmatrix} = \begin{bmatrix} \rho(\alpha)\rho(z) \\ u_{02} \end{bmatrix}.$$

By definition of the isomorphism and action,

$$\begin{aligned}
\begin{bmatrix} \rho(\alpha z) \\ u_{02} \end{bmatrix} &= R \begin{bmatrix} \alpha z \\ u_{01} \end{bmatrix} \mu(\alpha z)^{-1} \\
&= R\alpha \begin{bmatrix} z \\ u_{01} \end{bmatrix} \lambda(\alpha, z)^{-1} \mu(\alpha z)^{-1} \\
&= \rho(\alpha) \begin{bmatrix} \rho(z) \\ u_{02} \end{bmatrix} \mu(z)\lambda(\alpha, z)^{-1} \mu(\alpha z)^{-1} \\
&= \begin{bmatrix} \rho(\alpha)\rho(z) \\ u_{02} \end{bmatrix} \lambda(\rho(\alpha), \rho(z))\mu(z)\lambda(\alpha, z)^{-1} \mu(\alpha z)^{-1}.
\end{aligned}$$

We must have $\lambda(\rho(\alpha), \rho(z))\mu(z)\lambda(\alpha, z)^{-1}\mu(\alpha z)^{-1} = 1$ and our desired result follows which we also obtain (2). (3) can be computed similarly by definition of η. □

3.1.2 Symmetric spaces for Case IV

We now apply the above discussions to some explicit realizations of G_v. Note that the map $\mathfrak{i}$ defined in Example 2.1.1 induces the following isomorphism

$$\mathfrak{i} : G_v \xrightarrow{\sim} G_\infty := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^*\Phi_\infty g = \Phi_\infty,\ {}^t g \Psi_\infty g = \Psi_\infty\}, \tag{3.1.17}$$

with

$$\Phi_\infty = \begin{bmatrix} 0 & 0 & 0 & -1_{2m} \\ 0 & 0 & -1_r & 0 \\ 0 & 1_r & 0 & 0 \\ 1_{2m} & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_\infty = \begin{bmatrix} 0 & 0 & 0 & J'_m \\ 0 & 1_r & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ -J'_m & 0 & 0 & 0 \end{bmatrix}. \tag{3.1.18}$$

As in the last subsection

$$\Omega = \{U \in \mathrm{Mat}_{2n,n}(\mathbb{C}) : -iU^*\Phi_\infty U > 0,\ {}^t\!U\Psi_\infty U = 0\}, \tag{3.1.19}$$

and define the symmetric space by

$$\begin{gathered}\mathfrak{Z} := \mathfrak{Z}_n := \mathfrak{Z}_{m,r} = \{z \in \mathrm{Mat}_n(\mathbb{C}) : U(z) \in \Omega\}, \\ U(z) = \begin{bmatrix} z \\ u_0 \end{bmatrix}, \qquad u_0 = \begin{bmatrix} 0 & 1_r \\ 1_{2m} & 0 \end{bmatrix}.\end{gathered} \tag{3.1.20}$$

Explicitly,

$$\mathfrak{Z}_{m,r} = \left\{ z := \begin{bmatrix} u & v \\ w\,{}^t\!v J'_m & w \end{bmatrix} : \begin{array}{c} u \in \mathrm{Mat}_{2m}(\mathbb{C}), v \in \mathrm{Mat}_{2m,r}(\mathbb{C}), \\ uJ'_m + v\,{}^t\!v - J'_m\,{}^t\!u = 0, \\ w \in \mathrm{Mat}_r(\mathbb{C}),\ {}^t\!ww + 1 = 0, \\ i(z^* - z) > 0. \end{array} \right\}. \tag{3.1.21}$$

For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty$, the action of G_∞ on $\mathfrak{Z}$ and the automorphy factor are given by

$$gz = (az + bu_0)(cz + du_0)^{-1}u_0, \qquad \lambda(g, z) = u_0^{-1}(cz + du_0). \tag{3.1.22}$$

For $z_1, z_2 \in \mathfrak{Z}$, we set

$$\begin{aligned} \eta(z_1, z_2) &= i(z_1^* - z_2), & \delta(z_1, z_2) &= \nu(\eta(z_1, z_2)) \\ \eta(z) &= \eta(z, z), & \delta(z) &= \delta(z, z). \end{aligned} \tag{3.1.23}$$

We will take $z_0 = i \cdot 1_n$ to be the origin of $\mathfrak{Z}$ and K_∞ the subgroup of G_∞ fixing z_0. Then $g \mapsto \lambda(g, z_0)$ gives an isomorphism $K_\infty \cong \mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^*g = 1_n\}$ and our symmetric space $\mathfrak{Z} \cong G_\infty/K_\infty$.

We give another two useful realizations to compare with the symmetric spaces in other works. The group G_∞ is further isomorphic to

$$G'_\infty := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^*J_ng = J_n,\ {}^t\!gg = 1_{2n}\}. \tag{3.1.24}$$

Take $u_0 = 1$, the realization associated to this group is

$$\mathfrak{H} = \mathfrak{H}_n = \{z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t zz + 1 = 0, i(z^* - z) > 0\}. \tag{3.1.25}$$

This is an unbounded realization of type D domain in [Lanar]. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G'_\infty$, the action of G'_∞ on $\mathfrak{H}$ and the automorphy factor are given by

$$gz = (az + b)(cz + d)^{-1}, \qquad \lambda(g, z) = cz + d. \tag{3.1.26}$$

For $z_1, z_2 \in \mathfrak{H}$, we set $\eta(z_1, z_2) = i(z_1^* - z_2)$. We take $z_0 = i_n := i \cdot 1_n$ to be the origin of $\mathfrak{H}$ and K'_∞ the subgroup of G'_∞ fixing z_0. Since $\eta(gz_0) = \eta(z_0) = 2$ for $g \in K_\infty$, $g \mapsto \lambda(g, z_0)$ gives an isomorphism $K'_\infty \cong \mathrm{U}(n)$ and thus $\mathfrak{H} \cong G'_\infty / K'_\infty$.

Let $T' = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ and sending $g \mapsto T'^{-1} g T'$ we have an isomorphism

$$G'_\infty \cong G''_\infty := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^* \Phi''_\infty g = \Phi''_\infty,\ {}^t g \Psi''_\infty g = \Psi''_\infty\}, \tag{3.1.27}$$

with

$$\Phi''_\infty = \begin{bmatrix} i_n & 0 \\ 0 & -i_n \end{bmatrix}, \qquad \Psi''_\infty = \begin{bmatrix} 0 & -i_n \\ -i_n & 0 \end{bmatrix}. \tag{3.1.28}$$

Take $u_0 = 1$ the realization associated to this group is defined as

$$\mathfrak{B} = \mathfrak{B}_n = \{z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = -z, zz^* < 1_n\}. \tag{3.1.29}$$

This is a bounded domain of type $\mathfrak{R}_{\mathrm{III}}$ in [Hua63]. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G''_\infty$, the action of G''_∞ on $\mathfrak{B}$ and the automorphy factor is given by

$$gz = (az + b)(cz + d)^{-1}, \qquad \lambda(g, z) = cz + d. \tag{3.1.30}$$

For $z_1, z_2 \in \mathfrak{H}$, we set $\eta(z_1, z_2) = i(z_1^* z_2 - 1)$. We take $z_0 = 0$ to be the origin of $\mathfrak{B}$ and K''_∞ the subgroup of G''_∞ fixing z_0. Since $\eta(gz_0) = \eta(z_0) = -i$ for $g \in K_\infty$, $g \mapsto \lambda(g, z_0)$ gives an isomorphism $K''_\infty \cong \mathrm{U}(n)$ and thus $\mathfrak{H} \cong G''_\infty / K''_\infty$. The relation between $\mathfrak{H}$ and $\mathfrak{B}$ can be given explicitly by Cayley transform

$$\mathfrak{H} \xrightarrow{\sim} \mathfrak{B} : z \mapsto (z-i)(z+i)^{-1}. \tag{3.1.31}$$

Let $z_1, z_2 \in \mathfrak{B}_n, \alpha \in G_v$ as above and $dz = (dz_{hk})$ be a matrix of the same shape as $z \in \mathbb{C}_n^n$ whose entries are 1-forms dz_{hk}. Note that, on one hand,

$$\begin{aligned} &\begin{bmatrix} z_1 & 1 \\ 1 & -\overline{z}_1 \end{bmatrix}^* \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix} \begin{bmatrix} z_2 & 1 \\ 1 & -\overline{z}_2 \end{bmatrix} \\ =& \begin{bmatrix} z_1^* z_2 - 1 & z_1^* + \overline{z}_2 \\ z_2 + {}^t z_1 & 1 - {}^t z_1 \overline{z}_2 \end{bmatrix} = \begin{bmatrix} z_1^* z_2 - 1 & z_1^* - z_2^* \\ z_2 - z_1 & 1 - {}^t z_1 \overline{z}_2 \end{bmatrix}, \end{aligned} \tag{3.1.32}$$

and on the other hand,

$$\begin{aligned} &\begin{bmatrix} \alpha z_1 & 1 \\ 1 & -\overline{\alpha z_1} \end{bmatrix}^* \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix} \begin{bmatrix} \alpha z_2 & 1 \\ 1 & -\overline{\alpha z_2} \end{bmatrix} \\ =& \begin{bmatrix} (\alpha z_1)^*(\alpha z_2) - 1 & (\alpha z_1)^* - (\alpha z_2)^* \\ \alpha z_2 - \alpha z_1 & 1 - {}^t(\alpha z_1)(\overline{\alpha z_2}) \end{bmatrix}. \end{aligned} \tag{3.1.33}$$

Using the fact (which can be obtained from the property of $U(z)$)

$$\alpha \begin{bmatrix} z & 1 \\ 1 & -\overline{z} \end{bmatrix} = \begin{bmatrix} \alpha z & 1 \\ 1 & -\overline{\alpha z} \end{bmatrix} \begin{bmatrix} \lambda(\alpha, z) & 0 \\ 0 & \overline{\lambda(\alpha, z)} \end{bmatrix}, \tag{3.1.34}$$

we have

$$\alpha z_2 - \alpha z_1 = {}^t\lambda(\alpha, z_1)^{-1}(z_2 - z_1)\lambda(\alpha, z_2)^{-1}. \tag{3.1.35}$$

Therefore,

$$d(\alpha z) = {}^t\lambda(\alpha, z)^{-1} \cdot dz \cdot \lambda(\alpha, z)^{-1}. \tag{3.1.36}$$

Since the jacobian of the map $z \mapsto \alpha z$ is $j(\alpha, z)^{-n+1}$, the differential form

$$\mathbf{d}z = \delta(z)^{-n+1} \prod_{h \le k} [(i/2) dz_{hk} \wedge d\overline{z}_{hk}], \tag{3.1.37}$$

is an invariant measure. If we have another realization $\mathcal{H}$ (e.g. $\mathfrak{Z}, \mathfrak{H}$) with identification $\rho : \mathcal{H} \to \mathcal{B}$, we then define $\mathbf{d}z := \mathbf{d}(\rho(z))$ with $z \in \mathcal{H}$ to be the differential form on $\mathcal{H}$. Clearly, this is also an invariant measure.

3.1.3 Symmetric spaces for Case III

We can simply extend the above discussions to Case III. The map $\mathfrak{i}$ in Example 2.1.1 induces the following isomorphism

$$\mathfrak{i} : G_v \xrightarrow{\sim} G_\infty := \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : {}^t g \Phi_\infty g = \Phi_\infty\} \tag{3.1.38}$$

with

$$\Phi_\infty = \begin{bmatrix} 0 & 0 & 0 & -1_{2m} \\ 0 & 0 & -1_r & 0 \\ 0 & 1_r & 0 & 0 \\ 1_{2m} & 0 & 0 & 0 \end{bmatrix}. \tag{3.1.39}$$

Put

$$\Omega = \{U \in \mathrm{Mat}_{2n,n}(\mathbb{C}) : -iU^* \phi_\infty U > 0,\ {}^t U \phi_\infty U = 0\}, \tag{3.1.40}$$

and define the symmetric space by

$$\begin{gathered} \mathfrak{Z} := \mathfrak{Z}_n := \mathfrak{Z}_{m,r} = \{z \in \mathrm{Mat}_n(\mathbb{C}) : U(z) \in \Omega\}, \\ U(z) = \begin{bmatrix} z \\ u_0 \end{bmatrix}, \qquad u_0 = \begin{bmatrix} 0 & 1_r \\ 1_{2m} & 0 \end{bmatrix}. \end{gathered} \tag{3.1.41}$$

Explicitly,

$$\mathfrak{Z}_{m,r} = \left\{ z = \begin{bmatrix} u & v \\ {}^t v & w \end{bmatrix} : \begin{array}{c} u \in \mathrm{Mat}_{2m}(\mathbb{C}), v \in \mathrm{Mat}_{2m,r}(\mathbb{C}), \\ w \in \mathrm{Mat}_r(\mathbb{C}),\ {}^t z = z, i(z^* - z) > 0. \end{array} \right\}. \tag{3.1.42}$$

Define the action and $\lambda(g,z), j(g,z), \eta(z_1,z_2), \delta(z_1,z_2)$ in a same fashion as in Case IV. Explicitly, for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty$ and $z, z_1, z_2 \in \mathfrak{Z}$ we have

$$\begin{aligned} gz &= (az + bu_0)(cz + du_0)^{-1} u_0, & \lambda(g,z) &= u_0^{-1}(cz + du_0), \\ \eta(z_1,z_2) &= i(z_1^* - z_2), & \delta(z_1,z_2) &= \det(\eta(z_1,z_2)), \\ \eta(z) &= \eta(z,z), & \delta(z) &= \delta(z,z). \end{aligned} \tag{3.1.43}$$

We also set $j(g,z) = \nu(\lambda(g,z))$. They satisfies

$$\begin{aligned} \lambda(g,z_1)^*\eta(gz_1,gz_2)\lambda(g,z_2) &= \eta(z_1,z_2), \\ \overline{j(g,z_1)}\delta(gz_1,gz_2)j(g,z_2) &= \delta(z_1,z_2), \\ \lambda(g,z)^*\eta(gz)\lambda(g,z) &= \eta(z_1,z_2), \\ \delta(gz) &= |j(g,z)|^{-2}\delta(z). \end{aligned} \tag{3.1.44}$$

Take $z_0 = i \cdot 1_n$ to be the origin of $\mathfrak{Z}$ and K_∞ the subgroup of G_∞ fixing z_0. Then $g \mapsto \lambda(g,z_0)$ gives an isomorphism $K_\infty \cong \mathrm{U}(n)$ and our symmetric space $\mathfrak{Z} \cong G_\infty/K_\infty$.

Obviously, G_∞ is further isomorphic to the symplectic group

$$\mathrm{Sp}(2n,\mathbb{R}) := \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : {}^t g J_n g = J_n\}, \tag{3.1.45}$$

which acts on the usual Siegel upper half space

$$\mathfrak{H} = \mathfrak{H}_n = \{z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = z, i(z^* - z) > 0\}. \tag{3.1.46}$$

Furthermore, by the Caylay transform $\mathfrak{H}$ can be identified with a bounded domain

$$\mathfrak{B} = \mathfrak{B}_n = \{z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = z, zz^* < 1_n\}. \tag{3.1.47}$$

An invariant differential form on $\mathfrak{B}$ can be given by

$$\mathbf{d}z = \delta(z)^{-n-1} \prod_{h \le k} [(i/2)dz_{hk} \wedge d\overline{z}_{hk}]. \tag{3.1.48}$$

For other realizations $\mathcal{H}$ (e.g. $\mathfrak{Z}, \mathfrak{H}$) with identification $\rho : \mathcal{H} \to \mathfrak{Z}$ we then define $\mathbf{d}z := \mathbf{d}(\rho(z))$ with $z \in \mathcal{H}$ to be the invariant differential form on $\mathcal{H}$.

3.1.4 Symmetric spaces for the case $r = 0$

We end this section by considering the special case $r = 0$. In this special case, we can give a simpler and unified definition of symmetric spaces for groups in all four cases (see also [Jin22, Section 2.2]).

Let S_m be the additive algebraic group defined by

$$S_m(F) = \{\beta \in \mathrm{GL}_m(D) : \beta^* = -\epsilon\beta\}. \tag{3.1.49}$$

For an archimedean place v, denote $S^+_{m,v}$ for the subgroup of $S_{m,v}$ containing positive definite matrices. We define the symmetric space

$$\mathcal{H}_m := \{z = x + iy \in S_{m,v} \otimes \mathbb{C} : x \in S_{m,v}, y \in S^+_{m,v}\}. \tag{3.1.50}$$

The central point is chosen as $z_0 = i \cdot 1_m$. The action of $G_{m,0}$ on $\mathcal{H}_m$ is defined as

$$g.z = (az + b)(cz + d)^{-1} \text{ for } z \in \mathcal{H}_m, g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{m,0} \tag{3.1.51}$$

and $j(g, z) = \nu(cz+d)$. Recall that for archimedean places v, $D_v = \mathrm{Mat}_2(\mathbb{R})$ in Case III and $D_v = \mathbb{H}$ can be embedded into $\mathrm{Mat}_2(\mathbb{R})$ in Case IV. Then above space $\mathcal{H}_m$ can be embedded into $\mathrm{Mat}_{2m}(\mathbb{C})$ and one can show that $\mathcal{H}_m \cong \mathfrak{Z}_{m,0}$ is the symmetric spaces defined in above two subsections. In particular, the space $\mathcal{H}_m$ in Case III is same as the one for Case II with index $2m$.

3.2 Definition of modular forms

We review the definition of modular forms in this section. Both classical and adelic definitions are given and their relations are well known.

3.2.1 Modular forms on symmetric spaces

For $\gamma \in G(F)$, we naturally view it as an element of $G(\mathbb{A})$ and its action on $\mathfrak{Z}^d_{m,r}$ is given by $\gamma.z := (\gamma_v.z_v)_{v|\infty}$ for $z = (z_v)_{v|\infty} \in \mathfrak{Z}^d_{m,r}$. Fix a weight $\boldsymbol{l} = (l_v)_{v|\infty}$ with $l_v \in \mathbb{N}$ and denote

$$j(\gamma, z)^{\boldsymbol{l}} = \prod_{v|\infty} j(\gamma_v, z_v)^{l_v}. \tag{3.2.1}$$

Definition 3.2.1. A holomorphic function $\varphi : \mathfrak{Z}^d_{m,r} \to \mathbb{C}$ is called a modular form for a congruence subgroup $\Gamma \subset G(F)$ and weight $\boldsymbol{l}$ if for all $\gamma \in \Gamma$,

$$\varphi(\gamma.z) = j(\gamma, z)^{\boldsymbol{l}}\varphi(z), \qquad z = (z_v)_{v|\infty}. \tag{3.2.2}$$

Remark 3.2.2.

(1) In this work, we use the term 'modular form' as an analogue for the modular forms of GL_2 so in particular we only consider the holomorphic functions.

(2) When $m = 1$ we need further assume φ satisfies the cusp condition which is not necessary for $m \geq 2$ due the the Koecher principle [Kri85, Lemma 1.5] and [Shi00, Proposition 5.7].

(3) We are restrict ourselves to certain scalar weight modular forms. Also in unitary case, with notations in [Shi00, Section 5], there are indeed two automorphy factors $\lambda(g, z), \mu(g, z)$ and

$$j(g, z)^l = \nu(\mu(g, z))^{l_1}\nu(\lambda(g, z))^{l_2}, \text{ with } l = (l_1, l_2).$$

Here for simplicity we only consider $l = l_1, l_2 = 0$ to make our discussions consistent in all cases.

(4) Here we are using the realization $(G_\infty, \mathfrak{Z}_{m,r})$ for our symmetric space. In fact, the definition is independent of the choice of realizations in following sense. If we choose another realization $\mathcal{H}$ (e.g. $\mathfrak{H}_n, \mathfrak{B}_n$) with identification $\rho : \mathcal{H} \to \mathfrak{Z}_{m,r}$. Then with notation as in Equation (3.1.16), to a function $\varphi : \mathfrak{Z}^d_{m,r} \to \mathbb{C}$ we associate a function g on $\mathcal{H}^d$ by setting $\varphi'(z) = \prod_{v|\infty} \nu(\mu(z_v))^{-l_v}\varphi(\rho(z))$. Then $\varphi : \mathfrak{Z}^d_{m,r} \to \mathbb{C}$ is a modular form if and only if $\varphi' : \mathcal{H}^d \to \mathbb{C}$ is a modular form.

Denote $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$. We rephrase above definitions for functions $\phi : G(F_\infty) \to \mathbb{C}$. Set

$$\phi(g) = j(g_v, z_0)^{-\boldsymbol{l}}\varphi((g_v.z_0)_{v|\infty}). \tag{3.2.3}$$

Then clearly $\phi(gk) = \prod_{v|\infty} j(k_v, z_0)^{-l_v}\phi(g)$ for $k \in K$. We call ϕ a cusp form if

$$\int_{U(\mathbb{R})} \phi(ug)du = 0 \tag{3.2.4}$$

for every unipotent radical U of all proper parabolic subgroup of G.

We denote $M_{\boldsymbol{l}}^{m,r}(\Gamma)$ and $S_{\boldsymbol{l}}^{m,r}(\Gamma)$ for the space of modular forms and space of cusp forms. We use the superscript m, r to indicate their dependence on the group $G_{m,r}$ and omit the superscript for simplicity if the group G is clear from the context.

Let $\mathbf{d}z$ be the invariant differential form on $\mathfrak{Z}_{m,r}$. For two modular forms $\varphi_1, \varphi_2 \in M_{\boldsymbol{l}}(\Gamma)$, we define the Petersson inner product by

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\Gamma\backslash\mathfrak{Z}} \varphi_1(z)\overline{\varphi_2(z)}\delta(z)^{\boldsymbol{l}}\mathbf{d}z, \tag{3.2.5}$$

whenever the integral converges. For example, this is well defined when one of φ_1, φ_2 is a cusp form.

3.2.2 Adelic modular forms

Denote K_∞ be the maximal compact subgroup of $G(F_\infty) = \prod_{v|\infty} G(\mathbb{R})$ and K be any open compact subgroup of $\prod_{v\nmid\infty} G(F_v)$. Fix a weight $\boldsymbol{l} = (l_v)_{v|\infty}$ with $l_v \in \mathbb{N}$ as before.

Recall the following weak approximation of G

$$G(\mathbb{A}) = \coprod_i G(F)t_i K G(F_\infty). \tag{3.2.6}$$

For a function $\boldsymbol{f} : G(\mathbb{A}) \to \mathbb{C}$, we can associate a series of functions ϕ_i on $G(F_\infty)$ for each i defined by

$$\phi_i(g_\infty) = \boldsymbol{f}(t_i g_\infty) \qquad g_\infty \in G(F_\infty). \tag{3.2.7}$$

Definition 3.2.3. The space of weight $\boldsymbol{l}$ and level K (adelic) modular forms $\mathcal{M}_{\boldsymbol{l}}(K)$ contain functions $\boldsymbol{f} : G(\mathbb{A}) \to \mathbb{C}$ satisfying:

(1) $\boldsymbol{f}$ is left invariant under $G(F)$ and right invariant under K, i.e.

$$\boldsymbol{f}(\gamma g k) = \boldsymbol{f}(g) \text{ for } \gamma \in G(F), k \in K, \tag{3.2.8}$$

(2) The functions ϕ_i associated to $\boldsymbol{f}$ defined as above are weight $\boldsymbol{l}$ defined as in

Section 3.2.1. Especially,

$$\boldsymbol{f}(gk_\infty) = \prod_{v|\infty} j(k_v, z_0)^{l_v} \boldsymbol{f}(g) \text{ for } k_\infty = (k_v)_{v|\infty} \in K_\infty. \tag{3.2.9}$$

Furthermore, the subspace $\mathcal{S}_{\boldsymbol{l}}(K)$ of cusp forms consisting functions $\boldsymbol{f} \in \mathcal{M}_{\boldsymbol{l}}(K)$ satisfying

$$\int_{U(F)\backslash U(\mathbb{A})} \boldsymbol{f}(ug)du = 0, \tag{3.2.10}$$

for all unipotent radicals U of all proper parabolic subgroups of G. Equivalently, $\boldsymbol{f}$ is a cusp form if and only if all ϕ_i are cusp forms defined in Section 3.2.1. We may write $\mathcal{M}_{\boldsymbol{l}}^{m,r}(K)$ and $\mathcal{S}_{\boldsymbol{l}}^{m,r}(K)$ if we want to emphasize the index m, r.

The classical definition of modular forms in Section 3.2.1 and the above adelic definition are related by bijections

$$\mathcal{M}_{\boldsymbol{l}}(K) \cong \bigoplus_i M_{\boldsymbol{l}}(\Gamma_i), \qquad \mathcal{S}_{\boldsymbol{l}}(K) \cong \bigoplus_i S_{\boldsymbol{l}}(\Gamma_i), \tag{3.2.11}$$

given by the correspondence $\boldsymbol{f} \leftrightarrow \{\phi_i\}$ and $\Gamma_i = t_i K t_i^{-1} \cap G(F)$.

For two modular forms $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathcal{M}_{\boldsymbol{l}}(K)$, we define the Petersson inner product

$$\langle \boldsymbol{f}_1, \boldsymbol{f}_2 \rangle = \int_{G(F)\backslash G(\mathbb{A})/KK_\infty} \boldsymbol{f}_1(g)\overline{\boldsymbol{f}_2(g)}\mathrm{d}g, \tag{3.2.12}$$

whenever the integral converges. For example, this is well defined when one of $\boldsymbol{f}_1, \boldsymbol{f}_2$ is a cusp form. Here $\mathbf{d}g = \prod_v \mathbf{d}g_v$ is an invariant differential form of $G(\mathbb{A})$ given such that:

(1) for each nonarchimedean place v, $\mathbf{d}g_v$ is normalized such that the volume K_v is 1,

(2) for each archimedean place v, $\mathbf{d}g_v = \mathbf{d}(g_v z_0)$ with $\mathbf{d}(g_v z_0)$ an invariant differential form of $\mathfrak{Z}_{m,r}$.

3.2.3 The special case $r = 0$

It is well known that the modular forms defined in previous two subsections have a Fourier-Jacobi expansion (see for example [Shi97, Appendix 4]). When $r = 0$, it has a Fourier expansion of easier form which we are going to recall now.

Let $\varphi : \mathcal{H}_m^d \to \mathbb{C}$ be a modular form in $M_{\boldsymbol{l}}(\Gamma)$ with $\mathcal{H}_m$ defined in (3.1.50). Then for any $\gamma \in G$, φ has a Fourier expansion of the form

$$j(\gamma, z)^{-\boldsymbol{l}}\varphi(\gamma.z) = \sum_{\beta \in S_m(F)} c(\beta; \varphi, \gamma) \prod_{v|\infty} e^{2\pi i \tau(\beta z_v)}. \tag{3.2.13}$$

We call $c(\beta; \varphi, \gamma)$ the Fourier coefficients of φ and denote $c(\beta; \varphi) := c(\beta; \varphi, 1)$ for simplicity. We always have $c(\beta; \varphi, \gamma) = 0$ unless β is non-negative. In addition, φ is a cusp form if and only if for all γ, $c(\beta; \varphi, \gamma) = 0$ unless β is positive definite.

We can also reformulate above Fourier expansion in adelic language (see for example [Bou21, Proposition 2.4]). Let $e_{\mathbb{A}} = \prod_v e_v$ be the standard additive character of $\mathbb{A}$. That is $e_v(x) = e^{2\pi i x}$ for archimedean places v and $e_v(\varphi_v) = \prod_{v|\infty} e_v(-q_v^{-1})$ with φ_v the uniformizer of F_v and $|\varphi_v|_v = q_v^{-1}$. Also set $e_\infty = \prod_{v|\infty} e_v$. Let $\boldsymbol{f} : G(\mathbb{A}) \to \mathbb{C}$ be an (adelic) modular form in $\mathcal{M}_{\boldsymbol{l}}(K)$. Then for all $y \in \mathrm{GL}_n(\mathbb{A}_D)$ and $x \in S_m(\mathbb{A})$ we have a Fourier expansion of the form

$$\boldsymbol{f}\left(\begin{bmatrix} y & x\hat{y} \\ 0 & \hat{y} \end{bmatrix}\right) = \prod_{v|\infty} \nu(y_v^*)^{l_v} \cdot \sum_{\beta \in S_m(F)} \mathbf{c}(\beta; \boldsymbol{f}, y) e_\infty(\tau(iy^*\beta y)) e_{\mathbb{A}}(\tau(\beta x)). \tag{3.2.14}$$

We call $\mathbf{c}(\beta; \boldsymbol{f}, y)$ the Fourier coefficients of $\boldsymbol{f}$ and they have following properties:

(1) $\mathbf{c}(\beta; \boldsymbol{f}, y) = 0$ unless β is non-negative and $\prod_{v\nmid\infty} e_v(\tau(y^*\beta y x)) = 1$ for any $x \in S_m(\mathcal{O})$,

(2) $\mathbf{c}(\beta; \boldsymbol{f}, y) = \mathbf{c}(\beta; \boldsymbol{f}, \prod_{v\nmid\infty} y_v)$,

(3) $\mathbf{c}(b^*\beta b; \boldsymbol{f}, y) = \prod_{v|\infty} \nu(b^*)^{l_v} \mathbf{c}(\beta; \boldsymbol{f}, by)$ for any $b \in \mathrm{GL}_n(D)$,

(4) $\mathbf{c}(\beta; \boldsymbol{f}, yk) = \mathbf{c}(\beta; \boldsymbol{f}, y)$ for any $k \in \prod_{v\nmid\infty} \mathrm{GL}_n(\mathcal{O}_v)$,

(5) $\boldsymbol{f}$ is a cusp form if and only if for all y, $\mathbf{c}(\beta; \boldsymbol{f}, y) = 0$ unless β is positive definite.

3.3 Algebraic modular forms

In order to move from the analytic considerations discussed so far to algebraic questions, we need to discuss the notion of algebraic modular forms in our setting. For holomorphic modular forms on hermitian symmetric spaces, the notion of algebraic modular forms is well understood. There are mainly four characterizations of algebraic modular forms:

(1) In [Har85; Har86; Mil90], automorphic forms are interpreted as sections of certain automorphic vector bundles on Shimura varieties. The canonical model of Shimura varieties and automorphic vector bundles then define a subspace of algebraic automorphic forms.

(2) In [BJar; Gar77; Gar84a; Shi00], algebraic modular forms are defined via CM points.

(3) In [Gar81; Gar83; Gar84a], a characterization using Fourier-Jacobi expansion is given. In particular, in the special case $r = 0$, the modular forms have Fourier expansions and the algebraic modular forms are defined to be the one have algebraic Fourier coefficients. This generalizes the classical definition of algebraic modular forms of GL_2.

(4) In [Gar84a], there is yet another characterization using the pullback to classical modular forms over GL_2. Moreover, three definitions (2,3,4) are also proved to be equivalent there.

3.3.1 CM points

We will mainly define the algebraic modular forms via CM points. The symplectic and unitary groups are well studied in [Shi00]. The quaternionic orthogonal groups are considered in [Gar77]. In the following, we reviewed the CM points for quaternionic unitary groups discussed in [BJar]. Our approach for defining CM points and the underlying periods follows the idea of [Gar77; Shi67], where one "tensors" a given embedding $h : K_1 \times \ldots \times K_n \hookrightarrow G$, of CM fields K_i, with another CM field K, disjoint to the K_i's to obtain a fixed point whose associated abelian variety is of CM type (see also [Del71, proof of Theorem 6.4]). In this way we will be able to define and study the CM points in our case by considering an embedding of our group into a unitary group, after a choice of an imaginary quadratic field. However we show that our definition of CM points and the attached periods are independent of the choice of the auxiliary imaginary quadratic field.

Let (D,ρ) of type (b) with $D_v = \mathbb{H}$ for any archimedean place v. Consider the algebraic group

$$G(T) := \{g \in \mathrm{GL}_n(D) : gTg^* = T\}, \tag{3.3.1}$$

with $T^* = -T \in \mathrm{GL}_n(D)$ a skew-hermitian matrix. We assume $T = \mathrm{diag}[a_1, ..., a_n]$ is diagonal for simplicity. Diagonalizing Φ in (3.0.1), the group $G(\Phi)$ considered before is isomorphic to some $G(T)$.

We introduce the notion of CM points as [Shi00, Section 4.11]. Take a CM algebra $Y = K_1 \times ... \times K_n$ with each K_i are totally imaginary quadratic extension of F. Set $Y^1 = \{y \in Y : yy^\iota = 1\}$ with ι induced by the nontrivial involution (i.e. complex conjugations) on each K_i. Suppose there is an embedding $h : Y^1 \to G(T)$, then clearly $h(Y^1) \subset G(T,\mathbb{R})$ and $(Y^1 \otimes_F \mathbb{R})^\times$ is a compact subgroup of $G(T,\mathbb{R})$ and hence $h(Y^1)$ has a common fixed point in the hermitian symmetric space (with realization associated to $G(T,\mathbb{R})$). We call a point obtained as such fixed points a CM point. The existence of CM points can be easily shown by constructing an embedding $h : Y^1 \to G(T)$ as follows. Take totally imaginary quadratic extension $K_i := F(a_i)$ for $i = 1, ..., n$ and consider the CM algebra $Y = K_1 \times ... \times K_n$. We can define the embedding

$$h : Y^1 \to G(T), \qquad (y_1, ..., y_n) \mapsto \mathrm{diag}[y_1, ..., y_n]. \tag{3.3.2}$$

We then choose an imaginary quadratic field K which is different from the K_i's above, and splits D, i.e. $D \otimes_F K \cong \mathrm{Mat}_2(K)$. It is easy to see such a field K always exists. Fix an embedding $\mathrm{Mat}_n(D) \to \mathrm{Mat}_{2n}(K)$. Denote the image of T in $\mathrm{Mat}_{2n}(K)$ by $\mathcal{T}$ and define the unitary group

$$U(\mathcal{T}) := \{g \in \mathrm{GL}_{2n}(K) : g\mathcal{T}g^* = \mathcal{T}\}. \tag{3.3.3}$$

We note that

$$U(\mathcal{T},\mathbb{R}) \cong \left\{ g \in \mathrm{GL}_{2n}(\mathbb{C}) : g \begin{bmatrix} i \cdot 1_n & 0 \\ 0 & -i \cdot 1_n \end{bmatrix} g^* = \begin{bmatrix} i \cdot 1_n & 0 \\ 0 & -i \cdot 1_n \end{bmatrix} \right\}. \tag{3.3.4}$$

Its action on the bounded domain (see for example [Shi00])

$$\mathcal{B} = \{z \in M_n(\mathbb{C}) : 1 - z^* z > 0\}, \tag{3.3.5}$$

is defined by $gz = (az+b)(cz+d)^{-1}$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The two factors of automorphy are given by $\lambda(g,z) = \overline{c}\,{}^t z + \overline{d}$, and $\mu(g,z) = cz + d$. The embedding $\mathrm{Mat}_n(D) \to \mathrm{Mat}_{2n}(K)$ induces an embedding $\mathfrak{i} : G(T) \to U(\mathcal{T})$ which is compatible with natural inclusion $\iota : \mathfrak{B} \to \mathcal{B}$. Here recall that $\mathfrak{B}$ is the bounded realization given in (3.1.29). We will view $G(T)$ (resp. $\mathfrak{B}$) as a subgroup (resp. subspace) of $U(\mathcal{T})$ (resp. $\mathcal{B}$) under this embedding.

Lemma 3.3.1.

(1) Y is spanned by Y^1 over F. In particular there exists an element $\beta \in Y^1$ such that $Y = F[\beta]$ and $\beta_1, \ldots, \beta_n, \beta_1^\iota, \ldots, \beta_n^\iota$ are pairwise distinct.

(2) There is a unique $w \in \mathfrak{B}$ which is a common fixed point for $h(Y^1)$.

Proof. The first part can be shown exactly as [Shi00, Lemma 4.12], and for the second part we adapt an idea of the proof of that lemma. Without loss of generality we can assume that the origin 0 of $\mathfrak{B}$ is a fixed point for $h(Y^1)$ and our task is to show that it is the unique fixed point. We note that the maximal compact subgroup in $G(\Phi,\mathbb{R})$ fixing the origin is isomorphic to $\mathrm{U}(n) := \{g \in \mathrm{GL}_n(\mathbb{C}) : gg^* = 1_n\}$, and hence with respect to the embedding $G(T,\mathbb{R}) \hookrightarrow U(\mathcal{T},\mathbb{R})$ we have that $\mathrm{U}(n) \hookrightarrow \mathrm{U}(n) \times \mathrm{U}(n)$ embeds diagonally, i.e. $a \mapsto (a, \overline{a})$. In particular we have an embedding $h(Y^1) \hookrightarrow \mathrm{U}(n) \hookrightarrow \mathrm{U}(n) \times \mathrm{U}(n)$. Assume now there is another point $z \in \mathfrak{B}$ which is a fixed point of $h(Y^1)$. Then we must have that $z = az\overline{a}^{-1}$ for every element $\mathrm{diag}[a, \bar{a}] \in (\mathrm{U}(n) \times \mathrm{U}(n)) \cap h(Y^1)$. But for such a point we have that $a^* a = 1$ and hence $\overline{a}^{-1} = {}^t a$. That is $z = az\,{}^t a$. Diagnolize a and assume a has eigenvalues λ_i,

$i = 1, \ldots, 2n$ then we must have $z_{ij} = 0$ for every $\lambda_i \neq \lambda_j$. Taking a to be the element obtained from β above we have that z has to be the origin. $\square$

We are going to attach CM periods to our CM points. Here we employ the idea of [Shi67] (see also [Shi79, Section 7]) to relate our CM points to the CM points of unitary groups.

Let $w \in \mathfrak{B} \subset \mathcal{B}$ be a CM point fixed by $h(Y^1) \subset G(T) \subset U(\mathcal{T})$. Then for such a point we have that

$$\Lambda(\alpha, w)p(x, w) = p(x\alpha, w), \quad \alpha \in h(Y^1), \;\; x \in \mathbb{C}^{2n} \tag{3.3.6}$$

where $\Lambda(\alpha, w) \in \mathrm{GL}_{2n}(\mathbb{C})$ and $p(x, z) : \mathbb{C}^{2n} \times \mathcal{B} \to \mathbb{C}^{2n}$ are the maps defined in [Shi00, Section 4.7]. In this way we can obtain an embedding $Y \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2n})$ by sending $\alpha \mapsto \Lambda(\alpha, w)$ where we have used the fact that Y is spanned by Y^1 over F. We now extend this to an injection h of $K \otimes_F Y \cong \mathcal{S} := \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ into $\mathrm{End}(\mathbb{C}^{2n})$ where $\mathcal{S}_i = KK_i$. Indeed we set

$$h(\beta \otimes \alpha)p(x, w) = p(\beta x\alpha, w) = p(x\beta\alpha, w) = p(x\alpha\beta, w). \tag{3.3.7}$$

That is, the point w can be seen as a fixed point of $\mathcal{S}^1 \otimes_F \mathbb{R}$ where $\mathcal{S}^1 = \{s \in \mathcal{S} \mid ss^{\iota} = 1\}$ with ι the involution on $\mathcal{S}$ induced by the complex conjugation on KK_i. Hence w is a CM point in $\mathcal{B}$ defined in [Shi00, Section 4.11] for unitary groups. In particular, w has entries in F by [Shi00, Lemma 4.13].

Remark 3.3.2. Following [Shi00, Section 4], let $\Omega = \{K, \Psi, L, \mathcal{T}, \{u_i\}_{i=1}^{s}\}$ be a PEL-type and $\mathcal{F}(\Omega)$ family of polarised abelian varieties of PEL-type. The abelian varieties in $\mathcal{F}(\Omega)$ are parameterised by $\mathcal{B}$. More precisely, there is a bijection

$$\Gamma\backslash\mathcal{B} \xrightarrow{\sim} \mathcal{F}(\Omega), \quad \Gamma = \{\gamma \in U(\mathcal{T}) : L\gamma = L, u_i\gamma - u_i \in L\}.$$

As in [Shi63], we can define $\Omega' = \{\mathbb{B}, \Psi', L, T, \{u_i\}_{i=1}^{s}\}$ for quaternions and $\mathcal{F}(\Omega')$ are parameterized by $\mathfrak{B}$. The natural inclusion $\mathcal{F}(\Omega') \to \mathcal{F}(\Omega)$ is compatible with $\mathfrak{B} \to \mathcal{B}$. Moreover, similar to [Gar84a; Shi67] we actually have an embedding of canonical models between $\Gamma\backslash\mathfrak{B}$ and $\Gamma'\backslash\mathcal{B}$ for certain congruence subgroups Γ, Γ'.

As we have remarked, CM points for unitary groups have been extensively studied in [Shi00, Chapter II]. We recall some of their properties. For $\alpha \in \mathcal{S}^1$ we put $\psi(\alpha) := \lambda(h(\alpha), w) \in \mathrm{GL}_n(\mathbb{C})$, $\phi(\alpha) := \mu(h(\alpha), w) \in \mathrm{GL}_n(\mathbb{C})$, and $\Phi(\alpha) = \mathrm{diag}[\psi(\alpha), \phi(\alpha)] \in \mathrm{GL}_{2n}(\mathbb{C})$. We can then find $B, C \in \mathrm{GL}_n(\overline{\mathbb{Q}})$ (see [Shi00, page 78]) such that for all $\alpha \in \mathcal{S}$

$$\begin{aligned} B\psi(\alpha)B^{-1} &= \mathrm{diag}[\psi_1(\alpha), \dots, \psi_n(\alpha)], \\ C\phi(\alpha)C^{-1} &= \mathrm{diag}[\phi_1(\alpha), \dots, \phi_n(\alpha)], \end{aligned} \tag{3.3.8}$$

for some ring homomorphism $\phi_i, \psi_i : \mathcal{S} \to \mathbb{C}$, where we have F-linearly extended ψ and ϕ from $\mathcal{S}^1$ to $\mathcal{S}$. We set

$$\begin{aligned} \mathfrak{p}_{\infty}(w) &:= C^{-1}\mathrm{diag}[p_{\mathcal{S}}(\phi_1, \Phi), \dots, p_{\mathcal{S}}(\phi_n, \Phi)]C \in \mathrm{GL}_n(\mathbb{C}), \\ \mathfrak{p}_{\infty\iota}(w) &:= B^{-1}\mathrm{diag}[p_{\mathcal{S}}(\psi_1, \Phi), \dots, p_{\mathcal{S}}(\psi_n, \Phi)]B \in \mathrm{GL}_n(\mathbb{C}), \end{aligned} \tag{3.3.9}$$

where the CM-periods $p_{\mathcal{S}}(\psi_i, \Phi) \in \mathbb{C}^\times$ and $p_{\mathcal{S}}(\phi_i, \Phi) \in \mathbb{C}^\times$ are defined as in [Shi00, page 78]. Actually we should remark here that the periods $p_{\mathcal{S}}(\psi_i, \Phi)$, $p_{\mathcal{S}}(\phi_i, \Phi)$ are uniquely determined up to elements in $\overline{\mathbb{Q}}^\times$, but this is sufficient for our applications.

We now use the fact that $w \in \mathfrak{B} \subset \mathcal{B}$ is a CM point for both (Y, h) and also for $(\mathcal{S}, h)$. Note that $\psi(\alpha) = \phi(\alpha)$ for $\alpha \in Y^1 \subset \mathcal{S}^1$. Indeed, for $\alpha \in G(T, \mathbb{R})$ we have (see [Shi67, (2.18.9)]), $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix}$ and hence especially we have $\lambda(\alpha, z) = \mu(\alpha, z)$ since ${}^t z = -z$. In particular the values $\psi(\alpha) = \phi(\alpha) = \lambda(\alpha, w) = \mu(\alpha, w)$ for $\alpha \in Y^1$, that is the restrictions of ϕ and ψ to Y^1 are independent of the choice of the field K. Furthermore we note that $\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in K$ with $\alpha\bar{\alpha} = 1$ seen as elements of $U(\mathcal{T})$ i.e. $\alpha 1_{2n} \in U(\mathcal{T})$.

In the following lemma we use the notation $I_Y, J_Y, J_{\mathcal{S}_j}$ as defined in [Shi00, page 77].

Lemma 3.3.3. *With notation as above, for all $1 \leq i \leq n$, we have that*

$$p_{\mathcal{S}}(\psi_i, \Phi) = p_Y(Res_{\mathcal{S}/Y}(\psi_i), \Phi') = p_Y(Res_{\mathcal{S}/Y}(\phi_i), \Phi') = p_{\mathcal{S}}(\phi_i, \Phi),$$

where $\Phi' = Res_{\mathcal{S}/Y}\phi = Res_{\mathcal{S}/Y}\psi \in I_Y$.

Proof. Let us write $\Phi = \sum_{j=1}^{n} \Phi_j$ with $\Phi_j \in I_{\mathcal{S}_j}$ and $\Phi' = \sum_{j=1}^{n} \Phi'_j$, with $\Phi'_j \in I_{K_j}$. Then we have that $\Phi_j = \mathrm{Inf}_{\mathcal{S}_j/K_j}(\Phi'_j)$. Indeed first we observe that $\Psi = \sum_{j=1}^{n} \mathrm{Res}_{\mathcal{S}_j/K}\Phi_j \in I_K$ (see [Shi00, page 85]), where Ψ as in the Remark 3.3.2 above. Moreover we know that $\Phi = \phi + \psi$ with $\phi, \psi \in I_{\mathcal{S}}$ as above and we have seen that $\psi = \overline{\phi}$ when restricted to K via $K \hookrightarrow Y \otimes_F K = \mathcal{S}$. But on the other hand we have seen that $\psi = \phi$ when restricted to Y, from which we obtain that $\Phi_j = \Phi'_j \otimes \tau + \Phi'_j \otimes \overline{\tau}$, where τ a fixed embedding of $K \hookrightarrow \mathbb{C}$ (i.e. a CM type for K). Since $\mathcal{S}_j = K_j \otimes_F K$ the claim that $\Phi_j = \mathrm{Inf}_{\mathcal{S}_j/K_j}(\Phi'_j)$ now follows.

The statement of the Lemma is now obtained from the inflation-restriction properties of the periods (see [Shi00, page 84]):

$$p_{\mathcal{S}}(\psi_i, \Phi) = \prod_{j=1}^{n} p_{\mathcal{S}_j}(\psi_{ij}, \Phi_j) = \prod_{j=1}^{n} p_{K_j}(\mathrm{Res}_{\mathcal{S}_j/K_j}(\psi_{ij}), \Phi'_j) = p_Y(\mathrm{Res}_{\mathcal{S}/Y}(\psi_i), \Phi'),$$

where $\psi_{ij} \in J_{\mathcal{S}_j}$ induced by $\psi_i \in J_{\mathcal{S}} = \bigcup_{j=1}^{n} J_{\mathcal{S}_j}$. Other equality follow similarly. $\square$

The above lemma shows that we have $\mathfrak{p}_\infty(w) = \mathfrak{p}_{\infty\iota}(w)$ for $w \in \mathfrak{B}$ and they are independent of the choice of the imaginary quadratic field K we chose above (and hence of the embedding to the unitary group). We then simply define $\mathfrak{p}(w) = \mathfrak{p}_\infty(w) = \mathfrak{p}_{\infty\iota}(w)$ for the period attached to CM point $w \in \mathfrak{B}$. By [Shi00, Proposition 11.5] and the definition of periods we immediately have

(1) The coset $\mathfrak{p}(w)\mathrm{GL}_n(\overline{\mathbb{Q}})$ is determined by the point $w \in \mathfrak{B}$ independently of the embedding (Y, h) chosen above,

(2) $\mathfrak{p}(\gamma w)\mathrm{GL}_n(\overline{\mathbb{Q}}) = \lambda(\gamma, w)\mathfrak{p}(w)\mathrm{GL}_n(\overline{\mathbb{Q}})$ for all $\gamma \in G(T)$.

3.3.2 Definition and properties of algebraic modular forms

Let $\mathfrak{B}$ be the bounded realization of the symmetric space associated to $G(\mathbb{R})$ and denote $\mathcal{W}$ be a set of CM points which is dense in $\mathfrak{B}$. For a fixed integer l, set $\mathfrak{P}_l(w)$ be the CM period associated to $w \in \mathcal{W}$. It depends only on w and l. The definition of $\mathfrak{P}_l(w)$ can be found in [Gar77; Shi00] for Case II, III, V. For Case IV, with the CM period $\mathfrak{p}(w)$ defined in Section 3.3.1, we define $\mathfrak{P}_l(w) = \nu(\mathfrak{p}(w))^l$. Set $\mathfrak{P}_{\boldsymbol{l}}(w) = \prod_{v|\infty} \mathfrak{P}_{l_v}(w_v)$ for $w = (w_v)_{v|\infty} \in \mathcal{W}^d$ and $\boldsymbol{l} = (l_v)_{v|\infty}$ the fixed weight.

Definition 3.3.4. The subspace

$$M_{\boldsymbol{l}}(\Gamma, \overline{\mathbb{Q}}) \subset M_{\boldsymbol{l}}(\Gamma), \quad \text{resp. } S_{\boldsymbol{l}}(\Gamma, \overline{\mathbb{Q}}) \subset S_{\boldsymbol{l}}(\Gamma), \tag{3.3.10}$$

of algebraic modular forms (resp. algebraic cusp forms) consisting functions $\varphi : \mathfrak{B}^d \to \mathbb{C}$ such that $\varphi(w) \in \mathfrak{P}_{\boldsymbol{l}}(w)\overline{\mathbb{Q}}$ for any CM points $w \in \mathcal{W}^d$.

Remark 3.3.5. Note that here we are defining the algebraicity of modular forms using the bounded realization $\mathfrak{B}$. This definition is indeed independent of the choice of realizations. Suppose we are given another realization $\mathcal{H}$ (e.g. $\mathfrak{H}_n, \mathfrak{Z}_{m,r}$ in Section 3.1) with identification $\rho : \mathcal{H} \to \mathcal{B}$. A point $z \in \mathcal{H}$ is called a CM point if $\rho(z) \in \mathfrak{B}$ is a CM point and we also set $\mathfrak{P}_l(z) = \mathfrak{P}_l(\rho(z))$. A modular form $\varphi' : \mathcal{H}^d \to \mathbb{C}$ is called algebraic if $\varphi'(z) \in \mathfrak{P}_{\boldsymbol{l}}(z)\overline{\mathbb{Q}}$ for all $z \in \rho^{-1}(\mathcal{W}^d)$. As in (4) of Remark 3.2.2, there are one to one correspondence between modular form $\varphi : \mathfrak{B}^d \to \mathbb{C}$ and modular form $\varphi' : \mathcal{H}^d \to \mathbb{C}$. Then clearly φ is algebraic if and only if φ' is algebraic.

The important properties of algebraic modular forms are collected in the following proposition.

Proposition 3.3.6.

(1) There is a basis of $M_{\boldsymbol{l}}(\Gamma)$ consisting of algebraic modular forms. That is $M_{\boldsymbol{l}}(\Gamma) = M_{\boldsymbol{l}}(\Gamma, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. In particular, there is a well defined action of $\sigma \in \operatorname{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ on $M_{\boldsymbol{l}}(\Gamma)$ by acting on $\mathbb{C}$.

(2) We have $S_{\boldsymbol{l}}(\Gamma, \overline{\mathbb{Q}})^\sigma = S_{\boldsymbol{l}}(\Gamma, \overline{\mathbb{Q}})$ and $S_{\boldsymbol{l}}(\Gamma) = S_{\boldsymbol{l}}(\Gamma, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

(3) Assume $r = 0$ and $\varphi \in M_{\boldsymbol{l}}(\Gamma)$ has Fourier expansion

$$\varphi(z) = \sum_{\beta \in S_m(F)} c(\beta; \varphi) e^{2\pi i \tau(\beta x)} \tag{3.3.11}$$

as in (3.2.13). Then the action of $\sigma \in \operatorname{Aut}(\mathbb{C})$ on φ given by

$$\varphi^\sigma(z) = \sum_{\beta \in S_m(F)} c(\beta; \varphi)^\sigma e^{2\pi i \tau(\beta z)} \tag{3.3.12}$$

is well defined. In particular, we have $\varphi \in M_l(\Gamma, \overline{\mathbb{Q}})$ *if and only if* $c(\beta;\varphi) \in \overline{\mathbb{Q}}$.

Proof. This is proved for symplectic groups and unitary groups in [Shi00]. Other cases can be proved similarly. See also [Gar77; Gar84a; Mil90]. The proof for quaternionic unitary groups is also sketched in [BJar, Proposition 5.6]. □

We now reformulate above definition of algebraic modular forms in adelic language. An element $g \in G(\mathbb{A})$ is called a CM point if $g_v \cdot 0 \in \mathfrak{B}$ is a CM point for any archimedean place v of F with 0 the central point of $\mathfrak{B}$. Set $\mathcal{P}_l(g) = \prod_{v|\infty} \mathfrak{P}_{l_v}(g_v \cdot 0)$ to be the associated CM period.

Definition 3.3.7. The subspace

$$\mathcal{M}_l(K, \overline{\mathbb{Q}}) \subset \mathcal{M}_l(K), \quad \text{resp. } \mathcal{S}_l(K, \overline{\mathbb{Q}}) \subset \mathcal{S}_l(K), \tag{3.3.13}$$

of algebraic modular forms (resp. algebraic cusp forms) consisting functions $\boldsymbol{f}$: $G(\mathbb{A}) \to \mathbb{C}$ such that $\boldsymbol{f}(g) \in \mathcal{P}_l(g)\overline{\mathbb{Q}}$ for any CM points g.

By the relation between classical and adelic definition of modular forms in (3.2.11) we have

$$\mathcal{M}_l(K, \overline{\mathbb{Q}}) \cong \bigoplus_i M_l(\Gamma_i, \overline{\mathbb{Q}}), \qquad \mathcal{S}_l(K, \overline{\mathbb{Q}}) \cong \bigoplus_i S_l(\Gamma_i, \overline{\mathbb{Q}}). \tag{3.3.14}$$

The properties we need for algebraic modular forms are collected in the following proposition.

Proposition 3.3.8.

(1) There is a basis of $\mathcal{M}_l(K)$ *consisting of algebraic modular forms. That is* $\mathcal{M}_l(K) = \mathcal{M}_l(K, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. *In particular, there is a well defined action of* $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ *on* $\mathcal{M}_l(K)$ *by acting on* $\mathbb{C}$.

(2) We have $\mathcal{S}_l(K, \overline{\mathbb{Q}})^\sigma = \mathcal{S}_l(K, \overline{\mathbb{Q}})$ *and* $\mathcal{S}_l(K) = \mathcal{S}_l(K, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

(3) Assume $r = 0$ *and* $\boldsymbol{f} \in \mathcal{M}_l(K)$ *has Fourier expansion*

$$\boldsymbol{f}\left(\begin{bmatrix} y & x\hat{y} \\ 0 & \hat{y} \end{bmatrix}\right) = \prod_{v|\infty} \nu(y_v^*)^{l_v} \cdot \sum_{\beta \in S_m(F)} \mathbf{c}(\beta; \boldsymbol{f}, y) e_\infty(\tau(iy^*\beta y)) e_{\mathbb{A}}(\tau(\beta x)) \tag{3.3.15}$$

as in (3.2.14). *Then the action of* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *on* ϕ *characterized by*

$$\boldsymbol{f}^{\sigma}\left(\begin{bmatrix} y & x\hat{y} \\ 0 & \hat{y} \end{bmatrix}\right) = \prod_{v|\infty} \nu(y_v^*)^{l_v} \cdot \sum_{\beta \in S_m(F)} \mathbf{c}(\beta; \boldsymbol{f}, y)^{\sigma} e_{\infty}(\tau(iy^*\beta y)) e_{\mathbb{A}}(\tau(\beta x)) \quad (3.3.16)$$

is well defined. In particular, we have $\boldsymbol{f} \in \mathcal{M}_{\boldsymbol{l}}(K, \overline{\mathbb{Q}})$ *if and only if* $\mathbf{c}(\beta; \boldsymbol{f}, y) \in \overline{\mathbb{Q}}$ *for all* y.

3.4 Reformulating the integral representations

We keep the assumption of our global group as the beginning of this chapter. With the archimedean local integrals calculated in Section 3.4.1, we conclude the integral representation from Theorem 2.2.4 in Section 3.4.2. For later study of algebraic and p-adic properties, we reformulate our integral representations in Section 3.4.3.

3.4.1 The archimedean local integrals

Recall that

$$H := H(F) := \{h \in \mathrm{GL}_{2n}(D) : hJ_n h^* = J_n\}, \quad (3.4.1)$$

with a doubling embedding $G \times G \to H$ defined in (2.1.11). For an archimedean place v of F, denote $H_v = H(F_v)$ for the localization at v and $\mathcal{H}_n := \mathfrak{Z}_{n,0}$ the symmetric space associated to H_v. We also write $J(h, z)$ for the automorphy factor of H to distinguish the one for G. There is a doubling embedding (see for example [Shi97, Section 6, 7] and [BJar, Section 2.3])

$$\begin{aligned} \mathfrak{Z}_{m,r} \times \mathfrak{Z}_{m,r} &\to \mathcal{H}_n, \\ z_1, z_2 &\mapsto [z_1, z_2], \end{aligned} \quad (3.4.2)$$

compatible with the action, i.e. $(g_1, g_2) \cdot [z_0, z_0] = [g_1 z_0, g_2 z_0]$. In particular, we can fix the suitable z_0 and the embedding such that $[z_0, z_0] = i \cdot 1_n \in \mathcal{H}_n$. We simply write $i := i \cdot 1_n$ if it is clear from the context. This map is constructed in [Shi97; Shi00] for symplectic and unitary groups and in [BJar, Section 2.3] for quaternionic unitary groups. The doubling map for the quaternionic orthogonal groups can be similarly constructed as in [BJar].

Let ϕ be a cusp form of weight $\boldsymbol{l}$ and set $\phi_1 = \phi_2 = \phi$. Identifying $\phi = \otimes'_v \phi_v$ and simply write $\phi_\infty := \prod_{v|\infty} \phi_v$ in this subsection for simplicity. Assume $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ is a Hecke character of infinity type $\boldsymbol{l}$. That is $\chi_v(x) = x^{l_v}|x|^{-l_v}$ for any $v|\infty$. Define a section $f_s^\infty \in \mathrm{Ind}_{P(F_\infty)}^{H(F_\infty)}(\chi|\cdot|^s)$ by $f_s^\infty = \prod_{v|\infty} f_{s,v}^\infty$ with

$$\begin{aligned} f_{s,v}^\infty(h) &= J(h_v,i)^{-l_v}|J(h_v,i)|^{l_v-s-\kappa} && \text{Case II, III, IV,} \\ f_{s,v}^\infty(h) &= J(h_v,i)^{-l_v}|J(h_v,i)|^{l_v-2s-2\kappa} && \text{Case V.} \end{aligned} \tag{3.4.3}$$

and consider the archimedean integral

$$\mathcal{Z}(s;\phi_1,\phi_2,f_s^\infty) = \int_{G(F_\infty)} f_s^\infty(\delta(g,1))\langle \pi(g)\phi_\infty,\phi_\infty\rangle dg, \tag{3.4.4}$$

where

$$\delta = \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & -1_m & 0 & 1_m & 0 \\ 0 & \epsilon\cdot 1_m & 0 & 0 & 0 & 1_m \end{bmatrix}. \tag{3.4.5}$$

Proposition 3.4.1. *Assume* $\mathrm{Re}(s) + l_v > \kappa$ *for all* v. *We have*

$$\mathcal{Z}(s;\phi,\phi,f_s^\infty) = C(s)\cdot\prod_{v|\infty} c_{l_v}(s)\cdot\langle\phi_\infty,\pi(w)\phi_\infty\rangle. \tag{3.4.6}$$

Here w *is the Weyl element as in* (2.2.21), $C(s)$ *is a power of* 2 *depending on* s *and* $c_{l_v}(s)$ *is given by the following list:*

(Case II)

$$\pi^{\frac{m(m+1)}{2}}\prod_{i=0}^{m-1}\frac{\Gamma\left(\frac{1}{2}\left(s+l_v-\frac{1}{2}\right)-\frac{i}{2}\right)}{\Gamma\left(\frac{1}{2}\left(s+l_v+\frac{2m+1}{2}\right)-\frac{i}{2}\right)},$$

(Case III)

$$\pi^{\frac{n(n+1)}{2}}\prod_{i=0}^{n-1}\frac{\Gamma\left(\frac{1}{2}\left(s+l_v-\frac{1}{2}\right)-\frac{i}{2}\right)}{\Gamma\left(\frac{1}{2}\left(s+l_v+\frac{2n+1}{2}\right)-\frac{i}{2}\right)},$$

(Case IV)

$$\pi^{\frac{n(n-1)}{2}} \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{\Gamma\left(s + l_v + \frac{1}{2} - 2i\right)}{\Gamma\left(s + l_v + \frac{2n-1}{2} - 2i\right)},$$

(Case V)

$$\pi^{m(m+r)} \prod_{i=0}^{m-1} \frac{\Gamma\left(s + \frac{l_v}{2} - i\right)}{\Gamma\left(s + \frac{l_v}{2} + \frac{n}{2} - i\right)}$$

Proof. This is well known (see [Shi00] for symplectic and unitary case, [BJar] for quaternionic unitary case). Indeed, it suffices to calculate

$$\int_{G(F_\infty)} f_s^\infty(\delta(g,1))\overline{\phi_\infty(g'g)}dg.$$

Note that $J(\delta(g,1),[z_0,z_0]) = j(\delta,[gz_0,z_0])j(g,z_0)$ and rewrite above integral (for Case II, III, IV) as

$$\int_{G(F_\infty)} j(\delta,[gz_0,z_0])^{-l}|j(\delta,[gz_0,z_0])|^{l-s-\kappa}|j(g,z_0)|^{-l-s-\kappa}\overline{\phi_\infty(g'g)}dg.$$

This kind of integral is calculated in [Shi97, Appendix A.2] for symplectic and unitary case. The symmetric space for quaternionic orthogonal group is isomorphic to the one for symplectic group. All these cases including the quaternionic unitary group are treated in [Hua63, Theorem 2.2.1, 2.3.1, 2.4.1]. □

3.4.2 Summary of the integral representations

Let $\boldsymbol{l} = (l_v)_{v|\infty}$ be a tuple of positive integers indexed by archimedean places of F. Fix a specific prime $\boldsymbol{p}$ of $\mathfrak{o}$ and an integral ideal $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2 = \prod_v \mathfrak{p}_v^{\mathfrak{c}_v}$ with $\mathfrak{n}_1, \mathfrak{n}_2, \boldsymbol{p}$ coprime. Denote $\boldsymbol{\varpi}$ for the uniformizer of $\boldsymbol{p}$. Let $\boldsymbol{q}$ be the prime ideal of $\mathcal{O}$ above $\boldsymbol{p}$ and $\widetilde{\boldsymbol{\varpi}}$ the uniformizer of $\boldsymbol{q}$. We make the following assumptions:

(1) $2 \in \mathcal{O}_v^\times$ and $\theta \in \mathrm{GL}_r(\mathcal{O}_v)$ for all $v|\mathfrak{n}\boldsymbol{p}$.

(2) $\boldsymbol{f} \in \mathcal{S}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}))$ is an eigenform for the Hecke algebra $\mathcal{H}(K(\mathfrak{n}\boldsymbol{p}), \mathfrak{X})$ as in Section 2.3.4.

(3) $\boldsymbol{f}$ is an eigenform for the $U(\boldsymbol{p})$ operator with eigenvalue $\alpha(\boldsymbol{p}) \neq 0$.

(4) $\chi = \chi_1\boldsymbol{\chi}$ with χ_1 has conductor $\mathfrak{n}_2$ and $\boldsymbol{\chi}$ has conductor $\boldsymbol{p}^{\boldsymbol{c}}$ for some integer

$\boldsymbol{c} \geq 0$. We assume χ has infinity type $\boldsymbol{l}$. That is, $\chi_v(x) = x^{l_v}|x|^{-l_v}$ for all $v|\infty$.

(5) In Case V, all places $v|\mathfrak{n}\boldsymbol{p}$ are nonsplit in $\mathcal{O}$.

Denote $\eta_1, \eta_2 \in G(\mathbb{A})$ such that

$$(\eta_1)_v = \begin{cases} w & v|\mathfrak{n}_1 \\ 1 & \text{otherwise}, \end{cases}, \quad (\eta_2)_v = \begin{cases} w & v|\mathfrak{n}_2\boldsymbol{p} \\ 1 & \text{otherwise}, \end{cases}, \tag{3.4.7}$$

where

$$w = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & 1_r & 0 \\ \epsilon \cdot 1_m & 0 & 0 \end{bmatrix}$$

is an Weyl element.

Denote $E(h; f_s)$ be the Eisenstein series on $H(\mathbb{A})$ associated to f_s. Our global integral (2.2.10) can be written as

$$\begin{aligned} &\mathcal{Z}(s; \boldsymbol{f}, f_s) \\ &= \int_{(G\times G)(F)\backslash(G\times G)(\mathbb{A})} E((g_1, g_2); f_s)\overline{\boldsymbol{f}(g_1\eta_1)}\boldsymbol{f}(g_2\eta_2)\chi(\nu(g_2))^{-1}dg_1dg_2. \end{aligned} \tag{3.4.8}$$

The integral representation is summarized in the following theorem.

Theorem 3.4.2. *Take the section f_s to be*

$$\begin{aligned} f_s &= \prod_{v\nmid\mathfrak{n}\boldsymbol{p}\infty} f^0_{s,v} \cdot \prod_{v|\mathfrak{n}_1} f^{\dagger,\mathfrak{c}_v}_{s,v} \cdot \prod_{v|\mathfrak{n}_2} f^{\ddagger,\mathfrak{c}_v}_{s,v} \cdot f^{\ddagger,\boldsymbol{c}}_{s,\boldsymbol{p}} \cdot \prod_{v|\infty} f^{\infty}_{s,v}, && \boldsymbol{c} > 0, \\ f_s &= \prod_{v\nmid\mathfrak{n}\boldsymbol{p}\infty} f^0_{s,v} \cdot \prod_{v|\mathfrak{n}_1} f^{\dagger,\mathfrak{c}_v}_{s,v} \cdot \prod_{v|\mathfrak{n}_2} f^{\ddagger,\mathfrak{c}_v}_{s,v} \cdot f^{p}_{s,\boldsymbol{p}} \cdot \prod_{v|\infty} f^{\infty}_{s,v}, && \boldsymbol{c} = 0. \end{aligned} \tag{3.4.9}$$

with $f^0_{s,v}, f^{\dagger,\mathfrak{c}_v}_{s,v}, f^{\ddagger,\mathfrak{c}_v}_{s,v}, f^p_{s,v}$ are local sections defined in (2.4.6), (2.4.11), (2.4.15), (2.4.24) and $f^\infty_{s,v}$ the archimedean local section defined in (3.4.3). Then

$$\begin{aligned} &\mathcal{Z}(s; \boldsymbol{f}, f_s) = C' \cdot \prod_{v|\infty} c_{l_v}(s) \cdot L\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right) \cdot \langle \pi(\eta)\boldsymbol{f}|U'(\mathfrak{n}_1), \boldsymbol{f}\rangle, \quad \boldsymbol{c} > 0, \\ &\mathcal{Z}(s; \boldsymbol{f}, f_s) = C'' \cdot \prod_{v|\infty} c_{l_v}(s) \cdot L\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right) \cdot \langle \pi(\eta)\boldsymbol{f}|U'(\mathfrak{n}_1), \boldsymbol{f}\rangle \\ &\qquad \times M\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right), \qquad \boldsymbol{c} = 0. \end{aligned} \tag{3.4.10}$$

Here:

(a) $M(s, \boldsymbol{f} \times \chi)$ is the modification factor given in Proposition 2.4.6,

(b) $U'(\mathfrak{n}_1)$ is the Hecke operator defined by (2.3.13),

(c)

$$\eta = \begin{bmatrix} 0 & 0 & \boldsymbol{\varpi}^{-\boldsymbol{c}} \cdot 1_m \\ 0 & 1_r & 0 \\ \boldsymbol{\varpi}^{\boldsymbol{c}} \cdot 1_m & 0 & 0 \end{bmatrix} \cdot \prod_{v|\mathfrak{n}_2} \begin{bmatrix} 0 & 0 & \varpi_v^{-\mathfrak{c}_v} \cdot 1_m \\ 0 & 1_r & 0 \\ \varpi_v^{\mathfrak{c}_v} \cdot 1_m & 0 & 0 \end{bmatrix}, \quad \boldsymbol{c} > 0,$$
$$\eta = \begin{bmatrix} 0 & 0 & \widetilde{\boldsymbol{\varpi}}^{-1} \cdot 1_m \\ 0 & 1_r & 0 \\ \widetilde{\boldsymbol{\varpi}} \cdot 1_m & 0 & 0 \end{bmatrix} \cdot \prod_{v|\mathfrak{n}_2} \begin{bmatrix} 0 & 0 & \varpi_v^{-\mathfrak{c}_v} \cdot 1_m \\ 0 & 1_r & 0 \\ \varpi_v^{\mathfrak{c}_v} \cdot 1_m & 0 & 0 \end{bmatrix}, \quad \boldsymbol{c} = 0, \tag{3.4.11}$$

(d) The constants $c_{l_v}(s)$ are given in (3.4.6). Up to a power of 2 depending on s,

$$C' = \chi(\mathfrak{n}_1)^{m\mathbf{d}_1} |\mathfrak{n}_1|^{m\mathbf{d}_2(s+\kappa)} \mathrm{vol}(\mathrm{GL}_m(\mathcal{O})/\mathrm{GL}_m(\mathfrak{n}_2 \boldsymbol{p}^{\boldsymbol{c}} \mathcal{O})), \tag{3.4.12}$$

and

$$C'' = (-1)^m |\boldsymbol{\varpi}|_{\boldsymbol{p}}^{\mathbf{d}_3 \frac{m^2+m}{2}} \chi(\mathfrak{n}_1)^{m\mathbf{d}_1} |\mathfrak{n}_1|^{m\mathbf{d}_2(s+\kappa)} \mathrm{vol}(\mathrm{GL}_m(\mathcal{O})/\mathrm{GL}_m(\mathfrak{n}_2 \mathcal{O})), \tag{3.4.13}$$

with

$$\mathbf{d}_1 = \begin{cases} 1 & \textit{Case I, II, V}, \\ 2 & \textit{Case III, IV}, \end{cases} \qquad \mathbf{d}_2 = \begin{cases} 1 & \textit{Case I, II}, \\ 2 & \textit{Case III, IV, V}, \end{cases}$$
$$\mathbf{d}_3 = \begin{cases} 1 & \textit{Case I, II, V Ramified}, \\ 2 & \textit{Case III, IV, V Inert}. \end{cases} \tag{3.4.14}$$

3.4.3 Level lowering and the reformulation

For any integer $\boldsymbol{n} \geq 0$, let $\mathcal{K}(\boldsymbol{p}^{\boldsymbol{n}})$ (resp. $\mathcal{K}'(\boldsymbol{p}^{\boldsymbol{n}})$) be an open compact subgroup of $G(\mathbb{A})$ defined by $\mathcal{K}(\boldsymbol{p}^{\boldsymbol{n}}) = K'(\mathfrak{n}_1)K(\mathfrak{n}_2)K(\boldsymbol{p}^{\boldsymbol{n}})$ (resp. $\mathcal{K}'(\boldsymbol{p}^{\boldsymbol{n}}) = K(\mathfrak{n}_1)K'(\mathfrak{n}_2)K'(\boldsymbol{p}^{\boldsymbol{n}})$). Denote $w_\infty \in G(\mathbb{A})$ be an element such that $w_v = w$ for any archimedean place v and $w_v = 1$ for all non-archimedean places. Then for f_s as above, we have

$$\mathcal{E}(g_1, g_2; f_s) := \chi(\nu(g_2))^{-1} E((g_1, g_2^\iota); f_s) \in \mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}})) \otimes \mathcal{M}_{\boldsymbol{l}}(\mathcal{K}'(\boldsymbol{p}^{2\boldsymbol{n}})), \tag{3.4.15}$$

with

$$g^\iota = \begin{bmatrix} -1_m & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & 1_m \end{bmatrix}_\infty \cdot g \cdot \begin{bmatrix} -1_m & 0 & 0 \\ 0 & -1_r & 0 \\ 0 & 0 & 1_m \end{bmatrix}_\infty, \tag{3.4.16}$$

where the matrix with subscript ∞ means an element in $G(F_\infty)$. Here, in this section only, we abuse the notation by writing $\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}}))$ for the space of functions transforming as a modular form (i.e. satisfying (3.2.9) but may not necessary holomorphic). That is, $\mathcal{E}(g_1, g_2; f_s)$ transforms as a modular form in $\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}}))$ for the first variable and $\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}'(\boldsymbol{p}^{2\boldsymbol{n}}))$ for the second variable. Indeed, later in Section 4.1 and 4.2, we will specialize to the special points $s = s_0$ (as in (4.2.2)) in which case $\mathcal{E}(g_1, g_2; f_s)$ is holomorphic in both variables (follows from the Fourier expansion). Note that $\boldsymbol{n}$ can be took as any integer such that $\boldsymbol{n} \geq \boldsymbol{c}$ if $\boldsymbol{c} > 0$ and $\boldsymbol{n} \geq 1$ if $\boldsymbol{c} = 0$. We also remark that the involution ι is included in the second variable since our doubling embedding of the symmetric space (3.4.2) is holomorphic in the first variable and antiholomorphic in the second variable. To compare all these integral representations when varying the character $\boldsymbol{\chi}$ of different conductors, we further descend the level of Eisenstein series such that it is independent of $\boldsymbol{c}$. Our approach is an analogue of [BS00, Section 4].

Remark 3.4.3. In the following we actually assume $\boldsymbol{p}$ is nonsplit in D. As our argument is local, it directly extended to the split cases of Case III and IV by identifying the local group $G(F_{\boldsymbol{p}})$ with the group in Case I or Case II as in Section 2.3.3.

We will use the following general lemma to descend the level.

Lemma 3.4.4. *The Hecke operator $U(\boldsymbol{p}^{n-1})$ defined by (2.3.13) maps $\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}}))$ to $\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^2))$.*

Proof. We define a map $\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}})) \to \mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^2))$ in following steps. Let $f \in$

$\mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}}))$ and first set

$$f_1(g) := f\left(g\begin{bmatrix} \boldsymbol{\varpi}^{\boldsymbol{n}}\cdot 1_m & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & \boldsymbol{\varpi}^{-\boldsymbol{n}}\cdot 1_m \end{bmatrix}\right).$$

Then f_1 is fixed by

$$K''(\boldsymbol{p}^{\boldsymbol{n}}) := G(\mathfrak{o}_{\boldsymbol{p}}) \cap \begin{bmatrix} \mathrm{Mat}_m(\mathcal{O}_{\boldsymbol{p}}) & \mathrm{Mat}_{m,r}(\boldsymbol{p}^{\boldsymbol{n}}\mathcal{O}_{\boldsymbol{p}}) & \mathrm{Mat}_m(\boldsymbol{p}^{2\boldsymbol{n}}\mathcal{O}_{\boldsymbol{p}}) \\ \mathrm{Mat}_{r,m}(\boldsymbol{p}^{\boldsymbol{n}}\mathcal{O}_{\boldsymbol{p}}) & 1+\mathrm{Mat}_r(\boldsymbol{p}\mathcal{O}_{\boldsymbol{p}}) & \mathrm{Mat}_{r,m}(\boldsymbol{p}^{\boldsymbol{n}}\mathcal{O}_{\boldsymbol{p}}) \\ \mathrm{Mat}_m(\mathcal{O}_{\boldsymbol{p}}) & \mathrm{Mat}_{m,r}(\boldsymbol{p}^{\boldsymbol{n}}\mathcal{O}_{\boldsymbol{p}}) & \mathrm{Mat}_m(\mathcal{O}_{\boldsymbol{p}}) \end{bmatrix},$$

Secondly we define

$$f_2(g) := \sum_{\gamma\in K''(\boldsymbol{p})/K''(\boldsymbol{p}^{\boldsymbol{n}})} f_1(g\gamma),$$

where the representatives of $K''(\boldsymbol{p})/K''(\boldsymbol{p}^{\boldsymbol{n}})$ can be taken as

$$\begin{bmatrix} 1_m & -\boldsymbol{\varpi} b^*\theta^{-1} & \boldsymbol{\varpi}^2 c \\ 0 & 1_r & \boldsymbol{\varpi} b \\ 0 & 0 & 1_r \end{bmatrix},$$

with $b \in \mathrm{Mat}_{m,r}(\mathcal{O}_{\boldsymbol{p}}/\boldsymbol{p}^{\boldsymbol{n}-1}\mathcal{O}_{\boldsymbol{p}})$ and $c \in \mathrm{Mat}_m(\mathcal{O}_{\boldsymbol{p}}/\boldsymbol{p}^{2\boldsymbol{n}-2}\mathcal{O}_{\boldsymbol{p}})$ satisfying $\epsilon c + b^*\hat{\theta} b + c^* = 0$. Then $f_2 \in \mathcal{M}_{\boldsymbol{l}}(K''(\boldsymbol{p}))$. Finally we put

$$f_3(g) := f_2\left(g\begin{bmatrix} \boldsymbol{\varpi}^{-1}\cdot 1_m & 0 & 0 \\ 0 & 1_r & 0 \\ 0 & 0 & \boldsymbol{\varpi}\cdot 1_m \end{bmatrix}\right).$$

to obtain $f_3 \in \mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^2))$. Combining these three steps together, $f \mapsto f_3$ defines a map

$$\begin{aligned} \mathrm{Tr} : \mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^{2\boldsymbol{n}})) &\to \mathcal{M}_{\boldsymbol{l}}(\mathcal{K}(\boldsymbol{p}^2)) \\ f &\mapsto f|\mathrm{Tr}(g) := \sum_{\gamma} f(g\gamma) \end{aligned}$$

where γ runs through elements of the form

$$\begin{bmatrix} \boldsymbol{\varpi}^{\boldsymbol{n}-1}\cdot 1_m & -b^*\theta^{-1} & \epsilon\boldsymbol{\varpi}^{1-\boldsymbol{n}}c^* \\ 0 & 1_r & \boldsymbol{\varpi}^{1-\boldsymbol{n}}b \\ 0 & 0 & \boldsymbol{\varpi}^{1-\boldsymbol{n}}\cdot 1_m \end{bmatrix}$$

with $b \in \mathrm{Mat}_{m,r}(\mathcal{O}_{\boldsymbol{p}}/\boldsymbol{p}^{\boldsymbol{n}-1}\mathcal{O}_{\boldsymbol{p}})$ and $c \in \mathrm{Mat}_m(\mathcal{O}_{\boldsymbol{p}}/\boldsymbol{p}^{2\boldsymbol{n}-2}\mathcal{O}_{\boldsymbol{p}})$ satisfying $\epsilon c + b^*\hat{\theta}b + c^* = 0$. Comparing above matrix with the one in (2.3.11) for $U(\boldsymbol{p}^{\boldsymbol{n}-1})$ operator we obtain the lemma. $\square$

We apply above process for both variables and define

$$\mathbb{E}(h; f_s) := E(h; f_s)|\boldsymbol{U}(\boldsymbol{p}^{\boldsymbol{n}-1}) := \sum_{\gamma} E(h\gamma; f_s), \tag{3.4.17}$$

where γ runs through elements of the form

$$\begin{bmatrix} 1_r & 0 & 0 & 0 & \frac{\epsilon\boldsymbol{\varpi}^{1-\boldsymbol{n}}b_2}{2} & -\frac{\epsilon\boldsymbol{\varpi}^{1-\boldsymbol{n}}b_1}{2} \\ \epsilon b_2^*\theta^{-1} & \boldsymbol{\varpi}^{\boldsymbol{n}-1}\cdot 1_m & 0 & -\frac{b_2^*}{2} & -\boldsymbol{\varpi}^{1-\boldsymbol{n}}c_2 & 0 \\ -\epsilon b_1^*\theta^{-1} & 0 & \boldsymbol{\varpi}^{\boldsymbol{n}-1}\cdot 1_m & -\frac{b_1^*}{2} & 0 & \boldsymbol{\varpi}^{1-\boldsymbol{n}}c_1 \\ 0 & 0 & 0 & 1_r & -\boldsymbol{\varpi}^{1-\boldsymbol{n}}\theta^{-1}b_2 & \boldsymbol{\varpi}^{1-\boldsymbol{n}}\theta^{-1}b_1 \\ 0 & 0 & 0 & 0 & \boldsymbol{\varpi}^{1-\boldsymbol{n}}\cdot 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{\varpi}^{1-\boldsymbol{n}}\cdot 1_m \end{bmatrix} \tag{3.4.18}$$

with $b_1, b_2 \in \mathrm{Mat}_{m,r}(\mathcal{O}_{\boldsymbol{p}}/\boldsymbol{p}^{\boldsymbol{n}-1}\mathcal{O}_{\boldsymbol{p}})$ and $c_1, c_2 \in \mathrm{Mat}_m(\mathcal{O}_{\boldsymbol{p}}/\boldsymbol{p}^{2\boldsymbol{n}-2}\mathcal{O}_{\boldsymbol{p}})$ satisfying $\epsilon c_1 + b_1^*\hat{\theta}b_1 + c_1^* = 0, \epsilon c_2 + b_2^*\hat{\theta}b + c_2^* = 0$.

Take $\mathbb{E}(g_1, g_2; f_s)$ as with f_s as in (3.4.9). Then

$$\boldsymbol{E}(g_1, g_2; f_s) := \chi(\nu(g_2))^{-1}\mathbb{E}((g_1, g_2^{\iota}); f_s) \in \mathcal{M}_l(\mathcal{K}(\boldsymbol{p}^2)) \otimes \mathcal{M}_l(\mathcal{K}'(\boldsymbol{p}^2)). \tag{3.4.19}$$

We may also denote $\mathbb{E}(h; f_s, \chi, \boldsymbol{n})$ and $\boldsymbol{E}(g_1, g_2; f_s, \chi, \boldsymbol{n})$ to emphasize their dependence on $\chi, \boldsymbol{n}$. Consider the global integral

$$\begin{aligned} &\boldsymbol{Z}(s; \boldsymbol{f}, f_s) \\ &:= \int_{(G\times G)(F)\backslash(G\times G)(\mathbb{A})} \boldsymbol{E}((g_1, g_2^{\iota}); f_s)\overline{\boldsymbol{f}(g_1\eta_1\eta_{\boldsymbol{p}})}\boldsymbol{f}(g_2\eta_2)dg_1dg_2. \end{aligned} \tag{3.4.20}$$

with η_1, η_2 as (3.4.7) and

$$\eta_{\boldsymbol{p}} = \begin{bmatrix} 0 & 0 & \boldsymbol{\varpi}^{-1} \cdot 1_m \\ 0 & 1_r & 0 \\ \boldsymbol{\varpi} \cdot 1_m & 0 & 0 \end{bmatrix} \in G(F_{\boldsymbol{p}}). \tag{3.4.21}$$

Again we may also denote $\boldsymbol{Z}(s; \boldsymbol{f}, f_s, \chi, \boldsymbol{n})$ to emphasize its dependence on $\chi, \boldsymbol{n}$. By simply changing variables, we reformulate Theorem 3.4.2 in the following corollary. We remark that it is essential to assume that the eigenvalue $\alpha(\boldsymbol{p}) \neq 0$ otherwise the integral will be identically zero.

Corollary 3.4.5. *For $\boldsymbol{c} > 0$ we have*

$$\boldsymbol{Z}(s; \boldsymbol{f}, f_s) = \alpha(\boldsymbol{p})^{2\boldsymbol{n}-2} C' \prod_{v|\infty} c_{l_v}(s) \cdot L\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right) \cdot \langle \pi(\eta) \boldsymbol{f} | U'(\mathfrak{n}_1), \boldsymbol{f} \rangle, \tag{3.4.22}$$

and for $\boldsymbol{c} = 0$ we have

$$\begin{aligned} \boldsymbol{Z}(s; \boldsymbol{f}, f_s) = \alpha(\boldsymbol{p})^{2\boldsymbol{n}-2} C'' \prod_{v|\infty} c_{l_v}(s) \cdot L\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right) \cdot \langle \pi(\eta) \boldsymbol{f} | U'(\mathfrak{n}_1), \boldsymbol{f} \rangle \\ \times M\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right), \end{aligned} \tag{3.4.23}$$

Here the notations are same as Theorem 3.4.2 except

$$\eta = \prod_{v|\mathfrak{n}_2} \begin{bmatrix} 0 & 0 & \varpi_v^{-\mathfrak{c}_v} \cdot 1_m \\ 0 & 1_r & 0 \\ \varpi_v^{\mathfrak{c}_v} \cdot 1_m & 0 & 0 \end{bmatrix}. \tag{3.4.24}$$

Remark 3.4.6. If we denote $\boldsymbol{f}^1(g) := \boldsymbol{f}(g\eta_1\eta_{\boldsymbol{p}})$, $\boldsymbol{f}^2(g) := \boldsymbol{f}(g\eta_2)$ and let $\boldsymbol{V}$ be the operator defined by $\boldsymbol{f}^2 | \boldsymbol{V} := \pi(\eta) \boldsymbol{f}^2 | U'(\mathfrak{n}_1)$, then our computations in Section 2.4 also show that

$$\left\langle \boldsymbol{E}(g_1, g_2; f_s), \boldsymbol{f}^1(g_1) \right\rangle = \frac{\boldsymbol{Z}(s; \boldsymbol{f}, f_s)}{\langle \boldsymbol{f}^2 | \boldsymbol{V}, \boldsymbol{f}^2 \rangle} \cdot \overline{\boldsymbol{f}^2 | \boldsymbol{V}(g_2^\iota)}, \tag{3.4.25}$$

where the left hand side is the Petersson inner product respect to g_1. The integral (3.4.25) is the adelic version of the integral representation obtained in [BS00] and [Shi97; Shi00]. One can also further reformulate the integral in a classical setting as there (see also [Jin22, Section 4]). Indeed, recall that by the weak approximation

(3.2.6) of G there is finite number h such that

$$G(\mathbb{A}) = \coprod_{1\leq i\leq h} G(F)t_iKG(F_\infty).$$

For $1 \leq i, j \leq h$ and $z = g_\infty z_0 \in \mathfrak{Z}_{m,r}, w = g'_\infty z_0 \in \mathfrak{Z}_{m,r}$, we set

$$f_i^1(z) = \boldsymbol{f}^1(t_i g_\infty), \quad f_j^2(w) = \boldsymbol{f}^2|\boldsymbol{V}(t_j g'_\infty), \quad E_{ij}(z,w) = \boldsymbol{E}(t_i g_\infty, t_j g'_\infty; f_s).$$

Then the integral (3.4.20) can be written as

$$\sum_{i,j} \left\langle \left\langle E_{ij}(z, -\overline{w}), f_i^1(z) \right\rangle, f_j^2(w) \right\rangle \tag{3.4.26}$$

and (3.4.25) can be rewritten as

$$\sum_{i} \left\langle E_{ij}(z,w), f_i^1(z) \right\rangle = \frac{\boldsymbol{Z}(s; \boldsymbol{f}, f_s)}{\langle \boldsymbol{f}^2|\boldsymbol{V}, \boldsymbol{f}^2 \rangle} \cdot \overline{f_j^2(-\overline{w})}, \tag{3.4.27}$$

which is the pullback formula obtained in [Shi97; Shi00].

Chapter 4

The Eisenstein Series and Special Values of L-functions

In this chapter, we study the special values of L-functions utilizing our integral representations obtained in previous two chapters. Indeed, via the integral representation, the properties of the special L-values can be obtained from the properties of special values of Eisenstein series. We therefore calculate the Fourier expansion of the Eisenstein series explicitly in Section 4.1 and the properties of the special values of Eisenstein series can be simply read off from these Fourier coefficients. We conclude our main theorems on algebraicity of special L-values and construct the p-adic L-functions in Section 4.2.

This chapter is taken from [Jin23, Section 7-8]. In [BJar] and [Jin22], we also obtain some partial results for quaternionic unitary groups. However, the differential operators are applied in [BJar; Jin22] so that more critical values are studied. We omit these discussions and restrict to the study of a particular critical point here (see also Remark 4.2.7).

4.1 Fourier expansion of the Eisenstein series

We calculate the Fourier expansion of the Eisenstein series in this section. Then the properties of Eisenstein series can be directly obtained from the properties of Fourier coefficients.

4.1.1 Generalities

Let $e_{\mathbb{A}} = \prod_v e_v$ be the standard additive character of $\mathbb{A}$. That is, $e_v(x) = e^{2\pi i x}$ for an archimedean place v and $e_v(\varpi_v) = \prod_{v|\infty} e_v(-q_v)$ with ϖ_v the uniformizer of F_v and $|\varpi_v|_v = q_v^{-1}$. Denote S_n be the additive algebraic group such that

$$S_n(F) = \{\beta \in \mathrm{Mat}_n(D) : \beta^* = -\epsilon\beta\}. \tag{4.1.1}$$

The Eisenstein series $E(h; f_s)$ (2.2.9) on $H(\mathbb{A})$ has a Fourier expansion of the form

$$\begin{aligned} E(h; f_s) &= \sum_{\beta \in S_n(F)} E_\beta(h; f_s), \\ E_\beta(h; f_s) &= \int_{S_n(F)\backslash S_n(\mathbb{A})} E\left(\begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix} h; f_s\right) e_{\mathbb{A}}(-\tau(\beta S))dS. \end{aligned} \tag{4.1.2}$$

By the Iwasawa decomposition, the Eisenstein series is determined by its value at the parabolic element $q \in P(\mathbb{A})$. In particular, we can take $q_v = \mathrm{diag}[y, \hat{y}]$ for non-archimedean places v and for finitely many v we can assume $q_v = 1$. For an archimedean place v, we can take $q_v = \begin{bmatrix} y_v & x_v\hat{y}_v \\ 0 & \hat{y}_v \end{bmatrix}$ with $z_v = x_v + iy_vy_v^* \in \mathcal{H}_n$. We shall also denote such q as q_z to indicate its dependence on $z = (z_v)_{v|\infty}$.

Since our f_s is chosen such that, for at least one place v, the support of $f_{s,v}$ is in the big cell $P(F)J_nP(F)$, the Fourier coefficient $E_\beta(q; f_s)$ at parabolic element $q \in P(\mathbb{A})$ is factorizable. That is

$$\begin{aligned} E_\beta(q; f_s) &= \prod_v E_{\beta,v}(q; f_s), \\ E_{\beta,v}(q; f_s) &= \int_{S_n(F_v)} f_{s,v}\left(J_n \begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix} q\right) e_v(-\tau(\beta S))dS. \end{aligned} \tag{4.1.3}$$

For local sections $f_s = f_s^0, f_s^{\dagger,\mathfrak{c}}, f_s^{\ddagger,\mathfrak{c}}, f_s^p, f_s^\infty$ defined in Section 2.4, we calculate the local Fourier coefficients $E_{\beta,v}(q; f_s)$ place by place in next two subsections.

4.1.2 Non-archimedean computations

Let F be a non-archimedean local field and $\mathfrak{o}$ its ring of integers with the maximal ideal $\mathfrak{p}$. Fix uniformizer ϖ and the absolute value $|\cdot|$ on F normalized so that $|\varpi| = q^{-1}$ with q the cardinality of the residue field. We also fix a maximal order $\mathcal{O}$ of D such that $D = \mathcal{O} \otimes_{\mathfrak{o}} F$. Let $\mathfrak{q}$ be a prime in $\mathcal{O}$ above $\mathfrak{p}$ and fix $\widetilde{\varpi}$ a uniformizer of $\mathfrak{q}$.

We are going to calculate local Fourier coefficients

$$E_\beta(q; f_s) = \int_{S_n(F)} f_s\left(J_n \begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix} q\right) e(-\tau(\beta S))dS, \tag{4.1.4}$$

for various local sections $f_s^0, f_s^{\dagger,\mathfrak{c}}, f_s^{\ddagger,\mathfrak{c}}, f_s^p$ defined in (2.4.6), (2.4.11), (2.4.15), (2.4.24).

The unramified case

We first consider the local section f_s^0. Denote

$$S_n(\mathfrak{o})^* = \{\beta \in S_n(F) : \tau(\beta S) \in \mathfrak{o} \text{ for any } S \in S_n(\mathfrak{o})\}. \tag{4.1.5}$$

Proposition 4.1.1. *Set $t = \mathrm{rank}(\beta)$ and ${}^t\bar{b}\beta b = \mathrm{diag}[\beta', 0]$ with $b \in \mathrm{GL}_n(\mathcal{O})$ and $\beta' \in S_t(F)$. Let $q = \mathrm{diag}[a, \hat{a}]$, then $E_\beta(q; f_s^0)$ is nonzero only if $a^*\beta a \in S_t(\mathfrak{o})^*$. In this case, up to the term $\chi(\nu(a))|N_{E/F}(\nu(a))|^{s+\kappa}$ and a power of the discriminant of D, $E_\beta(q; f_s^0)$ is given by the following list.*

(Case I, Orthogonal) This case occurs as quaternionic unitary split case.

$$\prod_{i=1}^{\lfloor \frac{n-t}{2} \rfloor} L\left(2s - n + t + 2i, \chi^2\right) \cdot P_{a^*\beta a}(\chi(q)q^{-s-\kappa}),$$

(Case II, Symplectic Even) Assume t is even. Let λ_β be the quadratic character

associated to the quadratic field $F((-1)^{\frac{t}{2}}\nu(2\beta))$ over F.

$$L\left(s-\frac{n-1}{2}+\frac{t}{2},\chi\lambda_\beta\right)\cdot\prod_{i=1}^{\lfloor\frac{n-t}{2}\rfloor}L(2s-n+t+2i,\chi^2)\cdot P_{a^*\beta a}(\chi(q)q^{-s-\kappa}),$$

(Case II, Symplectic Odd) Assume t is odd. This case only occurs as quaternionic orthogonal split case.

$$\prod_{i=1}^{\lfloor\frac{n-t+1}{2}\rfloor}L(2s-n+t-1+2i,\chi^2)\cdot P_{a^*\beta a}(\chi(q)q^{-s-\kappa}),$$

(Case III, Quaternionic Orthogonal Nonsplit Even) Assume t is even. Let λ_β be the quadratic character associated to the quadratic field $F((-1)^{\frac{t}{2}}\nu(2\beta))$ over F.

$$L\left(s-\frac{2n-1}{2}+t,\chi\lambda_\beta\right)\cdot\prod_{i=1}^{n-t}L\left(2s-2n+2t+2i,\chi^2\right)\cdot P_{a^*\beta a}(\chi(q)q^{-s-\kappa}),$$

(Case III, Quaternionic Orthogonal Nonsplit Odd) Assume t is odd.

$$\prod_{i=1}^{n-t}L\left(2s-2n+2t+2i,\chi^2\right)\cdot P_{a^*\beta a}(\chi(q)q^{-s-\kappa}),$$

(Case IV, Quaternionic Unitary Nonsplit)

$$\prod_{i=1}^{n-t}L\left(2s-2n+2t+2i,\chi^2\right)\cdot P_{a^*\beta a}(\chi(q)q^{-s-\kappa}),$$

(Case V, Unitary)

$$\prod_{i=1}^{n-t}L(2s-i+1,\chi^0\chi_{E/F}^{n+i})\cdot P_{a^*\beta a}(\chi^0(q)q^{-2s-2\kappa}).$$

Here $L(s,\chi)$ means the local L-factor of Hecke L-functions and $P_{\alpha^*\beta\alpha}(X)\in\mathbb{Z}[X]$ is a polynomial with coefficients in $\mathbb{Z}$ whose constant term is 1.

Proof. Conjugate q to the left, we obtain

$$E_\beta(q;f_s^0)=\chi(\nu(a))|N_{E/F}(\nu(a))|^{s+\kappa}\int_{S_n(F)}f_s\left(J_n\begin{bmatrix}1_n & a^{-1}S\hat{a}\\ 0 & 1_n\end{bmatrix}q\right)e(-\tau(\beta S))dS.$$

The above integral is the Siegel series α studied in [Shi97, Chapter III] (see also [Fei89; Fei94]). The orthogonal, symplectic and unitary case are listed in [Shi97, Theorem 13.6]. (We remind the reader that we have already normalized the local section f_s^0 by $b(s,\chi)$). Two quaternionic cases can also be calculated in the same way as [Shi97, Section 13, 14, 15] (see also [Shi99b, Proposition 3.5] and [Yam17]). $\square$

The ramified case

We now assume $q=1$ and consider the local section $f_s^{\dagger,\mathfrak{c}}, f_s^{\ddagger,\mathfrak{c}}, f_s^p$. Denote

$$\begin{aligned} S_n(\mathfrak{o})^{\mathfrak{c}} &= S_n(\mathfrak{o}) \cap \begin{bmatrix} \mathrm{Mat}_r(\mathfrak{p}\mathcal{O}) & \mathrm{Mat}_{r,2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) \\ \mathrm{Mat}_{2m,r}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) & \mathrm{Mat}_{2m}(\mathfrak{p}^{\mathfrak{c}}\mathcal{O}) \end{bmatrix}, \\ S_n(\mathfrak{o})^{*,\mathfrak{c}} &= \{\beta \in S_n(F) : \tau(\beta S) \in \mathfrak{o} \text{ for any } S \in S_n(\mathfrak{o})^{\mathfrak{c}}\}. \end{aligned} \tag{4.1.6}$$

Proposition 4.1.2. *Assume $\nu(\beta) \neq 0$, then*

$$E_\beta(1; f_s^{\dagger,\mathfrak{c}}) = \begin{cases} 1 & \beta \in S_n(\mathfrak{o})^{*,\mathfrak{c}}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.1.7}$$

Proof. Consider

$$E_\beta(1; f_s^{\dagger,\mathfrak{c}}) = \int_{S_n(F)} f_s^{\dagger,\mathfrak{c}}\left(J_n \begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix}\right) e(-\tau(\beta S))dS.$$

By the definition of $f_s^{\dagger,\mathfrak{c}}$, the integrand vanishes unless $S \in S_n(\mathfrak{o})^{\mathfrak{c}}$ and the proposition easily follows. $\square$

For a character χ of F with conductor $\mathfrak{p}^{\mathfrak{c}}$, the local Gauss sum of χ is defined as

$$G(\chi) = \sum_{u \in \mathcal{O}/\mathfrak{p}^{\mathfrak{c}}\mathcal{O}} \chi(\nu(u)) e\left(\frac{\tau(u)}{\varpi^{\mathfrak{c}}}\right). \tag{4.1.8}$$

Lemma 4.1.3. *Consider*

$$G(\chi;\beta,m) = q^{\mathfrak{c}\mathbf{d}_1 \frac{m(m-1)}{2}} \sum_{u \in \mathrm{GL}_m(\mathcal{O}/\mathfrak{p}^{\mathfrak{c}}\mathcal{O})} \chi(\nu(u)) e\left(\frac{\tau(\beta u)}{\varpi^{\mathfrak{c}}}\right) \tag{4.1.9}$$

for a matrix $\beta \in \mathrm{Mat}_m(\mathcal{O})$. Then

$$G(\chi;\beta,m) = \begin{cases} \chi(\nu(\beta))G(\chi)^m & \beta \in \mathrm{GL}_m(\mathcal{O}), \\ 0 & \textit{otherwise.} \end{cases}. \tag{4.1.10}$$

Proof. This is an analogue of the computations in [BS00, Section 6]. Multiply by some matrix of $\mathrm{GL}_m(\mathcal{O}_v)$ on the left and right of β, it suffices to prove the lemma for diagonal $\beta = \mathrm{diag}[\beta_1, ..., \beta_m]$. In this case, we calculate that

$$G(\chi;\beta,m) = \nu(\varpi^{\mathfrak{c}})^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} \left(\sum_{u\in\mathcal{O}/\mathfrak{p}^{\mathfrak{c}}\mathcal{O}} \chi(\nu(u)) e\left(\frac{\tau(\beta_i u)}{\varpi^{\mathfrak{c}}}\right)\right).$$

By the property of Gauss sums, the sum in the bracket is nonzero if and only if $\beta_i \in \mathcal{O}^\times$ and in this case it equals $\chi(\nu(\beta_i))G(\chi)$. □

We write $\beta \in S_n(F)$ as

$$\beta = \begin{bmatrix} \beta_1 & -\epsilon\beta_2^* & -\epsilon\beta_3^* \\ \beta_2 & \beta_4 & -\epsilon\beta_5^* \\ \beta_3 & \beta_5 & \beta_6 \end{bmatrix} \tag{4.1.11}$$

with $\beta_1 \in S_r(F), \beta_4, \beta_6 \in S_m(F)$. Here recall that $n = 2m + r$ with m, r as in (3.0.1).

Proposition 4.1.4. *$E_\beta(1; f_s^{\ddagger,\mathfrak{c}}) = 0$ unless $\beta_5 \in \mathrm{GL}_m(\mathcal{O})$. In this case, if we further assume $\nu(\beta) \neq 0$, then*

$$E_\beta(1; f_s^{\ddagger,\mathfrak{c}}) = \begin{cases} q^{\mathfrak{c}\mathbf{d}_1\frac{m(m-1)}{2}}\chi(\nu(\beta_5))G(\chi)^m & \beta \in S_n(\mathfrak{o})^{*,0}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.1.12}$$

Proof. We need to compute

$$E_\beta(1; f_s^{\ddagger,\mathfrak{c}}) = \int_{S_n(F)} \sum_{u\in \mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O})} \chi^{-1}(\nu(u)) e(-\tau(\beta S))$$
$$\times f_s^{\dagger,0}\left(J_n \begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix} \begin{bmatrix} 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_m & 0 & 0 & 0 & \frac{u}{\varpi^{\mathfrak{c}}} \\ 0 & 0 & 1_m & 0 & -\frac{\epsilon u^*}{\varpi^{\mathfrak{c}}} & 0 \\ 0 & 0 & 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_m \end{bmatrix}\right) dS.$$

Changing variables

$$S \mapsto S - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{u}{\varpi^{\mathfrak{c}}} \\ 0 & -\frac{\epsilon u^*}{\varpi^{\mathfrak{c}}} & 0 \end{bmatrix},$$

we obtain

$$\int_{S_n(F)} f_s^{\dagger,0}\left(J_n \begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix}\right) e(-\tau(\beta S))dS$$

$$\times \sum_{u \in \mathrm{GL}_m(\mathcal{O})/\varpi^{\mathfrak{c}}\mathrm{GL}_m(\mathcal{O})} \chi^{-1}(\nu(u))e\left(\frac{2\tau(\beta_5 u)}{\varpi^{\mathfrak{c}}}\right).$$

The second line implies $\beta_5 \in \mathrm{GL}_m(\mathcal{O})$ by Lemma 4.1.3. In this case and $\nu(\beta) \neq 0$ the integral in the first line can be calculated as in Proposition 4.1.2. □

Proposition 4.1.5. *$E_\beta(1; f_s^p) = 0$ unless $\beta_5 \in \mathrm{GL}_m(\mathcal{O})$. In this case, if we further assume $\nu(\beta) \neq 0$, then*

$$E_\beta(1; f_s^p) = \begin{cases} 1 & \beta \in S_n(\mathfrak{o})^{*,0}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.1.13}$$

Proof. This is same as Proposition 4.1.5 except, rather than a Gauss sum, we obtain a term

$$\sum_{i=0}^{m}(-1)^i q^{\mathbf{d}_3\left(\frac{i(i-1)}{2}-im\right)} \sum_j \sum_{u \in \widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})\delta_{ij}^{-1}/\widetilde{\varpi}\mathrm{Mat}_m(\mathcal{O})} e(2\tau(\beta_5 u)),$$

which is nonzero unless $\beta_5 \in \mathrm{GL}_m(\mathcal{O})$. This can be shown by the property of exponential sums as in [BS00, page 1412]. □

4.1.3 Archimedean computations

We now turn to the archimedean setting. We fix an archimedean place v and omit it from the notation where we also abuse the notation by simply denoting $f_s^\infty := f_{s,v}^\infty$. Fix a positive integer l be our weight. Let $z = x + iyy^* \in \mathcal{H}_n$ and consider the local Fourier coefficients

$$E_\beta(z; f_s^\infty) = \int_{S_n(\mathbb{R})} f_s^\infty\left(J_n \begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix}\begin{bmatrix} y & x\hat{y} \\ 0 & \hat{y} \end{bmatrix}\right) e_\infty(-\tau(\beta S))dS. \tag{4.1.14}$$

For $y, \beta \in S_n(\mathbb{R})$ and $s_1, s_2 \in \mathbb{C}$ we define a function ξ_n by

$$\xi_n(y, \beta; s_1, s_2) = \int_{S_n(\mathbb{R})} \nu(s+iy)^{-s_1} \nu(s-iy)^{-s_2} e_\infty(\tau(\beta S)) dS. \tag{4.1.15}$$

By definition of f_s^∞, we have (for Case II, III, IV)

$$\begin{aligned}
&E_\beta(z, f_s^\infty) \\
=&\int_{S_n(\mathbb{R})} \nu(yi + x\hat{y} + S\hat{y})^{-l} |\nu(yi + x\hat{y} + S\hat{y})|^{l-s-\kappa} e_\infty(-\tau(\beta S)) dS \\
=&e_\infty(\tau(\beta x)) \nu(y)^{s+\kappa} \int_{S_n(\mathbb{R})} \nu(S + yy^* i)^{-\frac{s+\kappa+l}{2}} \nu(S - yy^* i)^{-\frac{s+\kappa-l}{2}} e_\infty(-\tau(\beta S)) dS \\
=&e_\infty(\tau(\beta x)) \nu(y)^{s+\kappa} \xi_n\left(yy^*, \beta; \frac{s+\kappa+l}{2}, \frac{s+\kappa-l}{2}\right).
\end{aligned} \tag{4.1.16}$$

Similarly, for Case V we have

$$\begin{aligned}
&E_\beta(z, f_s^\infty) \\
=&e_\infty(\tau(\beta x)) \nu(y^*)^l |\nu(y^*)|^{2s+2\kappa-l} \xi_n\left(yy^*, \beta; s+\kappa+\frac{l}{2}, s+\kappa-\frac{l}{2}\right).
\end{aligned} \tag{4.1.17}$$

Recall that in Case III, the symmetric space is same as the one for Case II. Denote $\xi_n^{\mathrm{II}}, \xi_n^{\mathrm{III}}$ to indicate above functions ξ_n in two cases. After identify β, yy^* with their image $\beta', (yy^*)'$ in $\{\beta' \in \mathrm{GL}_{2n}(\mathbb{R}) : {}^t\beta = \beta\}$ we have

$$\xi_n^{\mathrm{III}}\left(yy^*, \beta; \frac{s+\kappa+l}{2}, \frac{s+\kappa-l}{2}\right) = \xi_{2n}^{\mathrm{II}}\left((yy^*)', \beta'; s+\kappa+l, s+\kappa-l\right). \tag{4.1.18}$$

The function ξ_n is the confluent hypergeometric function studied in [Shi82]. We record some of its property in the following lemma.

Lemma 4.1.6. *Let $t = \mathrm{rank}(\beta)$ be the rank of β and t_+ (resp. t_-) the number of positive (resp. negative) eigenvalues of β. Then*

$$\xi(y, \beta; s_1, s_2) = \mathbf{\Gamma}(s_1, s_2) \times \omega(y, \beta; s_1, s_2) \tag{4.1.19}$$

where $\omega(y, \beta; s_1, s_2)$ is a holomorphic function in s_1, s_2 and $\mathbf{\Gamma}(s_1, s_2)$ is given by the following list.

(Case II, Symplectic)

$$\frac{\prod_{i=0}^{n-t-1}\Gamma\left(s_1+s_2-\frac{n+1+i}{2}\right)}{\prod_{i=0}^{n-t_--1}\Gamma\left(s_1-\frac{i}{2}\right)\prod_{i=0}^{n-t_+-1}\left(s_2-\frac{i}{2}\right)},$$

(Case IV, Quaternionic Unitary)

$$\frac{\prod_{i=0}^{n-t-1}\Gamma\left(2s_1+2s_2-2n+1-2i\right)}{\prod_{i=0}^{n-t_--1}\Gamma\left(2s_1-2i\right)\prod_{i=0}^{n-t_+-1}\Gamma\left(2s_2-2i\right)},$$

(Case V, Unitary)

$$\frac{\prod_{i=0}^{n-t-1}\Gamma\left(s_1+s_2-n-i\right)}{\prod_{i=0}^{n-t_--1}\Gamma\left(s_1-i\right)\prod_{i=0}^{n-t_+-1}\left(s_2-i\right)}.$$

In particular, if $\beta>0$, then

$$\omega(y,\beta;l,0)=2^n i^{-nl}\pi^{nl-\frac{\iota n(n-1)}{4}}\nu(2\beta)^{l-\kappa}e_\infty(i\tau(\beta y)), \tag{4.1.20}$$

with

$$\iota=\begin{cases}1 & \textit{Case II},\\ 4 & \textit{Case IV},\\ 2 & \textit{Case V}.\end{cases}$$

We are interested in the special value

$$s=s_0:=\begin{cases} l-\kappa & \text{Case II, III, IV},\\ \frac{l}{2}-\kappa & \text{Case V}.\end{cases}, \tag{4.1.21}$$

with

$$l\geq\begin{cases} m+1 & \text{Case II},\\ n+1 & \text{Case III, IV, V}.\end{cases} \tag{4.1.22}$$

and κ is defined as in (2.2.8). In this case we need to consider the special value of $\xi(y,\beta;s_1,s_2)$ at $s_1=l, s_2=0$.

Lemma 4.1.7. *The function $\xi(y,\beta;s_1,s_2)$ does not have a zero at $s_1=l,s_2=0$ only if $\beta>0$.*

Proof. We prove for Case IV and omit the same proof for Case II, V. Consider

$$\frac{\prod_{i=0}^{n-t-1}\Gamma\left(2l-2n+1-2i\right)}{\prod_{i=0}^{n-t_--1}\Gamma\left(2l-2i\right)\prod_{i=0}^{n-t_+-1}\Gamma\left(-2i\right)}.$$

and calculate the contributions of poles for each terms. By our assumption on l, $\prod_{i=0}^{n-t_--1}\Gamma(2l-2i)$ does not have poles. The numerator at most contributes $n-t$ poles while the denominator always has $n-t_+$ poles thus we must have $t=t_+$. □

We summarize the archimedean Fourier coefficients in the following proposition.

Proposition 4.1.8. *As a function of s, $E_\beta(z;f_s^\infty)$ does not have a zero at s_0 only if $\beta>0$. In this case, its value at $s=s_0$ is given by the following list.*

(Case II, Symplectic)

$$\frac{2^n i^{-nl}\pi^{nl-\frac{n(n-1)}{4}}}{\prod_{i=0}^{n-1}\Gamma\left(l-\frac{i}{2}\right)}\nu(2\beta)^{l-\frac{n+1}{2}}\nu(y)^l e_\infty(\tau(\beta z)),$$

(Case III, Quaternionic Orthogonal)

$$\frac{2^{2n}(-1)^{-nl}\pi^{2nl-\frac{n(2n-1)}{2}}}{\prod_{i=0}^{2n-1}\Gamma\left(l-\frac{i}{2}\right)}\nu(2\beta)^{l-\frac{2n+1}{2}}\nu(y)^l e_\infty(\tau(\beta z)),$$

(Case IV, Quaternionic Unitary)

$$\frac{2^n i^{-nl}\pi^{nl-n(n-1)}}{\prod_{i=0}^{n-1}\Gamma(2l-2i)}\nu(2\beta)^{l-\frac{2n-1}{2}}\nu(y)^l e_\infty(\tau(\beta z)),$$

(Case V, Unitary)

$$\frac{2^n i^{-nl}\pi^{nl-\frac{n(n-1)}{2}}}{\prod_{i=0}^{n-1}\Gamma(l-i)}\nu(2\beta)^{l-\frac{n}{2}}\nu(y^*)^l e_\infty(\tau(\beta z)).$$

4.1.4 The global Fourier expansion

We now study the global Fourier expansion of $\mathbb{E}(h;f_s,\chi,\boldsymbol{n})$ defined in (3.4.17). From now on, we assume the weight $\boldsymbol{l}=(l_v)_{v|\infty}$ is parallel. That is $l_v=l$ for all $v|\infty$. We also assume

$$\begin{aligned} &l\geq\begin{cases} m+1 & \text{Case II},\\ n+1 & \text{Case III, IV, V}.\end{cases} \quad \text{when } F\neq\mathbb{Q},\\ &l\geq\begin{cases} m+1 & \text{Case II},\\ n+r+1 & \text{Case III, IV, V},\end{cases} \quad \text{when } F=\mathbb{Q}.\end{aligned} \tag{4.1.23}$$

For such l, we are interested in the special value at

$$s = s_0 := \begin{cases} l - \kappa & \text{Case II, III, IV}, \\ \frac{l}{2} - \kappa & \text{Case V}. \end{cases} \tag{4.1.24}$$

Let $q_z \in G(\mathbb{A})$ be an element such that

$$(q_z)_v = \begin{cases} \mathrm{diag}[a_v, \hat{a}_v] & v \nmid \mathfrak{n}\boldsymbol{p}\infty, \\ 1 & v|\mathfrak{n}\boldsymbol{p}, \\ \begin{bmatrix} y_v & x_v\hat{y}_v \\ 0 & \hat{y}_v \end{bmatrix} & v|\infty, \end{cases} \tag{4.1.25}$$

for any $z = (z_v)_{v|\infty}$ with $z_v = x_v + y_v y_v^* i \in \mathcal{H}_n$. Denote

$$\boldsymbol{S} = S_n(F) \cap \prod_{v \nmid \mathfrak{n}\infty} \hat{a} S_n(\mathfrak{o}_v)^* a^{-1} \cdot \prod_{v|\mathfrak{n}_2\boldsymbol{p}} S_n(\mathfrak{o}_v)^{*,0} \cdot \prod_{v|\mathfrak{n}_1} S_n(\mathfrak{o}_v)^{*,\mathfrak{c}_v} \tag{4.1.26}$$

and

$$\boldsymbol{S}^{\boldsymbol{p}} = \left\{ \beta = \begin{bmatrix} \beta_1 & -\epsilon\beta_2^* & -\epsilon\beta_3^* \\ \beta_2 & \beta_4 & -\epsilon\beta_5^* \\ \beta_3 & \beta_5 & \beta_6 \end{bmatrix} \in \boldsymbol{S} : \begin{array}{c} \beta_2, \beta_3 \in \mathrm{Mat}_{m,r}(\boldsymbol{p}^{\boldsymbol{n}-1}\mathcal{O}_{\boldsymbol{p}}), \\ \beta_4, \beta_6 \in \mathrm{Mat}_m(\boldsymbol{p}^{2\boldsymbol{n}-2}\mathcal{O}_{\boldsymbol{p}}). \end{array} \right\}. \tag{4.1.27}$$

For a Hecke character $\chi = \prod_v \chi_v$, we define the Gauss sum $G(\chi) = \prod_v G(\chi_v)$ with $G(\chi_v)$ the local Gauss sum defined in (4.1.8). We may write $G^D(\chi)$ to indicate that the Gauss sum is defined for D. In Case II, IV, V, we always omit the superscript 'D' for simplicity as no confusion will occur. In Case III, we will need both $G^D(\chi)$ and $G^F(\chi)$ and we make the convention $G(\chi) := G^D(\chi)$.

Combining with the local computations in previous subsections, we summarize the Fourier expansion of $\mathbb{E}(h; f_s, \chi, \boldsymbol{n})$ in the following proposition.

Proposition 4.1.9. *At $s = s_0$, the Eisenstein series $\mathbb{E}(h; f_s, \chi, \boldsymbol{n})$ has a Fourier expansion of the form*

$$\mathbb{E}(q_z; f_s, \chi, \boldsymbol{n}) = \mathfrak{D} C(q_z) \nu(y'^*)^l \sum_{0 < \beta \in \boldsymbol{S}^{\boldsymbol{p}}} C(\beta, \chi) \cdot \mathbb{E}(\beta; \chi) e_\infty(\tau(\beta z')). \tag{4.1.28}$$

Here:

(1) $\mathfrak{D}$ *is a power of the discriminant of* D,

(2) $C(q_z) = \prod_{v\nmid\mathfrak{n}\infty} \chi(\nu(a_v))|N_{E/F}(\nu(a_v))|^{s_0+\kappa}$,

(3) $z' = x + iy'y'^*$ *with* $y' = \mathrm{diag}[1_r, \boldsymbol{\varpi}^{1-\boldsymbol{n}} \cdot 1_m, \boldsymbol{\varpi}^{1-\boldsymbol{n}} \cdot 1_m]y$.

(4) The constant $C(\beta, \chi)$ *is given by*

$$C(\beta,\chi) = \prod_{v|\mathfrak{n}_2} |\varpi_v|^{-\mathfrak{c}\mathbf{d}_1\frac{m(m-1)}{2}} \cdot |\boldsymbol{\varpi}|^{-\mathfrak{c}\mathbf{d}_1\frac{m(m-1)}{2}} G(\chi)^m \chi(\nu(\beta_5))\nu(2\beta)^{l-\kappa} \tag{4.1.29}$$

with $\mathbf{d}_1$ *as in* (2.4.14) *and* $\chi(\nu(\beta_5))$ *is understood as zero if* $\nu(\beta_5) \notin \mathcal{O}^\times_{\mathfrak{n}_2\boldsymbol{p}}$,

(5) $\mathbb{E}(\beta;\chi)$ *is given by the following list with* $d(F) = [F : \mathbb{Q}]$

(Case II, Symplectic)

$$\left(\frac{2^n i^{-nl}\pi^{nl-\frac{n(n-1)}{4}}}{\prod_{i=0}^{n-1}\Gamma\left(l-\frac{i}{2}\right)}\right)^{d(F)} \prod_{v\nmid\mathfrak{n}} L_v\left(s_0+\frac{1}{2},\chi\lambda_\beta\right)\cdot\prod_{v\nmid\mathfrak{n}} P_{a_v^*\beta a_v}(\chi(q_v)q_v^{-l}),$$

(Case III, Quaternionic Orthogonal with $r = 0$*)*

$$\left(\frac{2^{2n}(-1)^{-nl}\pi^{2nl-\frac{n(2n-1)}{2}}}{\prod_{i=0}^{2n-1}\Gamma\left(l-\frac{i}{2}\right)}\right)^{d(F)} \prod_{v\nmid\mathfrak{n}} L_v\left(s_0+\frac{1}{2},\chi\lambda_\beta\right)\cdot\prod_{v\nmid\mathfrak{n}} P_{a_v^*\beta a_v}(\chi(q_v)q_v^{-l}),$$

(Case III, Quaternionic Orthogonal with $r = 1$*)*

$$\left(\frac{2^{2n}(-1)^{-nl}\pi^{2nl-\frac{n(2n-1)}{2}}}{\prod_{i=0}^{2n-1}\Gamma\left(l-\frac{i}{2}\right)}\right)^{d(F)} \prod_{\substack{v\nmid\mathfrak{n}\\ v\,split}} L_v\left(s_0+\frac{1}{2},\chi\lambda_\beta\right)\prod_{v\nmid\mathfrak{n}} P_{a_v^*\beta a_v}(\chi(q_v)q_v^{-l}),$$

(Case IV, Quaternionic Unitary)

$$\left(\frac{2^n i^{-nl}\pi^{nl-n(n-1)}}{\prod_{i=0}^{n-1}\Gamma(2l-2i)}\right)^{d(F)} \prod_{v\nmid\mathfrak{n}} P_{a_v^*\beta a_v}(\chi(q_v)q_v^{-l}),$$

(Case V, Unitary)

$$\left(\frac{2^n i^{-nl}\pi^{nl-\frac{n(n-1)}{2}}}{\prod_{i=0}^{n-1}\Gamma(l-i)}\right)^{d(F)} \prod_{v\nmid\mathfrak{n}} P_{a_v^*\beta a_v}(\chi^0(q_v)q_v^{-l}).$$

Proof. We first show that only $\beta \in \boldsymbol{S}^{\boldsymbol{p}}$ can contribute a nonzero Fourier coefficient. Recall that the Eisenstein series $\mathbb{E}(h; f_s)$ is defined as

$$\mathbb{E}(h; f_s) := E(h; f_s)|\boldsymbol{U}(\boldsymbol{p}^{\boldsymbol{n}-1}) := \sum_\gamma E(h\gamma; f_s)$$

with γ running through elements of the form in (3.4.18). To ease the notation, we write

$$A = \begin{bmatrix} 1_r & 0 & 0 \\ \epsilon b_2^* \theta^{-1} & \boldsymbol{\varpi}^{\boldsymbol{n}-1} \cdot 1_m & 0 \\ -\epsilon b_1^* \theta^{-1} & 0 & \boldsymbol{\varpi}^{\boldsymbol{n}-1} \cdot 1_m \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{\epsilon \boldsymbol{\varpi}^{1-\boldsymbol{n}} b_2}{2} & -\frac{\epsilon \boldsymbol{\varpi}^{1-\boldsymbol{n}} b_1}{2} \\ -\frac{b_2^*}{2} & -\boldsymbol{\varpi}^{1-\boldsymbol{n}} c_2 & 0 \\ -\frac{b_1^*}{2} & 0 & \boldsymbol{\varpi}^{1-\boldsymbol{n}} c_1 \end{bmatrix},$$

so that $\gamma = \begin{bmatrix} A & B \\ 0 & \hat{A} \end{bmatrix}$. In the following, we write $\gamma_{\boldsymbol{p}}, A_{\boldsymbol{p}}, B_{\boldsymbol{p}}$ to indicate they are matrices with entries in $F_{\boldsymbol{p}}$. By straightforward computations,

$$\begin{aligned} \mathbb{E}_\beta(q_z; f_s) &= \int_{S_n(F)\backslash S_n(\mathbb{A})} \sum_{\gamma_{\boldsymbol{p}}} E\left(\begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix} q_z \gamma_{\boldsymbol{p}}; f_s \right) e_{\mathbb{A}}(-\tau(\beta S)) dS \\ &= \int_{S_n(F)\backslash S_n(\mathbb{A})} E\left(\begin{bmatrix} 1_n & S \\ 0 & 1_n \end{bmatrix} q_{z'}; f_s \right) e_{\mathbb{A}}(-\tau(\beta S)) dS \cdot \sum_{\gamma_{\boldsymbol{p}}} e_{\mathbb{A}}(\tau(\beta A_{\boldsymbol{p}}^{-1} B_{\boldsymbol{p}})). \end{aligned}$$

Here $q_{z'} \in G(\mathbb{A})$ is defined as in (4.1.25) for $z' = x + iy'y'^*$ with $y' = \mathrm{diag}[1_r, \boldsymbol{\varpi}^{1-\boldsymbol{n}} \cdot 1_m, \boldsymbol{\varpi}^{1-\boldsymbol{n}} \cdot 1_m]y$. The integral is the Fourier coefficients of $E(h; f_s)$ calculated in previous subsection. Hence, by Proposition 4.1.1, Proposition 4.1.2, Proposition 4.1.4, Proposition 4.1.5, this integral is nonzero unless $\beta \in \boldsymbol{S}$. The exponential sum is nonzero unless further $\beta \in \boldsymbol{S}^{\boldsymbol{p}}$.

Secondly, we show that only $\beta > 0$ can contribute a nonzero Fourier coefficient. Note that the condition $\beta \in \boldsymbol{S}^{\boldsymbol{p}}$ and Proposition 4.1.4, Proposition 4.1.5 imply that only such β with $\mathrm{rank}(\beta) \geq 2m$ can contribute a nonzero term. Under our assumptions on l, the L-functions occurring form Proposition 4.1.1 does not provide any poles. Therefore, the Fourier coefficients are nonvanishing if and only if the confluent hypergeometric function in archimedean computations does not have zeros. Then by Lemma 4.1.7, when specializing to $s = s_0$, $\mathbb{E}_\beta(q_z; f_s) \neq 0$ unless $\beta > 0$.

The proposition then follows from the explicit formulas in Proposition 4.1.1, Proposition 4.1.2, Proposition 4.1.4, Proposition 4.1.5 and Proposition 4.1.8. $\square$

Remark 4.1.10. We emphasize that we indeed obtain a better bound on l due the action of $\boldsymbol{U}(\boldsymbol{p})$ operator. Without such process, we have to consider β of all rank so that one need to assume $l \geq 2m+1$ in Case II and $2n+1$ in Case III, IV, V to avoid the occurrence of the poles in L-functions from unramified computations. When $r = 0$ or when $\boldsymbol{p}$ splits in Case III, we always have $\nu(\beta)$ is a square mod $\boldsymbol{p}$. This property is essential in the construction of p-adic L-functions for Case II, III (see also [Liu20, Section 3.5]).

From these explicit formulas of the Fourier coefficients, we immediately have

Corollary 4.1.11. *Up to a power of π, $\mathbb{E}(h; f_s, \chi, \boldsymbol{n})$ is an algebraic modular form on $H(\mathbb{A})$ at $s = s_0$. More precisely, we have*

$$\frac{\mathbb{E}(\beta,\chi)}{\pi^{d(F)d(\pi)}} \in \overline{\mathbb{Q}} \qquad \text{with } d(\pi) = \begin{cases} nl - \frac{n(n-1)}{4} + l - \frac{n}{2} & \text{Case II,} \\ 2nl - \frac{n(2n-1)}{2} + l - n & \text{Case III,} \\ nl - n(n-1) & \text{Case IV,} \\ nl - \frac{n(n-1)}{2} & \text{Case V.} \end{cases} \tag{4.1.30}$$

Furthermore, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$, we have

(Case II, Symplectic)

$$\left(\frac{\mathbb{E}(\beta;\chi)}{\pi^{d(F)\left(nl-\frac{n(n-1)}{4}\right)}(\pi i)^{d(F)\left(l-\frac{n}{2}\right)}G(\chi)}\right)^{\sigma} = \frac{\mathbb{E}(\beta;\chi^{\sigma})}{\pi^{d(F)\left(nl-\frac{n(n-1)}{4}\right)}(\pi i)^{d(F)\left(l-\frac{n}{2}\right)}G(\chi^{\sigma})},$$

(Case III, Quaternionic Orthognoal) In this case we denote $G^F(\chi)$ to indicate that the Gauss sum is defined for F.

$$\left(\frac{\mathbb{E}(\beta;\chi)}{\pi^{d(F)\left(2nl-\frac{n(2n-1)}{2}\right)}(\pi i)^{d(F)(l-n)}G^F(\chi)}\right)^{\sigma} = \frac{\mathbb{E}(\beta;\chi^{\sigma})}{\pi^{d(F)\left(2nl-\frac{n(2n-1)}{2}\right)}(\pi i)^{d(F)(l-n)}G^F(\chi^{\sigma})},$$

(Case IV, Quaternionic Unitary)

$$\left(\frac{\mathbb{E}(\beta;\chi)}{\pi^{d(F)(nl-n(n-1))}}\right)^{\sigma} = \frac{\mathbb{E}(\beta;\chi^{\sigma})}{\pi^{d(F)(nl-n(n-1))}},$$

(Case V, Unitary)

$$\left(\frac{\mathbb{E}(\beta;\chi)}{\pi^{d(F)\left(nl-\frac{n(n-1)}{2}\right)}}\right)^{\sigma} = \frac{\mathbb{E}(\beta;\chi^{\sigma})}{\pi^{d(F)\left(nl-\frac{n(n-1)}{2}\right)}}.$$

4.2 Properties of the special L-values

In this section, we apply our computations to study the properties of special L-values. Let $\boldsymbol{l} = (l, ..., l)$ be a parallel weight satisfying

$$\begin{aligned} &l \geq \begin{cases} m+1 & \text{Case II,} \\ n+1 & \text{Case III, IV, V.} \end{cases} \quad \text{when } F \neq \mathbb{Q}, \\ &l \geq \begin{cases} m+1 & \text{Case II,} \\ n+r+1 & \text{Case III, IV, V,} \end{cases} \quad \text{when } F = \mathbb{Q}. \end{aligned} \tag{4.2.1}$$

Fix a specific prime $\boldsymbol{p}$ of $\mathfrak{o}$ above an odd prime number p and an integral ideal $\mathfrak{n} = \mathfrak{n}_1\mathfrak{n}_2 = \prod_v \mathfrak{p}_v^{\mathfrak{c}_v}$ with $\mathfrak{n}_1, \mathfrak{n}_2, \boldsymbol{p}$ coprime. Denote $\boldsymbol{\varpi}$ for the uniformizer of $\boldsymbol{p}$. We make the following assumptions:

(1) $2 \in \mathcal{O}_v^\times$ and $\theta \in \mathrm{GL}_r(\mathcal{O}_v)$ for all $v|\mathfrak{n}\boldsymbol{p}$.

(2) $\boldsymbol{f} \in \mathcal{S}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}), \overline{\mathbb{Q}})$ is an algebraic eigenform for the Hecke algebra $\mathcal{H}(K(\mathfrak{n}\boldsymbol{p}), \mathfrak{X})$ as in Section 2.3.4.

(3) $\boldsymbol{f}$ is an eigenform for the $U(\boldsymbol{p})$ operator with eigenvalue $\alpha(\boldsymbol{p}) \neq 0$.

(4) $\chi = \chi_1\boldsymbol{\chi}$ with χ_1 has conductor $\mathfrak{n}_2$ and $\boldsymbol{\chi}$ has conductor $\boldsymbol{p}^{\boldsymbol{c}}$ for some integer $\boldsymbol{c} \geq 0$. We assume χ has infinity type $\boldsymbol{l}$. That is, $\chi_v(x) = x^l|x|^{-l}$ for all $v|\infty$.

(5) In Case V, all places $v|\mathfrak{n}\boldsymbol{p}$ are nonsplit in $\mathcal{O}$.

We study the special values of L-functions $L(s, \boldsymbol{f} \times \chi)$ at

$$s = s_0 := \begin{cases} l - \kappa & \text{Case II, III, IV,} \\ \frac{l}{2} - \kappa & \text{Case V.} \end{cases} \tag{4.2.2}$$

4.2.1 The algebraicity of special L-values

The following algebraic result is also studied in [BS00; Shi00] for symplectic and unitary groups and in [BJar] for quaternionic unitary groups. Comparing with our previous work in [BJar], here we obtain a better bound $l \geq n + r + 1$ rather than $l \geq 2n + 1$.

Theorem 4.2.1. *Let l and s_0 as in (4.2.1), (4.2.2). Then*

$$\begin{aligned} &\frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}\Omega\cdot\langle \boldsymbol{f},\boldsymbol{f}\rangle} \in \overline{\mathbb{Q}}, && \text{if } \boldsymbol{c}>0, \\ &\frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right) M\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}\Omega\cdot\langle \boldsymbol{f},\boldsymbol{f}\rangle} \in \overline{\mathbb{Q}}, && \text{if } \boldsymbol{c}=0. \end{aligned} \tag{4.2.3}$$

with

$$\boldsymbol{d}(\pi)=\begin{cases} nl-\frac{3m^2}{2}+l-\frac{n}{2} & \text{Case II,} \\ 2nl-\frac{3n^2}{2}+l-n & \text{Case III,} \\ nl-\frac{3}{2}n(n-1) & \text{Case IV,} \\ nl-\frac{n(n-1)}{2}-m(m+r) & \text{Case V.} \end{cases} \tag{4.2.4}$$

Here $\Omega=1$ in Case II, III, IV and in Case V, $\Omega\in\mathbb{C}^\times$ is the following CM period

$$\Omega=p_E\left(l\tau, r\tau\right), \tag{4.2.5}$$

where (E,τ) is a fixed CM type and p_E is the period notation given in [Shi00, Section 11.3].

Proof. The proof is similar to [BS00, Appendix] and [BJar; Shi00]. We remark that in [BJar; Shi00], one needs to use the fact that the space of algebraic modular forms is a direct sum of space of algebraic cusp forms and algebraic Eisenstein series. This result is proved in [Har84] when the Eisenstein series is absolutely convergent at $s=s_0$ which forces $l\geq n$ in Case I and $l\geq 2n+1$ in Case II, III, IV. This result is not necessary and not used in [BS00, Appendix] so that the special value below the absolutely convergence bound can be considered. However, the proof there need the assumption that the eigenvalue $\alpha(\boldsymbol{p})$ of the $U(\boldsymbol{p})$ operator for $\boldsymbol{f}$ is nonzero as we made here. We sketch the proof following [BS00, Appendix].

Let $\{\boldsymbol{f}_i\}$ be an orthogonal basis of $\mathcal{S}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}))$ consisting of eigenforms of the Hecke algebra $\mathcal{H}(K(\mathfrak{n}\boldsymbol{p}),\mathfrak{X})$, which without lossing generality we assume $\boldsymbol{f}_1=\boldsymbol{f}$. Take $\{\boldsymbol{h}_i\}$ be a basis of the orthogonal complement of $\mathcal{S}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}))$ in $\mathcal{M}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}))$. Denote $\boldsymbol{f}_i^1,\boldsymbol{f}_i^2$ (resp. $\boldsymbol{h}_i^1,\boldsymbol{h}_i^2$) such that $\boldsymbol{f}_i^1(g)=\boldsymbol{f}_i(g\eta_1\eta_{\boldsymbol{p}})$ (resp. $\boldsymbol{h}_i^1(g)=\boldsymbol{h}_i(g\eta_1\eta_{\boldsymbol{p}})$) and $\boldsymbol{f}_i^2(g)=\boldsymbol{f}_i(g\eta_2)$ (resp. $\boldsymbol{h}_i^2(g)=\boldsymbol{h}_i(g\eta_2)$) with η_1,η_2 in (3.4.7) and $\eta_{\boldsymbol{p}}$ in (3.4.21). Let

$\boldsymbol{V}$ be the operator defined by $\boldsymbol{f}|\boldsymbol{V} := \pi(\eta)\boldsymbol{f}|U'(\mathfrak{n}_1)$ and we use the superscript c to mean $\boldsymbol{f}^c(g) := \overline{\boldsymbol{f}(g^\iota)}$.

We can write the Eisenstein series $\boldsymbol{E}(g_1, g_2; f_s)$ as

$$\begin{aligned}\frac{\boldsymbol{E}(g_1, g_2; f_s)}{\pi^{d(F)d(\pi)}} &= \sum_{i,j} a_{ij}\boldsymbol{f}_i^1(g_1)\boldsymbol{f}_j^{2,c}|\boldsymbol{V}(g_2) + \sum_{i,j} b_{ij}\boldsymbol{h}_i^1(g_1)\boldsymbol{h}_j^{2,c}(g_2) \\ &+ \sum_{i,j} c_{ij}\boldsymbol{f}_i^1(g_1)\boldsymbol{h}_j^{2,c}(g_2) + \sum_{i,j} d_{ij}\boldsymbol{h}_i^1(g_1)\boldsymbol{f}_j^{2,c}|\boldsymbol{V}(g_2).\end{aligned} \tag{4.2.6}$$

We take the Petersson inner product on both sides of (4.2.6) with $\boldsymbol{f}_i^1$ for the first variable. Then the integral representation (3.4.25) shows that

$$\begin{aligned}&\frac{\boldsymbol{Z}(s; \boldsymbol{f}_i, f_s)}{\pi^{d(F)d(\pi)}\langle \boldsymbol{f}_i^2|\boldsymbol{V}, \boldsymbol{f}_i^2\rangle}\boldsymbol{f}_i^{2,c}|\boldsymbol{V}(g_2) \\ =&\sum_j a_{ij}\langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle \boldsymbol{f}_j^{2,c}|\boldsymbol{V}(g_2) + \sum_j c_{ij}\langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle \boldsymbol{h}_j^{2,c}(g_2).\end{aligned}$$

Clearly, we have $a_{ij} = 0$ if $j \neq i$ and $c_{ij} = 0$ for all j. Similarly taking the Petersson inner product on both sides of (4.2.6) with $\boldsymbol{f}_j^{2,c}|\boldsymbol{V}$ for the second variable, we conclude that $c_{ij} = d_{ij} = 0$ for all i, j and $a_{ij} \neq 0$ unless $i = j$ in which case

$$a_{ii} = \frac{\boldsymbol{Z}(s; \boldsymbol{f}_i, f_s)}{\pi^{d(F)d(\pi)}\langle \boldsymbol{f}_i, \boldsymbol{f}_i\rangle\langle \boldsymbol{f}_i^2|\boldsymbol{V}, \boldsymbol{f}_i^2\rangle}.$$

Hence we can write

$$\frac{\boldsymbol{E}(g_1, g_2; f_s)}{\pi^{d(F)d(\pi)}} = \sum_i a_{ii}\boldsymbol{f}_i^1(g_1)\boldsymbol{f}_i^{2,c}|\boldsymbol{V}(g_2) + \sum_{i,j} b_{ij}\boldsymbol{h}_i^1(g_1)\boldsymbol{h}_j^{2,c}(g_2). \tag{4.2.7}$$

Applying $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ on both sides of (4.2.7), we have

$$\left(\frac{\boldsymbol{E}(g_1, g_2; f_s)}{\pi^{d(F)d(\pi)}}\right)^\sigma = \sum_i a_{ii}^\sigma\boldsymbol{f}_i^{1,\sigma}(g_1)(\boldsymbol{f}_i^{2,c}|\boldsymbol{V})^\sigma(g_2) + \sum_{i,j} b_{ij}^\sigma\boldsymbol{h}_i^{1,\sigma}(g_1)\boldsymbol{h}_i^{2,c,\sigma}(g_2). \tag{4.2.8}$$

We now take the Petersson inner product on both sides of (4.2.8) with $\boldsymbol{f}_1^{1,\sigma}$ for the first variable g_1. For the left hand side, by Corollary 4.1.11, we have

$$\left(\frac{\boldsymbol{E}(g_1, g_2; f_s)}{\pi^{d(F)d(\pi)}}\right)^\sigma = \frac{\boldsymbol{E}(g_1, g_2; f_s)}{\pi^{d(F)d(\pi)}},$$

and the integral representation (3.4.25) shows that

$$\left\langle \frac{\boldsymbol{E}(g_1, g_2; f_s)}{\pi^{d(F)d(\pi)}}, \boldsymbol{f}_1^{1,\sigma}(g_1) \right\rangle = \frac{\boldsymbol{Z}(s, \boldsymbol{f}^\sigma, f_s)}{\pi^{d(F)d(\pi)} \langle \boldsymbol{f}_1^{2,\sigma} | \boldsymbol{V}, \boldsymbol{f}_1^{2,\sigma} \rangle} \cdot \left(\boldsymbol{f}_1^{2,\sigma} | \boldsymbol{V} \right)^c (g_2). \tag{4.2.9}$$

For the right hand side, we obtain

$$\begin{aligned} &\left(\frac{\boldsymbol{Z}(s; \boldsymbol{f}, f_s)}{\pi^{d(F)d(\pi)} \langle \boldsymbol{f}^\sigma, \boldsymbol{f}^\sigma \rangle \langle \boldsymbol{f}_1^{2,\sigma} | \boldsymbol{V}, \boldsymbol{f}_1^{2,\sigma} \rangle} \right)^\sigma \langle \boldsymbol{f}^\sigma, \boldsymbol{f}^\sigma \rangle \left(\boldsymbol{f}_1^{2,c} | \boldsymbol{V} \right)^\sigma (g_2) \\ &+ \sum_{i,j} b_{ij}^\sigma \langle \boldsymbol{h}_i^\sigma, \boldsymbol{f}_1^\sigma \rangle \boldsymbol{h}_i^{2,c,\sigma}(g_2). \end{aligned} \tag{4.2.10}$$

Our assumption on the algebraicity of $\boldsymbol{f}$ implies

$$\left(\boldsymbol{f}_1^{2,\sigma} \right)^c = \boldsymbol{f}_1^{2,c} \text{ and } \left(\Omega \cdot \boldsymbol{f}_1^{2,c} \right)^\sigma = \Omega \cdot \boldsymbol{f}_1^{2,c}.$$

Comparing (4.2.9) and (4.2.10) we conclude that

$$\frac{\boldsymbol{Z}(s, \boldsymbol{f}, f_s)}{\Omega \cdot \pi^{d(F)d(\pi)} \langle \boldsymbol{f}, \boldsymbol{f} \rangle \langle \boldsymbol{f}_1^2 | \boldsymbol{V}, \boldsymbol{f}_1^2 \rangle} = \left(\frac{\boldsymbol{Z}(s, \boldsymbol{f}, f_s)}{\Omega \cdot \pi^{d(F)d(\pi)} \langle \boldsymbol{f}, \boldsymbol{f} \rangle \langle \boldsymbol{f}_1^2 | \boldsymbol{V}, \boldsymbol{f}_1^2 \rangle} \right)^\sigma.$$

We finally conclude the theorem by the integral representation in Corollary 3.4.5. Assume $\boldsymbol{c} > 0$ (the case $\boldsymbol{c} = 0$ is similar), the term $\alpha(\boldsymbol{p})^{2\boldsymbol{n}-2}$ and the constant C' are algebraic so that

$$\frac{c_l(s)^{d(F)} L\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right)}{\Omega \cdot \pi^{d(F)d(\pi)} \langle \boldsymbol{f}, \boldsymbol{f} \rangle} = \left(\frac{c_l(s)^{d(F)} L\left(s + \frac{1}{2}, \boldsymbol{f} \times \chi\right)}{\Omega \cdot \pi^{d(F)d(\pi)} \langle \boldsymbol{f}, \boldsymbol{f} \rangle} \right)^\sigma.$$

The theorem then follows by the easy calculation of the power of π. □

When $r = 0$, we can define the action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on $\boldsymbol{f} \in \mathcal{S}_l(K(\mathfrak{n}\boldsymbol{p}), \overline{\mathbb{Q}})$ on the Fourier coefficients of $\boldsymbol{f}$. In this case we have the following refined version of above theorem.

Theorem 4.2.2. *Assume $r = 0$. Let l and s_0 as in (4.2.1), (4.2.2). For $\boldsymbol{c} > 0$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ we have*

(Case II, Symplectic)

$$\left(\frac{\chi(\mathfrak{n}_1)^{m\boldsymbol{d}_1} L\left(s_0 + \frac{1}{2}, \boldsymbol{f} \times \chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)} i^m G(\chi)^{m+1} \langle \boldsymbol{f}, \boldsymbol{f} \rangle} \right)^\sigma = \frac{\chi^\sigma(\mathfrak{n}_1)^{m\boldsymbol{d}_1} L\left(s_0 + \frac{1}{2}, \boldsymbol{f}^\sigma \times \chi^\sigma\right)}{\pi^{d(F)\boldsymbol{d}(\pi)} i^m G(\chi^\sigma)^{m+1} \langle \boldsymbol{f}^\sigma, \boldsymbol{f}^{c\sigma c} \rangle},$$

(Case III, Quaternionic Orthogonal)

$$\left(\frac{\chi(\mathfrak{n}_1)^{m\boldsymbol{d}_1}L\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}G^F(\chi)G(\chi)^m\langle\boldsymbol{f},\boldsymbol{f}\rangle}\right)^\sigma=\frac{\chi^\sigma(\mathfrak{n}_1)^{m\boldsymbol{d}_1}L\left(s_0+\frac{1}{2},\boldsymbol{f}^\sigma\times\chi^\sigma\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}G^F(\chi^\sigma)G(\chi^\sigma)^m\langle\boldsymbol{f}^\sigma,\boldsymbol{f}^{c\sigma c}\rangle},$$

(Case IV, Quaternionic Unitary)

$$\left(\frac{\chi(\mathfrak{n}_1)^{m\boldsymbol{d}_1}L\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}G(\chi)^m\langle\boldsymbol{f},\boldsymbol{f}\rangle}\right)^\sigma=\frac{\chi^\sigma(\mathfrak{n}_1)^{m\boldsymbol{d}_1}L\left(s_0+\frac{1}{2},\boldsymbol{f}^\sigma\times\chi^\sigma\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}G(\chi^\sigma)^m\langle\boldsymbol{f}^\sigma,\boldsymbol{f}^{c\sigma c}\rangle},$$

(Case V, Unitary)

$$\left(\frac{\chi(\mathfrak{n}_1)^{m\boldsymbol{d}_1}L\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}G(\chi)^m\Omega\cdot\langle\boldsymbol{f},\boldsymbol{f}\rangle}\right)^\sigma=\frac{\chi^\sigma(\mathfrak{n}_1)^{m\boldsymbol{d}_1}L\left(s_0+\frac{1}{2},\boldsymbol{f}^\sigma\times\chi^\sigma\right)}{\pi^{d(F)\boldsymbol{d}(\pi)}G(\chi^\sigma)^m\Omega\cdot\langle\boldsymbol{f}^\sigma,\boldsymbol{f}^{c\sigma c}\rangle}.$$

Here we use the superscript c to mean $\boldsymbol{f}^c(g)=\overline{\boldsymbol{f}(g^\iota)}$. When $\boldsymbol{c}=0$, one replace $L\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)$ by $L\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)M\left(s_0+\frac{1}{2},\boldsymbol{f}\times\chi\right)$ in above formulas.

Proof. We omit as it can be proved by the similar argument as in Theorem 4.2.1 (see also [BS00, Appendix]). □

Remark 4.2.3. We do not obtain above theorem in general for any r because we do not have a well defined action of $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. If for a field $\Psi\subset\overline{\mathbb{Q}}$, one can define the meaning of $\mathcal{M}_l(K,\Psi)\subset\mathcal{M}_l(K,\overline{\mathbb{Q}})$ properly such that $\mathcal{M}_l(K,\overline{\mathbb{Q}})=\mathcal{M}_l(K,\Psi)\otimes_\Psi\overline{\mathbb{Q}}$, then one can further define the action $\mathrm{Gal}(\overline{\mathbb{Q}}/\Psi)$ on $\mathcal{M}_l(K,\overline{\mathbb{Q}})$ by acting on $\overline{\mathbb{Q}}$. If this action preserves the subspace $\mathcal{S}_l(K,\Psi)$, then one can refine Theorem 4.2.1 to obtain similar formulas as in Theorem 4.2.2 for $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\Psi)$.

4.2.2 Preliminary on p-adic L-functions

We now turn to the p-adic interpolation of the special value $L(s_0+\frac{1}{2},\boldsymbol{f}\times\chi)$. For our specified prime $\boldsymbol{p}$, let p be the prime number under $\boldsymbol{p}$ and $\mathbb{C}_p=\widehat{\overline{\mathbb{Q}}}_p$ the completion of $\overline{\mathbb{Q}}_p$. Fix an embedding $\overline{\mathbb{Q}}\to\mathbb{C}_p$. The p-adic absolute value $|\cdot|_p$ naturally extends to $\mathbb{C}_p$ and we denote

$$\mathcal{O}_{\mathbb{C}_p}=\{x\in\mathbb{C}_p:|x|_p\leq 1\}. \tag{4.2.11}$$

Consider the p-adic analytic group

$$\mathrm{Cl}_E^+(\boldsymbol{p}^\infty) = E^\times \backslash \mathbb{A}_E^\times / U(\boldsymbol{p}^\infty) E_\infty^+ \tag{4.2.12}$$

where $U(\boldsymbol{p}^\infty)$ is the group of elements of $\hat{\mathfrak{o}}^\times$ that are congruent to 1 mod $\boldsymbol{p}^n$ for all integers n with $\hat{\mathfrak{o}}$ the completion of $\mathfrak{o}$ and E_∞^+ the connected component of the identity in $E_\infty = E \otimes_{\mathbb{Q}} \mathbb{R}$. We refer the reader to [BW19, Section 10.2] for more details on the geometry of this space and the locally analytic functions on this space. We denote by $\mathcal{A}(\mathrm{Cl}_E^+(\boldsymbol{p}^\infty), \mathbb{C}_p)$ the space of locally analytic functions on $\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)$ and the space of p-adic distributions $\mathcal{D}(\mathrm{Cl}_E^+(\boldsymbol{p}^\infty), \mathbb{C}_p)$ are defined as the topological dual of $\mathcal{A}(\mathrm{Cl}_E^+(\boldsymbol{p}^\infty), \mathbb{C}_p)$. Clearly there is a natural pairing

$$\begin{aligned} \mathcal{A}(\mathrm{Cl}_E^+(\boldsymbol{p}^\infty), \mathbb{C}_p) \times \mathcal{D}(\mathrm{Cl}_E^+(\boldsymbol{p}^\infty), \mathbb{C}_p) &\to \mathbb{C}_p, \\ (f, \mu) \mapsto \mu(f) &=: \int_{\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)} f d\mu. \end{aligned} \tag{4.2.13}$$

A p-adic distribution is called a p-adic measure if it is bounded.

The Hecke character $\boldsymbol{\chi}$ of conductor $\boldsymbol{p}^{\boldsymbol{c}}$ defines a locally analytic function on $\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)$ as in [BW19, Section 2.2.2]. Since they forms a dense subspace of $\mathcal{A}(\mathrm{Cl}_E^+(\boldsymbol{p}^\infty), \mathbb{C}_p)$, a p-adic distribution is uniquely determined by its value at all these Hecke characters. In following two subsections, we define the p-adic distribution interpolating the special value $L(s_0 + \frac{1}{2}, \boldsymbol{f} \times \chi)$ for p-ordinary $\boldsymbol{f}$ and prove that the distribution we constructed is indeed a p-adic measure. Note that Case II, III and Case IV, V are treated separately because of the occurrence of the Hecke L-function for the Fourier expansion in Proposition 4.1.9 for Case II, III.

We end up this subsection by the following important preliminary lemma.

Lemma 4.2.4. *Let $F(h)$ be a modular form on $H(\mathbb{A})$ with a Fourier expansion of the form*

$$F(q_z) = \nu(y^*)^l \sum_{\beta \in S} C(\beta) e_\infty(\tau(\beta z)) \tag{4.2.14}$$

with q_z as in 4.1.25. We further assume that $F(g_1, g_2) \in \mathcal{M}_l(\mathcal{K}(\boldsymbol{p}^2)) \otimes \mathcal{M}_l(\mathcal{K}'(\boldsymbol{p}^2))$ with notation as in Section 3.4.

(1) Let $\{\boldsymbol{f}_i\}$ *be a basis of* $\mathcal{M}_{\boldsymbol{l}}(K(\mathfrak{n}\boldsymbol{p}^2), \overline{\mathbb{Q}})$ *and denote* $\boldsymbol{f}_i^1$ *(resp.* $\boldsymbol{f}_i^2$*) such that* $\boldsymbol{f}_i^1(g) = \boldsymbol{f}_i(g\eta_1\eta_{\boldsymbol{p}})$ *(resp.* $\boldsymbol{f}_i^2(g) = \boldsymbol{f}_i(g\eta_2)$*) with* η_1, η_2 *in* (3.4.7) *and* $\eta_{\boldsymbol{p}}$ *in* (3.4.21)*. Then there exists some constants* a_{ij} *such that*

$$F(g_1, g_2) = \sum_{i,j} a_{ij} \boldsymbol{f}_i^1(g_1) \boldsymbol{f}_j^2(g_2). \tag{4.2.15}$$

(2) There exist a constant $\Omega_{\boldsymbol{p}} \in \overline{\mathbb{Q}}^\times$ *independent of* F *such that if* $C(\beta) \in \mathcal{O}_{\mathbb{C}_p}$ *then* $a_{ij} \in \Omega \cdot \Omega_{\boldsymbol{p}} \cdot \mathcal{O}_{\mathbb{C}_p}$ *where* $\Omega = 1$ *in Case II, III, IV and* Ω *is the CM period* (4.2.5) *in Case V.*

Proof. This lemma is a p-adic analogue of [Shi00, Lemma 24.11, Lemma 26.12]. The first part is already proved there and it is also shown that if $C(\beta) \in \overline{\mathbb{Q}}$ then $a_{ij} \in \Omega \cdot \overline{\mathbb{Q}}$. We descent the argument there to $\mathcal{O}_{\mathbb{C}_p}$.

Let $\{\boldsymbol{h}_i\}$ be a basis of $\mathcal{M}_{\boldsymbol{l}}^H(\mathcal{K}, \overline{\mathbb{Q}})$ (i.e. space of algebraic modular forms over H) where $\mathcal{K}$ is the image of $\mathcal{K}(\boldsymbol{p}^2) \times \mathcal{K}'(\boldsymbol{p}^2)$ under doubling map. We can write

$$F(h) = \sum_i A_i \cdot \boldsymbol{h}_i(h) \text{ for } A_i \in \mathbb{C}$$

and note that each $\boldsymbol{h}_i$ has a Fourier expansion of the form

$$\boldsymbol{h}_i(q_z) = \nu(y^*)^l \sum_{\beta \in S} c_i(\beta) e_\infty(\tau(\beta z)).$$

There exist a system $\{\beta_i\}$ such that the matrix $[c_i(\beta_k)]_{ik}$ is of full rank. For each i, k, $c_i(\beta_k) \in \overline{\mathbb{Q}}$ by the algebraicity of $\boldsymbol{h}_i$ and we can pick a constant Ω_1 depending on $\{\beta_i\}$ and $\{\boldsymbol{h}_i\}$ such that $c_i(\beta_k) \in \Omega_1 \mathcal{O}_{\mathbb{C}_p}$. Then $C(\beta) \in \mathcal{O}_{\mathbb{C}_p}$ implies $A_i \in \Omega_1^{-1} \mathcal{O}_{\mathbb{C}_p}$.

Choose a system of CM points $\{g_i\}$ of G such that the matrix $X = [\boldsymbol{f}_i(g_k)]_{ik}$ is of full rank. Note that for any k, (g_k, g_k) is a CM point of H so that $\boldsymbol{h}_i((g_k, g_k)) \in \mathcal{P}((g_k, g_k))\overline{\mathbb{Q}}$ where $\mathcal{P}((g_k, g_k))$ is the period of CM points over H. There exist a constant Ω_2 depending on $\{g_k\}$ and $\{\boldsymbol{h}_i\}$ such that

$$\boldsymbol{h}_i((g_k, g_k)) \in \Omega_2 \mathcal{P}((g_k, g_k)) \mathcal{O}_{\mathbb{C}_p}.$$

Then $C(\beta) \in \mathcal{O}_{\mathbb{C}_p}$ further implies

$$F((g_k, g_k)) \in \Omega_1^{-1}\Omega_2\mathcal{P}((g_k, g_k))\mathcal{O}_{\mathbb{C}_p}.$$

By the algebraicity of $\boldsymbol{f}_i$, we have $\boldsymbol{f}_i(g_k) \in \mathcal{P}(g_k)\overline{\mathbb{Q}}$ where $\mathcal{P}(g_k)$ is the period of CM points over G. We can choose a constant Ω_3 depending on $\{g_k\}$ and $\{\boldsymbol{f}_i\}$ such that

$$\boldsymbol{f}_i(g_k) \in \Omega_3\mathcal{P}(g_k)\mathcal{O}_{\mathbb{C}_p}.$$

Now write F as in (4.2.15) and compare the period $\mathcal{P}(g_k, g_k), \mathcal{P}(g_k)$ as in the proof of [Shi00, Lemma 26.12], we conclude that $C(\beta) \in \mathcal{O}_{\mathbb{C}_p}$ implies

$$a_{ij} \in \Omega \cdot \Omega_1^{-1}\Omega_2\Omega_3^{-1} \cdot \mathcal{O}_{\mathbb{C}_p}.$$

Take $\Omega_{\boldsymbol{p}} = \Omega_1^{-1}\Omega_2\Omega_3^{-1}$ which is clearly independent of F by our above constructions and the lemma follows. $\square$

4.2.3 p-adic L-functions for unitary and quaternionic unitary groups

Denote $\Omega = 1$ for quaternionic unitary groups and Ω is the CM period (4.2.5) for unitary groups. Fix χ_1 be a Hecke character of conductor $\mathfrak{n}_2$ and infinity type $\boldsymbol{l}$. We define a p-adic distribution $\mu(\boldsymbol{f})$ such that for any Hecke character $\boldsymbol{\chi}$ of conductor $\boldsymbol{p}^{\boldsymbol{c}}$,

$$\begin{aligned}\int_{\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) := {}& \alpha(\boldsymbol{p})^{2-2\boldsymbol{n}} C'^{-1} |\boldsymbol{\varpi}|^{\mathbf{c}\mathbf{d}_1 \frac{m(m-1)}{2}} G(\chi)^{-m} \pi^{d(F)d(\pi)} \\ & \times \left(\prod_{i=0}^{n-1} \Gamma(\mathbf{d}_1(l-i))\right)^{d(F)} \cdot \frac{\boldsymbol{Z}(s_0; \boldsymbol{f}, f_s, \chi, \boldsymbol{n})}{\Omega \cdot \langle \boldsymbol{f}, \boldsymbol{f}\rangle^2}.\end{aligned} \tag{4.2.16}$$

Here we are again denoting $\chi = \boldsymbol{\chi}\chi_1$ when $\boldsymbol{\chi}$ varying. The right hand side in above formula is indeed independent of $\boldsymbol{n}$ and $\mu(\boldsymbol{f})$ is a well-defined p-adic distribution.

Assume $\boldsymbol{\chi}$ is of finite order and χ has infinity type $\boldsymbol{l}$. By (3.4.22), (3.4.23) we have for $\boldsymbol{c} > 0$,

$$\begin{aligned}\int_{\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) = |\boldsymbol{\varpi}|^{\mathbf{c}\mathbf{d}_1 \frac{m(m-1)}{2}} G(\chi)^{-m} \pi^{d(F)d(\pi)} \\ \times \left(c_l(s_0) \prod_{i=0}^{n-1} \Gamma(\mathbf{d}_1(l-i)) \right)^{d(F)} \cdot \frac{L\left(s_0 + \frac{1}{2}, \boldsymbol{f} \times \chi\right)}{\Omega \cdot \langle \boldsymbol{f}, \boldsymbol{f} \rangle},\end{aligned} \tag{4.2.17}$$

and for $\boldsymbol{c} = 0$,

$$\begin{aligned}\int_{\mathrm{Cl}_E^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) = G(\chi)^{-m} \pi^{d(F)d(\pi)} M\left(s_0 + \frac{1}{2}, \boldsymbol{f} \times \chi\right) \\ \times \left(c_l(s_0) \prod_{i=0}^{n-1} \Gamma(\mathbf{d}_1(l-i)) \right)^{d(F)} \cdot \frac{L\left(s_0 + \frac{1}{2}, \boldsymbol{f} \times \chi\right)}{\Omega \cdot \langle \boldsymbol{f}, \boldsymbol{f} \rangle}.\end{aligned} \tag{4.2.18}$$

Theorem 4.2.5. *Assume $\boldsymbol{f}$ is $\boldsymbol{p}$-ordinary in the sense that $\alpha(\boldsymbol{p}) \in \mathcal{O}_{\mathbb{C}_p}^\times$. Then $\mu(\boldsymbol{f})$ defined above is a p-adic measure.*

Proof. The proof is similar to [BS00, Section 9]. Indeed, by Lemma 4.2.4, the boundness of the distribution $\mu(\boldsymbol{f})$ defined above follows from the boundness of the Fourier coefficients of Eisenstein series which can be checked straightforwardly from explicit formulas in Proposition 4.1.1. One can also verify $\mu(\boldsymbol{f})$ is a p-adic measure by checking the Kummer congruences following [CP04]. For more details see also [Jin22, Theorem 6.4], in which we prove the Kummer congruences for totally isotropic quaternionic unitary groups when $F = \mathbb{Q}$. $\square$

4.2.4 p-adic L-functions for symplectic and quaternionic orthogonal groups

As we have mentioned before, the Case II, III are different to Case IV, V because of the occurrence of the Hecke L-function for the Fourier expansion in Proposition 4.1.9. Therefore, we treat these two cases by the known p-adic interpolation of Hecke L-functions as in [BS00, Section 8] and [Liu20].

We first recall some fact about Hecke L-functions. Let $\psi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be any Hecke character trivial at infinity with conductor $c(\psi)$. We assume for simplicity that l, m has the same parity, i.e. $l \equiv m \bmod 2$ in Case II so that $s_0 + \frac{1}{2}$ is always even. In this case, there is a functional equation ([Shi00, Theorem 18.12])

$$L\left(s_0+\frac{1}{2},\psi\right)=\left(\frac{(2\pi i)^{s_0+\frac{1}{2}}}{2\Gamma\left(s_0+\frac{1}{2}\right)}\right)^{d(F)}\frac{\mathfrak{D}_F^{1/2}G^F(\psi)}{N_{F/\mathbb{Q}}(c(\psi))^{s_0-\frac{1}{2}}}\cdot L\left(\frac{1}{2}-s_0,\psi^{-1}\right) \tag{4.2.19}$$

We denote

$$L_{\boldsymbol{p}}\left(s_0+\frac{1}{2},\psi\right)=\begin{cases} 1 & \boldsymbol{p}|c(\psi), \\ \left(1-\psi(\boldsymbol{\varpi})|\boldsymbol{\varpi}|^{s_0+\frac{1}{2}}\right)^{-1} & \boldsymbol{p}\nmid c(\psi), \end{cases} \tag{4.2.20}$$

for the local L-factor at $\boldsymbol{p}$.

When $c(\psi)$ is coprime to $\boldsymbol{p}$, there is a p-adic measure $\mu(\psi)$ (see for example [Bar78; Cas79; DR80]) such that for all Hecke character $\boldsymbol{\chi}$ of conductor $\boldsymbol{p^c}$,

$$\int_{\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)}\boldsymbol{\chi}\mu(\psi)=L_{\boldsymbol{p}}\left(\frac{1}{2}-s_0,\psi^{-1}\boldsymbol{\chi}\right)\cdot L\left(\frac{1}{2}-s_0,\psi^{-1}\boldsymbol{\chi}\right). \tag{4.2.21}$$

The existence of such measure is equivalent to the existence of Kummer congruences ([CP04, Proposition 1.7]). In particular, for some constant $C\in\mathbb{C}_p$ with $C\cdot\boldsymbol{\chi}(x)\in\mathcal{O}_{\mathbb{C}_p}$ for all $x\in\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)$, we have

$$C\cdot L_{\boldsymbol{p}}\left(\frac{1}{2}-s_0,\psi^{-1}\boldsymbol{\chi}\right)\cdot L\left(\frac{1}{2}-s_0,\psi^{-1}\boldsymbol{\chi}\right)\in\mathcal{O}_{\mathbb{C}_p} \tag{4.2.22}$$

Fix χ_1 be a Hecke character of conductor $\mathfrak{n}_2$ and infinity type $\boldsymbol{l}$. We define a p-adic distribution $\mu(\boldsymbol{f})$ such that for any Hecke character $\boldsymbol{\chi}$ of conductor $\boldsymbol{p^c}$,

$$\begin{aligned}\int_{\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)}\boldsymbol{\chi}d\mu(\boldsymbol{f}):=&\,\alpha(\boldsymbol{p})^{2-2\boldsymbol{n}}C'^{-1}|\boldsymbol{\varpi}|^{\mathbf{cd}_1\frac{m(m-1)}{2}}N_{F/\mathbb{Q}}(\boldsymbol{p})^{\mathbf{c}\left(s_0-\frac{1}{2}\right)}\pi^{d(F)d(\pi)}\\ &\times G(\chi)^{-m}G^F(\chi)^{-1}\left(\Gamma\left(s_0+\frac{1}{2}\right)\prod_{i=0}^{n\mathbf{d}_1-1}\Gamma\left(l-\frac{i}{2}\right)\right)^{d(F)}\\ &\times\frac{L_{\boldsymbol{p}}\left(s_0+\frac{1}{2},\chi\right)}{L_{\boldsymbol{p}}\left(\frac{1}{2}-s_0,\chi^{-1}\right)}\cdot\frac{\boldsymbol{Z}(s_0;\boldsymbol{f},f_s,\chi,\boldsymbol{n})}{\langle\boldsymbol{f},\boldsymbol{f}\rangle}.\end{aligned} \tag{4.2.23}$$

Assume $\boldsymbol{\chi}$ is of finite order and χ has infinity type $\boldsymbol{l}$. By (3.4.22), (3.4.23) we have for $\boldsymbol{c} > 0$,

$$\int_{\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) = |\boldsymbol{\varpi}|^{\mathbf{cd}_1 \frac{m(m-1)}{2}} N_{F/\mathbb{Q}}(\boldsymbol{p})^{\boldsymbol{c}\left(s_0-\frac{1}{2}\right)} G(\chi)^{-m} G^F(\chi)^{-1} \pi^{d(F)d(\pi)}$$
$$\times \left(c_l(s_0)\Gamma\left(s_0+\frac{1}{2}\right)\prod_{i=0}^{n\mathbf{d}_1-1}\Gamma\left(l-\frac{i}{2}\right)\right)^{d(F)} \cdot \frac{1-\chi^{-1}(\boldsymbol{\varpi})|\boldsymbol{\varpi}|^{\frac{1}{2}-s_0}}{1-\chi(\boldsymbol{\varpi})|\boldsymbol{\varpi}|^{s_0+\frac{1}{2}}}$$
$$\times \frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\langle \boldsymbol{f}, \boldsymbol{f}\rangle}, \tag{4.2.24}$$

and for $\boldsymbol{c} = 0$,

$$\begin{aligned}&\int_{\mathrm{Cl}_F^+(\boldsymbol{p}^\infty)} \boldsymbol{\chi} d\mu(\boldsymbol{f}) \\ =&|\boldsymbol{\varpi}|^{\mathbf{cd}_1 \frac{m(m-1)}{2}} N_{F/\mathbb{Q}}(\boldsymbol{p})^{\boldsymbol{c}\left(s_0-\frac{1}{2}\right)} G(\chi)^{-m} G^F(\chi)^{-1} \pi^{d(F)d(\pi)} \\ \times&\left(c_l(s_0)\Gamma\left(s_0+\frac{1}{2}\right)\prod_{i=0}^{n\mathbf{d}_1-1}\Gamma\left(l-\frac{i}{2}\right)\right)^{d(F)} \\ \times& M\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)\cdot\frac{L\left(s_0+\frac{1}{2}, \boldsymbol{f}\times\chi\right)}{\langle \boldsymbol{f}, \boldsymbol{f}\rangle}.\end{aligned} \tag{4.2.25}$$

Theorem 4.2.6. *Assume $l \equiv m \bmod 2$ in Case II and $\boldsymbol{p}$ splits in Case III when $r = 1$. Assume $\boldsymbol{f}$ is $\boldsymbol{p}$-ordinary in the sense that $\alpha(\boldsymbol{p}) \in \mathcal{O}_{\mathbb{C}_p}^\times$. Then $\mu(\boldsymbol{f})$ defined above is a p-adic measure.*

Proof. The argument for checking $\mu(\boldsymbol{f})$ is again similar to [BS00, Section 9] or [CP04]. We have also proved the Kummer congruences for isotropic quaternionic orthogonal groups when $F = \mathbb{Q}$ in [Jin22, Theorem 6.5]. The main difference between Case II, III and Case IV, V is the occurrence of following term

$$\prod_{v\nmid\mathfrak{n}} L_v\left(s_0+\frac{1}{2}, \chi\lambda_\beta\right)$$

for the Fourier expansion in Proposition 4.1.9. We now explain how to p-adically interpolate this term. When $r = 1$ in Case III, we take the product only for those v splits in D, but this does not change our argument. Comparing (4.2.16) and (4.2.23), notice that we have multiply a term

$$N_{F/\mathbb{Q}}^{\boldsymbol{c}\left(s_0-\frac{1}{2}\right)} G^F(\chi)^{-1}\Gamma\left(s_0+\frac{1}{2}\right)^{d(F)} \frac{L_{\boldsymbol{p}}\left(s_0+\frac{1}{2},\chi\right)}{L_{\boldsymbol{p}}\left(\frac{1}{2}-s_0,\chi^{-1}\right)}.$$

By the functional equation (4.2.19), after multiplying above term and cancel out the power of π it remains to consider

$$\frac{G^F(\chi\lambda_\beta)}{G^F(\chi)}\cdot\prod_{v|\mathfrak{n}}L_v\left(s_0+\frac{1}{2},\chi\lambda_\beta\right)^{-1}\cdot\frac{L_{\boldsymbol{p}}\left(s_0+\frac{1}{2},\chi\right)}{L_{\boldsymbol{p}}\left(\frac{1}{2}-s_0,\chi^{-1}\right)}L\left(\frac{1}{2}-s_0,\chi^{-1}\lambda_\beta^{-1}\right).$$

Under our assumption ($\boldsymbol{p}$ splits in Case III when $r=1$), $\nu(\beta)$ is always a square mod $\boldsymbol{p}$ (Remark 4.1.10) so that $\lambda_\beta(\boldsymbol{\varpi})=1$. Using the p-adic interpolation of above Hecke L-function, especially (4.2.22), one checks that above term is in $\mathcal{O}_{\mathbb{C}_p}$ up to a bounded constant. Then our theorem follows from the explicit formulas for Fourier expansion of Eisenstein series in Proposition 4.1.1 and Lemma 4.2.4. □

We give a remark on what we have not done in this thesis.

Remark 4.2.7.

(1) In this thesis we have only consider one critical point at s_0. Of course one may also discuss other critical points by the standard process of applying differential operators. There are two approaches for applying the differential operators. One is following [CP04; Liu20; Shi00], in which the differential operator studied in [Shi94] is applied. This kind of differential operators are defined for all classical groups discussed here but one need to consider the nearly holomorphic Eisenstein series and apply the holomorphic projection. Another approach is following [BS00], in which the holomorphic differential operator constructed in [Böc85] is used and the holomorphic projection is avoided. The differential operator constructed there can also generalized to other groups with $r=0$ (see for example [Jin22] for the quaternionic unitary case). However, we do not know whether one can construct such differential operators for general groups with $r>0$.

(2) For the p-adic L-functions, we have only computed the interpolation at χ as assumed at the beginning of Section 4.2. In particular, we have only considered χ of infinity type $\boldsymbol{l}$ coincided with the weight of modular forms $\boldsymbol{f}$. This is because in our integral representation, we need the weight of Eisenstein series (which equals the infinity type of χ) coincide with the weight of modular forms. To consider Hecke

characters of other infinity type, we will need applying the differential operators on the Eisenstein series to shift the weight.

(3) We have only considered the parallel weight in this chapter. Especially, our archimedean computations have only done for scalar weight. For the general weight, [EL20] and [Liu21] have computed the archimedean integrals for unitary and symplectic groups. Also in [PSS21; HPSS22] the archimedean integrals are calculated for sympletic group in a different way.

Chapter 5

Integral Representations for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$

In this chapter, we study the (standard) tensor product L-functions for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$. In [CFGK19], Cai-Friedberg-Ginzburg-Kaplan present an integral representation for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ using the generalized doubling method. Comparing with the (classical) doubling method applied in previous chapters, they use the generalized Speh representations as inducing data for the Eisenstein series. Following the strategy and extending a previous result of [GS20], we derive new integrals of $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ for any positive even integer n and any positive integer k. We show that these new integrals unfold to non-unique models on Sp_{2n}. We carry out the unramified computation and show that these integrals represent the tensor product L-function for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ via the New Way method, generalizing a previous result on $\mathrm{Sp}_4 \times \mathrm{GL}_2$ [Yan23].

This chapter is independent of previous chapter and is taken from [JY23] which is joint with Pan Yan.

5.1 Introduction

The Rankin-Selberg method is a fruitful way to study L-functions of automorphic forms or automorphic representations. In [PR87], Piatetski-Shapiro and Rallis discovered a family of Rankin-Selberg integrals that represent the standard L-functions for split classical groups. Their construction, known as the doubling method, unfolds to a global matrix coefficient on the classical group. This construction does not depend on a model, and hence opens the door to the possibility of a wide range of applications. Around the same time, in [PR88], they also discovered another family of global integrals, known as the "New Way" integrals (named after the title of [PR88]), which represent the standard L-function for any cuspidal automorphic representation of $\mathrm{Sp}_{4n}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field F. In contrast to the usual constructions of Rankin-Selberg integrals where certain uniqueness result (such as a unique model or a unique global matrix coefficient) is involved (see [Pia97; Fur93; Fil13; GRS98; GJRS11] for more examples), the global integrals in [PR88] unfold to a non-unique model.

In [CFGK19], the method of [PR87] was generalized by Cai, Friedberg, Ginzburg and Kaplan. This construction, known as the generalized doubling method, gives a family of global integrals which represent the tensor product L-function $L(s+\frac{1}{2}, \pi \times \tau)$ where π and τ are irreducible automorphic cuspidal representations of $G(\mathbb{A})$ and $\mathrm{GL}_k(\mathbb{A})$ respectively, and G is a split classical group defined over F.

Recently, in [GS20], Ginzburg and Soudry revisited the integrals considered in [PR88], and showed that one can derive these New Way integrals in [PR88], from the doubling integrals for Sp_{4n} considered in [PR87] by a relatively simple global argument. Moreover, they applied the same idea to the global generalized doubling integral (after unfolding) in [CFGK19] for $\mathrm{Sp}_4 \times \mathrm{GL}_2$, and used the process of

(1) global root exchange,

(2) identities between Eisenstein series, proved in [GS21],

to derive and obtain a "simpler" integral. Furthermore, they conjectured that this new integral represents the standard L-function for $\mathrm{Sp}_4 \times \mathrm{GL}_2$ via the New Way method. This conjecture is proved to hold in [Yan23].

The purpose of this chapter is to generalize these results of Ginzburg and Soudry [GS20] and Yan [Yan23]. The first main result is that we derive new integrals for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ for any positive even integer n and any positive integer k, extending a result in [GS20] where the case $n = k = 2$ was considered. Throughout this chapter, we assume n is even. This assumption is made in order to avoid the use of the Eisenstein series on metaplectic groups (the same assumption also appeared in [PR88]). Let π and τ be irreducible automorphic cuspidal representations of $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\mathrm{GL}_k(\mathbb{A})$ respectively. Starting from the generalized doubling integral for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ in [CFGK19], we apply the same procedure as in [GS20] to derive the following new integral:

$$\begin{aligned}\mathcal{Z}(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s}) = &\int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h) \\ &\times \theta^{\Phi}_{\psi,n^2}(\alpha^k_T(u) i_T(1,h)) E(ut(1,h); f_{n,k,s}) \psi_k(u) du dh.\end{aligned} \tag{5.1.1}$$

Here:

(1) $\phi \in V_\pi$ is a non-zero cusp form;

(2) $N_{n^{k-1},kn}$ is a certain unipotent subgroup of Sp_{2kn} and ψ_k is a character on $N_{n^{k-1},kn}(\mathbb{A})$ which is trivial on $N_{n^{k-1},kn}(F)$;

(3) θ^{Φ}_{ψ,n^2} is a theta series associated to the dual pair $(\mathrm{SO}_{T_0}, \mathrm{Sp}_{2n})$ inside Sp_{2n^2}, where $T_0 \in \mathrm{GL}_n(F) \cap \mathrm{Sym}_n(F)$;

(4) E is an Eisenstein series on $\mathrm{Sp}_{2kn}(\mathbb{A})$ associated to a smooth section

$$f_{n,k,s} \in \mathrm{Ind}^{\mathrm{Sp}_{2kn}(\mathbb{A})}_{P_{kn}(\mathbb{A})}(\Delta(\tau \otimes \chi_T, n)|\det\cdot|^s), \tag{5.1.2}$$

where $T \in \mathrm{GL}_n(F)$ depends on T_0, χ_T is the character $\chi_T(x) = (x, \det(T))$ given by the global Hilbert symbol, and $\Delta(\tau \otimes \chi_T, n)$ is the generalized Speh representation of $\mathrm{GL}_{kn}(\mathbb{A})$ associated to $\tau \otimes \chi_T$.

We refer the reader to Section 5.2 for the precise definitions of the notations. See Theorem 5.3.6 for the precise statement of this result.

We remark that when $k = n$, the global integral $\mathcal{Z}(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s})$ also appeared in [Yan23, Section 7]. Based on the work in [PR88; GS20; Yan23], we expect that, for any positive even integer n and any positive integer k, the integral $\mathcal{Z}(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s})$ unfolds to a non-unique model of π and represents $L^S(s + \frac{1}{2}, \pi \times \tau)$ via the New Way method; this is Conjecture 5.3.7.

Our next goal of this chapter is to provide more evidence for Conjecture 5.3.7 in addition to [PR88; GS20; Yan23]. We will prove that Conjecture 5.3.7 holds for any $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$. Let S be a finite set of places (defined in Section 5.4) and let $f^*_{v,n,k,s}(g) = d^{\mathrm{Sp}_{2kn}}_{\tau_v \otimes \chi_T} \cdot f_{v,n,k,s}(g)$ if $v \notin S$, where $d^{\mathrm{Sp}_{2kn}}_{\tau_v \otimes \chi_T}$ is given by (5.2.29) or (5.2.30) depending on the parity of k. For $v \in S$, we let $f^*_{v,n,k,s}(g) = f_{v,n,k,s}(g)$. Put $f^{*,S}_{n,k,s} = \prod_v f^*_{v,n,k,s}(g)$ and

$$E(g, f^{*,S}_{n,k,s}) = \sum_{\gamma \in P_{kn}(F) \backslash \mathrm{Sp}_{2kn}(F)} f^{*,S}_{n,k,s}(\gamma g). \tag{5.1.3}$$

This is the partially normalized (outside S) Eisenstein series. Our second main result is the following.

Theorem 5.1.1. *There exists a choice of a nonzero cusp form $\phi \in V_\pi$, a matrix T_0, a theta series θ^{Φ}_{ψ,n^2}, and a section $f_{n,k,s} \in \mathrm{Ind}^{\mathrm{Sp}_{2kn}(\mathbb{A})}_{P_{kn}(\mathbb{A})}(\Delta(\tau \otimes \chi_T, n)|\det \cdot|^s)$ such that*

$$\mathcal{Z}(\phi, \theta^{\Phi}_{\psi,n^2}, f^{*,S}_{n,k,s}) = L^S(s + \frac{1}{2}, \pi \times \tau) \cdot \mathcal{Z}_S(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s}), \tag{5.1.4}$$

where $\mathcal{Z}_S(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s})$ is meromorphic in s. Moreover, for any $s_0 \in \mathbb{C}$, the data can be chosen such that $\mathcal{Z}_S(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s})$ is nonzero at s_0.

As applications, we reprove the meromorphic continuation of the partial L-function $L^S(s, \pi \times \tau)$ see [CFK18, Theorem 60]. We also deduce the result on the largest poles of $L^S(s, \pi \times \tau)$, and the relation between the existence of the poles and the non-vanishing of certain period integrals in Theorem 5.4.8 from the poles of fully normalized Eisenstein series [JLZ13, Theorem 5.2]. This generalizes the proposition in [PR88, p.120] where a necessary and sufficient condition for the existence of the poles is given in terms of the theta correspondence.

Now we give a summary of our proof. The first step is to unfold the integral $\mathcal{Z}(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s})$. We show that $\mathcal{Z}(\phi, \theta^{\Phi}_{\psi,n^2}, f_{n,k,s})$ unfolds to the Fourier coefficient of ϕ given by

$$\int_{N_n(F)\backslash N_n(\mathbb{A})} \phi(nh)\, \psi_T(n) dn, \tag{5.1.5}$$

where N_n is the unipotent radical of the Siegel parabolic subgroup of Sp_{2n}, and ψ_T is the character on $N_n(F)\backslash N_n(\mathbb{A})$ given by

$$\psi_T\left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix}\right) = \psi(\mathrm{tr}(Tz)). \tag{5.1.6}$$

See Proposition 5.4.1. We point out that the existence of T such that the integral (5.1.5) is non-zero is due to [Li92]. In general, the model of π corresponding to (5.1.5) is not unique (see [PR88]), that is the integral in (5.1.5) do not factor into local integrals, thus the New Way method is required to analyze the global integral. For more examples of New Way integrals, we refer the reader to [BFG95; PS17; PS18; GS15].

The next step is to carry out the local unramified computation. We show that at a finite place $v \notin S$ (hence all data are unramified), for any local functional corresponding to (5.1.5), the local integral produces the local L-function. This result is the heart of the New Way method. See Theorem 5.4.2. The main idea is to compare the unramified integral with the one from the generalized doubling method.

We can also control the local zeta integral at a place $v \in S$. This is done in Proposition 5.4.4 and Proposition 5.4.5. Then Theorem 5.1.1 follows from Theorem 5.4.3, Proposition 5.4.4 and Proposition 5.4.5.

Finally, we give an overview of the structure of the rest of this chapter. In Section 5.2, after fixing some notations we recall the definitions of theta series, Eisenstein series, and (k, c)-representations. In Section 5.4, we review the global and local integrals from the generalized doubling method and derive new Rankin-Selberg integrals following a strategy of [GS20]. The main new result in this section is Theorem 5.3.6. In Section 5.4, we state our main results on the new integrals we study in this chapter, while delaying the proofs to later sections, to give a more streamlined presentation. In Section 5.5, we give the global unfolding computation and in Section 5.6, we carry out the local unramified computation.

5.2 Preliminaries

5.2.1 Notations

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. Denote $\mathrm{Mat}_{m,n}$ for the additive algebraic group of all matrices of size $m \times n$ and $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$. Let 1_n be the $n \times n$ identity matrix. Set J_n for the $n \times n$ matrix with ones on the antidiagonal and zeros everywhere else. We denote ${}^t x$ for the transpose and $x^* = J_n {}^t x J_n$, $\hat{x} = (x^*)^{-1} = J_n {}^t x^{-1} J_n$ (if x is invertible). The symplectic group Sp_{2n} is realized as

$$\mathrm{Sp}_{2n} = \left\{ g \in \mathrm{GL}_{2n} : {}^t g \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\}. \tag{5.2.1}$$

Let $\mathbf{r} = (r_1, ..., r_m) \in \mathbb{Z}_{\geq 0}^m$ be a m-tuple with $0 \leq r_1 + ... + r_m \leq n$ and denote $|\mathbf{r}| = r_1 + ... + r_m$. Let $P_{\mathbf{r},n}$ be the standard parabolic subgroup of Sp_{2n} with Levi decomposition $P_{\mathbf{r},n} = M_{\mathbf{r},n} \ltimes N_{\mathbf{r},n}$, where $M_{\mathbf{r},n} \cong \mathrm{GL}_{r_1} \times ... \times \mathrm{GL}_{r_m} \times \mathrm{Sp}_{2(n-|\mathbf{r}|)}$. If $r_1 = ... = r_m = r \in \mathbb{Z}_{\geq 0}$ we also denote the tuple $\mathbf{r}$ by r^m. If $m = 1$ we omit it from the notation and simply write r. In particular, for $m = 1, r = n$ we obtain the Siegel parabolic subgroup $P_n := P_{n,n}$. Let $M_n := M_{n,n}, N_n := N_{n,n}$. Then we have $P_n = M_n \ltimes N_n$ where

$$\begin{aligned} M_n &= \left\{ m(x) = \begin{bmatrix} x & 0 \\ 0 & \hat{x} \end{bmatrix} : x \in \mathrm{GL}_n \right\}, \\ N_n &= \left\{ n(z) = \begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} : z \in \mathrm{Mat}_n^0 \right\}. \end{aligned} \tag{5.2.2}$$

Here

$$\mathrm{Mat}_n^0 = \{A \in \mathrm{Mat}_n : A^* J_n = J_n A\}. \tag{5.2.3}$$

For an integer $k \geq 2$, we will frequently use the following two unipotent subgroups $N_{n^{k-1},kn}$ and $N_{n^k,2kn}$.

The unipotent subgroup $N_{n^{k-1},kn}$ contains elements of the form

$$\begin{bmatrix} 1_n & u_{1,2} & * & * & * & * & * & * & * & * \\ 0 & \ddots & \ddots & * & * & * & * & * & * & * \\ 0 & 0 & 1_n & u_{k-2,k-1} & * & * & * & * & * & * \\ 0 & 0 & 0 & 1_n & x & y & z & * & * & * \\ 0 & 0 & 0 & 0 & 1_n & 0 & y^* & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1_n & -x^* & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u^*_{k-2,k-1} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u^*_{1,2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix} \in \mathrm{Sp}_{2kn}. \tag{5.2.4}$$

By sending elements of the above form to its central $2n \times 2n$ block we have a map

$$N_{n^{k-1},kn} \to N_{n,2n} = \left\{ u(x,y,z) = \begin{bmatrix} 1_n & x & y & z \\ 0 & 1_n & 0 & y^* \\ 0 & 0 & 1_n & -x^* \\ 0 & 0 & 0 & 1_n \end{bmatrix} \right\}. \tag{5.2.5}$$

For $u(x,y,z) \in N_{n,2n}$ we may also use the same notation for all its pre-images in $N_{n^{k-1},kn}$ and denote $u^0(x,y,z)$ to emphasize the one in $N_{n^{k-1},kn}$ obtained by natural embedding $N_{n,2n} \to N_{n^{k-1},kn}$.

The unipotent subgroup $N_{n^k,2kn}$ contains elements of the form

$$\begin{bmatrix} 1_n & u_{1,2} & * & * & * & * & * & * & * \\ 0 & \ddots & \ddots & * & * & * & * & * & * \\ 0 & 0 & 1_n & u_{k-1,k} & * & * & * & * & * \\ 0 & 0 & 0 & 1_n & y & z & * & * & * \\ 0 & 0 & 0 & 0 & 1_{2kn} & y' & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1_n & -u^*_{k-1,k} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u^*_{1,2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix} \in \mathrm{Sp}_{4kn}. \tag{5.2.6}$$

5.2.2 Theta series

We fix a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Let $T_0 = \mathrm{diag}[t_1, ..., t_n] \in \mathrm{GL}_n(F)$ be a diagonal matrix and set $T = J_n T_0$. We define a character $\chi_T : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ by $\chi_T(x) = (x, \det(T))$ where $(\cdot,\cdot)$ is the global Hilbert symbol. Denote $\mathcal{H}_n = \mathrm{Mat}_n \times \mathrm{Mat}_n \times \mathrm{Mat}_1$ for the Heisenberg group (of $2n^2+1$ elements) with multiplication

$$\begin{aligned} &(X_1, Y_1, z_1)(X_2, Y_2, z_2) \\ :=&(X_1 + X_2, Y_1 + Y_2, z_1 + z_2 + \mathrm{tr}(T(X_1Y_2^* - Y_1X_2^*))), \end{aligned} \tag{5.2.7}$$

for $X_1, Y_1, X_2, Y_2 \in \mathrm{Mat}_n$ and $z_1, z_2 \in \mathrm{Mat}_1$. We identify $N_{n,2n}$ with $\mathcal{H}_n$ by the map

$$\alpha_T : N_{n,2n} \to \mathcal{H}_n, \quad u(x, y, z) \mapsto (x, y, \mathrm{tr}(Tz)), \tag{5.2.8}$$

where $u(x, y, z)$ is of the form in (5.2.5). For an integer $k \geq 2$, we extend α_T to a map

$$\alpha_T^k : N_{n^{k-1},kn} \to N_{n,2n} \to \mathcal{H}_n \tag{5.2.9}$$

by taking the composite with the map in (5.2.5).

Consider the dual pair $(\mathrm{SO}_{T_0}, \mathrm{Sp}_{2n})$ inside Sp_{2n^2} with

$$\mathrm{SO}_{T_0} = \{g \in \mathrm{SL}_n : {}^t g T_0 g = T_0\}. \tag{5.2.10}$$

We embed $\mathrm{SO}_{T_0} \times \mathrm{Sp}_{2n}$ inside Sp_{4n} via $(m, h) \mapsto \mathrm{diag}[m, h, \hat{m}]$ and further embed $t : \mathrm{Sp}_{4n} \to \mathrm{Sp}_{2kn}$ by $t(g) = \mathrm{diag}[1_{(k-2)n}, g, 1_{(k-2)n}]$. We also denote (m, h) for its image in Sp_{4n} and $t(m, h)$ its image in Sp_{2kn}. We always assume n is even so that $\mathrm{SO}_{T_0}(\mathbb{A}) \times \mathrm{Sp}_{2n}(\mathbb{A})$ splits in the metaplectic double cover $\widetilde{\mathrm{Sp}}_{2n^2}(\mathbb{A})$. We fix such a splitting i_T and consider the restriction of the Weil representation $\omega_\psi := \omega_{\psi,n^2}$ of $\widetilde{\mathrm{Sp}}_{2n^2}(\mathbb{A})$, corresponding to the character ψ, to $\mathrm{SO}_{T_0}(\mathbb{A}) \times \mathrm{Sp}_{2n}(\mathbb{A})$ under i_T. For a Schwartz function $\Phi \in \mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$, we have following formulas for ω_ψ (see [GRS11; Kud96]):

$$\begin{aligned}
\omega_\psi(\alpha_T^k(u(x,y,z)))\Phi(\xi) &= \psi(\mathrm{tr}(Txy^*))\psi(2\mathrm{tr}(T\xi y^*))\psi(\mathrm{tr}(Tz))\Phi(\xi + x),\\
\omega_\psi(i_T(1, m(g)))\Phi(\xi) &= \chi_T(\det(g))|\det g|^{\frac{n}{2}}\Phi(\xi g),\\
\omega_\psi(i_T(1, b(w)))\Phi(\xi) &= \psi(\mathrm{tr}(T\, {}^t\xi w \xi))\Phi(\xi).
\end{aligned} \tag{5.2.11}$$

Here, $u(x, y, z) \in N_{n^{k-1},kn}(\mathbb{A}), g \in \mathrm{GL}_n(\mathbb{A}), w \in \mathrm{Mat}_n^0(\mathbb{A})$.

Given $\Phi \in \mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$, we form the theta series

$$\begin{aligned}
\theta_\psi^\Phi(\alpha_T^k(v) i_T(m, h)) &:= \theta_{\psi,n^2}^\Phi(\alpha_T^k(v) i_T(m, h))\\
&:= \sum_{\xi \in \mathrm{Mat}_n(F)} \omega_\psi(\alpha_T^k(v) i_T(m, h))\Phi(\xi),
\end{aligned} \tag{5.2.12}$$

with $v \in N_{n^{k-1},kn}(\mathbb{A}), m \in \mathrm{SO}_{T_0}(\mathbb{A}), h \in \mathrm{Sp}_{2n}(\mathbb{A})$.

We also need another kind of theta series. Let $\mathcal{H}_{k,n} = \mathrm{Mat}_{n,2kn} \times \mathrm{Mat}_1$ be the Heisenberg group of $2kn^2 + 1$ variables. Then $N_{n^k,2kn}$ has a structure of Heisenberg group $\mathcal{H}_{k,n}$ via the map

$$l_T^0(u) = (y, \mathrm{tr}(Tz)) \tag{5.2.13}$$

for u of the form in (5.2.6). Consider the dual pair $(\mathrm{SO}_{T_0}, \mathrm{Sp}_{2kn})$ inside Sp_{2kn^2} and fix a splitting $i_T^0 : \mathrm{SO}_{T_0} \times \mathrm{Sp}_{2kn} \to \widetilde{\mathrm{Sp}}_{2kn^2}$ inside the metaplectic double cover. We may realize the Weil representation ω_{ψ,kn^2} in $\mathcal{S}(\mathrm{Mat}_{n,kn}(\mathbb{A}))$ and define the theta series θ_{ψ,kn^2}^Φ for $\Phi \in \mathcal{S}(\mathrm{Mat}_{n,kn}(\mathbb{A}))$ similarly as above.

5.2.3 Representations of (k, c) type

We recall the definition and properties of (k, c) representations in [CFGK19; Cai21; CFGKar; CFKar], both locally and globally. See also the summary in [Yan23, Section 2.3].

Let k and c be positive integers. Let P_{c^k} be the standard parabolic subgroup of GL_{kc} whose Levi component is isomorphic to $\mathrm{GL}_c \times ... \times \mathrm{GL}_c$ with k copies so that its unipotent radical U_{c^k} consists of elements of the form

$$u = \begin{bmatrix} 1_c & u_{1,2} & * & * \\ 0 & \ddots & \ddots & * \\ 0 & 0 & 1_c & u_{k-1,k} \\ 0 & 0 & 0 & 1_c \end{bmatrix} \in \mathrm{GL}_{kc}. \tag{5.2.14}$$

Define a character

$$\begin{aligned} \psi_{c^k} : U_{c^k}(F)\backslash U_{c^k}(\mathbb{A}) &\to \mathbb{C}^\times, \\ u &\mapsto \psi\left(\sum_{i=1}^{k-1} \mathrm{tr}(u_{i,i+1})\right). \end{aligned} \tag{5.2.15}$$

For an automorphic function ϕ on $\mathrm{GL}_{kc}(F)\backslash \mathrm{GL}_{kc}(\mathbb{A})$, we consider the following Fourier coefficient

$$\Lambda(\phi) = \int_{U_{c^k}(F)\backslash U_{c^k}(\mathbb{A})} \phi(u)\psi_{c^k}^{-1}(u)du. \tag{5.2.16}$$

Definition 5.2.1. ([CFGK19, Definition 3]) An irreducible automorphic representation ρ of $\mathrm{GL}_{kc}(\mathbb{A})$ is called a (k, c) representation if the following holds.

(1) The Fourier coefficient $\Lambda(\phi)$ does not vanish identically on the space of ρ, and moreover, for all unipotent orbits greater than or noncomparable with (k^c), all corresponding Fourier coefficients are zero for all choices of data.

(2) Let ρ_v denote the irreducible constituent of ρ at a place v. Then for all unipotent orbits greater than or noncomparable with (k^c), the corresponding twisted Jacquet module of ρ_v vanishes. Moreover, $\mathrm{Hom}_{U_{c^k}(F_v)}(\rho_v, \psi_{c^k,v})$ (continuous morphisms if v is archimedean) is one-dimensional.

Let (ρ, V_ρ) be a (k, c) representation. Then the space $\mathcal{W}(\rho, \psi)$ of functions on $\mathrm{GL}_{kc}(\mathbb{A})$,

$$g \mapsto \Lambda(\rho(g)\phi), \qquad \phi \in V_\rho \tag{5.2.17}$$

is a unique model of ρ. If we write $\rho \cong \otimes'_v \rho_v$ as a restricted tensor product with respect to a system of spherical vectors $\{\xi_v^0\}_{v \notin S}$, then the space $\mathrm{Hom}_{U_{c^k}(F_v)}(\rho_v, \psi_{c^k,v})$ is one-dimensional. We fix a nonzero functional $\Lambda_v \in \mathrm{Hom}_{U_{c^k}(F_v)}(\rho_v, \psi_{c^k,v})$ and denote by $\mathcal{W}(\rho_v, \psi)$ the local unique model consisting of functions on $\mathrm{GL}_{kc}(F_v)$ given by

$$g \mapsto \Lambda_v(\rho_v(g)\xi_v), \qquad \xi_v \in V_{\rho,v}. \tag{5.2.18}$$

Proposition 5.2.2. *Let $\phi = \otimes'_v \xi_v \in V_\rho$ be a decomposable vector. For each place v of F, there exists a functional $\Lambda_v \in \mathrm{Hom}_{U_{c^k}(F_v)}(\rho_v, \psi_{c^k,v})$ such that $\Lambda_v(\xi_v^0) = 1$ for all $v \notin S$ and for all $g \in \mathrm{GL}_{kc}(\mathbb{A})$, we have*

$$\Lambda(\rho(g)\phi) = \prod_v \Lambda_v(\rho(g_v)\xi_v). \tag{5.2.19}$$

Proof. [Bum97, Theorem 3.5.2], [Sha74, §4], and [Cai21, Lemma 2.15]. □

Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$. Denote $\Delta(\tau, c)$ for the generalized Speh representation of $\mathrm{GL}_{kc}(\mathbb{A})$ associated to τ whose definition will be recalled in the following example. This is a (k, c) representation and we only consider such (k, c) representation in this chapter.

Example 5.2.3. Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$, $\underline{s} = (s_1, ..., s_c) \in \mathbb{C}^c$ and $E(g; \xi, \underline{s})$ denote the Eisenstein series associated with the induced representation

$$\mathrm{Ind}_{P_{k^c}(\mathbb{A})}^{\mathrm{GL}_{kc}(\mathbb{A})} \left(|\det \cdot|^{s_1} \tau \otimes ... \otimes |\det \cdot|^{s_c} \tau \right),$$

where ξ is a standard section. Let

$$\underline{s}_0 = \left(\frac{c-1}{2}, \frac{c-3}{2}, ..., \frac{1-c}{2} \right).$$

Then the Eisenstein series $E(g;\xi,\underline{s})$ has a simple multi-residue at $\underline{s}_0$,

$$E_{\underline{s}_0}(g;\xi) = \lim_{\underline{s}\to\underline{s}_0} \prod_{i=1}^{c-1}(s_i - s_{i+1} - 1)M(w_0,\underline{s})\xi(g;\underline{s}),$$

where $M(w_0,\underline{s})$ is the intertwining operator defined by

$$M(w_0,\underline{s})\xi(g;\underline{s}) = \int_{U_{k^c}(\mathbb{A})} \xi(w_0 ug,\underline{s})du, \qquad w_0 = \begin{bmatrix} & & & 1_c \\ & & 1_c & \\ & \iddots & & \\ 1_c & & & \end{bmatrix}.$$

The generalized Speh representation $\Delta(\tau,c)$ associated to τ is the automorphic representation of $\mathrm{GL}_{kc}(\mathbb{A})$ generated by all the residue functions $E_{\underline{s_0}}(g;\xi)$.

Write $\tau \cong \otimes'_v \tau_v$ and $\Delta(\tau,c) \cong \otimes'_v \Delta(\tau_v,c)$. Let v be a place such that τ_v is unramified and thus can be written in the form

$$\tau_v = \mathrm{Ind}_{B_{\mathrm{GL}_k(F_v)}}^{\mathrm{GL}_k(F_v)}(\chi_1 \otimes ... \otimes \chi_k). \tag{5.2.20}$$

Here B_{GL_k} is the standard Borel subgroup of GL_k consisting of upper triangular matrices and $\chi_1,...,\chi_k$ are unramified quasi-characters of $F_v^\times$. Then by [CFGK19, Claim 9],

$$\Delta(\tau_v,c) = \mathrm{Ind}_{P_{c^k}(F_v)}^{\mathrm{GL}_{kc}(F_v)}(\chi_1 \circ \det \otimes ... \otimes \chi_k \circ \det). \tag{5.2.21}$$

We simply denote the model of $\Delta(\tau,c)$ and $\Delta(\tau_v,c)$ by

$$\mathcal{W}(\tau,c,\psi) := \mathcal{W}(\Delta(\tau,c),\psi), \qquad \mathcal{W}(\tau_v,c,\psi) := \mathcal{W}(\Delta(\tau_v,c),\psi). \tag{5.2.22}$$

We have the following global (resp. local) invariance property for the unique functional Λ (resp. Λ_v).

Proposition 5.2.4. *For $g \in \mathrm{GL}_c$, denote $g^\Delta = \mathrm{diag}[g,...,g]$ for its diagonal embedding in GL_{kc}. Then for any $g \in \mathrm{SL}_c(\mathbb{A})$ and ϕ in the space of $\Delta(\tau,c)$, we have*

$$\Lambda(\Delta(\tau,c)(g^\Delta)\phi) = \Lambda(\phi). \tag{5.2.23}$$

For any $g \in \mathrm{GL}_c(F_v)$ and any ξ_v in the space of $\Delta(\tau_v, c)$, we have

$$\Lambda_v(\Delta(\tau_v, c)(g^{\Delta})\xi_v) = \tau_v(\det(g)1_k)\Lambda_v(\xi_v). \tag{5.2.24}$$

Proof. [CFGK19, Claim 8 and Proposition 24] and [CFK18, Lemma 14]. □

5.2.4 Eisenstein series

Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$ and $\Delta(\tau, c)$ the generalized Speh representation of $\mathrm{GL}_{kc}(\mathbb{A})$ associated to τ and c. Consider the induced representation

$$\mathrm{Ind}_{P_{kc}(\mathbb{A})}^{\mathrm{Sp}_{2kc}(\mathbb{A})}(\Delta(\tau, c)|\det\cdot|^s). \tag{5.2.25}$$

Its space consists of functions $\tilde{f}_{c,k,s} : \mathrm{Sp}_{2kc}(\mathbb{A}) \to V$ satisfying

$$\tilde{f}_{c,k,s}\left(\begin{bmatrix} a & \\ & \hat{a} \end{bmatrix}\begin{bmatrix} 1_{kc} & b \\ & 1_{kc} \end{bmatrix} g\right) = |\det(a)|^{s+\frac{kc+1}{2}}\Delta(\tau, c)(a)\tilde{f}_{c,k,s}(g), \tag{5.2.26}$$

where V is the space of automorphic forms of $\Delta(\tau, c)$. We identify it with the space of functions $f_{c,k,s} : \mathrm{Sp}_{2kc}(\mathbb{A}) \to \mathbb{C}$ by setting $f_{c,k,s}(g) = \tilde{f}_{c,k,s}(g)(1_{kc})$. For a smooth section $f_{c,k,s}$, we define an Eisenstein series on $\mathrm{Sp}_{2kc}(\mathbb{A})$ by

$$E(g; f_{c,k,s}) = \sum_{\gamma \in P_{kc}(F)\backslash \mathrm{Sp}_{2kc}(F)} f_{c,k,s}(\gamma g). \tag{5.2.27}$$

In this chapter we will only consider the cases where $c = 2n$ and $c = n$. When choosing the sections $f_{c,k,s}$ we will need the following normalizing factors (see [GS21, (1.47)(1.48)]):

$$d_{\tau_v}^{\mathrm{Sp}_{4kn}}(s) = L(s + k + \frac{1}{2}, \tau_v)\prod_{j=1}^{k} L(2s + 2j, \tau_v, \wedge^2)L(2s + 2j - 1, \tau_v, \mathrm{Sym}^2), \tag{5.2.28}$$

and

$$d_{\tau_v}^{\mathrm{Sp}_{2kn}}(s) = L(s + \frac{k}{2} + \frac{1}{2}, \tau_v)\prod_{j=1}^{\frac{k}{2}} L(2s + 2j, \tau_v, \wedge^2)L(2s + 2j - 1, \tau_v, \mathrm{Sym}^2), \tag{5.2.29}$$

if k is even and

$$d_{\tau_v}^{\mathrm{Sp}_{2kn}}(s) = L(s + \frac{k}{2} + \frac{1}{2}, \tau_v) \prod_{j=1}^{\frac{k+1}{2}} L(2s + 2j - 1, \tau_v, \wedge^2) L(2s + 2j, \tau_v, \mathrm{Sym}^2), \quad (5.2.30)$$

if k is odd.

5.3 New integrals derived from the generalized doubling method

In this section, we recall the generalized doubling construction in [CFGK19] and explain how to derive new Rankin-Selberg integrals from the generalized doubling method following a strategy of [GS20]. The main new result of this section is Theorem 5.3.6. We will also review the local unramified integrals from the generalized doubling method, which will be used in Section 5.6.

5.3.1 The generalized doubling construction

We first recall the generalized doubling construction in [CFGK19]. Let $G = \mathrm{Sp}_{2n}$ and $H = \mathrm{Sp}_{4kn}$. Define an embedding

$$\begin{aligned} &G \times G \to H, \\ &(g_1, g_2) \mapsto g_1 \times g_2 = \mathrm{diag}[g_1, ..., g_1, \begin{bmatrix} g_{1,1} & 0 & g_{1,2} \\ 0 & g_2 & 0 \\ g_{1,3} & 0 & g_{1,4} \end{bmatrix}, \hat{g}_1, ..., \hat{g}_1] \end{aligned} \quad (5.3.1)$$

where $g_1 = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix}$, $g_{1,i} \in \mathrm{Mat}_n$ and g_1 appears $k-1$ times. Let $P := P_{2kn}$ be the Siegel parabolic subgroup of H with Levi decomposition $P = M_P \ltimes U_P$ and $Q := P_{(2n)^{k-1},2kn}$ be the parabolic subgroup with $Q = M \ltimes U$ such that $U := N_{(2n)^{k-1},2kn}$ contains elements of the form

$$\begin{bmatrix} 1_{2n} & u_{1,2} & * & * & * & * & * & * & * \\ 0 & \ddots & \ddots & * & * & * & * & * & * \\ 0 & 0 & 1_{2n} & u_{k-2,k-1} & * & * & * & * & * \\ 0 & 0 & 0 & 1_{2n} & u_0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1_{4n} & u_0' & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1_{2n} & -u_{k-2,k-1}^* & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2n} & -u_{1,2}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2n} \end{bmatrix} \in \mathrm{Sp}_{4kn}. \tag{5.3.2}$$

Fix a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. For $u \in U(\mathbb{A})$ as in (5.3.2) with $u_0 = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$, $a_i, c_i \in \mathrm{Mat}_n(\mathbb{A})$ and $b_i \in \mathrm{Mat}_{n,2n}(\mathbb{A})$, we define a character $\psi_U : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\psi_U(u) = \psi\left(\sum_{i=1}^{k-2} \mathrm{tr}(u_{i,i+1}) + \mathrm{tr}(a_1 + c_2)\right). \tag{5.3.3}$$

Let (π, V_π) be an irreducible cuspidal representation of $G(\mathbb{A})$ and (τ, V_τ) be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$. Consider the generalized Speh representation $\Delta(\tau, 2n)$ of $\mathrm{GL}_{2kn}(\mathbb{A})$ associated to τ and the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\Delta(\tau, 2n)|\det\cdot|^s). \tag{5.3.4}$$

For a standard section $f_{2n,k,s}$ of above induced representation, we form an Eisenstein series on $H(\mathbb{A})$ by

$$E(h; f_{2n,k,s}) = \sum_{\gamma \in P(F)\backslash H(F)} f_{2n,k,s}(\gamma h). \tag{5.3.5}$$

For cusp forms $\phi_1, \phi_2 \in \pi$, the global zeta integral considered in [CFGK19] is

$$\begin{aligned} Z(s, \phi_1, \phi_2, f_{2n,k,s}) = \int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \phi_1(g_1)\overline{\phi_2({}^\iota g_2)} \\ \times E(u(g_1 \times g_2); f_{2n,k,s})\psi_U(u)du dg_1 dg_2. \end{aligned} \tag{5.3.6}$$

Here ${}^{\iota}g := \iota g \iota^{-1}$ with $\iota = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}$.

Set $U_0' = U \cap U_P$, and

$$\delta_0 = \begin{bmatrix} 0 & 1_{2kn} \\ -1_{2kn} & 0 \end{bmatrix} \begin{bmatrix} 1_{2n(k-1)} & 0 & 0 & 0 \\ 0 & 1_{2n} & 1_{2n} & 0 \\ 0 & 0 & 1_{2n} & 0 \\ 0 & 0 & 0 & 1_{2n(k-1)} \end{bmatrix}. \tag{5.3.7}$$

Let

$$\langle \phi_1, \phi_2 \rangle = \int_{G(F)\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}dg \tag{5.3.8}$$

be the standard inner product on $G(\mathbb{A})$.

The basic properties of the global integral $Z(s, \phi_1, \phi_2, f_{2n,k,s})$ are summarized below.

Theorem 5.3.1. *[CFGK19, Theorem 1] The integral $Z(s, \phi_1, \phi_2, f_{2n,k,s})$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to the plane. For $\mathrm{Re}(s) \gg 0$, it unfolds to*

$$\int_{G(\mathbb{A})} \int_{U_0'(\mathbb{A})} \langle \phi_1, \pi(g)\phi_2 \rangle f_{\mathcal{W}(\tau,2n,\psi_{(2n)^k}),s}(\delta_0 u_0(1 \times {}^{\iota}g))\psi_U(u_0)du_0dg, \tag{5.3.9}$$

where

$$f_{\mathcal{W}(\tau,2n,\psi_{(2n)^k}),s}(h) = \int_{U_{(2n)^k}(F)\backslash U_{(2n)^k}(\mathbb{A})} f_{2n,k,s}(vh)\psi_U^{-1}(v)dv. \tag{5.3.10}$$

Moreover, the integral (5.3.9) is Eulerian.

Remark 5.3.2. The Eulerian of the integral follows from the fact that $\mathcal{W}(\tau, 2n, \psi_{(2n)^k})$ is a unique model and Proposition 5.2.2. In [CFGK19, Section 3], one only obtain the 'almost Euler product' due to the lack of Proposition 5.2.2 (see also [Yan23, Remark 2.3] for more explanation).

5.3.2 The unramified computation of the generalized doubling integrals

In this subsection, we state the unramified computation from [CFGK19].

Theorem 5.3.3. *[CFGK19, Theorem 29] Let v be a finite place such that π_v and τ_v are unramified. Assume the character ψ is unramified. Let $\omega^0_{\pi_v}$ be the unramified matrix coefficient of π_v normalized such that $\omega^0_{\pi_v}(1_{2n}) = 1$. Let*

$$f^0_{\mathcal{W}(\tau_v,2n,\psi_{(2n)^k}),s} \in \mathrm{Ind}^{\mathrm{Sp}_{4kn}(F_v)}_{P_{2kn}(F_v)}(\mathcal{W}(\tau_v,2n,\psi_{(2n)^k})|\det\cdot|^s) \tag{5.3.11}$$

be the unramified section normalized such that

$$f^0_{\mathcal{W}(\tau_v,2n,\psi_{(2n)^k}),s}(1_{4kn}) = d^{\mathrm{Sp}_{4kn}}_{\tau_v}(s). \tag{5.3.12}$$

Then

$$\begin{aligned}&\int_{G(F_v)}\int_{U_0'(F_v)} \omega^0_{\pi_v}(g) f^0_{\mathcal{W}(\tau_v,2n,\psi_{(2n)^k}),s}(\delta_0 u_0 (1\times {}^\iota g))\psi_U(u_0)du_0dg\\ =&L(s+\frac{1}{2},\pi_v\times\tau_v).\end{aligned} \tag{5.3.13}$$

We will use a slightly different version of the above unramified integrals, which will be more convenient for our local unramified computation in Section 5.6. Denote

$$\delta = \begin{bmatrix} 0 & 1_{2n} & 0 & 0\\ 0 & 0 & 0 & 1_{2n(k-1)}\\ -1_{2n(k-1)} & 0 & 0 & 0\\ 0 & 1_{2n} & 1_{2n} & 0\end{bmatrix}. \tag{5.3.14}$$

Then we can rewrite (5.3.13) as (see [GS21, Proposition 4.8])

$$\begin{aligned}&\int_{G(F_v)}\int_{U_0(F_v)} \omega^0_{\pi_v}(g) f^0_{\mathcal{W}(\tau_v,2n,\psi_{(2n)^k}),s}(\delta u_0 (1\times g))\psi_U(u_0)du_0dg\\ =&L(s+\frac{1}{2},\pi_v\times\tau_v).\end{aligned} \tag{5.3.15}$$

Here $U_0 := N^0_{(2n)^{k-1},2kn}$ is the subgroup of $N_{(2n)^{k-1},2kn}$ consisting of elements of the form

$$\begin{bmatrix} 1_{2n(k-1)} & * & 0 & * \\ 0 & 1_{2n} & 0 & 0 \\ 0 & 0 & 1_{2n} & * \\ 0 & 0 & 0 & 1_{2n(k-1)} \end{bmatrix} \in \mathrm{Sp}_{4kn}. \tag{5.3.16}$$

5.3.3 New integrals derived from the generalized doubling method

In this subsection, we follow an argument of Ginzburg and Soudry in [GS20, Section 4.2] and explain how to derive new integrals for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$. We assume that n is even. Let

$$\xi(\phi, f_{2n,k,s})(g) = \int_{G(F)\backslash G(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \phi(h) E(u(g \times h); f_{2n,k,s}) \psi_U(u) du dh, \tag{5.3.17}$$

and consider the integral

$$\mathcal{L}(\phi, f_{2n,k,s}) = \int_{\mathrm{Mat}_n^0(F)\backslash \mathrm{Mat}_n^0(\mathbb{A})} \xi(\phi, f_{2n,k,s}) \left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} \right) \psi(\mathrm{tr}(Tz)) dz. \tag{5.3.18}$$

Clearly, $\mathcal{L}(\phi, f_{2n,k,s})$ equals

$$\begin{aligned} &\int_{G(F)\backslash G(\mathbb{A})} \int_{\mathrm{Mat}_n^0(F)\backslash \mathrm{Mat}_n^0(\mathbb{A})} \phi(h) \\ &\times \int_{U(F)\backslash U(\mathbb{A})} E\left(u \left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} \times h \right); f_{2n,k,s} \right) \psi_U(u) \psi(\mathrm{tr}(Tz)) du dz. \end{aligned} \tag{5.3.19}$$

Performing the root exchanging process for the integral along $U(F)\backslash U(\mathbb{A})$ and conjugate by certain Weyl elements as in the proof of [GS20, Theorem 4], we can find a suitable section $f'_{2n,k,s}$ such that

$$\begin{aligned} &\mathcal{L}(\phi, f_{2n,k,s}) \\ &= \int_{G(F)\backslash G(\mathbb{A})} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \psi_k(v) \phi(h) \\ &\times \int_{N^0_{n^k,2kn}(F)\backslash N^0_{n^k,2kn}(\mathbb{A})} E(u\tilde{v}(1_{2n} \times h); f'_{2n,k,s}) \psi_{N^0_{n^k,2kn}}(u) du dv dh. \end{aligned} \tag{5.3.20}$$

Here $N^0_{n^k,2kn}$ is the subgroup of $N_{n^k,2kn}$ containing elements of the form in (5.2.6) with $y = [\ 0_{n\times(k+1)n} \quad y_0 \quad y_0'\ ], y_0 \in \mathrm{Mat}_n, y_0' \in \mathrm{Mat}_{n,(k-2)n}$ and $\tilde{v} = \mathrm{diag}[1_{kn}, v, 1_{kn}]$.

The characters ψ_k and $\psi_{N^0_{n^k,2kn}}$ are given by

$$\begin{aligned}\psi_k(v) &= \psi\left(\sum_{i=1}^{k-2} \mathrm{tr}(v_{i,i+1})\right),\\ \psi_{N^0_{n^k,2kn}}(v) &= \psi\left(\sum_{i=1}^{k-1} \mathrm{tr}(v_{i,i+1}) - \mathrm{tr}(y_0) + \mathrm{tr}(z)\right),\end{aligned} \tag{5.3.21}$$

for v of the form in (5.2.4) or (5.2.6). We omit the lengthy computations but give an example of the case $k = 3$ to illustrate how the process is carried out.

Example 5.3.4. (The case $k = 3$) We view $12n \times 12n$ matrices as 12×12 block matrices where each block is of size $n \times n$. Let $e_{i,j}$ be the elementary matrix which has one at the (i, j) entry. Let

$$\begin{aligned} e'_{i,j} &= e_{i,j} - e_{13-j,13-i}, \qquad 1 \le i, j \le 6,\\ e'_{i,j} &= e_{i,j} + e_{13-j,13-i}, \quad 1 \le i \le 6, j > 6.\end{aligned}$$

Define

$$\begin{aligned} X_1 &= \{1 + x_{1,2}e'_{1,2} : x_{1,2} \in \mathrm{Mat}_n\}, \quad X_2 = \{1 + x_{3,4}e'_{3,4} : x_{3,4} \in \mathrm{Mat}_n\},\\ Y_1 &= \{1 + x_{2,3}e'_{2,3} : y_{2,3} \in \mathrm{Mat}_n\}, \qquad Y_2 = \{1 + x_{4,5}e'_{4,5} : y_{4,5} \in \mathrm{Mat}_n\}.\end{aligned}$$

Take $B_1 = U$ and C_1 the subgroup of B_1 generated by root subgroups in U that do not lie in Y_1 so that $B_1 = C_1Y_1$. Set $D_1 = C_1X_1$ and perform the root exchanging process for $(B_1, C_1, D_1, X_1, Y_1)$ as in [GS20, Section 2.4]. Then let $B_2 = D_1$ and similarly define C_2 so that $B_2 = C_2Y_2$. Set $D_2 = C_2X_2$ and perform the root exchanging process for $(B_2, C_2, D_2, X_2, Y_2)$ again. Conjugating by the Weyl element

$$w = \begin{bmatrix} w_0 & 0 \\ 0 & \hat{w}_0 \end{bmatrix}, \qquad \text{with } w_0 = \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix},$$

the inner integral over $U(F)\backslash U(\mathbb{A})$ in (5.3.19) is of the form

$$\int_{D_2'(F)\backslash D_2'(\mathbb{A})} E\left(uw\left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} \times h\right) w^{-1}; f''_{2n,3,s}\right) \psi_{D_2'}(u)du,$$

where D_2' consists of elements of the form

$$\begin{bmatrix} 1_n & u_{1,2} & * & * & * & * & * & * & * & * & * \\ 0 & 1_n & 0 & u_{2,4} & * & * & * & * & * & * & * \\ 0 & 0 & 1_n & * & u_{3,5} & * & * & * & * & * & * \\ 0 & 0 & 0 & 1_n & 0 & 0 & y_0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1_n & * & * & -y_0^* & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1_{2n} & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & -u_{3,5}^* & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & * & -u_{2,4}^* & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u_{1,2}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}$$

and

$$\psi_{D_2'}(u) = \psi\left(\mathrm{tr}(u_{1,2} + u_{2,4} + u_{3,5} - y_0)\right).$$

We then take $B_3 = D_2', X_3 = Y_1, Y_3 = X_2$ and define C_3 so that $B_3 = C_3Y_3$. Set $D_3 = C_3X_3$ and perform the root exchange process for $(B_3, C_3, D_3, X_3, Y_3)$. Then we conjugate by the Weyl element $w' = \begin{bmatrix} w_0' & 0 \\ 0 & \hat{w}_0' \end{bmatrix}$ where

$$w_0' = \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix},$$

we see that the integral over $U(F)\backslash U(\mathbb{A})$ becomes the form

$$\int_{D_3'(F)\backslash D_3'(\mathbb{A})} E\left(uw'w\left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} \times h\right) w^{-1}w'^{-1}; f'''_{2n,3,s}\right)\psi_{D_3'}(u)du,$$

where D_3' consists of elements of the form

$$\begin{bmatrix} 1_n & u_{1,2} & * & * & * & * & * & * & * & * & * \\ 0 & 1_n & u_{2,3} & * & * & * & * & * & * & * & * \\ 0 & 0 & 1_n & 0 & 0 & 0 & y_0 & * & 0 & * & * \\ 0 & 0 & 0 & 1_n & u_{4,5} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1_n & * & * & * & -y_0^* & * & * \\ 0 & 0 & 0 & 0 & 0 & 1_{2n} & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u_{4,5}^* & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u_{2,3}^* & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & -u_{1,2}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}$$

and

$$\psi_{D_3'}(u) = \psi\left(\mathrm{tr}(u_{1,2} + u_{2,3} + u_{4,5} - y_0)\right).$$

Computing $w'w\left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} \times h\right) w^{-1}w'^{-1}$ one easily obtain the integral in (5.3.20).

Now we continue with a general k. Write $N^0_{n^k,2kn} = U_0 Y_0$ such that U_0 contains elements with $y_0 = 0, y_0' = 0$ and Y_0 contains elements such that all entries above diagonal are zero except y_0, y_0'. Also denote ψ_{U_0} and ψ_{Y_0} for the restriction of $\psi_{N^0_{n^k,2kn}}$ to U_0 and Y_0. Then the integral in the second line of (5.3.20) can be factorized as

$$\int_{Y_0(F)\backslash Y_0(\mathbb{A})} \int_{U_0(F)\backslash U_0(\mathbb{A})} E(uy\tilde{v}(1_{2n} \times h); f'_{2n,k,s})\psi_{U_0}(u)\psi_{Y_0}(y)dudy. \tag{5.3.22}$$

Applying a theorem of Ikeda [Ike94] as in the proof of [GS20, Theorem 4], we can find certain section $f''_{2n,k,s}$ and $\Phi_1, \Phi_2 \in \mathcal{S}(\mathrm{Mat}_{n,kn}(\mathbb{A}))$ such that the integral over $U_0(F)\backslash U_0(\mathbb{A})$ equals

$$\begin{aligned} &\theta^{\Phi_1}_{\psi,kn^2}(l^0_T(y)i^0_T(1,v\tilde{h})) \int_{N_{n^k,2kn}(F)\backslash N_{n^k,2kn}(\mathbb{A})} \overline{\theta^{\Phi_2}_{\psi,kn^2}(l^0_T(u)i^0_T(1,v\tilde{h}))} \\ &\times E(u\tilde{v}(1_{2n} \times h); f''_{2n,k,s})\psi_{U_0}(u)du. \end{aligned} \tag{5.3.23}$$

Here we also denote $\tilde{h} = \mathrm{diag}[1_{(k-1)n}, h, 1_{(k-1)n}]$ for its embedding in Sp_{2kn}.

Lemma 5.3.5. *Let $f_{2n,k,s} \in \mathrm{Ind}^{\mathrm{Sp}_{4kn}(\mathbb{A})}_{P_{2kn}(\mathbb{A})}(\Delta(\tau, 2n)|\det\cdot|^s)$ be a smooth holomorphic section and $\Phi \in \mathcal{S}(\mathrm{Mat}_{n,kn}(\mathbb{A}))$. There exists a smooth, meromorphic section $\lambda(f_{2n,k,s}, \Phi) \in \mathrm{Ind}^{\mathrm{Sp}_{2kn}(\mathbb{A})}_{P_{kn}(\mathbb{A})}(\Delta(\tau \otimes \chi_T, n)|\det\cdot|^s)$ such that*

$$\begin{aligned} &E(g; \lambda(f_{2n,k,s}, \Phi)) \\ =&\int_{N_{n^k,2kn}(F)\backslash N_{n^k,2kn}(\mathbb{A})} \theta^{\Phi}_{\psi,kn^2}(l^0_T(u)i^0_T(1,g))E(\tilde{g}; f_{2n,k,s})\psi_{N_{n^k,2kn}}(u)du \end{aligned} \tag{5.3.24}$$

for $g \in \mathrm{Sp}_{2kn}$. Here we denote $\tilde{g} = \mathrm{diag}[1_{kn}, g, 1_{kn}]$. The character $\psi_{N_{n^k,2kn}}$ is given by

$$\psi_{N_{n^k,2kn}}(u) = \psi\left(\sum_{i=1}^{k-1} \mathrm{tr}(u_{i,i+1})\right) \tag{5.3.25}$$

for u of the form in (5.2.6).

Proof. The proof is similar to the one for [GS20, Lemma 2]. We sketch the proof and give the section $\lambda(f_{2n,k,s}, \Phi)$ explicitly in (5.3.29).

We start by unfolding the Eisenstein series on the right hand side of (5.3.24). That is, we need to compute

$$\sum_{\eta\in P_{2kn}(F)\backslash\mathrm{Sp}_{4kn}(F)/N_{n^k,2kn}(F)\widehat{\mathrm{Sp}}_{2kn}(F)}\ \sum_{\gamma\in S_\eta\backslash N_{n^k,2kn}(F)\widehat{\mathrm{Sp}}_{2kn}(F)} f_{2n,k,s}(\eta\gamma\hat{g}) \\ \times\int_{N_{n^k,2kn}(F)\backslash N_{n^k,2kn}(\mathbb{A})}\theta^{\Phi}_{\psi,kn^2}(l^0_T(u)i^0_T(1,g))\psi_{N_{n^k,2kn}}(u)du, \tag{5.3.26}$$

where $\widehat{\mathrm{Sp}_{2kn}}$ is its image in Sp_{4kn} under the embedding $g\mapsto\tilde{g}$ and $S_\eta=\eta^{-1}P_{2kn}\eta\cap N_{n^k,2kn}(F)\widehat{\mathrm{Sp}}_{2kn}(F)$ is the stabilizer of η. One shows that only the orbit represented by

$$w_0=\begin{bmatrix}0 & 1_{kn} & 0 & 0\\ 0 & 0 & 0 & 1_{kn}\\ -1_{kn} & 0 & 0 & 0\\ 0 & 0 & 1_{kn} & 0\end{bmatrix}$$

is nonzero. In this case $S_{w_0}=U_0\widehat{P}_{kn}$ with U_0 the subgroup of $N_{n^k,2kn}$ consisting of elements of the form

$$\begin{bmatrix}1_n & * & * & 0 & * & 0 & 0 & 0\\ 0 & \ddots & * & 0 & * & 0 & 0 & 0\\ 0 & 0 & 1_n & 0 & b & 0 & 0 & 0\\ 0 & 0 & 0 & 1_{kn} & 0 & b' & * & *\\ 0 & 0 & 0 & 0 & 1_{kn} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1_n & * & *\\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & *\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n\end{bmatrix}\in\mathrm{Sp}_{4kn}.$$

We further factor $U_0=U_1U_2$ such that U_1 contains elements of the above form with $b=0$ and $U_2=\{u_b\}$ contains elements of the above form such that all $*$'s are zero. Denote $N^0_{n^k,2kn}=U_0\backslash N_{n^k,2kn}$. Then our expression (5.3.26) becomes

$$\sum_{\gamma\in P_{kn}(F)\backslash\mathrm{Sp}_{2kn}(F)}\int_{N^0_{n^k,2kn}(\mathbb{A})}\int_{U_2(F)\backslash U_2(\mathbb{A})}\theta^{\Phi}_{\psi,kn^2}(l^0_T(u_bu)i^0_T(1,\gamma g)) \\ \times\int_{U_1(F)\backslash U_1(\mathbb{A})} f_{2n,k,s}(w_0u_1u_bu\widetilde{\gamma g})\psi_1(u_1)\psi_0(u)du_1du_bdu, \tag{5.3.27}$$

where ψ_0,ψ_1 are the restriction of $\psi_{N_{n^k,2kn}}$ to $N^0_{n^k,2kn}$ and U_1, respectively.

Let U_1' be the group containing elements of the form

$$\begin{bmatrix} 1_{kn} & x & * & * & * \\ 0 & 1_n & u'_{2,3} & * & * \\ 0 & 0 & \ddots & \ddots & * \\ 0 & 0 & 0 & 1_n & u'_{k,k+1} \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix} \in \mathrm{GL}_{2kn},$$

and

$$\psi_1'(u_1') = \psi\left(\sum_{i=1}^{k-1} \mathrm{tr}(u'_{i+1,i+2})\right).$$

Let U_1'' be the subgroup of U_1' containing elements of above form with $x = 0$ and ψ_1'' the restriction of ψ_1' to U_1''. Changing variables $u_1 \mapsto w_0^{-1} u_1 w_0$, the integral over $U_1(F)\backslash U_1(\mathbb{A})$ can be written as

$$\int_{U_1''(F)\backslash U_1''(\mathbb{A})} f_{2n,k,s}(\mathrm{diag}[u_1'', \hat{u}_1'']w_0 u(b) v\widetilde{\gamma g})\psi_1''(u_1'')du_1''.$$

Using the following identity (for details see the proof of [GS21, Proposition 2.4]):

$$\begin{aligned} f^{\psi_1'}_{2n,k,s}(g) &:= \int_{U_1'(F)\backslash U_1'(\mathbb{A})} f_{2n,k,s}(\mathrm{diag}[u_1', \hat{u}_1']g)\psi_1'(u_1')du_1' \\ &= \int_{U_1''(F)\backslash U_1''(\mathbb{A})} f_{2n,k,s}(\mathrm{diag}[u_1'', \hat{u}_1'']g)\psi_1''(u_1'')du_1'', \end{aligned}$$

the expression (5.3.27) becomes

$$\begin{aligned} &\sum_{\gamma \in P_{kn}(F)\backslash \mathrm{Sp}_{2kn}(F)} \int_{N^0_{n^k,2kn}(\mathbb{A})} \psi_0(u) \\ &\times \int_{U_2(F)\backslash U_2(\mathbb{A})} \theta^{\Phi}_{\psi,kn^2}(l^0_T(u_b u) i^0_T(1,\gamma g)) f^{\psi_1'}_{2n,k,s}(w_0 u_b u \widetilde{\gamma g}) du_b du. \end{aligned} \tag{5.3.28}$$

Changing variables $u(b) \mapsto w_0^{-1} u_b w_0$ and the variables x in U_1', the integral in the second line of (5.3.28) equals

$$\int_{U_2(F)\backslash U_2(\mathbb{A})} \theta^{\Phi}_{\psi,kn^2}(l^0_T(u_b u) i^0_T(1,\gamma g)) f^{\psi_1'}_{2n,k,s}(w_0 u \widetilde{\gamma g}) du_b.$$

Unfolding the theta series, this becomes

$$\omega_{\psi,kn^2}(l^0_T(u) i^0_T(1,\gamma g))\Phi(0) f^{\psi_1'}_{2n,k,s}(w_0 u \widetilde{\gamma g}).$$

Define

$$\lambda(f_{2n,k,s},\Phi)(g)=\int_{N^0_{n^k,2kn}(\mathbb{A})}\omega_{\psi,kn^2}(l^0_T(u)i^0_T(1,g))\Phi(0)f^{\psi'_1}_{2n,k,s}(w_0u\tilde{g})\psi_0(u)du. \quad (5.3.29)$$

One checks that this is a section of the induced representation $\mathrm{Ind}^{\mathrm{Sp}_{2kn}(\mathbb{A})}_{P_{kn}(\mathbb{A})}(\Delta(\tau\otimes\chi_T,n)|\det\cdot|^s)$ and hence the expression (5.3.28) equals

$$\sum_{\gamma\in P_{kn}(F)\backslash\mathrm{Sp}_{2kn}(F)}\lambda(f_{2n,k,s},\Phi)(\gamma g),$$

which is an Eisenstein series as desired. □

By Lemma 5.3.5, the second line in (5.3.23) is an Eisenstein series. Choosing $f_{2n,k,s}$ similar to [GS20, (4.35), (4.36)], we see that there exists a section $f_{n,k,s}$ such that the integral (5.3.22) becomes

$$\int_{Y_0(F)\backslash Y_0(\mathbb{A})}\theta^{\Phi_1}_{\psi,kn^2}(l^0_T(y)i^0_T(1,v\tilde{h}))E(\tilde{v}(1_{2n}\times h);f_{n,k,s})\psi_{Y_0}(y)dy. \quad (5.3.30)$$

Unfolding the theta series we have

$$\int_{Y_0(F)\backslash Y_0(\mathbb{A})}\sum_{\xi\in\mathrm{Mat}_{n,kn}}\omega^{\Phi_1}_{\psi,kn^2}(l^0_T(y)i^0_T(1,v\tilde{h}))\Phi(\xi)E(\tilde{v}(1_{2n}\times h);f_{n,k,s})\psi_{Y_0}(y)dy. \quad (5.3.31)$$

Recall that y is of the form $\begin{bmatrix}0_{n,kn} & 0_n & y_0 & y'_0\end{bmatrix}$ with $y_0\in\mathrm{Mat}_n, y'_0\in\mathrm{Mat}_{n,(k-2)n}$. Write $\xi=\begin{bmatrix}\xi_1 & \xi_2 & \xi_3\end{bmatrix}$ with $\xi_1\in\mathrm{Mat}_{n,(k-2)n},\xi_2\in\mathrm{Mat}_n,\xi_3\in\mathrm{Mat}_n$. Note that $\omega_{\psi,kn^2}(l^0_T(y))$ provides a character

$$\psi(\mathrm{tr}(2T(\xi_1y'^*_0+\xi_2y^*_0))). \quad (5.3.32)$$

Thus the above integral is nonvanishing unless $\xi_1=0,\xi_2=(2T)^{-1}$ and one obtains a theta series θ^{Φ}_{ψ,n^2} with $\Phi(\xi)=\Phi_1(\begin{bmatrix}0 & (2T)^{-1} & \xi\end{bmatrix})$. We summarize our results in the following theorem.

Theorem 5.3.6. *Let n be a positive even integer. Given $\Phi \in \mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$, there are nontrivial choices of sections*

$$\begin{aligned} f_{2n,k,s} &\in \mathrm{Ind}_{P_{2kn}(\mathbb{A})}^{\mathrm{Sp}_{4kn}(\mathbb{A})}(\Delta(\tau, 2n)|\det\cdot|^s), \\ f_{n,k,s} &\in \mathrm{Ind}_{P_{kn}(\mathbb{A})}^{\mathrm{Sp}_{2kn}(\mathbb{A})}(\Delta(\tau\otimes\chi_T, n)|\det\cdot|^s), \end{aligned} \tag{5.3.33}$$

such that the integral (5.3.18) is equal to

$$\begin{aligned} \mathcal{L}(\phi, f_{2n,k,s}) = &\int_{\mathrm{Sp}_{2n}(F)\backslash\mathrm{Sp}_{2n}(\mathbb{A})}\int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \psi_k(u)\phi(h) \\ &\times \theta^{\Phi}_{\psi,n^2}(\alpha_T(u)i_T(1,h))E(ut(1,h); f_{n,k,s})dudh. \end{aligned} \tag{5.3.34}$$

Motivated by the Theorem 5.3.6, we propose the following.

Conjecture 5.3.7. *Let n be a positive even integer. The integral in (5.3.34) is Eulerian in the sense of the New Way method, and represents the tensor product L-function $L^S(s+\frac{1}{2}, \pi\times\tau)$.*

The rest of this chapter is devoted to providing evidence for Conjecture 5.3.7. When $k = 1$, Conjecture 5.3.7 is proven to hold by Piatetski-Shapiro and Rallis [PR88] (see also [GS20, Section 4.1]). From now on we may assume $k \geq 2$. We remark that when $n = k = 2$, Theorem 5.3.6 recovers [GS20, Theorem 4], and in this case, Conjecture 5.3.7 reduces to the conjecture in [GS20], which is proven to hold in [Yan23]. When n is even and $k = n$, Conjecture 5.3.7 also appeared in [Yan23, Conjecture 7.1]. We will show that Conjecture 5.3.7 holds for any pair of positive integers n, k where n is even.

5.4 The global zeta integral and statement of theorems

Let F be a number field with the ring of adeles $\mathbb{A}$. We assume $n, k \geq 2$ are integers with n even. We fix a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ and let T_0, T, χ_T

be as before. Let (π, V_π) be an irreducible cuspidal automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\phi \in V_\pi$ be a cusp form. Define the (T-)Fourier coefficient of ϕ by

$$\phi_{\psi,T}(h) = \int_{\mathrm{Mat}_n^0(F)\backslash \mathrm{Mat}_n^0(\mathbb{A})} \phi\left(\begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} h\right)\psi(\mathrm{tr}(Tz))dz. \tag{5.4.1}$$

We always assume T is chosen such that $\phi_{\psi,T} \neq 0$ which is possible by [Li92]. In general, the models on π corresponding to (5.4.1) are not unique.

Let $\theta_\psi^\Phi := \theta_{\psi,n^2}^\Phi$ be the theta series associated with the Weil representation ω_{ψ,n^2} and the Schwartz function $\Phi \in \mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$ defined in (5.2.12). Let

$$f_s := f_{n,k,s} \in \mathrm{Ind}_{P_{kn}(\mathbb{A})}^{\mathrm{Sp}_{2kn}(\mathbb{A})}(\Delta(\tau \otimes \chi_T, n)|\det \cdot|^s), \tag{5.4.2}$$

be a section obtained as in Lemma 5.3.5 and we form the Eisenstein series $E(g; f_s)$. For $u \in N_{n^{k-1},kn}$ of the form in (5.2.4) we define a character

$$\begin{aligned} \psi_k : N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A}) &\to \mathbb{C}^\times, \\ u &\mapsto \psi\left(\sum_{i=1}^{k-2} \mathrm{tr}(2Tu_{i,i+1})\right). \end{aligned} \tag{5.4.3}$$

Recall that the global integral in (5.3.34) is

$$\begin{aligned} \mathcal{Z}(\phi, \theta_{\psi,n^2}^\Phi, f_s) := \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h) \\ \times \theta_{\psi,n^2}(\alpha_T(u) i_T(1,h)) E(ut(1,h); f_s)\psi_k(u) du dh. \end{aligned} \tag{5.4.4}$$

Now we state the basic properties of the integral $\mathcal{Z}(\phi, \theta_{\psi,n^2}^\Phi, f_s)$. Let $N_{n^{k-1},kn}^0$ be the subgroup of $N_{n^{k-1},kn}$ containing elements of the form

$$\begin{bmatrix} 1_{(k-1)n} & * & 0 & * \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 1_n & * \\ 0 & 0 & 0 & 1_{(k-1)n} \end{bmatrix}, \tag{5.4.5}$$

and let

$$\eta = \begin{bmatrix} 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & -1_{(k-1)n} \\ 1_{(k-1)n} & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 \end{bmatrix}. \tag{5.4.6}$$

We have the following.

Proposition 5.4.1. *The integral $\mathcal{Z}(\phi, \theta_\psi^\Phi, f_s)$ converges absolutely when* $\mathrm{Re}(s) \gg 0$ *and can be meromorphically continued to all $s \in \mathbb{C}$. For* $\mathrm{Re}(s) \gg 0$*, it unfolds to*

$$\begin{aligned}\int_{N_n(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N^0_{n^{k-1},kn}(\mathbb{A})} & \phi_{\psi,T}(h)\omega_\psi(\alpha_T^k(u)i_T(1,h))\Phi(1_n) \\ & \times f_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta u t(1,h))dudh.\end{aligned} \tag{5.4.7}$$

Here,

$$f_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(g) = \int_{U_{n^k}(F)\backslash U_{n^k}(\mathbb{A})} f_s\left(\begin{bmatrix} v & 0 \\ 0 & \hat{v}\end{bmatrix} g\right)\psi_{2T}^{-1}(u)du, \tag{5.4.8}$$

and

$$\psi_{2T}(u) = \psi\left(\sum_{i=1}^{k-1}\mathrm{tr}(2Tu_{i,i+1})\right), \tag{5.4.9}$$

with u of the form in (5.2.14).

Proposition 5.4.1 will be proved in Section 5.5.

We will take both Φ and f_s to be factorizable so that we can write

$$f_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(g) = \prod_v f_{\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T}),s}(g_v) \tag{5.4.10}$$

where $f_{\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T}),s} \in \mathrm{Ind}_{P_{kn}(F_v)}^{\mathrm{Sp}_{2kn}(F_v)}(\mathcal{W}(\tau_v \otimes \chi_T, n, \psi_{2T})|\det\cdot|^s)$. However, the integral $\mathcal{Z}(\phi, \theta_\psi^\Phi, f_s)$ is still not factorizable in the usual sense since $\phi_{\psi,T}$ corresponds to a non-unique model in general (i.e. it does not factor into an Euler product). This requires us to use the New Way method of Piatetski-Shapiro and Rallis, first appeared in [PR88], to analyze the integral $\mathcal{Z}(\phi, \theta_\psi^\Phi, f_s)$.

For a finite place v, we denote by $\mathcal{O}_{F_v}$ the ring of integers of F_v. We take S to be a finite set of of places such that $v \notin S$ if and only if $v \nmid 2, 3, \infty$; π_v, τ_v, ψ_v are unramified and all the diagonal coordinates of T_0 are in $\mathcal{O}_{F_v}^\times$.

For a place $v \notin S$, let $\Phi_v^0 = \mathbf{1}_{\mathrm{Mat}_n(\mathcal{O}_{F_v})}$ be the characteristic function of $\mathrm{Mat}_n(\mathcal{O}_{F_v})$. Let

$$f^0_{\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T}),s} \in \mathrm{Ind}_{P_{kn}(F_v)}^{\mathrm{Sp}_{2kn}(F_v)}(\mathcal{W}(\tau_v \otimes \chi_T, n, \psi_{2T})|\det\cdot|^s) \tag{5.4.11}$$

be the unramified section normalized such that

$$f^0_{\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T}),s}(1_{2kn}) = d_{\tau_v}^{\mathrm{Sp}_{4kn}}(s). \tag{5.4.12}$$

Theorem 5.4.2. *For a place $v \notin S$, take $\Phi_v^0, f^0_{\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T}),s}$ as above and fix a non-zero unramified vector $v_0 \in V_{\pi_v}$. Let $l_T : V_{\pi_v} \to \mathbb{C}$ be a linear functional on V_{π_v} such that*

$$l_T\left(\pi_v \begin{bmatrix} 1_n & z \\ 0 & 1_n \end{bmatrix} \xi\right) = \psi^{-1}(\mathrm{tr}(Tz))l_T(\xi) \tag{5.4.13}$$

for all $\xi \in V_{\pi_v}, z \in \mathrm{Mat}_n^0(F_v)$. Denote

$$\begin{aligned} \mathcal{Z}_v^*(l_T, s) = &\int_{N_n(F_v)\backslash \mathrm{Sp}_{2n}(F_v)} \int_{N^0_{n^{k-1},kn}(F_v)} l_T(\pi_v(h)v_0) \\ &\times \omega_{\psi,v}(\alpha_T^k(u) i_T(1,h))\Phi_v^0(1_n) f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta u t(1,h))du dh. \end{aligned} \tag{5.4.14}$$

Then for $\mathrm{Re}(s) \gg 0$ we have

$$\mathcal{Z}_v^*(l_T, s) = L(s + \frac{1}{2}, \pi_v \times \tau_v) \cdot l_T(v_0). \tag{5.4.15}$$

The unramified computations and the proof of the above theorem will be carried out in Section 5.6.

Globally, we choose the global section $f_s^{*,S}$ such that its local unramified counterpart is f_s^0 chosen above. By applying the strategy of [PR88] (see also the proofs of [PS17, Corollary 3.4], [Yan23, Theorem 3.4]) we obtain the following result.

Theorem 5.4.3. *Fix an isomorphism $\pi \cong \otimes'_v \pi_v$ and identify $\phi \in V_\pi$ with $\otimes'_v \xi_v, \xi_v \in V_{\pi_v}$. For $\mathrm{Re}(s) \gg 0$, we have*

$$\mathcal{Z}(\phi, \theta_\psi^\Phi, f_s^{*,S}) = L^S(s + \frac{1}{2}, \pi \times \tau) \cdot \mathcal{Z}_S(\phi, \theta_\psi^\Phi, f_s), \tag{5.4.16}$$

where

$$\begin{aligned}\mathcal{Z}_S(\phi,\theta_\psi^\Phi,f_s)=&\int_{N_n(\mathbb{A}_S)\backslash\mathrm{Sp}_{2n}(\mathbb{A}_S)}\int_{N^0_{n^{k-1},kn}(\mathbb{A}_S)}\omega_{\psi,S}(\alpha_T^k(u)i_T(1,h))\Phi_S(1_n)\\&\times\phi_{\psi,T}(h)f_{S,\mathcal{W}(\tau_S\otimes\chi_T,n,\psi_{2T}),s}(\eta ut(1,h))dudh.\end{aligned}\tag{5.4.17}$$

The local zeta integral at finite ramified places and archimedean places can be controlled by the following two propositions. We omit their proof as they are the same as the proof of [Yan23, Proposition 3.5, Proposition 3.6] (see also the proof of [GRS98, Proposition 6.6, Proposition 6.7]).

Proposition 5.4.4. *Let v be a finite place and K_0 be an open compact subgroup of $\mathrm{Sp}_{2n}(F_v)$. There is a choice of $\Phi_0\in\mathcal{S}(\mathrm{Mat}_n(F_v))$ and $f_{0,s}\in\mathrm{Ind}_{P_{kn}(F_v)}^{\mathrm{Sp}_{2kn}(F_v)}(\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T})|\det\cdot|^s)$ such that for any irreducible admissible representation π_v of $\mathrm{Sp}_{2n}(F_v)$, any vector $\xi_0\in V_{\pi_v}$ stabled under K_0 and any linear functional $l_T:V_{\pi_v}\to\mathbb{C}$ satisfying (5.4.5) we have*

$$\begin{aligned}&\int_{N_n(F_v)\backslash\mathrm{Sp}_{2n}(F_v)}\int_{N^0_{n^{k-1},kn}(F_v)}l_T(\pi_v(h)\xi_0)\\&\times\omega_{\psi,v}(\alpha_T^k(u)i_T(1,h))\Phi_0(1_n)f_{0,s}(\eta ut(1,h))dudh\\&=l_T(\xi_0).\end{aligned}\tag{5.4.18}$$

Proposition 5.4.5. *For any complex number $s_0 \in \mathbb{C}$, there is a choice of data $(\phi_j, \Phi_j, f_{j,s})$ such that the finite sum*

$$\begin{aligned}
&\sum_j \mathcal{Z}_\infty(\phi_j, \Phi_j, f_{j,s}) \\
=&\sum_j \int_{N_n(\mathbb{A}_\infty)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_\infty)} \int_{N^0_{n^{k-1},kn}(\mathbb{A}_\infty)} \phi_{j,\psi,T}(h) \\
&\times \omega_{\psi,\infty}(\alpha^k_T(u) i_T(1,h))\Phi_j(1_n) f_{j,s}(\eta u t(1,h)) du dh
\end{aligned} \tag{5.4.19}$$

admits meromorphic continuation to the whole complex plane and its meromorphic continuation is nonzero at s_0.

As applications to above theorems, we now study the analytic properties of the partial L-function $L^S(s, \pi \times \tau)$. By the meromorphic continuation of Eisenstein series, we reprove the following theorem in [CFK18, Theorem 60].

Corollary 5.4.6. *With S a finite set of places picked as above, the partial L-function $L^S(s, \pi \times \tau)$ admits meromorphic continuation to the whole complex plane.*

With the section $f^{*,S}_s \in \mathrm{Ind}^{\mathrm{Sp}_{2kn}(\mathbb{A})}_{P_{kn}(\mathbb{A})}(\Delta(\tau \otimes \chi_T, n)|\det \cdot|^s)$ chosen as above, we further define a section f^*_s by

$$f^*_s = \prod_{v \in S} d^{\mathrm{Sp}_{2kn}}_{\tau_v \otimes \chi_T}(s) \cdot f^{*,S}_s. \tag{5.4.20}$$

The location of possible poles of the fully normalized Eisenstein series $E(g; f^*_s)$ is determined in [JLZ13, Theorem 5.2]. We recall their result as follow.

Theorem 5.4.7. *Assume τ is a self-dual irreducible unitary automorphic cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$. The Eisenstein series $E(g;f_s^*)$ is holomorphic for $\mathrm{Re}(s)\geq 0$ except possibly at most simple poles in following cases:*

(1) If $L(s,\tau\otimes\chi_T,\wedge^2)$ has a pole at $s=1$, and $L(\frac{1}{2},\tau\otimes\chi_T)\neq 0$, then $E(g;f_s^*)$ has a simple pole at $s=1,2,...,\frac{n}{2}$,

(2) If $L(s,\tau\otimes\chi_T,\wedge^2)$ has a pole at $s=1$, and $L(\frac{1}{2},\tau\otimes\chi_T)=0$, then $E(g;f_s^*)$ has a simple pole at $s=1,2,...,\frac{n-2}{2}$ (if $n=2$ then $E(g;f_s^*)$ is holomorphic),

(3) If both $L(s,\tau\otimes\chi_T,\mathrm{Sym}^2)$ and $L(s,\tau\otimes\chi_T)$ have a pole at $s=1$ (this case occurs only if $k=1$ and $\tau\otimes\chi_T$ is the trivial character of $\mathrm{GL}_1(\mathbb{A})$), then $E(g;f_s^*)$ has a simple pole at $s=\frac{1}{2},\frac{3}{2},...,\frac{n+1}{2}$,

(4) If $L(s,\tau\otimes\chi_T,\mathrm{Sym}^2)$ has a poles at $s=1$, then $E(g;f_s^*)$ has a simple pole at $s=\frac{1}{2},\frac{3}{2},...,\frac{n-1}{2}$.

If $E(g;f_s^*)$ has a simple pole at $s=s_0$, denote by $\mathcal{R}(s_0,\Delta(\tau\otimes\chi_T,n))$ the space generated by the residues of $E(g;f_s^*)$ at $s=s_0$ as the section f_s^* varies in s. The elements $R\in\mathcal{R}(s_0,\Delta(\tau\otimes\chi_T,n))$ are automorphic forms on $\mathrm{Sp}_{2kn}(\mathbb{A})$. We have the following theorem on the poles of $L^S(s,\pi\times\tau)$, and the relation between the existence of the poles and the non-vanishing of certain period integrals. This theorem is a generalization of [PR88, p.120 Proposition] (see also [Yan22, Section 3.3]).

Theorem 5.4.8. *Assume τ is a self-dual irreducible unitary automorphic cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$. Then $L^S(s,\pi\times\tau)$ is holomorphic for $\mathrm{Re}(s)>\frac{n+2}{2}$, and admits at most a simple pole at $s_0=1,\frac{3}{2},...,\frac{n+2}{2}$. Moreover, for such s_0, if*

$$\mathrm{Res}_{s=s_0}L^S(s,\pi\times\tau)\neq 0, \tag{5.4.21}$$

then there exist a Schwartz function $\Phi\in\mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$, and a residue $R\in\mathcal{R}(s_0,\Delta(\tau\otimes$

$\chi_T, n)$), *such that the period integral*

$$\int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h)\theta_\psi^\Phi(\alpha_T^k(v)i_T(1,h))R(vt(1,h))dvdh \tag{5.4.22}$$

is not identically zero.

Proof. The proof is same as the proof of [Yan22, Theorem 1.4]. By Theorem 5.4.3, we have

$$\mathcal{Z}(\phi, \theta_\psi^\Phi, f_s^*) = L^S(s+\frac{1}{2}, \pi\times\tau)\cdot \mathcal{Z}_S(\phi,\theta_\psi^\Phi, f_s^*)\cdot \prod_{v\in S} d_{\tau_v\otimes\chi_T}^{\mathrm{Sp}_{2kn}}(s).$$

By Proposition 5.4.4 and 5.4.5, the section f_s^* can be chosen such that $\mathcal{Z}_S(\phi,\theta_\psi^\Phi, f_s^*)$ is non-vanishing for any s. One can also show that $d_{\tau_v\otimes\chi_T}^{\mathrm{Sp}_{2kn}}(s)\neq 0$ for any $v\in S$ and any s. Then the theorem follows from our integral representation and Theorem 5.4.7. □

5.5 Unfolding

In this section, we unfold the global zeta integral $\mathcal{Z}(\phi, \theta_{\psi,n^2}^\Phi, f_s)$ and prove Proposition 5.4.1. We start by unfolding the Eisenstein series. For $\mathrm{Re}(s)\gg 0$, we have

$$\mathcal{Z}(\phi, \theta_{\psi,n^2}^\Phi, f_s) = \sum_{\gamma\in P_{kn}(F)\backslash \mathrm{Sp}_{2kn}(F)/P_{n^{k-1},kn}(F)} I(\gamma), \tag{5.5.1}$$

where

$$I(\gamma) = \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h)\theta_\psi^\Phi(\alpha_T^k(u)i_T(1,h)) \times \sum_{g\in H_\gamma(F)\backslash P_{n^{k-1},kn}(F)} f_s(\gamma g u t(1,h))\psi_k(u)dudh, \tag{5.5.2}$$

with $H_\gamma = \gamma^{-1}P_{kn}\gamma\cap P_{n^{k-1},kn}$ the stabilizer of the orbit represented by γ. The orbits of $P_{kn}(F)\backslash \mathrm{Sp}_{2kn}(F)/P_{n^{k-1},kn}(F)$ and their stabilizers are described in the following lemma.

Lemma 5.5.1. *The representatives of $P_{kn}(F)\backslash \mathrm{Sp}_{2kn}(F)/P_{n^{k-1},kn}(F)$ are given by*

$$\gamma_{r_1,...,r_{k-1}} = \begin{bmatrix} \mu'_{k-1} & 0 & 0 & 0 & 0 & 0 & \epsilon'_{k-1} \\ 0 & \ddots & 0 & 0 & 0 & \iddots & 0 \\ 0 & 0 & \mu'_1 & 0 & \epsilon'_1 & 0 & \\ 0 & 0 & 0 & 1_{2n} & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 & \mu_1 & 0 & 0 \\ 0 & \iddots & 0 & 0 & 0 & \ddots & 0 \\ \epsilon_{k-1} & 0 & 0 & 0 & 0 & 0 & \mu_{l-1} \end{bmatrix}, \tag{5.5.3}$$

where $\mu_i, \epsilon_i, \mu'_i, \epsilon'_i$ are $n \times n$ matrices

$$\begin{aligned} \mu_i &= \begin{bmatrix} 1_{r_i} & 0 \\ 0 & 0 \end{bmatrix}, & \epsilon_i &= \begin{bmatrix} 0 & 0 \\ 1_{n-r_i} & 0 \end{bmatrix}, \\ \mu'_i &= \begin{bmatrix} 0 & 0 \\ 0 & 1_{r_i} \end{bmatrix}, & \epsilon'_i &= \begin{bmatrix} 0 & -1_{n-r_i} \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{5.5.4}$$

with $0 \le r_i \le n$. Denote $H_{r_1,...,r_{k-1}}$ for the stabilizer of $\gamma_{r_1,...,r_{k-1}}$. Then $H_{r_1,...,r_{k-1}} = M_{r_1,...,r_{k-1}} \ltimes N_{r_1,...,r_{k-1}}$, where $M_{r_1,...,r_{k-1}}$ consists of elements

$$m(h_0, g_1, ..., g_{k-1}) = \mathrm{diag}[g_{k-1}, ..., g_1, h_0, \hat{g}_1, ..., \hat{g}_{k-1}] \tag{5.5.5}$$

with $h_0 \in P_n$ and for $1 \le i \le k-1$,

$$g_i \in B^-_{r_i,n} = \left\{ \begin{bmatrix} * & 0_{(n-r_i)\times r_i} \\ * & * \end{bmatrix} \in \mathrm{GL}_n \right\}. \tag{5.5.6}$$

Proof. Let $(V, \langle\cdot,\cdot\rangle)$ be the underlying skew-symmetric space of the group Sp_{2kn} with Witt decomposition $V = I \oplus I'$ into two maximal isotropic subspace so that P_{kn} is the parabolic subgroup of Sp_{2kn} fixing I. The parabolic subgroup $P_{n^{k-1},kn}$ is the one fixing some flag of isotropic subspaces

$$0 \subset I_1 \subset I_2 \subset ... \subset I_{k-1} \subset V$$

with $I_i \subset I$ of rank ni. Then the double coset $P_{kn}\backslash \mathrm{Sp}_{2kn}/P_{n^{k-1},kn}$ is parameterized by tuple $(\kappa_1, ..., \kappa_{k-1})$ where $\kappa_i = \dim(I\gamma \cap I_i)$. One can easily pick the representatives as in the lemma and their stabilizers can be obtained by straightforward matrix computations. □

Denote $I_{r_1,...,r_{k-1}} = I(\gamma_{r_1,...,r_{k-1}})$. Then by Lemma 5.5.1,

$$\begin{aligned}
&I_{r_1,...,r_{k-1}} \\
&= \int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{1\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F) \\ h_0\in P_n(F)\backslash \mathrm{Sp}_{2n}(F)}} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \\
&\sum_{u_0\in N_{r_1,...,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(F)} f_s(\gamma_{r_1,...,r_{k-1}} u_0 m(h_0, g_1, ..., g_{k-1}) u t(1,h)) \\
&\times \phi(h)\theta^\Phi_\psi(\alpha^k_T(u) i_T(1,h))\psi_k(u) du dh.
\end{aligned} \tag{5.5.7}$$

Now we change variables $u \mapsto m(h_0, 1, ..., 1)^{-1} u m(h_0, 1, ..., 1)$. Clearly, ψ_k is preserved under this change, and note that

$$\begin{aligned}
m(h_0, 1, ..., 1)^{-1} u(x, y, z) m(h_0, 1, ..., 1) &= u([x, y]h_0, z), \\
\theta^\Phi_\psi(\alpha^k_T(u([x, y]h_0, z)) i_T(1, h)) &= \theta^\Phi_\psi(\alpha^k_T(u(x, y, z)) i_T(1, h_0 h)).
\end{aligned} \tag{5.5.8}$$

Then we obtain

$$\begin{aligned}
&I_{r_1,...,r_{k-1}} \\
&= \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{1\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \\
&\sum_{u_0\in N_{r_1,...,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(F)} f_s(\gamma_{r_1,...,r_{k-1}} u_0 m(1, g_1, ..., g_{k-1}) u t(1,h)) \\
&\times \phi(h)\theta^\Phi_\psi(\alpha^k_T(u) i_T(1,h))\psi_k(u) du dh.
\end{aligned} \tag{5.5.9}$$

Lemma 5.5.2. *$I_{r_1,...,r_{k-1}} = 0$ unless $r_1 = 0$.*

Proof. Let $N^c_{n^{k-1},kn}$ be a normal subgroup of $N_{n^{k-1},kn}$ containing elements of the form

$$u(z) = \begin{bmatrix} 1_{(k-2)n} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & z & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{(k-2)n} \end{bmatrix}.$$

Using the formulas of the Weil representation, we have

$$\theta_\psi^\Phi(\alpha_T^k(u(z))\alpha_T^k(u')i_T(1,h)) = \psi(\mathrm{tr}(Tz))\theta_\psi^\Phi(\alpha_T^k(u')i_T(1,h)),$$

for $u(z) \in N^c_{n^{k-1},kn}(\mathbb{A}), u' \in N_{n^{k-1},kn}(\mathbb{A}), h \in \mathrm{Sp}_{2n}(\mathbb{A})$.

Therefore,

$$\begin{aligned} & I_{r_1,\ldots,r_{k-1}} \\ = & \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{1\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N^c_{n^{k-1},kn}(\mathbb{A})N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \\ \times & \int_{\mathrm{Mat}^0_n(F)\backslash \mathrm{Mat}^0_n(\mathbb{A})} \phi(h)\psi(\mathrm{tr}(Tz))\theta_\psi^\Phi(\alpha_T(u)i_T(1,h))\psi_k(u) \\ \times & \sum_{u_0 \in N_{r_1,\ldots,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(F)} f_s(\gamma_{r_1,\ldots,r_{k-1}}u_0 m(1,g_1,\ldots,g_{k-1})u(z)ut(1,h))dzdudh. \end{aligned}$$

Note that

$$m(1,g_1,\ldots,g_{k-1})^{-1}u(z)m(1,g_1,\ldots,g_{k-1}) = u(g_1^{-1}z\hat{g}).$$

Changing variables $z \mapsto g_1^{-1}z\hat{g}_1$ we obtain

$$\begin{aligned} & I_{r_1,\ldots,r_{k-1}} \\ = & \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{1\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N^c_{n^{k-1},kn}(\mathbb{A})N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \\ \times & \int_{\mathrm{Mat}^0_n(F)\backslash \mathrm{Mat}^0_n(\mathbb{A})} \phi(h)\psi(\mathrm{tr}(Tg_1^{-1}z\hat{g}_1))\theta_\psi^\Phi(\alpha_T(u)i_T(1,h))\psi_k(u) \\ \times & \sum_{u_0 \in N_{r_1,\ldots,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(F)} f_s(\gamma_{r_1,\ldots,r_{k-1}}u_0 u(z)m(1,g_1,\ldots,g_{k-1})ut(1,h))dzdudh. \end{aligned}$$

Write $z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$ with $z_1 \in \mathrm{Mat}_{n-r_1,r_1}, z_2 \in \mathrm{Mat}_{n-r_1}, z_3 \in \mathrm{Mat}_{r_1}, z_4 \in \mathrm{Mat}_{r_1,n-r_1}$

and note that $\gamma_{r_1,\ldots,r_{k-1}}$ commutes with $u\left(\begin{bmatrix} 0 & 0 \\ z_3 & 0 \end{bmatrix}\right)$.

Then

$$
\begin{aligned}
&I_{r_1,\ldots,r_{k-1}} \\
&= \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{1\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N^c_{n^{k-1},kn}(\mathbb{A})N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \\
&\times \phi(h)\theta^\Phi_\psi(\alpha_T(u)i_T(1,h)) \int_{z_1,z_2,z_4} \psi\left(\mathrm{tr}\left(Tg_1^{-1}\begin{bmatrix} z_1 & z_2 \\ 0 & z_4 \end{bmatrix}\hat{g}_1\right)\right)\psi_k(u) \\
&\times \sum_{u_0 \in N_{r_1,\ldots,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(F)} f_s\left(\gamma_{r_1,\ldots,r_{k-1}} u\left(\begin{bmatrix} z_1 & z_2 \\ 0 & z_4 \end{bmatrix}\right) u_0 m(1,g_1,\ldots,g_{k-1})ut(1,h)\right) \\
&\times \int_{\mathrm{Mat}_{r_1}(F)\backslash \mathrm{Mat}_{r_1}(\mathbb{A})} \psi\left(\mathrm{tr}\left(Tg_1^{-1}\begin{bmatrix} 0 & 0 \\ z_3 & 0 \end{bmatrix}\hat{g}_1\right)\right) dz_3dz_1dz_2dz_4dudh.
\end{aligned}
$$

The lemma follows as the integral in the last line vanishes if $r_1 > 0$. $\square$

We now assume $r_1 = 0$ and omit it from our notation. We need to consider the integral

$$
\begin{aligned}
&I_{r_2,\ldots,r_{k-1}} \\
&= \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{2\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N_{n^{k-1},kn}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \\
&\times \sum_{u_0 \in N_{r_2,\ldots,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(F)} f_s(\gamma_{r_2,\ldots,r_{k-1}} u_0 m(1,1,g_2,\ldots,g_{k-1})ut(1,h)) \\
&\times \phi(h)\theta^\Phi_\psi(\alpha^k_T(u)i_T(1,h))\psi_k(u)dudh.
\end{aligned}
\tag{5.5.10}
$$

Lemma 5.5.3. *$I_{r_2,\ldots,r_{k-1}} = 0$ unless $r_2 = \ldots = r_{k-1} = 0$.*

Proof. Changing variables $u \mapsto u' = m(1,1,g_2,\ldots,g_{k-1})^{-1}um(1,1,g_2,\ldots,g_{k-1})$ we obtain

$$\begin{aligned}
&I_{r_2,\dots,r_{k-1}} \\
&= \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{2\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N_{r_2,\dots,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h)\theta_\psi^\Phi(\alpha_T^k(u)i_T(1,h)) \\
&\times f_s(\gamma_{r_2,\dots,r_{k-1}} u m(1,1,g_2,\dots,g_{k-1})t(1,h))\psi_k(u')dudh.
\end{aligned}$$

Note that $N_{r_2,\dots,r_{k-1}}$ contains a subgroup N_2 consisting of elements of the form

$$\begin{bmatrix}
1_{(k-3)n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1_{n-r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{r_2} & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_{n-r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{r_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1_{2n} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1_{r_2} & 0 & -x^* & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{n-r_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{r_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{n-r_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{(k-3)n}
\end{bmatrix}.$$

Thus

$$\begin{aligned}
&I_{r_2,\dots,r_{k-1}} \\
&= \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \sum_{\substack{2\le i\le k-1 \\ g_i \in B^-_{r_i,n}(F)\backslash \mathrm{GL}_n(F)}} \int_{N_2(\mathbb{A})N_{r_2,\dots,r_{k-1}}(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h)\psi_k(u') \\
&\times \theta_\psi^\Phi(\alpha_T^k(u)i_T(1,h)) f_s(\gamma_{r_2,\dots,r_{k-1}} u m(1,1,g_2,\dots,g_{k-1})t(1,h)) \\
&\times \int_{N_2(F)\backslash N_2(\mathbb{A})} \psi_k(u_2)du_2dudh.
\end{aligned}$$

Since $\psi_k(u_2)$ is a nontrivial character on $N_2(F)\backslash N_2(\mathbb{A})$ the integral in the last line is zero unless $r_2 = 0$. The lemma then follows by induction on $r_3, \dots, r_{k-1}$ using the same argument. $\square$

We then assume $r_2 = \dots = r_{k-1} = 0$ and omit it from our notation (so $\gamma = \gamma_{0,\dots,0}$, $N =$

$N_{0,\dots,0}$). Our integral becomes

$$I = \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h)\theta_\psi^\Phi(\alpha_T^k(u) i_T(1,h)) f_s(\gamma u t(1,h))\psi_k(u)\,du\,dh. \tag{5.5.11}$$

We next unfold the theta series. The general linear group GL_n acts on Mat_n by right multiplication. For $\xi \in \mathrm{Mat}_n(F)$ denote $G_\xi = \{g \in \mathrm{GL}_n : \xi g = \xi\}$ for its stabilizer. Then

$$I = \sum_{\xi \in \mathrm{Mat}_n(F)/\mathrm{GL}_n(F)} I_\xi, \tag{5.5.12}$$

where

$$I_\xi = \int_{P_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h) \times \sum_{a \in G_\xi \backslash \mathrm{GL}_n(F)} \omega_\psi(\alpha_T^k(u) i_T(1,h))\Phi(\xi a) f_s(\gamma u t(1,h))\psi_k(u)\,du\,dh. \tag{5.5.13}$$

Lemma 5.5.4. $I_\xi = 0$ *unless* $\xi = 1_n$.

Proof. We can pick ξ of the form $\xi = \begin{bmatrix} 0 & x \\ 0 & 1_r \end{bmatrix}$ so that $G_\xi = \left\{ \begin{bmatrix} * & * \\ 0 & 1_r \end{bmatrix} \in \mathrm{GL}_n(F) \right\}$. Recall that GL_n is embedded in Sp_{2n} via $a \mapsto m(a) = \mathrm{diag}[a, \hat{a}]$. Denote $\widetilde{G}_\xi$ for its image in Sp_{2n}. Clearly the representatives of $G_\xi \backslash \mathrm{GL}_n(F)$ can be taken in $\mathrm{SL}_n(F)$ so that

$$\omega_\psi(\alpha_T^k(u) i_T(1,h))\Phi(\xi a) = \omega_\psi(\alpha_T^k(u) i_T(1, m(a)h))\Phi(\xi)$$

for $a \in G_\xi \backslash \mathrm{GL}_n(F)$. Changing variables $u \mapsto t(1, m(a))^{-1} u t(1, m(a))$ we obtain

$$I_\xi = \int_{\widetilde{G}_\xi N_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h) \times \omega_\psi(\alpha_T^k(u) i_T(1,h))\Phi(\xi) f_s(\gamma u t(1,h))\psi_k(u)\,du\,dh.$$

Let N_n^r be a normal subgroup of $\widetilde{G}_\xi N_n$ consisting of elements of the form

$$\begin{bmatrix} 1_{n-r} & x & y & z \\ 0 & 1_r & 0 & y^* \\ 0 & 0 & 1_r & -x^* \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix}$$

and write

$$I_\xi=\int_{\widetilde{G}_\xi N_n(F)N_n^r(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})}\int_{N_n^r(F)\backslash N_n^r(\mathbb{A})}\phi(nh)$$
$$\times\omega_\psi(\alpha_T^k(u)i_T(1,nh))\Phi(\xi)f_s(\gamma ut(1,nh))\psi_k(u)dndudh.$$

Changing variables $u\mapsto t(1,n)ut(1,n)^{-1}$ and noting that $\gamma t(1,n)\gamma^{-1}\in N_n(F)$, we have

$$I_\xi=\int_{\widetilde{G}_\xi N_n(F)N_n^r(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})}\int_{N_n^r(F)\backslash N_n^r(\mathbb{A})}\phi(nh)$$
$$\times\ \omega_\psi(i_T(1,n)\alpha_T^k(u)i_T(1,h))\Phi(\xi)f_s(\gamma ut(1,h))\psi_k(u)dndudh.$$

Using formulas of the Weil representation, we have

$$\begin{aligned}
&\omega_\psi(i_T(1,n)\alpha_T^k(u)i_T(1,h))\Phi(\xi)\\
=&\omega_\psi\left(\alpha_k^T\left(u^0\left(\begin{bmatrix}0&x\\0&0\end{bmatrix},0,0\right)\right)i_T(1,n)\alpha_T^k(u)i_T(1,h)\right)\Phi\left(\begin{bmatrix}0&0\\0&1_r\end{bmatrix}\right)\\
=&\omega_\psi\left(i_T(1,n)\alpha_T^k\left(u^0\left(\begin{bmatrix}0&x\\0&0\end{bmatrix},\begin{bmatrix}0&x'\\0&0\end{bmatrix},0\right)\right)\alpha_T^k(u)i_T(1,h)\right)\Phi\left(\begin{bmatrix}0&0\\0&1_r\end{bmatrix}\right)\\
=&\omega_\psi\left(\alpha_T^k\left(u^0\left(0,\begin{bmatrix}0&x'\\0&0\end{bmatrix},0\right)\right)\alpha_T^k(u)i_T(1,h)\right)\Phi(\xi).
\end{aligned}$$

Changing variables $u\mapsto u^0\left(0,\begin{bmatrix}0&x'\\0&0\end{bmatrix},0\right)^{-1}u$ and since $u^0\left(0,\begin{bmatrix}0&x'\\0&0\end{bmatrix},0\right)\in N(\mathbb{A})$ the integral becomes

$$I_\xi=\int_{\widetilde{G}_\xi N_n(F)N_n^r(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})}\int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})}\omega_\psi(\alpha_T^k(v)i_T(1,h))\Phi(\xi)f_s(\gamma ut(1,h))\psi_k(u)$$
$$\times\int_{N_n^r(F)\backslash N_n^r(\mathbb{A})}\phi(nh)dndudh.$$

The lemma follows as the integral in the last line vanishes by the cuspidality of ϕ. $\square$

We then assume $\xi=1_n$ and omit it from our notation. It remains to consider the integral

$$\begin{aligned}
I =& \int_{N_n(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi(h) \\
&\times \omega_\psi(\alpha_T^k(u) i_T(1,h))\Phi(1_n) f_s(\gamma u t(1,h))\psi_k(u) du dh \\
&= \int_{N_n(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \int_{N_n(F)\backslash N_n(\mathbb{A})} \phi(nh) \\
&\times \omega_\psi(\alpha_T^k(u) i_T(1,nh))\Phi(1_n) f_s(\gamma u t(1,nh)) dn \psi_k(u) du dh.
\end{aligned} \tag{5.5.14}$$

Changing variables $u \mapsto t(1,n)ut(1,n)^{-1}$ and noting that

$$\omega_\psi(i_T(1,n(z))\alpha_T^k(u) i_T(1,h)) = \psi(\mathrm{tr}(Tz))\omega_\psi(\alpha_T^k(u) i_T(1,h)), \tag{5.5.15}$$

we have

$$\begin{aligned}
I =& \int_{N_n(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(F)\backslash N_{n^{k-1},kn}(\mathbb{A})} \int_{N_n(F)\backslash N_n(\mathbb{A})} \phi_{\psi,T}(h) \\
&\times \omega_\psi(\alpha_T^k(u) i_T(1,h))\Phi(1_n) f_s(\gamma u t(1,h))\psi_k(u) du dh \\
&= \int_{N_n(\mathbb{A})\backslash \mathrm{Sp}_{2n}(\mathbb{A})} \int_{N(\mathbb{A})\backslash N_{n^{k-1},kn}(\mathbb{A})} \phi_{\psi,T}(h) \times \int_{N(F)\backslash N(\mathbb{A})} \\
&\omega_\psi(\alpha_T^k(u_0)\alpha_T^k(u) i_T(1,h))\Phi(1_n) f_s(\gamma u_0 u t(1,h))\psi_k(u_0) du_0 du dh.
\end{aligned} \tag{5.5.16}$$

By straightforward computation, $N = N_{0,\dots,0}$ consists of elements of the form

$$\begin{bmatrix}
1_n & v_{1,2} & * & * & 0 & * & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & * & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 1_n & v_{k-2,k-1} & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_n & 0 & y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_n & 0 & y^* & * & * & * \\
0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1_n & -v_{k-2,k-1}^* & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & \ddots & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -v_{1,2}^* \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n
\end{bmatrix}. \tag{5.5.17}$$

Denote elements of the above form by $u_0(y)$. The integral of u_0 over $N(F)\backslash N(\mathbb{A})$

can be written as

$$\begin{aligned}
&\int_{N(F)\backslash N(\mathbb{A})} \omega_\psi(\alpha_T^k(u_0(y)u)i_T(1,h))\Phi(1_n) \\
&\quad \times f_s(\gamma u_0(y)vt(1,h))\psi_k(u_0(y))du_0(y) \\
=&\omega_\psi(\alpha_T^k(v)i_T(1,h))\Phi(1_n) \\
&\quad \times \int_{N(F)\backslash N(\mathbb{A})} f_s(\gamma u_0(y)ut(1,h))\psi_k(u_0(y))\psi(\mathrm{tr}(2Ty))du_0(y).
\end{aligned} \tag{5.5.18}$$

Let

$$\eta_0 = \begin{bmatrix} 0 & 0 & 1_n & 0 & 0 & 0 \\ 0 & \iddots & 0 & 0 & 0 & 0 \\ 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n \\ 0 & 0 & 0 & 0 & \iddots & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 \end{bmatrix}, \eta = \begin{bmatrix} 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & -1_{(k-1)n} \\ 1_{(k-1)n} & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 \end{bmatrix}. \tag{5.5.19}$$

For $u_0 \in N$ as in (5.5.17), note that $\eta_0\gamma u_0$ can be written as

$$\begin{bmatrix}
1_n & -y^* & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 1_n & -v^*_{k-2,k-1} & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & \ddots & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_n & -v^*_{1,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1_n & v_{1,2} & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & v_{k-2,k-1} & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n
\end{bmatrix}\eta. \tag{5.5.20}$$

Denote above matrix as u_0' so that $\eta_0\gamma u_0 = u_0'\eta$. Therefore, the integral over

$N(F)\backslash N(\mathbb{A})$ becomes

$$\begin{aligned}&\int_{N(F)\backslash N(\mathbb{A})} f_s(u_0'(y)\eta u t(1,h))\psi_k(u_0(y))\psi(\mathrm{tr}(2Ty))du_0(y)\\ =&\int_{U_{n^k}(F)\backslash U_{n^k}(\mathbb{A})} f_s\left(\begin{bmatrix} v & 0\\ 0 & \hat{v}\end{bmatrix}\eta u t(1,h)\right)\psi_{2T}^{-1}(v)dv.\end{aligned} \tag{5.5.21}$$

Here v is of the form

$$\begin{bmatrix} 1_n & v_{1,2} & * & * & *\\ 0 & \ddots & \ddots & * & *\\ 0 & 0 & 1_n & v_{k-2,k-1} & 0\\ 0 & 0 & 0 & 1_n & y\\ 0 & 0 & 0 & 0 & 1_n\end{bmatrix} \tag{5.5.22}$$

and

$$\psi_{2T}^{-1}(v)=\psi\left(\mathrm{tr}(2Ty)+\sum_{i=1}^{k-2}\mathrm{tr}(2Tv_{i,i+1})\right). \tag{5.5.23}$$

This completes the proof of Proposition 5.4.1.

5.6 The unramified computation

In the rest of this chapter, we study the local zeta integral corresponding to the integral $\mathcal{Z}(\phi,\theta^{\Phi}_{\psi,n^2},f_s)$ at $v\notin S$. For simplicity, we omit the symbol v. Therefore, let F be a non-archimedean local field with the ring of integers $\mathcal{O}$. Fix a nontrivial additive unramified character ψ of F and fix T_0,T,χ_T as before where χ_T is a quadratic character on $F^\times$. Let $\Phi^0=\mathbf{1}_{\mathrm{Mat}_n(\mathcal{O}_F)}$ be the characteristic function of $\mathrm{Mat}_n(\mathcal{O}_F)$ and

$$f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}\in \mathrm{Ind}_{P_{kn}(F)}^{\mathrm{Sp}_{2kn}(F)}(\mathcal{W}(\tau_v\otimes\chi_T,n,\psi_{2T})|\det\cdot|^s) \tag{5.6.1}$$

the unramified section normalized such that

$$f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(1_{2kn})=d_\tau^{\mathrm{Sp}_{4kn}}(s). \tag{5.6.2}$$

Let (π,V_π) be an irreducible admissible unramified representation of $\mathrm{Sp}_{2n}(F)$ with a fixed non-zero unramified vector $v_0 \in V_\pi$. Let (τ, V_τ) be an irreducible unramified principle series representation

$$\tau = \mathrm{Ind}_{B_{\mathrm{GL}_k}(F)}^{\mathrm{GL}_k(F)}(\chi_1 \otimes ... \otimes \chi_k) \tag{5.6.3}$$

where $\chi_1, ..., \chi_k$ are unramified quasi-characters of $F^\times$. Hence for any positive integer c,

$$\Delta(\tau \otimes \chi_T, c) = \mathrm{Ind}_{P_{c^k}(F)}^{\mathrm{GL}_{ck}(F)}(\chi_1\chi_T \circ \det \otimes ... \otimes \chi_k\chi_T \circ \det). \tag{5.6.4}$$

The aim of this section is to prove Theorem 5.4.2 utilizing the unramified local integrals (5.3.15) from the generalized doubling method. The idea of the computation is similar to the one in [Yan23] except when $k > 2$ we need to deal with bigger matrices.

5.6.1 Relation between unramified sections

Recall that for any character $\psi : U_{n^k}(F) \to \mathbb{C}^\times$, the model $\mathcal{W}(\tau \otimes \chi_T, n, \psi)$ consists of functions $W_\xi : \mathrm{GL}_{kn}(F) \to \mathbb{C}$ of the form

$$W_\xi(g) = \Lambda(\Delta(\tau \otimes \chi_T, n)(g)\xi) \tag{5.6.5}$$

where ξ is in the space of $\Delta(\tau \otimes \chi_T, n)$ and Λ can be realized as

$$\xi \mapsto \int_{U_{n^k}(F)} \xi(w_{k,n}u)\psi^{-1}(u)du, \quad w_{k,n} = \begin{bmatrix} 0 & 0 & 0 & 1_n \\ 0 & 0 & 1_n & 0 \\ 0 & \iddots & 0 & 0 \\ 1_n & 0 & 0 & 0 \end{bmatrix} \in \mathrm{GL}_{kn}. \tag{5.6.6}$$

By abusing the notation we denote the extension of our fixed character $\psi : F \to \mathbb{C}^\times$ to $U_{n^k}(F) \to \mathbb{C}^\times$ (see (5.2.15)) also as ψ. Recall that we have also defined a character ψ_{2T} on $U_{n^k}(F)$. Given $g_1, ..., g_k \in \mathrm{GL}_n(F)$, we define another character $\psi_{g_1,...,g_k}$ on $U_{n^k}(F)$ by

$$\psi_{g_1,...,g_k}(u) = \psi\left(\sum_{i=1}^{k-1} \mathrm{tr}(2g_iTg_{i+1}^{-1}u_{i,i+1})\right), \tag{5.6.7}$$

with u of the form in (5.2.14). Let $W_\xi^{g_1,\dots,g_k}$ be the function in $\mathcal{W}(\tau\otimes\chi_T,n,\psi_{g_1,\dots,g_k})$ corresponding to $\xi\in\Delta(\tau\otimes\chi_T,n)$, the linear functional Λ, and the character $\psi_{g_1,\dots,g_k}$ as above. Then $\psi_{2T}=\psi_{1,\dots,1}$ and we simply denote $W_\xi=W_\xi^{1,\dots,1}$.

Lemma 5.6.1. *If W_ξ is unramified then so is $W_\xi^{g_1,\dots,g_k}$. More precisely,*

$$\begin{aligned}&W_\xi\left(\mathrm{diag}[g_1,\dots,g_k]g\right)\\ =&\left|\frac{\det g_1}{\det g_k}\right|^n\prod_{i=1}^k\chi_i\chi_T(\det g_{k+1-i})|\det g_{k+1-i}|^{\frac{(k-2i+1)n}{2}}W_\xi^{g_1,\dots,g_k}(g).\end{aligned}\tag{5.6.8}$$

Proof. By the definition of W_ξ, we have

$$\begin{aligned}&W_\xi(\mathrm{diag}[g_1,\dots,g_k]g)\\ =&\int_{U_{n^k}(F)}\Delta(\tau\otimes\chi_T,n)(g)\xi\left(w_{k,n}u\mathrm{diag}[g_1,\dots,g_k]\right)\psi_{2T}^{-1}(u)du.\end{aligned}$$

Changing variables $u\mapsto\mathrm{diag}[g_1,..,g_k]u\mathrm{diag}[g_1,\dots,g_k]^{-1}$, we note that $u_{i,i+1}$ is changed to $g_iu_{i,i+1}g_{i+1}^{-1}$ and the above integral equals

$$\begin{aligned}&\left|\frac{\det g_1}{\det g_k}\right|^n\int_{U_{n^k}(F)}\Delta(\tau\otimes\chi_T,n)(g)\xi\left(\mathrm{diag}[g_k,\dots,g_1]w_{k,n}u\right)\psi_{g_1,\dots,g_k}^{-1}(u)du\\ =&\left|\frac{\det g_1}{\det g_k}\right|^n\prod_{i=1}^k\chi_i\chi_T(\det g_{k+1-i})|\det g_{k+1-i}|^{\frac{(k-2i+1)n}{2}}\\ \times&\int_{U_{n^k}(F)}\Delta(\tau\otimes\chi_T,n)(g)\xi\left(w_{k,n}u\right)\psi_{g_1,\dots,g_k}^{-1}(u)du\end{aligned}$$

as desired. □

Corollary 5.6.2. *For $g_1,\dots,g_k\in\mathrm{GL}_n(F)$, there exists an unramified section*

$$f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{g_1,\dots,g_k}),s}\in\mathrm{Ind}_{P_{2kn}(F)}^{\mathrm{Sp}_{kn}}(\mathcal{W}(\tau\otimes\chi_T,n,\psi_{g_1,\dots,g_k})|\det\cdot|^s)\tag{5.6.9}$$

determined by $f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}$ such that

$$\begin{aligned}&f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(m(\mathrm{diag}[g_1,\dots,g_k])g)\\ =&\left|\frac{\det g_1}{\det g_k}\right|^n\prod_{i=1}^k\chi_i\chi_T(\det g_{k+1-i})|\det g_{k+1-i}|^{s+(k-i)n+\frac{n+1}{2}}\\ &\times f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{g_1,\dots,g_k}),s}(g).\end{aligned}\tag{5.6.10}$$

In particular, if we take $g_i=(-2T)^{i-1}$ *then* $\psi_{g_1,\dots,g_k}=\psi^{-1}$ *and we have*

$$f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(m(\mathrm{diag}[g_1,...,g_k])g)=f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}(g). \tag{5.6.11}$$

5.6.2 Reformulating the unramified integral from the generalized doubling method

Let $v_0\in V_\pi$ be an unramified vector and l_T any linear functional on V_π satisfying (5.4.13). It follows from the unramified local zeta integral (5.3.15) ([GS21, Proposition 4.8]) of the generalized doubling method that we have

$$\begin{aligned}&\int_{\mathrm{Sp}_{2n}(F)}\int_{N^0_{(2n)^{k-1},2kn}(F)}\\&\qquad l_T(\pi(h)v_0)f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}(\delta u_0(1\times h))\psi^{-1}_{N_{(2n)^{k-1},2kn}}(u_0)du_0dh\\=&L(s+\frac{1}{2},\pi\times\tau)\cdot l_T(v_0).\end{aligned}\tag{5.6.12}$$

Our strategy of proving Theorem 5.4.2 is to compare our unramified local integral with the integral in above equation. The aim of this subsection is to reformulate above integral in the following simpler form

$$\int_{\mathrm{GL}_n(F)\cap\mathrm{Mat}_n(\mathcal{O}_F)}l_T(\pi(m(a))v_0)|\det a|^{-2n-1}\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(m(a))da, \tag{5.6.13}$$

where $\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})$ defined in (5.6.23) is a function on $\mathrm{Sp}_{2n}(F)$. In particular, we showed that $\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})$ is an unramified section of $\mathrm{Ind}^{\mathrm{Sp}_{2n}(F)}_{P_n(F)}(\chi_1(\det(\cdot))|\det\cdot|^{s+\frac{(2k-1)n}{2}})$ whose value at 1_{2n} is $d^{\mathrm{Sp}_{4kn}}_\tau(s)$.

By the Iwasawa decomposition of $\mathrm{Sp}_{2n}(F)$, we consider

$$\begin{aligned}&\int_{\mathrm{GL}_n(F)}\int_{\mathrm{Mat}^0_n(F)}l_T(\pi(m(a))v_0)\psi^{-1}(\mathrm{tr}(Tz))|\det a|^{-n-1}\\\times&\int_{N^0_{(2n)^{k-1},2kn}(F)}f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}(\delta u_0(1\times n(z)m(a)))\psi^{-1}_{N^0_{(2n)^{k-1},2kn}}(u_0)du_0dzda.\end{aligned}\tag{5.6.14}$$

The integral over $N^0_{(2n)^{k-1},2kn}(F)$ is

$$\int_{N^0_{(2n)^{k-1},2kn}(F)} f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}(\delta u_0\delta^{-1}\cdot\delta({}^\iota n(-z)\times m(a))\delta^{-1})\psi^{-1}_{N^0_{(2n)^{k-1},2kn}}(u_0)du_0. \tag{5.6.15}$$

Recall that $N^0_{(2n)^{k-1},2kn}$ contains elements of the form

$$\begin{bmatrix} 1_{2n(k-2)} & 0 & 0 & -a_1 & -a_2 & 0 & 0 & b_1 & b_2 & c \\ 0 & 1_n & 0 & -x_1 & -x_2 & 0 & 0 & z_1 & z_2 & b_2^* \\ 0 & 0 & 1_n & -x_3 & -x_4 & 0 & 0 & z_3 & z_1^* & b_1^* \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n & 0 & x_4^* & x_2^* & a_2^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & x_3^* & x_1^* & a_1^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2n(k-2)} \end{bmatrix} \tag{5.6.16}$$

and the character $\psi^{-1}_{N^0_{(2n)^{k-1},2kn}}$ is defined by

$$\psi^{-1}_{N^0_{(2n)^{k-1},2kn}}(u_0) = \psi(\mathrm{tr}(x_1)), \tag{5.6.17}$$

for u_0 as in (5.6.16). Conjugate it by δ and denote $N^{0,\delta}_{(2n)^{k-1},2kn} = \delta N^0_{(2n)^{k-1},2kn}\delta^{-1}$. It contains elements of the form

$$\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{2n(k-2)} & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & b_1 & b_2 & c & 1_{2n(k-2)} & 0 & 0 & 0 & 0 \\ x_1 & x_2 & z_1 & z_2 & b_2^* & 0 & 1_n & 0 & 0 & 0 \\ x_3 & x_4 & z_3 & z_1^* & b_1^* & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & x_4^* & x_2^* & a_2^* & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & x_3^* & x_1^* & a_1^* & 0 & 0 & 0 & 0 & 1_n \end{bmatrix} \tag{5.6.18}$$

and for u as in (5.6.18) we define

$$\psi_{N^{0,\delta}_{(2n)^{k-1},2kn}}(u)=\psi(\mathrm{tr}(x_1)). \tag{5.6.19}$$

We rewrite integral (5.6.15) as

$$\int_{N^{0,\delta}_{(2n)^{k-1},2kn}(F)} \psi^{-1}_{N^{0,\delta}_{(2n)^{k-1},2kn}}(u_0)$$
$$\times f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(u_0 \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1_{4n(k-1)} & 0 & 0 \\ 0 & 0 & 0 & \hat{a} & 0 \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix} \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & \tilde{z} & 0 & 0 \\ a^*-1 & 0 & 0 & 1_n & 0 \\ -z & a-1 & 0 & 0 & 1_n \end{bmatrix}\right) du_0. \tag{5.6.20}$$

Here we denote $\tilde{z}=\mathrm{diag}[{}^{\iota}n(z),...,{}^{\iota}n(z),{}^{\iota}n(-z),...,{}^{\iota}n(-z)]$ with ${}^{\iota}n(z)$ and ${}^{\iota}n(-z)$ appearing $k-1$ times respectively.

Lemma 5.6.3. *The integral* (5.6.20) *vanishes unless* $a \in \mathrm{GL}_n(F) \cap \mathrm{Mat}_n(\mathcal{O}_F)$.

Proof. We translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by

$$\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{4n(k-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n & 0 & -r^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}$$

for $r \in \mathrm{Mat}_n(\mathcal{O}_F)$.

Then we conjugate the above matrix to the left and change variables in u_0 to obtain the matrix

$$\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & -arz & ar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{4n(k-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n & 0 & -r^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & (arz)^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}.$$

This contributes $\psi^{-1}(\mathrm{tr}(ar))$ and the changing of variables in u_0 contributes $\psi^{-1}(\mathrm{tr}(ar))$. Hence the integral vanishes unless $\psi^{-1}(\mathrm{tr}(2ar)) = 1$ for all $r \in \mathrm{Mat}_n(\mathcal{O}_F)$, which implies $a \in \mathrm{Mat}_n(\mathcal{O}_F)$. $\square$

The integral (5.6.14) then equals

$$
\begin{aligned}
&\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} \int_{\mathrm{Mat}_n^0(F)} \int_{N^{0,\delta}_{(2n)^{k-1},2kn}(F)} l_T(\pi(m(a))v_0)|\det a|^{-n-1} \\
&\times f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(u_0 \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1_{4n(k-1)} & 0 & 0 \\ 0 & 0 & 0 & \hat{a} & 0 \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix} \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & \tilde{z} & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 \\ -z & 0 & 0 & 0 & 1_n \end{bmatrix} \right) \\
&\times \psi^{-1}(\mathrm{tr}(Tz))\psi^{-1}_{N^{0,\delta}_{(2n)^{k-1},2kn}}(u_0)du_0dzda.
\end{aligned}
\tag{5.6.21}
$$

Lemma 5.6.4. *The integral (5.6.21) vanishes unless* $z \in \mathrm{Mat}_n^0(\mathcal{O}_F)$.

Proof. We translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by

$$
\begin{bmatrix}
1_n & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\
0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{4n(k-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 & -r^* \\
0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n
\end{bmatrix}
$$

for $r \in \mathrm{Mat}_n(\mathcal{O}_F)$.

Conjugating it to the left and changing variables we obtain

$$\begin{bmatrix} 1_n & 0 & -rz & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{4n(k-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 & -r^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & (rz)^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}.$$

which contributes a character $\psi(\mathrm{tr}(rz))$. □

Then (5.6.21) becomes

$$\int_{\mathrm{GL}_n(F)\cap\mathrm{Mat}_n(\mathcal{O}_F)} \int_{\mathrm{Mat}_n^0(F)} \int_{N^{0,\delta}_{(2n)^{k-1},2kn}(F)} l_T(\pi(m(a))v_0)|\det a|^{-n-1}$$
$$\times f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(u_0\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1_{4n(k-1)} & 0 & 0 \\ 0 & 0 & 0 & \hat{a} & 0 \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix}\right)\psi^{-1}_{N^{0,\delta}_{(2n)^{k-1},2kn}}(u_0)du_0dzda. \tag{5.6.22}$$

Now we write u_0 as the form (5.6.18) in variables $a_1, a_2, b_1, b_2, c, x_1, x_2, x_3, x_4, z_2, z_2, z_3$.

Lemma 5.6.5. *The inner integral in* (5.6.22) *vanishes unless* $x_2a \in \mathrm{Mat}_n(\mathcal{O}_F)$.

Proof. We translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the left by

$$\mathrm{diag}\left[n'(ar), ..., n'(ar), \widehat{n'(ar)}, ..., \widehat{n'(ar)}\right]$$

for $r \in \mathrm{Mat}_n(\mathcal{O}_F)$ with $n'(ar) = \begin{bmatrix} 1_n & 0 \\ ar & 1_n \end{bmatrix}$ appearing k times. This is invariant by Proposition 5.2.4. Conjugating it to the right and making a change of variables we obtain a matrix in $\mathrm{Sp}_{2kn}(\mathcal{O}_F)$ and the change of variables produces a factor $\psi(\mathrm{tr}(x_2ar))$. $\square$

By changing variables, we can take x_2 out of the integrand. For $g = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \in \mathrm{Sp}_{2n}(F)$, we define

$$\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(g) = \int_{u_0} \psi(\mathrm{tr}(x_1)) \times f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(u_0 \begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & g_2 & 0 \\ 0 & 0 & 1_{4n(k-1)} & 0 & 0 \\ 0 & g_3 & 0 & g_4 & 0 \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix}\right) du_0 \tag{5.6.23}$$

where the integral is taking over $u_0 \in N^{0,\delta}_{(2n)^{k-1},2kn}$ with $x_2 = 0$. Then (5.6.22) can be written as

$$\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} l_T(\pi(m(a))v_0)|\det a|^{-2n-1}\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(m(a))da. \tag{5.6.24}$$

Lemma 5.6.6. *Write*

$$\tau = \mathrm{Ind}^{\mathrm{GL}_k(F)}_{B_{\mathrm{GL}_k}(F)}(\chi_1 \otimes ... \otimes \chi_k),$$

for unramified quasi-characters $\chi_1, ..., \chi_k$ *of* $F^\times$*, so that*

$$\Delta(\tau, 2n) = \mathrm{Ind}^{\mathrm{GL}_{2kn}(F)}_{P_{(2n)^k}(F)}(\chi_1 \circ \det \otimes ... \otimes \chi_k \circ \det).$$

Then $\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})$ *is an unramified section of*

$$\mathrm{Ind}^{\mathrm{Sp}_{2n}(F)}_{P_n(F)}(\chi_1(\det(\cdot))|\det\cdot|^{s+\frac{(2k-1)n}{2}}).$$

Proof. Clearly $\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})$ is left invariant under $N_n(F)$ and right invariant under $\mathrm{Sp}_{2n}(\mathcal{O}_F)$. Take $g = m(a)$ and conjugate it to the left. We obtain $|\det a|^{-n(k-1)}$ from the changing of variables in u_0 and

$$\chi_1(\det a)|\det a|^{(k-1)n}|\det a|^{s+\frac{2kn+1}{2}}$$

from the section $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$. Indeed, for any $h \in \mathrm{Sp}_{2kn}(F)$,

$$\begin{aligned}
&f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1_{4n(k-1)} & 0 & 0 \\ 0 & 0 & 0 & \hat{a} & 0 \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix} h\right) \\
=&|\det a|^{s+\frac{2kn+1}{2}}\Delta(\tau,2n)\left(\begin{bmatrix} 1_n & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1_{2n(k-1)} \end{bmatrix}\right) f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}(h) \\
=&|\det a|^{s+\frac{2kn+1}{2}}|\det(a)|^{(k-1)n}\chi_1(\det a) f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}(h).
\end{aligned}$$

□

Lemma 5.6.7. *We have*

$$\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(1_{2n}) = d_\tau^{\mathrm{Sp}_{4kn}}(s). \tag{5.6.25}$$

Proof. We calculate

$$\begin{aligned}
\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(1_{2n}) =& \int_{u_0} \psi(\mathrm{tr}(x_1)) \\
&\times f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(\begin{bmatrix}
1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{2n(k-2)} & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & b_1 & b_2 & c & 1_{2n(k-2)} & 0 & 0 & 0 & 0 \\
x_1 & 0 & z_1 & z_2 & b_2^* & 0 & 1_n & 0 & 0 & 0 \\
x_3 & x_4 & z_3 & z_1^* & b_1^* & 0 & 0 & 1_n & 0 & 0 \\
0 & 0 & x_4^* & 0 & a_2^* & 0 & 0 & 0 & 1_n & 0 \\
0 & 0 & x_3^* & x_1^* & a_1^* & 0 & 0 & 0 & 0 & 1_n
\end{bmatrix}\right) du_0
\end{aligned}$$

in the following five steps. The first three steps of computations are similar to the ones in [Yan23] for $k = 2$, and the last two steps are needed for $k > 2$.

(1) We translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by

$$\begin{bmatrix} 1_n & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{4n(k-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_n & r^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & r^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}$$

and conjugate it to the left. We get that x_3 is supported in $\mathrm{Mat}_n(\mathcal{O}_F)$.

(2) Let

$$g_0 = \begin{bmatrix} 1_i & 0 & 0 & 0 \\ r & 1_{n-i} & 0 & 0 \\ 0 & 0 & 1_{n-i} & 0 \\ 0 & 0 & -r^* & 1_i \end{bmatrix}$$

with r has entries in $\mathcal{O}_F$. Write

$$x_1 = \begin{bmatrix} x_{11} & x_{12} \\ x_{13} & x_{14} \end{bmatrix}, x_{11} \in \mathrm{Mat}_i, x_{14} \in \mathrm{Mat}_{n-i}.$$

We translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the left by $\mathrm{diag}[g_0, ..., g_0, \hat{g}_0, ..., \hat{g}_0]$ with k copies of g_0 and conjugate it to the right. We obtain that entries of x_{12} are supported in $\mathcal{O}_F$. Similarly, translating $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by

$$g_0 = \begin{bmatrix} 1_i & r & 0 & 0 \\ 0 & 1_{n-i} & 0 & 0 \\ 0 & 0 & 1_{n-i} & -r^* \\ 0 & 0 & 0 & 1_i \end{bmatrix}$$

and take $1 \leq i \leq n-1$, we obtain that x_1 is supported in $\mathrm{Mat}_n(\mathcal{O}_F)$.

(3) We translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by

$$\begin{bmatrix} 1_n & 0 & 0 & r & 0 \\ 0 & 1_n & 0 & 0 & r^* \\ 0 & 0 & 1_{4n(k-1)} & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 1_n \end{bmatrix}$$

for $r \in \mathrm{Mat}_n(\mathcal{O}_F)$ and conjugate it to the left. This shows that x_4 is supported in $\mathrm{Mat}_n(\mathcal{O}_F)$.

We conclude that in this stage we have

$$\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(1_{2n})$$
$$= \int_{u_0} f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}\left(\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{2n(k-2)} & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & b_1 & b_2 & c & 1_{2n(k-2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & z_2 & b_2^* & 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & z_3 & z_1^* & b_1^* & 0 & 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2^* & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & a_1^* & 0 & 0 & 0 & 0 & 1_n \end{bmatrix}\right) du_0.$$

We re-denote the unipotent elements in the integrand in the following form

$$\begin{bmatrix} 1_{2n} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{2n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{2n(k-2)} & 0 & 0 & 0 \\ a & b & c & 1_{2n(k-2)} & 0 & 0 \\ 0 & z & b' & 0 & 1_{2n} & 0 \\ 0 & 0 & a' & 0 & 0 & 1_{2n} \end{bmatrix}, a = \begin{bmatrix} a_{k-2} \\ \vdots \\ a_1 \end{bmatrix}, b = \begin{bmatrix} b_{k-2} \\ \vdots \\ b_1 \end{bmatrix}.$$

The following two additional steps are used to show that in the case $k > 2$ the integral vanishes unless all these entries are in $\mathcal{O}_F$.

(4) For each $i = 0, 1, ..., k-2$, we denote a matrix

$$g_i = \begin{bmatrix} 1_{2n} & 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 1_{2ni} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{2n} & 0 & 0 & 0 & 0 & r^* \\ 0 & 0 & 0 & 1_{2n(k-2-i)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{2n(k-2-i)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{2n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_{2ni} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2n} \end{bmatrix}$$

with $r \in \mathrm{Mat}_{2n}(\mathcal{O}_F)$. Translating $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by g_0 and conjugating it to the left show that a_1 and z are supported in $\mathrm{Mat}_{2n}(\mathcal{O}_F)$. Then translating $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by g_i for $i = 1, ..., k-3$ (in this order) and conjugating it to the left show that a_{i+1} and b_i are supported in $\mathrm{Mat}_{2n}(\mathcal{O}_F)$. Finally translating $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by g_{k-2} shows that b_{k-2} is supported in $\mathrm{Mat}_{2n}(\mathcal{O}_F)$.

(5) We finally use the same process as in Step (4) to show that $c \in \mathrm{Mat}_{2n(k-2)}(\mathcal{O}_F)$. For each $j = 1, 2, ..., k-2$ and $i = 0, 1, ..., k-2-j$, we use the matrix

$$g_{ji} = \begin{bmatrix} 1_{2nj} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{2n} & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 1_{2ni} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{2n} & 0 & 0 & 0 & 0 & r^* & 0 \\ 0 & 0 & 0 & 0 & 1_{2n(k-2-i-j)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{2n(k-2-i-j)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_{2n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2ni} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2n} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2nj} \end{bmatrix}$$

with $r \in \mathrm{Mat}_{2n}(\mathcal{O}_F)$. For each fixed $j = 1, 2, ..., k-2$ (in this order) we translate $f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}$ on the right by g_{ji} for $i = 0, ..., k-2-j$ (in this order) and we conjugate it to the left. This shows that the entries of the j-th column (viewed in $2n \times 2n$ blocks) of c are supported in $\mathcal{O}_F$ and thus c is supported in $\mathrm{Mat}_{2n(k-2)}(\mathcal{O}_F)$.

Therefore, we conclude that

$$\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(1_{2n}) = f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s}(1_{4kn}) = d_\tau^{\mathrm{Sp}_{4kn}}(s)$$

as desired. $\square$

5.6.3 Proof of Theorem 5.4.2

Recall that

$$\begin{aligned}\mathcal{Z}^*(l_T, s) = &\int_{N_n(F)\backslash \mathrm{Sp}_{2n}(F)} \int_{N^0_{n^{k-1},kn}(F)} l_T(\pi(h)v_0) \\ &\times \omega_\psi(\alpha^k_T(u) i_T(1,h))\Phi^0(1_n) f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta u t(1,h)) du dh.\end{aligned} \tag{5.6.26}$$

To finish the proof of Theorem 5.4.2, it suffices to show that

$$\begin{aligned}&\mathcal{Z}^*(l_T, s) \\ = &\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} l_T(\pi(m(a))v_0) |\det a|^{-2n-1} \lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(m(a)) da.\end{aligned} \tag{5.6.27}$$

We will show that our integral (5.6.26) can be written in the simpler form

$$\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3}{2}n-1} \lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(m(a)) da, \tag{5.6.28}$$

where $\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})$ defined in (5.6.37) is a function on $\mathrm{Sp}_n(F)$. In particular, we show that $\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})$ is an unramified section of $\mathrm{Ind}^{\mathrm{Sp}_{2n}(F)}_{P_n(F)}(\chi_1\chi_T(\det\cdot)|\det\cdot|^{s+n})$ whose value at 1_{2n} is $d^{\mathrm{Sp}_{4kn}}_\tau(s)$. Then two equations (5.6.26) and (5.6.27) can be compared by comparing two unramified sections $\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})$ and $\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})$.

By the Iwasawa decomposition decomposition of $\mathrm{Sp}_{2n}(F)$, we have

$$\begin{aligned}\mathcal{Z}^*(l_T, s) = &\int_{\mathrm{GL}_n(F)} \int_{N^0_{n^{k-1},kn}(F)} l_T(\pi(m(a))v_0) \\ &\times \omega_\psi(\alpha^k_T(u) i_T(1,m(a)))\Phi^0(1_n) \\ &\times f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta u t(1,m(a)))|\det a|^{-n-1} du da.\end{aligned} \tag{5.6.29}$$

Changing variables $u \mapsto t(1,m(a))ut(1,m(a))$ and using the formulas of the Weil representation, we obtain that $\mathcal{Z}^*(l_T, s)$ is equal to

$$\begin{aligned}&\int_{\mathrm{GL}_n(F)} \int_{N^0_{n^{k-1},kn}(F)} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3n}{2}-1} \\ &\times\omega_\psi(\alpha^k_T(u))\Phi^0(a) f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta t(1,m(a))u) du da.\end{aligned} \tag{5.6.30}$$

We write $u = u(x,0,z)$ as in (5.2.4) and (5.2.5). Then the inner integral over

$N^0_{n^{k-1},kn}(F)$ in (5.6.30) is

$$\begin{aligned}
&\int_{N^0_{n^{k-1},kn}(F)} \psi(\mathrm{tr}(Tz))\Phi^0(a+x)f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta t(1,m(a))u(x,0,z))du\\
=&\int_{N^0_{n^{k-1},kn}(F)} \psi(\mathrm{tr}(Tz))\Phi^0(x)f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta t(1,m(a))u(x-a,0,z))du\\
=&\int_{N^0_{n^{k-1},kn}(F)} \psi(\mathrm{tr}(Tz))f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta t(1,m(a))u(-a,0,z))du.
\end{aligned} \tag{5.6.31}$$

Thus integral (5.6.30) becomes

$$\begin{aligned}
&\int_{\mathrm{GL}_n(F)}\int_{N^{0,a}_{n^{k-1},kn}(F)} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3n}{2}-1}\\
&\times f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}(\eta t(1,m(a))u(-a,0,z))\psi(\mathrm{tr}(Tz))duda.
\end{aligned} \tag{5.6.32}$$

Here, $N^{0,a}_{n^{k-1},kn}(F)$ is the subgroup of $N^0_{n^{k-1},kn}(F)$ containing elements of the form

$$\begin{bmatrix}
1_{(k-2)n} & 0 & -b & 0 & -c & d\\
0 & 1_n & -a & 0 & z & c^*\\
0 & 0 & 1_n & 0 & 0 & 0\\
0 & 0 & 0 & 1_n & a^* & b^*\\
0 & 0 & 0 & 0 & 1_n & 0\\
0 & 0 & 0 & 0 & 0 & 1_n
\end{bmatrix}. \tag{5.6.33}$$

With $u \in N^{0,a}_{n^{k-1},kn}(F)$ of the above form, we write (5.6.32) as

$$\begin{aligned}
&\int_{\mathrm{GL}_n(F)}\int_{N^{0,a}_{n^{k-1},kn}(F)} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3n}{2}-1}\psi^{-1}(\mathrm{tr}(Tz))\\
&\times f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi_{2T}),s}\left(\begin{bmatrix} a & 0 & 0\\ 0 & 1_{2n(k-1)} & 0\\ 0 & 0 & \hat{a}\end{bmatrix}\begin{bmatrix}
1_n & 0 & 0 & 0 & 0 & 0\\
0 & 1_n & 0 & 0 & 0 & 0\\
0 & 0 & 1_{(k-2)n} & 0 & 0 & 0\\
b & c & d & 1_{(k-2)n} & 0 & 0\\
-a & z & c^* & 0 & 1_n & 0\\
0 & -a^* & b^* & 0 & 0 & 1_n
\end{bmatrix}\right)duda.
\end{aligned} \tag{5.6.34}$$

Applying Corollary 5.6.2 we see that the above integral is equal to

$$\int_{\mathrm{GL}_n(F)}\int_{N^{0,a}_{n^{k-1},kn}(F)} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3n}{2}-1}\psi^{-1}(\mathrm{tr}(4T^2z))$$
$$\times f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}\left(\begin{bmatrix} a & 0 & 0\\ 0 & 1_{2n(k-1)} & 0\\ 0 & 0 & \hat{a}\end{bmatrix}\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0\\ 0 & 1_n & 0 & 0 & 0 & 0\\ 0 & 0 & 1_{(k-2)n} & 0 & 0 & 0\\ b & c & d & 1_{(k-2)n} & 0 & 0\\ -a & z & c^* & 0 & 1_n & 0\\ 0 & -a^* & b^* & 0 & 0 & 1_n\end{bmatrix}\right) duda. \tag{5.6.35}$$

Lemma 5.6.8. *The above inner integral vanishes unless* $a\in\mathrm{Mat}_n(\mathcal{O}_F)$.

Proof. Translate $f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}$ by $u^0(r,0,0)$ for $r\in\mathrm{Mat}_n(\mathcal{O}_F)$ on the right and conjugate it to the left. □

Using the above lemma, the integral (5.6.32) becomes

$$\int_{\mathrm{GL}_n(F)\cap\mathrm{Mat}_n(\mathcal{O}_F)}\int_{b,c,d,z} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3n}{2}-1}\psi^{-1}(\mathrm{tr}(4T^2z))$$
$$\times f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}\left(\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0\\ 0 & 1_n & 0 & 0 & 0 & 0\\ 0 & 0 & 1_{(k-2)n} & 0 & 0 & 0\\ b & c & d & 1_{(k-2)n} & 0 & 0\\ 0 & z & c^* & 0 & 1_n & 0\\ 0 & 0 & b^* & 0 & 0 & 1_n\end{bmatrix}\begin{bmatrix} a & 0 & 0\\ 0 & 1_{2n(k-1)} & 0\\ 0 & 0 & \hat{a}\end{bmatrix}\right) duda. \tag{5.6.36}$$

For $g = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \in \mathrm{Sp}_{2n}(F)$, we define

$$\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(g) = \int_{b,c,d,z} \psi^{-1}(\mathrm{tr}(4T^2 z))\times$$
$$f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}\left(\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{(k-2)n} & 0 & 0 & 0 \\ b & c & d & 1_{(k-2)n} & 0 & 0 \\ 0 & z & c^* & 0 & 1_n & 0 \\ 0 & 0 & b^* & 0 & 0 & 1_n \end{bmatrix}\begin{bmatrix} g_1 & 0 & g_2 \\ 0 & 1_{2n(k-1)} & 0 \\ g_3 & 0 & g_4 \end{bmatrix}\right) du. \tag{5.6.37}$$

We can further write (5.6.32) as

$$\begin{aligned}\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} & l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3}{2}n-1} \\ & \times\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(m(a))da.\end{aligned} \tag{5.6.38}$$

Lemma 5.6.9. *Write*

$$\tau = \mathrm{Ind}^{\mathrm{GL}_k(F)}_{B_{\mathrm{GL}_k}(F)}(\chi_1\otimes ...\otimes\chi_k),$$

for unramified quasi-characters $\chi_1, ..., \chi_k$ *of* $F^\times$*, so that*

$$\Delta(\tau\otimes\chi_T, n) = \mathrm{Ind}^{\mathrm{GL}_{kn}(F)}_{P_{n^k}(F)}(\chi_1\chi_T\circ\det\otimes ...\otimes\chi_k\chi_T\circ\det).$$

Then $\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})$ *is an unramified section of*

$$\mathrm{Ind}^{\mathrm{Sp}_{2n}(F)}_{P_n(F)}(\chi_1\chi_T(\det(\cdot))|\det\cdot|^{s+n}).$$

Proof. The proof is similar to the one of Lemma 5.6.6. Clearly $\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})$ is left invariant under $N_n(F)$ and right invariant under $\mathrm{Sp}_{2n}(\mathcal{O}_F)$. Take $g=m(a)$ and conjugate it to the left. We obtain $|\det a|^{-n(k-2)}$ from the change of variables in u and

$$\chi_1\chi_T(\det a)|\det a|^{\frac{(k-1)n}{2}}|\det a|^{s+\frac{kn+1}{2}}$$

from the section $f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}$. Indeed, for any $h\in\mathrm{Sp}_{\mathrm{kn}}(F)$

$$\begin{aligned}
&f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}\left(\begin{bmatrix} a & 0 & 0\\ 0 & 1_{2n(k-1)} & 0\\ 0 & 0 & \hat{a}\end{bmatrix}h\right)\\
=&|\det a|^{s+\frac{kn+1}{2}}\Delta(\tau\otimes\chi_T,n)\left(\begin{bmatrix} a & 0\\ 0 & 1_{n(k-1)}\end{bmatrix}\right)f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}(h)\\
=&|\det a|^{s+\frac{kn+1}{2}}|\det a|^{\frac{(k-1)n}{2}}\chi_1\chi_T(\det a)f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}(h).
\end{aligned}$$

□

Moreover, we have the following.

Lemma 5.6.10.

$$\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(1_{2n})=d_\tau^{\mathrm{Sp}_{4kn}}(s). \tag{5.6.39}$$

Proof. We need to calculate

$$\begin{aligned}
\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(g)=&\int_{b,c,d,z}\psi^{-1}(\mathrm{tr}(4T^2z))\times\\
&f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}\left(\begin{bmatrix} 1_n & 0 & 0 & 0 & 0 & 0\\ 0 & 1_n & 0 & 0 & 0 & 0\\ 0 & 0 & 1_{(k-2)n} & 0 & 0 & 0\\ b & c & d & 1_{(k-2)n} & 0 & 0\\ 0 & z & c^* & 0 & 1_n & 0\\ 0 & 0 & b^* & 0 & 0 & 1_n\end{bmatrix}\right)du.
\end{aligned}$$

By the same argument as Steps (4-5) in the proof of Lemma 5.6.7 we can show that the integral vanishes unless all entries of b,c,d,z are in $\mathcal{O}_F$. Indeed, for each

$j = 0, 1, 2, ..., k-2$ and $i = 0, 1, ..., k-2-j$, consider the matrix

$$g_{ji} = \begin{bmatrix} 1_{nj} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 1_{ni} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_n & 0 & 0 & 0 & 0 & r^* & 0 \\ 0 & 0 & 0 & 0 & 1_{n(k-2-i-j)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{n(k-2-i-j)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{ni} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{nj} \end{bmatrix}$$

with $r \in \mathrm{Mat}_n(\mathcal{O}_F)$. For each fixed $j = 0, 1, 2, ..., k-2$ (in this order) we translate $f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}$ on the right by g_{ji} for $i = 0, ..., k-2-j$ (in this order) and conjugate it to the left. This gives the desired result and hence

$$\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(1_{2n}) = f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s}(1_{2kn}) = d_\tau^{\mathrm{Sp}_{4kn}}(s).$$

$\square$

We now conclude our computations as follows. Comparing Lemma (5.6.6) with Lemma (5.6.9) we have

$$\lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(m(a)) = |\det a|^{-\frac{n}{2}}\chi_T(\det a)\lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(m(a)). \tag{5.6.40}$$

Comparing (5.6.22) with (5.6.38) we have

$$\begin{aligned}
&\mathcal{Z}^*(l_T, s)\\
=&\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} l_T(\pi(m(a))v_0)\chi_T(\det a)|\det a|^{-\frac{3}{2}n-1}\\
&\qquad \times \lambda(f^0_{\mathcal{W}(\tau\otimes\chi_T,n,\psi^{-1}),s})(m(a))da\\
=&\int_{\mathrm{GL}_n(F)\cap \mathrm{Mat}_n(\mathcal{O}_F)} l_T(\pi(m(a))v_0)|\det a|^{-2n-1}\\
&\qquad \times \lambda(f^0_{\mathcal{W}(\tau,2n,\psi^{-1}),s})(m(a))da\\
=&L(s+\frac{1}{2},\pi\times\tau)\cdot l_T(v_0).
\end{aligned} \tag{5.6.41}$$

This completes the proof of Theorem 5.4.2.

Appendix A

L-function for Maass Forms on General Linear Groups

In this appendix, we study the L-function for general linear groups which is not covered in previous chapters. For a cuspidal automorphic representation of GL_n, its L-function can be defined via an integral representation constructed by Godement and Jacquet in [GJ72; Jac79]. See also [GH11; GJ21] for a summary. A Rankin-Selberg type integral representation for $\mathrm{GL}_n \times \mathrm{GL}_m$ is also provided in [Cog04].

We restrict ourselves to Maass forms (defined in Definition A.1.1 and Definition A.1.3 following [Gol15; GH11]) and we define the L-function more classically as a Dirichlet series of Hecke eigenvalues (1.1.17). The aim of this appendix (Theorem A.2.2) is to present an integral representation of a certain L-function via the doubling method following [Haz22] and [PR87]. The unfolding of the global integral (1.2.9) and the unramified computations are already done in [Haz22]. Our contribution in this appendix is to make the choice of each local section of the Eisenstein series and calculate the local integrals explicitly at all places (including ramified and archimedean cases). In particular, our choice of local sections is related to the Godement-Jacquet construction as in [PR87, Proposition 3.2] and is also inspired by [Hum21; Lin18].

A.1 Maass forms on general linear groups

We fix the following general notations throughout the appendix. For an associative ring R with identity, denote by $\mathrm{Mat}_{m,n}(R)$ the R-module of all $m \times n$ matrices with entries in R. Set $\mathrm{Mat}_n(R) = \mathrm{Mat}_{n,n}(R)$ and $\mathrm{GL}_n(R) = \mathrm{Mat}_n(R)^\times$. For $x \in \mathrm{Mat}_{m,n}(R)$, denote ${}^t x$ for its transpose. Denote by 1_n and 0_n, or even 1 and 0 if their sizes are clear from the context, for the identity and zero matrix in $\mathrm{Mat}_n(R)$.

We fix our base field to be $\mathbb{Q}$ and denote $\mathbb{A}$ to be the adele ring. For a place v, either corresponding to a prime p or the archimedean place ∞, denote $\mathbb{Q}_v$ to be the localization and write $\mathbb{Z}_p$ for the ring of integers of $\mathbb{Q}_p$. Write $\mathbb{A} = \mathbb{A}_\mathrm{f} \cdot \mathbb{R}$ with $\mathbb{A}_\mathrm{f}$ the ring of finite adeles. For a general linear group GL_n, we mean a $\mathbb{Q}$-algebraic group whose R-points is $\mathrm{GL}_n(R)$ for any $\mathbb{Q}$-algebra R. We also write SL_n for the special linear group containing elements of GL_n of determinant one and PGL_n is the projective linear group defined by GL_n modulo its center Z_n. Denote $\mathrm{O}_n := \{g \in \mathrm{GL}_n : {}^t g g = 1_n\}$ for the orthogonal group.

A.1.1 Definition of Maass forms

We start by reviewing the definition of Maass forms on GL_n, both classically and adelically, following [Gol15] and [GH11]. In [Gol15], the Fourier expansions, Hecke operators and L-functions are studied for Maass forms of full level. We will consider the Maass forms with any level. For more general automorphic forms of GL_n, the reader can refer to [GH11].

We shall always assume $n \geq 2$. The generalized upper half plane is defined to be

$$\mathfrak{h}^n = \{z = x \cdot y \in \mathrm{Mat}_n(\mathbb{R})\}, \tag{1.1.1}$$

where x, y are of the form

$$x = \begin{bmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 0 & 1 & x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & x_{n-1,n} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; y = \begin{bmatrix} y_1 y_2 \ldots y_{n-1} & 0 & 0 & 0 & 0 \\ 0 & y_1 y_2 \ldots y_{n-2} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \le i < j \le n$ and $0 < y_i \in \mathbb{R}$ for $1 \le i \le n-1$. The left invariant measure on $\mathfrak{h}^n$ is given by

$$\begin{aligned} \mathbf{d}z &= \mathbf{d}x \cdot \mathbf{d}y, \\ \mathbf{d}x &= \prod_{1 \le i < j \le n} dx_{i,j}, \qquad \mathbf{d}y = \prod_{i=1}^{n-1} y_i^{-i(n-i)-1} dy_i. \end{aligned} \tag{1.1.2}$$

By the Iwasawa decomposition, every element $g \in \mathrm{GL}_n(\mathbb{R})$ can be written as $g = \tilde{g} \cdot d \cdot k$ with $\tilde{g} \in \mathfrak{h}^n, k \in \mathrm{O}_n(\mathbb{R})$ and $d \in Z_n(\mathbb{R})$. Take $g \in \mathrm{GL}_n(\mathbb{R})$ and $z \in \mathfrak{h}^n$, we have $gz = \widetilde{gz} \cdot \kappa(g, z) \cdot d$ for uniquely determined $\widetilde{gz}$ and $\kappa(g, z) \in \mathrm{O}_n(\mathbb{R}), d \in Z_n(\mathbb{R})$. We then define the action of $g \in \mathrm{GL}_n(\mathbb{R})$ on $z \in \mathfrak{h}^n$ by setting $g.z := \widetilde{gz}$.

Denote $\mathfrak{g}_n = \mathfrak{gl}_n(\mathbb{C})$ and $U(\mathfrak{g}_n)$ the universal enveloping algebra of $\mathfrak{g}_n$ which is identified with the space of invariant differential operators as in [Gol15, Chapter 2]. Denote $Z(U(\mathfrak{g}_n))$ for the center of the universal enveloping algebra. For $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, we define a function $I_\nu : \mathfrak{h}^n \to \mathbb{C}$ by

$$I_\nu(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij}\nu_j}, \qquad b_{ij} = \begin{cases} ij & i + j \le n, \\ (n-i)(n-j) & i + j > n. \end{cases} \tag{1.1.3}$$

These are eigenfunctions for all invariant differential operators $D \in Z(U(\mathfrak{g}_n))$ ([Gol15, Section 2.4]). The type ν Harish-Chandra character

$$\lambda_\nu : Z(U(\mathfrak{g})) \to \mathbb{C} \tag{1.1.4}$$

is defined such that $DI_\nu = \lambda_\nu(D) I_\nu$.

For a fixed positive integer $\mathfrak{n}$, we define a congruence subgroup

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{bmatrix} A & B \\ C & d \end{bmatrix} \in \mathrm{SL}_n(\mathbb{Z}) : \begin{array}{cc} A \in \mathrm{Mat}_{n-1}(\mathbb{Z}) & B \in \mathrm{Mat}_{n-1,1}(\mathbb{Z}) \\ C \in \mathrm{Mat}_{1,n-1}(\mathfrak{n}\mathbb{Z}) & d \in \mathbb{Z} \end{array} \right\}. \quad (1.1.5)$$

Definition A.1.1. Fix a positive integer $\mathfrak{n}$ and $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$. A **Maass form** of level $\mathfrak{n}$, type ν, is a smooth function $f : \mathfrak{h}^n \to \mathbb{C}$ satisfying:

(1) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma_0(\mathfrak{n})$ and $z \in \mathfrak{h}^n$,

(2) $Df = \lambda_\nu f$ for all $D \in Z(U(\mathfrak{g}_n))$,

(3) f is of moderate growth in the sense of [GH11, Definition 12.3.10],

(4) $\int_{\Gamma_0(\mathfrak{n})\backslash\mathfrak{h}^n} |f(z)|^2 \mathbf{d}z < \infty$.

We further call f a **(Maass) cusp form** if

$$\int_{(\Gamma_0(\mathfrak{n})\cap U(\mathbb{R}))\backslash U(\mathbb{R})} f(uz)du = 0, \qquad \text{for any } z \in \mathfrak{h}^n \quad (1.1.6)$$

for any unipotent radical U of any proper parabolic subgroup P of GL_n. We denote the space of Maass forms as $M_\nu(\mathfrak{n})$ and the subspace of cusp forms by $S_\nu(\mathfrak{n})$.

Remark A.1.2. We note that in most works, including [Gol15], the term 'Maass form' actually means the Maass cusp form defined above.

We now rephrase the definition of Maass forms in the adelic language. For a fixed positive integer $\mathfrak{n} = \prod_p p^{\mathfrak{n}_p}$, we define an open compact subgroup $K_0(\mathfrak{n}) \subset \mathrm{GL}_n(\mathbb{A}_\mathrm{f})$ as an adelic analogue of $\Gamma_0(\mathfrak{n})$ as follows.

$$\begin{aligned} K_0(\mathfrak{n}) &= \prod_p K_p(p^{\mathfrak{n}_p}), \\ K_p(p^{\mathfrak{n}_p}) &= \mathrm{GL}_n(\mathbb{Z}_p) \cap \begin{bmatrix} \mathrm{Mat}_{n-1}(\mathbb{Z}_p) & \mathrm{Mat}_{n-1,1}(\mathbb{Z}_p) \\ \mathrm{Mat}_{1,n-1}(p^{\mathfrak{n}_p}\mathbb{Z}_p) & \mathbb{Z}_p^\times \end{bmatrix}. \end{aligned} \quad (1.1.7)$$

Recall that by the strong approximation of GL_n, we have

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{Q}) \cdot K_0(\mathfrak{n}) \cdot \mathrm{GL}_n(\mathbb{R}). \quad (1.1.8)$$

Definition A.1.3. Fix a positive integer $\mathfrak{n}$ and $\nu = (\nu_1, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$. An (adelic) **Maass form** of level $\mathfrak{n}$, type ν is a smooth function $\boldsymbol{f} : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$

such that $f(z) := \boldsymbol{f}(g_\infty) \in M_\nu(\mathfrak{n})$ is a Maass form defined in Definition A.1.1 for $z = g_\infty.1 \in \mathfrak{h}^n$. Here, for $g \in \mathrm{GL}_n(\mathbb{A})$, we write $g = g_f \cdot g_\infty$ with $g_f \in \mathrm{GL}_n(\mathbb{A}_\mathrm{f})$ and $g_\infty \in \mathrm{GL}_n(\mathbb{R})$. In particular,

$$\boldsymbol{f}(\gamma g k k_\infty) = \boldsymbol{f}(g) \tag{1.1.9}$$

for any $g \in \mathrm{GL}_n(\mathbb{A})$ and $\gamma \in \mathrm{GL}_n(\mathbb{Q}), k \in K_0(\mathfrak{n}), k_\infty \in \mathrm{O}_n(\mathbb{R})$. We further call $\boldsymbol{f}$ a **(Maass) cusp form** if

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \boldsymbol{f}(ug)du = 0 \tag{1.1.10}$$

for any unipotent radical U of any proper parabolic subgroup P of GL_n. We denote the space of such automorphic forms as $\mathcal{M}_\nu(\mathfrak{n})$ and the subspace of cusp forms by $\mathcal{S}_\nu(\mathfrak{n})$.

From the above definition, we have a map $\mathcal{M}_\nu(\mathfrak{n}) \to M_\nu(\mathfrak{n}), \boldsymbol{f} \mapsto f$. Conversely, for $f \in M_\nu(\mathfrak{n})$ we define its adelic lift

$$\boldsymbol{f}(g) := f(g_\infty.1) \tag{1.1.11}$$

for $g = \gamma g_\infty k$ with $\gamma \in \mathrm{GL}_n(\mathbb{Q}), g_\infty \in \mathrm{GL}_n(\mathbb{R}), k \in K_0(\mathfrak{n})$. One checks that there are bijections

$$\mathcal{M}_\nu(\mathfrak{n}) \cong M_\nu(\mathfrak{n}), \qquad \mathcal{S}_\nu(\mathfrak{n}) \cong S_\nu(\mathfrak{n}), \qquad \boldsymbol{f} \leftrightarrow f. \tag{1.1.12}$$

A.1.2 Hecke operators and L-functions

We are now going to define the action of Hecke operators on Maass forms and the L-function for Maass forms. For positive integers $e_1 \geq ... \geq e_n \geq 0$, we consider the double coset

$$K^p_{e_1,\dots,e_n} = K_p(p^{\mathfrak{n}_p}) \begin{bmatrix} p^{e_1} & & & \\ & p^{e_2} & & \\ & & \ddots & \\ & & & p^{e_n} \end{bmatrix} K_p(p^{\mathfrak{n}_p}). \tag{1.1.13}$$

We always take $e_n = 0$ if $p|\mathfrak{n}$. We denote $[K^p_{e_1,\ldots,e_n}]$ to be the Hecke operator associated to the double coset $K^p_{e_1,\ldots,e_n}$ whose action on $\boldsymbol{f} \in \mathcal{M}_\nu(\mathfrak{n})$ is given by

$$\begin{aligned}\boldsymbol{f}|[K^p_{e_1,\ldots,e_n}](g) &:= \int_{K^p_{e_1,\ldots,e_n}} \boldsymbol{f}(gk)dk \\ &= \sum_{k \in K^p_{e_1,\ldots,e_n}/K_p(p^{\mathfrak{n}_p})} \boldsymbol{f}(gk).\end{aligned} \tag{1.1.14}$$

where the measure dk is normalized such that $K_p(p^{\mathfrak{n}_p})$ has volume 1. We call $\boldsymbol{f}$ an eigenform if it is an eigenfunction under the action of all these Hecke operators $[K^p_{e_1,\ldots,e_n}]$. That is

$$\boldsymbol{f}|[K^p_{e_1,\ldots,e_n}] = \lambda(\boldsymbol{f}; K^p_{e_1,\ldots,e_n})\boldsymbol{f} \tag{1.1.15}$$

for some $\lambda(\boldsymbol{f}; K^p_{e_1,\ldots,e_n}) \in \mathbb{C}^\times$.

Let $0 \neq \boldsymbol{f} \in \mathcal{S}_\nu(\mathfrak{n})$ be an eigenform. We always assume $\boldsymbol{f}$ is normalized such that $\lambda(\boldsymbol{f}; K^p_{0,\ldots,0}) = 1$. For a positive integer $\mathfrak{m} = \prod_p p^{\mathfrak{m}_p}$, define the Hecke operator

$$T(\mathfrak{m}) = \prod_{p|\mathfrak{m}} \sum_{\substack{e_1 \geq \ldots \geq e_n \geq 0 \\ e_1 + \ldots + e_n = \mathfrak{m}_p \\ e_n = 0 \text{ if } p|\mathfrak{n}}} [K^p_{e_1,\ldots,e_n}], \tag{1.1.16}$$

and denote the eigenvalue of $\boldsymbol{f}$ associated to $T(\mathfrak{m})$ by $\lambda(\mathfrak{m})$. We define the L-function as the Dirichlet series of these eigenvalues, generalizing the classical definition of L-function for modular forms on GL_2. That is, for an eigenform $\boldsymbol{f}$, we define

$$L(s, \boldsymbol{f}) = \sum_{\mathfrak{m} \geq 0} \lambda(\mathfrak{m})\mathfrak{m}^{-s}, \tag{1.1.17}$$

which has an Euler product expression

$$\begin{aligned}L(s, \boldsymbol{f}) &= \prod_p L_p(s, \boldsymbol{f}), \\ L_p(s, \boldsymbol{f}) &= \sum_{e_1 \geq \ldots \geq e_n \geq 0} \lambda(\boldsymbol{f}; K^p_{e_1,\ldots,e_n}) p^{-(e_1 + \ldots + e_n)s}.\end{aligned} \tag{1.1.18}$$

A.2 The doubling method and the integral representation

In this section, we review the doubling method for GL_n following [Haz22] (see also [PR87, Section 3]). We choose the local sections of the Eisenstein series and calculate the local integrals explicitly at all places.

A.2.1 The global integral and the main theorem

We closely follow the notations in [Haz22]. For two positive integers m, n, write $P_{m,n} \subset \mathrm{GL}_{m+n}$ be the parabolic subgroup containing elements of the form

$$q = \begin{bmatrix} A_q & B_q \\ 0 & D_q \end{bmatrix} \text{ with } A_q \in \mathrm{GL}_m,\ D_q \in \mathrm{GL}_n. \tag{1.2.1}$$

Denote $U_{m,n}$ be the unipotent radical of $P_{m,n}$. Let $\delta_{P_{m,n}}$ be the modulus character of $P_{m,n}(\mathbb{A})$ defined by

$$\delta_{P_{m,n}}(q) = |\det A_q|^n |\det D_q|^{-m} \tag{1.2.2}$$

for $q \in P_{m,n}(\mathbb{A})$ written in the form (1.2.1).

We define a doubling embedding

$$\tau : \mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{n^2}, \qquad (g_1, g_2) \mapsto g_1 \otimes g_2, \tag{1.2.3}$$

where for $A = (a_{ij}) \in \mathrm{Mat}_n$ and $B = (b_{ij}) \in \mathrm{Mat}_n$, we write $A \otimes B$ for the Kronecker product defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \cdots & \cdots & \cdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \in \mathrm{Mat}_{nn}. \tag{1.2.4}$$

Fix a positive integer $\mathfrak{n} = \prod_{p|\mathfrak{n}} p^{\mathfrak{n}_p}$ and let $\boldsymbol{f} \in \mathcal{S}_\nu(\mathfrak{n})$ be an eigenform on GL_n. Let

f_s be a smooth section of the normalized parabolic induction

$$\mathrm{Ind}_{P_{n^2-1,1}(\mathbb{A})}^{\mathrm{GL}_{n^2}(\mathbb{A})}\delta_{P_{n^2-1,1}}^{s-\frac{1}{2}}. \tag{1.2.5}$$

In particular,

$$f_s(qh) = \left(|\det q||D_q|^{-n^2}\right)^s f_s(h), \tag{1.2.6}$$

for $q \in P_{n^2-1,1}(\mathbb{A})$ written as in (1.2.1) and $h \in \mathrm{GL}_{n^2}(\mathbb{A})$. We form the Eisenstein series on $\mathrm{GL}_{n^2}(\mathbb{A})$ associated to f_s by

$$E(h; f_s) = \sum_{\gamma \in P_{n^2-1,1}(\mathbb{Q})\backslash \mathrm{GL}_{n^2}(\mathbb{Q})} f_s(\gamma h). \tag{1.2.7}$$

Let

$$w_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}_n(\mathbb{Q}), \tag{1.2.8}$$

be a Weyl element. For an element $g \in \mathrm{GL}_n$, we denote $g^* = w_n\, {}^t g^{-1} w_n$. The global integral considered in this appendix is

$$\mathcal{Z}(g_1; \boldsymbol{f}, f_s) = \int_{(Z_n(\mathbb{A})\mathrm{GL}_n(\mathbb{Q}))\backslash \mathrm{GL}_n(\mathbb{A})} \boldsymbol{f}(g_2) E(\tau(g_1^*, g_2 w_n); f_s) dg_2, \tag{1.2.9}$$

for any $g_1 \in \mathrm{GL}_n(\mathbb{A})$.

The following proposition follows from the main theorem of [Haz22] (see also [PR87, Section 9]).

Proposition A.2.1. *The global integral* (1.2.9) *unfolds to*

$$\mathcal{Z}(g_1; \boldsymbol{f}, f_s) = \int_{\mathrm{PGL}_n(\mathbb{A})} \boldsymbol{f}(g_1 g_2) f_s(\delta \cdot \tau(1, g_2 w_n)) dg_2, \tag{1.2.10}$$

which converges absolutely for $\mathrm{Re}(s)$ *sufficiently large and has a meromorphic continuation to* $\mathbb{C}$*. Here*

$$\delta = \begin{bmatrix} 1_{n-1} & 0 & 0 \\ 0 & 0 & 1_{(n-1)n} \\ 0 & 1 & \mathfrak{e} \end{bmatrix}, \qquad \mathfrak{e} = \begin{bmatrix} e_{n-1} & e_{n-2} & \cdots & e_1 \end{bmatrix}, \tag{1.2.11}$$

where, for $1 \leq j \leq n-1$*,* e_j *are row vectors with* 1 *in the* j*-th entry and zero elsewhere.*

The aim of this section is to calculate the integral (1.2.10) explicitly for a specific section f_s which we are going to describe. We call this section the Godement-Jacquet section, due to its relation with the Godement-Jacquet L-function as in [PR87, Proposition 3.2].

We identify Mat_n as a free module of rank n^2. A basis of Mat_n can be chosen as $\{e_{jk}\}_{\substack{1\leq j\leq n\\ 1\leq k\leq n}}$ where e_{jk} is the $n\times n$ matrix with 1 on the (j,k)-entry and 0 elsewhere. We label this basis as $\{\epsilon_i\}$ for $1\leq i\leq n^2$ such that $\epsilon_{n(j-1)+k} = e_{jk}$. Then the (right) action of GL_{n^2} on Mat_n can be described via this basis. In particular, for $x\in \mathrm{Mat}_n$ and $g_1, g_2 \in \mathrm{GL}_n$, the action of $\tau(g_1,g_2)$ on x is given by $x.\tau(g_1,g_2) = {}^tg_1\cdot x\cdot g_2$.
Let $x_0 = \begin{bmatrix} 0_{n-1,n-1} & 0 \\ 0 & 1\end{bmatrix}$ so that the parabolic subgroup $P_{n^2-1,1}\subset \mathrm{GL}_{n^2}$ is the subgroup fixing the one-dimensional submodule generated by x_0. Also note that, for δ as in (1.2.11), we have $x_0.\delta = w_n$. For a Bruhat-Schwartz function $\Phi\in\mathcal{S}(\mathrm{Mat}_n(\mathbb{A}))$ and $h\in \mathrm{GL}_{n^2}(\mathbb{A})$, we define a section

$$f_s^{\Phi} \in \mathrm{Ind}_{P_{n^2-1,1}(\mathbb{A})}^{\mathrm{GL}_{n^2}(\mathbb{A})}\delta_{P_{n^2-1,1}}^s, \tag{1.2.12}$$

by setting

$$f_s^{\Phi}(h) = |\det h|^s\int_{\mathbb{A}^\times}\Phi(a\cdot x_0.h)|a|^{n^2s}da. \tag{1.2.13}$$

One checks that for $q\in P_{n^2-1,1}(\mathbb{A})$ written as the form in (1.2.1), we have

$$\begin{aligned} f_s^{\Phi}(qh) &= |\det qh|^s\int_{\mathbb{A}^\times}\Phi(aD_q\cdot x_0.h)|a|^{n^2s}da\\ &= |\det qh|^s\int_{\mathbb{A}^\times}\Phi(a\cdot x_0.h)|aD_q^{-1}|^{n^2s}da\\ &= |\det q|^s|D_q|^{-n^2s}f_s^{\Phi}(h;s).\end{aligned}$$

The global integral is related to the Godement-Jacquet construction in [PR87, Proposition 3.2] when one takes the section of the Eisenstein series as f_s^{Φ}. Our contribution in this appendix is to make an explicit choice of the Bruhat-Schwartz Φ and to calculate the local integrals at all places. In particular, our computations cover the ramified and archimedean cases.

We take the Bruhat-Schwartz function Φ to be

$$\Phi = \prod_{p\nmid\mathfrak{n}} \Phi_p^0 \cdot \prod_{p\mid\mathfrak{n}} \Phi_p^\dagger \cdot \Phi_\infty, \tag{1.2.14}$$

with $\Phi_p^0, \Phi_p^\dagger, \Phi_\infty$ defined in (1.2.17), (1.2.21) and (1.2.26). Combining the local computations in Proposition A.2.3, A.2.4 and A.2.5, we state our main theorem of the appendix in the following.

Theorem A.2.2. *Let $\boldsymbol{f} \in \mathcal{S}_\nu(\mathfrak{n})$ be an eigenform on GL_n. Take the section*

$$f_s^\Phi = \prod_{p\nmid\mathfrak{n}} f_{s,p}^0 \cdot \prod_{p\mid\mathfrak{n}} f_{s,p}^\dagger \cdot f_{s,\infty} \tag{1.2.15}$$

be the Godement-Jacquet section associated to Φ as in (1.2.14) with $f_{s,p}^0, f_{s,p}^\dagger, f_{s,\infty}$ defined in (1.2.18), (1.2.23), (1.2.27). Then, for any $g_1 \in \mathrm{GL}_n(\mathbb{A})$,

$$\mathcal{Z}(g_1; \boldsymbol{f}, f_s^\Phi) = \frac{\prod_{i=1}^{n-1} \Gamma\left(\frac{ns-i}{2}\right)}{2^{n-1}\pi^{\frac{n(n-1)\left(s-\frac{1}{2}\right)}{2}}} \cdot L(ns, \boldsymbol{f})\boldsymbol{f}(g_1). \tag{1.2.16}$$

A.2.2 Unramified nonarchimedean local integrals

Let p be a prime number such that $p \nmid \mathfrak{n}$. Define

$$\Phi_p^0(x) = \begin{cases} 1 & x \in \mathrm{Mat}_n(\mathbb{Z}_p), \\ 0 & x \notin \mathrm{Mat}_n(\mathbb{Z}_p), \end{cases} \tag{1.2.17}$$

and denote

$$f_{s,p}^0(h) = |\det h|^s \int_{\mathbb{Q}_p^\times} \Phi_p^0(a \cdot x_0.h)|a|^{n^2 s} da, \qquad x_0 = \begin{bmatrix} 0_{n-1,n-1} & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.2.18}$$

for the associated local section. Note that

$$f_{s,p}^0(1) = \int_{\mathbb{Q}_p^\times} \Phi_p^0(ax_0)|a|^{n^2 s} da = \sum_{i=0}^\infty p^{-in^2 s} = \zeta_p(n^2 s),$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$ is the local Euler factor of the Riemann zeta function. That is, $f^0_{s,p}$ is the local section such that

$$f^0_{s,p}(qk) = \left(|\det q||\det D_q|^{-n^2}\right)^s \zeta_p(n^2 s), \tag{1.2.19}$$

for $q \in P_{n^2-1,1}(\mathbb{Q}_p)$ written in the form (1.2.1) and $k \in \mathrm{GL}_n(\mathbb{Z}_p)$.

Proposition A.2.3. *Let p be a prime number such that $p \nmid \mathfrak{n}$ and $f^0_{s,p}$ the local section defined as in (1.2.18). Then, for any $g_0 \in \mathrm{GL}_n(\mathbb{A})$, we have*

$$\int_{\mathrm{PGL}_n(\mathbb{Q}_p)} \boldsymbol{f}(g_0 g) f^0_{s,p}(\delta \cdot \tau(1,g))dg = L_p(ns, \boldsymbol{f})\boldsymbol{f}(g_0). \tag{1.2.20}$$

Proof. By the Cartan decomposition, we can write

$$\mathrm{PGL}_n(\mathbb{Q}_p) = \coprod_{e_1 \geq \ldots \geq e_{n-1} \geq 0} K^p_{e_1,\ldots,e_{n-1},0},$$

where $K^p_{e_1,\ldots,e_n}$ is the double coset in (1.1.13).

Then

$$\begin{aligned}
&\int_{\mathrm{PGL}_n(\mathbb{Q}_p)} \boldsymbol{f}(g_0 g) f^0_{s,p}(\delta \cdot \tau(1,g))dg \\
&= \sum_{e_1 \geq \ldots \geq e_{n-1} \geq 0} \int_{K^p_{e_1,\ldots,e_{n-1},0}} \boldsymbol{f}(g_0 g) f^0_{s,p}(\delta \cdot \tau(1,g))dg \\
&= \sum_{e_1 \geq \ldots \geq e_{n-1} \geq 0} \int_{K^p_{e_1,\ldots,e_{n-1},0}} \boldsymbol{f}(g_0 g) p^{-n(e_1+\ldots+e_{n-1})s} \zeta_p(n^2 s)dg.
\end{aligned}$$

Note that by (1.1.18) and the definition of Hecke operators,

$$\begin{aligned}
L_p(s,\boldsymbol{f}) &= \sum_{e_1 \geq \ldots \geq e_n \geq 0} \lambda(\boldsymbol{f}; K_{e_1,\ldots,e_n}) p^{-(e_1+\ldots+e_n)s} \\
&= \sum_{e_1 \geq \ldots \geq e_n \geq 0} \lambda(\boldsymbol{f}; K_{e_1-e_n,\ldots,e_{n-1}-e_n,0}) p^{-(e_1+\ldots+e_n)s} \\
&= \sum_{e_n \geq 0} p^{-ne_n s} \cdot \sum_{e_1 \geq \ldots \geq e_{n-1} \geq 0} \lambda(\boldsymbol{f}; K_{e_1,\ldots,e_{n-1},0}) p^{-(e_1+\ldots+e_{n-1})s}.
\end{aligned}$$

The proposition then follows by comparing above two expressions. $\square$

A.2.3 Ramified nonarchimedean local integrals

Let p be a prime number such that $p|\mathfrak{n}$. Define

$$\Phi_p^\dagger(x) = \begin{cases} 1 & x \in \mathrm{Mat}_n(\mathbb{Z}_p),\ x_{1,1} \in \mathbb{Z}_p^\times,\ x_{1,2}, ..., x_{1,n} \in p^{\mathfrak{n}_p}\mathbb{Z}_p, \\ 0 & \text{otherwise}\ , \end{cases} \tag{1.2.21}$$

where we write

$$x = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \cdots & \cdots & \cdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} \in \mathrm{Mat}_n(\mathbb{Q}_p). \tag{1.2.22}$$

Denote

$$f_{s,p}^\dagger(h) = |\det h|^s \int_{\mathbb{Q}_p^\times} \Phi_p^\dagger(a \cdot x_0.h)|a|^{n^2 s} da. \tag{1.2.23}$$

for the associated local section. Denote the last row of $h \in \mathrm{GL}_{n^2}(\mathbb{Q}_p)$ be $\begin{bmatrix} h_1 & \cdots & h_{n^2} \end{bmatrix}$. Then $f_{s,p}^\dagger(h) \neq 0$ unless

$$h_1 \in \mathbb{Q}_p^\times,\ \text{and}$$
$$h_1^{-1}h_2, ..., h_1^{-1}h_n \in p^{\mathfrak{n}_p}\mathbb{Z}_p,\ \text{and}$$
$$h_1^{-1}h_{n+1}, ..., h_1^{-1}h_{n^2} \in \mathbb{Z}_p.$$

Hence, $f_{s,p}^\dagger$ is the local section supported on $P_{n^2-1,1}(\mathbb{Q}_p)w_{n^2}N_{n-1}$ with

$$f_{s,p}^\dagger(qw_{n^2}u) = \left(|\det q||\det D_q|^{-n^2}\right)^s, \tag{1.2.24}$$

for $q \in P_{n^2-1,1}(\mathbb{Q}_p)$ and $u \in N_{n-1}$. Here, $N_{n-1} \subset U_{1,n^2-1}(\mathbb{Z}_p)$ is a subgroup consisting of elements of the form

$$\begin{bmatrix} 1 & u_1 & \cdots & u_{n^2-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \qquad \begin{matrix} u_1, ..., u_{n-1} \in p^{\mathfrak{n}_p}\mathbb{Z}_p, \\ u_n, ..., u_{n^2-1} \in \mathbb{Z}_p \end{matrix}.$$

Proposition A.2.4. *Let p be a prime number such that $p|\mathfrak{n}$ and $f_{s,p}^\dagger$ the local section* *defined as in* (1.2.23). *Then, for any* $g_0 \in \mathrm{GL}_n(\mathbb{A})$, *we have*

$$\int_{\mathrm{PGL}_n(\mathbb{Q}_p)} \boldsymbol{f}(g_0 g) f^{\dagger}_{s,p}(\delta \cdot \tau(1, gw_n))dg = L_p(ns, \boldsymbol{f})\boldsymbol{f}(g_0). \tag{1.2.25}$$

Proof. For $g \in \mathrm{PGL}_n(\mathbb{Q}_p)$, we write

$$g = \begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ \cdots & \cdots & \cdots \\ g_{n,1} & \cdots & g_{n,n} \end{bmatrix}.$$

Then the last row of $\delta \cdot \tau(1, gw_n)$ is

$$\begin{bmatrix} g_{n,n} & \cdots & g_{n,1} & g_{n-1,n} & \cdots & g_{n-1,1} & \cdots & g_{1,n} & \cdots & g_{1,1} \end{bmatrix}.$$

As g is an element in $\mathrm{PGL}_n(\mathbb{Q}_p)$, we may take $g_{n,n} = 1$. Then $f^{\dagger}_{s,p}(\delta \cdot \tau(1, gw_n)) \neq 0$ unless

$$\begin{aligned} g_{i,j} &\in \mathbb{Z}_p \qquad \text{for } 1 \leq i \leq n-1, 1 \leq j \leq n, \\ g_{n,j} &\in p^{\mathfrak{n}_p}\mathbb{Z}_p \qquad \text{for } 1 \leq j \leq n-1. \end{aligned}$$

These conditions are equivalent to $g \in K_{e_1,\dots,e_{n-1},0}$ and for such g we have

$$f^{\dagger}_{s,p}(\delta \cdot \tau(1, gw_n)) = p^{-ns(e_1+\dots+e_{n-1})}.$$

Therefore,

$$\begin{aligned} &\int_{\mathrm{PGL}_n(\mathbb{Q}_p)} \boldsymbol{f}(g_0 g) f^{\dagger}_{s,p}(\delta \cdot \tau(1, gw_n))dg \\ =& \sum_{e_1 \geq \dots \geq e_{n-1} \geq 0} \int_{K_{e_1,\dots,e_{n-1},0}} \boldsymbol{f}(g_0 g)dg \cdot p^{-ns(e_1+\dots+e_{n-1})} \\ =& \sum_{e_1 \geq \dots \geq e_{n-1} \geq 0} \lambda(\boldsymbol{f}; K_{e_1,\dots,e_{n-1},0}) p^{-ns(e_1+\dots+e_{n-1})} \boldsymbol{f}(g_0) \\ =& L_p(ns, \boldsymbol{f})\boldsymbol{f}(g_0). \end{aligned}$$

□

A.2.4 Archimedean local integrals

Define

$$\Phi_\infty(x) = e^{-\pi \mathrm{tr}({}^t x x)}, \tag{1.2.26}$$

and

$$f_{s,\infty}(h) = |\det h|^s \int_{\mathbb{R}^\times} \Phi_\infty(a \cdot x_0.h)|a|^{n^2 s} da, \tag{1.2.27}$$

for the associated local section. Note that

$$f_{s,\infty}(1) = \int_{\mathbb{R}^\times} e^{-\pi a^2}|a|^{n^2 s} da = \pi^{-\frac{n^2 s}{2}} \Gamma\left(\frac{n^2 s}{2}\right).$$

Thus $f_{s,\infty}$ is the local section such that

$$f_{s,\infty}(qk) = \pi^{-\frac{n^2 s}{2}} \Gamma\left(\frac{n^2 s}{2}\right) \cdot \left(|\det q||\det D_q|^{-n^2}\right)^s \tag{1.2.28}$$

for $q \in P_{n-1,1}(\mathbb{R})$ written as the form of (1.2.1) and $k \in \mathrm{O}_{n^2}(\mathbb{R})$.

Proposition A.2.5. *Let $f_{s,\infty}$ be the local section defined as in (1.2.27). Then for any $g_0 \in \mathrm{GL}_n(\mathbb{A})$, we have*

$$\int_{\mathrm{PGL}_n(\mathbb{R})} \boldsymbol{f}(g_0 g) f_{s,\infty}(\delta \cdot \tau(1,g)) dg = \frac{\prod_{i=1}^{n-1} \Gamma\left(\frac{ns-i}{2}\right)}{2^{n-1}\pi^{\frac{n(n-1)\left(s-\frac{1}{2}\right)}{2}}} \cdot \boldsymbol{f}(g_0). \tag{1.2.29}$$

Proof. Using the Iwasawa decomposition and the definition of $f_{s,\infty}$, we have

$$\begin{aligned}
&\int_{\mathrm{PGL}_n(\mathbb{R})} \boldsymbol{f}(g_0 g) f_{s,\infty}(\delta \cdot \tau(1,g)) dg \\
=&\int_{\mathrm{PGL}_n(\mathbb{R})} \boldsymbol{f}(g_0 g)|\det g|^{ns} \int_{\mathbb{R}^\times} \Phi_\infty(a w_n z)|a|^{n^2 s} da dz \\
=&\int_{\mathrm{GL}_n(\mathbb{R})} \boldsymbol{f}(g_0 g)|\det g|^{ns} e^{-\pi \mathrm{tr}({}^t g g)} dg.
\end{aligned}$$

By [Shi00, Theorem A2.2], above integral equals

$$\int_{\mathfrak{h}^n} |\det z|^{ns} e^{-\pi \mathrm{tr}({}^t z z)} dz \cdot \boldsymbol{f}(g_0).$$

Write $z = x \cdot y$ as in (1.1.1), we need to calculate

$$\int_{\mathfrak{h}^n} e^{-\sum_{j=1}^{n-1} \pi (y_1 \ldots y_{n-j})^2 \left(1+\sum_{i=1}^{j-1} x_{i,j}^2\right)} \left(y_1^{n-1} y_2^{n-2} \ldots y_{n-1}\right)^{ns} \mathbf{d}x\mathbf{d}y.$$

Note that

$$\int_{-\infty}^{\infty} e^{-\pi (y_1 \ldots y_{n-j})^2 x_{i,j}^2} dx_{i,j} = (y_1 \ldots y_{n-j})^{-1}.$$

The integrals over all $x_{i,j}$ contribute

$$\prod_{j=1}^{n-1}(y_1...y_{n-j})^{1-j} = \prod_{i=1}^{n-1} y_i^{\frac{(1-n+i)(n-i)}{2}}.$$

It remains to calculate

$$\int_{y_i \in \mathbb{R}_{>0}} e^{-\sum_{j=1}^{n-1}\pi(y_1...y_{n-j})^2} \prod_{i=1}^{n-1} y_i^{\frac{(n-i)(2ns-n-i+1)}{2}-1} dy_i.$$

Note that for $1 \leq i \leq n-1$ and $c_i \in \mathbb{C}$,

$$\int_0^\infty e^{-\pi(y_1...y_{i-1})^2 y_i^2} y_i^{c_i} dy_i = \frac{1}{2\pi^{\frac{c_i+1}{2}}(y_1...y_{i-1})^{c_i+1}}\Gamma\left(\frac{c_i}{2}+1\right).$$

Our integral equals

$$\frac{1}{2^{n-1}\pi^{\frac{c_1+...+c_{n-1}+n-1}{2}}}\prod_{i=1}^{n-1}\Gamma\left(\frac{c_i+1}{2}\right),$$

where $c_{n-1} = ns - n$ and

$$c_i = \frac{(n-i)(2ns-n-i+1)}{2} - 1 - (c_{i+1}+1) - ... - (c_{n-1}+1).$$

It is not difficult to calculate that $c_i = ns - i - 1$ and the proposition follows. □

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